The modularity of arithmetic generating series of special cycles on $\mathcal{X}_0(N)$

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Abstract

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In the late 1990s, Stephen S. Kudla initiated an influential program to study the special cycles on orthogonal and unitary Shimura varieties. A major conjecture is the modularity of generating series of special cycles. In this thesis we focus on the case of modular curve $\mathcal{X}_0(N)$ where N is an arbitrary positive integer. We prove the modularity of the generating series of special cycles values in the arithmetic Chow groups.

For the generating series valued in the codimension 2 arithmetic Chow group, the modularity is proved by establishing the arithmetic Siegel–Weil formula on the modular curve $\mathcal{X}_0(N)$, i.e., we relate the arithmetic degrees of special cycles on $\mathcal{X}_0(N)$ to the derivatives of Fourier coefficients of a genus 2 Eisenstein series. The identity is proved by combining "difference formulae" on both the geometric and analytic sides. When N is odd and square-free, our work gives a different proof of the main results of Sankaran, Shi and Yang [1]. For the generating series valued in the codimension 1 arithmetic Chow group, the modularity is proved by combining the known results in codimension 2 and analyzing the reduction to primes $p|N$ of cusps of $\mathcal{X}_0(N)$. This generalizes the previous work of Du and Yang [2].

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Chapter 1: Introduction and Background

1.1 Background

The classical Siegel–Weil formula relates certain Siegel Eisenstein series to the arithmetic of quadratic forms, namely it expresses special values of these series as theta functions – generating series of representation numbers of quadratic forms. Kudla initiated an influential program to establish the arithmetic Siegel–Weil formula relating certain Siegel Eisenstein series to objects in arithmetic geometry.

In this article, we study the case of modular curves. Let N be a positive integer, the classical modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ over $\mathbb{C}$ is defined as the following smooth 1-dimensional complex curve,

$$\mathcal{Y}_0(N)_{\mathbb{C}} \coloneqq \mathrm{GL}_2(\mathbb{Q})\backslash \mathbb{H}_1^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/\Gamma_0(N)(\hat{\mathbb{Z}}) \simeq \Gamma_0(N)\backslash \mathbb{H}_1^{\pm},$$

where $\mathbb{H}_1^{\pm} = \mathbb{C}\backslash\mathbb{R}$. The group $\Gamma_0(N)(\hat{\mathbb{Z}})$ is the following open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$,

$$\Gamma_0(N)(\hat{\mathbb{Z}}) = \left\{ x = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}) \ : \ a, b, c, d \in \hat{\mathbb{Z}} \right\},$$

and $\Gamma_0(N) = \Gamma_0(N)(\hat{\mathbb{Z}}) \bigcap \mathrm{GL}_2(\mathbb{Z})$. Notice that the determinant of an element in the group $\Gamma_0(N)$ can be either 1 or -1 rather than only 1 in the classical setting because the space $\mathbb{H}_1^{\pm}$ has two connected component.

The smooth curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ is not proper, its compactification $\mathcal{X}_0(N)_{\mathbb{C}} \coloneqq \mathcal{Y}_0(N)_{\mathbb{C}} \cup \{\text{cusps}\}$ is a smooth projective curve over $\mathbb{C}$. It is a classical fact that the curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ parameterizes cyclic isogenies between elliptic curves over $\mathbb{C}$, here an isogeny $\pi : E \longrightarrow E'$ between two elliptic curves over $\mathbb{C}$ is called cyclic if $\ker(\pi)$ is a cyclic group.

Katz and Mazur [3] extends the concept of cyclic isogeny to arbitrary base scheme, an isogeny $\pi : E \to E'$ between two elliptic curves is called cyclic if $\ker(\pi)$ is a cyclic group scheme (cf. Definition 4.1.2). They also defined the $\Gamma_0(N)$-level structures on elliptic curves. In this article, we will mainly work on a 2-dimensional regular flat Deligne–Mumford stack $\mathcal{X}_0(N)$, defined by Česnavičius in [4], which is the moduli stack of generalized elliptic curves with $\Gamma_0(N)$-level structures and whose fiber over $\mathbb{C}$ is $\mathcal{X}_0(N)_{\mathbb{C}}$. We define the (arithmetic) special cycles on $\mathcal{X}_0(N)$ and study their intersection numbers. Finally, we prove that these intersection numbers are identified with the derivatives of Fourier coefficients of certain Siegel Eisenstein series of genus 2.

When N is an odd, square-free positive integer, the relation has already been obtained for all the pair (T, y) in the work of Sankaran, Shi, and Yang [1, Theorem 2.14] by computing both sides explicitly based on the previous works of Yang [5] and Kudla, Rapoport and Yang [6]. In this article, we use a different method and work with arbitrary level N.

When T is nonsingular, we introduce a formal scheme $\mathcal{N}_0(N)$ which is the Rapoport–Zink space associated to $\mathcal{X}_0(N)$. Via formal uniformization of the supersingular locus of the stack $\mathcal{X}_0(N)$ and its special cycles, we reduce the identity, which relates intersection numbers on $\mathcal{X}_0(N)$ and derivatives of Fourier coefficients of Eisenstein series, to a local identity between local arithmetic intersection numbers on $\mathcal{N}_0(N)$ and derivatives of local densities of quadratic forms. Now the key observation is that, both sides of the local identity, regardless of the level N, can be related to another intersection problem on Rapoport–Zink space of 1 dimension higher, but in a hyperspecial level, while the computation of the latter has been done in the works of Gross and Keating [7, Proposition 5.4], Wedhorn [8, §2.16] and Rapoport [9, Theorem 1.1] (see also the work of Li and Zhang [10, Theorem 1.2.1]).

When the matrix T is singular of rank 1. On the analytic side, we use the analytic difference formula of local densities to relate the singular Fourier coefficients of an Eisenstein series of genus 2 to the nonsingular Fourier coefficients of an Eisenstein series of genus 1. On the geometric side, the cycle $\widehat{\mathcal{Z}}(T, \mathsf{y})$ is essentially the intersection of a codimension 1 cycle and the metrized Hodge line bundle on $\mathcal{X}_0(N)$. We compute this intersection number by investigating the irreducible components of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p} := \mathcal{X}_0(N) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ of the model $\mathcal{X}_0(N)$, and the reduction mod p of the cuspidal divisor on the curve $\mathcal{X}_0(N)$.

When the matrix $T = \mathbf{0}_2$, both the analytic side and geometric side can be computed explicitly. Here again, the computation of the analytic side is based on the difference formulas of local densities, while the computation of the geometric side is based on the intersection of irreducible components of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$.

1.2 Summary of the main results

Let $\Delta(N)$ be the following rank 3 quadratic lattice over $\mathbb{Z}$,

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : \ a, b, c \in \mathbb{Z} \right\} \tag{1.1}$$

equipped with the quadratic form $x \mapsto \det(x)$.

We use v to denote a place of $\mathbb{Q}$. For every finite place v, let $\delta_v(N) = \Delta(N) \otimes_{\mathbb{Z}} \mathbb{Z}_v$ be a rank 3 quadratic lattice over $\mathbb{Z}_v$. Let $\mathbb{A}$ be the ring of adèles over $\mathbb{Q}$. Let $\mathbb{V} = \{\mathbb{V}_v\}$ be the incoherent collection of quadratic spaces of $\mathbb{A}$ of rank 3 nearby $\Delta(N)$ at ∞, i.e.,

$$\mathbb{V}_v = \delta_v(N) \otimes \mathbb{Q}_v \text{ if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.} \tag{1.2}$$

There is a classical incoherent Eisenstein series $E(\mathsf{z}, s, \Delta(N)^2)$ (cf. §3.1) on the Siegel upper half space of genus 2,

$$\mathbb{H}_2 = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} \mid \mathsf{x} \in \mathrm{Sym}_2(\mathbb{R}), \mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}\}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series. The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(\mathsf{z}, 0, \Delta(N)^2) = 0$ by the incoherence.

We thus consider its central derivative

$$\partial\mathrm{Eis}(\mathsf{z},\Delta(N)^2) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E(\mathsf{z},s,\Delta(N)^2).$$

Associated to the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$, it has a decomposition into the central derivatives of the Fourier coefficients

$$\partial\mathrm{Eis}(\mathsf{z},\Delta(N)^2) = \sum_{T\in\mathrm{Sym}_2(\mathbb{Q})} \partial\mathrm{Eis}_T(\mathsf{z},\Delta(N)^2)$$

On the geometric side, there is a regular integral model of the modular curve $\mathcal{Y}_0(N)_\mathbb{C}$ over $\mathbb{Z}$ defined by Katz and Mazur: for a scheme S, the groupoid $\mathcal{Y}_0(N)(S)$ consists of objects $(E \xrightarrow{\pi} E')$ where E, E' are elliptic curves over S and π is a cyclic isogeny such that $\pi^\vee \circ \pi = N$. They proved that $\mathcal{Y}_0(N)$ is 2-dimensional regular flat Deligne–Mumford stack (cf. [3, Theorem 5.1.1]). Česnavičius constructed a moduli stack $\mathcal{X}_0(N)$ in [4] which serves as a "compactification" of $\mathcal{Y}_0(N)$. It is a proper regular flat 2-dimensional Deligne–Mumford stack that contains $\mathcal{Y}_0(N)$ as an open substack, so we can consider the arithmetic intersection theory on $\mathcal{X}_0(N)$ following the lines of Gillet in [11].

1.2.1 Special cycles of codimension 1

The key concept is that of a special cycle. For positive integers t, we define a closed substack of $\mathcal{Y}_0(N)$ as follows: For an object $(E \xrightarrow{\pi} E')$ of $\mathcal{Y}_0(N)(S)$, the stack $\mathcal{Z}(t,\Delta(N))$ parameterizes isogenies j between E and E' with $j^\vee \circ j = t$ and orthogonal to the cyclic isogeny π. It can be shown that the stack $\mathcal{Z}(t,\Delta(N))$ is a generalized Cartier divisor even on the stack $\mathcal{X}_0(N)$ when $t > 0$ (Proposition 5.1.7). For non-positive integers t, the special cycle is a weighted summation of the cuspidal divisor on $\mathcal{X}_0(N)$. Finally for all pairs (t,y) where t is an integer and y a positive real number, we define the modified special divisor $\mathcal{Z}^*(t,y,\Delta(N))$ in (5.4).

For every pair (t,y) such that t is a nonzero integer and $y > 0$, a green function $\mathfrak{g}(t,y,\Delta(N))$ of the divisor $\mathcal{Z}^*(t,y,\Delta(N))$ is constructed by Kudla [12, (12.21)] (see also [2, §5]). We will recall the construction in §5.2.1. Finally, we define the following element in the codimension 1 arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$ of $\mathcal{X}_0(N)$ (Definition 5.2.4):

$$\widehat{\mathcal{Z}}(t, y, \Delta(N)) = (\mathcal{Z}^*(t, y, \Delta(N)), \mathfrak{g}(t, y, \Delta(N))). \tag{1.3}$$

These elements $\widehat{\mathcal{Z}}(t, y, \Delta(N))$ are invariant under the Atkin–Lehner involution W_N of the stack $\mathcal{X}_0(N)$, i.e., $W_N^*\widehat{\mathcal{Z}}(t, y, \Delta(N)) = \widehat{\mathcal{Z}}(t, y, \Delta(N))$ (Lemma 5.2.5).

When $t = 0$, the definition of $\widehat{\mathcal{Z}}(0, y, \Delta(N))$ is slightly different from (1.3). Recall that there is a metrized Hodge line bundle $\widehat{\omega}_N$ on the stack $\mathcal{X}_0(N)$ (Example 4.3.2), however, this bundle is not invariant under the Atkin–Lehner involution W_N (Corollary 4.8.4). Therefore we consider the following element

$$\widehat{\omega} = -\widehat{\omega}_N - W_N^*\widehat{\omega}_N,$$

the element $\widehat{\omega}$ is clearly invariant under the Atkin–Lehner involution. We define

$$\widehat{\mathcal{Z}}(0, y, \Delta(N)) = \widehat{\omega} + (\mathcal{Z}^*(0, y, \Delta(N)), \mathfrak{g}(0, y, \Delta(N))) - (0, \log y). \tag{1.4}$$

Now for a rational number t which is not an integer, we simply define $\widehat{\mathcal{Z}}(t, y, \Delta(N)) = 0 \in \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$. Let $\tau = x + iy \in \mathbb{H}_1$, we consider the following generating series with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,

$$\widehat{\phi}_1(\tau) = \sum_{t\in\mathbb{Q}} \widehat{\mathcal{Z}}(t, y, \Delta(N)) \cdot q^t \tag{1.5}$$

where $q^t = e^{2\pi i t\tau}$.

Theorem 1.2.1. *Let N be a positive integer. The generating series $\widehat{\phi}_1$ is a nonholomorphic Siegel modular form of genus 1, weight $\frac{3}{2}$ and level $\Gamma_0(4N)$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$.*

Remark 1.2.2. When N is square-free, the above theorem has been proved by Du and Yang [2, Theorem 1.1]. Similar results for Shimura curves are proved by Kudla, Rapoport, and Yang [6, Theorem A]. We summarize briefly the proof here: the group splits into the following direct sum

[6, Proposition 4.1.2]:

$$\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N)) = J_0(N)(\mathbb{C}) \oplus (\mathbb{C}\cdot\widehat{\omega}\oplus \mathrm{Vert}) \oplus \mathrm{C}^\infty(\mathcal{X}_0(N)(\mathbb{C}))_0.$$

where Vert consists of elements $(Y,0)$ where Y is an irreducible component of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ for some prime number p, the vector space $\mathrm{C}^\infty(\mathcal{X}_0(N)(\mathbb{C}))_0$ consists of smooth functions on $\mathcal{X}_0(N)(\mathbb{C})$ orthogonal to the hyperbolic metric $[\Omega] = \frac{\mathrm{d}x\wedge\mathrm{d}y}{2\pi y^2}$. Correspondingly,

$$\widehat{\phi}_1 = \widehat{\phi}_1^{J_0(N)} + \widehat{\phi}_1^0 + \widehat{\phi}_1^\infty,$$

The modularity of $\widehat{\phi}_1^\infty$ is proved essentially by Du and Yang [2, Theorem 8.4], the modularity of $\widehat{\phi}_1^{J_0(N)}$ in the Mordell–Weil component is the main result of Gross–Kohner–Zagier [13], or Borcherds' modularity result [6, Theorem 4.5.1]. Therefore it suffices to prove the modularity of $\widehat{\phi}_1^0$. By the decomposition in [6, Proposition 4.1.4], the modularity of $\widehat{\phi}_1^0$ is equivalent to the modularity to the geometric degree $\deg(\widehat{\phi}_1)$ (Proposition 7.1.1), the pairings $\langle\widehat{\phi}_1,\widehat{\omega}\rangle$ (Corollary 7.1.10) and $\langle\widehat{\phi}_1,(Y,0)\rangle$ (Proposition 7.1.7).

1.2.2 Special cycles of codimension 2

We will special cycles in the codimension 2 arithmetic Chow group $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$. Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a symmetric matrix. If $T = \mathbf{0}_2$, for every positive definite matrix $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$, we define

$$\widehat{\mathcal{Z}}(\mathbf{0}_2,\mathsf{y}) = \widehat{\omega}\cdot\widehat{\omega} + (0, \log\det\mathsf{y}\cdot[\Omega]) \in \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)),$$

recall that here $[\Omega]$ is the restriction of the $(1,1)$-form $\frac{\mathrm{d}x\wedge\mathrm{d}y}{2\pi y^2}$ on $\mathbb{H}_1$ to $\mathcal{X}_0(N)_{\mathbb{C}}$.

If the rank of T is 1, there exists a matrix $g \in \mathrm{GL}_2(\mathbb{Z})$ such that ${}^t gTg = \mathrm{diag}\{0,t\}$ for some nonzero rational number t, let $\mathsf{y} = \mathrm{diag}\{y_1,y_2\}$ be a symmetric matrix with $y_1, y_2 > 0$, we define the following element in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$

$$\widehat{\mathcal{Z}}(T,\mathsf{y}) = \widehat{\mathcal{Z}}(t,y_2,\Delta(N))\cdot\widehat{\omega} - \left(0, \log y_1\cdot\delta_{\mathcal{Z}^*(t,y_2,\Delta(N))_{\mathbb{C}}}\right).$$

If T is nonsingular, a detailed definition of the element $\widehat{\mathcal{Z}}(T, \mathsf{y})$ can be found in §5.2.2, especially (5.9). Now for an element $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$, we consider the following generating series with coefficients in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$,

$$\widehat{\phi}_2(\mathsf{z}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \widehat{\mathcal{Z}}(T, \mathsf{y}) \cdot q^T$$

where $q^T = e^{2\pi i \,\mathrm{tr}\,(T\mathsf{z})}$.

There is an isomorphism $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) \simeq \mathbb{C}$ given by the arithmetic degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) \to \mathbb{C}$$

constructed by Kudla, Rapoport and Yang [6, §2.4] (see also (4.6)). Our main result is the following

Theorem 1.2.3 (Theorem 5.3.1)**.** *Let N be a positive integer. The generating series $\widehat{\phi}_2$ is a non-holomorphic Siegel modular form of genus 2 and weight $\frac{3}{2}$. More precisely, under the isomorphism $\widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) \xrightarrow{\sim} \mathbb{C}$*

$$\widehat{\phi}_2(\mathsf{z}) = \frac{\psi(N)}{24} \cdot \partial \mathrm{Eis}(\mathsf{z}, \Delta(N)^2),$$

here $\psi(N) = N \cdot \prod_{p|N} (1 + p^{-1})$.

Remark 1.2.4. When N is square-free, the above theorem has been proved by Sankaran, Shi, and Yang [1]. Similar results for Shimura curves are proved by Kudla, Rapoport, and Yang [6, Theorem B].

1.3 Key ingredients and the strategy of the proof

In this section, we mainly explain the proof of Theorem 1.2.3 in this section, while Theorem 1.2.1 will come as a byproduct.

We will prove Theorem 1.2.3 term-by-term, i.e., for all symmetric matrices $T \in \mathrm{Sym}_2(\mathbb{Q})$, we prove that for all $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$,

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathsf{y}) \cdot q^T = \frac{\psi(N)}{24} \cdot \partial \mathrm{Eis}_T(\mathsf{z}, \Delta(N)^2). \tag{1.6}$$

We refer to both sides of (1.6) as nonsingular (resp. singular) terms if the matrix T is nonsingular (resp. singular). The proof for nonsingular terms and singular terms are different in nature. We will explain them separately.

1.3.1 Nonsingular terms

We first focus on the case that T is nonsingular. If T is not positive definite, the equality (1.6) is proved in [1, §4.2], see also the work of Bruinier and Yang [14, Theorem 7.1]. The main difficulty lies in the case that T is positive definite. We prove (1.6) by combining the "local" arithmetic Siegel–Weil formula on the Rapoport–Zink space $\mathcal{N}_0(N)$ associated to the modular curve $\mathcal{X}_0(N)$ and the formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$ by the formal scheme $\mathcal{N}_0(N)$.

The local arithmetic Siegel–Weil formula with level N

Fix a prime number p. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let W be the integer ring of the completion of the maximal unramified extension of $\mathbb{Q}_p$.

On the geometry side, let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2. Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$, then $\mathrm{End}^0(\mathbb{X}) \simeq \mathbb{B}$ as quadratic spaces. The Rapoport–Zink space associated to $\mathcal{X}_0(N)$ is the following deformation space $\mathcal{N}_0(N)$: for a W-scheme S where p is locally nilpotent, and an element $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, the set $\mathcal{N}_0(N)(S)$ consists of elements $(X \xrightarrow{\pi} X')$ where X, X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (cf. §6.1.1), the morphism π is a cyclic isogeny deforming x_0 and $\pi^\vee \circ \pi = N$.

Let $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$ be the subspace of quasi-isogenies which are orthogonal to x_0. For an element $x \in \mathbb{W}$, there is a closed formal subscheme $\mathcal{Z}(x)$ of $\mathcal{N}_0(N)$ over which the quasi-isogeny x lifts to an isogeny. This is an example of special cycle (cf. Definition 6.1.6) on $\mathcal{N}_0(N)$. For a rank 2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M, then define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M)=\chi(\mathcal{N}_0(N),\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}}\mathcal{O}_{\mathcal{Z}(x_2)}).$$

This number is independent of the choice of the basis $\{x_1,x_2\}$ of M.

On the analytic side, for two integral quadratic $\mathbb{Z}_p$-lattices L and M. Let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, an $\mathbb{Z}_p$-scheme such that for any $\mathbb{Z}_p$-algebra R, $\mathrm{Rep}_{M,L}(R)=\mathrm{QHom}(L\otimes_{\mathbb{Z}_p}R,M\otimes_{\mathbb{Z}_p}R)$, where QHom denotes the set of quadratic module homomorphisms. The local density of integral representations is defined to be

$$\mathrm{Den}(M,L)=\lim_{d\to\infty}\frac{\#\mathrm{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d\cdot\dim(\mathrm{Rep}_{M,L})_{\mathbb{Q}_p}}}.$$

Let $H_2^+=\mathbb{Z}_p^2$ be the rank 2 quadratic $\mathbb{Z}_p$-lattice equipped with the quadratic form $q_{H_2^+}(x,y)=xy$. For positive integers $k\geq 0$, let $H_{2k}^+:=(H_2^+)^{\oplus n}$ be a rank $2k$ quadratic $\mathbb{Z}_p$-lattice. For a $\mathbb{Z}_p$-lattice $M\subset\mathbb{W}$ of rank 2, define the local density of M with level N to be the polynomial $\mathrm{Den}_{\delta_p(N)}(X,M)$ such that for all $k\geq 0$,

$$\mathrm{Den}_{\delta_p(N)}(X,M)\,\big|_{X=p^{-k}}=\begin{cases}\dfrac{\mathrm{Den}(\delta_p(N)\oplus H_{2k}^+,M)}{\mathrm{Nor}^+(p^{-k},1)}, & \text{when } p\mid N;\\[2ex] \dfrac{\mathrm{Den}(\delta_p(N)\oplus H_{2k}^+,M)}{\mathrm{Nor}^{(N,p)_p}(p^{-k},2)}, & \text{when } p\nmid N.\end{cases}\tag{1.7}$$

where $(\cdot,\cdot)_p$ is the Hilbert symbol at p, the polynomials $\mathrm{Nor}^{\varepsilon}(X,n)$ are normalizing factors defined in Definition 2.3.1. Then $\mathrm{Den}_{\delta_p(N)}(1,M)=0$ since M can't be isometrically embedded into the quadratic lattice $\Delta_p(N)$, we define the derived local density of M with level N to be

$$\partial\mathrm{Den}_{\delta_p(N)}(M)=-\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1}\mathrm{Den}_{\delta_p(N)}(X,M).$$

The local arithmetic Siegel–Weil formula with level N is an exact identity between the two integers just defined.

Theorem 1.3.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\mathrm{Den}_{\delta_p(N)}(M).$$

We refer to $\mathrm{Int}_{\mathcal{N}_0(N)}(M)$ as the geometric side of the identity (related to the geometry of Rapoport–Zink spaces and Shimura varieties) and $\partial\mathrm{Den}_{\delta_p(N)}(M)$ the analytic side (related to the derivatives of Eisenstein series and L-functions).

Formal uniformization

For a prime number p, let $\mathcal{X}_0(N)_p^{\mathrm{ss}}$ be the supersingular locus of the stack $\mathcal{X}_0(N)$, i.e., those $\mathbb{F}$-points of $\mathcal{X}_0(N)$ which is isogenous to a supersingular elliptic curve. Let B be the unique quaternion algebra over $\mathbb{Q}$ which ramifies exactly at p and ∞. Let $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_p^{\mathrm{ss}})}$ be the completion of the stack $\mathcal{X}_0(N)$ along the closed substack $\mathcal{X}_0(N)_p^{\mathrm{ss}}$. Let $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ be the group $\prod_{v\neq\infty,p}\Gamma_0(N)(\mathbb{Z}_v)$. We have the following formal uniformization theorem of the stack $\mathcal{X}_0(N)$.

Proposition 1.3.2. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_p^{\mathrm{ss}})} \underset{\sim}{\overset{\Theta_{\mathcal{X}_0(N)}}{\longrightarrow}} B^\times(\mathbb{Q})_0\backslash[\mathcal{N}_0(N)\times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)]$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

The proposition was previously known only in the case that N is odd and square-free (see the work of Kim [15, Theorem 4.7] for the case that $p \nmid N$, and the work of Oki [16, Theorem 6.1] for the case that $p \mid N$). As a corollary, let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T,\varphi)$ be the completion of $\mathcal{Z}(T,\varphi)$ along its supersingular locus $\mathcal{Z}^{\mathrm{ss}}(T,\varphi) \coloneqq \mathcal{Z}(T,\varphi)\times_{\mathcal{X}_0(N)}\mathcal{X}_0(N)_p^{\mathrm{ss}}$. Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that:

1. It is positive definite at ∞;

2. For finite prime $l \neq p$, $\Delta(N)^{(p)}\otimes\mathbb{Q}_l$ is isometric to $\Delta_l(N)\otimes\mathbb{Q}_l$;

3. $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

For a pair of vectors $\boldsymbol{x} = (x_1, x_2) \in \left(\Delta(N)^{(p)}\right)^2$, let $T(\boldsymbol{x}) = (\frac{1}{2}(x_i, x_j))$ be the inner product matrix. We have the following formal uniformization theorem of the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$.

Corollary 1.3.3. *Let* $T \in \mathrm{Sym}_2(\mathbb{Q})$ *be a nonsingular symmetric matrix, and* $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. *Let* $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ *be a* T*-admissible Schwartz function. Let* $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_p^{\mathrm{ss}})})$ *be the Grothendieck group of coherent sheaves of* $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_p^{\mathrm{ss}})}}$*-modules. Then we have the following identity in* $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_p^{\mathrm{ss}})})$,

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta_{\mathcal{X}_0(N)}^{-1}(\mathcal{Z}(\boldsymbol{x}), g),$$

where $B_{\boldsymbol{x}}^\times \subset B^\times$ *is the stabilizer of* $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

1.3.2 Singular terms

Next we consider the case that T is singular. For the analytic side of (1.6), we relate the singular Fourier coefficients of Eisenstein series $E(\mathsf{z}, s, \Delta(N)^2)$ to the nonsingular Fourier coefficients of another Eisenstein series of lower genus. For the geometric side, computing the arithmetic degrees of the special cycle $\mathcal{Z}(T, \mathsf{y})$ amounts to computing the height of a certain codimension 1 cycle on $\mathcal{X}_0(N)$ with respect to the Hodge line bundle on $\mathcal{X}_0(N)$.

The singular coefficients of Eisenstein series

Our primary goals on the analytic side are:

(a). When the rank of T is 1, we relate the singular Fourier coefficients of the Eisenstein series $E_T(\mathsf{z}, s, \Delta(N)^2)$ to nonsingular Fourier coefficients of another Eisenstein series of lower genus.

(b). When $T = 0$, we compute the exact value of $\partial\mathrm{Eis}_{\mathbf{0}_2}(\mathsf{z}, \Delta(N)^2)$.

Proposition 1.3.4. *Let* $T \in \mathrm{Sym}_2(\mathbb{Q})$ *be a* 2×2 *singular matrix.*

- *If the rank of* T *is 1 and can be diagonalized to* $\mathrm{diag}\{0, t\}$ *for some* $t \in \mathbb{Q}^\times$ *under* $\mathrm{GL}_2(\mathbb{Q})$*. Let* $\mathsf{y} = \mathrm{diag}\{y_1, y_2\}$ *be a positive definite symmetric matrix, then for all complex numbers* k*, we have*

$$\begin{aligned} E_T(i\mathsf{y}, k, \Delta(N)^2) &= y_1^{k/2} E_t(iy_2, \frac{1}{2} + k, \Delta(N)) \\ &+ y_1^{-k/2} \cdot \frac{k-1}{k+1} \cdot N^{-k} \cdot \frac{\Lambda(2-2k)}{\Lambda(2+2k)} \cdot \prod_{p|N} \beta_p(k) \cdot E_t(iy_2, \frac{1}{2} - k, \Delta(N)), \end{aligned} \tag{1.8}$$

where $\beta_p(s)$ *is a rational function in* p^{-s}*.*

- *If* $T = \mathbf{0}_2$*, let* $n_p = v_p(N)$*, we have*

$$\partial \mathrm{Eis}_{\mathbf{0}_2}(\mathsf{z}, \Delta(N)^2) = \log\det(\mathsf{y}) + 2 - 4 \cdot \frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N} \frac{-n_p p^{n_p+1} + 2p^{n_p} + n_p p^{n_p-1} - 2}{p^{n_p-1}(p^2-1)} \cdot \log p. \tag{1.9}$$

Vertical components of the Hodge line bundle

The primary goal on the geometric side is to compute the height pairings of (metrized) special divisors $\widehat{\mathcal{Z}}(t, y, \Delta(N))$ and the modified metrized Hodge line bundle $\widehat{\omega}$, and the self-intersection numbers of the metrized line bundle $\widehat{\omega}$.

The main difficulty comes from the fact that the bundle $\widehat{\omega}$ contains vertical components. These components should be a linear combination of irreducible components of the reduction mod p of the stack $\mathcal{X}_0(N)$ for prime numbers p. We need to know the explicit multiplicities of these irreducible components appearing in $\widehat{\omega}$ and their intersections.

We first study the special fiber of the stack $\mathcal{X}_0(N)$. Let p be a prime number, by the works of Katz and Mazur, if $n_p = v_p(N) \geq 0$ is the p-adic valuation of the number N, then $\mathcal{X}_0(N)_p := \mathcal{X}_0(N) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ has $n_p + 1$ irreducible components $\mathcal{X}_p^a(N)$ and they meet each other at every supersingular point, where the index a satisfies that $-n_p \leq a \leq n_p$ and has the same parity with n. Moreover, every component $\mathcal{X}_p^a(N)$ has two natural morphisms to the stack $\mathcal{X}_0(Np^{-n_p})_p$, one of them is an isomorphism while the other one is finite flat of degree $p^{|a|}$ (see Theorem 4.6.3). Later in §4.7, we compute the intersection numbers between these irreducible components based on the explicit local equations of these components at supersingular points obtained by Katz and Mazur [3, Theorem 13.4.7] (we also give a new proof of this result, see Corollary 6.2.10).

After that, in §4.8 we use an explicit rational section Δ_N of the bundle $\widehat{\omega}_N^{\otimes 12\varphi(N)}$ to give an explicit expression of this bundle in the Chow group $\mathrm{CH}^1(\mathcal{X}_0(N))$ by computing the element $\mathrm{div}(\Delta_N) \in \mathrm{CH}^1(\mathcal{X}_0(N))$ (Theorem 4.8.3), this idea originates from the work of Du and Yang [2], where they consider the case that N is squarefree.

Our idea of computing the element $\mathrm{div}(\Delta_N)$ for general N comes from the observation that there is a correspondence between cusps and the special fibers of the stack $\mathcal{X}_0(N)$. Let's explain this in the specific case that $N = p^n$ for some integer $n \geq 0$. There is a cuspidal divisor $\mathsf{Cusp}(\mathcal{X}_0(p^n))$ on the stack $\mathcal{X}_0(p^n)$, it is a disjoint union of $n+1$ connected components (Proposition 4.5.1),

$$\mathsf{Cusp}(\mathcal{X}_0(p^n)) = \coprod_{\substack{-n\leq a\leq n \\ a\equiv n \bmod 2}} \mathsf{C}^a(p^n).$$

Note that the index set is exactly the same as the index set for the irreducible components of the stack $\mathcal{X}_0(p^n)_p$! Actually we will prove that the connected component $\mathsf{C}^a(p^n)$ pulls back to the cusp of the curve $\mathcal{X}_p^a(p^n)$ (Proposition 4.6.8). Then we pick a specific cusp lying in the component $\mathsf{C}^a(p^n)$ and consider the local expansion of the section Δ_{p^n} around this point, the multiplicity of p appearing in the local expansion gives the multiplicity of the vertical component of $\mathcal{X}_p^a(p^n)$ in the element $\mathrm{div}(\Delta_{p^n})$. For general positive integers N, the final result is

$$\mathrm{div}(\Delta_N) = \psi(N)\varphi(N)P_\infty(N) + \sum_{p|N} f_p(N),$$

where $P_\infty(N)$ is the connected component of the cuspidal divisor containing the cusp ∞, and for any $p|N$, $f_p(N)$ is the following vertical divisor,

$$f_p(N) = 12p^{n_p-1}\varphi(N_p) \sum_{\substack{-n_p\leq a<n_p \\ a\equiv n_p \bmod 2}} \left(\frac{1-p}{2}(n_p-a)-1\right)\varphi(p^{(n_p-|a|)/2}) \cdot \mathcal{X}_p^a(N).$$

This expression is the main tool to compute the self intersection of the modified Hodge bundle $\widehat{\omega}$ and the intersection pairing $\widehat{\mathcal{Z}}(t, y, \Delta(N)) \cdot \widehat{\omega}$.

1.3.3 Strategy: Difference formulas

Our strategy of proving the "local" arithmetic Siegel–Weil formula (Theorem 1.3.1), formal uniformization (Proposition 1.3.2) and the singular Fourier coefficients (Proposition 1.3.4) is combining the geometric and analytic difference formulae.

Geometric difference formulae

Let $\mathcal{N}$ be the following deformation functor: for a W-scheme S where p is locally nilpotent, the set $\mathcal{N}(S)$ consists of elements (X, X') where both X and X' are deformations over S of $\mathbb{X}$ with certain restrictions on polarizations (cf. §6.1.1). For a nonzero integral element $x \in \mathbb{B}$, i.e., $0 \leq \nu_p(x^\vee \circ x) < \infty$, there is a closed formal subscheme $\mathcal{Z}^\sharp(x)$ of $\mathcal{N}$ over which the quasi-isogeny x lifts to an isogeny. This is an example of special cycle (cf. Definition 6.1.2) on $\mathcal{N}$.

For a rank 3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L, then define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\mathrm{Int}^\sharp(L) = \chi(\mathcal{N}, O_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{O_\mathcal{N}} O_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{O_\mathcal{N}} O_{\mathcal{Z}^\sharp(x_3)}).$$

This number is independent of the choice of the basis $\{x_1, x_2, x_3\}$ of the lattice L.

The special cycle $\mathcal{Z}^\sharp(x)$ is cut out by a single element $f_x \in \mathfrak{m} = (p, t_1, t_2) \subset W[[t_1, t_2]]$, and when $\nu_p(x^\vee \circ x) \geq 2$, $f_{p^{-1}x}|f_x$. We define $d_x = f_x/f_{p^{-1}x} \in W[[t_1, t_2]]$ when $\nu_p(x^\vee \circ x) \geq 2$,

$d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1. The following divisor

$$\mathcal{D}(x) \coloneqq \operatorname{Spf} W[[t_1, t_2]]/d_x,$$

is called the difference divisor associated to x (cf. Definition 6.2.4), which is originally introduced by Terstiege in [17].

Fix $x_0 \in \mathbb{B}$ such that $x_0^\vee \circ x_0 = N$, recall that we have defined the deformation function $\mathcal{N}_0(N)$. In Theorem 6.2.6, we prove that $\mathcal{N}_0(N)$ is identified with the difference divisor $\mathcal{D}(x_0)$, i.e., there is an isomorphism of formal schemes,

$$\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N). \tag{1.10}$$

Let $x_0^{\mathrm{univ}} : X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ be the universal isogeny deforming x_0 over the special cycle $\mathcal{Z}^\sharp(x_0)$. We will prove that the base change of x_0^{univ} to $\mathcal{D}(x_0)$ is cyclic, therefore there is a natural morphism $\mathcal{D}(x_0) \to \mathcal{N}_0(N)$. The natural morphism is an isomorphism because both sides of the morphism are closed formal subschemes of $\mathcal{N}$ and are represented by 2-dimensional regular local rings. The identification of $\mathcal{D}(x_0)$ and $\mathcal{N}_0(N)$ implies the following difference formula of local arithmetic intersection numbers,

Theorem 1.3.5. *For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0). \tag{1.11}$$

We refer to formulae (1.10) and (1.11) as the geometric difference formulae. These formulae help us reduce the computations of geometric quantities such as intersection numbers and height pairings on the modular curve $\mathcal{X}_0(N)$ to that on the surface $\mathcal{H} \coloneqq \mathcal{X}_0(1) \times_{\mathbb{Z}} \mathcal{X}_0(1)$, which is 1 dimension higher but geometrically much simpler than $\mathcal{X}_0(N)$.

Moreover, we give a new proof of the description of the local ring of $\mathcal{X}_0(N)$ at a supersingular point [3, Theorem 13.4.6, Theorem 13.4.7] based on the geometric difference formula (1.10). By the Windows theory developed by Zink in [18], if $n_p := \nu_p(N) \geq 1$, we prove that the special fiber $\mathcal{Z}(x_0)_p$ of $\mathcal{Z}(x_0)$ has the following explicit description (cf. Theorem 6.2.9, Corollary 6.2.10),

$$\mathcal{Z}(x_0)_p \simeq \operatorname{Spf} \mathbb{F}[[t_1, t_2]] \Big/ \left(\prod_{\substack{a+b=n_p \\ a,b \geq 0}} (t_1^{p^a} - t_2^{p^b}) \right).$$

Based on the isomorphism (1.10): $\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N)$, the special fiber $\mathcal{N}_0(N)_p$ of $\mathcal{N}_0(N)$ can be described by

$$\mathcal{N}_0(N)_p \simeq \operatorname{Spf} \mathbb{F}[[t_1, t_2]] \Big/ \left((t_1 - t_2^{p^{n_p}}) \cdot (t_2 - t_1^{p^{n_p}}) \cdot \prod_{\substack{a+b=n_p \\ a,b \geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \right).$$

Analytic difference formulae

For a rank 3 quadratic $\mathbb{Z}_p$-lattice $L \subset \mathbb{B}$, define the local density of L to be the polynomial $\operatorname{Den}(X, L) \in \mathbb{Z}[X]$ such that for all $k \geq 0$,

$$\operatorname{Den}(X, L)\,\big|_{X=p^{-k}} = \frac{\operatorname{Den}(H_{2k+4}^{+}, L)}{\operatorname{Nor}^{+}(p^{-k}, 3)}.$$

then $\operatorname{Den}(1, L) = 0$ since L can't be isometrically embedded into the quadratic lattice H_4^{+}, we define the derived local density of L to be

$$\partial\operatorname{Den}(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \operatorname{Den}(X, L).$$

Theorem 1.3.6. *For a rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,*

$$\operatorname{Den}_{\delta_p(N)}(X, M) = \operatorname{Den}(X, M \obot \mathbb{Z}_p \cdot x_0) - X^2 \cdot \operatorname{Den}(X, M \obot \mathbb{Z}_p \cdot p^{-1}x_0). \tag{1.12}$$

Since the lattice $M \obot \mathbb{Z}_p \cdot x_0$ *can't be isometrically embedded into the lattice* H_4^+*,*

$$\partial\mathrm{Den}_{\delta_p(N)}(M) = \partial\mathrm{Den}(M \obot \mathbb{Z}_p \cdot x_0) - \partial\mathrm{Den}(M \obot \mathbb{Z}_p \cdot p^{-1}x_0). \tag{1.13}$$

The theorem is proved in a more general form in Theorem 2.5.8. We refer to formulae (1.12) and (1.13) as the analytic difference formulae.

We will explain briefly how do we deduce Proposition 1.3.4 from the above analytic difference formulae. As is well-known, the Fourier coefficients of Eisenstein series are (linear combinations) of Whittaker functions which are essentially products of local representation densities of quadratic lattices. The formula (1.12) roughly says that for any number $l \in \mathbb{Z}_p$ and any quadratic lattice M, the following two local densities are related

$$\mathrm{Den}(\langle -l\rangle \obot H_{2k+2}^+, M) \leftrightarrow \mathrm{Den}(H_{2k+4}^+, M \obot \langle l\rangle). \tag{1.14}$$

Taking the limit $\nu_p(l) \longrightarrow \infty$, the right-hand side of (1.14) converges to the Whittaker function with singular coefficient, while the left-hand side of (1.14) converges to $\mathrm{Den}(H_{2k+2}^+, M)$ (Lemma 2.4.1), i.e., we have the following relation,

$$\mathrm{Den}(H_{2k+2}^+, M) = \lim_{\nu_p(l)\to\infty} \mathrm{Den}(H_{2k+4}^+, M \obot \langle l\rangle). \tag{1.15}$$

When the rank of T is 1, we assume $T = \mathrm{diag}\{0, t\}$ with $t \neq 0$ for simplicity, let $\mathsf{y} = \mathrm{diag}\{y_1, y_2\}$ be a positive definite symmetric matrix, then there is a well-known relation (see [19, Lemma 5.4])

$$E_T(i\mathsf{y}, s, \Delta(N)^2) = y_1^{s/2} y_2^{-3/4} W_t(g_{iy_2}, s + \frac{1}{2}, \Delta(N)) + (y_1y_2)^{-3/4} W_T(g_{i\mathsf{y}}, s, \Delta(N)^2),$$

where both of the terms $W_t(g_{iy_2}, s+\frac{1}{2}, \Delta(N))$ and $W_T(g_{i\mathsf{y}}, s, \Delta(N)^2)$ are product of local Whittaker functions (see (3.1) and (3.5) for the precise definition).

Since $t \neq 0$, the term $W_t(g_{iy_2}, s + \frac{1}{2}, \Delta(N))$ is already the nonsingular Fourier coefficient of an Eisenstein series of the lower genus. Let's now consider the term $W_T(g_{iy}, s, \Delta(N)^2)$, which is the product of local Whittaker functions $W_{T,v}$ over all the places v of $\mathbb{Q}$. The relation between $W_{T,v}$ and $W_{t,v}$ is sketched in the following way: Let p be a finite prime, for any integer m, let $T_m = \mathrm{diag}\{p^m, t\}$ be a 2×2 nonsingular matrix. By the well-known relation between Whittaker functions and local densities (Proposition 3.2.3), we have

$$W_{T,p}(1, k, 1_{\delta_p(N)^2}) = \lim_{m \to \infty} W_{T_m,p}(1, k, 1_{\delta_p(N)^2}) \leftrightarrow \mathrm{Den}(\delta_p(N) \oplus H_{2k}^+, \langle t \rangle \oplus \langle p^m \rangle).$$

Applying (1.14) twice,

$$\begin{aligned} \mathrm{Den}(\delta_p(N) \oplus H_{2k}^+, \langle t \rangle \oplus \langle p^m \rangle) &\overset{l=N}{\leftrightarrow} \mathrm{Den}(H_{2k+4}^+, \langle t \rangle \oplus \langle p^m \rangle \oplus \langle N \rangle) \\ &\overset{l=p^m}{\leftrightarrow} \mathrm{Den}(\langle -p^m \rangle \oplus H_{2k+2}^+, \langle t \rangle \oplus \langle N \rangle). \end{aligned} \tag{1.16}$$

Taking limit $m \to \infty$, the formula (1.15) tells us that

$$\lim_{m \to \infty} \mathrm{Den}(\langle -p^m \rangle \oplus H_{2k+2}^+, \langle t \rangle \oplus \langle N \rangle) = \mathrm{Den}(H_{2k+2}^+, \langle t \rangle \oplus \langle N \rangle). \tag{1.17}$$

Applying (1.14) again, we get

$$\mathrm{Den}(H_{2k+2}^+, \langle t \rangle \oplus \langle N \rangle) \overset{l=N}{\leftrightarrow} \mathrm{Den}(\delta_p(N) \oplus H_{2k-2}^+, \langle t \rangle). \tag{1.18}$$

The last term $\mathrm{Den}(\delta_p(N) \oplus H_{2k-2}^+, \langle t \rangle)$ is clearly related to $W_{t,p}(1, k - \frac{1}{2}, 1_{\delta_p(N)})$ by Proposition 3.2.3. Therefore we have successfully built the bridge from $W_{T,v}$ and $W_{t,v}$ for a finite prime v.

However, we warn the readers that all the symbols "↔" are not exact equal, it just means that there are some relations between the two sides of the symbol. Detailed calculations based on the principles above are given in §3.5 and our final result is Proposition 1.3.4.

1.4 Structure of the thesis

In Part I, we focus on the Eisenstein series and local densities:

- In Chapter 2, we introduce notations on quadratic lattices and local densities (§2.1-2.4), then we prove the difference formula of local densities (§2.5). Then we establish a difference formula of local density (Theorem 1.3.6) and apply it to cases that we are interested (§2.5).
- In Chapter 3, we introduce the Siegel Eisenstein series and Whittaker functions (§3.1-3.4). We study the singular Fourier coefficients of Eisenstein series and prove Proposition 1.3.4 in §3.5.

In Part II, we study the global and local geometry of the modular curve $\mathcal{X}_0(N)$:

- In Chapter 4, we review the $\Gamma_0(N)$-level structures of elliptic curves and the integral model $\mathcal{X}_0(N)$ defined by Katz and Mazur (§4.1-4.4). We study the cusps and (intersections of) special fibers of this integral model (§4.5-4.8).
- In Chapter 5, we introduce and compare the notions of (arithmetic) special cycles on $\mathcal{X}_0(N)$ and the surface $\mathcal{Y}_0(1) \times \mathcal{Y}_0(1)$ in §5.1. We give detailed definitions of the element $\widehat{\mathcal{Z}}(T, \mathsf{y})$ in §5.2.
- In Chapter 6, we establish the geometric difference formulae (especially the isomorphism (1.10)) (§6.1-6.2) and prove the identity (1.6) when the matrix T is nonsingular by combining the geometric and analytic difference formulae (1.11), (1.13) (§6.3).
- In Chapter 7, we compute the arithmetic intersection pairings between special cycles of codimension 1 and the Hodge bundle by (a global variant of) the geometric difference formula and relate it to the Eisenstein series (§7.1). This leads to the proof of the modularity of (1.5) and the identity (1.6) when the matrix T is singular (§7.2).

Part I

Analytic side

Chapter 2: Quadratic lattices and local densities

2.1 Notations on quadratic lattices

Let p be a prime number. Let F be a nonarchimedean local field of residue characteristic p, with ring of integers O_F, residue field $\kappa = \mathbb{F}_q$ of size q, and uniformizer π. Let $\nu_\pi : F \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the valuation on F and $|\cdot| : F \longrightarrow \mathbb{R}_{\geq 0}$ be the normalized absolute value on F. Let $(\cdot,\cdot)_F$ be the Hilbert symbol on the local field F. Let $\chi_F = \left(\frac{\cdot}{\pi}\right)_F : F^\times/(F^\times)^2 \longrightarrow \{\pm 1, 0\}$ be the (extended) quadratic residue symbol.

A quadratic space (U, q_U) over F is a finite dimensional vector space U over F equipped with a quadratic form $q_U : U \longrightarrow F$, the quadratic form q_U induces a symmetric bilinear form given by

$$\begin{aligned} (\cdot,\cdot) : U \times U &\longrightarrow F, \\ (x, y) &\longmapsto q_U(x+y) - q_U(x) - q_U(y). \end{aligned} \tag{2.1}$$

An isometry between two quadratic spaces (U, q_U) and $(U', q_{U'})$ is a linear isomorphism $\phi : U \longrightarrow U'$ preserving quadratic forms, i.e., $q_{U'}(\phi(x)) = q_U(x)$ for any $x \in U$. In that case, we say U and U' are isometric.

A quadratic lattice (L, q_L) is a finite free O_F-module equipped with a quadratic form $q_L : L \longrightarrow F$. The quadratic form q_L also induces a symmetric bilinear form $L \times L \xrightarrow{(\cdot,\cdot)} F$ by similar formula (2.1). Let $L^\vee = \{x \in L \otimes_{O_F} F : (x, L) \subset O_F\}$. We say a quadratic lattice is integral if $q_L(x) \in O_F$ for all $x \in L$, is self-dual if it is integral and $L = L^\vee$.

Let's assume that $\dim_F U = n$ and the symmetric bilinear form $(\cdot,\cdot)$ is nondegenerate. Let $\{x_i\}_{i=1}^n$ be a basis of U, and $t_{ij} = \frac{1}{2}(x_i, x_j)$, we define the discriminant of the quadratic space U to be:

$$\mathrm{disc}(U) = (-1)^{n(n-1)/2}\det\big((t_{ij})\big) \in F^\times/(F^\times)^2.$$

If $\{x_i\}_{i=1}^n$ is an orthogonal basis of U then $t_{ij} = 0$ if $i \neq j$ and $t_{ii} \neq 0$ by the nondegeneracy of $(\cdot,\cdot)$. The Hasse invariant of the quadratic space U is

$$\epsilon(U) = \prod_{i<j}(t_{ii}, t_{jj})_F,$$

For a quadratic lattice L, we use $\mathrm{disc}(L)$ and $\epsilon(L)$ to denote the corresponding invariants on the quadratic space $L_F = L \otimes_{O_F} F$. Recall that when p is odd, quadratic spaces U over F are classified by the following three invariants:

$$\dim_F U, \quad \mathrm{disc}(U), \quad \epsilon(U).$$

i.e., two quadratic spaces U and U' are isometric if and only if the above three invariants for U and U' are the same.

For a quadratic space U, define $\chi_F(U) \coloneqq \chi_F(\mathrm{disc}(U))$. For a quadratic lattice L, define $\chi_F(L) = \chi_F(L \otimes_{O_F} F)$. When p is odd, the quadratic space U admits a self-dual sub-lattice if and only if $\epsilon(U) = +1$ and $\chi_F(U) \neq 0$, we will use H_k^ε to denote the unique self-dual lattice of rank k and

$$\chi_F(H_k^\varepsilon) = \varepsilon.$$

When $p = 2$, let $H_{2n}^+ = (H_2^+)^{\oplus n}$ be a self-dual lattice of rank $2n$, where the quadratic form on $H_2^+ = O_F^2$ is given by $(x, y) \in O_F^2 \mapsto xy$.

Example 2.1.1. Let $N \in O_F$. Let $\delta_F(N)$ be the following rank 3 quadratic lattice over O_F,

$$\delta_F(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : \ a, b, c \in O_F \right\}.$$

equipped with the quadratic form induced by $x \mapsto \det(x)$. Under the following basis of $\delta_F(N)$,

$$e_1 = \begin{pmatrix} -N & \\ & 1 \end{pmatrix}, \; e_2 = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \; e_3 = \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

the quadratic form can be represented by the following symmetric matrix,

$$T = \begin{pmatrix} -N & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

therefore $\mathrm{disc}(\delta_F(N)) = -N/4$, and $\epsilon(\delta_F(N)) = (N, -1)_F$. Moreover,

$$\delta_F(N)^\vee = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : \; a \in \frac{1}{2N}O_F, b, c \in O_F \right\}.$$

therefore $\delta_F(N)^\vee / \delta_F(N) \simeq O_F/2N$.

Throughout this article, we will mainly focus on the case that $F = \mathbb{Q}_p$. In this case, we simply use $\delta_p(N)$ to denote the lattice $\delta_{\mathbb{Q}_p}(N)$ (as we did in the introduction).

2.2 Local densities

Definition 2.2.1. Let L, M be two quadratic O_F-lattices. Let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, an O_F-scheme such that for any O_F-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{O_F} R, M \otimes_{O_F} R),$$

where QHom denotes the set of injective module homomorphisms which preserve the quadratic forms. The local density of integral representations is defined to be

$$\mathrm{Den}(M,L)=\lim_{d\to\infty}\frac{\#\mathrm{Rep}_{M,L}(O_F/\pi^d)}{q^{d\cdot\dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.2. If L, M have rank n, m respectively and the generic fiber $(\mathrm{Rep}_{M,L})_F \neq \varnothing$, then $n \le m$ and

$$\dim(\mathrm{Rep}_{M,L})_F=\dim\mathrm{O}_m-\dim\mathrm{O}_{m-n}=\tbinom{m}{2}-\tbinom{m-n}{2}=mn-\frac{n(n+1)}{2}.$$

Definition 2.2.3. Let L, M be two quadratic O_F-lattices. Let $\mathrm{PRep}_{M,L}$ be the O_F-scheme of primitive integral representations such that for any O_F-algebra R,

$$\begin{aligned}&\mathrm{PRep}_{M,L}(R)\\&=\{\phi\in\mathrm{Rep}_{M,L}(R):\phi\text{ is an isomorphism between }L_R\text{ and a direct summand of }M_R\}\end{aligned}$$

where L_R (resp. M_R) is $L\otimes_{O_F}R$ (resp. $M\otimes_{O_F}R$). The primitive local density is defined to be

$$\mathrm{Pden}(M,L)=\lim_{d\to\infty}\frac{\#\mathrm{PRep}_{M,L}(O_F/\pi^d)}{q^{d\cdot\dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.2.4. For any positive integer d, a homomorphism $\phi\in\mathrm{Rep}_{M,L}(O_F/\pi^d)$ or $\mathrm{Rep}_{M,L}(O_F)$ is primitive if and only if $\overline{\phi}:=\phi \bmod \pi\in\mathrm{PRep}(O_F/\pi)$, which is equivalent to $\dim_{\mathbb{F}_q}(\phi(L)+\pi\cdot M)/\pi\cdot M=\mathrm{rank}_{O_F}(L)$.

Example 2.2.5. For a nonzero element $N\in O_F$, we use $(\langle N\rangle, q_{\langle N\rangle})$ to denote the rank 1 quadratic lattice over O_F with a O_F-generator l_N such that $q_{\langle N\rangle}(l_N)=N$. When p odd, it has been calculated explicitly that for any $N \in O_F$ (cf. [10, (3.3.2.1)]),

$$\mathrm{Pden}(H_k^{\varepsilon}, \langle N\rangle) = \begin{cases} 1 - q^{1-k}, & \text{when } k \text{ is odd and } \pi \mid N; \\ 1 + \varepsilon\chi_F(N) q^{(1-k)/2}, & \text{when } k \text{ is odd and } \pi \nmid N; \\ (1 - \varepsilon q^{-k/2})(1 + \varepsilon q^{1-k/2}), & \text{when } k \text{ is even and } \pi \mid N; \\ 1 - \varepsilon q^{-k/2}, & \text{when } k \text{ is even and } \pi \nmid N. \end{cases} \tag{2.2}$$

When $p = 2$, the same formula makes sense and holds true only in the case that k is even and $\varepsilon = +1$.

Lemma 2.2.6. *Let H be a self-dual quadratic lattice. Let L be a quadratic O_F-lattice, k is any positive integer, then we have the following stratification,*

$$\mathrm{Rep}_{H,L}(O_F) = \bigsqcup_{L \subset L' \subset L'^{\vee}} \mathrm{PRep}_{H,L'}(O_F).$$

Proof. This is proved by Cho and Yamauchi in [20, (3.1)]. □

2.3 Functional equations of local density functions

Definition 2.3.1. Let $n \geq 0$, for $\varepsilon \in \{\pm 1\}$, we define the normalizing factors to be

$$\mathrm{Nor}^{\varepsilon}(X, n) = \left(1 - \frac{1 + (-1)^{n+1}}{2} \cdot \varepsilon q^{-(n+1)/2} X\right) \prod_{1 \leq i < (n+1)/2} (1 - q^{-2i} X^2).$$

Lemma 2.3.2. *Let L be a quadratic lattice of rank n. Let $N \in O_F$. When p is odd, there exist polynomials* $\mathrm{Den}(X, L)$ *and* $\mathrm{Den}^{\flat}(X, L)$*, such that for positive integers k,*

$$\mathrm{Den}(X, L)\big|_{X=q^{-k}} = \frac{\mathrm{Den}(H^{+}_{2k+n+1}, L)}{\mathrm{Nor}^{+}(q^{-k}, n)}, \quad \mathrm{Den}^{\flat}(X, L)\big|_{X=q^{-k}} = \frac{\mathrm{Den}(H^{+}_{2k+n}, L)}{\mathrm{Nor}^{+}(q^{-k}, n-1)} \cdot (1 - \chi(L) q^{-k}).$$

We define the derived local density of L to be

$$\partial\mathrm{Den}(L) = -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1}\mathrm{Den}(X, L).$$

When $p = 2$ and n is odd, the polynomial $\mathrm{Den}^{+}(X, L)$ *with the same evaluation formulas at* $X = q^{-k}$ *exist, the derived local density* $\partial\mathrm{Den}(L)$ *of L is defined similarly.*

Proof. This is a combination of [10, Lemma 3.3.2, Definition 3.4.1, Definition 3.5.2] □

We have the following functional equation for $\mathrm{Den}(X, L)$.

Theorem 2.3.3. *When the residue characteristic of O_F is not 2. Let L be a quadratic lattice of rank n. Then*

$$\begin{aligned}\mathrm{Den}(X, L) &= w(L)\cdot X^{\mathrm{val}(L)}\cdot \mathrm{Den}\left(X^{-1}, L\right),\\ \mathrm{Den}^{\flat}(X, L) &= (q^{1/2}X)^{2[\frac{\mathrm{val}(L)}{2}]}\cdot \mathrm{Den}^{\flat}\left((qX)^{-1}, L\right).\end{aligned}$$

where $\mathrm{val}(L)$ *is the π-adic valuation of the moment matrix of L under an O_F-basis, and the sign of the functional equation is equal to*

$$w(L) \coloneqq (\det L, -(-1)^{\binom{n+1}{2}})_F\cdot \epsilon(L)\in\{\pm 1\}.$$

Proof. Both of these two formulas are proved by Ikeda [21, Theorem 4.1 (1), (2)] when n is odd, the same proof works in general. □

Remark 2.3.4. When the field F is dyadic, similar functional equation has been proved by Ikeda and Katsurada [22, Proposition 2.1]. In this article, we will need the functional equation in the following case: Let $F = \mathbb{Q}_2$ and $L = \langle N, t\rangle$ for some $N, t \in \mathbb{Z}$. When $-tN$ is not a square, let $-d$ be the fundamental discriminant of the field extension $\mathbb{Q}(-tN)/\mathbb{Q}$. When $-tN$ is a square, let $d = -1$, then there exists a positive number c such that

$$4Nt = c^2 d.$$

We have the following functional equation:

$$\mathrm{Den}^\flat(X, L) = (2^{1/2}X)^{2\nu_2(c)} \cdot \mathrm{Den}^\flat\left((2X)^{-1}, L\right). \tag{2.3}$$

2.4 Some simple relations

Lemma 2.4.1. *Let L, M be two quadratic O_F-lattices such that the quadratic form on L is nondegenerate, then there exists a positive number m such that for any $a \in O_F$ with $\nu_\pi(a) \geq m$, the following identity holds,*

$$\mathrm{Den}(\langle a\rangle \obot M, L) = \mathrm{Den}(M, L).$$

Proof. This is proved in [23, Proposition 2.5]. □

Lemma 2.4.2. *Let L be a nondegenerate quadratic O_F-lattice with quadratic form q. Let $\{L_n\}_{n\geq 1}$ be a sequence of quadratic lattices with quadratic forms $\{q_n\}_{n\geq 1}$ such that $L_n = L$ as O_F-lattices and $\lim_{n\to\infty} q_n = q$ as functions from L to F. Then for all quadratic O_F-lattices M, we have*

$$\mathrm{Den}(M, L) = \lim_{n\to\infty} \mathrm{Den}(M, L_n). \tag{2.4}$$

Proof. For all positive integers N, there exists an integer M such that for all $n \geq M$, we have $q(x) - q_n(x) \in \pi^N O_F$ for all $x \in L$. Therefore the quadratic lattice L_n is isometric to L when n is large enough, hence $\mathrm{Den}(M, L) = \lim_{n\to\infty} \mathrm{Den}(M, L_n)$. □

Remark 2.4.3. The above lemma is not true in general if L is degenerate. We refer the readers to

Remark 3.2.2 for details.

2.5 Analytic difference formula I

2.5.1 Primitive decomposition

Let $N \in F$, recall that we use $(\langle N\rangle, q_{\langle N\rangle})$ to denote the rank 1 quadratic lattice over O_F with a O_F-generator l_N such that $q_{\langle N\rangle}(l_N) = N$, then $\langle N\rangle$ is an integral quadratic lattice if and only if $N \in O_F$. Let $n = \nu_\pi(N)$, all the rank 1 integral quadratic lattice L' containing $\langle N\rangle$ has the following form,

$$L' = \pi^{-i}\langle N\rangle \simeq \langle \pi^{-2i}N\rangle, \text{ for } i = 0, 1, \cdots, [\frac{n}{2}].$$

Let H be a self-dual quadratic O_F-lattice of finite rank. Since $q_H(x) \in O_F$ for every $x \in H$, Lemma 2.2.6 gives the following decomposition,

$$\mathrm{Rep}_{H,\langle N\rangle}(O_F) = \bigsqcup_{i=0}^{[n/2]} \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(O_F).$$

Now for every $0 \leq i \leq [\frac{n}{2}]$, we pick an arbitrary $\phi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(O_F)$ and consider the following sub-lattice of H,

$$H(\phi) \coloneqq \{x \in H : (x, \phi(l_N)) = 0\}.$$

Lemma 2.5.1. *The isometric class of the quadratic lattice $H(\phi)$ is independent of the choice of $\phi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(O_F)$.*

Proof. Let $\phi' \in \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(O_F)$ be another element. The homomorphims ϕ and ϕ' are totally determined by $x \coloneqq \phi(l_{\pi^{-2i}N})$ and $x' \coloneqq \phi'(l_{\pi^{-2i}N})$. The fact that ϕ and ϕ' are primitive implies that $x \notin \pi \cdot H$ and $x' \notin \pi \cdot H$. Therefore,

$$(x, H) = O_F, \quad (x', H) = O_F.$$

where we use $(\cdot, \cdot)$ to denote the associated bilinear form on H. Since H is self-dual, then by the work of Morin-Strom [24, Theorem 5.3], there exists $\varphi \in \mathrm{O}(H)(\mathcal{O}_F)$ such that $\varphi(x) = x'$. The homomorphism φ also induces an isometric between $H(\phi)$ and $H(\phi')$ because $H(\phi) = x^{\perp} \cap H$ and $H(\phi') = x'^{\perp} \cap H$. □

Let $N \in \mathcal{O}_F$ be an element of valuation n, for every $0 \le i \le [\frac{n}{2}]$ and $\phi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N\rangle}(\mathcal{O}_F)$, we use $H(N,i)$ to denote the quadratic lattice $H(\phi)$.

Example 2.5.2. Let $N \in \mathcal{O}_F$ has valuation n. When $k > 4$, we have an orthogonal decomposition,

$$H_k^{\varepsilon} \simeq H_4^{+} \oplus H_{k-4}^{\varepsilon}.$$

Recall that the symbol H_k^{ε} is understood in the following way: when p is odd, k can be any positive integer, and $\varepsilon \in \{\pm 1\}$ is arbitrary; when $p = 2$, k is even and $\varepsilon = +1$. The lattice $\mathrm{M}_2(\mathcal{O}_F)$ is equipped with the quadratic form induced by the determinant, it is self-dual and $\chi_F(\mathrm{M}_2(\mathcal{O}_F)) = 1$, hence we can view $\mathrm{M}_2(\mathcal{O}_F)$ as a model lattice for H_4^{+}.

Let's consider the following element $\phi \in \mathrm{PRep}_{H_k^{\varepsilon},\langle \pi^{-2i}N\rangle}(\mathcal{O}_F)$

$$\begin{aligned} \phi_i : \langle \pi^{-2i}N \rangle &\longrightarrow \mathrm{M}_2(\mathcal{O}_F) \simeq H_4^{+} \hookrightarrow H_k^{\varepsilon}, \\ l_{\pi^{-2i}N} &\longmapsto \begin{pmatrix} \pi^{-2i}N & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

the corresponding element in $\mathrm{Rep}_{H_k^{\varepsilon},\langle N\rangle}(\mathcal{O}_F)$ sends l_N to $\begin{pmatrix} \pi^{-i}N & 0 \\ 0 & \pi^{i} \end{pmatrix}$.

Lemma 2.5.1 implies that the following quadratic lattices are isometric,

$$H_k^{\varepsilon}(N,i) = H_k^{\varepsilon}(\phi_i) \simeq \phi_i(l_{\pi^{-2i}N})^{\perp} \oplus H_{k-4}^{\varepsilon},$$

where $\phi_i(l_{\pi^{-2i}N})^{\perp}$ is the space of elements in $\mathrm{M}_2(\mathcal{O}_F)$ that are orthogonal to $\phi_i(l_{\pi^{-2i}N})$, it can be described explicitly as follows,

$$\phi_i(l_{\pi^{-2i}N})^\perp = \left\{ x = \begin{pmatrix} -\pi^{-2i}Na & b \\ c & a \end{pmatrix} : \ a, b, c \in O_F \right\}.$$

It is exactly the lattice $\Delta_F(\pi^{-2i}N)$ we defined in Example 2.1.1, therefore $H_k^\varepsilon(N,i) \simeq \delta_F(\pi^{-2i}N) \obot H_{k-4}^\varepsilon$.

2.5.2 Difference formula of local densities

Theorem 2.5.3. *Let H be a self-dual quadratic O_F-lattice of finite rank k. Let M be an integral quadratic O_F-lattice of finite rank r. Let $N \in O_F$ be an element of valuation n, then*

$$\mathrm{Den}(H, M \obot \langle N\rangle) = \sum_{i=0}^{[n/2]} q^{(2-k+r)i} \cdot \mathrm{Pden}(H, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(H(N,i), M).$$

The proof of this theorem is based on the following lemmas.

Lemma 2.5.4. *Let H be a self-dual quadratic O_F-lattice. Let $N \in O_F$ be an element of valuation n, then there is a bijective map D between the following sets when the positive integer d is large enough,*

$$D : \mathrm{Rep}_{H,\langle N\rangle}(O_F/\pi^d) \xrightarrow{\sim} \bigsqcup_{i=0}^{[n/2]} \bigsqcup_{\overline{x}\in O_F/\pi^i} \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(O_F/\pi^{d-i}).$$

Proof. Let l_N be a generator of the rank 1 O_F-module $\langle N\rangle$ such that $q_{\langle N\rangle}(l_N) = N$. Any $f \in \mathrm{Rep}_{H,\langle N\rangle}(O_F/\pi^d)$ is determined by $f(\overline{l_N}) \in H/\pi^d H$. There is a natural filtration on $H/\pi^d H$ given as follows,

$$0 \subset \pi^{d-1}H/\pi^d H \subset \pi^{d-2}H/\pi^d H \subset \cdots \subset \pi^2 H/\pi^d H \subset \pi H/\pi^d H \subset H/\pi^d H.$$

Let i be the minimal integer such that $f(\overline{l_N}) \in \pi^i H/\pi^d H$, then $0 \le i \le [\frac{n}{2}]$ since $\nu_\pi(N) = n$. Then there exists $l \in H$ such that $f(\overline{l_N}) = \overline{\pi^i l} \in \pi^i H/\pi^d H$, the image of l in $H/\pi^{d-i}H$ is uniquely determined by f. Let q be the quadratic form on H, then

$$N \bmod \pi^d = \overline{q_{\langle N\rangle}}(\overline{l_N}) = \overline{q}(f(\overline{l_N})) = \pi^{2i}\overline{q}(\bar{l}) = \overline{\pi^{2i}q(l)},$$

hence $\overline{\pi^{2i}q(l)} \equiv \pi^{-2i}N \bmod \pi^{d-2i}$. Therefore there exists $x \in \mathcal{O}_F$ such that $q(l) = \pi^{-2i}N + \pi^{d-2i}x$. Next we show that $\overline{x} \in \mathcal{O}_F/\pi^i$ is independent of the choice of $l \in H$ when d is large enough. Suppose l' is another element of H such that $f(\overline{l_N}) = \overline{\pi^i l'}$, then there exists $\delta \in H$ such that $l' - l = \pi^{d-i}\delta$. Therefore when d is large enough,

$$q(l') = q(l + \pi^{d-i}\delta) = q(l) + \pi^{d-i}(l,\delta) + \pi^{2d-2i}q(\delta) \equiv q(l) \bmod \pi^{d-i}.$$

Suppose $q(l') = \pi^{-2i}N + \pi^{d-2i}x'$ for some $x' \in \mathcal{O}_F$, the above congruence formula between $q(l)$ and $q(l')$ implies that $x' \equiv x \bmod \pi^i$. The above construction gives the following element $D(f)$ in $\mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$: the homomrphism $D(f)$ sends the generator $\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}$ of $\langle \pi^{-2i}N + \pi^{d-2i}x\rangle/\pi^{d-i}\langle \pi^{-2i}N + \pi^{d-2i}x\rangle$ to $\bar{l} \in H/\pi^{d-i}H$.

Now for any element $\varphi \in \mathrm{PRep}_{H,\langle \pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, we consider the following morphism,

$$\begin{aligned} \tilde{\varphi} : \langle N\rangle/\pi^d\langle N\rangle &\longrightarrow H/\pi^d H, \\ \overline{l_N} &\longmapsto \pi^i\varphi(\overline{l_{\pi^{-2i}N+\pi^{d-2i}x}}). \end{aligned}$$

then $\tilde{\varphi} \in \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d)$ because $\overline{q}(\tilde{\varphi}(\overline{l_N})) = \overline{\pi^{2i}(\pi^{-2i}N + \pi^{d-2i}x)} = N \bmod \pi^d$. This construction gives the inverse map of D. □

Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank. Let $N \in \mathcal{O}_F$ be an element of valuation n. Let $M^\sharp = M \oplus \langle N\rangle$ be a quadratic $\mathcal{O}_F$-lattice of 1 rank higher than M, there is a natural restriction map as follows for any positive integer d and any self-dual quadratic $\mathcal{O}_F$-lattice H,

$$\mathrm{res} : \mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d) \to \mathrm{Rep}_{H,\langle N\rangle}(\mathcal{O}_F/\pi^d),$$

given by composing any element in the set $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ with the natural inclusion $\langle N\rangle/\pi^d\langle N\rangle \hookrightarrow M^\sharp/\pi^d M^\sharp$. The next lemma describes the fiber of the map $D\circ\mathrm{res}$.

Lemma 2.5.5. *Let H be a self-dual quadratic $\mathcal{O}_F$-lattice. Let M be an integral quadratic $\mathcal{O}_F$-lattice of finite rank r. Let $N\in\mathcal{O}_F$ be an element of valuation n. Let $M^\sharp = M\obot\langle N\rangle$ be a quadratic $\mathcal{O}_F$-lattice of rank $r+1$. Let $0\le i\le[\frac{n}{2}]$ be an integer. Let $\varphi\in\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, then for d large enough,*

$$\#(D\circ\mathrm{res})^{-1}(\varphi) = q^{ir}\cdot\#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d).$$

Proof. Let f be an element in $\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)$ such that $D\circ\mathrm{res}(f)=\varphi$, by the proof of Lemma 2.5.4, there exists $l'_N\in H\backslash\pi H$ such that $f(\overline{l_N})=\overline{\pi^i l'_N}$, and $q(l'_N)=\pi^{-2i}N$ when d is large enough.

Let $\{e_i\}_{i=1}^r$ be an $\mathcal{O}_F$-basis of M, then f is determined by $\{x_i\coloneqq f(\overline{e_i})\in H/\pi^d H\}_{i=1}^r$. Therefore $(D\circ\mathrm{res})^{-1}(\varphi)$ can be described by the following set,

$$(D\circ\mathrm{res})^{-1}(\varphi)=\Big\{(x_1,\cdots,x_r)\in(H/\pi^dH)^r : (x_i,\overline{\pi^i l'_N})=0,\ (x_i,x_j)=(\overline{e_i},\overline{e_j})\text{ for } i\neq j, \quad \text{and }\ \overline{q}(x_i)=\overline{q_M}(\overline{e_i})\ \text{ for every } i.\Big\}. \tag{2.5}$$

Let L be the rank 1 sub-lattice of H generated by l'_N. We have the following exact sequence,

$$0\longrightarrow L\oplus H(N,i)\xrightarrow{\theta}H\longrightarrow Q\coloneqq H/L\oplus H(N,i)\longrightarrow 0.$$

where θ is the natural inclusion map. After tensoring the above exact sequence with $\mathcal{O}_F/\pi^d$, we get the following exact sequence by the flatness of H over $\mathcal{O}_F$,

$$0\longrightarrow\mathrm{Tor}^1_{\mathcal{O}_F}(Q,\mathcal{O}_F/\pi^d)\longrightarrow L/\pi^dL\ \oplus\ H(N,i)/\pi^dH(N,i)\xrightarrow{\overline{\theta}}H/\pi^dH\longrightarrow Q/\pi^dQ\longrightarrow 0. \tag{2.6}$$

Claim: Let $K=\Big\{x\in H/\pi^dH : (x,\overline{\pi^i l'_N})=0\Big\}$. When d is large enough, for every $\overline{x}\in K$, there exists $x'\in L$ and $x''\in H(N,i)$ such that the image of $\overline{x'}+\overline{x''}\in L/\pi^dL\ \oplus\ H(N,i)/\pi^dH(N,i)$

under $\overline{\theta}$ in $H/\pi^d H$ is $\overline{x}$.

Proof of the claim: We have the following decomposition in the quadratic space $V = H \otimes_{O_F} F$,

$$x = x' + x'',$$

where $x' \in L_F \coloneqq L \otimes_{O_F} F$ and $x'' \in (L_F)^\perp \subset V$.

The fact that $\overline{x} \in K$ implies that $(x', l_N) = (x, l_N) \in (\pi^d)$, therefore $x' \in (\pi^{d-n}) \cdot l_N \in L_F$. It turns out that $x' \in L \subset H$ when d is large enough, hence $x' = x - x' \in H \cap \{l_N\}^\perp = H(N,i)$. $\square$

We get the following description of the inverse image of the set $(D \circ \mathrm{res})^{-1}(\varphi)$ under $\Theta \coloneqq \overline{\theta} \times \cdots \times \overline{\theta}$ by (2.5)

$$\Theta^{-1}((D \circ r)^{-1}(\varphi)) = (\pi^{d+i-n} L/\pi^d L)^r \times \mathrm{Rep}_{H(N,i),M}(O_F/\pi^d). \tag{2.7}$$

and the claim implies that the map $\Theta^{-1}((D \circ \mathrm{res})^{-1}(\varphi)) \xrightarrow{\Theta} (D \circ \mathrm{res})^{-1}(\varphi)$ is surjective.

Now we compute $\#\ker(\Theta)$, which equlas $(\#\ker(\theta))^r$ by definition. By the exact sequence (2.6), $\#\ker(\overline{\theta}) = \#\mathrm{Tor}^1_{O_F}(Q, O_F/\pi^d) = \#Q/\pi^d Q$. Therefore when d is large enough, $Q/\pi^d Q = Q$. Since $l'_N \notin \pi H$, there exists $y \in H$ such that $(l'_N, y) = 1$. The existence of y implies the following exact sequence,

$$0 \longrightarrow H(N,i) \xrightarrow{\theta} H \longrightarrow L^\vee \longrightarrow 0,$$
$$x \longmapsto l(x) : v \in L \mapsto (x, v).$$

Therefore $H \simeq L^\vee \oplus H(N,i)$ as O_F-modules, and $Q \simeq L^\vee/L \simeq \pi^{2i-n}L/L$. Then by (2.7)

$$\#(D \circ \mathrm{res})^{-1}(\varphi) = \frac{q^{r(n-i)}}{q^{r(n-2i)}} \cdot \#\mathrm{Rep}_{H(N,i),M}(O_F/\pi^d) = q^{ir} \cdot \#\mathrm{Rep}_{H(N,i),M}(O_F/\pi^d).$$

$\square$

Proof of Theorem 2.5.3: By Lemma 2.5.4 and Lemma 2.5.5, we only need to know the size of the set $\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$ when $x\in\mathcal{O}_F$. We first show that when d is large enough, the following identity holds for any $x\in\mathcal{O}_F$,

$$\#\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})=\#\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i}).$$

Because when d is large enough, for an element $x\in\mathcal{O}_F$, we can find $c_x\in\mathcal{O}_F^\times$ such that $c_x^{-2}=1+\pi^dN^{-1}x \bmod \pi^{d-i}$, then for any element $l\in\mathrm{PRep}_{H,\langle\pi^{-2i}N+\pi^{d-2i}x\rangle}(\mathcal{O}_F/\pi^{d-i})$, $c_x\cdot l\in\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})$. Let $M^\sharp=M\obot\langle N\rangle$, we have

$$\begin{aligned}\mathrm{Den}(H,M\obot\langle N\rangle)&=\lim_{d\to\infty}\frac{\#\mathrm{Rep}_{H,M^\sharp}(\mathcal{O}_F/\pi^d)}{q^{d(k(r+1)-(r+1)(r+2)/2)}}\\&=\lim_{d\to\infty}\sum_{i=0}^{[n/2]}q^i\cdot\frac{\#\mathrm{PRep}_{H,\langle\pi^{-2i}N\rangle}(\mathcal{O}_F/\pi^{d-i})}{q^{(d-i)(k-1)}}\cdot\frac{q^{ir}}{q^{i(k-1)}}\cdot\frac{\#\mathrm{Rep}_{H(N,i),M}(\mathcal{O}_F/\pi^d)}{q^{d((k-1)r-r(r+1)/2)}}\\&=\sum_{i=0}^{[n/2]}q^{(2-k+r)i}\cdot\mathrm{Pden}(H,\langle\pi^{-2i}N\rangle)\cdot\mathrm{Den}(H(N,i),M).\end{aligned}$$

□

Corollary 2.5.6. *Let M be a quadratic lattice of rank r over $\mathcal{O}_F$. Let $k>r+2$ be a positive integer and $\varepsilon\in\{\pm1\}$. For all $a\in\mathcal{O}_F$, we have the following identity,*

$$\lim_{m\to\infty}\mathrm{Den}(H_k^\varepsilon,M\obot\langle a\pi^m\rangle)=\mathrm{Den}(H_{k-2}^\varepsilon,M)\cdot\frac{\mathrm{Pden}(H_k^\varepsilon,\langle\pi\rangle)}{1-q^{2-k+r}}.$$

Proof. By Theorem 2.5.3 and Example 2.5.2, we know that for any positive integer m,

$$\begin{aligned}\mathrm{Den}(H_k^\varepsilon,M\obot\langle a\pi^m\rangle)-&q^{2-k+r}\mathrm{Den}(H_k^\varepsilon,M\obot\langle a\pi^{m-2}\rangle)\\&=\mathrm{Pden}(H_k^\varepsilon,\langle a\pi^m\rangle)\cdot\mathrm{Den}(\langle-a\pi^m\rangle\obot H_{k-2}^\varepsilon,M).\end{aligned}$$

Notice that $\mathrm{Pden}(H_k^\varepsilon,\langle a\pi^m\rangle)$ is independent of the positive integer m by Example 2.2.5. Taking

$m \to \infty$, Lemma 2.4.1 implies the following,

$$(1 - q^{2-k+r}) \cdot \lim_{m\to\infty} \mathrm{Den}(H_k^{\varepsilon}, M \oplus \langle a\pi^m\rangle) = \mathrm{Den}(H_{k-2}^{\varepsilon}, M) \cdot \mathrm{Pden}(H_k^{\varepsilon}, \langle \pi\rangle).$$

□

Definition 2.5.7. Let $N \in O_F$. Let M be a quadratic lattice of rank $r \geq 2$ over O_F. Define the local density of M with level N to be a polynomial $\mathrm{Den}_{\delta_F(N)}(X, M)$ satisfying

$$\mathrm{Den}_{\delta_F(N)}(X, M)\big|_{X=q^{-m}} = \begin{cases} \dfrac{\mathrm{Den}(\delta_F(N) \oplus H^+_{2m+r-2}, M)}{\mathrm{Nor}^+(q^{-m}, r-1)}, & \text{When } \pi \mid N; \\ \dfrac{\mathrm{Den}(\delta_F(N) \oplus H^+_{2m+r-2}, M)}{\mathrm{Nor}^{\chi_F(N)}(q^{-m}, r)}, & \text{When } \pi \nmid N. \end{cases}$$

for $m \geq 0$. Moreover, if the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the self-dual lattice H^+_{r+2}. Define the derived local density of M with level N to be

$$\partial\mathrm{Den}_{\delta_F(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}_{\delta_F(N)}(X, M).$$

Theorem 2.5.8. *Let $N \in O_F$. Let M be a quadratic lattice of rank $r \geq 2$ over O_F. Then we have*

$$\mathrm{Den}_{\delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle).$$

Moreover, if the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the self-dual lattice H^+_{r+2}.

$$\partial\mathrm{Den}_{\delta_F(N)}(M) = \partial\mathrm{Den}(M \oplus \langle N\rangle) - \partial\mathrm{Den}(M \oplus \langle \pi^{-2}N\rangle).$$

Proof. Recall the definition of the polynomial $\mathrm{Nor}^{\varepsilon}(X, n)$ in Definition 2.3.1, we can verify immediately by formula (2.2.5) that, for any $x \in O_F$,

$$\mathrm{Nor}^+(q^{-m}, r+1) = \mathrm{Pden}(H^{\varepsilon}_{2m+r+2}, \langle x\rangle) \cdot \mathrm{Nor}^{\chi_F(x)}(q^{-m}, r), \quad \text{when } \pi \nmid x;$$

$$\mathrm{Nor}^+(q^{-m}, r+1) = \mathrm{Pden}(H^{\varepsilon}_{2m+r+2}, \langle x\rangle) \cdot \mathrm{Nor}^{+}(q^{-m}, r-1), \quad \text{when } \pi \mid x.$$

Let $n = \nu_\pi(N)$, Theorem 2.5.3 and Example 2.5.2 imply the following decomposition,

$$\begin{aligned} &\mathrm{Den}(H^+_{2m+r+2}, M \oplus \langle N\rangle) \\ &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H^+_{2m+r+2}, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(H^+_{2m+r+2}(N,i), M) \\ &= \sum_{i=0}^{[n/2]} q^{-2mi} \cdot \mathrm{Pden}(H^+_{2m+r+2}, \langle \pi^{-2i}N\rangle) \cdot \mathrm{Den}(\delta_F(\pi^{-2i}N) \oplus H^+_{2m+r-2}, M). \end{aligned}$$

By Definition 2.5.7, when p is odd, the following formula holds

$$\mathrm{Den}(X, M \oplus \langle N\rangle) = \sum_{i=0}^{[n/2]} X^{2i} \cdot \mathrm{Den}_{\delta_F(\pi^{-2i}N)}(X, M). \tag{2.8}$$

When $n = 0$ or 1, $\mathrm{Den}(X, M \oplus \langle N\rangle) = \mathrm{Den}_{\delta_F(N)}(X, M)$ and $\mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle) = 0$ since $\pi^{-2}N$ is not in O_F, therefore $\mathrm{Den}_{\delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle)$. When $n \geq 2$, $\mathrm{Den}_{\delta_F(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle \pi^{-2}N\rangle)$ follows from the formula (2.8).

The fact that the lattice $M \oplus \langle N\rangle$ can't be isometrically embedded into the quadratic space H^+_{r+2} implies that $\mathrm{Den}(1, M \oplus \langle N\rangle) = \mathrm{Den}(1, M \oplus \langle \pi^{-2}N\rangle) = 0$, the second formula in the theorem follows from the first one and the definitions of the symbols $\partial\mathrm{Den}_{\delta_F(N)}$ and $\partial\mathrm{Den}$. □

Now we apply Theorem 2.5.8 to the case that we are interested, i.e., $F = \mathbb{Q}_p$ and $r = 2$, Let $N \in \mathbb{Z}_p$, we get the following difference formula of local density functions,

$$\mathrm{Den}_{\delta_p(N)}(X, M) = \mathrm{Den}(X, M \oplus \langle N\rangle) - X^2 \cdot \mathrm{Den}(X, M \oplus \langle p^{-2}N\rangle). \tag{2.9}$$

Note that the lattice $M \obar \langle N\rangle$ is a sub-lattice of $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$ which is the unique division quaternion algebra over $\mathbb{Q}_p$, hence the lattice $M \obar \langle N\rangle$ can't be isometrically embedded into the quadratic space $H_4^+ \otimes \mathbb{Q}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$, therefore Theorem 2.5.8 implies the following difference formula of the derivatives of local densities,

$$\partial\mathrm{Den}_{\delta_p(N)}(M) = \partial\mathrm{Den}(M \obar \langle N\rangle) - \partial\mathrm{Den}(M \obar \langle p^{-2}N\rangle). \tag{2.10}$$

2.5.3 Example

Assume p is odd. In the following example, we compute an explicit example of local densities and compare our formulas with known formulas given in [8] and [1].

Example 2.5.9. Let $N = N_0$ is a positive integer with $\nu_p(N_0) = 0$ or 1. Let M be a rank 2 $\mathbb{Z}_p$-lattice such that M is isometrically embedded into $\mathbb{W}$ and is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to the matrix $\mathrm{diag}\{\varepsilon_1 p^2, \varepsilon_2 p^3\}$ where $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_p^\times$. Let $N_k = p^{2k}N_0$ where N_0 is a positive integer with $\nu_p(N_0) = 0$ or 1 and $k \geq 1$ is an integer. By the formula in [8, §2.11],

$$\begin{aligned}\mathrm{Den}(X, M \obar \langle N_k\rangle) =& 1 + pX + (p+p^2)X^2 + p^2X^3 + p^2X^4 - p^2X^{2k+1+\nu_p(N_0)} - p^2X^{2k+2+\nu_p(N_0)} \\ &- (p+p^2)X^{2k+3+\nu_p(N_0)} - pX^{2k+4+\nu_p(N_0)} - X^{2k+5+\nu_p(N_0)}, \text{ when } k \geq 3.\end{aligned}$$

the formula (2.9) implies when $k \geq 3$,

$$\mathrm{Den}_{\delta_p(N_k)}(X, M) = 1 + pX + (p^2+p-1)X^2 + (p^2-p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6.$$

when $k \geq 3$. Therefore $\partial\mathrm{Den}_{\delta_p(N_0)}(M) = 2 + 4p + 6p^2$ when $k \geq 3$.

We will double check our results by comparing it with the formulas of local density given in [5, Theorem 7.1]. The theorem implies that for sufficiently large positive integer m,

$$\mathrm{Den}(\delta_p(N_k) \obar H_{2m}^+, M) = 1 + R_{1,k}(X) + R_{2,k}(X)\big|_{X=p^{-m}}.$$

where $R_{1,k}(X) = \sum_{i=1}^{8} I_{1,i,k}(X)$ and $R_{2,k}(X) = (1-p^{-1})\sum_{i=1}^{8} I_{2,i,k}(X) + p^{-1}I_{2,6,k}(X)$. $I_{1,i,k}(X)$ and $I_{2,i,k}(X)$ are explicitly constructed polynomials in the beginning of section 7 of [5]. In our case, when $k \geq 3$,

$$I_{1,1,k}(X) = (p-p^{-1})X + (p^2-1)X^2,\ I_{1,2,k}(X) = -X^3,\ I_{1,3,k}(X) = 0,\ I_{1,4,k} = -p^2X^4.$$

$$I_{2,1,k}(X) = (p^2-1)X^3,\ I_{2,3,k}(X) = I_{2,5,k}(X) = I_{2,7,k}(X) = 0.$$

$$I_{2,2,k}(X) = -pX^4 - pX^5,\ I_{2,4,k}(X) = -p^2X^5 - p^2X^6,\ I_{2,6,k}(X) = pX^7,\ I_{2,8,k}(X) = pX^2 + p^2X^4.$$

therefore when m is sufficiently large,

$$\begin{aligned}\mathrm{Den}(\delta_p(N_k) \oplus H_{2m}^+, M) = 1 &+ (p-p^{-1})X + (p^2+p-2)X^2 + (p^2-2p+p^{-1}-1)X^3 \\ &- (2p-1)X^4 + (1-p^2)X^5 + (p-p^2)X^6 + pX^7 \big|_{X=p^{-m}}.\end{aligned}$$

By Definition 2.5.7, when $k \geq 3$,

$$\begin{aligned}&\mathrm{Den}_{\delta_p(N_k)}(X, M)\big|_{X=p^{-m}} = \frac{\mathrm{Den}(\delta_p(N_k) \oplus H_{2m}^+, M)}{1-p^{-m-1}} \\ &= 1 + pX + (p^2+p-1)X^2 + (p^2-p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6 \big|_{X=p^{-m}}.\end{aligned}$$

hence $\mathrm{Den}_{\delta_p(N_k)}(X, M) = 1 + pX + (p^2+p-1)X^2 + (p^2-p)X^3 - pX^4 - p^2X^4 - p^2X^5 - p^2X^6$ when $k \geq 3$, this agrees with our previous calculations.

Chapter 3: Fourier coefficients of Eisenstein series

3.1 Incoherent Eisenstein series

For an positive integer r, let W_r be the standard symplectic space over $\mathbb{Q}$ of dimension $2r$. Let $P = MN \subset \mathrm{Sp}(\mathsf{W}_r)$ be the standard Siegel parabolic subgroup, which take the following form under the standard basis of W_r,

$$M(\mathbb{Q}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathrm{GL}_r(\mathbb{Q}) \right\},$$

$$N(\mathbb{Q}) = \left\{ n(b) = \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} : b \in \mathrm{Sym}_r(\mathbb{Q}) \right\}.$$

Let $\mathbb{A}$ be the adèle ring over $\mathbb{Q}$. There is an isomorphism $\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}}) \xrightarrow{\sim} \mathrm{Sp}(\mathsf{W}_r)(\mathbb{A}) \times \mathbb{C}^1$ with the multiplication on the latter is given by the global Rao cycle, therefore we can write an element of $\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}})$ as (g, t) where $g \in \mathrm{Sp}(\mathsf{W}_r)(\mathbb{A})$ and $t \in \mathbb{C}^1$.

Let $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$ be the standard Siegel parabolic subgroup of $\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}})$ where

$$M(\mathbb{A}) = \left\{ (m(a), t) : a \in \mathrm{GL}_r(\mathbb{A}), t \in \mathbb{C}^1 \right\},$$

$$N(\mathbb{A}) = \left\{ n(b) : b \in \mathrm{Sym}_r(\mathbb{A}) \right\}.$$

Recall the following incoherent collection of rank 3 quadratic spaces $\mathbb{V} = \{\mathbb{V}_v\}$ over $\mathbb{A}$ we defined in (1.2),

$$\mathbb{V}_v = \delta_v(N) \otimes \mathbb{Q}_v \ \text{ if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.}$$

then we can verify immediately that $\prod_v \epsilon(\mathbb{V}_v) = -1$.

Let $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the quadratic character given by $\chi(x) = \prod_{v\leq\infty} \chi_v(x_v) = \prod_{v\leq\infty} (x_v, -N)_v$ for all $x = (x_v) \in \mathbb{A}^\times$. Fix the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ such that $\psi_\infty(x) = e^{2\pi i x}$. We may view χ as a character on $M(\mathbb{A})$ by

$$\chi(m(a), t) = \chi(\det(a)) \cdot \gamma(\det(a), \psi)^{-1} \cdot t.$$

and extend it to $P(\mathbb{A})$ trivially on $N(\mathbb{A})$. Here $\gamma(\det(a), \psi)$ is the Weil index (see the work of Kudla [12, p. 548]). We define the degenerate principal series to be the unnormalized smooth induction

$$I(s, \chi) \coloneqq \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}})}(\chi \cdot |\cdot|_{\mathbb{Q}}^{s+(r+1)/2}), \quad s \in \mathbb{C}.$$

For a standard section $\Phi(-, s) \in I(s, \chi)$ (i.e., its restriction to the standard maximal compact subgroup of $\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}})$ is independent of s), we define the associated Siegel Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{Sp}(\mathbb{Q})} \Phi(\gamma g, s),$$

which converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$.

Recall that $\mathscr{S}(\mathbb{V}^r)$ is the space of Schwartz functions on $\mathbb{V}^r$. The fixed choice of χ and ψ gives a Weil representation $\omega = \omega_{\chi,\psi}$ of $\mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}}) \times \mathrm{O}(\mathbb{V})$ on $\mathscr{S}(\mathbb{V}^r)$. For $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}^r)$, define a function

$$\Phi_{\boldsymbol{\varphi}}(g) \coloneqq \omega(g)\boldsymbol{\varphi}(0), \quad g \in \mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}}).$$

Then $\Phi_{\boldsymbol{\varphi}}(g) \in I(0, \chi)$. Let $\Phi_{\boldsymbol{\varphi}}(-, s) \in I(s, \chi)$ be the associated standard section, known as the standard Siegel–Weil section associated to $\boldsymbol{\varphi}$. For $\boldsymbol{\varphi} \in \mathscr{S}(\mathbb{V}^r)$, we write $E(g, s, \boldsymbol{\varphi}) \coloneqq E(g, s, \Phi_{\boldsymbol{\varphi}})$.

In this thesis, we will mainly work on the case that $r = 1$ or 2.

3.2 Whittaker functions and local densities

Let v be a finite place of $\mathbb{Q}$. Define the local degenerate principal series to be the unnormalized smooth induction

$$I_v(s,\chi_v) \coloneqq \mathrm{Ind}_{P(\mathbb{Q}_v)}^{\mathrm{Mp}(\mathsf{W}_{r,v})}(\chi_v \cdot |\cdot|_v^{s+(r+1)/2}), \quad s \in \mathbb{C}.$$

Let $N = \mathrm{Sym}_r$ be the group of symmetric matrices. For all functions $\Phi_v \in I_v(s,\chi_v)$ and matrices $T \in N(\mathbb{Q}_v)$, define the local Whittaker function by the following integral,

$$W_{T,v}(g_v,s,\Phi_v) = \int\limits_{N(\mathbb{Q}_v)} \Phi_v\left(w_r^{-1}n(b)g_v,s\right)\psi_v\left(-\mathrm{tr}\left(\frac{1}{2}Tb\right)\right)\mathrm{d}n(b), \quad w_r = \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}. \tag{3.1}$$

The integral converges absolutely when $\mathrm{Re}(s) \gg 0$, and it has meromorphic continuation to $s \in \mathbb{C}$.

Lemma 3.2.1. *Let $\Phi_v \in I_v(s,\chi_v)$. Let $\{T_n\}_{n\geq 1} \subset \mathrm{Sym}_n(\mathbb{Q}_v)$ be a sequence of symmetric matrices such that $\lim\limits_{n\to\infty} T_n = T \in \mathrm{Sym}_r(\mathbb{Q}_v)$, i.e., the sequence $\{T_n\}_{n\geq 1}$ converges to T l-adically. Then we have the following equality when* $\mathrm{Re}(s) \gg 0$*,*

$$W_{T,v}(g_v,s,\Phi_v) = \lim_{n\to\infty} W_{T_n,v}(g_v,s,\Phi_v). \tag{3.2}$$

Proof. Let $\epsilon > 0$ be a number. Let $s_0 > 0$ such that the integral (3.1) converges absolutely when $\mathrm{Re}(s) > s_0$. Therefore there exists an integer $n_0 > 0$ such that for all integers $n > n_0$, we have

$$\int\limits_{v^{-n}N(\mathbb{Z}_v)\backslash v^{-n_0}N(\mathbb{Z}_v)} \left|\Phi_v(w_r^{-1}n(b)g_v,s)\right|\mathrm{d}n(b) < \epsilon. \tag{3.3}$$

There also exists an integer N_0 such that for all $n \geq N_0$, we have $T_n - T \in v^{n_0}N(\mathbb{Z}_v)$. Therefore for all $n \geq N_0$, we have

$$\begin{aligned}
&\left| W_{T,v}(g_v, s, \Phi_v) - W_{T_n,v}(g_v, s, \Phi_v) \right| \\
&\qquad \leq \left| W_{T,v}(g_v, s, \Phi_v) - \int\limits_{v^{-n_0} N(\mathbb{Z}_v)} \Phi_v(w_r^{-1} n(b) g_v, s) \psi_v(-\mathrm{tr}(\frac{1}{2} T b)) \mathrm{d}n(b) \right| \\
&\qquad + \left| \int\limits_{v^{-n_0} N(\mathbb{Z}_v)} \Phi_v(w_r^{-1} n(b) g_v, s) \psi_v(-\mathrm{tr}(\frac{1}{2} T b))(1 - \psi(\mathrm{tr}(\frac{1}{2}(T - T_n)) b)) \mathrm{d}n(b) \right| \\
&\qquad + \left| W_{T_n,v}(g_v, s, \Phi_v) - \int\limits_{v^{-n_0} N(\mathbb{Z}_v)} \Phi_v(w_r^{-1} n(b) g_v, s) \psi_v(-\mathrm{tr}(\frac{1}{2} T b)) \mathrm{d}n(b) \right| .
\end{aligned}$$

Notice that the middle one is 0. By the estimation (3.3), we have the following inequality for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > s_0$,

$$\left| W_{T,v}(g_v, s, \Phi_v) - W_{T_n,v}(g_v, s, \Phi_v) \right| \leq 2\epsilon, \quad \text{for all } n \geq N_0.$$

Hence the limit (3.2) holds. □

Remark 3.2.2. By the well-known relations between special values of Whittaker functions and local densities, e.g. [12, Proposition A.4] and Lemma 3.2.1, we would expect that the formula (2.4) holds also for degenerate quadratic lattice L. But this is not true in general! The formula (2.4) amounts to saying that the following two limits are equal for some integer k,

$$\lim_{s \to k} \lim_{n \to \infty} W_{T_n,v}(g_v, s, \Phi_v) \overset{?}{=} \lim_{n \to \infty} \lim_{s \to k} W_{T_n,v}(g_v, s, \Phi_v). \tag{3.4}$$

It is true if $T = \lim\limits_{n \to \infty} T_n$ is nondegenerate because the limit process for $n \to \infty$ would stabilize. But this is not true if T is degenerate, hence (3.4) may not be true. For example, by Yang's formula in [25, Theorem 5.7 (i)], the two sides of (3.4) would differ by a nonzero multiple of $\mu_p(T)$ (see the notations in *loc. cit.*).

The fixed choice of χ_v and ψ_v gives a local Weil representation $\omega_v = \omega_{\chi_v,\psi_v}$ of $\mathrm{Mp}(\mathsf{W}_{r,v}) \times \mathrm{O}(\mathbb{V}_v)$ on the Schwartz function space $\mathscr{S}(\mathbb{V}_v^r)$. We define the local Whittaker function associated to $\boldsymbol{\varphi}_v$ and $T \in \mathrm{Sym}_r(\mathbb{Q}_v)$ to be

$$W_{T,v}(g_v, s, \boldsymbol{\varphi}_v) := W_{T,v}(g_v, s, \Phi_{\boldsymbol{\varphi}_v}), \tag{3.5}$$

where $\Phi_{\boldsymbol{\varphi}_v}(g_v) := \omega_v(g_v)\boldsymbol{\varphi}_v(0) \in I_v(0,\chi_v)$ and $\Phi_{\boldsymbol{\varphi}_v}(-,s)$ is the associated standard section.

The relationship between Whittaker functions and local densities is encoded in the following proposition.

Proposition 3.2.3. *Suppose $v \neq \infty$. Let L be an integral quadratic $\mathbb{Z}_v$-lattice of rank 1 or 2. Suppose that the quadratic form of L is represented by a number $t \in \mathbb{Q}_v$ (when the rank of L is 1) or a matrix $T \in \mathrm{Sym}_r(\mathbb{Q}_v)$ after a choice of $\mathbb{Z}_v$-basis of L, we have the following identity,*

$$W_{t,v}(1, k+\frac{1}{2}, 1_{\delta_v(N)}) = (-1,-N)_v|2N|_v^{1/2}\cdot\gamma(\mathbb{V}_v)\cdot\mathrm{Den}(\delta_v(N)\oplus H_{2k}^+, L), \quad \text{when the rank of } L \text{ is } 1. \tag{3.6}$$

$$W_{T,v}(1, k, 1_{\delta_v(N)^2}) = |2|_v^{3/2}\cdot|N|_v\cdot\gamma(\mathbb{V}_v)^2\cdot\mathrm{Den}(\delta_v(N)\oplus H_{2k}^+, L), \quad \text{when the rank of } L \text{ is } 2. \tag{3.7}$$

where the constant $\gamma(\mathbb{V}_v) = \gamma(\det(\mathbb{V}_v),\psi_v)^{-1}\cdot\epsilon(\mathbb{V}_v)\cdot\gamma(\psi_v)^{-3}$, $\gamma(\det(\mathbb{V}_v),\psi_v)$ and $\gamma(\psi_v)$ are Weil indexes (cf. Appendix of [26]).

Proof. Both of the two formulas are proved in [6, Lemma 5.7.1]. □

For the place $v = \infty$, there is also a well-developed theory about the Whittaker function at ∞. Let $W_{T,\infty}(g,s,\Delta(N)^2)$ (resp. $W_{t,\infty}(g,s,\Delta(N))$) be the Whittaker functions for the Gaussian function on the space $\mathbb{V}_\infty^2$ (resp. $\mathbb{V}_\infty$), both of them can be computed explicitly, for example, the works of Kudla, Rapoport and Yang [27, Lemma 8.6], [6, Theorem 5.2.7, Proposition 5.7.7].

Definition 3.2.4. For a symmetric matrix $T \in \mathrm{Sym}_r(\mathbb{Q})$, and a factorizable section $\Phi = \otimes_v\Phi_v \in I(s,\chi)$ where Φ_∞ is the standard section associated to the Gaussian function on the space $\mathbb{V}_\infty^r$, for a complex number $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \gg 0$, we define the global Whittaker function to be

$$W_T(g, s, \varphi) = W_{T,\infty}(g_\infty, s, \Delta(N)^r) \cdot \prod_{v<\infty} W_{T,v}(g_v, s, \varphi_v), \ \ g = (g_v)_v \in \mathrm{Mp}(\mathrm{W}_{r,\mathbb{A}}).$$

the above product can be meromorphically continued to the whole complex plane.

Remark 3.2.5. In this thesis, we use the simplified notation $W_T(g, s, \Delta(N)^n)$ to denote the global Whittaker function $W_T(g, s, \varphi)$ on $\mathrm{Mp}(\mathrm{W}_{n,\mathbb{A}})$ where $n = 1$ or 2 and φ_f is the Schwartz function $\otimes_{v<\infty} 1_{\delta_p(N)^n}$.

3.3 Fourier expansion

We have a Fourier expansion of the Siegel Eisenstein series defined above,

$$E(g, s, \varphi) = \sum_{T \in \mathrm{Sym}_r(\mathbb{Q})} E_T(g, s, \varphi),$$

where

$$E_T(g, s, \varphi) = \int_{\mathrm{Sym}_r(\mathbb{Q}) \backslash \mathrm{Sym}_r(\mathbb{A})} E(n(b)g, s, \varphi)\psi(-\mathrm{tr}(Tb))\mathrm{d}n(b),$$

the Haar measure $\mathrm{d}n(b)$ is normalized to be self-dual with respect to ψ.

Lemma 3.3.1. *Let $T \in \mathrm{Sym}_r(\mathbb{Q})$ be a nonsingular matrix, let $\varphi \in \mathcal{S}(\mathbb{V}^r)$ be a factorizable Schwartz function, we have*

$$E_T(g, s, \varphi) = W_T(g, s, \varphi).$$

Proof. This lemma follows from unravelling the integral defining the function E_T, see [10, §11.2].

□

Combining with (3.1), we have the following decomposition of the derivative of a nonsingular Fourier coefficient,

$$E'_T(g, s, \Phi) = \sum_v E'_{T,v}(g, s, \Phi),$$

where T is nonsingular and

$$E'_{T,v}(g,s,\Phi) = W'_{T,v}(g_v,s,\Phi_v)\cdot\prod_{v'\neq v} W_{T,v'}(g_{v'},s,\Phi_{v'}). \tag{3.8}$$

3.4 Classical incoherent Eisenstein series

The hermitian symmetric domain for $\mathrm{Sp}(\mathsf{W}_r)$ is the Siegel upper half space

$$\mathbb{H}_r = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} \mid \mathsf{x}\in \mathrm{Sym}_r(\mathbb{R}), \mathsf{y}\in \mathrm{Sym}_r(\mathbb{R})_{>0}\}.$$

When $r = 1$, let $\tau = x + iy \in \mathbb{H}^+$ with $x, y \in \mathbb{R}$ and y is positive. Define the classical incoherent Eisenstein series to be

$$E(\tau,s,\boldsymbol{\varphi}) = y^{-3/4}\cdot E(g_\tau,s,\boldsymbol{\varphi}),\ \ g_\tau = [n(x)m(y^{1/2}),1]\in \mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}}). \tag{3.9}$$

In this paper, we will focus on the case that $\boldsymbol{\varphi} = 1_{\Delta(N)\otimes\hat{\mathbb{Z}}}\otimes\boldsymbol{\varphi}_\infty \in \mathscr{S}(\mathbb{V})$, where $1_{\Delta(N)\otimes\hat{\mathbb{Z}}}$ is the characteristic function of the rank 3 $\hat{\mathbb{Z}}$-lattice $\Delta(N)\otimes\hat{\mathbb{Z}}$ and $\boldsymbol{\varphi}_\infty$ is the Gaussian function $\boldsymbol{\varphi}_\infty(x) = e^{-\pi q_\infty(x)}$. For the fixed choice of the Schwartz function $\boldsymbol{\varphi}$ above and any $t\in\mathbb{Q}$, we write

$$E(\tau,s,\Delta(N)) \coloneqq E(\tau,s,1_{\Delta(N)\otimes\hat{\mathbb{Z}}}\otimes\boldsymbol{\varphi}_\infty),\ \ E_t(\tau,s,\Delta(N)) \coloneqq E_t(\tau,s,1_{\Delta(N)\otimes\hat{\mathbb{Z}}}\otimes\boldsymbol{\varphi}_\infty). \tag{3.10}$$

We will also need the normalized genus 1 Eisenstein series $\mathcal{E}(\tau,s,\Delta(N))$ defined as follows,

$$\mathcal{E}(\tau,s,\Delta(N)) = C_N(s)E(\tau,s-\frac{1}{2},\Delta(N)),\ \text{ where } C_N(s) = -\frac{s}{4\pi}\Lambda(2s)\left(\prod_{p\mid N}(1-p^{-2s})\right)N^{(3s+1)/2}. \tag{3.11}$$

Similarly, for any $t\in\mathbb{Q}$, $\mathcal{E}_t(\tau,s,\Delta(N)) = C_N(s)E_t(\tau,s-\frac{1}{2},\Delta(N))$.

When $r = 2$, let $\mathsf{z} = \mathsf{x}+i\mathsf{y}\in\mathbb{H}_2$ with $\mathsf{x},\mathsf{y}\in\mathrm{Sym}_2(\mathbb{R})$ and $\mathsf{y} = {}^t a\cdot a$ is positive definite. Define the classical incoherent Eisenstein series to be

$$E(\mathsf{z},s,\varphi)=\chi_\infty(m(a))^{-1}|\det(m(a))|^{-3/2}\cdot E(g_{\mathsf{z}},s,\varphi),\ \ g_{\mathsf{z}}=[n(x)m(a),1]\in \mathrm{Mp}(\mathsf{W}_{r,\mathbb{A}}).\tag{3.12}$$

We write the central derivatives as,

$$\partial\mathrm{Eis}(\mathsf{z},\varphi):=E'(\mathsf{z},0,\varphi),\ \ \partial\mathrm{Eis}_T(\mathsf{z},\varphi):=E_T'(\mathsf{z},0,\varphi).\tag{3.13}$$

Then we have a Fourier expansion

$$\partial\mathrm{Eis}(\mathsf{z},\varphi)=\sum_{T\in\mathrm{Sym}_2(\mathbb{Q})}\partial\mathrm{Eis}_T(\mathsf{z},\varphi).$$

In this paper, we will focus on the case that $\varphi=1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty\in\mathscr{S}(\mathbb{V}^2)$, where $\varphi=1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}$ is the characteristic function of the rank 3 $\hat{\mathbb{Z}}$-lattice $\Delta(N)\otimes\hat{\mathbb{Z}}$ and φ_∞ is the Gaussian function $\varphi_\infty(x)=e^{-\pi\,\mathrm{tr}\,T(x)}$. For the fixed choice of the Schwartz function φ above and any $T\in\mathrm{Sym}_2(\mathbb{Q})$, we write

$$E(\mathsf{z},s,\Delta(N)^2):=E(\mathsf{z},s,1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty),\ \ E_T(\mathsf{z},s,\Delta(N)^2):=E_T(\mathsf{z},s,1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty),\tag{3.14}$$

$$\partial\mathrm{Eis}(\mathsf{z},\Delta(N)^2):=\partial\mathrm{Eis}(\mathsf{z},1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty),\ \ \partial\mathrm{Eis}_T(\mathsf{z},\Delta(N)^2):=\partial\mathrm{Eis}_T(\mathsf{z},1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty).\tag{3.15}$$

Similarly, for any $T\in\mathrm{Sym}_2(\mathbb{Q})$, $W_T(g,s,\Delta(N)^2):=W_T(g,s,1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}\otimes\varphi_\infty)$.

Lemma 3.4.1. *Let* $T=\mathrm{diag}\{0,t\}$ *with* $t\neq 0$, *let* $\mathsf{y}=\mathrm{diag}\{y_1,y_2\}$ *be a positive definite symmetric matrix, then*

$$E_T(i\mathsf{y},s,\Delta(N)^2)=y_1^{s/2}y_2^{-3/4}W_t(g_{iy_2},s+\frac{1}{2},\Delta(N))+(y_1y_2)^{-3/4}W_T(g_{i\mathsf{y}},s,\Delta(N)^2).$$

Proof. This lemma follows from [1, Lemma 4.11, formula (38)]. □

3.5 Singular Fourier coefficients of Eisenstein series

In this section, we study the Fourier coefficient $E_T(\mathsf{z}, s, \Delta(N)^2)$ of the Eisenstein series defined in §3.4 when the matrix T is singular.

We first fix some notations. For any prime number p, we define the local zeta function $\zeta_p(s) = (1-p^{-s})^{-1}$, we also define $\zeta_\infty(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$. Let $\Lambda(s) = \prod\limits_v \zeta_v(s)$ be the completed zeta function where v ranges over all the places of $\mathbb{Q}$, then $\Lambda(s)$ can be meromorphically continued to the whole complex plane and satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$.

3.5.1 The rank of T is 0.

In this case, the matrix $T = \mathbf{0}_2$.

Lemma 3.5.1. *Let $n = v_p(N) \geq 0$ be the p-adic valuation of N, let $A_p(s)$ be the following function for $s \in \mathbb{C}$,*

$$A_p(s) = |N|_p \cdot \begin{cases} \frac{1}{1+p^{-1-s}}, & \text{if } n = 0; \\ \frac{1}{1+p^{-1-s}} \cdot \frac{1-p^{(1-s)(n+1)}}{1-p^{1-s}} - p^{-2s}\frac{1}{1+p^{-1-s}} \cdot \frac{1-p^{(1-s)(n-1)}}{1-p^{1-s}}, & \text{if } n \geq 1. \end{cases}$$

The following identity holds,

$$W_{\mathbf{0}_2,p}(1, k, 1_{\delta_p(N)^2}) = |2|_p^{3/2}(N, -1)_p \cdot \frac{\zeta_p(2k-1)}{\zeta_p(2k+2)} \cdot A_p(k).$$

Proof. For a rank 2 nondegenerate quadratic lattice M and integer $k \geq 0$,

$$W_{T,p}(1, k, 1_{\delta_p(N)^2}) = |2|_p^{3/2}|N|_p(N, -1)_p \cdot \mathrm{Nor}^+(X, 1) \cdot \mathrm{Den}_{\delta(N)}(X, M)\big|_{X=p^{-k}}. \tag{3.16}$$

Suppose the Gross-Keating invariant of the quadratic lattice M is $\mathrm{GK}(M) = (a, b)$ for integers $a \leq b$ and $a, b \gg n$.

- If $n \equiv a \pmod 2$, by the formula in [8, §2.11], we have

$$\begin{aligned}\mathrm{Den}(X, M \ominus \langle N\rangle) = {} & \frac{(1-(pX)^{n+1})(1-(pX^2)^{(n+a)/2-i})}{(1-pX)(1-pX^2)} + p^{(n+a)/2}X^a \frac{(1-X^{n+1})(1-(\tilde{\xi}X)^{b-a+1})}{(1-X)(1-(\tilde{\xi}X))} \\ & - \frac{p^{(n+a)/2-1}X^{b+2}}{1-p^{-1}X^2}\left(\frac{1-X^{n+1}}{1-X} - \frac{(p^{-1}X^2)^{(n+a)/2} - p^{(n-a)/2+1}X^{a-1}}{1-pX^{-1}}\right).\end{aligned}$$

where $\tilde{\xi} \in \{\pm 1\}$ depends on the lattice M.

- If $n \not\equiv a \pmod 2$, by the formula in [8, §2.11], we have

$$\begin{aligned}\mathrm{Den}(X, M \ominus \langle N\rangle) = {} & \frac{1-(pX)^{n+1}}{(1-pX)(1-pX^2)} - \frac{p^{(n+a+1)/2}(X^{n+a+1}-X^a)}{(1-X^{-1})(1-pX^2)} \\ & - \frac{p^{(n+a-1)/2}X^{b+1}}{1-p^{-1}X^2}\left(\frac{1-X^{n+1}}{1-X} - \frac{(p^{-1}X^2)^{(n+a+1)/2} - p^{(n-a+1)/2}X^{a}}{1-pX^{-1}}\right).\end{aligned}$$

In both cases, when $0 < X \leq p^{-1}$, we have the following formula

$$\lim_{a,b\to\infty} \mathrm{Den}(X, M \ominus \langle N\rangle) = \frac{1-(pX)^{n+1}}{(1-pX)(1-pX^2)}.$$

therefore when $0 < X \leq p^{-1}$,

$$\begin{aligned}\lim_{a,b\to\infty} \mathrm{Den}_{\delta_p(N)}(X, M) &= \lim_{a,b\to\infty}\Big(\mathrm{Den}(X, M \ominus \langle N\rangle) - X^2\mathrm{Den}(X, M \ominus \langle p^{-2}N\rangle)\Big) \\ &= \frac{1-X^2+(p^{n-1}-p^{n+1})X^{n+1}}{(1-pX)(1-pX^2)}.\end{aligned}$$

Note that $\mathrm{Nor}^+(X,1) = 1-p^{-1}X$, the lemma follows by combining (3.16) and the following formula,

$$W_{\mathbf{0}_2,p}(1,k,1_{\delta_p(N)^2}) = |2|_p^{3/2}|N|_p(N,-1)_p \cdot (1-p^{-k-1}) \cdot \lim_{a,b\to\infty} \mathrm{Den}_{\delta(N)}(p^{-k}, M)$$

□

In the following, let $\Lambda(s) = \Gamma(s)\pi^{-s/2}\zeta(s)$ be the normalized Riemann zeta function.

Proposition 3.5.2. *Let* $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$ *and* $n_p = \nu_p(N)$*, we have*

$$\partial \mathrm{Eis}_{\mathbf{0}_2}(\mathsf{z}, \Delta(N)^2) = \log\det(\mathsf{y}) + 2 - 4\frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N} \frac{-n_p p^{n_p+1} + 2p^{n_p} + n_p p^{n_p-1} - 2}{p^{n_p-1}(p^2-1)} \cdot \log p$$

Proof. By similar arguments in [1, §4.4], we have

$$(\det\mathsf{y})^{3/4} \cdot E_{\mathbf{0}_2}(\mathsf{z}, s, \Delta(N)^2) = W_{\mathbf{0}_2}(g_{\mathsf{z}}, s, \Delta(N)^2) + \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} B(m(\gamma)g, s) + \Phi_{1_{\Delta(N)^2}}(g, s)$$

where the derivative of the middle term $\sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} B(m(\gamma)g, s)$ at $s = 0$ is 0, and

$$W_{\mathbf{0}_2}(g_{\mathsf{z}}, s, \Delta(N)^2) = \det(\mathsf{y})^{-s/2+3/4} \frac{(s-1)\Lambda(2s-1)}{(s+1)\Lambda(2s+2)} \cdot \prod_{p|N} A_p(s), \quad \Phi_{1_{\Delta(N)^2}}(g, s) = \det(\mathsf{y})^{s/2+3/4}.$$

The proposition follows from combining the above formulas and Lemma 3.5.1. □

3.5.2 The rank of T is 1.

Let N be a positive integer, let t be a nonzero integer, we fix a 2×2 matrix $T = \mathrm{diag}\{0, t\}$. Now we are going to define a quadratic character $\chi_t : \mathbb{A}^\times \to \mathbb{C}^\times$: When $-tN$ is not a square in $\mathbb{Q}$, the character χ_t is the quadratic Dirichlet character attached to the quadratic field extension $\mathbb{Q}(-tN)/\mathbb{Q}$; when $-tN$ is a square, let χ_t be the trivial character. When $-tN$ is not a square, let $-d$ be the fundamental discriminant of the field extension $\mathbb{Q}(-tN)/\mathbb{Q}$. When $-tN$ is a square, let $d = -1$, then there exists a positive number c such that

$$4Nt = c^2 d.$$

For any prime number p, we define the local zeta function associated to χ_t to be $L_p(s, \chi_t) =$

$(1-\chi_t(p)p^{-s})^{-1}$ and $L_\infty(s,\chi_t)$ to be the following,

$$L_\infty(s,\chi_t) = |d|^{s/2}\cdot\begin{cases}\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2}), & \text{if } t>0;\\ \pi^{-s/2}\Gamma(\frac{s}{2}), & \text{if } t<0.\end{cases}$$

Let $\Lambda(s,\chi_t) \coloneqq \prod_v L_v(s,\chi_t)$ be the completed L-function of the quadratic character χ_t, then $\Lambda(s,\chi_t)$ can be meromorphically continued to the whole complex plane and satisfies the functional equation $\Lambda(s,\chi_t)=\Lambda(1-s,\chi_t)$.

Define the following function for positive integers k,

$$g_p(k) = \frac{\mathrm{Den}(H^+_{2k+4},\langle t, Np^{-2}\rangle)}{\mathrm{Den}(H^+_{2k+4},\langle t, N\rangle)}, \tag{3.17}$$

Notice that if the p-adic valuation of N is 0 or 1, the function $g_p(k)=0$.

Lemma 3.5.3. *The function $g_p(k)$ is a rational function in p^{-k}, hence it can be meromorphically defined over $\mathbb{C}$. The function g_p also has the following functional equation,*

$$g_p(k) = p^{2k+1} g_p(-k-1).$$

Proof. We only consider the case that the p-adic valuation of N is greater or equal to 2. By definition made in Lemma 2.3.2, when k is a positive integer,

$$g_p(k) = \frac{\mathrm{Den}(H^+_{2k+4},\langle t, p^{-2}N\rangle)}{\mathrm{Den}(H^+_{2k+4},\langle t, N\rangle)} = \frac{\mathrm{Den}^\flat(p^{-k-1},\langle t, p^{-2}N\rangle)}{\mathrm{Den}^\flat(p^{-k-1},\langle t, N\rangle)} = \left.\frac{\mathrm{Den}^\flat(X,\langle t, p^{-2}N\rangle)}{\mathrm{Den}^\flat(X,\langle t, N\rangle)}\right|_{X=p^{-k-1}}.$$

Therefore $g_p(k)$ is a rational function in p^{-k} by Lemma 2.3.2, hence it can be meromorphically defined over $\mathbb{C}$. The functional equation is a consequence of Theorem 2.3.3. □

Lemma 3.5.4. *Let p be a prime number, for any positive integers k,*

$$\frac{W_{T,p}(1,k,1_{\delta_p(N)^2})}{\mathrm{Den}^\flat(p^{-k},\langle t,N\rangle)} = |2|_p|N|_p^{1/2}\gamma(\mathbb{V}_p)^2\cdot\frac{1-p^{-2k}g_p(k-1)}{1-\chi_t(p)p^{-k}}\cdot\begin{cases}1-p^{-2k-2}, & \text{if } p\nmid N;\\ 1-p^{-k-1}, & \text{if } p\mid N.\end{cases}, \tag{3.18}$$

$$\frac{W_{t,p}(1,k+\frac{1}{2},1_{\delta_p(N)})}{\mathrm{Den}^\flat(p^{-k-1},\langle t,N\rangle)} = |2N|_p^{1/2}\gamma(\mathbb{V}_p)\cdot(-1,N)_p\cdot\frac{1-p^{-2k-1}g_p(k)}{1-\chi_t(p)p^{-k-1}}\cdot\begin{cases}1-p^{-2k-2}, & \text{if } p\nmid N;\\ 1-p^{-k-1}, & \text{if } p\mid N.\end{cases}. \tag{3.19}$$

Proof. By Proposition 3.2.3 and Theorem 2.5.3, we know that

$$\begin{aligned}W_{T,p}(1,k,1_{\delta_p(N)^2}) &= |2|_p^{3/2}|N|_p\gamma(\mathbb{V}_p)^2\cdot\lim_{m\to\infty}\mathrm{Den}(\delta_p(N)\oplus H_{2k}^+,\langle t,p^m\rangle)\\ = |2|_p^{3/2}|N|_p\gamma(\mathbb{V}_p)^2\cdot\lim_{m\to\infty}&\frac{\mathrm{Den}(H_{2k+4}^+,\langle N,t,p^m\rangle)-p^{-2k}\mathrm{Den}(H_{2k+4}^+,\langle Np^{-2},t,p^m\rangle)}{\mathrm{Pden}(H_{2k+4}^+,\langle N\rangle)}.\end{aligned}$$

By Lemma 2.5.6, we have

$$\begin{aligned}&W_{T,p}(1,k,1_{\delta_p(N)^2})\\ &= |2|_p^{3/2}|N|_p\gamma(\mathbb{V}_p)^2\cdot\frac{\mathrm{Pden}(H_{2k+4}^+,\langle p\rangle)}{\mathrm{Pden}(H_{2k+4}^+,\langle N\rangle)}\cdot\frac{\mathrm{Den}(H_{2k+2}^+,\langle N,t\rangle)-p^{-2k}\mathrm{Den}(H_{2k+2}^+,\langle Np^{-2},t\rangle)}{1-p^{-2k}}\\ &= |2|_p^{3/2}|N|_p\gamma(\mathbb{V}_p)^2\cdot\mathrm{Den}(H_{2k+2}^+,\langle N,t\rangle)\cdot\frac{\mathrm{Pden}(H_{2k+4}^+,\langle p\rangle)}{\mathrm{Pden}(H_{2k+4}^+,\langle N\rangle)}\cdot(1-p^{-2k}g_p(k)).\end{aligned}$$

Then the formula (3.18) follows from the definition of local density polynomial in Lemma 2.3.2.

For the formula (3.19), we have the following identity by Proposition 3.2.3 and Theorem 2.5.3,

$$
\begin{aligned}
&W_{t,p}(1, k+\frac{1}{2}, 1_{\delta_p(N)}) \\
&= |2N|_p^{1/2}\gamma(\mathbb{V}_p)\cdot(-1,N)_p\cdot\frac{\mathrm{Den}(H_{2k+4}^{+},\langle N,t\rangle)-p^{-2k-1}\mathrm{Den}(H_{2k+4}^{+},\langle Np^{-2},t\rangle)}{\mathrm{Pden}(H_{2k+4}^{+},\langle N\rangle)} \\
&= |2N|_p^{1/2}\gamma(\mathbb{V}_p)\cdot(-1,N)_p\cdot\mathrm{Den}(H_{2k+4}^{+},\langle N,t\rangle)\cdot\frac{1-p^{-2k-1}g_p(k)}{\mathrm{Pden}(H_{2k+4}^{+},\langle N\rangle)}.
\end{aligned}
$$

Then the formula (3.19) also follows from the definition of local density polynomial in Lemma 2.3.2. □

For any prime number p, we consider the following meromorphic function for $s\in\mathbb{C}$,

$$\beta_p(s)=\frac{1+p^{s-1}}{1+p^{-s-1}}\cdot\frac{1-p^{-2s}g_p(s-1)}{1-g_p(s-1)}. \tag{3.20}$$

Corollary 3.5.5. *Let p be a prime number, the following identity holds for any integer k,*

$$
\begin{aligned}
\frac{W_{T,p}(1,k,1_{\delta_p(N)^2})\cdot\frac{\zeta_p(2k)}{L_p(k,\chi_t)}}{W_{t,p}(1,\frac{1}{2}-k,1_{\delta_p(N)})\cdot\frac{\zeta_p(2-2k)}{L_p(1-k,\chi_t)}} =&|2|_p|N|_p^{1/2}(-1,N)_p\gamma(\mathbb{V}_p)\cdot p^{(1-2k)\nu_p(c)}\cdot\frac{\zeta_p(2k)}{\zeta_p(2k+2)} \\
&\times\begin{cases}1, & \text{if } p\nmid N;\\ \beta_p(k), & \text{if } p\mid N.\end{cases}
\end{aligned}
$$

Proof. By Theorem 2.3.3 and the formula (2.3) in Remark 2.3.4, there is a functional equation,

$$\mathrm{Den}^\flat(p^{-k},\langle N,t\rangle)=p^{(1-2k)\nu_p(c)}\cdot\mathrm{Den}^\flat(p^{k-1},\langle N,t\rangle).$$

The corollary follows from the functional equation above and Lemma 3.5.4. □

Proposition 3.5.6. *Let T be a 2×2 matrix of rank 1 which can be diagonalized to* $\mathrm{diag}\{0,t\}$ *with $t\neq 0$. Let* $\mathsf{y}=\mathrm{diag}\{y_1,y_2\}$ *be a positive definite symmetric matrix, then for any complex number* *k, we have*

$$E_T(i\mathrm{y}, k, \Delta(N)^2) = y_1^{k/2} E_t(iy_2, \frac{1}{2} + k, \Delta(N)) \\ + y_1^{-k/2} \cdot \frac{k-1}{k+1} \cdot N^{-k} \cdot \frac{\Lambda(2-2k)}{\Lambda(2+2k)} \cdot \prod_{p|N} \beta_p(k) \cdot E_t(iy_2, \frac{1}{2} - k, \Delta(N)).$$

Proof. When $-tN$ is not a square, let $-d$ be the fundamental discriminant of the field extension $\mathbb{Q}(-tN)/\mathbb{Q}$. When $-tN$ is a square, let $d = -1$, then there exists a positive number c such that

$$4Nt = c^2 d.$$

For any complex number k, the following identity is proved in [1, Lemma 4.14] (see also [6, Proposition 5.7.7]),

$$W_{T,\infty}(g_{i\mathrm{y}}, k, \Delta(N)^2) \cdot \frac{\zeta_\infty(2k)}{L_\infty(k, \chi_t)} = -2iy_1^{-k/2+3/4} \cdot \frac{1-k}{1+k} \cdot \frac{\zeta_\infty(2k)}{\zeta_\infty(2k+2)} \cdot c^{2k-1} N^{1/2-k} \\ \times W_{t,\infty}(g_{iy_2}, \frac{1}{2} - k, \Delta(N)) \cdot \frac{\zeta_\infty(2-2k)}{L_\infty(1-k, \chi_t)}. \tag{3.21}$$

Recall that by Definition 3.2.4, $W_T(g_{i\mathrm{y}}, k, \Delta(N)^2) = W_{T,\infty}(g_{i\mathrm{y}}, k, \Delta(N)^2) \cdot \prod_p W_{T,p}(1, k, 1_{\delta_p(N)^2})$ and $W_t(g_{iy_2}, k, \Delta(N)) = W_{t,\infty}(g_{iy_2}, k, \Delta(N)) \cdot \prod_p W_{t,p}(1, k, 1_{\delta_p(N)})$.

Notice that both of the functions $W_T(g_{i\mathrm{y}}, k, \Delta(N)^2)$ and $W_t(g_{iy_2}, k, \Delta(N))$ can be meromorphically continued to whole complex plane. By combining the above identity (3.21), Corollary 3.5.5, and the functional equations $\Lambda(s) = \Lambda(1-s)$, $\Lambda(s, \chi_t) = \Lambda(1-s, \chi_t)$, we get

$$W_T(g_{i\mathrm{y}}, k, \Delta(N)^2) = y_1^{-k/2+3/4} \cdot \frac{k-1}{k+1} \cdot N^{-k} \cdot \frac{\Lambda(2-2k)}{\Lambda(2+2k)} \cdot \prod_{p|N} \beta_p(k) \cdot W_t(g_{iy_2}, \frac{1}{2} - k, \Delta(N)).$$

Then Lemma 3.4.1, Lemma 3.3.1 and formulas (3.9), (3.12) imply that

$$\begin{aligned} E_T(i\mathsf{y}, k, \Delta(N)^2) = y_1^{k/2} E_t(iy_2, \frac{1}{2} + k, \Delta(N)) \\ + y_1^{-k/2} \cdot \frac{k-1}{k+1} \cdot N^{-k} \cdot \frac{\Lambda(2-2k)}{\Lambda(2+2k)} \cdot \prod_{p|N} \beta_p(k) \cdot E_t(iy_2, \frac{1}{2} - k, \Delta(N)) \end{aligned}$$

□

Corollary 3.5.7. *Let T be a 2×2 matrix of rank 1 which can be diagonalized to* $\operatorname{diag}\{0, t\}$ *with* $t \neq 0$. *Let* $\mathsf{y} = \operatorname{diag}\{y_1, y_2\}$ *be a positive definite symmetric matrix, we have*

$$\begin{aligned} E_T'(i\mathsf{y}, 0, \Delta(N)^2) =& 2E_t'(iy_2, \frac{1}{2}, \Delta(N)) \\ &+ \left(\log y_1 + 2 + \log N + \frac{4\Lambda'(2)}{\Lambda(2)} - \sum_{p|N} \beta_p'(0) \right) E_t(iy_2, \frac{1}{2}, \Delta(N)). \end{aligned}$$

Proof. This follows from Proposition 3.5.6. □

Part II

Geometric side

Chapter 4: Arithmetic intersection theory on $\mathcal{X}_0(N)$

4.1 Cyclic group schemes

Let S be a scheme. Let G/S be a finite locally free group scheme over S. On every connected component of S, the rank of G is a constant, if the rank is a same number N for every connected component, we say that G has order N.

Let $\mathcal{O}_S$ be the structure sheaf of the scheme S. Let G/S be a finite locally free group scheme of order N, then the structure sheaf $\mathcal{O}_G$ of G is finite locally free of rank N as an $\mathcal{O}_S$-module. Any element $f \in \mathcal{O}_G$ acts on itself by left multiplication, this defines an $\mathcal{O}_S$-linear endomorphism of $\mathcal{O}_G$, the characteristic polynomial of this endomorphism

$$\det(T - f) = T^N - \operatorname{tr}(f)T^{N-1} + \cdots + (-1)^N \mathrm{N}(f),$$

is a monic polynomial in $\mathcal{O}_S[T]$ of degree N.

Definition 4.1.1. We say that a set of N not necessarily distinct points $\{P_i\}_{i=1}^N$ in $G(S)$ is a full set of sections of G/S if the following condition is fulfilled: for any element $f \in \mathcal{O}_G$, the following equality of polynomials with coefficients in $\mathcal{O}_S$ holds,

$$\det(T - f) = \prod_{i=1}^{N} (T - f(P_i)).$$

Definition 4.1.2. We say a finite locally free group scheme G/S of rank N is cyclic over S if there exits a section $P \in G(S)$ such that $\{aP\}_{a=1}^N$ forms a full set of sections of G/S, we say P is a generator of G over S. We say G/S is cyclic if G_T is cyclic over T after some fppf covering by some scheme $T \to S$.

Remark 4.1.3. The cyclicity of a group scheme is preserved under base change by the definition, i.e., if G/S is cyclic, then for any morphism $S' \to S$, the base change group scheme $G \times_S S'/S'$ is also cyclic.

Proposition 4.1.4. *Let S be a scheme. Let E/S be an elliptic curve over S. Let $G \subset E[N]$ be a finite locally free group scheme of order N over S. Then there exists a closed subscheme $S^{\mathrm{cyc}} \subset S$ which is universal for the condition "G is cyclic", in the sense that for any morphism $T \longrightarrow S$, the base change G_T/T is cyclic if and only if the morphism $T \longrightarrow S$ factors through the closed subscheme S^{cyc}.*

Proof. This is proved in [3, Theorem 6.4.1]. □

Lemma 4.1.5. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be a reduced, noetherian, quasi-separated and flat scheme over W. Let G be a finite locally free group scheme of order p^n over S, which is also embedded into an elliptic curve E/S. If for every generic point ξ of S, G_ξ doesn't factor through the multiplication-by-p morphism of E_ξ, then G is a cyclic group scheme.*

Proof. Since S is quasi-separated, quasi-compact and flat over W, then $S[\pi^{-1}]$ is dense in S since the scheme-theoretic image commutes with flat base change, therefore every generic point ξ lies in the open dense subscheme $S[\pi^{-1}]$. Let $\kappa(\xi)$ be the residue field of ξ, it has characteristic 0.

The group scheme G_ξ is of order p^n over the characteristic 0 field $\kappa(\xi)$, hence $G_\xi \simeq \prod_{i=1}^{k} \mathbb{Z}/p^{a_i}\mathbb{Z}$ where $\sum_{i=1}^{k} a_i = n$. The fact that G_ξ doesn't factor through multiplication-by-p morphism of E_ξ is equivalent to saying that $E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2 \not\subset G$. Hence the only possibility is $k = 1$ and $G_\xi \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Let S^{cyc} be the closed subscheme described by Proposition 4.1.4, we know that every generic point is contained in the closed subscheme S^{cyc}, hence $S^{\mathrm{cyc}} = S$ since S is reduced. □

Corollary 4.1.6. *Let W be a discrete valuation ring with residue characteristic p and uniformizer π. Let S be an integral noetherian scheme, quasi-separated and flat over W. Let G be a finite* *locally free group scheme of order p^n over S, which is also embedded into an elliptic curve E/S. If the isogeny $\pi_G : E \longrightarrow E/G$ doesn't factor through multiplication-by-p morphism of E, then G is a cyclic group scheme.*

Proof. The isogeny $\pi_G : E \longrightarrow E/G$ factors through the multiplication-by-p morphism of E if and only if $\ker([p]_E)$ is contained (as a Cartier divisor on E) in G, this is a closed condition on the base scheme S by [3, Lemma 1.3.4]. We use $\mathcal{I} \neq 0$ (since the morphism π_G doesn't factor through the multiplication-by-p morphism of E) to denote the ideal sheaf of this closed subscheme of S, it is functorial with respect to the base change of S.

Let ξ be the only generic point of S, then G_ξ doesn't factor through the multiplication-by-p morphism because otherwise $\mathcal{I}_\xi = 0$, but the injection $\mathcal{I} \longrightarrow \mathcal{I}_\xi$ will imply that $\mathcal{I} = 0$, which is a contradiction. Then the corollary follows from Lemma 4.1.5. □

4.2 $\Gamma_0(N)$-structures on elliptic curves

Let S be a scheme. We say a scheme C over S is a smooth curve over S if the structure morphism $C \longrightarrow S$ is a smooth proper morphism of relative dimension 1.

Definition 4.2.1. A closed immersion $i : D \longrightarrow C$ is called an effective Cartier divisor if the following conditions hold,

(i) The closed subscheme D is flat over S;

(ii) The ideal sheaf $\mathcal{I}(D)$ defining D is an invertible $\mathcal{O}_C$-module.

Lemma 4.2.2. *If C/S is a smooth curve, then any section $s \in C(S)$ defines an effective Cartier divisor on C, denoted by $[s]$.*

Proof. This is proved in [3, Lemma 1.2.2]. □

Given two effective Cartier divisors D and D' on C/S, we can define their sum $D + D'$. It is an effective Cartier divisor on C/S defined locally by the product of the defining equations of D and D'. Explicitly, if $S = \operatorname{Spec} R$ and if over an affine open subscheme $\operatorname{Spec} A$ of C, the Cartier divisor D (resp. D') is defined by an element $f \in A$ (resp. $g \in A$), then the Cartier divisor $D + D'$ is defined by the equation fg.

Lemma 4.2.3. *Suppose E/S and E'/S are two elliptic curves over S, $\pi : E \longrightarrow E'$ is an isogeny, i.e., π is surjective and* $\ker(\pi)$ *is a finite flat group scheme locally of finite presentation over S. Then* $\ker(\pi) \longrightarrow E$ *is an effective Cartier divisor.*

Proof. By cancellation theorem of morphisms of locally finite presentation, any morphism between abelian schemes are locally of finite presentation. Hence π is locally of finite presentation, and therefore $\ker(\pi)$ is also locally of finite presentation over S. Then the lemma follows from of [3, Lemma 1.2.3]. □

Definition 4.2.4. We say an isogeny $\pi : E \longrightarrow E'$ between two elliptic curves E and E' is a cyclic N-isogeny if $\pi^\vee \circ \pi = N$, and there exists an fppf covering of S by a scheme $T \longrightarrow S$ with a point $P \in \ker(\pi)(T)$ such that the following equality of Cartier divisors on E_T holds:

$$\ker(\pi)_T = \sum_{a=1}^{N} [aP].$$

A $\Gamma_0(N)$-structure on an elliptic curve E/S is a cyclic N-isogeny $E \xrightarrow{\pi} E'$.

Lemma 4.2.5. *Let $\pi : E \longrightarrow E'$ be an isogeny between two elliptic curves E and E', the isogeny π is a N-cyclic isogeny if and only if* $\ker(\pi)$ *is a cyclic group scheme of order N.*

Proof. By [3, Theorem 1.10.1], the set $\{aP\}_{a=1}^{N}$ (where $P \in \ker(\pi)(S)$) forms a full set of sections of $\ker(\pi)$ if and only if we have the following equality of effective Cartier divisors in E/S,

$$\ker(\pi) = \sum_{a=1}^{N} [aP],$$

which is exactly the definition of the cyclicity of a N-isogeny. □

Example 4.2.6. (a) Suppose $\tau = x + iy \in \mathbb{H}_1$, we consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, and a finite subgroup K generated by $1/N$ inside E_τ, then $\pi : E_\tau \longrightarrow E_\tau/K$ is a cyclic isogeny.

(b) Suppose E/S is an elliptic curve over a $\mathbb{F}_p$-scheme S, then for any $n \geq 1$, the n^{th} iterated relative Frobenius

$$F^n : E \to E^{(p^n)},$$

is a cyclic p^n-isogeny. The origin $P = 0$ is a generator of $\ker(F^n)$ because $\ker(F^n) \simeq \mathcal{O}_S[T]/(T^{p^n})$ Zariski locally (cf. [3, Lemma 12.2.1]).

Let $\mathcal{E}ll$ be the stack of elliptic curves, i.e., for an arbitrary scheme S, $\mathcal{E}ll(S)$ is a groupoid whose objects are elliptic curves $p : E \to S$ and morphisms are isomorphisms of elliptic curves over S. We use $\mathcal{Y}_0(N)$ to denote the stack which consists of all the $\Gamma_0(N)$-structures on elliptic curves, i.e., for a scheme S, $\mathcal{Y}_0(N)(S)$ is a groupoid whose objects are cyclic N-isogenies $(E \xrightarrow{\pi} E')$ where E and E' are elliptic curves over S, a morphism between two cyclic isogenies $(E_1 \xrightarrow{\pi_1} E_1')$ and $(E_2 \xrightarrow{\pi_2} E_2')$ is a pair of isomorphisms of elliptic curves $a : E_1 \xrightarrow{\sim} E_2$ and $a' : E_1' \xrightarrow{\sim} E_2'$ such that $a' \circ \pi_1 = \pi_2 \circ a$. We have the following functors,

$$\begin{aligned} s : \mathcal{Y}_0(N) &\longrightarrow \mathcal{E}ll, \\ (E/S \xrightarrow{\pi} E'/S) &\longmapsto E/S. \end{aligned}$$

Lemma 4.2.7. *Both $\mathcal{Y}_0(N)$ and $\mathcal{E}ll$ are 2-dimensional Deligne–Mumford stacks. The functor $s : \mathcal{Y}_0(N) \to \mathcal{E}ll$ is finite flat of degree $\psi(N) = N \cdot \prod_{l|N}(1 + l^{-1})$, and representable by schemes, s is also étale over* $\operatorname{Spec} \mathbb{Z}[1/N]$.

Proof. This is proved in [3, Theorem 5.1.1]. The key input is that a finite order group scheme is automatically étale if the order is invertible in the base scheme. □

For a $\mathbb{Z}_{(p)}$-scheme S, a geometric point $\overline{s}$ of S and an elliptic curve E over S. Let $E_{\overline{s}}$ be the base change of E to $\overline{s}$. Let $T^p(E_{\overline{s}})$ (resp. $V^p(E_{\overline{s}})$) be the integral (resp. rational) Tate module of the elliptic curve $E_{\overline{s}}$. A $\mathbb{Z}_{(p)}^\times$-isogeny $f : E \to E'$ over S is a quasi-isogeny and there exists a prime-to-p number M, such that $M \circ f$ is an isogeny. Let $V^p(f)$ be the homomorphism on rational Tate modules induced by f.

Lemma 4.2.8. *Let $\mathcal{E}ll_{(p)}$ be the localization of the stack $\mathcal{E}ll$ to* $\operatorname{Spec}\mathbb{Z}_{(p)}$. *Then $\mathcal{E}ll_{(p)}$ can be described by the following stack: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{E}ll_{(p)}(S)$ is a groupoid whose objects are pairs $(E/S, \overline{\eta^p})$, where $\overline{\eta^p}$ is a $\pi_1(S, \overline{s})$-invariant* $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$*-equivalence class of isomorphism*

$$\eta^p : V^p(E_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$$

A morphism between two objects $(E/S, \overline{\eta^p})$ and $(E'/S, \overline{\eta'^p})$ is a $\mathbb{Z}_{(p)}^{\times}$-isogeny $f : E \longrightarrow E'$ over S such that $\overline{\eta^p} = \overline{V^p(f) \circ \eta'^p}$.

Proof. We temporarily use $\mathcal{E}ll'$ to denote the stack described in the lemma. It suffices to show that for a connected scheme S over $\operatorname{Spec}\mathbb{Z}_{(p)}$, there is a category equivalence between $\mathcal{E}ll(S)$ and $\mathcal{E}ll'(S)$. We first construct a functor F from $\mathcal{E}ll(S)$ to $\mathcal{E}ll'(S)$. Given an elliptic curve E over S, and a geometric point $\overline{s}$ of S, we choose an isomorphism

$$\eta^p : T^p(E_{\overline{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$$

then clearly the $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-orbit of $\overline{\eta^p}$ is $\pi_1(S, \overline{s})$-invariant (because $\pi_1(S, \overline{s})$ acts linear on $T^p(E_{\overline{s}})$). We define $F(E) = (E, \overline{\eta^p})$, this functor is independent of the choice of η^p.

Now we prove that this functor is essentially surjective and fully faithful. For essential surjectivity, we pick an arbitrary object $(E/S, \overline{\eta^p})$ of $\mathcal{E}ll'(S)$, by the work of Lan [28, Corollary 1.3.5.4], there is a $\mathbb{Z}_{(p)}^{\times}$-isogeny $f : E' \longrightarrow E$, such that $\eta'^p = \eta^p \circ V^p(f) : V^p(E'_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}_f^p)^2$ maps $T^p(E'_{\overline{s}})$ to $(\hat{\mathbb{Z}}^p)^2$. Therefore the object $(E/S, \overline{\eta^p})$ is isomorphic to $(E'/S, \overline{\eta'^p})$, which is the essential image of $E' \in Ob\,\mathcal{E}ll(S)$.

Next we show that there is an isomorphism,

$$\mathrm{Hom}_{\mathcal{E}ll(S)}(E, E') \simeq \mathrm{Hom}_{\mathcal{E}ll'(S)}((E, \overline{\eta^p}), (E', \overline{\eta'^p})) \tag{4.1}$$

This is clearly injective by the above discussion. Now we pick an arbitrary element f from the right hand side, then f is a $\mathbb{Z}_{(p)}^{\times}$-isogeny, and $\eta'^p = \eta^p \circ V^p(f)$. There exists an integer M prime to p, such that $\tilde{f} = M \circ f$ is an isogeny from E to E'. We claim that this isogeny factors through the multiplication-by-M map, i.e., f itself is an isogeny. By the relation $\eta'^p = \eta^p \circ V^p(f)$ and the construction above, $V^p(f)$ maps $T^p(E_{\overline{s}})$ isomorphically to $T^p(E'_{\overline{s}})$, then obviously $\tilde{f}$ maps $E'_{\overline{s}}[M] \simeq E'[M]_{\overline{s}}$ to 0, this holds for every geometric point $\overline{s}$ of S, then since S is a $\mathbb{Z}_{(p)}$-scheme and rigidity result proved by Mumford, Fogarty and Kirwan [29, Proposition 6.1], we know the isogeny $\tilde{f}$ vanishes on $E'[M]$, hence f itself is an isogeny. Now $\ker(f)$ is a finite flat group scheme over S of order prime to p, but since $V^p(f)$ maps $T^p(E_{\overline{s}})$ isomorphically to $T^p(E'_{\overline{s}})$, this group scheme must be trivial, i.e., f is an isomorphism, therefore it comes from an element of the left hand side of (4.1). □

Remark 4.2.9. We consider the following Deligne–Mumford stack,

$$\mathcal{H}^\circ = \mathcal{E}ll \times_{\mathbb{Z}} \mathcal{E}ll \tag{4.2}$$

For any prime p, we use $\mathcal{H}^\circ_{(p)}$ to denote the localization of $\mathcal{H}^\circ$ to $\operatorname{Spec}\mathbb{Z}_{(p)}$.

There is a similar description of the stack $\mathcal{H}^\circ_{(p)}$ as follows, for any $\mathbb{Z}_{(p)}$-scheme S, the groupoid $\mathcal{H}^\circ_{(p)}(S)$ consists of pairs $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}))$, where $\overline{\eta^p}$ (resp. $\overline{\eta'^p}$) is a $\pi_1(S, \overline{s})$-invariant $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$-equivalence class of isomorphism $V^p(E_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}^p_f)^2$ (resp. $V^p(E'_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}^p_f)^2$).

For any $N \in \mathbb{Z}_{>0}$, let w_N be the following 2×2 matrix,

$$w_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the following stack $\mathcal{Y}_0(N)'_{(p)}$ over $\operatorname{Spec}\mathbb{Z}_{(p)}$: for every $\mathbb{Z}_{(p)}$-scheme S, $\mathcal{Y}_0(N)'_{(p)}(S)$ is a groupoid whose objects are pairs $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)})$, where $E \xrightarrow{\pi} E'$ is a cyclic N-isogeny and $\overline{(\eta^p, \eta'^p)}$ is a pair of $\pi_1(S, \overline{s})$-invariant $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-equivalence class (we will specify the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ in (4.4)) of isomorphisms

$$\eta^p : V^p(E_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}^p_f)^2, \ \ \eta'^p : V^p(E'_{\overline{s}}) \xrightarrow{\simeq} (\mathbb{A}^p_f)^2.$$

which maps $T^p(E_{\overline{s}})$ and $T^p(E'_{\overline{s}})$ to $(\hat{\mathbb{Z}}^p)^2$, and the isomorphism η'^p is determined by η^p by the following commutative diagram,

$$\begin{tikzcd}
V^p(E_{\overline{s}}) \arrow[r, "\eta^p"] \arrow[d, "V^p(\pi)"] & (\mathbb{A}_f^p)^2 \arrow[d, "w_N"] \\
V^p(E'_{\overline{s}}) \arrow[r, "\eta'^p"] & (\mathbb{A}_f^p)^2.
\end{tikzcd} \tag{4.3}$$

A morphism from $(E_1 \overset{\pi_1}{\to} E'_1, \overline{(\eta_1^p, \eta_1'^p)})$ to $(E_2 \overset{\pi_2}{\to} E'_2, \overline{(\eta_2^p, \eta_2'^p)})$ is a pair (f, f') of isomorphisms $f : E_1 \to E_2$ and $f' : E'_1 \to E'_2$ such that $f' \circ \pi_1 = \pi_2 \circ f$ and $\overline{(\eta_1^p, \eta_1'^p)} = \overline{(\eta_2^p \circ V^p(f), \eta_2'^p \circ V^p(f'))}$ as $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbits. The action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ on the pair (η^p, η'^p) is given by

$$g \cdot ((\eta^p, \eta'^p)) = (g \circ \eta^p, w_N g w_N^{-1} \circ \eta'^p). \tag{4.4}$$

Lemma 4.2.10. *Let* $\mathcal{Y}_0(N)_{(p)}$ *be the localization of* $\mathcal{Y}_0(N)$ *to* $\mathbb{Z}_{(p)}$. *There is an isomorphism* $G : \mathcal{Y}_0(N)_{(p)} \to \mathcal{Y}_0(N)'_{(p)}$ *of stacks over* $\operatorname{Spec} \mathbb{Z}_{(p)}$.

Proof. Let S be a scheme over $\operatorname{Spec} \mathbb{Z}_{(p)}$, and an object $(E \overset{\pi}{\to} E')$ in the groupoid $\mathcal{Y}_0(N)_{(p)}(S)$. Let $\overline{s}$ is a geometric point of S, the cyclicity of π implies that $\pi_{\overline{s}}$ is also cyclic, since l is invertible in $\operatorname{Spec} \kappa(\overline{s})$ if $l \neq p$, there exist isomorphisms $\eta^p : T^p(E_{\overline{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ and $\eta'^p : T^p(E'_{\overline{s}}) \simeq (\hat{\mathbb{Z}}^p)^2$ such that $\omega_N \circ \eta^p = \eta'^p \circ T^p(\pi)$. Now we consider a different choice of (η^p, η'^p), say $(\tilde{\eta}^p, \tilde{\eta}'^p)$, satisfying the above conditions. Then $\tilde{\eta}'^p$ differs η^p by an element $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$, i.e. $\tilde{\eta}^p = g \circ \eta^p$, correspondingly $\tilde{\eta}'^p = \omega_N g \omega_N^{-1} \circ \eta'^p$. However, $\omega_N g \omega_N^{-1} \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ since both η'^p and $\tilde{\eta}'^p$ give isomorphisms from $T^p(E_{\overline{s}})$ to $(\hat{\mathbb{Z}}^p)^2$, therefore $g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \cap \omega_N^{-1} \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \omega_N = \Gamma_0(N)(\hat{\mathbb{Z}}^p)$, thus the $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$-orbit $\overline{(\eta^p, \eta'^p)}$ is well-defined. We define $G((E \overset{\pi}{\to} E')) = ((E \overset{\pi}{\to} E'), \overline{(\eta^p, \eta'^p)})$. For a pair of isomorphisms (f, f'), where $f : E_1 \to E'_1$ and $f' : E_2 \to E'_2$, define $G((f, f')) = (f, f')$.

Now it suffices to show that for a connected scheme S over $\operatorname{Spec} \mathbb{Z}_{(p)}$, the functor $G(S) : \mathcal{Y}_0(N)_{(p)}(S) \to \mathcal{Y}_0(N)'_{(p)}(S)$ is an equivalence of categories. This functor is essentially surjective by definition, now we show that it is fully faithful, i.e., the following morphism between sets is bijective,

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}((E_1 \xrightarrow{\pi_1} E_1'),((E_2 \xrightarrow{\pi_2} E_2')))\\
&\qquad\qquad \xrightarrow{G} \mathrm{Hom}_{\mathcal{Y}_0(N)_{(p)}(S)}((E_1 \xrightarrow{\pi_1} E_1', \overline{(\eta_1^p, \eta_1'^p)}), (E_2 \xrightarrow{\pi_2} E_2', \overline{(\eta_2^p, \eta_2'^p)})),\\
&\qquad (f, f') \longmapsto (f, f').
\end{aligned}$$

but this is clearly bijective by the definition. □

There is a natural morphism from $\mathcal{Y}_0(N)_{(p)}$ to $\mathcal{H}^\circ_{(p)}$, i.e., $(E \xrightarrow{\pi} E') \longrightarrow (E, E')$. By Remark 4.2.9 and Lemma 4.2.10, we can also describe it as follows

$$\begin{aligned}
\mathcal{Y}_0(N)_{(p)} &\longrightarrow \mathcal{H}^\circ_{(p)},\\
(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}) &\longmapsto ((E, E'), (\overline{\eta^p}, \overline{\eta'^p})).
\end{aligned} \tag{4.5}$$

4.2.1 Compactification of $\mathcal{Y}_0(N)$

Next we introduce the compactification of the moduli stack $\mathcal{Y}_0(N)$. Let S be a scheme, we first introduce the notion of Néron n-gons.

Definition 4.2.11. For any integer $n \geq 1$ and an scheme S, the Néron n-gon over S is the coequalizer of

$$\bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} S \rightrightarrows \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} \mathbb{P}^1_S.$$

where the top (resp. the bottom) closed immersion includes the i^{th} copy of S as the 0 (resp. the ∞) section of the i^{th} (resp. $(i+1)^{\text{st}}$) copy of $\mathbb{P}^1_S$.

Definition 4.2.12. A generalized elliptic curve over a scheme S is the data of

- A proper, flat, finitely presented morphism $E \to S$ each of whose geometric fibers is either a smooth connected curve of genus 1 or a Néron n-gon for some $n \geq 1$;
- An S-morphism $E^{\text{sm}} \times_S E \xrightarrow{+} E$ that restricts to a commutative S-group scheme structure on

E^{sm} for which + becomes an S-group action such that via the pullback of line bundles the action + induces the trivial action of E^{sm} on $\mathrm{Pic}^0_{E/S}$.

We will use $\mathcal{X}$ to denote the moduli stack consisting of generalized elliptic curves whose degenerate fibers are Néron 1-gons, i.e., for a scheme S, $\mathcal{X}(S)$ is a groupoid whose objects are generalized elliptic curves E over S and whose geometric fibers are either elliptic curves or Néron 1-gons. The following result is proved in [4].

Lemma 4.2.13. *$\mathcal{X}$ is a proper smooth 2-dimensional Deligne–Mumford stack.*

Proof. This is proved in Theorem 3.1.6 of [4]. □

We have a natural morphism of Deligne–Mumford stacks $\mathcal{E}ll \to \mathcal{X}$, which sends an elliptic curve E over S to itself. This morphism is an open immersion, i.e., the stack $\mathcal{E}ll$ is an open substack of the stack $\mathcal{X}$. Recall that we have a finite flat representable morphism $\mathcal{Y}_0(N) \to \mathcal{E}ll$ by Lemma 4.2.7, let $\mathcal{X}_0(N)$ be the normalization of $\mathcal{Y}_0(N)$ with respect to this morphism. A moduli description of $\mathcal{X}_0(N)$ in terms of level structures on the generalized elliptic curves can be found in [4, section 5.9]. The stack $\mathcal{Y}_0(N)$ can be realized as an open substack of the stack $\mathcal{X}_0(N)$ based on this description. Notice that there are two natural morphisms from $\mathcal{Y}_0(N)$ to $\mathcal{Y}_0(1)$ given as follows

$$\begin{aligned} p_1, p_2 : \mathcal{Y}_0(N) &\longrightarrow \mathcal{Y}_0(1) \\ (E_1 \xrightarrow{\pi} E_2) &\xmapsto{p_1} E_1; \\ (E_1 \xrightarrow{\pi} E_2) &\xmapsto{p_2} E_2. \end{aligned}$$

Both of the morphism $p_1, p_2 : \mathcal{Y}_0(N) \to \mathcal{Y}_0(1)$ extends to morphisms from $\mathcal{X}_0(N)$ to $\mathcal{X}_0(1)$, we still use the symbols p_1 and p_2 to denote them.

Theorem 4.2.14. *The stack $\mathcal{X}_0(N)$ is a regular proper 2-dimensional Deligne–Mumford stack, both of the morphisms p_1 and p_2 are finite flat of degree $\psi(N) := N \cdot \prod_{p|N} (1 + p^{-1})$ over $\mathcal{X}_0(1)$. Moreover, the stack $\mathcal{X}_0(N)$ is smooth over $\mathbb{Z}[\frac{1}{N}]$*

Proof. This is proved in [4, Theorem 5.13], the degree of p_1 and p_2 are computed in [3, (13.4.9)]. □

Lemma 4.2.15. *Let* $N = M_1 \cdot M_2$ *where* $\text{g.c.d.}(M_1, M_2) = 1$*, there is a natural isomorphism*

$$\mathcal{X}_0(N) \simeq \mathcal{X}_0(M_1) \times_{p_1, \mathcal{X}_0(1), p_1} \mathcal{X}_0(M_2).$$

Proof. This follows from [3, Corollary 1.10.15]. □

4.3 Arithmetic Chow groups $\widehat{\mathbf{CH}}^{\bullet}(\mathcal{X}_0(N))$

We apply the arithmetic intersection theory developed by Gillet and Soulé in [11], [30] and [31] to the regular proper flat Deligne–Mumford stack $\mathcal{X}_0(N)$. We obtain the following arithmetic Chow ring of $\mathcal{X}_0(N)$,

$$\widehat{\mathrm{CH}}^{\bullet}(\mathcal{X}_0(N)) = \bigoplus_{n=0}^{2} \widehat{\mathrm{CH}}^{n}(\mathcal{X}_0(N)).$$

Roughly speaking, a class in $\widehat{\mathrm{CH}}^{n}(\mathcal{X}_0(N))$ is represented by an arithmetic cycle $(\mathcal{Z}, g_{\mathcal{Z}})$, where $\mathcal{Z}$ is a codimension n closed substack of $\mathcal{X}_0(N)$, with $\mathbb{C}$-coefficients, and $g_{\mathcal{Z}}$ is a Green current for $\mathcal{Z}(\mathbb{C})$, i.e., $g_{\mathcal{Z}}$ is a current on the proper smooth complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$ of degree $(n-1, n-1)$ for which there exists a smooth form ω such that

$$dd^c(g) + \delta_{\zeta} = [\omega].$$

holds; here $[\omega]$ is the current defined by integration against the smooth form ω. The rational arithmetic cycles are those of the form $\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), \iota_*[-\log(|\tilde{f}|^2)])$, where $f \in \kappa(\mathcal{Z})^{\times}$ is a rational function on a codimension $n-1$ integral substack $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}_0(N)$, together with classes of the form $(0, \partial\eta + \bar{\partial}\eta')$. By definition, the arithmetic Chow group $\mathrm{CH}^n(\mathcal{X}_0(N))$ is the quotient of the space of arithmetic cycles by the $\mathbb{C}$-subspace spanned by those rational cycles.

Let $\mathcal{Z}$ be an irreducible codimension 2 cycle on $\mathcal{X}_0(N)$, then $\mathcal{Z}$ is a Deligne–Mumford stack over $\mathbb{F}_p$ for some prime number p and the groupoid $\mathcal{Z}(\overline{\mathbb{F}}_p)$ would be a singleton with a finite automorphism group $\mathrm{Aut}(\mathcal{Z})$, the rational function field $\kappa(\mathcal{Z})$ of $\mathcal{Z}$ is a finite extension of $\mathbb{F}_p$. Clearly $\delta_{\mathcal{Z}} = 0$ because $\mathcal{Z}(\mathbb{C}) = \emptyset$.

Let $(\mathcal{Z}, g) = (\sum\limits_i n_i[\mathcal{Z}_i], g)$ be an arithmetic cycle of codimension 2 where each $\mathcal{Z}_i$ is an irreducible codimension 2 cycle on $\mathcal{X}_0(N)$. We define the degree map as follows,

$$\begin{aligned} \widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) &\longrightarrow \mathbb{C}, \\ [(\mathcal{Z}, g)] &\longmapsto \sum_i n_i \cdot \frac{\log|\kappa(\mathcal{Z}_i)|}{|\mathrm{Aut}(\mathcal{Z}_i)|} + \frac{1}{2} \int\limits_{\mathcal{X}_0(N)(\mathbb{C})} g. \end{aligned} \tag{4.6}$$

here the integration $\int\limits_{\mathcal{X}_0(N)_\mathbb{C}} g$ is the integration of the constant function 1 on $\mathcal{X}_0(N)_\mathbb{C}$ against the $(1,1)$-current g. It is a finite number since the stack $\mathcal{X}_0(N)$ is proper. This number is independent of the choice of representing element $(\mathcal{Z}, g)$ as a consequence of the product formula (cf. [30, §3.4.3]).

4.3.1 Extended arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S})$

Definition 4.3.1. Let $\mathcal{S}$ be the set of cusps on the complex curve $\mathcal{X}_0(N)_\mathbb{C}$. For $P \in \mathcal{S}$ and $\varepsilon > 0$, denote by $B_\varepsilon(P) \subset \mathcal{X}_0(N)_\mathbb{C}$ the open disk of radius ε centered at P and $\mathcal{X}_0(N)_\varepsilon = \mathcal{X}_0(N)_\mathbb{C} \setminus \bigcup\limits_{P \in \mathcal{S}} B_\varepsilon(P)$. Let t be a local parameter at a point $P \in \mathcal{S}$, for a line bundle $\mathcal{L}$ on $\mathcal{X}_0(N)$, a singular metric $h_\mathcal{L}$ on the induced complex line bundle $\mathcal{L}_\infty$ on $\mathcal{X}_0(N)_\mathbb{C}$ is called hermitian, logarithmically singular (with respect to $\mathcal{S}$), if the following two conditions hold:

(a). $h_\mathcal{L}$ is a smooth, hermitian metric on $\mathcal{L}_\infty$ restricted to $\mathcal{Y}_0(N)_\mathbb{C}$;

(b). For each $P \in \mathcal{S}$ and any section l of $\mathcal{L}$, there exists a real number $\alpha_{\widehat{\mathcal{L}},l,P}$ and a positive, continuous function $\varphi_{\widehat{\mathcal{L}},l,P}$ defined on $B_\varepsilon(P)$ and smooth away from the origin such that the equality

$$h_\mathcal{L}(l(t)) = -\log(|t|^2)^{\alpha_{\widehat{\mathcal{L}},l,P}} \cdot |t|^{\mathrm{ord}_P(l)} \cdot \varphi_{\widehat{\mathcal{L}},l,P}(t)$$

holds for all $t \in B_\varepsilon(P)\backslash\{P\}$; furthermore, there exist positive constants $\beta_{\widehat{\mathcal{L}},l,P}$ and $\rho_{\widehat{\mathcal{L}},l,P}$ such that the inequalities

$$\left|\frac{\partial\varphi_{\widehat{\mathcal{L}},l,P}(t)}{\partial t}\right| \leq \frac{\beta_{\widehat{\mathcal{L}},l,P}}{|t|^{1-\rho_{\widehat{\mathcal{L}},l,P}}}, \quad \left|\frac{\partial\varphi_{\widehat{\mathcal{L}},l,P}(t)}{\partial \bar{t}}\right| \leq \frac{\beta_{\widehat{\mathcal{L}},l,P}}{|t|^{1-\rho_{\widehat{\mathcal{L}},l,P}}}, \quad \left|\frac{\partial^2\varphi_{\widehat{\mathcal{L}},l,P}(t)}{\partial t\partial \bar{t}}\right| \leq \frac{\beta_{\widehat{\mathcal{L}},l,P}}{|t|^{2-\rho_{\widehat{\mathcal{L}},l,P}}}.$$

Example 4.3.2. Let $\pi^{\mathrm{univ}} : E^{\mathrm{univ}} \to E'^{\mathrm{univ}}$ be the universal cyclic N-isogeny between generalized elliptic curves $E^{\mathrm{univ}} \overset{p^{\mathrm{univ}}}{\longrightarrow} \mathcal{X}_0(N)$ and $E'^{\mathrm{univ}} \overset{p'^{\mathrm{univ}}}{\longrightarrow} \mathcal{X}_0(N)$ over the modular curve $\mathcal{X}_0(N)$. Let $\omega_N \coloneqq p_*^{\mathrm{univ}}\left(\Omega^1_{E^{\mathrm{univ}}/\mathcal{X}_0(N)}\right)$. This bundle can be metrized on the complex curve $\mathcal{Y}_0(N)_{\mathbb{C}}$ in the following way: let f be a section of ω_N, for any $\tau = u + iv \in \mathcal{H}^+$, the metric $\|\cdot\|$ at τ is determined by the formula

$$\|f\|_\tau^2 = 2\sqrt{\pi}e^{-\frac{\gamma}{2}}v \cdot |f(\tau)|^2, \tag{4.7}$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant, this metric is hermitian and logarithmically singular with respect to the set $\mathcal{S}$ by the work of Du and Yang [2, Theorem 5.1]. In the rest of this paper, we will use $\widehat{\omega}_N = (\omega_N, \|\cdot\|)$ to denote this hermitian, logarithmically singular line bundle.

Let $\widehat{\mathrm{Pic}}(\mathcal{X}_0(N), \mathcal{S})$ be the group of isomorphism classes of hermitian, logarithmically singular line bundle on $\mathcal{X}_0(N)$ with respect to the set $\mathcal{S}$, we also denote it by $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S})$ and call it the extended arithmetic Chow group of $\mathcal{X}_0(N)$ with respect to the set $\mathcal{S}$.

Definition 4.3.3. Let $\widehat{\mathcal{L}} = (\mathcal{L}, h_{\mathcal{L}})$ and $\widehat{\mathcal{M}} = (\mathcal{M}, h_{\mathcal{M}})$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}_0(N)$ with respect to the set $\mathcal{S}$, let l, m be non-trivial, global sections, whose induced divisors on $\mathcal{X}_0(N)_{\mathbb{C}}$ have no points in common. Then the generalized arithmetic intersection number $\widehat{\mathcal{L}} \cdot \widehat{\mathcal{M}}$ is defined by

$$\widehat{\mathcal{L}} \cdot \widehat{\mathcal{M}} \coloneqq (l, m)_{\mathrm{fin}} + \langle l, m\rangle_\infty;$$

here $(l, m)_{\mathrm{fin}}$ is defined by Serre's Tor-formula, which specializes to

$$(l, m)_{\mathrm{fin}} = \sum_{x \in \mathcal{X}_0(N)} \log \#\mathcal{O}_{\mathcal{X}_0(N),x}/(l_x, m_x),$$

where l_x, m_x are the local equations of l, m respectively at the point $x \in \mathcal{X}_0(N)$ and

$$\langle l, m\rangle_\infty = -\log(h_{\mathcal{M}}(m))(\mathrm{div}(l) - \sum_{P\in\mathcal{S}} \mathrm{ord}_P(l)\cdot P) + \sum_{P\in\mathcal{S}} \mathrm{ord}_P(l)\left(\alpha_{\widehat{\mathcal{M}},m,P} - \log(\varphi_{\widehat{\mathcal{M}},m,P}(0))\right) \tag{4.8}$$

$$-\lim_{\varepsilon\to 0}\left(\sum_{P\in\mathcal{S}} \mathrm{ord}_P(l)\cdot \alpha_{\widehat{\mathcal{M}},m,P}\cdot \log(-\log \varepsilon^2) + \int_{\mathcal{X}_0(N)_\varepsilon} \log h_{\mathcal{L}}(l)\cdot c_1(\widehat{\mathcal{M}})\right).$$

Proposition 4.3.4. *The formula (4.8) induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S}) \times \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S}) \longrightarrow \mathbb{C}.$$

Proof. This is proved by Kühn in [32]. □

Example 4.3.5. The pairing $\widehat{\omega}_N \cdot \widehat{\omega}_N$ has been computed by Kühn [32] and Bost, adjusting for the normalization of the metric in [2, Lemma 7.3], the result is

$$\widehat{\omega}_N \cdot \widehat{\omega}_N = \langle \widehat{\omega}_N, \widehat{\omega}_N\rangle = \frac{\psi(N)}{24}\left(\frac{1}{2} - \frac{\Lambda'(-1)}{\Lambda(-1)}\right).$$

4.4 The Atkin–Lehner involution on $\mathcal{X}_0(N)$

Lemma 4.4.1. *Let S be a scheme, and $\pi : E \to E'$ be a cyclic N-isogeny between elliptic curves over S, then $\pi^\vee : E' \to E$ is also a cyclic N-isogeny.*

Proof. We only need to show that the order N quotient group scheme $E[N]/\ker(\pi)$ is cyclic, and this is proved in [3, Corollary 5.5.4(3)]. □

With Lemma 4.4.1, we can define the Atkin–Lehner involution W_N on the stack $\mathcal{Y}_0(N)$ by the following,

$$\begin{aligned} W_N : \mathcal{Y}_0(N) &\longrightarrow \mathcal{Y}_0(N) \\ (E \xrightarrow{\pi} E') &\longmapsto (E' \xrightarrow{\pi^\vee} E) \end{aligned}$$

Lemma 4.4.2. *The Atkin–Lehner involution W_N extends to the compactified stack $\mathcal{X}_0(N)$.*

Proof. By [4, Proposition 4.2.7 (c)], the cyclicity condition of a N-isogeny is closed, hence $\mathcal{X}_0(N)$ is a closed substack of $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$. Let $\iota : \mathcal{X}_0(1) \times \mathcal{X}_0(1) \to \mathcal{X}_0(1) \times \mathcal{X}_0(1)$ be the involution which switches the two copies of $\mathcal{X}_0(1)$, let $\mathcal{X}_0(N)' = \mathcal{X}_0(N)_{\mathcal{X}_0(1)\times\mathcal{X}_0(1),\iota}\mathcal{X}_0(1)\times\mathcal{X}_0(1)$ be the image of $\mathcal{X}_0(N)$ under the involution W_N, it is also a closed substack of $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$, but $\mathcal{X}_0(N)$ and $\mathcal{X}_0(N)'$ have common dense open dense substack $\mathcal{Y}_0(N)$, hence $\mathcal{X}_0(N) = \mathcal{X}_0(N)'$, therefore the Atkin–Lehner involution W_N extends to the compactified stack $\mathcal{X}_0(N)$. □

4.5 Cusps of the modular curve $\mathcal{X}_0(N)$

Let $\mathcal{H} := \mathcal{X}_0(1) \times \mathcal{X}_0(1)$ be the product of the smooth Deligne–Mumford stack $\mathcal{X}_0(1)$. It can be viewed as the compactification of the stack $\mathcal{H}^\circ$ defined in (4.2). Let $P_\infty(1) : \operatorname{Spec}\mathbb{Z} \to \mathcal{X}_0(1)$ be the cuspidal divisor of the stack $\mathcal{X}_0(1)$ which corresponds to the standard 1-gon over $\mathbb{Z}$ (we refer to [33, §2.1] for detailed definitions of standard 1-gon). We can view $P_\infty(1)$ as an element in $\mathrm{CH}^1(\mathcal{X}_0(1))$, then we define the following element in $\mathrm{CH}^1(\mathcal{H})$:

$$\mathsf{Cusp}(\mathcal{H}) := P_\infty(1) \times \mathcal{X}_0(1) + \mathcal{X}_0(1) \times P_\infty(1).$$

It's easy to see that the two curves $P_\infty(1) \times \mathcal{X}_0(1)$ and $\mathcal{X}_0(1) \times P_\infty(1)$ intersect transversally at the point $(P_\infty(1), P_\infty(1))$.

The natural morphism $\mathcal{X}_0(N) \xrightarrow{(p_1,p_2)} \mathcal{H}$ is a closed immersion. We define the cuspidal divisor of $\mathcal{X}_0(N)$ to be the pullback of $\mathsf{Cusp}(\mathcal{H})$, i.e.,

$$\mathsf{Cusp}(\mathcal{X}_0(N)) = (p_1, p_2)^*\mathsf{Cusp}(\mathcal{H}) \in \mathrm{CH}^1(\mathcal{X}_0(N)).$$

Let $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$ be the formal completion of $\mathcal{X}_0(N)$ along the cuspidal divisor $\mathrm{Cusp}(\mathcal{X}_0(N))$. It is well-known that $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1)) \simeq \mathrm{Spf}\,\mathbb{Z}[[q]]$. The two morphisms $p_1, p_2 : \mathcal{X}_0(N) \to \mathcal{X}_0(1)$ induce two morphisms from $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$ to $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$, we still use p_1 and p_2 to denote them, both of them are finite flat of degree of $\psi(N)$.

Proposition 4.5.1. *Let $n \geq 0$ be an integer, the formal scheme $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n))$ is a disjoint union as follows:*

$$\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n)) = \coprod_{\substack{-n\leq a\leq n \\ a\equiv n \bmod 2}} \mathsf{C}^a(p^n),$$

where every $\mathsf{C}^a(p^n)$ is finite flat over $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$ via the morphisms p_1 and p_2, such that $W_N(\mathsf{C}^a(p^n)) = \mathsf{C}^{-a}(p^n)$. Let $\deg_1(\mathsf{C}^a(p^n))$ (resp. $\deg_2(\mathsf{C}^a(p^n))$) be the finite flat degree of $\mathsf{C}^a(p^n)$ over $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$ via the morphism p_1 (resp. p_2), then

$$\deg_1(\mathsf{C}^a(p^n)) = \begin{cases} \varphi(p^{(n-a)/2}), & \text{if } 0 \leq a \leq n; \\ p^{-a}\varphi(p^{(n+a)/2}), & \text{if } -n \leq a < 0. \end{cases}$$

$$\deg_2(\mathsf{C}^a(p^n)) = \begin{cases} p^{a}\varphi(p^{(n-a)/2}), & \text{if } 0 \leq a \leq n; \\ \varphi(p^{(n+a)/2}), & \text{if } -n \leq a < 0. \end{cases}$$

More explicitly, if we view $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n))$ as a $\mathbb{Z}[[q]]$-formal scheme via the morphism p_1, then

$$\mathsf{C}^a(p^n) \simeq \begin{cases} \mathrm{Spf}\,\mathbb{Z}[\zeta_{p^{(n-a)/2}}][[q]], & \text{if } 0 \leq a \leq n; \\ \mathrm{Spf}\,\mathbb{Z}[\zeta_{p^{(n+a)/2}}][[q]][z]/(z^{p^{-a}} - \zeta_{p^{(n+a)/2}}q), & \text{if } -n \leq a < 0, \end{cases} \tag{4.9}$$

where for any $k \geq 1$, ζ_{p^k} is a primitive p^k-th root of unity.

Proof. It is proved by Edixhoven in [34, §1.2.2] that if we view $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n))$ as a $\mathbb{Z}[[q]]$-formal scheme via the morphism p_1, then

$$\begin{aligned}\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n)) \simeq \operatorname{Spf}\mathbb{Z}[[q]] &\coprod \operatorname{Spf}\mathbb{Z}[[q^{p^{-n}}]] \\ &\coprod \coprod_{\substack{c+d=n\\ c\geq d>0}} \operatorname{Spf}\mathbb{Z}[\zeta_{p^d}][[q]] \\ &\coprod \coprod_{\substack{c+d=n\\ d>c>0}} \operatorname{Spf}\mathbb{Z}[\zeta_{p^c}][[q]][z]/(z^{p^{d-c}} - \zeta_{p^c}q),\end{aligned} \tag{4.10}$$

where z is the parameter of the cusp of the second copy $\mathcal{X}_0(1)$ in $\mathcal{H}$. Therefore $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n))$ is a disjoint union of $n+1$ formal schemes, let $\mathsf{C}^a(p^n)$ be one of the formal schemes according to formula (4.9), then $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n)) = \coprod\limits_{\substack{-n\leq a\leq n\\ a\equiv n \bmod 2}} \mathsf{C}^a(p^n)$ by (4.10). The finite flat degree of $\mathsf{C}^a(p^n)$ over $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$ via the morphisms p_1 and p_2 can be computed explicitly by (4.9).

Since we have $p_1 \circ W_N = p_2$ and $p_2 \circ W_N = p_1$, the finite flat degrees of $W_N(\mathsf{C}^a(p^n))$ over $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$ via the morphisms p_1 and p_2 equal to that of $\mathsf{C}^{-a}(p^n)$. Hence we have $W_N(\mathsf{C}^a(p^n)) = \mathsf{C}^{-a}(p^n)$. □

Remark 4.5.2. Proposition 4.5.1 can be easily generalized to arbitrary positive integer N. Let $N = \prod\limits_{i=1}^{r} q_i^{n_i}$ be the prime decomposition of N. Then for every positive integer $M|N$, let $M = \prod\limits_{i=1}^{r} q_i^{m_i}$, there is an irreducible component $\mathsf{C}^M(N) \simeq \mathsf{C}^{n_1-2m_1}(q_1^{n_1}) \times_{\mathsf{C}^0(1)} \cdots \times_{\mathsf{C}^0(1)} \mathsf{C}^{n_r-2m_r}(q_r^{n_r})$ such that

$$\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N)) = \coprod_{M|N} \mathsf{C}^M(N).$$

The finite flat degree of $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$ over $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(1))$ via p_1 and p_2 are given by

$$\deg_1(\mathsf{C}^M(N)) = \prod_{i=1}^{r} \deg_1(\mathsf{C}^{n_i-2m_i}(q_i^{n_i})), \quad \deg_2(\mathsf{C}^M(N)) = \prod_{i=1}^{r} \deg_2(\mathsf{C}^{n_i-2m_i}(q_i^{n_i})).$$

The component $\mathsf{C}^1(N)$ is isomorphic to $\operatorname{Spf}\mathbb{Z}[[q]]$ via the morphism p_1 by (4.9). Denote the corresponding cusp point by $P_\infty(N)$, it corresponds to a morphism $P_\infty(N) : \operatorname{Spec}\mathbb{Z} \to \mathcal{X}_0(N)$.

When $N = 1$, this definition agrees with our previous definition of $P_\infty(1)$.

Let W_N be the Atkin–Lehner involution of $\mathcal{X}_0(N)$, it induces an isomorphism $\mathsf{C}^M(N) \xrightarrow{W_N} \mathsf{C}^{N/M}(N)$. Let $P_0(N) = W_N \circ P_\infty(N) : \operatorname{Spec}\mathbb{Z} \to \mathcal{X}_0(N)$ be the composition of the automorphism W_N of $\mathcal{X}_0(N)$ and the $\mathbb{Z}$-point $P_\infty(N)$. When $N = 1$, we have $P_\infty(1) = P_0(1)$.

There is an explicit description of the cusps of the complex modular curve $\mathcal{X}_0(N)_{\mathbb{C}}$ (cf. [35, §3.8]),

$$P_{\frac{aM}{N}}, \text{ where } M|N \text{ and } a \in (\mathbb{Z}/(M, N/M))^\times.$$

Especially, when $N = p^n$ for some non-negative integer n, the cusps of the complex modular curve $\mathcal{X}_0(p^n)_{\mathbb{C}}$ are

$$P_{\frac{a}{p^k}}, \text{ where } 0 \le k \le n \text{ and } a \in (\mathbb{Z}/p^{\min\{k,n-k\}})^\times.$$

Lemma 4.5.3. *Let P be a cusp of the complex modular curve $\mathcal{X}_0(p^n)_{\mathbb{C}}$, let* $\mathrm{ra}_1(P)$ *(resp.* $\mathrm{ra}_2(P)$*) be the ramification degree of the morphism p_1 (resp. p_2) at P. Suppose $P = P_{\frac{a}{p^k}}$ for some integer $0 \le k \le n$ and $a \in (\mathbb{Z}/p^{\min\{k,n-k\}})^\times$, then*

$$\mathrm{ra}_1(P_{a/p^k}) = \begin{cases} 1, & \text{if } \frac{n}{2} \le k \le n; \\ p^{n-2k}, & \text{if } 0 \le k < \frac{n}{2}. \end{cases} \tag{4.11}$$

$$\mathrm{ra}_2(P_{a/p^k}) = \begin{cases} p^{2k-n}, & \text{if } \frac{n}{2} \le k \le n; \\ 1, & \text{if } 0 \le k < \frac{n}{2}. \end{cases} \tag{4.12}$$

Proof. For any cusp point P, let $\mathrm{Stab}(P) \in \Gamma(p^n)$ be the stabilizer of the cusp. For the cusp point $P_\infty(p^n)$, we have

$$\mathrm{Stab}(P_\infty(p^n)) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}. \right\}.$$

Let $A \in \mathbb{Z}$ be a lift of a to $\mathbb{Z}$, then A is prime to p, hence there exists integers b, d such that

$Ab - dp^k = 1$. Let

$$\gamma = \begin{pmatrix} A & d \\ p^k & b \end{pmatrix},$$

then $\mathrm{Stab}(P_{\frac{a}{p^k}}) = \gamma \mathrm{Stab}(P_\infty(p^n))\gamma^{-1} \cap \Gamma_0(p^n)$. By simple calculations,

$$\gamma \mathrm{Stab}(P_\infty(p^n))\gamma^{-1} \cap \Gamma_0(p^n) = \left\{ \begin{pmatrix} 1 - p^k Ax & xA^2 \\ -p^{2k}x & 1 + p^k Ax \end{pmatrix} : x \in p^{n-2k}\mathbb{Z} \cap \mathbb{Z}. \right\},$$

the formula (4.11) follows from the calculations above. Notice that $p_2 = p_1 \circ W_N$, hence the formula (4.12) follows from (4.11). □

Corollary 4.5.4. *Let* $P_{\frac{a}{p^k}}$ *be a cusp of the complex modular curve* $\mathcal{X}_0(p^n)_{\mathbb{C}}$*, where* $0 \le k \le n$ *and* $a \in (\mathbb{Z}/p^{\min\{k,n-k\}})^\times$*, then* $P_{\frac{a}{p^k}}$ *belongs to the component* $\mathsf{C}^{2k-n}(p^n)$ *of* $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(p^n))$*, i.e.,* $P_{\frac{a}{p^k}} \in \mathsf{C}^{2k-n}(p^n)(\mathbb{C})$.

Proof. This follows from formula (4.9) in Proposition 4.5.1, and formulas (4.11) and (4.12) in Lemma 4.5.3. □

Remark 4.5.5. Corollary 4.5.4 can also be generalized to arbitrary N. Let $N = \prod_{i=1}^{r} q_i^{n_i}$ be the prime decomposition N. Then the modular curve $\mathcal{X}_0(N)$ decomposes into product of $\mathcal{X}_0(q_i^{n_i})$ by Lemma 4.2.15. Let $P_{\frac{aM}{N}}$ be a cusp point of $\mathcal{X}_0(N)$ where $M|N$ and $a \in (\mathbb{Z}/(M, N/M))^\times$. Suppose $M = \prod_{i=1}^{r} q_i^{m_i}$. Let $a_i \in (\mathbb{Z}/q_i^{\min\{n_i - m_i, m_i\}})^\times$ be the image of a. The cusp point $P_{\frac{aM}{N}}$ decomposes into the product of cusp points $P_{\frac{a_i}{q^{n_i - m_i}}} \in \mathcal{X}_0(q_i^{n_i})$, hence it belongs to the component $\mathsf{C}^M(N)$ of $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$.

4.6 Reduction mod p of $\mathcal{X}_0(N)$

Let p be a fixed prime, the reduction mod p of the stack $\mathcal{X}_0(N)$ has been studied extensively in [3, Chapter 13]. For an $\mathbb{F}$-scheme S, an integer $m \ge 0$ and an elliptic curve over S, let $E^{(p^m)}$ be the m-th Frobenius twist of E, we use $F^m : E \to E^{(p^m)}$ to denote the m-th iterated Frobenius morphism which has degree p^m, and $V^m : E^{(p^m)} \to E$ be the m-th iterated Verschiebung morphism which is also the dual morphism of F^m.

For all integers a, define a closed substack $\mathcal{Y}^a \subset \mathcal{Y}_0(1) \times \mathcal{Y}_0(1)$ over $\mathbb{F}_p$ as follows:

- $a \geq 0$: For an $\mathbb{F}_p$-scheme S, the groupoid $\mathcal{Y}^a(S)$ consists of objects $(E, E^{(p^a)})$.
- $a < 0$: For an $\mathbb{F}_p$-scheme S, the groupoid $\mathcal{Y}^a(S)$ consists of objects $(E^{(p^{-a})}, E)$.

Let $n \geq 0$ be an integer, for all integers a such that $n \geq |a|$ and $n \equiv a \pmod 2$. Define the following morphism

$$\pi_{a,E} := \begin{cases} E \xrightarrow{F^{(n+a)/2}} E^{(p^{(n+a)/2})} \xrightarrow{V^{(n-a)/2}} E^{(p^a)}, & \text{if } a \geq 0; \\ E^{(p^{-a})} \xrightarrow{F^{(n+a)/2}} E^{(p^{(n-a)/2})} \xrightarrow{V^{(n-a)/2}} E, & \text{if } a < 0. \end{cases}$$

Lemma 4.6.1. *Let $n \geq 0$ be an integer, for all integers a such that $n \geq |a|$ and $n \equiv a \pmod 2$. The morphisms $\pi_{a,E}$ are cyclic of order p^n.*

Proof. This is proved in [3, Theorem 13.3.5]. □

Remark 4.6.2. The above lemma implies that the closed immersion $\mathcal{Y}^a \to \mathcal{Y}_0(1) \times \mathcal{Y}_0(1)$ factors through the closed substack $\mathcal{Y}_0(p^n)$ if $n \geq |a|$ and $n \equiv a \pmod 2$.

Theorem 4.6.3. *Let $n \geq 0$ be an integer, let $\mathcal{Y}_0(p^n)_{\mathbb{F}} := \mathcal{Y}_0(p^n) \times_{\mathbb{Z}} \mathbb{F}$ be the base change to $\mathbb{F} = \overline{\mathbb{F}}_p$ of the stack $\mathcal{Y}_0(p^n)$. For all integers a such that $n \geq |a|$ and $n \equiv a \pmod 2$, the stack $\mathcal{Y}^a$ has the following properties:*

(a). The stack $\mathcal{Y}^a$ is a 1-dimensional Deligne–Mumford stack.

(b). The composite morphism $\mathcal{Y}^a \to \mathcal{Y}_0(p^n)_{\mathbb{F}_p} \xrightarrow{p_1} \mathcal{Y}_0(1)_{\mathbb{F}_p}$ is an isomorphism if $a \geq 0$; is finite flat of degree p^{-a} if $a \leq 0$.

(c). The following equality holds: $W_N(\mathcal{Y}^a) = \mathcal{Y}^{-a}$.

(d). The stack $\mathcal{Y}^a$ contains every supersingular point of $\mathcal{Y}_0(p^n)_{\mathbb{F}_p}$. Let $P = (E \xrightarrow{\pi} E')$ be a supersingular $\mathbb{F}$-point of $\mathcal{Y}_0(p^n)$. Let O_P be the completed local ring of the stack $\mathcal{Y}_0(1)_{\mathbb{F}} \times \mathcal{Y}_0(1)_{\mathbb{F}}$ at P, let $O_{a,P}$ be the completed local ring of $\mathcal{Y}^a_{\mathbb{F}} := \mathcal{Y}^a \times_{\mathbb{F}_p} \mathbb{F}$ at the point corresponding to the

pair (E,E'), then there exists an isomorphism $\mathcal{O}_P \simeq \mathbb{F}[[t_1,t_2]]$ such that the closed immersion $\mathcal{Y}^a_{\mathbb{F}} \to \mathcal{Y}_0(1)_{\mathbb{F}} \times \mathcal{Y}_0(1)_{\mathbb{F}}$ induces the following isomorphism,

$$\mathcal{O}_{a,P} \simeq \begin{cases} \mathbb{F}[[t_1,t_2]]/(t_1 - t_2^{p^a}), & \text{if } a \geq 0; \\ \mathbb{F}[[t_1,t_2]]/(t_1^{p^{-a}} - t_2), & \text{if } a \leq 0. \end{cases}$$

(e). Over the open substack $\mathcal{Y}_0(p^n)^{\mathrm{ord}}_{\mathbb{F}_p} = \mathcal{Y}_0(p^n)_{\mathbb{F}_p} - \{$supersingular points$\}$ of $\mathcal{Y}_0(p^n)_{\mathbb{F}_p}$, the morphism $\bigsqcup_{\substack{-n\leq a\leq n \\ n\equiv a \pmod 2}} \mathcal{Y}^a \to \mathcal{Y}_0(p^n)^{\mathrm{red}}_{\mathbb{F}_p}$ is an isomorphism.

Proof. These are the main results of [3, §13]. □

Remark 4.6.4. Let $\mathcal{X}^a$ be the scheme-theoretic closure of $\mathcal{Y}^a$ inside $\mathcal{H}$, then Theorem 4.6.3 is still true if we replace all the symbol $\mathcal{Y}$ by the symbol $\mathcal{X}$. The closed immersion $\mathcal{X}^a \to \mathcal{H}$ factors through the closed substack $\mathcal{X}_0(p^n)$ if $n \geq |a|$ and $n \equiv a \pmod 2$.

Definition 4.6.5. Let N be a positive integer. Let $n = v_p(N)$ be the p-adic valuation of N, let $N_p = p^{-n}N$. For all integers a such that $n \geq |a|$ and $n \equiv a \pmod 2$, define $\mathcal{Y}^a_p(N) = \mathcal{Y}^a \times_{p_1,\mathcal{Y}_0(1),p_1} \mathcal{Y}_0(N_p)$ and $\mathcal{X}^a_p(N) = \mathcal{X}^a \times_{p_1,\mathcal{X}_0(1),p_1} \mathcal{X}_0(N_p)$. It is a closed substack of $\mathcal{X}_0(N)$.

Remark 4.6.6. The stack $\mathcal{X}^a_p(N)$ (resp. $\mathcal{Y}^a_p(N)$) is a closed substack of $\mathcal{H}$ (resp. $\mathcal{H}^\circ$) as long as $n \geq |a|$ and $n \equiv a \pmod 2$. The stack $\mathcal{X}^a_p(N)$ is the scheme-theoretic closure of $\mathcal{Y}^a_p(N)$ in $\mathcal{H}$. Both $\mathcal{X}^a_p(N)$ and $\mathcal{Y}^a_p(N)$ are independent of the p-adic valuation $v_p(N)$ of N since the definition of $\mathcal{X}^a$ and $\mathcal{Y}^a$ are independent of $v_p(N)$ by Remark 4.6.4.

The independence can also be understood from the following moduli description: Let S be an $\mathbb{F}_p$-scheme, the groupoid $\mathcal{Y}^a_p(N)(S)$ consists of the following objects:

- $a \geq 0$: $\left((E,E'), E^{(p^a)} \xrightarrow{j} E'\right)$ where j is a cyclic isogeny of order N_p.
- $a < 0$: $\left((E^{(p^{-a})},E'), E \xrightarrow{j} E'\right)$ where j is a cyclic isogeny of order N_p.

Theorem 4.6.7. *Let $n = v_p(N)$ be the p-adic valuation of N, let $N_p = p^{-n}N$. For any integer a such that $-n \leq a \leq n$ and $n \equiv a \pmod 2$, let $\mathcal{X}_p^a(N) = \mathcal{X}^a \times_{p_1, \mathcal{X}_0(1), p_1} \mathcal{X}_0(N_p)$ be a closed substack of $\mathcal{X}_0(N)$, then we have the following identity in* $\mathrm{CH}^1(\mathcal{X}_0(N))$*:*

$$\mathrm{div}(p) = \sum_{\substack{-n \leq a \leq n \\ n \equiv a \pmod 2}} \varphi(p^{(n-|a|)/2}) \cdot \mathcal{X}_p^a(N).$$

Proof. The multiplicities of the irreducible components $\mathcal{X}_p^a(N)$ are byproducts of [3, Theorem 13.3.5, Theorem 13.4.7]. □

We now consider the reduction mod p of cusps. Let $n = v_p(N)$ be the p-adic valuation of N. For any integer a such that $-n \leq a \leq n$ and $n \equiv a \pmod 2$, let $N_p = p^{-n}N$. Let $\mathsf{C}^a(p^n, N_p) := \mathsf{C}^a(p^n) \times_{p_1, \mathcal{X}_0(1), p_1} \mathcal{X}_0(N_p)$, and $\mathsf{C}_p^a(p^n, N_p)$ be its reduction mod p. Let $C_p^n(p^n, N_p)$ (resp. $C_p^{-n}(p^n, N_p)$) be the curve $\mathcal{X}_p^n(N)$ (resp. $\mathcal{X}_p^{-n}(N)$) over $\mathbb{F}_p$, and $C_p^a(p^n, N_p)$ be the non-reduced curve over $\mathbb{F}_p$ corresponding to $p^{(n-|a|)/2-1}(p-1)\mathcal{X}_p^a(N)$ when $-n < a < n$.

Proposition 4.6.8. *Let $n = v_p(N)$ be the p-adic valuation of N, let $N_p = p^{-n}N$. For any integer a such that $-n \leq a \leq n$ and $n \equiv a \pmod 2$, the formal scheme $\mathsf{C}_p^a(p^n, N_p)$ is the formal completion of the curve $C_p^a(p^n, N_p)$ along its cuspidal locus.*

Proof. By Proposition 4.5.1, the formal completion of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ along its cuspidal locus is a disjoint union of $n+1$ formal schemes $\mathsf{C}_p^a(p^n, N_p)$ by the definition of $\mathsf{C}_p^a(p^n, N_p)$ above. It is finite flat over $\mathsf{C}_p^0(1,1)$ (which is the reduction mod p of the completion of the cuspidal divisor on $\mathcal{X}_0(1)$) via the morphisms p_1 and p_2, with degrees equal to $\psi(N_p) \cdot \deg_1(\mathsf{C}^a(p^n))$ and $\psi(N_p) \cdot \deg_2(\mathsf{C}^a(p^n))$ respectively.

For $i = 1$ or 2, let $\deg_i(C_p^a(p^n, N_p))$ be the finite flat degree of the curve $C_p^a(p^n, N_p)$ over $\mathcal{X}_p(1)$ via the morphism p_i. By the definition of the curve $C_p^a(p^n, N_p)$, we have

$$\deg_1(C_p^a(p^n, N_p)) = \psi(N_p) \cdot \begin{cases} \varphi(p^{(n-a)/2}), & \text{if } 0 \leq a \leq n; \\ p^{-a}\varphi(p^{(n+a)/2}), & \text{if } -n \leq a < 0. \end{cases}$$

$$\deg_2(C_p^a(p^n, N_p)) = \psi(N_p) \cdot \begin{cases} p^a \varphi(p^{(n-a)/2}), & \text{if } 0 \leq a \leq n; \\ \varphi(p^{(n+a)/2}), & \text{if } -n \leq a < 0. \end{cases}$$

The formal completion of the curve $C_p^a(p^n, N_p)$ is a closed formal subscheme of the formal completion of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ along its cuspidal locus, with the same finite flat degrees over $\mathsf{C}_p^0(1,1)$ via the morphisms p_1 and p_2 as the closed formal subscheme $\mathsf{C}_p^a(p^n, N_p)$, hence they equals to each other. □

As a short summary, we take the example that $N = p^n$ and draw the following table.

Cusp points	$\frac{a}{p^n}$	$\frac{a}{p^{n-1}}$	$\cdots$	$\frac{a}{p^k}$	$\cdots$	$\frac{a}{p}$	a
Choice of a	1	$(\mathbb{Z}/p\mathbb{Z})^\times$	$\cdots$	$(\mathbb{Z}/(p^k, p^{n-k})\mathbb{Z})^\times$	$\cdots$	$(\mathbb{Z}/p\mathbb{Z})^\times$	1
Ramification via p_1	1	1	$\cdots$	$\max\{1, p^{n-2k}\}$	$\cdots$	p^{n-2}	p^n
Ramification via p_2	p^n	p^{n-2}	$\cdots$	$\max\{1, p^{2k-n}\}$	$\cdots$	1	1
Components	$\mathsf{C}^n(p^n)$	$\mathsf{C}^{n-2}(p^n)$	$\cdots$	$\mathsf{C}^{2k-n}(p^n)$	$\cdots$	$\mathsf{C}^{2-n}(p^n)$	$\mathsf{C}^{-n}(p^n)$
Mod p	$C_p^n(p^n)$	$C_p^{n-2}(p^n)$	$\cdots$	$C_p^{2k-n}(p^n)$	$\cdots$	$C_p^{2-n}(p^n)$	$C_p^{-n}(p^n)$

4.7 Intersection numbers of irreducible components of $\mathcal{X}_0(N)_{\mathbb{F}_p}$

Let $\widehat{\mathcal{X}}_p^a(N) = (\mathcal{X}_p^a(N), 0)$ be the corresponding class in the codimension 1 arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$. Recall that there is a pairing $\langle \cdot, \cdot \rangle : \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N)) \times \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N)) \to \mathbb{C}$ given by Definition 4.3.3.

Lemma 4.7.1. *Let $n = v_p(N)$ be the p-adic valuation of N, let $N_p = p^{-n}N$, for integers a, b such that $-n \leq a \neq b \leq n$ and $a, b \equiv n \pmod 2$, we have*

$$\langle \widehat{\mathcal{X}}_p^a(N), \widehat{\mathcal{X}}_p^b(N) \rangle = \log(p) \cdot \begin{cases} \frac{\psi(N_p)(p-1)}{24} p^{\min\{|a|,|b|\}}, & \text{if } ab \geq 0; \\ \frac{\psi(N_p)(p-1)}{24}, & \text{if } ab \leq 0. \end{cases} \tag{4.13}$$

For integer a such that $-n \leq a \leq n$ and $a \equiv n \pmod 2$, we have

$$\langle \widehat{\mathcal{X}}_p^a(N), \widehat{\mathcal{X}}_p^a(N) \rangle = \log(p) \cdot \begin{cases} -\frac{\psi(N_p)p^{|a|}}{12}, & \text{if } |a| \neq n; \\ -\frac{\psi(N_p)(p-1)p^{n-1}}{24}, & \text{if } |a| = n. \end{cases} \tag{4.14}$$

Proof. Let first assume $a, b \geq 0$ and $a \neq b$, then by the definition of the pairing $\langle \cdot, \cdot \rangle$ and (d), (e) of Theorem 4.6.3,

$$\begin{aligned} \langle \widehat{\mathcal{X}}_p^a(N), \widehat{\mathcal{X}}_p^b(N) \rangle &= \sum_{P \in \mathcal{X}_0^{\mathrm{ss}}(N)(\mathbb{F})} \frac{1}{\#\mathrm{Aut}(P)} \cdot \mathrm{Length}\left(\mathbb{F}[[t_1, t_2]]/(t_1 - t_2^{p^a}, t_1 - t_2^{p^b})\right) \cdot \log(p) \\ &= \sum_{P \in \mathcal{X}_0^{\mathrm{ss}}(N)(\mathbb{F})} \frac{1}{\#\mathrm{Aut}(P)} \cdot p^{\min\{a,b\}} \cdot \log(p). \end{aligned}$$

By (e) of Theorem 4.6.3, the stack $\mathcal{X}_0(p^n)$ has the same number of supersingular $\mathbb{F}$-points as the stack $\mathcal{X}_0(1)$, while the later one has been computed explicitly by the following formula (for example, see [3, Corollary 12.4.6]),

$$\sum_{P \in \mathcal{X}_0^{\mathrm{ss}}(1)(\mathbb{F})} \frac{1}{\#\mathrm{Aut}(P)} = \frac{p-1}{24}.$$

Moreover, by Lemma 4.2.15 we know that $\mathcal{X}_0(N)_{\mathbb{F}_p} \simeq \mathcal{X}_0(p^n)_{\mathbb{F}_p} \times_{\mathcal{X}_0(1)_{\mathbb{F}_p}} \mathcal{X}_0(N_p)_{\mathbb{F}_p}$, recall the fact that $\mathcal{X}_0(N_p)_{\mathbb{F}_p}$ is finite flat of degree $\psi(N_p)$ over $\mathcal{X}_0(1)_{\mathbb{F}_p}$, hence

$$\sum_{P \in \mathcal{X}_0^{\mathrm{ss}}(N)(\mathbb{F})} \frac{1}{\#\mathrm{Aut}(P)} = \frac{\psi(N_p)(p-1)}{24},$$

therefore $\langle \widehat{\mathcal{X}}_p^a(N), \widehat{\mathcal{X}}_p^b(N) \rangle = \log(p) \cdot \frac{\psi(N_p)(p-1)}{24} p^{\min\{a,b\}}$. The other cases of formula (4.13) can be proved similarly.

Recall that the principal arithmetic divisor associated to the constant function p is

$$\widehat{\mathrm{div}}(p) = (\mathrm{div}(p), -\log(p^2)).$$

It is 0 in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$, hence for any integer a such that $-n \leq a \leq n$ and $a \equiv n \pmod 2$,

$$\langle \widehat{\mathcal{X}}_p^a(N), \mathcal{X}_p^n(N) + \mathcal{X}_p^{-n}(N) + \sum_{\substack{-n<b<n \\ n\equiv b \pmod 2}} p^{(n-|b|)/2-1}(p-1)\mathcal{X}_p^b(N)\rangle = \langle \widehat{\mathcal{X}}_p^a(N), (\mathrm{div}(p), 0)\rangle \tag{4.15}$$

$$= \langle \widehat{\mathcal{X}}_p^a(N), (0, \log(p^2))\rangle = 0.$$

therefore formula (4.14) can be proved by combining (4.13) and (4.15). □

Let $n = \nu_p(N) \geq 0$ be an integer and $N_p = p^{-n}N$, let $\widehat{\mathcal{X}}_p(N)$ be the following element in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,

$$\widehat{\mathcal{X}}_p(N) = \frac{n}{2}\widehat{\mathcal{X}}_p^n(N) - \frac{n}{2}\widehat{\mathcal{X}}_p^{-n}(N) + \sum_{\substack{-n<a<n \\ a\equiv n \bmod 2}} \frac{a}{2}\cdot p^{(n-|a|)/2-1}(p-1)\widehat{\mathcal{X}}_p^a(N). \tag{4.16}$$

Corollary 4.7.2. *We have $W_N^*(\widehat{\mathcal{X}}_p(N)) = -\widehat{\mathcal{X}}_p(N)$. Let $n_p = \nu_p(N) \geq 0$ be an integer, then*

$$\langle \widehat{\mathcal{X}}_p(N), \widehat{\mathcal{X}}_p(N)\rangle = \frac{\psi(N)}{24}\cdot\frac{-n_p p^{n_p+1} + 2p^{n_p} + n_p p^{n_p-1} - 2}{p^{n_p-1}(p^2-1)}\cdot\log(p).$$

Proof. The fact $W_N^*(\widehat{\mathcal{X}}_p(N)) = -\widehat{\mathcal{X}}_p(N)$ follows from the definition of the element $\widehat{\mathcal{X}}_p(N)$. The intersection number can be computed by Lemma 4.7.1. □

4.8 An explicit section of the Hodge line bundle

In this section, we define and study the associated divisor class of an explicit rational section of the line bundle $\widehat{\omega}_N^{\otimes 12\varphi(N)}$.

For any positive integer N, define the following function a_N on the set of positive integers,

$$a_N(t) = \sum_{r|t}\mu(\frac{t}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)},$$

where $\mu(\cdot)$ is the Möbius function and r ranges over all the positive integers dividing t.

Lemma 4.8.1. *The function a_N has the following properties,*

(a). $\sum\limits_{t|N} a_N(t) = \varphi(N)$ and $\sum\limits_{t|N} t^{-1}a_N(t) = 0$.

(b). Let p be a prime number such that $p|N$, for any positive integer t such that $pt|N$,

$$a_N(pt) = pa_{p^{-1}N}(t).$$

Proof. The part (a) has been proved in [2, Lemma 3.2]. We prove part (b) as follows. Let $n = v_p(N) \geq 1$ be the p-adic valuation of the integer N, and $N_p = p^{-n}N$. By definition,

$$\begin{aligned}
a_N(pt) &= \sum_{r|pt} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} \\
&= \sum_{\substack{r|pt \\ p\nmid r}} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} + \sum_{\substack{r|pt \\ p|r}} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} \\
&= \sum_{\substack{r|pt \\ p\nmid r}} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} + \sum_{r'|t} \mu(\frac{t}{r'})\mu(\frac{p^{-1}N}{r'})\frac{\varphi(N)}{\varphi(p^{-1}N/r')}
\end{aligned}$$

If $n \geq 2$, we have $\mu(\frac{N}{r}) = 0$ when $p \nmid r$, then

$$\begin{aligned}
a_N(pt) &= \sum_{r'|t} \mu(\frac{t}{r})\mu(\frac{p^{-1}N}{r'})\frac{\varphi(N)}{\varphi(p^{-1}N/r')} \\
&= p\sum_{r'|t} \mu(\frac{t}{r'})\mu(\frac{p^{-1}N}{r'})\frac{\varphi(p^{-1}N)}{\varphi(p^{-1}N/r')} = pa_{p^{-1}N}(t).
\end{aligned}$$

If $n = 1$, we have $t | p^{-1}N = N_p$, especially, $p \nmid t$,

$$\begin{aligned} a_N(pt) &= \sum_{r|pt} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} \\ &= \sum_{r|t} \mu(\frac{pt}{r})\mu(\frac{N}{r})\frac{\varphi(N)}{\varphi(N/r)} + \sum_{r'|t} \mu(\frac{t}{r'})\mu(\frac{p^{-1}N}{r'})\frac{\varphi(N)}{\varphi(p^{-1}N/r')} \\ &= \sum_{r|t} \mu(\frac{t}{r})\mu(\frac{p^{-1}N}{r})\frac{\varphi(p^{-1}N)}{\varphi(p^{-1}N/r)} + \sum_{r'|t} \mu(\frac{t}{r'})\mu(\frac{p^{-1}N}{r'})\frac{(p-1)\varphi(p^{-1}N)}{\varphi(p^{-1}N/r')} = pa_{p^{-1}N}(t). \end{aligned}$$

□

Let Δ be the modular discriminant function on the upper half plane $\mathbb{H}^+$, it is a cusp form of weight 12 and level 1 with expansion at ∞ given as follows,

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}, \;\; q = e^{2\pi i z} \text{ where } z \in \mathbb{H}^+. \tag{4.17}$$

The modular form $\Delta(z)$ is a global section of the line bundle $\widehat{\omega}_1^{\otimes 12}$ by the definition made in Katz's work [36]. Similarly, for any positive integer t, the modular form $\Delta(tz)$ is a section of the line bundle $\widehat{\omega}_t^{\otimes 12}$. Now we construct an explicit section Δ_N of the line bundle $\widehat{\omega}_N^{\otimes 12\varphi(N)}$ following the lines in [2, §1],

$$\Delta_N(z) = \prod_{t|N} \Delta(tz)^{a_N(t)}. \tag{4.18}$$

where t ranges over all the positive integers dividing N. Let W_N be the following matrix:

$$W_N = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}. \tag{4.19}$$

The element W_N induces the Atkin–Lehner involution we defined in §4.4, hence the notation. Define the following section:

$$\Delta_N^0(z) := \Delta_N\big|_{W_N, 12\varphi(N)}(z) = \Delta_N(W_N z) \cdot (Nz^2)^{-6\varphi(N)}.$$

It is also a rational section of the metrized line bundle $\widehat{\omega}_N^{\otimes 12\varphi(N)}$.

Lemma 4.8.2. *Let p be a prime number such that $p|N$, let $n = \nu_p(N) \geq 1$ and $N_p = p^{-n}N$, we have*

$$\Delta_N(z) = \frac{\Delta_{N_p}(p^n z)^{p^n}}{\Delta_{N_p}(p^{n-1}z)^{p^{n-1}}}.$$

Proof. By the definition of Δ_N, we have

$$\Delta_N(z) = \prod_{t|N_p} \Delta(tz)^{a_N(t)} \cdot \prod_{p|t|N} \Delta(tz)^{a_N(t)}.$$

If $n \geq 2$, we have $a_N(t) = 0$ when $t|N_p$. Combining Lemma 4.8.1, we have

$$\Delta_N(z) = \prod_{t'|p^{-1}N} \Delta(pt'z)^{a_N(pt')} = \left(\prod_{t'|p^{-1}N} \Delta(pt'z)^{a_{p^{-1}N}(t')} \right)^p = \Delta_{p^{-1}N}(pz)^p. \tag{4.20}$$

If $n = 1$, we have $a_N(t) = -a_{N_p}(t)$ when $t|N_p$. Combining Lemma 4.8.1, we have

$$\Delta_N(z) = \left(\Delta_{N_p}(z)\right)^{-1} \cdot \prod_{t'|N_p} \Delta(pt'z)^{a_N(pt')} = \frac{\Delta_{N_p}(pz)^p}{\Delta_{N_p}(z)}. \tag{4.21}$$

Therefore the lemma follows by induction based on formulas (4.20) and (4.21). □

Recall the definition of the cusps $P_\infty(N)$ and $P_0(N)$ in Remark 4.5.2. We have the following theorem which describes the divisors associated to the rational sections Δ_N and Δ_N^0 explicitly.

Theorem 4.8.3. *For any positive integer N, and any prime number p, let $n = \nu_p(N) \geq 0$ be the p-adic valuation of the integer N, then as an element in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$, we have*

$$\mathrm{div}(\Delta_N) = \psi(N)\varphi(N)P_\infty(N) + \sum_{p|N} f_p(N),$$

$$\mathrm{div}(\Delta_N^0) = \psi(N)\varphi(N)P_0(N) + \sum_{p|N} f_p^0(N),$$

where for any $p|N$*,*

$$f_p(N) = 12p^{n-1}\varphi(N_p) \sum_{\substack{-n\leq a<n \\ a\equiv n \bmod 2}} \left(\frac{1-p}{2}(n-a)-1\right)\varphi(p^{(n-|a|)/2})\cdot \mathcal{X}_p^a(N). \tag{4.22}$$

$$f_p^0(N) = -6n\varphi(N)\mathcal{X}_p^{-n}(N) + 12p^{n-1}\varphi(N_p) \sum_{\substack{-n<a\leq n \\ a\equiv n \bmod 2}} \varphi(p^{(n-|a|)/2})\left(\frac{1-p}{2}n-1\right)\cdot \mathcal{X}_p^a(N). \tag{4.23}$$

Proof. The expansion of Δ_N at ∞ is computed in [2, Proposition 3.3], there exist some integer $C_N(n)$ such that

$$\Delta_N(z) = q^{\psi(N)\varphi(N)}\prod_{n\geq 1}(1-q^n)^{24C_N(n)},$$

hence the vanishing order of Δ_N at ∞ is $\psi(N)\varphi(N)$, and there is no vertical component of $\mathrm{div}(\Delta_N)$ at the cusp $\mathsf{C}^n(p^n, N_p)$. Let k be a positive integer such that $0 \leq k < n$, we first study the expansion of the modular form $\Delta_{N_p,p^n}(z) \coloneqq \Delta_{N_p}(p^n z)$ at the component $\mathsf{C}^{2k-n}(p^n, N_p)$ of $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$, since the point $P_{\frac{1}{p^k}}$ belongs to this cusp by Corollary 4.5.4, let

$$\gamma = \begin{pmatrix} 1 & 0 \\ p^k & 1 \end{pmatrix},\ \gamma_k = \begin{pmatrix} p^{n-k} & -1 \\ 1 & 0 \end{pmatrix},\ n_k = \begin{pmatrix} p^k & 1 \\ 0 & p^{n-k} \end{pmatrix}.$$

Note that $p^n \cdot \gamma z = \gamma_k n_k z$ for any $z \in \mathbb{C}$.

$$\begin{aligned} \Delta_{N_p,p^n}\big|_{\gamma,12\varphi(N_p)}(z) &= \Delta_{N_p}(p^n\gamma z)\cdot j(\gamma,z)^{-12\varphi(N_p)} \\ &= \Delta_{N_p}(\gamma_k n_k z)\cdot j(\gamma_k, n_k z)^{-12\varphi(N_p)}\cdot\left(\frac{j(\gamma_k, n_k z)}{j(\gamma,z)}\right)^{12\varphi(N_p)} \\ &= p^{-12\varphi(N_p)(n-k)}\cdot \Delta_{N_p}\big|_{\gamma_k,12\varphi(N_p)}(n_k z). \end{aligned}$$

Moreover, for any $z \in \mathbb{H}^+$,

$$\begin{aligned}\Delta_{N_p}\big|_{\gamma_k,12\varphi(N_p)}(z) &= \Delta_{N_p}(\frac{p^{n-k}z-1}{z}) \cdot j(\gamma_k,z)^{-12\varphi(N_p)}\\ &= \Delta_{N_p}(-\frac{1}{z}) \cdot z^{-12\varphi(N_p)}\\ &= \prod_{t|N_p} \Delta(-\frac{t}{z})^{a_{N_p}(t)} \cdot z^{-12\varphi(N_p)}\\ &= \prod_{t|N_p} \Delta(\frac{z}{t})^{a_{N_p}(t)} \cdot \prod_{t|N_p} t^{-12a_{N_p}(t)}.\end{aligned}$$

Let $C_{N_p} = \prod\limits_{t|N_p} t^{-12a_{N_p}(t)}$, we know that $v_p(C_{N_p}) = 0$, and

$$\Delta_{N_p,p^n}\big|_{\gamma,12\varphi(N_p)}(z) = p^{-12\varphi(N_p)(n-k)} \cdot C_{N_p} \cdot \prod_{t|N_p} \Delta\left(\frac{p^k z+1}{p^{n-k}t}\right)^{a_{N_p}(t)}.$$

Similarly, for $\Delta_{N_p,p^{n-1}}(z) = \Delta_{N_p}(p^{n-1}z)$, we have

$$\Delta_{N_p,p^n}\big|_{\gamma,12\varphi(N_p)}(z) = p^{-12\varphi(N_p)(n-k-1)} \cdot C_{N_p} \cdot \prod_{t|N_p} \Delta\left(\frac{p^k z+1}{p^{n-k-1}t}\right)^{a_{N_p}(t)}.$$

By definition,

$$\begin{aligned}\Delta_N\big|_{\gamma,12\varphi(N)}(z) &= \frac{\Delta_{N_p,p^n}^{p^n}}{\Delta_{N_p,p^{n-1}}^{p^{n-1}}}\Bigg|_{\gamma,12\varphi(N)}(z) = \frac{\left(\Delta_{N_p,p^n}\big|_{\gamma,12\varphi(N_p)}(z)\right)^{p^n}}{\left(\Delta_{N_p,p^{n-1}}\big|_{\gamma,12\varphi(N_p)}(z)\right)^{p^{n-1}}}\\ &= p^{-12\varphi(N_p)\left(p^n(n-k)-p^{n-1}(n-k-1)\right)} \cdot C_{N_p}^{p^n-p^{n-1}} \cdot \prod_{t|N_p}\left(\frac{\Delta\left(\frac{p^kz+1}{p^{n-k}t}\right)^{p^n}}{\Delta\left(\frac{p^kz+1}{p^{n-k-1}t}\right)^{p^{n-1}}}\right)^{a_{N_p}(t)}.\end{aligned}\tag{4.24}$$

By the expansion of Δ at ∞ given in the formula (4.17), we know that the last term in formula (4.24) doesn't vanish. Recall that the cusp $P_{\frac{1}{p^k}}$ lies in the component $\mathsf{C}^{2k-n}(p^n, N_p)$ of $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$ and mod p reduction $\mathsf{C}_p^{2k-n}(p^n, N_p)$ of the formal scheme $\mathsf{C}^{2k-n}(p^n, N_p)$ is the completion of curve $C_p^{2k-n}(p^n, N_p) = p^{(n-|2k-n|)/2-1}(p-1)\mathcal{X}_p^{2k-n}(N)$ along its cuspidal locus, hence the multiplicity of $C_p^{2k-n}(p^n, N_p)$ in $\operatorname{div}(\Delta_N)$ is $-12\varphi(N_p)\left(p^n(n-k) - p^{n-1}(n-k-1)\right)$. □

Corollary 4.8.4. *For any positive integer N, we have the following identity in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,*

$$\widehat{\omega}_N - W_N^* \widehat{\omega}_N = \sum_{p|N} \widehat{\mathcal{X}}_p(N).$$

Proof. We know that

$$\widehat{\omega}_N = (\operatorname{div}(\Delta_N), -\log \|\Delta_N\|^2) = (\operatorname{div}(\Delta_N^0), -\log \|\Delta_N^0\|^2),$$

therefore

$$\begin{aligned}\widehat{\omega}_N - W_N^* \widehat{\omega}_N &= (\operatorname{div}(\Delta_N^0), -\log \|\Delta_N^0\|^2) - W_N^*(\operatorname{div}(\Delta_N), -\log \|\Delta_N\|^2) \\ &= (\sum_{p|N} \left(f_p^0(N) - W_N^* f_p(N)\right), 0) = \sum_{p|N} \widehat{\mathcal{X}}_p(N).\end{aligned}$$

□

Chapter 5: Special cycles

5.1 Special cycles on $\mathcal{H}^\circ$ and $\mathcal{X}_0(N)$

Let p be a prime number, we first define the special cycles on the stack $\mathcal{H}^\circ_{(p)}$.

Definition 5.1.1. For every symmetric $n \times n$ matrix $T = (T_{ik})$. Let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the following objects,

$$((E, E'), (\overline{\eta^p}, \overline{\eta'^p}), \boldsymbol{j}),$$

where $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}))$ is an object in $\mathcal{H}^\circ_{(p)}(S)$, $\boldsymbol{j} = (j_1, j_2, \cdots, j_n) \in (\mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})^n$ and $\eta^*(\boldsymbol{j}) := \eta'^p \circ V^p(\boldsymbol{j}) \circ (\eta^p)^{-1} \in \tilde{\boldsymbol{\omega}}^p$. Moreover,

$$T_{ik} = \frac{1}{2}\left(\deg(j_i + j_k) - \deg(j_i) - \deg(j_k)\right) = \frac{1}{2}(j_i \circ j_k^\vee + j_k \circ j_i^\vee).$$

The special cycle $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$ may be empty.

For every symmetric $n \times n$ matrix T, we have a natural finite unramified morphism $i_n^\sharp : \mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p) \to \mathcal{H}^\circ_{(p)}$ by forgetting the extra morphisms $\boldsymbol{j}$ of an object $((E, E'), (\overline{\eta^p}, \overline{\eta'^p}), \boldsymbol{j})$ of $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p)$. Recall the following definition of generalized Cartier divisor appeared in the work of Howard and Madapusi [37, Definition 2.4.1].

Definition 5.1.2. Suppose $D \to X$ is any finite, unramified, and relatively representable morphism of Deligne–Mumford stacks, then there is an étale cover $U \to X$ by a scheme such that the pullback $D_U \to U$ is a finite disjoint union

$$D_U = \bigsqcup_i D_U^i$$

with each map $D^i_U \to U$ a closed immersion. If each of these closed immersions is an effective Cartier divisor on U in the usual sense (the corresponding ideal sheaves are invertible), then we call $D \to X$ a generalized Cartier divisor.

Proposition 5.1.3. *Let $\tilde{\varphi}^p$ be the characteristic function of an open compact subset $\tilde{\omega}^p$ of* $\mathrm{M}_2(\mathbb{A}^p_f)$ *invariant under the action of* $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. *For any positive number $d \in \mathbb{Q}$, the finite unramified morphism $i^\sharp_1 : \mathcal{Z}^\sharp(d, \tilde{\varphi}^p) \to \mathcal{H}^\circ_{(p)}$ is a generalized Cartier divisor*

Proof. This is proved by Howard and Madapusi in [38, Proposition 6.5.2] (see also [37, Proposition 2.4.3]). □

Now let's come to the special cycles on the stack $\mathcal{X}_0(N)_{(p)}$ and $\mathcal{Y}_0(N)_{(p)}$, we first introduce the notion of special morphisms for the moduli stack $\mathcal{Y}_0(N)_{(p)}$.

Definition 5.1.4. Let S be a scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$, for an object $((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$ in $\mathcal{Y}_0(N)_{(p)}(S)$, a special morphism of this object is an element $j \in \mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ satisfying

$$j \circ \pi^\vee + \pi \circ j^\vee = 0.$$

we denote this space by $S(E, \pi)$.

Definition 5.1.5. For every symmetric $n \times n$ matrix $T = (T_{ik})$. Let $\boldsymbol{\varphi}^p$ be the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}^p_f)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. We consider the stack $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$, whose fibered category over a $\mathbb{Z}_{(p)}$-scheme S consists of the following objects,

$$((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)}, \boldsymbol{j}),$$

where $((E \xrightarrow{\pi} E'), \overline{(\eta^p, \eta'^p)})$ is an object in $\mathcal{Y}_0(N)_{(p)}(S)$, $\boldsymbol{j} = (j_1, j_2, \cdots, j_n) \in S(E, \pi)^n$ and $\eta^*(\boldsymbol{j}) := \eta'^p \circ V^p(\boldsymbol{j}) \circ (\eta^p)^{-1} \in \boldsymbol{\omega}^p$. Moreover,

$$T_{ik} = \frac{1}{2}\left(\deg(j_i + j_k) - \deg(j_i) - \deg(j_k)\right) = \frac{1}{2}(j_i \circ j_k^\vee + j_k \circ j_i^\vee).$$

Notice that the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ may be empty.

For every symmetric $n \times n$ matrix T, we have a natural morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ by forgetting the special morphisms.

Remark 5.1.6. Let $T \in \mathrm{Sym}_n(\mathbb{Q})$. Let $\tilde{\boldsymbol{\varphi}}^p$ be the characteristic function of an open compact subset $\tilde{\boldsymbol{\omega}}^p$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$ invariant under the action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$. Let $\boldsymbol{\varphi}^p$ be the restriction of $\tilde{\boldsymbol{\varphi}}^p$ to the subspace $(\mathbb{V}_f^p)^n$ of $\mathrm{M}_2(\mathbb{A}_f^p)^n$, then $\boldsymbol{\varphi}^p$ is the characteristic function of an open compact subset $\boldsymbol{\omega}^p$ of $(\mathbb{V}_f^p)^n$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$, then the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi}^p$ is a union of some connected components of the fiber product $\mathcal{Z}^\sharp(T, \tilde{\boldsymbol{\varphi}}^p) \times_{\mathcal{H}_{(p)}} \mathcal{Y}_0(N)_{(p)}$. Therefore the morphism $i_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is also finite unramified. In particular, when $n = 1$, let $T = d \in \mathbb{Q}_{>0}$, the morphism $i_1 : \mathcal{Z}(d, \boldsymbol{\varphi}^p) \to \mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Proposition 5.1.3.

We will show next that the composite $\tilde{i}_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \xrightarrow{i_n} \mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also finite unramified. We start with the case that $n = 1$.

Proposition 5.1.7. *Let φ^p be the characteristic function of an open compact subset ω^p of $\mathbb{V}_f^p$ invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. For any positive number $d \in \mathbb{Q}$, the morphism $\tilde{i}_1 : \mathcal{Z}(d, \varphi^p) \to \mathcal{X}_0(N)_{(p)}$ is finite unramified, and $\mathcal{Z}(d, \varphi^p)$ is a generalized Cartier divisor.*

Proof. The morphism $\tilde{i}_1$ is unramified since i_1 is unramified and the open immersion $\mathcal{Y}_0(N)_{(p)} \to \mathcal{X}_0(N)_{(p)}$ is also unramified, therefore we only need to show the finiteness of $\tilde{i}_1$.

We first prove that the stack $\mathcal{Z}(d, \varphi^p)$ is flat over $\mathbb{Z}_{(p)}$, since the morphism $\mathcal{Z}(d, \varphi^p)\mathcal{Y}_0(N)_{(p)}$ is a generalized Cartier divisor by Remark 5.1.6, the flatness of $\mathcal{Z}(d, \varphi^p)$ is equivalent to the fact that its local equation is not divisible by p since the stack $\mathcal{Y}_0(N)_{(p)}$ is flat over $\mathbb{Z}_{(p)}$. We assume the converse and suppose that there exists a point $z \in \mathcal{Z}(d, \varphi^p)(\overline{\mathbb{F}}_p)$ such that the equation of $\mathcal{Z}(d, \varphi^p)$ in the étale local ring $O^{\text{ét}}_{\mathcal{Y}_0(N),z}$ is divisible by p, then the stack $\mathcal{Z}(d, \varphi^p)$ contains an irreducible component of $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in an étale neighbourhood of z, let $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)})$ be the object corresponding to the generic point of this irreducible component, then $\mathrm{End}(E) \simeq \mathbb{Z}$ since the j-invariant of E must be transcendental over $\mathbb{F}_p$ (by the description of the stack $\mathcal{Y}_0(N)_{\mathbb{F}_p}$ in [3, Proposition 13.4.5, Theorem 13.4.7]). There also exists an isogeny $j \in \mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $j^\vee \circ \pi + \pi^\vee \circ j = 0$. Let $\alpha = j^{-1} \circ \pi \in \mathrm{End}^\circ(E) \coloneqq \mathrm{End}(E) \otimes \mathbb{Q} \simeq \mathbb{Q}$, then $\alpha^2 = -Nd^{-1} < 0$, this contradicts to the fact that $\mathrm{End}^\circ(E) \simeq \mathbb{Q}$.

Therefore the stack $\mathcal{Z}(d, \varphi^p)$ is flat over $\mathbb{Z}_{(p)}$, hence equals to the flat closure of its generic fiber $\mathcal{Z}(d, \varphi^p)_{\mathbb{Q}} \coloneqq \mathcal{Z}(d, \varphi^p) \times_{\mathbb{Z}_{(p)}} \mathbb{Q}$. The stack $\mathcal{Z}(d, \varphi^p)_{\mathbb{Q}}$ consists of finitely many points whose residue fields are finite extensions of $\mathbb{Q}$, therefore the structure sheaf $O_{\mathcal{Z}(d,\varphi^p)}$ of $\mathcal{Z}(d, \varphi^p)$ is a finite product of subrings of the integer rings of these residue fields, hence the stack $\mathcal{Z}(d, \varphi^p)$ is finite over $\mathbb{Z}_{(p)}$, then $\tilde{i}_1 : \mathcal{Z}(d, \varphi^p) \to \mathcal{X}_0(N)_{(p)}$ is proper since $\mathcal{X}_0(N)_{(p)}$ is proper over $\mathbb{Z}_{(p)}$. The morphism $\tilde{i}_1$ is obviously quasi-finite by the finiteness of $\mathcal{Z}(d, \varphi^p)$ over $\mathbb{Z}_{(p)}$, hence $\tilde{i}_1$ is finite.

We already know that the morphism $\tilde{i}_1$ is a generalized Cartier divisor over the open substack $\mathcal{Y}_0(N)_{(p)}$ of $\mathcal{X}_0(N)_{(p)}$. Moreover, étale locally around a cusp point of $\mathcal{X}_0(N)_{(p)}$, the stack $\mathcal{Z}(d, \varphi^p)$ is cut out by 1 since $\tilde{i}_1$ factors through the non-cuspidal locus $\mathcal{Y}_0(N)_{(p)}$, hence the finite unramified morphism $\tilde{i}_1 : \mathcal{Z}(d, \varphi^p) \to \mathcal{X}_0(N)_{(p)}$ is a generalized Cartier divisor on the stack $\mathcal{X}_0(N)_{(p)}$. □

Corollary 5.1.8. *Let* $\boldsymbol{\varphi}^p = \prod_{i=1}^{n} \varphi_i^p$ *be the characteristic function of an open compact subset* $\boldsymbol{\omega}^p$ *of* $(\mathbb{V}_f^p)^n$ *invariant under the action of* $\Gamma_0(N)(\hat{\mathbb{Z}}^p)$. *For any matrix* $T \in \mathrm{Sym}_n(\mathbb{Q})_{>0}$, *the morphism* $\tilde{i}_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{X}_0(N)_{(p)}$ *is finite unramified.*

Proof. Suppose the diagonal elements of T are $d_1, \cdots, d_n$. Proposition 5.1.7 implies that the morphism $\mathcal{Z}(d_1, \varphi_1^p) \times_{\mathcal{X}_0(N)_{(p)}} \times \cdots \times_{\mathcal{X}_0(N)_{(p)}} \mathcal{Z}(d_n, \varphi_n^p) \to \mathcal{X}_0(N)_{(p)}$ is finite unramified. The stack $\mathcal{Z}(T, \boldsymbol{\varphi}^p)$ is a connected component of $\mathcal{Z}(d_1, \varphi_1^p) \times_{\mathcal{X}_0(N)_{(p)}} \times \cdots \times_{\mathcal{X}_0(N)_{(p)}} \mathcal{Z}(d_n, \varphi_n^p)$, hence the morphism $\tilde{i}_n : \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{X}_0(N)_{(p)}$ is finite unramified. □

We will mainly focus on the case that T is a nonsingular 2×2 symmetric matrix with coefficients in $\mathbb{Q}$. For every such matrix T, define the difference set to be $\mathrm{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l.\}$.

Proposition 5.1.9. *Let* $T \in \mathrm{Sym}_2(\mathbb{Q})$ *be a nonsingular matrix. If* $\mathcal{Z}(T, \boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p) \neq \varnothing$ *for some* *prime p, then T is positive definite, and*

$$\mathrm{Diff}(T,\Delta(N))=\{p\}.$$

Moreover, in this case, the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is supported in the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$.

Proof. Since $\mathcal{Z}(T,\boldsymbol{\varphi}^p)(\overline{\mathbb{F}}_p)\neq\emptyset$, Corollary 5.1.8 implies that there are two elliptic curves $\mathbb{E}$ and $\mathbb{E}'$ over $\overline{\mathbb{F}}_p$, a cyclic isogeny $\pi\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')$, and two isogenies $x_1,x_2\in\mathrm{Hom}(\mathbb{E},\mathbb{E}')_{(p)}$ such that

$$T=(\frac{1}{2}(x_i,x_j)),\ \text{ and } (x_1,\pi)=(x_2,\pi)=0.$$

therefore T must be positive definite and both $\mathbb{E}$ and $\mathbb{E}'$ are supersingular elliptic curves over $\overline{\mathbb{F}}_p$ since $\dim_{\mathbb{Q}}\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}\geq 3$. The quadratic space $\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p$ is isometric to the underlying quadratic space of the unique division quaternion algebra $\mathbb{B}$ over $\mathbb{Q}_p$.

The isogenies $x_1,x_2\in\{\pi\}^{\perp}\subset\mathrm{Hom}(\mathbb{E},\mathbb{E}')\otimes\mathbb{Q}_p\simeq\mathbb{B}$ where $\pi^{\vee}\circ\pi=N$. However, $\{\pi\}^{\perp}$ and $\Delta(N)\otimes\mathbb{Q}_p$ have the same discriminant $-N$ but opposite Hasse invariants, therefore $p\in\mathrm{Diff}(T,\Delta(N))$. At the same time, by choosing some level structures on $\mathbb{E}$ and $\mathbb{E}'$ away from p, we get that T can be realized in $\Delta(N)\otimes\mathbb{Q}_l$ for any finite prime $l\neq p$. Therefore p is the only prime in the set $\mathrm{Diff}(T,\Delta(N))$. □

Remark 5.1.10. Proposition 5.1.9 implies that the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is also finite unramified over the stack $\mathcal{X}_0(N)_{(p)}$ because the scheme-theoretic image $\tilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{X}_0(N)_{(p)}$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$, which equals the supersingular locus of the special fiber $\mathcal{Y}_0(N)_{\mathbb{F}_p}$, hence $\tilde{\mathcal{Z}}(T,\boldsymbol{\varphi}^p)$ is contained in $\mathcal{Y}_0(N)_{(p)}$, and therefore equals to the scheme-theoretic image of $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ in $\mathcal{Y}_0(N)_{(p)}$, over which $\mathcal{Z}(T,\boldsymbol{\varphi}^p)$ is finite unramified.

For any nonsingular 2×2 symmetric matrix $T\in\mathrm{Sym}_2(\mathbb{Q})$, we say a Schwartz function $\boldsymbol{\varphi}=\bigotimes\limits_{v<\infty}\boldsymbol{\varphi}_v\in\mathcal{S}(\mathbb{V}_f^2)$ is T-admissible if $\boldsymbol{\varphi}$ is invariant under the action of $\Gamma_0(N)(\hat{\mathbb{Z}})$, $\boldsymbol{\varphi}=\varphi_1\times\varphi_2$ where $\varphi_i\in\mathcal{S}(\mathbb{V}_f)$ and

- T is not positive definite, or
- T is positive definite, and $|\mathrm{Diff}(T, \Delta(N))| \neq 1$, or
- T is positive definite, $\mathrm{Diff}(T, \Delta(N)) = \{p\}$ for some prime number p, and $\boldsymbol{\varphi} = \boldsymbol{\varphi}^p \otimes \boldsymbol{\varphi}_p$ where $\boldsymbol{\varphi}^p \in \mathcal{S}((\mathbb{V}_f^p)^2)$ and $\boldsymbol{\varphi}_p = c \cdot \mathbf{1}_{\Delta_p(N)^2}$ for some $c \in \mathbb{C}$.

Definition 5.1.11. For a nonsingular 2×2 matrix $T \in \mathrm{Sym}_2(\mathbb{Q})$ and a T-admissible Schwartz function $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ which is also a characteristic function of a $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$, we define a stack finite unramified over $\mathcal{X}_0(N)$ as follows,

$$\mathcal{Z}(T, \boldsymbol{\varphi}) \coloneqq \mathcal{Z}(T, \boldsymbol{\varphi}^p) \to \mathcal{X}_0(N)_{(p)} \hookrightarrow \mathcal{X}_0(N),$$

where $p \in \mathrm{Diff}(T, \Delta(N))$. If $|\mathrm{Diff}(T, \Delta(N))| \neq 1$, we define $\mathcal{Z}(T, \boldsymbol{\varphi}) = \varnothing$.

Remark 5.1.12. By Proposition 5.1.9, $\mathcal{Z}(T, \boldsymbol{\varphi})$ is nonempty only if $|\mathrm{Diff}(T, \Delta(N))| = 1$, therefore the above definition makes sense.

Remark 5.1.13. If we view $\mathcal{Z}(T, \boldsymbol{\varphi})$ as an element in $\mathrm{CH}^2(\mathcal{X}_0(N))$, we can drop the restrictions in Definition 5.1.11 that the Schwartz functions $\boldsymbol{\varphi}$ is the characteristic function of an open compact subset of $\mathbb{V}_f^2$. Since any T-admissible Schwartz functions $\boldsymbol{\varphi}$ on $\mathbb{V}_f^2$ is a finite linear combination of $\Gamma_0(N)(\hat{\mathbb{Z}})$-invariant characteristic functions of some open compact subsets, we can define $\mathcal{Z}(T, \boldsymbol{\varphi})$ as the corresponding linear combination of elements in $\mathrm{CH}^2(\mathcal{X}_0(N))$.

5.1.1 Comparison with [1, §2.2]

Another kind of special cycles of $\mathcal{X}_0(N)$ is defined in [1, §2.2] as follows,

Definition 5.1.14. For $m \in \mathbb{Z}$, let $\mathcal{Z}(m)$ denote the moduli stack whose S points, for a base scheme S, are given by

$$\mathcal{Z}(m)(S) \coloneqq \{(E \xrightarrow{\pi} E', \alpha)\}$$

where $(E \xrightarrow{\pi} E') \in \mathcal{Y}_0(N)(S)$ where $\alpha \in \mathrm{End}(E)$ satisfies the following conditions:
(a). $\alpha^\vee \circ \alpha = mN$ and $\alpha^\vee + \alpha = 0$;

(b). $\alpha \circ \pi^{-1} \in \mathrm{Hom}(E', E)$;

(c). $\pi \circ \alpha \circ \pi^{-1} \in \mathrm{End}(E')$.

Lemma 5.1.15. *For every prime number p, let* $\mathcal{Z}(m)_{(p)} := \mathcal{Z}(m) \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$, *then we have an isomorphism of stacks as follows,*

$$\begin{aligned} T : \mathcal{Z}(m)_{(p)} &\xrightarrow{\sim} \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}), \\ (E \xrightarrow{\pi} E', \alpha) &\longmapsto (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee). \end{aligned}$$

Proof. We first prove that T is well-defined. For any connected $\mathbb{Z}_{(p)}$-scheme S, let $\overline{s}$ be a geometric point of S, we can choose trivilizations $\eta^p : V^p(E_{\overline{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ and $\eta'^p : V^p(E'_{\overline{s}}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ such that $T^p(E_{\overline{s}})$ and $T^p(E'_{\overline{s}})$ are mapped isomorphically to $(\hat{\mathbb{Z}}^p)^2$, and $\eta'^p \circ V^p(\pi) \circ (\eta^p)^{-1} = w_N$ by the cyclicity of π. Moreover,

$$\begin{aligned} (\alpha \circ \pi^{-1})^\vee \circ \pi + \pi^\vee \circ (\alpha \circ \pi^{-1}) &= \frac{1}{N}\pi^\vee \circ \alpha^\vee \circ \pi + \frac{1}{N}\pi^\vee \circ \alpha \circ \pi \\ &= \frac{1}{N}\pi^\vee \circ (\alpha^\vee + \alpha) \circ \pi = 0. \end{aligned}$$

hence $(\alpha \circ \pi^{-1})^\vee \in S(E, \pi)$, then (b) implies that $\eta'^p \circ V^p((\alpha \circ \pi^{-1})^\vee) \circ (\eta^p)^{-1} \in \Delta(N) \otimes \hat{\mathbb{Z}}^p \subset \mathbb{V}_f^p$, therefore $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, (\alpha \circ \pi^{-1})^\vee) \in \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$.

We define the following morphism,

$$\begin{aligned} R : \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p}) &\longrightarrow \mathcal{Z}(m)_{(p)}, \\ (E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j) &\longmapsto (E \xrightarrow{\pi} E', j^\vee \circ \pi). \end{aligned}$$

The morphism R is well-defined: For a connected $\mathbb{Z}_{(p)}$-scheme S, an object $(E \xrightarrow{\pi} E', \overline{(\eta^p, \eta'^p)}, j)$ $\in \mathcal{Z}(m, \mathbf{1}_{\Delta(N)\otimes\hat{\mathbb{Z}}^p})(S)$ means that $j \in \mathrm{Hom}(E, E') \otimes \mathbb{Z}_{(p)}$ and $j^\vee \circ \pi + \pi^\vee \circ j = 0$, the fact that $\eta'^p \circ V^p(j) \circ (\eta^p)^{-1} \in \Delta(N) \otimes \hat{\mathbb{Z}}^p$ implies that $j \in \mathrm{Hom}(E, E')$, then $j^\vee \circ \pi \in \mathrm{End}(E)$, $(j^\vee \circ \pi)^\vee \circ (j^\vee \circ \pi) = \pi^\vee \circ j \circ j^\vee \circ \pi = mN$ and $(j^\vee \circ \pi)^\vee + j^\vee \circ \pi = \pi^\vee \circ j + j^\vee \circ \pi = 0$, which is exactly (a). Moreover, (b) and (c) are easily verified, hence $(E \xrightarrow{\pi} E', j^\vee \circ \pi) \in \mathcal{Z}(m)(S)$, then the morphism R is well-defined. It's easy to see that T and R are inverse to each other, therefore the lemma is proved. $\square$

5.2 Arithmetic special cycles on $\mathcal{X}_0(N)$

Now we are going to construct the Green functions for special cycles, following the work of Kudla [12]. We first fix some notations. Let $\mathbb{D} = \{z \in \Delta(N) \otimes_{\mathbb{Z}} \mathbb{C} : (z,z) = 0,\ (z,\bar{z}) < 0\} / \mathbb{C}^* \subset \mathbb{P}(\Delta(N) \otimes \mathbb{C})$, for any $x \in \Delta(N) \otimes \mathbb{R}$ and $[z] \in \mathbb{D}$, let $\mathrm{R}(x,[z]) = -|(x,z)|^2 \cdot (z,\bar{z})^{-1}$. For any element in $\tau = x + iy \in \mathbb{H}_1$, we let

$$h(\tau) \coloneqq \frac{1}{\sqrt{N}y} \begin{pmatrix} Nx & N(x^2+y^2) \\ -1 & -x \end{pmatrix}.$$

We have the following $\mathrm{GL}_2(\mathbb{R})$-equivariant identification,

$$\begin{aligned} c : \mathbb{H}_1^{\pm} &\xrightarrow{\sim} \mathbb{D}, \\ \tau &\longmapsto c(\tau) \coloneqq \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} -N\tau & -N\tau^2 \\ 1 & \tau \end{pmatrix} \right\}. \end{aligned}$$

5.2.1 The codimension 1 case: Arithmetic special divisors in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$

In the section, we define an element in the codimension 1 arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$ of $\mathcal{X}_0(N)$ for any pair (t,y), where t is an integer and y is a positive real number. Before giving the full definition, we modify the bundle $\widehat{\omega}_N$ following [2],

$$\widehat{\omega} \coloneqq -\widehat{\omega}_N - W_N^* \widehat{\omega}_N = -2\widehat{\omega}_N + \sum_{p|N} \widehat{\mathcal{X}}_p(N), \tag{5.1}$$

notice that the last identity follows from Corollary 4.8.4. We have $\widehat{\omega} = (\mathrm{div}(\widehat{\omega}), \mathfrak{g}_{\widehat{\omega}})$ as an element in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$.

Definition 5.2.1. Let t be an integer. Define the following element in $\mathrm{CH}^1(\mathcal{X}_0(N))$:

$$\mathcal{Z}(t, \Delta(N)) = \begin{cases} \mathcal{Z}(t), & \text{if } t > 0; \\ \operatorname{div}(\widehat{\omega}), & \text{if } t = 0; \\ 0, & \text{if } t < 0. \end{cases}$$

These elements need to be modified due to the fact that $\mathcal{Y}_0(N)$ is not compact. The modification is given by adding some boundary components to $\mathcal{Z}(t, \Delta(N))$ when $t \leq 0$. For a real number s, define the function β_s to be

$$\beta_s(r) = \int_1^{\infty} e^{-rt} t^{-s} \mathrm{d}t, \text{ where } r > 0. \tag{5.2}$$

Let $y > 0$ be a positive real number, t be an integer, define

$$g(t, y, \Delta(N)) = \begin{cases} \frac{\sqrt{N}}{2\pi\sqrt{y}} \beta_{3/2}(-4\pi t y), & \text{if } -Nt > 0 \text{ is a square}; \\ \frac{\sqrt{N}}{2\pi\sqrt{y}}, & \text{if } t = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

Recall that $\mathsf{Cusp}(\mathcal{X}_0(N))$ is the cuspidal divisor of the stack $\mathcal{X}_0(N)$, the completion of $\mathcal{X}_0(N)$ along $\mathsf{Cusp}(\mathcal{X}_0(N))$ has the following decomposition by Remark 4.5.2:

$$\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N)) = \coprod_{M|N} \mathsf{C}^M(N),$$

where M ranges over all positive divisors of N. Define $\mathsf{Cusp}^M(\mathcal{X}_0(N)) = \mathsf{C}^M(N) \times_{\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))} \mathsf{Cusp}(\mathcal{X}_0(N))$. We have the following equality in $\mathrm{CH}^1(\mathcal{X}_0(N))$:

$$\mathsf{Cusp}(\mathcal{X}_0(N)) = \sum_{M|N} \mathsf{Cusp}^M(\mathcal{X}_0(N)).$$

For a positive divisor M of N, define

$$\sigma(N,M)=\prod_{i=1}^{r}p_i^{|n_i-2m_i|},\ \text{ if } N=\prod_{i=1}^{r}q_i^{n_i} \text{ and } M=\prod_{i=1}^{r}q_i^{m_i}.$$

Define the modified special divisor to be

$$\mathcal{Z}^*(t,y,\Delta(N))=\mathcal{Z}(t,\Delta(N))+\sum_{M|N}g(t,y,\Delta(\sigma(N,M)))\cdot\mathsf{Cusp}^M(\mathcal{X}_0(N)). \tag{5.4}$$

It is an element in $\mathrm{CH}^1(\mathcal{X}_0(N))$.

Green functions

Now we construct Green forms for the special cycles in Definition 5.2.1. For any pair (t,y) where t is an integer and y is a positive real number, we define

$$\mathfrak{g}(t,y,\Delta(N))([z])=\sum_{\substack{x\in\Delta(N),x\neq0\\(x,x)=t}}\beta_1(2\pi\mathrm{R}(y^{1/2}x,[z])).$$

It is invariant under the action of $\Gamma_0(N)$, hence the function $\mathfrak{g}(d,y,\Delta(N))([z])$ is smooth on the open subset $\mathcal{Y}_0(N)_{\mathbb{C}}-\mathcal{Z}(t)_{\mathbb{C}}$. Define the modified Green forms to be

$$\mathfrak{g}^*(t,y,\Delta(N))([z])=\begin{cases}\mathfrak{g}(t,y,\Delta(N))([z]), & \text{if } t\neq0;\\ \mathfrak{g}(0,y,\Delta(N))([z])+\mathfrak{g}_{\widehat{\omega}}-\log(y), & \text{if } t=0.\end{cases}$$

For the pair (t,y), we also consider the following differential on the upper half plane

$$\omega(t,y,\Delta(N))=\sum_{\substack{x\in\Delta(N),x\neq0\\(x,x)=t}}\left(y(x,h(\tau))^2-\frac{1}{2\pi}\right)e^{-2\pi\mathrm{R}(x,c(\tau))}\cdot\frac{\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}}{-2i\mathrm{Im}(\tau)^2}\in\Omega^{1,1}(\mathbb{H}^+). \tag{5.5}$$

It is invariant under the action of $\Gamma_0(N)$ and can be extended to the cusps, hence the differential $(1,1)$-form $\omega(t,y,\Delta(N))$ can be descended to a differential $(1,1)$-form on the complex modular curve $\mathcal{X}_0(N)_{\mathbb{C}}$. The following theorem reveals that the modification of the Green forms $\mathfrak{g}(t, y, \Delta(N))$ is designed to align with the modification of the special cycle $\mathcal{Z}^*(t, y, \Delta(N))$.

Lemma 5.2.2. *For a cusp point $P = P_{aM/N}$ of $\mathcal{X}_0(N)$, let $l_P \in \Delta(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ be an isotropic vector corresponding to P. Then*

$$\#\Gamma_0(N) \cap \mathrm{Stab}_P \backslash \{v : \ v \in \Delta(N), q(v) = t, (v, l_P) = 0\} = \begin{cases} 4\sqrt{-\sigma(N, M)t}, & \text{if } -Nt \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider another realization of the lattice $\Delta(N)$ in the matrix group $\mathrm{M}_2(\mathbb{Q})^{\mathrm{tr}=0}$ (cf. [2, (1.2)]):

$$\Delta(N) \coloneqq \left\{ x = \begin{pmatrix} a & -b/N \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

with the quadratic form given by $N \cdot \det$. The advantage of using this realization of the lattice $\Delta(N)$ is that the group $\mathrm{GSpin}(\Delta(N))(\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{Q})$ acts on it simply by conjugation. The isotropic vector corresponding to the cusp point $P_{aM/N}$ is

$$l_{P_{aM/N}} = \begin{pmatrix} aM/N & -a^2M^2/N^2 \\ 1 & -aM/N \end{pmatrix}, \ l_{P_\infty} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We only give the proof when $N = p^n$, the idea for the general case is similar. Suppose $M = p^k$ with $0 \leq k \leq n$ and $a \in (\mathbb{Z}/p^{\min\{k,n-k\}})^\times$. The cusp point $P_{a/p^{n-k}}$ is mapped to P_{-a^{-1}/p^k} under the Atkin–Lehner involution W_{p^n} which is given by the matrix (4.19). The element W_{p^n} also preserves the lattice $\Delta(p^n)$ and the group $\Gamma_0(p^n)$. Therefore the left hand side of the equality in the lemma are the same for $P_{a/p^{n-k}}$ and P_{-a^{-1}/p^k}, while the right hand side are also the same for $M = p^k$ and p^{n-k}, hence we only need to consider the case that $P = P_{a/p^k}$ where $[n/2] < k \leq n$. We can also assume $a = 1$ without loss of generality. Define

$$\gamma_k = \begin{pmatrix} 1 & 0 \\ -p^k & 1 \end{pmatrix}.$$

Then γ_k maps the cusp $P = P_{1/p^k}$ to P_∞. The map $v \longrightarrow \gamma_k \cdot v \cdot \gamma_k^{-1}$ defines a bijection

$$\{v : \ v \in \Delta(p^n), q(v) = t, (v, l_P) = 0\} \overset{\sim}{\longrightarrow} \{v : \ v \in \gamma_k \Delta(p^n) \gamma_k^{-1}, q(v) = t, (v, l_\infty) = 0\}. \quad (5.6)$$

The action of the group $\Gamma_0(p^n) \cap \mathrm{Stab}_P$ on the left hand side transforms to the action of the group $\gamma_k \Gamma_0(p^n) \gamma_k^{-1} \cap \mathrm{Stab}_{P_\infty}$. The assumption that $[n/2] \leq k \leq n$ implies that

$$\gamma_k \Gamma_0(p^n) \gamma_k^{-1} \cap \mathrm{Stab}_{P_\infty} = \left\{ \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}. \quad (5.7)$$

An element v on the right hand side of (5.6) has the following form

$$v = \begin{pmatrix} \frac{a}{p^k} & \frac{a+p^k b}{p^{2k}} \\ 0 & -\frac{a}{p^k} \end{pmatrix}, \ a, b \in \mathbb{Z} \text{ and } -p^{n-2k} a^2 = t. \quad (5.8)$$

Hence the set on the right hand side of (5.6) is empty if $-p^n t$ is not a square. Now Let's assume $-p^n t$ is a square. The element $g = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ acts on the element (5.8) in the following way:

$$v = \begin{pmatrix} \frac{a}{p^k} & \frac{a+p^k b}{p^{2k}} \\ 0 & -\frac{a}{p^k} \end{pmatrix} \mapsto g v g^{-1} = \begin{pmatrix} \frac{a}{p^k} & \frac{a+p^k(b-2ax)}{p^{2k}} \\ 0 & -\frac{a}{p^k} \end{pmatrix}$$

Hence for a given a, the element b has $2|a| = 2\sqrt{-p^{2k-n} t}$ choices. Therefore

$$\#\Gamma_0(p^n) \cap \mathrm{Stab}_P \backslash \{v : \ v \in \Delta(p^n), q(v) = t, (v, l_P) = 0\} = 4\sqrt{-p^{2k-n} t}.$$

□

Theorem 5.2.3. *For any pair (t, y) such that t is an integer and y is a positive number,*

(a). The function $\mathfrak{g}(t, y, \Delta(N))([z])$ is a Green function for the modified divisor $\mathcal{Z}^*(t, y, \Delta(N))$ if $t \neq 0$; for the boundary divisor $\sum_{M|N} g(0, y, \Delta(\sigma(N, M))) \cdot \mathsf{Cusp}^M(\mathcal{X}_0(N))$ if $t = 0$.

(b). The element $(\sum_{M|N} g(0, y, \Delta(\sigma(N, M))) \cdot \mathsf{Cusp}^M(\mathcal{X}_0(N)), \mathfrak{g}(0, y, \Delta(N))([z]))$ and all the elements $(\mathcal{Z}^*(t, y, \Delta(N)), \mathfrak{g}(t, y, \Delta(N))([z]))$ for nonzero t belong to the extended arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S})$, more precisely,

$$[\omega(t, y, \Delta(N))] = \begin{cases} \mathrm{dd}^c \mathfrak{g}(t, y, \Delta(N)) + \delta_{\mathcal{Z}^*(t,y,\Delta(N))_{\mathbb{C}}}, & \text{if } t \neq 0; \\ \mathrm{dd}^c \mathfrak{g}(0, y, \Delta(N)) + \delta_{\sum\limits_{M|N} g(0,y,\Delta(\sigma(N,M)))\cdot \mathsf{Cusp}^M(\mathcal{X}_0(N))}, & \text{if } t = 0. \end{cases}$$

- *When $-Nt > 0$ is a square, the function $\mathfrak{g}(t, y, \Delta(N))([z])$ has log singularity at the cusps;*
- *When $t = 0$, the function $\mathfrak{g}(0, y, \Delta(N))([z])$ has log-log singularity at the cusps;*
- *Otherwise the function $\mathfrak{g}(t, y, \Delta(N))([z])$ decreases exponentially at the cusps.*

(c). The element $(\mathcal{Z}^*(0, y, \Delta(N)), \mathfrak{g}^*(0, y, \Delta(N))) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S})$ lies in the usual arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$.

Proof. The case $-Nt$ is not a square has been proved in [2, Theorem 5.1 (1)]. We only need to consider the case $-Nt$ is a square.

When $-Nt > 0$ is a square, let P be a cusp point and q_P be the local parameter of $\mathcal{X}_0(N)$ at P. It is also proved in *loc. cit.* that $\omega(t, y, \Delta(N))$ have the following expansian near the point P:

$$\omega(t, y, \Delta(N)) = -g_P(t, y, \Delta(N)) \cdot (\log |q_P|^2) + o(1),$$

here

$$g_P(t, y, \Delta(N)) = \frac{\alpha_P(t, y, \Delta(N))}{8\pi\sqrt{-ty}} \cdot \beta_{3/2}(-4\pi t y).$$

Let l_P be the isotropic vector in the vector space $\Delta(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to the cusp P, the number $\alpha_P(t, y, \Delta(N))$ is the cardinality of the following set:

$$\Gamma_0(N)\backslash\{(v, l) : \ v \in \Delta(N), q(v) = t, (v, l) = 0, l \text{ is } \Gamma_0(N)\text{-equivalent to } l_P\}.$$

The above set can be mapped to the following set bijectively:

$$\Gamma_0(N) \cap \mathrm{Stab}_P\backslash\{v : \ v \in \Delta(N), q(v) = t, (v, l_P) = 0\}.$$

The cardinality of the above set has been calculated in Lemma 5.2.2. Therefore if $P = P_{aM/N}$ for some positive integer $M|N$, we have

$$g_P(t, y, \Delta(N)) = \begin{cases} \frac{\sqrt{\sigma(N,M)}}{2\pi\sqrt{y}} \cdot \beta_{3/2}(-4\pi t y), & \text{if } -Nt > 0 \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

Hence $g_P(t, y, \Delta(N)) = g(t, y, \Delta(\sigma(N, M)))$. This concludes the proof of (a) and (b) for $t \neq 0$. The proof of (a) and (b) for the case $t = 0$ is similar to the $-Nt > 0$ is a square case, so we omit it. While the last statement (c) is proved in [2, Proposition 6.6]. □

Definition 5.2.4. For any integer $t \in \mathbb{Z}$ and real number $y > 0$, define the following element in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,

$$\widehat{\mathcal{Z}}(t, y, \Delta(N)) = (\mathcal{Z}^*(t, y, \Delta(N)), \mathfrak{g}^*(t, y, \Delta(N)))$$

Lemma 5.2.5. *For any integer $t \in \mathbb{Z}$ and real number $y > 0$, we have the following identity in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,*

$$W_N^* \widehat{\mathcal{Z}}(t, y, \Delta(N)) = \widehat{\mathcal{Z}}(t, y, \Delta(N)).$$

Proof. When $t = 0$, this follows from the definition of $\widehat{\omega}$ in (5.1). When $t \neq 0$, this is proved in [2, Proposition 6.8]. □

5.2.2 The codimension 2 case: Arithmetic special cycles in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$

The rank of T is 2

Next we associate to any nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ and a T-admissible Schwartz function $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ an element in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$. Let $\mathsf{y} = {}^t a \cdot a \in \mathrm{Sym}_2(\mathbb{R})$ be a positive definite matrix, where $a \in \mathrm{GL}_2(\mathbb{R})$.

- For a positive definite T and a T-admissible Schwartz function $\boldsymbol{\varphi}$, we consider the following element

$$\widehat{\mathcal{Z}}(T, \mathsf{y}, \boldsymbol{\varphi}) = (\mathcal{Z}(T, \boldsymbol{\varphi}), 0) \in \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)).$$

- For other nonsingular T which is not positive definite, we apply the general machine developed by Garcia and Sankaran in [19] which is made explicit in [1]. We define an element in $\mathcal{S}(\mathbb{V}_\infty^2) \otimes \mathcal{A}^{1,1}(\mathbb{H}_1^{\pm})$ by the following description: for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{V}_\infty^2$ and $[z] \in \mathbb{D}$,

$$\nu(\boldsymbol{x}, [z]) = \left(-\pi^{-1} + 2\sum_{i=1}^{2}(\mathrm{R}(x_i, [z]) + (x_i, x_i))\right) e^{-2\pi(\sum\limits_{i=1}^{2}(\mathrm{R}(x_i,[z]) + \frac{1}{2}(x_i,x_i)))} \cdot \frac{dx \wedge dy}{y^2}.$$

then we define a smooth $(1,1)$-form $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ on $\mathbb{D}$ as follows, its value at the point $[z] \in \mathbb{D}$ is

$$\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})([z]) = \sum_{\substack{\boldsymbol{x} \in (\Delta(N) \otimes \mathbb{Q})^2 \\ T(\boldsymbol{x}) = T}} \boldsymbol{\varphi}(\boldsymbol{x}) \cdot \int_1^\infty \nu(t^{\frac{1}{2}} \boldsymbol{x} \cdot {}^t a, [z]) \cdot \frac{\mathrm{d}t}{t}.$$

the sum converges absolutely, and descends to a smooth $(1,1)$-form on the modular curve $\mathcal{Y}_0(N)_{\mathbb{C}}$. The following is proved in [1].

Lemma 5.2.6. *For nonsingular $T \in \mathrm{Sym}_2(\mathbb{R})$ which is not positive definite. The form $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ is absolutely integrable on $\mathcal{X}_0(N)_{\mathbb{C}}$, hence $\mathfrak{g}(T, \mathsf{y}, \boldsymbol{\varphi})$ defines a $(1,1)$-current on $\mathcal{X}_0(N)_{\mathbb{C}}$.*

Proof. This is proved in [1, Lemma 2.9]. □

To sum up, let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ is a

T-admissible Schwartz function, we define

$$\widehat{\mathcal{Z}}(T, \mathsf{y}, \varphi) = \begin{cases} ([\mathcal{Z}(T, \varphi)], 0), & \text{when } T \text{ is positive definite;} \\ (0, \mathfrak{g}(T, \mathsf{y}, \varphi)), & \text{when } T \text{ is not positive definite.} \end{cases} \tag{5.9}$$

it is an element in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$.

The rank of T is 1

Lemma 5.2.7. *Let* $T \in \mathrm{Sym}_2(\mathbb{Q})$ *be a half-integral* 2×2 *symmetric matrix such that* $\mathrm{rank}(T) = 1$, *then there exists* $t \in \mathbb{Z}$ *and an element* $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ *such that*

$$T = {}^t\gamma \cdot \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \cdot \gamma.$$

Moreover, the integer t *is uniquely determined by* T, *and is invariant upon replacing* T *by* ${}^t\sigma T \sigma$ *with* $\sigma \in \mathrm{GL}_2(\mathbb{Z})$.

Proof. This is proved in [1, Lemma 2.10]. □

Definition 5.2.8. Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a half-integral 2×2 symmetric matrix such that $\mathrm{rank}(T) = 1$, let $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$ is a positive definite 2×2 matrix. Let t be the integer associated to T by Lemma 5.2.7. Let $y_2 = t^{-1}\mathrm{tr}(T\mathsf{y})$ and $y_1 = \frac{\det \mathsf{y}}{y_2}$, both of y_1 and y_2 are positive numbers. Define

$$\widehat{\mathcal{Z}}(T, \mathsf{y}) = \widehat{\mathcal{Z}}(t, y_2, \Delta(N)) \cdot \widehat{\omega} - \left(0, \log y_1 \cdot \delta_{\mathcal{Z}^*(t, y_2, \Delta(N))_{\mathbb{C}}}\right)$$

where $\mathcal{Z}^*(t, y_2, \Delta(N))_{\mathbb{C}}$ is the complex points of the modified special cycle defined in (5.4).

Remark 5.2.9. When the rank of T is 1. We notice two invariance properties, which both follows from the definitions:

$$\widehat{\mathcal{Z}}({}^t\gamma T \gamma, \mathsf{y}) = \widehat{\mathcal{Z}}(T, \gamma \mathsf{y}{}^t\gamma), \text{ for any } \gamma \in \mathrm{GL}_2(\mathbb{Z}). \tag{5.10}$$

$$\widehat{\mathcal{Z}}\left(\begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix}, \theta \mathsf{y}^t\theta\right) = \widehat{\mathcal{Z}}\left(\begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix}, \mathsf{y}\right),\ \text{ for any } \theta = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \text{ where } x \in \mathbb{R}. \tag{5.11}$$

The rank of T is 0

In this case, the matrix $T = \mathbf{0}_2$.

Definition 5.2.10. Let $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})$ be a positive definite matrix, we define the following element in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$,

$$\widehat{\mathcal{Z}}(\mathbf{0}_2, \mathsf{y}) = \widehat{\omega} \cdot \widehat{\omega} + (0, \log \det \mathsf{y} \cdot [\Omega]).$$

Recall that $[\Omega]$ is the restriction of the $(1,1)$-form $\frac{\mathrm{d}x \wedge \mathrm{d}y}{2\pi y^2}$ to $\mathcal{X}_0(N)_{\mathbb{C}}$.

5.3 Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$

For a rational number t which is not an integer, we simply define $\widehat{\mathcal{Z}}(t, y) = 0 \in \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$. Let $\tau = x + iy \in \mathbb{H}_1$, we consider the following generating series with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$,

$$\widehat{\phi}_1(\tau) = \sum_{t \in \mathbb{Q}} \widehat{\mathcal{Z}}(t, y, \Delta(N)) \cdot q^t$$

where $q^t = e^{2\pi i t\tau}$.

For an element $\mathsf{z} = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_2$, we consider the following generating series with coefficients in $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$,

$$\widehat{\phi}_2(\mathsf{z}) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} \widehat{\mathcal{Z}}(T, \mathsf{y}) \cdot q^T$$

where $q^T = e^{2\pi i \,\mathrm{tr}\,(T\mathsf{z})}$. Recall that we have defined a degree map $\widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) \xrightarrow{\sim} \mathbb{C}$ in §4.3. Now we are able to state the main theorem of this thesis.

Theorem 5.3.1. *Let N be a positive integer.*

- *The modularity theorem for arithmetic generating series of codimension 1:*

The generating series $\widehat{\phi}_1$ *is a nonholomorphic Siegel modular form of genus 1, weight* $\frac{3}{2}$ *and level* $\Gamma_0(4N)$ *with values in* $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$.

- *The modularity theorem for arithmetic generating series of codimension 2:*

  The generating series $\widehat{\phi}_2$ *is a nonholomorphic Siegel modular form of genus 2 and weight* 2. *More precisely, under the isomorphism* $\widehat{\deg} : \widehat{\mathrm{CH}}^2(\mathcal{X}_0(N)) \xrightarrow{\sim} \mathbb{C}$

  $$\widehat{\phi}_2(\mathsf{z}) = \frac{\psi(N)}{24} \cdot \partial\mathrm{Eis}(\mathsf{z}, \Delta(N)^2),$$

  here $\psi(N) = N \cdot \prod_{p|N} (1 + p^{-1})$.

The proof of Theorem 5.3.1 will be given in the next two chapters.

Chapter 6: Proof of the main theorem: nonsingular terms

In this chapter, we aim to prove the following identity when the matrix $T \in \mathrm{Sym}_2(\mathbb{Q})$ is nonsingular:

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathsf{y}) \cdot q^T = \frac{\psi(N)}{24} \cdot \partial \mathrm{Eis}_T(\mathsf{z}, \Delta(N)^2). \tag{6.1}$$

6.1 Rapoport–Zink spaces and special cycles

6.1.1 $\Gamma_0(N)$-structures on p-divisible groups

Let p be a prime number. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let W be the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let Nilp_W be the category of schemes S over $\mathrm{Spec}\, W$ such that p is locally nilpotent on S, let $\overline{S}$ be the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$. For a p-divisible group X, we use $X^\vee$ to denote the dual p-divisible group. We will introduce two Rapoport–Zink spaces in this chapter, they are essentially isomorphic to the completed local rings of supersingular points in characteristic p of the moduli stacks $\mathcal{H}$ and $\mathcal{X}_0(N)$.

Let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}$ of dimension 1 and height 2, the associated filtered isocrystal $\mathbb{D}(\mathbb{X})_{\mathbb{Q}}$ has pure slope $\frac{1}{2}$, e.g., we can take $\mathbb{X}$ to be $\mathbb{E}[p^\infty]$ where $\mathbb{E}$ is a supersingular elliptic curve over $\mathbb{F}$. Let $\lambda_0 : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ be a principal polarization. We consider the following functor $\mathcal{N}$ on the category Nilp_W: for any $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}(S)$ is the isomorphism classes of tuples $((X, \rho, \lambda), (X', \rho', \lambda'))$, where

(1) X and X' are two p-divisible group over S, ρ and ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X \times_S \overline{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X' \times_S \overline{S}$.

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\overline{S}$,

we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$.

Proposition 6.1.1. *The functor $\mathcal{N}$ is represented by the formal scheme* $\operatorname{Spf} W[[t_1, t_2]]$ *over* $\operatorname{Spf} W$.

Proof. When p is odd, this is explained in [10, Example 4.5.3 (ii)]. In general, the deformation space of the supersingular elliptic curve $\mathbb{E}$ is isomorphic to $\operatorname{Spf} W[[t]]$. By Serre-Tate theorem, this is also the deformation space of the p-divisible group $\mathbb{X}$ with certain restrictions on the polarization as in the definition of the deformation functor $\mathcal{N}$. Therefore $\mathcal{N} \simeq \operatorname{Spf} W[[t_1]] \times_{\operatorname{Spf} W} \operatorname{Spf} W[[t_2]] \simeq \operatorname{Spf} W[[t_1, t_2]]$. □

Let $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ be the universal p-divisible group over $\mathcal{N} = \operatorname{Spf} W[[t_1, t_2]]$. By Lemma 6.2.3 below, the category of p-divisible groups over $\operatorname{Spf} W[[t_1, t_2]]$ is equivalent to the category of p-divisible groups over $\operatorname{Spec} W[[t_1, t_2]]$, we use

$$((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$$

to denote the corresponding p-divisible group over $\operatorname{Spec} W[[t_1, t_2]]$.

Next we fix a N-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$, i.e., $x_0 \circ x_0^\vee = N$. $\mathcal{N}_0(N)$ is a contravariant set-valued functor defined over Nilp_W, for every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of the isomorphism classes of elements of the following form $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda'))$, where

(1) X and X' are two p-divisible group over S, ρ and ρ' are two quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X \times_S \overline{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X' \times_S \overline{S}$.

(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\overline{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.$$

for some $c(\rho), c(\rho') \in \mathbb{Z}_p^\times$.

(3) $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

We will prove it later that the functor $\mathcal{N}_0(N)$ is represented by a closed formal subscheme of $\mathrm{Spf}\, W[[t_1, t_2]]$ cut out by a single equation (cf. Theorem 6.2.6).

6.1.2 Special cycles on $\mathcal{N}$ and $\mathcal{N}_0(N)$

Now we give the definition of special cycles on the formal schemes $\mathcal{N}$ and $\mathcal{N}_0(N)$. Recall that $((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}, \lambda^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}}, \lambda'^{\mathrm{univ}}))$ is the universal p-divisible group over $\mathcal{N}$, and $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$ is the unique division quaternion algebra over $\mathbb{Q}_p$, whose Hasse invariant as a quadratic space is -1.

Definition 6.1.2. For any subset $L \subset \mathbb{B}$, define the special cycle $\mathcal{Z}^\sharp(L)$ to be the following closed formal subscheme of $\mathcal{N}$: for an object $S \in \mathrm{Nilp}_W$, the groupoid $\mathcal{Z}^\sharp(L)(S)$ consists of pairs $((X, \rho, \lambda), (X', \rho', \lambda')) \in \mathcal{N}(S)$ such that the quasi-isogeny $\rho' \circ x \circ \rho^{-1}$ is an isogeny from X to X'.

Remark 6.1.3. The special cycle $\mathcal{Z}^\sharp(L)$ only depends on the $\mathbb{Z}_p$-linear span of L in $\mathbb{B}$, and is nonempty only when this span is an integral quadratic $\mathbb{Z}_p$-lattice in $\mathbb{B}$.

Proposition 6.1.4. *Let $x \in \mathbb{B}$ be a nonzero and integral element, i.e., $0 \leq v_p(x^\vee \circ x) < \infty$. Then $\mathcal{Z}^\sharp(x)$ is a Cartier divisor on $\mathcal{N}$, i.e., it is defined by a single nonzero element $f_x \in W[[t_1, t_2]]$. Moreover, $\mathcal{Z}^\sharp(x)$ is also flat over* $\mathrm{Spf}\, W$*, i.e., $p \nmid f_x$.*

Proof. When p is odd, the formal scheme $\mathcal{N}$ is an example of GSpin Rapoport–Zink space (cf. [10, Example 4.5.3 (ii)]), and the proposition has been proved for every GSpin Rapoport–Zink space in [10, Proposition 4.10.1]. For all p (especially $p = 2$), this is proved in [3, Theorem 6.8.1]. □

Now let's come to the special cycles on $\mathcal{N}_0(N)$. Firstly, we give the definition of the space of special quasi-isogenies, recall that we have fixed a N-isogeny x_0 when we define the formal scheme $\mathcal{N}_0(N)$.

Definition 6.1.5. We call a quasi-isogeny $x \in \mathbb{B} = \mathrm{End}^0(\mathbb{X})$ is special to x_0 if the following condition holds,

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

By definition, the space of quasi-isogenies special to x_0 is just the quadratic space $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$. By Witt's theorem, it is a 3-dimensional quadratic space over $\mathbb{Q}_p$ whose isometric class is independent of the choice of the N-isogeny x_0.

Definition 6.1.6. Let $(\breve{X} \xrightarrow{\breve{x}} \breve{X}', (\breve{\rho}, \breve{\rho}'), (\breve{\lambda}, \breve{\lambda}'))$ be the universal object over $\mathcal{N}_0(N)$. For any subset $M \subset \mathbb{W}$, define the special cycle $\mathcal{Z}(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the following conditions,

$$\breve{\rho}' \circ x \circ \breve{\rho}^{-1} \in \mathrm{Hom}(\breve{X}, \breve{X}').$$

for any $x \in M$.

For a subset $M \subset \mathbb{W} \subset \mathbb{B}$, we have the following Cartesian diagram,

$$\begin{tikzcd}
\mathcal{Z}(M) \arrow[r] \arrow[d] & \mathcal{N}_0(N) \arrow[d] \\
\mathcal{Z}^\sharp(M) \arrow[r] & \mathcal{N}.
\end{tikzcd}$$

6.1.3 Formal uniformization of $\mathcal{X}_0(N)$ and the special cycle $\mathcal{Z}(T, \varphi)$

Let B be the unique quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. Then $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$ is the unique division quaternion algebra over $\mathbb{Q}_p$. Let $\mathbb{E}$ be a supersingular elliptic curve over $\mathbb{F}$ and $\mathbb{X} = \mathbb{E}[p^\infty]$ is the p-divisible group of $\mathbb{E}$. Then $B \simeq \mathrm{End}^0(\mathbb{E})$ and $\mathbb{B} \simeq \mathrm{End}^0(\mathbb{X})$. Suppose $x_0 \in \mathbb{B}$ comes from a cyclic N-isogeny of $\mathbb{E}$ under the above isomorphism $\mathrm{End}^0(\mathbb{E}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{B}$.

We first state and explain the formal uniformization theorem of the supersingular locus $\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}}$ of $\mathcal{H}_{\mathbb{F}_p}$. We use $\hat{\mathcal{H}}/_{(\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}})}$ to denote the completion of $\mathcal{H}$ along the closed substack $\mathcal{H}_{\mathbb{F}_p}^{\mathrm{ss}}$.

Theorem 6.1.7. *There is an isomorphism of formal stacks over W*

$$\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})} \underset{\sim}{\xrightarrow{\Theta_{\mathcal{H}}}} B^{\times}(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2 / \mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2], \tag{6.2}$$

where $B^{\times}(\mathbb{Q})_0$ is the subgroup of $B^{\times}(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Theorem 6.1.7 is proved by Rapoport and Zink [39, Theorem 6.24]. Here we only describe the isomorphism, especially the group action on the right hand side of (6.2). Let $\eta_0^p : V^p(\mathbb{E}) \xrightarrow{\sim} (\mathbb{A}_f^p)^2$ be a prime-to-p level structure of $\mathbb{E}$. Let $\tilde{\mathbb{E}}$ be a deformation of $\mathbb{E}$ to W, and let $\tilde{\mathbb{X}} := \tilde{\mathbb{E}}[p^\infty]$ be the corresponding deformation of $\mathbb{X}$ to W. For some object $S \in \mathrm{Nilp}_W$, we pick an object $((X, \rho, \lambda), (X', \rho', \lambda'), (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2$. The quasi-isogeny ρ (resp. ρ') gives rise to a quasi-isogeny $\tilde{\rho} : \tilde{\mathbb{X}}_S \to X$ (resp. $\tilde{\rho}' : \tilde{\mathbb{X}}_S \to X'$). Then there exists an elliptic curve E (resp. E') up to prime-to-p isogeny over S and a quasi-isogeny $\rho_E : \tilde{\mathbb{E}}_S \to E$ (resp. $\rho_{E'} : \tilde{\mathbb{E}}_S \to E'$), such that $E[p^\infty] \simeq X$ (resp. $E'[p^\infty] \simeq X'$) and ρ_E (resp. $\rho_{E'}$) induces $\tilde{\rho}$ (resp. $\tilde{\rho}'$) under this isomorphism. The object $((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))$ is mapped to

$$((E, E'), (\overline{g^{-1}\eta_0^p \circ V^p(\rho_E^{-1})}, \overline{g'^{-1}\eta_0^p \circ V^p(\rho_{E'}^{-1})})) \in \mathcal{H}(S).$$

The group action is given as follows, for a pair of elements $(b, b') \in B^{\times}(\mathbb{Q})_0 \times B^{\times}(\mathbb{Q})_0$, by the following map,

$$B(\mathbb{Q}) \to B(\mathbb{Q}_p) \simeq \mathrm{End}^0(\mathbb{X}) \xrightarrow{\rho^*} \mathrm{End}^0(X) \text{ (resp. } \xrightarrow{\rho'^*} \mathrm{End}^0(X')),$$

and a fixed isomorphism $B(\mathbb{A}_f^p) \simeq \mathrm{GL}_2(\mathbb{A}_f^p)$. We obtain another triple

$$(b, b')_*(((X, \rho, \lambda), (X', \rho', \lambda'), (g, g'))) := ((X, \rho \circ b^{-1}, \lambda), (X', \rho' \circ b'^{-1}, \lambda'), (bg, b'g')).$$

Now let $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$) be the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)_{\mathbb{F}_p}$).

Let $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$ (resp. $\hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$) be the completion of $\mathcal{X}_0(N)$ (resp. $\mathcal{Y}_0(N)$) along the closed substack $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ (resp. $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$). By the definition of $\mathcal{X}_0(N)$, we have $\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} = \mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$ and therefore $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Proposition 6.1.8. *There is an isomorphism of formal stacks over W,*

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \underset{\sim}{\overset{\Theta_{\mathcal{X}_0(N)}}{\longrightarrow}} B^{\times}(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)], \tag{6.3}$$

where $B^{\times}(\mathbb{Q})_0$ is the subgroup of $B^{\times}(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

Proof. The following diagram is Cartesian with all arrows closed immersions.

$$\begin{tikzcd}
\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p} \arrow[r] \arrow[d] & \mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p} \arrow[d] \\
\mathcal{Y}_0(N) \arrow[r] & \mathcal{H}.
\end{tikzcd}$$

this diagram gives a closed immersion $i : \hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \simeq \hat{\mathcal{Y}}_0(N)/_{(\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})} \to \hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Recall that we have the following isomorphism,

$$\hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})} \underset{\sim}{\overset{\Theta_{\mathcal{H}}}{\longrightarrow}} B^{\times}(\mathbb{Q})_0^2 \backslash [\mathcal{N} \times \mathrm{GL}_2(\mathbb{A}_f^p)^2/\mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2].$$

Let S be an object in Nilp_W, let $(z, (g, g')) \in \mathcal{N}(S) \times \mathrm{GL}_2(\mathbb{A}_f^p)^2$ be a point in the closed formal substack $\mathcal{Y}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$, then clearly $z \in \mathcal{N}_0(N)(S)$. Suppose z corresponds to a cyclic isogeny $E \xrightarrow{\pi} E'$ by our description of the isomorphism $\Theta_{\mathcal{H}}$, then g' is determined by g by the following diagram,

$$\begin{tikzcd}
V^p(E_{\bar{s}}) \arrow[r, "g^{-1}\eta_0^p \circ V^p(\rho_E^{-1})"] \arrow[d, "V^p(\pi)"] & (\mathbb{A}_f^p)^2 \arrow[d, "w_N"] \\
V^p(E'_{\bar{s}}) \arrow[r, "g'^{-1}\eta_0^p \circ V^p(\rho_{E'}^{-1})"] & (\mathbb{A}_f^p)^2.
\end{tikzcd} \tag{6.4}$$

thus we only focus on the pair $(z,g)\in\mathcal{N}_0(N)(S)\times\mathrm{GL}_2(\mathbb{A}_f^p)$. Consider the following morphism

$$\begin{aligned}\Theta:\mathcal{N}_0(N)\times\mathrm{GL}_2(\mathbb{A}_f^p) &\longrightarrow \hat{\mathcal{H}}/_{(\mathcal{H}^{\mathrm{ss}}_{\mathbb{F}_p})},\\ (z,g)&\longmapsto \Theta_{\mathcal{H}}^{-1}(z,(g,g')).\end{aligned}$$

the image of Θ lies in the closed formal substack $\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}$.

Let (z_1,g_1), $(z_2,g_2)\in\mathcal{N}_0(N)(S)\times\mathrm{GL}_2(\mathbb{A}_f^p)$ be two points, then $\Theta(z_1,g_1)=\Theta(z_2,g_2)$ if and only if there exists $b,b'\in B^\times(\mathbb{Q})_0$ and $k_1,k_1'\in\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ such that $(z_2,(g_2,g_2'))=((b,b')_*z_1,(bg_1k_1,b'g_1'k_1'))$. We still use $E\xrightarrow{\pi}E'$ to denote the corresponding point of z_2 under $\Theta_{\mathcal{H}}$. Notice that $(z_2,(g_2,g_2'))=(z_2,(bg_1,b'g_1'))$ in the quotient stack $[\mathcal{N}\times\mathrm{GL}_2(\mathbb{A}_f^p)^2/\mathrm{GL}_2(\hat{\mathbb{Z}}^p)^2]$, therefore $\Theta_{\mathcal{H}}(z_2,(g_2,g_2'))=\Theta_{\mathcal{H}}(z_2,(bg_1,b'g_1'))\in\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p})}(S)$, hence both ($g_2=bg_1k_1$, $g_2'=b'g_1'k_1'$) and $(bg_1,b'g_1')$ satisfy the commutative diagram (6.4), then

$$k_1'=w_Nk_1w_N^{-1},$$

since both k_1 and k_1' belongs to $\mathrm{GL}_2(\hat{\mathbb{Z}}^p):=\prod_{v\neq\infty,p}\mathrm{GL}_2(\mathbb{Z}_v)$, there exists $a,b,c,d\in\hat{\mathbb{Z}}^p$ such that

$$k_1=\begin{pmatrix}a & b\\ Nc & d\end{pmatrix}\in\Gamma_0(N)(\hat{\mathbb{Z}}^p).$$

Moreover, the element b' is also determined by b by the diagram (6.4). Therefore $\Theta(z_1,g_1)=\Theta(z_2,g_2)$ if and only if there exists $b\in B^\times(\mathbb{Q})_0$ and $k\in\Gamma_0(N)(\hat{\mathbb{Z}}^p)$ such that

$$(z_2,g_2)=(b_*z_1,bg_1k)$$

. □

Let $\hat{\mathcal{Z}}^{\mathrm{ss}}(T,\varphi)$ be the completion of $\mathcal{Z}(T,\varphi)$ along its supersingular locus $\mathcal{Z}^{\mathrm{ss}}(T,\varphi):=\mathcal{Z}(T,\varphi)\times_{\mathcal{X}_0(N)}\mathcal{X}_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}$. Let $\Delta(N)^{(p)}$ be the unique quadratic space over $\mathbb{Q}$ (up to isometry) such that:

(1) It is positive definite at ∞;

(2) For finite prime $l \neq p$, $\Delta(N)^{(p)} \otimes \mathbb{Q}_l$ is isometric to $\Delta_l(N) \otimes \mathbb{Q}_l$;

(3) $\Delta(N)^{(p)} \otimes \mathbb{Q}_p$ is isometric to $\mathbb{W}$.

As a corollary of the formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$ (cf. Proposition 6.1.8), we have the following formal uniformization of the special cycles on $\mathcal{X}_0(N)$.

Corollary 6.1.9. *Let* $T \in \mathrm{Sym}_2(\mathbb{Q})$ *be a nonsingular symmetric matrix, and* $\mathrm{Diff}(T, \Delta(N)) = \{p\}$. *Let* $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ *be a* T*-admissible Schwartz function. Let* $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})})$ *be the Grothendieck group of coherent sheaves of* $\mathcal{O}_{\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})}}$*-modules. Then we have the following identity in* $K_0'(\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)_{\mathbb{F}_p}^{\mathrm{ss}})})$,

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \Theta_{\mathcal{X}_0(N)}^{-1}(\mathcal{Z}(\boldsymbol{x}), g),$$

where $B_{\boldsymbol{x}}^\times \subset B^\times$ *is the stabilizer of* $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$.

Proof. We only need to prove the corollary when $\boldsymbol{\varphi}$ is the characteristic function of some open compact subset $\boldsymbol{\omega}$ of $\mathbb{V}_f^2$. Let S be an object in Nilp_W. Suppose $\Theta_{\mathcal{X}_0(N)}^{-1}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ for some $z \in \mathcal{N}_0(N)(S)$, then z gives rise to a cyclic isogeny $E \xrightarrow{\pi} E'$, along with two isogenies $x_1, x_2 \in \mathrm{Hom}(E, E')_{(p)}$ such that

$$T = (\frac{1}{2}(x_i, x_j)), \text{ and } (x_1, \pi) = (x_2, \pi) = 0.$$

then x_1, x_2 and π induce endomorphisms of the correponding p-divisible groups, and hence endomorphims of $\mathbb{X}$. We still use x_1, x_2 to denote the endomorphisms of $\mathbb{X}$, let $T(\boldsymbol{x}) := (\frac{1}{2}(x_i, x_j))$ be the inner product matrix of $\boldsymbol{x} = (x_1, x_2))$, we have

$$T = T(\boldsymbol{x}), \text{ and } (x_1, x_0) = (x_2, x_0) = 0.$$

i.e., $x_1, x_2 \in \{x_0\}^\perp = \mathbb{W} \simeq \Delta(N)^{(p)} \otimes_\mathbb{Q} \mathbb{Q}_p$. We can also identify x_1 and x_2 as elements in $\Delta(N)^{(p)} \otimes \mathbb{A}_f^p$ via the level structures $\eta_0^p \circ V^p(\rho_E^{-1})$ and $\eta_0^p \circ V^p(\rho_{E'}^{-1})$ of E and E'. The positivity assumption on T makes it embeddable into $\Delta(N)^{(p)} \otimes_\mathbb{Q} \mathbb{R}$. By carefully choosing the isometry $\mathbb{W} \simeq \Delta(N)^{(p)} \otimes_\mathbb{Q} \mathbb{Q}_p$, we can find $\boldsymbol{x} \in (\Delta(N)^{(p)})^2$ which induces x_1 and x_2 locally at every place of $\mathbb{Q}$.

Then the condition $\Theta^{-1}_{\mathcal{X}_0(N)}(z, g) \in \hat{\mathcal{Z}}^{\mathrm{ss}}(T, \boldsymbol{\varphi})(S)$ implies that

$$z \in \mathcal{Z}(\boldsymbol{x}) \text{ and } g^{-1}\boldsymbol{x} \in \boldsymbol{\omega} \text{ (here } g \in \mathrm{GL}_2(\mathbb{A}_f) \text{ with } g_p = 1).$$

and this is exactly the meaning of the identity in the theorem. □

6.2 Difference formula at the geometric side

6.2.1 p-divisible groups over adic noetherian rings

Definition 6.2.1. A topological ring R is an adic noetherian ring if it is noetherian as a ring and it has a topological basis consisting of all translations of the neighborhoods of zero of the form I^n ($n > 0$) where $I \subset R$ is a fixed ideal of R, and R is Hausdorff and complete in that topology. A choice of such an ideal is said to be the defining ideal of the topological ring R.

Lemma 6.2.2. *Let A be an adic noetherian local ring whose defining ideal is the maximal ideal $\mathfrak{m}$, then any ideal $I \subset A$ is complete in the topological ring A, i.e.,*

$$I = \bigcap_n (I + \mathfrak{m}^n).$$

Moreover, A/I is an adic noetherian ring with defining ideal $\mathfrak{m}/I$.

Proof. Nakayama lemma implies that $\bigcap_n \mathfrak{m}^n I = 0$, then we can apply [40, Lemma 031B] to conclude that I is $\mathfrak{m}$-adically complete, i.e., $I \simeq \hat{I} := \varprojlim_n I/\mathfrak{m}^n I$.

We have the following exact sequence,

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0.$$

since A is noetherian, after taking completion with respect to the maximal ideal $\mathfrak{m}$, we get

$$0 \longrightarrow \hat{I} \longrightarrow \hat{A} \longrightarrow \widehat{A/I} \longrightarrow 0.$$

However, $A = \hat{A}$ and $I = \hat{I}$, hence $\widehat{A/I} \simeq A/I$. We conclude A/I is an adic noetherian ring.

By definition, $\widehat{A/I} = \varprojlim_n A/(I + \mathfrak{m}^n)$, hence $\widehat{A/I} \simeq A/I$ implies that $I = \bigcap_n (I + \mathfrak{m}^n)$. □

Lemma 6.2.3. *Let A be an adic noetherian ring whose defining ideal is I, then the following functor*

$$\{\text{Category of } p\text{-divisible groups over } \operatorname{Spec} A\} \to \{\text{Category of } p\text{-divisible groups over } \operatorname{Spf}(A)\},$$
$$G = (G_n/A) \mapsto (G_k = (G_k(n) = G(n) \times_A A/I^k))_{k \geq 1}.$$

is an equivalence.

Proof. This is proved by de Jong in [41, Lemma 2.4.4]. □

6.2.2 Difference Divisors on $\mathcal{N}$

Recall that for every nonzero integral $x \in \mathbb{B}$, we define the special divisor $\mathcal{Z}^\sharp(x)$ on $\mathcal{N}$ as the closed formal subscheme of $\mathcal{N}$ over where x lifts to an isogeny (cf. Definition 6.1.2 and Proposition 6.1.4). It is cut out by an element $f_x \in W[[t_1, t_2]]$.

For any nonzero $x \in \mathbb{B}$ such that $\nu_p(x^\vee \circ x) \geq 2$, there is a closed immersion:

$$i : \mathcal{Z}^\sharp(p^{-1}x) \longrightarrow \mathcal{Z}^\sharp(x),$$

by composing every deformation of $p^{-1}x$ with the multiplication-by-p morphism. Since the ring

$W[[t_1,t_2]]$ is a unique factorization domain, we get $f_{p^{-1}x}|f_x$. Define $d_x = f_x/f_{p^{-1}x} \in W[[t_1,t_2]]$ when $\nu_p(x^\vee \circ x) \geq 2$, $d_x = f_x$ when $\nu_p(x^\vee \circ x) = 0$ or 1.

Definition 6.2.4. Let $x \in \mathbb{B}$ be a nonzero and integral element. The difference divisor associated to x to be

$$\mathcal{D}(x) = \operatorname{Spf} W[[t_1,t_2]]/d_x.$$

The notion of difference divisor is first introduced by Terstiege in [17]. Proposition 6.1.4 implies that $p \nmid f_x$, hence $p \nmid d_x$, therefore the difference divisor $\mathcal{D}(x)$ is flat over $\operatorname{Spf} W$. The following theorem asserts that $\mathcal{D}(x)$ is regular.

Theorem 6.2.5. *Let $x \in \mathbb{B}$ be a nonzero and integral element. Let $\mathfrak{m} = (p, t_1, t_2)$ be the maximal ideal of $W[[t_1,t_2]]$, then $d_x \in \mathfrak{m}\backslash\mathfrak{m}^2$, i.e., the difference divisor $\mathcal{D}(x)$ is regular. Moreover, for any $i \geq 1$, d_x and $d_{p^{-i}x}$ are coprime to each other if $p^{-i}x$ is also integral.*

Proof. Let $n \geq 0$ be the p-adic valuation of $x^\vee \circ x$. We first prove this result when $n = 0$, in this case the result follows from the work of Li and Zhu [42, Lemma 3.2.2] (p odd) and [9, Lemma 3.1] ($p = 2$), and $W[[t_1,t_2]]/f_x \simeq W[[t]]$ is even smooth over W.

Now we suppose $n \geq 1$, we can always find an element $x' \in \mathbb{B}$ such that $x'^\vee \circ x'$ has p-adic valuation 0, and $(x, x') = 0$. We consider the formal closed subscheme $\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x')$, it is cut out by the ideal $(f_x, f_{x'}) \subset \mathfrak{m}$; it is also a formal closed subscheme of $\mathcal{Z}^\sharp(x') \simeq \operatorname{Spf} W[[t]]$ cut out by the image $\tilde{f_x}$ of f_x under the surjective map $A \rightarrow W[[t]]$. By [7, (5.10)] (see also [10, p. 5.1]), we have the following decomposition of $\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x')$ into Cartier divisors on $\mathcal{Z}^\sharp(x')$,

$$\mathcal{Z}^\sharp(x) \times_{\mathcal{N}} \mathcal{Z}^\sharp(x') = \sum_{i=0}^{[n/2]} \mathcal{Z}_i, \tag{6.5}$$

each $\mathcal{Z}_i \simeq \operatorname{Spf} O_{\breve{K},i}$, where $O_{\breve{K},i}$ is the ring of integers of some nonarchimedean local field, hence it is a regular local ring, and they are different from each other. Let $d_i \in W[[t]]$ be the function defining the divisor $\mathcal{Z}_i$ on $\mathcal{Z}^\sharp(x')$. Then we have the following identity,

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_i. \tag{6.6}$$

the regularity of $O_{\breve{K},i}$ implies that $d_i \in (p,t) \backslash (p,t)^2$.

Let $\tilde{d}_{p^{-i}x}$ be the image of $d_{p^{-i}x}$ under the surjective map $A \longrightarrow A/(f_{x'}) \simeq W[[t]]$. By definition we have $f_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} d_{p^{-i}x}$, therefore,

$$\tilde{f}_x = (\text{unit}) \times \prod_{i=0}^{[n/2]} \tilde{d}_{p^{-i}x}. \tag{6.7}$$

We induct on n to conclude that $\tilde{d}_x = (\text{unit}) \times d_{[n/2]} \in (p,t)\backslash(p,t)^2$. When $n = 1$, we simply get $\tilde{d}_x = (\text{unit}) \times d_0 \in (p,t)\backslash(p,t)^2$. Let's assume the claim is true for $n < m$ for some $m \geq 2$, we will prove the result for $n = m$. For this, we just need to compare (6.6) and (6.7) for $p^{-1}x$ and x.

Therefore we have proved that $A/(d_x, f_{x'})$ is a regular local ring, hence we conclude that $d_x \in \mathfrak{m}\backslash\mathfrak{m}^2$, and $\mathcal{D}(x) \simeq \operatorname{Spf} A/(d_x)$ is regular. Moreover, since every piece on the right hand side of (6.5) is different from each other, we conclude that d_x and $d_{p^{-i}x}$ are coprime to each other. □

Fix a N-isogeny $x_0 \in \mathbb{B}$, recall that we have defined the deformation functor $\mathcal{N}_0(N)$ in §6.1.1. Compare the moduli intepretations of $\mathcal{N}_0(N)$ and $\mathcal{Z}^\sharp(x_0)$, we have a natural functor,

$$\begin{gathered} i : \mathcal{N}_0(N) \longrightarrow \mathcal{Z}^\sharp(x_0), \\ (X \xrightarrow{x \text{ cyclic}} X', (\rho,\rho'), (\lambda,\lambda')) \longmapsto (X \xrightarrow{x} X', (\rho,\rho'), (\lambda,\lambda')). \end{gathered}$$

Theorem 6.2.6. *The natural functor i is a closed immersion, and induces an isomorphism:*

$$\mathcal{N}_0(N) \xrightarrow{\sim} \mathcal{D}(x_0).$$

Proof. By Proposition 6.1.4, $\mathcal{Z}^\sharp(x_0)$ is represented by $\operatorname{Spf} W[[t_1,t_2]]/f_{x_0}$. We consider the maximal ideal $\mathfrak{m} = (p, t_1, t_2)$ of $W[[t_1, t_2]]$ and a projective system of rings $\varprojlim_n R_n$ where $R_n = W[[t_1, t_2]]/(f_{x_0} + \mathfrak{m}^n)$. We use $(X_n \xrightarrow{x_n} X'_n, (\rho_n, \rho'_n), (\lambda_n, \lambda'_n))$ to denote the corresponding object in $\mathcal{Z}^\sharp(x_0)(R_n)$ by the natural morphism $W[[t_1, t_2]]/f_{x_0} \longrightarrow R_n$, which is essentially the base change from $\mathcal{Z}^\sharp(x_0)$ to Spec R_n of the universal object $(X^{\text{univ}} \overset{x_0^{\text{univ}}}{\longrightarrow} X'^{\text{univ}}, (\rho^{\text{univ}}, \rho'^{\text{univ}}), (\lambda^{\text{univ}}, \lambda'^{\text{univ}}))$. The following diagram is commutative,

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "x_n"] & X_{n+1} \arrow[d, "x_{n+1}"] \\
X'_n \arrow[r] & X'_{n+1}.
\end{tikzcd}$$

By [41, Lemma 2.4.4], x_n fits together to be an isogeny of p-divisible groups $x_0^{\text{univ}} : X^{\text{univ}} \longrightarrow X'^{\text{univ}}$ over Spec $W[[t_1, t_2]]/f_{x_0}$.

Now we apply Serre-Tate theorem (cf. [43]) to the projective system $\varprojlim_n R_n$, we obtain a direct system of elliptic curves E_n, E'_n over Spec R_n and $\tilde{x}_n \in \text{End}_{R_n}(E_n, E'_n)$ such that,

(i) There exist isomorphisms $i_n : E_n[p^\infty] \simeq X_n$ and $i'_n : E'_n[p^\infty] \simeq X'_n$;

(ii) $x_n = i'_n \circ \tilde{x}_n[p^\infty] \circ i_n^{-1}$.

Since every elliptic curve is equipped with a canonical ample line bundle given by the unit section, we can apply Grothendieck's algebraization theorem (cf. [40, Theorem 089A, Lemma 0A42]) to obtain a triple $(E^{\text{univ}} \overset{\tilde{x}_0^{\text{univ}}}{\longrightarrow} E'^{\text{univ}}, (\rho^{\text{univ}}, \rho'^{\text{univ}}), (\lambda^{\text{univ}}, \lambda'^{\text{univ}}))$ where E^{univ} and E'^{univ} are two elliptic curves over Spec $W[[t_1, t_2]]/f_{x_0}$ with the following isomorphisms

$$i^{\text{univ}} : E^{\text{univ}}[p^\infty] \simeq X^{\text{univ}}, \ i'^{\text{univ}} : E'^{\text{univ}}[p^\infty] \simeq X'^{\text{univ}}.$$

and $x_0^{\text{univ}} = i'^{\text{univ}} \circ \tilde{x}_0^{\text{univ}}[p^\infty] \circ (i^{\text{univ}})^{-1}$. Then we have

$$\ker(x_0^{\text{univ}}) \simeq \ker(\tilde{x}^{\text{univ}}[p^\infty]) = \ker(\tilde{x}_0^{\text{univ}})[p^\infty] \hookrightarrow E^{\text{univ}}.$$

Where $\ker(\tilde{x}_0^{\text{univ}})[p^\infty]$ is the p-torsion subgroup scheme of the finite locally free group scheme $\ker(\tilde{x}_0^{\text{univ}})$. Therefore the universal kernel $\ker(x_0^{\text{univ}})$ is embedded into an elliptic curve, we can apply Proposition 4.1.4 and conclude that there is an ideal $\mathcal{I}^{cyc}(x_0) \subset W[[t_1, t_2]]$ containing f_{x_0} such that for an object $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda')) \in \mathcal{Z}^\sharp(x_0)(S)$ where $S \in \mathrm{Nilp}_W$, the isogeny x is a cyclic isogeny if and only if the morphism $S \to \operatorname{Spf} W[[t_1, t_2]]/f_{x_0}$ factors through the closed formal subscheme $\operatorname{Spf} W[[t_1, t_2]]/\mathcal{I}^{cyc}(x_0)$. We conclude from here that $\mathcal{N}_0(N)$ is represented by the formal scheme $\operatorname{Spf} W[[t_1, t_2]]/\mathcal{I}^{cyc}(x_0)$ and the natural functor i is a closed immersion.

Recall that we use d_{x_0} to denote the equation that cuts out the difference divisor $\mathcal{D}(x_0)$. In the following we use $\mathcal{D}$ to denote the difference divisor $\mathcal{D}(x_0)$. Let $x_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ be the base change of $x_0^{\mathrm{univ}} : X^{\mathrm{univ}} \to X'^{\mathrm{univ}}$ to $\mathcal{D}$. We first show that $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$. Let's assume the converse, i.e., $x_{\mathcal{D}} = p \circ x'_{\mathcal{D}}$ where $x'_{\mathcal{D}} : X_{\mathcal{D}} \to X'_{\mathcal{D}}$ is an isogeny. Let $\mathcal{D}_n = \operatorname{Spec} W[[t_1, t_2]]/(d_{x_0} + \mathfrak{m}^n)$, the base change of $x'_{\mathcal{D}}$ from $\mathcal{D}$ to $\mathcal{D}_n$ is a deformation of $p^{-1}x_0$, hence the natural morphism $\mathcal{D}_n \to \mathcal{Z}^\sharp(x_0)$ factors through $\mathcal{Z}^\sharp(p^{-1}x_0) \simeq \operatorname{Spf} W[[t_1, t_2]]/(f_{p^{-1}x_0})$. Since $W[[t_1, t_2]]/(d_{x_0}) \simeq \varprojlim_n W[[t_1, t_2]]/(d_{x_0} + \mathfrak{m}^n)$ by Lemma 6.2.2, we get a ring homomorphism $W[[t_1, t_2]]/(f_{p^{-1}x_0}) \to W[[t_1, t_2]]/(d_{x_0})$. However, d_{x_0} is coprime to $f_{p^{-1}x_0}$ by Theorem 6.2.5, this is a contradiction. Hence $x_{\mathcal{D}}$ doesn't factor through the multiplication-by-p morphism of $X_{\mathcal{D}}$.

Lemma 4.1.5 and Corollary 4.1.6 imply that $\ker(x_{\mathcal{D}})$ is a cyclic group scheme since $\mathcal{D}$ is an integral noetherian scheme which is also separated and flat over W, hence there exists a natural morphism from $\operatorname{Spec} W[[t_1, t_2]]/d_{x_0}$ to $\operatorname{Spec} W[[t_1, t_2]]/\mathcal{I}^{cyc}(x_0)$. Therefore we conclude that $\mathcal{I}^{cyc}(x_0) \subset (d_{x_0}) \subset W[[t_1, t_2]]$. This shows that the closed immersion $\mathcal{D}(x_0) \to \mathcal{Z}^\sharp(x_0)$ decomposes in the following way:

$$\mathcal{D}(x_0) \to \mathcal{N}_0(N) \to \mathcal{Z}^\sharp(x_0).$$

Therefore we have an inclusion of ideals, $(f_{x_0}) \subset \mathcal{I}^{cyc}(x_0) \subset (d_{x_0}) \in W[[t_1, t_2]]$. [3, Theorem 6.6.1] (see also Case II of 5.3.2.1 of *loc.cit*) asserts that $W[[t_1, t_2]]/\mathcal{I}^{cyc}(x_0)$ is a 2-dimensional regular local ring. Recall that we have already proved in Theorem 6.2.5 that $W[[t_1, t_2]]/d_{x_0}$ is also a regular local ring, hence we must have $\mathcal{I}^{cyc}(x_0) = (d_{x_0})$, i.e., $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$. □

Special Fibers

In this part we use the identification $\mathcal{N}_0(N) \xrightarrow{\sim} \mathcal{D}(x_0)$ to explicitly describe the special fiber of the local ring $\mathcal{N}_0(N)$. The main results of this part will not be used in the following calculations, readers can skip on first reading.

Let $\mathfrak{a} = (t_1, t_2) \subset W[[t_1, t_2]]$. Let $\overline{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\mathbb{F}[[t_1, t_2]]$. Let $A_n = W[[t_1, t_2]]/\mathfrak{a}^n$ and $R_n = \mathbb{F}[[t_1, t_2]]/\overline{\mathfrak{a}}^n$. Let $A_0 = W[[t_1, t_2]]$ and $R_0 = \mathbb{F}[[t_1, t_2]]$. Equip each A_n with a morphism σ which extends the Frobenius morphism on W and maps t_1 to t_1^p, t_2 to t_2^p. Then A_n is a frame for R_n in the sense of [18, Definition 1]. For any $n \geq 0$, let (M, M_1, Φ) be an A_n-window in the sense of [18, Definition 2]. Since $\Phi(M_1) \subset p \cdot M$ and p is not a zero-divisor in A_n, we define $\Phi_1 : M_1 \to M$ to be $p^{-1}\Phi$. The morphism Φ_1 is σ-linear and induces an isomorphism $\Phi_1^\sigma : M_1^\sigma \to M$ because both sides are free A_n-module of the same rank and Φ_1^σ is surjective by the definition of windows ([18, Definition 2(ii)]). Let α be the following injective A_n-morphism,

$$\alpha : M_1 \hookrightarrow M \xrightarrow{(\Phi_1^\sigma)^{-1}} M_1^\sigma.$$

Theorem 6.2.7. *For any $n \geq 0$, we have the following category equivalences,*

$$\{A_n\text{-window } (M, M_1, \Phi)\} \xleftrightarrow{\sim} \{\text{formal } p\text{-divisible groups over } R_n\}$$

Moreover, both these two categories are equivalent to the following category.

$$\{\text{pairs } (M_1, \alpha : M_1 \to M_1^\sigma), \text{ such that } \mathrm{Coker}(\alpha) \text{ is a free } R_n\text{-module.}\}$$

where the functor from A_n-window (M, M_1, Φ) to pairs $(M_1, \alpha : M_1 \to M_1^\sigma)$ is given by the constructions above.

Proof. This is proved in [18, Theorem 4]. □

Let $((\overline{X}, \overline{\rho}, \overline{\lambda}), (\overline{X}', \overline{\rho}', \overline{\lambda}'))$ be the universal object in $\mathcal{N}(\mathbb{F}[[t_1, t_2]])$, i.e., the base change of the universal object $((X^{\text{univ}}, \rho^{\text{univ}}, \lambda^{\text{univ}}), (X'^{\text{univ}}, \rho'^{\text{univ}}, \lambda'^{\text{univ}}))$ over $W[[t_1, t_2]]$ to $\mathbb{F}[[t_1, t_2]]$. The corresponding window can be described as follows, let $\mathbb{D} = W \cdot e + W \cdot f$ be the Dieudonne module of $\mathbb{X}$, where $Fe = Ve = f$, $Ff = Vf = p \cdot e$ (F, V are the Frobenius and Verschiebung morphisms on $\mathbb{D}$). Then we let $M = \mathbb{D} \otimes_W W[[t]]$, and $M_1 = (W \cdot f + pW \cdot e) \otimes_W W[[t]]$. We still use σ to denote the Frobenius action on $W[[t]]$ which sends t to t^p and extends the Frobenius morphism on W. Let Φ be the σ-linear map from M to M such that $\Phi(e \otimes 1) = t \cdot (e \otimes 1) + f \otimes 1, \Phi(f \otimes 1) = p \cdot (e \otimes 1)$. Then (M, M_1, Φ) is the $W[[t]]$-window corresponding to the universal deformation of $\mathbb{X}$ over $\mathbb{F}[[t]]$ (cf. [44, (86)]). Let (M', M'_1, Φ') be the correpsonding window for $\mathbb{X}'$, then the $W[[t_1, t_2]]$-window corresponding to the universal deformation of $\mathbb{X} \times_{\mathbb{F}} \mathbb{X}'$ over $\mathbb{F}[[t_1, t_2]]$ is given by $(M \oplus M', M_1 \oplus M'_1, \Phi \oplus \Phi')$, or $(M_1 \oplus M'_1, \alpha)$ where under the basis $\{p \cdot (e \otimes 1), f \otimes 1, p(e' \otimes 1), f' \otimes 1\}$, the map α is given by the following matrix

$$\alpha = \begin{pmatrix} & 1 & & \\ p & -t_1 & & \\ & & & 1 \\ & & p & -t_2 \end{pmatrix}.$$

Any quasi-isogeny $x \in \mathbb{B}$ induces the following endomorphism of the window $M_1 \oplus M'_1$ of $\mathbb{X} \times_{\mathbb{F}} \mathbb{X}'$ under the basis $\{p \cdot e, f, p \cdot e', f'\}$,

$$\mathbb{D}(x) = \begin{pmatrix} & & \sigma(a) & -\sigma(b) \\ & & -p \cdot b & a \\ a & \sigma(b) & & \\ p \cdot b & \sigma(a) & & \end{pmatrix},$$

where $a, b \in \mathbb{Q}_{p^2}$.

Let $M_1(n) = M_1 \otimes_{A_0} A_n$, $M'_1(n) = M'_1 \otimes_{A_0} A_n$, $\alpha(n) = \alpha \otimes_{A_0} A_n$. By Theorem 6.2.7, a quasi-isogeny x lifts to an isogeny over R_n if and only if there exists $x(n) \in \operatorname{End}((M_1(n) \oplus M'_1(n), \alpha(n)))$

such that $x(1) = \mathbb{D}(x)$ and the following diagram commutes

$$\begin{tikzcd}
M_1(n) \oplus M_1'(n) \arrow[r, "\alpha(n)"] \arrow[d, "x(n)"] & M_1(n)^\sigma \oplus M_1'(n)^\sigma \arrow[d, "\sigma(x(n))"] \\
M_1(n) \oplus M_1'(n) \arrow[r, "\alpha(n)"] & M_1(n)^\sigma \oplus M_1'(n)^\sigma.
\end{tikzcd}$$

Under the basis $\{p \cdot (e \otimes 1), f \otimes 1, p(e' \otimes 1), f' \otimes 1\}$, the morphism $x(n)$ has the following form

$$x(n) = \begin{pmatrix} A(n) & Y(n) \\ X(n) & B(n) \end{pmatrix}.$$

where $X(n), Y(n), A(n), B(n) \in \mathrm{M}_2(A_n)$ satisfy the following equations,

$$X(n) = p^{-1}U'(t_2) \cdot \sigma(X(n)) \cdot U(t_1), \;\; Y(n) = p^{-1}U'(t_1) \cdot \sigma(Y(n)) \cdot U(t_2);$$
$$A(n) = p^{-1}U'(t_1) \cdot \sigma(A(n)) \cdot U(t_1), \;\; B(n) = p^{-1}U'(t_2) \cdot \sigma(B(n)) \cdot U(t_2).$$

where $U(t) = \begin{pmatrix} & 1 \\ p & -t \end{pmatrix}$ and $U'(t) = \begin{pmatrix} t & 1 \\ p & \end{pmatrix}$. Since $A(1) = B(1) = 0$, we conclude (by comparing degrees of t_1 and t_2) that $A(n) = B(n) = 0$.

For any $A \in \mathrm{M}_2(A_n \otimes_{\mathbb{Z}} \mathbb{Q})$, the matrix $\sigma(A)$ is a well-defined element in $\mathrm{M}_2(A_{pn} \otimes_{\mathbb{Z}} \mathbb{Q})$. Therefore, starting from $X(1)$ and $Y(1)$, we can define successively

$$X(p^{l+1}) = p^{-1}U'(t_2) \cdot \sigma(X(p^l)) \cdot U(t_1), \;\; Y(p^{l+1}) = p^{-1}U'(t_1) \cdot \sigma(Y(p^l)) \cdot U(t_2). \tag{6.8}$$

Since the local ring $\mathcal{O}_{\mathcal{Z}(x)}$ only depends (up to noncanonical isomorphisms) on the valuation of x, we will take the following specific choice of x and $\mathbb{D}(x)$ in the following computations.

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k$ for some $k \geq 0$, we take

$$X(1) = Y(1) = \begin{pmatrix} p^k & \\ & p^k \end{pmatrix}.$$

By computation based on the recursion formula (6.8), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{\frac{p^{l-1}-1}{p-1}}(t_2^{p^{l-1}} - t_1^{p^{l-1}}) \\ 0 & 0 \end{pmatrix} + p \cdot C\right);$$

$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{\frac{p^{l-1}-1}{p-1}}(t_2^{p^{l-1}} - t_1^{p^{l-1}}) \\ 0 & 0 \end{pmatrix} + p \cdot D\right).$$

for some matrices $C, D \in \mathrm{M}_2(A_{p^l})$.

- When $\mathrm{ord}_p(x^\vee \circ x) = 2k+1$ for some $k \geq 0$, we take

$$X(1) = -Y(1) = \begin{pmatrix} & p^k \\ p^{k+1} & \end{pmatrix}.$$

By computation based on the recursion formula (6.8), it turns out that for any $l \geq 1$,

$$X(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l}(t_1t_2)^{\frac{p^{l}-1}{p-1}} \\ 0 & 0 \end{pmatrix} + p \cdot C'\right);$$

$$Y(p^l) = \frac{1}{p^{l-k}}\left(\begin{pmatrix} 0 & (-1)^{l-1}(t_1t_2)^{\frac{p^{l}-1}{p-1}} \\ 0 & 0 \end{pmatrix} + p \cdot D'\right).$$

for some matrices $C', D' \in \mathrm{M}_2(A_{p^l})$.

Proposition 6.2.8. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above, let $f_x \in W[[t_1, t_2]]$ be the element cutting out $\mathcal{Z}(x)$, then*

$$\overline{f}_x := f_x \mod p = (\text{unit}) \times \begin{cases} (t_1t_2)^{\frac{p^{n/2}-1}{p-1}}(t_2^{p^{n/2}} - t_1^{p^{n/2}}) \mod (t_1, t_2)^{p^{n/2+1}} & \text{when } n \text{ is even;} \\ (t_1t_2)^{\frac{p^{(n+1)/2}-1}{p-1}} \mod (t_1, t_2)^{p^{(n+1)/2}} & \text{when } n \text{ is odd.} \end{cases} \tag{6.9}$$

Proof. By the above formula for $X(p^l)$ and $Y(p^l)$, we can conclude that x can be lifted to an isogeny over $R_{p^{[n/2]}}$ but not over $R_{p^{[n/2]+1}}$. Then the formula for $X(p^{[n/2]+1})$ and $Y(p^{[n/2]+1})$ imply the equation above. ◻

Theorem 6.2.9. *Let $x \in \mathbb{B}$ be an integral nonzero element which has valuation n and induces $X(1)$ and $Y(1)$ as described above, let $f_x \in W[[t_1, t_2]]$ be the element cutting out $\mathcal{Z}(x)$, then $\overline{f}_x$ is divisible by*

$$t_1 - t_2^{p^a},\;\; t_1^{p^a} - t_2 \;\text{ for } 0 \le a \le n \text{ and } a \equiv n \bmod 2.$$

Moreover, $\overline{f}_x$ has no other irreducible factors and the multiplicity of $t_1 - t_2^{p^a}, t_1^{p^a} - t_2$ in $\overline{f}_x$ is $p^{(n-a)/2}$.

Proof. We first prove that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\overline{f}_x$, where $k_1, k_2 \ge 0$ and $k_1 + k_2 = n$. We will prove this by showing that $X(p^l), Y(p^l) \bmod (t_1^{p^{k_1}} - t_2^{p^{k_2}}) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}}))$ for any $l \ge 0$.

- When $n = 2k$ is even, the recursion formula (6.8) implies that,

  $$X(p^l) = p^{k-l} U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{l-1}}) U(t_1^{p^{l-1}}) \cdots U(t_1^p) U(t_1);$$
  $$Y(p^l) = p^{k-l} U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{l-1}}) U(t_2^{p^{l-1}}) \cdots U(t_2^p) U(t_2).$$

  Let's assume $k_1 \le k_2$. For any $0 \le t \le l - k_1$, we have the relation $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}}$, hence $U'(t_2^{p^{l-t}}) U(t_1^{p^{k_2-k_1+l-t}}) = p \cdot I_2$. Moreover, when $1 \le t \le k_2 - k_1$, $t_2^{p^{l-t}} = t_1^{p^{k_2-k_1+l-t}} = 0$, hence $U(t_2^{p^{l-t}}) = U(0)$.

  $$X(p^l) = U'(t_2) U'(t_2^p) \cdots U'(t_2^{p^{k_2-1}}) U(t_1^{p^{k_1-1}}) \cdots U(t_1^p) U(t_1) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}}));$$
  $$Y(p^l) = U'(t_1) U'(t_1^p) \cdots U'(t_1^{p^{k_1-1}}) U(t_2^{p^{k_2-1}}) \cdots U(t_2^p) U(t_2) \in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}} - t_2^{p^{k_2}})).$$

  The proof of the case $k_1 > k_2$ is similar and we get the same formula for $X(p^l)$ and $Y(p^l)$ as above, therefore we conclude that $t_1^{p^{k_1}} - t_2^{p^{k_2}}$ divides $\overline{f}_x$ when $k_1, k_2 \ge 0$ and $k_1 + k_2 = 2k$ by Theorem 6.2.7, hence $\overline{f}_x$ is divisible by the following polynomial,

  $$(t_1 - t_2)^{p^k} \cdot \prod_{a=1}^{k} \left((t_1 - t_2^{p^{2a}})(t_2 - t_1^{p^{2a}}) \right)^{p^{k-a}}.$$

We also know that $(t_1-t_2)^{p^k}\cdot\prod_{a=1}^{k}\left((t_1-t_2^{p^{2a}})(t_2-t_1^{p^{2a}})\right)^{p^{k-a}} \equiv (t_1t_2)^{\frac{p^k-1}{p-1}}\cdot(t_2^{p^k}-t_1^{p^k})$ mod $(t_1,t_2)^{p^{k+1}}$, the lemma follows by comparing this formula with (6.9).

- When $n=2k+1$ is odd, the recursion formula (6.8) implies that,

$$X(p^l)=p^{k-l}U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{l-1}})\begin{pmatrix}0&1\\p&0\end{pmatrix}U(t_1^{p^{l-1}})\cdots U(t_1^p)U(t_1);$$

$$Y(p^l)=p^{k-l}U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{l-1}})\begin{pmatrix}0&1\\p&0\end{pmatrix}U(t_2^{p^{l-1}})\cdots U(t_2^p)U(t_2).$$

Let's assume $k_1<k_2$. For any $0\leq t\leq l-k_1$, we have the relation $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}$, hence $U'(t_2^{p^{l-t}})U(t_1^{p^{k_2-k_1+l-t}})=p\cdot I_2$. Moreover, when $1\leq t\leq k_2-k_1$, $t_2^{p^{l-t}}=t_1^{p^{k_2-k_1+l-t}}=0$, hence $U(t_2^{p^{l-t}})=U(0)$.

$$X(p^l)=U'(t_2)U'(t_2^p)\cdots U'(t_2^{p^{k_2-1}})U(t_1^{p^{k_1-1}})\cdots U(t_1^p)U(t_1)\in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}}));$$

$$Y(p^l)=U'(t_1)U'(t_1^p)\cdots U'(t_1^{p^{k_1-1}})U(t_2^{p^{k_2-1}})\cdots U(t_2^p)U(t_2)\in \mathrm{M}_2(A_{p^l}/(t_1^{p^{k_1}}-t_2^{p^{k_2}})).$$

therefore we conclude that $t_1^{p^{k_1}}-t_2^{p^{k_2}}$ divides $\overline{f}_x$ when $k_1,k_2\geq 0$ and $k_1+k_2=2k+1$ by Theorem 6.2.7, hence $\overline{f}_x$ is divisible by the following polynomial,

$$\prod_{a=0}^{k}\left((t_1-t_2^{p^{2a+1}})(t_2-t_1^{p^{2a+1}})\right)^{p^{k-a}}.$$

We also know that $\prod_{a=0}^{k}\left((t_1-t_2^{p^{2a+1}})(t_2-t_1^{p^{2a+1}})\right)^{p^{k-a}} \equiv (t_1t_2)^{\frac{p^{k+1}-1}{p-1}}$ mod $(t_1,t_2)^{p^{k+1}}$, the lemma follows by comparing this formula with (6.9).

□

Corollary 6.2.10. *Let $x\in\mathbb{B}$ be an integral nonzero element which has valuation $n\geq 1$. Let $\mathcal{Z}(x)_p$*

be special fiber of $\mathcal{Z}(x)$*, then*

$$\mathcal{Z}(x)_p \simeq \operatorname{Spf} \mathbb{F}[[t_1, t_2]] / \left(\prod_{\substack{a+b=n \\ a,b \geq 0}} (t_1^{p^a} - t_2^{p^b}) \right).$$

Let $\mathcal{D}(x)_p$ *(resp.* $\mathcal{N}_0(N)_p$*) be the base change of* $\mathcal{D}(x)$ *(resp.* $\mathcal{N}_0(N)$*) to* $\mathbb{F}[[t_1, t_2]]$*, then*

$$\mathcal{N}_0(N)_p \simeq \mathcal{D}(x)_p \simeq \operatorname{Spf} \mathbb{F}[[t_1, t_2]] / \left((t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \\ a,b \geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \right).$$

Proof. The statement for $\mathcal{Z}(x)_p$ follows from Theorem 6.2.9. The statement for $\mathcal{D}(x)_p$ follows from the definition of difference divisors. □

Remark 6.2.11. The same formula has been proved in [3, Theorem 13.4.6, Theorem 13.4.7] by a totally different method.

6.2.3 Local arithmetic intersection numbers

We give the definition of the local arithmetic intersection numbers.

Definition 6.2.12. For a rank 3 lattice $L \subset \mathbb{B}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2, x_3\}$ of L. Let $O_{\mathcal{Z}^\sharp(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}^\sharp(x_i)$. Let $O_{\mathcal{N}}$ be the structure sheaf of the formal scheme $\mathcal{N}$. Let $- \otimes^{\mathbb{L}}_{O_{\mathcal{N}}} -$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}$. Define the local arithmetic intersection number of L on $\mathcal{N}$ to be

$$\operatorname{Int}^\sharp(L) = \chi(\mathcal{N}, O_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{O_{\mathcal{N}}} O_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{O_{\mathcal{N}}} O_{\mathcal{Z}^\sharp(x_3)}).$$

This number is finite and independent of the choice of the basis $\{x_i\}_{i=1}^3$ of L because of the following result.

Lemma 6.2.13. *Let* $x, y \in \mathbb{B}$ *be two linearly independent elements, then the tor sheaves*

$\underline{\mathrm{Tor}}_i^{\mathcal{O}_{\mathcal{N}}}(\mathcal{O}_{\mathcal{Z}^\sharp(x)}, \mathcal{O}_{\mathcal{Z}^\sharp(y)})$ *vanish for all* $i \geq 1$. *In particular,*

$$\mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)} = \mathcal{O}_{\mathcal{Z}^\sharp(x)} \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(y)}.$$

Moreover, the same formula holds if $\mathcal{Z}^\sharp(x)$ *or* $\mathcal{Z}^\sharp(y)$ *or both are replaced by* $\mathcal{D}(x)$ *resp.* $\mathcal{D}(y)$.

Let $L \subset \mathbb{B}$ *be an integral quadratic lattice of rank 3 over* $\mathbb{Z}_p$ *with basis* $\{x_1, x_2, x_3\}$, *then the derived tensor product* $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}$ *is independent of the choice of the basis.*

Proof. This is proved in [17, Lemma 4.1, Proposition 4.2]. □

Now let's come to the local arithmetic intersection numbers on $\mathcal{N}_0(N)$. For a fixed N-isogeny x_0 of $\mathbb{X}$, recall that we have defined the space of quasi-isogenies of $\mathbb{X}$ special to x_0 (cf. Definition 6.1.5) to be those $x \in \mathbb{B}$ such that

$$x \circ x_0^\vee + x_0 \circ x^\vee = 0.$$

Recall that we use $\mathbb{W}$ to denote this space.

Definition 6.2.14. For any rank 2 lattice $M \subset \mathbb{W}$, we choose a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of M. Let $\mathcal{O}_{\mathcal{Z}(x_i)}$ be the structure sheaf of the special cycle $\mathcal{Z}(x_i)$. Let $\mathcal{O}_{\mathcal{N}_0(N)}$ be the structure sheaf of the formal scheme $\mathcal{N}_0(N)$. Let $- \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} -$ be the derived tensor product functor in the derived category of coherent sheaves on $\mathcal{N}_0(N)$. Define the local arithmetic intersection number of M on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}).$$

This number is independent of the choice of the basis $\{x_1, x_2\}$ of M because of Lemma 6.2.13 and Theorem 6.2.6, we will relate it to the derivative of the local density of the quadratic lattice M with level N. The following theorem is the starting point of our calculation.

Theorem 6.2.15. *For any prime number p, let* $L \subset \mathbb{B}$ *be a* $\mathbb{Z}_p$*-lattice of rank 3, then*

$$\mathrm{Int}^\sharp(L) = \partial\mathrm{Den}(L).$$

Proof. In [7, §4], the Gross–Keating invariants (a_1, a_2, a_3) of the rank 3 quadratic lattice L is defined, then the local arithmetic intersection number $\mathrm{Int}^\sharp(L)$ is computed explicitly in terms of these invariants (see also [9, Theorem 1.1]). In [8, §2.11], the local density $\mathrm{Den}(X, L)$ is also expressed explicitly in terms of the Gross-Keating invariants (a_1, a_2, a_3), hence the derived local density $\partial\mathrm{Den}(L)$, the theorem is proved by comparing the expressions of both sides in terms of (a_1, a_2, a_3) (see [8, §2.16]). The readers can also see [10] for a recent new proof when p is odd. $\square$

6.2.4 Difference formula of the local arithmetic intersection numbers

Fix an N-isogeny $x_0 \in \mathrm{End}(\mathbb{X})$, recall that $\mathbb{W} = \{x_0\}^\perp \longrightarrow \mathbb{B}$.

Theorem 6.2.16. *For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot p^{-1}x_0).$$

Proof. Let $\{x_1, x_2\}$ be a basis of M. By Lemma 6.2.13 and the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we have the following isomorphism as complexes of coherent sheaves on $\mathcal{N}$,

$$\begin{aligned} \mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{D}(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}. \end{aligned}$$

When $v_p(N) = 0$ or 1, the difference divisor $\mathcal{D}(x_0)$ is just $\mathcal{Z}^\sharp(x_0)$, hence $\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot x_0)$ and $\mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot p^{-1}x_0) = 0$ since $p^{-1}x_0$ is not integral, therefore $\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot p^{-1}x_0)$.

When $v_p(N) \geq 2$, we have the following exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)} \xrightarrow{\times d_{x_0}} \mathcal{O}_{\mathcal{Z}^\sharp(x_0)} \longrightarrow \mathcal{O}_{\mathcal{D}(x_0)} \simeq \mathcal{O}_{\mathcal{N}_0(N)} \longrightarrow 0.$$

Tensoring the above exact sequence with the complex $\mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}$ in the derived category of coherent sheaves on $\mathcal{N}$, we get an exact triangle

$$\begin{aligned}\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} &\to \mathcal{O}_{\mathcal{Z}^\sharp(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \\ &\to \mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \to,\end{aligned}$$

hence the following identity,

$$\begin{aligned}\chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}) &= \chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}) \\ &\quad + \chi(\mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}).\end{aligned}$$

We already know that $\mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \simeq \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}$ since $\mathcal{N}_0(N) \simeq \mathcal{D}(x_0)$, hence

$$\begin{aligned}\mathrm{Int}_{\mathcal{N}_0(N)}(M) &= \chi(\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{D}(x_0)}} \mathcal{O}_{\mathcal{Z}(x_2)}) = \chi(\mathcal{O}_{\mathcal{N}_0(N)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}) \\ &= \chi(\mathcal{O}_{\mathcal{Z}^\sharp(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}) - \chi(\mathcal{O}_{\mathcal{Z}^\sharp(p^{-1}x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)}) \\ &= \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).\end{aligned}$$

□

6.3 Proof of Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$: the nonsingular terms

6.3.1 Local arithmetic Siegel–Weil formula with level N

Let p be a prime number. The difference formulas at the analytic side and the geometric side are combined together to prove the following theorem.

Theorem 6.3.1. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\mathrm{Den}_{\delta_p(N)}(M). \tag{6.10}$$

Proof. Theorem 6.2.16 gives the following difference formula of local arithmetic intersection numbers,

$$\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^\sharp(M \obot \mathbb{Z}_p \cdot p^{-1}x_0).$$

We also have the difference formula of the derived local densities (cf.(2.10)),

$$\partial\mathrm{Den}_{\delta_p(N)}(M) = \partial\mathrm{Den}(M \obot \mathbb{Z}_p \cdot x_0) - \partial\mathrm{Den}(M \obot \mathbb{Z}_p \cdot p^{-1}x_0).$$

Theorem 6.2.15 implies that $\mathrm{Int}^\sharp(L) = \partial\mathrm{Den}(L)$ for any rank 3 lattice $L \subset \mathbb{B}$. Therefore $\mathrm{Int}_{\mathcal{N}_0(N)}(M) = \partial\mathrm{Den}_{\delta_p(N)}(M)$ holds by combining the above two difference formulas. □

6.3.2 Intersection Numbers and Whittaker functions

Proposition 6.3.2. *Let $M \subset \mathbb{W}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$W'_T(1, 0, 1_{\delta_p(N)^2}) = c_p \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(M) \cdot \log(p) \tag{6.11}$$

where the constant c_p is given as follows

$$c_p = \begin{cases} (1-p^{-1}) \cdot (N,-1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \mid N; \\ (1-p^{-2}) \cdot (N,-1)_p \cdot |N|_p \cdot |2|_p^{3/2} & \text{when } p \nmid N. \end{cases}$$

Proof. Recall that $\delta_p(N)^\vee/\delta_p(N) \simeq \mathbb{Z}_p/2N\mathbb{Z}_p$ (cf. Example 2.1.1). By Proposition 3.2.3 and the explicit formula given by Rao in the appendix of [26],

$$\begin{aligned} W_T(1, k, 1_{\delta_p(N)^2}) &= |2N|_p \cdot \gamma(\delta_p(N) \otimes \mathbb{Q}_p)^2 \cdot |2|_p^{1/2} \cdot \mathrm{Den}(\delta_p(N) \oplus H_{2k}^+, M) \\ &= |N|_p \cdot (N,-1)_p \cdot |2|_p^{3/2} \cdot \mathrm{Den}(\delta_p(N) \oplus H_{2k}^+, M). \end{aligned} \tag{6.12}$$

Taking derivatives of both sides of (6.12),

$$W_T'(1,0,1_{\delta_p(N)^2}) = c_p \cdot \partial\mathrm{Den}_{\delta_p(N)}(M) \cdot \log(p).$$

The formula (6.11) follows from Theorem 6.3.1. $\square$

6.3.3 Proof of the identity (6.1)

Proposition 6.3.3. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix. Let $\boldsymbol{\varphi} \in \mathcal{S}(\mathbb{V}_f^2)$ be a T-admissible Schwartz function. Suppose $\boldsymbol{\varphi} = \varphi_1 \times \varphi_2$ where $\varphi_i \in \mathcal{S}(\mathbb{V}_f)$, then for all $\mathsf{y} \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, we have*

$$\widehat{\deg}(\widehat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi})) = \begin{cases} \chi\left(\mathcal{Z}(T,\boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}\right) \cdot \log(p), & \text{when } \mathrm{Diff}(T,\Delta(N)) = \{p\}; \\ 0, & \text{when } \#\mathrm{Diff}(T,\Delta(N)) \neq 1. \end{cases}$$

Proof. By definition (cf. (5.9)), the arithmetic special cycle $\widehat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi}) = ([\mathcal{Z}(T,\boldsymbol{\varphi})], 0)$, therefore $\widehat{\deg}(\widehat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi}))$ is independent of y. We can assume $\mathrm{Diff}(T,\Delta(N)) = \{p\}$ for some prime number p since otherwise both sides are 0 since $\mathcal{Z}(T,\boldsymbol{\varphi})$ would be an empty stack.

Let $x \in \mathcal{Z}(T,\boldsymbol{\varphi})(\overline{\mathbb{F}}_p)$ be a geometric point, it is contained in $\mathcal{Y}_0(N)$ by Corollary 5.1.8, hence the special divisors $\mathcal{Z}(t_1,\varphi_1)$ and $\mathcal{Z}(t_2,\varphi_2)$ intersect properly at x because T is nonsingular. Then $\chi(\mathcal{Z}(T,\boldsymbol{\varphi}), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2,\varphi_2)}) \cdot \log(p)$ is the sum of the length of local rings $\mathcal{O}_{\mathcal{X}_0(N),x}$ cut out by these two divisors times $\log(p)$, which is exactly $\widehat{\deg}(\widehat{\mathcal{Z}}(T,\mathsf{y},\boldsymbol{\varphi}))$ by the definition of the degree homomorphism. $\square$

Proof of the identity (6.1): We first consider the case that T is positive definite. By Proposition 5.1.9, we only need to consider the case $\mathrm{Diff}(T,\Delta(N)) = \{p\}$ for some prime number p because otherwise both sides are 0. The same proposition and Corollary 5.1.8 imply that the special cycle $\mathcal{Z}(T,\boldsymbol{\varphi})$ lies in the supersingular locus of $\mathcal{X}_0(N)_{\mathbb{F}_p}$. Then by the definition of special cycles and the formal uniformization of the special cycle $\mathcal{Z}(T, \boldsymbol{\varphi})$ (cf. Corollary 6.1.9),

$$\chi(\mathcal{Z}(T,\boldsymbol{\varphi}), O_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(t_2,\varphi_2)}) \cdot \log(p) = \\ \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x})=T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x}) \cdot \log(p).$$

It is known that (cf. (6.11))

$$W_T'(1, 0, 1_{\delta_p(N)^2}) = c_p \cdot \mathrm{Int}_{\mathcal{N}_0(N)}(\boldsymbol{x}) \cdot \log(p).$$

with constants c_p given by (6.3.2).

There exists a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f^p)$ such that

$$\sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x})=T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} \boldsymbol{\varphi}(g^{-1}\boldsymbol{x}) \\ = \frac{1}{\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p))} \cdot \int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\mathrm{d}g.$$

By definition, the last integral is a product of “local" integrals

$$\int_{\mathrm{SO}(\Delta(N)^{(p)})(\mathbb{A}_f^p)} \boldsymbol{\varphi}^p(g^{-1}\boldsymbol{x})\mathrm{d}g = \prod_{v \neq p,\infty} \int_{\mathrm{SO}(\delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\mathrm{d}g_v.$$

By the classical local Siegel–Weil formula which is made explicit in the work of Kudla, Rapoport and Yang [6, Proposition 5.3.3], for every place v of $\mathbb{Q}$, there exists a number $d_v \in \mathbb{R}^\times$ such that

$$\int_{\mathrm{SO}(\Delta_v(N))(\mathbb{Q}_v)} \boldsymbol{\varphi}_v(g_v^{-1}\boldsymbol{x})\mathrm{d}g_v = d_v \cdot W_{T,v}(1, 0, \boldsymbol{\varphi}_v),$$

with $\prod_{v \le \infty} d_v = 1$. Moreover, [6, Lemma 5.3.9] implies the following,

$$\mathrm{vol}(\Gamma_0(N)_v, dg_v) = d_v \cdot \gamma(\Delta_v(N))^2 \cdot |2|_v^{3/2} \cdot \begin{cases} (1 - v^{-2}), & \text{when } v \nmid N; \\ |N|_v^{-1}(1 + v^{-1}) & \text{when } v | N. \end{cases}$$

then it can be checked immediately that

$$\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}}^p)) \cdot d_p d_\infty \cdot c_p = 2^{-1/2} \psi(N)^{-1} \cdot \frac{3}{\pi^2}.$$

Suppose $\mathsf{z} = \mathsf{x} + i\mathsf{y}$, it's a classical result that

$$W_{T,\infty}(g_{\mathsf{z}}, 0, \Phi_\infty^{3/2}) = -2^{7/2}\pi^2 \cdot \det(\mathsf{y})^{3/4} q^T.$$

Combining these together with the definitions made in previous sections (cf. (3.8) and (3.13)) and Proposition 6.3.3, we get the formula stated in the theorem.

When T is not positive definite, the equality follows from [1, §4.2] and our computations of the volume of $\mathrm{vol}(\Gamma_0(N)(\hat{\mathbb{Z}})) = \prod_{v < \infty} \mathrm{vol}(\Gamma_0(N)_v, dg_v)$ above.

For the identity (6.1), we only need to take $\boldsymbol{\varphi}$ to be the characteristic function of the $\widehat{\mathbb{Z}}$-lattice $\left(\Delta(N) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}\right)^2$. □

Chapter 7: Proof of the main theorem: singular terms

In this chapter, we aim to prove the following identity when the matrix $T \in \mathrm{Sym}_2(\mathbb{Q})$ is singular:

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathsf{y}) \cdot q^T = \frac{\psi(N)}{24} \cdot \partial \mathrm{Eis}_T(\mathsf{z}, \Delta(N)^2). \tag{7.1}$$

As a side effect, we will show that the generating series $\widehat{\phi}_1$ is a nonholomorphic Siegel modular form of genus 1, weight $\frac{3}{2}$ and level $\Gamma_0(4N)$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$ (Theorem 7.1.11).

7.1 Heights of arithmetic special divisors

7.1.1 Degrees of arithmetic special divisors

Let $(\mathcal{Z}, g)$ be an element in the extended arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S})$, then there exists a smooth $(1,1)$-form ω on the complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$ such that we have the following identity as Green currents on the complex curve $\mathcal{X}_0(N)_{\mathbb{C}}$,

$$\mathrm{dd}^c(g) + \delta_{\mathcal{Z}_{\mathbb{C}}} = [\omega].$$

we define the following degree map,

$$\begin{aligned} \deg : \widehat{\mathrm{CH}}^1(\mathcal{X}_0(N), \mathcal{S}) &\longrightarrow \mathbb{C} \\ (\mathcal{Z}, g) &\longmapsto \deg(\mathcal{Z}, g) = \int_{\mathcal{X}_0(N)_{\mathbb{C}}} \omega = \langle (\mathcal{Z}, g), (0, 2) \rangle. \end{aligned}$$

Proposition 7.1.1. *Let (t, y) be a pair such that t is an integer and y is a positive number. Let*

$\tau = x + iy \in \mathbb{H}^+$ and $q = e^{2\pi i\tau}$, we have

$$\deg \widehat{\mathcal{Z}}(t, y, \Delta(N)) \cdot q^t = \frac{2}{\varphi(N)} \mathcal{E}_t(\tau, 1, \Delta(N)).$$

Proof. This is proved in [2, Proposition 6.7]. $\square$

7.1.2 Height pairings of special divisors and vertical fibers

Recall that $\mathcal{H} = \mathcal{X}_0(1) \times \mathcal{X}_0(1)$ is the product of the smooth Deligne–Mumford stack $\mathcal{X}_0(1)$ over $\mathbb{Z}$. Let p be a prime number and $\mathcal{H}_p := \mathcal{H} \times \operatorname{Spec} \mathbb{F}_p$ be the reduction mod p of $\mathcal{H}$, it is a 2-dimensional smooth Deligne–Mumford stack over $\mathbb{F}_p$. Let $i_p : \mathcal{H}_p \to \mathcal{H}$ be the closed immersion. Recall that $\mathsf{Cusp}(\mathcal{H}) \in \mathrm{CH}^1(\mathcal{H})$ is the cuspidal divisor of the stack $\mathcal{H}$, we define

$$\mathsf{Cusp}(\mathcal{H}_p) := i_p^* \mathsf{Cusp}(\mathcal{H}) \in \mathrm{CH}^1(\mathcal{H}_p).$$

Now we are going to construct an element $\mathcal{Z}^\sharp(t)$ in $\mathrm{CH}^1(\mathcal{H})$ for every integer t.

Definition 7.1.2. Let t be a nonzero integer.

+ When $t > 0$, the stack $\mathcal{Z}^\sharp(t)$ is defined as follows, its fibered category over a scheme S consists of the following objects,

$$((E, E'), j),$$

where (E, E') is an object in $\mathcal{H}(S)$, i.e., a pair of generalized elliptic curves. The element $j \in \operatorname{Hom}_S(E, E')$ is an isogeny such that $j^\vee \circ j = t$.

– When the integer $t < 0$, define

$$\mathcal{Z}^\sharp(t) := \mathsf{Cusp}(\mathcal{H}).$$

Lemma 7.1.3. *Let p be a prime number, let $n = v_p(N)$ be the p-adic valuation of N. Let (t, y) be a pair such that t is a nonzero integer and y is a positive number, the following identities hold for all integers a such that $n \geq |a|$ and $a \equiv n \bmod 2$.*

(a) If $t > 0$,

$$\langle \widehat{\mathcal{Z}}(t, y, \Delta(N)), \widehat{\mathcal{X}}_p^a(N) \rangle = \left(\sum_{x \in \mathcal{Z}(t,\Delta(N)) \cap \mathcal{Y}_p^a(N)(\overline{\mathbb{F}}_p)} \frac{\log p}{\mathrm{Aut}(x)} \cdot \mathrm{length}\, \mathcal{O}_{\mathcal{H}_p,x}/(f_{t,x}, f_{a,x}) \right),$$

where $f_{t,x}$ *and* $f_{a,x}$ *are the local equations of the divisors* $\mathcal{Z}^\sharp(t)$ *and* $\mathcal{X}_p^a(N)$ *respectively.*

(b) If $t < 0$,

$$\langle \widehat{\mathcal{Z}}(t, y, \Delta(N)), \widehat{\mathcal{X}}_p^a(N) \rangle = \sum_{M | N_p}$$
$$\left(\sum_{x \in \mathsf{Cusp}^{p^{(n+a)/2}M}(\mathcal{X}_0(N)) \cap \mathcal{X}_p^a(N)(\overline{\mathbb{F}}_p)} \frac{g(t, y, \Delta(p^{|a|}\sigma(N_p, M))) \cdot \log p}{\mathrm{Aut}(x)} \cdot \mathrm{length}\, \mathcal{O}_{\mathcal{H}_p,x}/(f_{c,x}, f_{a,x}) \right)$$

where $f_{c,x}$ *and* $f_{a,x}$ *are the local equations of the divisors* $\mathsf{Cusp}(\mathcal{H}_p)$ *and* $\mathcal{X}_p^a(N)$ *respectively.*

Proof. We first prove (a). The closed substack $\mathcal{Z}(t, \Delta(N))$ of $\mathcal{X}_0(N)$ actually lies in the stack $\mathcal{Y}_0(N)$ by Proposition 5.1.7, hence $\mathcal{Z}(t, \Delta(N)) \cap \mathcal{X}_p^a(N) = \mathcal{Z}(t, \Delta(N)) \cap \mathcal{Y}_p^a(N)$. Let $x \in \mathcal{Z}(t, \Delta(N)) \cap \mathcal{X}_p^a(N)(\overline{\mathbb{F}}_p)$ be a point. Let $\mathcal{O}_{\mathcal{Z}(t,\Delta(N))\cap\mathcal{X}_p^a(N),x}$ and $\mathcal{O}_{\mathcal{H}_p,x}$ be the local rings of the stacks $\mathcal{Z}(t, \Delta(N)) \cap \mathcal{X}_p^a(N)$ and $\mathcal{H}_p$ at x respectively. We will show that $\mathcal{O}_{\mathcal{Z}(t,\Delta(N))\cap\mathcal{X}_p^a(N),x} \simeq \mathcal{O}_{\mathcal{H}_p,x}/(f_{t,x}, f_{a,x})$ and it is an Artinian local ring. Let (E, E') be the object in $\mathcal{H}_p(\overline{\mathbb{F}}_p)$ corresponding to x where both E and E' are elliptic curves over $\overline{\mathbb{F}}_p$. There exists a cyclic isogeny $\pi : E \longrightarrow E'$ of order N and an isogeny $j : E \longrightarrow E'$ of order t such that $\pi^\vee \circ j + j^\vee \circ \pi = 0$. Let $f_{j,x}, f_{\pi,x} \in \mathcal{O}_{\mathcal{H}_p,x}$ be the equations cutting out the closed subschemes where j, π deform to isogenies, then $f_{j,x} = f_{t,x}$ and $f_{a,x} | f_{\pi,x}$. Since j and π are orthogonal to each other, we have

$$\mathcal{O}_{\mathcal{Z}(t,\Delta(N))\cap\mathcal{X}_p^a(N),x} = \mathcal{O}_{\mathcal{X}_p^a(N),x}/(\overline{f_{j,x}}) = \mathcal{O}_{\mathcal{H}_p,x}/(f_{t,x}, f_{a,x}).$$

Moreover, the two elements $f_{j,x}$ and $f_{\pi,x}$ have no common factors in the regular local ring $\mathcal{O}_{\mathcal{H}_p,x}$ by [9]. Hence $\mathcal{O}_{\mathcal{H}_p,x}/(f_{t,x}, f_{a,x})$ is an Artinian local ring. The equality in (a) follows from the intersection pairings in Definition 4.3.3 and Serre's Tor formula.

Now we consider the case $t < 0$. Let $N = \prod_{i=i}^{r} q_i^{n_i}$ be the prime decomposition of N where $q_1 = p$ and $n_1 = n$. Define $N_p = Np^{-n}$. For a positive integer $M = \prod_{i=2}^{r} q_i^{m_i}$ of N_p, the component $\mathsf{Cusp}^{p^m M}(\mathcal{X}_0(N))$ of the cuspidal divisor $\mathcal{X}_0(N)$ intersects with $\mathcal{X}_p^a(N)$ if and only if $2m - n = a$. Then the formula follows from the definition of the divisor $\mathcal{Z}^*(t, y, \Delta(N))$ in (5.4) and the fact that the cuspidal divisor $\mathsf{Cusp}(\mathcal{X}_0(N))$ intersects with $\mathcal{X}_p^a(N)$ properly. □

Lemma 7.1.4. *Let N_p be a positive integer prime to p. Let t be a positive integer. Let a be an integer. Let n be an integer such that $n \geq |a|$ and $n \equiv a \bmod 2$. Let S be an $\mathbb{F}_p$-scheme, the groupoid $\mathcal{Z}(t, \Delta(p^n N_p)) \cap \mathcal{X}_p^a(p^n N_p)(S)$ consists of the following objects:*

- $a \geq 0$*:* $\left((E, E'), E^{(p^a)} \xrightarrow{j} E', E \xrightarrow{\pi} E'\right)$*, where E, E' are elliptic curves over S and $j \circ F^a$ and π are orthogonal to each other;*
- $a < 0$*:* $\left((E^{(p^{-a})}, E'), E \xrightarrow{j} E', E^{(p^{-a})} \xrightarrow{\pi} E'\right)$*, where E, E' are elliptic curves over S and $j \circ V^a$ and π are orthogonal to each other.*

Proof. We only give the proof for the case $a \geq 0$ because the case $a < 0$ is similar. By the proof of Lemma 7.1.3, we know that $\mathcal{Z}(t, \Delta(p^n N_p)) \cap \mathcal{X}_p^a(p^n N_p) = \mathcal{Z}(t, \Delta(p^n N_p)) \cap \mathcal{Y}_p^a(p^n N_p)$. For an object $x \in \mathcal{Z}(t, \Delta(p^n N_p)) \cap \mathcal{X}_p^a(p^n N_p)(S)$, it corresponds to a pair of elliptic curves (E, E') over S. By Remark 4.6.6 and the definition of $\mathcal{Z}(t, \Delta(N))$, there is a cyclic isogeny $j : E^{(p^a)} \longrightarrow E'$ of order N_p and an isogeny $\pi : E \longrightarrow E'$ of order t. As an object in $\mathcal{X}_p^a(N)(S) \subset \mathcal{X}_0(N)(S)$, the object x corresponds to the following cyclic isogeny of order N:

$$p^{(n-a)/2} \cdot j \circ F^a : E \xrightarrow{p^{(n-a)/2} \cdot F^a} E^{(p^a)} \xrightarrow{j} E'.$$

Since x is also an object of $\mathcal{Z}(t, \Delta(N))(S)$, the isogeny π and $j \circ F^a$ are orthogonal to each other by Definition 5.1.5. □

Remark 7.1.5. Let t be an integer. Let a be an integer. Let N be an integer such that the p-adic valuation $n := v_p(N)$ of N satisfying that $n \geq |a|$ and $n \equiv a \bmod 2$.

- $t > 0$: the stack $\mathcal{Z}(t, \Delta(N)) \cap \mathcal{X}_p^a(N) = \mathcal{Z}(t, \Delta(N)) \cap \mathcal{Y}_p^a(N)$ is independent of n by the above lemma.
- $t < 0$: the stack $\mathsf{Cusp}^{p^{(n+a)/2}M}(\mathcal{X}_0(N)) \cap \mathcal{X}_p^a(N)$ is independent of n for a fixed positive integer $M|N_p$:

$$\begin{aligned}
&\mathsf{Cusp}^{p^{(n+a)/2}M}(\mathcal{X}_0(N)) \cap \mathcal{X}_p^a(N) \\
&\qquad \simeq \left(\mathsf{Cusp}^{p^{(n+a)/2}}(\mathcal{X}_0(p^n)) \times_{\mathsf{Cusp}(\mathcal{X}_0(1))} \mathsf{Cusp}^M(\mathcal{X}_0(N_p))\right) \cap \mathcal{X}_p^a(N) \\
&\qquad \simeq \left(\mathsf{Cusp}^{p^{(n+a)/2}}(\mathcal{X}_0(p^n)) \cap \mathcal{X}_p^a\right) \times_{\mathsf{Cusp}(\mathcal{X}_0(1))} \mathsf{Cusp}^M(\mathcal{X}_0(N_p)) \\
&\qquad \simeq \left(\mathsf{Cusp}(\mathcal{H}_p) \cap \mathcal{X}_p^a\right) \times_{\mathsf{Cusp}(\mathcal{X}_0(1))} \mathsf{Cusp}^M(\mathcal{X}_0(N_p)).
\end{aligned}$$

The last isomorphism follows from Proposition 4.6.8.

Corollary 7.1.6. *Let p be a prime that $n = v_p(N) \geq 2$, let a be an integer such that $-n < a < n$ and $a \equiv n \bmod 2$, then for any pair (t, y) such that t is an integer and $y > 0$,*

$$\langle \widehat{\mathcal{Z}}(t, y, \Delta(N)), \widehat{\mathcal{X}}_p^a(N) \rangle_{\mathcal{X}_0(N)} = \langle \widehat{\mathcal{Z}}(t, y, \Delta(Np^{-2})), \widehat{\mathcal{X}}_p^a(Np^{-2}) \rangle_{\mathcal{X}_0(Np^{-2})}. \tag{7.2}$$

Notice that we add subscript here to emphasize that the pairing on the left hand side happens in the group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$ while the right hand side happens in the group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(Np^{-2}))$.

Proof. We first prove this when $t \neq 0$. We have the following equalities by Remark 7.1.5,

- $t > 0$: $\mathcal{Z}(t, \Delta(N)) \cap \mathcal{Y}_p^a(N)(\overline{\mathbb{F}}_p) = \mathcal{Z}(t, \Delta(Np^{-2})) \cap \mathcal{Y}_p^a(Np^{-2})(\overline{\mathbb{F}}_p)$.
- $t < 0$: $\mathsf{Cusp}^{p^{(n+a)/2}M}(\mathcal{X}_0(N)) \cap \mathcal{X}_p^a(N)(\overline{\mathbb{F}}_p) = \mathsf{Cusp}^{p^{(n-2+a)/2}M}(\mathcal{X}_0(Np^{-2})) \cap \mathcal{X}_p^a(Np^{-2})(\overline{\mathbb{F}}_p)$.

Therefore (7.2) follows from Lemma 7.1.3 and the definition of the function $g(t, y, \Delta(N))$ in (5.3).

Next we consider the case that $t = 0$. The same arguments as above for $t < 0$ implies that the following pairing

$$\left\langle\left(\sum_{M|N} g(0,y,\Delta(\sigma(N,M)))\cdot \mathsf{Cusp}^{M}(\mathcal{X}_0(N)), \mathfrak{g}(0,y,\Delta(N))([z])\right), \widehat{\mathcal{X}}_p^a(N)\right\rangle$$

is independent of the p-adic valuation $n = v_p(N)$ as long as $n > |a|$. We only need to show that the following pairing is also independent of n:

$$\left\langle(\operatorname{div}(\widehat{\omega}), \mathfrak{g}_{\widehat{\omega}} - \log(y)), \widehat{\mathcal{X}}_p^a(N)\right\rangle = \left\langle(\operatorname{div}(\widehat{\omega}), \mathfrak{g}_{\widehat{\omega}}), \widehat{\mathcal{X}}_p^a(N)\right\rangle = \langle\widehat{\omega}, \widehat{\mathcal{X}}_p^a(N)\rangle.$$

We know that $12\varphi(N)\cdot\widehat{\omega}_N = (\psi(N)\varphi(N)P_\infty(N) + \sum\limits_{p|N} f_p(N), -\log \parallel \Delta_N \parallel^2)$, hence

$$\begin{aligned}\langle\widehat{\omega}, \widehat{\mathcal{X}}_p^a(N)\rangle &= -\frac{1}{12\varphi(N)}\langle \widehat{f}_p(N) + W_N^*\widehat{f}_p(N), \widehat{\mathcal{X}}_p^a(N)\rangle \\ &= -\frac{n(1-p)-2}{p-1}\cdot\langle(\operatorname{div}(p),0), \widehat{\mathcal{X}}_p^a(N)\rangle = 0.\end{aligned}$$

Therefore (7.2) is also true for $t = 0$. □

Proposition 7.1.7. *Let p be a prime number, let $n = v_p(N)$ be the p-adic valuation of N. For any pair (t, y) where t is an integer and y is a positive number, let $\tau = x + iy \in \mathbb{H}_1$ and $q = e^{2\pi i\tau}$, we have*

$$\begin{aligned}&\left\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)), \widehat{\mathcal{X}}_p^n(N)\right\rangle\cdot q^t \\ &\qquad = \frac{1}{\varphi(N)}\mathcal{E}_t(\tau,1,\Delta(N))\log p - \sum_{i=1}^{[n/2]}\frac{p-1}{\varphi(Np^{-2i})}\mathcal{E}_t(\tau,1,\Delta(Np^{-2i}))\log p.\end{aligned}$$

Proof. By Theorem 4.6.7, we have the following identity in the codimension 1 arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$:

$$(\operatorname{div}(p|_{\mathcal{X}_0(N)}), 0) = \sum_{\substack{-n\le a\le n \\ n\equiv a \ (\mathrm{mod}\ 2)}} \varphi(p^{(n-|a|)/2})\cdot\widehat{\mathcal{X}}_p^a(N) = (0, \log p^2).$$

We also know that

$$\widehat{\mathcal{X}}_p^n(N)+\widehat{\mathcal{X}}_p^{-n}(N)=(\mathrm{div}(p|_{\mathcal{X}_0(N)},0)-(p-1)\sum_{i=1}^{[n/2]}(\mathrm{div}(p|_{\mathcal{X}_0(Np^{-2i})}),0)$$

By Lemma 5.2.5 we know that $W_N^*\widehat{\mathcal{Z}}(t,y,\Delta(N))=\widehat{\mathcal{Z}}(t,y,\Delta(N))$ and $W_N^*\widehat{\mathcal{X}}_p^n(N)=\widehat{\mathcal{X}}_p^{-n}(N)$, combining with the formula (7.2) in Corollary 7.1.6, we get

$$\begin{aligned}
&\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{\mathcal{X}}_p^n(N)\rangle\cdot q^t=\frac{1}{2}\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{\mathcal{X}}_p^n(N)+\widehat{\mathcal{X}}_p^{-n}(N)\rangle\cdot q^t\\
&=\frac{1}{2}\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),(\mathrm{div}(p|_{\mathcal{X}_0(N)},0)-(p-1)\sum_{i=1}^{[n/2]}(\mathrm{div}(p|_{\mathcal{X}_0(Np^{-2i})}),0)\rangle\cdot q^t\\
&=\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),(0,\log p)\rangle_{\mathcal{X}_0(N)}\cdot q^t-(p-1)\sum_{i=1}^{[n/2]}\langle\widehat{\mathcal{Z}}(t,y,\Delta(Np^{-2i})),(0,\log p)\rangle_{\mathcal{X}_0(Np^{-2i})}\cdot q^t.
\end{aligned}$$

In the last line we add the subscript $\mathcal{X}_0(Np^{-2i})$ to indicate that the pairing happens in the group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(Np^{-2i}))$. The proposition then follows from Proposition 7.1.1. □

7.1.3 Heights of arithmetic special divisors

There is an explicit rational section Δ_N of the bundle $\widehat{\omega}_N^{\otimes 12\varphi(N)}$ in §4.8, recall that

$$\mathrm{div}(\Delta_N)=\psi(N)\varphi(N)P_\infty(N)+\sum_{p|N}f_p(N),$$

where $f_p(N)$ is given explicitly in (4.22). Let $\widehat{f_p}(N)=(f_p(N),0)$, we define the following element in the extended arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N),\mathcal{S})$ as follows,

$$\widehat{\Delta}_N=(\psi(N)\varphi(N)P_\infty(N),-\log\|\Delta_N\|^2)=12\varphi(N)\widehat{\omega}_N-\sum_{p|N}\widehat{f_p}(N).$$

Lemma 7.1.8. *The self pairing of the element* $\widehat{\Delta}_N$ *is*

$$\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle = 6\psi(N)\varphi(N)^2 \left(\frac{1}{2} - \frac{\Lambda'(-1)}{\Lambda(-1)} \right) - \sum_{p|N} \langle \widehat{f}_p(N), \widehat{f}_p(N) \rangle. \tag{7.3}$$

where the pairing $\langle \widehat{f}_p(N), \widehat{f}_p(N) \rangle$ *can be computed as follows: let* $n = v_p(N)$ *be the* p*-adic valuation of the integer* N*,*

$$\langle \widehat{f}_p(N), \widehat{f}_p(N) \rangle = -6\psi(N)\varphi(N)^2 \cdot \frac{np^2 + 1 - n}{p^2 - 1} \log p. \tag{7.4}$$

Proof. The cusp $P_\infty(N) : \operatorname{Spec} \mathbb{Z} \to \mathcal{X}_0(N)$ factors through the component $\mathsf{C}^n(p^n, N_p)$ of the formal scheme $\widehat{\mathsf{Cusp}}(\mathcal{X}_0(N))$, hence its reduction mod p factors through the irreducible component $\mathcal{X}_p^n(N)$,

$$\langle P_\infty(N), \widehat{f}_p(N) \rangle = 0,$$

since the coefficient of the irreducible component $\mathcal{X}_p^n(N)$ in $f_p(N)$ is 0 by (4.22). Hence

$$144\varphi(N)^2 \langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = \langle \widehat{\Delta}_N + \sum_{p|N} \widehat{f}_p(N), \widehat{\Delta}_N + \sum_{p|N} \widehat{f}_p(N) \rangle = \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle + \sum_{p|N} \langle \widehat{f}_p(N), \widehat{f}_p(N) \rangle.$$

In Example 4.3.5, we get

$$\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = \frac{\psi(N)}{24} \left(\frac{1}{2} - \frac{\Lambda'(-1)}{\Lambda(-1)} \right).$$

Then the formula (7.3) follows.

The formula (7.4) can be obtained by combining Lemma 4.7.1 and (4.22). □

Proposition 7.1.9. *Let* $n_p = v_p(N)$ *be the* p*-adic valuation of the integer* N*. Let* (t, y) *be a pair such that* t *is an integer and* y *is a positive number. Let* $\tau = x + iy \in \mathbb{H}^+$ *and* $q = e^{2\pi i \tau}$*, we have*

$$\langle \widehat{\mathcal{Z}}(t, y, \Delta(N)), \widehat{\Delta}_N \rangle \cdot q^t = 12\mathcal{E}'_t(\tau, 1, \Delta(N)).$$

Proof. We only give the detailed proof when $t = 0$, the other cases follow easily. For any prime number p, let $n_p = v_p(N)$ be the p-adic valuation of the integer N. Recall the definition of the modified special divisor $\mathcal{Z}^*(0, y, \Delta(N)) = g(0, y, \Delta(N)) \cdot \mathsf{Cusp}(\mathcal{X}_0(N))$ in §5.2.1. Let

$$\mathcal{Z}_1(0, y, \Delta(N)) = \mathcal{Z}^*(0, y, \Delta(N)) - g(0, y, \Delta(N))P_\infty(N).$$

and $\widehat{\mathcal{Z}}_1(0, y, \Delta(N)) = (\mathcal{Z}^*(0, y, \Delta(N)), \mathfrak{g}(0, y, \Delta(N))) - \frac{g(0,y,\Delta(N))}{\psi(N)\varphi(N)} \cdot \widehat{\Delta}_N$. By definition, the element $\widehat{\mathcal{Z}}(0, y, \Delta(N)) = \widehat{\omega} + (\mathcal{Z}^*(0, y, \Delta(N)), \mathfrak{g}(0, y, \Delta(N))) - (0, \log y)$, we have

$$\begin{aligned}&\langle \widehat{\mathcal{Z}}(0, y, \Delta(N)), \widehat{\Delta}_N \rangle \\ &\qquad = \langle \widehat{\mathcal{Z}}_1(0, y, \Delta(N)), \widehat{\Delta}_N \rangle + \frac{g(0, y, \Delta(N))}{\psi(N)\varphi(N)} \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle + \langle \widehat{\omega}, \widehat{\Delta}_N \rangle - \langle (0, \log y), \widehat{\Delta}_N \rangle.\end{aligned} \tag{7.5}$$

Notice that

$$\langle (0, \log y), \widehat{\Delta}_N \rangle = \log y \cdot \langle (0, 1), 12\varphi(N)\widehat{\omega}_N - \sum_{p|N} \widehat{f}_p(N) \rangle = 12\varphi(N) \log y \cdot \langle (0, 1), \widehat{\omega}_N \rangle.$$

By Definition 4.3.3, the pairing $\langle (0, 1), \widehat{\omega}_N \rangle$ is the volume of the modular curve $\mathcal{X}_0(N)_\mathbb{C}$ under the Hermitian metric specified in (4.7). Therefore $\langle (0, 1), \widehat{\omega}_N \rangle = \psi(N)/24$. Hence the last term in (7.5) is $\langle (0, \log y), \widehat{\Delta}_N \rangle = \frac{1}{2}\varphi(N)\psi(N) \log y$.

Next we compute the pairing $\langle \widehat{\omega}, \widehat{\Delta}_N \rangle$:

$$\langle \widehat{\omega}, \widehat{\Delta}_N \rangle = \langle -2\widehat{\omega}_N + \sum_{p|N} \widehat{\mathcal{X}}_p(N), \widehat{\Delta}_N \rangle = \langle -\frac{\widehat{\Delta}_N + \sum_{p|N} \widehat{f}_p(N)}{6\varphi(N)}, \widehat{\Delta}_N \rangle + \sum_{p|N} \langle \widehat{\mathcal{X}}_p(N), \widehat{\Delta}_N \rangle.$$

Recall that the coefficient of $\mathcal{X}_p^n(N)$ in $f_p(N)$ is 0, but the reduction mod p of the cusp $P_\infty(N)$ lies in the curve $\mathcal{X}_p^n(N)$ by Corollary 4.5.4 and Proposition 4.6.8, hence $\langle \widehat{f}_p(N), \widehat{\Delta}_N \rangle = 0$.

Moreover, the coefficient of $\widehat{\mathcal{X}}_p^n(N)$ in $\widehat{\mathcal{X}}_p(N)$ is $\frac{n}{2}$ (cf. (4.16)), hence we have $\langle \widehat{\mathcal{X}}_p(N), \widehat{\Delta}_N \rangle = \frac{n}{2}\varphi(N)\psi(N) \log p$, therefore

$$\langle \widehat{\omega}, \widehat{\Delta}_N \rangle = -\frac{\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle}{6\varphi(N)} + \frac{1}{2}\varphi(N)\psi(N) \log N.$$

Hence we have

$$\langle\widehat{\mathcal{Z}}(0,y,\Delta(N)),\widehat{\Delta}_N\rangle = \langle\widehat{\mathcal{Z}}_1(0,y,\Delta(N)),\widehat{\Delta}_N\rangle + \frac{1}{2}\varphi(N)\psi(N)\log\left(\frac{N}{y}\right) + \frac{6g(0,y,\Delta(N))-\psi(N)}{6\psi(N)\varphi(N)}\langle\widehat{\Delta}_N,\widehat{\Delta}_N\rangle. \tag{7.6}$$

By similar arguments in [2, §7.2], we have

$$\begin{aligned}\langle\widehat{\mathcal{Z}}_1(0,y,\Delta(N)),\widehat{\Delta}_N\rangle = {}& \frac{1}{2}\psi(N)\varphi(N)\log\left(\frac{y}{N}\right) - \int_{\mathcal{X}_0(N)_{\mathbb{C}}}\log\|\Delta_N\|\left(\omega(0,y,\Delta(N)) - \frac{\mathrm{d}x\wedge\mathrm{d}y}{2\pi y^2}\right) \\ & + \varphi(N)\left(\psi(N)-6g(0,y,\Delta(N))\right)\left(24\zeta'(-1) - 1 + \log 4\pi + \gamma + 2\sum_{p|N}\frac{n_p p^2 + 1 - n_p}{p^2-1}\log p\right)\end{aligned} \tag{7.7}$$

where the differential form $\omega(0,y,\Delta(N))$ is defined in (5.5).

The term $\langle\widehat{\Delta}_N,\widehat{\Delta}_N\rangle$ has been computed in Lemma 7.1.8. Then the case $t=0$ of the proposition is proved by combining (7.6), (7.7) and [2, Theorem 1.6] which states that

$$\int_{\mathcal{X}_0(N)_{\mathbb{C}}}\log\|\Delta_N\|\left(\omega(0,y,\Delta(N)) - \frac{\mathrm{d}x\wedge\mathrm{d}y}{2\pi y^2}\right) = -12\mathcal{E}_0'(\tau,1,\Delta(N)).$$

□

Corollary 7.1.10. *Let* (t,y) *be a pair such that* t *is an integer and* y *is a positive number. Let* $\tau = x+iy\in\mathbb{H}^+$ *and* $q=e^{2\pi i\tau}$, *we have*

$$\begin{aligned}&-\frac{24}{\psi(N)}\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{\omega}_N\rangle\cdot q^t \\ &= E_t'(\tau,\frac{1}{2},\Delta(N)) + \left(1+\frac{2\Lambda'(2)}{\Lambda(2)}+\frac{\log N}{2}-\sum_{p|N}\frac{\beta_p'(0)}{2}\right)E_t(\tau,\frac{1}{2},\Delta(N)).\end{aligned}$$

Proof. Recall that we have

$$\widehat{\Delta}_N = 12\varphi(N)\widehat{\omega}_N - \sum_{p|N}\widehat{f}_p(N).$$

By Proposition 7.1.9, we only need to compute $\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{f_p}(N)\rangle$ for every prime $p|N$. Let $n=\nu_p(N)$ be the p-adic valuation of the number N. Since the arithmetic special divisor $\widehat{\mathcal{Z}}(t,y,\Delta(N))$ is invariant under the Atkin–Lehner involution W_N^*, we have

$$\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{f_p}(N)\rangle=\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),W_N^*\widehat{f_p}(N)\rangle=\frac{1}{2}\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{f_p}(N)+W_N^*\widehat{f_p}(N)\rangle$$

By formula (4.22), we have

$$\widehat{f_p}(N)+W_N^*\widehat{f_p}(N)=12p^{n-1}\varphi(N_p)\left(\widehat{\mathcal{X}}_p^n(N)+\widehat{\mathcal{X}}_p^{-n}(N)+(n(1-p)-2)\,\widehat{\operatorname{div}}(p)\right).$$

Recall that we have calculated $\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{\mathcal{X}}_p^n(N)\rangle$ in Proposition 7.1.7, then

$$\begin{aligned}&\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{f_p}(N)\rangle\cdot q^t\\&=(n-1-np)\,\frac{12}{p-1}\mathcal{E}_t(\tau,1,\Delta(N))\log(p)-\sum_{i=1}^{[n/2]}\frac{12p^{n-1}(p-1)}{\varphi(p^{n-2i})}\mathcal{E}_t(\tau,1,\Delta(Np^{-2i}))\log(p).\end{aligned}$$

By definition of the function $g_p(k)$ in (3.17) and $C_N(s)$ in the formula (3.11), we can easily verify that

$$\sum_{i=1}^{[n/2]}\frac{p^{n-1}(p-1)}{\varphi(p^{n-2i})}\mathcal{E}_t(\tau,1,\Delta(Np^{-2i}))\log(p)=C_N(1)E_t(\tau,1,\Delta(N))\cdot\frac{p^{-1}g_p(0)}{1-p^{-1}g_p(0)}\log(p).$$

Therefore,

$$\begin{aligned}&\varphi(N)\langle\widehat{\mathcal{Z}}(t,y,\Delta(N)),\widehat{\omega}_N\rangle\cdot q^t \qquad (7.8)\\=&\mathcal{E}_t'(\tau,1,\Delta(N))+\sum_{p|N}\left(\frac{n-1-np}{p-1}\mathcal{E}_t(\tau,1,\Delta(N))-C_N(1)E_t(\tau,1,\Delta(N))\cdot\frac{p^{-1}g_p(0)}{1-p^{-1}g_p(0)}\log(p)\right)\\=&C_N(1)\left(E_t'(\tau,1,\Delta(N))+\left(\frac{C_N'(1)}{C_N(1)}+\sum_{p|N}\left(\frac{n-1-np}{p-1}-\frac{p^{-1}g_p(0)}{1-p^{-1}g_p(0)}\right)\log(p)\right)E_t(\tau,1,\Delta(N))\right).\end{aligned}$$

By simple calculations, we get

$$\frac{C_N'(s)}{C_N(s)} = \frac{1}{s} + \frac{2\Lambda'(2s)}{\Lambda(2s)} + 2\sum_{p|N} \frac{p^{-2s}\log(p)}{1-p^{-2s}} + \frac{3}{2}\log(N). \tag{7.9}$$

Moreover, by the definition of the function $\beta_p(s)$ in (3.20), for any prime p dividing N,

$$\beta_p'(0) = \left(\frac{2}{1+p} + \frac{2g_p(-1)}{1-g_p(-1)}\right)\log(p) = \left(\frac{2}{1+p} + \frac{2p^{-1}g_p(0)}{1-p^{-1}g_p(0)}\right)\log(p), \tag{7.10}$$

here we use the functional equation $g_p(k) = p^{2k+1}g_p(-k-1)$ proved in Lemma 3.5.3, then the Corollary is proved by combining formulas (7.8), (7.9) and (7.10). □

Theorem 7.1.11. *Let N be a positive integer. Let $\tau = x+iy \in \mathbb{H}_1^+$ and $q^t = e^{2\pi i t\tau}$, the following generating series with coefficients in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$*

$$\widehat{\phi}_1(\tau) = \sum_{t\in\mathbb{Q}} \widehat{\mathcal{Z}}(t, y, \Delta(N)) \cdot q^t$$

is a nonholomorphic Siegel modular form of genus 1, weight $\frac{3}{2}$ and level $\Gamma_0(4N)$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N))$.

Proof. Let $n = \nu_p(N)$ be the p-adic valuation of the integer N. The Corollary 7.1.6 and Proposition 7.1.7 implies that for any integer a such that $-n \leq a \leq n$ and $a \equiv n \bmod 2$, the pairing $\langle \widehat{\phi}_1(\tau), \widehat{\mathcal{X}}_p^a(N)\rangle$ is a Siegel modular form of genus 1, weight $\frac{3}{2}$ and level $\Gamma_0(4N)$. Proposition 7.1.1 and Corollary 7.1.10 imply that $\deg \widehat{\phi}_1(\tau)$ and $\langle \widehat{\phi}_1(\tau), \widehat{\omega}_N\rangle$ are both Siegel modular forms of genus 1, weight $\frac{3}{2}$ and level $\Gamma_0(4N)$. Then the theorem follows the same proof of Theorem 8.4 of [2]. □

Remark 7.1.12. Du and Yang constructed a vector-valued version of the generating series $\widehat{\phi}_1(\tau)$ in [2], with values in $\widehat{\mathrm{CH}}^1(\mathcal{X}_0(N)) \otimes_{\mathbb{C}} C[\Delta(N)^\vee/\Delta(N)]$ under the assumption that N is odd and squarefree. That generating series is modular under the full metaplectic group. We expect that similar construction should also exist for general N.

7.2 Proof of the identity (7.1)

7.2.1 The rank of T is 1

By the invariance properties (5.10) and (5.11), we reduce the proof to the case that

$$T = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \text{ and } \mathsf{y} = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}.$$

where t is a nonzero integer and y_1, y_2 are positive real numbers.

In this case, by the definition of the element $\widehat{\mathcal{Z}}(T, \mathsf{y})$, we have

$$\widehat{\mathcal{Z}}(T, \mathsf{y}) = -2\widehat{\mathcal{Z}}(t, y_2, \Delta(N)) \cdot \widehat{\omega}_N - (0, \log y_1 \cdot \delta_{\mathcal{Z}^*(t, y_2, \Delta(N))_{\mathbb{C}}}).$$

Then the identity (7.1) follows from combining Corollary 3.5.7 and Corollary 7.1.10.

7.2.2 The rank of T is 0

Notice that $\langle \widehat{\omega}, \widehat{\mathcal{X}}_p(N) \rangle = \langle W_N^* \widehat{\omega}, W_N^* \widehat{\mathcal{X}}_p(N) \rangle = \langle \widehat{\omega}, -\widehat{\mathcal{X}}_p(N) \rangle$, hence $\langle \widehat{\omega}, \widehat{\mathcal{X}}_p(N) \rangle = 0$, then

$$\widehat{\omega} \cdot \widehat{\omega} = \langle -2\widehat{\omega}_N + \widehat{\mathcal{X}}_p(N), \widehat{\omega} \rangle = -2\langle \widehat{\omega}_N, \widehat{\omega} \rangle = -2\langle \widehat{\omega}_N, -2\widehat{\omega}_N + \sum_{p|N} \widehat{\mathcal{X}}_p(N) \rangle.$$

We also notice that

$$\begin{aligned} \langle \widehat{\omega}_N, \widehat{\mathcal{X}}_p(N) \rangle = \langle W_N^* \widehat{\omega}_N, W_N^* \widehat{\mathcal{X}}_p(N) \rangle &= \langle \widehat{\omega}_N - \widehat{\mathcal{X}}_p(N), -\widehat{\mathcal{X}}_p(N) \rangle \\ &= -\langle \widehat{\omega}_N, \widehat{\mathcal{X}}_p(N) \rangle + \langle \widehat{\mathcal{X}}_p(N), \widehat{\mathcal{X}}_p(N) \rangle. \end{aligned}$$

Therefore $\widehat{\omega} \cdot \widehat{\omega} = 4\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle - \langle \widehat{\mathcal{X}}_p(N) \rangle, \widehat{\mathcal{X}}_p(N) \rangle$. By Example 4.3.5, we know that

$$\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = \frac{\psi(N)}{24} \left(\frac{1}{2} - \frac{\Lambda'(-1)}{\Lambda(-1)} \right).$$

Let $n_p = \nu_p(N)$, Corollary 4.7.2 implies that

$$\widehat{\omega} \cdot \widehat{\omega} = \frac{\psi(N)}{24}\left(2 - 4 \cdot \frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N} \frac{-n_p p^{n_p+1} + 2p^{n_p} + n_p p^{n_p-1} - 2}{p^{n_p-1}(p^2-1)} \cdot \log(p)\right)$$

Finally, we observe that

$$\widehat{\deg}(0, \log \det \mathsf{y} \cdot [\Omega]) = \frac{\log \det \mathsf{y}}{2} \int_{\mathcal{X}_0(N)_{\mathbb{C}}} \frac{\mathrm{d}x \wedge \mathrm{d}y}{2\pi y^2} = \frac{\psi(N)}{24} \cdot \log \det \mathsf{y}.$$

The identity (7.1) follows from the above computations and Proposition 3.5.2.

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