LIMITING PROPERTIES OF CERTAIN GEOMETRIC FLOWS IN COMPLEX GEOMETRY

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ABSTRACT

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In this thesis, we study convergence results of certain non-linear geometric flows on vector bundles over complex manifolds.

First we consider the case of a semi-stable vector bundle E over a compact Kähler manifold X of arbitrary dimension. We show that in this case Donaldson's functional is bounded from below. This allows us to construct an approximate Hermitian-Einstein structure on E along the Donaldson heat flow, generalizing a classic result of Kobayashi for projective manifolds to the Kähler case.

Next we turn to general unstable bundles. We show that along a solution of the Yang-Mills flow, the trace of the curvature $\Lambda F(A_t)$ approaches in L^2 an endomorphism with constant eigenvalues given by the slopes of the quotients from the Harder-Narasimhan filtration of E. This proves a sharp lower bound for the Hermitian-Yang-Mills functional and thus the Yang-Mills functional, generalizing to arbitrary dimension a formula of Atiyah and Bott first proven on Riemann surfaces. Furthermore, we show any reflexive extension to all of X of the limiting bundle E_∞ is isomorphic to $Gr^{hns}(E)^{**}$, verifying a conjecture of Bando and Siu. Our work on semi-stable bundles plays an important part of this result.

For the final section of this thesis, we show that, in the case where X is an arbitrary Hermitian manifold equipped with a Gauduchon metric, given a stable Higgs bundle the Donaldson heat flow converges along a subsequence of times to a Hermitian-Einstein connection. This allows us to extend to the non-Kähler case the correspondence between stable Higgs bundles and (possibly) non-unitary Hermitian-Einstein connections first proven by Simpson on Kähler manifolds.

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1 Introduction

All of the problems considered in this thesis lie within complex differential geometry, a field which combines both analytic and algebraic techniques to study a wide range of geometric problems. These problems generally involve solving non-linear partial differential equations that describe certain optimal curvature conditions, and the existence and regularity of solutions to these PDEs can relate intimately to the underlying geometric structure. In many cases, this relationship with the geometry is not a hindrance, but in fact an essential tool to help overcome the difficulties associated with the non-linearity of these equations.

Many techniques have been successfully employed in studying these types of problems. In this thesis I focus on geometric flows, a technique which has gained in prominence after Perelman's solution of the Poincaré conjecture ([32], [33], [34]) using Hamilton's Ricci flow [24]. Perelman's pioneering work on Ricci flow also led to many important estimates in the compact Kähler setting [41], and these results have pushed the Kähler-Ricci flow to the forefront of many geometric problems. Such problems including a geometric approach to the Minimal Model program ([47], [48]), and the Kähler-Einstein problem in the Fano case (see [37], [36]).

This thesis involves a robust investigation of the technique and applications of certain geometric flows. We restrict ourselves to the case where E is a holomorphic vector bundle over a fixed compact complex manifold X. Let ω be a Hermitian $(1,1)$ form on X normalized so that X has volume one. In this setting, perhaps the most natural geometric problem to consider is the Hermitian Einstein problem. Specifically one looks for a metric H on E whose curvature F satisfies the following differential equation:

$$\Lambda F = \mu(E) I$$

where $\mu(E)$ is the slope of E. This problem has been solved, first by Narasimhan and Seshadri in the case of curves [31], then for algebraic surfaces by Donaldson [13], and for higher dimensional Kähler manifolds by Uhlenbeck and Yau [52]. Simpson studied this equation on Higgs bundles and certain non-compact cases [43]. Buchdahl extended Donaldson's result to arbitrary complex surfaces in [6], and Li and Yau generalized the Donaldson-Uhlenbeck-Yau theorem to any compact complex Hermitian manifold in [27]. A detailed account of the case of Gauduchon metrics can also be found in the book of Lübke and Teleman [28]. Finally, in [5] Bando and Siu were able to extend this theory to metrics on the locally free part of coherent sheaves.

In all cases, the existence of a Hermitian-Einstein metric requires an algebraic notion of stability. We say E is stable (in the sense of Mumford-Takemoto) if for every proper coherent subsheaf $\mathcal{F} \subset E$,

$$\mu(\mathcal{F}) := \frac{deg(\mathcal{F})}{rk(\mathcal{F})} < \frac{deg(E)}{rk(E)} =: \mu(E).$$

With this definition, any irreducible vector bundle E admits a Hermitian-Einstein metric if and only if it is stable. Furthermore the proof of Simpson, and the proof of Siu in [45] rely on the fact that a certain functional is bounded from below. This functional, introduced by Donaldson in [13], is defined on Kähler manifolds and compares two metrics H_0 and H on E. We denote it by $M(H_0, H)$, and for a fixed H_0, its gradient flow is given by:

$$H^{-1}\dot{H} = -(\Lambda F - \mu I). \tag{1.0.1}$$

We refer to this flow as the Donaldson heat flow. One can see that at a critical point the metric will be Hermitian-Einstein. With this functional in mind, we now state the following version of the Donaldson-Uhlenbeck-Yau Theorem:

Theorem 1. *Let E be an indecomposable vector bundle over a compact Kähler manifold X. Then the following conditions are equivalent:*

i) E is stable.

ii) For any fixed metric H_0 on E, the Donaldson functional $M(H_0, H)$ is bounded from below and proper.

iii) E admits a Hermitian-Einstein metric.

While this is an extremely powerful theorem, it leaves many questions unanswered, in particular: What if the bundle is not stable? Does there exist some sort of canonical metric in this case?

Recently this problem has been addressed over surfaces by Daskalopoulos and Wentworth in [10]. In their paper, they study the Yang Mills flow on E, which is gauge equivalent to the Donaldson heat flow. They show that on a Kähler surface X, along the Yang-Mills flow the trace of the curvature approaches in L^p an endomorphism with locally constant eigenvalues corresponding to the Harder-Narasimhan type of E. Furthermore they prove that away from a bubbling set and along a subsequence, the Yang-Mills flow converges (up to gauge transformations) to a limiting Yang-Mills connection on a new bundle E_∞ with a possibly different topology. E_∞ extends over the singular set, and Daskalopoulos and Wentworth prove this extention is isomorphic to the bundle $Gr^{hns}(E)^{**}$, the double dual of the graded quotients of the Harder-Narasimhan-Seshadri filtration. In this way they were able to verify a conjecture of Bando and Siu in the surface case.

The main result of this thesis is to prove the conjecture of Bando and Siu in arbitrary dimension. As a first step, we need to understand the limiting properties of the curvature in the semi-stable case. We say that E is semi-stable if for every proper coherent subsheaf $\mathcal{F} \subset E$,

$$\mu(\mathcal{F}) \leq \mu(E).$$

Our first result is to show the condition of semi-stability is equivalent to the existence of an approximate Hermitian-Einstein structure, which means for all $\epsilon > 0$, there exists a metric H on E with curvature F such that:

$$\sup_X |\Lambda F - \mu(E)I|_{C^0} < \epsilon.$$

The proof of this result makes up Section 3 of this thesis. We state the full theorem here:

Theorem 2. *Let E be a holomorphic vector bundle over a compact Kähler manifold X. Then the following conditions are equivalent:*

i) E is semi-stable.

ii) For any fixed metric H_0 on E, the Donaldson functional $M(H_0, H)$ is bounded from below.

iii) E admits an approximate Hermitian-Einstein structure.

When X is a projective manifold, this theorem was first proven by Kobayashi in [26]. There Kobayashi also conjectures that the result should be true for general compact Kähler manifolds, the main difficulty being finding a proof of the lower bound of $M(H_0, H)$ from semi-stability without using certain algebraic facts. We present such a proof in this paper, and are thus able to extend Kobayashi's theorem to the Kähler case.

A particularly important feature of Theorem 2 is that the *analytic* property of the lower boundedness of a functional is deduced directly from the *algebraic* property of semi-stability. This may be useful for the analogous question in the problem of constant scalar curvature Kähler metrics. The analogue of the Donaldson functional is in this case the Mabuchi K-energy [29], and several analogues of Mumford-Takemoto stability have also been introduced, including Chow-Mumford stability, K-stability (Tian [50], Donaldson [16]), uniform K-stability (Szekelyhidi [49]), slope-stability (Ross-Thomas [39]), b-stability (Donaldson [18]), as well as infinite-dimensional notions (Donaldson [19], Phong-Sturm [37, 38] and references therein). Donaldson [17] has shown that Chow-Mumford stability implies the lower boundedness of the K-energy. It would be very instructive if similar implications can be established directly from the other notions of stability. The lower boundedness of the K-energy is an important geometric property in itself. It implies the vanishing of the Futaki invariant, and plays an important role in the Kähler-Ricci flow, which is a more non-linear analogue of the gradient flow of the Donaldson functional.

Another potentially interesting feature of Theorem 2 is its proof: a fundamental use is made of the regularization of sheaves, building on the works of Buchdahl [6] and Bando-Siu [5]. And while using blowups to regularize sheaves is not a new idea, the proof contains detailed computations of how induced metrics on quotient sheaves change during each blowup, which could be further developed and potentially useful in many other geometric problems.

We briefly describe the proof of Theorem 2. The proof of the lower bound for $M(H_0, H)$ is a direct generalization of Donaldson's proof that $M(H_0, H)$ is bounded from below in the semi-stable case if X is a curve. It relies on the fact that for any semi-stable vector bundle E, one can find a destabilizing subbundle S with quotient bundle Q such that S is stable and Q is semi-stable. Then the functional M on E breaks up into the corresponding Donaldson functionals on S and Q. Since S is stable, that piece is bounded from below. Q is semi-stable, and of strictly less rank than E, so by induction we can keep going until we have rank one bundles, which are stable and thus the Donaldson functional is bounded from below.

For us the key difficulty is that the destabilizing objects S and Q may not be vector bundles (as in the case of curves), but only torsion free sheaves. Thus the bulk of the work goes into defining the functional and corresponding terms on a torsion-free subsheaf S with quotient Q. We view these sheaves as holomorphic vector bundles off their singular locus, and the main difficulty is that the induced metrics on these sheaves blow up or degenerate as we approach the singular set. The key tool to help us through this difficulty is an explicit regularization procedure which generalizes a procedure of Buchdahl (from [6]). After a finite number of blowups, denoted $\pi : \tilde{X} \longrightarrow X$, we can pull back and alter these subsheaves to get smooth vector bundles $\tilde{S}$ and $\tilde{Q}$ on $\tilde{X}$. Degenerate metrics on S and Q can now be indentified with smooth metrics on $\tilde{S}$ and $\tilde{Q}$, allowing many of the desired terms to be computed in this smooth setting, including the Donaldson functional. It also helps with the induction step since now we break apart the Donaldson functional on $\tilde{Q}$, which is a smooth vector bundle with smooth metric, so we only have to worry about subsheaves of smooth vector bundles, and not subsheaves of torsion free sheaves. The proof also relies heavily on the work of Bando and Siu [5], and since we use a different regularization procedure than they used, we find it useful to go over some of the important estimates in our case.

The proof of Theorem 2 and a few applications concludes Section 3. In Section 4 we generalize the work Daskalopoulos and Wentworth from [10] and thus prove of the Bando-Siu conjecture for arbitrary dimensional Kähler manifolds, as well as the generalized Atiyah-Bott formula. Here we give some brief motivation for considering the Yang-Mills flow.

Given a holomorphic vector bundle E, the Yang-Mills flow provides a natural approach to constructing Yang-Mills connections on E. Aside from their original application to particle physics, Yang-Mills connections are of fundamental interest due to how they reflect the topology of the original bundle. If X is a complex surface, then the moduli space of Yang-Mills connections reflects deep topological information about X (see [21]). Specifically for us, on a compact Kähler manifold X of general dimension, if A is a smooth Yang-Mills connection on E then the trace of the curvature $\Lambda F(A)$ will have locally constant eigenvalues determined by the Harder-Narasimhan type of E. In fact any Yang-Mills connection will decompose E into a direct sum of stable bundles whose slopes corresponds to the slopes of the quotients of this natural filtration [26]. Because of this behavior one would expect existence of Yang-Mills connections to be intimately related to the slope and stability of the original bundle, and this expectation ends up being correct, since if E is indecomposable and stable, a Yang-Mills connection A must be Hermitian-Einstein.

Of course if E is indecomposable and not stable, then the flow can not converge (or else this would contradict the Donaldson-Uhlenbeck and Yau Theorem). However, we show that nevertheless the limiting properties of the Yang-Mills flow once again reflects many of the geometric properties of E, and in many of the same ways as does a Yang-Mills connection. Specifically, equip E with a Hermitian metric H. Let Q^i be the quotients of the Harder-Narasimhan filtration, and let π^i denote orthogonal projections onto the subsheaves of this filtration. Define the endomorphism:

$$\Psi_H = \sum_i \mu(Q^i)(\pi^i - \pi^{i-1}). \tag{1.0.2}$$

This is an endomorphism with locally constant eigenvalues determined by the slopes of the quotients of the Harder-Narsimhan filtration. We have the following theorem:

Theorem 3. *Let E be a holomorphic vector bundle over a compact Kähler manifold X. Given a fixed metric H and any initial integrable connection A on E, let A_t be a smooth solution of the Yang-Mills flow starting with this initial connection. Then for all $\epsilon > 0$, there exists a time t_0 such that for $t > t_0$, we have*

$$||\Lambda F(A_t) - \Psi_H||^2_{L^2} < \epsilon.$$

The existence of such a connection for each $\epsilon > 0$ is called an L^2 approximate Hermitian structure on E (see Definition 8 below). As an immediate consequence we get a sharp lower bound for the Hermitian-Yang-Mills functional $||\Lambda F(\cdot)||^2_{L^2}$, and since this functional is related to the Yang-Mills functional by a topological constant, we get a sharp lower bound for the Yang-Mills functional as well. In fact we are able to generalize a formula of Atiyah and Bott from [2]. Let $\mathcal{F}$ be a slope decreasing filtration of E, and let $\mathcal{Q}^i$ be the quotients of this filtration. Then we define:

$$\Phi(\mathcal{F})^2 = \sum_{i=0}^{q} \mu(\mathcal{Q}^i)^2 rk(\mathcal{Q}^i).$$

Normalize ω to have volume one, and let A be an integrable connection. We have the following result:

Theorem 4. *For all holomorphic vector bundles E over X the following formula holds:*

$$\inf_A ||\Lambda F(A)||^2_{L^2} = \sup_{\mathcal{F}} \Phi(\mathcal{F})^2.$$

We note that the supremum on the right is attained by the Harder-Narasimhan filtration of E. This formula is the higher dimensional generalization of a formula first proven on Riemann surfaces by Atiyah and Bott in [2]. We also direct the reader to the paper of Donaldson [15], in which he states the Atiyah-Bott formula and proves a generalization relating the Calabi functional to test configurations.

We now explain our main result of this thesis, which is an identification of the limit of the Yang-Mills flow. First, given a sequence of connections A_j along the Yang-Mills flow, we define the analytic bubbling set by:

$$Z_{an} = \bigcap_{r>0} \{x \in X \,|\, \liminf_{j\to\infty} r^{4-2n} \int_{B_r(x)} |F(A_j)|^2 \omega^n \geq \epsilon\}.$$

This set is the same singular set used by Hong and Tian in [25]. Our complete result is as follows:

Theorem 5. *Let E be a holomorphic vector bundle over a compact Kähler manifold X. Let A_t be a connection on E evolving along the Yang-Mills flow. Then there exists a subsequence of times t_j such that on $X \backslash Z_{an}$, the sequence A_{t_j} converges (modulo gauge transformations) in C^∞ to a limiting connection A_∞ on a limiting bundle E_∞. E_∞ extends to all of X as a reflexive sheaf $\hat{E}_\infty$ which is isomorphic to the double dual of the stable quotients of the graded Harder-Narasimhan-Seshadri filtration, denoted $Gr^{hns}(E)^{**}$, of E.*

In [25], Hong and Tian prove that away from Z_{an}, a subsequence along the Yang-Mills flow A_j converges smoothly to a limiting Yang-Mills connection on a limiting bundle E_∞. They also prove that Z_{an} is a holomorphic subvariety of X, although we do not utilize this result. By the work of Bando and Siu [5], we know E_∞ extends to all of X as a reflexive sheaf $\hat{E}_\infty$. Our contribution is to construct an explicit isomorphism between $\hat{E}_\infty$ and $Gr^{hns}(E)^{**}$.

Here we remark that these results are not a full generalization of the work of Daskalopoulos and Wentworth. Theorem 3 stated above is the direct analogue of

Theorem 3.11 from [10], however they prove the existence of an L^p approximate Hermitian structure as opposed to L^2. We are unable to improve L^2 to L^p for $2 \leq p < \infty$, except in the semi-stable case, where in fact an L^∞ version of the estimate is given by the approximate Hermitian-Einstein structure of Therorem 2. Also, in [11], the authors prove that the bubbling set Z_{an} is in fact equal to the singular set of $Gr^{hns}(E)$, in other words they show the Yang-Mills flow bubbles precisely where the sheaf $Gr^{hns}(E)$ fails to be locally free. While this is an extremely amazing and attractive fact, as of yet we can not generalize it to higher dimensions.

We briefly describe the proofs of these results, all of which are contained in Section 4. First, we must construct an L^2 approximate Hermitian structure on E, and then use such a structure to show that in fact one is realized along the Yang-Mills flow, proving Theorem 3. This first step is highly nontrivial, and takes up the bulk of the section. We use a similar to the method to the one utilized in the proof of Theorem 2, modified to fit our particular case.

First, we define a new relative functional on the space of Hermitian metrics, denoted $P(H_0, H)$, which is closely related to Donaldson's functional. For a fixed metric H_0, the P-functional is designed so that if H_t is a smooth path of metrics satisfying:

$$H_t^{-1}\dot{H}_t = -(\Lambda F_t - \Psi_{H_t}), \tag{1.0.3}$$

then the derivative of the P-functional along this path is given by:

$$\dot{P}(H_0, H_t) = -||\Lambda F_t - \Psi_{H_t}||^2_{L^2}.$$

The difference between this flow and the Donaldson heat flow (1.0.1) comes from replacing $\mu(E)I$ with Ψ_H. Now, it follows that we can construct an L^2 approximate Hermitian structure on E by showing $\dot{P}(H_0, H_t)$ goes to zero along a solution of (1.0.3). To accomplish this we need to prove that the P-functional is bounded below and that a solution to (1.0.3) exists for all time. These two facts, along with a simple differential inequality, show that $\dot{P}(H_0, H_t)$ tends to zero as t tends to infinity.

The lower bound the P-functional is proven in a similar fashion to the lower bound of the Donaldson functional for semi-stable bundles (Section 3). The key difficulty lies in adapting the blowup procedure to regularize the quotients of the Harder-Narasimhan filtration. Once we have this adaptation, we show the value of the functional is preserved during this regularization, and take advantage of the fact that on the regularized filtration the P-functional decomposes into positive terms plus the sum of the Donaldson functionals on the quotients of the filtration. We know the Donaldson functional is bounded below on the semi-stable quotients, and thus the P-functional is bounded below.

To show long time existence of (1.0.3), we follow closely the arguments of Donaldson and Simpson which demonstrate long time existence of the Donaldson heat flow. As we have mentioned, the only difference between (1.0.3) and the Donaldson heat flow is that the zeroth order terms from (1.0.3) are non-constant. This does not pose a large problem in proving long time existence since all these terms are controlled. However this difference is significant enough to prevent proving the existence of an L^∞ approximate Hermitian structure on E. Thus using our methods, an L^2 appoximate Hermitian structure is the best we can hope for.

Once we have established the existence of an L^2 appoximate Hermitian structure, showing that such a structure is realized along the Yang-Mills flow requires following a distance decreasing argument from [10]. This proves Theorem 3, and Theorem 4 follows as a result. The proof of Theorem 5 requires explicit construction of an isomorphism between $\hat{E}_\infty$ and $Gr^{hns}(E)^{**}$. We use Theorem 3, in combination with a modification of the Chern-Weil formula, to produce the necessary estimate to show that the second fundamental forms associated to the Harder-Narasimhan filtration go to zero in L^2. This proves that in the limit we get a holomorphic splitting of E_∞ into a direct sum of semi-stable quotients. Furthermore, using the approximate Hermitian-Einstein structure on the semi-stable quotients, we show the second fundamental form of any destabilizing subsheaves of these quotients must go to zero, creating a holomorphic splititng of the limiting bundle into a direct sum of stable bundles.

Now, utilizing an idea which goes back to Donaldson in [13] (and is used by Daskalapolus and Wentworth in [10]), we can show the holomorphic inclusion maps of the subsheaves from the filtration into E converge to limiting holomorphic maps. Following a stability argument from [26] these limiting maps can be shown to be isomorphisms, thus construting an isomorphism between $Gr^{hns}(E)$ and E_∞ on $X\backslash Z_{an}$. Theorem 5 follows from the uniqueness of the reflexive extention $\hat{E}_\infty$.

The main difficulty we encounter is showing that the limiting holomorphic map f_∞ is non-trivial. Since we only have uniform bounds for A_j on compact subsets K away from the bubbling set, at first we only have convergence of the holomorphic maps on K. Thus even though we assume the global L^2 norm of f_j is normalized, it could be that the maps f_j concentrate on $X\backslash K$ causing f_∞ to be zero on K. To address this concern we prove the following estimate:

$$||f_j||_{C^0(X)} \leq C||f_j||_{L^2(K)},$$

which bounds the L^2 norm of f_j on K from below. The difficulty with this estimate is that we need to bound a global C^0 norm by the L^2 norm on a subset of X with boundary. We accomplish this by exploiting the rigidity of holomorphic functions, and show that in fact a standard estimate on K can be extended across the boundary. This completes the proof of Theorem 5 and concludes Section 4.

In Section 5 we shift our focus away from Kähler manifolds, and consider the case where X is a general compact Hermitian manifold equipped with a Gauduchon metric. A Gauduchon metric is a generalization of a Kähler metric, and while Kähler metrics may not always exists, there exists a Gauduchon metric on every compact Hermitian manifolds. As we have already stated, in [27] Li and Yau solved the Hermitian-Einstein problem in this case. However, rather than using the parabolic approach of the Donaldson heat flow, they utilized the elliptic approach of the method of continuity. In this final section, we reprove Li and Yau's result using the heat flow technique, and extend their result to the more general case of Higgs bundles.

In [43], Simpson first solved the Hermitian Einstein problem for (possibly) non unitary connections using Higgs bundles. We briefly review his result here. Let X be a compact Kähler manifold with volume one. A Higgs bundle is a vector bundle E, together with an endomorphism valued one form:

$$\theta : E \longrightarrow \Lambda^{1,0}(E),$$

which we call the Higgs field. If $\theta^\dagger$ is the adjoint of θ with respect to H, and if ∇ is the usual unitary-Chern connection on E, we can define a new connection $D := \nabla + \theta + \theta^\dagger$, and look for a solution of the Hermitian-Einstein problem:

$$\Lambda F_\theta = \mu(E) I, \tag{1.0.4}$$

where here F_θ is the curvature of D. Note that if $\theta \neq 0$ then our connection D is not unitary. Now, given the extra assumption that θ be holomorphic and $\theta \wedge \theta = 0$, Simpson was able to construct a solution to (1.0.4) in the case that E is stable. Here stability is defined as before, with the restriction that each subsheaf $\mathcal{F}$ be preserved by the Higgs field.

Our main result from this final section is to extend Simpson's proof to the non-Kähler setting. We state the result here:

Theorem 6. *Let X be a compact, complex Hermitian manifold, and let E be an irreducible Higgs bundle over X. Assume the Higgs field is holomorphic and satisfies the integrability condition $\theta \wedge \theta = 0$. Then there exists a Gauduchon metric g on X such that a solution to* (1.0.4) *exists if and only if E is stable.*

We note that with the two stated assumptions on the Higgs field, the curvature F_θ of D takes a special form, and (1.0.4) reduces to solving:

$$\Lambda F - \mu(E) I = g^{j\bar{k}} [\theta^\dagger_{\bar{k}}, \theta_j],$$

where here F is the curvature of ∇. In this form we can see how this equation generalizes the original Hermitian-Einstein equation. Of course, assuming that θ is holomorphic and $\theta \wedge \theta = 0$ may seem arbitrary, yet in certain cases these assumptions do arise naturally from the geometry of E. Specifically, if $c_1(E) = c_2(E) \cdot [\omega]^{n-2} = 0$, then any connection D which satisfies (1.0.4) must be flat, and in this case we say D is stable if E admits no non-trivial D-invariant subbundles. Then if X is Kähler, using the existence of harmonic metrics ([8], [12], [20]), and a Bochner type formula of Siu [46] and Sampson [40], it follows that if D is stable than $\bar{\partial}\theta = \theta \wedge \theta = 0$ (for details see [9]). Thus these two assumptions arise naturally from stability and equation (1.0.4). Of course, this argument uses that X is Kähler in a fundamental way, and it would be interesting to know if the corresponding statement that D stable implies $\bar{\partial}\theta = \theta \wedge \theta = 0$ is true in the non-Kähler case.

We prove Theorem 6 by following the parabolic approach used by Donaldson [13] and Simpson [43]. The main idea is to define the following generalization of the Donaldson heat flow:

$$H^{-1}\dot{H} = -(\Lambda F_\theta - \mu(E)I), \tag{1.0.5}$$

and show that along a subsequence of times a solution of this flow converges to a solution of (1.0.4). Aside from having to be careful with torsion terms after integrating by parts, a surprising number of difficulties arise when X is not Kähler. The most striking is that the evolution equation (1.0.5) is not the gradient flow of any functional. Both Simpson and Donaldson rely heavily on the fact that in the Kähler setting (1.0.5) is the gradient flow of $M(H_0, H)$. Thus we must modify our proof substantially to get around this difficulty. The first place this comes up is in showing a C^0 bound for H along the flow. In our case we cannot follow Simpson's proof, so we adapt the elliptic C^0 estimate of Uhlenbeck and Yau to our parabolic setting. The second difficulty that arises is in showing that ΛF_θ converges to $\mu(E)I$ in L^2. When X is Gauduchon the functional $M(H_0, H)$ is not path independent in the space of Hermitian metrics, so we define its value to be the integral along a specific path, and then compute the variation of $M(H_0, H)$ along this path (which will have a complicated derivative). We then show that the extra terms we get are in fact bounded by terms we can control, and achieve L^2 convergence in this fashion.

It was shown by Biswas in [4] by an explicit example that the correspondence between stable Higgs bundles and representations of the fundamental group for compact Kähler manifolds does not extend to the non-Kähler case. Thus we have no hope of extending Simpson's famous correspondence to Gauduchon manifolds. However, we hope the existence of Hermitian-Einstein metrics on Higgs bundles in this setting will provide insight into other geometric problems in non-Kähler geometry.

2 Basic Complex Geometry

We begin with some basic facts about complex manifolds and vector bundles. Let X be a compact complex manifold of complex dimension n. Let $T_{\mathbf{C}}X := TX \otimes \mathbf{C}$ be the complexified tangent bundle of X. Since X is a complex manifold, X admits an integrable complex structure:

$$J : T_{\mathbf{C}}X \longrightarrow T_{\mathbf{C}}X,$$

such that $J^2 = -I$. This endormorphism allows us to decompose $T_{\mathbf{C}}X = T^{1,0}X \oplus T^{0,1}X$ into the eigenspaces of $\pm i$. We refer to $T^{1,0}X$ as the holomorphic tangent bundle of X, which in local coordinates has a basis given by $\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}\}$. Let $g_{\bar{k}j}$ be a Hermitian metric on $T^{1,0}X$. Then we define the fundamental $(1,1)$ form ω by

$$\omega = \frac{i}{2}\, g_{\bar{k}j}\, dz^j \wedge d\bar{z}^k.$$

The metric g is said to be *Kähler* if $d\omega = 0$, *semi-Kähler* if $d(\omega^{n-1}) = 0$, and *Gauduchon* if $\partial\bar{\partial}(\omega^{n-1}) = 0$.

Note that g also defines a metric on $\Lambda^{p,q}(X)$, the space of (p,q) forms on X, for all p, q. Let Λ denote the adjoint of wedging with ω. If η is a $(p+1, q+1)$ form, then $\Lambda\eta$ is a (p,q) form with local coeficients $g^{j\bar{k}}\eta_{\bar{Q}P\bar{k}j}$, where P and Q are multi-indices of length p and q. The volume form on X is given by $\dfrac{\omega^n}{n!}$. For simplicity we write ω^n for the volume form and denote $\dfrac{\omega^{n-1}}{n-1!}$ by ω^{n-1}. One can check that for a $(1,1)$ form ζ, we have $(\Lambda\zeta)\,\omega^n = \zeta \wedge \omega^{n-1}$. Throughout this thesis, we always assumed ω is normalized so that:

$$Vol(X) := \int_X \omega^n = 1.$$

2.1 Holomorphic vector bundles

Let E be a holomorphic vector bundle over X, which carries a smooth Hermitian metric H. On a local holomorphic trivialization, for any section $\phi^\alpha \in \Gamma(X, E)$ we define the unitary-Chern connection ∇ by:

$$\nabla_{\bar{k}}\phi^\alpha = \partial_{\bar{k}}\phi^\alpha \qquad \text{and} \qquad \nabla_j\phi^\alpha = \partial_j\phi^\alpha + H^{\alpha\bar{\beta}}\partial_j H_{\bar{\beta}\gamma}\phi^\gamma.$$

The curvature of this connection is an endomorphism valued two form:

$$F := \frac{i}{2} F_{\bar{k}j}{}^{\alpha}{}_{\gamma}\, dz^j \wedge d\bar{z}^k,$$

where $F_{\bar{k}j}{}^{\alpha}{}_{\gamma} = -\partial_{\bar{k}}(H^{\alpha\bar{\beta}}\partial_j H_{\bar{\beta}\gamma})$. By convention, we always use Latin indicies for the base manifold X and Greek indicies for the bundle E. The *degree* of E can be computed as follows:

$$deg(E) = \int_X \mathrm{Tr}(F) \wedge \omega^{n-1}, \tag{2.1.1}$$

and this definition is independent of a choice of metric as long as g is Gauduchon, since for a fixed holomorphic structure on E the difference between the trace of the curvature tensors of two metrics is $\partial\bar{\partial}$-exact. In the case that g is Kähler or semi-Kähler, then degree is a topological invariant, as it depends only on $c_1(E)$ (this is because the difference between the trace of the curvature tensors of any two unitary connections is d-exact). We define the *slope* of E to be

$$\mu(E) := \frac{deg(E)}{rk(E)}.$$

Given a torsion-free subsheaf $\mathcal{F} \subset E$, we can view $\mathcal{F}$ as a holomorphic subbundle off the singular set $Z(\mathcal{F})$ where $\mathcal{F}$ fails to be locally free. We know from [26] that $Z(\mathcal{F})$ is a holomorphic subvariety of X of codimension at least two. Then on $X \backslash Z(\mathcal{F})$ we have a metric on the bundle $\mathcal{F}$ induced from the metric H on E, and the curvature of this metric is at least in in L^1 (see section 3.1 for details). Thus the degree and slope of the subsheaf $\mathcal{F}$ can be defined in the same was as E, by just computing away from the singular set $Z(\mathcal{F})$.

We say E is *stable* if $\mu(\mathcal{F}) < \mu(E)$ for all propers torsion free subsheaves $\mathcal{F} \subset E$. E is defined to be *semi-stable* if the weak inequality $\mu(\mathcal{F}) \leq \mu(E)$ holds for all propers torsion free subsheaves $\mathcal{F} \subset E$.

3 The Donaldson Heat Flow on Semi-Stable Bundles

In this section we prove Theorem 2, which states that E is semi-stable if and only if it admits an approximate Hermitian-Einstein structure. Throughout this section we assume the manifold X carries a fixed Kähler metric g. We begin with a discussion of torsion free subsheaves of E.

3.1 Induced metrics on sheaves

Given a torsion free subsheaf S of E, we can construct the following short exact sequence:

$$0 \longrightarrow S \xrightarrow{f} E \xrightarrow{p} Q \longrightarrow 0, \tag{3.1.1}$$

where we assume that the quotient sheaf Q is torsion free (by saturating S if necessary). We define the singular set of Q to be $Z := \{x \in X \,|\, Q_x \text{ is not free}\}$. Then on $X \backslash Z$, we can view (3.1.1) as a short exact sequence of holomorphic vector bundles. Here, a smooth metric H on E induces a metric J on S and a metric K on Q. For sections ψ, ϕ of S, we define the metric J as follows:

$$\langle \phi, \psi \rangle_J = \langle f(\phi), f(\psi) \rangle_H.$$

In order to define K on Q, we note that a choice of a metric H on E gives a splitting of (3.1.1):

$$0 \longleftarrow S \xleftarrow{\lambda} E \xleftarrow{p^\dagger} Q \longleftarrow 0. \tag{3.1.2}$$

Here λ is the orthogonal projection from E onto S with respect to the metric H. For sections v, w of Q, we define the metric K as:

$$\langle v, w \rangle_K = \langle p^\dagger(v), p^\dagger(w) \rangle_H.$$

Definition 1. On $X \backslash Z$ both S and Q are holomorphic vector bundles. We define an *induced metric* on either Q or S to be one constructed as above.

Once we have sequence (3.1.2), the second fundamental form $\gamma \in \Gamma(X, \Lambda^{0,1} \otimes Hom(Q, S))$ is given by:

$$\gamma = \bar{\partial} \circ p^{\dagger}.$$

We know that for any $q \in \Gamma(X \backslash Z, Q)$, $\gamma(q)$ lies in S since p is holomorphic and $p \circ p^{\dagger} = I$, thus $p\,(\bar{\partial} \circ p^{\dagger}(q)) = 0$. Now, because the maps f and p vanish on Z, any induced metric will degenerate or blow up as we approach the singular set, causing curvature terms to blow up. However, these singularities are not too bad, and the following proposition tells us what control we can expect.

Proposition 1. *The second fundamental form of an induced metric is in L^2, i.e.*

$$\int_{X \backslash Z} g^{j\bar{k}} \mathrm{Tr}\,(\gamma_j^{\dagger} \gamma_{\bar{k}})\, \omega^n \leq C.$$

We prove this proposition in section 3.2. Working on $X \backslash Z(Q)$, we now turn to the decomposition of connections and curvature onto subbundles and quotient bundles, which is described in detail in [22]. Let ∇^S and ∇^Q be the unitary-Chern connections on S and Q with respect to the metrics J and K. In a local coordinate patch, any section Φ of E decomposes onto the bundles S and Q, denoted $\Phi = \phi + q$. We now have the following decomposition of the unitary-Chern connection ∇:

$$\nabla(\Phi) = \begin{pmatrix} \nabla^S & \gamma \\ -\gamma^{\dagger} & \nabla^Q \end{pmatrix} \begin{pmatrix} \phi \\ q \end{pmatrix}. \tag{3.1.3}$$

Now, denote the curvature of the induced metric J by F^S and the curvature of the induced metric K by F^Q. The full curvature tensor F now decomposes as follows:

$$F(\Phi) = \begin{pmatrix} F^S + \gamma^{\dagger} \wedge \gamma & \nabla\gamma \\ -(\nabla\gamma)^{\dagger} & F^Q - \gamma^{\dagger} \wedge \gamma \end{pmatrix} \begin{pmatrix} \phi \\ q \end{pmatrix}. \tag{3.1.4}$$

Summing up on S and Q explicitly we have:

$$F^S = F|_S - \gamma^{\dagger} \wedge \gamma \tag{3.1.5}$$

and

$$F^Q = F|_Q + \gamma^\dagger \wedge \gamma. \tag{3.1.6}$$

Combining these two formulas with the fact that F is smooth implies the following result:

Proposition 2. *The curvature of an induced metric is in L^1.*

With this proposition we see that formula (2.1.1) is well defined for an induced metric, and which justifies using this formula to compute the degree of S and Q.

3.2 Regularization of sheaves

In this section we give a procedure to regularize the short exact sequence (3.1.1). This procedure generalizes a procedure of Buchdahl from [6] to the higher dimensional case. The main difference is that we do not attempt to regularize arbitrary torsion free sheaves over a Hermitian manifold, we only address the specific case where we have a subsheaf of a vector bundle E. In fact, one can view this procedure as a way to regularize the map f so its rank does not drop, allowing us to define a new holomorphic subbundle and quotient bundle. We go over a simple example first which illustrates many of the key points.

Consider the ideal sheaf $\mathcal{I}$ of holomorphic functions vanishing at the origin in $\mathbf{C}^2$. We can write it as the following holomorphic quotient:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\ f\ } \mathcal{O}^2 \xrightarrow{\ p\ } \mathcal{I} \longrightarrow 0,$$

where the maps are given in matrix form by:

$$f = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad p = \begin{pmatrix} -z_2 & z_1 \end{pmatrix}.$$

We blowup at the origin $\pi : \tilde{\mathbf{C}}^2 \longrightarrow \mathbf{C}^2$, and let $D = \pi^{-1}(0)$. Pulling back the short exact sequence we get:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\ \pi^* f\ } \mathcal{O}^2 \xrightarrow{\ \pi^* p\ } \pi^*\mathcal{I} \longrightarrow 0$$

(here we are implicitly using the fact that $\pi^*\mathcal{O}_{\mathbf{C}^2} \cong \mathcal{O}_{\tilde{\mathbf{C}}^2}$). $\tilde{\mathbf{C}}^2$ can be covered by two coordinate patches $U_i := \{z_i \neq 0\}$, $i = 1, 2$. On U_1 we have coordinates $w_1 = z_1$ and $w_2 = \dfrac{z_2}{z_1}$, and we can write our maps as:

$$\pi^* f = \begin{pmatrix} 1 \\ w_2 \end{pmatrix} w_1 \qquad , \qquad \pi^* p = \begin{pmatrix} -w_2 & 1 \end{pmatrix} w_1.$$

Now, multiplication by w_1 gives us a map from $\mathcal{O}$ to $\mathcal{O}(-D)$, and since w_1 factors out of the map $\pi^* f$, the map $\tilde{f} := \dfrac{1}{w_1}\pi^* f$ defines a holomorphic inclusion of $\mathcal{O}(-D)$ into $\mathcal{O}^2$. Thus we get a new short exact sequence:

$$0 \longrightarrow \mathcal{O}(-D) \xrightarrow{\tilde{f}} \mathcal{O}^2 \xrightarrow{\tilde{p}} \mathcal{O}^2/\mathcal{O}(-D) \longrightarrow 0,$$

which we say is regularized since now the rank of $\tilde{f}$ does not drop anywhere. Since we know what $\tilde{p}$ is on each coordinate patch, we can explicitly compute the transition functions of $\mathcal{O}^2/\mathcal{O}(-D)$ in this construction. Given a section (η_1, η_2) of $\mathcal{O}^2$, then on U_1 $\tilde{p}((\eta_1, \eta_2)) = -\dfrac{z_2}{z_1}\eta_1 + \eta_2$ and on U_2 we have $\tilde{p}((\eta_1, \eta_2)) = -\eta_1 + \dfrac{z_1}{z_2}\eta_2$. Thus the transition function from U_1 to U_2 is multiplication by $\dfrac{z_2}{z_1}$, so in this case $\mathcal{O}^2/\mathcal{O}(-D) \cong \mathcal{O}(D)$. Now the regularized sequence can be expressed as:

$$0 \longrightarrow \mathcal{O}(-D) \xrightarrow{\tilde{f}} \mathcal{O}^2 \xrightarrow{\tilde{p}} \mathcal{O}(D) \longrightarrow 0.$$

With this example in mind, we now turn to the general procedure.

Once again consider the short exact sequence over X:

$$0 \longrightarrow S \xrightarrow{f} E \xrightarrow{p} Q \longrightarrow 0,$$

with E locally free and Q torsion free. Suppose S has rank s, E has rank r, and Q has rank q. In section 3.1 we defined the singular set Z of Q, and off this set we can view this sequence as a short exact sequence of holomorphic vector bundles. After choosing coordinates, off of Z we view f as an $r \times s$ matrix of holomorphic functions with full rank. Since Z is a subset of codimension 2 or more, we can extend f over the singular set to get a matrix of holomorphic functions defined on our entire coordinate patch. On points in Z the rank of f may drop, and it is exactly this behavior that we need to regularize before we can carry out the analysis in later sections.

Let Z_k be the subset of Z where $rk(f) \leq k$. For the smallest such k, on Z_k we can choose coordinates so that f can be expressed as

$$f = \begin{pmatrix} I_k & 0 \\ 0 & g \end{pmatrix},$$

where g vanishes identically on Z_k. Blowing up along Z_k by the map $\pi : \tilde{X} \longrightarrow X$, we choose coordinate patches $\{U_\alpha\}$ on $\tilde{X}$. On a given coordinate patch let w define the exceptional divisor. Then the pullback of f can be decomposed as follows:

$$\pi^* f = \begin{pmatrix} I_k & 0 \\ 0 & \tilde{g} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & w^a I_{s-k} \end{pmatrix}, \tag{3.2.7}$$

where a is the largest power of w we can pull out of the $\pi^* g$. Denote the matrix on the left as $\tilde{f}$ and the matrix on right as t. We would like to define $\tilde{S}$ as the image of the sheaf S under the map t. Explicitly, we note that off of $\pi^{-1}(Z)$, $\pi^* S$ is a holomorphic vector bundle with transition functions $\{\Phi_{\alpha\beta}\}$ so that for a section ψ^ρ of $\pi^* S$,

$$\psi^\rho|_{U_\alpha} = \Phi_{\alpha\beta}{}^\rho{}_\gamma \psi^\gamma|_{U_\beta}.$$

With this, the transition functions $\{\tilde{\Phi}_{\alpha\beta}\}$ of $\tilde{S}$ can be expressed as:

$$\tilde{\Phi}_{\alpha\beta}{}^\rho{}_\gamma = \frac{{w_\alpha}^{a_\gamma}}{{w_\beta}^{a_\rho}} \Phi_{\alpha\beta}{}^\rho{}_\gamma.$$

Here a_γ is equal to 0 if $\gamma \leq k$ or a if $\gamma > k$. Although these transition functions may blow up as we approach $\pi^{-1}(Z)$, they are useful in understanding how the map t twists up S. Now the map $\tilde{f}$ defines a new holomorphic inclusion of the sheaf $\tilde{S}$ into the bundle $\pi^* E$, with a new quotient $\tilde{Q}$. Of course, the rank of $\tilde{f}$ may still drop, but one of two things has happened. Either $rk(\tilde{f}) > k$ on $\pi^*(Z_k)$, or for all $x \in Z_k$, if m_x is the maximal ideal at the point x, then the smallest power p such that m_x^p sits inside the ideal generated by the vanishing of $\tilde{g}$ is smaller than that of g. In either case we have improved the regularity of f. After a finite number of blowups we can conclude that $rk(\tilde{f}) > k$ everwhere. Thus we can next blowup along Z_{k+1} and continue this process until the rank of $\tilde{f}$ does not drop.

After a finite number of blowups we have that the map $\tilde{f}$ is holomorphic and has constant rank on $\tilde{X}$. It defines a holomorphic subbundle $\tilde{S}$ of π^*E with holomorphic quotient $\tilde{Q}$. Summing up, we have proven the following Proposition:

Proposition 3. *Over a complex-Hermitian manifold X, let S be a torsion free subsheaf of E with torsion free quotient, so locally the inclusion of S into E is given by a matrix of holomorphic functions f_0 with transition functions on the overlaps. Then there exists a finite number of blowups*

$$\tilde{X}_N \xrightarrow{\pi_N} \tilde{X}_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} \tilde{X}_1 \xrightarrow{\pi_1} X,$$

and matrices of holomorphic functions f_k over $\tilde{X}_k$ with the the following properties:

i) On each $\tilde{X}_k$ there exists coordinates so that if w defines the exceptional divisor, there exists a diagonal matrix of monomials in w (denoted t) so that

$$\pi_{k-1}^* f_{k-1} = f_k\, t.$$

ii) The rank of f_N is constant on $\tilde{X}_N$, thus it defines a holomorphic subbundle of $\pi_N^* \circ \cdots \circ \pi_1^* E$ with a holomorphic quotient bundle.

We note that this procedure is consistent with another viewpoint found in Uhlenbeck and Yau [52]. In their paper they view a torsion free sheaf locally as a rational map from X to the Grassmanian $Gr(s, r)$ (this is our map f). By Hironaka's Theorem we know this map can be regularized after a finite number of blowups. We follow our procedure in order to find coordinates which let us keep track of how that map changes at each step, and in doing so we can work out how the induced metrics on $\tilde{S}$ and $\tilde{Q}$ change during each step.

3.2.1 Induced metrics on regularizations

We now compute how induced metrics change during regularization. First we need a good local description of these metrics. Recall the short exact sequence (3.1.1). Fix an open set $U \subset X$, and fix a holomorphic trivialization of E over U. Since we view S and Q as holomorphic vector bundles off Z, we consider local trivializations for these bundles over $U \backslash Z$. In these coordinates the map f is a matrix of holomorphic functions. For any section $\phi^\alpha \in \Gamma(X, S)$, we have $f(\phi) = f^\gamma{}_\alpha \phi^\alpha \in \Gamma(X, E)$. The induced metric $J_{\bar{\beta}\alpha}$ is given by

$$J_{\bar{\beta}\alpha}\phi^\alpha \overline{\phi^\beta} = H_{\bar{\rho}\nu} f^\nu{}_\alpha \phi^\alpha \overline{f^\rho{}_\beta \phi^\beta},$$

so

$$J_{\bar{\beta}\alpha} := H_{\bar{\rho}\nu} f^\nu{}_\alpha \overline{f^\rho{}_\beta}. \tag{3.2.8}$$

The induced metric $K_{\bar{\beta}\alpha}$ is defined similarly. Let $q^\alpha \in \Gamma(X, Q)$. If we recall the splitting (3.1.2), then in local coordinates the metric $K_{\bar{\beta}\alpha}$ is given by

$$K_{\bar{\beta}\alpha} q^\alpha \overline{q^\beta} = H_{\bar{\rho}\nu}\, p^{\dagger^\nu}{}_\alpha q^\alpha\, \overline{p^{\dagger\rho}{}_\beta q^\beta},$$

so

$$K_{\bar{\beta}\alpha} := H_{\bar{\rho}\nu}\, p^{\dagger^\nu}{}_\alpha \overline{p^{\dagger\rho}{}_\beta}. \tag{3.2.9}$$

In many cases it will be easier to work with the projection λ as opposed to $p^\dagger$. Using the fact that p is surjective we write $q = p(V)$ for some $V \in \Gamma(X, E)$. Then $p^\dagger(q) = p^\dagger p(V) = (I - \lambda)V$. Thus the formula

$$|q|_K^2 = |(I - \lambda)V|_H^2$$

describes the metric K along with (3.2.9). We note that V is not unique, however given another V' such that $p(V') = q$, then $p(V - V') = 0$, and since (3.1.1) is exact we know $(I - \lambda)(V - V') = 0$. This justifies the alternate definition of K.

Proposition 4. *Consider a single blowup from the regularization procedure $\pi : \tilde{X} \longrightarrow X$. Let J and K be metrics induced by f and $\tilde{J}$ and $\tilde{K}$ be metrics induced by $\tilde{f}$, where $\tilde{f}$ is defined by* (3.2.7)*. Then if w locally defines the exceptional divisor, there exists natural numbers a_α so that:*

$$\pi^* J_{\bar{\beta}\alpha} = w^{a_\alpha}\overline{w^{a_\beta}}\tilde{J}_{\bar{\beta}\alpha} \qquad , \qquad \pi^* K_{\bar{\beta}\alpha} = \frac{1}{w^{a_\alpha}\overline{w^{a_\beta}}}\tilde{K}_{\bar{\beta}\alpha}.$$

Proof. By (3.2.7) we know how $\pi^* f$ decomposes, thus from (3.2.8) we can see that:

$$\pi^* J_{\bar{\beta}\alpha} = \pi^* H_{\bar{\rho}\nu}\pi^* f^\rho{}_\alpha \overline{\pi^* f^\nu{}_\beta} = \pi^* H_{\bar{\rho}\nu} w^{a_\alpha}\, \tilde{f}^\rho{}_\alpha \overline{w^{a_\beta}\tilde{f}^\nu{}_\beta} = w^{a_\alpha}\overline{w^{a_\beta}}\tilde{J}_{\bar{\beta}\alpha}.$$

This tells us how $J_{\bar{\beta}\alpha}$ changes during each step of the regularization. How $K_{\bar{\beta}\alpha}$ changes is a little more difficult to see. We note that at each point in $\tilde{X}$ the projection λ from $\pi^* E$ onto the image of $\pi^* f$ is equal to the projection $\tilde{\lambda}$ onto the image of $\tilde{f}$. This follows because the only difference between the matrices $\pi^* f$ and $\tilde{f}$ is multiplication by the diagonal matrix t (from (3.2.7)), which only changes the length of each column vector, not the span of the columns. Thus for $V \in \Gamma(\tilde{X}, \pi^* E)$, we have

$$(I - \lambda)(V) = (I - \tilde{\lambda})(V).$$

We need a formula for how $p^\dagger$ changes under regularization. First we note that on Q the map $p \circ p^\dagger$ is the identity, so for q a section of $\pi^* Q$ we have:

$$\pi^* p\, \pi^* p^\dagger(q) = q.$$

We now write $\pi^* p = \tilde{w}\tilde{p}$, where $\tilde{w}$ is a diagonal matrix defining the exceptional divisor. So $\tilde{w}\tilde{p}\,\pi^* p^\dagger(q) = q$, and because $\tilde{w}$ is invertible it follows:

$$\tilde{p}\,\pi^* p^\dagger(q) = \tilde{w}^{-1} q. \tag{3.2.10}$$

Now since the metric $\pi^* H$ on $\pi^* E$ gives a splitting of the following sequence:

$$0 \longrightarrow \tilde{S} \xrightarrow{\tilde{f}} \pi^* E \xrightarrow{\tilde{p}} \tilde{Q} \longrightarrow 0,$$

we have a map $\tilde{p}^\dagger : \tilde{Q} \to \pi^* E$. Applying this map to each side of (3.2.10) we get:

$$\begin{aligned} \tilde{p}^\dagger \tilde{w}^{-1} q &= \tilde{p}^\dagger \tilde{p}\, \pi^* p^\dagger(q) \\ &= (I - \tilde{\lambda}) \pi^* p^\dagger(q) \\ &= (I - \lambda) \pi^* p^\dagger(q) \\ &= \pi^* p^\dagger(q), \end{aligned}$$

where the last line follows from the fact that $\pi^* p^\dagger$ is already perpendicular to the image of $\pi^* f$. Thus we have shown $\pi^* p^\dagger = \tilde{p}^\dagger \tilde{w}^{-1}$, and plugging this into the formula for the metric we have:

$$\begin{aligned} \pi^* K_{\bar{\beta}\alpha} s^\alpha \overline{s^\beta} &= H_{\bar{\nu}\rho}\, \pi^* {p^\dagger}^\rho{}_\alpha\, s^\alpha\, \overline{\pi^* {p^\dagger}^\nu{}_\beta\, s^\beta} \\ &= \frac{1}{w^{a_\alpha} \overline{w^{a_\beta}}} H_{\bar{\nu}\rho}\, {\tilde{p}^\dagger}{}^\rho{}_\alpha\, s^\alpha\, \overline{{\tilde{p}^\dagger}{}^\nu{}_\beta\, s^\beta} \\ &= \frac{1}{w^{a_\alpha} \overline{w^{a_\beta}}} \tilde{K}_{\bar{\beta}\alpha} s^\alpha \overline{s^\beta}. \end{aligned}$$

This completes the proof of the proposition. □

3.2.2 Transformation of curvature terms

Now that we know how induced metrics change after each step in the regularization procedure, we can compute how the associated curvature terms change. In this section all computations are local, and we restrict ourselves to working with the sheaf Q with induced metric K, since all computation involving the subsheaf S are similar. From now on let F denote the curvature of K. First we compute how the trace of curvature changes under regularization.

Lemma 1. *For a single blowup in the regularization procedure $\pi : \tilde{X} \longrightarrow X$, let w locally define the exceptional divisor. Then the following decomposition holds in the sense of currents:*

$$\pi^* \mathrm{Tr}(F) = \sum_\alpha a_\alpha \partial\bar{\partial} \log |w|^2 + \mathrm{Tr}(\tilde{F}).$$

Along the course of proving the lemma we will also give a formula for π^*F in terms of $\tilde{F}$.

Proof. We work in a local trivialization and apply Proposition 4:

$$\begin{aligned}\pi^* F_{\bar{k}j}{}^{\alpha}{}_{\beta} &= -\partial_{\bar{k}}(\pi^* K^{\alpha\bar{\gamma}}\partial_j \pi^* K_{\bar{\gamma}\beta}) \\ &= -\partial_{\bar{k}}(\tilde{K}^{\alpha\bar{\gamma}} w^{a_\alpha}\overline{w^{a_\gamma}}\partial_j(\frac{1}{w^{a_\beta}\overline{w^{a_\gamma}}}\tilde{K}_{\bar{\gamma}\beta})).\end{aligned}$$

Now since $\overline{w}^{a_\gamma}$ is anti-holomorphic, it follows that

$$\begin{aligned}\pi^* F_{\bar{k}j}{}^{\alpha}{}_{\beta} &= -\partial_{\bar{k}}(\tilde{K}^{\alpha\bar{\gamma}} w^{a_\alpha}\partial_j(\frac{1}{w^{a_\beta}}\tilde{K}_{\bar{\gamma}\beta})) \\ &= -\partial_{\bar{k}}(w^{a_\alpha}\partial_j(\frac{1}{w^{a_\beta}})\tilde{K}^{\alpha\bar{\gamma}}\tilde{K}_{\bar{\gamma}\beta} + \frac{w^{a_\alpha}}{w^{a_\beta}}\tilde{K}^{\alpha\bar{\gamma}}\partial_j\tilde{K}_{\bar{\gamma}\beta}) \\ &= a_\alpha\partial_j\partial_{\bar{k}}\log|w|^2\delta^{\alpha}{}_{\beta} - \partial_{\bar{k}}(\frac{w^{a_\alpha}}{w^{a_\beta}}\tilde{K}^{\alpha\bar{\gamma}}\partial_j\tilde{K}_{\bar{\gamma}\beta}). \qquad (3.2.11)\end{aligned}$$

We can use this last line as a formula for the transformation of F. Taking the trace proves the lemma. □

Because we need to deal with the pullback of Kähler forms under the blowup map, we extend the definition of degree to include these degenerate metrics.

Definition 2. Let E be a vector bundle on $\tilde{X}$, where $\tilde{X}$ is given by a blowup map $\pi : \tilde{X} \longrightarrow X$. Let F^E be the curvature of a given metric H on E, and let ω be a Kähler metric on X. Then the *degree* of E with respect to $\pi^*\omega$ is given by:

$$deg(E, \pi^*\omega) = \int_{\tilde{X}} \mathrm{Tr}(F^E) \wedge \pi^*\omega^{n-1}.$$

Even though the metric $\pi^*\omega$ is degenerate on the exceptional divisor, since $\pi^*\omega$ is closed this definition is independent of the choice of metric on E. Once again if Q is a torsion free sheaf and the curvature of Q is L^1 on the locally free part of Q, then this definition extends from vector bundles to torsion free sheaves.

Lemma 2.

$$deg(Q, \omega) = deg(\tilde{Q}, \pi^*\omega).$$

Proof. By Proposition 2 we see the degree of Q is given by:

$$deg(Q,\omega) = \int_X \mathrm{Tr}(F) \wedge \omega^{n-1}.$$

We now pullback this quantity by the blowup map and regularize Q. During each step in the procedure we have:

$$\begin{aligned}
\int_X \mathrm{Tr}(F) \wedge \omega^{n-1} &= \int_{\tilde{X}} \pi^* \mathrm{Tr}(F) \wedge \pi^* \omega^{n-1} \\
&= \int_{\tilde{X}} (\sum_\alpha a_\alpha \partial\bar{\partial} \log |w|^2 + \mathrm{Tr}(\tilde{F})) \wedge \pi^* \omega^{n-1} \\
&= \int_{\tilde{X}} \mathrm{Tr}(\tilde{F}) \wedge \pi^* \omega^{n-1},
\end{aligned}$$

since $\pi^*\omega$ becomes degenerate along the support of $\partial\bar{\partial} \log |w|^2$. We continue the regularization procedure and after a finite number of blowups $\tilde{F}$ will be smooth. The integral stays the same after each step. □

Proposition 1 also follows from Lemma 1.

Proof of proposition 1. To prove this result we show that after each step in the regularization procedure $||\gamma||^2_{L^2} = ||\tilde{\gamma}||^2_{L^2}$, thus after a finite number of blowups $||\tilde{\gamma}||^2_{L^2}$ will be an integral on a smooth vector bundle over a compact manifold and thus bounded. From (3.1.6) it follows that that:

$$\mathrm{Tr}\,(\gamma^\dagger \wedge \gamma) = \mathrm{Tr}\,(F) - \mathrm{Tr}\,((I-\lambda) \circ F^E).$$

Pulling back onto the blowup we compute:

$$\begin{aligned}
\pi^* \mathrm{Tr}\,(\gamma^\dagger \wedge \gamma) \wedge \pi^* \omega^{n-1} &= (\mathrm{Tr}\,(\tilde{F}) + \sum_\alpha a_\alpha \partial\bar{\partial} \log |w|^2) \wedge \pi^* \omega^{n-1} \\
&\quad - \mathrm{Tr}\,((I-\lambda) \circ F^E) \wedge \pi^* \omega^{n-1} \\
&= \mathrm{Tr}\,(\tilde{F}) \wedge \pi^* \omega^{n-1} - \mathrm{Tr}\,((I-\tilde{\lambda}) \circ F^E) \wedge \pi^* \omega^{n-1} \\
&= \mathrm{Tr}\,(\tilde{\gamma}^\dagger \wedge \tilde{\gamma}) \wedge \pi^* \omega^{n-1}.
\end{aligned}$$

Here we used the fact that the projection $\tilde{\lambda}$ is equal to the projection λ, which we saw in the proof of Proposition 4. Integrating this last equality proves $||\gamma||^2_{L^2} = ||\tilde{\gamma}||^2_{L^2}$. □

3.2.3 The Donaldson functional on regularizations

In this subsection we extend the definition of the Donaldson functional to include metrics on torsion free subsheaves S and Q. This definition only works for induced metrics, and does not extend to arbitrary metrics defined on the locally free parts of S and Q. First we go over the definition of the Donaldson functional on the vector bundle E.

Fix a reference metric H_0 on E. For any other metric H define the endomorphism $h = H_0^{-1}H$. Let $Herm^+(E)$ denote the space of positive definite hermitian endomorphisms of E. For $t \in [0,1]$, consider any path $h_t \in Herm^+(E)$ with $h_0 = I$ and $h_1 = h$, and let F_t be the curvature of the metric $H_t := H_0\, h_t$ along the path. Then the Donaldson functional is given by:

$$M(H_0, H, \omega) = \int_0^1 \int_X \mathrm{Tr}(F_t h_t^{-1} \partial_t h_t) \wedge \omega^{n-1}\, dt - \mu(E) \int_X \log\det(h_1)\, \omega^n.$$

One can check that this definition is independent of the choice of path (for instance see [45]). Given a blowup map $\pi : \tilde{X} \longrightarrow X$, one can also define the Donaldson functional on a vector bundle over $\tilde{X}$ by integrating with respect to the degenerate metric $\pi^*\omega$. Since $\pi^*\omega$ is closed the functional will still be independent of path. We now define the Donaldson functional on the sheaves S and Q as follows:

Definition 3. For a subsheaf S of E, we define the *Donaldson functional* on S to be:

$$M_S(H_0, H, \omega) := M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega),$$

for any regularization $\tilde{S}$. Similarly we define the *Donaldson functional* on the quotient sheaf Q to be:

$$M_Q(H_0, H, \omega) := M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega),$$

for the regularization $\tilde{Q}$ corresponding to $\tilde{S}$.

Here $M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega)$ and $M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega)$ are the Donaldson functionals for the vector bundles $\tilde{S}$ and $\tilde{Q}$ defined using the degenerate metric $\pi^*\omega$. We note that the domains of the functionals M_S and M_Q are metrics on the vector bundle E, thus this definition only applies to induced metrics and does not extend to arbitrary metrics on S and Q. In the following proposition we show that this definition is well defined.

Proposition 5. *M_S and M_Q are well defined functionals for any pair of metrics on E and are independent of the choice of regularization.*

Proof. Since the regularization procedure is not unique, we show the functional gives the same value independent of the sequence of blowups chosen. Once again, we prove this proposition for the quotient sheaf Q, as the argument works the same for S.

As we have seen, a choice of metrics H_0 and H on E induce metrics K_0 and K on Q. Furthermore if we regularize Q we get corresponding induced metrics $\tilde{K}_0$ and $\tilde{K}$ on $\tilde{Q}$. Set $\tilde{k} = \tilde{K}_0^{-1}\tilde{K}$ as the endomorphism relating these two metrics, and let $\tilde{k}_t$, $t \in [0,1]$, be any path in $Herm^+(\tilde{Q})$ connecting the identity to $\tilde{k}$. Then we have defined the Donaldson functional on Q to be the following integral:

$$M_Q(H_0, H, \omega) = \int_0^1 \int_{\tilde{X}} \mathrm{Tr}(\tilde{F}_t \tilde{k}_t^{-1} \partial_t \tilde{k}_t)\, \pi^*\omega^{n-1} dt - \mu(E) \int_{\tilde{X}} \log\det(\tilde{k}_1)\pi^*\omega^n. \quad (3.2.12)$$

We note that the path $\tilde{k}_t$ defines a path $k_t := \dfrac{w^{a_\alpha}}{w^{a_\gamma}}\tilde{k}^\alpha{}_\gamma$ which is an endomorphism of the quotient sheaf one step back in the regularization procedure. Similarly the metrics $(K_0)_{\bar{\beta}\alpha} := \dfrac{1}{w^{a_\alpha}\overline{w^{a_\beta}}}(\tilde{K}_0)_{\bar{\beta}\alpha}$ and $K_t := K_0 k_t$ are defined one step back in the regularization procedure. Let F_t be the curvature of K_t. Then we can compute using formula (3.2.11):

$$\begin{aligned} \mathrm{Tr}(F_t k_t^{-1}\partial_t k_t) \wedge \pi^*\omega^{n-1} &= (F_t)^\alpha{}_\beta (k_t^{-1})^\beta{}_\gamma (\partial_t k_t)^\gamma{}_\alpha \wedge \pi^*\omega^{n-1} \\ &= -\bar{\partial}(\frac{w^{a_\alpha}}{w^{a_\beta}} \tilde{K}^{\alpha\bar{\nu}} \partial \tilde{K}_{\bar{\nu}\beta}) \frac{w^{a_\beta}}{w^{a_\gamma}} \tilde{k}^{-1\beta}{}_\gamma \frac{w^{a_\gamma}}{w^{a_\alpha}} \partial_t \tilde{k}^\gamma{}_\alpha \wedge \pi^*\omega^{n-1} \\ &= \mathrm{Tr}\,(\tilde{F}\tilde{k}^{-1}\partial_t \tilde{k}) \wedge \pi^*\omega^{n-1}, \end{aligned}$$

since w is holomorphic. Thus the first integral does not change at any step in the regularization procedure and we get the following equality:

$$\int_0^1 \int_{\tilde{X}} \mathrm{Tr}(\tilde{F}_t \tilde{k}_t^{-1} \partial_t \tilde{k}_t)\, \pi^*\omega^{n-1} dt = \int_0^1 \int_X \mathrm{Tr}(F_t k_t^{-1} \partial_t k_t)\, \omega^{n-1} dt. \quad (3.2.13)$$

Here the integral on the right is only in terms of the initial induced metrics K_0 and K, where the path k_t is such that $k_0 = I$ and $k_1 = K_0^{-1}K$. Since the integral in (3.2.12) is independent of path, we conclude that the integral in (3.2.13) is independent of regularization and depends only on the choice of metrics H_0 and H on E. We now do the same for the second integral of line (3.2.12).

It helps to write the formula for k_1 in matrix notation $k_1 = t^{-1}\tilde{k}_1\, t$, where t is the matrix defined in (3.2.7). Thus it is clear that $\det(k_1)$=$\det(\tilde{k}_1)$ for each blowup in the regularization procedure, so once again we can write

$$\int_{\tilde{X}} \log\det(\tilde{k}_1)\pi^*\omega^n = \int_X \log\det(k_1)\omega^n,$$

where the integral on the right only depends on K_0 and K. Thus our definition of the Donaldson functional on Q only depends on the choice of metrics H_0 and H on E. □

Now that we have this definition, we can state a decomposition result which plays a major role in the proof of our main theorem. First we assume that S and Q are genuine holomorphic vector bundles, which have the same slope as E. In [13] Donaldson proved:

$$M(H_0, H, \omega) = M_S(J_0, J, \omega) + M_Q(K_0, K, \omega) + ||\gamma||^2_{L^2} - ||\gamma_0||^2_{L^2},$$

where $M(H_0, H, \omega)$ is the Donaldson functional on E, and $M_S(J_0, J, \omega)$, $M_Q(K_0, K, \omega)$ are the corresponding Donaldson functionals on S and Q. In fact, we can see right away that this decomposition extends to induced metrics on sheaves. Since $M(H_0, H, \omega) = M(\pi^*H_0, \pi^*H, \pi^*\omega)$, we can pull back the functional and look at the decomposition onto the regularized vector bundles $\tilde{S}$ and $\tilde{Q}$. We get the following decompoistion:

$$M(\pi^*H_0, \pi^*H, \pi^*\omega) = M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega) + M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega) + ||\tilde{\gamma}||^2_{L^2} - ||\tilde{\gamma}_0||^2_{L^2}.$$

Now since the L^2 norm of the second fundamental form is independent of regularization we get the following lemma:

Lemma 3. *Let S be a torsion free subsheaf of E with torsion free quotient Q. If S, E, and Q all have the same slope then we have the following decomposition:*

$$M(H_0, H, \omega) = M_S(H_0, H, \omega) + M_Q(H_0, H, \omega) + ||\gamma||^2_{L^2} - ||\gamma_0||^2_{L^2}.$$

3.3 A lower bound for the Donaldson functional

In this subsection we prove a lower bound for the Donaldson functional on E under the assumption that $M_S(H_0, H, \omega)$ is bounded from below for S stable, a fact we shall prove in the next section. We first define a notion of slope and stability with respect to a degenerate metric, using Definition 2:

Definition 4. Let B be a vector bundle on $\tilde{X}$, where $\tilde{X}$ is given by a blowup map $\pi : \tilde{X} \longrightarrow X$. Then the *slope* of B with respect to $\pi^*\omega$ is given by:

$$\mu(B, \pi^*\omega) = \frac{deg(B, \pi^*\omega)}{rk(B)}.$$

Definition 5. We say B is *stable* with respect to $\pi^*\omega$ if for all proper torsion free subsheaves $\mathcal{F} \subset B$, we have

$$\mu(\mathcal{F}, \pi^*\omega) < \mu(B, \pi^*\omega).$$

We say B is *semi-stable* with respect to $\pi^*\omega$ if

$$\mu(\mathcal{F}, \pi^*\omega) \leq \mu(B, \pi^*\omega).$$

The following theorem is the main ingredient of Theorem 2, and remained the final step in generalizing Kobayishi's proof in [26] from projective manifolds to all compact Kähler manifolds.

Theorem 7. *If E is a semi-stable vector bundle over X compact Kähler, than the Donaldson functional is bounded from below on E.*

Proof. E is a semi-stable vector bundle on X, so all destabilizing subsheaves have the same slope as E. We restrict ourselves to subsheaves which have torsion free quotients. Choose the one with the lowest rank, which we call S. Then S is stable since any proper subsheaf of S would be a subsheaf of E and thus would have lower slope (since S was chosen with minimal rank). The torsion free quotient Q has the same slope as S (and E), and is semi-stable.

We now decompose the Donaldson functional into functionals on S and Q using Lemma 4.2.6. In the next section we show $M_S(H_0, H, \omega)$ is bounded from below since S is stable. $||\gamma||_{L^2}^2$ is a positive term and $||\gamma_0||_{L^2}^2$ is fixed (having only to do with the fixed initial metric H_0), so the only remaining term to check is $M_Q(H_0, H, \omega)$. Since $M_Q(H_0, H, \omega) = M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega)$ for some regularization $\tilde{Q}$, we choose to show the latter term is bounded from below, which is helpful since $\tilde{K}_0$ and $\tilde{K}$ are now smooth metrics on a holomorphic vector bundle $\tilde{Q}$. We need to show $\tilde{Q}$ is semi-stable with respect to $\pi^*\omega$, that way we can continue this process of decomposing the functional and use induction on rank. First we prove a few lemmas.

Lemma 4. *If E is semi-stable with respect to ω, then π^*E is semi-stable with respect to $\pi^*\omega$ on $\tilde{X}$.*

We note this lemma is false if we use the Kähler metic $\omega_\epsilon = \pi^*\omega + \epsilon\sigma$ on $\tilde{X}$ (where σ is the pullback of the Fubini-Study metric on the exceptional divisor times a suitable bump function). It only works since $\pi^*\omega$ degenerates (see [6]).

Proof of lemma. Suppose π^*E is not semi-stable with respect to $\pi^*\omega$. Then it contains a proper subsheaf $\mathcal{F}$ of rank $p < r$ such that $\mu(\mathcal{F}, \pi^*\omega) > \mu(\pi^*E, \pi^*\omega)$ (here r is the rank of E). Since π is an isomorphism off the exceptional divisor, we have that $\mu(\pi_*\mathcal{F}, \omega) > \mu(E, \omega)$, which would contradict the fact that E is semi-stable if we can show $\pi_*\mathcal{F}$ is a proper subsheaf of E. Clearly away from Z this is true, and since it is a set of codimension ≥ 2, off of Z we can view $\pi_*\mathcal{F}$ as a rational map from X into the Grassmanian $Gr(p, r)$ (see [52]). We can extend this rational map over Z since

E is locally free, thus $\pi_*\mathcal{F}$ is a subsheaf of E. □

Lemma 5. *If $\tilde{Q}$ is a torsion free quotient with the same slope as π^*E, then $\tilde{Q}$ is semi-stable with respect to $\pi^*\omega$.*

Proof. Suppose $\mathcal{G}$ is a subsheaf of $\tilde{Q}$ with $\mu(\mathcal{G}, \pi^*\omega) > \mu(\tilde{Q}, \pi^*\omega)$. Then since we have the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \tilde{Q} \longrightarrow \tilde{Q}/\mathcal{G} \longrightarrow 0,$$

by [26] Lemma (7.3) we know $\mu(\tilde{Q}/\mathcal{G}, \pi^*\omega) < \mu(\tilde{Q}, \pi^*\omega) = \mu(\pi^*E, \pi^*\omega)$. We define $\mathcal{B} := Ker(\pi^*E \to \tilde{Q}/\mathcal{G})$. Then $\mathcal{B}$ is included in the following exact sequence:

$$0 \longrightarrow \mathcal{B} \longrightarrow \pi^*E \longrightarrow \tilde{Q}/\mathcal{G} \longrightarrow 0.$$

Now once again by [26] Lemma (7.3) we see $\mu(\mathcal{B}, \pi^*\omega) > \mu(\pi^*E, \pi^*\omega)$, contradicting the semi-stability of π^*E. □

So $\tilde{Q}$ is semi-stable with respect to $\pi^*\omega$, and we continue this process. Recall that the vector bundle $\tilde{Q}$ has smooth metrics $\tilde{K}$ and $\tilde{K}_0$ induced from H and H_0 on E. Among all subsheaves of $\tilde{Q}$ with the same slope, let S_1 be a subsheaf of minimal rank. Then S_1 is stable with quotient Q_1, and in the next section we show $M_{S_1}(\tilde{K}_0, \tilde{K}, \pi^*\omega)$ is bounded from below. Using Lemma 4.2.6, we can reduce the problem to showing the Donaldson functional is bounded from below on Q_1. Blowing up again $\pi_1 : \tilde{X}_1 \longrightarrow \tilde{X}$ and constructing the regularization $\tilde{Q}_1$ over $\tilde{X}_1$, we see $\tilde{Q}_1$ is semi-stable with respect to $\pi_1^* \circ \pi^*(\omega)$ by the previous two Lemmas. Since $\tilde{Q}_1$ has stricly lower rank than $\tilde{Q}$, after a finite number of steps the process will terminate since all rank one sheaves are stable. This proves the lower bound for $M(H_0, H, \omega)$. □

3.3.1 A lower bound on stable sheaves

In this subsection we show that the Donaldson functional $M_S(H_0, H, \omega)$ is bounded from below if S is constructed as in the proof of Theorem 7. This result relies heavily on [5], in which Bando and Siu prove that any stable sheaf admits a Hermitian-Einstein metric off its singular locus. However, the lower bound on the Donaldson functional is not a consequence of this result, but rather it is an essential step in the proof. This is important from the point of view of this paper, since we want the proof of the main theorem to only rely on stability conditions, and not on the existence of any canonical metric. Now, because we use a different regularization procedure than the procedure described in [5], we choose to go over parts of the proof here in order to confirm that the necessary details carry over in our case. Furthermore, our proof of the lower bound is different, especially in the induction step used in the proof of Theorem 7.

At this point we have only defined the functional $M_S(H_0, H, \omega)$ for induced metrics on S (see Section 3.2.3). However, showing this functional is bounded from below is by definition equivalent to showing $M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega)$ is bounded from below for smooth metrics $\tilde{J}_0$ and $\tilde{J}$. We have that the functional $M_{\tilde{S}}$ is actually defined for any pair of metrics on $\tilde{S}$, allowing us to evolve $\tilde{J}$ by the gradient flow of $M_{\tilde{S}}$. Assume that ω is normalized so that $\int_X \omega^n = 1$.

As a first step to defining the gradient flow we compute the Euler-Lagrange equation for $M_{\tilde{S}}$. First we only consider a single blowup, and towards the end of the section we consider the case when we have a finite number of blowups. Consider the fixed metric $\tilde{J}_0$ and suppose we have a one parameter family of metrics $\tilde{J}_s$ with $\tilde{J}_1 = \tilde{J}$. Since $M_{\tilde{S}}$ is defined via integration along a path and the integral is path independent, assume we are integrating along the path $\tilde{h}_s \in Herm^+(\tilde{X}, \tilde{S})$ which corresponds to $\tilde{J}_s = \tilde{J}_0\tilde{h}_s$. Let $\tilde{F}_s$ be the curvature of $\tilde{J}_s$. We now compute:

$$\delta M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega) = \partial_s M(\tilde{J}_0, \tilde{J}_s, \pi^*\omega)|_{s=1} = \int_{\tilde{X}} \mathrm{Tr}((\Lambda_0\tilde{F}_1 - \mu(S,\omega)I)\tilde{h}_1^{-1}\partial_s\tilde{h}_1)\pi^*\omega^n$$

(For details of this computation see [45]). Here Λ_0 refers to the trace with respect to the degenerate metric $\pi^*\omega$. Thus at a critical point of M we have $\Lambda_0\tilde{F} = \mu(S,\omega)I$.

We can now consider the flow of metrics on $\tilde{S}$ given by:

$$\tilde{J}_t^{-1}\partial_t \tilde{J}_t = -\Lambda_0 \tilde{F}_t + \mu(S,\omega)I. \tag{3.3.14}$$

If J_t is any solution to this flow and we define $M_{\tilde{S}}(t) = M_{\tilde{S}}(\tilde{J}_0, \tilde{J}_t, \pi^*\omega)$, then

$$\partial_t M_{\tilde{S}}(t) = -\int_{\tilde{X}} \mathrm{Tr}((\Lambda_0 \tilde{F} - \mu(S,\omega)I)^2)\pi^*\omega^n \leq 0,$$

which is clearly decreasing. Since the flow decreases the value of $M_{\tilde{S}}$, if it is bounded below along the flow it is bounded from below in general. A priori it is not clear that the degenerate flow (3.3.14) is well defined since $\pi^* g^{j\bar{k}}$ blows up along the exceptional divisor. Thus our first step is to prove existence of a solution.

Theorem 8 (Bando, Siu)**.** *Let $\tilde{S}$ be a vector bundle over $\tilde{X}$, where $\pi : \tilde{X} \to X$ is the blowup of the Kähler manifold X along an analytic subvariety. Let $\pi^*\omega$ be the degenerate Kähler metric pulled back from X. Then there exists a metric H_0 and a family of metrics $H(t)$ on $\tilde{S}$ such that $H(0) = H_0$ and $H(t)$ satisfies* (3.3.14)*.*

We prove existence by showing the flow is in fact the limit of existing flows. Let $\pi : \tilde{X} \longrightarrow X$ be the blowup of X on which we construct $\tilde{S}$. On $\tilde{X}$, define the metric $\omega_0 := \pi^*\omega$. This metric is degenerate along the exceptional divisor, so we adjust it by adding on a small bump function times the pullback of Fubini-Study metric from the exceptional divisor, which we call σ (for details see [5],[6],[7]). This gives us a family of Kähler forms ω_ϵ on $\tilde{X}$, given by $\omega_\epsilon = \omega_0 + \epsilon\sigma$. Consider $g_\epsilon^{j\bar{k}}$, which is the inverse of the metric associated to ω_ϵ. Since ω_ϵ no longer degenerates we know that $g_\epsilon^{j\bar{k}}$ is smooth. We now can define the standard Donaldson heat flow on $\tilde{S}$ with respect to this new base metric ω_ϵ. We prove uniform bounds in ϵ, showing we can take a subsequence as $\epsilon \longrightarrow 0$ which converges to our degenerate flow (3.3.14).

We start out by defining an appropriate conformal change. Set $J_{\epsilon,0} = e^{\phi_\epsilon}\tilde{J}$, where ϕ_ϵ is defined by the equation

$$\Delta_\epsilon \phi_\epsilon = \mathrm{Tr}(-\Lambda_\epsilon \tilde{F} + \mu(\tilde{S}, \omega_\epsilon)I).$$

This equation admits a smooth solution for $\epsilon > 0$ since the right hand side integrates to zero against the volume form ω_ϵ^n. With these initial starting metrics $J_{\epsilon,0}$, the family of flows is given by:

$$J_{\epsilon,t}^{-1}\partial_t J_{\epsilon,t} = -\Lambda_\epsilon F_{\epsilon,t} + \mu(\tilde{S}, \omega_\epsilon)I. \tag{3.3.15}$$

These flows give a family of metrics $J_{\epsilon,t}$ that depend on ϵ and t. As before, we let $h_{\epsilon,t} = J_{\epsilon,0}^{-1}J_{\epsilon,t}$. From this point on we may from time to time drop the subscripts on $J_{\epsilon,t}$ and refer to the metric simply as J. To show these flows converge along a subsequence we need the following uniform bounds for the full curvature tensor independent of ϵ:

$$||F_{\epsilon,t}||_{C^k} \leq C, \tag{3.3.16}$$

for all k. This is possible when $0 < t_1 \leq t \leq t_2 < \infty$. In fact, we cannot do better then this, and the bounds fall apart if we send t_1 to zero or t_2 to infinity. Our first step is an L^1 bound.

Proposition 6. *For all time $t \geq 0$, we have the estimate*

$$||\Lambda_\epsilon F_{\epsilon,t}||_{L^1} \leq C,$$

independent of ϵ and t.

Proof. To prove the proposition, we need to work out how $\Lambda_\epsilon F_{\epsilon,t}$ and its norms evolve along the flow. We drop subscripts. Along the Donaldson heat flow we have:

$$\partial_t\langle\Lambda F, \Lambda F\rangle = \langle\partial_t\Lambda F, \Lambda F\rangle + \langle\Lambda F, \partial_t\Lambda F\rangle.$$

To compute the evolution of curvature, we use a formula from [45]:

$$\partial_t F_{\bar{m}l} = \partial_t(F_{\bar{m}l} - F^0_{\bar{m}l}) = -\partial_t\nabla_{\bar{m}}(\nabla_l h\, h^{-1}) = -\nabla_{\bar{m}}\nabla_l(h^{-1}\partial_t h).$$

Plugging our flow into this equation, we see

$$\partial_t\Lambda F = g^{l\bar{m}}\partial_t F_{\bar{m}l} = -g^{l\bar{m}}\nabla_{\bar{m}}\nabla_l(J^{-1}\partial_t J) = g^{l\bar{m}}\nabla_{\bar{m}}\nabla_l(\Lambda F) = \overline{\Delta}\Lambda F = \Delta\Lambda F.$$

The last equality holds because we are taking the Laplacian of the specific endomorphism ΛF. We now compute how the norm squared evolves:

$$\begin{aligned} \partial_t|\Lambda F|^2 = \partial_t\langle \Lambda F, \Lambda F\rangle &= \langle \partial_t \Lambda F, \Lambda F\rangle + \langle \Lambda F, \partial_t \Lambda F\rangle. \\ &= \langle \Delta\Lambda F, \Lambda F\rangle + \langle \Lambda F, \overline{\Delta}\Lambda F\rangle. \end{aligned}$$

Also,

$$\Delta|\Lambda F|^2 = \langle \Delta\Lambda F, \Lambda F\rangle + \langle \Lambda F, \overline{\Delta}\Lambda F\rangle + |\nabla\Lambda F|^2 + |\overline{\nabla}\Lambda F|^2.$$

Putting these two lines together we see:

$$\partial_t|\Lambda F|^2 = \Delta|\Lambda F|^2 - |\nabla\Lambda F|^2 - |\overline{\nabla}\Lambda F|^2.$$

which implies

$$\partial_t|\Lambda F|^2 \leq \Delta|\Lambda F|^2.$$

We also have the following inequality from [5]:

$$\partial_t|\Lambda F| \leq \Delta|\Lambda F|.$$

Of course because $|\Lambda F|$ is not smooth where the function hits zero, $\Delta|\Lambda F|$ is only defined in the distributional sense. However once we integrate both sides the leftover mass is the correct sign, which gives:

$$\int_{\tilde{X}} \partial_t|\Lambda F|\omega_\epsilon^n \leq 0.$$

Now, if we have an L^1 bound for $\Lambda_\epsilon F_{\epsilon,0}$ we can pull the derivative out of the integral:

$$\partial_t \int_{\tilde{X}} |\Lambda F|\, \omega_\epsilon^n \leq 0.$$

Since for all ϵ the L^1 norm decreases in time, all we need to do is show that the L^1 bound for $\Lambda_\epsilon F_{\epsilon,0}$ is independent of ϵ. To see this we note

$$\Lambda_\epsilon F_{\epsilon,0} = \Delta_\epsilon \phi_\epsilon I + \Lambda_\epsilon \tilde{F},$$

so

$$\begin{aligned}\int_{\tilde{X}} |\Lambda_\epsilon F_{\epsilon,0}|\omega_\epsilon^n &\leq \int_{\tilde{X}} |\Delta_\epsilon \phi_\epsilon|\omega_\epsilon^n + \int_{\tilde{X}} |\Lambda_\epsilon \tilde{F}|\omega_\epsilon^n \\ &= \int_{\tilde{X}} |\mathrm{Tr}(-\Lambda_\epsilon \tilde{F} + \mu_\epsilon I)|\omega_\epsilon^n + \int_{\tilde{X}} |\Lambda_\epsilon \tilde{F}|\omega_\epsilon^n \\ &\leq 2\int_{\tilde{X}} |\Lambda_\epsilon \tilde{F}|\omega_\epsilon^n + C.\end{aligned}$$

Thus to complete the proof we need to show $||\Lambda_\epsilon \tilde{F}||_{L^1}$ is bounded independent of ϵ. Since $\Lambda_\epsilon \tilde{F}$ is smooth for $\epsilon > 0$, if we can show the bound for $\epsilon = 0$ (the degenerate case) we will be done.

First we note that $\mathrm{Tr}(\Lambda_0 \tilde{F}) = \pi^*\mathrm{Tr}(\Lambda F)$ since $\mathrm{Tr}(\tilde{F}) \wedge \omega_0^{n-1} = \pi^*\mathrm{Tr}(F) \wedge \omega_0^{n-1}$. Then since $\pi^*\mathrm{Tr}(F)$ is in L^1 by Proposition 2, we have

$$\int_{\tilde{X}} |Tr(\Lambda_0 \tilde{F})|\omega_\epsilon^n \leq C. \tag{3.3.17}$$

Furthermore since $\tilde{J}$ is induced from a metric π^*H on π^*E, we have by (3.1.6)

$$\Lambda_0 \tilde{F} = \pi^*(\Lambda F^E|_S) + \Lambda_0(\gamma^\dagger \wedge \gamma).$$

Now even though the endomorphism $\Lambda_0(\gamma^\dagger \wedge \gamma)$ is unbounded, we do know it is positive. Thus since $\pi^*\Lambda F^E|_S$ is the pullback of a smooth endomorphism it follows that the eigenvalues of $\Lambda_0 \tilde{F}$ are bounded from below. This fact, along with (3.3.17), give the desired L^1 bound for $\Lambda_0 \tilde{F}$. Thus the L^1 norm of $\Lambda_\epsilon \tilde{F}$ is independent of ϵ. □

With this uniform L^1 bound, we can now get a L^∞ estimate for $\Lambda_\epsilon F_{\epsilon,t}$.

Proposition 7. *For all $t > 0$, the following bound is independent of ϵ:*

$$|\Lambda_\epsilon F_{\epsilon,t}|_{L^\infty} \leq C.$$

Proof. This bound cannot be extended to $t = 0$, since in this case we know that $\Lambda_0 F_{0,0}$ is not in L^∞. However, for all times $t > 0$ we use a heat kernel estimate. We have seen that this endomorphism evolves by a heat equation $\Lambda_\epsilon \partial_t F_{\epsilon,t} = \Delta_\epsilon \Lambda_\epsilon F_{\epsilon,t}$, thus using the heat kernal $\Phi_{\epsilon,t}$ we have

$$\Lambda_\epsilon F_{\epsilon,t} = \int_{\tilde{X}} \Phi_{\epsilon,t} \Lambda_\epsilon F_{\epsilon,0}\, \omega_\epsilon^n.$$

Now Proposition 2 of [5] gives a L^∞ estimate for the heat kernel independent of ϵ:

$$0 \leq \Phi_{\epsilon,t} \leq C(t^{-n} + 1).$$

Using this estimate, for any $t > 0$ we have

$$\begin{aligned} |\Lambda_\epsilon F_{\epsilon,t}|_{L^\infty} &\leq \int_{\tilde{X}} |\Phi_{\epsilon,t} \Lambda_\epsilon F_{\epsilon,0}|\, \omega_\epsilon^n \\ &\leq C(t^{-n} + 1) \int_{\tilde{X}} |\Lambda_\epsilon F_{\epsilon,0}|\, \omega_\epsilon^n \\ &\leq C(t^{-n} + 1) \end{aligned}$$

since we have a uniform L^1 bound. This proves the proposition.

□

Our next step is to prove a uniform bound in ϵ for $Tr(h_{\epsilon,t})$. Once we get this bound, standard theory for the Donaldson heat flow will give us control of F in C^k for all k.

Proposition 8. *$Tr(h_{\epsilon,t})$ is bounded for all time t where $0 < t_1 \leq t \leq t_2 < \infty$ independent of ϵ.*

Proof. Dropping subscripts we have that

$$\partial_t Tr(h) = Tr(\partial_t h) = -Tr(h(\Lambda F - \mu I)).$$

Since $t \geq t_1 > 0$, by the previous proposition $|\Lambda F| \leq C$ for some large positive constant C. Then

$$\partial_t Tr(h) \leq C\, Tr(h).$$

Set $Tr(h) = f$. We have

$$\partial_t f - Cf \leq 0.$$

multiplying both sides of the equation by e^{-Ct} we get

$$e^{-Ct}\partial_t f - Ce^{-Ct} f = \partial_t(e^{-Ct} f) \leq 0.$$

Integrating both sides gives

$$e^{-Ct}Tr(h_{\epsilon,t}) \leq Tr(h_{\epsilon,0})e^{C\cdot 0} = 1.$$

Thus

$$Tr(h_{\epsilon,t}) \leq e^{Ct} \leq e^{Ct_2},$$

which is independent of ϵ . □

The conformal change we made in defining $J_{\epsilon,0}$ guarantees that $\det h = 1$ along the flow. Since the trace of h is bounded from above it follows that all the eigenvalues of h are bounded away from zero, and thus h^{-1} is in L^∞. At this point standard theory gives the desired C^k bounds of the curvature independent of ϵ. After going to a subsequence ϵ_i, $\epsilon_i \longrightarrow 0$ as $i \to \infty$, the flows converge to a flow $J_{0,t}$ for $t \in [t_1, t_2]$. This flow is the degenerate flow (3.3.14) we hoped to define. It is not unique (It may change if we take a different subsequence or if we choose a different time interval $[t_1', t_2']$), however we can still prove long time existence.

Proposition 9 (Long time existence)**.** *Given $\tilde{J}$ at time t_0, once we choose positive times t_1 and t_2 to get a degenerate flow for $t \in [t_1, t_2]$, we can extend the flow for all time.*

Proof. Recall that we choose subsequence of flows as $\epsilon \to 0$ to define the degenerate flow. Now choose a sequence of times $\{t_n\}$ going to infinity. We extend the flow to intervals, $[t_1, t_3], ..., [t_1, t_n], ...,$ always taking subsequences of the defining sequence from the previous step. Since the standard Donaldson heat flow exists for all time, each flow $J_{\epsilon,t}$, $\epsilon > 0$, exists for all time, and we can continue this process and get a degenerate flow as t_n goes to infinity. □

Now that we have the degenerate flow defined for one blowup, we briefly describe the case of multiple blowups. Let $\pi_1, ...\pi_k$ be the sequence of blowups given in the regularization procedure for S. Assume that π_1 is the first blowup in the procedure, and thus it is on the singular set with highest codimension. On the final blowup, we have the following Kähler form:

$$\omega = \omega_0 + \epsilon_k \sigma_k + ... + \epsilon_1 \sigma_1.$$

If we define the Donaldson heat flow with respect to this Kähler form, then letting ϵ_1 go to zero will correspond with the previous work in this section. Thus after going to a subsequence we get a smooth flow for times $t \in [t_1', \infty)$, $t_1' > t_1$, with respect to the metric

$$\omega = \omega_0 + \epsilon_k \sigma_k + ... + \epsilon_2 \sigma_2.$$

We now repeat this process, which is possible since the L^1 bound from Proposition 6 is independent of all ϵ_i, including ϵ_2. Thus we get the bounds we need to send ϵ_2 to zero, and along a subsequence get a smooth flow for $t \in [t_2', \infty)$, where $t_2' > t_1'$. This process continues and after a finite number of steps we have the desired degenerate flow defined for $t \in [t_k', \infty)$. Choose $t^* \in [t_k', \infty)$, then Theorem 8 is proved by choosing the initial metric J_{0,t^*}.

Proposition 10. *Let E be a semi-stable vector bundle of rank r over X, and let S be a subsheaf of minimal rank among all sheaves with the property $\mu(S, \omega) = \mu(E, \omega)$. Then the Donaldson functional $M_{\tilde{S}}$ on the regularized vector bundle $\tilde{S}$ over $\tilde{X}$ is bounded from below.*

Proof. We have previously shown the degenerate flow is defined for all time with initial metric J_{0,t^*}. We now follow the proof of Simpson from [43] to show $M_{\tilde{S}}(t)$ is bounded from below. Suppose we choose S as in Theorem 7, so that it has minimal rank among all sheaves with the property $\mu(S) = \mu(E)$. We actually work along a subsequence of times, which we call t_i. Denote $h_i := h_{0,t_i}$ for simplicity, and let

s_i =log(h_i). We now use a different form of $M_{\tilde{S}}$, introduced by Donaldson in [20]. Here, by explicit computation along a specific path, one sees that the functional is given by:

$$M_{\tilde{S}}(t_i) = \int_{\tilde{X}} \mathrm{Tr}(F_0\, s_i)\omega_0^{n-1} + \int_{\tilde{X}} \sum_{\alpha\gamma} |\bar{\partial} s_{i\ \alpha}^{\gamma}|^2 \frac{e^{\lambda_\gamma - \lambda_\alpha} - (\lambda_\gamma - \lambda_\alpha) - 1}{(\lambda_\gamma - \lambda_\alpha)^2}\omega_0^n,$$

where λ_α are the eigenvalues of s_i. Now, because ω is degenerate along the exceptional divisor, we consider the pushforward sheaf $\pi_*\tilde{S}$. Recall π is an isomorphism off Z, thus $\pi_*\tilde{S}$ is a vector bundle on $X\backslash Z$. Since the set $\pi^{-1}Z$ has measure zero the Donaldson functional can now be expressed as:

$$M_{\tilde{S}}(t_i) = \int_{X\backslash Z} \mathrm{Tr}(F_0\, s_i)\omega^{n-1} + \int_{X\backslash Z} \sum_{\alpha\gamma} |\bar{\partial} s_{i\ \alpha}^{\gamma}|^2 \frac{e^{\lambda_\gamma - \lambda_\alpha} - (\lambda_\gamma - \lambda_\alpha) - 1}{(\lambda_\gamma - \lambda_\alpha)^2}\omega^n.$$

Now we can apply the argument of Simpson. His argument works in this case because the non-compact manifold $X\backslash Z$ satisfies all the assumptions Simpson imposes on the base, and the key assumption on the vector bundle, that ΛF_0 is in L^∞, is satisfied by Proposition 7 and the fact that π is an isomorphsim off Z. We assume by contradiction that there do not exist large constants C_1, C_2 so that the following estimate holds:

$$||s_i||_{L^1} \leq C_1 + C_2 M_{\tilde{S}}(t_i). \tag{3.3.18}$$

Then using the blowup argument of Simpson we can construct a proper torsion free subsheaf $\mathcal{F}$ of $\pi_*\tilde{S}$, such that $\mu(\mathcal{F},\omega) \geq \mu(\pi_*\tilde{S},\omega)$ and $rk(\mathcal{F}) < rk(\tilde{S})$. Denote $rk(\mathcal{F}) = p$. Of course, we assumed that S was stable, not $\pi_*\tilde{S}$, so we do not arrive at a contradiction just yet. However, because $\tilde{S}$ is a subbundle of π^*E, we have $\pi_*\tilde{S}$ and thus $\mathcal{F}$ is a subsheaf of E off Z. Once again because Z has codimension two we can view $\mathcal{F}$ as locally a rational map into the Grassmanian $Gr(p,r)$ and complete this map over Z. So $\mathcal{F}$ is a subsheaf of E, and since E is semi-stable we know $\mu(\mathcal{F}) = \mu(E)$. However $\mathcal{F}$ has rank strictly less than $\tilde{S}$ and thus S, contradicting our choice of S as the subsheaf of E with the same slope and minimal rank.

With this contradiction inequality (3.3.18) follows, and we can conclude:

$$M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi^*\omega) \geq -\frac{C_1}{C_2}.$$

By definition $M_S(H_0, H)$ is bounded from below as well. □

As a final step, we need alter the previous proposition so it can be applied to the induction step in the proof of Theorem 7.

Proposition 11. *Let E be a semi-stable vector bundle of rank r over $\tilde{X}$, where $\tilde{X}$ is given by a blowup $\pi : \tilde{X} \longrightarrow X$. Let S be a subsheaf of minimal rank among all sheaves with the property $\mu(S, \pi^*\omega) = \mu(E, \pi^*\omega)$. Then the functional $M_S(H_0, H, \pi^*\omega)$ is bounded from below.*

Proof. First we construct the regularization $\tilde{S}$ on the blowup $\pi_1 : \tilde{X}_1 \longrightarrow \tilde{X}$. As before, we have the degenerate flow defined for all time for some initial metric J_{0,t^*} on $\tilde{S}$. Assume that along a subsequence of times estimate (3.3.18) does not hold. We view the sheaf $\pi_*\pi_{1*}\tilde{S}$ as a vector bundle on $X \backslash Z$, and just as in the proof of the previous proposition we use the argument of Simpson from [43] to construct a proper torsion free subsheaf $\mathcal{F}$ of $\pi_*\pi_{1*}\tilde{S}$ such that $\mu(\mathcal{F}, \omega) \geq \mu(\pi_*\pi_{1*}\tilde{S}, \omega)$. From this fact we derive our contradiction.

Since $\tilde{S}$ is a holomorphic subbundle of π_1^*E, it follows that on $X \backslash Z$, $\mathcal{F}$ is a subsheaf of π_*E. Thus we get a map $\mathcal{F} \longrightarrow \pi_*E$ defined on all of X given by the composition of restriction to $X \backslash Z$ followed by inclusion. It follows that the natural map $\pi^*\pi_*E \longrightarrow E$ gives us a map:

$$j : \pi^*\mathcal{F} \longrightarrow \pi^*\pi_*E \longrightarrow E.$$

Of course this map may not be injective, however if we quotient out by the kernel of j, we can construct a proper subsheaf of E:

$$0 \longrightarrow \pi^*\mathcal{F}/Ker(j) \longrightarrow E.$$

Because π is an isomorphsim off Z, we see j is injective off $\pi^{-1}(Z)$, so $Ker(j)$ is a torsion sheaf supported on $\pi^{-1}(Z)$. We will arrive at a contradiction if we can show $\mu(\pi^*\mathcal{F}/Ker(j), \pi^*\omega) = \mu(E, \pi^*\omega)$, since $rk(\pi^*\mathcal{F}/Ker(j)) < rk(S)$ and S was chosen to be minimal. Consider the short exact sequence:

$$0 \longrightarrow Ker(j) \longrightarrow \pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{F}/Ker(j) \longrightarrow 0. \tag{3.3.19}$$

$Ker(j)$ is a torsion sheaf, so by Proposition 6.14 from [26], the determinant line bundle $\det Ker(j)$ admits a non-trivial holomorphic section ζ, which can only vanish along the support of $Ker(j)$. Let V be the vanishing locus of ζ. It follows that:

$$deg(Ker(j), \pi^*\omega) = \int_V \pi^*\omega^{n-1} = 0,$$

and the integral on the right is equal to zero since $\pi^*\omega$ is degenerate along the exceptional divisor (which contains V). Thus by (3.3.19) we have $deg(\pi^*\mathcal{F}/Ker(j), \pi^*\omega) = deg(\pi^*\mathcal{F}, \pi^*\omega)$, and since both sheaves have the same rank it follows that:

$$\mu(\pi^*\mathcal{F}/Ker(j), \pi^*\omega) = \mu(\pi^*\mathcal{F}, \pi^*\omega) \geq \mu(\tilde{S}, \pi_1^*\pi^*\omega) = \mu(S, \pi^*\omega) = \mu(E, \pi^*\omega).$$

E is semi-stable with respect to $\pi^*\omega$, so $\mu(\pi^*\mathcal{F}/Ker(j), \pi^*\omega) = \mu(E, \pi^*\omega)$, and we have our contradiction. We can now conclude:

$$M_{\tilde{S}}(\tilde{J}_0, \tilde{J}, \pi_1^*\pi^*\omega) \geq -\frac{C_1}{C_2}.$$

By definition $M_S(H_0, H, \pi^*\omega)$ is bounded from below as well.

□

3.4 Applications

In this final subsection we provide some applications of Theorem 2. First we define an approximate Hermitian-Einstein structure on a holomorphic vector bundle E.

Definition 6. We say E admits an *approximate Hermitian-Einstein structure* if for all $\epsilon > 0$, there exists a metric H on E with curvature F such that:

$$\sup_X |\Lambda F - \mu(E)I|_{C^0} < \epsilon.$$

With this definition, we now prove Theorem 2 as stated in the introduction.

Proof. This theorem is proven in [26] in the case where X is a projective algebraic manifold. The only part of that proof which does not extend to the Kähler case is the proof that condition *i*) implies condition *ii*). This is exactly what we prove for X Kähler in Sections 3.2.3 and 3.3.1. For a proof that condition *ii*) implies condition *iii*) and that condition *iii*) implies condition *i*) we direct the reader to [26]. □

We now state the following applications. In each application X is always assumed to be Kähler. The proofs of the first four Corollaries can be found in [26], chapter IV Section 5, under the assumption that E admits an approximate Hermitian-Einstein structure. We use Theorem 5 to identify existence of an approximate Hermitian-Einstein structure with E semi-stable. We note that Corollaries 2-4 are not original results, however Theorem 5 provides a natural proof of these statements. For example Corollary 2 is also proven in [1].

Corollary 1. *If E is semi-stable, so is the symmetric tensor product S^pE, and the exterior product Λ^pE.*

Corollary 2. *If E_1 and E_2 are semi-stable vector bundles, so is $E_1 \otimes E_2$.*

Corollary 3. *Let $\hat{X}$ be a finite unramified covering of X with projection $p : \hat{X} \longrightarrow X$. If E is a semi-stable vector bundle over X, then p^*E is a semi-stable vector bundle over $\hat{X}$. Also if $\hat{E}$ is a semi-stable vector bundle over $\hat{X}$, then $p_*\hat{E}$ is a semi-stable vector bundle over E.*

Corollary 4. *Let E be a semi-stable vector bundle of rank r over X. Then*

$$\int_X ((r-1)c_1(E)^2 - 2r\, c_2(E)) \wedge \omega^{n-2} \leq 0.$$

4 Identifying the Limit of the Yang-Mills Flow

In this section we prove Theorem 5, the main result of this thesis, verifying a conjecture of Bando and Siu. Once again throughout this section we assume X is Kähler. We begin with a discussion of the natural filtrations on E which will allow us to describe the limit space of the Yang-Mills flow.

4.1 Preliminaries

We begin with a few Preliminaries specific to our problem.

4.1.1 A natural filtration on E

Let E be a holomorphic vector bundle over X compact Kähler. Since we have no stability assumptions, we consider the following proposition, a proof of which can be found in [26]:

Proposition 12. *Any torsion-free sheaf E carries a unique filtration of subsheaves:*

$$0 = S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^p = E, \tag{4.1.1}$$

called Harder-Narasimhan filtration of E, such that the quotients of this filtration $Q^i = S^i/S^{i-1}$ are torsion-free and semi-stable. The quotients are slope decreasing $\mu(Q^i) > \mu(Q^{i+1})$, and the associated graded object $Gr^{hn}(E) := \bigoplus_i Q^i$ is uniquely determined by the isomorphism class of E.

Let f^i denote the holomorphic inclusion of the sheaf S^i into E. Also, let π^i denote the orthogonal projection of E onto S^i with respect to H. We note this projection only exists where S^i is locally free.

We also need an analogous filtration for semi-stable sheaves. For a torsion-free sheaf $\mathcal{Q}$ which is semi-stable but not stable, we can always assume there is at least one proper subsheaf $\mathcal{F}$ of $\mathcal{Q}$ such that $\mu(\mathcal{F}) = \mu(\mathcal{Q})$. In general there may be many such subsheaves.

Definition 7. Given a semi-stable sheaf $\mathcal{Q}$, a *Seshadri filtration* is a filtration of torsion free subsheaves

$$0 \subset \tilde{S}^0 \subset \tilde{S}^1 \subset \cdots \subset \tilde{S}^q = \mathcal{Q}, \tag{4.1.2}$$

such that $\mu(\tilde{S}^i) = \mu(\mathcal{Q})$ for all i, and each quotient $\tilde{Q}^i = \tilde{S}^i/\tilde{S}^{i-1}$ is torsion-free and stable.

While such a filtration may not be unique, we do have the following proposition, once again from [26].

Proposition 13. *Given a Seshadri filtration of a torsion free sheaf $\mathcal{Q}$, the direct sum of the stable quotients, denoted $Gr^s(\mathcal{Q}) := \bigoplus_i \tilde{Q}^i$, is canonical and uniquely determined by the isomorphism class of $\mathcal{Q}$.*

Given our initial holomorphic vector bundle E, let Q^k denote the k-th quotient of the Harder-Narasimhan filtration. Then $Gr^s(Q^k)$ can be denoted by $\bigoplus_i \tilde{Q}^i_k$. Putting these two propositions together, there exists a double filtration of E such that the corresponding graded object:

$$Gr^{hns}(E) := \bigoplus_k \bigoplus_i \tilde{Q}^i_k$$

is canonical and depends only on the isomorphism class of E. We now define the algebraic singular set of E as

$$Z_{alg} := \{x \in X \,|Gr^{hns}(E)_x \text{ is not free}\}.$$

Since the sheaf $Gr^{hns}(E)$ is torsion-free, we know Z_{alg} is of complex codimension two.

Finally, let r be the rank of E. We construct an r-tuple of real numbers:

$$(\mu(Q^1), \cdots, \mu(Q^1), \mu(Q^2), \cdots, \mu(Q^2), \cdots, \mu(Q^p), \cdots, \mu(Q^p)),$$

where the multiplicity of each number $\mu(Q^i)$ is determined by $rk(Q^i)$. We call this r-tuple the Harder-Narasimhan type of E. Now, recall from (1.0.2) the endomorphism

Ψ_H, whose eigenvalues are defined to be the Harder-Narasimhan type of E. We note the dependence on the metric H comes from metric dependence on the orthogonal projections $\pi^i : E \longrightarrow S^i$.

Definition 8. *We say E carries an L^p approximate Hermitian structure if for all $\epsilon > 0$, there exists a metric H on E such that:*

$$||\Lambda F - \Psi_H||_{L^p} < \epsilon.$$

We conclude this section by recalling the second fundamental form, which we introduced in Section 3.1. Let $S \subset E$ be a torsion free subsheaf, and let π and $q^\dagger$ be as in (3.1.2). Then as we saw the second fundamental form γ is given by $\bar{\partial} \circ p^\dagger$. Since $p^\dagger \circ p = I - \pi$, we have $\gamma \circ p = \bar{\partial}(p^\dagger) \circ p = \bar{\partial}(p^\dagger \circ p) = \bar{\partial}(I - \pi) = -\bar{\partial}\pi$. Thus $||\gamma||^2_{L^2} = ||\bar{\partial}\pi||^2_{L^2}$, and as a consequence of Proposition 1 we have $\pi \in L^2_1$. Conversely, as proven by Uhlenbeck and Yau in [52], any weakly holomorphic L^2_1 projection defines a coherent subsheaf of E. Thus, throughout the thesis we will go back and forth between working with a subsheaf S and the L^2_1 projection π that defines the subsheaf, and we will do this for all the sheaves involved in both the Harder Narasimhan filtration and the Seshadri filtraion.

4.1.2 The Yang-Mills functional and the Yang-Mills flow

In Section 2.1 we computed the curvature of the unitary-Chern connection, simply denoted F, with respect to a fixed metric H_0. We now consider more general connections A on E. Every connection is an endomorphism valued 1-form, and hence will decompose into $(1,0)$ and $(0,1)$ parts since X is a complex manifold. So $A = A' + A''$, where A'' represents the $(0,1)$ part of A. Define $\partial_A := \partial + A'$ and $\bar{\partial}_A := \bar{\partial} + A''$. We say A is *integrable* if $\bar{\partial}^2_A = 0$ (thus A defines a holomorphic structure), and we denote the space of integrable unitary connections by $\mathcal{A}^{1,1}$. The curvature of such a connection only has a $(1,1)$ component, and is defined by:

$$F_A := \bar{\partial}A' + \partial A'' + A'' \wedge A' + A' \wedge A''.$$

The Yang-Mills functional $YM : \mathcal{A}^{1,1} \longrightarrow \mathbf{R}$ can now be expresed:

$$YM(A) := ||F_A||^2_{L^2}.$$

On a general complex manifold, the Yang-Mills flow is the gradient flow of this functional, and is given by:

$$\dot{A} = -d^*_A F_A.$$

However, because we are on a Kähler manifold, Bianchi's second identity ($d_A F_A = 0$) and the Kähler identities allow us to express $d^*_A F_A$ in a simpler form:

$$\begin{aligned} d^*_A F_A &= \partial^*_A F_A + \bar{\partial}^*_A F_A \\ &= i[\Lambda, \bar{\partial}_A] F_A - i[\Lambda, \partial_A] F_A \\ &= -i\bar{\partial}_A \Lambda F_A + i\partial_A \Lambda F_A. \end{aligned}$$

Thus the Yang-Mills flow can be expressed as:

$$\dot{A} = i\bar{\partial}_A \Lambda F_A - i\partial_A \Lambda F_A. \tag{4.1.3}$$

From this formulation one can check that the Yang-Mills flow stays inside $\mathcal{A}^{1,1}$ if we start with an integrable connection.

In fact our approach to the Yang-Mills flow can be developed further. We follow the viewpoint taken by Donaldson in [13]. For details we refer the reader to [13], and just present the setup here. Recall we have fixed an initial metric H_0 on E. Any other metric H defines an endormphism $h \in Herm^+(E)$ by $h = H_0^{-1}H$. Then Donaldson heat flow is a flow of endomorphisms $h = h(t)$ given by:

$$h^{-1}\dot{h} = -(\Lambda F - \mu I),$$

where F is the curvature of the metric $H(t) = H_0 h(t)$. We set the initial condition $h(0) = I$. A unique smooth solution of the flow exists for all $t \in [0, \infty)$, and on any stable bundle this solution will converge to a smooth Hermitian-Einstein metric [13], [14], [43], [45]. In our case E is not stable, so we do not expect the flow to converge. However, it is useful in that it allows us to construct a solution to the Yang-Mills flow.

Working in a unitary frame with respect to H_0, let A_0 be an initial connection in $\mathcal{A}^{1,1}$. We have the decomposition $A_0 = A_0' + A_0''$. Now, starting with our initial holomorphic structure $\bar{\partial}_0 = \bar{\partial} + A_0''$, we consider the flow of holomorphic structures $\bar{\partial}_t = \bar{\partial} + A_t''$, where A_t'' is defined by the action of $w = h^{1/2}$ on A_0''. Explicitly, this action is given by:

$$A_t'' = wA_0''w^{-1} - \bar{\partial}ww^{-1}. \tag{4.1.4}$$

Using this flow of holomorphic structures and H_0 to define a flow of unitary connections A_t, one can check that A_t evolves by the Yang-Mills flow. Conversely, any path in $\mathcal{A}^{1,1}$ along the Yang-Mills flow defines an orbit of the complexified gauge group, which gives rise to a solution of the Donaldson heat flow.

Given this setup, the curvature of F along the Donaldson heat flow is related to the curvature F_A along the Yang-Mills flow by the following relation:

$$F_A = w\, F\, w^{-1}. \tag{4.1.5}$$

We now state the convergence result of Hong and Tian from [25]. Consider a sequence of connections A_j evolving along the Yang-Mills flow. Then, on $X \backslash Z_{an}$, along a subsequence the connections A_j converge in C^∞, modulo unitary gauge transformations, to a Yang-Mills connection A_∞. Thus, always working on $X \backslash Z_{an}$, we have a sequence of holomorphic structures $(E, \bar{\partial}_j)$ which converge in C^∞ to a holomorphic structure on a (possibly) different bundle $(E_\infty, \bar{\partial}_\infty)$. By the work of Bando and Siu, the bundle $(E_\infty, \bar{\partial}_\infty)$ extends to all of X as a reflexive sheaf $\hat{E}_\infty$. Once again the main goal of this thesis is to identify $\hat{E}_\infty$ with $Gr^{hns}(E)^{**}$, proving this limit is canonical and independent of subsequence.

4.2 The P-functional

An important and difficult step to proving Theorem 3 is to show there exists an L^2 approximate Hermitian structure on E. The proof of this step is similar to our previous work. First, we define a functional called P-functional and describe some basic properties. We then show it is bounded from below along it's gradient flow, which is the content of this section. Using this result, in the next section we can construct the desired L^2 approximate Hermitian structure along the gradient flow.

First we define the P-functional. Fix an initial metric H_0 on E. Then for any other metric H we can define the endomorphism $h = H_0^{-1}H$. Consider any path h_t in $Herm^+(E)$, $t \in [0,1]$ such that $h_0 = I$ and $h_1 = h$. The P-functional is defined by:

$$P(H_0, H) = \int_0^1 \int_X \mathrm{Tr}((\Lambda F_t - \Psi_t)h_t^{-1}\dot{h}_t)\,\omega^n\,dt,$$

Where F_t is the curvature of the metric $H_t = H_0 h_t$. The above integral converges, for even though the projections π^i that make up Ψ_t are only defined on $X\backslash Z_{alg}$, we know that they are at least in L_1^2, as we saw in section 4.1.1. We now check the P-functional is well defined independent of path.

Proposition 14. *The P-functional is path independent for any pair of metrics H_0, H on E.*

Proof. We note that the first term

$$\int_0^1 \int_X \mathrm{Tr}(\Lambda F_t h_t^{-1}\dot{h}_t)\,\omega^n\,dt,$$

appears in the Donaldson functional and is thus path independent (for a proof we refer the reader to [44]). Therefore we turn our attention to the second term:

$$\int_0^1 \int_X \mathrm{Tr}(\Psi_t h_t^{-1}\dot{h}_t)\,\omega^n\,dt = \sum_i \mu(Q^i) \int_0^1 \int_X \mathrm{Tr}((\pi_t^i - \pi_t^{i-1})h_t^{-1}\dot{h}_t)\,\omega^n\,dt.$$

Note that $\mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t) = \mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t f^i \pi_t^i) = \mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t f^i)$, where $\pi_t^i h_t^{-1}\dot{h}_t f^i$ is now an endomorphism of the bundle S^i. We need the following lemma:

Lemma 6. *Dropping the subscript t for simplicity, we have:*

$$\pi^i h^{-1}\dot{h} f^i = (h^i)^{-1}\dot{h}^i,$$

where J_0^i, J^i are the induced metrics on the subbundle S^i defined by H_0 and H, and h^i is the endomorphism of S^i defined by $h^i = (J_0^i)^{-1}J^i$.

Proof. First we note that $h^{-1}\dot{h}$ can be defined using the derivative of the metric H:

$$\partial_t\langle\cdot,\cdot,\rangle_H = \partial_t\langle h(\cdot),\cdot,\rangle_{H_0} = \langle\dot{h}(\cdot),\cdot,\rangle_{H_0} = \langle h^{-1}\dot{h}(\cdot),\cdot,\rangle_H.$$

Thus for any two sections ψ, ϕ of S^i, we define $(h^i)^{-1}\dot{h}^i$ by:

$$\partial_t\langle\psi,\phi\rangle_{J^i} = \langle(h^i)^{-1}\dot{h}^i\psi,\phi\rangle_{J^i}.$$

However by definition of the induced metric we have

$$\partial_t\langle\psi,\phi\rangle_{J^i} = \partial_t\langle f^i\psi, f^i\phi\rangle_H = \langle h^{-1}\dot{h}f^i\psi, f^i\phi\rangle_H = \langle\pi^i h^{-1}\dot{h}f^i\psi,\phi\rangle_{J^i},$$

concluding the lemma. □

Of course the lemma is only true where S^i is locally free, thus we restrict ourself to $X\backslash Z_{alg}$. On this set we have $\mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t) = \mathrm{Tr}((h_t^i)^{-1}\dot{h}_t^i) = \partial_t \log\, \det(h_t^i)$, so

$$\begin{aligned}
\int_0^1\int_X \mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t)\,\omega^n\,dt &= \int_0^1\int_{X\backslash Z_{alg}} \mathrm{Tr}(\pi_t^i h_t^{-1}\dot{h}_t)\,\omega^n\,dt \\
&= \int_0^1 \partial_t \int_{X\backslash Z_{alg}} \log\det(h_t^i)\,\omega^n\,dt \\
&= \int_{X\backslash Z_{alg}} \log\det(h_1^i)\,\omega^n.
\end{aligned}$$

Thus the integral is path independent. □

The goal of the next few sections is to prove the following Theorem:

Theorem 9. *For a fixed reference metric H_0, the functional $P(H_0, H)$ is bounded below for all other Hermitian metrics H.*

This theorem is a major step in the proof of Theorem 3. As a first step towards its proof we must regularize the Harder-Narasimhan filtration.

4.2.1 Regularization of the Harder-Narasimhan filtration

In this section we expand upon our sheaf regularization procedure in order to regularize the Harder-Narasimhan filtration of E (4.1.1). Recall that $f^i : S^i \longrightarrow E$ denotes the holomorphic inclusion of S^i into E, and let $l^i : S^i \longrightarrow S^{i+1}$ be the holomorphic inclusion of each subsheaf S^i into the the corresponding sheaf of next lowest rank S^{i+1}. Then we have that $f^{p-1} = l^{p-1}$, $f^{p-2} = l^{p-1} \circ l^{p-2}$, and in general $f^i = l^{p-1} \circ \cdots \circ l^i$. To regularize this filtration, we begin by regularizing each subsheaf, starting with S^1 and then working with subsheaves of successively higher rank. We describe the process as follows.

Given S^i from the filtration, for each i we have a sequence of blowups $\pi_i : \tilde{X}^i \longrightarrow \tilde{X}^{i-1}$ and a corresponding holomorphic inclusion map $\tilde{f}^i : \tilde{S}^i \longrightarrow {\pi_i}^* E$ such that the rank of $\tilde{f}^i$ does not drop. From (3.2.7) we have that $\tilde{f}^i$ is defined by ${\pi_i}^* f^i = \tilde{f}^i \circ t$, where t is some diagonal matrix of powers of the exceptional divisor. Since $f^i = l^{p-1} \circ \cdots \circ l^i$, and $\tilde{f}^i = {\pi_i}^* f^i \circ t^{-1}$, we can define $\tilde{l}^i := {\pi_i}^* l^i \circ t^{-1}$. Because after a finite number of steps the rank of $\tilde{f}^i$ does not drop, we know the rank of $\tilde{l}^i$ does not drop, thus

$$0 \longrightarrow (\pi_{i+1})^* \tilde{S}^i \xrightarrow{\tilde{l}^i} \tilde{S}^{i+1}$$

defines a holomorphic inclusion of vector bundles, and the regularized quotient $\tilde{Q}^{i+1}$ is a holomorphic vector bundle. Following this construction for all i we have a finite sequence of blowups that regularizes each sheaf in the Harder-Narasimhan filtration of E, such that the quotients $\tilde{Q}^i$ are all locally free. Summing up we have proved the following proposition:

Proposition 15. *Given a holomorphic vector bundle E over a complex-Hermitian manifold X, let*

$$0 = S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^p = E$$

be the Harder-Narasimhan filtration of E. The inclusion maps $l_0^i : S^i \longrightarrow S^{i+1}$ can be defined by matrices of holomorphic functions with transition functions on the overlaps.

Then there exists a finite number of blowups

$$\tilde{X}_N \xrightarrow{\pi_N} \tilde{X}_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} \tilde{X}_1 \xrightarrow{\pi_1} X,$$

and matrices of holomorphic functions l^i_k over $\tilde{X}_k$ with the the following properties:

i) On each $\tilde{X}_k$ there exists coordinates so that if w defines the exceptional divisor, there exists a diagonal matrix of monomials in w (denoted t) so that

$$\pi^*_{k-1} l^i_{k-1} = l^i_k \circ t.$$

ii) The rank of l^i_N is constant for each i, thus it defines a holomorphic subbundle of $\tilde{S}^{i+1}$ with a holomorphic quotient bundle.

With our regularization process understood, we now turn back to the proof of Theorem 9. Our method is to re-write the P-functional and writing it as a sum of objects which we know are bounded from below. Specifically, if we let $M_i(H_0, H, \omega)$ denote the Donaldson functional on the quotient sheaf Q^i with induced metrics from E (see Section 3.2.3), then we will prove that:

$$P(H_0, H) = \sum_i M_i(H_0, H, \omega) + ||\gamma^i||^2_{L^2} - ||\gamma^i_0||^2_{L^2}. \tag{4.2.6}$$

Here γ^i is the second fundamental form of the short exact sequence:

$$0 \longrightarrow S^{i-1} \longrightarrow S^i \longrightarrow Q^i \longrightarrow 0,$$

associated to the metric H, and γ^i_0 is the second fundamental form associated to H_0. Thus to prove Theorem 9 we have to complete two steps. First we show that all the terms in (4.2.6) are well defined for induced metrics on the sheaves Q^i, and secondly we need to show that the functional does indeed satisfy the decomposition (4.2.6). In this subsection we will focus on showing all the terms are well defined.

We have shown that there exists a regularized Harder-Narasimhan filtration:

$$0 = \tilde{S}^0 \subset \tilde{S}^1 \subset \cdots \subset \tilde{S}^{p-1} \subset \tilde{S}^p = \pi^* E,$$

such that the rank of the holomorphic inclusion maps $\tilde{f}^i$ does not drop on $\tilde{X}$ (here $\pi : \tilde{X} \longrightarrow X$ is the sequence of blowups needed to construct the regularization). So given $\pi^* H$ on $\pi^* E$, the smooth induced metric on $\tilde{S}^i$ is defined by:

$$\tilde{J}^i_{\bar{\beta}\alpha} := (\tilde{f}^i)^\rho{}_\alpha \overline{(\tilde{f}^i)^\gamma{}_\beta} H_{\bar{\gamma}\rho}.$$

Also, because the rank of $\tilde{l}^i : \tilde{S}^i \longrightarrow \tilde{S}^{i+1}$ does not drop, we have an exact sequence of holomorphic vector bundles:

$$0 \longrightarrow \tilde{S}^i \xrightarrow{\tilde{l}^i} \tilde{S}^{i+1} \xrightarrow{p^i} \tilde{Q}^{i+1} \longrightarrow 0.$$

The metric $\tilde{J}^{i+1}$ gives a splitting of the short exact sequence:

$$0 \longleftarrow \tilde{S}^i \xleftarrow{\lambda^i} \tilde{S}^{i+1} \xleftarrow{p^{i\dagger}} \tilde{Q}^{i+1} \longleftarrow 0,$$

and it follows that the metric $\tilde{K}_{\bar{\beta}\alpha}$ on $\tilde{Q}^{i+1}$ defined by:

$$\tilde{K}_{\bar{\beta}\alpha} := (p^{i\dagger})^\rho{}_\alpha \overline{(p^{i\dagger})^\gamma{}_\beta} \tilde{J}^i_{\bar{\gamma}\rho}$$

is smooth.

Our main concern is that the integrals that make up each term in (4.2.6) might not be finite, which is a reasonable concern because along Z_{alg}, curvature terms will blow up. We show these terms are in fact controlled by using formulas describing the change during each step in the regularization procedure, and prove that in fact the desired terms do not change during regularization, just as in our proof of the lower bound for the Donaldson function. Then once we are working with the regularized filtration we know the induced metrics are smooth, so each term will be finite.

Recall Proposition 4, in which we proved how induced metrics change during each blowup in our regularization procedure. Specifically, given a single blowup $\pi : \tilde{X} \longrightarrow X$, let J and K be induced metrics on S^i and Q^i, respectively. Then if w locally defines the exceptional divisor, there exists natural numbers a_α so that:

$$\pi^* J_{\bar{\beta}\alpha} = w^{a_\alpha} \overline{w^{a_\beta}} \tilde{J}_{\bar{\beta}\alpha} \qquad , \qquad \pi^* K_{\bar{\beta}\alpha} = \frac{1}{w^{a_\alpha} \overline{w^{a_\beta}}} \tilde{K}_{\bar{\beta}\alpha}.$$

Using these formulas we can compute how the induced curvature changes, just as we did in Lemma 1. For simplicity we restrict ourselves to working with K on the quotient Q^i, and denote the curvature of K by F. The following formula holds in the sense of currents:

$$\pi^*\mathrm{Tr}(F) = \sum_{\alpha} a_\alpha \partial\bar{\partial}\mathrm{log}|w|^2 + \mathrm{Tr}(\tilde{\mathrm{F}}).$$

A similar formula holds for induced metrics on S^i. Now, just as in the proof of Proposition 1, given the above formula it follows that $||\gamma^i||^2_{L^2} = ||\tilde{\gamma}^i||^2_{L^2}$ for all i. Thus the L^2 norm of the second fundamental form terms does not change during regularization. After a finite number of steps these forms are smooth, so since X is compact we know the each term is finite. Thus the two right hand terms in (4.2.6) are finite.

Next we show that the Donaldson functional $M_i(H_0, H, \omega)$ is well defined on any quotient sheaf Q^i arising from the filtration. Given a blowup map $\pi : \tilde{X} \longrightarrow X$, one can also define the Donaldson functional on a vector bundle over $\tilde{X}$ by integrating with respect to the degenerate metric $\pi^*\omega$. Since $\pi^*\omega$ is closed the functional will still be independent of path. We define the Donaldson functional on the sheaves Q^i as follows:

Definition 9. For any quotient sheaf Q^i arising from the Harder-Narasimhan filtration of E, we define the *Donaldson functional* on Q^i to be:

$$M_i(H_0, H, \omega) := M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega),$$

for any regularization $\tilde{Q}^i$.

Here $M_{\tilde{Q}}(\tilde{K}_0, \tilde{K}, \pi^*\omega)$ is the Donaldson functional for the vector bundles $\tilde{Q}$ defined using the degenerate metric $\pi^*\omega$. We note that the domains of the functionals M_i are metrics on the vector bundle E, thus this definition only applies to induced metrics and does not extend to arbitrary metrics on Q^i. The following proposition proves that this definition is well defined.

Proposition 16. *For each i the functional M_i is well defined for any pair of metrics on E, and is independent of the choice of regularization.*

For a proof we direct the reader to Proposition 5. Immediately we see the P-functional is well defined on the subsheaves S^i as well, and that its value is independent of regularization. Now all three terms on the right hand side of (4.2.6) are well defined for induced metrics on the quotient sheaves Q^i. The next step is to show that the decompostion formula does indeed hold.

4.2.2 Decomposition of the P-functional

In this subsection we prove the decomposition formula (4.2.6) holds. We begin by considering the proper subsheaf of highest rank in the filtration, S^{p-1}. In the proof of Proposition 14, we found the following formula for P:

$$P(H_0, H) = \int_0^1 \int_X \mathrm{Tr}(\Lambda F_t h_t^{-1} \dot{h}_t)\, \omega^n \, dt - \sum_i \mu(Q^i) \int_X (\log\det(h_1^i) - \log\det(h_1^{i-1}))\, \omega^n,$$

Where h^i is the endomorphism defined by induced metrics J^i and J_0^i on S^i. Dropping the subscript t, we note that by Proposition 16 we have:

$$\int_0^1 \int_X \mathrm{Tr}(\Lambda F h^{-1} \dot{h}) \omega^n = \int_0^1 \int_{\tilde{X}} \mathrm{Tr}(\pi^*(\Lambda F h^{-1} \dot{h})) \pi^* \omega^n,$$

where $\pi : \tilde{X} \longrightarrow X$, is a sequence of blowups which regularizes the Harder-Narasimhan filtration. Now, the regularized $\tilde{S}^{p-1}$ and $\tilde{Q}^p$ are holomorphic subbundles and quotient bundles of $\pi^* E$, and with the metric $\pi^* H$ we can identify the following splitting:

$$0 \longleftarrow \tilde{S}^{p-1} \longleftarrow \pi^* E \xleftarrow{\;p^\dagger\;} \tilde{Q}^p \longleftarrow 0.$$

Following Section 3.1 we have the decomposition of curvature:

$$\Lambda F = \begin{pmatrix} \Lambda F^{\tilde{S}^{p-1}} + \pi^* g^{j\bar{k}} \gamma_{\bar{k}} \gamma_j^\dagger & \pi^* g^{j\bar{k}} \nabla_j \gamma_{\bar{k}} \\ \pi^* g^{j\bar{k}} \nabla_{\bar{k}} \gamma_j^\dagger & \Lambda F^{\tilde{Q}^p} - \pi^* g^{j\bar{k}} \gamma_j^\dagger \gamma_{\bar{k}} \end{pmatrix}.$$

Define $V := p^\dagger - p_0^\dagger$. Using the description of $h^{-1}\dot{h}$ from the proof of Lemma 6 we can see how $h^{-1}\dot{h}$ decomposes:

$$h^{-1}\dot{h} = \begin{pmatrix} (h^{-1}\dot{h})^{p-1} & -\dot{V} \\ -\dot{V}^\dagger & (h^{-1}\dot{h})^{p} \end{pmatrix}.$$

Here $(h^{-1}\dot{h})^{p-1}$ and $(h^{-1}\dot{h})^{p}$ are the induced endomorphims on $\tilde{S}^{p-1}$ and $\tilde{Q}^p$. Thus we now have:

$$\begin{aligned} \mathrm{Tr}(\pi^*(\Lambda F h^{-1}\dot{h})) &= \mathrm{Tr}\,(\Lambda F^{\tilde{S}^{p-1}}(h^{-1}\dot{h})^{p-1} + \pi^*\Lambda F^{\tilde{Q}^p}(h^{-1}\dot{h})^{p}) + \\ &\quad \pi^* g^{j\bar{k}}\,\mathrm{Tr}(\gamma_{\bar{k}}\gamma_j^\dagger(h^{-1}\dot{h})^{p-1} - \nabla_j\gamma_{\bar{k}}\dot{V}^\dagger - \nabla_{\bar{k}}\gamma_j^\dagger\dot{V} - \gamma_j^\dagger\gamma_{\bar{k}}(h^{-1}\dot{h})^{p}). \end{aligned}$$

We note now that the term:

$$\int_0^1\int_{\tilde{X}} \mathrm{Tr}(\pi^*\Lambda F^{\tilde{Q}^p}(h^{-1}\dot{h})^{p})\pi^*\omega^n,$$

combines with:

$$-\mu(Q^p)\int_{\tilde{X}}(\log\det(h_1) - \log\det(h_1^{p-1}))\pi^*\omega^n,$$

to give $M_p(H_0, H, \omega)$. Also the term:

$$\int_0^1\int_{\tilde{X}} \mathrm{Tr}(\pi^*\Lambda F^{S^{p-1}}(h^{-1}\dot{h})^{p-1})b^*\pi^n,$$

combines with

$$-\sum_{i=1}^{p-1}\mu(Q^i)\int_0^1\int_X \mathrm{Tr}((\pi_t^i - \pi_t^{i-1})h_t^{-1}\dot{h}_t)\,\pi^*\omega^n\,dt,$$

to give $P_{|_{\tilde{S}^{p-1}}}(H_0, H)$. Thus the remaining term to identify is

$$\int_0^1\int_{\tilde{X}} \pi^* g^{j\bar{k}}\mathrm{Tr}(\gamma_{\bar{k}}\gamma_j^\dagger(h^{-1}\dot{h})^{p-1} - \nabla_j\gamma_{\bar{k}}\dot{V}^\dagger - \nabla_{\bar{k}}\gamma_j^\dagger\dot{V} - \gamma_j^\dagger\gamma_{\bar{k}}(h^{-1}\dot{h})^{p})\pi^*\omega^n\,dt.$$

Now, since $\gamma_{\bar{k}} = \partial_{\bar{k}}p^\dagger$, we have $\dot{\gamma}_{\bar{k}} = \partial_t(\gamma_{\bar{k}} - (\gamma_{\bar{k}})_0) = \partial_t(\partial_{\bar{k}}(p^\dagger - p_0^\dagger)) = \nabla_{\bar{k}}\dot{V}$. Thus we can integrate by parts to get:

$$\int_0^1\int_{\tilde{X}} \pi^* g^{j\bar{k}}\mathrm{Tr}(\gamma_{\bar{k}}\gamma_j^\dagger(h^{-1}\dot{h})^{p-1} + \gamma_{\bar{k}}\dot{\gamma}_j^\dagger + \gamma_j^\dagger\dot{\gamma}_{\bar{k}} - \gamma_j^\dagger\gamma_{\bar{k}}(h^{-1}\dot{h})^{p})\pi^*\omega^n\,dt.$$

Consider the following formula, which can be found in [13]:

$$\partial_t(\gamma_j^\dagger) = \dot{\gamma}_j^\dagger + \gamma_j^\dagger(h^{-1}\dot{h})^{p-1} - (h^{-1}\dot{h})^p\gamma_j^\dagger.$$

The final term now becomes:

$$\int_0^1 \int_{\tilde{X}} \partial_t(\pi^* g^{j\bar{k}}\mathrm{Tr}(\gamma_{\bar{k}}\gamma_j^\dagger)\pi^*\omega^n)\, dt = ||\gamma^p||_{L^2}^2 - ||\gamma_0^p||_{L^2}^2.$$

This completes the first step of the decomposition. We can continue the process on $P_{|_{\tilde{S}^{p-1}}}(H_0, H)$ to prove the desired decomposition formula (4.2.6). We are now ready to prove Theorem 9.

Proof. By (4.2.6), we know the P-functional is the sum over all i of three terms. The two second fundamental form terms are bounded below since $||\gamma^i||_{L^2}^2$ is positive and $-||\gamma_0^i||_{L^2}^2$ is fixed and only depends on our initial metric H_0. To see that $M_i(H_0, H, \omega)$ is bounded below, notice that this functional is equivalent to the Donaldson functional defined on some regularization $\tilde{Q}^i$. This regularization is a holomorphic vector bundle over $\tilde{X}$, and is semi-stable with respect to the pull back form $\pi^*\omega$. Thus this term is bounded from below by the proof of Theorem 7.

□

4.3 The modified Donaldson heat flow

We now turn to the next step in the construction of an L^2 approximate Hermitian structure on E. In this section we prove long time existence of the gradient flow of the P-functional, which we call the modified Donaldson heat flow:

$$H^{-1}\dot{H} = -(\Lambda F - \Psi_H). \tag{4.3.7}$$

We can also express this flow as a flow of endomorphisms h. Let $\hat{\nabla}$ be the unitary-Chern connection with respect to the fixed metric H_0. Then we have:

$$\dot{h} = \hat{\Delta} h - g^{j\bar{m}}\hat{\nabla}_{\bar{m}} h h^{-1}\hat{\nabla}_j h - h\Lambda\hat{F} + h\Psi_H. \tag{4.3.8}$$

At this point once again we note that this flow is similar to the Donaldson Heat flow, namely that the two flows differ only by terms of order zero. As a result many proofs in this section follow the proofs of Donaldson [13] and Simpson [43] on the long time existence of equation (1.0.1). In these cases we will simply state our result and direct the reader to the relevant references. However, some differences arise, mostly relating to the fact that $\mu(E)I$ (which is the zeroth order term from the Donaldson heat flow) is constant in time and space while Ψ_H varies in both. We also note that because Ψ_H is only in L_1^2, we follow Simpson's and work on $X\backslash Z_{alg}$, where we know Ψ_H is smooth. Simpson's arguments, including using an exhaustion function in order to use maximum principle arguments, carry over to our case in the same fashion, and we refer the reader to [43] for details. Since in the end we are only concerned with achieving and L^2 estimate for the curvature, working away from Z_{alg} is justified. We begin with short time existence:

Proposition 17. *For any initial metric H_0, there exists a time T such that a solution to the modified Donaldson heat flow* (4.3.7) *exists for short time $t \in [0, T)$.*

Proof. Let M be the differential operator:

$$\begin{aligned} M(h) &= \hat{\Delta} h - g^{j\bar{m}} \hat{\nabla}_{\bar{m}} h h^{-1} \hat{\nabla}_j h - h \Lambda \hat{F} + h \Psi_H. \\ &= \hat{\Delta} h - g^{j\bar{m}} \hat{\nabla}_{\bar{m}} h h^{-1} \hat{\nabla}_j h - h \Lambda \hat{F} + \sum_i \mu(Q^i)(\pi^i - \pi^{i-1}). \end{aligned}$$

We need a different formulation of Ψ_H. Using the fact that $\mu(Q^i) - \mu(Q^{i+1}) = c_i > 0$ (since the Harder-Narasimhan filtration is slope decreasing), we have:

$$\Psi_H = \sum_{i=1}^{p} c_i \pi^i.$$

So

$$M(h) = \hat{\Delta} h - g^{j\bar{m}} \hat{\nabla}_{\bar{m}} h h^{-1} \hat{\nabla}_j h - h \Lambda \hat{F} + \sum_i c_i \pi^i.$$

To see that (4.3.8) is parabolic, we compute the linearized operator $M_h(\cdot)$ at the point h. First we compute the derivative of π^i along some path. Note that if ϕ lies in S^i

and ψ lies in $S^{i\perp}$, then $\langle \phi, \psi \rangle_H = 0$. Of course, since the perpendicular space $S^{i\perp}$ changes with H, we need to choose a vector that lies in $S^{i\perp}$ for all time, so we choose $(I - \pi^i)V$ for some fixed section V of E. Thus $\langle \phi, (I - \pi^i)V \rangle_H = 0$ for all time. We compute the time derivative of this expression:

$$0 = \partial_t \langle \phi, (I - \pi^i)V \rangle_H = \langle \phi, h^{-1}\dot{h}(I - \pi^i)V \rangle_H - \langle \phi, \dot{\pi}^i V \rangle_H.$$

Thus $\dot{\pi}^i$ is the component of $h^{-1}\dot{h}$ which sends $S^{i\perp}$ to S^i. So when we compute the derivative of M at $t = 0$ of the path $h + t\eta$, we have $\dot{\pi}^i = \pi^i h^{-1}\eta(I - \pi^i)$. The linearized operator is now given by:

$$\begin{aligned} M_h(\eta) &= \hat{\Delta}\eta - g^{j\bar{m}}(\hat{\nabla}_{\bar{m}}\eta h^{-1}\hat{\nabla}_j h + \hat{\nabla}_{\bar{m}} h h^{-1}\hat{\nabla}_j \eta) \\ &\quad + g^{j\bar{m}}\hat{\nabla}_{\bar{m}} h h^{-1}\eta h^{-1}\hat{\nabla}_j h + \eta\Lambda\hat{F} + \sum_i c_i \pi^i h^{-1}\eta(I - \pi^i). \end{aligned}$$

The highest order term of this linear operator is just the laplacian, thus (4.3.8) is parabolic. Short time existence follows. □

We now turn our attention to long time existence. Following Donaldson, we introduce a notion of distance between two metrics on E.

Definition 10. *We define the following two quantities for any two Hermitian metrics H,K on E:*

$$\begin{aligned} \tau(H, K) &= \mathrm{Tr}(H^{-1}K) \\ \sigma(H, K) &= \tau(H, K) + \tau(K, H) - 2\, rk(E). \end{aligned}$$

Just as in [13] we see that $\sigma(H, K) \geq 0$ with equality if and only if $H = K$.

Lemma 7. *If H_t, K_t are two solutions to the heat flow* (4.3.7)*, then we have:*

$$\sigma(H_t, K_t) \leq e^{Ct} \sup_X \sigma(H_0, K_0),$$

for some constant C depending only on X, E.

Proof. This proof relies on an application of the maximum principle. First we need to compute the evolution equation for σ.

$$\begin{aligned}\partial_t \mathrm{Tr}(H^{-1}K) &= \mathrm{Tr}(-H^{-1}\dot{H}H^{-1}K + H^{-1}\dot{K}) \\ &= \mathrm{Tr}(-H^{-1}\dot{H}H^{-1}K + H^{-1}HH^{-1}\dot{K}) \\ &= \mathrm{Tr}(H^{-1}K(K^{-1}\dot{K} - H^{-1}\dot{H})).\end{aligned}$$

Thus setting $k = H^{-1}K$, we have:

$$\partial_t \tau(H,K) = \mathrm{Tr}(k(-\Lambda F_K + \Lambda F_H + \Psi_K - \Psi_H).$$

Now, because k is a positive definite matrix, and $\Psi_K - \Psi_H$ consists of the difference between sums of projections times topological constants, it follows that:

$$\mathrm{Tr}(k(\Psi_K - \Psi_H)) \leq C\mathrm{Tr}(\mathrm{k}) = C\,\tau(H,K).$$

Also, because $-\Lambda F_K + \Lambda F_H = g^{j\bar{m}}\nabla_{\bar{m}}(k^{-1}\nabla_j k)$, where ∇ is the covariant derivative with respect to H, we have:

$$\begin{aligned}\mathrm{Tr}(k(-\Lambda F_K + \Lambda F_H)) &= \mathrm{Tr}(\Delta k) - \mathrm{Tr}(g^{j\bar{m}}\nabla_{\bar{m}}kk^{-1}\nabla_j k) \\ &\leq \Delta\tau(H,K).\end{aligned}$$

Putting these facts together we see:

$$\partial_t \tau(H,K) \leq \Delta\tau(H,K) + C\,\tau(H,K).$$

It is now clear from the definition of $\sigma(H,K)$ that:

$$\partial_t \sigma(H,K) \leq \Delta\sigma(H,K) + C\,\sigma(H,K).$$

Now, it follows that:

$$\partial_t(e^{-Ct}\sigma(H,K)) = e^{-Ct}(\partial_t\sigma(H,K) - C\,\sigma(H,K)) \leq \Delta(e^{-Ct}\sigma(H,K)).$$

So by the maximum principle, we see:

$$\sup_X e^{-Ct}\sigma(H,K) \le \sup_X \sigma(H_0,K_0),$$

from which the statement of the lemma follows.

□

Corollary 5 (Uniqueness)**.** *Given two solutions H_t, K_t to the heat flow* (4.3.7)*, if $H_0 = K_0$ then the two solutions agree for all time $t \in [0,T)$.*

Proof. By Lemma 7, we have:

$$\sigma(H_t,K_t) \le e^{Ct} \sup_X \sigma(H_0,K_0) = 0,$$

since the metrics agree at time $t = 0$. Uniqueness follows. □

Corollary 6. *Suppose that a solution H_t to* (4.3.7) *exists for $t \in [0,T)$. Then H_t converges in C^0 to some continuous metric H_T as $t \to T$.*

Proof. Once again by Donaldson we know the space of metrics can be identified with the symmetric space $GL(r,\mathbf{C})/U(r)$, and from this space we inherit a distance function $d(H,K)$. This metric space is complete, and moreover the function $\sigma(H,K)$ compares uniformly to $d(H,K)$. Thus, to complete the proof of the corollary, it suffices to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\sup_X \sigma(H_t,H_{t'}) < \epsilon \qquad \text{for all} \quad t,t' \in (T-\delta,T).$$

Now, $\sigma(H_0,H_0) = 0$, so by continuity of σ we know there exists a $\delta > 0$ such that:

$$\sup_X \sigma(H_0,H_\rho) < \frac{\epsilon}{e^{CT}} \qquad \text{for all} \quad \rho < \delta.$$

So by Lemma 7 we see:

$$\sup_X \sigma(H_t,H_{t+\rho}) \le e^{Ct} \sup_X \sigma(H_0,H_\rho) < \epsilon,$$

as long as $\rho < \delta$. This completes the proof of the corollary.

□

Lemma 8. *Along the modified Donaldson Heat flow, for any finite time T the trace of the curvature stays uniformly bounded. Explicitly:*

$$||\Lambda F_t||_{L^\infty} \leq C_T.$$

Proof. We begin by computing the evolution of ΛF along the flow. Since along any path of metrics we have:

$$\partial_t \Lambda F = -\partial_t g^{j\bar{k}} \nabla_{\bar{k}} (\nabla_j h h^{-1}) = -g^{j\bar{k}} \nabla_{\bar{k}} \nabla_j (h^{-1} \dot{h}).$$

In our case we see:

$$\partial_t \Lambda F = \bar{\Delta}(\Lambda F - \Psi).$$

Now we show that along (4.3.7) the following formula holds:

$$\partial_t \langle \Lambda F, \Lambda F \rangle = 2\langle \Lambda \dot{F}, \Lambda F \rangle, \tag{4.3.9}$$

where the inner product is taken with respect to the evolving metric H. To see this we note that for the adjoint $(\Lambda F)^*$, we have the following evolution equation:

$$\partial_t (\Lambda F)^* = (\Lambda \dot{F})^* - H^{-1}\dot{H}(\Lambda F)^* + (\Lambda F)^* H^{-1} \dot{H}.$$

So it follows that

$$\begin{aligned} \partial_t \langle \Lambda F, \Lambda F \rangle &= \partial_t \mathrm{Tr}(\Lambda F (\Lambda F)^*) \\ &= \mathrm{Tr}(\Lambda \dot{F} (\Lambda F)^* + \Lambda F (\Lambda \dot{F})^* - \Lambda F H^{-1} \dot{H} (\Lambda F)^* + \Lambda F (\Lambda F)^* H^{-1} \dot{H}) \\ &= 2\langle \Lambda \dot{F}, \Lambda F \rangle + \mathrm{Tr}(-\Lambda F H^{-1} \dot{H} (\Lambda F)^* + \Lambda F (\Lambda F)^* H^{-1} \dot{H}). \end{aligned}$$

Now the leftover terms on the right are given by:

$$\mathrm{Tr}(\Lambda F (\Lambda F - \Psi)(\Lambda F)^* - \Lambda F (\Lambda F)^* (\Lambda F - \Psi)),$$

and the terms involving only ΛF clearly cancel each other. We are left with:

$$\mathrm{Tr}(-\Lambda F \Psi (\Lambda F)^* + \Lambda F (\Lambda F)^* \Psi)),$$

which vanishes in a unitary frame using the fact that curvature is skew-adjoint, implying $(\Lambda F)^* = \Lambda F$. Since trace is independent of a choice of frame we have shown (4.3.9).

Now, we have

$$\Delta\langle \Lambda F, \Lambda F\rangle = \langle \Delta\Lambda F, \Lambda F\rangle + \langle \Lambda F, \bar{\Delta}\Lambda F\rangle + |\nabla \Lambda F|^2 + |\bar{\nabla}\Lambda F|^2.$$

Using the fact that $\Delta\Lambda F = \bar{\Delta}\Lambda F$ we can compare $\Delta\langle \Lambda F, \Lambda F\rangle$ with $\partial_t\langle \Lambda F, \Lambda F\rangle$.

$$\begin{aligned}
\partial_t\langle \Lambda F, \Lambda F\rangle &= 2\langle \Lambda\dot{F}, \Lambda F\rangle \\
&= 2\langle \bar{\Delta}(\Lambda F - \Psi), \Lambda F\rangle \\
&= \Delta\langle \Lambda F, \Lambda F\rangle - |\nabla \Lambda F|^2 - |\bar{\nabla}\Lambda F|^2 - 2\langle \bar{\Delta}\Psi, \Lambda F\rangle. \\
&\leq \Delta\langle \Lambda F, \Lambda F\rangle - 2\langle \bar{\Delta}\Psi, \Lambda F\rangle.
\end{aligned}$$

We will be done with the lemma if we can estimate $-\langle \bar{\Delta}\Psi, \Lambda F\rangle$ by $\langle \Lambda F, \Lambda F\rangle$. Recall the following formulation of Ψ from Proposition 17:

$$\Psi = \sum_{i=1}^{p} c_i \pi^i.$$

Thus

$$\langle \bar{\Delta}\Psi, \Lambda F\rangle = \sum_i c_i \langle \bar{\Delta}\pi^i, \Lambda F\rangle.$$

Consider the subbundle S^i with projection $\pi^i : E \longrightarrow S^i$. If γ^i is the second fundamental form associated to S^i, then as stated in Section 4.1.1 we have $\gamma^i_{\bar{k}} \circ p = -\nabla_{\bar{k}}\pi^i$. If we let $S^{i\perp}$ be the perpendicular space to S^i defined by the metric H, we can check how ΛF decomposes onto these orthogonal subspaces of E. Explicitly the component that sends $S^{i\perp}$ to S^i is given by $-g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j \pi^i = -\bar{\Delta}\pi^i$. We then see that:

$$-\langle \bar{\Delta}\pi^i, \Lambda F\rangle = \langle (I - \pi^i)\Lambda F \pi^i, \Lambda F\rangle = \langle (I - \pi^i)\Lambda F \pi^i, (I - \pi^i)\Lambda F \pi^i\rangle.$$

Thus

$$(\partial_t - \Delta)\langle \Lambda F, \Lambda F\rangle \leq C\langle \Lambda F, \Lambda F\rangle,$$

and the lemma follows from the maximum principle.

□

Recall that $h_t = H_0^{-1} H_t$. We can now bound $\mathrm{Tr}(h)$ for finite time.

Lemma 9. *There exist a constant C_T depending only on H_0 such that for a given finite time T, we have*

$$\mathrm{Tr}(h) \leq C_T. \tag{4.3.10}$$

Proof. The proof is once again by the maximum principle, which we see after computing how $\mathrm{Tr}(h)$ evolves with time. For details see Proposition 8. □

Now we exploit the special properties of the flow to show that given any initial metric K on E, there exists a metric H_0 in the conformal class of K such that $\det(h) = 1$ along the flow. This fact, along with the previous lemma, tells us that for all finite time the metrics H_0 and H_t are equivalent, as that H_t cannot degenerate in finite time.

Lemma 10. *For any initial metric K on E, there exists a C^∞ function ϕ such that if we set $H_0 = e^\phi K$, then $\det(h) = 1$ along the modified Donaldson heat flow.*

Proof. The proof of this fact is exactly the same as for the Donaldson heat flow. See [30] for details. □

We now have that if the flow exists up to some finite time T, then H_t converges to some non-degenerate limit metric H_T in C^0. We need higher order derivative estimates to show that in fact this convergence is smooth. Once we have this, using short time existence starting with H_T we get a solution of the flow up to time $T + \epsilon$, proving long time existence. Note that the previous lemma along with the finite time bound for $\mathrm{Tr}(h)$ show that the metrics H_0 and H_t define equivalent norms. Thus for the remainder of the section we compute norms with respect to H_0 knowing we can get the same estimates using H.

Our next goal is to gain C^0 control of the connection for finite time, for which we will need the following two lemmas:

Lemma 11. *Define* $S = |\nabla h\, h^{-1}|^2_{H_0}$*. Then along the modified Donaldson heat flow there exists a constant* C *such that*

$$(\Delta - \partial_t)S \geq -C\,S$$

Proof. This computation is straightforward and parallels the case of the Donaldson heat flow, see [30]. The only difference are the terms involving $\nabla\Psi$, yet as we have seen in the proof of Lemma 8, these terms are related to the second fundamental forms of the filtration and are in fact bounded by S. □

We note that an analogue of Lemma 11 was worked out for the Kähler-Ricci flow in [35], where it was also pointed out that it can be viewed as a more precise, parabolic version of the Calabi identity [54].

Lemma 12. *Along the modified Donaldson heat flow there exists positive constants* C_1 *and* C_2 *(which depend on the maximal time of existence* T*), such that*

$$(\Delta - \partial_t)\mathrm{Tr}(h) \geq C_1 S - C_2$$

Proof. The proof of this Lemma is another computation, and the result follows just as in the Donaldson heat flow case (once again see [30]). □

Now we combine the previous two lemmas to prove the following proposition:

Proposition 18. *Let* A_t *be a the unitary-Chern connection evolving along the modified Donaldson heat flow. Then we have*

$$||\nabla h h^{-1}||_{L^\infty} \leq C_T,$$

Where C_T *depends on the maximal existence time* T*.*

Proof. The proof of this fact follows from the maximum principle and the finite time bound on $\mathrm{Tr}(h)$.

□

Our next goal is to show that for finite time we have C^0 control of the full curvature tensor. First we prove L^p control for any p, which is the subject of the following lemma

Lemma 13. *Up to a finite time T, we have*

$$||F||_{L^p} < C_T$$

for any $1 \leq p < \infty$.

Proof. Recall that if $\hat{F}$ is the curvature of the initial metric H_0, and $\hat{\nabla}$ the initial unitary-Chern connection, then

$$\begin{aligned} F_{\bar{k}j} - \hat{F}_{\bar{k}j} &= -\hat{\nabla}_{\bar{k}}(h^{-1}\hat{\nabla}_j h) \\ &= -h^{-1}\hat{\nabla}_{\bar{k}}\hat{\nabla}_j h + h^{-1}\hat{\nabla}_{\bar{k}} h h^{-1}\hat{\nabla}_j h. \end{aligned}$$

We have

$$\hat{\Delta} h = h(\Lambda\hat{F} - \Lambda F) + g^{j\bar{k}}\hat{\nabla}_{\bar{k}} h h^{-1}\hat{\nabla}_j h.$$

From here we see up to finite time that the right hand side is uniformly bounded in C^0. By standard L^p theory of elliptic PDE's, it follows that for any $1 \leq p < \infty$ we have

$$||h||_{W^{2,p}} < \infty.$$

Since the curvature F is given by a formula involving two derivatives of h, we have the desired result. □

Lemma 14. *Along the modified Donaldson heat flow, we have the following inequality:*

$$\partial_t |F|^2 \leq \Delta |F|^2 + C(|F|^2 + |F|^3) \qquad (4.3.11)$$

Proof. This case the same as the proof of the Donaldson heat flow case in [13]. Once again we use the fact that $\nabla_{\bar{k}}\nabla_j\Psi = \sum_i c_i\nabla_{\bar{k}}\nabla_j\pi^i$, where $\nabla_{\bar{k}}\nabla_j\pi^i$ is the component of $F_{\bar{k}j}$ that maps S^i to $S^{i\perp}$. □

From the previous two lemmas we can now get L^∞ control of F.

Proposition 19. *Along the modified Donaldson heat flow, we have the following L^∞ control of the full curvature tensor:*

$$||F||_{L^\infty} \leq C_T,$$

where C_T depends on the maximal existence time T.

Proof. Using equation (4.3.11) and the heat kernel $\Phi(x, y, t)$ on X, we get the following estimate:

$$|F_t|^2(x) \leq \int_X \Phi(x, y, t)|F_0|^2(y)\omega^n(y) + C\int_0^t\int_X \Phi(x, y, t-s)(|F_s|^2 + |F_s|^3)(y)\omega^n(y)ds.$$

The proposition follows from Lemma 13. □

Once we have C^0 control, standard theory gives that F is bounded in C^∞ for finite time (for instance see [53]). Thus the limiting metric H_T is smooth, and the flow can be carried on past H_T. Thus we have shown the following theorem:

Theorem 10. *A solution to the modified Donaldson heat flow exists for all time $t \in [0, \infty)$.*

4.4 Two L^2 approximate Hermitian structures

We are now ready to construct an L^2 approximate Hermitian structure on E using the modified Donaldson heat flow. We then prove Theorem 3 by showing an L^2 approximate Hermitian structure can be realized along the Yang-Mills flow.

Proposition 20. *Along the modified Donaldson heat flow we can construct an L^2 approximate Hermitian structure on E.*

Proof. We begin by noting that along the flow P-functional is non-increasing:

$$\partial_t P(t) = \partial_t P(H_0, H_t) = -\int_X \mathrm{Tr}((\Lambda F - \Psi)^2)\,\omega^n \leq 0.$$

By Theorem 10 we know a solution to the modified Donaldson heat flow exists for all time. Set

$$Y(t) = \int_X \mathrm{Tr}((\Lambda F - \Psi)^2)\,\omega^n = ||\Lambda F_H - \Psi_H||^2_{L^2}.$$

First, because P is bounded from below by Theorem 9, we have the integral of $Y(t)$ over all time is bounded:

$$\int_0^\infty Y(t)dt = -\int_0^\infty \partial_t P(t)dt = P(0) - \lim_{T\longrightarrow\infty} P(T) \leq C.$$

Hence there exists a sequence of times $t_m \in [m, m+1)$ with $Y(t_m) \longrightarrow 0$. Next we show

$$\dot{Y} \leq CY. \tag{4.4.12}$$

To see this, we compute out

$$\dot{Y}(t) = 2\int_X \mathrm{Tr}((\Lambda F - \Psi)(\Lambda\dot{F} - \dot{\Psi}))\omega^n = 2\int_X \mathrm{Tr}((\Lambda F - \Psi)\Lambda\dot{F})\omega^n - 2\int_X \mathrm{Tr}((\Lambda F - \Psi)\dot{\Psi})\omega^n.$$

The first term on the right is in fact negative. Recall that $\Lambda\dot{F} = g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j(\Lambda F - \Psi)$, so after integration by parts we see:

$$\int_X \mathrm{Tr}((\Lambda F - \Psi)g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j(\Lambda F - \Psi))\omega^n = -\int_X g^{j\bar{k}}\mathrm{Tr}(\nabla_{\bar{k}}(\Lambda F - \Psi)\nabla_j(\Lambda F - \Psi))\omega^n \leq 0.$$

For the second term we recall the following formulation of Ψ:

$$\Psi = \sum_{i=1}^{p} c_i \pi^i.$$

Because $c_i > 0$ we only need to bound $-\int_X \mathrm{Tr}((\Lambda F - \Psi)\dot{\pi}^i)\omega^n$ by CY for some constant C. Recall from Proposition 17 that $\dot{\pi}^i$ is the component of $-\Lambda F + \Psi$ which sends $S^{i\perp}$ to S^i. Thus

$$-\int_X \mathrm{Tr}((\Lambda F - \Psi)\dot{\pi}^i)\omega^n = \int_X \mathrm{Tr}((\Lambda F - \Psi)\pi^i(\Lambda F - \Psi)(I - \pi^i))\omega^n \leq CY,$$

proving (4.4.12). This implies $Y(t) \leq Y(s)e^{C(t-s)}$ for $t \geq s$. So using our subsequence t_m from before we have $Y(t) \leq Y(t_m)e^{2C}$ for $t_m \in [m+1, m+2)$. It follows that $Y(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Because we have long time existence along the flow by Theorem 10, given $\epsilon > 0$, we can always pick a time t such that $Y(t) < \epsilon$. Of course, since we constructed a solution to the modified Donaldson heat flow on $X \backslash Z_{alg}$, our metric H is not smooth everywhere. However, once we have the L^2 estimate on the curvature, we can use the fact that smooth functions are dense in L^2_2 to construct a smooth metric $\tilde{H}$ on all of X with curvature $\Lambda\tilde{F}$ as close to ΛF as we like, thus we still have an L^2 apporximate Hermitian structure on E.

□

Next we show how to use Proposition 20 to construct a L^2 approximate Hermitian structure along the Yang-Mills flow. Because, in a sense, the Yang-Mills flow is gauge equivalent to the Donaldson heat flow (see equation (4.1.5)), we first prove the following proposition for the Donaldson heat flow, which then extends to the Yang-Mills flow.

Proposition 21. *Given a family of connections A_t along the Donaldson heat flow, for all $\epsilon > 0$ there exists a $T > 0$ large enough such that $||\Lambda F_t - \Psi_t||^2_{L^2} < \epsilon$ for $t > T$.*

Proof. To prove this proposition, we utilize the method of Daskalopoulos-Wentworth from [10], which is the method of continuity applied to the space of Hermitian metrics. Let $\mathcal{H}_\epsilon$ be the set of all Hermitian metrics H_0 such that, if we consider H_0 to be an initial metric along the Donaldson heat flow, there exists a $T > 0$ such that:

$$||\Lambda F_t - \Psi_t||^2_{L^2} < \epsilon$$

for all $t > T$. We show that $\mathcal{H}_\epsilon$ is open, closed, and nonempty, thus all Hermitian metrics lie in $\mathcal{H}_\epsilon$ and the proposition is proved.

First we state a fact about adjoints. In local coordinates, an the adjoint of an endomorpism A with respect to the metric H is given by

$$A^{*\alpha}{}_{\beta} = H^{\alpha\bar{\gamma}}\overline{A^{\rho}{}_{\gamma}}H_{\bar{\rho}\beta}.$$

Notice now that if we wanted to compute the adjoint with respect to the metric H_0, denoted by $*_0$, we have

$$A^{*_0\alpha}{}_{\beta} = H_0^{\alpha\bar{\gamma}}\overline{A^{\rho}{}_{\gamma}}H_{0\bar{\rho}\beta} = H_0^{\alpha\bar{\gamma}}H_{\bar{\gamma}\nu}H^{\nu\bar{\kappa}}\overline{A^{\sigma}{}_{\kappa}}H_{\bar{\sigma}\eta}H^{\eta\bar{\rho}}H_{0\bar{\rho}\beta} = h^{\alpha}{}_{\nu}A^{*\nu}{}_{\eta}h^{-1\eta}{}_{\beta}.$$

Thus in matrix notation we have $A^{*_0} = hA^*h^{-1}$. We now see that $||\Lambda F||^2_{L^2(H)} = ||\Lambda F_A||^2_{L^2(H_0)}$, since by (4.1.5) we have

$$\begin{aligned} ||\Lambda F||^2_{L^2(H)} &= \int_X \mathrm{Tr}(\Lambda F \Lambda F^*)\omega^n = \int_X \mathrm{Tr}(w^{-1}\Lambda F_A w(w^{-1}\Lambda F_A w)^*)\omega^n \\ &= \int_X \mathrm{Tr}(\Lambda F_A h \Lambda F_A^* h^{-1})\omega^n = \int_X \mathrm{Tr}(\Lambda F_A \Lambda F_A^{*_0})\omega^n = ||\Lambda F_A||^2_{L^2(H_0)}. \end{aligned}$$

We now consider the following result, which can be found in [10], Proposition 2.8:

Proposition 22. *Let X be a compact Kähler manifold, and E a holomorphic vector bundle over X with Hermitian metric H. Then the critical values of the Yang-Mills functional on $\mathcal{A}^{1,1}$ are discrete.*

In fact, it this proposition is proved by noticing that the set of all possible Harder-Narasimhan types bounded above of is discrete (see equation (4.3) in [10]). Thus, there exists an $\delta_0 > 0$ such that if $||\Lambda F_t - \Psi_t||^2_{L^2} < \delta_0$, then $\lim_{t\longrightarrow\infty} ||\Lambda F_t - \Psi_t||^2_{L^2} = 0$, since that quantity can not stop before reaching zero without violating Proposition 22.

Without loss of generality, we assume $\epsilon < \delta_0$. We know $\mathcal{H}_\epsilon$ is non-empty by Proposition 20. $\mathcal{H}_\epsilon$ is also open by continuous dependence of the flow on initial conditions. Thus we must show that $\mathcal{H}_\epsilon$ is closed. Let H^j be a sequence of metrics in $\mathcal{H}_\epsilon$ which converges to some metric K in C^∞. We will show that there exists a time T along the flow such that

$$||\Lambda F_{K(T)} - \Psi_{K(T)}||^2_{L^2} < \epsilon. \tag{4.4.13}$$

Then since $\epsilon < \delta_0$ the inequality (4.4.13) holds for all $t > T$, proving $K \in \mathcal{H}_\epsilon$ which implies $\mathcal{H}_\epsilon$ is closed. Now, because $H^j \in \mathcal{H}_\epsilon$, we have a sequence T_j such that if we pick $t_j > T_j$ then

$$||\Lambda F_{H^j(t_j)} - \Psi_{H^j(t_j)}||^2_{L^2} < \frac{\epsilon}{3}.$$

Here we were able to decrease ϵ to $\frac{\epsilon}{3}$ since $\epsilon < \delta_0$. Without loss of generality assume t_j is increasing and $t_j \longrightarrow \infty$. We claim there exists a j large enough so that

$$||\Lambda F_{K(t_j)} - \Psi_{K(t_j)}||^2_{L^2} < \epsilon, \tag{4.4.14}$$

which proves (4.4.13). We define the endomorhpsim h_j by

$$H^j_{t_j} = h_j K_{t_j}.$$

If ∇^j denotes the covariant derivative with respect to the connection $A_j = K^{-1}_{t_j}\partial K_{t_j}$, then we have

$$F_{H^j_{t_j}} - F_{K_{t_j}} = -\bar{\nabla}(h_j^{-1}\nabla^j h_j).$$

Our first goal is to show $||\bar{\nabla}(h_j^{-1}\nabla^j h_j)||_{L^1} \longrightarrow 0$. We need two key facts. First, because $H^j \longrightarrow K$ in C^∞, we have $\sigma(H^j, K) \longrightarrow 0$, and since $(\partial_t - \Delta)\sigma \leq 0$ along the Donaldson heat flow (see [13]), we know $\sigma(H^j_{t_j}, K_{t_j}) \longrightarrow 0$ as j tends to infinity. In particular

$$\sup_X |h_j - I| \longrightarrow 0. \tag{4.4.15}$$

Second, since A_j is a sequence of metrics along the Yang-Mills flow, by [25] we know there exists a singular set Z such that along some subsequence (still denoted A_j) these connections converge to a limiting connection A_∞ in C^∞ on all compact sets $K \subset X\backslash Z$. We pass to such a subsequence. Also, we note that in [25] Z is shown to be of real Hausdorff codimension 4, so in particular $L^1(X\backslash Z) = L^1(X)$. Let $\phi \in \Lambda^{2,0}(\mathrm{End}(E))$ be a smooth test form supported in K. Then we have

$$(\bar{\nabla}(h_j^{-1}\nabla^j h_j), \phi)_{L^2} = (h_j^{-1}\bar{\nabla}\nabla^j h_j, \phi)_{L^2} - (h_j^{-1}\bar{\nabla}h_j h_j^{-1}\nabla^j h_j, \phi)_{L^2}. \tag{4.4.16}$$

We show each term goes to zero. For notational simplicity assume we are taking the absolute value of each inner product so that all terms are positive. For the first term, by (4.4.15) we know:

$$\begin{aligned}(h_j^{-1}\bar{\nabla}\nabla^j h_j, \phi)_{L^2} &\leq C(\bar{\nabla}\nabla^j h_j, \phi)_{L^2} \\ &\leq C(\bar{\nabla}(A_j - A_\infty)h_j, \phi)_{L^2} + (\bar{\nabla}\nabla^\infty h_j, \phi)_{L^2}.\end{aligned}$$

Here, the first term goes zero since $A_j \longrightarrow A_\infty$ in C^∞. For the second term, using the fact that $(\bar{\nabla}\nabla^\infty h_j, \phi)_{L^2} = (h_j, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2}$ we see

$$\begin{aligned}(h_j, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2} \longrightarrow (I, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2} &= \int_X \mathrm{Tr}(\nabla^{\infty *}\bar{\nabla}^*\phi)\omega^n \\ &= \int_X \partial^*\bar{\partial}^*\mathrm{Tr}(\phi)\omega^n = 0.\end{aligned}$$

Thus $(h_j^{-1}\bar{\nabla}\nabla^j h_j, \phi)_{L^2}$ goes to zero as j tends to infinity. For the second term in (4.4.16), once again by (4.4.15) we have

$$\begin{aligned}(h_j^{-1}\bar{\nabla}h_j h_j^{-1}\nabla^j h_j, \phi)_{L^2} &\leq C(\bar{\nabla}h_j\nabla^j h_j, \phi)_{L^2}. \\ &\leq C(h_j\bar{\nabla}\nabla^j h_j, \phi)_{L^2} + C(h_j\nabla^j h_j, \bar{\nabla}^*\phi)_{L^2}.\end{aligned}$$

The first term goes to zero as before. For the second term we have

$$(h_j\nabla^j h_j, \bar{\nabla}^*\phi)_{L^2} \leq C(\nabla^j h_j, \bar{\nabla}^*\phi)_{L^2} = C(h_j, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2}.$$

As we have seen

$$(h_j, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2} \longrightarrow (I, \nabla^{\infty *}\bar{\nabla}^*\phi)_{L^2} = 0.$$

This shows $||\bar{\nabla}(h_j^{-1}\nabla^j h_j)||_{L^1} \longrightarrow 0$ (since its integral against all test forms are zero). It follows that $||\Lambda\bar{\nabla}(h_j^{-1}\nabla^j h_j)||_{L^1} \longrightarrow 0$. However, we need the L^2 norm to go to zero, rather than the L^1. We use the fact that that:

$$||\Lambda\bar{\nabla}(h_j^{-1}\nabla^j h_j)||_{L^\infty} \leq C,$$

uniformly in j, which is a consequence of the fact that $\Lambda F_{H^j(t_j)}$ and $\Lambda F_{K(t_j)}$ are uniformly bounded in L^∞ along the Yang-Mills flow. The desired L^2 convergence now follows from the following elementary lemma

Lemma 15. *Let $f_j \geq 0$ be a sequence of positive functions such that $\int_X f_j \longrightarrow 0$ as $j \longrightarrow \infty$. If $f_j \leq C$ uniformly in j, then $\int_X f_j^2 \longrightarrow 0$ as $j \longrightarrow \infty$ as well.*

Proof. We will show for all $\epsilon > 0$, for j sufficiently large we have $\int_X f_j^2 < \epsilon$. Fix j large enough so that $\int_X f_j < \epsilon/C$, where C is as in the statement of the lemma. Then

$$\int_X f_j^2 \leq C \int_X f_j < \epsilon.$$

□

Thus we have shown that there exists a j large enough so that

$$||\Lambda F_{H^j(t_j)} - \Lambda F_{K(t_j)}||_{L^2}^2 < \frac{\epsilon}{3}.$$

Also, by equation 4.4.15 and the fact that all projections are bounded in L_1^2 and thus L^2, we know $\Psi_{H^j(t_j)}$ converges to $\Psi_{K(t_j)}$ in L^2 as j tends to infinity. Thus by the triangle inequality we can conclude for j large enough we have:

$$\begin{aligned} ||\Lambda F_{K(t_j)} - \Psi_{K(t_j)}||_{L^2}^2 &\leq ||\Lambda F_{H^j(t_j)} - \Lambda F_{K(t_j)}||_{L^2}^2 + ||\Lambda F_{H^j(t_j)} - \Psi_{H^j(t_j)}||_{L^2}^2 \\ &\quad + ||\Psi_{H^j(t_j)} - \Psi_{K(t_j)}||_{L^2}^2 < \epsilon \end{aligned}$$

which finishes the proof of (4.4.14) and thus the Proposition. □

We now show how Theorem 3 follows simply from the preceding proposition. First we need to relate our projections evolving along the Donaldson heat flow to projections evolving along the Yang-Mills flow. In the case of the Donaldson heat flow, the orthogonal projection π_t onto a fixed subsheaf $S \subset E$ evolves due to the fact that the metric H is changing. Along the Yang-Mills flow, our metric H_0 is fixed, however the subsheaf S is acted on by the complexified gauge transformation w (defined in Section 4.1.2). Thus the projection π_w onto $w(S)$ evolves as well.

Lemma 16. *The two evolving projections are related as follows*

$$\pi_w = w \pi_t w^{-1}$$

Proof. It is immediately clear that $(w\pi_t w^{-1})^2 = w\pi_t w^{-1}$, so $w\pi_t w^{-1}$ is a projection onto the subsheaf $w(S)$. We complete the lemma by showing it is unitary with respect to H_0.

$$(w\pi_t w^{-1})^{*_0} = w^{-1}(\pi_t)^{*_0} w = h^{-1/2} h \pi_t^* h^{-1} h^{1/2} = w\pi_t w^{-1}.$$

□

From this lemma we see that $w\Psi_t w^{-1} = \Psi_w$, where Ψ_t is evolving along the Donaldson Heat flow and Ψ_w is evolving along the Yang-Mills flow. It follows that

$$\begin{aligned} ||\Lambda F_t - \Psi_t||^2_{L^2(H)} &= \int_X \mathrm{Tr}((w^{-1}\Lambda F_A w - \Psi_t)(w^{-1}\Lambda F_A w - \Psi_t)^*)\omega^n \\ &= \int_X \mathrm{Tr}(\Lambda F_A - w\Psi_t w^{-1})h(\Lambda F_A - w\Psi_t w^{-1})^* h^{-1})\omega^n \\ &= \int_X \mathrm{Tr}(\Lambda F_A - \Psi_w)(\Lambda F_A - \Psi_w)^{*_0})\omega^n \\ &= ||\Lambda F_A - \Psi_w||^2_{L^2(H_0)}. \end{aligned}$$

Thus Proposition 21 implies that the Yang-Mills flow realizes an L^2 approximate Hermitian structure, proving Theorem 3. From this point on we abuse notation and also refer to the endomorphism evolving along the Yang-Mills flow as Ψ_t, and which endomorphism we are using will be clear from context.

At this point we can now prove Theorem 4 as stated in the introduction, generalizing a result of Atiyah and Bott. First we review some notation. Consider a flag $\mathcal{F}$ of subbundles:

$$0 = E^0 \subset E^1 \subset \cdots \subset E^q = E.$$

Define $\mathcal{F}$ to be slope decreasing if $\mu(E^1) > \mu(E^2) > ... > \mu(E)$. Let $\mathcal{Q}^i = E^i/E^{i-1}$, and recall that

$$\Phi(\mathcal{F})^2 = \sum_{i=0}^{q} \mu(\mathcal{Q}^i)^2 rk(\mathcal{Q}^i).$$

We now prove that for all $\mathcal{F}$ slope drecreasing:

$$\inf_A ||\Lambda F(A)||^2_{L^2} = \sup_{\mathcal{F}} \Phi(\mathcal{F})^2.$$

Proof. First we show that $\sup_{\mathcal{F}} \Phi(\mathcal{F})^2 = ||\Psi_H||^2_{L^2}$ independent of any metric H. Since we already know the supremum is attained if $\mathcal{F}$ is the Harder-Narasimhan filtration of E, we need to show $||\Psi_H||^2_{L^2} = \sum_{i=0}^{p} \mu(Q^i)^2 rk(Q^i)$, where as before Q^i are the quotients coming from the Harder-Narasimhan filtration. We check that

$$\Psi_H^2 = \sum_i \mu(Q^i)^2(\pi^i - \pi^{i-1}). \tag{4.4.17}$$

To see this note that for any $k > 0$, we have $\pi^{i-k}\pi^i = \pi^i\pi^{i-k} = \pi^{i-k}$ since the subbundles are ordered by inclusion. Thus $(\pi^i - \pi^{i-1})^2 = \pi^{i2} - \pi^{i-1}\pi^i - \pi^i\pi^{i-1} + \pi^{i-1^2} = \pi^i - \pi^{i-1}$. Also, all the cross terms in Ψ_H^2 vanish, since

$$\begin{aligned}
(\pi^{i-k} - \pi^{i-k-1})(\pi^i - \pi^{i-1}) &= (\pi^{i-k}(\pi^i - \pi^{i-1}) - \pi^{i-k-1}(\pi^i - \pi^{i-1})) \\
&= (\pi^{i-k} - \pi^{i-k} - \pi^{i-k-1} + \pi^{i-k-1}) \\
&= 0.
\end{aligned}$$

This proves (4.4.17). Now since Ψ_H is self adjoint, we have:

$$||\Psi_H||^2_{L^2} = \int_X \mathrm{Tr}(\Psi_H^2)\omega^n = \sum_i \int_X \mathrm{Tr}(\mu(Q^i)^2(\pi^i - \pi^{i-1}))\omega^n = \sum_i \mu(Q^i)^2\, rk(Q^i).$$

Thus $||\Psi||^2_{L^2} = \sup_{\mathcal{F}} \Phi(\mathcal{F})^2$, where we have dropped the H from Ψ_H since this norm is independent of metric. The statement $\inf_A ||\Lambda F(A)||^2_{L^2} = ||\Psi||^2_{L^2}$ follows from the fact we have an L^2 approximate Hermitian structure along the Yang-Mills flow. □

4.5 Construction of an isomorphism

We now begin the proof of Theorem 5. First we recall our basic setup. Let A_t be a family of connections evolving along the Yang-Mills flow. By a result of Hong and Tian [25], there exists a subsequence of connections A_j which converge in C^∞ (on $X \backslash Z_{an}$ and modulo unitary gauge transformations), to a Yang-Mills connection A_∞. Thus, always working on $X \backslash Z_{an}$, we have a sequence of holomorphic structures $(E, \bar{\partial}_j)$ which converge in C^∞ to a holomorphic structure on a (possibly) different bundle $(E_\infty, \bar{\partial}_\infty)$.

We can now identity the Harder-Narasimhan type of the limiting connection A_∞. Since A_∞ is Yang-Mills, we have that ΛF_∞ solves the following equation:

$$-i\bar{\partial}_\infty \Lambda F_\infty + i\partial_\infty \Lambda F_\infty = 0.$$

In particulart ΛF_∞ has locally constant eigenvalues, which means that about any point in $X \backslash Z_{an}$, we can choose coordinates so that ΛF_∞ has the following form:

$$\Lambda F_\infty = \begin{pmatrix} \lambda_1 I_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p I_p \end{pmatrix}. \tag{4.5.18}$$

Here I_i are identity matrices whose rank is determined by the multiplicity of each eigenvalue λ_i. Assume that the eigenvalues are decreasing $\lambda_1 > \lambda_2 > \cdots > \lambda_q$. Now because E realizes an L^2 approximate Hermitian structure along the Yang-Mills flow, we can precisely identify the eigenvalues ΛF_∞, so $\lambda_i = \mu(Q^i)$, and $rk(I_i) = rk(Q^i)$.

Furthermore, because ΛF_∞ is of this special form, from [26] we know it will decompose E_∞ into a direct sum of stable bundles:

$$E_\infty = \hat{Q}^1_\infty \oplus \hat{Q}^2_\infty \oplus \cdots \oplus \hat{Q}^q_\infty, \tag{4.5.19}$$

each admitting an induced smooth Hermitian-Einstein connection. Let $Z = Z_{an} \cup Z_{alg}$. Working on $X \backslash Z$, we prove the direct sum (4.5.19) is isomorphic to the graded double filtration $Gr^{hns}(E)$, which is the subject of the following proposition:

Proposition 23. *Working with* (4.5.19) *above, on $X \backslash Z$ each $\hat{Q}^i_\infty$ is isomorphic to a specific stable quotient from $Gr^{hns}(E)$.*

We prove this proposition over the next few subsections. First we need some convergence results. Fix a metric H_0 on E, and consider the L^2_1 projections which define the Harder-Narasimhan filtration of E:

$$0 \subset \pi^0 \subset \pi^1 \subset \pi^2 \cdots \subset \pi^p \subset E. \tag{4.5.20}$$

Recall that along the Yang-Mills flow we have a sequence of endomorphisms w_j which define the action given by (4.1.4). The action of w_j also produces a sequence of filtrations $\{\pi^i_j\}$, where each π^i_j is defined by orthogonal projection onto the subsheaf $w_j(\pi^i)$. Our first goal is to show that this sequence of filtrations converges along a subsequence.

We use two main assumptions to show convergence of a sequence of projections, then show these assumptions hold in our particular case.

Proposition 24. *Let π be the L^2_1 projection associated to a subsheaf $\mathcal{F} \subset E$, and let $Z(\mathcal{F})$ be the singular set of $\mathcal{F}$. Let A_j be a sequence of connections along the Yang-Mills flow. The action of w_j produces a sequence of projections $\{\pi_j\}$ defined by orthogonal projection onto the subsheaf $w_j(\pi)$. Assume that:*

i) For any compact subset $K \subset X \backslash (Z_{an} \cup Z(\mathcal{F}))$, we have $A_j \longrightarrow A_\infty$ in $C^\infty(K)$.

ii) $||\bar{\partial}_j \pi_j||^2_{L^2} \longrightarrow 0$.

Then there exists a subsequence of projections (still denoted π_j) which converges in L^2_1 to a limiting subsheaf π_∞. Furthermore, under the same assumptions the limiting projection π_∞ is smooth away from $Z_{an} \cup Z(\mathcal{F})$.

We note that assumption ii) gives that π_∞ splits E_∞ holomorphically.

Proof. By assumption ii) we have $||\bar{\partial}_j \pi_j||^2_{L^2}$ goes to zero as $j \to \infty$. Beacuse $\pi_j = \pi_j^*$ it follows that $|\bar{\partial}_j \pi_j|^2 = |\partial_j \pi_j|^2$, thus we have $\partial_j \pi_j$ is uniformly bounded in L^2 and π_j converges along a subsequence to a weak limit π_∞ in L^2_1. We must show that π_∞ is a weakly holomorphic subbundle as defined in [43] or [52], and thus represents a coherent subsheaf. This means we have to show $(I - \pi_\infty)\bar{\partial}_\infty \pi_\infty = 0$ in L^2. Working on a compact set K specified in assumption i), we have:

$$\bar{\partial}_\infty \pi_j = \bar{\partial}_j \pi_j + (\bar{\partial}_\infty - \bar{\partial}_j)\pi_j,$$

so it follows that

$$\begin{aligned} ||\bar{\partial}_\infty \pi_j||_{L^2(K)} &\leq ||\bar{\partial}_j \pi_j||_{L^2(K)} + ||(\bar{\partial}_\infty - \bar{\partial}_j)\pi_j||_{L^2(K)} \\ &\leq ||\bar{\partial}_j \pi_j||_{L^2(K)} + ||A_j - A_\infty||_{L^\infty(K)} ||\pi_j||_{L^2(K)}. \end{aligned}$$

We have that $A_j \to A_\infty$ in $L^\infty(K)$ by assumption i). Because $||\bar{\partial}_j \pi_j||_{L^2} \to 0$ it follows that $||\bar{\partial}_\infty \pi_j||_{L^2(K)} \to 0$. Finally, from the simple formula:

$$\bar{\partial}_\infty \pi_\infty = \bar{\partial}_\infty \pi_j + \bar{\partial}_\infty(\pi_\infty - \pi_j),$$

we see that

$$\begin{aligned} ||\bar{\partial}_\infty \pi_\infty||_{L^2(K)} &\leq ||\bar{\partial}_\infty \pi_j||_{L^2(K)} + ||\bar{\partial}_\infty(\pi_\infty - \pi_j)||_{L^2(K)} \\ &= ||\bar{\partial}_\infty \pi_j||_{L^2(K)} + ||\pi_\infty - \pi_j||_{L^2_1(K)}. \end{aligned}$$

The left hand side is independent of j, so we would like to send j to infinity proving $||\bar{\partial}_\infty \pi_\infty||_{L^2(K)} = 0$. We have to be careful about the second term on the right since π_j only converges to π_∞ weakly in L^2_1. However, we can achieve strong L^2_1 convergence along a subsequence. Now equation (3.1.3) describes how a connection decomposes onto subbundles π_j with quotient Q_j. From this formula we see that the second fundamental form is just one component of the connection A_j, so we have:

$$\int_K |\tilde{\nabla}_j(\bar{\partial}_j \pi_j)|^2 \omega^n \leq \int_K |\nabla_j(A_j)|^2 \omega^n \leq C,$$

where $\tilde{\nabla}_j$ is the induced connection on $Hom(Q_j, \pi_j)$. The bound on the right follows from assumption i). Thus π_j is bounded in L^2_2, and thus along a subsequence we have strong convergence in L^2_1. It follows that $||\bar{\partial}_\infty \pi_\infty||_{L^2(K)} = 0$. This holds independent of which compact set K we choose, so

$$||\bar{\partial}_\infty \pi_\infty||_{L^2(X \setminus Z_{an})} = ||\bar{\partial}_\infty \pi_\infty||_{L^2(X)} = 0,$$

since $Z_{an} \cup Z(\mathcal{F})$ has complex codimension two. Thus π_∞ is a weakly holomorphic L^2_1 subbundle of $(E_\infty, \bar{\partial}_\infty)$. Furthermore, because the eigenvalues of the projections π_j are either zero or one, we know that rk(π_∞)=rk(π_j). It also follows that $\mu(\pi) = \mu(\pi_\infty)$, since degree does not depend on a choice of metric.

We now prove π_∞ is smooth away from $Z_{an} \cup Z(\mathcal{F})$. By (3.1.3) we know the C^k norm of the second fundamental form γ^j associated to each subbundle is less than the C^k norm of the connection ∇:

$$||\gamma^j(\phi)||_{C^k} \leq ||\nabla(\phi)||_{C^k}.$$

In particular, considering our convergent subsequence of connections along the Yang-Mills flow, we have on compact subsets K away from the singular set:

$$||(\gamma^j - \gamma^\infty)(\phi)||_{C^k(K)} \leq ||(\nabla^j - \nabla^\infty)(\phi)||_{C^k(K)} \longrightarrow 0.$$

Thus we have smooth convergence of second fundamental forms, and because $\gamma^j = \bar{\partial}_j \pi_j$, we know we get smooth convergence of the projections π_j.

□

4.5.1 The semi-stable case

Before working with general unstable bundles, we first need to prove Proposition 23 in the case that E is semi-stable but not stable. We also prove a convergence result for holomorphic maps of our subsheaves into E, and the analysis needed for these arguments lies at the heart of our construction of an explicit isomorphism.

Note in this case the Harder-Narasimhan filtration of E is just given by E itself. Let

$$0 \subset \tilde{\pi}^0 \subset \tilde{\pi}^1 \subset \tilde{\pi}^2 \cdots \subset \tilde{\pi}^p \subset E \tag{4.5.21}$$

be the Seshadri filtration of E. Let $Gr^s(E) := \bigoplus_i \tilde{Q}^i$ be the graded sum of stable quotients. If we consider a sequence of connections A_j on E along the Yang-Mills flow, then we can see right away that the assumptions of Proposition 24 hold for each subsheaf $\tilde{\pi}^i_j$ along the flow. Assumption i) holds by the convergence results of Hong and Tian. Assumption ii) is clear by the construction of the aproximate

Hermitian structure given in Section 3. Indeed, note the following modification of the Chern-Weil formula:

$$\mu(\tilde{\pi}^i_j) = \mu(E) + \frac{1}{rk(\tilde{\pi}^i_j)}\Big(\int_X \mathrm{Tr}((\Lambda F_{A_j} - \mu(E)I)\circ\tilde{\pi}^i_j)\omega^n - ||\bar{\partial}_j\tilde{\pi}^i_j||^2_{L^2}\Big) \tag{4.5.22}$$

Now since $\mu(E) = \mu(\tilde{\pi}^i_j)$, we have that:

$$||\bar{\partial}_j\tilde{\pi}^i_j||^2_{L^2} = \int_X \mathrm{Tr}((\Lambda F_{A_j} - \mu(E)I)\circ\tilde{\pi}^i_j)\,\omega^n.$$

Recall that $F_{A_j} = w_j F_j w_j^{-1}$, where F_j evolves along the Donaldson heat flow. This allows us to write:

$$\begin{aligned} ||\bar{\partial}_j\tilde{\pi}^i_j||^2_{L^2} &= \int_X \mathrm{Tr}((\Lambda F_j - \mu(E)I)\circ w_j^{-1}\tilde{\pi}^i_j w_j)\,\omega^n \\ &\leq \int_X |\Lambda F_j - \mu(E)I|\,\omega^n. \end{aligned} \tag{4.5.23}$$

Since E admits an apporximate Hermitian-Einstein structure, we see $||\bar{\partial}_j\tilde{\pi}^i_j||^2_{L^2}$ goes to zero as $j\to\infty$.

For notational simplicity let $\tilde{\pi}^1_j$ be denoted by $\tilde{\pi}_j$. We have $\tilde{\pi}_j$ converges in L^2_1 to a limiting subsheaf $\tilde{\pi}_\infty$. The next step is to show convergence of the holomorphic inclusion maps into E. For notational simplicity we let $\tilde{S}$ be the sheaf of lowest rank from (4.5.21), and let f_0 be the holomorphic inclusion of $\tilde{S}$ into E:

$$0 \longrightarrow \tilde{S} \xrightarrow{f_0} (E,\bar{\partial}_0) \longrightarrow Q \longrightarrow 0. \tag{4.5.24}$$

We note that although a priori f_0 is only defined where $\tilde{S}$ is locally free, since this set has codimension two by the Riemann extension theorem we can extend f_0 to a holomorphic map defined on all of X. Now, the composition $w_j\circ f_0$ defines an inclusion of the sheaf $\tilde{S}$ into $(E,\bar{\partial}_j)$. We define $f_j := \tilde{w}_j\circ f_0$, where $\tilde{w}_j$ is a normalization of w_j chosen so that $||f_j||_{L^2(X)} = 1$.

Lemma 17. *For all j, the map f_j defines a holomorphic inclusion of $\tilde{S}$ into $(E,\bar{\partial}_j)$.*

Proof. Using the defining equation for the action of $\tilde{w}_j$, we know that $A_j'' = \tilde{w}_j A_0'' \tilde{w}_j^{-1} - \bar{\partial}\tilde{w}_j \tilde{w}_j^{-1}$. Multiplying on the right by $\tilde{w}_j \circ f_0$ and rearranging terms we get the equation:

$$(\bar{\partial}\tilde{w}_j) f_0 + A_j'' \tilde{w}_j f_0 - \tilde{w}_j A_0'' f_0 = 0. \tag{4.5.25}$$

Now f_0 is holomorphic with respect to $\bar{\partial}_0$, thus it satisfies $\bar{\partial} f_0 + A_0'' f_0 - f_0 A_0'' = 0$, which we can write as $A_0'' f_0 = f_0 A_0'' - \bar{\partial} f_0$. Plugging this into equation (4.5.25) proves that f_j solves

$$\bar{\partial}_j f_j = \bar{\partial} f_j + A_j'' f_j - f_j A_0'' = 0,$$

and is indeed holomorphic. □

We are now ready to show convergence of the maps f_j along a subsequence. The main difficulty is that we do not have control of the connections A_j as we approach the analytic singular set Z_{an}, forcing us to work on compact subsets off Z_{an}. Thus we only at first have convergence on a compact subset of X, and since the normalization $\tilde{w}_i$ only controls a global L^2 norm of f_j, we must make sure that in fact f_j does not go to zero on this compact set. We get around this difficulty by exploiting the rigidity of holomorphic functions, and by showing that in fact the estimate of the C^0 norm of f_j does not depend on any bump functions or the boundary of the compact subset we choose.

Proposition 25. *Let f_0 be the holomorphic inclusion of the sheaf $\tilde{S}$ into $(E, \bar{\partial}_0)$. Given a sequence of holomorphic structures $\bar{\partial}_j$ along the Yang-Mills flow, we have a sequence $f_j = \tilde{w}_j \circ f_0$ of holomorphic inclusions of $\tilde{S}$ into $(E, \bar{\partial}_j)$, which converge (after passing to a subsequence) in L_2^p to a holomorphic map f_∞, which is non-trivial.*

Proof. Consider the family of compact subsets $K(r) \subset X \backslash Z_{an}$, defined to be the complement of a tube around Z_{an}:

$$K(r) := X \backslash \bigcup_{x \in Z_{an}} B_r(x).$$

Our first goal is to show that for some $r_0 > 0$,

$$||f_j||_{C^0(X)} \leq C\,||f_j||_{L^2(K(r_0))}, \tag{4.5.26}$$

where the constant C is uniform in j. Since f_j is holomorphic it solves the equation:

$$\bar{\partial}^{\dagger}(\bar{\partial} f_j + A''_j f_j - f_j A''_0) = 0, \tag{4.5.27}$$

which is an elliptic equation in divergence form. After multiplying f_j by a suitable bump function which vanishes along $\partial K(r)$ and is identically one in $K(2r)$, we can use a Moser iteration technique to get the following estimate:

$$||f_j||_{C^0(K(2r))} \leq C||f_j||_{L^2(K(r))}, \tag{4.5.28}$$

(for details see [23] Theorem 8.15). However, since we used a bump function this estimate will depend on r, and will blow up if we send r to zero. We get around this difficulty as follows.

Note we have uniform control of f_j in the set $K(2r)$ where the bump function is identically one. The trick is to use this bound to control the C^0 norm of f_j on all of X. We consider the most general assumption on the singular set, that Z_{an} is closed and has finite (2n-4)-Hausdorff measaure, denoted $H^{2n-4}(Z_{an}) \leq C$. It follows that $H^{2(n-1)}(Z_{an}) = 0$, and by a result of Shiffman from [42] we know that for every point $x \in X \backslash Z_{an}$, almost all complex lines through x do not intersect Z_{an}. Thus working in local coordinates, we can construct a lattice L with edges formed by complex lines such that $L \cap Z_{an} = \emptyset$. Since both L and Z_{an} are compact, they are separated by a finite distance, so in particular there exists an r_0 such that $L \cap X \backslash K(r_0) = \emptyset$. Thus by (4.5.28) the maps f_j are uniformly bounded in C^0 along the complex lines that form L. Since the f_j are holomorphic maps and defined on all of X, locally they are given by a matrix of holomorphic functions. By applying Cauchy's integral formula along each line to each entry of the matrix we can extend this uniform bound to each face of L, and after repeated applications of Cauchy's integral formula we can extend the

C^0 bound to the interior of L. Since X is compact it can be covered by the interiors of finitely many lattices, thus there exists a single r_0 that works for all of X. The C^0 bound on all of X follows, proving (4.5.26).

Now, if we choose a bump function which is identically one in $K(2r_0)$ and zero on $\partial K(\rho)$ for any $\rho < r_0$, then even if ρ is arbitrarily small we still get the same uniform estimate:

$$||f_j||_{C^0(X)} \leq C||f_j||_{L^2(K(\rho))}, \tag{4.5.29}$$

since in this case the derivative of the bump function will always be bounded. Also, once we have these C^0 bounds on all of X, Cauchy's estimates give us control of all the derivatives of f_j from the C^0 bound.

Next we rearrange equation (4.5.27) to get the following equality:

$$\bar{\partial}^\dagger \bar{\partial} f_j = \bar{\partial}^\dagger (A_j'' f_j - f_j A_0'').$$

Working on $K(\rho)$, which is compact and supported away from Z_{an}, we have uniform C^1 bounds for A_j and f_j and thus the right hand side is bounded in L^∞ independent of j. Ellipticity of $\bar{\partial}^\dagger \bar{\partial}$ gives that f_j are uniformly bounded in L_2^p for any p, thus the f_j converge weakly in L_2^p (and strongly in L_1^p) to a limiting map f_∞. We see that:

$$\bar{\partial}_\infty f_\infty = \bar{\partial}_j (f_\infty - f_j) - (\bar{\partial}_j - \bar{\partial}_\infty) f_\infty,$$

so it follows:

$$||\bar{\partial}_\infty f_\infty||_{L^p(K(\rho))} \leq ||f_\infty - f_j||_{L_1^p(K(\rho))} + ||A_j - A_\infty||^{\lambda}_{L^q(K(\rho))} ||f_\infty||^{1-\lambda}_{L^r(K(\rho))},$$

where q, r, and λ are given by Holder's inequality. The left hand side is independent of j, so sending j to infinity we see

$$||\bar{\partial}_\infty f_\infty||_{L^p(K(\rho))} = 0$$

for any p. By elliptic regularity we have that f_∞ is smooth, and thus holomorphic.

Next we show f_∞ is not identically zero on $K(\rho)$. From the C^0 estimate (4.5.29) one can show $||f_j||^2_{L^2(X\backslash K(\rho))} \leq C{\cdot}\mathrm{Vol}(X\backslash K(\rho))$, where C is independent of j and the choice of ρ. We can now choose ρ small enough so that $\mathrm{Vol}(X\backslash K(\rho)) < 1/(2C)$. It then follows that

$$||f_j||^2_{L^2(K(\rho))} = ||f_j||^2_{L^2(X)} - ||f_j||^2_{L^2(X\backslash K(\rho))} \geq \frac{1}{2},$$

using our normalization $||f_j||^2_{L^2(X)} = 1$. Since $f_j \longrightarrow f_\infty$ strongly in L^2, we know f_∞ is not identically zero.

In fact we can show f_∞ is holomorphic on all of $X\backslash Z_{an}$. Pick any point $x_0 \in X\backslash Z_{an}$. Then there exists a $\rho' < \rho$ such that $K(\rho')$ contains x_0. By choosing the sequence f_j from above, and repeating the convergence argument for the compact set $K(\rho')$ as opposed to $K(\rho)$, we get convergence along a subsequence to a new holomorphic map f'_∞ defined on all of $K(\rho')$. Since we choose a subsequence of our original sequence we know $f'_\infty = f_\infty$ on $K(\rho)$, thus f_∞ extends holomorphically to all of $K(\rho')$. We can do this for each point in $X\backslash Z_{an}$, thus f_∞ is holomorphic everywhere in $X\backslash Z_{an}$. Since $H^{2n-2}(Z_{an}) = 0$, by Lemma 3 in [42] we know f_∞ extends to a holomorphic map on all of X.

□

Since E is semi-stable, we know the limiting Yang-Mills connection A_∞ on E_∞ over $X\backslash Z_{an}$ is Hermitian-Einstein (by equation (4.5.18)), and thus will decompose E_∞ into a direct sum of stable bundles:

$$E_\infty = Q^1_\infty \oplus Q^2_\infty \oplus \cdots \oplus Q^p_\infty, \tag{4.5.30}$$

each admitting an induced smooth Hermitian-Einstein connection. We now prove a version of Proposition 23 for semi-stable bundles:

Proposition 26. *In the case that E is semi-stable, off Z there exists an isomorphism between E_∞ and $Gr^s(E)$.*

Proof. Recall that $Gr^s(E)$ is the direct sum of the stable quotients from the Seshadri filtration of E. Working by induction on the rank, we will show each quotient $\tilde{Q}_0^i$ is isomorphic to its corresponding limit $\tilde{Q}_\infty^i$ (given by the L_1^2 convergence of $\tilde{\pi}_j^i$ to some limiting subsheaf $\tilde{\pi}_\infty^i$). This will prove that the subsheaves $\tilde{\pi}_\infty^i$ form a Seshadri filtration of E_∞, and since Seshadri filtrations are unique, it follows that $\oplus_i \tilde{Q}_\infty^i$ must be isomorphic to $\oplus_i Q_\infty^i$.

Consider a sequence of holomorphic structures $\{\bar{\partial}_j\}$ evolving along by the Yang-Mills flow. Let $\tilde{\pi}_j^1$ be the sequence of projections given by the first sheaf in the Seshadri filtration, with holomorphic inclusion given by f_j. By the previous two propositions we can construct a limiting projection $\tilde{\pi}_\infty^1$ and a limiting holomorphic map f_∞. As before we know $\mathrm{rk}(\tilde{\pi}_0^1) = \mathrm{rk}(\tilde{\pi}_\infty^1)$, and $\tilde{\pi}_\infty^1$ defines a coherent sub-sheaf $\tilde{S}_\infty^1$ of E_∞. Now, because along the flow $\pi_j^1 \circ f_j = f_j$, we see that in the limit $\tilde{\pi}_\infty^1 \circ f_\infty = f_\infty$. Thus we know f_∞ is a holomorphic map not just into E_∞, but one which includes into $\tilde{S}_\infty^1$:

$$f_\infty : S^1 \longrightarrow \tilde{S}_\infty^1.$$

We also know that $\mu(S^1) = \mu(\tilde{S}_\infty^1)$. Now, because in a Seshadri filtration S^1 is chosen to have minimal rank among all subsheaves of E with that property $\mu(S^1) = \mu(E)$, we know S^1 is stable. Thus by [26] Chapter V (7.11), since S^1 is stable and $\tilde{S}_\infty^1$ semi-stable, the holomorphic map f_∞ is injective and $\mathrm{rk}(S^1) = \mathrm{rk}(f_\infty(S^1))$. Thus by Corollary 7.12 from [26] the map f_∞ is an isomorphism between S^1 and $\tilde{S}_\infty^1$. We note that although the quoted results from [26] are only stated for compact Kähler manifolds, the arguments needed trivially carry over for the manifold $X \backslash Z_{an}$. It now follows that $\tilde{S}_\infty^1$ is stable and indecomposable. It is indecomposable because S^1 is indecomposable. To show $\tilde{S}_\infty^1$ is stable assume there exists a subsheaf $\mathcal{F}$ such that $\mu(\mathcal{F}) = \mu(\tilde{S}_\infty^1)$. Since $\tilde{S}_\infty^1$ admits an admissible Hermitian-Einstein metric, by formula (4.5.35) the second fundamental form associated to $\mathcal{F}$ is zero, creating a holomorphic splitting of $\tilde{S}_\infty^1$, which contradicts the fact that it is indecomposable.

We now continue this process. Since $\tilde{S}^1$ is stable we know its quotient $\tilde{Q}^1$ is semi-stable, and that along the Yang Mills flow the holomorphic structure $(\tilde{Q}^1, \bar{\partial}_j)$ converges to a limiting holomorphic structure on $(\tilde{Q}^1_\infty, \bar{\partial}_\infty)$. If p denotes the holomorphic projection from E onto $\tilde{Q}^1$, and $p^\dagger$ is the adjoint of p in the fixed metric H_0, then the sequence of induced connections on $\tilde{Q}^1$ is given by:

$$p \circ (d + A_j) \circ p^\dagger.$$

From this formula it is clear that these induced connections satisfy the same bounds as A_j and converge on compact subsets away from Z_{an} along a subsequence. The final thing we need to check in order to repeat the argument is that the second fundamental form associated to any destabilizing subsheaf of $\tilde{Q}^1$ goes to zero in L^2, and by estimate (4.5.23) we see this follows if $||\Lambda F^1_j - \mu(\tilde{Q}^1)I||_{L^1}$ goes to zero, where F^1_j is the curvature of the induced connection on $\tilde{Q}^1$. Using our decomposition formula of curvature onto quotient bundles we have:

$$F^1_j = F|_{\tilde{Q}^1} + (\bar{\partial}_j \tilde{\pi}^1_j)^\dagger \wedge (\bar{\partial}_j \tilde{\pi}^1_j).$$

Thus:

$$||\Lambda F^1_j - \mu(\tilde{Q}^1)I||_{L^1} \leq ||\Lambda F_j - \mu(E)I||_{L^1} + ||\bar{\partial}_j \tilde{\pi}^1_j||^2_{L^2}.$$

Now by estimate (4.5.23) and the existence of an approximate Hermitian-Einstein structure it follows that $||\Lambda F^1_j - \mu(\tilde{Q}^1)I||_{L^1}$ goes to zero. Thus we can pick a destabilizing subsheaf of $\tilde{Q}^1$ and the argument can be repeated inductively, allowing us to construct an isomorphism from each quotient $\tilde{Q}^i$ in $Gr^s(E)$ into one of the indecomposable stable bundles from

$$E_\infty = \tilde{Q}^1_\infty \oplus \cdots \oplus \tilde{Q}^p_\infty. \tag{4.5.31}$$

Since each $\tilde{Q}^i_\infty$ is stable, this proves the $\tilde{S}^i_\infty$ form a Seshadri filtraton of E_∞, and thus (4.5.31) must be isomorphic to (4.5.30).

□

4.5.2 The general unstable case

Now that we have proven Proposition 23 in the case that E is semi-stable, we turn to the general unstable case. First we need to show that the assumptions of Proposition 24 hold for the projections that make up the Harder-Narasimhan filtration (4.1.1). By taking a subsequence of connections along the Yang-Mills flow, assumption i) is clear from the result of Hong and Tian [25] (also, see [51]). For assumption ii), we need a modification of the Chern-Weil formula. Once again recall that ω is normalized so $\int_X \omega^n = 1$. We have

$$\int_X \mathrm{Tr}(\Psi \circ \pi^i)\omega^n = \sum_k \int_X \mathrm{Tr}(\mu(Q^k)(\pi^k - \pi^{k-1}) \circ \pi^i)\omega^n.$$

However, if $k \geq i$, then because the Harder-Narasimhan filtration is ordered by inclusion we know $\pi^k \circ \pi^i = \pi^i$, so

$$\begin{aligned} \int_X \mathrm{Tr}(\Psi \circ \pi^i)\omega^n &= \sum_{k \leq i} \int_X \mathrm{Tr}(\mu(Q^k)(\pi^k - \pi^{k-1}))\omega^n. \\ &= \sum_{k \leq i} \mu(Q^k)\, rk(Q^k) = \sum_{k \leq i} \deg(Q^k). \end{aligned}$$

We note that $\deg(Q^k) = \deg(S^k) - \deg(S^{k-1})$. The Chern-Weil formula relates the degree of a subbundle to the second fundamental form:

$$\deg(S^k) = \int_X \mathrm{Tr}(\Lambda F^{S^k})\omega^n = \int_X \mathrm{Tr}(\Lambda F \circ \pi^k) - ||\bar{\partial}\pi^k||^2_{L^2}.$$

Now, the sum

$$\sum_{k \leq i} \deg(Q^k) = \sum_{k \leq i} (\deg(S^k) - \deg(S^{k-1})),$$

is a telescoping sum, so the only contribution is the term coming from $k = i$. Thus

$$\int_X \mathrm{Tr}(\Psi \circ \pi^i)\omega^n = \sum_{k \leq i} \deg(Q^k) = \int_X \mathrm{Tr}(\Lambda F \circ \pi^i) - ||\bar{\partial}\pi^i||^2_{L^2}.$$

Therefore, for each projection π^i_j in our sequence along the Yang-Mills flow, we have the following formula:

$$||\bar{\partial}_j\pi^i_j||^2_{L^2} = \int_X \mathrm{Tr}((\Lambda F_j - \Psi_j)\circ\pi^i_j)\omega^n.$$

And because the eigenvalues of π^i_j are either 0 or 1, it follows that

$$||\bar{\partial}_j\pi^i_j||^2_{L^2} \leq \int_X |\Lambda F_j - \Psi_j|\omega^n. \tag{4.5.32}$$

E admits an L^2 approximate Hermitian structure along the Yang-Mills flow, thus assumption $i)$ holds for all subsheaves in the Harder Narasimhan filtration (4.1.1). We therefore get convergence to a limiting filtration away from Z:

$$\pi^1_\infty \subset \cdots \subset \pi^p_\infty = E_\infty.$$

In the following lemma we prove two important facts about the quotients $Q^i_\infty = \pi^i_\infty/\pi^{i-1}_\infty$:

Lemma 18. *Each quotient $Q^i_\infty = \pi^i_\infty/\pi^{i-1}_\infty$ is semi-stable. Furthermore, E_∞ splits as a direct sum:*

$$E_\infty = Q^1_\infty \oplus \cdots \oplus Q^{p-1}_\infty. \tag{4.5.33}$$

Proof. We begin with the subsheaf of highest rank π^{p-1}_∞. Because the second fundamental form $||\bar{\partial}\pi^{p-1}_\infty||^2_{L^2} = 0$, the induced curvature on Q^p_∞ is just

$$\Lambda F^{Q^p}_\infty = (I - \pi^{p-1}_\infty)\circ\Lambda F_\infty \circ (I - \pi^{p-1}_\infty).$$

Thus, because $rk(I_{p-1}) = rk(I - \pi^{p-1}_\infty)$, we know $\Lambda F^{Q^p}_\infty = \lambda_p I_p$ (where the eigenvaule λ_p is defined in (4.5.18)). So Q^p_∞ admits a Hermitian-Einstein connection, and thus it is semi-stable.

Now, $||\bar{\partial}\pi^{p-1}_\infty||^2_{L^2} = 0$ implies that E_∞ splits as a direct sum $E_\infty = \pi^{p-1}_\infty \oplus Q^p_\infty$. This splitting, along with the fact that $||\bar{\partial}\pi^{p-2}_\infty||^2_{L^2} = 0$, implies the second fundamental form with respect to the inclusion $\pi^{p-2}_\infty \subset \pi^{p-1}_\infty$ is zero, from which it follows that:

$$\Lambda F^{Q^{p-1}}_\infty = (\pi^{p-1}_\infty - \pi^{p-2}_\infty)\circ\Lambda F_\infty \circ (\pi^{p-1}_\infty - \pi^{p-2}_\infty).$$

We continue in this way down the entire filtration. Each Q^i_∞ admits a Hermitian-Einstein connection, and thus it is semi-stable. The decomposition (4.5.33) follows as well.

□

Because each Q^i_∞ is semi-stable admitting a Hermitian-Einstein connection, we know Q^i_∞ will decompose into a direct sum of stable bundles. These stable bundles make up the direct sum (4.5.19), and it is on this level that we must construct the isomorphism with $Gr^{hns}(E)$.

Lemma 19. *Given a sequence of connections A_j along the Yang-Mills flow, the induced connections on Q^i realize an L^1 approximate Hermitian-Einstein structure.*

Proof. For a subbundle π^i in the Harder-Narasimhan filtration, the induced curvature satisfies the following inequality:

$$\int_X |\Lambda F^{S^i}|\omega^n \leq \int_X |\pi^i \circ \Lambda F \circ \pi^i|\omega^n + ||\bar{\partial}\pi^i||^2_{L^2}.$$

Now, because the second fundamental form for the inclusion $\pi^{i-1} \subset \pi^i$ is given by $\bar{\partial}\pi^{i-1} - \bar{\partial}\pi^i$, the induced curvature on $Q^i = S^i/S^{i-1}$ satisfies the following:

$$\begin{aligned}\int_X |\Lambda F^{Q^i}|\omega^n \quad &\leq \quad \int_X |(I-\pi^{i-1}) \circ \Lambda F^{S^i} \circ (I-\pi^{i-1})|\omega^n + ||\bar{\partial}\pi^i||^2_{L^2} \\ &\quad + ||\bar{\partial}\pi^{i-1}||^2_{L^2} + 2||\bar{\partial}\pi^i||_{L^2}||\bar{\partial}\pi^{i-1}||_{L^2}.\end{aligned}$$

Putting the last two inequalities together we see:

$$\begin{aligned}\int_X |\Lambda F^{Q^i}|\omega^n \quad &\leq \quad \int_X |(\pi^i-\pi^{i-1}) \circ \Lambda F \circ (\pi^i-\pi^{i-1})|\omega^n + 2||\bar{\partial}\pi^i||^2_{L^2} \\ &\quad + ||\bar{\partial}\pi^{i-1}||^2_{L^2} + 2||\bar{\partial}\pi^i||_{L^2}||\bar{\partial}\pi^{i-1}||_{L^2}.\end{aligned}$$

Thus, along a subsequence A_j we have the following:

$$\begin{aligned}\int_X |\Lambda F^{Q^i}_j - \mu(Q^i)I|\omega^n \quad &\leq \quad \int_X |(\pi^i-\pi^{i-1}) \circ (\Lambda F_j - \Psi_j) \circ (\pi^i-\pi^{i-1})|\omega^n + 2||\bar{\partial}_j\pi^i_j||^2_{L^2} \\ &\quad + ||\bar{\partial}_j\pi^{i-1}_j||^2_{L^2} + 2||\bar{\partial}_j\pi^i_j||_{L^2}||\bar{\partial}_j\pi^{i-1}_j||_{L^2}.\end{aligned}$$

Now we apply (4.5.32) to get the desired estimate:

$$\int_X |\Lambda F_j^{Q^i} - \mu(Q^i)I|\omega^n \leq 6\int_X |\Lambda F_j - \Psi_j|\omega^n.$$

This completes the lemma. □

We now turn to convergence of the Seshadri filtrations. Since each quotient Q^i in the Harder-Narasimhan filtration is semi-stable, it admits a Seshadri filtration

$$0 \subset \tilde{S}_i^1 \subset \tilde{S}_i^2 \subset \cdots \subset \tilde{S}_i^q = Q^i, \tag{4.5.34}$$

where $\mu(\tilde{S}_i^k) = \mu(Q^i)$ for all k, and each quotient $\tilde{Q}_i^k = \tilde{S}_i^k/\tilde{S}_i^{k-1}$ is torsion free and stable. Here, just as in Section 4.1.1, the subscript i on $\tilde{S}_i^k$ denotes that we are working with the Seshadri filtration from the i-th quotient from the Harder-Narasimhan filtration. Let

$$0 \subset \tilde{\pi}_i^1 \subset \tilde{\pi}_i^2 \subset \cdots \subset \tilde{\pi}_i^{q-1} \subset Q^i$$

be the filtration of L_1^2 projections corresponding to (4.5.34). We show for all k that the sequence of projections $(\tilde{\pi}_i^k)_j$ converges to a limiting projection $(\tilde{\pi}_i^k)_\infty$ in L_1^2 along a subsequence. To do so we need to check that this sequence satisfies the assumptions of Proposition 24. For the first assumption, it is enough to note that by Proposition 24 the projections π^i are smooth away from the singular set Z, and thus the induced connections on the quotients Q^i converge in C^∞ on compact subsets away from Z. For assumption ii), we use the following modification of the Chern-Weil formula:

$$\mu(\tilde{S}_i^k) = \mu(Q^i) + \frac{1}{rk(\tilde{S}_i^k)}\left(\int_X \mathrm{Tr}((\Lambda F_j^{Q^i} - \mu(Q^i)I) \circ (\tilde{\pi}_i^k)_j)\omega^n - ||\bar{\partial}_j(\pi_i^k)_j||_{L^2}^2\right). \tag{4.5.35}$$

Because $\mu(\tilde{S}_i^k) = \mu(Q^i)$, we have

$$||\bar{\partial}_j(\tilde{\pi}_i^k)_j||_{L^2}^2 = \int_X \mathrm{Tr}((\Lambda F_j^{Q^i} - \mu(Q^i)I) \circ (\tilde{\pi}_i^k)_j)\omega^n,$$

which goes to zero by Lemma 19. Thus we can apply Proposition 24 to $(\tilde{\pi}_i^k)_j$, and get that the Seshadri filtration converges to a limiting filtration:

$$0 \subset (\tilde{\pi}_i^1)_\infty \subset (\tilde{\pi}_i^2)_\infty \subset \cdots \subset (\tilde{\pi}_i^{q-1})_\infty \subset Q_\infty^i.$$

Since the norms of the second fundamental forms go to zero, this filtration decomposes Q^i_∞ into a direct sum of quotients $(\tilde{Q}^k_i)_\infty$. At this point the proof of Proposition 23 under the assumption that E is semi-stable from the last subsection carries over, allowing us to construct an isomorphism between $\bigoplus_k \tilde{Q}^k_i$ and $\bigoplus_k (\tilde{Q}^k_i)_\infty$. Thus, on $X \backslash Z$ we have an isomorphism between Q^i_∞ and $Gr^s(Q^i)$. Recall that the proof of this fact starts by considering the subsheaf of lowest rank from $Gr^s(Q^i)$, and working with subsheaves of higher and higher rank until an isomorphism has been constructed for the entire filtration. Since this process is independent of i, applying this argument inductively to each quotient sheaf Q^i, we construct an isomorphism between $Gr^{hns}(E)$ and $\bigoplus_i \bigoplus_k (\tilde{Q}^k_i)_\infty$. Since the direct sum of quotients from any Seshadri filtration is unique, we know $\bigoplus_i \bigoplus_k (\tilde{Q}^k_i)_\infty$ is isomorphic to $\bigoplus_p \hat{Q}^p_\infty$ (from (4.5.19)), proving Proposition 23 in the general unstable case.

4.5.3 Extension over the singular set

We have now constructed, on $X \backslash Z$, the following isomorphism:

$$Gr^{hns}(E) \cong E_\infty. \tag{4.5.36}$$

In order to prove Theorem 5, we need to show this isomorphism can be extended to an isomorphism between $Gr^{hns}(E)^{**}$ and the Bando-Siu extension $\hat{E}_\infty$ on all of X. As a first step we show that E_∞ can be extended over Z as the reflexive sheaf $Gr^{hns}(E)^{**}$. To do so, notice that:

$$\Gamma(X \backslash Z, Gr^{hns}(E)) \cong \Gamma(X \backslash Z, Gr^{hns}(E)^{**}), \tag{4.5.37}$$

since $Gr^{hns}(E)$ is locally free on $X \backslash Z$. Since all holomorphic functions can be extended over Z by a result of Shiffman from [42], and because $Gr^{hns}(E)^{**}$ is reflexive and thus defined by $Hom(Gr^s(E)^*, \mathcal{O})$, we have that:

$$\Gamma(X \backslash Z, Gr^{hns}(E)^{**}) \cong \Gamma(X, Gr^{hns}(E)^{**}).$$

Combining this isomorphism with (4.5.37) we have

$$\Gamma(X\backslash Z, Gr^{hns}(E)) \cong \Gamma(X, Gr^{hns}(E)^{**}). \tag{4.5.38}$$

Thus, using (4.5.36), we see that E_∞ extends over the singular set Z as the reflexive sheaf $Gr^{hns}(E)^{**}$.

As proven in [5], the existence of a Bando-Siu extension $\hat{E}_\infty$ is a consequence of Bando's removable singularity theorem [3] and Siu's slicing theorem [44]. For completeness, we give a brief description of Bando and Siu's result. We work on a coordinate chart $U \subset \mathbf{C}^n$. It follows from the final corollary in [42] that given a point $x \in U$, since $H^{2n-4}(Z_{an}) \leq C$ almost all complex two planes through x intersect Z_{an} at a finite number of points. Thus if we choose a countable dense set $A \subset U$, for each point in A we can choose a complex plane with a fixed normal vector that intersects Z_{an} at a finite number of points. Change coordinates so that this plane P is given by $z_3 = z_4 = \cdots = z_n = c$. Let Δ^2 be a polydisk in P. Given a domain $D \subset U$ which lies in $U \cap \mathbf{C}^{n-2}$, let A' be the projection of A onto D. We are now ready to apply the slicing theorem. E_∞ is a locally free sheaf defined on $(D \times \Delta^2) \cap Z_{an}$. For each $t \in A'$, we have that $(\{t\} \times \Delta^2) \cap Z_{an}$ is a finite number of points, thus by Theorem 10 from [3] E_∞ can be extended to a locally free sheaf $E_\infty(t)$ on $\{t\} \times \Delta^2$. Thus the assumptions of the slicing Theorem [44] hold in our case and E_∞ can be uniquely extended to a reflexive sheaf $\hat{E}_\infty$ on $D \times \Delta^2$.

The uniqueness condition stated in [44] is characterized by the fact that given any other reflexive extension (in our case $Gr^s(E)^{**}$), there exists a sheaf isomorphsim $\phi : \hat{E}_\infty \longrightarrow Gr^s(E)^{**}$ on X, which restricts to the isomorphism constructed in Proposition 26 on $X\backslash Z_{an}$. This completes the proof of Theorem 5. Thus even through we do not know whether Z_{an} depends on the subsequence A_j, the limiting reflexive sheaf $\hat{E}_\infty$ (defined on all of X) is canonical and does not depend on the choice of subsequence. We have the following corollary of Theorem 5:

Corollary 7. *The algebraic singular set* Z_{alg} *is contained in the analytic singular set* Z_{an}.

Proof. We prove $Z_{alg} \subseteq Z_{an}$. Suppose there exists a point $x_0 \in Z_{alg}$ which is not in Z_{an}. We know there exists a quotient $\tilde{Q}_i^k$ from $Gr^{hns}(E)$ such that $\tilde{Q}_i^k$ is not locally free at x_0. Yet by Theorem 5 we know Q^i is isomorphic to some Q^i_∞ from the direct sum $E_\infty = \oplus_p \hat{Q}^p_\infty$, and since E_∞ is a vector bundle off Z_{an} we know $\tilde{Q}_i^k$ is locally free there.

□

It would be quite valuable to know the other set inclusion, proving that in fact $Z_{alg} = Z_{an}$. This would show that the bubbling set Z_{an} is unique and canonical, and does not depend on the subsequence we choose along the Yang-Mills flow. However, to do so, a much more detailed analysis of the singular set is needed.

5 Stable Higgs Bundles on Non-Kähler Manifolds

5.1 Preliminaries

We first go over some background that we will use throughout this section.

5.1.1 Higgs bundles

Let X be a compact complex manifold equipped with a Gauduchon metric g and associated form ω, thus $\partial\bar{\partial}(\omega^{n-1}) = 0$. Once again assume ω is normalized so that X has volume one. Let E be a holomorphic vector bundle over X of rank r. We say E is a *Higgs bundle* if there exists a holomorphic map $\theta : E \longrightarrow \Lambda^{1,0}(E)$, which we call the *Higgs field*. Given a metric H on E, let $\theta^\dagger$ be the adjoint of θ with respect to H. In a local frame $\theta^\dagger$ is given by ${\theta^\dagger}^\alpha{}_\rho = H^{\alpha\bar{\beta}}\overline{\theta^\gamma{}_\beta}H_{\bar{\gamma}\rho}$. Given the usual unitary-Chern connection $\nabla = \nabla^{1,0} + \bar{\partial}$, we consider the following connection on E:

$$D = \nabla + \theta + \theta^\dagger.$$

Let F_θ denote the curvature of D. If F is the curvature of ∇, and $\theta \wedge \theta = 0$, we can express F_θ as follows:

$$F_\theta = F + \nabla^{1,0}\theta + \bar{\partial}\theta^\dagger + \theta \wedge \theta^\dagger + \theta^\dagger \wedge \theta.$$

Now, we have $F_\theta \wedge \omega^{n-1} = \Lambda F_\theta \, \omega^n$. Because $\nabla^{1,0}\theta$ is a $(2,0)$-form and $\bar{\partial}\theta^\dagger$ a $(0,2)$-form, it follows that $(\nabla^{1,0}\theta + \bar{\partial}\theta^\dagger) \wedge \omega^{n-1} = 0$. Thus, in local coordinates ΛF_θ is given by:

$$\Lambda F_\theta = \Lambda F - g^{j\bar{k}}[\theta^\dagger_{\bar{k}}, \theta_j].$$

The connection D can be broken into two parts $D = D' + D''$, defined by:

$$D' = \nabla^{1,0} + \theta^\dagger \qquad \text{and} \qquad D'' = \bar{\partial} + \theta.$$

We note that this is *not* the usual decomposition of a connection into $(1,0)$ and $(0,1)$ parts. However, this decomposition has the feature that if the metric $H(t)$ on E is evolving with time, D' evolves as well, yet D'' remains fixed. This fact turns out to be extremely important in proving many useful formulas needed in later sections. Also, because of the assumptions that θ is holomorphic and $\theta \wedge \theta = 0$, we see that $(D'')^2 = (D')^2 = 0$, thus using this decomposition we have:

$$F_\theta = D' \circ D'' + D'' \circ D'.$$

This fact proves useful in later sections as well.

Here we define our notion of stability, which was considered by Simpson in [43]. Given a vector bundle E with metric H, we define its degree as:

$$deg(E) = \int_X \mathrm{Tr}(F_\theta) \wedge \omega^{n-1}.$$

Of course we would get the same degree if we used the curvature of unitary-Chern connection ∇ in our definition as opposed to D. However, for the purposes of our paper it will be more useful to consider D. Note that g Gauduchon is the minimum assumption we need for degree to be well defined. That is, given a different metric $\hat{H}$ on E, there is a smooth function ψ so that difference $\mathrm{Tr}(F_\theta)\wedge\omega^{n-1} - \mathrm{Tr}(\hat{F}_\theta)\wedge\omega^{n-1} = \partial\bar{\partial}\psi \wedge \omega^{n-1}$, which integrates to zero in the Gauduchon case.

Consider a proper torsion free subsheaf $\mathcal{F} \subset E$ with torsion free quotient. We say $\mathcal{F}$ is a *sub-Higgs sheaf* of E if θ preserves $\mathcal{F}$. Since $\mathcal{F}$ is a vector bundle off a singular set $Z(\mathcal{F})$ of codimension 2, off this set we can consider the orthogonal projection $\pi : E \longrightarrow \mathcal{F}$. Let ϕ be a section of $\mathcal{F}$. The connection D induces a connection on $\mathcal{F}$, which is given by $D_{\mathcal{F}}(\phi) = \pi \circ D(\phi)$. The second fundamental form associated to this connection is defined by:

$$(D - D_{\mathcal{F}})\phi = (I - \pi)D\phi.$$

Because both $\bar{\partial}$ and θ preserve $\mathcal{F}$, we know $(I - \pi)\circ D'' = 0$, so the second fundamental form can in fact be expressed $(I - \pi)D'\phi$. We also compute:

$$D'(\pi)\phi = D'(\pi\phi) - \pi D'(\phi) = (I - \pi)D'\phi,$$

thus $D'(\pi)$ represents the second fundamental form associated to D. Now, using how curvature decomposes onto subbundles (see [22]), we have:

$$F_\theta^{\mathcal{F}} = \pi F_\theta \pi - D''(\pi) \wedge D'(\pi).$$

Although this formula only holds on $X \backslash Z(\mathcal{F})$, by Proposition 2 we know that the induced curvature is at least in L^1, and since $Z(\mathcal{F})$ has zero measure the degree of $\mathcal{F}$ can once again be defined by integrating over X:

$$deg(\mathcal{F}) = \int_X \mathrm{Tr}(\pi F_\theta) \wedge \omega^{n-1} - ||D'\pi||_{L^2}^2.$$

This is the well know Chern-Weil formula. The slope of a sheaf $\mu(\mathcal{F})$ is the quotient of the degree of $\mathcal{F}$ by the rank of $\mathcal{F}$. We say E is stable if, given any proper coherent sub-Higgs sheaf $\mathcal{F}$, we have $\mu(\mathcal{F}) < \mu(E)$. This is the stability condition we need in order to solve our equation 1.0.4.

5.1.2 Gauduchon and semi-Kähler assumptions

In this subsection we explore the intricacies of our non-Kähler setting. First we note that if X is a compact Hermitian manifold with metric g_0 on $T^{1,0}X$, then there exists a smooth function ψ so that $e^\psi g_0$ is Gauduchon [28]. Thus the assumption that X admit a Gauduchon metric is completely general, as opposed to the existence of Kähler or semi-Kähler metrics, which may not always exist. Because we will mostly compute in local coordinates, we will derive here coordinate versions of the Gauduchon and semi-Kähler conditions.

The torsion tensor T is defined by the relation

$$T_{l\bar{k}j} = \partial_l g_{\bar{k}j} - \partial_j g_{\bar{k}l}.$$

Note T vanishes on Kähler manifolds. We will need to compute the derivative of the inverse metric times the volume form:

$$\begin{aligned}\partial_{\bar{k}}(g^{p\bar{k}}\omega^n) &= -g^{p\bar{m}}\partial_{\bar{k}}g_{\bar{m}l}g^{l\bar{k}}\omega^n + g^{p\bar{k}}\partial_{\bar{k}}g_{\bar{q}l}g^{l\bar{q}}\omega^n. \\ &= g^{p\bar{k}}\overline{T_{k\bar{l}q}}g^{l\bar{q}}\omega^n \\ &= g^{p\bar{k}}\overline{T_k{}^q{}_q}\omega^n.\end{aligned}$$

This will be useful in the following computations. Let $\star$ be the Hodge star operator. We have the following two lemmas:

Lemma 20. *For any* $(0,1)$ *form* ψ*, we have the following identity:*

$$\psi_{\bar{j}}g^{k\bar{j}}T_k{}^p{}_p = i\star(\partial(\omega^{n-1})\wedge\psi_{\bar{p}}d\bar{z}^p).$$

Proof. Let $\phi \in \mathbf{C}^\infty(X)$, $\psi \in \Lambda^{0,1}(X)$, and let $(\cdot,\cdot)$ be the L^2 inner product on differential forms over X. Then we have:

$$\begin{aligned}(\phi, i\star(\partial(\omega^{n-1})\wedge\psi_{\bar{p}}d\bar{z}^p)) &= \int_X \phi\wedge\star\overline{i\star(\partial(\omega^{n-1})\wedge\psi_{\bar{p}}d\bar{z}^p)} \\ &= -i\int_X \phi(-1)^{2n}\overline{\partial(\omega^{n-1})\wedge\psi} \\ &= i\int_X \bar{\partial}(\phi\overline{\psi})\wedge\omega^{n-1} \\ &= -\int_X g^{j\bar{k}}\partial_{\bar{k}}(\phi\overline{\psi_{\bar{j}}})\omega^n.\end{aligned}$$

We integrate by parts again and see:

$$\begin{aligned}-\int_X g^{j\bar{k}}\partial_{\bar{k}}(\phi\overline{\psi_{\bar{j}}})\omega^n &= \int_X \phi\,\overline{\psi_{\bar{j}}\partial_k(g^{k\bar{j}}\omega^n)} \\ &= \int_X \phi\,\overline{\psi_{\bar{j}}g^{k\bar{j}}T_k{}^p{}_p}\omega^n \\ &= (\phi, \psi_{\bar{j}}g^{k\bar{j}}T_k{}^p{}_p),\end{aligned}$$

completing the proof of the lemma. □

Lemma 21. *The following identity holds on all compact Hermitian manifolds:*

$$-i\star(\partial\bar{\partial}\omega^{n-1}) = \nabla^k T_k{}^m{}_m + g^{k\bar{j}}T_k{}^m{}_m\overline{T_j{}^p{}_p}.$$

Proof. The proof is similar to the previous lemma. We integrate by parts twice, apply the same contraction formula, and integrate by parts twice again. □

We now have the following corollaries, which we will use throughout the paper to take advantage of our assumptions on g.

Corollary 8. *If g is semi-Kähler, then for any $(0,1)$ form ψ, we have* $\psi_{\bar{j}} g^{k\bar{j}} T_k{}^p{}_p = 0$.

Corollary 9. *If g is Gauduchon, then* $\nabla^k T_k{}^m{}_m + g^{k\bar{j}} T_k{}^m{}_m \overline{T_j{}^p{}_p} = 0$.

Here we note that semi-Kähler condition is the minimum assumption we need to integrate by parts without torsion terms. To see this, let $f \in C^\infty(X)$ and $\phi \in \Lambda^{1,0}(X)$. Then if ω is semi-Kähler we have:

$$
\begin{aligned}
\int_X g^{j\bar{k}} \nabla_{\bar{k}} f \phi_j \, \omega^n &= -\int_X f \nabla_{\bar{k}} (\phi_j \, g^{j\bar{k}} \omega^n) \\
&= -\int_X f \nabla_{\bar{k}} \phi_j \, g^{j\bar{k}} \omega^n - \int_X f \phi_j \, \nabla_{\bar{k}} (g^{j\bar{k}} \omega^n) \\
&= -\int_X f \nabla_{\bar{k}} \phi_j \, g^{j\bar{k}} \omega^n - \int_X f \phi_j \, g^{j\bar{k}} \overline{T_k{}^p{}_p} \omega^n .
\end{aligned}
$$

The second term on the right vanishes by Corollary 8.

5.2 The Donaldson heat flow on Higgs bundles

In this section we introduce the Donaldson heat heat flow on Higgs bundles, prove some basic evolution equations, and introduce the Donaldson functional. For an initial metric H_0, we define the flow of endomorphsims $h = h(t)$ by:

$$
h^{-1} \dot{h} = -(\Lambda F_\theta - \mu(E) I), \tag{5.2.1}
$$

where $h(0) = I$ and $F_\theta = F_\theta(t)$ is the curvature of the metric $H(t) = H_0 h(t)$. The goal of this paper is to show the flow converges to a solution of (1.0.4) along a subsequence of times. First we compute the evolution of a few key terms, which we will need for later sections. Let $\hat{\nabla}$ be the unitary Chern connection associated to the fixed metric

H_0. We denote all other terms associated to H_0 in a similar fashion. The following formula relates the two unitary-Chern connections:

$$\nabla - \hat{\nabla} = \nabla^{1,0} h h^{-1}.$$

Note we also have

$$\theta^\dagger(h)h^{-1} = \theta^\dagger h h^{-1} - h\theta^\dagger h^{-1} = \theta^\dagger - \hat{\theta}^\dagger.$$

Thus $(\nabla + \theta^\dagger) - (\hat{\nabla} + \hat{\theta}^\dagger) = D'hh^{-1}$. Now, recall the difference between the two unitary-Chern curvature tensors:

$$F - \hat{F} = \bar{\partial}(\nabla^{1,0} h h^{-1}).$$

In fact, we have a corresponding formula for the curvature of D. We compute:

$$D''(D'hh^{-1}) = \bar{\partial}(\nabla - \hat{\nabla} + \theta^\dagger - \hat{\theta}^\dagger) + \theta(\nabla - \hat{\nabla} + \theta^\dagger - \hat{\theta}^\dagger) = F_\theta - \hat{F}_\theta. \tag{5.2.2}$$

Of course, deriving these formula's using h^{-1} instead of h, we also get the corresponding formula involving the covariant derivative with respect to H_0:

$$F_\theta - \hat{F}_\theta = D''(h^{-1}\hat{D}'h). \tag{5.2.3}$$

We now use these formulas to help compute the evolution equations for key terms along the flow. From [45], we have that along any path of endomorphisms $h(t)$, the unitary-Chern connection evolves by $\partial_t(\nabla) = \partial_t(\nabla^{1,0} h\, h^{-1}) = \nabla^{1,0}(h^{-1}\dot{h})$. Also, by the definition of adjoint we have $\partial_t(\theta^\dagger) = \theta^\dagger h^{-1}\dot{h} - h^{-1}\dot{h}\theta^\dagger$. Thus $\partial_t(D'hh^{-1}) = D'(h^{-1}\dot{h})$. We can now compute how curvature evolves with time:

$$\dot{F}_\theta = \partial_t D''(D'h\, h^{-1}) = D''D'(h^{-1}\dot{h}) = -D''D'(\Lambda F_\theta).$$

From this equation we also see how the trace of the curvature evolves:

$$\Lambda\dot{F}_\theta = g^{j\bar{k}} D''_{\bar{k}} D'_j(\Lambda F_\theta) = \overline{\Delta}_D(\Lambda F_\theta) = \Delta_D(\Lambda F_\theta),$$

since in this special case, $\overline{\Delta}_D = \Delta_D$. Here Δ_D is an elliptic operator, so as an immediate consequence of the last formula, by the maximum principle we have:

$$\sup_X |\Lambda F_\theta(t)| < C, \tag{5.2.4}$$

uniformly in time. We will need the following lemma:

Lemma 22. *We can pick an initial metric on E so that $det(h) = 1$ for all time along the flow.*

Proof. Let K be a fixed metric on E. We want to find a function ϕ so that $H_0 = e^\phi K$ will satisfy $\mathrm{Tr}(\Lambda\hat{F}_\theta - \mu(E)I) = 0$. Following [30], we compute out how the conformal change effects the curvature of the unitary-Chern connection:

$$\mathrm{Tr}(\Lambda\hat{F}) = ir\Delta\phi + \mathrm{Tr}(\Lambda F(K)).$$

Here we note that Δ is the complex Laplacian on functions, defined by $\Delta := g^{j\bar{k}}\partial_j\partial_{\bar{k}}$, which is not equal to the standard Levi-Civita Laplace-Beltrami operator since we are on a non-Kähler manifolds. However, for our analysis the complex Laplacian works fine. Now because the conformal factor does not affect $\theta^\dagger$, this implies

$$\mathrm{Tr}(\Lambda\hat{F}_\theta) = ir\Delta\phi + \mathrm{Tr}(\Lambda F_\theta(K)).$$

We want to choose ϕ so that $\mathrm{Tr}(\Lambda\hat{F}_\theta - \mu(E)I) = 0$, and this will be true if $\Delta\phi = \frac{i}{r}Tr(\Lambda F_\theta(K) - \mu(E)I)$. By Corollary 1.2.9 in [28], we can pick such a ϕ as as long as:

$$\int_X \mathrm{Tr}(\Lambda F_\theta(K) - \mu(E)I)\,\omega^n = 0,$$

which we know holds by the definition of slope. Now pick H_0 as the initial metric on E, and let H be the evolving metric along (5.2.1). We can compute:

$$\partial_t(\log\det h) = \mathrm{Tr}(h^{-1}\dot{h}) = -\mathrm{Tr}(\Lambda F_\theta - \mu(E)I).$$

Now at time t=0 we have $\mathrm{Tr}(\Lambda\hat{F}_\theta - \mu(E)I) = 0$. And we also saw

$$\partial_t\mathrm{Tr}(\Lambda F_\theta - \mu I) = \Delta\mathrm{Tr}(\Lambda F_t - \mu I).$$

So by the maximum principle $\partial_t(\log\det h) = 0$ for all time t. Since at time $t = 0$ we have $\log\det h = \log\det I = 0$, this implies $\log\det h = 0$ for all t. $\square$

From this point on, H_0 will always be the initial fixed metric on E satisfying Lemma 22, and H will denote the metric on E evolving by the flow.

We now consider the Donaldson functional on Higgs bundles. In the Kähler and semi-Kähler cases, the Donaldson heat flow is the gradient flow of this functional. If g is only Gauduchon, then it is not even clear that the functional decreases along the flow. However, even in this case the functional plays an important role in the proof of Theorem 6, so we take the time to introduce it here.

Recall we have a fixed metric H_0 on E. Consider the space of positive definite Hermitian endomorphisms, denoted $Herm^+(E)$. For any other metric H on E, as before we define $h \in Herm^+(E)$ by $h = H_0^{-1}H$, and since h is positive definite it can be expressed as $h = e^s$ for some endomorphism s. Now, consider the path in $Herm^+(E)$ connecting I to h given by $h_u = e^{u\cdot s}$ for $u \in [0,1]$, and let $F_{\theta,u}$ be defined using the metric $H_u = H_0 h_u$. We now define the Donaldson functional:

$$M(H_0, H) = \int_0^1 \int_X \mathrm{Tr}(F_{\theta,u} s)\, \omega^{n-1} du - \mu(E) \int_X \log\det(h_1)\omega^n. \tag{5.2.5}$$

Here we note that this definition is slightly different to the usual definition in the Kähler case, in that we chose a specific path connecting I to h, instead of allowing any path. We do this because unlike in the Kähler case, if g is Gauduchon then $M(H_0, H)$ is not path independent. Our specific path was chosen to make certain computations easier in later sections. Now, if g is semi-Kähler then $M(H_0, H)$ is indeed path independent. For details we direct the reader to [45], and note that the proof carries over to the semi-Kähler case precisely because we are able to integrate by parts without creating torsion terms.

Let $H = H(t)$ be a path of metrics on E, and let $M(t) := M(H_0, H(t))$ denote the Donaldson functional evolving with time. We have the following proposition:

Proposition 27. *If the metric on X is semi-Kähler then*

$$\partial_t M(t) = \int_X \mathrm{Tr}((\Lambda F_\theta - \mu(E)I)h^{-1}\dot{h})\,\omega^n.$$

Thus along the Donaldson heat flow $M(t)$ is non-increasing, and at a critical point of M the metric satisfies the Hermitian-Einstein equation.

Proof. The proof of the proposition follows easily from the fact that $M(H_0, H)$ path independent of semi-Kähler manifolds. □

We compute the derivative of $M(t)$ in the Gauduchon case in the next section, as it is an integral part of the proof of Step 1.

5.3 Convergence properties of the flow

The goal of this section is to prove the Theorem 6, under the assumption that along the Donaldson heat flow the function $\mathrm{Tr}(h)$ is bounded in C^0 independent of time. Since our initial metric H_0 was chosen so that $\det(h) = 1$, the bound on $\mathrm{Tr}(h)$ implies every eigenvalue λ_i of h satisfies $0 < c \leq \lambda_i \leq C$ uniformly. Thus we can take norms with respect to either metric H or H_0 and get equivalent bounds. Here we also note that long time existence of the flow, first proven by Donaldson in [13], carries over with no changes in the proof to the non-Kähler setting.

Consider the following proposition:

Proposition 28. *Under the assumption that $||\mathrm{Tr}(h)||_{C^0} \leq C$, we have $||D'hh^{-1}||_{C^0} \leq C$ independent of time.*

The proof of the proposition consists of several local computations and an application of the maximum principle. Because it does not make use of the global structure of X, the proof for g Gauduchon is exactly the same as in the Kähler case. We direct the reader to [30] for details.

Proposition 29. *Under the assumption that $||\mathrm{Tr}(h)||_{C^0} \leq C$, we have the following L^p bound on the full curvature tensor F_θ, which holds for any $1 \leq p < \infty$:*

$$||F_\theta||_{L^p} < C.$$

Proof. Recall from Section 5.2 that

$$\begin{aligned}(F_\theta)_{\bar{k}j} - (\hat{F}_\theta)_{\bar{k}j} &= -D''_{\bar{k}}(h^{-1}\hat{D}'_j h) \\ &= -h^{-1}D''_{\bar{k}}\hat{D}'_j h + h^{-1}D''_{\bar{k}} h h^{-1}\hat{D}'_j h.\end{aligned}$$

Thus we have

$$\hat{\Delta}_D h = h(\Lambda\hat{F}_\theta - \Lambda F_\theta) + g^{j\bar{k}}D''_{\bar{k}} h h^{-1}\hat{D}'_j h.$$

From here we see the right hand side is uniformly bounded in C^0, which imples $||\hat{\Delta}_D h||_{C^0} < C$. By standard L^p theory of elliptic PDE's, it follows that for any $1 \leq p < \infty$ we have

$$||h||_{W^{2,p}} < \infty.$$

Since the curvature F_θ is given by a formula involving two derivatives of h, we have the desired result. □

Our next goal is to show that along the Donaldson heat flow, $\Lambda F_\theta - \mu(E)I$ goes to zero strongly in L^2 as t goes to infinity. First we need a lemma bounding the Donaldson functional above and below.

Lemma 23. *Assume the bound $||\mathrm{Tr}(h)||_{C^0} \leq C$ holds uniformly in time along the Donaldson heat flow. Then the Donaldson functional $M(t)$ is uniformly bounded from above and below along the flow.*

Proof. Recall H_0 was chosen so that $\det(h) = 1$, which implies the second term in the definition of the Donaldson functional (5.2.5) vanishes. Thus along the flow we have:

$$M(t) = \int_0^1 \int_X \mathrm{Tr}(F_{\theta,u}\, s)\, \omega^{n-1}\, du$$

Since $s = \log(h)$, and all the eigenvalues of h are uniformly bounded from above and away from zero, we know s is bounded in C^0. The lemma now follows from the L^p bound on the curvature F_θ.

□

We can now prove L^2 convergence:

Proposition 30. *Along the Donaldson heat flow, for any $\epsilon > 0$ there exists a time T so that for $t > T$ we have:*

$$||\Lambda F_\theta(t) - \mu(E)I||^2_{L^2} < \epsilon.$$

Proof. If the metric g on X is at least semi-Kähler, then this lemma follows easily. As we saw in Proposition 27, in this case $M(t)$ is constructed so that

$$\partial_t M(t) = \int_X \mathrm{Tr}((\Lambda F_\theta - \mu(E)I)h^{-1}\dot{h})\omega^n.$$

Thus along the flow $\partial_t M(t) \leq 0$. Using the semi-Kähler assumption once more, we can integrate by parts to see:

$$\begin{aligned}\partial_t^2 M(t) &= -2\int_X \mathrm{Tr}((\Lambda F_\theta - \mu(E)I)\Delta_D(\Lambda F))\omega^n \\ &= 2\int_X |D'(\Lambda F)|^2\omega^n \geq 0.\end{aligned}$$

Now the fact that M is non-increasing, convex, and bounded below implies $\partial_t M(t) \longrightarrow 0$ as t approaches infinity, which gives the desired result. Now, if g is only Gauduchon, then the preceding argument does not work, and more delicate analysis needs to be done. First we introduce the following lemma:

Lemma 24. *If g is Gauduchon, then the derivative of the Donaldson functional along any path $H(t)$ is given by:*

$$\begin{aligned}\dot{M}(t) &= \int_X \mathrm{Tr}((\Lambda F_\theta - \mu(E)I)h^{-1}\dot{h})\omega^n \\ &\quad - \int_0^1\int_X (\bar{\partial}\,\mathrm{Tr}(sD'(h^{-1}\dot{h})) - \partial\,\mathrm{Tr}(sD''(h^{-1}\dot{h}))) \wedge \omega^{n-1} du\end{aligned}$$

First we show how to complete the proposition using this lemma. Define $Y(t) = ||\Lambda F_\theta - \mu(E)I||^2_{L^2}$. The lemma gives that along the Donaldson heat flow we have:

$$\dot{M}(t) = -Y(t) - \int_0^1 \int_X (\bar\partial \operatorname{Tr}(sD'(\Lambda F_\theta)) - \partial \operatorname{Tr}(sD''(\Lambda F_\theta))) \wedge \omega^{n-1} du. \tag{5.3.6}$$

Our goal is to show the integral $\int_0^\infty Y(t)dt$ is bounded. We concentrate first on the second term on the right hand side. After integrating by parts we have the following bound:

$$\int_0^1 \int_X (g^{j\bar k}\operatorname{Tr}(sD'_j(\Lambda F_\theta))\overline{T_k{}^q{}_q} - \operatorname{Tr}(sD''_{\bar k}(\Lambda F_\theta))T_j{}^p{}_p)\, \omega^n du \leq C||D'(\Lambda F_\theta)||^2_{L^2}.$$

Here we have used that s is bounded in C^0 and that the torsion terms depend only on the base metric g and are therefore bounded. In fact we can identify $||D'(\Lambda F_\theta)||^2_{L^2}$ as being equal to $\dot Y(t)$. We compute as follows:

$$\begin{aligned}
\dot Y(t) &= 2\int_X \operatorname{Tr}((\Lambda F_\theta - \mu(E)I)\Delta_D(\Lambda F_\theta))\omega^n \\
&= -2||D'(\Lambda F_\theta)||^2_{L^2} - 2\int_X \operatorname{Tr}((\Lambda F_\theta - \mu(E)I)D''_{\bar k}(\Lambda F_\theta))g^{j\bar k}T_j{}^p{}_p\omega^n.
\end{aligned}$$

We need to show the second term on the right equals zero. By Corollary 9 we have:

$$\begin{aligned}
0 &= \int_X \operatorname{Tr}((\Lambda F_\theta - \mu(E)I)^2)(\nabla^j T_j{}^p{}_p + g^{j\bar k}T_j{}^m{}_m\overline{T_k{}^p{}_p})\omega^n \\
&= -\int_X \nabla_{\bar k}\operatorname{Tr}((\Lambda F_\theta - \mu(E)I)^2)g^{j\bar k}T_j{}^p{}_p)\omega^n \\
&= -2\int_X \operatorname{Tr}((\Lambda F_\theta - \mu(E)I)D''_{\bar k}(\Lambda F_\theta))g^{j\bar k}T_j{}^p{}_p\omega^n.
\end{aligned}$$

Thus $\dot Y(t) = -2||D'(\Lambda F_\theta)||^2_{L^2}$. This allows us to return to inequality (5.3.6):

$$\dot M(t) \leq -Y(t) + C||D'(\Lambda F_\theta)||^2 = -Y(t) - \frac{C}{2}\dot Y(t).$$

Which gives:

$$Y(t) \leq -\dot M(t) - C\dot Y(t).$$

Now we can integrate from zero to infinity to get:

$$\int_0^\infty Y(t)dt \leq M(0) + C\,Y(0) - \lim_{b\longrightarrow\infty}(M(b) + Y(b)) < \infty.$$

This integral is finite since $M(t)$ is uniformly bounded from below by Lemma 23 and $Y(t)$ is non-negative. The finiteness of this integral, along with the fact that $\dot{Y}(t) = -2||D'(\Lambda F_\theta)||^2_{L^2} \leq 0$, implies $Y(t)$ goes to zero as t goes to infinity. Now, to finish the proof of the proposition, we prove Lemma 24.

We follow the proof of Lemma (3.6) from [26]. Consider the operator $\tilde{d}$ defined by:

$$\tilde{d}f = \partial_t f\, dt + \partial_u f\, du$$

for $f \in Herm^+(E)$. We also define ϕ =Tr$(F_\theta h^{-1}\tilde{d}h)$. Recall that the Donaldson functional M was defined by integrating along the specific path $h_u = e^{u\cdot s}$. Now if we let Δ be the region in $Herm^+(E)$ defined by $0 \leq u \leq 1,\ \ t_0 \leq t \leq t_0 + \delta$, then by Stokes Theorem we have:

$$\int_{\partial\Delta} \phi = \int_\Delta \tilde{d}\phi.$$

Noting that $\phi|_{u=0} = 0$, we compute the left hand side:

$$\begin{aligned} \int_{\partial\Delta} \phi &= \int_{u=0}^{u=1} \phi|_{t_0} + \int_{t_0}^{t_0+\delta} \phi|_{u=1} - \int_{u=0}^{u=1} \phi|_{t_0+\delta} \\ &= \int_0^1 \mathrm{Tr}(F_\theta s(t_0))dv + \int_{t_0}^{t_0+\delta} \mathrm{Tr}(F_\theta h^{-1}\partial_t h)dt - \int_0^1 \mathrm{Tr}(F_\theta s(t_0+\delta))dv. \end{aligned}$$

Integrating over X, applying Stokes Theorem, and adding appropriate terms to both sides we have:

$$\begin{aligned} & M(t_0) + \int_{t_0}^{t_0+\delta} \int_X \mathrm{Tr}(F_\theta h^{-1}\dot{h})\,\omega^{n-1} dt - M(t_0+\delta) \\ = & \int_X \int_\Delta \tilde{d}\phi\,\omega^{n-1} - \mu(E) \int_X \log\det(h(t_0))\omega^n + \mu(E)\int_X \mathrm{logdet}(h(t_0+\delta))\omega^n. \end{aligned}$$

This allows us compute the derivative of M at t_0:

$$\begin{aligned} \partial_t M(t_0) &= \lim_{\delta\to 0} \frac{M(t_0+\delta) - M(t_0)}{\delta} \\ &= \int_X \mathrm{Tr}((\Lambda F_\theta - \mu(E)I)h^{-1}\dot{h})\,\omega^n - \lim_{\delta\to 0}\frac{1}{\delta}\int_X\int_\Delta \tilde{d}\phi\,\omega^{n-1}. \end{aligned}$$

So to prove the lemma we need to understand the last term on the right. Using the fact that $\partial_t F_\theta = D''D'(h^{-1}\partial_t h)$, we can follow [26] and [45] to compute:

$$\tilde{d}\phi = -\bar{\partial}\partial\,\mathrm{Tr}(h^{-1}\dot{h}s) + \bar{\partial}\,\mathrm{Tr}(sD'(h^{-1}\dot{h})) - \partial\,\mathrm{Tr}(sD''(h^{-1}\dot{h})).$$

Since, g is Gauduchon, integrating over X kills the $\bar{\partial}\partial$-term, and we have:

$$\int_X \int_\Delta \tilde{d}\phi = \int_X \int_\Delta (\bar{\partial}\,\mathrm{Tr}(sD'(h^{-1}\dot{h})) - \partial\,\mathrm{Tr}(sD''(h^{-1}\dot{h})))du \wedge dt\,\omega^{n-1}.$$

Taking the limit as δ goes to zero and dividing by δ reduces the integral over the whole region to the integral over the slice $\{t = t_0\}$. Thus:

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_X \int_\Delta \tilde{d}\phi\,\omega^{n-1} = \int_0^1 \int_X (\bar{\partial}\,\mathrm{Tr}(sD'(h^{-1}\dot{h})) - \partial\,\mathrm{Tr}(sD''(h^{-1}\dot{h})))\,\omega^{n-1} du.$$

□

We are now ready to prove Theorem 6.

Proof. Let t_i be a subsequence of times along the Donaldson heat flow. By Proposition 28 we have $||H(t_i)||_{C^1} \leq C$, so by going to a subsequence (still denoted t_i) we have that $H(t_i)$ converge strongly in C^0 to a limiting metric H_∞. This metric is non-degenerate by the eigenvalue bounds $0 < c \leq \lambda_i \leq C$ independent of time. Equation (5.3.6) proves a $W^{2,p}$ bound for H, so taking yet another subsequence we have $H(t_i)$ converge weakly in $W^{2,p}$, and the weak limit must be equal to H_∞. By Banach-Alaglu $H_\infty \in W^{2,p}$. This implies the curvature F_θ^∞ of H_∞ is well defined and in L^p. At this point we have

$$(\Lambda F_\theta(t_i) - \mu(E)I) \longrightarrow (\Lambda F_\theta^\infty - \mu(E)I)$$

weakly in L^p as $i \longrightarrow \infty$. Yet, because $\Lambda F_\theta(t_i) - \mu(E)I$ goes to zero strongly in L^2, it follows that $\Lambda F_\theta^\infty - \mu(E)I$ is weakly zero in L^p. Thus $\Lambda F_\theta^\infty = \mu(E)I$ in this weak sense. Now by standard elliptic regularity results H_∞ is in fact smooth. Thus we have found a smooth solution to equation (1.0.4) and proven Theorem 6 under the assumption that $\mathrm{Tr}(h)$ is bounded. □

5.4 The C^0 bound from stability

In this section we prove $\mathrm{Tr}(h)$ is uniformly bounded in C^0 along the Donaldson heat flow, under the assumptions that g is Gauduchon and E is stable. This step is perhaps the most geometrically meaningful, since we have to use the algebraic-geometric condition of stability to prove a uniform bound along a PDE. Simpson proves this bound in the Kähler case in Proposition 5.3 from [43], and we state his proposition here:

Proposition 31. *If E is stable and $h(t) = e^{s(t)}$ evolves by the Donaldson heat flow, then for all time:*

$$\sup_X |s| \leq C_1 + C_2 M(t). \tag{5.4.7}$$

This proposition is attractive not only because it gives the desired bound on h, but it also gives an explicit lower bound on the Donaldson functional $M(t)$ that does not require existence of any canonical metric. The proof of this proposition carries over to the case where g is semi-Kähler, once again because we are able to integrate by parts. However, in the case that g is Gauduchon, we cannot hope to generalize this result. For even if one could find a proof of (5.4.7) in this case, it is not clear that $M(t)$ is decreasing along the flow, thus we would not even get a bound on h.

Instead we adapt the C^0 bound from the elliptic approach of Uhlenbeck and Yau, suitably modified to fit our parabolic case. We prove the bound by contradiction, which is the content of the following proposition:

Proposition 32. *Let H_i be a sequence of metrics along the Donaldson heat flow, and set $h_i = H_0^{-1} H_i$, where H_0 is our initial fixed metric on E. Assume that the metric g is Gauduchon and that*

$$M_i := \sup_X \mathrm{Tr}(h_i) \longrightarrow \infty.$$

Set $\tilde{h}_i = h_i / M_i$. Then we have the following two conclusions:

i) $\tilde{h}_i$ tends weakly in $L^2_1(E, End(E))$ to a non-trivial endomorphism h_∞, after going to a subsequence;

ii) There exists a subsequence of times t_i so that $\pi := \lim_{\sigma \to 0} \lim_{i \to \infty} (I - h_i^\sigma)$ represents a destabilizing coherent sub-Higgs sheaf of E.

We begin by proving i). This step is the same as for the elliptic case, yet we include it here for the readers convenience. First we prove L^2_1 bounds for $\tilde{h}^\sigma_i$, which will then allow us to take a weak limit. For notational simplicity, we suppress the i from h_i. Set $\langle\cdot,\cdot\rangle$ to be the inner product on E defined by H_0, and let $\hat{D}'$ denote the covarient derivative with respect to this fixed metric. As in [28], we consider the following inequality, which holds for $0<\sigma\leq 1$:

$$g^{j\bar{k}}\langle h^{-1}\hat{D}'_j h, \hat{D}'_k h^\sigma\rangle \geq |h^{-\frac{\sigma}{2}}\hat{D}'h^\sigma|^2. \tag{5.4.8}$$

Next, we will need the following inequality to establish an L^2_1 bound for $\tilde{h}^\sigma$:

$$\int_X |h^{-\frac{\sigma}{2}}\hat{D}'h^\sigma|^2\,\omega^n \leq C\int_X |h^\sigma|\omega^n. \tag{5.4.9}$$

We begin by integrating inequality (5.4.8), followed by integration by parts:

$$\begin{aligned}\int_X |h^{-\frac{\sigma}{2}}\hat{D}'h^\sigma|^2\,\omega^n &\leq \int_X \mathrm{Tr}(h^{-1}\hat{D}'_j h\hat{D}''_{\bar{k}}h^\sigma)g^{j\bar{k}}\omega^n\\ &\leq -\int_X \mathrm{Tr}(D''_{\bar{k}}(h^{-1}\hat{D}'_j h)h^\sigma)g^{j\bar{k}}\omega^n - \int_X \mathrm{Tr}(h^{\sigma-1}\hat{D}'_j h)g^{j\bar{k}}\overline{T_k{}^p{}_p}\omega^n.\\ &= \int_X \mathrm{Tr}((\Lambda F_\theta - \Lambda\hat{F}_\theta)h^\sigma)g^{j\bar{k}}\omega^n.\end{aligned} \tag{5.4.10}$$

In the last line we used the fact that the second term on the right vanishes by Corollary 9:

$$\begin{aligned}\int_X \mathrm{Tr}(h^{\sigma-1}\hat{D}'_j h)g^{j\bar{k}}\overline{T_k{}^p{}_p}\omega^n &= \int_X \mathrm{Tr}(\hat{D}'_j(h^\sigma))g^{j\bar{k}}\overline{T_k{}^p{}_p}\omega^n\\ &= -\int_X \mathrm{Tr}(h^\sigma)(\nabla^k T_k{}^p{}_p + g^{j\bar{k}}T_j{}^q{}_q\overline{T_k{}^p{}_p})\omega^n = 0.\end{aligned}$$

Now, using the C^0 bound for ΛF_θ given by (5.2.4), we see the desired bound (5.4.9) follows from (5.4.10). This bound holds independent of i. Up to now we have not considered the normalized endomorphisms $\tilde{h}_i$. However, simply by dividing by M_i, all the previous inequalities will be true for $\tilde{h}_i$. By definition of the normalization we have $\tilde{h}\leq I$, which in turn implies $\tilde{h}^{-\frac{\sigma}{2}}\geq I$. Thus it follows that:

$$\int_X |\hat{D}'\tilde{h}^\sigma|^2\omega^n \leq \int_X |\tilde{h}^{-\frac{\sigma}{2}}\hat{D}'\tilde{h}^\sigma|^2\omega^n \leq C\int_X |\tilde{h}^\sigma|\omega^n \leq C\int_X \omega^n \leq C,$$

completing the L_1^2 bound for $\tilde{h}_i^\sigma$. Thus, for each σ, there exists a weak limit h_∞^σ and a subsequence of endomorphisms (still denoted $\tilde{h}_i^\sigma$) that converges weakly in L_1^2 to h_∞^σ. Choose a sequence $\sigma_i \longrightarrow 0$. Now, following a diagonalization argument, we can assume that a single subsequence $\tilde{h}_i^\sigma$ converges for all σ_i.

Next, we show that the limit metric h_∞ is not completely degenerate. To do so we need to the following lemma:

Lemma 25. *Along the Donaldson heat flow, the following inequality holds uniformly in i:*

$$\sup_X \operatorname{Tr}(\tilde{h}_i) \leq C||\tilde{h}_i||_{L^1}.$$

This lemma requires a Green's function argument, along with the fact that ΛF_θ is uniformly bounded in time. In [43] Simpson proves an inequality similar to:

$$\Delta \log \operatorname{Tr}(h) \geq -C(|\Lambda F_\theta| + |\Lambda \hat{F}_\theta|),$$

which involves a local computation and thus holds in the non-Kähler setting. From this inequality the Green's function argument in Lemma 4.2 from [53] carries over to our case. Using the lemma, by definition of the normalization we have:

$$1 = \sup_X \operatorname{Tr}(\tilde{h}_i) \leq C||\tilde{h}_i||_{L^1} \leq C||\tilde{h}_i||_{L^2} \cdot vol(X).$$

Which implies:

$$||\tilde{h}_i||_{L^2} \geq \frac{1}{C} > 0.$$

Thus the limit metric h_∞ is not completely degenerate. This proves i) from Proposition 32. We now turn our attention to ii), which requires us to prove the endomorphism π is a weakly holomorphic sub-Higgs bundle. We say π is a weakly holomorphic sub-Higgs bundle if $\pi^* = \pi$, $\pi^2 = \pi$ and $(I - \pi)D''\pi = 0$ in L^1. For a proof of these three fact, we direct the reader to [28], as the result carries over directly to our case.

We now invoke the following theorem of Uhlenback and Yau from [52]:

Theorem 11. *Given a weakly holomorphic subbundle π of E, there exists a coherent subsheaf $\mathcal{F}$ of E, and an analytic subset $S \subset X$ with the following properties:*

i) $codim_X S \geq 2$

ii) $\pi|_{X \setminus S}$ is $\mathbf{C}^\infty$ and satisfies both $\pi^* = \pi = \pi^2$ and $(I - \pi)\bar{\partial}\pi = 0$

iii) $\mathcal{F}' := \mathcal{F}|_{X \setminus S}$ is a holomorphic subbundle.

The condition that $(I - \pi)D''\pi = 0$ implies both $(I - \pi)\bar{\partial}\pi = 0$, which allows us to invoke Theorem 11, and $(I - \pi)\theta\pi = 0$, which shows that in fact our limiting sheaf $\mathcal{F}$ is a sub-Higgs sheaf of E. Thus, using this theorem, in order to finish Proposition 32 we must show $\mathcal{F}$ is proper and destabilizing. We first show $\mathcal{F}$ is a proper subsheaf of E.

Since $\tilde{h}_i$ converge strongly in L^2 to $h_\infty \neq 0$ almost everwhere, we know h_∞ has a strictly positive eigenvalue. Thus almost everywhere h^0_∞ has at least one nonzero eigenvalue, and hence $rk\,(h^0_\infty) \geq 1$. This implies:

$$rk(\mathcal{F}) = rk(\pi) = rk\,(I - h^0_\infty) \leq r - 1.$$

On the other hand, since $\det(h(t)) = 1$ along the Donaldson heat flow, because we are assuming that $\sup_X \mathrm{Tr}(h)$ goes to ∞ we must have an eigenvalue of h that goes to zero. Thus almost everywhere h_∞ has an eigenvalue equal to zero, which implies $rk(\mathcal{F}) > 0$. So $\mathcal{F}$ is indeed a proper subsheaf of E.

We now prove the $\mathcal{F}$ is destabilizing, thus we have to show the inequality $\mu(\mathcal{F}) \geq \mu(E)$. Recall from section 5.1.1 the Chern-Weil formula, which we apply using the fixed connection associated to H_0:

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} \int_X \mathrm{Tr}(\pi \hat{F}_\theta) \wedge \omega^{n-1} - ||\hat{D}'\pi||^2_{L^2}.$$

We modify the formula slightly to include $\mu(E)$:

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} \left(\int_X \mathrm{Tr}((\Lambda \hat{F}_\theta - \mu(E)I) \circ \pi)\, \omega^n - |\hat{D}'\pi||^2_{L^2} \right) + \mu(E).$$

Thus to show $\mu(\mathcal{F}) \geq \mu(E)$, we must verify:

$$\int_X \mathrm{Tr}((\Lambda\tilde{F}_\theta - \mu(E)I) \circ \pi)\omega^n \geq ||\hat{D}'\pi||^2_{L^2}.$$

Now, by definition π is given by the following L^2 limit $\pi = \lim_{\sigma\to 0}\lim_{i\to\infty}(I - \tilde{h}_i^\sigma)$, and we have $\mathrm{Tr}(\Lambda\hat{F}_\theta - \mu(E)I) = 0$ (from the proof of Lemma 22). Thus it follows that:

$$\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I) \circ \pi)\omega^n = -\lim_{\sigma\to 0}\lim_{i\to\infty}\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I) \circ \tilde{h}_i^\sigma)\omega^n. \quad (5.4.11)$$

At this point we use introduce the evolving curvature ΛF_θ using equality (5.2.3):

$$\mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)Id) \circ \tilde{h}_i^\sigma) \;=\; g^{j\bar{k}}\,\mathrm{Tr}(D''_{\bar{k}}(\tilde{h}_i^{-1}\hat{D}'_j\tilde{h}_i)\,\tilde{h}_i^\sigma) + \mathrm{Tr}((\Lambda F_\theta(t_i) - \mu(E)I) \circ \tilde{h}_i^\sigma).$$

This next step is where our proof differs from the elliptic case. Because h is evolving along the Donaldson heat flow we have $-\mathrm{Tr}((\Lambda F_\theta(t_i) - \mu(E)I) \circ \tilde{h}_i^\sigma) = \mathrm{Tr}((h_i^{-1}\dot{h}_i) \circ \tilde{h}_i^\sigma)$. We show we can choose a subsequence (still denoted $\tilde{h}_i^\sigma$), so that

$$\lim_{i\to\infty}\int_X \mathrm{Tr}(h_i^{-1}\dot{h}_i\tilde{h}_i^\sigma))\omega^n = 0. \quad (5.4.12)$$

Notice that:

$$\mathrm{Tr}(h_i^{-1}\dot{h}_i\tilde{h}_i^\sigma)) = \mathrm{Tr}(\tilde{h}_i^{-1}\partial_t\tilde{h}_i\tilde{h}_i^\sigma) = \mathrm{Tr}(\tilde{h}_i^{\sigma-1}\partial_t\tilde{h}_i) = \frac{1}{\sigma}\partial_t\,\mathrm{Tr}(\tilde{h}_i^\sigma).$$

Also, by inequality (5.2.4), we have that:

$$\begin{aligned}\int_X |\partial_t\mathrm{Tr}(\tilde{h}_i^\sigma)|\omega^n \;&\leq\; \int_X |\mathrm{Tr}((\Lambda F_\theta(t_i) - \mu(E)I)\tilde{h}_i^\sigma)|\omega^n \\ &\leq\; C\int_X |\mathrm{Tr}(\tilde{h}_i^\sigma)|\omega^n \leq C,\end{aligned}$$

so by the Lebesgue dominated convergence theorem we can take limits and derivatives in and out of the integral sign. We define the function $f : \mathbf{R}^+ \longrightarrow \mathbf{R}$ by

$$f(t) = \frac{1}{\sigma}\int_X \mathrm{Tr}(\tilde{h}(t)^\sigma)\frac{\omega^n}{n!}.$$

Our goal is to show $\dot{f}(t_i)$ goes to zero along a subsequence. The normalization along with the fact that h is positive definite imply $0 \leq f \leq Vol(X)$ for all t. In fact, since $\tilde{h}_i^\sigma$ converges strongly in L^2, we have:

$$\lim_{t\to\infty} f(t) = \frac{1}{\sigma}\int_X \mathrm{Tr}(h_\infty^\sigma)\omega^n = C.$$

Thus we can now integrate $\dot{f}$ from zero to infinity:

$$\int_0^\infty \dot{f}\, dt = \lim_{t\to\infty} f(t) - f(0) < \infty.$$

This implies we can pick out a subsequence of times such that $\dot{f}(t_i)$ goes to zero, establishing 5.4.12 for this subsequence.

Thus along this subsequence we have the following equality:

$$-\lim_{\sigma\to 0}\lim_{i\to\infty}\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I)\circ \tilde{h}_i^\sigma)\omega^n = -\lim_{\sigma\to 0}\lim_{i\to\infty}\int_X g^{j\bar{k}}\,\mathrm{Tr}(D''_{\bar{k}}(\tilde{h}_i^{-1}\hat{D}'_j\tilde{h}_i)\,\tilde{h}_i^\sigma)\omega^n. \tag{5.4.13}$$

If we integrate by parts the right hand side becomes:

$$\lim_{\sigma\to 0}\lim_{i\to\infty}\left(\int_X g^{j\bar{k}}\,\mathrm{Tr}(\tilde{h}_i^{-1}\hat{D}_j\tilde{h}_i D''_{\bar{k}}\tilde{h}_i^\sigma)\omega^n + \int_X \mathrm{Tr}(\tilde{h}_i^{-1}D'_j\tilde{h}_i\tilde{h}_i^\sigma)g^{j\bar{k}}\overline{T_k{}^q{}_q}\omega^n\right). \tag{5.4.14}$$

Note the last term on the right vanishes by of our Gauduchon assumption:

$$\begin{aligned}
\int_X \mathrm{Tr}((\tilde{h}_i^{-1}\hat{D}'_j\tilde{h}_i)\,\tilde{h}_i^\sigma)g^{j\bar{k}}\overline{T_k{}^q{}_q}\,\omega^n &= \int_X \mathrm{Tr}(\tilde{h}_i^{\sigma-1}\hat{D}'_j\tilde{h}_i)g^{j\bar{k}}\overline{T_k{}^q{}_q}\,\omega^n \\
&= \frac{1}{\sigma}\int_X \hat{\nabla}_j\mathrm{Tr}(\tilde{h}_i^\sigma)g^{j\bar{k}}\overline{T_k{}^q{}_q}\,\omega^n \\
&= \frac{1}{\sigma}\int_X \mathrm{Tr}(\tilde{h}_i^\sigma)(\nabla^{\bar{k}}\overline{T_k{}^q{}_q} + g^{j\bar{k}}T_j{}^p{}_p\overline{T_k{}^q{}_q})\omega^n = 0.
\end{aligned}$$

We also estimate the term on the left from (5.4.14):

$$\begin{aligned}
\int_X g^{j\bar{k}}\,\mathrm{Tr}(\tilde{h}_i^{-1}\hat{D}'_j\tilde{h}_i D''_{\bar{k}}\tilde{h}_i^\sigma)\,\omega^n &\geq \int_X |\tilde{h}_i^{-\frac{\sigma}{2}}\hat{D}'\tilde{h}_i^\sigma|^2\,\omega^n \\
&\geq ||\hat{D}'(\tilde{h}_i^\sigma)||_{L^2}^2 = ||\hat{D}'(I - \tilde{h}_i^\sigma)||_{L^2}^2.
\end{aligned}$$

Combining this last inequality with (5.4.14) and (5.4.13) gives:

$$\lim_{\sigma\to 0}\lim_{i\to\infty}||\hat{D}'(I-\tilde{h}_i^\sigma)||_{L^2}^2 \leq -\lim_{\sigma\to 0}\lim_{i\to\infty}\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I)\circ\tilde{h}_i^\sigma)\omega^n.$$

Thus we can conclude:

$$\begin{aligned}
||\hat{D}'\pi||_{L^2}^2 &= \lim_{\sigma\to 0}\lim_{i\to\infty}(\hat{D}'\pi, \hat{D}'(I-\tilde{h}_i^\sigma))_{L^2} \\
&\leq \lim_{\sigma\to 0}\lim_{i\to\infty}||\hat{D}'\pi||_{L^2}||\hat{D}'(I-\tilde{h}_i^\sigma)||_{L^2} \\
&\leq ||\hat{D}'\pi||_{L^2}\left(-\lim_{\sigma\to 0}\lim_{i\to\infty}\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I)\circ\tilde{h}_i^\sigma)\omega^n\right)^{\frac{1}{2}} \\
&\leq ||\hat{D}'\pi||_{L^2}\left(\int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I)\circ\pi)\omega^n\right)^{\frac{1}{2}},
\end{aligned}$$

where in the last inequality we used 5.4.11. This gives the desired inequality:

$$||\hat{D}'\pi||_{L^2}^2 \leq \int_X \mathrm{Tr}((\Lambda\hat{F}_\theta - \mu(E)I)\circ\pi)\omega^n,$$

and thus proves $\mathcal{F}$ is destabilizing. Even though we only constructed a destabilizing subsheaf along a subsequence (since we only proved (5.4.12) for a subsequence), we did this by assuming that $\mathrm{Tr}(h_i)$ goes to infinity. Thus, if E is indeed stable we must have that $\mathrm{Tr}(h_i)$ is bounded for all subsequences along the Donaldson heat flow. This proves Proposition 32 and as a result Theorem 6.

Just as in [43], as a corollary we get the following Bogomolov-Gieseker inequality:

Corollary 10. *Suppose E is a stable Higgs bundle on X such that $\bar{\partial}\theta = \theta\wedge\theta = 0$. Then*

$$\left(2r\,c_2(E) - (r-1)c_1(E)^2\right)[\omega]^{n-2} \geq 0.$$

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