Singular theta lifts and near-central special values of Rankin-Selberg L-functions

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ABSTRACT

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In this thesis we study integrals of a product of two automorphic forms of weight 2 on a Shimura curve over $\mathbb{Q}$ against a function on the curve with logarithmic singularities at CM points obtained as a Borcherds lift. We prove a formula relating periods of this type to a near-central special value of a Rankin-Selberg L-function. The results provide evidence for Beilinson's conjectures on special values of L-functions.

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To my parents

Chapter 1

Introduction

1.1 Statement of the main theorem

Let X be a projective smooth variety over a number field F. Attached to it there are its motivic L-functions $L(H^k(X), s)$ for $0 \leq k \leq 2\dim(X)$; they are complex analytic functions defined by an Euler product converging on a right half plane $Re(s) > \frac{k}{2} + 1$. The relations between algebraic cycles on X and special values of L-functions form one of the deepest and most fascinating chapters of modern number theory.

We begin this introduction by stating the main result of this thesis (cf. Theorem 12). The rest of the introduction will be devoted to providing motivation and background for this result.

Let B be a non-split quaternion algebra over $\mathbb{Q}$ of odd discriminant $D(B)$; we assume that B is indefinite, so that $B \otimes \mathbb{R} \cong M_2(\mathbb{R})$ and fix a maximal Eichler order $R \subset B$. Let $H^0 = PB^\times$ and let $\pi_1 = \otimes'_v \pi_{1,v}, \pi_2 = \otimes'_v \pi_{2,v}$ be automorphic representations of $H^0(\mathbb{A})$ and assume the following:

- $\pi_1 \ncong \pi_2^\vee$,
- If p does not divide $D(B)$, then $\pi_{i,p}$ is unramified for $i = 1, 2$,
- If $p | D(B)$, then $\pi_1 \cong \pi_2 \cong 1_{H_v^0}$,

- The representations $\pi_{i,\infty}$ of $PGL_2(\mathbb{R})$ are discrete series of weight ± 2.

The reduced norm $n : B \to \mathbb{Q}$ endows $V = B$ with the structure of a quadratic vector space. Moreover $GSO(V) \cong \mathbb{G}_m \backslash (B^\times \times B^\times)$ so that $\pi_1 \boxtimes \pi_2$ is an irreducible automorphic representation of $GSO(V)$. We also denote by $\pi_1 \boxtimes \pi_2$ its unique extension to $GO(V)$ appearing in the local theta correspondence with GSp_4 (see [Roberts, 2001]).

Let B^0 the set of elements in B of reduced trace 0. The reduced norm $n : B \to \mathbb{Q}$ endows $V^0 = B^0$ with the structure of a quadratic vector space and we write $H = O(V^0)$ (note that then $SO(V) \cong H^0$). For an open compact subgroup $K \subset H(\mathbb{A}_f)$, we write:

$$X_K = H(\mathbb{Q}) \backslash H(\mathbb{A}) / K K_\infty$$

where $K_\infty \subset H(\mathbb{R})$ is a fixed maximal compact group.

Consider the even lattice $L = R \subset V^0$ and its dual $L^\vee \supset L$. Recall that the metaplectic group $\widetilde{SL_2(\mathbb{Z})}$ acts on the group ring $\mathbb{C}[L^\vee/L]$; we denote this representation by ρ_L. For any integer k, let $M^!_{k/2,\rho_L}$ denote the space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$ defined on the upper half plane $\mathbb{H}$ of weight $k/2$ and type ρ_L that are meromorphic at the cusp $i\infty$; such forms admit a Fourier expansion $f(q) = \sum_{n\in\mathbb{Q}} \sum_{\lambda \in L^\vee/L} a_{n,\lambda}(f) q^n \lambda$ with $a_{n,\lambda}(f) \in \mathbb{C}$. We denote by $S^!_{k/2,\rho_L,\mathbb{Z}} \subset M^!_{k/2,\rho_L}$ the subgroup defined by the conditions $a_{0,0}(f) = 0$ and $a_{n,\lambda}(f) \in \mathbb{Z}$ for $n \leq 0$. Let $K \subset H(\mathbb{A}_f)$ be open compact such that K preserves $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and acts trivially on $L^\vee/L$. In [Borcherds, 1998], Borcherds constructs a homomorphism

$$f^! \mapsto \Psi_{f^!} : S^!_{1-\frac{n}{2},\rho_L,\mathbb{Z}} \to \mathbb{C}(X_K)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

that intertwines the action of $H(\mathbb{A}_f)$ and such that $div(\Psi_{f^!})$ is supported on certain CM points on X_K; we refer to $\Psi_{f^!}$ as a Borcherds lift.

Let $K \subset H(\mathbb{A}_f)$ be the maximal compact subgroup of $H(\mathbb{A}_f)$ determined by the order R. Let $f_i \in \pi_i$ be holomorphic of weight 2 and level K so that

$$\theta_\varphi(f) = f_1 \boxtimes \overline{f_2}$$

for a certain cusp form $f \in L^2(Z(\mathbb{A})GSp_4(\mathbb{Q})\backslash GSp_4(\mathbb{A}))$ and a certain Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$ (see Section 2.6.4 for the details). We can now state our main theorem.

Theorem 1. *Under the hypothesis 5.2 we have:*

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} f_1(h)\overline{f_2(h)} \log|\Psi_{f^!}(h)|dh = q \cdot L'(\pi_1 \otimes \pi_2, 0) \cdot \lim_{(s,s')\to(0,\frac{3}{2})} I(s,s')$$

where $q \in \mathbb{Q}^\times$ *and*

$$I(s,s') = \sum_{m>0,\mu} a_{-m,\mu}(f^!) \frac{Vol(K_0(D))Vol(K_f)}{Vol(K_f \cap H_m(\mathbb{A}_f))} \cdot \frac{L(1,\eta_m)\zeta(s)}{j_{T(m)}(s)d(s)} I_{m,\infty}(s,s').$$

Here $L(\pi_1 \otimes \pi_2, s) = \prod_{p<\infty} L(\pi_{1,p} \otimes \pi_{2,p}, s)$ *and*

$$I_{m,\infty}(s,s')$$

$$= s \cdot \int_{H_m(\mathbb{R})\backslash H(\mathbb{R})} \phi_\infty(h^{-1}v_m, s') \int_{N(\mathbb{R})\backslash Sp_4(\mathbb{R})} l_{T_m}(f)(g)(\omega(g,h)\varphi_\infty)(v_1, v_{m'})|a(g)|^{s+\frac{1}{2}} dg dh.$$

1.2 Outline of the thesis

We now describe the contents of each Chapter of this thesis.

In the rest of Chapter 1, we set the context for the main result of this thesis by giving a brief description of (one of) Beilinson's conjectures on special values of L-functions. First we review the relevant definitions concerning motivic L-functions and motivic cohomology groups in a general setting and after that we explain what the conjectures predict for the Rankin-Selberg L-functions considered in this thesis. We then describe an explicit example of a class in a motivic cohomology group of a product of Shimura curves constructed using special divisors and meromorpic functions on them with divisors supported on CM points. The chapter ends by fixing the notations that we will use throughout the thesis.

Chapter 2 describes some basic facts about the theta correspondence for reductive dual pairs of the form $(Sp_4, O(V))$, where $O(V)$ is a quadratic vector space of dimension 4 and discriminant 1. The results in this chapter are crucial for the main result, as the computation of the integral in the main theorem relies on the fact that the cusp form in the integrand is a global theta lift from Sp_4. Hence we start the chapter by first reviewing the local correspondence for unramified representations, including the correspondence of local L-parameters. Then we discuss some known results about the correspondence for certain representations with Iwahori-fixed vectors. After that we address the archimedean correspondence and a result of Roberts ([Roberts, 1999]) that ensures the non-vanishing of the global theta lift. Finally, we discuss a specific choice of Schwartz functions at every place v ensuring that the theta lift has the right weight and level. We also state the version of the Siegel-Weil formula that we will need in the proof.

In Chapter 3 we give a very quick review of some of the key results in the theory of singular theta lifts. Namely, we first review the definition of weakly holomorphic modular forms valued in the Weil representation of a lattice. Then, after recalling the definition of Siegel theta series, we state the main results of Borcherds giving a lifting of weakly holomorphic cusp forms to meromorphic functions on orthogonal Shimura varieties. These results are formulated here in adelic language closely following [Kudla, 2003]. After that we review the work of Bruinier, which constitutes a generalization of Borcherds's work. Again we quote some key results of Bruinier that will be used in the proof of the theorem.

Chapter 4 starts with a brief review of the local doubling integrals introduced by Piatetskii-Shapiro and Rallis. Although the main integral does not directly unfold to an Euler product of local doubling integrals, its calculation reduces to them. We state the basic result in [Piatetski-Shapiro and Rallis, 1986] for unramified representations and then review the results of [Zorn, 2011] concerning doubling integrals for representations of $Sp_4(\mathbb{Q}_p)$ having Iwahori-fixed vectors. These results are then used in the second part of the chapter, where some local computations are performed that allow us to generalize the results in [Piatetski-Shapiro and Rallis, 1988] for unramified representations to a certain local ramified representation π_{ng} of $Sp_4(\mathbb{Q}_p)$. This will be used in the proof of the main theorem to compute the local integrals at primes dividing the discriminant of the quaternion algebra B.

Finally, Chapter 5 is simply a calculation leading to the formula that consitutes the main theorem of the thesis, using tools described and developed in the previous chapters.

1.3 Beilinson's conjectures

1.3.1 Motivic L-functions

We start by reviewing the definition and expected properties of motivic L-functions. We will be very brief as there are many excellent sources for this material, see e.g. [Ramakrishnan, 1989], [Rapoport *et al.*, 1988]. Our presentation follows [Rapoport *et al.*, 1988] closely.

Let X be a smooth projective variety over $\mathbb{Q}$. We denote by $X_{\overline{\mathbb{Q}}} = X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ its base change to $\overline{\mathbb{Q}}$. One has then a number of cohomology groups attached to it, namely:

- For any prime number l and integer k with $0 \leq k \leq 2\dim(X)$ one can define an étale cohomology group $H^k_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$. This is a finite dimensional $\mathbb{Q}_l$ vector space on which the group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts linearly.

- For $0 \leq k \leq 2\dim(X)$, one can consider $H^k_B(X^{an}_{\mathbb{C}}, \mathbb{Q})$, the singular cohomology group of the complex analytic manifold attached to $X_{\mathbb{C}}$. These groups are $\mathbb{Q}$-Hodge structures of weight k, that is, there is a Hodge decomposition:

$$H^k_B(X^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=k} H^{p,q}$$

  satisfying $H^{q,p} = \overline{H^{p,q}}$.

Choosing a Galois closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ yields an inclusion $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \subset Gal(\overline{\mathbb{Q}}/\mathbb{Q})$; a different choice of Galois closure will result in a conjugate embedding. Thus for every prime p there is a well defined conjugacy class of subgroups isomorphic to $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Denote by I_p the inertia subgroup of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and by Fr_p the Frobenius element at p (i.e., the topological generator of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)/I_p \cong \hat{\mathbb{Z}}$ defined by $Fr_p^{-1}(x) = x^p$) and define:

$$P_p(s) = \det(1 - Fr_p \cdot p^{-s} | H^k_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_p})$$

It is a deep fact (following from the Weil conjectures) that P_p is a polynomial in $\mathbb{Z}[p^{-s}]$ when X has good reduction at p (note that this is true for all but finitely many p); in fact this is conjectured to be true for all p. Assuming this, the local L-factor at p is then defined by:

$$L_p(H^k(X), s) = \frac{1}{P_p(s)}$$

and the L-function $L(H^k(X), s)$ is defined as the Euler product:

$$L(H^k(X), s) = \prod_p L_p(H^k(X), s)$$

It is known that the product is absolutely convergent whenever $Re(s) > \frac{k}{2} + 1$. Moreover, it is expected that $L(H^k(X), s)$ has a meromorphic continuation to $s \in \mathbb{C}$ and satisfies a functional equation. To state it, it is necessary to introduce a local factor $L_\infty(H^k(X), s)$ at the archimedean place. Its form depends on the Hodge decomposition on $H^k_B(X)$ as follows. Let $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$. Note that complex conjugation F_∞ acts on $H^{p,p}$ giving a decomposition $H^{p,p} = H^{p,+} \oplus H^{p,-}$; we write $h^{p,\pm} = \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$. Define:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

$$\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s)$$

Then one defines:

$$L_\infty(H^{2k+1}, s) = \prod_{\substack{p<q \\ p+q=2k+1}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}}$$

$$L_\infty(H^{2k}, s) = \prod_{\substack{p<q \\ p+q=2k}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \Gamma_{\mathbb{R}}(s-k)^{h^{k,+}} \cdot \Gamma_{\mathbb{R}}(s-k+1)^{h^{k,-}}$$

(In this work, we will only be interested in L-functions attached to even-dimensional cohomology groups).

Finally, one defines a completed L-function $\Lambda(H^k(X), s) = L(H^k(X), s) \cdot L_\infty(H^k(X), s)$. It is conjectured that it admits meromorphic continuation to $s \in \mathbb{C}$ and that (for $k = \dim(X)$) it satisfies a functional equation relating s and $k+1-s$.

Note that the function $L_\infty(H^k(X), s)$ is a meromorphic function on $\mathbb{C}$ without zeroes. Its poles lie at integral values of s such that $Re(s) \leq k/2$. Especially important for us will be the order of these poles. Observe that:

$$-ord_{s=k} L_\infty(H^{2k}(X), s) = \dim_{\mathbb{C}} H^{k,k}(X).$$

1.3.2 Motivic cohomology groups

For a smooth quasi-projective variety X over a field K, Beilinson ([Beĭlinson, 1984]) has defined motivic cohomology groups $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$ using the higher K-theory of X. These groups were later shown by Bloch to agree with the higher Chow groups $CH^k(X, n)$ introduced in [Bloch, 1986]. The advantage of higher Chow groups is their concrete nature: they are obtained as the cohomology groups of a complex built from algebraic cycles on $X \times \mathbb{A}^n$. We next recall the expression of $H^{2k+1}_{\mathcal{M}}(X, \mathbb{Q}(k+1))$ for k a non-negative integer in terms of algebraic cycles following the exposition in [Voisin, 2002].

Consider first the group $\mathcal{Z}^{k+1}(X, 1)$: an element of this group is given by a finite formal sum $\sum n_i(Z_i, \phi_i, f_i)$ with $n_i \in \mathbb{Q}$ and where Z_i is a normal variety of dimension $\dim(X) - k$, f_i is a meromorphic function on Z_i and $\phi_i : Z_i \to X$ is a generically finite proper map such that

$$\sum n_i(\phi_i)_*(div(f_i)) = 0$$

as a codimension $k+1$ cycle on X.

Some elements of this group can be constructed using tame symbols as follows. Let $\phi : W \to X$ a generically finite proper map with W a normal variety of dimension $\dim(X) + 1 - k$ and let f_1, f_2 be meromorphic functions on X. Given an irreducible subvariety $D \subset W$ of codimension 1 with normalization $\nu_{\tilde{D}} : \tilde{D} \to D \subset W$, we define the tame symbol

$$T(f_1, f_2)_D := (-1)^{ord_D(f_1) ord_D(f_2)} \nu^*_{\tilde{D}} \left(\frac{f_1^{ord_D(f_2)}}{f_2^{ord_D(f_1)}} \right) \in k(\tilde{D})^\times.$$

Note that $T(f_1, f_2)_D \neq 1$ only for a finite number of subvarieties D. It is easy to check that the formal sum

$$T(W, f_1, f_2) := \sum_D (\tilde{D}, \phi \circ \nu_{\tilde{D}}, T(f_1, f_2)_D)$$

is an element of $\mathcal{Z}^{k+1}(X, 1)$. The subgroup generated by elements of the form $T(W, f_1, f_2)$ will be denoted by $\mathcal{B}^{k+1}(X, 1)$. We define:

$$H^{2k+1}_{\mathcal{M}}(X, \mathbb{Q}(k+1)) := \frac{\mathcal{Z}^{k+1}(X, 1)}{\mathcal{B}^{k+1}(X, 1)}.$$

(This group is also denoted by $CH^{k+1}(X,1)$). In other words, denoting by $X^{(m)}$ the set of points of X of codimension m, the group $CH^{k+1}(X,1)$ is the H^1 of the Gersten complex

$$\bigoplus_{x\in X^{(k-1)}} K_2(k(x)) \xrightarrow{T} \bigoplus_{x\in X^{(k)}} k(x)^\times \xrightarrow{div} \bigoplus_{x\in X^{(k+1)}} \mathbb{Z}$$

where K_2 stands for the Milnor K_2-group and where the arrows are given by the tame symbol and divisor maps.

Let us now consider the case of varieties X defined over $\mathbb{Q}$. Even for such varieties, the groups we have just defined can be infinite-dimensional over $\mathbb{Q}$, e.g for $X = Spec(\mathbb{Q})$ one finds $H^1_{\mathcal{M}}(X,\mathbb{Q}(1)) = \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. A (conjecturally) better behaved subspace $H^{2k+1}_{\mathcal{M}}(X,\mathbb{Q}(k+1))_{\mathbb{Z}} \subset H^{2k+1}_{\mathcal{M}}(X,\mathbb{Q}(k+1))$ of integral motivic cohomology classes can be defined as follows. Assume first that the variety X is projective, smooth and that it admits a regular proper flat model $\mathcal{X}$ over $\mathbb{Z}$. One can then similarly define a group $H^{2k+1}_{\mathcal{M}}(\mathcal{X},\mathbb{Q}(k+1))$ by considering cycles and meromorphic functions on $\mathcal{X}$. Restriction to the generic fiber defines a map

$$H^{2k+1}_{\mathcal{M}}(\mathcal{X},\mathbb{Q}(k+1)) \to H^{2k+1}_{\mathcal{M}}(X,\mathbb{Q}(k+1))$$

and we define $H^{2k+1}_{\mathcal{M}}(X,\mathbb{Q}(k+1))_{\mathbb{Z}}$ to be its image. For X smooth and projective, it was shown by Beilinson ([Beĭlinson, 1984, 2.4.2]) that $H^i_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}}$ is independent of the choice of model $\mathcal{X}$. A definition of $H^i_{\mathcal{M}}(M,\mathbb{Q}(j))_{\mathbb{Z}}$ for any Chow motive M defined over $\mathbb{Q}$ generalizing this was given later by Scholl in [Scholl, 2000] using J. de Jong's alterations. The following conjecture (a special case of conjectures of Bass on K-theory) is wide open.

Conjecture 1. *The motivic cohomology groups $H^i_{\mathcal{M}}(M,\mathbb{Q}(j))_{\mathbb{Z}}$ are finite dimensional over $\mathbb{Q}$.*

For an integer n, we define $\mathbb{R}(n) := \mathbb{R}\cdot(2\pi i)^n \subset \mathbb{C}$. To a projective smooth variety X defined over $\mathbb{R}$ one can attach a bigraded cohomology theory $H^i_{\mathcal{D}}(X,\mathbb{R}(n))$ known as Deligne cohomology (see [Rapoport *et al.*, 1988] for the definition). These cohomology groups lie in a long exact sequence

$$\ldots \to H^{i-1}_{DR}(X) \to H^i_{\mathcal{D}}(X,\mathbb{R}(n)) \to F^n H^i_{DR}(X) \oplus H^i(X,\mathbb{R}(n)) \to H^i_{DR}(X) \to \ldots$$

where $H^i_{DR}(X)$ stands for the algebraic de Rham cohomology of X and $H^i(X, \mathbb{R}(n))$ is singular cohomology of the analytic space over $\mathbb{R}$ associated with X. It follows that one can think of classes in $H^i_{\mathcal{D}}(X, \mathbb{R}(n))$ as singular cohomology classes of degree i lying in the n-th step of the Hodge filtration. We are interested in the case where $2n - i = 1$; then an application of Hodge theory (see [Rapoport *et al.*, 1988, p. 8]) shows that this long exact sequence simplifies to:

$$0 \to F^{k+1}H^{2k}_{DR}(X) \to H^{2k}(X(\mathbb{C}), \mathbb{R}(k))^{(-1)^k} \to H^{2k+1}_{\mathcal{D}}(X, \mathbb{R}(k)) \to 0 \tag{1.1}$$

where the superscript $(-1)^k$ indicates the sign of action of complex conjugation. It follows that Deligne cohomology groups have the simpler description in terms of Hodge theory:

$$H^{2k+1}_{\mathcal{D}}(X, \mathbb{R}(k+1)) = H^{k,k}(X(\mathbb{C}))^+$$

(see [Ramakrishnan, 1989]). Here the superscript stands for invariants under complex conjugation.

Let X be a projective smooth variety over $\mathbb{Q}$. In [Beĭlinson, 1984], Beilinson defines a regulator map

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(n)).$$

When $2n - i = 1$, this map sends a cycle $\sum n_i(Z_i, \phi_i, f_i) \in H^{2k+1}_{\mathcal{M}}(X, \mathbb{R}(k+1))$ to the current

$$r_{\mathcal{D}}\left(\sum n_i(Z_i, \phi_i, f_i)\right) = \sum n_i(\phi_i)_*(\log|f_i|\delta_{Z_i}) \in (H^{n-k,n-k}(X(\mathbb{C})))^\vee$$

where δ_Z is defined to be the current given by integration on Z. Note that this current is $\partial\bar{\partial}$-closed by the condition $\sum n_i(\phi_i)_*(div(f_i)) = 0$ and hence its cohomology class in $H^{k,k}(X(\mathbb{C}))^+$ is well defined.

Denote by $C^k_{hom}(X)$ the group of algebraic cycles on X of codimension k modulo homological equivalence. Then Beilinson also defines an extended regulator map

$$\tilde{r}_{\mathcal{D}} : H^{2k+1}_{\mathcal{M}}(X, \mathbb{Q}(k+1))_{\mathbb{Z}} \oplus (C^k_{hom}(X) \otimes \mathbb{Q}) \to H^{k,k}(X(\mathbb{C}))^+$$

as the direct sum of the map $r_{\mathcal{D}}$ (restricted to $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$) and the usual cycle class map.

Note also that since X is defined over $\mathbb{Q}$, the first two terms in the short exact sequence (1.1) have a natural $\mathbb{Q}$-structure; this way we obtain a $\mathbb{Q}$-structure

$$\mathcal{B}_{2k,k+1} := \det(H^{2k}(X(\mathbb{C}), \mathbb{Q}(k))^{(-1)^k}) \otimes (\det(F^{k+1}H^{2k}_{DR}(X)))^\vee$$

on $\det(H^{2k+1}_{\mathcal{D}}(X, \mathbb{R}(k)))$. After these preliminaries, we can now state Beilinson's conjectures for the near central points.

Conjecture 2. *[Beĭlinson, 1984, 6.5.7] Let X be a smooth projective variety defined over $\mathbb{Q}$ and let $k \geq 0$. Denote by $L^*(H^{2k}(X), k)$ the first non-vanishing coefficient of the expansion of $L(H^{2k}(X), s)$ around $s = k$. Then:*

i) The image of $\tilde{r}_{\mathcal{D}}$ is a $\mathbb{Q}$-lattice in $H^{2k+1}_{\mathcal{D}}(X, \mathbb{R}(k+1))$.

ii) $Ker(\tilde{r}_{\mathcal{D}}) = 0$.

iii) $\det(Im(\tilde{r}_{\mathcal{D}})) = L^*(H^{2k}(X), k)\mathcal{B}_{2k,k+1}$.

Remark 1. *Parts i) and ii) can be combined into the claim that*

$$\tilde{r}_{\mathcal{D}} \otimes \mathbb{R} : (H^{2k+1}_{\mathcal{M}}(X, \mathbb{Q}(k+1))_{\mathbb{Z}} \otimes \mathbb{R}) \oplus (C^k_{hom}(X) \otimes \mathbb{R}) \to H^{2k+1}_{\mathcal{D}}(X, \mathbb{R}(k+1))$$

is an isomorphism. Note that the Tate conjecture predicts that $\dim_{\mathbb{Q}} C^k_{hom}(X) \otimes \mathbb{Q}$ is equal the order of the pole of $L(H^{2k}(X), s)$ at $s = k+1$. Hence, in order to be compatible with the Tate conjecture and the expected functional equation of $L(H^{2k}(X), s)$, we must have:

$$\dim_{\mathbb{Q}} H^{2k+1}_{\mathcal{M}}(X, \mathbb{Q}(k+1))_{\mathbb{Z}} = ord_{s=k} L(H^{2k}(X), s).$$

Remark 2. *Note that the ring of correspondences on X acts on motivic cohomology groups and on Deligne cohomology and the regulator map intertwines these actions. As a consequences, one can define motivic cohomology of pure motives and state an analogous conjecture for motivic L-functions.*

1.4 The case of Rankin-Selberg L-functions

Let us describe more precisely the statement of the conjecture in the case of Rankin-Selberg L-functions. Let $\pi_l = \mathcal{D}_l$ be a discrete series representation of $G = GL_2(\mathbb{R})$ of even weight l

and consider the local Rankin-Selberg L-function $L(\pi \times \widetilde{\pi}, s)$. Note that (see e.g. [Shimura, 1975, p. 80])

$$L(\pi_l \times \pi_l, s) = \Gamma_{\mathbb{C}}(s + l - 1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1)$$

where we write $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s)$.

Remark 3. *One way to see that this is the right L-function is to use Serre's recipe for the factors at infinity of motivic L-functions as follows. Let f be a holomorphic cusp form of weight l and some level N. The Hodge structure V_f attached to f has type $\{(l-1, 0), (0, l-1)\}$. Then $L(f \times f, s) = L(V_f \otimes V_f, s)$ and the L-function on the RHS is determined by the type of the Hodge structure $V_f \otimes V_f$ as in [Rapoport* et al.*, 1988, , p. 4], giving:*

$$L(V_f \otimes V_f, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{R}}(s - (l - 1))\Gamma_{\mathbb{R}}(s - (l - 1) + 1))$$

Note that the variable s here is normalized following the convention for motivic L-functions, so that the global L-function should satisfy a functional equation relating s to $2l-1-s$. Thus we need to shift s to change to the automorphic normalization (where the global functional equation will relate s to $1-s$) and we get

$$L(\pi_l \times \pi_l, s) = L(f \times f, s + l - 1) = L(V_f \otimes V_f, s + l - 1) = \Gamma_{\mathbb{C}}(s + l - 1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1)$$

as claimed.

Now consider two automorphic representations π_1, π_2 of $GL_2(\mathbb{A})$. Assume that (i) both have trivial central character, (ii) $\pi_1 \ncong \widetilde{\pi_2}$ and that $\pi_{1,\infty} \cong \pi_{2,\infty} \cong \mathcal{D}_2$, the discrete series representation of $GL_2(\mathbb{R})$ of weight 2. Let $L(\pi_1 \otimes \pi_2, s) = \prod_p L(\pi_{1,p} \otimes \pi_{2,p}, s)$ be the global Rankin-Selberg L-function. Then $L(\pi_1 \otimes \pi_2, s)$ is regular and non-vanishing at $s = 1$ and $L(\pi_{1,\infty} \otimes \pi_{2,\infty}, s)$ has a pole of order 1 at $s = 0$. By the functional equation, it follows that $ord_{s=0}L(\pi_1 \otimes \pi_2, s) = 1$. Beilinson's conjecture predicts the existence of a non-trivial class in $H^3_{\mathcal{M}}(X, \mathbb{Q}(2))_{\mathbb{Z}}$ where X is a product of modular curves of appropriate level.

We now describe some of the previous work on this conjecture, with the goal of providing context for this work. We will focus on the cases where the underlying variety X is a Shimura variety. In the paper [Beĭlinson, 1984] where he first made his conjectures, Beilinson considers the self-product $X = X_1(N) \times X_1(N)$ of the modular curve $X_1(N)$. Using the

Manin-Drinfel'd theorem (which states that the difference of any two cusps $P, Q \in X_1(N)$ is torsion in $Jac(X_1(N))$, he is able to produce motivic cohomology classes in $H^3(X, \mathbb{Q}(2))_{\mathbb{Z}}$. Using these classes and the Rankin-Selberg method for $GL(2) \times GL(2)$, he is able to prove part of Conjecture 2.

In [Ramakrishnan, 1986], Ramakrishnan considers the case where X is a product of two Shimura curves attached to an indefinite (non-split) quaternion algebra over $\mathbb{Q}$. Note that these Shimura curves are compact, so that there are no cusps and hence no direct analogue of Beilinson's construction. Using the Jacquet-Langlands theorem and Faltings's Isogeny Theorem, he managed to prove part of Conjecture 2 for X by reducing to the case considered by Beilinson. However, note that his solution does not produces any explicit motivic cohomology elements and leaves open the question of finding them. Moreover, this method does not seem to generalize to products of Shimura curves over a general totally real field F.

In fact, Shimura curves come with a dense collection of special points: these are defined over ring class fields of quadratic imaginary fields and are dense in X. In [Ramakrishnan, 1990], Ramakrishnan raised the question whether it is possible to use meromorphic functions on Shimura curves whose divisors consist of special points to construct motivic cohomology classes. At that time, no systematic construction of such functions was known and hence the question remained difficult to approach. However, in his seminal papers [Borcherds, 1998], [Borcherds, 1995], Borcherds showed how to construct many such functions using theta functions. We will review Borcherds's work in Section 3.1.

1.5 An example

Consider an indefinite, non-split quaternion algebra B over $\mathbb{Q}$ and let X_B the associated Shimura curve. As a Riemann surface, it is obtained as follows: let $\mathcal{O}_B^1$ be the group of elements of reduced norm 1 in a maximal order $\mathcal{O}_B \subset B$ and fix an isomorphism ι : $B \otimes_{\mathbb{Q}} \mathbb{R} \to M_2(\mathbb{R})$. Then $\iota(\mathcal{O}_B^1) \subset SL_2(\mathbb{R})$ is a discrete subgroup and we define:

$$X_B^{an} := \iota(\mathcal{O}_B^1) \backslash \mathbb{H}$$

where $\mathbb{H}$ is the Poincaré upper half plane endowed with its usual action of $SL_2(\mathbb{R})$. This is a compact Riemann surface and Shimura's theory of canonical models shows that the corresponding algebraic curve over $\mathbb{C}$ has a model X_B over $\mathbb{Q}$. The curve X_B has a special abelian subgroup $W_B \subseteq Aut(X_B)$ of involutions known as the Atkin-Lehner group; if $2r$ is the number of primes dividing the discriminant of B, one has $W_B \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$ with generators $w_1, \ldots, w_{2r}$. For $w \in W_B$, we will denote by $X_w \subset X_B \times X_B$ the graph of w. One can try to construct cycles of the form $\sum_w n_w(X_w, f_w) \in CH^2(X_B^2, 1)$; this is sometimes possible when one can find $f \in \mathbb{Q}(X_B)^\times$ supported in the fixed points of some Atkin-Lehner involutions.

Let us give a concrete example. Let B be of discriminant $74 = 2 \cdot 37$; according to [Ogg, 1983], the curve X_B is hyperelliptic over $\mathbb{Q}$ with hyperelliptic involution $w_{74} := w_2 w_{37}$. Let $Z_2 = \mathrm{CM}(\mathbb{Z}[\sqrt{-2}])$ and $Z_{74} = \mathrm{CM}(\mathbb{Z}[\sqrt{-74}])$ be the divisors of fixed points of w_2 and w_{74}. Since they are defined over $\mathbb{Q}$ and are stable under w_{74}, one can find a non-constant function $f \in \mathbb{Q}(X_B)^\times$ with $div(f) = aZ_2 + bZ_{74}$ by pulling back a meromorphic function on $\mathbb{P}^1_{\mathbb{Q}}$; this function is unique up to multiples by a rational number. It is easy to see that the cycle

$$Z_f = (X_1, f) - (X_2, f) - (X_{74}, f) + (X_{37}, f)$$

belongs to $CH^2(X_B^2, 1)$. Moreover, Z_f is essentially an integral class:

Claim 1. *One can choose f so that the cycle $Z_f \in CH^2(X_B^2, 1)_{\mathbb{Z}}$.*

Proof. Consider the model $\mathcal{X}_B/\mathbb{Z}$ of $X_B/\mathbb{Q}$ as described in [Kudla *et al.*, 2006] (we take $\mathcal{X}_B/\mathbb{Z}$ to be the coarse moduli scheme attached to the moduli problem $\mathcal{M}$ described in op.cit.). Recall that $\mathcal{X}$ has good reduction at p for every prime $p \neq 2, 37$; for $p = 2$ or $p = 37$ the special fiber $\mathcal{X}_{\mathbb{F}_p}$ has two isomorphic irreducible components that are permuted by the action of some $w \in W_B$.

Note that, since $div(f)$ is invariant under W_B, we have $w^*(f) = \pm f$. Hence, if we write $div_{\mathcal{X}}(f)$ for the divisor of f on $\mathcal{X}$, we find that $div_{\mathcal{X}}(f)$ is invariant under W_B. By the previous remark on the bad fibers of $\mathcal{X}$, this implies that the vertical part of $div_{\mathcal{X}}(f)$ is of the form $a_1\mathcal{X}_{\mathbb{F}_{p_1}} + \ldots + a_k\mathcal{X}_{\mathbb{F}_{p_k}}$. The claim follows easily from this. □

Moreover (see [Ogg, 1983]), on X one can find two holomorphic eigenforms ω_1, ω_2 defined over $\mathbb{Q}$ with common Atkin-Lehner eigenvalues $a_2 = a_{74} = -1$. For this pair, we obtain:

$$< r_{\mathcal{D}}(Z), \omega_1 \boxtimes \overline{\omega_2} >= \frac{4}{2\pi i} \int_{X_B} \omega_1(z) \wedge \overline{\omega_2(z)} \log |f(z)| \tag{1.2}$$

In Chapter 5, we will show that this expression is obtained as a special value of a Rankin-Selberg integral related to the L-function attached to $\omega_1 \boxtimes \omega_2$.

It seems possible to construct other motivic cohomology classes supported on graphs of Atkin-Lehner involutions; it would be interesting to know when these classes are non-trivial. The main result of this thesis (Theorem 12) computing an integral like 1.2 can be seen as a first step in this direction.

1.6 Notations and measures

1.6.1

For any field k, the vector space k^4 has a standard symplectic structure given by the matrix:

$$J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

The group $GL_4(k)$ acts on k^4 by right multiplication (i.e., we consider k^4 as a space of row vectors). The symplectic group Sp_4 is the algebraic subgroup of GL_4 defined by:

$$Sp_4(F) = \{g \in GL_4(F) | {}^t g J g = J\}.$$

The group GSp_4 is the algebraic subgroup of GL_4 preserving the symplectic form up to a similitude factor:

$$Sp_4(F) = \{g \in GL_4(F) | {}^t g J g = \lambda(g) J, \quad \lambda(g) \in F^\times\}.$$

We denote by X, $Y \subset k^4$ the maximal isotropic subspaces:

$$X = \{(f, 0) \in k^4 | f \in k^2\},$$

$$Y = \{(0, f) \in k^4 | f \in k^2\}.$$

The stabilizer of Y in Sp_4 is a maximal parabolic P_4 known as the Siegel parabolic; we denote it simply by P. We write $P = MN$ with N the unipotent radical of P and M a Levi subgroup. In terms of matrices:

$$P(k) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

$$M(k) = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mid a \in GL_2(k) \right\}$$

$$N(k) = \left\{ \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} \mid b \in Sym_2(k) \right\}$$

where we denote by $Sym_2(k)$ the additive group of symmetric matrices of size 2. For $a \in GL_2(k)$, $b \in Sym_2(k)$, we write:

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \quad n(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix}$$

$$w = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

We will also use the following representatives for generators of the Weyl group of Sp_4:

$$w_\alpha = m(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$w_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

If k is a local field with norm $|\cdot|$, using the Iwasawa decomposition $Sp_4(k) = N(k)M(k)K$ we define a function $|a(g)| : Sp_4(k) \to \mathbb{R}_{>0}$ by

$$|a(n(b)m(A)k)| = |\det(A)|$$

and one checks easily that this is well defined.

1.6.2

Let k be a local field of characteristic 0. We denote by $\psi : k \to S^1$ the additive character in Tate's thesis:

- $k = \mathbb{Q}_p$: consider the quotient map $p : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}$. Then $\psi(x) = \psi_{\mathbb{Q}_p}(x) := e^{-2\pi i p(x)}$
- $k = \mathbb{R}$: define $\psi(x) := e^{2\pi i x}$
- $k = \mathbb{C}$: define $\psi(z) = e^{4\pi i Re(z)}$
- k a finite extension of $\mathbb{Q}_p$: define $\psi_k = \psi_{\mathbb{Q}_p}(tr_{k/\mathbb{Q}_p}(\cdot))$

We also fix an additive Haar measure on k as follows:

- If $k = \mathbb{R}$, then dx is Lebesgue measure,
- If $k = \mathbb{C}$, then dx is twice Lebesgue measure,
- If k is non-archimedean, then dx is the measure assigning to $\mathcal{O}_k$ the value $d_k^{-1/2}$, where d_k is the discriminant of k.

Note that this Haar measure is characterized by being self-dual for the Fourier transform, i.e. if we define:

$$\hat{\varphi}(y) = \int_k \varphi(x)\psi(xy)dx$$

then $\hat{\hat{\varphi}}(x) = -\varphi(x)$.

For a quadratic vector space (V, q) over a local field $k \neq \mathbb{C}$, we define a space of Schwartz functions $\mathcal{S}(V)$ as follows:

- If k is non-archimedean, then $\mathcal{S}(V)$ consists of all locally constant functions $f : V \to k$ with compact support;
- If k is archimedean, then fix an orthogonal decomposition $V = V^+ \oplus V^-$ $\mathcal{S}(V)$ with V^+ (resp. V^-) positive (resp. negative) definite; we will write $v = v^+ + v^- \in V$. Define a positive definite quadratic form by $q_0(v) = q(v^+) - q(v^-)$ consists of all functions $f : V \to k$ of the form $f(v) = p(v)e^{-2\pi q_0(v)}$ where p is a polynomial function.

Note that there is a natural inclusion $\mathcal{S}(V_1) \otimes \mathcal{S}(V_2) \to \mathcal{S}(V_1 \oplus V_2)$; the image is dense (and surjective when k is non-archimedean).

If (V, q) is a quadratic vector space over a totally real field k, the Schwartz space $\mathcal{S}(V(\mathbb{A})) = \otimes_v' \mathcal{S}(V_v)$ is a restricted tensor product defined as follows. Pick any lattice $L \subset V$; one obtains distinguished vectors $\varphi_p^0 = 1_{L \otimes \mathcal{O}_v} \in \mathcal{S}(V(k_v))$ for every prime p. The space $\mathcal{S}(V(\mathbb{A}))$ is the restricted tensor product with respect to these vectors. It is easy to check that $\mathcal{S}(V(\mathbb{A}))$ does not depend on the lattice L.

Representations of p-adic groups will always be assumed to be smooth.

For an abelian group M, we write $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$.

Chapter 2

Theta Correspondence for (Sp_4, O_4)

In this section we review the theory of Yoshida lifts. We start by recalling basic facts about four-dimensional quadratic spaces and their orthogonal groups over local and global fields. After that we explain how functoriality and the Arthur conjectures predict the existence of lifts from automorphic representations of such orthogonal groups to automorphic representations of GSp_4. In the rest of the section we explain how the theta correspondence allows to construct the expected lifts. To do this, we first recall the formulas defining the Weil representation of $Sp_4 \times O(V)$ in the local Schrodinger model. Next we state without proof some facts about the correspondence of local representations. Finally, we review a result of Roberts about non-vanishing of the global theta correspondence; this result confirms the prediction made by functoriality and gives explicit formulas for the corresponding automorphic forms. We finish by describing some specific Schwartz forms of interest used for the global lift.

2.1 Four-dimensional quadratic spaces and their orthogonal groups

Assume that k is a local field of characteristic 0 and that $k \neq \mathbb{C}$. Let (V, q) be a quadratic space over k with $\dim V = 4$ and discriminant 1. Recall that, up to isometry, there are two such spaces V^+ and V^- that are classified by their Hasse invariant $\epsilon(V^{\pm}) = \pm 1$.

More concretely, consider the space $M_2(k)$ of two by two matrices over k and define $q(v) = \det(v)$. Then $(V^+, q) \cong (M_2(k), \det)$. Similarly, let B be the unique non-split quaternion algebra over k and let $n : B \to k$ be the reduced norm; then $(V^-, q) \cong (B, n)$.

The group $GL_2(k) \times GL_2(k)$ (resp. $B^\times \times B^\times$ acts on V^+ (resp. V^-) by sending $x \in V^+$ to $g_1 x g_2^{-1} \in V^+$ (resp. $x \in V^-$ to $g_1 x g_2^{-1} \in V^-$). These maps fit into short exact sequences:

$$1 \to k^\times \to GL_2(k) \times GL_2(k) \to GSO(V^+) \to 1$$

$$1 \to k^\times \to B^\times(k) \times B^\times(k) \to GSO(V^-) \to 1$$

Hence, defining

$$(GL_2 \times GL_2)^0 := \{(g_1, g_2) \in GL_2 \times GL_2 | \det(g_1) = \det(g_2)\}$$

$$(B^\times \times B^\times)^0 := \{(g_1, g_2) \in B^\times \times B^\times | n(g_1) = n(g_2)\}$$

one sees that

$$SO(V^+) = \mathbb{G}_m \backslash (GL_2 \times GL_2)^0, \quad SO(V^-) = \mathbb{G}_m \backslash (B^\times \times B^\times)^0.$$

Thus irreducible representations of $GSO(V^+)$ are of the form $\pi = \pi_1 \boxtimes \pi_2$, where π_1 and π_2 are irreducible representations of $GL(2)$ with central characters ω_1 and ω_2 satisfying $\omega_1 \omega_2 = 1$. Similarly, irreducible representations of $GSO(V^-)$ are of the form $\pi = \pi_1^B \boxtimes \pi_2^B$, where π_1^B and π_2^B are irreducible representations of $B^\times$ with $\omega_1^B \omega_2^B = 1$.

The groups $GSO(V^+)$ and $GSO(V-)$ admit an automorphism $\iota(g_1, g_2) = (g_2, g_1)$. One has:

$$GO(V^+) = GSO(V^+) \rtimes \{1, \iota\}$$

$$GO(V^-) = GSO(V^-) \rtimes \{1, \iota\}$$

Assume now that k is a totally real number field and let V be a four dimensional quadratic vector space with discriminant 1. Then $(V, q) \cong (B, n)$ for a unique quaternion algebra over k with reduced norm n. This quaternion algebra is completely determined by the condition that B ramifies at v if and only if $\epsilon(V_v) = -1$.

2.2 Yoshida lifts

For a given reductive algebraic group G over a field k, denote its dual group by $\hat{G}$. This is a complex reductive Lie group whose root datum is dual to the one of $G_{\overline{k}}$ (see [Borel, 1976] for details). Let V be a four dimensional quadratic vector space over a field k. Then we have:

$$\widehat{GSO(V)} = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}))^0, \quad \widehat{GSp_4} = GSp_4(\mathbb{C})$$

Note that there is an inclusion map:

$$\widehat{GSO(V)} \to \widehat{GSp_4} = GSp_4(\mathbb{C})$$

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

The local Langlands conjecture (proved for GSp_4; see [Gan and Takeda, 2011a]) yields the existence, for every irreducible representation $\pi \cong \pi_1 \boxtimes \pi_2$ of $GSO(V)(\mathbb{Q}_v)$, of an L-packet $\Pi(\pi_1, \pi_2)$ consisting of a finite number of representations of $GSp_4(\mathbb{Q}_v)$. In fact, the local L-packet will always consist of 1 or 2 elements: the latter case will happen precisely when both π_1 and π_2 are discrete series representations. In that case the L-packet $\Pi(\pi_1, \pi_2) = \{\Pi(\pi_1, \pi_2)_{gen}, \Pi(\pi_1, \pi_2)_{ng}\}$ will consist of one generic and one non-generic representation.

According to the Langlands functoriality conjecture, this map of dual groups should induce a transfer of automorphic representations from $GSO(V)(\mathbb{A})$ to $GSp_4(\mathbb{A})$. Moreover, Arthur's conjectures give a refinement by specifying the multiplicity with which a given element of the global Arthur packet should appear in the discrete spectrum of $GSp_4(\mathbb{A})$.

2.2.1

We explain the structure of the L-packet $\Pi(\pi_1, \pi_2)$ in a special case relevant to this work. Consider the automorphic representations π_1 and π_2 of $GL_2(\mathbb{A})$ generated by two cusp forms f_1 and f_2 of weight 2 and square-free levels N_1 and N_2. Let $N = gcd(N_1, N_2)$ and let $T = \{\infty\} \cup T_f$, where T_f the set of primes dividing N. The local L-packet at v has 2 elements precisely when $v \in T$. It follows that the cardinality of the global L-packet is 2^{d+1}. Moreover, for a given element Π in the global L-packet, let $T(\Pi) \subset T$ the set of places v where Π_v is non-generic. Then Arthur's conjecture predicts that the multiplicity $m(\Pi)$ of Π in the discrete spectrum of $GSp_4(\mathbb{Q})\backslash GSp_4(\mathbb{A})$ is given by:

$$m(\Pi) = \begin{cases} 1 & \text{if } |T(\Pi)| \text{ is even,} \\ 0 & \text{if } |T(\Pi)| \text{ is odd.} \end{cases}$$

This prediction has been confirmed by Roberts in [Roberts, 1999], where he also explains how to construct Π as a global theta lift.

2.3 Weil representation

Consider k^4 as a symplectic vector space with its standard symplectic form. Then the vector space $W := k^4 \otimes V$ has a natural symplectic structure on it ; this gives an embedding $Sp_4 \times O(V) \hookrightarrow Sp(W)$. There is a representation ω_ψ of $\widetilde{Sp}(W)$ called the Weil representation; we denote it simply by ω (recall that we denote by ψ the standard character introduced in section 1.6.1). It can be realized in the space $\mathcal{S}(V^2)$ of Schwartz functions on V^2. For a Schwartz function φ, the action of $Sp_4 \times O(V)$ is given by the following formulas:

$$\omega(1, h)\varphi(v) = \varphi(h^{-1}v)$$

$$\omega(m(a), 1)\varphi(v) = \det(a)^2\varphi(v \cdot a)$$

$$\omega(n(b), 1)\varphi(v) = \psi\frac{1}{2}tr(b(v, v))\varphi(v)$$

$$\omega(w, 1)\varphi(v) = \gamma(V, q)\hat{\varphi}(v)$$

$$\omega(1, h)\varphi(v) = \varphi(h^{-1}v)$$

2.4 Non-archimedean correspondence

Let us now describe some well-known results about the local theta correspondence. For proofs of the results of this section, see [Gan and Takeda, 2011b].

Recall that to a character

$$(\chi_1, \chi_2, \chi_0) : \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_0 t_1^{-1} & \\ & & & t_0 t_2^{-1} \end{pmatrix} \to \chi_1(t_1)\chi_2(t_2)\chi_0(t_0)$$

of the standard torus of GSp_4, one attaches a principal series representation $I_B^{GSp_4}(\chi_1, \chi_2; \chi_0)$ obtained by normalized induction from the standard Borel subgroup of GSp_4.

Recall that there is a standard parabolic Q or GSp_4 (usually called the Klingen parabolic):

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \right\} \subset GSp_4$$

with Levi subgroup isomorphic to $GL_1 \times GL_2$. For a representation (χ, τ) of $GL_1 \times GL_2$, we denote by $I_Q(\chi, \tau)$ the normalized induced representation on GSp_4.

For the proofs of the following facts, we refer to [Gan and Takeda, 2011b].

Proposition 1. *Let k be non-archimedean and let $\pi_1 = \pi(\chi_1', \chi_1)$ and $\pi_2 = \pi(\chi_2', \chi_2)$ be spherical representations of $GL_2(k)$. Then*

$$\theta(\pi_1 \boxtimes \pi_2) = J_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1).$$

Proposition 2. *Let k be non-archimedean, $1_{B^\times}$ be the trivial representation of $B^\times$ and St be the Steinberg representation of $GL_2(k)$. Then the induced representation $I_Q(1, St)$ is a direct sum of two irreducible tempered representations π_{ng} and π_{gen} and:*

$$\theta(1_{B^\times} \boxtimes 1_{B^\times}) = \pi_{ng},$$

$$\theta(St \boxtimes St) = \pi_{gen}.$$

Here π_{ng} is not generic and π_{gen} is generic.

2.4.1 Correspondence of local L-factors

Let k be non-archimedean. Recall that to a spherical representation π of $GSp_4(k)$, the Satake isomorphism attaches a semisimple class $\phi_\pi \in ({}^L G)^0 = GSp_4(\mathbb{C})$. This class is explicitly given as follows: let $\pi = Ind_B(\chi_1, \chi_2; \chi)$, where χ_1, χ_2 and χ are unramified characters. Then

$$\phi_\pi = \begin{pmatrix} \chi_1(\varpi) & & & \\ & \chi_2(\varpi) & & \\ & & \chi(\varpi)\chi_1(\varpi)^{-1} & \\ & & & \chi(\varpi)\chi_2(\varpi)^{-1} \end{pmatrix}$$

Given a finite dimensional representation (V, ρ) of $GSp_4(\mathbb{C})$ and $s \in \mathbb{C}$, one defines the associated local L-factor by:

$$L(\pi, \rho, s) = \det(1 - \rho(\phi_\pi) q^{-s} | V)^{-1}.$$

There is a natural representation of $GSp_4(\mathbb{C})$ on a certain 5-dimensional quadratic space; the resulting map $std : GSp_4(\mathbb{C}) \to SO_5(\mathbb{C})$ induces an isomorphism $PGSp_4(\mathbb{C}) \cong SO_5(\mathbb{C})$. The corresponding local L-factor

$$L(\pi, std, s) = (1 - q^{-s})^{-1} \prod_{i=1,2} (1 - \chi_i(\omega) q^{-s})^{-1} (1 - \chi_i(\omega)^{-1} q^{-s})^{-1}$$

is known as the standard L-function of π.

The description of the local theta correspondence in the unramified case given above shows that for π_1 and π_2 spherical representations of $GL_2(k)$:

$$L(\theta(\pi_1 \boxtimes \pi_2), std, s) = \zeta_k(s) L(\pi_1 \boxtimes \pi_2, s)$$

where $L(\pi_1 \boxtimes \pi_2, s)$ denotes the Rankin-Selberg L-function of $GL(2) \times GL(2)$.

2.4.2 A non-vanishing result

The non-vanishing of global theta lifts for the dual pair $(Sp_4, O(V))$ has been established by [Roberts, 1999]. We quote here his result restricted to our context.

Theorem 2. *[Roberts, 1999, Cor. 1.3] Let V be an even dimensional, nondegenerate symmetric bilinear space over $\mathbb{Q}$. Suppose that the signature of V is of the form $(2n, 2)$. Let σ be an irreducible cuspidal automorphic representation of $O(X, \mathbb{A})$ with $\sigma \cong \otimes_v \sigma_v$, and let V_σ be a realization of σ in the space of cusp forms. Assume σ_v is tempered and occurs in the theta correspondence for $O(V, \mathbb{Q}_v)$ and $Sp(2n+2, \mathbb{Q}_v)$ for all places v. If $L^S(\sigma, s)$ does not vanish at $s = 1$, then $\Theta_n(V_\sigma) \neq 0$.*

The results described above show that the hypothesis concerning the local components are satisfied in our case, hence Roberts's theorem implies that the global theta lift is non-zero.

2.5 Archimedean correspondence

2.5.1 (Limits of) discrete series on $Sp_4(\mathbb{R})$.

We briefly review some results concerning the Harish-Chandra parametrization of (limits of) discrete series on $Sp_4(\mathbb{R})$. In this section we write $G := Sp_4(\mathbb{R})$ and denote by K the maximal compact subgroup of G defined by $K = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G\}$. The map $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$ yields an isomorphism $K \cong U(2)$. Let us briefly review the classification by highest weight of the finite-dimensional irreducible representations of $U(2)$. Consider the embedding $\iota : SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to G$ given by:

$$\iota(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}$$

Write $k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Given a representation τ of $U(2)$ and integers l, l', we say that $v \in \pi$ has weight (l, l') if:

$$\tau(\iota(k_\theta, k_{\theta'}))v = e^{i(l\cdot\theta + l'\theta')} \cdot v$$

We parametrize irreducible representations of $U(2)$ by $\{(l, l') \in \mathbb{Z}^2 | l \geq l'\}$: we denote by $\tau_{(l,l')}$ the unique irreducible representation of $U(2)$ having highest weight (l, l'). The dimension of $\tau_{(l,l')}$ is $l - l' + 1$.

Note that $Lie(K)$ consists of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where B is symmetric and A anti-symmetric. We choose a compact Cartan subalgebra

$$\mathfrak{H} = \left\{ t = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} | B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \subset Lie(K)$$

and define $e_1(t) = a$ and $e_2(t) = b$. Define an inner product $(\cdot, \cdot)$ on $\mathfrak{H}^*_{\mathbb{C}}$ by $(e_i, e_j) = \delta_{ij}$. The adjoint action of $\mathfrak{H}_{\mathbb{C}}$ on $\mathfrak{sp}_{4,\mathbb{C}}$ defines a root system Φ of type C_2 given by

$$\Phi = \{\pm(e_1 - e_2), \pm(e_1 + e_2), \pm 2e_1, \pm 2e_2\}.$$

Note that the compact roots are $\Phi_c = \{\pm(e_1 - e_2)\}$; we define $\Phi_c^+ = \{e_1 - e_2\}$. There are four positive root systems Φ^+ containing Φ_c^+. They are given by:

$$\Phi_I^+ = \{e_1 - e_2, e_1 + e_2, 2e_1, 2e_2\}$$

$$\Phi_{II}^+ = \{e_1 - e_2, e_1 + e_2, 2e_1, -2e_2\}$$

$$\Phi_{III}^+ = \{e_1 - e_2, -e_1 - e_2, 2e_1, -2e_2\}$$

$$\Phi_{IV}^+ = \{e_1 - e_2, -e_1 - e_2, -2e_1, -2e_2\}$$

Let $J \in \{I, II, III, IV\}$. We identify the lattice of integral weights $\Lambda = \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$ with $\mathbb{Z}^2$ by sending $l \cdot e_1 + l' \cdot e_2 \in \Lambda$ to $(l, l') \in \mathbb{Z}^2$. For each J, let $\Phi_{J,nc}^+ = \Phi_J^+ \backslash \Phi_c^+$ be the set of non-compact roots in Φ_J^+. We denote by Λ_J the set of integral weights $\lambda \in \Lambda$ such that (i) $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Phi_{J,nc}^+$ and (ii) $(\lambda, e_1 - e_2) > 0$ if $e_1 - e_2$ is a simple root in Φ_J^+.

According to Harish-Chandra's parametrization of (limits of) discrete series ([Knapp, 2001, Thm 12.26]) these are parametrized by the set

$$\{(\lambda, J) | \lambda \in \Lambda_J\}$$

and we write $\pi = \pi(\lambda, J)$ accordingly. If $(\lambda, \alpha) > 0$ for all $\alpha \in \Phi_J^+$, then J is uniquely determined by λ and $\pi(\lambda, J) = \pi(\lambda)$ is a discrete series representation. Otherwise $\pi(\lambda, J)$ is a limit of discrete series. Write $\delta_{J,nc}$ (resp. δ_K) for half the sum of the non-compact (resp. compact) roots in Φ_J^+. The K-types of $\pi(\lambda, J)$ are of the form

$$\lambda + \delta_{J,nc} - \delta_K + \sum_{\alpha \in \Phi_J^+} n_\alpha \alpha$$

where n_α are non-negative integers. Moreover the K-type $\lambda + \delta_{J,nc} - \delta_K$ has multiplicity one in $\pi(\lambda, J)$ and is known as the minimal K-type of $\pi(\lambda, J)$.

2.5.2 Archimedean theta correspondence.

We are interested in a limit of discrete series arising in the theta correspondence for the reductive dual pair $(G, O_{2,2}(\mathbb{R}))$; here $O_{2,2}(\mathbb{R})$ is the group of orthogonal transformations of the split four-dimensional quadratic space V^+ over $\mathbb{R}$ described in section 2.1. Consider the representation $\pi((2,0)_+)$ of $O_{2,2}(\mathbb{R})$; it is one of the constituents of the restriction of the discrete series representation $(\mathcal{D}_2 \boxtimes \mathcal{D}_2)^+$ of $GO_{2,2}(\mathbb{R}) \cong \mathbb{R}^\times \backslash (GL_2(\mathbb{R}) \times GL_2(\mathbb{R})) \rtimes \{1, \iota\}$. For a detailed analysis of the theta correspondence in this case we refer to [Mœglin, 1989]. Recall that the minimal $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$-type of this representation is of the form $(2) \boxtimes (-2) + (-2) \boxtimes (2)$ (see [Harris and Kudla, 1992, 4.2]). Then we have

$$\theta(\pi((2,0)_+)) = \pi((1,0), II)$$

and in particular $\theta(\pi((2,0)_+))$ is a limit of discrete series representation of G. It follows from the previous description of the K-types in $\pi((1,0), II)$ that its minimal K-type has highest weight $(2,0)$; we denote by v_∞ a highest weight vector in this K-type.

Consider now the Schrödinger model $\mathcal{S}(V(\mathbb{R})^2)$ of the Weil representation of $G \times O_{2,2}(\mathbb{R})$. Recall that the functions in $\mathcal{S}(V(\mathbb{R})^2)$ are of the form $\varphi(v_1, v_2) = P(v_1, v_2)e^{-\pi(q_0(v_1)+q_0(v_2))}$ where P is a polynomial on $V(\mathbb{R})^2$ and q_0 is a (positive-definite) majorant of q. For a non-negative integer d, denote by $\mathcal{S}(V(\mathbb{R})^2)_d$ the space of such functions φ where the polynomial P has degree d. Note that $\mathcal{S}(V(\mathbb{R})^2)_d$ is stable under the action of $K \times K_{O_{2,2}(\mathbb{R})}$. If τ is an irreducible finite-dimensional representation of K, we define the degree $\deg(\tau)$ or τ to be the smallest d such that τ is a subrepresentation of $\mathcal{S}(V(\mathbb{R})^2)_d$ (we say that $\deg(\tau) = \infty$ if no such d exists). By [Harris and Kudla, 1992, Prop. 4.2.1], the degree of the K-type with highest weight $(2, 0)$ equals 2. In the next section we will choose a specific Schwartz function of degree 2 in this K-type.

2.6 Special Schwartz forms

In this section we describe choices for the Schwartz functions $\varphi_v \in \mathcal{S}(V(\mathbb{Q}_v)^2)$ for each place v.

2.6.1

Suppose v corresponds to a prime p that does not divide $disc(B)$. Recall that we have fixed an isomorphism $B \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$; via this isomorphism we define:

$$\varphi_v(v_1, v_2) = 1_{M_2(\mathbb{Z}_p)^2}(v_1, v_2) \tag{2.1}$$

Note that this Schwartz function is invariant under $Sp_4(\mathbb{Z}_p)$.

2.6.2

Suppose v corresponds to a prime p that divides $disc(B)$ and let $\mathcal{O}$ be the maximal order of $B \otimes \mathbb{Q}_p$. We define:

$$\varphi_v(v_1, v_2) = 1_{\mathcal{O}^2}(v_1, v_2) \tag{2.2}$$

Recall that we denote by P_1 the Siegel parahoric group; thus P_1 is defined as the inverse image of the Siegel parabolic $P(\mathbb{F}_p) \subset Sp_4(\mathbb{F}_p)$ under the reduction map $Sp_4(\mathbb{Z}_p) \to Sp_4(\mathbb{F}_p)$:

$$P_1 = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}_p) \mid C \equiv 0 \ (mod\ p) \right\}.$$

Lemma 1. *1. The function $\varphi_v = 1_{\mathcal{O}^2}$ is P_1-invariant.*

2. $\widehat{1_{\mathcal{O}^2}} = p^{-1} 1_{\mathcal{D}^2}$

Proof. Recall the Iwahori decomposition: we have $P_1 = N^-(p\mathbb{Z}_p)M(\mathbb{Z}_p)N(\mathbb{Z}_p)$, where $N^- = wNw^{-1}$ and w is the long element in the Weyl group. Invariance under $M(\mathbb{Z}_p)N(\mathbb{Z}_p)$

is obvious; thus it suffices to check it under $N^-(p\mathbb{Z}_p)$. Writing $\hat{f}$ for the Fourier transform of f, we have $\hat{\varphi} = 1_{\mathcal{D}^2}$, where $\mathcal{D}$ denotes the inverse different of $\mathcal{O}$, defined by:

$$\mathcal{D} := \{x \in B | tr(x\mathcal{O}) \subseteq \mathbb{Z}_p\}$$

The claim then follows from the fact that $n(\mathcal{D}) = p^{-1}\mathbb{Z}_p$ ([Vignéras, 1980, p. 35]), so that $1_{\mathcal{D}^2}$ is invariant under $N(p\mathbb{Z}_p)$. □

2.6.3

Finally, let us describe the explicit Schwartz forms of interest at the archimedean place. Recall that $V_{\mathbb{R}} = M_2(\mathbb{R})$ with quadratic form given by $q(x) = \det(x)$. Let

$$V^+ =< \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} >$$

$$V^- =< \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} >$$

and note that V^+ is positive definite, V^- is negative definite and $V = V^+ \oplus V^-$ is an orthogonal sum. For a vector $v \in V$, we write $v = v^+ + v^-$ accordingly. For $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define

$$\varphi^0(v) = e^{-2\pi(q(v^+)-q(v^-))} = e^{-\pi tr(v\cdot v^t)} = e^{-\pi(a^2+b^2+c^2+d^2)},$$

a Schwartz function in $\mathcal{S}(V(\mathbb{R}))$. More generally, let p be the polynomial on V given by:

$$p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + ib + ic - d$$

and for any positive integer l, define

$$\varphi_\infty^l(v) = p(v)^l e^{-\pi tr(v\cdot v^t)}.$$

Then one checks:

$$\omega(k_\theta; (k_{\theta_1}, k_{\theta_2}))\varphi_\infty^l = e^{i\pi l\theta} e^{i\pi l(\theta_1-\theta_2)}\varphi_\infty^l$$

Define a Schwartz function $\varphi_\infty \in \mathcal{S}(V(\mathbb{R})^2)$ by:

$$\varphi_\infty(v_1, v_2) = p(v_1)^2\varphi_0(v_1)\varphi_0(v_2)$$

Then φ_∞ has degree 2 and weight $(2,0)$ under the action of K and moreover the previous computation shows that

$$\omega(1;(k_{\theta_1},k_{\theta_2}))\varphi_\infty = e^{2i\cdot(\theta_1-\theta_2)}\varphi_\infty$$

2.6.4

We now consider the global correspondence. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D = D(B)$ and consider automorphic representations π_1, π_2 of $B(\mathbb{A})^\times$. For simplicity we assume that $D(B)$ is odd. We assume the following:

- Both π_1 and π_2 have trivial central character,
- $\pi_1 \ncong (\pi_2)^\vee$,
- If $p|D(B)$, then $\pi_{1,p} \cong \pi_{2,p} \cong 1_{B_p^\times}$,
- If $p \nmid D(B)$, then both $\pi_{1,p}$ and $\pi_{2,p}$ are unramified,
- At the archimedean place both π_1 and π_2 are discrete series of weight 2.

Thus if we denote by (V,q) the 4-dimensional quadratic space over $\mathbb{Q}$ attached to B, we can consider $\pi_1 \boxtimes \pi_2$ as an automorphic representation of $GSO(V) = \mathbb{G}_m \backslash B^\times \times B^\times$ (see Section 2.1). Consider now its theta lift

$$\sigma = \theta(\pi_1 \boxtimes \pi_2) = \otimes_v^{'} \theta_v(\pi_{1,v} \boxtimes \pi_{2,v})$$

to an automorphic representation of GSp_4; we will denote by $V_\sigma \subset L^2_{cusp}(Z(\mathbb{A})GSp_4(\mathbb{Q})\backslash GSp_4(\mathbb{A}))$ its realization in the space of L^2-automorphic forms (note that this is unique as the multiplicity of σ in the discrete spectrum is 1 as proved by Roberts). Recall that we denote by $P_1 \subset GSp_4(\mathbb{Q}_p)$ the Siegel parahoric subgroup. Consider the open compact subgroup $K_0(D) = \prod_p K_0(D)_p \subset GSp_4(\mathbb{A}_f)$ defined by:

$$K_0(D)_p = \begin{cases} GSp_4(\mathbb{Z}_p), & \text{if } p \nmid D; \\ P_1 \subset GSp_4(\mathbb{Q}_p), & \text{if } p|D. \end{cases}$$

We write

$$K_0(D)' := K_0(D) \cap Sp_4(\mathbb{A}_f).$$

Note that by the local results above, the space of $K_0(D)$-invariants in σ_f has dimension 1. Let $f \in V_\sigma$ be an automorphic form such that

- $f(gk) = f(g)$ for all $k \in K_0(D)$;
- The form f defines a highest weight vector of weight $(2, 0)$ in the minimal K-type of the representation $\sigma_\infty|_{Sp_4(\mathbb{R})}$.

Such a form f is then unique up to scalars and for the choice of Schwartz functions given above we have:

$$\theta_\varphi(\overline{f}) = f_1 \boxtimes \overline{f_2} \in \pi_1 \boxtimes \pi_2$$

where f_1 and f_2 are holomorphic cusp forms of maximal level and weight 2.

2.7 Siegel-Weil formula

We review the statement of the Siegel-Weil formula in the case we need. This formula equates a value of an Eisenstein series on a symplectic group with the integral of a theta function over an orthogonal group. Such formulas were first proved by Siegel and in a more general form by [Weil, 1964] and were later generalized by Kudla and Rallis and others: see [Kudla and Rallis, 1988] for the case needed in this thesis.

Let k be a number field and V an orthogonal quadratic space of even dimension m. We assume V to be anisotropic and denote its orthogonal group by $H = O(V)$. Let $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$ and recall that we denote by $\omega = \omega_\psi$ the Weil representation of $Sp_4(\mathbb{A}) \times H(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A})^2)$. For $g \in Sp_4(\mathbb{A})$ and $h \in H(\mathbb{A})$, define

$$\theta_\varphi(g, h) = \sum_{x \in V^2} \omega(g, h)\varphi(x).$$

For $s \in \mathbb{C}$, define

$$\Phi(g, s) = (\omega(g)\varphi)(0) \cdot |a(g)|^{s - s_0} \in Ind_{P(\mathbb{A})}^{Sp_4(\mathbb{A})}(\chi_V |\cdot|^s)$$

and let

$$E(g, \Phi, s) = \sum_{\gamma \in P(k) \backslash Sp_4(k)} \Phi(\gamma g, s)$$

be the Eisenstein series constructed with Φ. This series is absolutely convergent for $Re(s) > \frac{3}{2}$. Assuming φ is K-finite (here $K = \prod Sp_4(\mathcal{O}) \times U(2)$ is a maximal compact subgroup of $Sp_4(\mathbb{A})$)), it has a meromorphic continuation to $s \in \mathbb{C}$.

Theorem 3. *[Kudla and Rallis, 1988] Assume* φ *is* K*-finite. Let* dh *be the invariant measure on* $H(\mathbb{A})$ *such that* $Vol(H(\mathbb{Q})\backslash H(\mathbb{A}), dh) = 1$. *Then:*

$$E(g, \Phi, s_0) = \kappa \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta_\varphi(g, h) dh$$

where $\kappa = 1$ *for* $m > 2$; $\kappa = 2$ *for* $m = 2$.

Chapter 3

Singular theta lifts

3.1 Borcherds lifts

In this section we review the definition and basic properties of Borcherds lifts. We then reformulate these properties in adelic language following Kudla's exposition in [Kudla, 2003].

3.1.1 The Weil representation of a lattice

Let V be a a quadratic space over $\mathbb{Q}$ of dimension $m+2 = dim(V)$. We assume that the signature of $V_{\mathbb{R}} = V \otimes \mathbb{R}$ equals $(m, 2)$. Let $L \subset V$ an even lattice and denote by

$$L^{\vee} = \{v \in V | (v, l) \in \mathbb{Z} \forall l \in L\}$$

its dual lattice. Note that $L \subset L^{\vee}$ and that the group $L^{\vee}/L$ is finite. Its cardinality is called the discriminant of L. We denote by

$$S_L := \mathbb{C}[L^{\vee}/L] = \oplus_{\mu \in L^{\vee}/L} \mathbb{C}\varphi_{\mu}$$

the group ring of $L^{\vee}/L$. Here φ_{μ} denotes the characteristic function of the coset $\mu + L \in L^{\vee}/L$.

Recall that the group $SL_2(\mathbb{Z})$ admits a non-trivial double cover fitting in an exact sequence:

$$1 \to \{\pm 1\} \to \widetilde{SL_2}(\mathbb{Z}) \to SL_2(\mathbb{Z}) \to 1$$

Denote by $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) = c\tau + d$ the standard automorphy factor. It is customary to denote elements of $\widetilde{SL_2}(\mathbb{Z})$ by pairs $(g, \sqrt{j(g,\tau)})$, where $g \in SL_2(\mathbb{Z})$ and $\sqrt{j(g,\tau)})$ is a choice of square root of $j(g,\tau)$. Multiplication in $\widetilde{SL_2}(\mathbb{Z})$ is given by:

$$(g_1, f_1(\tau)) \cdot (g_2, f_2(\tau)) = (g_1 g_2, f_1(g_2\tau) f_2(\tau)).$$

The group $\widetilde{SL_2}(\mathbb{Z})$ is known as the metaplectic cover of $SL_2(\mathbb{Z})$. It acts on the group ring S_L and this representation (which we denote by ρ_L) is known as the Weil representation attached to the lattice L. Let us quote some explicit formulas for this action.

$$\rho_L(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}), 1)\varphi_\mu = e^{\pi i(\mu,\mu)}\varphi_\mu$$

$$\rho_L(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}), \sqrt{\tau})\varphi_\mu = \frac{\sqrt{i}^{n-2}}{\sqrt{L^\vee/L}} \sum_{\lambda \in L^\vee/L} e^{-2\pi i(\lambda,\mu)} e_\lambda.$$

Since the elements $(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}), 1)$ and $(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}), \sqrt{\tau})$ generate $\widetilde{SL_2}(\mathbb{Z})$, these formulas determine the representation ρ_L completely. Explicit formulas for the action of general elements of $\widetilde{SL_2}(\mathbb{Z})$ can be found in [Shintani, 1975] and [Weil, 1964].

3.1.2 Vector valued modular forms

Recall the well-known action of the group $SL_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}$: given an integer k, one defines:

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d})$$

A holomorphic modular form satisfying $f|_k g = f$ for all $g \in SL_2(\mathbb{Z})$ is called a weak modular form of (even) weight k. Using the metaplectic group $\widetilde{SL_2}(\mathbb{Z})$, one can also define modular forms of half-integral weight.

Definition 1. *Let k be in $\frac{1}{2}\mathbb{Z}$ and (V, ρ) be a complex representation of $\widetilde{SL_2}(\mathbb{Z})$. A holomorphic function $f : \mathbb{H} \to V$ is said to be a weakly holomorphic modular form of weight k valued in V it is satisfies:*

- *f is holomorphic on $\mathbb{H}$;*
- *$f(g\tau) = \sqrt{j(g,\tau)}^{2k}\rho(g, \sqrt{j(g,\tau)})f(\tau)$ for all $(g, \sqrt{j(g,\tau)}) \in \widetilde{SL_2}(\mathbb{Z})$;*
- *The components of f have at most poles at the cusp.*

In particular, one can consider weakly holomorphic modular forms of weight k valued in S_L. We will denote the space of such forms by $S^!_{k,L}$. Note that for any form $f \in S^!_{k,L}$, there exists a unique vector valued polynomial

$$P_f(\tau) = \sum_{\substack{a\in\mathbb{Q}_{\geq 0} \\ \mu\in L^\vee/L}} c(-a,\mu)e^{-2\pi i a\tau}\varphi_\mu$$

such all the components of $f(\tau) - P_f(\tau)$ are rapidly decreasing as τ approaches the cusp.

3.1.3 Siegel theta series

The weakly holomorphic forms introduced in the previous paragraph serve as the input for a regularized theta lift. To describe, we first recall the definition of Siegel theta series.

Recall that V denotes a quadratic space over $\mathbb{Q}$ with signature $(n, 2)$. For every place v of $\mathbb{Q}$, there is a quadratic space $V_v = V \otimes \mathbb{Q}_v$ over $\mathbb{Q}_v$. One can define a space of Schwartz functions $\mathcal{S}(V(\mathbb{Q}_v))$ as follows:

- For any prime number p, the space $\mathcal{S}(V(\mathbb{Q}_p))$ consists of all locally constant, complex valued functions on $V(\mathbb{Q}_p)$ with compact support;
- The space $\mathcal{S}(V(\mathbb{R}))$ consists of all C^∞ functions f on $V(\mathbb{R})$ such that f and all its derivatives are rapidly decreasing.

These spaces can be tensored together to give an adelic Schwartz space

$$\mathcal{S}(V(\mathbb{A})) = \otimes'_v\mathcal{S}(V(\mathbb{Q}_v)).$$

This is a restricted tensor product defined as follows: pick any lattice $L \subset V$; one obtains distinguished vectors $\varphi_p^0 = 1_{L\otimes\mathbb{Z}_p} \in \mathcal{S}(V(\mathbb{Q}_p))$ for every prime p. The space $\mathcal{S}(V(\mathbb{A}))$ is the restricted tensor product with respect to these vectors. It is easy to check that $\mathcal{S}(V(\mathbb{A}))$ does not depend on the lattice L.

Recall that there is an action of the metaplectic cover $\widetilde{SL_2}(\mathbb{R})$ of $SL_2(\mathbb{R})$ on $\mathcal{S}(V(\mathbb{R})$. We denote by $\widetilde{SO_2}(\mathbb{R})$ the inverse image of $SO_2(\mathbb{R})$ in $\widetilde{SL_2}(\mathbb{R})$ and define a character $\chi_{1/2} : \widetilde{SO_2}(\mathbb{R}) \to \mathbb{C}^\times$ by

$$\chi_{1/2}(k_\theta, \phi) = \pm e^{i\theta/2}$$

An element of $\mathcal{S}(V(\mathbb{R}))$ transforming under $\widetilde{SO_2}(\mathbb{R})$ by $\chi_{1/2}^l$ will be said to be of weight l.

Fix a decomposition $V = V^+ \oplus V^-$ of V into an orthogonal sum of a positive definite and a negative definite subspace. For a vector $v \in V(\mathbb{R})$, we write $v = v^+ + v^-$. Define the function

$$\varphi_\infty(v) = e^{-2\pi(q(v^+)-q(v^-))}$$

and note that it belongs to $\mathcal{S}(V(\mathbb{R}))$. It has weight $k_0 := \frac{n}{2} - 1$.

There is a functional on the adelic Schwartz space

$$\theta : \mathcal{S}(V(\mathbb{A})) \to \mathbb{C}$$

defined by

$$\theta(\varphi) = \sum_{v \in V} \varphi(x)$$

called the theta functional. Note that there is an action ω of the group $O(V)(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ defined by $\omega(h)\varphi(v) = \varphi(h^{-1}v)$. Together with the $\widetilde{SL_2}(\mathbb{R})$ action described above, this yields a function

$$\theta_\varphi : \widetilde{SL_2}(\mathbb{R}) \times O(V)(\mathbb{A}) \to \mathbb{C}$$

defined by $\theta_\varphi(g, h) = \sum_{v \in V} \omega(g, h)\varphi(v)$. In particular, given a lattice $L \subset V$, we obtain the Siegel theta series of L by choosing $\varphi_f = 1_{\hat{L}}$ so that:

$$\theta_L(g) := \theta_{1_{\hat{L}} \otimes \varphi_\infty}(g, 1) = \sum_{l \in L} \omega(g)\varphi_\infty(l)$$

More generally, for $\mu \in L^\vee/L$ one can define

$$\theta_\mu(g) = \theta_{1_{\mu+\hat{L}} \otimes \varphi_\infty}(g, 1) = \sum_{l \in \mu + L} \omega(g)\varphi_\infty(l)$$

Finally, we define $\theta_\varphi(\tau,h)$ by pushing down $\theta_\varphi(g,h)$ to the upper half plane so that we obtain:

$$\theta_\varphi(\tau,h)=\sum_{v\in V}\varphi_f(h_f^{-1}v)e(\tau\cdot q((h_\infty^{-1}v)_{z^\perp})-\overline{\tau}\cdot q((h_\infty^{-1}v)_z)).$$

3.1.4 Regularized theta lifts

Given a weakly holomorphic modular form $f^!$ in $S^!_{-k_0,L}$, one can formally define its theta lift as a function on $O(V(\mathbb{A}))$ by:

$$\theta_{f^!}(h)=\sum_{\mu\in L^\vee/L}\int_{\mathcal{F}} f^!_\mu(\tau)\theta_\mu(\tau,h)d\mu(\tau),$$

where $\mathcal{F}=\{\tau\in\mathbb{H}||Re(\tau)|<\frac{1}{2},\quad |\tau|>1\}$ denotes the standard fundamental domain of $SL_2(\mathbb{Z})$ in $\mathbb{H}$. However the integrals in this expression are divergent. A way to regularize this lift is described in [Harvey and Moore, 1996] and [Borcherds, 1998] and we recall it next. For a real number $T>0$, consider a truncation of the fundamental domain $\mathcal{F}_T=\{\tau\in\mathcal{F}|Im(\tau)\le T\}$. One can then show that the limit

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} f^!_\mu(\tau)\theta_\mu(\tau,h)y^{-s}d\mu(\tau)$$

exists for $Re(s)>>0$. Moreover, one proves that the resulting functions admit meromorphic continuation to the whole complex plane. Hence we can define:

$$\int_{\mathcal{F}}^{reg} f^!_\mu(\tau)\theta_\mu(\tau,h)d\mu(\tau)=CT_{s=0}\lim_{T\to\infty}\int_{\mathcal{F}_T} f^!_\mu(\tau)\theta_\mu(\tau,h)y^{-s}d\mu(\tau)$$

and a regularized theta lift:

$$\theta_{f^!}(h):=\sum_{\mu\in L^\vee/L}\int_{\mathcal{F}}^{reg} f^!_\mu(\tau)\theta_\mu(\tau,h)d\mu(\tau).$$

Borcherds's main result has been reformulated in adelic language by Kudla; the following are the main theorems concerning regularized theta lifts and are taken from [Kudla, 2003].

Theorem 4. *Let $f^!\in S^!_{-k_0,L}$ be a weakly holomorphic modular form and assume that the Fourier coefficients $c_\varphi(m)$ for $m\le 0$ are integers. Then there exists a meromorphic modular form $\Psi_{f^!}$ on $O(V)(\mathbb{A})$, of weight $k=\frac{1}{2}c_0(0)$ such that*

$$\theta_{f^!}(h)=-2\log|\Psi_{f^!}(h)|-c_0(0)(2\log|y|+\log(2\pi)+\Gamma'(1))$$

Borcherds also described the divisor of the modular form $\Psi_{f^!}$; it turns out to be determined by the principal part $P_{f^!}$ and to be a sum of special divisors whose definition we recall next. We denote by D the space of negative oriented 2-planes in $\mathbb{R}$; this is the Hermitian symmetric space associated with the group $H(\mathbb{R})$, where H denotes the group $O(V)$. For $x \in V$ a vector with positive norm, let

$$D_x = \{z \in D | z \perp x\}.$$

Denote by H_x the stabilizer of x in H. For $h \in H(\mathbb{A}_f)$ and $K \subset H(\mathbb{A}_f)$ open compact, consider the map of Shimura varieties

$$H_x(\mathbb{Q})\backslash D_x \times H_x(\mathbb{A}_f)/(H_x(\mathbb{A}_f) \cap hKh^{-1}) \to H(\mathbb{Q})\backslash D \times H(\mathbb{A}_f)/K =: X_K$$

sending (z, h') to $(z, h'h)$ whose image defines a divisor in X_K that we denote by $Z(x, h)_K$. The divisors of the meromorphic functions $\Psi_{f^!}$ on X_K constructed by Borcherds are certain weighted combinations of the $Z(x, h)_K$ defined as follows. For a positive rational number m, let

$$\Omega_m(\mathbb{A}_f) = \{v \in V(\mathbb{A}_f) | q(v) = m\} \subset V(\mathbb{A}_f)$$

and assume that $\Omega_m(\mathbb{A}_f) \neq \emptyset$. The group $H(\mathbb{A}_f)$ acts on $\Omega_m(\mathbb{A}_f)$ and the number of orbits under any compact open subgroup $K \subset H(\mathbb{A}_f)$ is finite. Thus choosing $x_0 \in \Omega_m(\mathbb{A}_f)$ we can write

$$\Omega_m(\mathbb{A}_f) = \coprod_{i \in I} K \cdot h_i^{-1} x_0$$

where I is a finite set and $h_i \in H(\mathbb{A}_f)$ for all $i \in I$. For m a positive rational number, $K \subset H(\mathbb{A}_f)$ open compact and $\varphi \in \mathcal{S}(V(\mathbb{A})_f)^K$, define

$$Z(m, \varphi)_K = \sum_{i \in I} \varphi(h_i^{-1} x_0) Z(x_0, h_i)_K.$$

(This is assuming $\Omega_m(\mathbb{A}_f) \neq \emptyset$; otherwise define $Z(m, \varphi)_K = 0$). We can now state Borcherds's theorem describing the divisor of $\Psi_{f^!}$.

Theorem 5. *Let $f^! \in S_{-k_0,L}$ be as in the statement of Theorem 4. Let $K \subset O(V(\mathbb{A})_f)$ be a compact open subgroup such that the image of $\mathbb{C}[L^\vee/L] \to \mathcal{S}(V(\mathbb{A}_f))$ is contained in*

$\mathcal{S}(V(\mathbb{A}_f))^K$. Denote by $\Psi_{f^!}$ the Borcherds lift of $f^!$ regarded as a meromorphic function on X_K. Then:

$$div(\Psi_{f^!}^2) = \sum_{\substack{a>0 \\ \mu \in L^\vee/L}} c(-a,\mu) Z(a,\mu,K).$$

3.2 Harmonic Whittaker forms

To compute integrals of automorphic forms against the Borcherds lifts defined in the previous section, it will be more convenient to use as an input a class of functions more general than that of weakly holomorphic modular forms. These functions are known as harmonic Whittaker forms. We briefly review their definition and the main properties of the associated regularized theta lift; see [Bruinier, 2012] for a more detailed exposition and proofs of all the facts in this section.

We start by defining Whittaker functions that will depend on a complex parameter s. Consider Kummer's hypergeometric function:

$$M(a,b,z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $(a)_0 = 1$. We then define the function:

$$M_{\nu,\mu}(z) = e^{-z/2} z^{1/2+\mu} M(\frac{1}{2} + \mu - \nu, 1 + 2\mu, z)$$

This is a solution of the Whittaker differential equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - 1/4}{z^2}\right) w = 0$$

characterized by its asymtptotic behaviour, given by:

$$M_{\nu,\mu}(z) = z^{\mu+1/2}(1 + O(z)) \qquad \text{when } z \to 0,$$

$$M_{\nu,\mu}(z) = \frac{\Gamma(1+2\mu)}{\Gamma(\mu - \nu + 1/2)} e^{z/2} z^{-v} (1 + O(z^{-1})), \qquad \text{when } z \to \infty$$

For $s \in \mathbb{C}$ and $v \in \mathbb{R}$, define:

$$\mathcal{M}_s(v) = |v|^{-k/2} M_{sgn(v)k/2, s/2}(|v|)$$

For a positive rational number m, we define the Whittaker function $f_m(\cdot, s) : \mathbb{H} \to \mathbb{C}$ of parameter s as

$$f_m(\tau, s) = \Gamma(s+1)^{-1}\mathcal{M}_s(-4\pi mv)\psi_{\mathbb{R}}(-m\overline{\tau})$$
$$= \Gamma(s+1)^{-1}\mathcal{M}_s(-4\pi mv)e^{-2\pi m(v+iu)}$$

where we write $\tau = u + iv$. We would like to define the theta lift of such a Whittaker function as

$$\int_{\tilde{\Gamma}_\infty\backslash\mathbb{H}} f_m(\tau, s)\cdot v\theta_\mu(\tau, h)d\mu(\tau).$$

However, this integral does not converge due to the exponential growth of $f_m(\tau, s)$. Bruinier shows that it can be regularized by writing it as an iterated integral. We will be especially interested in the value of this lift at

$$s_0 := \frac{\dim(V) - 2}{2}.$$

Definition 2. *Let $\mu \in L^\vee/L$ and $m \in q(\mu + L)$ be positive. The regularized theta lift $\Phi_{m,\mu}(h, s)$ is defined for $Re(s) > s_0 + 2$ by:*

$$\Phi_{m,\mu}(h, s) = \int^{reg}_{\tilde{\Gamma}_\infty\backslash\mathbb{H}} f_m(\tau, s)\cdot v\theta_\mu(\tau, h)d\mu(\tau)$$
$$= \int_{\mathbb{R}_{>0}}\int_{\mathbb{Z}\backslash\mathbb{R}} f_m(\tau, s)\theta_\mu(\tau, h)du\ \frac{dv}{v}.$$

Theorem 6. *[Bruinier, 2012][Thm. 5.2] For $Re(s) > s_0+2$, the integral defining $\Phi_{m,\mu}(h, s)$ converges outside a set of measure zero and defines an integrable function on $X_K := H(\mathbb{Q})\backslash H(\mathbb{A})/KK_\infty$.*

Note that for s as in the theorem, we have

$$\Phi_{m,\mu}(h, s) = \sum_{v\in V_m} \phi_\mu(h, v, s) \tag{3.1}$$

where $\phi_\mu(h, v, s) = (\omega(h_f)\varphi_\mu)(v)\cdot\phi_\infty(h_\infty^{-1}v, s)$ and

$$\phi_\infty(v, s) = \Gamma(s+1)^{-1}\int_0^{+\infty}\mathcal{M}_s(-4\pi my)e^{-2\pi my}e^{-2\pi y(q(v^+)-q(v^-))}\frac{dy}{y}.$$

In fact, the sum $\sum_{v\in V_m}\phi_\mu(h, v, s)$ converges to an integrable function on X_K for any s with $Re(s) > s_0$ ([Bruinier, 2012, Thm. 5.3]).

Theorem 7. *[Bruinier, 2012, Thm. 5.12] For any $h \in X_K \backslash Z(m,\mu)_K$, the function $\Phi_{m,\mu}(h,s)$ admits a meromorphic continuation to $s \in \mathbb{C}$ with a simple pole at $s = s_0$ with residue*

$$A(m,\mu) := Res_{s=s_0} \Phi_{m,\mu}(h,s) = 2\frac{\deg(Z(m,\mu))}{Vol(X_K)}.$$

Outside the set of poles the resulting function $\Phi(\cdot, s)$ is real analytic in $X_K \backslash Z(m,\mu)_K$.

Let Y be a Cartier divisor Y in a complex manifold X. We say that a function F on $X \backslash Y$ has a logarithmic singularity along Y if for every $y \in Y$ there is a neighborhood U of y and a local equation G of $Y \cap U$ such that $F - \log |G|$ extends to a continuous function on U.

Definition 3. *Let $f = \sum c(m,\mu) q^m \varphi_\mu$ be a polynomial valued in $\mathbb{C}[L^\vee/L]$ with coefficients $c(m,\mu) \in \mathbb{Z}$. Define:*

$$Z(f)_K = \sum c(m,\mu) Z(m,\mu)_K$$
$$\Phi_f(h,s) = \sum c(m,\mu) \Phi_{m,\mu}(h,s)$$

The regularized Green function $\Phi_f(h)$ is defined as the constant term in the Laurent expansion of $\Phi_f(h,s)$ at $s = s_0$:

$$\Phi_f(h) = CT_{s=s_0} \Phi_f(h,s).$$

By [Bruinier, 2012, Thm. 5.14], Φ_f a real analytic function on $X_K \backslash Z(f)_K$ with a logarithmic singulariy along $-2Z(f)_K$.

The relation with the Borcherds lift defined above is as follows. Let $f^! \in S^!_{-k_0,L}$ be a weakly holomorphic modular form of weight $-k_0$ valued in $\mathbb{C}[L^\vee/L]$ with $c(0,0) = 0$. Then its principal part $Pf^! = \sum c(-a,\mu; f) q^a \varphi_\mu$ is a polynomial valued in $\mathbb{C}[L^\vee/L]$ and for a certain constant $C \in \mathbb{R}$ we have:

$$\log |\Psi_{f^!}(h)| = -\frac{1}{2} \Phi_{Pf^!}(h) + C$$

This follows from the fact that both sides define harmonic (by [Bruinier, 2012, Remark 5.17]) functions on X_K with the same logarithmic singularity and hence their difference is constant. It follows that for $f^! \in S^!_{-k_0,L}$ we can write

$$\log |\Psi_{f^!}(h)| = -\frac{1}{2} CT_{s=s_0} \sum c(-m,\mu) \Phi_{m,\mu}(h,s) + C$$

(for a constant $C \in \mathbb{R}$) and since in fact the residue of the right hand side vanishes by Theorem 7 this can be written simply as:

$$\log|\Psi_{f^!}(h)| = -\frac{1}{2}\lim_{s\to s_0}\sum c(-m,\mu)\Phi_{m,\mu}(h,s) + C. \tag{3.2}$$

Chapter 4

Local Zeta integrals

4.1 Doubling Integrals

We will now briefly describe the doubling method for admissible representations of $Sp_{2n}(K))$ and the main result in [Piatetski-Shapiro and Rallis, 1986] relating these zeta integrals to local L-functions.

Let k be a non-archimedean local field of characteristic 0. Let $V = k^{2n}$ be endowed with its standard symplectic form (see Notations) and let $G = Sp(V)$ be the group of symplectic transformations of V. Fix a polarization $V = X \oplus Y$ (here X and Y are as in 1.6.1) and let $Q \subset G$ be the parabolic subgroup fixing Y; we write $Q = MN$ with Levi subgroup $M \cong GL_n(k)$ and unipotent radical $N \cong Sym_n(k)$. Consider the vector space $W = V \oplus V$ endowed with the symplectic form

$$((v_1, v_2), (v_1', v_2')) = (v_1, v_1') - (v_2, v_2').$$

Let $H = Sp(W)$ and denote by $\iota : G \times G \to W$ the natural embedding given by $\iota(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$. We denote by $P \subset H$ the parabolic subgroup of H fixing $Y \oplus Y \subset H$. For a character $\chi : k^\times \to \mathbb{C}^\times$ and $s \in \mathbb{C}$, consider the representation $Ind_P^H(\chi|\cdot|^s)$: this is the space of functions $\Phi : H \to \mathbb{C}$ satisfying

$$\Phi(n(b)m(a)g) = \chi(\det(a))|\det(a)|^{s+n+\frac{1}{2}}\Phi(g)$$

and the action of H is given by $(\rho(h)\Phi)(h') = \Phi(h'h)$.

Now let π be an irreducible representation of G and denote by $\pi^\vee$ its contragredient. For $v \in \pi$, $v^\vee \in \pi^\vee$ and $\Phi(\cdot, \chi, s) \in Ind_P^H(\chi|\cdot|^s)$, consider the integral:

$$Z(v, v^\vee, \Phi, s) = \int_G (\pi(g)v, v^\vee)\Phi(\delta\iota(g,1), \chi, s)dg$$

where the left-invariant measure dg is chosen so that the maximal compact subgroup $K = Sp_{2n}(\mathcal{O})$ has measure 1 and

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2}1_n & \frac{1}{2}1_n \\ \frac{1}{2}1_n & \frac{1}{2}1_n & 0 & 0 \\ 1_n & -1_n & 0 & 0 \\ 0 & 0 & 1_n & 1_n \end{pmatrix}.$$

This integral is known as the doubling zeta integral. Its value for the spherical matrix coefficient of an unramified representation π was computed in [Piatetski-Shapiro and Rallis, 1986]; we include it for the convenience of the reader.

Theorem 8. *[Piatetski-Shapiro and Rallis, 1986] Let π be an unramified representation of G, $v_0 \in \pi$ and $v_0^\vee \in \pi^\vee$ be vectors fixed by K such that $(v_0, v_0^\vee) = 1$. Let $\Phi^0 \in Ind_P^H(|\cdot|^s)$ the unique $Sp_{4n}(\mathcal{O})$-fixed vector such that $\Phi^0(1) = 1$. Then:*

$$Z(v_0, v_0^\vee, \Phi^0, s) = \frac{L(\pi, std, s + \frac{1}{2})}{d(s)}$$

where $L(\pi, std, s)$ denotes the local Langlands L-function attached to the standard representation of ${}^LG^0$ and

$$d(s) = \zeta(s + n + \frac{1}{2}) \prod_{j=0}^{n-1} \zeta(2s + 2j + 1).$$

One can construct functions $\Phi \in Ind_P^H(|\cdot|^s)$ using the Weil representation as follows. Consider a quadratic vector space V of even dimension m over k. There is a quadratic character χ_V attached to V defined by

$$\chi_V(x) = (x, (-1)^{\frac{m}{2}} \det(V))_k,$$

where $(\cdot,\cdot)_k =$ denotes the Hilbert symbol of k and $\det(V) \in k^\times/(k^\times)^2$ is the determinant of the matrix $((v_i, v_j))_{1\le i,j\le m}$ for any orthogonal basis $(v_i)_{1\le m}$.

For $\varphi \in \mathcal{S}(V^{2n})$ and $s \in \mathbb{C}$, consider the function $\Phi(\cdot, s)$ defined by

$$\Phi(g,s) = (\omega(g)\varphi)(0) \cdot |a(g)|^{s-s_0}$$

where $s_0 = \frac{m}{2} - \frac{n+1}{2}$. The assignment $\varphi \mapsto \Phi(\cdot, s_0)$ defines a map

$$\lambda : \mathcal{S}(V^n) \to Ind_P^H(\chi_V|\cdot|^{s_0})$$

intertwining the action of H.

The following result was proved in [Li, 1992a].

Proposition 3. *[Li, 1992a] For $\varphi, \varphi' \in \mathcal{S}(V^n)$, let $\Phi(\cdot, s) = \lambda(\varphi \otimes \varphi') \in Ind_P^H(\chi_v|\cdot|^s)$. Then:*

$$\Phi(\delta\iota(g,1), s_0) = \gamma_V^{-n} \int_{V^n} (\omega(g)\varphi)(x)\varphi'(-x)dx.$$

Proof. Note that

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2}1_n & \frac{1}{2}1_n \\ \frac{1}{2}1_n & \frac{1}{2}1_n & 0 & 0 \\ 1_n & -1_n & 0 & 0 \\ 0 & 0 & 1_n & 1_n \end{pmatrix} = \iota(\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}, 1)m(\begin{pmatrix} 1_n & -1_n \\ \frac{1}{2}1_n & \frac{1}{2}1_n \end{pmatrix}).$$

Hence

$$\begin{aligned}\Phi(\delta\iota(g,1), s_0) &= \gamma_V^{-n} \int_{V^n} \omega(m(\begin{pmatrix} 1_n & -1_n \\ \frac{1}{2}1_n & \frac{1}{2}1_n \end{pmatrix}))(\omega(g)\varphi) \otimes \varphi')(x,0)dx \\ &= \gamma_V^{-n} \int_{V^n} (\omega(g)\varphi)(x)\varphi'(-x)dx.\end{aligned}$$

□

4.1.1 Representations of $Sp_4(k)$ with Iwahori-fixed vectors

We will denote by I the Iwahori subgroup of $Sp_4(k)$ given by:

$$I = \left\{ g \in Sp_4(k) | g \in \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix} \right\} \subset Sp_4(\mathcal{O}).$$

Note that I is just the inverse image of the standard Borel subgroup $B(\mathbb{F}_q) \subset Sp_4(\mathbb{F}_q)$ under the reduction map $Sp_4(\mathcal{O}) \to Sp_4(\mathbb{F}_q)$.

We will be interested in studying some representations V of $Sp_4(k)$ having Iwahori-fixed vectors. Such representations can be studied through the action of the Iwahori-Hecke algebra $\mathcal{H}(G, I)$.

Definition 4. *Let $X \subset G$ be a bi-I-invariant subset of G. We denote by $\mathcal{H}(X, I)$ the algebra of compactly supported, bi-I-invariant functions on G, with multiplication given by convolution:*

$$(f_1 * f_2)(g) = \int_I f_1(gh) f_2(h^{-1}) dh$$

where dh is the Haar measure on $Sp_4(k)$ giving I measure 1. The algebra $\mathcal{H}(G, I)$ is called the Iwahori-Hecke algebra.

Note that this is a unital associative algebra with unit $\phi_I := \text{char}(I)$ (characteristic function of I). There is an inclusion of algebras $\mathcal{H}(K, I) \to \mathcal{H}(G, I)$. Recall the Bruhat decomposition:

$$K = \coprod_{w \in W} IwI$$

where the disjoint union is over the Weyl group $W = N_K(T)/T$. Thus the algebra $\mathcal{H}(K, I)$ has dimension $|W|$ and a natural basis (as a vector space) indexed by W and given by $\phi_w = \text{char}(IwI)$. We will consider ϕ_w also as an element of $\mathcal{H}(G, I)$ via the inclusion given above.

The main interest of Iwahori-Hecke algebras in the theory of admissible representations of p-adic groups is the following fundamental fact due to Borel.

Theorem 9. *[Borel, 1976] Let G be the group of rational points of a connected semi-simple algebraic group $\mathcal{G}$ defined over a locally compact non-archimedean field k and let $I \subset G$ be an Iwahori subgroup. Denote by $Adm(G)^I$ the abelian category of admissible representations of G over a fixed field of characteristic 0 having non-zero Iwahori-fixed vectors and by $\mathcal{H}(G, I)$-mod the abelian category of finite-dimensional $\mathcal{H}(G, I)$-modules. Then the functor*

$$\begin{aligned} Adm(G)^I \to & \quad \mathcal{H}(G, I) - mod \\ V \mapsto & \quad V^I \end{aligned}$$

is an exact equivalence of categories.

4.1.2

We next review some results concerning the representation π_{ng} of $Sp_4(k)$ (see Section 2.4).

Let B the non-split quaternion algebra over k; The reduced norm $n : B \to k$ makes $V = B$ a quadratic vector space. We would like to have an explicit model for the theta correspondence for the reductive dual pair $(Sp_4, O(V))$. For a detailed description of this explicit correspondence see [Li, 1989]. Fix an irreducible representation π of $O(V)$ and consider the $Sp_4 \times O(V) \times Sp_4 \times O(v)$-equivariant map:

$$I : \pi \otimes \mathcal{S}(V) \otimes \pi^\vee \otimes \mathcal{S}(V) \to C^\infty(Sp_4(k))$$

given by

$$I(v, \varphi, v^\vee, \varphi')(g) = \int_{O(V)} (\pi(h)v, v^\vee)(\omega(g, h)\varphi, \varphi')dh.$$

Note that the integral converges since $O(V)$ is compact.

Proposition 4. *1. The map I factors through $\pi \otimes \mathcal{S}(V) \otimes \pi^\vee \otimes \mathcal{S}(V) \to \theta(\pi) \otimes \theta(\pi^\vee)$.*

2. If $\theta(\pi) \neq 0$, then $I \neq 0$.

Proof. This is Prop.16.1 in [Gan and Ichino, to appear], combined with Prop. 8.1 in [Gan and Takeda, 2011b] which implies that $\Theta(\pi) = \theta(\pi)$ in this case. □

Using this explicit construction, we can study vectors of minimal level in π_{ng}. The following result can be found in the literature (see [Schmidt, 2007] or [Roberts and Schmidt, 2007], for example), but the proof given here is shorter and we have decided to include for the sake of completeness.

Proposition 5. *Let* $\pi = \pi_{ng} = \theta(1_{B^\times} \boxtimes 1_{B^\times})$ *and* $\varphi_0 = 1_{\mathcal{O}^2} \in \mathcal{S}(V^2)$.

1. *The function*
$$f(g) = \int_{V^2} (\omega(g)\varphi_0)(x)\varphi_0(x)dx$$
is a non-zero matrix coefficient of π.

2. *The space* π^{P_1} *of vectors fixed by the Siegel parahoric has dimension* 1*; the action of the Hecke algebra* $\mathcal{H}(K,I)$ *on a non-zero vector* $v_0 \in \pi^{P_1}$ *is given by*
$$\phi_{w_\alpha} v_0 = v_0,$$
$$\phi_{w_\beta} v_0 = -q^{-1} v_0.$$

Proof. The first statement follows directly from the previous Proposition. Lemma 1 shows that the matrix coefficient $f(g)$ is bi-P_1-invariant, hence the space π^{P_1} contains non-zero vectors and $f(g) = (\pi(g)v_0, v_0^\vee)$ for $v_0 \in \pi^{P_1}$ and $v_0^\vee \in (\pi^\vee)^{P_1}$. Let us compute the action of $\mathcal{H}(K,I)$ on $f(g)$. Since $\omega(w_\alpha)\varphi_0 = \varphi_0$,we find:
$$f(w_\alpha) = f(1)$$
and
$$f(w_\beta) = \int_{V^2} (\omega(w_\beta)\varphi_0)(x)\varphi_0(x)dx =$$
$$-q^{-1}\int_{V^2} (1_{\mathcal{O}\oplus\mathcal{D}})(x)\varphi_0(x)dx = -q^{-1}f(1)$$
and the claim follows. □

4.1.3

In the paper [Zorn, 2011], Zorn studies representations π of $Sp_4(k)$ (for $p \neq 2$) with Iwahori-fixed vectors and finds explicit test vectors $v, v^\vee, \Phi$ such that $Z(v, v^\vee, \Phi, s)$ equals $L(\pi, std, s)$ up to an explicit product of degree 1 L-functions (where $L(\pi, std, s)$ was defined in [Lusztig, 1983]). We review his result here and then state the detailed result for the particular representation π_{ng}.

Let $\mathbb{H}$ be a hyperbolic plane: the quadratic vector space underlying the quadratic form represented by $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Choose a basis $\{e, f\}$ of $\mathbb{H}$ of isotropic vectors such that (e, f)=1 and let $L_{\mathbb{H}} = \mathcal{O}e \oplus \mathcal{O}f$. Given a quadratic vector space V and a non-negative integer r, define a quadratic vector space $V_r := V \oplus \mathbb{H}^r$ (orthogonal direct sum). For a Schwartz form $\varphi \in \mathcal{S}(V^4)$, let $\varphi^{(r)} = \varphi \otimes (1_{L^r_{\mathbb{H}}})^{\otimes 4} \in \mathcal{S}(V_r^4)$. Let $s_0 = \frac{dim(V)-5}{2}$ and define a function $\Phi_{\varphi^{(r)}}(\cdot) \in Ind_P^{Sp_8}(\chi_V, s_0 + r)$ by

$$\Phi_{\varphi^{(r)}}(g) = (\omega(g)\varphi^{(r)})(0).$$

Note that $\Phi_{\varphi^{(r)}}(\cdot)$ is the special value at $s = r$ of the standard section $\Phi_\varphi(\cdot, s) \in Ind_P^{Sp_8}(\chi_V, s)$ given by

$$\Phi_\varphi(g, s) = (\omega(g)\varphi)(0) \cdot |a(g)|^{s-s_0}.$$

Theorem 10. *[Zorn, 2011] Let π be a representation of Sp_4 with non-zero Iwahori-fixed vectors. Denote by $\rho' : W_k' \to PGSp_4(\mathbb{C})$ its local Langlands parameter and let $std : PGSp_4(\mathbb{C}) \to GL_5(\mathbb{C})$ be the standard representation of $PGSp_4(\mathbb{C})$. For an unramified quadratic character χ, there exists a six-dimensional quadratic space (V, Q) with $\chi_V = \chi$ and explicit choices of $\varphi \in \mathcal{S}(V^4)$ and $v \in \pi^I$, $v^\vee \in (\pi^\vee)^I$ such that for all non-negative integers r we have:*

$$\int_{Sp_4} (\pi(g)v, v^\vee)\Phi_{\varphi^{(r)}}(\delta\iota(g, 1))dg = \frac{L(r + 1, \kappa_V \otimes (std \circ \rho'))}{d_{\Phi,\pi}(r + \frac{1}{2})}$$

where $\kappa_V : W_k \to \mathbb{C}^\times$ is the character of W_k associated with χ_V by class field theory and $d_{\Phi,\pi}$ is an explicit product of degree 1 L-factors.

We focus now on the representation $\pi = \pi_{ng}$ of Sp_4. Recall (see [Gan and Takeda, 2011b]) that π is the theta lift of the trivial representation of $O(V)$ for the 4-dimensional quadratic vector space V with $\chi_V = 1$ and Hasse invariant $\epsilon = -1$. Thus V can be identifed with the non-split quaternion algebra B over k and there is a natural lattice $L = \mathcal{O}_B$ in V given by the maximal order of B. It can also be described as the non-generic direct summand of the representation $I_Q^{Sp_4}(1, St)$ induced from the Klingen parabolic subgroup.

In Proposition 5 we have defined Iwahori-fixed vectors $v_0 \in \pi^I$ and $v_0^\vee \in (\pi^\vee)^I$ affording a certain representation of $\mathcal{H}(G, I)$. Consider the Schwartz function $\varphi_0 = \mathrm{char}(\mathcal{O}_B^2)$.

Zorn's result shows that with these choices one has, for $\pi = \pi_{ng}$ and an explicit constant $C \in \mathbb{Q}^\times$:

$$\int_{Sp_4} (\pi(g)v_0, v_0^\vee)\Phi_{(\varphi_0\otimes\varphi_0)^{(r)}}(\delta\iota(g,1))dg = C \cdot \frac{L(r,\pi,std)}{d(r-\frac{1}{2})}.$$

where:

$$L(s,\pi,std) = \zeta(s)^2\zeta(s+1) = \zeta(s)L(St \times St, s)$$

$$d(s) = \zeta(s+\frac{3}{2})\zeta(2s+1).$$

Remark 4. *An explicit formula (generalizing MacDonald's formula for spherical functions) for the matrix coefficient $(\pi(g)v_0, v_0^\vee)$ has been established by Jian-Shu Li in [Li, 1992b]; this provides an alternative way of proving Zorn's result.*

4.2 Non-unique models

Let us first recall the main result of [Piatetski-Shapiro and Rallis, 1988], specialized here to the group Sp_4. Let k be a non-archimedean field with ring of integers $\mathcal{O}$ and consider the representation $Ind_{P(k)}^{Sp_4(k)}(|\cdot|^s)$ with normalized standard spherical vector $\Phi(\cdot, s)$ defined by:

$$\Phi(n(X)m(A)k, s) = |det(A)|^{s+\frac{5}{2}}$$

for any $k \in Sp_4(\mathcal{O})$. For a symmetric two-by-two matrix $T \in Sym_2(k)$, define:

$$j_T(s) = \int_{N(k)} \Phi(w_0 n(X), s)\psi(-tr({}^tTX))dX$$

where w_0 denotes the long element of the Weyl group of Sp_4.

Theorem 11. *[Piatetski-Shapiro and Rallis, 1988] Let k be a local non-archimedean field, $\mathcal{O}$ its ring of integers and ϖ a uniformizer. Consider an irreducible unramified representation (π, V) of $Sp_4(k)$ and denote by v_0 an unramified vector in V. Let l_T be a linear functional on V satisfying:*

$$l_T(\pi(n(X))v) = \overline{\psi(tr(TX))}v, \quad v \in V, \quad X \in Sym_2(k),$$

where $T \in Sym_2(\mathcal{O})$ *is a fixed integral two by two matrix of rank* 2 *such that* $T \notin \varpi Sym_2(\mathcal{O})$*. Then:*

$$\int_{M_{2,2}(\mathcal{O}_k)\cap GL_2(k)} l_T(m(g)v_0)|\det(g)|^{s-\frac{1}{2}}dg = \frac{L(\pi,\rho,s+\frac{1}{2})}{j_T(s)d(s)} l_T(v_0),$$

where ρ *is the standard representation of* ${}^LSp_4 = SO_5(\mathbb{C}) \times W_k$ *in* $GL_5(\mathbb{C})$ *and* $d_4(s)$ *is given by*

$$d(s) = \zeta(s+\frac{5}{2})\zeta(2s+1)\zeta(2s+3).$$

If T *is unimodular, then*

$$j_T(s)d(s) = L(\omega_T, s+\frac{3}{2})\zeta(2s+1)$$

where $\omega_T = (\cdot, -\det(T))$ *is the quadratic character attached to* T*.*

Using Zorn's result and the uniqueness of Iwahori-fixed vectors for the representation π_{ng} we can generalize Lemma 11 to the ramified representation π_{ng}.

Proposition 6. *Let* v_0 *be an Iwahori-fixed vector in* (π_{ng}, V)*. Let* l_T *be a linear functional on* V *satisfying:*

$$l_T(\pi(n(X))v) = \overline{\psi(tr(TX))}v, \quad v \in V, \quad X \in Sym_2(k),$$

where $T \in Sym_2(\mathcal{O})$ *is of the form* $T = k \cdot \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \cdot {}^t k$ *for* $k \in GL_2(\mathcal{O})$ *and* $d \in \mathcal{O}^\times \cup \varpi\mathcal{O}^\times$*. Let* $v_1, v_d \in \mathcal{O}_B$ *such that* $Q[(v_1, v_d)] = T$*. Then, for* s *with large enough real part we have*

$$\int_{N(k)\backslash Sp_4(k)} l_T(\pi(g)v_0)\cdot\omega(g)\varphi_{\mathcal{O}_B^2}(v_1,v_d)|a(g)|^s dg = C\cdot l_T(v_0)\frac{L(s,\pi_{ng},std)}{j_T(s-\frac{1}{2})d(s-\frac{1}{2})},$$

where $C \in \mathbb{Q}^\times$ *and*

$$L(s,\pi_{ng},std) = \zeta(s)L(St\times St, s), \qquad d(s) = \zeta(s+\frac{3}{2})\zeta(2s+1),$$

$$j_T(s) = \int_{N(k)} \Phi_{\varphi_{\mathcal{O}_B^2}\otimes\varphi_{\mathcal{O}_B^2}}(\delta\iota(n(X),1),s)\psi(-tr(TX))dX.$$

Proof. Let $\varphi = \varphi_{\mathcal{O}_B^2}$. For r a large positive integer, consider the integral

$$I(r) = \int_{Sp_4(k)} l_T(\pi(g)v_0)\Phi_{(\varphi\otimes\varphi)^{(r)}}(\delta\iota(g,1))dg$$

where the measure dg is normalized by $Vol(I) = 1$. Due to the uniqueness of Iwahori-fixed vectors, we have for such a measure:

$$\int_I l_T(\pi(kg)v_0)dk = l_T(v_0) \cdot (\pi(g)v_0, v_0^\vee).$$

where $v_0 \in (\pi_{ng}^\vee)^I$ and $(v_0, v_0^\vee) = 1$. Hence, for a constant $C \in \mathbb{Q}^\times$:

$$I(r) = C \cdot l_T(v_0) \cdot \frac{L(r, \pi, std)}{d(r - \frac{1}{2})}$$

Now we compute $I(r)$ using the Iwasawa decomposition. Writing $Sp_4(k) = NMK$ with $K = Sp_4(\mathcal{O})$, we find:

$$I(r) = \int_{GL_2(k)} \int_K l_T(\pi(m(A)k)v_0) \int_{N(k)} (\omega(nm(A)k)\varphi^{(r)}, \varphi^{(r)})\psi(-Tn)dndk|\det(A)|^{-3}dA$$

Since both v_0 and φ are eigenvectors for the Hecke algebra $\mathcal{H}(K, I)$, to prove the claim for $s = r$ it suffices to show that

$$\omega(m(A))\varphi(v_1, v_d) \cdot |\det(A)|^r \cdot j_T(s) = \int_{N(k)} (\omega(nm(A))\varphi^{(r)}, \varphi^{(r)})\psi(-Tn)dn$$

for any $A \in GL_2(k)$. Note that

$$\omega(m(A))\varphi(v_1, v_d) = \begin{cases} 0, & \text{if } A \notin M_2(\mathcal{O}); \\ |\det(A)|^2, & \text{if } A \in M_2(\mathcal{O}). \end{cases}$$

Assume first that $A \notin M_2(\mathcal{O})$. We can assume in fact that $A = \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{pmatrix}$ where $n < 0$ or $m < 0$; we assume that $m < 0$ from now on (the case when $n < 0$ is similar). Then:

$$\int_{N(k)} (\omega(nm(A))\varphi^{(r)}, \varphi^{(r)})\psi(-Tn)dn$$

$$= |\det(A)|^{2+r} \cdot \int_{N(k)} \int_{V(r)^2} \varphi^{(r)}((x_1, x_2) \cdot A)\varphi^{(r)}(-(x_1, x_2))\psi(tr(nQ(x)))\psi(-Tn)dxdn$$

But the inner integral is invariant under $\begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix} N(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$ and hence this expression vanishes.

If $A \in M_2(\mathcal{O})$, then we have:

$$\int_{N(k)} (\omega(nm(A))\varphi^{(r)}, \varphi^{(r)})\psi(-Tn)dn$$

$$= |\det(A)|^{2+r} \cdot \int_{N(k)} \int_{V(r)^2} \varphi^{(r)}((x_1, x_2) \cdot A)\varphi^{(r)}(-(x_1, x_2))\psi(tr(nQ(x)))\psi(-Tn)dxdn$$

$$= |\det(A)|^{2+r} j_T(r)$$

and this proves the claim for $s = r$. Since both sides define bounded analytic functions on $Re(s) > N$ for large enough N that are periodic under $s \mapsto s + \frac{2\pi i}{\log(q)}$, this implies the result for all s with large enough real part.

□

Remark 5. *Using the results in [Zorn, 2011], it seems straightforward to generalize this result to other representations of $Sp_4(\mathbb{Q}_p)$ having a one-dimensional space of Iwahori-fixed vectors. This would allow us to state the main result in this thesis (Theorem 1) for a larger family of representations π_1 and π_2 including the case (for $p|D(B)$) where $\pi_{i,p}$ for $i = 1, 2$ is the sign representation of $B^\times$ or the case (for p not dividing $D(B)$) where $\pi_{i,p}$ is a Steinberg or quadratic twist of Steinberg representation.*

Chapter 5

Global computation

Our goal is to compute the integral

$$R = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} f_1(h)\overline{f_2(h)} \log |\Psi_{f^!}(h)| dh \tag{5.1}$$

where $f_1 \boxtimes \overline{f_2} = \theta_\varphi(\overline{f})$ are certain holomorphic eigenforms of weight 2 with trivial character and maximal level on $B^\times(\mathbb{A})$ obtained as a global theta lift from Sp_4 (see Section 2.6.4) and $\Psi_{f^!}$ is the Borcherds lift of a weakly holomorphic modular form $f^!$ (see Section 3.1).

By equation (3.2), we have:

$$\log |\Psi_{f^!}(h)| = -\frac{1}{2}\Phi_{Pf^!}(h, s'_0) + C$$

with $C \in \mathbb{R}$ a constant, $s'_0 = \frac{3}{2}$ and where for $Re(s') > s'_0$ we write

$$\Phi_{Pf^!}(h, s') = \sum_{\substack{m>0 \\ \mu \in L^\vee/L}} a_{-m,\mu}(f^!)\Phi_{m,\mu}(h, s').$$

By (3.1), for every $m \in \mathbb{Q}_{>0}$ and $\mu \in L^\vee/L$, the function $\Phi_{m,\mu}(h, s')$ is integrable on $H(\mathbb{Q})\backslash H(\mathbb{A})$ and given by:

$$\Phi_{m,\mu}(h, s') = \sum_{v \in V_m} \phi_\mu(h, v, s').$$

For $m \neq 0$ and assuming $V_m \neq \emptyset$, choose a vector $v_m \in V_m$ and let $H_m = Stab_{v_m}(H)$. Then we can write:

$$\Phi_{m,\mu}(h, s') = \sum_{\gamma \in H_m(\mathbb{Q})\backslash H(\mathbb{Q})} \phi_\mu(\gamma h, v_m, s').$$

Substituting this expression in (5.1) (note that the contribution of the constant C vanishes since $Hom_{H(\mathbb{A})}(\pi_1 \otimes \pi_2, \mathbb{C}) = 0$ by assumption), we find that $R = -\frac{1}{2}R(s_0')$ where

$$R(s') = \sum_{m,\mu} a_{-m,\mu}(f^!) \int_{H_m(\mathbb{Q})\backslash H(\mathbb{A})} \phi_\mu(h, v_m, s') f_1(h)\overline{f_2(h)} dh$$

$$= \sum_{m,\mu} a_{-m,\mu}(f^!) \int_{H_m(\mathbb{A})\backslash H(\mathbb{A})} \phi_\mu(h, v_m, s') \int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} f_1(h'h)\overline{f_2(h'h)} dh' dh.$$

Denote by $I_m(h)$ the inner integral:

$$I_m(h) = \int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} f_1(h'h)\overline{f_2(h'h)} dh'.$$

Next recall that the form $f_1 \boxtimes \overline{f_2}$ is obtained as a global theta lift; for $(h_1, h_2) \in (B^\times(\mathbb{A}) \times B^\times(\mathbb{A}))^0$, we have:

$$f_1(h_1)\overline{f_2(h_2)} = \int_{Sp_4(\mathbb{Q})\backslash Sp_4(\mathbb{A})} \overline{f(g)} \theta_\varphi(g, h_1, h_2) dg$$

where φ is given explicitly in Section 2.6. Interchanging the integrals, we can thus rewrite $I_m(h)$ as:

$$I_m(h) = \int_{Sp_4(\mathbb{Q})\backslash Sp_4(\mathbb{A})} \overline{f(g)} \int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} \theta_{\omega(g,h)\varphi}(h') dh'.$$

The inner integral can be computed using Theorem 3.

Lemma 2. *Let $V(m) = \mathbb{Q}v_1 \oplus \mathbb{Q}v_m \subset V$ be the (positive-definite) quadratic space spanned by v_1 and v_m and let η be the quadratic character attached to $V(m)$. For $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$, define*

$$I(g, \varphi, s) = \sum_{\gamma \in P(\mathbb{Q})\backslash Sp_4(\mathbb{Q})} \sum_{(v,v') \in V(m)^2} (\omega(\gamma g)\varphi)(v, v') |a(\gamma g)|^{s+\frac{1}{2}}.$$

Then the sum converges for $Re(s) > \frac{3}{2}$ and the function admits meromorphic continuation to $s \in \mathbb{C}$. Let dh' be the Tamagawa measure on H_m. Then we have:

$$\int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} \theta_{\omega(g)\varphi}(h') dh' = L(1, \eta) I(g, \varphi, -\frac{1}{2}).$$

Proof. Write $V = V(m) \oplus V(m)'$ where $V(m)'$ is the orthogonal complement of $V(m)$ in V. First assume that $\varphi = \varphi^1 \otimes \varphi^2$ where $\varphi_1 \in \mathcal{S}(V(m)(\mathbb{A})^2)$ and $\varphi_2 \in \mathcal{S}(V(m)'(\mathbb{A})^2)$. In that case we have

$$\theta_{\omega(g)\varphi}(h') = \theta_{\omega(g)\varphi^1}(h') \cdot \theta_{\omega(g)\varphi^2}(1)$$

for every $h' \in H_m(\mathbb{A})$, so that:

$$\int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} \theta_{\omega(g)\varphi}(h')dh' = \theta_{\omega(g)\varphi^2}(1) \cdot \int_{H_m(\mathbb{Q})\backslash H_m(\mathbb{A})} \theta_{\omega(g)\varphi^1}(h')dh'.$$

By the Siegel-Weil formula (Theorem 3) this is the value at $s = -\frac{1}{2}$ of the expression

$$= L(1,\eta)\theta_{\omega(g)\varphi^2}(1) \sum_{\gamma \in P(\mathbb{Q})\backslash Sp_4(\mathbb{Q})} (\omega(\gamma g)\varphi^1(0)\ |a(\gamma g)|^{s+\frac{1}{2}}.$$

Since $\theta_{\omega(g)\varphi^2}(1)$ is left invariant under $Sp_4(\mathbb{Q})$, we can rewrite this as

$$L(1,\eta) \sum_{\gamma \in P(\mathbb{Q})\backslash Sp_4(\mathbb{Q})} \theta_{\omega(\gamma g)\varphi^2}(1)(\omega(\gamma g)\varphi^1)(0)\ |a(\gamma g)|^{s+\frac{1}{2}}$$

$$= L(1,\eta) \sum_{\gamma \in P(\mathbb{Q})\backslash Sp_4(\mathbb{Q})} \sum_{(v,v') \in V(m)^2} (\omega(\gamma g)\varphi^1)(0)(\omega(\gamma g)\varphi^2)(v,v')\ |a(\gamma g)|^{s+\frac{1}{2}}$$

$$= L(1,\eta) \sum_{\gamma \in P(\mathbb{Q})\backslash Sp_4(\mathbb{Q})} \sum_{(v,v') \in V(m)^2} (\omega(\gamma g)\varphi)(v,v')\ |a(\gamma g)|^{s+\frac{1}{2}}.$$

By linearity, this proves the claim assuming $\varphi \in \mathcal{S}(V(m)(\mathbb{A})^2) \otimes \mathcal{S}(V(m)'(\mathbb{A})^2)$; this suffices as both sides of the equation depend continuously on φ and $\mathcal{S}(V(m)(\mathbb{A})^2) \otimes \mathcal{S}(V(m)'(\mathbb{A})^2)$ is dense in $\mathcal{S}(V(\mathbb{A})^2)$. □

Thus the lemma shows that $I_m(h) = I_m(h, -\frac{1}{2})$ where we define:

$$I_m(h,s) = L(1,\eta) \int_{Sp_4(\mathbb{Q})\backslash Sp_4(\mathbb{A})} f(g)I(g, \omega(h)\varphi, s)dg.$$

For $Re(s) > \frac{3}{2}$, this can be rewritten unfolding the sum:

$$I_m(h,s) = L(1,\eta) \int_{P(\mathbb{Q})\backslash Sp_4(\mathbb{A})} f(g) \sum_{(v,v') \in V(m)^2} (\omega(g,h)\varphi)(v,v')|a(g)|^{s+\frac{1}{2}}dg.$$

Next note that we can write $V(m)^2$ as a disjoint union $V(m)^2 = (V(m)^2)_{rk=2} \cup (V(m)^2)_{rk<2}$ where

$$(V(m)^2)_{rk=2} = \{(v,v') \in V(m)^2 | \mathrm{Span}(v,v') = V(m)\},$$

$$(V(m)^2)_{rk<2} = \{(v,v') \in V(m)^2 | \mathrm{Span}(v,v') \subsetneq V(m)\}.$$

Arguing as in [Piatetski-Shapiro and Rallis, 1988], one shows that the contribution of $(V(m)^2)_{rk<2}$ to $I_m(h,s)$ vanishes since f is a cusp form. The Levi subgroup $M(\mathbb{Q})$ ($\cong GL_2(\mathbb{Q})$) acts simply transitively on $(V(m))^2_{rk=2}$; since $M \cong P/N$, we can unfold further to

$$I_m(h,s) = L(1,\eta) \int_{N(\mathbb{Q})\backslash Sp_4(\mathbb{A})} \overline{f(g)}(\omega(g,h)\varphi)(v_1, v_{m'})|a(g)|^{s+\frac{1}{2}}dg.$$

where $v_{m'} = dv_m$ is a rational multiple of v_m such that $q(v_{m'}) = m' = d^2m$ is a square-free positive integer.Let $T_m = \begin{pmatrix} 1 & 0 \\ 0 & m' \end{pmatrix}$ be the matrix representing the quadratic form on $V(m)$ in the basis $\{v_1, dv_m\}$. Then

$$(\omega(n(X)g, h)\varphi)(v_1, v_{m'}) = \psi(tr(T_m X))$$

and this gives the expression

$$I_m(h, s) = L(1, \eta) \int_{N(\mathbb{A})\backslash Sp_4(\mathbb{A})} l_{T_m}(f)(g)(\omega(g, h)\varphi)(v_1, v_{m'})|a(g)|^{s+\frac{1}{2}} dg$$

where for a matrix $T \in Sym_2(\mathbb{Q})$ we define

$$l_T(f)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{f(ng)}\psi(tr(Tn))dn.$$

5.0.1

Note that, if $D = D(B)$ denotes the discriminant of the quaternion algebra B (note that we assume D odd), then when considering divisors $Z(m, \mu)$ we can assume that $m \cdot 2D$ is an integer (otherwise $Z(m, \mu) = 0$ for every μ since $L^\vee$ does not contain vectors of norm m). To simplify the final expression, we will assume from now on the following condition:

$$a_{-m,\mu}(f^!) \neq 0 \Rightarrow mD \text{ is a square-free integer.} \tag{5.2}$$

Interpreting $Z(m, \mu)$ as a divisor consisting of CM points, this amounts to considering Borcherds lifts whose divisor is supported on CM points of conductor 1.

Note that under this hypothesis, every vector $v \in L^\vee$ of norm m is primitive, hence the set $V_m(\mathbb{A}) \cap (L^\vee \otimes \hat{\mathbb{Z}})$ forms a single orbit under the action of the maximal compact group $K_f = (R \otimes \hat{Z})^\times$ of $B(\mathbb{A}_f)^\times$. Since φ is invariant under K_f, we have:

$$\int_{H_m(\mathbb{A})\backslash H(\mathbb{A})} \phi_\mu(h, v_m, s') I_m(h, s) dh$$

$$= Vol((K_f \cap H_m(\mathbb{A}_f))\backslash K_f, dh) \cdot \int_{H_m(\mathbb{R})\backslash H(\mathbb{R})} \phi_\mu(h, v_m, s') I_m(h, s) dh$$

where $Vol((K_f \cap H_m(\mathbb{A}_f)\backslash K_f, dh)$ denotes the volume of $(K_f \cap H_m(\mathbb{A}_f))\backslash K_f \subset H_m(\mathbb{A}_f)\backslash H(\mathbb{A}_f)$ with the measure induced by the Tamagawa measures on H and H_m (see Notations).

Recall that for $T = \begin{pmatrix} 1 & 0 \\ 0 & m' \end{pmatrix}$ and any prime p there is a local factor $j_{T,p}(s)$ defined in Propositions 11 (when $p \nmid D(B)$) and 6 (when $p|D(B)$). Similarly, for every prime p there are local factors $d_p(s)$ defined by:

$$d_p(s) = \begin{cases} \zeta_p(s+\frac{5}{2})\zeta_p(2s+1)\zeta_p(2s+3), & \text{if } p \nmid D(B); \\ \zeta_p(s+\frac{3}{2})\zeta_p(2s+1), & \text{if } p|D(B). \end{cases}$$

Define:

$$d(s) = \prod_{p<\infty} d_p(s),$$
$$j_T(s) = \prod_{p<\infty} j_{T,p}(s).$$

Theorem 12. *Under the hypothesis 5.2 we have:*

$$R = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} f_1(h)\overline{f_2(h)} \log|\Psi_{f^!}(h)|dh = q \cdot L'(\pi_1 \otimes \pi_2, 0) \cdot \lim_{(s,s')\to(0,\frac{3}{2})} I(s,s')$$

where $q \in \mathbb{Q}^\times$ *and*

$$I(s,s') = \sum_{m>0,\mu} a_{-m,\mu}(f^!) \frac{Vol(K_0(D))Vol(K_f)}{Vol(K_f \cap H_m(\mathbb{A}_f))} \cdot \frac{L(1,\eta_m)\zeta(s)}{j_{T(m)}(s)d(s)} I_{m,\infty}(s,s').$$

Here $L(\pi_1 \otimes \pi_2, s) = \prod_{p<\infty} L(\pi_{1,p} \otimes \pi_{2,p}, s)$ *and*

$$I_{m,\infty}(s,s')$$
$$= s \cdot \int_{H_m(\mathbb{R})\backslash H(\mathbb{R})} \phi_\infty(h^{-1}v_m, s') \int_{N(\mathbb{R})\backslash Sp_4(\mathbb{R})} l_{T_m}(f)(g)(\omega(g,h)\varphi_\infty)(v_1, v_{m'})|a(g)|^{s+\frac{1}{2}} dgdh.$$

Proof. This follows from the unfolding described above, together with the local computations in Proposition 11 and Proposition 6 that show that there is a constant $q \in \mathbb{Q}^\times$ (a product of the local constants in Proposition 6 for all primes p dividing $D(B)$) such that:

$$I_m(h,s) = q \cdot Vol(K_0(D)) \cdot \frac{L(1,\eta_m)\zeta(s)L(\pi_1 \otimes \pi_2, s)}{j_{T(m)}(s)d(s)} I_{m,\infty}(h,s)$$

with

$$I_{m,\infty}(h,s)$$
$$= s \cdot \int_{H_m(\mathbb{R})\backslash H(\mathbb{R})} \phi_\infty(h^{-1}v_m, s') \int_{N(\mathbb{R})\backslash Sp_4(\mathbb{R})} l_{T_m}(f)(g)(\omega(g,h)\varphi_\infty)(v_1, v_{m'})|a(g)|^{s+\frac{1}{2}} dgdh.$$

□

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