Hydrodynamic limits and fluctuations for interacting particle systems

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Abstract

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In this thesis, we study the limiting behavior of several different models in the KPZ universality class.

In Chapters 1 and 2, we study the hydrodynamic limits of the t-PNG model and the inhomogeneous stochastic six-vertex model with periodic weights using soft techniques. We show that the height functions of both models converge almost surely to a deterministic limit shape. The key element of the proofs is the construction of colored versions of both models, which allows us to apply the superadditive ergodic theorem. Along the way, we also construct the stationary t-PNG model and prove a version of Burke's theorem for it.

In Chapter 3, we consider the stochastic six-vertex model with step initial conditions and a single second-class particle at the origin. We show almost sure convergence of the speed of the second-class particle to a random limit. This allows us to define the stochastic six-vertex model speed process, whose law we show to be ergodic and stationary for the dynamics of the multi-class stochastic six-vertex process. The proof requires the development of precise bounds on the fluctuations of the height function of the stochastic six-vertex model around its limit shape using methods from integrable probability. As part of the proof, we also obtain a novel geometric stochastic domination result that states that a second-class particle to the right of any number of third-class particles will at any fixed time be overtaken by at most a geometric number of third-class particles.

In Chapters 4 and 5, we show that under a certain moderate deviation scaling, the multiplicative-noise stochastic heat equation describes the fluctuations of the quenched density of two different models of diffusion in a time-dependent random environment. The first model is the motion of a particle under a continuum random environment whose distribution is given by the Howitt-Warren flow. The second model is a 1D nearest-neighbor random walk in a space-time random environment. The proofs rely on certain Girsanov transforms and showing that the quenched density solves a prelimiting SPDE that resembles the stochastic heat equation. For the random walk model, we also show that independent noise is generated in the limit, in the sense that the prelimiting noise field does not converge to the driving noise of the limiting SPDE.

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Dedication

To Sabba

Introduction

0.1 Overview

In classical probability theory, a system of many independent random walks typically gives rise to Gaussian behavior. In contrast, the study of *interacting particle systems*, which is concerned with the behavior of randomly moving particles that have some nontrivial interactions with each other, gives rise to other rich universality classes, including the famous *KPZ universality class*.

An interacting particle system is a Markov process whose state space consists of particle configurations. Let S be a countable set of sites (e.g., $S = \mathbb{Z}^d$) and let W be a compact metric space representing the phase space at each site. Typically, we will take $W = \{0, 1\}$ where 1 represents the presence of a particle at a given site and 0 represents a hole, or the absence of a particle. We will then consider discrete or continuous time Markov processes on the state space $X = W^S$ endowed with the product topology. What differentiates interacting particle systems from systems of independent particles is that the future position of a particle does not just depend on its current position, but rather on the current position of the other particles as well. See [1] for a comprehensive overview of this subject.

Many types of tools have been developed to study interacting particle systems, and we will broadly divide them into two categories. The first consists of probabilistic techniques such as coupling methods and the construction of explicit stationary measures. The second category consists

of algebraic techniques. Certain particle systems have some special underlying algebraic structure that gives rise to many additional and powerful tools coming from representation theory. Some famous examples are the totally asymmetric exclusion process (TASEP), the stochastic six-vertex model, and the Hammersley process. We say that such models are *integrable* or *exactly solvable*. In this thesis, we will study the limiting behavior of both integrable and nonintegrable interacting particle systems in the KPZ universality class.

To introduce KPZ universality, we first consider *random growth models*, which are closely related to interacting particle systems. A random growth model is represented by a height function $h(x,t)$ which gives the height of a growing surface at position x at time t. We will be interested in models where the growth at different sites is not independent, again giving rise to non-Gaussian universality classes. Many such random growth models can also be viewed as interacting particle systems by interpreting the level lines of the height function in space-time coordinates as particle trajectories. In what follows, we often take the dual perspective of viewing our models as both random growth models and interacting particle systems.

Two natural questions we can ask about a random growth model are:

1. What is the limit shape of the height function, i.e., the limit of $h(Nx,Nt)/N$ as $N \to \infty$?

2. What are the random fluctuations of h around this limit shape?

While the limit shape is highly dependent on the specific choice of model, the fluctuations are believed to be universal for a large class of models whose large-scale behavior is similar to that of the solution of the *KPZ equation*. The KPZ equation is a stochastic PDE first introduced by Kardar, Parisi, and Zhang in [2]:

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}(\partial_x \mathcal{H}(t,x))^2 + \sqrt{2D_0}\xi(t,x), \tag{1}$$

where $D_0 > 0$ is the noise coefficient and $\xi(t,x)$ is a space-time white noise (i.e., a Gaussian noise with $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$). The solution $\mathcal{H}(t,x)$ represents the height function of a randomly growing surface whose growth at a point x depends on the height at nearby points. We

say that models whose height functions exhibit the same large-time scaling limits as $\mathcal{H}(t, x)$ are in the *KPZ universality class*. In general, it is difficult to prove that models belong to this class—most work in this direction has focused on exactly solvable models. However, it is of great interest to go beyond these special models, which has been a driving force of the following research.

In Chapters 1 and 2, we study the limit shapes of two positive temperature interacting particle systems: the t-PNG model and a family of inhomogeneous stochastic six-vertex models. We use coupling methods from interacting particle systems to prove almost sure convergence of these models to their limit shapes. While the t-PNG and stochastic six-vertex models are exactly solvable, the inhomogeneous generalization of the stochastic six-vertex model that we consider is not exactly solvable for a generic choice of inhomogeneities, showing the broader applicability of our techniques beyond exactly solvable models.

In Chapter 3 we construct the stochastic six-vertex model speed process, a process that tracks the asymptotic speeds of particles in the multi-class six-vertex model. We do so by proving tail bounds to control the fluctuations of the height function around its limit shape, contributing to a growing body of work developing tools to prove tail bounds for positive temperature models in the KPZ universality class. This work combines techniques arising from integrable probability with certain coupling-based techniques. In particular, the proof of the tail bounds makes use of certain integrable identities that connect the stochastic six-vertex model to the *Meixner ensemble*, a determinantal point process. We then relate these tail bounds to the speed of the second-class particle via a novel stochastic geometric domination result, which allows us to control the second-class particle with a large number of third-class particles.

In Chapters 4 and 5, we then turn to studying the extremal behavior of systems of diffusions in random environments. We prove that the fluctuations of the maximal particle for random walks in a dynamical random environment are governed by the KPZ equation in the moderate deviations regime. We also prove an analogous result for sticky Brownian motions, a continuum limit of random walks in a random environment. These works prove a form of KPZ universality for a large class of models that are not exactly solvable. The techniques used in this project are mostly derived

from stochastic analysis and the theory of stochastic partial differential equations.

We now give a brief overview of the KPZ equation, followed by an overview of the results of each chapter.

0.2 Solving the KPZ equation

While the KPZ equation is formally defined as in (1), making sense of this equation is quite challenging due to its nonlinearity and the poor regularity of the noise term. The notion of solution that we will work with in this thesis is the *Cole-Hopf solution*. If $\xi(t,x)$ were more regular (e.g. smooth), we would have that $\mathcal{U}(t,x) = e^{\mathcal{H}(t,x)}$ solves the *multiplicative stochastic heat equation* (SHE)

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{U}(t,x) + \sqrt{2D_0}\mathcal{U}(t,x)\xi(t,x). \tag{2}$$

However, unlike the KPZ equation, the SHE still has a well-defined solution when we take $\xi(t,x)$ to be space-time white noise. Therefore, as long as $\mathcal{U}(t,x) > 0$, we can define $\mathcal{H}(t,x) := \log \mathcal{U}(t,x)$ to be the solution to the KPZ equation. We call this the Cole-Hopf solution, and it coincides with other existing notions of solutions (see e.g. [3, 4, 5, 6, 7, 8]), under suitable assumptions.

The works [9, 10] showed that if the initial conditions to the SHE are nonnegative and not identically zero, then the solution is strictly positive for all positive times, i.e., $\mathcal{U}(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ almost surely. Therefore, the Cole-Hopf solution of the KPZ equation is well-defined for such initial conditions. We will mostly be interested in the solution of the SHE with Dirac delta initial conditions $\mathcal{U}(0,x) = \delta_0(x)$. The corresponding initial conditions for the KPZ equation are called the *narrow wedge initial conditions* and are formally given by $-\infty$ everywhere except at $x = 0$.

The notion of solution to the SHE that we will work with is due to Walsh [11] and relies on a theory of stochastic integration with respect to *martingale measures*. Alternatively, there is the solution theory developed by Da Prato and Zabczyk (see e.g. [12]) involving infinite-dimensional

stochastic evolutions. See [13] for a comparison of these two approaches.

We say that $\mathcal{U}$ is a *mild solution* (also called a Duhamel solution) of the SHE with initial conditions $\mathcal{U}(0,x)$ if it satisfies the following integral equation:

$$\mathcal{U}(t,x) = \int_{-\infty}^{\infty} q(t,x-y)\mathcal{U}(0,y)dy + \sqrt{2D_0}\int_0^t \int_{-\infty}^{\infty} q(t-s,x-y)\mathcal{U}(s,y)\xi(ds,dy). \tag{3}$$

where $q(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is the heat kernel and where the integral on the right can be rigorously defined as an integral with respect to a martingale measure [11].

In the seminal work [14], Amir, Corwin, and Quastel showed that the large-time behavior of the KPZ equation with narrow wedge initial conditions is described by the *Tracy-Widom GUE distribution*, a distribution first introduced in [15] to model the fluctuations of the largest eigenvalue of the *Gaussian unitary ensemble*.

On the other hand, the short-time behavior of the KPZ equation is Gaussian. Heuristically, taking time to 0 corresponds (under rescaling the equation) to putting a parameter ε in front of the nonlinear term in the KPZ equation and taking $\varepsilon \to 0$. This results in the additive stochastic heat equation (also called the *Edwards-Wilkinson equation*)

$$\partial_t H(t,x) = \frac{1}{2}\partial_{xx}H(t,x) + \sqrt{2D_0}\xi(t,x). \tag{4}$$

The mild solution to this equation is given by

$$H(t,x) = \int_{-\infty}^{\infty} q(t,x-y)H(0,y)dy + \sqrt{2D_0}\int_0^t \int_{-\infty}^{\infty} q(t-s,x-y)\xi(ds,dy).$$

Note that, unlike in (3), the right-hand side no longer depends on $H(t,x)$, so this can be taken as the definition of the solution H. Furthermore, H is a Gaussian process as the second integral is the integral of a deterministic function against a Gaussian white noise.

While the above argument was non-rigorous, this can also be seen rigorously through the Cole-

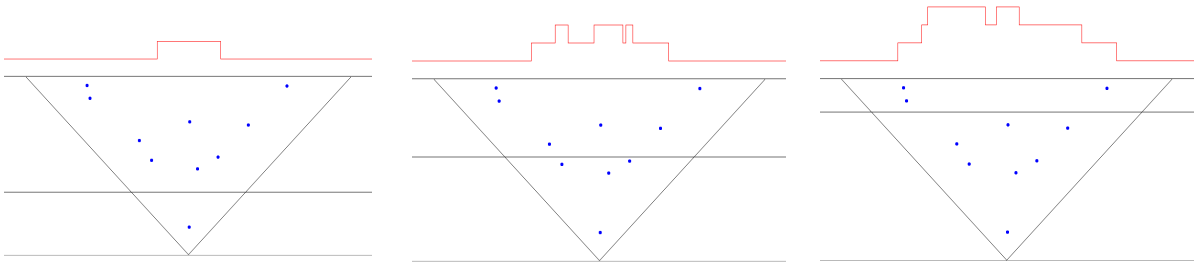


Figure 1: The time evolution of the PNG model. The red curve denotes the growing surface. The cone at the bottom contains the space-time nucleations, and the black bar denotes the time evolution. Image reprinted with permission from [16], based on simulations by Patrik Ferrari (`https://wt.iam.uni-bonn.de/ferrari/research/animationpng`).

Hopf solution to the KPZ equation by considering the corresponding small time limit of the SHE. Therefore, the one-point distributions of the KPZ equation *crossover* from the Gaussian distribution to the Tracy-Widom GUE distribution as we transition from small-time to large-time.

0.3 Limit shape of the t-PNG model

In a series of projects with Yier Lin [17, 18] (the content of Chapters 1 and 2, respectively), we prove *almost sure* convergence to the limit shape for the t-PNG model and a class of inhomogeneous stochastic six-vertex models. These are *positive temperature* KPZ models for which it is not straightforward to apply the subadditive ergodic theorem. To overcome this difficulty, we develop new colored versions of these models that give couplings under which we can apply the ergodic theorem.

The polynuclear growth (PNG) model, for which the limit shapes were first computed in the celebrated works of [19, 20], is a model with a space-time Poisson point process of *nucleations* in the light cone $\{(x, s) \in \mathbb{R} \times \mathbb{R}_{\geq 0} : |x| \leq s\}$. When a nucleation occurs at position x at time s, an island of height one and infinitesimal width nucleates on top of the existing one-dimensional surface at position x. This island then spreads out laterally with unit speed until it merges with another island of the same height. See Figure 1 for a simulation of this model. The t-PNG model

is a deformation of the PNG model recently introduced by Aggarwal, Borodin, and Wheeler in [16]. It is defined similarly to the PNG model, but when two islands merge, a new island nucleates at the merging location with probability t, for some fixed parameter $t \in [0, 1]$. Note that t does not represent time. The reason that we use the parameter t as the deformation parameter (following the convention in [16]) is due to the model's connection to the Hall-Littlewood polynomials, which use the parameter t.

We will now introduce an alternative description of the t-PNG model that comes from rotating space-time coordinates by $45°$. Although the above description is more clearly connected to the notion of a randomly growing surface, the following definition will be easier to work with and will relate our model to *Poissonian last passage percolation* and the *Hammersley Process*.

Fix $t \in [0, 1]$. First, we place a Poisson point process with intensity 1 on the upper-right quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ representing *nucleations*. We draw lines emanating from each of these nucleations in both the upward and rightward directions until they collide with each other. We call this collision point an *intersection point*. Given the Poisson nucleations, we sample the outcomes of the intersection points (lines will either cross or annihilate each other) starting with the intersection point which has the smallest sum of x- and y- coordinates and moving sequentially outward. At an intersection point, the two lines will cross each other with probability t and will annihilate each other with probability $1 - t$, forming a corner. We call these two types of intersection points *crossing points* and *corner points*, respectively. Note that when lines cross, they might generate new intersection points. See Figure 2 for a sampling of the t-PNG model. See also Figure 3 for computer simulations of the model for different values of t.

Taking $t = 0$, we recover the usual PNG model. In this case, we get a family of non-intersecting down-right paths.

We can now rigorously define the height function, which we denote by $N(x, y)$. We set the height function at the origin to be zero. Whenever we cross a line from left to right or from bottom to top, the height function increases by 1. See Figure 2 for the values of the height function for different sampled configurations.

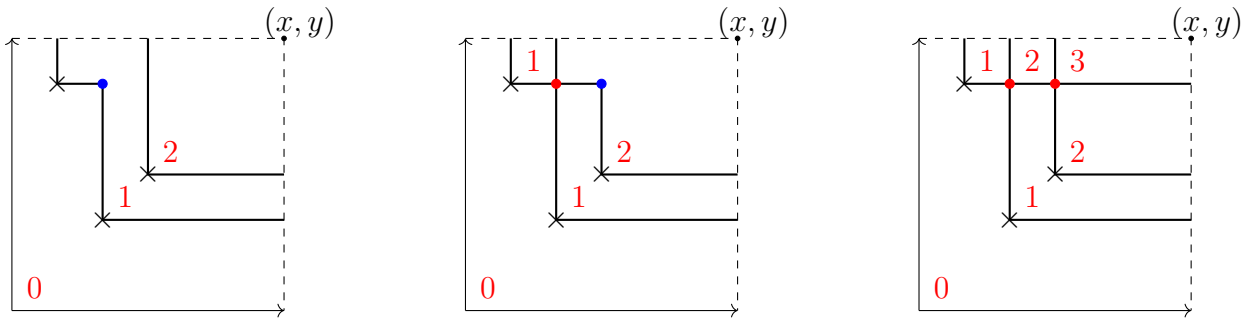


Figure 2: Possible samplings of the t-PNG model. $\times$ are the Poisson nucleations. The red points are the crossing points, and the blue points are the corner points. The numbers denote the height function in each section. Left panel: With probability $1 - t$, the two colliding lines form a corner point. Center panel: With probability $t(1 - t)$, we have a crossing point and a corner point. Right panel: Finally, with probability t^2, we have two crossing points.

For $t = 0$, $N(x, y)$ can alternatively be defined as the maximal number of Poissonian nucleations that can be collected along an upright path from $(0, 0)$ to (x, y). It is this perspective that relates the PNG model to the problem of *last passage percolation* (essentially the problem of finding a path with maximal energy through a random environment) and why this model is sometimes called Poissonian last passage percolation.

Inspired by this, we can equivalently define $N(x, y)$ for $t > 0$ to be the maximal number of *both* Poissonian nucleations and crossing points that an upright path can collect (see Figure 4). It is this reinterpretation of the height function that allows us to adapt tools that were developed for the PNG model. Note that the point process obtained by taking the union of Poissonian nucleations and crossing points no longer has the property that points occur independently of one another, as the locations of crossing points depend on the locations of other crossing points and nucleations to their bottom-left.

With the above interpretation, we can also connect the t-PNG model to *Ulam's problem* [21] of finding the length of the longest increasing subsequence of a permutation chosen uniformly at random from the symmetric group S_n. Let L_n be the random variable denoting the length of the longest increasing subsequence of a permutation chosen uniformly from S_n. Then the question posed by Ulam was to determine the limit of $\frac{\mathbb{E}[L_n]}{n^{1/2}}$. Hammersley [22] converted this problem to the problem of studying the limit shape of the height function of the PNG model through the realization

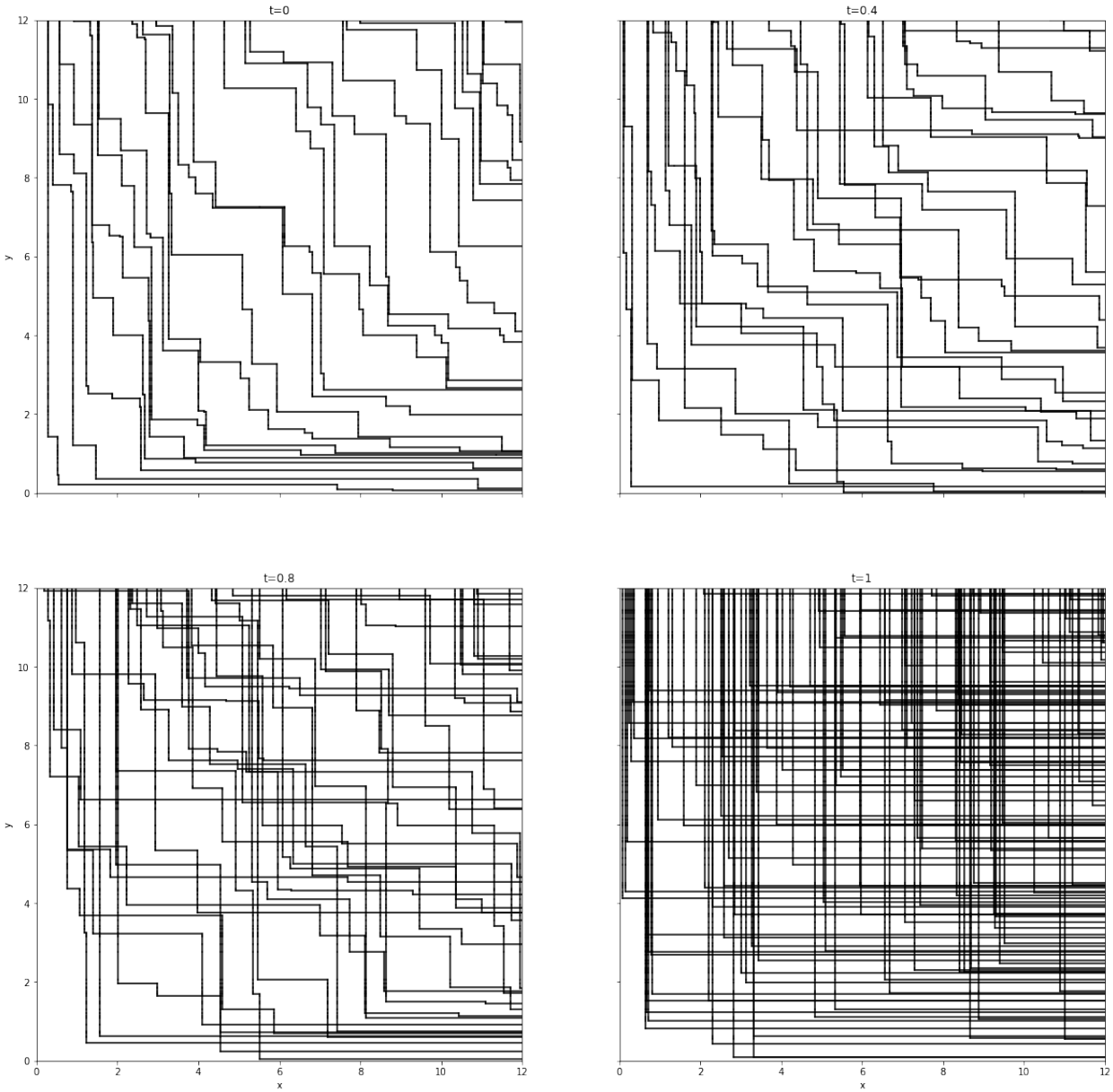


Figure 3: Simulation of the t-PNG model for $t = 0, 0.4, 0.8,$ and 1.

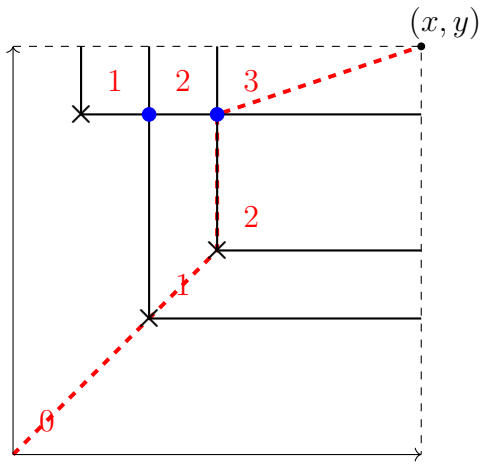


Figure 4: $N(x, y)$ equals the largest number of Poisson points ($\times$) and crossing points ($\bullet$) we can collect along an up-right path from $(0, 0)$ to (x, y).

that $N(n, n)$ is equal in distribution to L_M where M is a Poisson random variable with mean n^2.

Furthermore, the level lines of the PNG height function can be interpreted as the piles of cards that appear when *patience sorting* a uniformly shuffled deck of cards with a Poissonian number of cards. Patience sorting is a solitaire card game that can be used to find the length of the longest increasing subsequence in a given permutation (as represented by a deck of cards), see [23]. Similarly, the level lines in the t-PNG model can be interpreted as the piles for *patience sorting with errors* as introduced in [16]. This is similar to the usual patience sorting, but with probability t, when considering a card that should be placed onto an existing pile, the sorter makes an error and starts a new pile. These errors correspond to the crossing points and increase the height function (or, correspondingly, the erroneously calculated length of the longest increasing subsequence).

For PNG, which has space-time independent nucleations, [22, 24] used Kingman's subadditive ergodic theorem to prove that the height function converges almost surely to its limit shape. However, for t-PNG, this is more complicated since the locations of new nucleations depend on the previous ones, and the resulting nucleation field is no longer ergodic.

To remedy this, we view t-PNG as an interacting particle system and construct a new *colored t-PNG model*. We then couple the height function $N(x, y)$ with this colored model in such a way that we recover ergodicity and can apply the subadditive ergodic theorem. We obtain the following

result:

Theorem 0.3.1. *For $t \in [0,1)$, the height function of the t-PNG model converges almost surely to its limit shape, i.e., almost surely for all $x, y > 0$: $\lim_{s\to\infty} \frac{N(sx,sy)}{s} = \frac{2\sqrt{xy}}{\sqrt{1-t}}$.*

Weak convergence to the limit shape follows from the fluctuation result in [16] where they generalized the Baik-Deift-Johansson result [25] for the PNG model to prove that for all $t \in [0,1)$, as $s \to \infty$,

$$\frac{N(sx,sy) - \frac{2s\sqrt{xy}}{\sqrt{1-t}}}{(1-t)^{-\frac{1}{6}} s^{\frac{1}{3}} (xy)^{\frac{1}{6}}} \Rightarrow F_2 \tag{5}$$

where F_2 is the Tracy-Widom GUE distribution. Therefore, it just remains to prove almost sure convergence. However, since the proof of the fluctuation result relies on integrable techniques, we also give an alternative derivation of the limit shape using "soft" techniques coming from interacting particle systems, inspired by the works [22, 23, 26, 27].

Note that the limit shape blows up as $t \to 1$. At $t = 1$, the t-PNG model is trivial (the lines emanating from each Poissonian nucleation keep going forever, see the last picture in Figure 3) and the height function at (x, y) just counts the number of Poisson nucleations in the box $[0, x] \times [0, y]$. Then, by the usual law of large numbers and the central limit theorem, we know that

$$\frac{N(sx,sy)}{s^2} \to xy; \quad \frac{N(sx,sy) - s^2xy}{s} \Rightarrow \mathcal{N}(0,1)$$

where $\mathcal{N}(0,1)$ is a standard normal distribution.

We know from (5) that for $t < 1$ the model exhibits Tracy-Widom distributions. Therefore, one can expect that as $t \to 1$ under a suitable scaling, we should expect some crossover fluctuations that interpolate between the Tracy-Widom and Gaussian distributions. In [16], they proved that this crossover is given by the KPZ equation at the level of one-point distributions, but proving the process-level convergence remains an open question.

0.3.1 The colored t-PNG model

The key tool that we use to prove the almost sure convergence of the t-PNG model to its limit shape is the construction of a new *colored t-PNG model*. This is a generalization of the t-PNG model, wherein lines (which can be thought of as particle trajectories) are assigned colors with varying priorities. Note that in the vertex model literature, the term *color* is used while the same concept is also called *class* in the interacting particle system literature. We will use both terms depending on the model and context.

A color will be represented by an integer, with smaller integers having higher priority over larger integers. The colored model is constructed to satisfy the following three properties:

Property 1 (Color Ignorance): Lines with higher priority colors ignore those with lower priority colors. For instance, the lines with colors that belong to $\{1, \dots, m\}$ ignore the behavior of lines with colors greater than m. This means that if we sample the n-colored model and ignore the lines with colors greater than m, the remaining lines will behave as the m-colored model.

Property 2 (Projection from n colors to m colors): Suppose we start with colors $1, \dots, n$. Fix arbitrary integers $1 \leq r_1 < \cdots < r_m \leq n$. Then there is an explicit projection procedure that allows us to project the colors in $\{r_{k-1} + 1, \dots, r_k\}$ to color k for each $k \in \{1, \dots, m\}$ so that we recover the m-colored model. In particular, we can project the colored model down to the single-colored model.

Property 3 (Monotonicity of the Height Function): This property says that under certain conditions, adding a second color to the model does not decrease the height function. In other words, suppose that $H^1(x, y)$ is the height function of a single-colored model. If we allow lines of a lower priority color to enter from the bottom or left boundaries and then project both colors back to a single-colored model, then the height function $H^2(x, y)$ that we obtain is at least as large as $H^1(x, y)$.

It is precisely these three properties that will allow us to apply the subadditive ergodic theorem to the colored model. Due to color ignorance, the colored model will satisfy *ergodicity*, as we

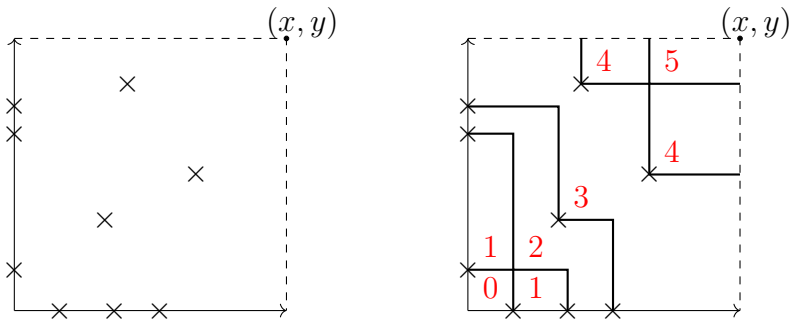


Figure 5: A sampling of the t-PNG model with sources on the bottom boundary and sinks on the left boundary.

will be able to ignore the effects of lines originating from the left and below a given region by assigning them a lower priority color. The projection property will allow us to couple the colored model to the height function of the single-colored model, which is the object we are trying to study. Finally, the monotonicity property will ensure that the colored model still satisfies *superdditivity* (it is straightforward to show that the height function of the single-colored model is superadditive, but adding in more colors can potentially break this).

0.3.2 Stationary t-PNG model and derivation of the limit shape

As part of our proof, we also construct a stationary version of the t-PNG model, generalizing the results in [27] for the PNG model.

To construct the stationary t-PNG model, we will first need to introduce the notion of *boundary data* for the t-PNG model. Fix a vector of locations on the positive x-axis which we will call *sources* and a vector of locations on the positive y-axis which we will call *sinks* such that on any rectangle $[0, m] \times [0, n]$ there are only finitely many sources and sinks on the bottom-left boundary. We treat the sources and sinks as additional nucleations and sample the model as before, ignoring lines that go along either the x-axis or y-axis (see Figure 5). The height function of the t-PNG model with boundary data is defined in the same way as before.

We now fix $\lambda, T_1, T_2 > 0$ and consider the t-PNG model on $[0, T_1] \times [0, T_2]$ with the following boundary data: a Poisson point process of sources of intensity λ on the bottom boundary and a

Poisson point process of sinks of intensity $\frac{1}{\lambda(1-t)}$ on the left boundary.

Given the t-PNG model, we can naturally obtain an interacting particle system called the t-*Hammersley process* as follows: Let $(X_\tau)_{0\leq\tau\leq T_2}$ be the configuration of particle locations in $[0, T_1]$ at time τ. The particle locations in this interpretation are the locations x such that (x, τ) belongs to a vertical line segment in the t-PNG model. More precisely, X_τ is a Markov chain on the state space E consisting of all finite point configurations on $[0, T_1]$.

Due to our choice of boundary data, X_0 is a Poisson process with intensity λ. We show that the Markov process X is stationary, meaning that the point configuration X_τ remains a Poisson point process with intensity λ for all $\tau \in [0, T_2]$.

Using this, we can now derive the limit shape. Let N_λ^{stat} be the height function of the stationary model with boundary parameter λ. Using the stationarity property, it is not hard to show that

$$\mathbb{E}[N_\lambda^{\text{stat}}(x, y)] = \frac{y}{\lambda(1-t)} + \lambda x. \tag{6}$$

Minimizing over all possible boundary parameters λ for the stationary model yields an upper bound on the limit shape for the original model. This is because adding in sources and sinks on the boundary can only increase the height function. The right-hand size of (6) is minimized for $\lambda = \frac{\sqrt{y}}{\sqrt{x(1-t)}}$. It follows that

$$\frac{\mathbb{E}[N(x, y)]}{\sqrt{xy}} \leq \frac{2}{\sqrt{1-t}} \tag{7}$$

which gives an upper bound on the limit shape.

It turns out that this upper bound is sharp and equals the limit shape. The intuition for this is as follows: $N_\lambda^{\text{stat}}(x, y)$ is counting the largest number of Poisson points and crossing points that can be accumulated along an up-right path from $(0, 0)$ to (x, y). As we have Poisson processes along the bottom and left boundaries, the optimal path will first choose one of the boundaries to go along for some time before entering into the interior of the quadrant. Changing the value of λ will change which boundary is likely to be favored by the optimal path and for how long the optimal path will cling to the boundary. At the specific choice $\lambda = \frac{\sqrt{y}}{\sqrt{x(1-t)}}$, both boundaries are equally

likely to be chosen. Therefore, the optimal path won't stick to the boundary for that long, making the height function very close to the height function of the model without sources and sinks.

To actually prove the lower bound, however, we employ a different approach inspired by [26] which relates the limit shape to a law of large numbers for the number of nucleation points in the box $[0, s] \times [0, s]$. We generalize this to the case where $t > 0$ by proving a law of large numbers for the number of Poisson nucleations and crossing points in $[0, s] \times [0, s]$.

0.4 Limit shape of the stochastic six-vertex model

Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_1	$1 - b_1$	b_2	$1 - b_2$

Figure 6: The six allowed configurations for the S6V model

In Chapter 2, we extend these results to the *stochastic six-vertex* (S6V) model. The S6V model was first introduced by Gwa and Spohn in [28] as a specialization of the six-vertex model, a classical model from equilibrium statistical mechanics going back to [29]. We use the observation from [16] that t-PNG can be viewed as a scaling limit of the S6V model to lift our techniques to that setting to prove almost sure convergence to the limit shape for the S6V model. These results also extend to an inhomogeneous generalization of the S6V model. Notably, for generic inhomogeneities, the S6V model is no longer exactly solvable, showing the robustness of our methods.

The S6V model with step initial data is a stochastic path ensemble on the discrete upper-right quadrant with a particle entering from each vertex on the left boundary. We fix two parameters $b_1, b_2 \in [0, 1]$. Starting from $(1, 1)$ and progressing sequentially away from the origin, we tile each vertex with one of the six configurations in Figure 6, according to their weight (see the left panel of Figure 7). We can think of lines entering a vertex from the bottom or left as *inputs* and the lines exiting from the top and right as *outputs*. The S6V model is *stochastic* in the sense that if we fix

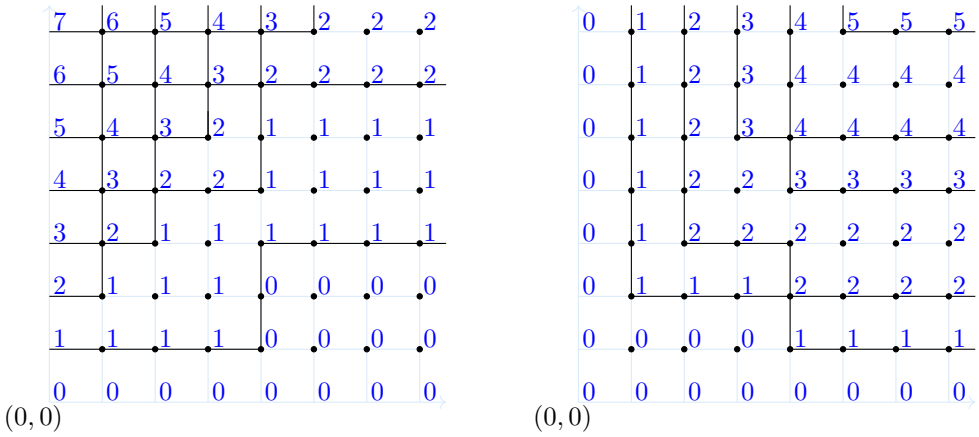


Figure 7: Left panel: A sampling of the S6V model on the first quadrant, with the blue numbers denoting the height function. Right panel: The CS6V model obtained by horizontally complementing the S6V model.

the inputs to a given vertex, then the weights of all configurations with those inputs add up to 1.

We define the height function $h(x, y)$ as the number of lines that pass through or to the right of (x, y), see the left panel of Figure 7. We can also view these lines as particle trajectories. We call an edge *occupied* if there is a particle there and *unoccupied* if not. If an edge is unoccupied, we can also call it a *hole*.

The model exhibits different behavior depending on whether b_1 or b_2 is larger. If $b_1 > b_2$, then particles tend to move up more than to the right. This leads to a sharp transition, known as a *shock*, between the region above the diagonal where the particle density is 1 and the region below the diagonal where the density is 0. See the left panel of Figure 8 for a simulation of this regime. On the other hand, if $b_1 < b_2$, then particles want to move right more than up, and thus they spread out. Three regions form: one above the line $x = \left(\frac{1-b_2}{1-b_1}\right) y$, where the density of particles is 1, one below the line $x = \left(\frac{1-b_1}{1-b_2}\right) y$, where the density is 0, and one in between, where the density decreases continuously from 1 to 0. The middle section is known as the *rarefaction fan*. See the right panel of Figures 8 and 11 for simulations of this regime.

We can already see that $\frac{1-b_1}{1-b_2}$ is an important quantity, and due to this, we will define an alter-

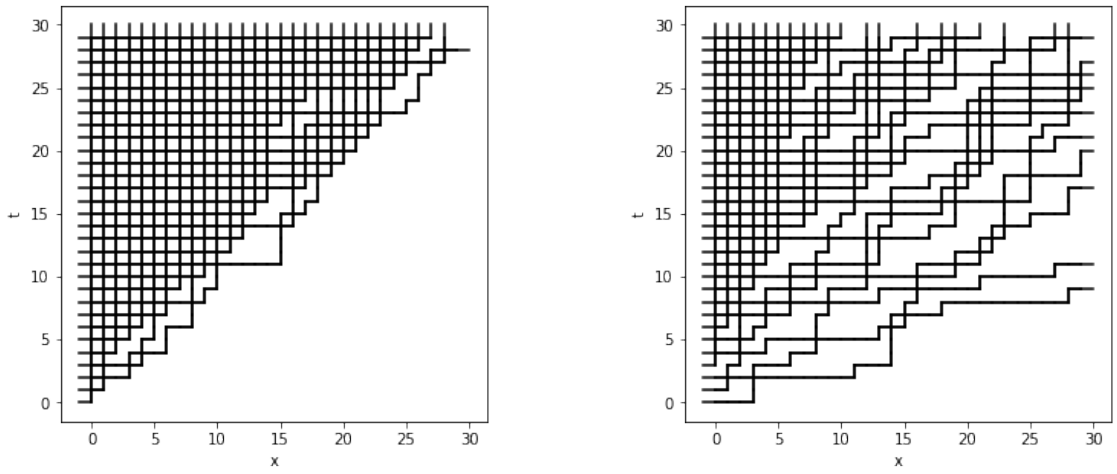


Figure 8: Left panel: A sampling of the S6V model with $b_1 = 0.7$ and $b_2 = 0.3$. Right Panel: A sampling of the S6V model with $b_1 = 0.3$ and $b_2 = 0.7$.

native parameterization of the S6V model by parameters $q, \kappa > 0$ defined as

$$q := \frac{b_1}{b_2} \qquad \kappa := \frac{1 - b_1}{1 - b_2}\,.$$

Using this notation, the rarefaction fan is the interval of slopes in $[\kappa^{-1}, \kappa]$.

More generally, we can allow the parameters b_1 and b_2 to vary from vertex to vertex as long as they satisfy certain periodicities. In other words, we consider vertex weights $b_1(i, j)$ and $b_2(i, j)$ that depend on the vertex (i, j) subject to the assumption that there exist periods $I, J \in \mathbb{Z}_{\geq 1}$ such that $b_k(x + I, y) = b_k(x, y)$ and $b_k(x, y + J) = b_k(x, y)$ for arbitrary x, y and $k \in \{1, 2\}$. We call this generalization the *inhomogeneous S6V model* and its height function can be defined in an analogous way to that of the homogeneous S6V model. The behavior of the inhomogeneous model will depend in an intricate way on the choice of inhomogeneities.

The key observation that allows us to use the tools from the t-PNG model to study the S6V model is the following: Consider the S6V model after *horizontal complementation*. If there is a horizontal line, we erase it; if there is no horizontal line, we add it, see Figure 7 for an example. This procedure converts an ensemble of upright paths to an ensemble of downright paths. Note that this model has appeared in [16] without a specific name, and we call it the *complemented S6V*

Type	I	II	III	IV	V	VI
Configuration at (i, j)						
Weight	1	1	b_1	$1 - b_1$	b_2	$1 - b_2$

Figure 9: Six types of configurations for the CS6V model

(CS6V) model. The six possible vertex configurations in the CS6V model are depicted in Figure 9.

Let H be the height function of the CS6V model defined on $\mathbb{Z}^2_{\geq 0}$, where $H(x, y)$ records the number of lines that pass through or to the left of (x, y). It is straightforward that $H(x, y) = y - h(x, y)$, see the right panel of Figure 7.

Horizontal complementation is a purely bijective procedure, however, it allows us to more directly compare the S6V model with models of last passage percolation. The height function in the complemented model can now be redefined as the maximum number of vertices with configurations of types III and VI in Figure 9 that can be collected along an up-right path from $(0, 0)$ to (x, y).

After performing horizontal complementation, it is possible to take a scaling limit of the complemented S6V model to the t-PNG model as explained in [16]. However, we do *not* use this limit to prove our results. Instead, we prove them directly for the complemented S6V model, using the previous work for the t-PNG model as an analogy.

We prove the following for the height function of the inhomogeneous S6V model:

Theorem 0.4.1. *There exists a Lipschitz function g such that with probability* 1*,*

$$\lim_{n\to\infty} \frac{h(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g(x, y), \qquad \forall x, y \in \mathbb{R}_{\geq 0}. \tag{8}$$

We do not know the formula for $g(x, y)$ for general inhomogeneities. For the homogeneous S6V model, the limit shape has been computed explicitly in [30, 31] at the level of convergence in

probability and is given by the following formulas: If $b_1 \leq b_2$, we have

$$g(x,y) = \begin{cases} \frac{\left(\sqrt{y(1-b_1)} - \sqrt{x(1-b_2)}\right)^2}{b_2 - b_1} & \frac{1-b_2}{1-b_1} < \frac{x}{y} < \frac{1-b_1}{1-b_2}, \\ 0 & \frac{x}{y} \geq \frac{1-b_1}{1-b_2}, \\ y - x & \frac{x}{y} \leq \frac{1-b_2}{1-b_1}. \end{cases} \tag{9}$$

This corresponds to the rarefaction fan regime. On the other hand, if $b_1 \geq b_2$, we have

$$g(x,y) = \begin{cases} 0 & x \geq y, \\ y - x & x \leq y. \end{cases} \tag{10}$$

This represents the shock regime. Our result strengthens the convergence in [30, 31] to almost sure convergence.

As a corollary of Theorem 0.4.1, we also obtain almost sure convergence to the limit shape for the higher spin S6V model, as the higher spin model can be obtained from the usual S6V model with a certain choice of inhomogeneities via a procedure called *fusion*. This procedure goes back to [32] (see also [33] for a more probabilistic explanation).

To prove Theorem 0.4.1, we construct a colored version of the complemented S6V model. It is important to note that this model is *not* the same as the usual colored S6V model studied in [34, 35, 36, 37, 38], even after performing horizontal complementation. The colored model we introduce is new and can be thought of as a generalization of the colored models in the above works in that it contains a strictly larger number of allowed configurations.

0.5 The stochastic six-vertex model speed process

While the previous work addresses the macroscopic behavior of the S6V model, we are also interested in the microscopic fluctuations of the system around its limit shape. We can study these fluctuations using *second-class particles*, first introduced by Liggett and Harris in the 1970s [39,

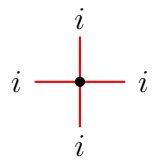
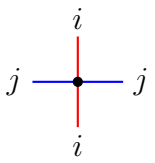
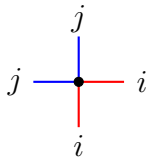
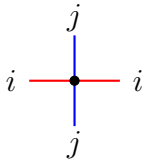
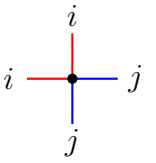
Configuration					
Weight	1	b_1	$1 - b_1$	b_2	$1 - b_2$

Figure 10: The allowed configurations for the multi-class S6V model, where red lines represent class i and blue lines represent class j for $i < j$.

40], and since used to study many different particle systems. With Levi Haunschmid-Sibitz [41], we show that the speed of a second-class particle in the S6V model converges almost surely to a random limit. This is the content of Chapter 3.

The multi-class S6V model is a multi-class generalization of the above model where each particle is assigned a class in $\mathbb{R} \cup \{-\infty, \infty\}$. Classes with higher values have lower priority. For example, a particle with class 1 (a first-class particle) will have priority over a particle with class 2 (a second-class particle). When a particle of class i and a particle of class j enter a vertex, the outputs only depend on the relative ordering of i and j. In particular, we sample the output according to the weights given in Figure 10.

The multi-class (or colored) version of the S6V model that we are considering here is the one studied in the works [34, 35, 36, 37, 38] and is not related to the new colored models discussed in the previous section. We can recover the single-class S6V model from the multi-class model by viewing occupied edges as particles with class 1 and empty edges as particles with class ∞.

We will consider the following initial conditions: we will have first-class particles entering from each vertex along the left boundary (as in the case of step initial conditions). Furthermore, we will have a single second-class particle entering from the bottom boundary at $(0, 0)$ as depicted in Figure 11. We can think of this setup as follows: we can first sample all of the first-class particles and then inject a single second-class particle into the system. This second-class particle moves like the rest of the particles but is restricted to the edges that remain unoccupied by first-class particles.

Denote by $\boldsymbol{X}_t$ the position of the second-class particle at time t, where the vertical axis represents time and the horizontal axis represents space (i.e., we now use the coordinates (x, t) instead

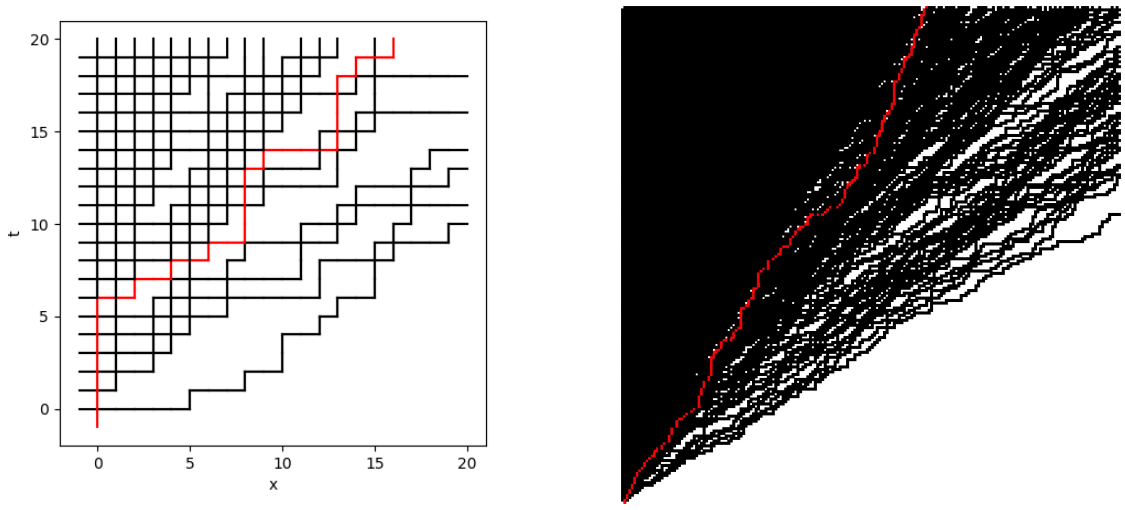


Figure 11: Simulations of the S6V model with a second-class particle at the origin in red.

of (x, y)). We show that the second-class particle's speed $\frac{\boldsymbol{X}_t}{t}$ converges almost surely to a random speed in the rarefaction fan. Note that the speed of the particle is just the reciprocal of its slope when plotted on the quadrant.

Theorem 0.5.1. *Let $0 < b_1 < b_2 < 1$, and consider the (homogeneous) S6V model with step initial positions with a second-class particle at the origin. Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t. Then almost surely*

$$\lim_{t\to\infty} \frac{\boldsymbol{X}_t}{t} = \boldsymbol{U} \tag{11}$$

where $\boldsymbol{U}$ is a continuous random variable taking values in the rarefaction fan $[\kappa^{-1}, \kappa]$ with density $\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}}$.

This means that a second-class particle will choose a random speed in the rarefaction fan according to $\boldsymbol{U}$ and will then follow this fixed speed almost surely.

We also derive a bound on the fluctuations around the limiting speed:

Theorem 0.5.2. *Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t as above and $\boldsymbol{U}$ its*

almost sure limit. Then for any $\delta > 0$, almost surely we have that

$$\lim_{t\to\infty} |\boldsymbol{X}_t - t\boldsymbol{U}| t^{-(\frac{7}{9}+\delta)} = 0\,. \tag{12}$$

The study of the convergence of second-class particles under step initial conditions began with the work of [42], where they showed the weak convergence of the speed of the second-class particle for the totally asymmetric simple exclusion process (TASEP) with step initial conditions. This was then strengthened to a.s. convergence by [43]. Analogous results were then proven for the Hammersley process [44] and the totally asymmetric zero range process (TAZRP) [45].

Weak convergence of the second-class particle follows from knowing its hydrodynamic limit. However, the proof of almost sure convergence is more complicated and for the above models relies crucially on the zero temperature nature of these models and, in particular, on connections between the models under consideration and last passage percolation models.

Since this no longer holds for the asymmetric simple exclusion process (ASEP), which can be thought of as a positive temperature model, new tools were required to prove the analogous result for ASEP under step initial conditions, and this was done in [46]. In particular, they used moderate deviation tail bounds coming from integrable probability together with Rezakhanlou's coupling for ASEP [47] to prove this result. The S6V model is an even more general positive temperature model than ASEP (in fact ASEP can be obtained as a specific scaling limit of the S6V model) and therefore our proof strategy for Theorem 0.5.1 is inspired by the ideas in [46] and relies on developing these same two ingredients for the S6V model.

In [48], it was shown that a second-class particle starting at the origin in the S6V model with stationary initial conditions fluctuates on the order of $t^{2/3}$. We expect that this should be the true order of fluctuations for a second-class particle starting from step initial conditions as well, so that the exponent $\frac{7}{9}$ in Theorem 0.5.2 should actually be $\frac{2}{3}$. However, we cannot reach this optimal exponent with our current methods.

Once we prove that the speed of a single second-class particle at the origin converges almost

surely, we can generalize this to the case where each particle in the S6V model starts off with a different class. In this setting, we are able to distinguish the speeds of each individual particle trajectory, and we can therefore ask about the simultaneous convergence of the speeds of each particle. This leads to what we call the *S6V model speed process*.

To define this, it is convenient to work with the S6V model defined on the entire upper half plane instead of just the upper right quadrant. It was shown by [31] that the model remains well-defined in this setting. We denote the *multi-class S6V process on the line* as $(\eta_t)_{t\geq 0}$, $\eta_t : \mathbb{Z} \to \mathbb{R} \cup \{\infty\}$, where $\eta_t(x) = i$ if at time t, there is a particle of class i at position x. Our initial conditions $\eta_0(x)$ now specify the class of particle entering from each position $x \in \mathbb{Z}$ along the bottom boundary.

We consider *packed initial conditions* where the particle entering from position x has class x. In this way, the priority of particles is increasing as you go from right to left. Denote by $\boldsymbol{X}_t(x)$ the position of the particle of class x at time t. It follows in a straightforward manner from Theorem 0.5.1 that we can simultaneously take the limit of each particle's speed:

Theorem 0.5.3. *$\left(\frac{\boldsymbol{X}_t(x)}{t}\right)_{x\in\mathbb{Z}}$ converges a.s. as $t \to \infty$ to a stochastic process $\boldsymbol{U}(x)$ called the **S6V speed process**.*

Speed processes were first constructed for TASEP [49], and have since been constructed for TAZRP [50] and ASEP [46].

The S6V model speed process has many interesting properties. For a fixed x, $\boldsymbol{U}(x) \sim \boldsymbol{U}$ where $\boldsymbol{U}$ has density $\frac{\sqrt{\kappa}}{2(\kappa-1)}x^{-\frac{3}{2}}$. However, the $U(x)$ are far from being independent for different choices of x. If they were independent, then the probability of the event $\boldsymbol{U}(x) = \boldsymbol{U}(x+1)$ would be zero. However, for speed processes, this event actually has a positive probability. This is related to a phenomenon known as *convoys*, where nearby particles tend to travel at the same speeds. This phenomenon was first studied in [49] for the TASEP speed process and was later studied in the setting of ASEP by [51].

Another important property of the speed process is that it is stationary under the dynamics of the multi-class S6V model. In particular, we prove the following result:

Proposition 0.5.4 (Stationarity of the Speed Process)**.** *Let U be sampled from the S6V speed process. Consider the multi-class S6V model with initial conditions given by $(-U(-x))_{x\in\mathbb{Z}}$. This process is stationary.*

The proof of this relies on the remarkable property of *color-position symmetry* for the stochastic-six vertex model. This property was proven in [52] and gives a distributional identity between the class of the particle at position x and the position of the particle of class x. This allows us to reinterpret the speed process, which is defined as a limit of the *positions* of the particles as the stationary behavior of the *classes* of the particles.

The S6V model lies in the KPZ universality class. Therefore, the S6V model speed process converges under a certain rescaling to the *stationary horizon* [53], a universal scaling limit for multi-class stationary measures of KPZ models. This was shown in [54] at the level of finite-dimensional distributions. It is still an open problem to prove the convergence of the ASEP and S6V model speed processes to the stationary horizon in the space $D(\mathbb{R}, C(\mathbb{R}))$.

0.5.1 Proof ideas

We now explain how we prove Theorem 0.5.1, which, as mentioned above, involves developing two new tools for the S6V model, both of which are of independent interest. In particular, we develop the following two ingredients:

- A geometric stochastic domination result that states that a second-class particle to the right of any number of third-class particles will at any fixed time be overtaken by at most a geometric number of third-class particles.
- Effective hydrodynamic estimates that quantify how close the height function of the S6V model started from step initial conditions will be to its limit shape.

These results will be used in the following way. We want to control the behavior of a single second-class particle. Hydrodynamic theory allows us to control the bulk behavior of a macroscopic number of particles, so we augment our system by filling up all empty positions to the left

of $\boldsymbol{X}_t$ with third-class particles. We then use our effective hydrodynamic estimates to control the union of the second- and third-class particles. Finally, we can revert this back to an estimate of the position of the second-class particle since we know that our second-class particle is to the left of at most a geometric number of the third-class particles. A similar argument can be made to bound the position of the second-class particle from the left.

We now state each of these two results in more detail.

0.5.2 Geometric stochastic domination result

Recall that $q = \frac{b_1}{b_2}$. By $X \sim \mathrm{Geo}(q)$ we denote the law given by

$$\mathbb{P}[X = k] = (1-q)q^k \text{ for } k \geq 0\,.$$

Theorem 0.5.5 (Geometric Stochastic Domination)**.** *Let $(\eta_t)_{t\geq 0}$ be a multi-class S6V process on the line with parameters $0 < b_1 < b_2 < 1$ and with the following initial conditions:*

- *There are some first-class particles (finitely or infinitely many).*
- *There is a single second-class particle.*
- *There are M third-class particles, all to the left of the second-class particle.*

Let $\boldsymbol{Z}_t(0) > \boldsymbol{Z}_t(1) > \cdots > \boldsymbol{Z}_t(M)$ be the ordered positions of the second- and third-class particles at time t. Further, let L_t be the number of third-class particles to the right of the second-class particle at time t. Then for any t, the law of L_t, conditioned on both $\boldsymbol{Z}$ and the space-time history of the first-class particle, is dominated by $\mathrm{Geo}\,(q)$.

Informally, this result says that at most a geometric number of third-class particles will overtake the second-class particle. This generalizes and significantly strengthens Rezakhanlou's coupling for ASEP [47], which states that a second-class particle in ASEP will stay to the right of a *uniformly* chosen third-class particle. That result gets weaker as M increases, whereas the geometric random variable in our result doesn't depend on M or t. Our result also extends to ASEP, where it can

also be viewed as a consequence of censoring inequalities for Markov chain mixing [55]. Such a connection is not known for the S6V model and could be interesting to explore further.

0.5.3 Moderate deviations tail bounds

We prove the following two tail bounds for the height function of the S6V model with step initial conditions in the rarefaction fan regime ($b_1 < b_2$). Recall that g is the limit shape of the S6V model as defined in (9).

Proposition 0.5.6 (Tail Bounds). *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[h(T\mu, T) \geq g(\mu, 1)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{3}{2}}}, \tag{13}$$

$$\mathbb{P}\left[h(T\mu, T) \leq g(\mu, 1)T - sT^{1/3}\right] \leq c^{-1}e^{-cs}, \tag{14}$$

and c can be chosen to weakly decrease in ε.

This result says that the height function in the rarefaction fan concentrates on the $T^{1/3}$ scale around its limit shape with exponential tail bounds. The power $T^{\frac{1}{3}}$ is optimal since on this scale the fluctuations of the height function have been shown in [30] to converge to the Tracy-Widom GUE distribution. The optimal exponents on the right-hand side, however, are expected to be s^3 for (13) and $s^{3/2}$ for (14), matching the tails of the Tracy-Widom distribution.

We call (13) a "lower tail" bound even though it seemingly describes the upper tail of the random variable $h(T\mu, T)$ since it corresponds to the lower tail of the Tracy-Widom distribution. Similarly, we call (14) an "upper tail" bound. This better matches the usage in the literature of the terms "upper" and "lower" tails for models in the KPZ universality class. The reason that the upper tail decays more slowly than the lower tail is that for the height function to be smaller than expected, we just need the position of the right-most particle in the S6V model to be small. On the other hand, for the height function to be larger than expected, we must have that the positions of

many particles are large. Since this requires more deviations to occur, the probability decays more quickly.

Proving lower tail bounds (the more challenging tail) for zero temperature KPZ models dates back to the seminal works [25, 56]. Our result contributes to a growing body of recent literature developing tools to prove lower tail bounds for *positive temperature* models, including [57, 58, 59, 60, 61, 62, 63, 64, 65, 66].

To prove the lower tail bound, we use a remarkable identity from [67] that expresses the q-Laplace transform of the height function in terms of an expectation with respect to the law of the Meixner ensemble. This identity allows us to bound the upper tail for the height function by the lower tail of the position of the smallest hole in the Meixner ensemble. The Meixner ensemble is a determinantal point process, so this tail can be expressed as a Fredholm determinant, which we then bound using Widom's trick [68]. The upper tail bound is more straightforward. We directly express the q-Laplace transform of the height function in terms of a Fredholm determinant and use Fredholm determinant estimates from [69].

The upper tail bound was also proven independently by [48] while our work was in progress, and they obtained the optimal tail exponent $s^{3/2}$ on the right-hand side of (14). Their proof method substantially differs from ours and relies on probabilistic couplings, building on ideas developed in [58, 62, 63].

Proof of lower tail

We now go into more detail on how we prove the lower tail bound. We call a subset $X \subseteq \mathbb{Z}_{\geq 0}$ a *point configuration*, and we define $\text{Conf}(\mathbb{Z}_{\geq 0}) = 2^{\mathbb{Z}_{\geq 0}}$ to be the set of all possible point configurations. A random configuration X is called a *random point process*. We also define the *n-point correlation function* as follows: for $A = \{x_1, ..., x_n\} \subseteq \mathbb{Z}_{\geq 0}$, let

$$\rho_n(A) = \rho_n(x_1, ..., x_n) := \mathbb{P}\left[A \subseteq X\right].$$

Definition 0.5.7 (Determinantal Point Process)**.** *A random point process X on $\mathbb{Z}_{\geq 0}$ is determinantal if there exists a kernel $K : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that for all $n \geq 1$ and for all $x_1, \ldots, x_n$ with $x_i \neq x_j$ for $i \neq j$, we have*

$$\rho_n(x_1, ..., x_n) = \det\left(K(x_i, x_j)_{i,j=1}^n\right). \tag{15}$$

We use the term *particles* to refer to the elements of X and use the term *holes* to refer to elements of $\mathbb{Z}_{\geq 0} \setminus X$. *Particle-hole involution* is an involution on point configurations that exchanges particles with holes. In other words, $X \mapsto X^\circ := \mathbb{Z}_{\geq 0} \setminus X$. If we start with an N-point ensemble, then particle-hole involution yields a point process with infinitely many particles. Suppose that X is a determinantal point process with kernel K. Then X° is a determinantal point process with kernel $1 - K$.

We can now define the Meixner ensemble. Fix two parameters: $\beta > 0$ and $\xi \in (0, 1)$, and then define the weight function $W : \mathbb{Z}_{\geq 0} \to \mathbb{R}$: $W(x) = \frac{\Gamma(\beta+x)}{\Gamma(\beta)x!}\xi^x$. Let $P_1, P_2, \ldots$ be the family of orthogonal polynomials on $\mathbb{Z}_{\geq 0}$ with respect to the weight function W. We define the Meixner ensemble Meixner(N, β, ξ) as an *N-point ensemble* (a random point process where $\mathbb{P}$ is supported on configurations with exactly N particles) with

$$\mathbb{P}(X = \{x_1, \ldots, x_N\}) \propto \prod_{i<j}(x_j - x_i)^2 \prod_{i=1}^N W(x_i).$$

The Meixner ensemble is determinantal with *Christoffel-Darboux* kernel

$$K_N(x, y) := (W(x)W(y))^{\frac{1}{2}} \sum_{n=0}^{N-1} P_n(x)P_n(y).$$

We will also need the following lemma, which relates the tail of a random variable to its q-Laplace transform (the right-hand side of (16)):

Lemma 0.5.8 (Lemma B.7 in [46])**.** *Let* $\mathbf{A}$ *be a real-valued random variable, $q \in [0, 1)$ and $b \in \mathbb{R}$.*

Then,

$$\mathbb{P}[\mathbf{A} \geq b] \leq e^{-q^b/(q-1)} \mathbb{E}\left[\prod_{i\geq 0} \frac{1}{1+q^{\mathbf{A}+i}}\right]. \tag{16}$$

Using this, we can reduce the problem of studying the tail of the height function to the problem of understanding its q-Laplace transform. We will then use the following identity from [67, 70] to study the q-Laplace transform of the height function. Let $\mathbb{E}_{\mathbf{6v}}$ refer to the expectation with respect to the S6V model.

Proposition 0.5.9 ([67, 70])**.** *Consider the S6V model parameterized by q and κ. Consider any integers $M > N \geq 1$. Then for any $\xi \notin -q^{\mathbb{Z}_{\leq 0}}$ we have*

$$\mathbb{E}_{\mathbf{6v}}\left[\prod_{i\geq 0} \frac{1}{1+\xi q^{H(M,N)+i}}\right] = \mathbb{E}_X\left[\prod_{x\in X} \frac{1}{1+\xi q^x}\right], \tag{17}$$

where the right-hand expectation is with respect to the point process $X \sim \mathrm{Meixner}^{\circ}(N, M-N, \kappa^{-1})$.

This says that the q-Laplace transform of the height function of the S6V model is equal to an expectation over the holes of the Meixner ensemble.

Let us denote the holes of the Meixner ensemble by $x_1, x_2, ...$ By throwing away all the terms on the right-hand side of (17) except for the one corresponding to the smallest hole x_1 of the Meixner ensemble, we ultimately obtain

$$\mathbb{P}\left[H(T\mu, T\nu) \geq g(\mu,\nu)T + sT^{1/3}\right] \leq e^{-1/(q-1)}\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] + e^{-1/(q-1)} q^{\frac{sT^{1/3}}{2}}. \tag{18}$$

In other words, we reduce the problem of studying the tail of the height function to studying the tail of x_1.

We can represent this tail probability as a Fredholm determinant. We have

$$\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] = \det(1-\widetilde{K})_{\ell^2(h,h-1,\ldots)}, \tag{19}$$

where $\widetilde{K}$ is a correlation kernel obtained as a dual of the kernel for the Meixner ensemble and $h = (g(\mu,\nu)-\nu)\,T + \frac{sT^{1/3}}{2}$.

To estimate the Fredholm determinant in (19), we will use a technique known as Widom's trick, first used by Widom in [68]. It consists of the observation that for a kernel K with eigenvalues in $[0,1]$ it holds that

$$\det(1-K) \leq \exp(-\text{Tr}(K))\,. \tag{20}$$

Therefore, obtaining an upper bound on $\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right]$ reduces to obtaining a lower bound for $\text{Tr}(\widetilde{K})$. We obtain this bound by a careful asymptotic analysis using a contour integral representation of the kernel $\widetilde{K}$.

0.5.4 Combining these tools to prove the main result

Using the geometric stochastic domination result and the moderate deviations tail bounds, the proof of Theorem 0.5.1 proceeds as follows. We will choose some time sequence S_n and some $\gamma > 0$ such that with high probability along this sequence we have

$$\left|\frac{\boldsymbol{X}_{S_n}}{S_n} - \frac{\boldsymbol{X}_{S_{n+1}}}{S_{n+1}}\right| \leq S_n^{-\gamma}. \tag{21}$$

If we choose γ so that $\sum_{n=1}^{\infty} S_n^{-\gamma} < \infty$, then it follows that the sequence $\frac{\boldsymbol{X}_{S_n}}{S_n}$ converges.

Suppose that $\frac{X_{S_n}}{S_n} = \mu$. Then our goal is to show that $X_{S_{n+1}}$ lies in a small neighborhood around μS_{n+1}. In particular, we will show that with high probability we have that

$$X_{S_{n+1}} \in \left[\mu S_{n+1} - S_n^{1-\gamma}, \mu S_{n+1} + S_n^{1-\gamma}\right].$$

This implies that

$$\frac{X_{S_{n+1}}}{S_{n+1}} \in \left[\mu - S_n^{-\gamma}, \mu + S_n^{-\gamma}\right]$$

which is equivalent to (21).

In order to show that $\boldsymbol{X}_{S_n} \geq \mu S_{n+1} - S_n^{1-\gamma}$ with high probability, we augment the system with third-class particles by filling up all empty positions to the left of $\boldsymbol{X}_{S_n}$ with third-class particles. We then use our moderate deviation tail bounds to show that at time S_{n+1}, with high probability, at least $S_n^{1/3}$ third-class particles are to the right of $\mu S_{n+1} - S_n^{1-\gamma}$. Finally, we use the geometric stochastic domination result to show that since the second-class particle is to the right of all but a geometric number of the third-class particle, it too is to the right of $\mu S_{n+1} - S_n^{1-\gamma}$ with high probability.

The above argument shows that $\frac{\boldsymbol{X}_{S_n}}{S_n}$ converges, but we want convergence of the sequence $\frac{\boldsymbol{X}_S}{S}$ for all times S. To prove convergence for general times S, we will use the monotonicity of $\boldsymbol{X}_S$ in S to bound $\boldsymbol{X}_S$ for $S \in [S_n, S_{n+1}]$. This yields the following trade-off: choosing a sequence S_n that grows more rapidly will allow us to better control the position of the second-class particle along that sequence, as the hydrodynamic estimates we will use are stronger for larger times S_n. On the other hand, choosing a sequence that grows more slowly will allow us to better control the position of the second-class particle for times in between S_n and S_{n+1}.

We will ultimately choose the time sequence to be of the form $S_{n+1} = S_n + S_n^{\beta}$ for some $\beta \in (0, 1)$. Choosing β close to 1 would give us a stronger bound on the fluctuations along the time sequence S_n and, in fact, would allow us to prove that

$$|X_{S_n} - S_n U| \leq S_n^{\frac{2}{3}+}.$$

However, this sequence grows too quickly to prove the finer statement in Theorem 0.5.2 as we will crucially use the fact that $|S_{n+1} - S_n| \leq S^{\beta}$ for $S \in [S_n, S_{n+1}]$ to bound the fluctuations between the times S_n. In the end, we will choose $\beta = 7/9$ to balance these two effects.

Note that in [46], they used the time sequence $S_{n+1} = S_n + \frac{S_n}{\log S_n}$, which grows like $e^{N^{1/2}}$. That sequence would also suffice to prove the almost sure convergence of the second-class particle, but again grows too fast to yield the fluctuations result.

0.6 Diffusions in dynamical random environments

In two joint works with Sayan Das and Shalin Parekh [71, 72] (the content of Chapters 4 and 5, respectively), we are motivated by the following question: *How does a system of diffusions in a dynamical random environment differ from a system of independent diffusions?* While the typical particles behave similarly in both systems, the extremal behavior is quite different. In particular, we will see that the fluctuations of the model in a certain moderate deviations regime are given by the KPZ equation, and we will also translate these results into fluctuation results for the position of the maximal particle.

Perhaps the simplest model of diffusion in a dynamical random environment is the *nearest-neighbor random walk in a dynamical random environment* (RWRE). To define it, let $(\omega_{t,x})_{t\in\mathbb{Z}_{\geq 0}, x\in\mathbb{Z}}$ be a family of i.i.d. $[0,1]$-valued random variables indexed in time and space. A random walk $R(t)$ in the environment ω jumps from position x to $x+1$ at time t with probability $\omega_{t,x}$ and to $x-1$ with probability $1-\omega_{t,x}$. Let $\mathsf{P}^\omega(R(t)=x)$ denote the probability that $R(t)=x$ given a realization of the environment ω. Note that this probability is itself a random variable, inheriting the randomness of ω.

If we run multiple walkers $R^1(t), \dots, R^k(t)$ in the same environment ω then they will behave as independent random walks when they are apart but will exhibit some sticky interactions when they meet. We can see this from the following calculation: Let Z denote the (random) probability that two walkers that are at the same location x at time t remain at the same location at time $t+1$. Also let $\sigma^2 := \mathrm{Var}(\omega_{t,x})$. We have

$$Z := \mathsf{P}^\omega\left(R^1(t+1) = R^2(t+1) \mid R^1(t) = R^2(t) = x\right) = \omega_{t,x}^2 + (1-\omega_{t,x})^2.$$

We can then see that $\mathbb{E}[Z] = \frac{1}{2} + 2\sigma^2 \geq \frac{1}{2}$, whereas if the two walks were independent, then we would get that $Z = \frac{1}{2}$.

The RWRE is actually a family of models that depends on the distribution of ω. We review several choices of ω to see the different possible behaviors of the model. From now on, we will always assume that $\mathbb{E}[\omega_{t,x}] = \frac{1}{2}$.

1. **Independent simple symmetric random walks ($\sigma^2 = 0$):** If we choose $\omega_{t,x} = \frac{1}{2}$ then $R^1(t), \ldots, R^k(t)$ will just be independent random walks.

2. **Coalescing random walks ($\sigma^2 = \frac{1}{4}$):** At the other extreme, we have coalescing random walks, which occur if we take $\omega_{t,x} \sim \text{Ber}(1/2)$. In this case, the $\omega_{t,x}$ can only take the values 0 or 1, so once two random walkers meet up, they will stay together for all later times.

3. **Beta RWRE:** This model was introduced in [73] and is obtained by taking $\omega_{t,x} \sim \text{Beta}(\alpha, \beta)$ for some $\alpha, \beta > 0$. The RWRE is *integrable* for this choice of weights in the sense that there are explicit contour integral formulas for the quenched transition probabilities.

We won't be interested in the case of coalescing random walks. Instead we will restrict ourselves to environments ω where $\text{Var}(\omega_{t,x}) = \sigma^2 \in [0, 1/4)$. It is also important to note that we are *not* restricting ourselves to the integrable case of the Beta RWRE.

We will consider the (quenched) random transition probabilities

$$\mathsf{P}^\omega(t, x) := \mathsf{P}^\omega(R(t) = x). \tag{22}$$

To first order, P^ω behaves asymptotically as $N \to \infty$ like a Gaussian, and it exhibits random fluctuations around that. Three different regimes have been studied, as depicted in Figure 12.

1. If we take $x \sim \sqrt{t}$, then these random fluctuations should themselves be Gaussian, and in particular are given by the Edwards-Wilkinson (EW) equation, a Gaussian process. This was shown in various manners in the works [74, 75, 76, 77, 78, 79, 80]. We call this the *diffusive regime*.

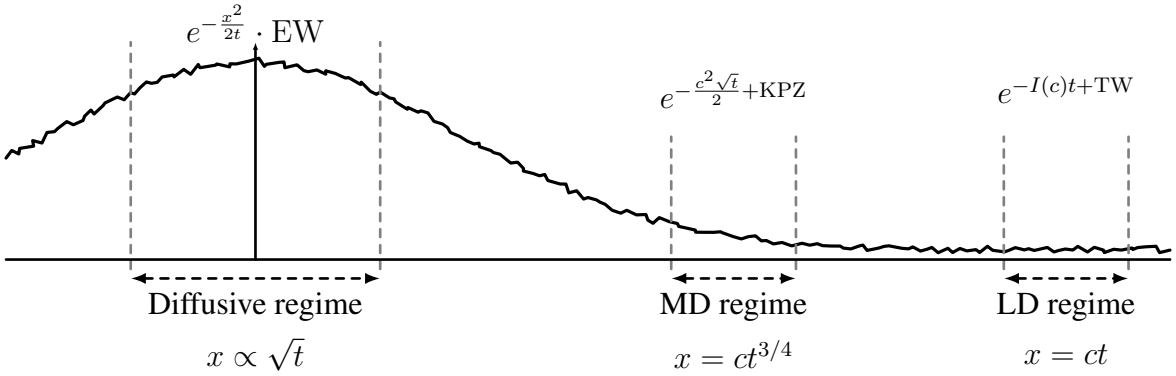


Figure 12: Schematic diagram of the quenched density $\mathsf{P}^\omega(t, x)$ of the position of a particle at time t.

If we take $x \sim t$, then we are in the *large deviations regime*. In this regime, it was shown in [81] that P^ω satisfies a large deviation principle with linear speed and a non-universal rate function I that depends on the environment. After factoring out the rate function, the fluctuations of $\log \mathsf{P}^\omega$ should be given by the Tracy-Widom GUE distribution. This was rigorously proven for the Beta RWRE [73] (see also [82]) using integrable methods, but is conjectured to hold for all random environments.

2. Finally, the papers [83, 84, 85] conjectured that there should be a crossover regime at $x \sim t^{3/4}$ at which $\log \mathsf{P}^\omega$ should have fluctuations given by the KPZ equation. The KPZ equation is a natural candidate as its fluctuations cross over between the Gaussian and Tracy-Widom GUE distributions as shown in the seminal work [14]. This regime is called the *moderate deviations regime*.

It is precisely this last regime that we will be interested in. In particular, we will resolve the physics conjectures made in [83, 84, 85]. We will show that in a window of size $t^{1/2}$ around the spatial location $x \sim t^{3/4}$, these transition kernels have fluctuations that are described by the multiplicative stochastic heat equation. In particular, this implies that the fluctuations of $\log \mathsf{P}^\omega$ are given by the KPZ equation, whose solution can be taken as the logarithm of the solution to the stochastic heat equation via the Cole-Hopf transform.

Consider $\mathsf{P}^\omega(Nt, N^{3/4}t + N^{1/2}x)$ for a fixed t and x and take $N \to \infty$. As discussed above,

this probability behaves to first order like the Gaussian distribution. To see any nontrivial fluctuations, we first need to renormalize by the first-order Gaussian behavior, which is contained in the following prefactor: For $t \geq 0$ and $x \in \mathbb{R}$ and $N \in \mathbb{N}$ define the constant

$$C_{N,t,x} := e^{N^{1/4}x + (N^{1/2} - N\log\cosh(N^{-1/4}))t}. \tag{23}$$

Informally, we want to prove that

$$C_{N,t,x}\mathsf{P}^{\omega}(Nt, N^{3/4}t + N^{1/2}x) \to \mathcal{U}(t,x)$$

where $\mathcal{U}(t,x)$ solves the SHE. However, P^{ω} is too noisy for convergence to hold pointwise for a fixed t and x, and we will need to smooth out P^{ω} with a spatial test function in order to see the SHE fluctuations.

We therefore define for each $N \geq 1$ and $t \in N^{-1}\mathbb{Z}_{\geq 0}$ a random measure $\mathscr{U}_N(t,\cdot)$ on $\mathbb{R}$, given by a superposition of Dirac masses as follows

$$\mathscr{U}_N(t,\cdot) := \sum_{x \in N^{-1/2}\mathbb{Z} - N^{1/4}t} C_{N,t,x} \cdot \mathsf{P}^{\omega}(Nt, N^{3/4}t + N^{1/2}x) \cdot \delta_x. \tag{24}$$

Here δ_x denotes a Dirac mass at spatial position x, and $a\mathbb{Z} + b := \{ax + b : x \in \mathbb{Z}\}$. The definition of $\mathscr{U}_N(t,\cdot)$ is extended to $t \in \mathbb{R}_+$ by linearly interpolating, i.e., taking an appropriate convex combination of the two measures at the two nearest points of $N^{-1}\mathbb{Z}_{\geq 0}$. We call $\mathscr{U}_N$ the *quenched density field*. Our main result is as follows.

Theorem 0.6.1. *Let ω denote a random environment as above with $\mathbb{E}[\omega_{t,x}] = \frac{1}{2}$ and $\mathrm{Var}(\omega_{t,x}) = \sigma^2 \in [0, 1/4)$. For any terminal time $T > 0$ we have the following:*

1. *The collection $\{\mathscr{U}_N\}_{N \geq 1}$ is tight with respect to the topology of $C([0,T], \mathcal{S}'(\mathbb{R}))$, where $S'(\mathbb{R})$ is the space of tempered distributions on $\mathbb{R}$.*

2. *Furthermore, any limit point as $N \to \infty$ lies in $C((0,T], C(\mathbb{R}))$ and coincides with the law*

of solution of the SHE started with δ_0 initial conditions at $t = 0$

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\partial_x^2 \mathcal{U}(t,x) + \sqrt{\frac{8\sigma^2}{1-4\sigma^2}} \cdot \mathcal{U}(t,x)\xi(t,x), \qquad t \geq 0, x \in \mathbb{R}. \tag{25}$$

Note that tightness and uniqueness of the limit point imply that $\mathscr{U}_N$ converges in distribution to $\mathcal{U}$ in the aforementioned topology, as $N \to \infty$.

As a corollary of the above result, we can obtain the asymptotic fluctuations for the maximal particle of k random walkers in the moderate deviations regime:

Theorem 0.6.2. *Fix $c, T > 0$ and $d \in \mathbb{R}$. Let $(R^1(r), \ldots, R^k(r))_{r \geq 0}$ be sampled according to the ω-averaged law of k independent random walk particles in the environment ω. Set the number of particles $k = k(N) := \lfloor \exp(\frac{1}{2}cN^{1/2} + dN^{1/4}) \rfloor$. Then we have the following convergence in distribution as $N \to \infty$:*

$$\max_{1 \leq i \leq k(N)} \left\{ N^{-\frac{1}{4}} R^i(Nt) \right\} - a_N(c,d,t) \xrightarrow{d} \sqrt{\tfrac{t}{c}} \big(G + \log \mathcal{U}(c,d) \big), \tag{26}$$

where

$$a_N(c,d,t) := \sqrt{ctN} - dN^{\frac{1}{4}} \sqrt{\tfrac{t}{c}} + \sqrt{\tfrac{c}{t}} \tfrac{1}{4} \log N - \frac{c^{3/2}}{6\sqrt{t}}.$$

Here G is a standard Gumbel random variable which is independent of $\mathcal{U}$, and $(t,x) \mapsto \mathcal{U}(t,x)$ solves (2) *with noise coefficient $D_0 = \frac{4\sigma^2 c}{(1-4\sigma^2)t}$.*

This theorem lets us tease apart the effects of the two sources of randomness in the model. The Gumbel distribution dictates the fluctuations of the maximal particle of independent diffusions, so it is not surprising to see it here. The KPZ term, however, is new and accounts for the randomness of the environment.

0.6.1 Proof ideas

There are several different characterizations of the SHE that have been successfully employed to show that a given model converges to the solution of the SHE. The two main ones are the *martingale problem* and the *chaos expansion* characterizations. A third, but only partial, characterization of the SHE is given by its moments.

A remarkable property of the RWRE model is that the proof method of chaos expansions that was used to prove convergence to the SHE for directed polymer models [86, 87] fails in our setting, indicating that there are both microscopic and mesoscopic sources of noise in the model. Instead, we develop a new proof method using the martingale problem characterization of the stochastic heat equation, inspired by the work of [88]. Additionally, to further probe why the chaos method fails, we prove that the driving noise in the limiting stochastic heat equation consists of both the limit of the environment ω as well as some additional *independent noise created in the limit*. We believe this is the first known case of this phenomenon for a one-dimensional model, although similar behavior has been observed in higher dimensions (see e.g., [89]).

We will now discuss these three characterizations in more detail and discuss how they relate to our proof methodology. In this section, we will consider the SHE with a general noise coefficient D_0 started with δ_0 initial conditions:

$$\partial_t \mathcal{U} = \frac{1}{2}\partial_x^2 \mathcal{U}(t,x) + \sqrt{2D_0} \cdot \mathcal{U}(t,x)\dot{W}(x,t).$$

1: Chaos Series Expansion

The mild solution to the SHE with δ_0 initial conditions is given by $\mathcal{U}(t,x)$ satisfying

$$\mathcal{U}(t,x) = q(t,x) + \sqrt{2D_0}\int_0^t \int_{-\infty}^{\infty} q(t-s, x-y)\mathcal{U}(s,y)\xi(ds,dy) \tag{27}$$

where $q(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the usual heat kernel.

We can iterate this by replacing $\mathcal{U}(s,y)$ on the right-hand side with this same expression. This yields the following series expansion for $\mathcal{U}$:

$$\mathcal{U}(t,x) = q(t,x) + \sum_{k=1}^{\infty} (2D_0)^{k/2} \int_{\mathbb{R}^k \times \Delta_k(t)} \prod_{\ell=1}^{k+1} q(t_\ell - t_{\ell-1}, x_\ell - x_{\ell-1}) \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}). \tag{28}$$

The right-hand side is a k-fold iterated Wiener-Ito integral where $\Delta_k(t)$ is the simplex of ordered times $0 \le t_1 \le \cdots \le t_k \le t$.

In [86], they show that the partition function for the directed polymer admits a discrete polynomial chaos expansion that converges to the chaos series for the SHE. In our setting, $\mathsf{P}^\omega(r,y)$ also admits a polynomial chaos expansion as shown in [90], however the kth term (when appropriately rescaled) converges to

$$(8\sigma^2)^{k/2} \int_{\mathbb{R}^k \times \Delta_k(t)} \prod_{\ell=1}^{k+1} q(t_\ell - t_{\ell-1}, x_\ell - x_{\ell-1}) \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}).$$

This would imply that the limiting SHE has a noise coefficient $2D_0 = 8\sigma^2$ rather than the true noise coefficient $2D_0 = \frac{8\sigma^2}{1-4\sigma^2}$. It follows that while the chaos expansion converges term by term, the infinite sum over k does not commute with the term-by-term limit. Because of this, we cannot use the chaos series method in our setting.

Note that $8\sigma^2$ is strictly smaller than the actual noise coefficient $\frac{8\sigma^2}{1-4\sigma^2}$. This is due to a phenomenon that we call *creation of independent noise in the limit* and which is encapsulated in the following result which considers what happens when we keep track of the rescaled field $\mathscr{U}_N$ together with the noise variables $\omega_{t,x}$. We define the prelimiting noise field

$$\Xi_N(\cdot,\cdot) := (2N^{3/2}\sigma^2)^{-1/2} \sum_{t \in N^{-1}\mathbb{Z} \cap [0,T]} \sum_{\substack{x \in N^{-1/2}\mathbb{Z} - N^{1/4}t \\ \text{same parity}}} \left(\omega_{Nt, N^{1/2}x + N^{3/4}t} - \tfrac{1}{2}\right) \cdot \delta_{(t,x)}. \tag{29}$$

We might expect that Ξ_N will converge to the driving noise ξ of $\mathcal{U}$, but in fact, this is not the case! Instead, we have the following:

Theorem 0.6.3 (Creation of independent noise in the limit)**.** *Any joint limit point of* $(\Xi_N, \mathscr{U}_N)$ *is of the form* $(\xi_1\ ,\ \mathcal{U})$*, where* ξ_1 *is a standard space-time white noise, and*

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\partial_x^2 \mathcal{U}(t,x) + \left(\sqrt{8\sigma^2}\ \xi_1(t,x) + \sqrt{\frac{32\sigma^4}{1-4\sigma^2}}\ \xi_2(t,x)\right)\mathcal{U}(t,x), \qquad t \geq 0, x \in \mathbb{R}, \tag{30}$$

with $\mathcal{U}(0,x) = \delta_0(x)$*. Here* ξ_2 *is another space-time white noise,*independent *of* ξ_1*.*

2: The Martingale Problem for the SHE

Due to the failure of the chaos expansion method, we will instead characterize our limit using a martingale problem characterization. The solution to the SHE can be characterized uniquely by the following martingale problem.

Definition 0.6.4 (Definition 4.10 in [88])**.** *Let* $X_t(r)$ *denote the canonical process on* $C([0,\infty), C(\mathbb{R}))$*. Let* μ *be a probability measure on* $C([0,\infty), C(\mathbb{R}))$ *such that for all* $T > 0$*,*

$$\sup_{t\in[0,T]} \sup_{r\in\mathbb{R}} e^{-a|r|} \int X_t(r)^2 d\mu < \infty \tag{31}$$

for some $a > 0$*. Furthermore, let* θ_0 *be a random element of* $C(\mathbb{R})$ *such that for each* p *there exists* $a = a_p$ *such that*

$$\sup_{r\in\mathbb{R}} e^{-a|r|}\mathbb{E}(|\theta_0(r)|^p) < \infty. \tag{32}$$

We say that the measure μ *solves the* martingale problem for the SHE with initial conditions θ_0 *if* $\mu(X_0 \in A) = \mathbb{P}(\theta_0 \in A)$ *for all Borel sets* $A \subset C(\mathbb{R})$ *and if for all test functions* $\phi \in \mathcal{D}(\mathbb{R})$ *the process*

$$M_t(\phi) := (X_t, \phi)_{L^2(\mathbb{R})} - \frac{1}{2}\int_0^t (X_s, \phi'')_{L^2(\mathbb{R})} ds \tag{33}$$

is a μ-local martingale with quadratic variation given by

$$\langle M(\phi)\rangle_t = 2D_0 \int_0^t (X_t^2, \phi^2)_{L^2(\mathbb{R})} ds. \tag{34}$$

Theorem 0.6.5 (Proposition 4.11 in [88])**.** *For every random function $\theta_0 \in C(\mathbb{R})$ satisfying* (32)*, the above martingale problem has a unique solution μ which coincides with the law of the SHE with initial conditions given by θ_0.*

We show that the quenched heat kernel for the RWRE satisfies a prelimiting version of this martingale problem. Together with tightness estimates, we can then conclude that the limiting field satisfies this martingale problem and hence is the solution to the SHE. This approach is inspired by the approach used by Bertini and Giacomin in [88] to show that the weakly asymmetric simple exclusion process converges to the KPZ equation.

3. The Moment Problem

In order to prove the tightness of $\mathscr{U}_N$, we will need to control the moments of certain observables. The techniques that we use to show that these moments converge are based on Girsanov transforms and tilting arguments. These techniques also allow us to show that the moments of $\mathscr{U}_N$ converge to the moments of the SHE. Although the moments do not completely characterize the SHE as they grow too rapidly, showing that the moments of $\mathscr{U}_N$ converge to the moments of the SHE already gives strong evidence that $\mathscr{U}_N$ converges to the SHE. In fact, recent work by Shalin Parekh [91] shows that under several additional natural conditions, the SHE can indeed be characterized by just a *finite* number of moments.

Due to the centrality of these tilting arguments in our work, we demonstrate them below by sketching the proof that the first and second moments of $\mathscr{U}_N$ converge to the first and second moments of $\mathcal{U}(t,x)$. The second moment's convergence will justify the choice of the particular moderate deviations scaling $N^{3/4}t$ as the correct regime in which to see the SHE. It will also show how the denominator of the nontrivial noise coefficient $\frac{8\sigma^2}{1-4\sigma^2}$ arises from local time considerations coming from pairwise interactions in the 2-point motions.

We will use the following Feynman-Kac representation of the moments of the SHE as proven in [92]. We define the local time at $a \in \mathbb{R}$ of a continuous semimartingale X as

$$L_a^X(t) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon < X_s < a+\varepsilon\}} d\langle X, X \rangle_s. \tag{35}$$

Then the k-th moment of the SHE integrated against a spatial test function can be written as

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} \phi(x)\mathcal{U}(t,x)dx\right)^k\right] = \mathbf{E}\left[\Phi(\vec{B(t)}) \exp\left\{D_0 \sum_{i<j} L_0^{B^i - B^j}(t)\right\}\right] \tag{36}$$

where $\mathbb{E}$ on the left is the expectation over the noise η of the SHE, and on the right $\Phi(\vec{x}) := \phi(x_1)\cdots\phi(x_k)$, and the expectation $\mathbf{E}$ is over independent Brownian motions $B^1, \dots B^k$ starting from 0.

For each $k \in \mathbb{N}$ let $\mathbf{P}_{(k)}$ denote the *annealed* law on the canonical space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$ of k independent walks sampled from the environment ω, all started at 0. In other words, it is the law on the motion of k-walkers, after averaging out the randomness of ω. We call the process $(R^1, \dots, R^k)$ with law given by $\mathbf{P}_{(k)}$ the *k-point motion*.

In particular, we will make extensive use of the one and two-point motions. The one-point motion R^1 is simply a simple symmetric random walk on $\mathbb{Z}$. The law of the two-point motion (R^1, R^2) is slightly more complicated. The two walkers behave as two independent simple symmetric random walks when apart, but display sticky interactions when they meet. The transition probabilities for (R^1, R^2) when $R^1(t) = R^2(t)$ are depicted in Figure 13.

As discussed above, the prefactor $C_{N,t,x}$ encodes the first-order asymptotic behavior of P^ω. However, its precise mathematical form was chosen so that it can be interpreted as a Radon-Nikodym derivative that has the effect of exponentially tilting the random walks.

We first recall some basic facts about exponential tilting. Let $M(\lambda)$ and $K(\lambda) = \log M(\lambda)$ be the moment and cumulant generating functions of some distribution μ. Then we can define the exponentially tilted measures

$$\mu^\lambda(dx) := e^{\lambda x - K(\lambda)} \mu(dx). \tag{37}$$

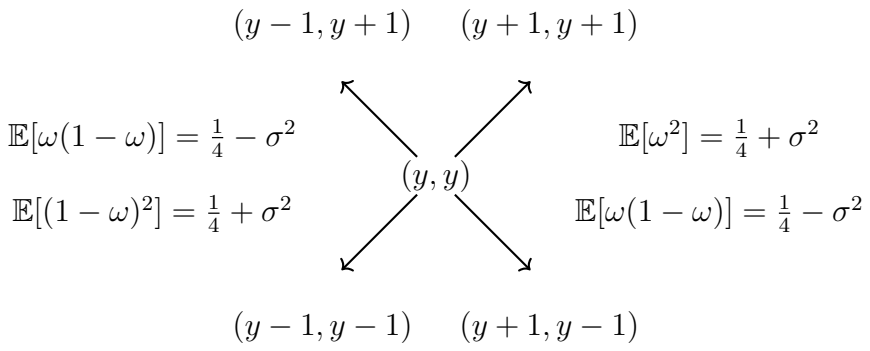


Figure 13: Markov chain transition diagram of the two-point motion (R^1, R^2) at points along the diagonal of $\mathbb{Z}^2$.

The mean of this tilted distribution is given by $\int_I x\mu^\lambda(dx) = K'(\lambda)$.

Let $R(t)$ denote the one-point motion, i.e., a simple symmetric random walk starting from 0. We will take $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ to be the distribution of the independent increments of R. Note that $C_{N,t,\frac{R(Nt)-N^{3/4}t}{N^{1/2}}} = \exp\left(N^{-1/4}R(Nt) - K(N^{-1/4})Nt\right)$ which has the effect of tilting our random walk so that $R(Nt)$ has mean $NtK'(N^{-1/4}) \approx N^{3/4}t$ under this tilted measure. Therefore, $\frac{R(Nt)-N^{3/4}t}{N^{1/2}}$ is asymptotically centered under the tilted measure.

More generally, if we want to study $\mathsf{P}^\omega(Nt, d_N t + N^{1/2}x)$ for some general location scale d_N, we could choose a prefactor $C_{N,t,x}$ so that $C_{N,t,\frac{R(Nt)-d_N t}{N^{1/2}}} = \exp\left(\lambda R(Nt) - K(\lambda)Nt\right)$ for some λ. This has the effect of tilting our random walk so that $R(Nt)$ has mean $NtK'(\lambda) = Nt\lambda$ under the tilted measure. If we pick λ so that $d_N \approx NK'(\lambda)$ then $\frac{R(Nt)-d_N t}{N^{1/2}}$ will be asymptotically centered under the tilted measures as $N \to \infty$. We will see soon why we need to choose the moderate deviation scale corresponding to choosing $(d_N, \lambda) = (N^{3/4}, N^{-1/4})$ in order to see the SHE emerge.

We can now study the first and second moments of $\mathscr{U}_N$.

Convergence of First Moment: Fix a Schwartz function ϕ and consider the first moment of

the pairing $(\mathscr{U}_N(t,\cdot),\phi)_{L^2(\mathbb{R})} =: \mathscr{U}_N(t,\phi)$. Note that for $t \in \mathbb{Z}/N$

$$\begin{aligned}\mathscr{U}_N(t,\phi) &= \sum_{x\in N^{-1/2}\mathbb{Z}-N^{1/4}t} C_{N,t,x}\cdot \mathsf{P}^{\omega}(Nt, N^{3/4}t+N^{1/2}x)\cdot\phi(x)\\ &= \sum_{x\in\mathbb{Z}} C_{N,t,\frac{x-N^{3/4}t}{N^{1/2}}}\mathsf{P}^{\omega}(Nt,x)\phi\left(\frac{x-N^{3/4}t}{N^{1/2}}\right)\\ &= \mathsf{E}^{\omega}\left[C_{N,t,\frac{R(Nt)-N^{3/4}t}{N^{1/2}}}\phi\left(\frac{R(Nt)-N^{3/4}t}{N^{1/2}}\right)\right], \end{aligned} \tag{38}$$

where E^{ω} denotes a quenched expectation operator given the realization of the environment ω. Thus after taking the *annealed* expectation (that is, averaging over all possible environments ω) we have

$$\mathbb{E}[\mathscr{U}_N(t,\phi)] = \mathbf{E}_{(1)}\left[C_{N,t,\frac{R(Nt)-N^{3/4}t}{N^{1/2}}}\phi\left(\frac{R(Nt)-N^{3/4}t}{N^{1/2}}\right)\right], \tag{39}$$

where now the expectation on the right is with respect to the one-point motion, a simple symmetric random walk.

Under the tilted law obtained from $C_{N,t,\frac{R(Nt)-N^{3/4}t}{N^{1/2}}}$, the process $\frac{R(Nt)-N^{3/4}t}{N^{1/2}}$ is centered and, by Donsker's principle, will converge in law to a standard Brownian motion. In summary, denoting $\widetilde{\mathbf{E}}$ as the expectation with respect to the tilted law, we have that

$$\mathbb{E}[\mathscr{U}_N(t,\phi)] = \widetilde{\mathbf{E}}\left[\phi\left(\frac{R(Nt)-N^{3/4}t}{N^{1/2}}\right)\right] \overset{N\to\infty}{\longrightarrow} \mathbf{E}_{BM}[\phi(B_t)]. \tag{40}$$

The right-hand side is indeed equal to $\mathbb{E}\big[\int_{\mathbb{R}}\mathcal{U}(t,x)\phi(x)dx\big]$ by the Feynman-Kac formula (36).

Convergence of Second Moment: We now compute the second moment of $\mathscr{U}_N$. We will initially write things out in terms of a general pair (d_N,λ) and point out where it will become necessary to choose the moderate deviations scale $(d_N,\lambda) = (N^{3/4}, N^{-1/4})$. Proceeding similarly

to the first moment computation, we obtain that

$$\mathbb{E}[\mathscr{U}_N(t,\phi)^2] = \mathbf{E}_{(2)}\left[C_{N,T,\frac{R^1(NT)-d_NT}{N^{1/2}}}C_{N,T,\frac{R^2(NT)-d_NT}{N^{1/2}}}\phi\left(\frac{R^1(NT)-d_NT}{N^{1/2}}\right)\phi\left(\frac{R^2(NT)-d_NT}{N^{1/2}}\right)\right], \quad (41)$$

where now the expectation on the right is with respect to the annealed two-point motion.

The prefactors $C_{N,T,\frac{R^1(NT)-d_NT}{N^{1/2}}}C_{N,T,\frac{R^2(NT)-d_NT}{N^{1/2}}}$ again have the effect of tilting the two-point motion so that it is centered, but only when $R^1(t) \neq R^2(t)$. Taking into account times when the two walkers meet, the correct tilting factor should actually be

$$C_{N,T,\frac{R^1(NT)-N^{3/4}T}{N^{1/2}}}C_{N,T,\frac{R^2(NT)-N^{3/4}T}{\mathbb{N}^{1/2}}}\exp\left\{-g\left(N^{-1/4}\right)\sum_{i=0}^{NT-1}\mathbb{1}_{\{R^1(i)=R^2(i)\}}\right\} \quad (42)$$

where

$$g(\lambda) := \log\left(\frac{1+4\sigma^2}{2}\cosh(2\lambda)+\frac{1-4\sigma^2}{2}\right)-2\log\cosh(\lambda), \quad (43)$$

which is the difference between the CGF of two independent random walks and the CGF of the two-point motion when the two walkers meet. Taking this correction into account, we obtain

$$\mathbb{E}[\mathscr{U}_N(t,\phi)^2] = \widetilde{\mathbf{E}}\left[\exp\left\{g\left(N^{-1/4}\right)\sum_{i=0}^{NT-1}\mathbb{1}_{\{R^1(i)=R^2(i)\}}\right\}\phi\left(\frac{R^1(NT)-d_NT}{N^{1/2}}\right)\phi\left(\frac{R^2(NT)-d_NT}{N^{1/2}}\right)\right]. \quad (44)$$

As $N\to\infty$, $\frac{R^1(NT)-d_NT}{N^{1/2}}$ and $\frac{R^2(NT)-d_NT}{N^{1/2}}$ will converge to two independent Brownian motions B^1 and B^2. Therefore, in order for this to converge to the second moment of the SHE, we will need

$$\exp\left\{g\left(N^{-1/4}\right)\sum_{i=0}^{NT-1}\mathbb{1}_{\{R^1(i)=R^2(i)\}}\right\}\to\exp\left\{\frac{4\sigma^2}{1-4\sigma^2}L_0^{B^1-B^2}(T)\right\}.$$

We have that $g(\lambda)=4\sigma^2\lambda^2+O(\lambda^4)$. It is here that we see why we need to take $\lambda=N^{-1/4}$, which corresponds to looking at the random walks at positions on the scale of $d_N=N^{3/4}$. We

know that for two independent simple symmetric random walks X^1 and X^2,

$$\frac{\sum_{i=0}^{NT-1} \mathbb{1}_{\{X^1(i)=X^2(i)\}}}{N^{1/2}} \to L_0^{B^1-B^2}(T). \tag{45}$$

Therefore, we need $g(\lambda)$ to be of order $N^{-1/2}$ in order for the left-hand side of (45) to converge to Brownian local time.

However, our random walks R^1 and R^2 are not actually independent and are more likely to stick together than two independent random walks. Therefore, we expect that $\sum_{i=0}^{NT-1} \mathbb{1}_{\{R^1(i)=R^2(i)\}}$ should be strictly larger than the corresponding sum for two independent random walks. Taking this stickiness into account, we show that under the tilted measures, we have the convergence

$$\frac{\sum_{i=0}^{NT-1} \mathbb{1}_{\{R^1(i)=R^2(i)\}}}{N^{1/2}} \to \frac{L_0^{B^1-B^2}(T)}{1-4\sigma^2}. \tag{46}$$

This explains why the noise coefficient ends up being $D_0 = \frac{4\sigma^2}{1-4\sigma^2}$ instead of the naive guess of $4\sigma^2$.

0.6.2 Generalizing the above

While above we have only described things in detail for the nearest-neighbor RWRE, these results can be generalized to a large class of models. In particular, we prove analogs of the above results for sticky Brownian motion, a continuum limit of the RWRE, in Chapter 4.

In a related physics work [93] with Eric Corwin, Ivan Corwin, and Jacob Hass, we also derive the noise coefficient D_0 for a class of non-nearest neighbor random walks and from this can compute the *extreme diffusion coefficient* for these models. This coefficient was recently introduced in [94] as a refinement of the classical *Einstein diffusion coefficient*. The Einstein diffusion coefficient is computed by taking the coefficient of the mean-squared displacement power-law for a *single* tracer particle in a random environment. This, in turn, yields a microscopic statistic of the environment, namely the square of the mean-free path length divided by the mean collision time. However, by examining the extremes of *multiple* tracer particles, we can reveal much more

information about the underlying environment since, as shown above, the extremal fluctuations are more sensitive to the disorder of the environment. This work is part of a program to ultimately compute this coefficient for real-world systems such as colloids and optical photons.

Chapter 1: Hydrodynamics of the t-PNG model via a colored t-PNG model

This chapter is adapted from the paper [17] and is coauthored with Yier Lin.

1.1 Introduction

1.1.1 Background

The distribution theory of the length of the longest increasing subsequence ℓ_n of a random permutation of the numbers $1, \dots, n$ with uniform measure has been under intense study in the past few decades. Now there are surveys and books [23, 95, 96] on this topic.

One object of particular interest is the $n \to \infty$ asymptotic behavior of ℓ_n. [22] proved a law of large numbers $\frac{\ell_n}{\sqrt{n}} \xrightarrow{p} \gamma$ (without identifying the constant γ) by considering a Poissonized version of ℓ_n and relating it to a last passage percolation model called the polynuclear growth (PNG) model or Hammersley's process. Then the superadditive ergodic theorem implies the law of large numbers. There have been various approaches to determining the constant γ, see [19, 20, 97, 23, 26, 27]. By detailed analysis of the exact expression of the distribution function, [25] (see also [98, 99]) proved a Tracy-Widom fluctuation limit theorem for ℓ_n.

The PNG model lies in the so-called Kardar-Parisi-Zhang universality (KPZ) class. It is natural to wonder whether there is a way to deform the PNG model so that its deformation also belongs to the KPZ universality class. [16] recently introduced a one-parameter deformation of the PNG model called the t-PNG model. The t-PNG model is also related to the length of the longest increasing subsequence from the perspective of patience sorting [23]. Using the method of integrable probability, the authors of [16] proved a Tracy-Widom fluctuation limit theorem for the t-PNG model that substantially generalizes the result of [25]. This shows that the t-PNG model also belongs to the KPZ universality class.

Inspired by [22, 23, 26, 27], the purpose of our paper is to study the t-PNG model using soft arguments. The main result is a shape theorem for the t-PNG model. We will view the t-PNG model from two different perspectives: as a single-colored projection of the colored t-PNG model and as an interacting particle system. We will see how a combination of these perspectives leads us to the main result.

1.1.2 The t-PNG model

Let us proceed to define the t-PNG model. Fix $t \in [0, 1]$. First, we place a Poisson point process with intensity 1 on the upper-right quadrant representing *nucleations*. We draw lines emanating from each of these nucleations in both the upward and rightward directions until they collide with one another. We call this collision point an *intersection point*. Given the Poisson nucleations, we sample the outcomes of the intersection points (lines will either cross or annihilate each other) starting with the intersection point which has the smallest sum of x- and y- coordinates and moving sequentially outward. At an intersection point, the two lines will cross each other with probability t and will annihilate each other with probability $1 - t$, forming a corner. We call these two types of intersection points *crossing points* and *corner points*, respectively. Note that when lines cross, they might generate new intersection points. We refer to Figure 1.1 for a sampling of the t-PNG model.

Taking $t = 0$, we recover the usual PNG model. The reason that we use the parameter t as the deformation parameter (following [16]) is due to the model's connection to the Hall-Littlewood polynomials, which use the parameter t.

The main object we are interested in studying is the height function. As with the PNG model, we define the height function at the origin to be zero. Whenever we cross a line from left to right or from bottom to top, the height function increases by 1 (see Figure 1.1). To avoid ambiguity, we let the height function be right-continuous. We use $N(x, y)$ to denote the height function at the location (x, y). Note that we hide the dependence on t in the notation since it will be clear from the context.

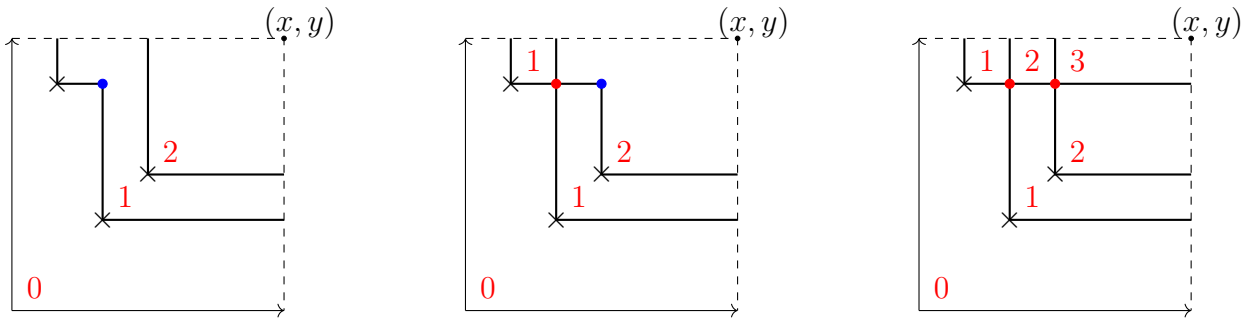


Figure 1.1: Possible samplings of the t-PNG model. $\times$ are the Poisson nucleations. The red points are the crossing points, and the blue points are the corner points. The numbers denote the height function in each section. Left panel: With probability $1 - t$, the two colliding lines form a corner point. Center panel: With probability $t(1 - t)$ we have a crossing point and a corner point. Right panel: Finally, with probability t^2, we have two crossing points.

We can extend the above definitions to define the t-PNG model with boundary data. Fix a vector of locations on the positive x-axis which we will call *sources* and a vector of locations on the positive y-axis which we will call *sinks* such that on any rectangle $[0, m] \times [0, n]$ there are only finitely many sources and sinks on the bottom-left boundary. We treat the sources and sinks as additional nucleations and sample the model as before, ignoring lines that go along either the x-axis or y-axis (see Figure 1.2). The height function of the t-PNG model with boundary data is defined in the same way as before.

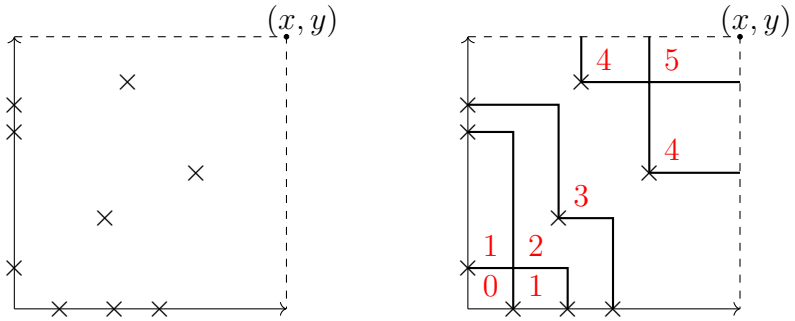


Figure 1.2: A sampling of the t-PNG model with sources on the bottom boundary and sinks on the left boundary

1.1.3 Statement of main result

We present our main result, which is a shape theorem for the t-PNG model.

Theorem 1.1.1. *Fix $t \in [0,1)$. Let $N(x,y)$ be the height function of the t-PNG model as defined above. Then the following shape theorem holds: With probability 1, we have for all fixed $x, y > 0$,*

$$\lim_{s\to\infty} \frac{N(sx,sy)}{s} = \frac{2\sqrt{xy}}{\sqrt{1-t}}.$$

As a consequence, we have $\frac{N(x,y)}{\sqrt{xy}} \xrightarrow{p} \frac{2}{\sqrt{1-t}}$ as $xy \to \infty$.

Remark 1.1.2. *When $t = 0$, our result recovers the shape theorem for the PNG model that was proved in [19, 20, 24, 97, 26, 27]. If we take $t = 1$, the right-hand side blows up. In that case, $N(x,y)$ equals the number of Poisson nucleations in the rectangle $[0,x] \times [0,y]$. The asymptotic behavior of $N(sx,sy)$ then follows from the central limit theorem for a Poisson random variable.*

Remark 1.1.3. *Using the methods of integrable probability, [16] showed that as $s \to \infty$,*

$$\frac{N(sx,sy) - \frac{2s\sqrt{xy}}{\sqrt{1-t}}}{(1-t)^{-\frac{1}{6}} s^{\frac{1}{3}} (xy)^{\frac{1}{6}}} \Rightarrow F_2$$

where F_2 is the Tracy-Widom distribution. Our theorem uses a softer technique to extract the first-order asymptotic of $N(sx,sy)$ at the almost sure level.

Remark 1.1.4. *A natural question to ask is whether one can prove anything about the local convergence of the model. When $t = 0$, [24] showed that*

$$\{N(as+x,s) - N(as,s), x \in (-\infty,\infty)\}$$

converges in distribution to a Poisson point process with intensity $\frac{1}{\sqrt{a}}$. The proof in [24] is via identifying the PNG model as an interacting particle system on the real line and classifying its stationary distributions as convex combinations of Poisson point processes.

We believe that for general $t \in [0,1)$, the same process converges in distribution to a Poisson point process with intensity $\frac{1}{\sqrt{(1-t)a}}$. The value $\frac{1}{\sqrt{(1-t)a}}$ comes from taking the spatial partial derivative of the hydrodynamic limit at $x = a$. The difficulty is that although the t-PNG model can

still be viewed as an interacting particle system on any finite interval (see Section 1.4), it is not clear how to extend the definition of the particle system to the entire real line due to its intricate dynamics.

A natural first step to proving the hydrodynamic limit theorem of the PNG model and other last passage percolation models is to apply the superadditive ergodic theorem. To apply the superadditive ergodic theorem (see Theorem 1.3.1), one needs to construct a family of superadditive random variables $\{X_{m,n} : 0 \leq m \leq n\}$ where $\{X_{0,n}, n \geq 0\}$ records the height function. A subfamily of the random variables also needs to be ergodic. For the PNG model, one can define $X_{m,n}$ as the length of the longest up-right path from (m, m) to (n, n) (we allow segments that go straight up or to the right), where the length of a path is defined as the number of nucleation points that it collects along its trajectory.

Let us introduce a few notions for the t-PNG model so that we can try to modify the approach above for the case where $t > 0$. We define two sets of points called α-points and β-points, which generalize the definitions in [26]. We define the set of α-points as the union of the Poisson nucleations and crossing points. We define the set of β-points as the union of corner points and crossing points. In particular, the intersection of the set of α-points and the set of β-points is the set of crossing points.

We redefine the length of an up-right path to be the number of α-points it collects. Then the height function of the t-PNG model is equal to the length of the longest up-right path. Hence, one can naively try to define the random variables $\{X_{m,n} : 0 \leq m \leq n\}$ in a similar way as above, but with our new definition of length. With this definition, the $X_{m,n}$ are superadditive as desired. However, they are no longer stationary since the number of crossing points in a box no longer just depends on the number of Poisson nucleations in that box—it also depends on the lines entering that box from the left and bottom boundaries. Therefore, we no longer even have stationarity and thus no ergodicity.

It turns out that there is indeed a way construct a family of random variables $\{X_{m,n} : 0 \leq m \leq n\}$ that satisfies the conditions required by the superadditive ergodic theorem. The family

$\{X_{m,n} : 0 \le m \le n\}$ comes from a colored version of the t-PNG model that we are going to introduce.

1.1.4 The colored t-PNG model

As we see from above, for the t-PNG model, two lines emanate rightward and upward from each Poisson nucleation point. The key rule for sampling the t-PNG model after fixing the Poisson nucleation points is that when two lines meet, they cross with probability t and annihilate each other with probability $1 - t$. This rule can be encoded into a stochastic matrix. We associate each intersection point with a 4-tuple $i, j, k, l \in \{0, 1\}$ which specifies the number of lines on the bottom, left, top, and right, respectively. We define a stochastic matrix $\mathcal{L}^1$ as follows.

Definition 1.1.5. *The matrix $\mathcal{L}^1$ is indexed by a 4-tuple $i, j, k, l \in \{0, 1\}$, where i, j, k, l denote the number of lines (either zero or one) on the bottom, left, top, and right of an intersection point. We define*

$$\mathcal{L}^1(1,1;1,1) = t, \qquad \mathcal{L}^1(1,1;0,0) = 1-t, \qquad \mathcal{L}^1(1,0;1,0) = 1,$$
$$\mathcal{L}^1(0,1;0,1) = 1, \qquad \mathcal{L}^1(0,0;0,0) = 1.$$

For all other $i, j, k, l \in \{0, 1\}$, we set $\mathcal{L}^1(i, j; k, l) = 0$. For fixed input lines $i, j \in \{0, 1\}$, $\mathcal{L}^1(i, j; \cdot, \cdot)$ is a probability measure on the output lines. See Figure 1.3 for illustration.

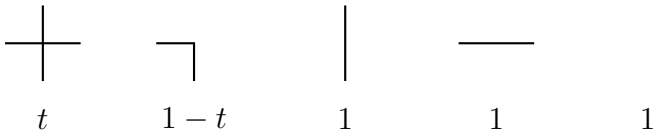


Figure 1.3: We draw all the intersection configurations that have non-zero weights.

In Figure 1.3, the first two configurations represent a vertical line intersecting with a horizontal line. The two lines cross with probability t and annihilate each other with probability $1 - t$. The third and fourth configurations depict that a horizontal (resp. vertical) line will continue as long as it does not meet a vertical (resp. horizontal) line. The last empty configuration represents the fact

that lines cannot emerge out of nowhere (except at a Poisson nucleation, but those have already been fixed before sampling the rest of the model).

For the colored t-PNG model, we denote the different colors by integers $i \in \mathbb{N}$. We allow multiple (but only finitely many) lines with different colors to travel together. We say that the color i has higher priority than the color j if $i < j$. The only restriction is that lines traveling together must have different colors. The colored t-PNG model is defined by specifying the sampling rule for when horizontal lines and vertical lines meet. The sampling rule is given by a family of stochastic matrices $\{\mathsf{L}^n, n \in \mathbb{N}\}$ that are consistent. More concretely, the matrix $\mathcal{L}^n$ has both rows and columns indexed by $\{0,1\}^n \times \{0,1\}^n$. The matrix elements are given by $\mathsf{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$, where the four vectors $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \{0,1\}^n$ specify the number of lines (either zero or one) of each color in $\{1,\dots,n\}$ on the bottom, left, top, and right of an intersection, respectively (see Figure 1.4). The stochastic matrices give a probability measure on the output lines $\mathbf{k},\mathbf{l}$ from an intersection point given the input lines $\mathbf{i},\mathbf{j}$.

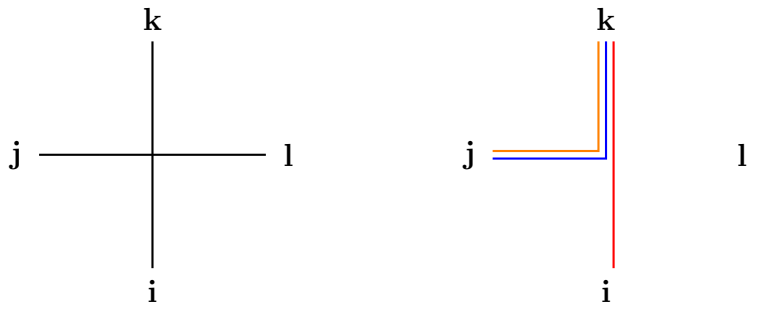


Figure 1.4: Left panel: Fix $n \in \mathbb{N}$. At an intersection point, we have lines with colors in $\{1,\dots,n\}$ in each of the four directions, but for any given direction there can be at most one line per color. Let $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \{0,1\}^n$ denote the number of lines on the bottom, left, top and right directions, respectively, where the m-th coordinate of each vector records the number of lines with color m. Right panel: Take $n = 3$. Let red, blue, and orange denote the colors 1, 2, and 3. We illustrate an example of the configuration with $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,1)$, $\mathbf{k} = (1,1,1)$, and $\mathbf{l} = (0,0,0)$.

We proceed to give a closed form to the matrices $\{\mathcal{L}^n : n \geq 1\}$. We first need to introduce some notation.

Definition 1.1.6 (r-fold Projection)**.** *For* $\mathbf{x} = (x_1,\dots,x_n) \in \{0,1\}^n$ *and* $r \in \{1,\dots,n\}$*, we*

define

$$\mathfrak{s}_r(\mathbf{x}) = \left(\sum_{m=1}^{r} x_m\right) \mod 2.$$

Then for a vertex configuration given by the 4-tuple $(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$*, we can define its* r*-fold projection to be the single-colored configuration* $(i,j;k,l) = (\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))$.

In other words, we are projecting the first r colors of $(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$ down to a single color and ignoring all colors greater than r. The projection $\mathfrak{s}_r(\mathbf{x})$ corresponds to replacing the colors $1,2,\ldots,r$ with a single color and then erasing every pair of lines. Then $\mathfrak{s}_r(\mathbf{x})$ denotes the number of lines that remain. This is also equivalent to replacing the total number of lines by itself mod 2.

As an example consider the following three-colored configuration. We adopt the convention that red, blue, and orange denote the colors 1, 2, and 3 respectively.

Its 1-fold, 2-fold, and 3-fold projections are given by the following table:

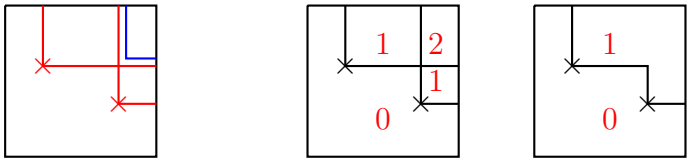
Procedure	1-fold projection	2-fold projection	3-fold projection
Step 1: Consider the first r colors.			
Step 2: Recolor everything black.			
Step 3: Replace the number of lines with the number of lines modulo 2.			

Figure 1.5: **r-fold Projections**: An example of how to take the r-fold projections of a given configuration. We recolor the first r colors black and delete all other colors. We then replace the number of lines with the number of lines modulo 2.

To compute the weight of an n-colored configuration $(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$, we will look at the weights of all of its r-fold projections for $r \in \{1,\ldots,n\}$, which can be computed using the matrix $\mathcal{L}^1$ that we have already defined in Definition 1.1.5. We want the r-fold projections of the colored model to have the same distribution as the single-colored t-PNG model. Therefore, if any of the r-fold projections have weight zero then we need the weight of $(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$ to be zero as well, as such a configuration should not be allowed.

Furthermore, we want to disallow the case where one of the r-fold projections has weight t and another one has weight $1-t$ since this would prevent the colored model from being superadditive. The reason for this is as follows: Superadditivity will follow from a certain monotonicity property of the height function (See Property 3 below). Namely, the height function of the 2-fold projection should be greater than or equal to the height function of the 1-fold projection.

If we allow r-fold projections with weights t and $1-t$, then we will allow the following sampling in Figure 1.6 whose 2-fold projection has a smaller height function than its 1-fold projection:



1-fold projection 2-fold projection

Figure 1.6: The weight of the 1-fold projection is t while the weight of the 2-fold projection is $1-t$. Notice how the height function in the top-right corner of the 2-fold projection is smaller than the height function in the top-right corner of the 1-fold projection.

Therefore, we will define the weight of $(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})$ to be the minimum of the weights of its r-fold projections; however, instead of using the classical minimum function, we use a modified version defined as follows:

Fix $t \in [0,1)$. Let $\min$ be a modification of the $\min$ function such that for $x_1,\ldots,x_n \in \{0,t,1-t,1\}$,

$$\min\big(x_1,\ldots,x_n\big) = \begin{cases} 0 & \text{if } x_i = t \text{ and } x_j = 1-t \text{ for some } i,j \in \{1,\ldots,n\}, \\ \min\big(x_1,\ldots,x_n\big) & \text{else.} \end{cases}$$

For example, we have $\min(t,1) = t$, $\min(1-t,1) = 1-t$ and $\min(t,1-t) = 0$. Note that we are treating t and $1-t$ as indeterminates, so we can ignore the case where $t = 1-t$. With this definition, a configuration will only have nonzero weight if all of its r-fold projections have nonzero weight and if t and $1-t$ are not both weights of different r-fold projections.

Definition 1.1.7. *Fix arbitrary $n \in \mathbb{Z}_{\geq 1}$. We define the matrix $\mathcal{L}^n$ via*

$$\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathfrak{min}_{r\in\{1,\ldots,n\}}\Big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\Big). \tag{1.1}$$

The table in Figure 1.7 illustrates how to compute the weights of some three-colored configurations. For the following configurations, we use the colors red, blue, and orange to represent the colors 1, 2, and 3 respectively. We also draw all of the two-colored configurations with non-zero weights in Section 1.6 for further illustration.

Configuration	1-fold projection	2-fold projection	3-fold projection	Total weight
	1	1	 $1-t$	$\mathfrak{min}(1,1,1-t) = 1-t$
	 t	1	 $1-t$	$\mathfrak{min}(t,1,1-t) = 0$

Figure 1.7: Examples of how to compute the weights of three-colored configurations

It is not a priori clear that $\mathcal{L}^n$ is stochastic; this will be proved in Section 1.2. The family of stochastic matrices $\{\mathcal{L}^n, n \geq 1\}$ satisfies the following three properties:

Property 1 (Color Ignorance): Lines with higher priority colors ignore lines with lower priority colors (see Figure 1.8 for illustration). For instance, the lines with colors belonging to $\{1,\ldots,m\}$ ignore the behavior of lines with colors greater than m. This means that if we sample the n-colored model and ignore the lines with color greater than m, the remaining lines will reduce to the m-colored model. In particular, if we ignore the lines of color 2 in the two-colored model, then the lines of color 1 have the same distribution as the single-colored t-PNG model. On the other hand, if we ignore the lines of color 1, the lines of color 2 do *not* have the same distribution as the single-colored t-PNG model. Because of this property, we will be able to define the $X_{m,n}$ in a way that maintains ergodicity, as we will see in Section 1.3.

Property 2 (Mod 2 Erasure): Fix arbitrary integers $1 \leq r_1 < \cdots < r_m \leq n$. We can project the

matrix $\mathcal{L}^n$ to $\mathcal{L}^m$ if we replace the colors in $\{r_{k-1}+1, \ldots, r_k\}$ with color k for each $k \in \{1, \ldots, m\}$ and then erase every pair of lines that has the same color (see Figure 1.8 for illustration). Because of this property, we can project the colored model down to the single-colored t-PNG model. This will ensure that the random variables $X_{0,n}$ will record the height function of the t-PNG model, which is the quantity that we are interested in studying.

Property 3 (Monotonicity of the Height Function): Suppose we have a sampling of the two-colored t-PNG configuration on a rectangle $[0, x] \times [0, y]$ where all nucleations are of the first color, and all sources and sinks are of the second color. Let $N^1(x, y)$ denote the height function at (x, y) of the 1-fold projection of all of the lines in the rectangle. Let $N^2(x, y)$ denote the height function at (x, y) of the 2-fold projection of the lines. Then

$$N^1(x, y) \leq N^2(x, y).$$

In other words, adding a second color to the model does not decrease the height function. This property will be crucial to proving the superadditivity of the random variables $X_{m,n}$.

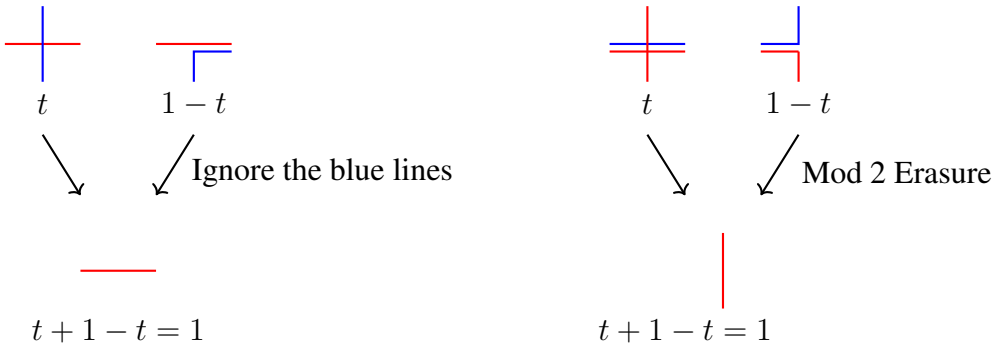


Figure 1.8: Left Panel: Color Ignorance. The top two configurations have the same input lines, and if we ignore the blue lines then they both equal the same single-colored configuration—a horizontal line. In fact, these are the only two-colored configurations with the given input lines whose first color is a horizontal line. Therefore the sum of their weights equals the weight of the horizontal line. Right Panel: Mod 2 Erasure. The top two configurations are the only two-colored configurations with the given input lines whose 2-fold projection is a vertical line. Therefore the sum of their weights equals the weight of the horizontal line.

The proofs of the first two properties are in Section 1.2. The third property follows from Lemma 1.4.3. Note that if we take $t = 0$ then the two-colored t-PNG model degenerates to the

two-colored PNG model defined in [27]. To our best knowledge, it seems that the sampling rule of the colored t-PNG model has not been defined earlier.

In [27], the authors take the viewpoint of interacting particle systems. The lines of the second color for the PNG model are the trajectory lines of second class particles for the Hammersley's process. This was one motivation for arriving at our definition of the colored model. We define our colored model so that the lines of the second color are the second class particles for the t-Hammersley process that we introduce in Section 1.4. A second motivation for our definition comes from the colored stochastic six vertex model as detailed in the next subsection.

Connection to the stochastic six vertex model

The stochastic six vertex (S6V) model is a classical model in two-dimensional statistical physics. The model was introduced in [28] as a special case of the six vertex model [100, 101]. We associate six possible configurations to each vertex in $\mathbb{Z}^2_{\geq 0}$ as illustrated in Figure 1.9. The weight of each configuration is parameterized by two parameters $b_1, b_2 \in [0, 1]$. We view the lines entering the vertex from the left and the bottom as input lines and view the lines leaving to the right and above as output lines. The S6V model is stochastic since if we are given the number of input lines from the left and bottom, the sum of the weights of all possible configurations with that input equals 1.

We view the S6V model as a stochastic path ensemble on $\mathbb{Z}^2_{\geq 0}$. We fix boundary conditions on the axis $\mathbb{Z}_{\geq 0} \times \{0\}$ (resp. $\{0\} \times \mathbb{Z}_{\geq 0}$) which indicate whether there is an input line entering each vertex along the axis from the bottom (resp. left). Starting from the vertex $(0, 0)$, we tile the given site with one of the six vertex configurations where we only consider configurations whose input lines match the input lines of the given vertex. We then assign an allowed configuration with probability given by the weight of the configuration. This tiling construction then progresses sequentially in the linear order $(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (1, 1), (0, 2), \ldots$ to the entire quadrant (see the left panel of Figure 1.11).

To relate this to the t-PNG model, we horizontally complement the S6V model. In other words, if there is a horizontal line, we erase it; if there is no horizontal line, we add it (see the right panel of

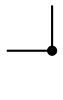
Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_1	$1-b_1$	b_2	$1-b_2$

Figure 1.9: Six types of configurations for the S6V model.

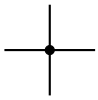
Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_1	$1-b_1$	b_2	$1-b_2$

Figure 1.10: Six types of configurations for the S6V model after horizontal complementation.

Figure 1.11). In Figure 1.10, we show the vertex configurations after horizontal complementation.

As done in [16], if we scale the weights $b_1 \to t$ and $b_2 \to 1$ in an appropriate way and simultaneously scale the discrete lattice to the continuum with certain boundary data, the complemented model in the previous paragraph converges to the t-PNG model. One can observe that the weights in Figure 1.10 reduce to that of the $\mathcal{L}^1$ matrix in the scaling limit.

It is natural to ask if our colored t-PNG model is related to the colored S6V model [38]. We focus on the case with two colors and recall the definition of the two-colored S6V model from [38, Section 2] (the multicolored S6V model was also defined therein). As usual, we use red to denote the higher priority color and blue to denote the lower priority color. The number of output lines for each color must equal the number of input lines for that color; however, unlike for the colored t-PNG model, there can be at most one line emanating from the vertex in each direction. The vertex weight of a two-colored S6V configuration is then defined to be the weight of the single-colored S6V configuration obtained by just considering the lines of the highest-priority color present in the configuration.

There is also a relation between the two-colored t-PNG model and the two-colored S6V model via horizontal complementation. Given a two-colored t-PNG configuration with at most one line

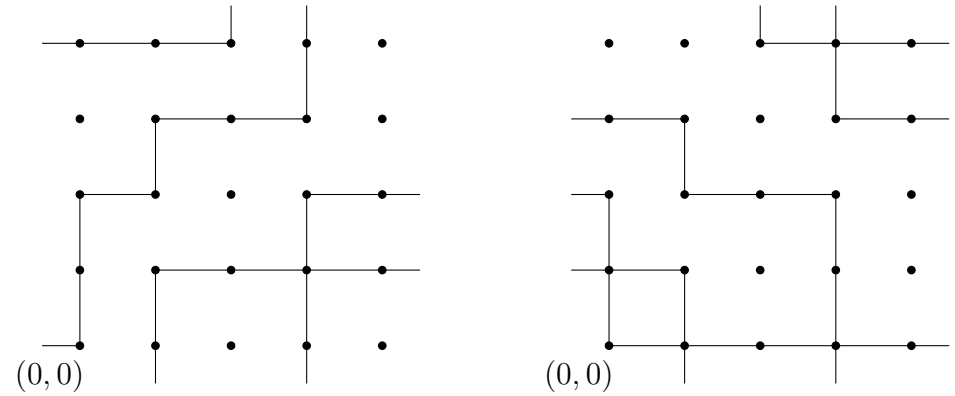


Figure 1.11: Left panel: A sampling of the S6V model on the first quadrant. Right panel: The S6V model after complementing the horizontal lines.

in the vertical direction, we can perform the following horizontal complementation (referred to as "hc" in the figures below). If there is a horizontal **red** line, we erase it; if there is no horizontal red line, we add it. We leave the blue lines unchanged. One can check that the resulting configuration is a configuration for the two-colored S6V model where $b_1 = t$ and $b_2 = 1$ and that the two configurations have the same weights in their respective models. However, this procedure will not work for two-colored t-PNG configurations with both blue and red lines in the vertical direction (see Figure 1.12). This is because the complemented configuration would still have two lines in the vertical direction, while the two-colored S6V model allows at most one line in each direction. Therefore, not every two-colored t-PNG configuration can be obtained through horizontal complementation of the two-colored S6V model.

Figure 1.12: Left panel: When there is at most one line in the vertical direction, one can obtain the two-colored t-PNG model by horizontally complementing the red lines. Moreover, the weights of the two-colored t-PNG and S6V configurations match. Right panel: When there are two lines in the vertical direction, one cannot obtain the t-PNG model via horizontal complementation, since the S6V model allows at most one line in the vertical direction.

1.1.5 A Burke's theorem for the t-PNG model

The original Burke's theorem [102] states that the departure process of an M/M/1 queue with a Poisson arrival process is a Poisson process. Generalizations of Burke's theorem have been proved for last passage models [27, 103, 104], polymers [105, 106, 107], and stochastic vertex models [108, 109, 110]. Note that these models all admit a stationary version and that Burke's theorem is a stronger property than stationarity.

The t-PNG model has a stationary version, which will be an important tool in the proof of our main theorem and is constructed in Section 1.4. As a natural extension, we also prove a Burke's theorem for the t-PNG model. Our result extends Theorem 3.1 in [27] from $t = 0$ to general $t \in [0, 1)$.

Theorem 1.1.8 (Burke's Theorem)**.** *Fix $T_1, T_2 > 0$. Consider the t-PNG model on $[0, T_1] \times [0, T_2]$ with a Poisson process of sources on of intensity λ on the bottom boundary, a Poisson process of sinks of intensity $\frac{1}{\lambda(1-t)}$ on the left boundary, and Poisson process of nucleations of intensity 1 in the interior of the box. We choose all three Poisson processes to be independent. Denote this process by L_λ and let L_λ^{corner} denote the set of corner points, let L_λ^{in} denote the entry points of paths on the right boundary, and let L_λ^{out} denote the exit points of paths on the top boundary. Then*

(i) L_λ^{corner} is a Poisson point process with intensity 1 in $[0, T_1] \times [0, T_2]$, L_λ^{in} is a Poisson point process of intensity $\frac{1}{\lambda(1-t)}$, and L_λ^{out} is a Poisson point process of intensity λ.

(ii) All three Poisson processes are independent.

The proof of Theorem 1.1.8 is given in Section 1.4.

1.1.6 Proof idea

As explained after Remark 1.1.4, unlike in the case of the PNG model, we can no longer directly apply the superadditive ergodic theorem to the t-PNG model using attractivity. Instead, we need to define a colored version of the t-PNG model which will allow us to construct a family

of random variables $\{X_{m,n} : 0 \leq m \leq n\}$ that satisfies the conditions of the superadditive ergodic theorem.

Definition 1.1.7 for the matrices $\{\mathcal{L}^n : n \geq 1\}$ follows from a sophisticated guess based on Properties 1–3; however, it is not immediately clear that these matrices are stochastic, and it also takes some work to show that all of the desired Properties 1–3 are indeed satisfied. The proof of stochasticity follows from an induction argument. Properties 1–3 then follow from a case-by-case study using the stochasticity and the fact that the entries of $\{\mathcal{L}^n : n \geq 1\}$ belong to $\{0, t, 1-t, 1\}$.

Having defined the stochastic matrices $\{\mathcal{L}^n : n \geq 1\}$, we proceed to prove the hydrodynamic limit theorem. The idea is to consider a colored t-PNG model and define $X_{m,n}$ to be a color-dependent version of the height function restricted to $[m, n] \times [m, n]$. We assign the Poisson nucleations with different colors so that points with smaller x-coordinates or y-coordinates have lower priority. We can now define $X_{m,n}$ in such a way so that it only depends on nucleations inside the box $[m, n] \times [m, n]$ and ignores all lines entering from the bottom or the left since those lines will have lower priority. Therefore, the random variables $X_{m,n}$ will be stationary and independent on disjoint boxes and hence ergodic. The superadditivity intrinsically follows from a coupling argument that says that allowing more lines to enter the boundary only increases the height function. We can now apply the superadditive ergodic theorem [111] to the family of random variables $\{X_{m,n} : 0 \leq m \leq n\}$. By the property of mod 2 erasure, the colored t-PNG model reduces to the single-colored t-PNG model after color projection. Hence, we have

$$\lim_{n\to\infty} \frac{N(n,n)}{n} = \sup_{n\geq 1} \frac{\mathbb{E}[N(n,n)]}{n} = \gamma, \qquad \text{a.s.}, \tag{1.2}$$

where γ is some constant in $[0, \infty]$.

To determine the value of this constant, we adopt an idea from [26]. Before carrying out that idea in the next paragraph, we need to first show that $\gamma \in (0, \infty)$. We do this by deriving an upper bound and lower bound for γ. The lower bound can be simply obtained via a coupling with the PNG model. To obtain the upper bound, we generalize [24] and view the t-PNG model as an

interacting particle system where the vertical lines play the role of particle trajectories. We add sources and sinks on the left and bottom boundary and obtain a stationary version of the t-PNG model. A coupling with the stationary model together with the attractive property provides an upper bound for γ.

In [26], the author proposed a soft argument for computing the constant γ in the case of $t = 0$. The key idea is to relate the computation of γ to the law of large numbers (LLN) for the number of α-points. When $t = 0$, the α-points are exactly the Poisson nucleations and the LLN follows immediately. For $t > 0$, the set of α-points is the union of Poisson nucleations and crossing points. We prove an LLN for the number of α-points using the fact that the number of α-points on the horizontal line emanating from a Poisson nucleation is asymptotically a geometric random variable.

1.1.7 Literature review

In [21], Ulam first posed the question of studying the average length of the longest increasing subsequence of a random permutation, which is now called “Ulam’s problem". In [22], Hammersley transferred the problem into a last-passage model now called the polynuclear growth (PNG) model and used the subadditive ergodic theorem to show that the limit γ exists. Logan and Shepp [19], and Vershik and Kerov [20] simultaneously proved that $\gamma = 2$. In fact, they proved a more general result concerning the limit shape of a Young diagram associated with the random permutation. In [24], an alternative proof is given using the perspective of interacting particle systems. [25] proved the entire fluctuation theorem through an analysis of exact formulas. [97, 26, 95, 27, 112] gave different proofs of $\gamma = 2$ using soft arguments.

Beyond the PNG model, the limit shapes of other last passage models have been well studied. For two discrete variants of the PNG model, the almost sure convergence to an explicit limit shape has been proved in [113, 114, 112]. The explicit limit shape for the exponential last passage percolation (LPP) model was first proved in [115] (also see the notes [116, 104, 117]), and similar results can be obtained for the geometric case. The fluctuations of the exponential and geometric

LPP models around their limit shapes are given in [118] using integrable methods. We remark that a common approach to deriving the explicit limit shape of an exactly solvable LPP or polymer model is to couple it with a stationary model and then solve a corresponding variational problem. This idea, however, no longer works for the t-PNG model since we do not have a similar coupling. For the LPP model with general i.i.d. weights, the limit shape is no longer explicit. [119] proved a general shape theorem and studied the continuity of the shape function as well as its asymptotic behavior near the edge. A shape theorem for the last passage percolation model on a two-dimensional compound Poisson process was proved in [120].

For the PNG model, the two-colored version has been studied in [27], where the paths of the second color can be viewed as the trajectories of the second class particles. The behavior of second class particles in the PNG model was studied in [121]. By a duality between the second class particle and the exit point, [122] shows that the fluctuation exponent of the stationary PNG model along its characteristic direction is $\frac{1}{3}$. A similar result was obtained in [123] for a discrete variant of the PNG model. The colored PNG model has been considered in [124], where it was obtained via the basic coupling of multiple particles. Note that it is not clear how to go from this perspective to our definition of the stochastic matrices in Definition 1.1.5.

The authors of [16] introduced the t-PNG model, which is a one-parameter deformation of the PNG model. They proved a one-point fluctuation limit theorem for the t-PNG model using integrable methods. They also proved one-point convergence to the KPZ equation. We remark that a different deformation of the t-PNG model was considered in [125]. The t-PNG model can be realized as a scaling limit of the S6V and its higher spin generalization after complementing the horizontal lines. The S6V model and its various generalizations have been studied recently in [33, 37, 126, 127, 38, 128, 129, 110, 130, 109, 131, 16] and references therein.

1.1.8 Outline

In Section 1.2 we prove that the matrices in Definition 1.1.5 are stochastic and satisfy Properties 1-2. In Section 1.3 we apply the superadditive ergodic theorem to prove the hydrodynamic limit

without identifying the constant γ. In Section 1.4 we prove an upper bound for γ by constructing a stationary version of the t-PNG model, we show that Property 3 is satisfied, and we also prove a version of Burke's Theorem. In Section 1.5 we identify γ by proving a law of large numbers for the number of α-points. In Section 1.6, we give all the two-colored configurations with non-zero weights. In Section 1.7, we provide some technical computations for Section 1.4.

1.2 Properties of the $\mathcal{L}$ matrices

In this section, we prove that the matrices $\{\mathcal{L}^n : n \geq 1\}$ defined in Definition 1.1.7 are stochastic and satisfy the properties of color ignorance and mod 2 erasure. The stochasticity is shown in Proposition 1.2.2. The properties of color ignorance and mod 2 erasure are respectively shown in Propositions 1.2.5 and 1.2.6.

Lemma 1.2.1. *Fix arbitrary $i, j \in \{0,1\}$. There exists a unique pair $(k,l) \in \{0,1\}^2$ such that $\mathcal{L}^1(i,j;k,l) \in \{t,1\}$. Similarly, there exists a unique pair $(k',l') \in \{0,1\}^2$ such that $\mathcal{L}^1(i,j;k',l') \in \{1-t,1\}$.*

Proof. This is straightforward from Definition 1.1.5. □

Let $A = \{r_1, \ldots, r_m\}$ be an ordered subset of $\{1, \ldots, n\}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$, we define $\mathbf{x}_A = (x_{r_1}, \ldots, x_{r_m})$. In particular, we define $\mathbf{x}_{[1,m]} = (x_1, \ldots, x_m)$.

Proposition 1.2.2 (Stochasticity)**.** *For arbitrary fixed $n \in \mathbb{Z}_{\geq 1}$, $\mathcal{L}^n$ is a stochastic matrix, i.e. the entries of $\mathcal{L}^n$ are non-negative, and for any $\mathbf{i}, \mathbf{j} \in \{0,1\}^n$,*

$$\sum_{\mathbf{k},\mathbf{l} \in \{0,1\}^n} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 1.$$

Proof. Let us prove the result by induction. When $n = 1$, we can easily see that $\mathcal{L}^1$ is stochastic. We use induction to prove the stochasticity for general n. We assume $\mathcal{L}^{N-1}$ is stochastic and use

this to show that $\mathcal{L}^N$ is also stochastic. Referring to (1.1), for $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \{0,1\}^N$, we have

$$\mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \min\Big(\mathcal{L}^{N-1}\big(\mathbf{i}_{[1,N-1]},\mathbf{j}_{[1,N-1]};\mathbf{k}_{[1,N-1]},\mathbf{l}_{[1,N-1]}\big), \mathcal{L}^1(\mathfrak{s}_N(\mathbf{i}),\mathfrak{s}_N(\mathbf{j});\mathfrak{s}_N(\mathbf{k}),\mathfrak{s}_N(\mathbf{l}))\Big). \tag{1.3}$$

Fix arbitrary $\mathbf{i},\mathbf{j} \in \{0,1\}^N$. $\mathcal{L}^{N-1}$ is a stochastic matrix whose entries take value in $\{0,t,1-t,1\}$, so we have one of the following two cases:

Case 1: There exist unique $\mathfrak{k},\mathfrak{l} \in \{0,1\}^{N-1}$ such that $\mathcal{L}^{N-1}(\mathbf{i}_{[1,N-1]},\mathbf{j}_{[1,N-1]};\mathfrak{k},\mathfrak{l}) = 1$.

Case 2: There exist unique $\mathfrak{k}_1,\mathfrak{l}_1,\mathfrak{k}_2,\mathfrak{l}_2 \in \{0,1\}^{N-1}$ such that

$$\mathcal{L}^{N-1}(\mathbf{i}_{[1,N-1]},\mathbf{j}_{[1,N-1]};\mathfrak{k}_1,\mathfrak{l}_1) = t, \qquad \mathcal{L}^{N-1}(\mathbf{i}_{[1,N-1]},\mathbf{j}_{[1,N-1]};\mathfrak{k}_2,\mathfrak{l}_2) = 1-t.$$

Let us prove the stochasticity of $\mathcal{L}^N$ in each case.

Proof for Case 1: Using $\mathcal{L}^{N-1}(\mathbf{i}_{[1,N-1]},\mathbf{j}_{[1,N-1]};\mathfrak{k},\mathfrak{l}) = 1$ and (1.3), we have

$$\mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^1(\mathfrak{s}_N(\mathbf{i}),\mathfrak{s}_N(\mathbf{j});\mathfrak{s}_N(\mathbf{k}),\mathfrak{s}_N(\mathbf{l}))$$

when $\mathbf{k}_{[1,N-1]} = \mathfrak{k}$ and $\mathbf{l}_{[1,N-1]} = \mathfrak{l}$. In addition, $\mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$ when $\mathbf{k}_{[1,N-1]} \neq \mathfrak{k}$ or $\mathbf{l}_{[1,N-1]} \neq \mathfrak{l}$. Note that $(\mathfrak{s}_N(\mathbf{k}),\mathfrak{s}_N(\mathbf{l}))$ equals each element of $\{0,1\}^2$ exactly once when we vary $\mathbf{k},\mathbf{l}$ under the restriction $\mathbf{k}_{[1,N-1]} = \mathfrak{k}$ and $\mathbf{l}_{[1,N-1]} = \mathfrak{l}$. Therefore,

$$\sum_{\mathbf{k},\mathbf{l}\in\{0,1\}^N} \mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \sum_{\substack{\mathbf{k}_{[1,N-1]}=\mathfrak{k}\\ \mathbf{l}_{[1,N-1]}=\mathfrak{l}}} \mathcal{L}^1(\mathfrak{s}_N(\mathbf{i}),\mathfrak{s}_N(\mathbf{j});\mathfrak{s}_N(\mathbf{k}),\mathfrak{s}_N(\mathbf{l})) = 1.$$

The last equality comes from the stochasticity of $\mathcal{L}^1$.

Proof for Case 2: By (1.3), $\mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$ when $(\mathbf{k}_{[1,N-1]},\mathbf{l}_{[1,N-1]}) \notin \{(\mathfrak{k}_1,\mathfrak{l}_1),(\mathfrak{k}_2,\mathfrak{l}_2)\}$. If $(\mathbf{k}_{[1,N-1]},\mathbf{l}_{[1,N-1]}) = (\mathfrak{k}_1,\mathfrak{l}_1)$, we have

$$\mathcal{L}^N(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \min\big(t, \mathcal{L}^1(\mathfrak{s}_N(\mathbf{i}),\mathfrak{s}_N(\mathbf{j});\mathfrak{s}_N(\mathbf{k}),\mathfrak{s}_N(\mathbf{l}))\big).$$

By Lemma 1.2.1, there exists only one pair of $(\mathfrak{s}_N(\mathbf{k}), \mathfrak{s}_N(\mathbf{l}))$ such that $\mathcal{L}^1(\mathfrak{s}_N(\mathbf{i}), \mathfrak{s}_N(\mathbf{j}); \mathfrak{s}_N(\mathbf{k}), \mathfrak{s}_N(\mathbf{l})) \in \{t, 1\}$. Using this and the equation above, there exists a unique pair $(\mathbf{k}^1, \mathbf{l}^1) \in \{0,1\}^N$ such that $\mathbf{k}^1_{[1,N-1]} = \mathfrak{k}_1$ and $\mathbf{l}^1_{[1,N-1]} = \mathfrak{l}_1$ and $\mathcal{L}^N(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = t$. Similarly, there exists only one pair of $(\mathbf{k}^2, \mathbf{l}^2) \in \{0,1\}^N$ such that $\mathbf{k}^2_{[1,N-1]} = \mathfrak{k}_2$, $\mathbf{l}^2_{[1,N-1]} = \mathfrak{l}_2$ and $\mathcal{L}^N(\mathbf{i}, \mathbf{j}; \mathbf{k}^2, \mathbf{l}^2) = 1 - t$. Hence, we have

$$\sum_{\mathbf{k},\mathbf{l} \in \{0,1\}^N} \mathcal{L}^N(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \mathcal{L}^N(\mathbf{i}, \mathbf{j}; \mathbf{k}^1, \mathbf{l}^1) + \mathcal{L}^N(\mathbf{i}, \mathbf{j}; \mathbf{k}^2, \mathbf{l}^2) = 1.$$

By (1.1), $\mathcal{L}^N$ is non-negative, hence it is a stochastic matrix. □

The following lemmas will be used to prove the properties of color ignorance and mod 2 erasure.

Lemma 1.2.3. *Fix positive integers $m \leq n$. Assume that for fixed $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l} \in \{0,1\}^m$ we have $\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathfrak{k}, \mathfrak{l}) \in \{t, 1-t\}$. Then for any fixed $\mathbf{i}, \mathbf{j} \in \{0,1\}^n$ satisfying $(\mathbf{i}_{[1,m]}, \mathbf{j}_{[1,m]}) = (\mathfrak{i}, \mathfrak{j})$, there exist unique $\mathbf{k}^1, \mathbf{l}^1$, $\mathbf{k}^2, \mathbf{l}^2 \in \{0,1\}^n$ satisfying*

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^1, \mathbf{l}^1) = t, \qquad \mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^2, \mathbf{l}^2) = 1 - t.$$

For $(\mathbf{k}, \mathbf{l})$ that does not equal either $(\mathbf{k}^1, \mathbf{l}^1)$ or $(\mathbf{k}^2, \mathbf{l}^2)$, we have $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = 0$.

Proof. When $n = m$, the claim is true due to the stochasticity of L^m. Now we prove the lemma for $m < n$. Using the stochasticity of $\mathcal{L}^m$, we know that there exist $\mathfrak{k}^1, \mathfrak{l}^1; \mathfrak{k}^2, \mathfrak{l}^2 \in \{0,1\}^m$ satisfying $\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathfrak{k}^1, \mathfrak{l}^1) = t$ and $\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathfrak{k}^2, \mathfrak{l}^2) = 1 - t$. Using (1.1) and $\mathbf{i}_{[1,m]} = \mathfrak{i}, \mathbf{j}_{[1,m]} = \mathfrak{j}$, we have

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \min\Big(\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathbf{k}_{[1,m]}, \mathbf{l}_{[1,m]}), \min_{r=m+1}^n \Big(\mathcal{L}^1 1(\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}); \mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l}))\Big)\Big). \quad (1.4)$$

If $(\mathbf{k}_{[1,m]}, \mathbf{l}_{[1,m]}) = (\mathfrak{k}^1, \mathfrak{l}^1)$, we have

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \min\Big(t, \min_{r=m+1}^n \Big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}); \mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l}))\Big)\Big).$$

Therefore, $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = t$ if and only if $\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}); \mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l})) \in \{t, 1\}$ for every $r \in \{m+$

$1, \dots, n\}$. By Lemma 1.2.1, for every $\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}) \in \{0,1\}$, there exist unique $\mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l}) \in \{0,1\}$ such that $\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}); \mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l})) \in \{t, 1\}$. Since $(\mathbf{k}, \mathbf{l})$ is uniquely determined by the value of $(\mathbf{k}_{[1,m]}, \mathbf{l}_{[1,m]})$ and $(\mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l}))_{r=m+1}^n$ (and vice versa), there is a unique pair $(\mathbf{k}^1, \mathbf{l}^1)$ such that $(\mathbf{k}^1_{[1,m]}, \mathbf{l}^1_{[1,m]}) = (\mathfrak{k}^1, \mathfrak{l}^1)$ and $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^1, \mathbf{l}^1) = t$. For $(\mathbf{k}, \mathbf{l})$ satisfying $(\mathbf{k}_{[1,m]}, \mathbf{l}_{[1,m]}) = (\mathfrak{k}^1, \mathfrak{l}^1)$ and $(\mathbf{k}, \mathbf{l}) \neq (\mathbf{k}^1, \mathbf{l}^1)$, we have $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = 0$.

For a similar reason, there exists a unique pair $(\mathbf{k}^2, \mathbf{l}^2)$ such that $(\mathbf{k}^2_{[1,m]}, \mathbf{l}^2_{[1,m]}) = (\mathfrak{k}^2, \mathfrak{l}^2)$ and $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^2, \mathbf{l}^2) = 1 - t$. By the stochasticity of $\mathcal{L}^n$, for $(\mathbf{k}, \mathbf{l}) \notin \{(\mathbf{k}^1, \mathbf{l}^1), (\mathbf{k}^2, \mathbf{l}^2)\}$, we have $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = 0$. This concludes the lemma. □

Let us prepare some notation for the next lemma. We call π a partition of $\{1, \dots, n\}$ if π takes the form of

$$\pi = \big\{\{1, \dots, r_1\}, \{r_1 + 1, \dots, r_2\}, \dots, \{r_{m-1} + 1, \dots, r_m\}\big\}$$

for some $m \leq n$ and $1 = r_1 < r_2 < \cdots < r_m = n$. We define a map g_π from $\{0,1\}^n \to \{0,1\}^m$ such that

$$g_\pi(x_1, \dots, x_n) = \left(\Big(\sum_{i=r_{k-1}+1}^{r_k} x_i \Big) \bmod 2 \right)_{k=1}^m.$$

We define $\ell(\pi) = m$ to be the length of the partition π.

Lemma 1.2.4. *Fix positive integers $m \leq n$. Fix a partition π of $\{1, \dots, n\}$ such that $\ell(\pi) = m$. Assume that for fixed $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l} \in \{0,1\}^m$, we have $\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathfrak{k}, \mathfrak{l}) \in \{t, 1-t\}$. Then for any fixed $\mathbf{i}, \mathbf{j} \in \{0,1\}^n$ satisfying $(g_\pi(\mathbf{i}), g_\pi(\mathbf{j})) = (\mathfrak{i}, \mathfrak{j})$, there exist $\mathbf{k}^1, \mathbf{l}^1, \mathbf{k}^2, \mathbf{l}^2 \in \{0,1\}^n$ satisfying*

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^1, \mathbf{l}^1) = t, \qquad \mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}^2, \mathbf{l}^2) = 1 - t.$$

For $(\mathbf{k}, \mathbf{l})$ that does not equal either $(\mathbf{k}^1, \mathbf{l}^1)$ or $(\mathbf{k}^2, \mathbf{l}^2)$, we have $\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = 0$.

Proof. The lemma is clearly true when $m = n$. We only need to prove it for $m < n$.

By stochasticity of $\mathcal{L}^m$, there exists $\mathfrak{k}^1, \mathfrak{l}^1, \mathfrak{k}^2, \mathfrak{l}^2 \in \{0,1\}^m$ satisfying $\mathcal{L}^m(\mathfrak{i}, \mathfrak{j}, \mathfrak{k}^1, \mathfrak{l}^1) = t$ and

$\mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k}^2,\mathfrak{l}^2) = 1-t$. We let

$$\pi = \big\{\{1,\dots,r_1\},\{r_1+1,\dots,r_2\},\dots,\{r_{m-1}+1,\dots,r_m\}\big\}$$

where $r_m = n$. Let $A_\pi = \{r_1,\dots,r_m\}$ and $B_\pi = \{1,\dots,n\}\backslash A_\pi$. Using (1.1), we have

$$\begin{aligned}\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) &= \min_{r=1}^n\big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\big)\\ &= \min\Big(\min_{r\in A_\pi}\big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\big),\min_{r\in B_\pi}\big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\big)\Big)\\ &= \min\Big(\mathcal{L}^m(g_\pi(\mathbf{i}),g_\pi(\mathbf{j});g_\pi(\mathbf{k}),g_\pi(\mathbf{l})),\min_{r\in B_\pi}\big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\big)\Big).\end{aligned}$$

Note that $(g_\pi(\mathbf{i}),g_\pi(\mathbf{j})) = (\mathfrak{i},\mathfrak{j})$. If $(g_\pi(\mathbf{k}),g_\pi(\mathbf{l})) = (\mathfrak{k}^1,\mathfrak{l}^1)$, then

$$\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \min\Big(t,\min_{r\in B_\pi}\big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\big)\Big).$$

In order for $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = t$, we need $\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l})) \in \{t,1\}$ for each $r \in B_\pi$; otherwise, $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Note that $\mathbf{i},\mathbf{j}$ are fixed. By Lemma 1.2.1, for each $r \in B_\pi$, there is only one choice for $(\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))$ such that $\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l})) \in \{t,1\}$. It is straightforward that we have a bijection from $\{0,1\}^n$ to itself, given by $\mathbf{x} \leftrightarrow (g_\pi(\mathbf{x}),(\mathfrak{s}_r(\mathbf{x}))_{r\in B_\pi})$. Hence, there exist unique $\mathbf{k}^1,\mathbf{l}^1$ satisfying $(g_\pi(\mathbf{k}^1),g_\pi(\mathbf{l}^1)) = (\mathfrak{k}^1,\mathfrak{l}^1)$ and $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) = t$.

Similarly, there exist unique $\mathbf{k}^2,\mathbf{l}^2$ such that $(g_\pi(\mathbf{k}^2),g_\pi(\mathbf{l}^2)) = (\mathfrak{k}^2,\mathfrak{l}^2)$ and $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^2,\mathbf{l}^2) = 1-t$. By stochasticity, for $(\mathbf{k},\mathbf{l}) \notin \{(\mathbf{k}^1,\mathbf{l}^1),(\mathbf{k}^2,\mathbf{l}^2)\}$, we have $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. This concludes the lemma. □

Proposition 1.2.5 (Color Ignorance). *Fix $m \in \{1,\dots,n\}$ and $\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l} \in \{0,1\}^m$. For all $\mathbf{i},\mathbf{j} \in \{0,1\}^n$ such that $\mathbf{i}_{[1,m]} = \mathfrak{i}$ and $\mathbf{j}_{[1,m]} = \mathfrak{j}$, we have*

$$\sum_{\substack{\mathbf{k}_{[1,m]}=\mathfrak{k},\\ \mathbf{l}_{[1,m]}=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}). \tag{1.5}$$

Proof. We have $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) \in \{0, t, 1-t, 1\}$. We prove the equality (1.5) for each possible value of $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l})$.

Case 1: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) = 0$. By (1.4), when $\mathbf{i}_{[1,m]} = \mathfrak{i}$, $\mathbf{j}_{[1,m]} = \mathfrak{j}$, $\mathbf{k}_{[1,m]} = \mathfrak{k}$, $\mathbf{l}_{[1,m]} = \mathfrak{l}$, we have $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Hence, (1.5) holds.

Case 2: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) = 1$. By stochasticity of $\mathcal{L}^m$ and (1.4), if we have $\mathbf{i}_{[1,m]} = \mathfrak{i}$, $\mathbf{j}_{[1,m]} = \mathfrak{j}$ and $(\mathbf{k}_{[1,m]}, \mathbf{l}_{[1,m]}) \neq (\mathfrak{k},\mathfrak{l})$, we have $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Using this and the stochasticity of $\mathcal{L}^n$,

$$\sum_{\substack{\mathbf{k}_{[1,m]}=\mathfrak{k}\\ \mathbf{l}_{[1,m]}=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \sum_{\mathbf{k},\mathbf{l}\in\{0,1\}^n} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 1 = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

Hence, (1.5) holds.

Case 3: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) = t$. Since $\mathbf{i}_{[1,m]} = \mathfrak{i}$ and $\mathbf{j}_{[1,m]} = \mathfrak{j}$, by Lemma 1.2.3, we know that there exist unique $\mathbf{k}^1, \mathbf{l}^1 \in \{0,1\}^n$ satisfying $\mathbf{k}^1_{[1,m]} = \mathfrak{k}$, $\mathbf{l}^1_{[1,m]} = \mathfrak{l}$ and $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) = t$. For all $(\mathbf{k},\mathbf{l}) \in \{0,1\}^n$ satisfying $\mathbf{k}_{[1,m]} = \mathfrak{k}$, $\mathbf{l}_{[1,m]} = \mathfrak{k}$ and $(\mathbf{k},\mathbf{l}) \neq (\mathbf{k}^1,\mathbf{l}^1)$, $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Hence,

$$\sum_{\substack{\mathbf{k}_{[1,m]}=\mathfrak{k}\\ \mathbf{l}_{[1,m]}=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) = t = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

Case 4: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) = 1-t$. The proof is similar to Case 3, and we omit it. This concludes the proof of the proposition. □

Proposition 1.2.6 (Mod 2 Erasure)**.** *Fix a partition π of $\{1,\dots,n\}$ such that $\ell(\pi) = m$. Fix $\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l} \in \{0,1\}^m$. For all $\mathbf{i},\mathbf{j} \in \{0,1\}^n$ satisfying $g_\pi(\mathbf{i}) = \mathfrak{i}$ and $g_\pi(\mathbf{j}) = \mathfrak{j}$, we have*

$$\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}). \tag{1.6}$$

Proof. Let $\pi = \big\{\{1,\dots,r_1\},\{r_1+1,\dots,r_2\},\dots,\{r_{m-1}+1,\dots,r_m\}\big\}$, where $1 \le r_1 < \cdots < r_m = n$. We again divide our proof into four cases.

Case 1: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = 0$. By (1.1), for $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}$ satisfying $g_\pi(\mathbf{i}) = \mathfrak{i}, g_\pi(\mathbf{j}) = \mathfrak{j}, g_\pi(\mathbf{k}) = \mathfrak{k}, g_\pi(\mathbf{l}) = \mathfrak{l}$, we have

$$\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) \leq \min_{i=1}^m \big(\mathcal{L}^1(\mathfrak{s}_{r_i}(\mathbf{i}),\mathfrak{s}_{r_i}(\mathbf{j});\mathcal{L}^1(\mathfrak{s}_{r_i}(\mathbf{k}),\mathfrak{s}_{r_i}(\mathbf{l}))\big) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = 0.$$

This implies that $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Hence,

$$\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0 = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

Case 2: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = 1$. By (1.1), we know that $(g_\pi(\mathbf{k}), g_\pi(\mathbf{l})) \neq (\mathfrak{k},\mathfrak{l})$ implies that $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 0$. Therefore,

$$\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k},\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \sum_{\mathbf{k},\mathbf{l}\in\{0,1\}^n} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = 1 = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

The second equality is due to the stochasticity of $\mathcal{L}^n$.

Case 3: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = t$. We fix $\mathbf{i},\mathbf{j}$ that satisfy $(g_\pi(\mathbf{i}), g_\pi(\mathbf{j})) = (\mathfrak{i},\mathfrak{j})$. By Lemma 1.2.4, there exist unique $\mathbf{k}^1,\mathbf{l}^1$ satisfying $(g_\pi(\mathbf{k}^1), g_\pi(\mathbf{l}^1)) = (\mathfrak{k},\mathfrak{l})$ and $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) \neq 0$. Moreover, $\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) = t$. Hence,

$$\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k},\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k}^1,\mathbf{l}^1) = t = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

Case 4: $\mathcal{L}^m(\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l}) = 1 - t$. The proof is similar to Case 3. □

Remark 1.2.7. *One can also use real numbers to label colors, not just positive integers. The rule is that for two colors with labels $a < b$, the color b has less priority than a. Hence, we can still sample the output of an intersection given finite input lines labeled by real numbers, using $\{\mathcal{L}^n : n \geq 1\}$. For our application of the colored model in the next section, we use negative integers to label the*

colors (see also Remark 1.3.3).

1.3 The colored model and superadditivity

We are going to construct $\{X_{m,n} : 0 \le m \le n\}$ as discussed in the introduction using the colored t-PNG model. Before doing that, let us recall Liggett's superadditive ergodic theorem from [111]. Note that the theorem was originally stated in the subadditive setting, but for our purposes, we formulate it in the superadditive setting by placing a negative sign where necessary.

Theorem 1.3.1. *Suppose $\{X_{m,n}\}$ is a collection of random variables that is indexed by integers $0 \le m \le n$ and satisfies:*

(i) Almost surely $X_{0,0} = 0$ and $X_{0,n} \ge X_{0,m} + X_{m,n}$ for $0 \le m \le n$.

(ii) For each $k \ge 1$, $\{X_{(n-1)k,nk} : n \ge 1\}$ is an ergodic process.

(iii) $\{X_{m,m+k} : k \ge 0\} \stackrel{d}{=} \{X_{m+1,m+k+1} : k \ge 0\}$ for each $m \ge 0$.

(iv) $\mathbb{E}[X_{0,1}^-] < \infty$ where $x^- = \max(-x, 0)$.

Then there exists a constant $\gamma = \sup_{n \ge 1} \frac{\mathbb{E}[X_{0,n}]}{n} \in (-\infty, \infty]$ satisfying

$$\gamma = \lim_{n \to \infty} \frac{X_{0,n}}{n} \text{ a.s.}$$

Definition 1.3.2 (Step colored t-PNG model)**.** *We consider a Poisson point process of nucleations on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ with density 1. We can assume that there are no nucleations with an integer x- or y-coordinate and that no two nucleations have the same x- or y-coordinates, since these events have probability zero. Fix arbitrary integers $m, n \in \mathbb{Z}_{\ge 0}$. We color the nucleations inside the unit square $[m, m+1] \times [n, n+1]$ with the color $-\min(m+1, n+1)$. In other words, we assign the color $-m-1$ to nucleations lying in the L-shape area $[m, \infty) \times [m, m+1] \cup [m, m+1] \times [m, \infty)$. For each nucleation with a given color k, the lines emanating from it in the upward and rightward directions also have color k. When horizontal and vertical lines intersect, the output lines emanate*

from the intersection following the stochastic matrix $\{\mathcal{L}^n : n \geq 1\}$ *defined in Definition 1.1.7 (see also Remark 1.2.7). The model is referred to as the* step colored t-PNG model. *See Figure 1.13.*

Remark 1.3.3. *The reason that we label the colors with negative integers instead of positive integers as in Section 1.2 is that to apply Theorem 1.3.1, we want to construct a model with infinite colors such that nucleations closer to the axes have lower priority.*

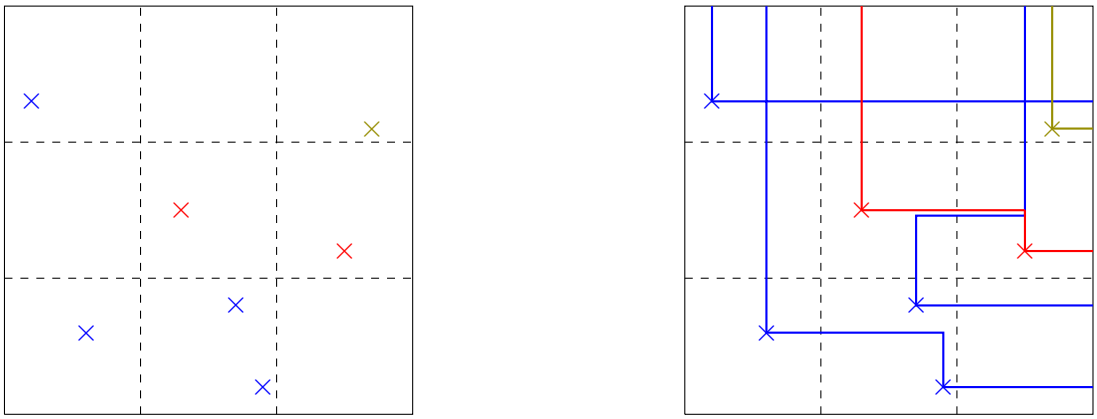


Figure 1.13: Left panel: We sample a Poisson point process with density 1 and assign the nucleations different colors using the aforementioned rules. Although this happens on the entire first quadrant, we just show a picture of the square $[0,3] \times [0,3]$. Right panel: We sample the step colored t-PNG model in $[0,3] \times [0,3]$ using these Poisson nucleations. Note that olive has a higher priority than red, and red has a higher priority than blue. Hence, the behavior of the olive lines does not depend on that of the red and blue lines. The behavior of the red lines does not depend on that of the blue lines.

We proceed to define the random variables $\{X_{m,n}, m, n \in \mathbb{Z}_{\geq 0}, m \leq n\}$. Let $H_{m,n}$ be the set of intersection points between the horizontal segment $[m,n] \times \{n\}$ and vertical lines in the step colored t-PNG model, where in the case of multiple lines traveling together, we count the intersection point just once. Let $v_z^{[m,n]}$ denote the number of vertical lines that go through z with colors in $\{-n, \ldots, -m-1\}$. We define

$$X_{m,n} = \sum_{z \in H_{m,n}} \left(v_z^{[m,n]} \bmod 2\right). \tag{1.7}$$

In other words, $X_{m,n}$ is the number of vertical lines with color $-n$ that cross the segment $[m,n] \times \{n\}$ after we recolor all lines with colors in $\{-n, \ldots, -m-1\}$ with the color $-n$ and apply the

mod 2 erasure procedure.

Proposition 1.3.4. *We have* $\{X_{0,k}, k \in \mathbb{Z}_{\geq 1}\} = \{N(k,k), k \in \mathbb{Z}_{\geq 1}\}$ *in distribution.*

Proof. It suffices to prove that for each $n \in \mathbb{N}$, $\{X_{0,k}, k = 1 \ldots, n\} \stackrel{d}{=} \{N(k,k), k = 1, \ldots, n\}$. The lines in the square $[0,n] \times [0,n]$ have colors that belong to $\{-n, \ldots, -1\}$. Replace these colors with a single color and apply the mod 2 erasure procedure. Taking $m = 1$ in Proposition 1.2.6, we see that the resulting model is just the single-colored t-PNG model. Note that $X_{0,k}$ in (1.7) is exactly the number of lines crossing the segment $[0,k] \times \{k\}$, which equals $N(k,k)$ for $k \in \{1, \ldots, n\}$. This concludes the proposition. □

Proposition 1.3.5. *The stochastic process* $\{X_{m,n} : m, n \in \mathbb{Z}_{\geq 0}, m \leq n\}$ *satisfies conditions (ii)-(iv) of Theorem 1.3.1.*

Proof. We first prove (ii). Consider the square $[m,n] \times [m,n]$. There are lines flowing inside through the left boundary $\{m\} \times [m,n]$ and the bottom boundary $[m,n] \times \{m\}$. These lines have colors that belong to $\{-m, \ldots, -1\}$. The Poisson nucleations in $[m,n] \times [m,n]$ have colors in $\{-n, \ldots, -m-1\}$. Note that the color i takes priority over j if $i < j$, so the lines that emanate from the Poisson nucleations in $[m,n] \times [m,n]$ have higher priority than the lines entering through the left and bottom boundaries. Therefore, by Proposition 1.2.5, we can sample the colored t-PNG model in the square $[m,n] \times [m,n]$ just using the colors $\{-n, \ldots, -m-1\}$ and ignore the lower priority lines entering from the left and bottom. Hence, the distribution of $X_{m,n}$ is independent of the number and location of the lines entering the bottom and left boundaries of $[m,n] \times [m,n]$. This implies that for each $k \geq 1$, the random variables $\{X_{(n-1)k,nk}, n \geq 1\}$ are independent. It is straightforward to see that $X_{(n-1)k,nk}$ has the same distribution as $N(k,k)$ for all $n \geq 1$, so therefore this sequence is i.i.d and hence ergodic.

We proceed to prove (iii). It suffices to show that for arbitrary $m \in \mathbb{Z}_{\geq 0}$,

$$\{X_{m,m+k}, k \geq 0\} \stackrel{d}{=} \{X_{0,k}, k \geq 0\}. \tag{1.8}$$

We look at the step colored t-PNG model restricted to $[m,\infty) \times [m,\infty)$. Note that there are lines

with colors in $\{-m, \dots, -1\}$ entering from the left and bottom boundaries of $[m, \infty) \times [m, \infty)$. By Proposition 1.2.5, the behavior of lines in $[m, \infty) \times [m, \infty)$ with colors less than $-m$ is unaffected by the lower priority lines entering from the boundary. This implies that after a diagonal shift by (m, m), the lines with colors $i_1, \dots, i_k \in \mathbb{Z}_{\leq -m-1}$ in $[m, \infty) \times [m, \infty)$ behave the same (in distribution) as the lines with colors $i_1 + m, \dots, i_k + m$ in $[0, \infty) \times [0, \infty)$. Hence, we conclude (1.8).

Finally, since $X_{0,1}$ is non-negative, $X_{0,1}^- = 0$. Hence, (iv) holds. □

Let us proceed to prove that $\{X_{m,n}, m, n \in \mathbb{Z}_{\geq 0}, m \leq n\}$ also satisfies the superadditive condition (i) in Theorem 1.3.1. We begin with some preparation. In the square $[0, n] \times [0, n]$, we replace the colors $\{-m, \dots, -1\}$ with the color -1 and replace the colors $\{-n, \dots, -m-1\}$ with the color -2. After that, as long as there are two lines with the same color that travel together, we erase them. By Proposition 1.2.6, the resulting model is a two-colored t-PNG model. In particular, the Poisson nucleations have color -1 in the L-shaped area $[0, m] \times [0, n] \cup [0, n] \times [0, m]$. The nucleation points have color -2 in the square $[m, n] \times [m, n]$.

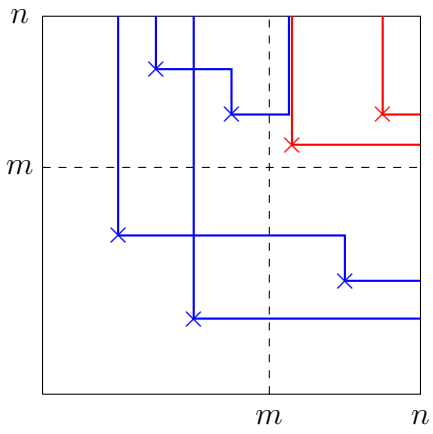


Figure 1.14: We provide a possible sampling of the two-colored t-PNG model in the square $[0, m+n] \times [0, m+n]$. The dashed lines $x = m$ and $y = m$ divide the square $[0, n] \times [0, n]$ into four rectangles. Blue represents the color -1, and red represents the color -2. In this example, we have $Q_1 = 2$ and $Q_2 = Q_{1,2} = P_1 = 1$.

For the resulting two-colored t-PNG model, let Q_1 be the number of be lines with color -1 that cross $[0, m] \times \{m\}$, let Q_2 be the number of vertical lines with color -2 that cross the segment

$[m, n] \times \{n\}$, and let P_1 be the number of horizontal lines with color -1 that cross $\{m\} \times [m, n]$. Finally, let $Q_{1,2}$ be the number of pairs of vertical lines of colors -1 and -2 that travel together and cross $[m, n] \times \{n\}$.

Consider the single-colored t-PNG model. For each Poisson nucleation or intersection point, the number of lines going upward or leftward equals the number of lines going downward or rightward (see Figure 1.3). The next lemma follows immediately.

Lemma 1.3.6. *Consider the (single-colored) t-PNG model. Fix an arbitrary rectangle. The number of lines that cross the top and left boundaries of the rectangle equals the number of lines that cross the bottom and right boundaries.*

Lemma 1.3.7. *The following result holds:*

$$X_{0,m} = Q_1, \tag{1.9}$$

$$X_{m,n} = Q_2, \tag{1.10}$$

$$X_{0,n} \geq Q_1 + P_1 + Q_2 - Q_{1,2}. \tag{1.11}$$

Proof. Recall that we obtain the two-colored t-PNG model in $[0, n] \times [0, n]$ by replacing the colors $\{-n, \ldots, -m-1\}$ with the color -2, replacing the colors $\{-m, \ldots, -1\}$ with the color -1 and erasing pairs of lines with the same color. The erasure corresponds to the mod 2 erasure procedure in (1.7). Hence, equations (1.9) and (1.10) directly follow from (1.7).

We proceed to prove (1.11). Note that $X_{0,n}$ is the number of single vertical lines (i.e. the line does not travel in a pair) that cross the segment $[0, n] \times \{n\}$ in the two-colored t-PNG model. We decompose

$$X_{0,n} = Y_1 + Y_2, \tag{1.12}$$

where Y_1 equals the number of vertical lines with the color -1 that cross $[0, m] \times \{n\}$ and Y_2 equals the number of single vertical lines of either color that cross $(m, n] \times \{n\}$, excluding pairs. Note that in the rectangle $[0, m] \times [m, n]$, there are only lines with color -1. Applying Lemma 1.3.6 to

the rectangle $[0, m] \times [m, n]$, we have $Y_1 = Q_1 + P_1$. By definition, $Y_2 \geq Q_2 - Q_{1,2}$. Using this together with (1.12), we conclude (1.11). □

Proposition 1.3.8. *We have* $X_{0,0} = 0$ *and* $X_{0,n} \geq X_{0,m} + X_{m,n}$ *for* $0 \leq m \leq n$. *Hence,* $\{X_{m,n}, m \leq n \in \mathbb{Z}_{\geq 0}\}$ *satisfies (i) of Theorem 1.3.1.*

Proof. By definition, we have $X_{0,0} = 0$. We proceed to show that $X_{0,n} \geq X_{0,m} + X_{m,n}$. By Lemma 1.3.7, it suffices to show that $P_1 \geq Q_{1,2}$. We restrict ourselves to the square $[m, n] \times [m, n]$. All lines entering this square from the bottom and left boundaries have color -1, and all Poisson nucleations inside the square have color -2. P_1 equals the number of color -1 lines that enter the left boundary. We can think of $Q_{1,2}$ as the number of color -2 lines that cross $[m, n] \times \{n\}$ and are erased by a color -1 line. By looking at the possible two color configurations in Section 1.6, we find that each color -1 line which erases a color -2 line and crosses $[m, n] \times \{n\}$ must enter $[m, n] \times [m, n]$ from the left boundary. This implies that $P_1 \geq Q_{1,2}$ and concludes the proposition. □

The following scaling property follows immediately from the corresponding scaling property of the Poisson nucleations.

Lemma 1.3.9 (Scaling)**.** *For a fixed* $0 < s < \infty$*, we have*

$$(N(x, y); x, y \geq 0) \stackrel{d}{=} (N(sx, y/s); x, y \geq 0).$$

The following proposition partially proves Theorem 1.1.1.

Proposition 1.3.10. *Let* $\gamma = \sup_{n \geq 1} \frac{\mathbb{E}[N(n,n)]}{n}$. *With probability* 1*, we have*

$$\lim_{s \to \infty} \frac{N(sx, sy)}{s} = \gamma \sqrt{xy}$$

for any $x, y > 0$.

Proof. We first prove that almost surely, $\frac{N(s,s)}{s} \to \gamma$. Using Proposition 1.3.5 and Proposition 1.3.8, we can now apply Theorem 1.3.1 to conclude that almost surely $\frac{X_{0,n}}{n} \to \gamma$ as $n \to \infty$, where $\gamma = \sup_{n \geq 1} \frac{\mathbb{E}[X_{0,n}]}{n}$. The convergence also holds in L^1 if γ is finite. Using Proposition 1.3.4, we have that $\lim_{n\to\infty} \frac{N(n,n)}{n} = \gamma$ almost surely. Note that since $N(s, s)$ is increasing in s, we also have that almost surely,

$$\lim_{t\to\infty} \frac{N(s, s)}{s} = \gamma.$$

This together with Lemma 1.3.9 implies that almost surely $\lim_{s\to\infty} \frac{N(sx,sy)}{s} = \gamma\sqrt{xy}$ for arbitrary fixed $x, y > 0$. We take a probability 1 event such that $\lim_{s\to\infty} \frac{N(sx,sy)}{s} = \gamma\sqrt{xy}$ for $x, y \in \mathbb{Q}_{>0}$. By the density of $\mathbb{Q}_{>0}$ in $\mathbb{R}_{>0}$ and the monotonicity of the height function, we know that on that event, $\lim_{s\to\infty} \frac{N(sx,sy)}{s} = \gamma\sqrt{xy}$ for all $x, y \in \mathbb{R}_{>0}$. □

To complete the proof of Theorem 1.1.1, it remains to identify the constant γ. This will be done in Section 1.5. Before carrying out the proof, we need to first show that γ is neither zero nor infinity. The fact that γ is non-zero can be easily seen by the following lemma.

Lemma 1.3.11. *We have* $\gamma \geq 2$.

Proof. Let $N_0(x, y)$ be the height function of the PNG model (where $t = 0$). We can couple together the t-PNG model and the PNG model so that they have the same Poisson nucleations. Under this coupling, we always have $N(x, y) \geq N_0(x, y)$. Using this together with the law of large numbers for the PNG model, we conclude that $\gamma \geq 2$. □

1.4 Stationary model and upper bound

In this section, we prove that γ is finite by constructing a stationary version of the t-PNG model and comparing it with the original t-PNG model. We remark that the existence of a stationary version of the t-PNG model is not surprising, since similar stationary models have been observed for the last passage models [27, 122, 104], polymers [105, 106, 107], and stochastic vertex models [108, 109, 110]. The results in this section can be viewed as generalizations of the results in Section 1.3 of [27].

Fix $\lambda, T_1, T_2 > 0$. Consider the t-PNG model on $[0, T_1] \times [0, T_2]$ with the following boundary data: a Poisson point process of sources of intensity λ on the bottom boundary and a Poisson point process of sinks of intensity $\frac{1}{\lambda(1-t)}$ on the left boundary. In order to study the stationarity of this model, it is helpful to understand the model as an interacting particle system.

Given the t-PNG model, we can naturally obtain an interacting particle system called the *t-Hammersley process* as follows: Let $(X_\tau)_{0 \le \tau \le T_2}$ be the configuration of particle locations in $[0, T_1]$ at time τ. To avoid ambiguity, we let X_τ be right continuous. The particle locations in this interpretation are the locations x such that (x, τ) belongs to a vertical line segment in the t-PNG model. More precisely, X_τ is a Markov chain on the state space E consisting of all finite point configurations on $[0, T_1]$. We can decompose $E = \bigsqcup_{n=0}^{\infty} E_n$ where each E_n consists of particle configurations with exactly n points:

$$E_n = \{(x_1, \ldots, x_n) : 0 \le x_1 \le \ldots \le x_n \le T_1\} \quad \text{when } (n \ge 1),$$

and $E_0 = \{\emptyset\}$, where $\emptyset$ is the empty configuration. We give each set E_n the usual topology so that E is a locally compact space. The reason that we allow multiple points at the same location is purely technical; we want E to be a Polish space if we identify the point configurations with Radon measures.

Due to our choice of boundary data, X_0 is a Poisson process with intensity λ. Using the infinitesimal generator of X, we will prove that the Markov process X is stationary, meaning that the point configuration X_τ will remain a Poisson point process with intensity λ for all $\tau \in [0, T_2]$.

We define the infinitesimal generator of X. Let us first define two families of operators: $(\mathcal{R}_z^i)_{i=1}^{\infty}$ and $(\mathcal{L}^i)_{i=1}^{\infty}$. For each $z \in (0, T_1)$ and $i \ge 1$ we can define the operator $\mathcal{R}_z^i : E \to E$ such that for $x_{m-1} < z \le x_m$ (take $x_0 = 0$ and $x_{n+1} = \infty$),

$$\mathcal{R}_z^i x = \begin{cases} (x_1, \ldots, x_{m-1}, z, x_m, \ldots, \widehat{x}_{m+i-1}, \ldots, x_n), & \text{if } x \in E_n, i \le n - m + 1, \\ (x_1, \ldots, x_{m-1}, z, x_m, \ldots, x_n), & \text{if } x \in E_n, i > n - m + 1, \end{cases} \tag{1.13}$$

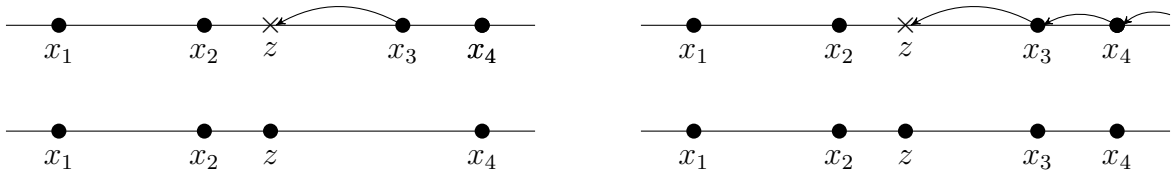


Figure 1.15: Example of $\mathcal{R}_z^i x$ with $n = 4$ and $m = 3$. On the left we take $i = 1$ and on the right we take $i > 2$. Note that for $i \leq n - m + 1 = 2$, the number of particles is preserved and for $i > 2$, the number of particles increases by 1.

where $\widehat{x}_k$ denotes that the particle at x_k is removed from the configuration. The operator $\mathcal{R}_z^i$ describes what happens when there is a nucleation at position z. It inserts a particle at z and removes the particle whose position is i positions to the right of z if there are at least i particles to the right of z. We can also think of $\mathcal{R}_z^i$ as sequentially shifting over the i particles at positions $x_m, \ldots, x_{m+i-1}$ to positions $z, x_m, \ldots, x_{m+i-2}$. If there are not i particles to shift over, then we shift over $x_m, \ldots, x_n$ and insert a new particle "from infinity" at x_n, as illustrated in Figure 1.15. The advantage of this viewpoint is that it maintains the ordering between the particles.

We also define $\mathcal{L}^i : E \to E$ such that

$$\mathcal{L}^i x = \begin{cases} (x_1, \ldots, \widehat{x_i}, \ldots, x_n), & \text{if } x \in E_n, i \leq n, \\ (x_1, \ldots, x_n), & \text{if } x \in E_n, i > n. \end{cases} \tag{1.14}$$

The operator $\mathcal{L}^i$ describes what happens when there is a sink on the left boundary. It removes the ith particle if there is one, and if not, does nothing.

Proposition 1.4.1. *Let X_τ be the t-Hammersley process for $t \in [0, 1)$ with Poisson sources of intensity λ and Poisson sinks of intensity $\frac{1}{\lambda(1-t)}$. Its generator G is given by the following formula when acting on $f \in C_0(E)$:*

$$Gf(x) = \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_0^{T_1} (f(\mathcal{R}_z^i x) - f(x))dz + \sum_{i=1}^{\infty} \frac{t^{i-1}}{\lambda} (f(\mathcal{L}^i x) - f(x)). \tag{1.15}$$

Proof. The first term on the right comes from moving the particle configuration from x to $\mathcal{R}_z^i x$, where i is sampled from a geometric distribution and z is chosen according to the uniform measure

on $[0, T_1]$. The second term comes from moving the particle configuration from x to $\mathcal{L}^i x$ for some i (i is again sampled from a geometric distribution) with rate $\frac{1}{\lambda(1-t)}$. A rigorous proof follows from a direct computation as in [27]. □

Let G^* be the dual of G with respect to μ. In other words, the operator satisfying

$$\int_E Gf(x)g(x)\mu(dx) = \int_E f(y)G^*g(y)\mu(dy) \text{ for all } f, g \in C_0(E). \tag{1.16}$$

To compute G^*, we will need to define two additional sets of operators. These operators are similar to the ones above except that they move particles from left to right instead of right to left. For each $s \in (0, T_1)$ and $i \geq 1$ we can define the operator $\mathcal{L}_s^i : E \to E$ such that for $x_{m-1} < s \leq x_m$,

$$\mathcal{L}_s^i x = \begin{cases} (x_1, \ldots, \widehat{x}_{m-i}, \ldots, x_{m-1}, s, x_m, \ldots, x_n), & \text{if } x \in E_n, i \leq m-1, \\ (x_1, \ldots, x_{m-1}, s, x_m, \ldots, x_n), & \text{if } x \in E_n, i > m-1. \end{cases} \tag{1.17}$$

This operator inserts a particle at position s and removes the particle whose position is i positions to the left of s if there are at least i particles to the left of s.

We also define $\mathcal{R}^i : E \to E$ such that

$$\mathcal{R}^i x = \begin{cases} (x_1, \ldots, \widehat{x}_{n-i+1}, \ldots, x_n), & \text{if } x \in E_n, i \leq n, \\ (x_1, \ldots, x_n), & \text{if } x \in E_n, i > n. \end{cases} \tag{1.18}$$

This operator removes the $(n-i+1)$th particle if there is one, and if not, does nothing.

Lemma 1.4.2. *For all $g \in C_0(E)$,*

$$G^*g(y) = \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_0^{T_1} (g(\mathcal{L}_s^i y) - g(x))ds + \sum_{i=1}^{\infty} \frac{t^{i-1}}{\lambda}(g(\mathcal{R}^i y) - g(y)). \tag{1.19}$$

Proof. The proof is similar to [27]; however, since the dynamics of the t-Hammersley process are

more intricate for $t > 0$, the computations are more complicated. We record the details in Lemma 1.7.1 in Section 1.7. □

We can see that $G^*1 = 0$. This proves that μ is stationary for the Markov process X. Using the particle system interpretation, we now prove Burke's theorem for the t-PNG model.

Proof of Theorem 1.1.8. The argument intrinsically follows Theorem 3.1 in [27]. We define the time-reversed Markov process $(\widetilde{X}_\tau)_{0\leq\tau\leq T_2}$ by

$$\widetilde{X}_\tau = \lim_{\tau'\downarrow\tau} X_{T_2-\tau'}.$$

We define this process using left limits to make sure that it is càdlàg. Since G^* and G are dual with respect to the stationary measure μ, it is standard that G^* is in fact the generator of the time-reversed process $\widetilde{X}_s$.

However, by a similar argument to Proposition 1.4.1, we see that G^* is also the generator of the process X_V obtained by a vertical reflection of all the space-time paths of L_λ across the vertical line segment $\{\frac{1}{2}T_1\} \times [0, T_2]$. Particles in X jump from right to left, while in X_V we reverse the direction and particles jump from left to right.

The time-reversed process $\widetilde{X}$ can be obtained from the original space-time paths of L_λ by performing a horizontal reflection across the horizontal line segment $[0, T_1] \times \{\frac{1}{2}T_2\}$. Accordingly, we can rename this process $X_H = \widetilde{X}$. The two reflected processes X_V and X_H share the same generator G^*. In addition, both processes start with Poisson initial data with intensity λ, so we can conclude that the two processes are equal in distribution.

In X_V, particles are inserted at the vertical reflection of the Poisson nucleations of the original process. In X_H, particles are inserted at the horizontal reflection of the corner points of the original process (see Figure 1.16). These two sets must be equal in distribution and since the Poisson process is invariant under reflections, this shows that the distribution of the corner points of L_λ is also a Poisson point process with intensity 1.

In the process X_V, paths exit on the right side according to a Poisson process with intensity

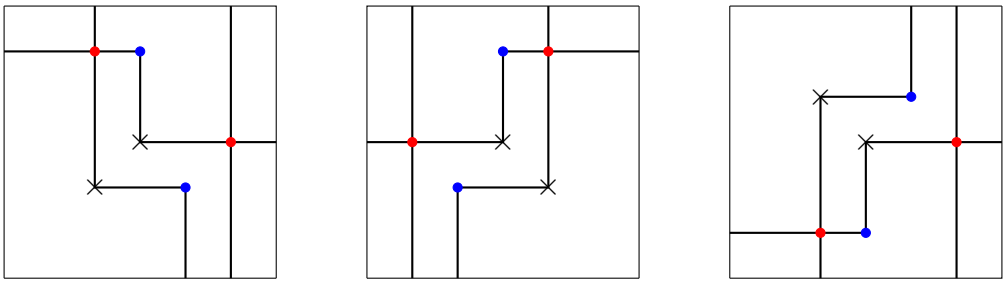


Figure 1.16: From left to right these are the processes X, X_V and $X_H = \widetilde{X}$. The latter two figures are obtained from the first by a vertical and horizontal reflection, respectively. The second and third pictures are equal in distribution, hence the blue circles (corner points of X) have the same distribution as the $\times$'s (nucleations of X).

$\frac{1}{\lambda(1-t)}$. In X_H, paths exit on the right side according to the horizontal reflection of L_λ^{in}. Hence L_λ^{in} is a Poisson process with intensity $\frac{1}{\lambda(1-t)}$. Similarly, for X_V, paths enter from the bottom with the vertical reflection of a Poisson process of intensity λ and this is equal in distribution to the paths entering X_H, but those are just L_λ^{out}. the independence of L_λ^{in}, L_λ^{out} and $L_\lambda^{\text{corner}}$ follows from the fact that these three processes are independent for X_V since they are equal to the sources, sinks and nucleations of L_λ. □

Using the stationarity of X, we can now compute the upper bound for γ. Note that we only use stationarity and do not need Burke's theorem. We will compare the t-PNG model with empty boundary data to the stationary model L_λ defined above. To this end, we need the following lemma showing that the t-PNG model is attractive as an interacting particle system (see [1]). It follows that the height function of the t-PNG model is monotone under adding sources and sinks.

Lemma 1.4.3 (Attractivity)**.** *We have*

(i) Let η_τ and ζ_τ denote two t-Hammersley processes on the rectangle $[0, T_1] \times [0, T_2]$ with sinks given by ξ and sources given by $\eta_0 \subseteq \zeta_0$. Then there exists a coupling of the two processes η_τ and ζ_τ such that $\eta_\tau \subseteq \zeta_\tau$ for all $\tau \in [0, T_2]$.

(ii) Let η_τ and $\tilde{\eta}_\tau$ denote two t-Hammersley processes on the rectangle $[0, T_1] \times [0, T_2]$ with sinks given by $\xi \subseteq \widetilde{\xi}$, respectively, and sources given by η_0. Then there exists a coupling of the

t-PNG processes η_τ and $\widetilde{\eta}_\tau$ such that for all $\tau \in [0, T_2]$, the number of particles in $\tilde{\eta}_\tau$ must be at least the number of particles in η_τ minus the number of sinks in $\tilde{\xi} \setminus \xi$ added up until time τ.

(iii) We conclude that if we add either sources or sinks to the t-PNG model then there is a coupling so that the height function weakly increases.

Proof. (i) We color all the sources in η_0 with color 1 and color the sources in $\zeta_0 \setminus \eta_0$ with color 2. We sample η_τ by ignoring the second color and sample ζ_τ by performing the mod 2 erasure procedure on both colors. Note that the sources of color 2 can only erase lines in the horizontal direction and therefore cannot erase any vertical lines which represent particles in η_τ (see the possible two-colored configurations in Section 1.6). Therefore $\eta_\tau \subseteq \zeta_\tau$.

(ii) Similarly, we color all the sinks in ξ with color 1 and color all the sinks in $\tilde{\xi} \setminus \xi$ with color 2. We sample η_τ by ignoring the second color and sample $\tilde{\eta}_\tau$ by performing the mod 2 erasure procedure on both colors. Each sink of color 2 can delete at most one vertical line, hence the number of particles decreases by at most the number of sinks in $\tilde{\xi} \setminus \xi$ added up until time τ.

(iii) Note that

$$N(x, y) = \text{number of sinks in } \{0\} \times [0, y] + \text{number of particles in } [0, x] \times \{y\}.$$

It follows from (i) and (ii) that the height function does not decrease if we add sources and sinks using the above couplings.

□

Lemma 1.4.4 (Upper Bound)**.** *We have*

$$\gamma \leq \frac{2}{\sqrt{1-t}}. \tag{1.20}$$

Proof. Consider the t-PNG model without boundary data as well as the stationary t-PNG model

L_λ defined above. Let N_λ^{stat} be the height function of the stationary model. We have

$$\mathbb{E}[N(x,y)] \le \mathbb{E}[N_\lambda^{\text{stat}}(x,y)] = \mathbb{E}[N_\lambda^{\text{stat}}(0,y)] + \mathbb{E}[N_\lambda^{\text{stat}}(x,y) - N_\lambda^{\text{stat}}(0,y)] = \frac{y}{\lambda(1-t)} + \lambda x.$$

The first inequality follows from Lemma 1.4.3. The third equality follows because $N_\lambda^{\text{stat}}(0,y)$ is just the number of points in a Poisson point process of intensity $\frac{1}{\lambda(1-t)}$ in an interval of length x, and $N_\lambda^{\text{stat}}(x,y) - N_\lambda^{\text{stat}}(0,y)$ is the number of points in a Poisson process of intensity λ in an interval of length y (due to the stationary of X). Taking $\lambda = \frac{\sqrt{y}}{\sqrt{x(1-t)}}$ (which minimizes the right-hand side above), we conclude that

$$\frac{\mathbb{E}[N(x,y)]}{\sqrt{xy}} \le \frac{2}{\sqrt{1-t}}. \tag{1.21}$$

Recall that $\gamma = \sup_{n\ge 1} \frac{\mathbb{E}[N(n,n)]}{n}$. The above inequality implies that $\gamma \le \frac{2}{\sqrt{1-t}}$. □

1.5 Proving $\gamma = \frac{2}{\sqrt{1-t}}$

In this section, we prove that $\gamma = \frac{2}{\sqrt{1-t}}$. Recall that when two lines intersect, the intersection point is called a crossing point if the two lines cross and a corner point if the two lines annihilate. The set of α-points is the union of the Poisson nucleations and crossing points, and the set of β-points is the union of corner points and crossing points. For the t-PNG model with empty boundary data, let $\mathsf{A}_1(x,y)$ be the number of α-points and $\mathsf{A}_2(x,y)$ be the number of β-points in the rectangle $[0,x]\times[0,y]$. As an abbreviation, we use $\mathsf{A}_1(s)$ to denote $\mathsf{A}_1(s,s)$.

1.5.1 Law of large numbers for the α-points

The main result in this subsection is the following law of large numbers:

Proposition 1.5.1. *We have* $\frac{1}{s^2}\mathsf{A}_1(s) \xrightarrow{p} \frac{1}{1-t}$, *as* $s \to \infty$.

We will need to prove a few lemmas before proving this proposition. Fix any rectangle $R = [x_1,x_2]\times[y_1,y_2]$. Define

$$h_R = \min\Big\{\#\text{intersections between } [x_1,x_2]\times\{y\} \text{ and the vertical lines in } t\text{-PNG model} : y \in [y_1,y_2]\Big\}.$$

In other words, as we move the horizontal segment vertically from $[x_1, x_2] \times \{y_1\}$ to $[x_1, x_2] \times \{y_2\}$, h_R is the minimum number of intersections between the vertical lines and this horizontal segment.

Lemma 1.5.2. *Fix any rectangle $R = [x_1, x_2] \times [y_1, y_2]$. Let ℓ be the number of horizontal lines that enter the rectangle from the left. Let b be the number of vertical lines that enter the rectangle R from the bottom. Then $h_R \geq (b - \ell)^+$.*

Proof. In the t-PNG model, a vertical line will go upward until it meets a horizontal line. Moreover, each horizontal line can annihilate at most one vertical line. This concludes the lemma. □

Fix $M \in \mathbb{Z}_{\geq 2}$. Let $R(s) = [(1 - \frac{1}{M})s, s] \times [\frac{s}{M}, s]$.

Lemma 1.5.3. *We have almost surely,*

$$\lim_{s \to \infty} h_{R(s)} \to \infty. \tag{1.22}$$

Proof. For each $i \in \{0, 1, \ldots, M^2 - 1\}$, let $b_i(s)$ be the number of lines entering the rectangle $R_i = [(1 - \frac{1}{M})s, s] \times [\frac{is}{M^2}, \frac{(i+1)s}{M^2}]$ from the bottom boundary. Let $\ell_i(s)$ be the number of lines entering the same rectangle from the left boundary. It is straightforward that

$$\begin{aligned} b_i(s) &= N\Big(s, \frac{is}{M^2}\Big) - N\Big((1 - \frac{1}{M})s, \frac{is}{M^2}\Big), \\ \ell_i(s) &= N\Big((1 - \frac{1}{M})s, \frac{(i+1)s}{M^2}\Big) - N\Big((1 - \frac{1}{M})s, \frac{is}{M^2}\Big). \end{aligned}$$

By Lemma 1.5.2, we know that $h_{R_i(s)} \geq b_i(s) - \ell_i(s)$. Since $R = \cup_{i=M}^{M^2-1} R_i$, we have $h_{R(s)} \geq \min_{i=M}^{M^2-1} \big(b_i(s) - \ell_i(s)\big)$.

By the law of large numbers in Proposition 1.3.10, we know that almost surely,

$$\begin{aligned} \lim_{s \to \infty} \frac{b_i(s)}{s} &= \gamma \frac{\sqrt{i}}{M}\Big(1 - \sqrt{1 - \frac{1}{M}}\Big), \\ \lim_{s \to \infty} \frac{\ell_i(s)}{s} &= \gamma \sqrt{1 - \frac{1}{M}}\Big(\frac{\sqrt{i+1}}{M} - \frac{\sqrt{i}}{M}\Big). \end{aligned}$$

Subtracting the second equation from the first above yields

$$\lim_{s\to\infty} \frac{b_i(s)-\ell_i(s)}{s} = \gamma\Big(\frac{\sqrt{i}}{M}\Big(1-\sqrt{1-\frac{1}{M}}\Big) - \sqrt{1-\frac{1}{M}}\Big(\frac{\sqrt{i+1}}{M}-\frac{\sqrt{i}}{M}\Big)\Big).$$

When $i \geq M$, it is straightforward to check that

$$\frac{\sqrt{i}}{M}\Big(1-\sqrt{1-\frac{1}{M}}\Big) - \sqrt{1-\frac{1}{M}}\Big(\frac{\sqrt{i+1}}{M}-\frac{\sqrt{i}}{M}\Big) > 0.$$

This together with $\gamma > 0$ (see Lemma 1.3.11) implies that for any $i \geq M$, we have $\lim_{s\to\infty} \frac{b_i(s)-\ell_i(s)}{s} > 0$. Since $h_{R(s)} \geq \min_{i=M}^{M^2-1} \big(b_i(s)-\ell_i(s)\big)$, we conclude that $\lim_{s\to\infty} \frac{h_{R(s)}}{s} > 0$. This concludes the lemma. □

Lemma 1.5.4. *We have*

$$\lim_{s\to\infty} s^{-2}\mathbb{E}\big[\mathsf{A}_1(s)\big] = \frac{1}{1-t}.$$

Proof. We prove the lemma by showing that

$$\limsup_{s\to\infty} s^{-2}\mathbb{E}[\mathsf{A}_1(s)] \leq \frac{1}{1-t}, \qquad \textbf{(upper bound)}$$

$$\liminf_{s\to\infty} s^{-2}\mathbb{E}[\mathsf{A}_1(s)] \geq \frac{1}{1-t}. \qquad \textbf{(lower bound)}$$

Proof of (upper bound): Let $K(s)$ be the total number of Poisson nucleations in $[0,s]\times[0,s]$. We label these points by $1,\dots,K(s)$. Let $\alpha_i(s)$ be the number of vertical lines in the t-PNG model that cross the the horizontal line emanating from the nucleation point i (including the vertical line that emanates from the point i). For every $n \in \mathbb{Z}_{\geq 1}$,

$$\mathbb{E}\big[\alpha_n(s) \,|\, K(s) \geq n\big] \leq \frac{1}{1-t}, \tag{1.23}$$

This is because if we condition on the event $\{K(s) \geq n\}$, $\alpha_n(s)$ is stochastically dominated by a

geometric random variable with parameter $1-t$. As a result, we have

$$\mathbb{E}\big[\mathsf{A}_1(s)\big] = \mathbb{E}\Big[\sum_{i=1}^{K(s)} \alpha_i(s)\Big] = \sum_{n=1}^{\infty} \mathbb{P}(K(s)\geq n)\mathbb{E}\big[\alpha_n(s)\,|\,K(s)\geq n\big] \leq \frac{1}{1-t}\mathbb{E}\big[K(s)\big] = \frac{s^2}{1-t}.$$

For the first equality, note that there are $\alpha_i(s)$ number of α-points that have the same y-coordinate as that of the nucleation point i, hence $\mathsf{A}_1(s) = \sum_{i=1}^{K(s)} \alpha_i(s)$. The first inequality above is due to (1.23). The third equality holds because $K(s)$ is a Poisson random variable with parameter s^2. Hence, we have shown that

$$\limsup_{s\to\infty} \frac{1}{s^2}\mathbb{E}\big[\mathsf{A}_1(s)\big] \leq \frac{1}{1-t}. \tag{1.24}$$

Proof of (lower bound): Fix $M, m \in \mathbb{Z}_{\geq 2}$. Define $c_M(\varepsilon) = (1-\frac{1}{M})^2 - \varepsilon$. We can choose $\varepsilon > 0$ small enough such that $c_M(\varepsilon) > 0$. Let $K_M(s)$ be the total number of Poisson nucleations in the square $[0,(1-\frac{1}{M})s)] \times [\frac{s}{M}, s]$. We have

$$\mathbb{E}\big[\mathsf{A}_1(s)\big] \geq \mathbb{E}\Big[\sum_{i=1}^{K_M(s)} \alpha_i(s)\Big] \geq \mathbb{E}\Big[\Big(\sum_{i=1}^{c_M(\varepsilon)s^2} \alpha_i(s)\Big)\mathbb{1}_{\{K_M(s)\geq c_M(\varepsilon)s^2, h_R(s)\geq m\}}\Big].$$

For a line that emanates rightward from a Poisson nucleation in $[0,(1-\frac{1}{M})s)] \times [(1-\frac{1}{M})s, s]$, there are potentially at least $h_R(s)$ number of vertical lines it can cross. Hence, there exist i.i.d. geometric random variables $\{X_i\}_{i\in\mathbb{Z}_{\geq 1}} \sim \text{Geo}(1-t)$ that are independent of the t-PNG model, and such that $\alpha_i(s) \geq X_i \wedge h_R(s)$. Therefore, we have

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{c_M(\varepsilon)s^2} \alpha_i\Big)\mathbb{1}_{\{K_M(s)\geq c_M(\varepsilon)s^2, h_R(s)\geq m\}}\Big] \geq c_M(\varepsilon)s^2\mathbb{E}\big[X_1\wedge m\big]\mathbb{P}\Big(K_M(s)\geq c_M(\varepsilon)s^2, h_R(s)\geq m\Big).$$

We divide both sides by s^2 and let $s\to\infty$. Note that $K_M(s)$ is a Poisson random variable with mean $(1-\frac{1}{M})^2 s^2$. By a standard large deviation bound, $\lim_{s\to\infty}\mathbb{P}\big(K_M(s)\geq c_M(\varepsilon)s^2\big) = 1$. Furthermore, by Lemma 1.22, we have $\lim_{s\to\infty}\mathbb{P}(h_R(s)\geq m) = 1$. Hence,

$$\lim_{s\to\infty} \mathbb{P}\Big(K_M(s)\geq c_M(\varepsilon)s^2, h_R(s)\geq m\Big) = 1.$$

This implies that

$$\liminf_{s\to\infty} \frac{\mathbb{E}\big[\mathsf{A}_1(s)\big]}{s^2} \geq c_M(\varepsilon)\mathbb{E}[X_1 \wedge m].$$

Letting $M, m \to \infty$ and $\varepsilon \to 0$, we get

$$\liminf_{s\to\infty} \frac{\mathbb{E}[\mathsf{A}_1(s)]}{s^2} \geq \frac{1}{1-t}. \qquad \square$$

To prove Proposition 1.5.1, it remains to prove the following lemma.

Lemma 1.5.5. *We have* $\lim_{s\to\infty} \mathbb{E}\Big[\big(\frac{1}{s^2}\mathsf{A}_1(s) - \frac{1}{1-t}\big)^+\Big] = 0$ *where* $x^+ := \max(x, 0)$.

Proof. Since each $\alpha_i(s)$ is stochastically dominated by a geometric random variable, it follows that the random variable $\mathsf{A}_1(s)$ is stochastically dominated by $\sum_{i=1}^{K(s)} X_i$, where $\{X_i\}_{i=1}^{\infty}$ are i.i.d. geometric random variables that are independent of $K(s)$. Hence, it suffices to show that

$$\lim_{s\to\infty} \mathbb{E}\bigg[\Big(\frac{1}{s^2}\sum_{i=1}^{K(s)} X_i - \frac{1}{1-t}\Big)^+\bigg] = 0. \tag{1.25}$$

We compute

$$\begin{aligned}\mathbb{E}\bigg[\Big(\frac{1}{s^2}\sum_{i=1}^{K(s)} X_i - \frac{1}{1-t}\Big)^2\bigg] &= \mathbb{E}\bigg[\Big(\frac{1}{s^2}\sum_{i=1}^{K(s)}(X_i - \frac{1}{1-t}) + \frac{1}{1-t}(\frac{K(s)}{s^2} - 1)\Big)^2\bigg] \\ &\leq 2\mathbb{E}\bigg[\Big(\frac{1}{s^2}\sum_{i=1}^{K(s)}(X_i - \frac{1}{1-t})\Big)^2\bigg] + \frac{2}{(1-t)^2}\mathbb{E}\Big[(\frac{K(s)}{s^2} - 1)^2\Big] = \mathcal{O}(s^{-2}).\end{aligned}$$

For the inequality, we use $(a+b)^2 \leq 2(a^2+b^2)$. The last equality follows from a direct computation since $K(s)$ is a Poisson random variable with mean s^2. This concludes (1.25). $\square$

Proof of Proposition 1.5.1. Using Lemma 1.5.4 and 1.5.5, we have $\lim_{s\to\infty} \mathbb{E}[|\frac{1}{s^2}\mathsf{A}_1(s) - \frac{1}{1-t}|] = 0$. This implies the convergence in probability. $\square$

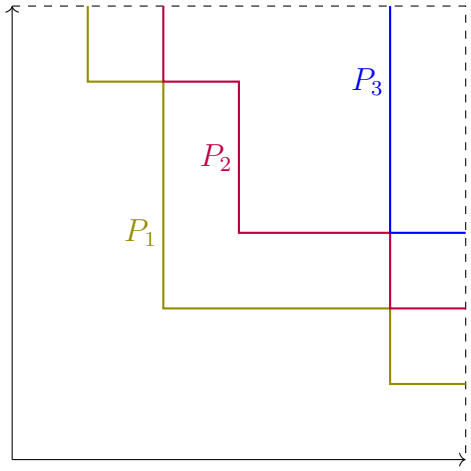


Figure 1.17: We decompose the t-PNG model in any rectangle into a sequence of down-right paths. To distinguish between them, we give them different colors (this has nothing to do with the colored model). We label them P_1, P_2 and P_3. As shown in Figure 1.1, these down-right paths play the role as level lines of the t-PNG height function.

1.5.2 Proving $\gamma = \frac{2}{\sqrt{1-t}}$

We proceed to prove that $\gamma = \frac{2}{\sqrt{1-t}}$ by building a connection between the number of α-points and the limit shape, using the same argument as in [26].

Lemma 1.5.6. *Fix $x, y > 0$. We have*

$$N(x,y) = \#\{\alpha \textit{ points in } [0,x] \times [0,y]\} - \#\{\beta \textit{ points in } [0,x] \times [0,y]\}. \tag{1.26}$$

Proof. For the t-PNG model, we can decompose the collection of paths into a number of down-right paths that do not cross each other, see Figure 1.17. $N(x,y)$ equals the number of down-right paths that one crosses in the north-east direction from $(0,0)$ to (x,y). We want to count the number of α- and β- points on each down-right path. To avoid ambiguity at crossing points, which are both α- and β-points, we treat the β-point as lying on the left down-right path passing through the crossing point and the α-point as lying on the right down-right path. Then, for each down-right path, it is straightforward to see that the difference between the number of α-points and the number of β-points lying on it equals 1. Hence, the right-hand side of (1.26) is exactly the number of down-right paths that one crosses as going from $(0,0)$ to (x,y), which equals $N(x,y)$. □

For $x, y > 0$, we denote $V_s(x,y) = s^{-1}N(sx, sy)$. We associate a random measure ξ_s to this random variable by letting

$$\xi_s(B) := \int_B dV_s(x,y) = s^{-1}\Big(\#\{\alpha \text{ points in } sB\} - \#\{\beta \text{ points in } sB\}\Big),$$

where $sB = \{(sx, sy) : (x,y) \in B\}$.

Let $B = [0,x] \times [0,y]$. We define

$$V_s^+(x,y) = s^{-1}\{\text{number of } \alpha \text{ points in } sB\},$$
$$V_s^-(x,y) = s^{-1}\{\text{number of } \beta \text{ points in } sB\},$$

and

$$\widetilde{V}_s(x,y) = \int_{[0,x)\times[0,y)} dV_s(u,v). \tag{1.27}$$

Note that in the integral above, we omit the upper edge of the rectangle $[0,x] \times [0,y]$ but not the lower edge. This is crucial to the establishment of the next lemma, which intrinsically goes back to [26, Lemma 2.1].

Lemma 1.5.7. *We have*

$$V_s(x,y)^2 = s^{-1}\big(V_s^+(x,y) + V_s^-(x,y)\big) + 2\int_{[0,x]\times[0,y]} \widetilde{V}_s(u,v) dV_s(u,v).$$

Proof. Consider the down-right paths we cross while going from $(0,0)$ to (sx, sy) in the north-east direction. We label these paths $P_1, P_2, ..., P_m$, where P_1 is the down-right path closest to the origin and P_m is the path farthest from the origin, see Figure 1.17. Note that we have $V_s(x,y) = \frac{m}{s}$. For an α-point (u,v) lying on P_i, we have $\widetilde{V}_s(u,v) = \frac{i-1}{s}$. For a β-point (u,v) lying on P_i, we have $\widetilde{V}_s(u,v) = \frac{i}{s}$. Recall that $\mathsf{A}_1(x,y)$ is the number of α-points and $\mathsf{A}_2(x,y)$ is the number of β-points

contained in $[0,x]\times[0,y]$. We have

$$\int_{[0,x]\times[0,y]} \widetilde{V}_s(u,v)dV_s(u,v) = s^{-2}\sum_{i=1}^{m}\Big((i-1)\#\{\alpha \text{ points on } P_i\} - i\#\{\beta \text{ points on } P_i\}\Big)$$
$$= s^{-2}\sum_{i=1}^{m}\Big((i-1)\#\{\alpha \text{ points on } P_i\} - (i-1)\#\{\beta \text{ points on } P_i\}\Big) - s^{-2}\mathsf{A}_2(sx,sy).$$

For each space-time curve P_i, contained in $[0,s]\times[0,s]$, we have

$$\#\{\alpha \text{ points on } P_i\} - \#\{\beta \text{ points on } P_i\} = 1$$

Using this together with $m = sV_s(x,y)$, we have

$$\begin{aligned}\int_{[0,x]\times[0,y]} \widetilde{V}_s(u,v)dV_s(u,v) &= (2s^2)^{-1}m(m-1) - s^{-2}\mathsf{A}_2(sx,sy)\\ &= \frac{1}{2}V_s(x,y)^2 - \frac{1}{2}s^{-1}V_s(x,y) - s^{-2}\mathsf{A}_2(sx,sy)\\ &= \frac{1}{2}V_s(x,y)^2 - \frac{1}{2}s^{-1}\big(V_s^+(x,y)+V_s^-(x,y)\big).\end{aligned}$$

In the last equality, we have used $s^{-1}\mathsf{A}_2(sx,sy) = V_s^-(x,y)$, and $V_s(x,y) = V_s^+(x,y) - V_s^-(x,y)$.

□

Proposition 1.5.8. *Recall γ from Proposition 1.3.10. We have $\gamma = \frac{2}{\sqrt{1-t}}$.*

Proof. Note that $V_s^-(x,y) = V_s^+(x,y) - V_s(x,y)$, and $sV_s^+(x,y) = \mathsf{A}_1(sx,sy)$. Using this together with Lemma 1.5.7, we have

$$V_s(x,y)^2 = 2s^{-2}\mathsf{A}_1(sx,sy) - s^{-1}V_s(x,y) + 2\int_{[0,x]\times[0,t]} \widetilde{V}_s(u,v)dV_s(u,v). \tag{1.28}$$

By Proposition 1.3.10, we have almost surely $\lim_{s\to\infty} V_s(u,v) = \gamma\sqrt{uv}$. Hence, we have as $s\to\infty$ (see [26, page 688] for detail)

$$\int_{[0,x]\times[0,y]} \widetilde{V}_s(u,v)dV_s(u,v) \to \int_{[0,x]\times[0,y]} \gamma\sqrt{uv}\cdot d(\gamma\sqrt{uv}) = \frac{\gamma^2}{4}xy.$$

By Proposition 1.5.1 and scaling, we have $\frac{1}{s^2}\mathsf{A}_1(sx, sy) \xrightarrow{p} \frac{xy}{1-t}$, as $y \to \infty$. Hence, there exists a sequence $\{s_n\}_{n\in\mathbb{Z}_{\geq 1}}$ that goes to infinity and satisfies $\lim_{n\to\infty} \frac{1}{s_n^2}\mathsf{A}_1(s_n x, s_n y) = \frac{xy}{1-t}$ almost surely. Take $s = s_n$ and let $n \to \infty$ on both sides of (1.28). The left-hand side converges to $\gamma^2 xy$ and the right-hand side converges to $\frac{2xy}{1-t} + \frac{\gamma^2}{2}xy$. Hence, we have $\gamma^2 xy = \frac{2}{1-t}xy + \frac{1}{2}\gamma^2 xy$. Using this together with Lemma 1.20 (which shows that γ is finite), we have $\gamma = \frac{2}{\sqrt{1-t}}$. □

Remark 1.5.9. *Since all of the terms containing V_s in* (1.28) *converge almost surely as $s \to \infty$, so does $\mathsf{A}(sx, sy)$. Hence, we can strengthen Proposition 1.5.1 to almost sure convergence.*

Proof of Theorem 1.1.1. The theorem is a direct consequence of Proposition 1.3.10 and Proposition 1.5.1. The convergence $\frac{N(x,y)}{\sqrt{xy}} \xrightarrow{p} \frac{2}{\sqrt{1-t}}$ follows from Lemma 1.3.9. □

1.6 Configurations for the two-colored t-PNG model

In Figure 1.18, we draw the two-colored configurations that have non-zero weights. In other words, we draw all of the two-colored configurations with $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \{0, 1\}^2$ such that $\mathcal{L}^2(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) \neq 0$.

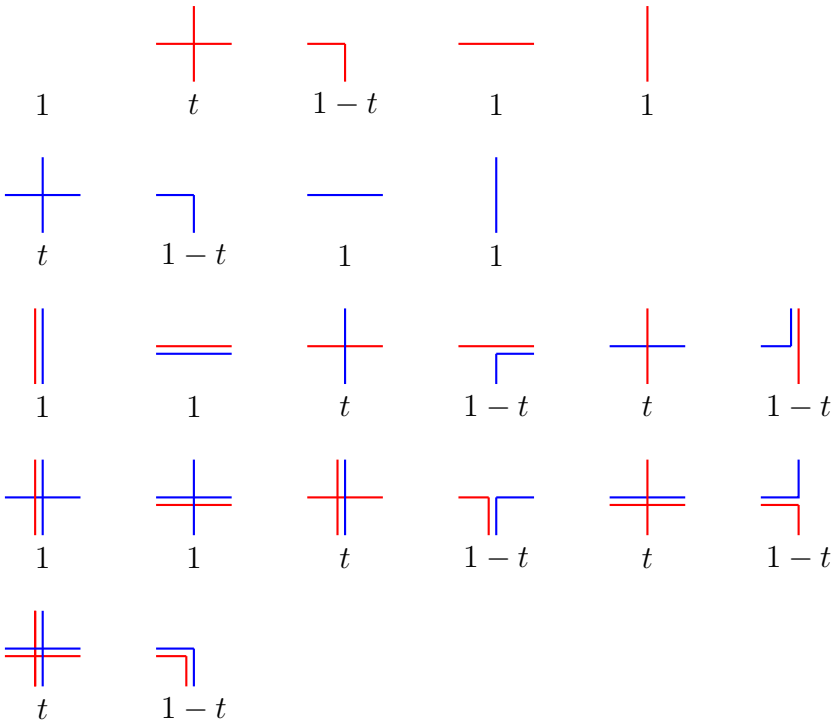


Figure 1.18: Let red be the color 1, and let blue be the color 2. The above figure shows all the configurations with non-zero weights in the two-colored t-PNG model given by Definition 1.1.7. Note that some of the configurations can be obtained from the two-colored S6V model by horizontal complementation of the red lines (i.e., deleting existing horizontal red lines and placing horizontal red lines where there aren't any). However, not all the configurations (e.g., the ones on the last row) can be obtained in this way.

1.7 Computing G^*

In this section, we prove the following lemma.

Lemma 1.7.1. *We have*

$$G^*g(y) = \sum_{i=1}^{\infty} t^{i-1}(1-t)\int_0^{T_1}(g(\mathcal{L}_s^i y) - g(x))ds + \sum_{i=1}^{\infty}\frac{t^{i-1}}{\lambda}(g(\mathcal{R}^i y) - g(y)).$$

Proof. Recall that μ is the measure induced on E by a Poisson process with intensity λ. Let μ_n be the restriction of μ to E_n, where E_n consists of configurations with n particles and where $E_0 = \{\emptyset\}$ consists of the empty configuration. Then we have $\mu_n(dx) = \lambda^n e^{-\lambda T_1}dx$.

We define

$$G_+ f = Gf + \left(\frac{1}{\lambda(1-t)} + T_1 \right) f. \tag{1.29}$$

For simplicity, we compute the dual of G_+ which will then yield the dual of G. For $f, g \in C_0(E)$ we have

$$\begin{aligned}
&\int_E G_+ f(x) g(x) \mu(dx) = \sum_{n=0}^{\infty} \int_{E_n} G_+ f(x) g(x) \mu_n(dx) \\
&= e^{-\lambda T_1} \sum_{n=0}^{\infty} \left[\lambda^n \int_{E_n} \int_0^{T_1} \sum_{i=1}^{\infty} t^{i-1}(1-t) f\left(\mathcal{R}_z^i x\right) g(x) dz dx + \lambda^{n-1} \int_{E_n} \sum_{i=1}^{\infty} t^{i-1} f(\mathcal{L}^i x) g(x) dx \right] \\
&= e^{-\lambda T_1} \sum_{n=0}^{\infty} A_n.
\end{aligned}$$

Suppose that $x = (x_1, \ldots, x_n) \in E_n, z \in [0, T_1]$, and there are $m-1$ particles in x to the left of z. Let $j(z) = n - m - 1$ equal the number of particles to the right of z. We can decompose A_n into

$$A_n = \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{x \in E_n, z \in [0,T_1], i \le j(z)\}} f\left(\mathcal{R}_z^i x\right) g(x) dx dz \tag{1.30}$$

$$+ \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{x \in E_n, z \in [0,T_1], i > j(z)\}} f\left(\mathcal{R}_z^i x\right) g(x) dx dz \tag{1.31}$$

$$+ \lambda^{n-1} \sum_{i=1}^{n} t^{i-1} \int_{E_n} f(\mathcal{L}^i x) g(x) dx \tag{1.32}$$

$$+ \lambda^{n-1} \sum_{i=n+1}^{\infty} t^{i-1} \int_{E_n} f(\mathcal{L}^i x) g(x) dx. \tag{1.33}$$

We will perform a change of variables by setting y to equal the input to the function f. In other words, we will set $y = \mathcal{R}_z^i x$ or $y = \mathcal{L}^i x$ in the integrals above. We then need to solve for x in terms of y where y belongs to the dual system. Let $k(s)$ equal the number of particles to the left of s in the dual system. Then we have the following four cases:

1. $y = \mathcal{R}_z^i x \in E_n$ for $i \le j(z)$. In this case, $x = \mathcal{L}_s^i y$ where $s = x_{m+i-1}$.

2. $y = \mathcal{R}_z^i x \in E_{n+1}$ for $i > j(z)$. In this case, $x = \mathcal{R}^{j(z)+1} y$.

3. $y = \mathcal{L}^i x \in E_{n-1}$ for $i \leq n$. In this case, $x = \mathcal{L}_s^k y$ for $s = x_i$ and for all $k > k(s)$.

4. $y = \mathcal{L}^i x \in E_n$ for $i > n$. In this case, $x = \mathcal{R}^k y = y$ for all $k > n$.

Therefore, when we perform the change of variables, the first and third terms (1.30) and (1.32) will correspond to the operators $\mathcal{L}_s^i$ and the second and fourth terms (1.31) and (1.33) will correspond to the operator $\mathcal{R}^i$. We compute the change of variables explicitly on each of the four terms.

First term: We have

$$\begin{aligned}
&\lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{x \in E_n, z \in [0,T_1], i \leq j(z)\}} f\left(\mathcal{R}_z^i x\right) g(x) dx dz \\
&= \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{y \in E_n, s \in [0,T_1], i \leq k(s)\}} f\left(y\right) g(\mathcal{L}_s^i y) dy ds.
\end{aligned}$$

Second term: We have

$$\begin{aligned}
&\lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{x \in E_n, z \in [0,T_1], i > j(z)\}} f\left(\mathcal{R}_z^i x\right) g(x) dx dz \\
&= \lambda^n \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} t^{i-1}(1-t) \int_{\{x \in E_n, z \in [0,T_1], j(z) = j\}} f\left(\mathcal{R}_z^i x\right) g(x) dx dz \\
&= \lambda^n \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} t^{i-1}(1-t) \int_{\{y \in E_{n+1}\}} f\left(y\right) g(\mathcal{R}^{j+1} y) dy.
\end{aligned}$$

We sum over i and shift the index j by one to get

$$\lambda^n \sum_{j=1}^{\infty} t^{j-1}(1-t) \int_{\{y \in E_{n+1}\}} f\left(y\right) g(\mathcal{R}^j y) dy.$$

Third term: We have

$$\lambda^{n-1} \sum_{i=1}^{n} t^{i-1} \int_{E_n} f(\mathcal{L}^i x) g(x) dx = \lambda^{n-1} \sum_{i=1}^{n} \sum_{j=i}^{\infty} t^{j-1}(1-t) \int_{E_n} f(\mathcal{L}^i x) g(x) dx$$

$$= \lambda^{n-1} \sum_{i=1}^{n} \sum_{j=i}^{\infty} t^{j-1}(1-t) \int_{\{y \in E_{n-1}, s \in [0,T_1], y_{i-1} \leq s \leq y_i\}} f(y) g(\mathcal{L}_s^j y) dy ds$$
$$= \lambda^{n-1} \sum_{j=1}^{\infty} t^{j-1}(1-t) \int_{\{y \in E_{n-1}, s \in [0,T_1], j > k(s)\}} f(y) g(\mathcal{L}_s^j y) dy ds.$$

Fourth term: We have

$$\lambda^{n-1} \sum_{i=n+1}^{\infty} t^{i-1} \int_{E_n} f(\mathcal{L}^i x) g(x) dx = \lambda^{n-1} \sum_{i=n+1}^{\infty} t^{i-1} \int_{E_n} f(y) g(\mathcal{R}^i y) dy.$$

We collect the four terms and relabel the index j by i if necessary to obtain

$$\begin{aligned} A_n = {} & \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{y \in E_n, s \in [0,T_1], i \leq k(s)\}} f(y)\, g(\mathcal{L}_s^i y) dy ds \\ & + \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{E_{n+1}} f(y)\, g(\mathcal{R}^i y) dy \\ & + \lambda^{n-1} \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{y \in E_{n-1}, s \in [0,T_1], i > k(s)\}} f(y) g(\mathcal{L}_s^i y) dy ds \\ & + \lambda^{n-1} \sum_{i=n+1}^{\infty} t^{i-1} \int_{E_n} f(y) g(\mathcal{R}^i y) dy. \end{aligned}$$

Finally, we re-index each term so that all the integrals are over E_n instead of E_{n-1} or E_{n+1}:

$$\begin{aligned} A_n = {} & \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{y \in E_n, s \in [0,T_1], i \leq k(s)\}} f(y)\, g(\mathcal{L}_s^i y) dy ds \\ & + \lambda^{n-1} \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{E_n} f(y)\, g(\mathcal{R}^i y) dy \\ & + \lambda^n \sum_{i=1}^{\infty} t^{i-1}(1-t) \int_{\{y \in E_n, s \in [0,T_1], i > k(s)\}} f(y) g(\mathcal{L}_s^i y) dy ds \\ & + \lambda^{n-1} \sum_{i=n+1}^{\infty} t^{i-1} \int_{E_n} f(y) g(\mathcal{R}^i y) dy. \end{aligned}$$

Multiplying everything back by $e^{-\lambda T_1}$ and summing over n, we get

$$\begin{aligned}
&\int_E G_+ f(x)g(x)\mu(dx) = e^{-\lambda T_1}\sum_{n=0}^{\infty} A_n \\
&= e^{-\lambda T_1}\sum_{n=0}^{\infty}\left[\lambda^n \int_{E_n}\int_0^{T_1}\sum_{i=1}^{\infty} t^{i-1}(1-t)f\,(y)\, g(\mathcal{L}_s^i y)dsdy + \lambda^{n-1}\int_{E_n}\sum_{i=1}^{\infty} t^{i-1}f(y)g(\mathcal{R}^i y)dy\right] \\
&= \int_E f(y)G_+^* g(y)\mu(dy),
\end{aligned}$$

where G_+^* is the dual of G_+. By (1.29), we have

$$G^* g = G_+^* g - \Big(\frac{1}{\lambda(1-t)} + T_1\Big)g,$$

which concludes the proof.

□

1.8 Acknowledgments

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Chapter 2: Strong law of large numbers for the stochastic six vertex model

This chapter is adapted from the paper [18] and is coauthored with Yier Lin.

2.1 Introduction

2.1.1 The S6V model

The stochastic six vertex (S6V) model [28] is a classical model in two-dimensional statistical physics. It is a stochastic version of the six vertex model [100, 101], which is a natural model for crystal lattices with hydrogen bonds. We associate six possible configurations to each lattice point on the first quadrant, as in Figure 2.1. We view the lines from the south and the west as *inputs* and the lines to the north and the east as *outputs*. The model is *stochastic* in the sense that when we fix the inputs, the weights of possible configurations with those inputs sum up to 1. The configurations chosen for neighboring vertices need to be compatible so that the output lines of a vertex are the input lines of the vertices to its immediate north and east.

It was predicted by Gwa and Spohn [28] and proved by Borodin, Corwin and Gorin [30] that the S6V model belongs to the Kardar-Parisi-Zhang (KPZ) universality class—a class of models that exhibit universal statistical behavior in their large time/large-scale limits. See [2, 132, 133] for an overview of this topic. Some recent works on the S6V model include [134, 108, 135, 136, 137, 31, 138, 139, 140, 141] and references therein.

In this paper, we view the S6V model as a stochastic path ensemble on the first quadrant with the boundary condition that there is a line entering each of the vertices in $\{1\} \times \mathbb{Z}_{\geq 1}$ from the left, and no lines entering $\mathbb{Z}_{\geq 1} \times \{1\}$ from the bottom. This boundary data is called *step initial data.* Fix a collection of parameters $\{b_1(i,j), b_2(i,j)\}_{i,j}$ that take values in $[0,1]$. Starting from $(1,1)$, we tile each vertex with one of the six vertex configurations in Figure 2.1, where we only consider

configurations whose input lines match the input lines of the given vertex. We assign an allowed configuration with probability given by the weight of the configuration. This tiling construction then progresses sequentially in the linear order $(2,1),(1,2),(3,1),(2,2),(1,3),\ldots$ to the entire quadrant. Note that we can view the stochastic path ensemble as a family of upright paths ordered in the northwest direction, as depicted in the left panel of Figure 2.2.

Let h be the height function of the S6V model defined on $\mathbb{Z}^2_{\geq 0}$ where $h(x,y)$ records the number of lines that pass through or to the right of (x,y), see the left panel of Figure 2.2.

Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	$b_1(i,j)$	$1-b_1(i,j)$	$b_2(i,j)$	$1-b_2(i,j)$

Figure 2.1: Six types of configurations for the S6V model

Assumption 2.1.1. *There exist periods $I, J \in \mathbb{Z}_{\geq 1}$ such that $b_k(x+I,y) = b_k(x,y)$ and $b_k(x,y+J) = b_k(x,y)$ for arbitrary x, y and $k \in \{1,2\}$.*

Theorem 2.1.2. *There exists a Lipschitz function g such that with probability 1,*

$$\lim_{n\to\infty} \frac{h(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g(x,y), \qquad \forall x, y \in \mathbb{R}_{\geq 0}. \tag{2.1}$$

Remark 2.1.3. *The S6V model considered here is inhomogeneous and not integrable in most cases. The parameters b_1 and b_2 have period I in space and period J in time. Hence, there are IJ degrees of freedom. If $I, J = 1$, the S6V model is homogeneous and its limit shape has been explicitly derived in [30, Theorem 1.1] and [31, Theorem 5.1] (also see [142] for a derivation of the limit shape using a variational principle). Note that the derivations of explicit limit shapes in these two papers rely respectively on integrability and the existence of explicit stationary distributions for the model. For general I, J, due to the lack of these properties, we don't think that there is an explicit formula for g.*

Remark 2.1.4. *Take $I = J = 1$ and consider the homogeneous S6V model where $b_1(i,j)$ and $b_2(i,j)$ do not depend on i, j. By [30, Theorem 1.1] or [31, Theorem 5.1], if $b_1 \le b_2$, we have*

$$g(x,y) = \begin{cases} \frac{\left(\sqrt{y(1-b_1)}-\sqrt{x(1-b_2)}\right)^2}{b_2-b_1} & \frac{1-b_2}{1-b_1} < \frac{x}{y} < \frac{1-b_1}{1-b_2}, \\ 0 & \frac{x}{y} \ge \frac{1-b_1}{1-b_2}, \\ y-x & \frac{x}{y} \le \frac{1-b_2}{1-b_1}. \end{cases} \tag{2.2}$$

If $b_1 \ge b_2$, we have

$$g(x,y) = \begin{cases} 0 & x \ge y, \\ y-x & x \le y. \end{cases}$$

With Theorem 2.1.2, we strengthen the convergence in probability in [30, 31] to the almost sure level.

2.1.2 The complemented S6V model

To prove Theorem 2.1.2, it is convenient to study the S6V model after *horizontal complementation*. If there is a horizontal line, we erase it; if there is no horizontal line, we add it, see Figure 2.2 for an example. This procedure converts an ensemble of upright paths to an ensemble of down-right paths. Note that this model has appeared in [16] without a specific name, and we call it the *complemented S6V (CS6V) model*.

Let H be the height function of the CS6V model defined on $\mathbb{Z}^2_{\ge 0}$, where $H(x,y)$ records the number of lines that pass through or to the left of (x,y). It is straightforward that $H(x,y) = y - h(x,y)$, see the right panel of Figure 2.2.

We can also sample the CS6V model on the first quadrant by sequentially tiling the vertices with one of the six possible configurations in Figure 2.3, which are obtained by horizontally complementing the configurations in Figure 2.1. Instead of step initial data, we consider *empty boundary data*—the boundary data obtained by horizontally complementing the step initial data. In other words, no lines enter the quadrant from either the left boundary $\{1\} \times \mathbb{Z}_{\ge 1}$ or the bottom boundary

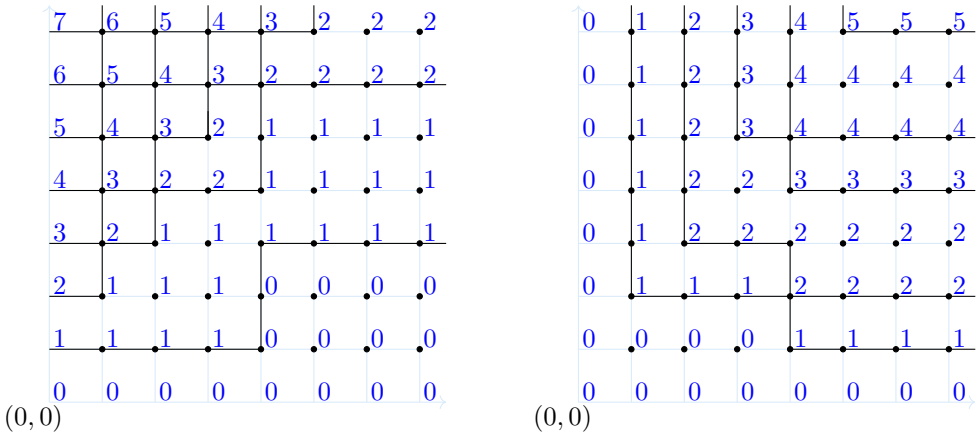


Figure 2.2: **Left panel:** A sampling of the S6V model on the first quadrant, with the blue numbers denoting the height function. **Right panel:** The CS6V model obtained by horizontally complementing the S6V model.

Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	$b_1(i,j)$	$1-b_1(i,j)$	$b_2(i,j)$	$1-b_2(i,j)$

Figure 2.3: Six types of configurations for the CS6V model

$\mathbb{Z}_{\geq 1} \times \{1\}$.

Due to the relationship $H(x,y) = y - h(x,y)$, proving Theorem 2.1.2 is equivalent to proving the following theorem.

Theorem 2.1.5. *There exists a Lipschitz function g such that with probability* 1,

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = y - g(x,y), \qquad \forall x, y \in \mathbb{R}_{\geq 0}.$$

2.1.3 Connections to the discrete Hammersley process and the t-PNG model

[113] studied Ulam's problem on the planar lattice. This model is also referred to as the discrete Hammersley process in [112, 123]. Consider the integer lattice on $\mathbb{Z}^2_{\geq 0}$ and a random set ξ of integer points chosen independently with probability p. Define the partial order $\prec$ such that $(x_1, y_1) \prec (x_2, y_2)$ if and only if $x_1 < x_2$ and $y_1 < y_2$.

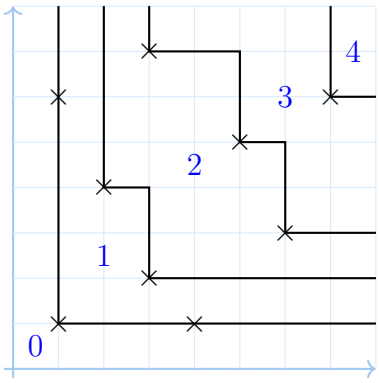


Figure 2.4: Illustration of the discrete Hammersley process. The $\times$'s are chosen independently on the lattice with probability p.

Let H^{d} denote the height function defined on $\mathbb{Z}^2_{\geq 0}$:

$$H^{\mathsf{d}}(x,y) = \max\Big\{L : \text{ there exist integer points } (x_1,y_1) \prec \cdots \prec (x_L,y_L) \in \xi \cap [1,x] \times [1,y]\Big\}.$$

As noticed in [112], we can construct a collection of downright paths which play the role of the level lines of H^{d}. We first use a downright path to connect the minimal points of ξ under the order $\prec$. We remove these points from ξ and connect the new minima to obtain the second line and so on. The height function starts from $H^{\mathsf{d}}(0,0) = 0$ and whenever we cross a downright path from southwest to northeast, the height function increases by 1, see Figure 2.4. It is straightforward to check that the CS6V model is a deformed version of the discrete Hammersley process:

Proposition 2.1.6. *Take $b_1 = 0$ and $b_2 = 1 - p$. The CS6V model degenerates to the discrete Hammersley process.*

Using this, we recover the almost sure limit shape of the discrete Hammersley process established in [113, Theorem 1] and [112, Theorem 1].

Corollary 2.1.7. *With probability 1, for all $x, y \geq 0$,*

$$\lim_{t\to\infty} \frac{H^{\mathsf{d}}(tx,ty)}{t} = \begin{cases} (1-p)^{-1}\big(2\sqrt{pxy} - (x+y)p\big) & \text{if } p < \min(\frac{x}{y}, \frac{y}{x}), \\ \min(x,y) & \text{else.} \end{cases}$$

Proof. One can readily check that the right-hand side above equals $y - g(x, y)$ when $b_1 = 0$ and $b_2 = 1 - p$, using the formula of g in Remark 2.1.4. The corollary then follows from a combination of Theorem 2.1.5 and Proposition 2.1.6. □

Just like the CS6V model is a deformation of the discrete Hammersley process, the t-PNG model introduced in [16] is a deformation of the polynuclear growth (PNG) model [143], a continuous version of the discrete Hammersley process. We will now discuss the relationship between these two deformed models. To define the t-PNG model, we fix a parameter $t \in [0, 1]$ and place a Poisson point process with intensity 1 on the upper-right quadrant representing *nucleations*. We draw lines emanating from each of these nucleations in both the upward and rightward directions until they collide with one another. We call these collision points *intersection points*. Given the Poisson nucleations, we sample the outcomes of the intersection points (lines will either cross or annihilate each other) starting with the intersection point which has the smallest sum of x- and y- coordinates and moving sequentially outward. At an intersection point, the two lines will cross each other with probability t and will annihilate each other with probability $1 - t$, forming a corner. Note that when lines cross, they might generate new intersection points. We refer to Figure 2.5 for a sampling of the t-PNG model.

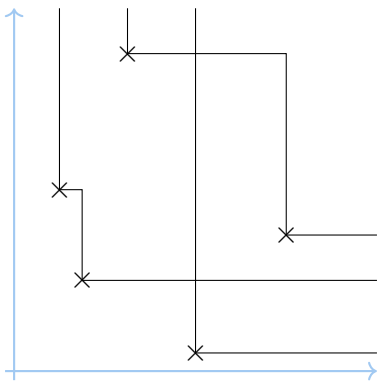


Figure 2.5: A sampling of the t-PNG model, where "$\times$" are the Poisson nucleations.

Consider the CS6V model where b_1 and b_2 do not depend on i, j. As noticed by [16], if we take $b_1 = t$ and $b_2 = 1 - \varepsilon^2$ and simultaneously scale the x- and y-axis by ε^{-1}, the CS6V model degenerates to the t-PNG model as $\varepsilon \to 0$.

Using integrable methods, [16] proved a weak law of large numbers and a fluctuation theorem for the height function of the t-PNG model. In [17], we proved a strong law of large numbers of the t-PNG model by constructing a colored t-PNG model and using Liggett's superadditive ergodic theorem (one can easily go from the usual subadditive ergodic theorem to the superadditive version by placing negative signs as needed).

2.1.4 Generalization to the stochastic higher spin six vertex (SHS6V) model

In the definition of the S6V model, we only allow at most one line traveling in the horizontal and vertical directions. Now we consider a generalization of the S6V model called the SHS6V model which relaxes this restriction and allows multiple lines to travel together. Our results will easily generalize to this setting.

The SHS6V model is a higher spin version of the S6V model that has been actively studied in recent years, see [144, 33, 145, 126, 146, 127, 128, 109, 110, 130] and the references therein. The model has two spin parameters $I, J \in \mathbb{Z}_{\geq 1}$, which will be fixed from now on. To define the SHS6V model, let us introduce the matrix $L^{I,J}_{\alpha,q}$ with rows and columns both indexed by $\{0, 1, \cdots, I\} \times \{0, 1, \cdots, J\}$. Fix two other parameters $\alpha, q \in \mathbb{R}$. For $(i_1, j_1), (i_2, j_2) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$, we define matrix entries

$$L^{I,J}_{\alpha,q}(i_1, j_1; i_2, j_2) := \mathbb{1}_{\{i_1+j_1=i_2+j_2\}} q^{\frac{2j_1-j_1^2}{4} - \frac{2j_2-j_2^2}{4} + \frac{i_2^2+i_1^2}{4} + \frac{i_2(j_2-1)+i_1 j_1}{2}} \tag{2.3}$$
$$\times \frac{\nu^{j_1-i_2}\alpha^{j_2-j_1+i_2}(-\alpha\nu^{-1};q)_{j_2-i_1}}{(q;q)_{i_2}(-\alpha;q)_{i_2+j_2}(q^{J+1-j_1};q)_{j_1-j_2}} {}_4\overline{\phi}_3\left(\begin{matrix} q^{-i_2}; q^{-i_1}, -\alpha q^J, -q\nu\alpha^{-1} \\ \nu, q^{1+j_2-i_1}, q^{J+1-i_2-j_2} \end{matrix} \middle| q, q\right),$$

where $\nu = q^{-I}$. The term $(a, q)_n$ that appears in the above expression is the q-Pochhammer symbol, and the function ${}_4\overline{\phi}_3$ is the regularized terminating basic hyper-geometric series. For more precise details see [33, Theorem 3.15].

We shall think of the four tuples $(i_1, j_1; i_2, j_2)$ as the number of lines on the bottom, left, top, and right of a vertex. We think of lines flowing into the vertex from the bottom and the left, and flowing out of the vertex from the top and the right, see Figure 2.6.

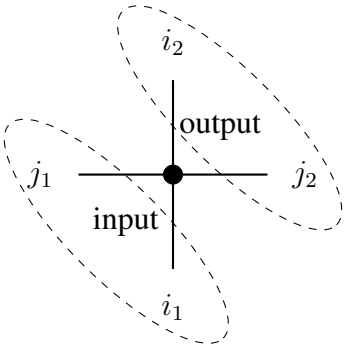


Figure 2.6: The vertex configuration labeled by the four tuple $(i_1, j_1; i_2, j_2)$ has weight $L_{\alpha}^{I,J}(i_1, j_1; i_2, j_2)$. The vertex absorbs i_1 input lines from the bottom, j_1 input line from the left, and produces i_2 output lines above and j_2 output lines to the right.

The indicator in (2.3) implies that the model is *conservative*, i.e. we only consider vertex configurations where the number of lines entering the vertex from the bottom and left is equal to that leaving the vertex above and to the right.

One can deduce from [33, Proposition 2.3] that $L_{\alpha,q}^{I,J}$ is a stochastic matrix if either 1) $q \in [0, 1), \alpha < q^{-I-J+1}$ or 2) $q > 1, \alpha \in (-q^{-I-J+1}, 0)$. In the following, we assume this is satisfied when we talk about the SHS6V model.

We can define the SHS6V model and its height function h^{hs} in a similar way to the S6V model using the stochastic matrix $L_{\alpha,q}^{I,J}$ to sequentially assign a configuration of lines to each vertex on the first quadrant. Let $i_{x,y}$ and $j_{x,y}$ denote the number of lines leaving the vertex (x, y) from above and from the right. Note that $i_{x,y}$ equals the number of lines that enter $(x, y + 1)$ from the bottom and $j_{x,y}$ equals the number of lines that enter $(x + 1, y)$ from the left. We define the *step initial condition* where we set $j_{0,y} = J$ for every $y \in \mathbb{Z}_{\geq 1}$ and $i_{x,0} = 0$ for every $x \in \mathbb{Z}_{\geq 1}$. In other words, we consider the boundary condition where we let J lines enter each vertex on the left boundary and let no lines enter the bottom boundary. The SHS6V model reduces to the S6V model with $b_1 = L_{\alpha,q}^{1,1}(1, 0; 1, 0) = \frac{1+\alpha q}{1+\alpha}$ and $b_2 = L_{\alpha,q}^{1,1}(0, 1; 0, 1) = \frac{\alpha+q^{-1}}{1+\alpha}$.

It turns out that using a procedure called fusion, the SHS6V model can be realized as a fused inhomogeneous S6V model with periodicities. This procedure goes back to [32]. We also recommend [33, Section 3] for a more probabilistic explanation.

Define $\beta(i,j) := (i \bmod I) + (j \bmod J)$ and

$$b_1(i,j) := L^{1,1}_{\alpha q^{\beta(i,j)},q}(1,0;1,0) = \frac{1+\alpha q^{\beta(i,j)+1}}{1+\alpha q^{\beta(i,j)}},$$
$$b_2(i,j) := L^{1,1}_{\alpha q^{\beta(i,j)},q}(0,1;0,1) = \frac{\alpha q^{\beta(i,j)}+q^{-1}}{1+\alpha q^{\beta(i,j)}}.$$

It is clear that $b_1(\cdot,\cdot), b_2(\cdot,\cdot)$ both have period I in the first coordinate and period J in the second coordinate. As in Section 2.1.1, we use h to denote the height function of the inhomogeneous S6V model with the parameters b_1, b_2 defined above. It turns out that h^{hs} and h have the same distribution if we fuse J number of rows into one row and I number of columns into one column for the inhomogeneous S6V model.

Lemma 2.1.8. *The stochastic process $\{h^{\mathsf{hs}}(x,y)\}_{x,y}$ is equal in distribution to $\{h(Ix,Jy)\}_{x,y}$.*

Proof. The result is due to fusion, see [33, Proposition 3.15] or [141, Proposition 8.2]. □

Using the above lemma together with Theorem 2.1.2, we obtain that the SHS6V model with the step initial condition converges almost surely to a deterministic limit shape.

Corollary 2.1.9. *There exists a deterministic Lipschitz function $g^{\mathsf{hs}}(x,y)$, such that with probability 1, we have for all $x, y \in \mathbb{R}_{\geq 0}$,*

$$\lim_{n\to\infty} \frac{h^{\mathsf{hs}}(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g^{\mathsf{hs}}(x,y).$$

Remark 2.1.10. *We did not see the expression of g^{hs} written down in the literature. However, using the techniques in [31] which exactly compute the limit shape of the S6V model, it is not hard to see that $g^{\mathsf{hs}}(x,y) = Jy - \int_0^x u(y,r)dr$, where $u(t,x)$ represents the density function of the vertical lines in the SHS6V model at time nt around location nx as $n \to \infty$. In particular, u is the unique entropy solution to Burger's equation*

$$\partial_t u = \partial_x(G(u)), \qquad u(0,x) = I\!\!1_{\{x\leq 0\}}.$$

Here $G(\rho)$ denotes the flux which records the density of horizontal lines if we start the SHS6V model from a stationary distribution where the density of the vertical lines is ρ. The function G can be explicitly computed since the stationary distribution of the SHS6V model has been explicitly given in [109, 110].

2.1.5 Proof ideas

In this section, we explain the idea behind the proof of Theorem 2.1.5. For the discrete Hammersley process defined in Section 2.1.3, the strong law of large numbers follows as a direct consequence of the superadditive ergodic theorem (see Theorem 2.3.1 for the full statement). To apply the superadditive ergodic theorem, one needs to construct a family of superadditive random variables $\{X_{m,n} : 0 \le m \le n\}$ where $\{X_{0,n}, n \ge 0\}$ records the height function. A subfamily of the random variables also needs to be ergodic. For the discrete Hammersley process, one can simply define

$$X_{m,n} := \max\Big\{L : \text{ there exist integer points } (x_1, y_1) \prec \cdots \prec (x_L, y_L) \in \xi \cap [m+1, n] \times [m+1, n]\Big\}. \tag{2.4}$$

It is straightforward to check that $\{X_{m,n} : 0 \le m \le n\}$ satisfies the condition required by the superadditive ergodic theorem.

For the CS6V model, one can also define a family of random variables $\{X_{m,n} : 0 \le m \le n\}$ by slightly modifying the above definition and setting

$$X_{m,n} := \max\Big\{L : \text{ there exist integer points } (x_1, y_1) \preceq \cdots \preceq (x_L, y_L) \in \eta \cap [m+1, n] \times [m+1, n]\Big\},$$

where $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$, and η represents the set of vertices of type III and VI in Figure 2.3. It is easy to verify that the definition of $X_{m,n}$ above coincides with that in

(2.4) when $b_1 = 0$ and $b_2 = 1 - p$.

We check whether the conditions in the superadditive ergodic theorem are still satisfied for this modified family of random variables. Indeed, the random variables are still superadditive, and $\{X_{0,n} : n \geq 0\}$ records the height function as desired. However, the ergodic (and even stationary) property no longer holds, since the random variable $X_{m,n}$ depends on the lines entering the box $[m+1, n] \times [m+1, n]$ from the left and bottom boundaries, and the distribution of these lines varies with m, n.

To overcome this difficulty, we apply a similar idea as in [17] and construct a colored CS6V model. The colored CS6V model is a generalization of the CS6V model, where each line is assigned a color. We denote the different colors by integers $i \in \mathbb{N}$, and we say that the color i has higher priority than the color j if $i < j$. We allow multiple (but only finitely many) lines with different colors to travel together, with the restriction that lines traveling together must have different colors. Note that although the SHS6V model introduced in Section 2.1.4 also allows multiple lines to travel together, it is a single-colored model and does not have much relation to the colored CS6V model discussed here.

The colored CS6V model is defined by specifying a sampling rule for when horizontal lines and vertical lines meet. The sampling rule is given by a family of stochastic matrices $\{\mathsf{L}^n, n \in \mathbb{N}\}$ that are consistent. More concretely, the matrix $\mathcal{L}^n$ has both rows and columns indexed by $\{0,1\}^n \times \{0,1\}^n$. The matrix elements are given by $\mathsf{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l})$, where the four vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \{0,1\}^n$ specify the number of lines (either zero or one) of each color in $\{1, \ldots, n\}$ on the bottom, left, top, and right of an intersection, respectively (see Figure 2.7). The stochastic matrices give a probability measure on the output lines $\mathbf{k}, \mathbf{l}$ from an intersection point given the input lines $\mathbf{i}, \mathbf{j}$. In Section 2.2, we will explicitly define the matrices $\{\mathcal{L}^n\}_{n \geq 1}$ and show that they satisfy three properties. We will now give a brief overview of these properties, and the precise statements will be given in Propositions 2.2.6, 2.2.7, and 2.2.8.

Property 1 (Color Ignorance): Lines with higher priority colors ignore those with lower priority colors. For instance, the lines with colors that belong to $\{1, \ldots, m\}$ ignore the behavior of lines

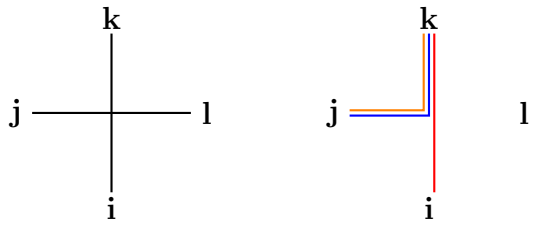


Figure 2.7: **Left panel:** Fix $n \in \mathbb{N}$. At an intersection point, we have lines with colors in $\{1, \dots, n\}$ in each direction, with at most one line per color in each direction. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \{0,1\}^n$ denote the number of lines on the bottom, left, top and right directions, respectively, where the m-th coordinate of each vector records the number of lines with color m. **Right panel:** Take $n = 3$. Let red, blue, and orange denote the colors 1, 2, and 3. We illustrate an example of the configuration with $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,1)$, $\mathbf{k} = (1,1,1)$, and $\mathbf{l} = (0,0,0)$.

with colors greater than m. This means that if we sample the n-colored model and ignore the lines with colors greater than m, the remaining lines will behave as the m-colored model. Because of this, we can define the random variables $\{X_{m,n}, 0 \le m \le n\}$ in a way that maintains ergodicity, as we will see in Section 2.3.

Property 2 (Mod 2 Erasure): Fix arbitrary integers $1 \le r_1 < \cdots < r_m \le n$. We can project the matrix $\mathcal{L}^n$ to $\mathcal{L}^m$ if we replace the colors in $\{r_{k-1}+1, \dots, r_k\}$ with color k for each $k \in \{1, \dots, m\}$ and then erase every pair of lines with the same color traveling together. Because of this property, we can project the colored CS6V model down to the single-colored CS6V model. This will ensure that the random variables $\{X_{0,n}, n \ge 0\}$ will record the height function of the CS6V model, which is the quantity that we are interested in studying.

Property 3 (Monotonicity of the Height Function): Suppose we have a sampling of the two-colored CS6V configuration on a rectangle $[1,x] \times [1,y]$ where we assign the vertex configurations in the rectangle using the matrix L^2, and we allow only second color lines to enter from the left and bottom boundary. We can naturally extend the definition of $H(x,y)$ to apply to the case of non-empty boundary data by defining $H(x,y)$ as the sum of the number of lines entering the box $[1,x] \times [1,y]$ from the left boundary and the number of lines exiting the box $[1,x] \times [1,y]$ from the top boundary. Let $H^1(x,y)$ be the height function when we only consider lines of the first color. Let $H^2(x,y)$ be the height function of the projection of both colors to the single-colored model

after we applied the procedure of mod 2 erasure. Then

$$H^1(x,y) \le H^2(x,y).$$

In other words, adding a second color to the model does not decrease the height function. This property will be important for proving the superadditivity of the random variables $\{X_{m,n}, 0 \le m \le n\}$.

Novelty of the paper

In [17], we studied the t-PNG model, which is a one-parameter model. The construction of the L-matrices therein comes from a sophisticated guess. The proof of stochasticity of $\{\mathsf{L}^n\}_{n\ge1}$ and Properties 1–3 for the t-PNG model is then based on a case by case check. The CS6V model is a two-parameter model, so defining the colored CS6V model via guessing is more challenging. One novelty of the paper is that we use a Boolean-type product to define the L-matrices, which significantly simplifies both the definition and the proof of the properties. This Boolean-type product construction can also be applied to give a simpler definition of the colored t-PNG model, see Section 2.4.

Other novelties come from working in a discrete setting and the inhomogeneity in the weights. For the t-PNG model, due to a scaling property, we only need to prove the strong law of large numbers in the diagonal direction. For the CS6V model, there is no similar scaling property, so we need to prove the strong law of large numbers in every direction. To this end, for each rational direction, we construct a colored model on the first quadrant and prove the strong law of large numbers in that direction. The convergence in all directions then follows from a density argument.

Another possible approach

We want to remark that it is also possible to prove Theorem 2.1.2 without complementing the S6V model. Indeed, [147, 148] prove a law of large numbers for a family of translation invariant

exclusion processes. Their proof also relies on the superadditive ergodic theorem, but in a different way. An important ingredient is to couple the exclusion processes so that we can run them under the same random environment, starting at different times and initial data. The S6V model is also an exclusion-type interacting particle system, and to apply their idea, we need to find such a coupling for the S6V model. Another interesting question is to study the almost sure convergence of the S6V model to the limit shape under general initial data. Note that this result has been proved for a family of finite-range exclusion processes in [149]. We leave both of these questions to future work.

Outline

In Section 2.2, we define a family of $\mathcal{L}$-matrices using a Boolean-type product. Then we prove that the $\mathcal{L}$-matrices are stochastic and satisfy Properties 1–3. In Section 2.3, we construct a colored CS6V model on the first quadrant using the $\mathcal{L}$-matrices. We apply Liggett's superadditive ergodic theorem to the model and prove Theorem 2.1.5 (which is equivalent to Theorem 2.1.2).

2.2 Definition of the L-matrices

In this section, we first define the $\mathcal{L}^1$-matrix, which encodes the weights of the usual single-colored CS6V model. Then, we introduce the Boolean-type product and use it to define the $\mathcal{L}^n$-matrices. In the following, we take generic parameters $b_1, b_2 \in [0,1]$.

Definition 2.2.1. *The matrix $\mathcal{L}^1$ is indexed by a 4-tuple $i,j,k,l \in \{0,1\}$, where i,j,k,l denote the number of lines (either zero or one) on the bottom, left, top, and right of a vertex. We define*

$$\mathcal{L}^1(1,0;1,0) = 1, \qquad \mathcal{L}^1(0,1;0,1) = 1, \qquad \mathcal{L}^1(0,0;0,0) = b_2,$$
$$\mathcal{L}^1(0,0;1,1) = 1-b_2, \qquad \mathcal{L}^1(1,1;1,1) = b_1, \qquad \mathcal{L}^1(1,1;0,0) = 1-b_1.$$

For all other $i,j,k,l \in \{0,1\}$, we set $\mathcal{L}^1(i,j;k,l) = 0$. For fixed input lines $i,j \in \{0,1\}$, $\mathcal{L}^1(i,j;\cdot,\cdot)$ is a probability measure on the output lines.

We proceed to give a closed form to the matrices $\{\mathcal{L}^n\}_{n\geq 1}$. Let

$$W' := \{0, 1, b_1, b_2, 1-b_1, 1-b_2\}$$

denote the set of possible weights for the CS6V model. Note that we treat the elements in W' as indeterminates, so the size of W' does not depend on the values of b_1 and b_2. We define a product $*$ on the extended space

$$W := \{0, 1, b_1, b_2, (1-b_1), (1-b_2), b_1b_2, b_1(1-b_2), (1-b_1)b_2, (1-b_1)(1-b_2)\}. \tag{2.5}$$

For $x, y \in W$, we define $[x]$ to be the set of factors of x. More precisely, $[x] = \{x\}$ if $x \in W'$, $[x] = \{b_1, b_2\}$ if $x = b_1b_2$ and similarly for other values. We define

$$x * y := \begin{cases} 0 & \text{if } \{b_1, 1-b_1\} \subseteq [x] \cup [y] \text{ or } \{b_2, 1-b_2\} \subseteq [x] \cup [y], \\ \prod_{z \in [x] \cup [y]} z & \text{otherwise.} \end{cases}$$

It is straightforward to check that W is closed under $*$. Moreover, we readily obtain the following.

Proposition 2.2.2. *The product $*$ satisfies the following three properties.*

1. *(commutativity) $a * b = b * a$, for $a, b \in W$.*
2. *(associativity) $a * (b * c) = (a * b) * c$, for $a, b, c \in W$.*
3. *(distributivity) Suppose that $a_1, ..., a_n \in W$ such that $\sum_{i=1}^n a_i \in W$. For any $c \in W$,*

$$\sum_{i=1}^n c * a_i = c * \sum_{i=1}^n a_i.$$

Alternatively, we can also define $*$ first on W' and extend it to W. For $x, y \in W'$, we define $x * x = x$, and $x * y = 0$ if $\{x, y\} = \{b_1, 1-b_1\}$ or $\{b_2, 1-b_2\}$. This is why we call $*$ a Boolean-type product. For other choices of $x, y \in W'$, we define $x * y = xy$. We then extend the product $*$

from W' to W using the commutative and associative properties.

Remark 2.2.3. *This Boolean product structure can also be found when taking the expectation of products of independent Bernoulli random variables. For example, if we let B_0, B_1, and B_2 be independent Bernoulli random variables with parameters $0, b_1$, and b_2 respectively, then we have $b_i * b_j = \mathbb{E}[B_i B_j]$ and $(1 - b_i) * b_j = E[(1 - B_i)B_j]$ for $0 \leq i \leq j \leq 2$.*

We proceed to define the $\mathcal{L}^n$-matrices, using W as the set of possible weights. For $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$ and $r \in \{1, \ldots, n\}$, we define the r**-fold projection**

$$\mathfrak{s}_r(\mathbf{x}) = \Big(\sum_{m=1}^{r} x_m \Big) \mod 2.$$

As an example, consider the vector $\mathbf{x} = (1, 0, 1, 1)$. We have

$$\mathfrak{s}_1(\mathbf{x}) = 1, \;\; \mathfrak{s}_2(\mathbf{x}) = 1, \;\; \mathfrak{s}_3(\mathbf{x}) = 0, \;\; \mathfrak{s}_4(\mathbf{x}) = 1.$$

Definition 2.2.4. *Fix arbitrary $n \in \mathbb{Z}_{\geq 1}$. The matrix $\mathcal{L}^n$ is indexed by a 4-tuple $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \{0, 1\}^n$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ denote the number of lines (either zero or one) of each color in $\{1, \ldots, n\}$ on the bottom, left, top, and right of an intersection, respectively. We define the matrix $\mathcal{L}^n$ via*

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) := \prod_{r \in \{1, \ldots, n\}}^{*} \mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}), \mathfrak{s}_r(\mathbf{j}); \mathfrak{s}_r(\mathbf{k}), \mathfrak{s}_r(\mathbf{l})), \tag{2.6}$$

where $\prod^*$ denotes the Boolean-type product of multiple terms under $*$.

2.2.1 Properties of $\mathcal{L}^n$

We proceed to state and prove that the matrices $\mathcal{L}^n$ are stochastic and that they satisfy the properties mentioned above. The key to the proofs of stochasticity, color ignorance, and mod 2 erasure is the distributivity of the Boolean-type product, which allows us to manipulate the sums and products as we normally would.

Proposition 2.2.5 (Stochasticity). *$\mathcal{L}^n$ is a stochastic matrix, i.e., the entries of $\mathcal{L}^n$ are non-negative, and for any $\mathbf{i}, \mathbf{j} \in \{0,1\}^n$,*

$$\sum_{\mathbf{k},\mathbf{l} \in \{0,1\}^n} \mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = 1.$$

Proof. We prove this by induction. The case $n = 1$ follows from examination. Suppose that $\mathcal{L}^{n-1}$ is stochastic. Using the definition for $\mathcal{L}^n$, we can factor it as follows:

$$\mathcal{L}^n(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}) = \mathcal{L}^{n-1}(\mathbf{i}_{[1,n-1]}, \mathbf{j}_{[1,n-1]}, \mathbf{k}_{[1,n-1]}, \mathbf{l}_{[1,n-1]}) * \mathcal{L}^1(\mathfrak{s}_n(\mathbf{i}), \mathfrak{s}_n(\mathbf{j}); \mathfrak{s}_n(\mathbf{k}), \mathfrak{s}_n(\mathbf{l})). \tag{2.7}$$

Note that $(\mathfrak{s}_n(\mathbf{k}), \mathfrak{s}_n(\mathbf{l}))$ equals each element of $\{0,1\}^2$ exactly once when we vary $\mathbf{k}, \mathbf{l}$ under the restriction $\mathbf{k}_{[1,n-1]} = \mathfrak{k}$ and $\mathbf{l}_{[1,n-1]} = \mathfrak{l}$. Applying this and Proposition 2.2.2 to the right-hand side of (2.7) yields

$$\begin{aligned}
&\sum_{\mathbf{k},\mathbf{l} \in \{0,1\}^n} \mathcal{L}^n(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}) \\
&= \left(\sum_{\mathfrak{k},\mathfrak{l} \in \{0,1\}^{n-1}} \mathcal{L}^{n-1}(\mathbf{i}_{[1,n-1]}, \mathbf{j}_{[1,n-1]}; \mathfrak{k}, \mathfrak{l}) \right) * \sum_{k,l \in \{0,1\}} \mathcal{L}^1(\mathfrak{s}_n(\mathbf{i}), \mathfrak{s}_n(\mathbf{j}); k, l) \\
&= 1.
\end{aligned}$$

The last equality follows from the stochasticity of both $\mathcal{L}^{n-1}$ and $\mathcal{L}^1$. □

Proposition 2.2.6 (Color Ignorance). *Fix $m \in \{1, \ldots, n\}$ and $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{l} \in \{0,1\}^m$. For all $\mathbf{i}, \mathbf{j} \in \{0,1\}^n$ such that $\mathbf{i}_{[1,m]} = \mathfrak{i}$ and $\mathbf{j}_{[1,m]} = \mathfrak{j}$, we have*

$$\sum_{\substack{\mathbf{k}_{[1,m]} = \mathfrak{k}, \\ \mathbf{l}_{[1,m]} = \mathfrak{l}}} \mathcal{L}^n(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \mathcal{L}^m(\mathfrak{i}, \mathfrak{j}; \mathfrak{k}, \mathfrak{l}). \tag{2.8}$$

Proof. By Definition 2.2.4, we have

$$\sum_{\substack{\mathbf{k}_{[1,m]}=\mathfrak{k},\\ \mathbf{l}_{[1,m]}=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) * \left(\sum_{\substack{\mathbf{k}_{[1,m]}=\mathfrak{k},\\ \mathbf{l}_{[1,m]}=\mathfrak{l}}} \prod_{i=m+1}^{n*} \mathcal{L}^1(\mathfrak{s}_i(\mathbf{i}),\mathfrak{s}_i(\mathbf{j});\mathfrak{s}_i(\mathbf{k}),\mathfrak{s}_i(\mathbf{l})) \right).$$

The last product can be rewritten as $\mathcal{L}^{n-m}(i,j,k,\ell)$ where $i = (\mathfrak{s}_{m+1}(\mathbf{i}),\cdots,\mathfrak{s}_n(\mathbf{i}))$, and where j,k, and ℓ are defined similarly. Summing $\mathcal{L}^{n-m}(i,j,k,\ell)$ over $\mathbf{k}_{[1,m]} = \mathfrak{k}$ and $\mathbf{l}_{[1,m]} = \mathfrak{l}$ reduces to summing the same object over $k,\ell \in \{0,1\}^{n-m}$, which equals 1 due to stochasticity. This concludes the result. □

We call π a partition of $\{1,\ldots,n\}$ if it takes the form of

$$\pi = \big\{\{1,\ldots,r_1\},\{r_1+1,\ldots,r_2\},\ldots,\{r_{m-1}+1,\ldots,r_m\}\big\}$$

for some $m \le n$ and $1 = r_1 < r_2 < \cdots < r_m = n$. We define a projection map $g_\pi : \{0,1\}^n \to \{0,1\}^m$ such that

$$g_\pi(x_1,\ldots,x_n) = \left(\Big(\sum_{i=r_{k-1}+1}^{r_k} x_i \Big) \bmod 2 \right)_{k=1}^{m}.$$

We define $\ell(\pi) = m$ to be the length of the partition π.

Proposition 2.2.7 (Mod 2 Erasure)**.** *Fix a partition π of $\{1,\ldots,n\}$ such that $\ell(\pi) = m$. Fix $\mathfrak{i},\mathfrak{j},\mathfrak{k},\mathfrak{l} \in \{0,1\}^m$. For all $\mathbf{i},\mathbf{j} \in \{0,1\}^n$ satisfying $g_\pi(\mathbf{i}) = \mathfrak{i}$ and $g_\pi(\mathbf{j}) = \mathfrak{j}$, we have*

$$\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).$$

Proof. Define $A = \{r_1,\ldots,r_m\}$. We define i,j to be the unique $n-m$ dimensional vectors such

that

$$(\mathfrak{s}_1(i),\dots,\mathfrak{s}_{n-m}(i)) = (\mathfrak{s}_1(\mathbf{i}),\dots,\widehat{\mathfrak{s}}_{r_1}(\mathbf{i}),\dots,\widehat{\mathfrak{s}}_{r_m}(\mathbf{i}),\dots,\mathfrak{s}_n(\mathbf{i})),$$
$$(\mathfrak{s}_1(j),\dots,\mathfrak{s}_{n-m}(j)) = (\mathfrak{s}_1(\mathbf{j}),\dots,\widehat{\mathfrak{s}}_{r_1}(\mathbf{j}),\dots,\widehat{\mathfrak{s}}_{r_m}(\mathbf{j}),\dots,\mathfrak{s}_n(\mathbf{j})),$$

where $\widehat{x}$ denotes the removal of x in the vector. We have

$$\begin{aligned}&\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l})\\ &= \sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \left(\prod_{i\in A}^{*} \mathcal{L}^1(\mathfrak{s}_i(\mathbf{i}),\mathfrak{s}_i(\mathbf{j});\mathfrak{s}_i(\mathbf{k}),\mathfrak{s}_i(\mathbf{l}))\right) * \left(\prod_{i\notin A, i\le n}^{*} \mathcal{L}^1(\mathfrak{s}_i(\mathbf{i}),\mathfrak{s}_i(\mathbf{j});\mathfrak{s}_i(\mathbf{k}),\mathfrak{s}_i(\mathbf{l}))\right).\end{aligned} \tag{2.9}$$

Since $g_\pi(\mathbf{k}) = \mathfrak{k}$ and $g_\pi(\mathbf{l}) = \mathfrak{l}$, we have $\prod^{*}_{i\in A} \mathcal{L}^1(\mathfrak{s}_i(\mathbf{i}),\mathfrak{s}_i(\mathbf{j});\mathfrak{s}_i(\mathbf{k}),\mathfrak{s}_i(\mathbf{l})) = \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l})$. Using this and Proposition 2.2.2, the right-hand side of (2.9) can be rewritten as

$$\begin{aligned}&\mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) * \left(\sum_{\substack{g_\pi(\mathbf{k})=\mathfrak{k}\\ g_\pi(\mathbf{l})=\mathfrak{l}}} \prod_{i\notin A, i\le n}^{*} \mathcal{L}^1(\mathfrak{s}_i(\mathbf{i}),\mathfrak{s}_i(\mathbf{j});\mathfrak{s}_i(\mathbf{k}),\mathfrak{s}_i(\mathbf{l}))\right)\\ &= \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}) * \left(\sum_{k,l\in\{0,1\}^{n-m}} \mathcal{L}^{n-m}(i,j,k,l)\right)\\ &= \mathcal{L}^m(\mathfrak{i},\mathfrak{j};\mathfrak{k},\mathfrak{l}).\end{aligned}$$

The first equality is due to (2.6). The last equality follows from the stochasticity of $\mathcal{L}^{n-m}$. □

The next proposition implies that if we start with a CS6V model on $[1,x]\times[1,y]$ with empty boundary data and wish to add some boundary data (i.e., lines entering the box from the bottom and the left), then we can use the colored model to sample the CS6V model with the boundary data in a way so that the height function at (x,y) cannot decrease with the addition of the boundary data.

Proposition 2.2.8 (Monotonicity of the height function)**.** *Suppose we have a two-colored CS6V model on* $[1, x] \times [1, y]$*, where the vertex configurations are assigned using the matrix* L^2*, and the only lines entering from the bottom and left boundaries are lines of the second color. Let* $H^1(x, y)$ *denote the height function when we only consider lines of the first color (i.e., the lines of the* 1*-fold projection). Let* $H^2(x, y)$ *denote the height function of the* 2*-fold projection of both colors to the single-colored model. Then*

$$H^1(x, y) \leq H^2(x, y).$$

Proof. Consider the rectangle $[1, x] \times [1, y]$. Let A_1 (resp. B_1) be the number of first-color lines leaving the top (resp. right) boundary. Let A_2 (resp. B_2) be the number of second-color lines that leave the same boundary and are accompanied by a first-color line. Let A_3 (resp. B_3) be the number of second-color lines that leave the same boundary without accompanying first-color lines. Let A_4 (resp. B_4) be the number of second-color lines entering the left (resp. bottom) boundary. We have

$$H^1(x, y) = A_1 = B_1, \qquad H^2(x, y) = A_1 - A_2 + A_3 + A_4 = B_1 - B_2 + B_3 + B_4.$$

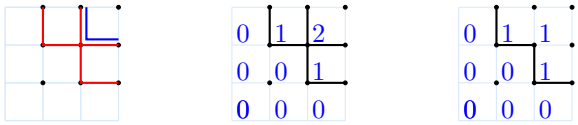
By examining all possible two-color configurations in Figure 2.11, we know that for a vertex configuration, the number of second-color lines entering from the left and bottom is no smaller than the number of lines exiting above and to the right. This implies that the number of second-color lines entering the rectangle from the left and bottom boundaries is no smaller than the number of second-color lines exiting the top and right boundaries. Hence, $A_4 + B_4 \geq A_2 + A_3 + B_2 + B_3$. Hence, we have either $A_4 \geq A_2 + A_3$ or $B_4 \geq B_2 + B_3$. Assume without loss of generality that the first inequality holds. Then we have

$$H^1(x, y) = A_1 \leq A_1 - A_2 + A_3 + A_4 = H^2(x, y). \qquad \square$$

Remark 2.2.9. *It is possible to define a colored model in the same way as in Definition 2.2.4, using*

ordinary multiplication instead of the Boolean-type product. This will still yield a model with the properties of color ignorance and mod 2 erasure. However, with this definition, the colored model will no longer satisfy the monotonicity of the height function. To see why this is, note that with the Boolean definition, we cannot have the configurations in Figure 2.8. This is because these configurations have weights $b_1 * (1 - b_1) = 0$ *and* $b_2 * (1 - b_2) = 0$*. However, with the ordinary product, these configurations would have non-zero weights. As demonstrated in Figure 2.9, these configurations will yield samplings that fail the monotonicity of the height function.*

Figure 2.8: The weights of the above configurations equal $b_1 * (1 - b_1) = b_2 * (1 - b_2) = 0$ using the Boolean-type product, and equal $b_1(1 - b_1)$, $b_2(1 - b_2)$ using ordinary multiplication.



1-fold projection 2-fold projection

Figure 2.9: A sampling that violates the monotonicity of the height function since the height function in the top-right corner of its 2-fold projection is smaller than the height function in the top-right corner of its 1-fold projection.

2.3 The colored model and proof of Theorem 2.1.5

With the help of the matrices $\{\mathsf{L}^n\}_{n\in\mathbb{Z}_{\geq 1}}$, we apply the superadditive ergodic theorem to the colored CS6V model and prove the existence of the limit shape. The argument is similar to [17, Section 3] with certain adaptations.

We are going to construct $\{X_{m,n} : 0 \leq m \leq n\}$ as discussed in the introduction, using the colored CS6V model. Before doing that, let us recall Liggett's superadditive ergodic theorem. For our purposes, we formulate it in the superadditive setting by placing negative signs where necessary.

Theorem 2.3.1 ([1, page 277]). *Suppose $\{X_{m,n}\}$ is a collection of random variables that is indexed by integers $0 \le m \le n$ and satisfies:*

(i) Almost surely $X_{0,0} = 0$ and $X_{0,n} \ge X_{0,m} + X_{m,n}$ for $0 \le m \le n$.

(ii) For each $k \ge 1$, $\{X_{(n-1)k,nk} : n \ge 1\}$ is a stationary and ergodic process.

(iii) $\{X_{m,m+k} : k \ge 0\} \stackrel{d}{=} \{X_{m+1,m+k+1} : k \ge 0\}$ for each $m \ge 0$.

(iv) $\mathbb{E}[X_{0,1}^-] < \infty$ where $x^- = \max(-x, 0)$.

Then there exists a constant $\gamma = \sup_{n\ge1} \frac{\mathbb{E}[X_{0,n}]}{n} \in (-\infty, \infty]$ satisfying

$$\gamma = \lim_{n\to\infty} \frac{X_{0,n}}{n} \text{ a.s.}$$

We proceed to define a colored CS6V model on the first quadrant. Recall that I, J are the periods of the weights of the CS6V model, as in Assumption 2.1.1. Fix a direction $(x, y) \in \mathbb{Q}^2_{>0}$, and define

$$N := \text{ the smallest positive integer such that } \frac{Nx}{I}, \frac{Ny}{J} \in \mathbb{N}. \tag{2.10}$$

We have omitted the dependence of N on x, y. Assign color $-k$ to the vertex (a, b) if either $a \in [N(k-1)x + 1, Nkx]$ or $b \in [N(k-1)y + 1, Nky]$, see the left panel of Figure 2.10. We use negative numbers to label the colors so that we can have infinitely many colors of increasing priority. Given the input lines of a vertex with color $-k$, we sample the output lines according to stochastic matrices $\mathcal{L}^k$ with parameters $b_1 = b_1(i, j)$ and $b_2 = b_2(i, j)$. The colored CS6V model is then sampled sequentially at the vertices $(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), \ldots$ to the entire quadrant, using the L-matrices assigned to different vertices. By definition, $\mathcal{L}^k$ is defined for the positive colors $1, \ldots, k$, so we just map the colors $-k, \ldots, -1$ to $1, \ldots, k$, preserving their order. One can check that if a vertex of color $-k$ has no input lines, then the only possible output lines emanating from that vertex can be of color $-k$. See the right panel of Figure 2.10 for a sampling of the colored CS6V model.

We proceed to define the random variables $\{X_{m,n}, m, n \in \mathbb{Z}_{\geq 0}, m \leq n\}$ that satisfy the condition of Theorem 2.3.1. Let $v_{z,Nny}$ denote the number of vertical lines exiting the vertex (z, Nny) with colors that belong to $\{-n, \dots, -m-1\}$. We define

$$X_{m,n} := \sum_{z=Nmx+1}^{Nnx} \big(v_{z,Nny} \bmod 2\big). \tag{2.11}$$

The next proposition encodes the height function of the CS6V model in terms of $\{X_{m,n} : 0 \leq m \leq n\}$.

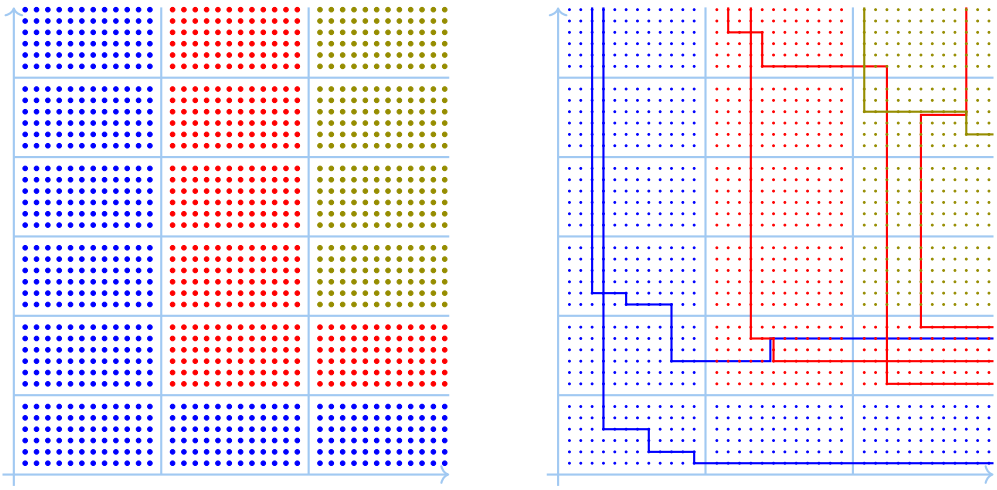


Figure 2.10: **Left panel:** We illustrate how we color the vertices on the first quadrant when $I = 3, J = 2, x = 2$, and $y = 1$. In this case, $N = 6$. Let blue, red, and olive represent the colors $-1, -2$, and -3. Each rectangle contains 12×6 vertices with the same color. **Right panel:** A sampling of the colored CS6V model. The lines emanating from a vertex inherit the color of that vertex. Due to the property of color ignorance, the red lines can be sampled ignoring the blue lines, and the olive lines can be sampled ignoring both the blue and red lines. Note that lines of different colors can travel together across the same vertices.

Proposition 2.3.2. *We have $\{X_{0,k}, k \in \mathbb{Z}_{\geq 0}\} = \{H(Nkx, Nky), k \in \mathbb{Z}_{\geq 0}\}$ where H is the height function of the single-colored CS6V model with periods I and J.*

Proof. It suffices to show that for each $n \in \mathbb{N}$, $\{X_{0,k}, k = 0, \dots, n\} \stackrel{d}{=} \{H(Nkx, Nky), k = 0, \dots, n\}$. The lines in the square $[\frac{1}{2}, Nnx + \frac{1}{2}] \times [\frac{1}{2}, Nny + \frac{1}{2}]$ have colors that belong to $\{-n, \dots, -1\}$. Replace these colors with a single color. Then, by the property of mod 2 erasure, the resulting model reduces to the single-colored CS6V model. Note that $X_{0,k}$ in (2.11) is

equal to the number of vertical lines that cross the segment $[\frac{1}{2}, Nkx + \frac{1}{2}] \times \{Nky + \frac{1}{2}\}$, which is $H(Nkx, Nky)$. This implies that $\{X_{0,k}, k = 0, \ldots, n\} \stackrel{d}{=} \{H(Nkx, Nky), k = 0, \ldots, n\}$. $\square$

Proposition 2.3.3. *The stochastic process* $\{X_{m,n} : m, n \in \mathbb{Z}_{\geq 0}, m \leq n\}$ *satisfies conditions (ii)–(iv) of Theorem 2.3.1.*

Proof. We first prove (ii). Consider the square $[Nmx+\frac{1}{2}, Nnx+\frac{1}{2}] \times [Nmy+\frac{1}{2}, Nny+\frac{1}{2}]$. There are lines flowing inside through the left boundary $\{Nmx+\frac{1}{2}\} \times [Nmy+\frac{1}{2}, Nny+\frac{1}{2}]$ and the bottom boundary $[Nmx+\frac{1}{2}, Nnx+\frac{1}{2}] \times \{Nmy+\frac{1}{2}\}$. These lines have colors belonging to $\{-m, \ldots, -1\}$. The vertices in $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$ have colors in $\{-n, \ldots, -m - 1\}$. Note that the color i takes priority over j if $i < j$, so the lines that emanate from the vertices in $[Nmx+\frac{1}{2}, Nnx+\frac{1}{2}] \times [Nmy+\frac{1}{2}, Nny+\frac{1}{2}]$ have higher priority than the lines entering through the left and bottom boundaries. Therefore, by Proposition 2.2.6, the behavior of the lines with colors $\{-n, \ldots, -m-1\}$ in the rectangle $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$ does not depend on the lower priority lines entering from the left and bottom. Hence, the distribution of $X_{m,n}$ is independent of the number and location of the lines entering the bottom and left boundaries of $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$. This implies that for each $k \geq 1$, the random variables $\{X_{(n-1)k,nk}, n \geq 1\}$ are independent. It is straightforward to see that $X_{(n-1)k,nk}$ has the same distribution as $H(Nkx, Nky)$ for all $n \geq 1$, therefore this sequence is i.i.d, and hence it is stationary and ergodic.

We proceed to prove (iii). It suffices to show that for arbitrary $m \in \mathbb{Z}_{\geq 0}$,

$$\{X_{m,m+k}, k \geq 0\} \stackrel{d}{=} \{X_{0,k}, k \geq 0\}. \tag{2.12}$$

We look at the colored CS6V model restricted to $[mNx + \frac{1}{2}, \infty) \times [mNy + \frac{1}{2}, \infty)$. Note that there are lines with colors in $\{-m, \ldots, -1\}$ entering from the left and bottom boundaries of $[mNx + \frac{1}{2}, \infty) \times [mNy + \frac{1}{2}, \infty)$. By Proposition 2.2.6, the behavior of lines in $[mNx + \frac{1}{2}, \infty) \times [mNy + \frac{1}{2}, \infty)$ with colors less than $-m$ is not affected by the lower priority lines entering from the boundary. Since $b_1(mNx + \cdot, mNy + \cdot) = b_1(\cdot, \cdot)$ and $b_2(mNx + \cdot, mNy + \cdot) =$

$b_2(\cdot,\cdot)$, this implies that after a shift by (mNx, mNy), the lines with colors $i_1, \dots, i_k \in \mathbb{Z}_{\le -m-1}$ in $[mNx + \frac{1}{2}, \infty) \times [mNy + \frac{1}{2}, \infty)$ behave the same (in distribution) as the lines with colors $i_1 + m, \dots, i_k + m$ in $[\frac{1}{2}, \infty) \times [\frac{1}{2}, \infty)$. Hence, we conclude (2.12).

Finally, we have $X^-_{0,1} = 0$ because $X_{0,1}$ is non-negative. Hence, (iv) holds. □

Let us proceed to prove that $\{X_{m,n}, m, n \in \mathbb{Z}_{\ge 0}, m \le n\}$ also satisfies the superadditive condition (i) in Theorem 2.3.1. We begin with some preparation. In the square $[\frac{1}{2}, Nnx + \frac{1}{2}] \times [\frac{1}{2}, Nny + \frac{1}{2}]$, we replace the colors $\{-m, \dots, -1\}$ with the color -1 and replace the colors $\{-n, \dots, -m-1\}$ with the color -2. After that, as long as there are two lines with the same color that travel together, we erase them. By Proposition 2.2.7, the resulting model is a two-colored CS6V model. In particular, the vertices have color -1 in the L-shaped area $[\frac{1}{2}, Nmx + \frac{1}{2}] \times [\frac{1}{2}, Nny + \frac{1}{2}] \cup [\frac{1}{2}, Nnx + \frac{1}{2}] \times [\frac{1}{2}, Nmy + \frac{1}{2}]$. The vertices have color -2 in the square $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$.

For the resulting two-colored CS6V model, let Q_1 be the number of vertical lines with color -1 that cross the segment $[\frac{1}{2}, Nmx + \frac{1}{2}] \times \{Nmy + \frac{1}{2}\}$, let Q_2 be the number of vertical lines with color -2 that cross the segment $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times \{Nny + \frac{1}{2}\}$, and let P_1 be the number of horizontal lines with color -1 that cross $\{Nmx + \frac{1}{2}\} \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$. Finally, let R be the number of single vertical lines of either color that cross $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times \{Nny + \frac{1}{2}\}$ (i.e., lines that do not travel in a pair).

Lemma 2.3.4. *The following result holds:*

$$X_{0,m} = Q_1, \tag{2.13}$$

$$X_{m,n} = Q_2, \tag{2.14}$$

$$X_{0,n} = Q_1 + P_1 + R. \tag{2.15}$$

Proof. Recall that we obtain the two-colored CS6V model in $[\frac{1}{2}, Nnx + \frac{1}{2}] \times [\frac{1}{2}, Nny + \frac{1}{2}]$ by replacing the colors $\{-n, \dots, -m-1\}$ with the color -2, replacing the colors $\{-m, \dots, -1\}$ with the color -1, and erasing pairs of lines with the same color. The erasure corresponds to the

mod 2 erasure procedure in (2.11). Hence, equations (2.13) and (2.14) directly follow from (2.11).

We proceed to prove (2.15). Note that $X_{0,n}$ is the number of single vertical lines that cross the segment $[\frac{1}{2}, Nnx + \frac{1}{2}] \times \{Nny + \frac{1}{2}\}$ in the two-colored CS6V model. We decompose

$$X_{0,n} = Y_1 + R, \tag{2.16}$$

where Y_1 equals the number of vertical lines with the color -1 that leave $[\frac{1}{2}, Nmx+\frac{1}{2}]\times\{Nny+\frac{1}{2}\}$. We turn to the rectangle $[\frac{1}{2}, Nmx+\frac{1}{2}]\times[Nmy+\frac{1}{2}, Nny+\frac{1}{2}]$, in which there are only lines with color -1. The CS6V model is conservative in the sense that the number of lines on the right and bottom of a vertex equals the number of lines on the top and left of that vertex. Therefore, the number of lines that cross the bottom and right boundaries of the rectangle $[\frac{1}{2}, Nmx+\frac{1}{2}]\times[Nmy+\frac{1}{2}, Nny+\frac{1}{2}]$ is equal to the number of lines that cross the top and left boundaries, hence,

$$Y_1 = Q_1 + P_1. \tag{2.17}$$

Using this together with (2.16) and (2.17), we conclude (2.15). □

Proposition 2.3.5. *We have $X_{0,0} = 0$ and $X_{0,n} \geq X_{0,m} + X_{m,n}$ for $0 \leq m \leq n$. Hence, $\{X_{m,n}, m \leq n \in \mathbb{Z}_{\geq 0}\}$ satisfies (i) of Theorem 2.3.1.*

Proof. By definition, we have $X_{0,0} = 0$. We proceed to show that $X_{0,n} \geq X_{0,m} + X_{m,n}$. By Lemma 2.3.4, this is equivalent to $R + P_1 \geq Q_2$. This follows from applying Proposition 2.2.8 to the rectangle $[Nmx + \frac{1}{2}, Nnx + \frac{1}{2}] \times [Nmy + \frac{1}{2}, Nny + \frac{1}{2}]$, where only lower priority lines enter from the bottom-left boundary. □

Proposition 2.3.6. *Fix arbitrary $x, y \in \mathbb{Q}_{\geq 0}$, there exists a constant $\widetilde{g}(x, y) \in \mathbb{R}_{\geq 0}$ such that almost surely,*

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = \widetilde{g}(x, y).$$

Proof. If $x = 0$ or $y = 0$, we know that $H(\lfloor nx \rfloor, \lfloor ny \rfloor) = 0$ for all n, thus $\widetilde{g}(x, y) = 0$. Now we can assume that $x, y \in \mathbb{Q}_{>0}$. Recall the definition of $X_{m,n}$ from (2.11). By Propositions 2.3.3

and 2.3.5, $\{X_{m,n}, m \leq n \in \mathbb{Z}_{\geq 0}\}$ satisfy the conditions of the superadditive ergodic theorem. Therefore, there exists a constant $\gamma(x,y)$ such that almost surely, $\lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma(x,y)$. By Proposition 2.3.2, we know that almost surely, $\lim_{n\to\infty} \frac{H(nNx,nNy)}{n} = \gamma(x,y)$. Recall that N is the positive integer defined in (2.10). Since the height function $h : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a Lipschitz function, we conclude that almost surely,

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = \gamma(x,y)/N := \widetilde{g}(x,y). \qquad \square$$

Proof of Theorem 2.1.5. Using the fact that $\mathbb{Q}^2_{\geq 0}$ is countable and Proposition 2.3.6, almost surely,

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = \widetilde{g}(x,y)$$

for all $(x,y) \in \mathbb{Q}^2_{>0}$. By definition, H is a Lipschitz function, hence $\widetilde{g} : \mathbb{Q}^2_{\geq 0} \to \mathbb{R}$ is also a Lipschitz function. We can thus extend $\widetilde{g}$ to be a Lipschitz function on $\mathbb{R}^2_{\geq 0}$. By the density of $\mathbb{Q}^2_{\geq 0}$ in $\mathbb{R}^2_{\geq 0}$ and the Lipschitz property, we have that almost surely,

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = \widetilde{g}(x,y)$$

for all $(x,y) \in \mathbb{R}^2_{\geq 0}$. Taking $g(x,y) = y - \widetilde{g}(x,y)$, we conclude the proof. $\square$

2.4 The Boolean-type product for the t-PNG model

Taking $b_1 = t$ and $b_2 = 1$, the set W in (2.5) reduces to $\{0, t, 1-t, 1\}$. The Boolean-type product reduces to the following: for $a, b \in \{0, t, 1-t, 1\}$,

$$a * b := \begin{cases} 0 & \text{if } \{a,b\} = \{t, 1-t\}, \\ a & \text{if } a = b, \\ ab & \text{else.} \end{cases}$$

Recall from [17, Definition 1.7] that the $\mathcal{L}^n$-matrix of the t-PNG model is defined as

$$\mathcal{L}^n(\mathbf{i},\mathbf{j};\mathbf{k},\mathbf{l}) := \min_{r\in\{1,\dots,n\}}\Big(\mathcal{L}^1(\mathfrak{s}_r(\mathbf{i}),\mathfrak{s}_r(\mathbf{j});\mathfrak{s}_r(\mathbf{k}),\mathfrak{s}_r(\mathbf{l}))\Big),$$

where $\min$ is a modified version of the minimum. For $x_1,\dots,x_n \in \{0,t,1-t,1\}$,

$$\min\big(x_1,\dots,x_n\big) := \begin{cases} 0 & \text{if } x_i = t \text{ and } x_j = 1-t \text{ for some } i,j\in\{1,\dots,n\},\\ \min\big(x_1,\dots,x_n\big) & \text{else.}\end{cases}$$

It is straightforward to verify that $\min(a,b) = a*b$ for $a,b\in\{0,t,1-t,1\}$, and thus $\min(x_1,\dots,x_n) = x_1*\cdots*x_n$. Hence, the definition [17, Definition 1.7] is equivalent to Definition 2.2.4 when $b_1 = t$ and $b_2 = 1$.

2.5 Two-colored CS6V configurations

In Figure 2.11, we list the configurations of the two-colored CS6V model. Note that the weights of the vertex configurations are given by the matrix $\mathcal{L}^2$.

2.6 Acknowledgments

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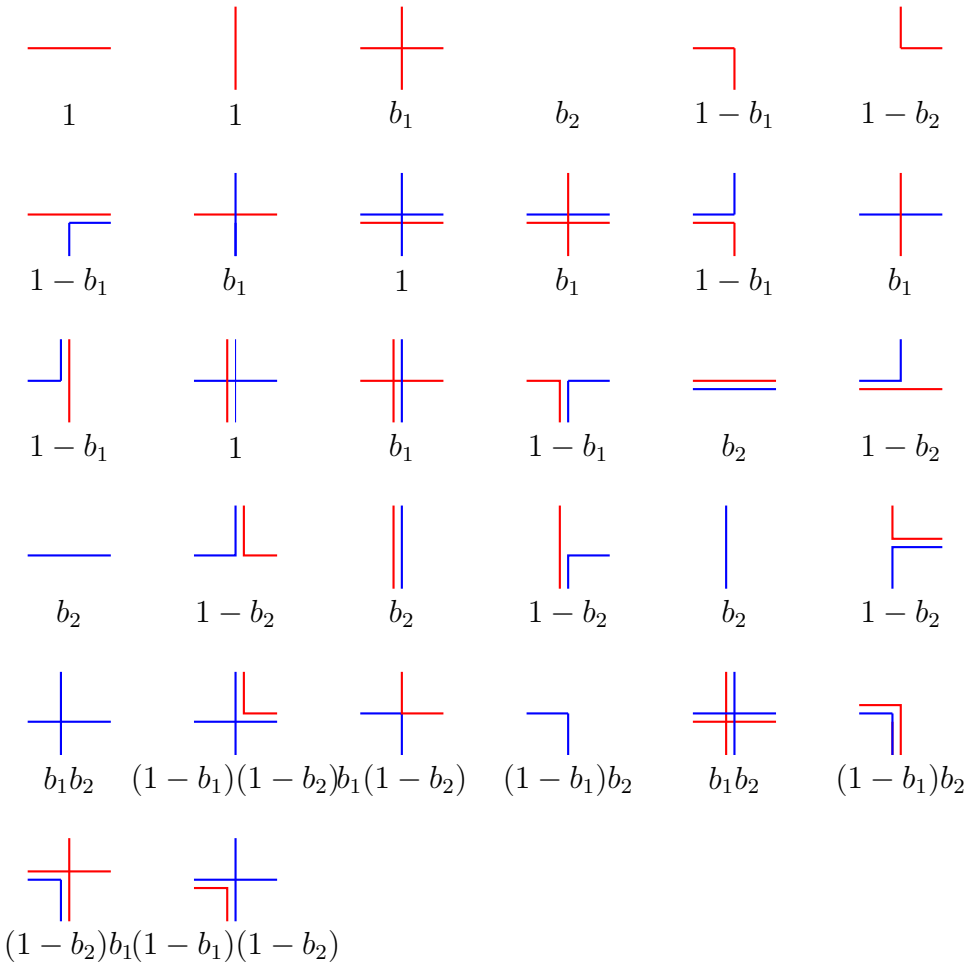


Figure 2.11: The two-colored CS6V configurations with non-zero weight given by L^2. Red has a higher priority than blue.

Chapter 3: The stochastic six-vertex model speed process

This chapter is adapted from the paper [41] and is coauthored with Levi Haunschmid-Sibitz.

3.1 Introduction

3.1.1 Preface

Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_1	$1-b_1$	b_2	$1-b_2$

Figure 3.1: The six allowed configurations for the stochastic six-vertex model

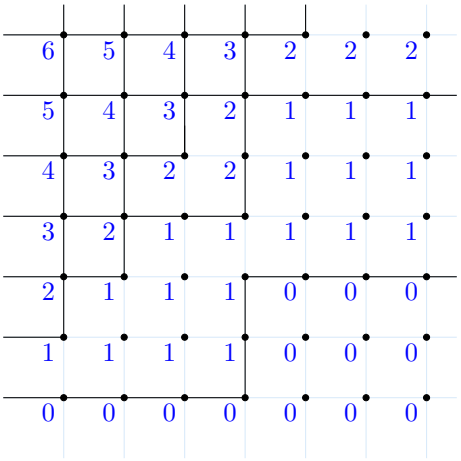


Figure 3.2: A possible sampling of the stochastic six-vertex model on the quadrant with step initial data. The height function is denoted in blue.

The stochastic six-vertex model was first introduced by Gwa and Spohn in [28] as a specialization of the six-vertex model, which is a classical model from equilibrium statistical mechanics going back to [29]. Recently there has been a lot of interest in this model. It is connected via a

suitable limit degeneration to ASEP [134], the Kardar-Parisi-Zhang equation [126, 138, 110], the stochastic telegraph equation [135, 137, 130] and lies in the (one-dimensional) KPZ universality class [28, 30, 139, 54]. Furthermore, it can be put into the more general setting of higher-spin vertex models, see [33, 127].

We will now define the stochastic six-vertex model on the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ as a stochastic path ensemble, where each path can be thought of as a particle trajectory. The model depends on two parameters $b_1, b_2 \in [0, 1]$ that we fix throughout the paper. We call the bottom and left edges incident to a vertex its **incoming** edges and the top and right edges its **outgoing** edges. We say that edges are **occupied** if they are part of a particle trajectory. Each edge will be occupied by at most one particle, and the model will satisfy a local conservation law, namely that the number of particles entering a vertex equals the number of particles exiting.

We first specify a boundary configuration on the incoming edges along the left and bottom boundaries of the quadrant, i.e., the incoming edges from the left at the vertices $\{0\} \times \mathbb{Z}_{\geq 0}$ and from the bottom at the vertices $\mathbb{Z}_{\geq 0} \times \{0\}$. This boundary configuration specifies a subset of edges along the boundary that are occupied, i.e., locations where particles enter the system. The most common boundary condition that we will work with is one where all incoming edges from the left boundary of the quadrant are occupied and all incoming edges from the bottom boundary are empty. We will refer to this boundary condition as **step initial conditions**.

Given the information about which incoming edges of a vertex are occupied, we will sample the outgoing edges using the weights in Figure 3.1. Note that the weights of configurations with the same occupied incoming edges add up to 1, so that the outgoing edges can be determined in a stochastic manner. We will begin from $(0, 0)$ and proceeding in an antidiagonal way (i.e, ordered by $x + y$), we will tile each of the vertices in the interior of the quadrant with one of the six possible configurations in Figure 3.1, according to their weights. This procedure defines a law on configurations of upright paths in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. See Figure 3.2 for a sampling of the stochastic six-vertex model with step initial conditions.

There is an alternative parameterization of the model by parameters $q, \kappa > 0$ defined as

$$q := \frac{b_1}{b_2} \qquad \kappa := \frac{1 - b_1}{1 - b_2}\,.$$

This parameterization will be quite useful to us, and these variables will appear in many formulas throughout the paper.

For a given configuration ω of the stochastic six-vertex model with step initial conditions, we define the height function $H(x,t) = H(x,t;\omega)$ for $x, t \in \mathbb{R}_{\geq 0}$ by setting $H(x,0;\omega) = 0$ for all x and increasing H whenever one crosses a path in the vertical direction, see Figure 3.2.

The model exhibits two very different behaviors depending on whether b_1 or b_2 is larger. If $b_1 > b_2$, then particles prefer moving up to moving to the right. Since the upper part of the quadrant is already packed, this leads to a sharp transition between a region with density 1 and a region with density 0, whose boundary stays close to the line $x = t$. This behavior is known as a **shock**. On the other hand if $b_1 < b_2$, particles want to move right more than up, and thus they spread out. Three regions form: one above the line $x = \kappa^{-1}t$, where the density of particles is 1, one below the line $x = \kappa t$, where the density is 0, and one in between, where the density decreases continuously from 1 to 0 (see the right-hand side of Figure 3.4 for a simulation). The middle section is known as the **rarefaction fan**. Both the shock and rarefaction fan regimes are interesting in their own rights, but our results concern the latter: from now on we always assume $b_1 < b_2$.

We will now introduce the multi-class stochastic six-vertex model. Instead of every edge being either occupied or unoccupied it will now be assigned a **class** from $\mathbb{R} \cup \{-\infty, \infty\}$. The classes assigned to the two outgoing edges equal the classes of the incoming edges, and the weight of a vertex depends on the classes, see Figure 3.3. Intuitively if $i < j$ then a particle of class i treats particles of class j as **holes**. The single-class stochastic six-vertex model can be obtained from the multi-class one by setting the class of unoccupied edges to 1 and the class of occupied edges to ∞. In the vertex model literature the multi-class stochastic six-vertex model is also called the *colored*

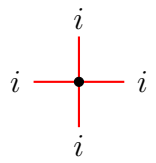
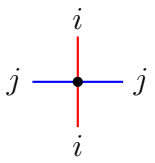
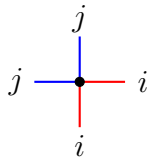
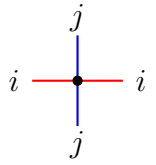
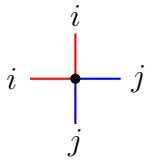
Configuration					
Weight	1	b_1	$1-b_1$	b_2	$1-b_2$

Figure 3.3: The allowed configurations for the multi-class stochastic six-vertex model, where red lines represent class i and blue lines represent class j for $i < j$.

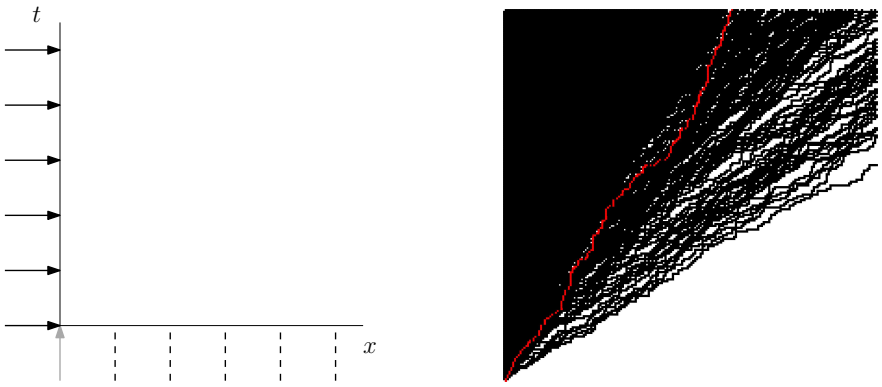


Figure 3.4: Left panel: step initial conditions with a second-class particle at the origin. Black arrows denote first-class particles, while the grey arrow denotes the second-class particle. Dashed lines denote holes. Right panel: a simulation of this process on a 200 by 200 square with $b_1 = 0.3$ and $b_2 = 0.6$ and with the second-class particle in red.

stochastic six-vertex model. See [34, 35, 36, 37, 38] for previous works on this model from an algebraic perspective.

Our main theorem concerns the following variant of step initial conditions, which we will call **step initial conditions with a second-class particle at the origin**. All particles coming in from the left have class 1, there is a single particle coming in from the bottom at $(0, 0)$, and all other incoming particles from the bottom have class ∞, i.e. are holes, see Figure 3.4. By the conservative property of the model, for every t there is exactly one x such that the vertical arrow leaving (x, t) has class 2. We call this x the **position of the second-class particle at time** t and denote it by $\boldsymbol{X}_t$. Our main result states that the speed $\frac{\boldsymbol{X}_t}{t}$ of the second-class particle converges a.s. to a random limit:

Theorem 3.1.1. *Let $0 < b_1 < b_2 < 1$, and consider the stochastic six-vertex model with step initial positions with a second-class particle at the origin. Let $\boldsymbol{X}_t$ be the position of the second-*

class particle at time t. Then almost surely

$$\lim_{t\to\infty} \frac{\boldsymbol{X}_t}{t} = \boldsymbol{U} \tag{3.1}$$

where $\boldsymbol{U}$ is a continuous random variable taking values in $[\kappa^{-1}, \kappa]$ with density $\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}}$.

Even the weak convergence of the speed of the second-class particle has not been stated in the literature, to the best of the authors' knowledge. However, it follows readily from the hydrodynamic limit proved in [31] in the same way as for ASEP using the arguments from [42]. For the convenience of the reader, we adapt this argument to our setting in Section 3.9. Strengthening the weak convergence to a.s. convergence is highly nontrivial and is the main contribution of this theorem.

For TASEP with step initial conditions, the weak convergence of the speed of a second-class particle at the origin was first proven in [42] and a.s. convergence was proven in [43] (see also [150] and [151] for alternative proofs). For the Hammersley process, a.s. convergence of the speed of a second-class particle at the origin was proven in [44] under suitable initial conditions, and for the totally asymmetric zero range process (TAZRP), this was proven in [45]. All of these proofs rely crucially on connections between the models under consideration and last passage percolation models (for example, TASEP can be coupled with exponential last passage percolation). Since this no longer holds for ASEP, new tools were required to prove the analogous result for ASEP under step initial conditions, and this was done in [46] using inputs from integrable probability as well as a coupling due to Rezakhanlou [47]. Since for the stochastic six-vertex model, last passage methods also do not apply, our proof strategy for Theorem 3.1.1 is inspired by the ideas in [46]. The speed of second-class particles for ASEP and the Hammersley process has also been studied for other classes of initial conditions in [152, 153, 154].

We also derive a bound on the fluctuations around the limiting speed:

Theorem 3.1.2. *Let $\boldsymbol{X}_t$ be the position of the second-class particle at time t as above and $\boldsymbol{U}$ its*

almost sure limit. Then for any $\delta > 0$, almost surely we have that

$$\lim_{t\to\infty} |\boldsymbol{X}_t - t\boldsymbol{U}| t^{-(\frac{7}{9}+\delta)} = 0\,. \tag{3.2}$$

The exponent $-\left(\frac{7}{9}+\delta\right)$ is not optimal. For both ASEP and the stochastic six-vertex model with stationary initial conditions, a second-class particle starting at the origin has been shown to fluctuate on the order of $t^{2/3}$. This was shown for ASEP in [155, 156] and for the stochastic six-vertex model in [48] (building on earlier ideas developed in [108]). However, the exact fluctuations of a second-class particle starting from step initial conditions are not known for either model. Because the fluctuations at stationarity are of the order $t^{2/3}$, the best exponent one could expect to achieve in (3.2) is $-(\frac{2}{3}+\delta)$, see also Remark 3.5.9. Our proof techniques can be used for ASEP as well, where they would give an analogous result to Theorem 3.1.2 with the same exponent of $-\left(\frac{7}{9}+\delta\right)$.

Going beyond adding a single second-class particle into our model, we can consider initial conditions where each incoming particle has a different class in $\mathbb{Z}$. Individually, each particle will have an asymptotic speed given by Theorem 3.1.1. By considering the joint speeds of all the particles simultaneously, we can construct the **stochastic six-vertex model speed process**. Speed processes have previously been constructed and studied for TASEP [49], TAZRP [50], and ASEP [46]. To define the speed process, we first need to discuss how we can view the stochastic six-vertex model as a particle system, as was first done in [28], see also [30, Section 2.2].

3.1.2 The stochastic six-vertex model as an interacting particle system

Until this point, we have treated the stochastic six-vertex model as a measure on up-right paths. However, it is also natural to consider it as a particle system with time as the vertical direction, as has been quite noticeable in the language we have been using and was already observed in [28]. Let us now introduce notation that emphasizes this connection. For a given configuration ω on

$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, define $\eta_t(x)$ for $x \in \mathbb{Z}_{\geq 0}$ by

$$\eta_t(x) = \begin{cases} 1 & \text{if the incoming vertex at } (x,t) \text{ from below in } \omega \text{ is occupied} \\ 0 & \text{else.} \end{cases}$$

Defined like this $(\eta_t)_{t \in \mathbb{Z}_{\geq 0}}$ is a Markov process with values in $\{0,1\}^{\mathbb{Z}_{\geq 0}}$. We call this a **stochastic six-vertex process**. The boundary conditions on the bottom give the initial condition η_0 and the boundary conditions on the left inject particles at specific times. The transition probabilities of this process can be described as follows: Particles stay in place with probability b_1 and start moving to the right with probability $(1 - b_1)$. If a particle starts moving, the amount it moves is the minimum of a Geo(b_2) distributed random variable and the distance to the nearest particle to its right. If it moves to the location of the neighboring particle to the right, that other particle then starts moving, following the above-described rules. See [30, Section 2.2] for these transition weights written out in more detail.

We now define the height function in this setting and show that it generalizes the definition of $H(x,t)$ above for the case of step initial conditions.

Definition 3.1.3 (Height Function)**.** *For a given stochastic six-vertex process $(\eta_t)_{t \geq 0}$, the height function $h_t(x) = h_t(x;\eta)$ is the unique function (up to a global shift) that satisfies*

$$h_t(x;\eta) - h_t(x+1;\eta) = \eta_t(x) \tag{3.3}$$

$$h_{t+1}(0;\eta) - h_t(0;\eta) = \begin{cases} 1 & \textit{if there is an incoming arrow from the left at } (0,t) \\ 0 & \textit{else.} \end{cases} \tag{3.4}$$

Since the height function is only unique up to a global shift, unless otherwise specified the choice of height function is made by setting $h_0(0) = 0$, but in some places, it will be convenient to choose some other $h_0(0)$. For a configuration ω of the stochastic six-vertex model with step initial conditions, one recovers the definition of $H(x,t)$ above, since by (3.3), $h_0(x) = 0$ for all x.

Definition 3.1.4. *As shown in [31], these dynamics can be extended to processes $\eta_t : \mathbb{Z} \to \{0,1\}$. We call this the* ***stochastic six-vertex process on the line****.*

Given an initial condition $\eta_0 : \mathbb{Z} \to \{0,1\}$ that satisfies $\eta_0(x) = \mathbf{1}_{x<0}$, the restriction $(\eta_t(x))_{x,t\in\mathbb{Z}_{\geq 0}}$ of the stochastic six-vertex process on the line to $x \geq 0$ agrees with the process on the quadrant with step initial conditions. It is this process that we will be considering in Sections 3.2 to 3.6. The height function is still defined by (3.3) and (3.4).

This extension is also compatible with the multi-class stochastic six-vertex process. While the single-class processes $\eta_t : \mathbb{Z} \to \{0,1\}$ have occupation variables in $\{0,1\}$ with 0 encoding holes and 1 encoding particles, we will let the multi-class processes have occupation variables in $\mathbb{R} \cup \{\infty\}$, with ∞ encoding holes and all other values encoding particles of different classes. In other words, we define the **multi-class stochastic six-vertex process on the line** as $\eta_t : \mathbb{Z} \to \mathbb{R}\cup\{\infty\}$, where $\eta_t(x) = i$ if at time t, there is a particle of class i at position x. To avoid confusion, we will always specify in the text whether we are considering a single- or multi-class process.

We can now define the speed process whose existence will be obtained as a corollary of Theorem 3.1.1.

Corollary 3.1.5 (Existence of the Speed Process)**.** *Consider the multi-class stochastic six-vertex model on the line with initial conditions $\eta_0(x) = x$ for all $x \in \mathbb{Z}$, i.e. at position x there is a particle of class x. We call this* ***packed initial conditions****. Denote by $\boldsymbol{X}_t(x)$ the position of the unique particle of class x at time t. Then the process $\left(\frac{\boldsymbol{X}_t(x)}{t}\right)_{x\in\mathbb{Z}}$ converges a.s. as $t \to \infty$ to a process $U(x)$. We call $U(x)$ the* ***stochastic six-vertex model speed process****.*

Now that the stochastic six-vertex model speed process is defined, we can study some of its properties. In Section 3.8, we will prove that the speed process is ergodic and stationary with respect to the dynamics of the multi-class stochastic six-vertex model. The uniqueness of multi-class stationary measures with a given marginal for ASEP is known, see [39, 157] and was already believed to hold for the stochastic six-vertex model as well, see [158, Remark 7.9]. Since we were not able to find a proof of this in the literature we state it in Proposition 3.8.5 and provide a proof

in Section 3.10. It then follows from a close examination of the construction of such stationary measures in [158] that the ergodic stationary measures for the multi-class stochastic six-vertex model on the line are the same as the ergodic stationary measures for the multi-class ASEP. This then implies that the stochastic six-vertex speed process is related to the ASEP speed process by a deterministic map given by a pointwise composition with a specific map, see Proposition 3.8.7.

There are also many avenues for further work on these processes. In particular, the article [159] shows that the suitably rescaled TASEP speed process converges weakly to a process known as the stationary horizon. The stationary horizon was first introduced in [53] and is expected to be a universal scaling limit for multi-class invariant measures of models in the KPZ universality class. Then in [160], they develop a more general framework to show convergence to the stationary horizon. In particular, they show that if a model converges to the directed landscape under suitable rescaling, then the stationary measures of the associated multi-class process converge to the stationary horizon at the level of finite-dimensional projections. In [54] they prove the convergence of the stochastic six-vertex model and ASEP to the directed landscape, and hence using the results from [160], they obtain as a corollary [54, Corollary 2.14] that the stationary measures for the multi-class ASEP converge to marginals of the stationary horizon. As mentioned above, these stationary measures are the same as for the multi-class stochastic six-vertex model. It is still an open problem to prove the convergence of the ASEP and stochastic six-vertex model speed processes to the stationary horizon in the space $D(\mathbb{R}, C(\mathbb{R}))$.

In this paper, we deal with step and packed initial conditions for the stochastic six-vertex model. These initial conditions play the role of “fundamental solutions" from which more general initial conditions can then be studied. Step initial conditions have long been of interest in the study of interacting particle systems and have been particularly important in the study of KPZ universality for the stochastic six-vertex model, see, e.g., [30, 161, 139, 162]. Packed initial conditions (which can be viewed as a coupling of different step initial conditions) have recently been the subject of [54] where they showed that stochastic six-vertex model and ASEP with packed initial conditions converge to the Airy Sheet. In [163], this convergence was extended to more general classes of

initial conditions, using the convergence for packed initial conditions as a key input.

3.1.3 Proof ideas

The proof of the main theorem uses a variety of tools. We follow the general strategy developed in [46], which requires certain model-specific inputs that have not yet been developed for the stochastic six-vertex model. In particular, we need the following two ingredients, which are the key novelties of this paper:

- A geometric stochastic domination result that states that a second-class particle to the right of any number of third-class particles will at any fixed time be overtaken by at most a geometric number of third-class particles.
- Effective hydrodynamic estimates that quantify how close the height function of the stochastic six-vertex model started from step initial conditions will be to its limit shape.

These results will be used in the following way. We want to control the behavior of a single second-class particle. Hydrodynamic theory allows us to control the bulk behavior of many particles, so we augment our system by filling up all empty positions to the left of $\boldsymbol{X}_t$ with third-class particles. We then use our effective hydrodynamic estimates to control the union of the second- and third-class particles. Finally, we can revert this back to an estimate of the position of the second-class particle since we know that our second-class particle is to the left of at most a geometric number of the third-class particles. A similar argument can be made to bound the position of the second-class particle from the left.

We now state these two results in detail. The first will be the content of Theorem 3.1.6 and the second, the content of Propositions 3.1.7 and 3.1.8.

Controlling a second-class particle by third-class particles

The following theorem allows us to control the behavior of a single second-class particle by controlling the behavior of a large number of third-class particles inserted to the left of the second-

class particle.

Recall that $q = \frac{b_1}{b_2}$. By $X \sim \mathrm{Geo}(q)$ we denote the law given by

$$\mathbb{P}[X = k] = (1 - q)q^k \text{ for } k \geq 0\,.$$

Theorem 3.1.6 (Geometric Stochastic Domination)**.** *Let $(\eta_t)_{t\geq 0}$ be a multi-class stochastic six-vertex process on the line with parameters $0 < b_1 < b_2 < 1$ and with the following initial conditions:*

- *There are some first-class particles (finitely or infinitely many).*
- *There is a single second-class particle.*
- *There are M third-class particles, all to the left of the second-class particle.*

Let $\boldsymbol{Z}_t(0) > \boldsymbol{Z}_t(1) > \cdots > \boldsymbol{Z}_t(M)$ be the ordered positions of the second- and third-class particles at time t. Further, let L_t be the number of third-class particles to the right of the second-class particle at time t. Then for any t, the law of L_t, conditioned on both $\boldsymbol{Z}$ and the space-time history of the first-class particle, is dominated by $\mathrm{Geo}(q)$.

Let us briefly compare this result with Rezakhanlou's coupling from [47], which was used to control a second-class particle in ASEP in [46]. In [47] an auxiliary label process on the second and third-class particles is defined, which has the following properties.

- Every second- and third-class particle has a unique label from 0 to M, which can change over time.
- The law of this labeling process at any fixed time is that of a uniform permutation, and it is stationary.
- It is coupled to the dynamics of the multi-class ASEP, such that at any time, the particle with label 1 is to the left of the single second-class particle.

This allows us to control the second-class particle with a uniformly chosen third-class particle, see [46, Proposition 5.4].

One can construct an analogous coupling for the stochastic six-vertex model,[1] but only for the case $b_1 < \frac{1}{2}$. This is essentially due to the fact that certain monotonicity properties that hold for ASEP, do not hold for the stochastic six-vertex model (see the discussion before Proposition 3.2.6). Theorem 3.1.6 takes a different approach and works for all $b_1 < b_2$. There are two key differences between these approaches: Firstly, Theorem 3.1.6 does not proceed via a coupling. Secondly, the bound in Theorem 3.1.6 is significantly stronger for large M. Intuitively, the result from [47] shows that the number of third-class particles that *do not* pass the second-class particle grows linearly in the number of third-class particles, while Theorem 3.1.6 shows that the number that *do* pass is of order 1.

Since the statement of Theorem 3.1.6 is entirely insensitive to scaling time or space, it can be carried over to ASEP, with $q = \frac{b_1}{b_2}$ fixed. For ASEP this result could also be obtained from the censoring inequality [55] (as pointed out to the authors by Dominik Schmid). See also [164, 156] where labeling processes on second- and third- class particles for ASEP were studied using blocking measures.

Tail bounds for the height function

In this subsection, we state effective hydrodynamic estimates for the fluctuations of the height function $H(x,t)$ of the stochastic six-vertex model with step initial conditions. To do so we first state the law of large numbers for H.

With probability one it holds that

$$\lim_{n\to\infty} \frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = g(x,y), \qquad \forall x, y \in \mathbb{R}_{\geq 0}, \tag{3.5}$$

[1] Such a coupling was presented by Ivan Corwin at the 2022 PIMS-CRM Summer School in Probability.

where for $b_1 \leq b_2$, we have

$$g(x,y) = \begin{cases} y - x & \text{if } \frac{x}{y} \leq \kappa^{-1} \\ \frac{\left(\sqrt{x} - \sqrt{\kappa y}\right)^2}{\kappa - 1} & \text{if } \kappa^{-1} < \frac{x}{y} < \kappa \\ 0 & \text{if } \frac{x}{y} \geq \kappa \end{cases} \tag{3.6}$$

and for $b_1 \geq b_2$, we have

$$g(x,y) = \begin{cases} 0 & \text{if } x \geq y \\ y - x & \text{if } x \leq y. \end{cases}$$

This was proven at the level of weak convergence in [30] and [31] and was strengthened to almost sure convergence in [18].

Let $g(x) := g(x,1)$. We prove the following two tail bounds on the fluctuations of the height function H around its limit shape g.

Proposition 3.1.7 (Lower Tail Bound)**.** *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{3}{2}}}, \tag{3.7}$$

and c can be chosen to weakly decrease in ε.

Proposition 3.1.8 (Upper Tail Bound)**.** *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and for any $T \geq 1$, $s \geq 0$,*

$$\mathbb{P}\left[H(T\mu, T) \leq g(\mu)T - sT^{1/3}\right] \leq c^{-1}e^{-cs},$$

and c can be chosen to weakly decrease in ε.

Remark 3.1.9. *The power $T^{\frac{1}{3}}$ on the left-hand side of Propositions 3.1.7 and 3.1.8 is optimal since on this scale the fluctuations of the height function have been shown to converge to the*

Tracy-Widom GUE distribution, see [30, Theorem 1.2]. The optimal exponents on the right-hand side however, are expected to be s^3 for Proposition 3.1.7 and $s^{3/2}$ for Proposition 3.1.8, as was obtained for the longest increasing subsequence of a permutation in [165, 166]. The optimality of $s^{3/2}$ for the upper tail was recently confirmed in [48], see the discussion below. The parameters μ_1 and μ_2 need to be bounded away from the edge of the rarefaction fan in order to obtain a uniform constant $c(\varepsilon)$.

We call Proposition 3.1.7 a "lower tail" bound even though it seemingly describes the upper tail of the random variable $H(T\mu, T)$ since it corresponds to the lower tail of the Tracy-Widom distribution. Similarly, we call Proposition 3.1.8 an "upper tail" bound. This better matches the usage in the literature of the terms "upper" and "lower" tails for models in the KPZ universality class. The reason that the upper tail decays more slowly than the lower tail is that for the height function to be smaller than expected, we just need the position of the right-most particle in the stochastic six-vertex model to be small. On the other hand, for the height function to be larger than expected, we must have that the positions of many particles are large. Since this requires more deviations to occur, the probability decays more quickly.

We prove the lower tail bound by using an identity from [67] that expresses the q-Laplace transform of the height function in terms of an expectation with respect to the law of the Meixner ensemble. This identity allows us to bound the upper tail for the height function by the lower tail of the position of the smallest hole in the Meixner ensemble. The Meixner ensemble is a determinantal point process, so this tail can be expressed as a Fredholm determinant, which we then bound using Widom's trick [68]. The upper tail bound is more straightforward. We directly express the q-Laplace transform of the height function in terms of a Fredholm determinant and use Fredholm determinant estimates from [69].

The above tail bounds are in the "moderate deviations" regime since we are considering fluctuations of order $T^{1/3}$, in contrast with the "large deviations" regime, which considers fluctuations of order T. For TASEP, tail bounds in the moderate deviations regime have been obtained in [167, 168] (with some of the key ideas originating in [169]). For ASEP, these bounds were obtained in

[46].

While our paper was in progress, [48] considered the stochastic six-vertex model under stationary initial conditions and obtained tail bounds in the moderate deviations regime. They also obtained [48, Theorem 2.7] an upper tail bound with the optimal exponent of $s^{3/2}$ and optimal constants for step initial conditions by developing the Rains-EJS formula for the stochastic six-vertex model. Their proof differs from our proof of Proposition 3.1.8 in that it does not rely on integrable methods, and instead uses probabilistic couplings, building on ideas developed in [58, 62, 63]. We still include our proof since the methods are substantially different. There is also an upcoming work [170] that will prove tight upper and lower tail bounds with the optimal exponents in the moderate deviations regime using Riemann-Hilbert techniques. Going to the large deviations regime, the recent work [162] obtains a large deviation principle for the lower tail of the stochastic six-vertex model under step initial conditions.

Let us briefly summarize some other fluctuation results for the stochastic six-vertex model with step initial conditions. As mentioned earlier, the stochastic six-vertex model belongs to the KPZ universality class (see [133] for a survey of this area), and in particular, this means that the height function exhibits fluctuations of scale $T^{\frac{1}{3}}$ and correlations of scale $T^{\frac{2}{3}}$ on a domain of size T. In [30] they proved that the one-point fluctuations of the height function around its limit shape are of the order $T^{1/3}$ and are asymptotically given by the Tracy-Widom distribution. To go beyond a one-point result, we can view the height function as a spatial process with space rescaled by $T^{2/3}$ and the fluctuations by $T^{1/3}$. This process was shown to be tight in [161] and its two-point distribution converges to the two-point distribution of the Airy process [139]. Finally, the recent groundbreaking work of [54] proved the convergence of the height function when viewed as a four-parameter field to the directed landscape, fully confirming that the stochastic six-vertex model is in the KPZ universality class.

Proof sketch

We now sketch the proof of Theorem 3.1.1 using the above two ingredients. To show that the speed $\frac{\boldsymbol{X}_t}{t}$ converges a.s., we will introduce a sequence of times S_n and prove that as long as we are not too close to the edge of the rarefaction fan, then with high probability,

$$\left| \frac{\boldsymbol{X}_{S_n}}{S_n} - \frac{\boldsymbol{X}_{S_{n+1}}}{S_{n+1}} \right| \leq S_n^{-\gamma} \tag{3.8}$$

for some positive γ.

For this to imply convergence of the sequence $\frac{\boldsymbol{X}_{S_n}}{S_n}$, we need the right-hand side to be summable. For general times $S_n \leq t \leq S_{n+1}$, one can then use the monotonicity of $\boldsymbol{X}_t$ to bound $\left| \frac{\boldsymbol{X}_t}{t} - \frac{\boldsymbol{X}_{S_n}}{S_n} \right|$ as long as the sequence S_n does not grow too quickly. We will take the sequence $S_{n+1} = S_n + T(S_n) := S_n + S_n^{\frac{7}{9}}$ and prove (3.8) for this sequence in Proposition 3.5.2.

To prove Proposition 3.5.2, we want to control the behavior of the second-class particle after some large initial time S_0. However, the effective hydrodynamic bounds in Propositions 3.1.7 and 3.1.8 only allow us to control the behavior of a large number of particles, not of an individual one since they are mesoscopic statements as opposed to microscopic ones. Therefore, we fill up all empty positions to the left of $\boldsymbol{X}_S$ with third-class particles and control the union of the second- and third-class particles by Propositions 3.1.7 and 3.1.8. Letting $T = T(S) = S^{\frac{7}{9}}$, Theorem 3.1.6 will guarantee that only a small number of these third-class particles will be to the right of $\boldsymbol{X}_{S+T}$ at time $S+T$, so that controlling the union of the second- and third-class particles gives us a bound on $\boldsymbol{X}_{S+T}$.

We split the proof of (3.8) into an upper and a lower bound, which are treated analogously. Theorem 3.1.6 reduces the lower bound to showing that a large number of these second- and third-class particles are to the right of $\boldsymbol{X}_S + \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma}$ at time $S+T$. To do so denote by $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ the augmented (single-class) stochastic six-vertex model containing the union of all first-, second- and third-class particles and by $\boldsymbol{\mathcal{B}}^{(1)}$ the process with only the first-class particles. Additionally, we introduce an auxiliary third process $\boldsymbol{\mathcal{B}}^{\text{step}}$ which is started at time S from the initial condition

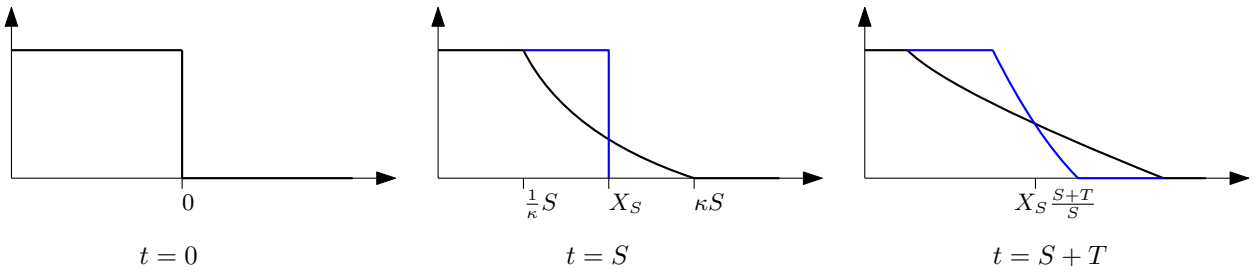


Figure 3.5: A sketch of the densities of the processes $\boldsymbol{\mathcal{B}}^{(1)}$ in black at times $0, S$ and $S + T$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ in blue at times S and $S + T$. At time S the process $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ is given exactly by the maximum of the two processes $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$, while at time $S + T$ it is at least the maximum of $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$.

$\boldsymbol{\mathcal{B}}^{\text{step}}_S(x) = \mathbf{1}_{x \le \boldsymbol{X}_S}$. At time S these three processes satisfy

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_S(x) = \max(\boldsymbol{\mathcal{B}}^{(1)}_S(x), \boldsymbol{\mathcal{B}}^{\text{step}}_S(x)) \,. \tag{3.9}$$

The multi-class stochastic six-vertex process allows us to couple $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ such that at any later time $S + T$ it holds that $\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+t}(x) \ge \boldsymbol{\mathcal{B}}^{\text{step}}_{S+t}(x)$. Since $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ and $\boldsymbol{\mathcal{B}}^{(1)}$ are already coupled in such a way, this implies that for any $t \ge 0$

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+t}(x) \ge \max(\boldsymbol{\mathcal{B}}^{(1)}_{S+t}(x), \boldsymbol{\mathcal{B}}^{\text{step}}_{S+t}(x)) \,. \tag{3.10}$$

Note that this also couples $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ in some non-trivial way. See Figure 3.5 for a sketch of the particle densities of the processes $\boldsymbol{\mathcal{B}}^{(1)}$ and $\boldsymbol{\mathcal{B}}^{\text{step}}$ at times $0, S$, and $S + T$.

By using the effective hydrodynamic estimates together with a recent approximate monotonicity result from [54], we show that with high probability $\boldsymbol{\mathcal{B}}^{(1)}$ is still close to the hydrodynamic limit at time $S + T$, uniformly over all possible configurations of $\boldsymbol{\mathcal{B}}^1_S$ when on a certain event H_S, which also occurs with high probability. Since the process $\boldsymbol{\mathcal{B}}^{\text{step}}$ is started from step initial conditions, it is also close to a hydrodynamic limit at time $S + T$, which is obtained by translating the hydrodynamic limit for standard step initial conditions. By the coupling above

$$\boldsymbol{\mathcal{B}}^{(1,2,3)}_{S+T}(x) - \boldsymbol{\mathcal{B}}^{(1)}_{S+T}(x) \ge \boldsymbol{\mathcal{B}}^{\text{step}}_{S+T}(x) - \boldsymbol{\mathcal{B}}^{(1)}_{S+T}(x). \tag{3.11}$$

Using the hydrodynamic estimates for the two processes on the right-hand side, this gives a lower bound for the number of third-class particles to the right of $\frac{X_S}{S}(S+T) - S^{1-\gamma}$, as desired.

The proof of Theorem 3.2 uses similar arguments as above to bound $|\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}|$. To bound $|\boldsymbol{X}_t - s\boldsymbol{U}|$ for general times $S_n \leq t \leq S_{n+1}$, we will need to bound $|S_{n+1} - S_n|$ and it is here that we make crucial use of the exact form of the sequence $S_{n+1} = S_n + S_n^{\frac{7}{9}}$ to obtain the $\frac{7}{9}$ fluctuation exponent in Theorem 3.1.2.

This proof represents a significant streamlining of the strategy employed in [46], which is possible in part because Theorem 3.1.6 is stronger than Rezakhanlou's coupling. In particular, the fact that Theorem 3.1.6 does not get worse with the number of particles allows us to fill in all empty positions to the left of the second-class particle with third-class particles. In [46] only a small number of positions were filled, which made it necessary to deal with more complicated "φ-distributed" Bernoulli initial conditions and introduced a further approximation step. Furthermore, we are able to obtain Theorem 3.1.2 due to taking a more refined time scale, and more careful analysis. See Remark 3.5.9 for further discussion of this.

3.1.4 Summary

In summary, the purpose of this paper is four-fold:

- We obtain a new tool in Theorem 3.1.6 for studying the stochastic six-vertex model and ASEP by adding higher class particles. Since this article was announced, this tool has already been used in [163] to study the convergence of the stochastic six-vertex model and ASEP to the KPZ fixed point for general initial conditions.

- We give bounds on the deviations of the height-function of the stochastic six vertex model in the moderate deviation regime in Proposition 3.1.7 and Proposition 3.1.8. Our results contribute to a growing body of recent literature developing tools to prove lower tail bounds for *positive temperature* KPZ models, see e.g., [57, 58, 59, 60, 61, 62, 63, 64, 65, 66].

- We give bounds on the fluctuations of the second class particle started in step initial con-

ditions in Theorems 3.1.1 and 3.1.2, in particular showing almost sure convergence of its speed.

- We show the existence of the stochastic six-vertex speed process, and show that this process is ergodic and stationary for the dynamics of the stochastic six-vertex model in Section 3.8.

3.1.5 Structure

In Section 3.2, we recall some couplings and properties of the stochastic six-vertex model, including the approximate monotonicity result from [54] which is stated in Proposition 3.2.6. The two core ingredients are proved in Sections 3.3 and 3.4 respectively—in Section 3.3 we prove Theorem 3.1.6 and in Section 3.4 we prove Propositions 3.1.7 and 3.1.8.

These results are then used in Sections 3.5, 3.6 and 3.7 to prove the main theorem. In order these sections show that

- the main theorem follows if one can show that with high probability the second-class particle does not deviate too much from its current speed in a given time frame,
- which follows if one can show that the augmented progress with additional third-class particles does not deviate too much from its hydrodynamic limit with high probability,
- which follows from the effective hydrodynamics from Section 3.4 together with approximate monotonicity.

Finally in Section 3.8 the existence of the speed process is deduced from Theorem 3.1.1, and we study some of its properties.

3.1.6 Notation

Throughout the paper, many floor functions are dropped when we consider large integers. We use

$$[\![A, B]\!] = [A, B] \cap \mathbb{Z}$$

for intervals of integers.

Our convention for geometric random variables is that a random variable $X \sim \text{Geo}(q)$ satisfies

$$\mathbb{P}[X = k] = (1 - q)q^k .$$

We consider both single-class and multi-class processes by considering their occupation variables. Single-class processes have occupation variables in $\{0, 1\}$ with 0 encoding holes and 1 encoding particles, while multi-class processes have occupation variables in $\mathbb{R} \cup \{\infty\}$, with ∞ encoding holes and all other values encoding particles of different classes.

The parameters b_1 and b_2 are fixed throughout the paper and therefore all constants can depend on them freely even if this is not explicitly mentioned.

3.1.7 Acknowledgments

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3.2 The basic coupling and some properties of the stochastic six-vertex model

We consider the following construction of the single-class stochastic six-vertex model, which also allows us to couple multiple stochastic six-vertex models with varying boundary conditions. We will first state it on the quadrant.

Definition 3.2.1 (Basic Coupling)**.** *Consider two independent families* $(\chi^1(x,t))_{x,t\geq 0}$ *and* $(\chi^2(x,t))_{x,t\geq 0}$ *of i.i.d. Bernoulli*(b_1) *and Bernoulli*(b_2) *random variables respectively. Given such random variables, we can sample the stochastic six-vertex model in the following way. If at a given vertex there are either two incoming arrows or no incoming arrows then there is only a single possible outcome. If there is a single incoming vertical arrow at* (x,t) *and* $\chi^1(x,t)=1$*, then the outgoing arrow is vertical. If* $\chi^1(x,t)=0$*, then the outgoing arrow is horizontal. Similarly, if there is a single incoming vertical arrow and* $\chi^2(x,t)=1$*, then the outgoing arrow is vertical. If* $\chi^2(x,t)=0$*, then the outgoing arrow is horizontal.*

Given boundary conditions on the left and bottom edge of $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ *the random variables* $(\chi^1(x,t))_{x,t\geq 0}$ *and* $(\chi^2(x,t))_{x,t\geq 0}$ *uniquely define a configuration, which can be obtained by updating the vertices along the anti-diagonal lines* $\{(x,t) : x+t=k\}$ *with increasing* k*. Note also that the order of updates does not matter. Using the same* $(\chi^1(x,t))_{x,t\geq 0}$ *and* $(\chi^2(x,t))_{x,t\geq 0}$ *for different boundary conditions gives a coupling of stochastic six-vertex models, which we call the* ***basic coupling****.*

Some useful properties of this coupling were recently developed in [54]. Before we discuss the properties of this coupling, let us show how it can be used to define the stochastic six-vertex-process on the line in a way that is similar to both the construction in [31, Section 2.1] using a different coupling of the stochastic six-vertex model and to the graphical construction of Harris for ASEP on $\mathbb{Z}$ in [40].

Proposition 3.2.2 (Extension to $\mathbb{Z}$)**.** *The construction in Definition 3.2.1 can be extended to the domain* $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$*. More specifically given two independent families* $(\chi^1(x,t))_{x\in\mathbb{Z},t\geq 0}$ *and* $(\chi^2(x,t))_{x\in\mathbb{Z},t\geq 0}$ *of i.i.d. Bernoulli*(b_1) *and Bernoulli*(b_2) *and any boundary conditions on the incoming edges of*

$\mathbb{Z} \times \{0\}$*, there is almost surely a unique configuration on* $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ *that is coherent with the boundary conditions and that at each vertex satisfies the rules in Definition 3.2.1, i.e. if there is only one incoming arrow, the configuration at the vertex* (x, t) *is given by the values of* $\chi^1(x, t)$ *and* $\chi^2(x, t)$*. Furthermore, the law of this unique configuration is given by the stochastic six-vertex model.*

Proof. We will construct the configuration line by line. Consider first the random variables $\chi^1(x, 0)$ and $\chi^2(x, 0)$. We call a vertex $(x, 0)$ such that $\chi^1(x, 0) = \chi^2(x, 0) = 0$ a **cut-vertex**. Almost surely, there are infinitely many cut-vertices both to the left and to the right of the origin since each vertex $(x, 0)$ has an independent positive probability of $(1 - b_1)(1 - b_2)$ to be a cut-vertex. Notice that at a cut-vertex, the outgoing horizontal edge is occupied if and only if the incoming vertical edge is occupied, and the outgoing vertical edge is occupied if and only if the incoming horizontal edge is occupied. Therefore, if $(x_0, 0)$ and $(x_1, 0)$ with $x_0 < x_1$ are cut-vertices, the configuration of all vertices $(x, 0)$ with $x_0 < x \leq x_1$ is determined by the incoming arrows at these vertices and the Bernoulli variables $\chi^1(x, 0)$ and $\chi^2(x, 0)$ for $x_0 \leq x \leq x_1$. Therefore on the probability 1 event that there are cut-vertices infinitely far to the left, the configuration is uniquely determined. □

Again, using the same Bernoulli random variables for different initial conditions gives a coupling of stochastic six-vertex processes. Let us now consider several properties of this coupling starting with attractivity. As mentioned in the introduction, we will use the notation $(\eta_t(x))_{x \in \mathbb{Z}, t \geq 0}$ for the occupation variables, i.e. $\eta_t(x) = 1$ if the vertical incoming edge is occupied. The initial conditions are then given by a function $\eta_0(x) : \mathbb{Z} \to \{0, 1\}$.

Lemma 3.2.3 (Attractivity)**.** *Given a collection of initial conditions* η_0^k *for* $k = 1, \ldots, n$*, such that* $\eta_0^i(x) \leq \eta_0^j(x)$ *for* $i \leq j$ *and all* $x \in \mathbb{Z}$*, under the basic coupling it will hold that* $\eta_t^i(x) \leq \eta_t^j(x)$ *for all* $t \in \mathbb{Z}_{\geq 0}$ *and* $x \in \mathbb{Z}$*.*

Proof. Let us consider η^i and η^j. Assume that the desired property is true until updating a specific vertex. If at this vertex the incoming arrows are identical for η^i and η^j, by the coupling the outgoing arrows will also be identical. If they are not, since the property holds for all the previous steps,

either there are two incoming arrows in η^i or no incoming arrows in η^j. In either case, the outgoing arrows will also still satisfy the desired condition. □

Remark 3.2.4. *Note that the basic coupling with initial conditions η_0^k for $k = 1, \dots, n$, such that $\eta_0^i(x) \leq \eta_0^j(x)$ for $i \leq j$ and all $x \in \mathbb{Z}$, exactly corresponds to the $n+1$-class stochastic six-vertex model with classes $\{1, \dots, n, \infty\}$ in the following way. Define*

$$\eta_t^{mult}(x) = \min\{i \in \{1, \dots n\} : \eta_t^i(x) = 1\},$$

where the convention is used that the minimum of the empty set is ∞. By considering the possible situations at a single vertex, one easily checks that η_t^{mult} is a multi-class stochastic six-vertex process.

The attractivity property also has the following analogue for the multi-class process.

Lemma 3.2.5 (Merging)**.** *Let $(\eta_t)_{t\in\mathbb{Z}_{\geq 0}}$ be a multi-class stochastic six-vertex model with classes in $\mathbb{Z} \cup \{-\infty, \infty\}$, i.e. $\eta_t : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, \infty\}$. Then for any weakly increasing function $\phi : \mathbb{Z} \cup \{-\infty, \infty\} \to \mathbb{Z} \cup \{-\infty, \infty\}$, the process $(\phi \circ \eta_t)_{t\in\mathbb{Z}_{\geq 0}}$ is also a multi-class stochastic six-vertex model.*

Proof. This is an immediate consequence of the weights in Figure 3.3 only depending on the incoming classes i and j via their ordering. Consider a vertex for which an update is about to be performed. If the two incoming classes i and j are equal, they will also be equal after applying the map, and in either case, there is exactly one outcome which then of course has probability 1. If the two incoming classes i and j are different, i.e. $i < j$ (note that we do not assume whether i is the horizontal or vertical incoming arrow), then either $\phi(i) < \phi(j)$ or $\phi(i) = \phi(j)$. In the first case, there are two possible outcomes for both a vertex with incoming arrows i and j and a vertex with incoming arrows $\phi(i)$ and $\phi(j)$ and the probabilities match, since the relative order of the incoming arrows is the same. In the second case there are two possible outcomes before applying ϕ but only one outcome after applying ϕ. Since the two possibilities before applying ϕ are complementary,

their probabilities sum up to 1, which is the probability of the one possible outcome after applying ϕ. □

The basic coupling for the stochastic six-vertex model is not monotone. As monotonicity properties will be crucial in our proof, we will use a recent result from [54] which gives an approximate form of monotonicity for the basic coupling. Recall that given a stochastic six-vertex process $(\eta_t(x))_{x\in\mathbb{Z},t\geq 0}$, there is a height function $h_t(x;\eta)$ defined up to a global shift defined in Definition 3.1.3 The following proposition is Lemma D.3 of [54].

Proposition 3.2.6 (Approximate Monotonicity)**.** *Consider two single-class initial conditions $\eta^1 : \mathbb{Z} \to \{0,1\}$ and $\eta^2 : \mathbb{Z} \to \{0,1\}$ both with at most N particles . Further consider height functions $h_t(x;\eta^1)$ and $h_t(x;\eta^2)$ satisfying $h_t(x;\eta^1) = h_t(x;\eta^2) = 0$ for x large enough. If $M \geq (logN)^2$ and $|h_0(x;\eta^1) - h_0(x;\eta^2)| < K$ for all $x \in \mathbb{Z}$, and $t \geq 0$, then with probability at least $1 - c^{-1}e^{-cM}$, and for all $x \in \mathbb{Z}$ it holds that*

$$|h_t(x;\eta^1) - h_t(x;\eta^2)| \leq K + M\,.$$

Remark 3.2.7. *In [54] this is stated without the absolute value. However, the basic coupling has the following property: If (η^1, η^2) are two stochastic six-vertex processes coupled using the basic coupling so are (η^2, η^1). Note that this is a property that the monotone coupling in [31, Proposition 2.6] does not have. Additionally, the conditions on η^1 and η^2 are symmetric, and therefore the statement with the absolute value follows from the statement without the absolute value by a simple union bound.*

Another property that we will need is a special case of [54, Lemma D.4], and the proof is quite similar to [31, Proposition 2.17].

Proposition 3.2.8 (Finite Speed of Discrepancies)**.** *There exists a constant $c = c(b_2) > 0$ depending only on b_2 such that the following holds. Consider two particle configurations η_0 and ξ_0 with height function $h_0(x;\eta)$ and $h_0(x;\xi)$which are equal on some interval $[\![A, B]\!]$. Then, under the*

basic coupling, with probability at least $1 - c^{-1}e^{-cT}$ *it holds that* $h_t(x;\eta) = h_t(x;\xi)$ *for all* $t \leq T$ *and all* $x \in \left[\!\left[A + \frac{2T}{1-b_2} + 1, B\right]\!\right]$.

Using Propositions 3.2.6 and 3.2.8 together, we can show that given two initial conditions with height functions close on an interval, the height functions will stay close on a smaller interval for some time.

Lemma 3.2.9 (Approximate Monotonicity on Intervals)**.** *There exists a constant* $c = c > 0$*, depending only on* $b_1, b_2 \in (0,1)$*, such that the following holds. Consider two particle configurations* η_0 *and* ξ_0 *with height functions* $h(x;\eta_0)$ *and* $h(x;\xi_0)$ *such that for* $x \in [\![A, B]\!]$ *we have* $|h_0(x;\xi) - h_0(x;\eta)| \leq K$*. Let* $M \geq \log(B-A)^2$*. Then we can couple them such that with probability at least* $1 - c^{-1}(e^{-cT} + e^{-cM})$ *it holds that* $|h_T(x;\xi) - h_T(x;\eta)| \leq 3K + M$ *for all* $x \in [\![A + \frac{2T}{1-b_2} + 1, B]\!]$.

Proof. This will follow from Propositions 3.2.6 and 3.2.8. Let $\widetilde{\eta}_0$ be the particle configuration obtained from η_0 by setting

$$\widetilde{\eta}_0(x) = \begin{cases} 0 & \text{if } x < A \\ \eta(x) & \text{if } x \in [\![A, B]\!] \\ 0 & \text{if } x > B\,, \end{cases} \tag{3.12}$$

and define $\widetilde{\xi}_0$ in the same way. Couple $\eta, \xi, \widetilde{\eta}$ and $\widetilde{\xi}$ all with one basic coupling (i.e. all using the same i.i.d. Bernoulli random variables). Let the height functions $h_0(x;\widetilde{\eta})$ and $h_0(x;\widetilde{\xi})$ be chosen such that $h_0(B;\widetilde{\eta}) = h_0(B;\widetilde{\xi}) = 0$, i.e. $h_0(x;\widetilde{\eta}) = h_0(x;\eta) - h_0(B,\eta)$ for $x \in [\![A, B]\!]$ and the same for ξ. Note that $h_0(x;\eta) - h_0(B,\eta)$ is a height function for η_0, and therefore by applying Proposition 3.2.8 twice, once for η and once for ξ, and a union bound, we obtain that

$$h_t(x,\widetilde{\eta}) = h_t(x,\eta) - h_0(B,\eta) \text{ and } h_t(x,\widetilde{\xi}) = h_t(x,\xi) - h_0(B,\xi) \tag{3.13}$$

holds for all $t \leq T$ and all $x \in [\![A + \frac{2T}{1-b_2}, B]\!]$ with probability at least $1 - c^{-1}e^{-cT}$.

Further note that at time 0, for all x

$$|h_0(x;\widetilde{\eta}) - h_0(x;\widetilde{\xi})| = |(h_0(x;\eta) - h_0(B,\eta)) - (h_0(x;\xi) - h_0(B;\xi))| \leq 2K\,. \tag{3.14}$$

Therefore we can apply Proposition 3.2.6 to $\widetilde{\xi}$ and $\widetilde{\eta}$ since they are coupled with the basic coupling. Indeed both $\widetilde{\xi}$ and $\widetilde{\eta}$ have at most $B - A$ particles each, so we will have with probability at least $1 - c^{-1}e^{-cM}$ that

$$|h_T(x;\widetilde{\eta}) - h_T(x;\widetilde{\xi})| \leq 2K + M \text{ for all } x \in \mathbb{Z}. \tag{3.15}$$

By a union bound, with probability at least $1 - c^{-1}(e^{-cT} + e^{-cM})$ both events (3.13) and (3.15) take place. On this event it holds for all $x \in [\![A + \frac{2T}{1-b_2} + 1, B]\!]$ that

$$\begin{aligned}
|h_T(x,\eta) - h_T(x,\xi)| &\leq |h_T(x,\eta) - h_T(x,\widetilde{\eta}) + h_T(x,\widetilde{\xi}) - h_T(x,\xi) + h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\
&\leq |h_T(x,\eta) - h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi}) + h_T(x,\xi)| + |h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\
&= |-h_0(B,\eta) + h_0(B,\xi)| + |h_T(x,\widetilde{\eta}) - h_T(x,\widetilde{\xi})| \\
&\leq K + 2K + M = 3K + M\,,
\end{aligned}$$

where we are used a triangular inequality, (3.13), (3.15) and that $|-h_0(B,\eta) + h_0(B,\xi)| < K$ by the assumption on the height functions at time 0. □

Remark 3.2.10. *The factor 3 in the term $3K + M$ in the previous step is an artifact of Proposition 3.2.6 being stated only for height functions which are 0 far enough to the right. This restriction could be removed, which would remove the factor 3. However, for our purposes the above is sufficient.*

This property will be used in Proposition 3.7.2, to show that if a stochastic six-vertex process η is close to its hydrodynamic limit at time S, it will still be close to its hydrodynamic limit at time $S + T$ with high probability, even conditioned on its full configuration at time S.

Finally, the stochastic six-vertex model has the following two symmetries which are often used together.

Proposition 3.2.11 (Particle-Hole Inversion)**.** *If we interchange all particles and holes in a stochastic six-vertex process, we obtain another stochastic six-vertex process, but with* b_1 *and* b_2 *swapped.*

Proposition 3.2.12 (Space Inversion)**.** *If we exchange the two coordinate axes in a stochastic six-vertex process, we obtain another stochastic six-vertex process, but with* b_1 *and* b_2 *swapped.*

Proof. Both of these can be seen by looking at what happens to the six configurations in Figure 3.1 under this inversion. □

Using both of these symmetries on the quadrant, which is symmetric with respect to the line $x = t$, we obtain a symmetry of one stochastic six-vertex model with itself. In particular one can see that the law of the stochastic six-vertex model started from step initial conditions on the quadrant is invariant after applying both inversions. Furthermore, the step initial condition with a single particle coming in at the origin from the left is dual to step initial conditions with a single particle coming in at the origin from the bottom. Therefore it suffices to prove the main theorem for this kind of initial condition.

3.3 Number of overtaking third-class particles

The purpose of this subsection is to prove Theorem 3.1.6 which will allow us to control an individual second-class particle by controlling a large number of third-class particles.

Proof of Theorem 3.1.6. As stated we will condition both on the paths of the first-class particles and on $\boldsymbol{Z}$ and prove the statement for any given realization of these. After conditioning on the paths of the first-class particles, one can run the stochastic six-vertex dynamics as follows: Assume all vertices (x, t) with $t \leq t_0$ and $x \in \mathbb{Z}$ have already been updated. Let $x_{\text{min}} := Z_{t_0}(M)$ and $x_{\text{max}} := Z_{t_0}(0)$ be the position of left-most and the right-most second- or third-class particles, respectively. Since we have conditioned on the paths of the first-class particles, the configurations of the vertices $(x, t_0 + 1)$ for $x < x_{\text{min}}$ are already determined. Starting with $x = x_{\text{min}}$ one can update each vertex $(x, t_0 + 1)$ sequentially. The only time the result of this update is random is when the incoming particles are the second-class particle and a third-class particle. All other

updates are determined by either the paths of the first-class particles or $\boldsymbol{Z}$. Therefore after updating $(x_{\max}, t_0 + 1)$ all remaining vertices $(x, t_0 + 1)$ are determined. One can then continue with the next line $(x, t_0 + 2)$.

Using this system of updating we will redefine L_t to refer to the number of third-class particles to the right of the second-class particle after t updates have been performed. The sequence L_t where t now refers to an update is a refinement of the original sequence L_t indexed by times t since there are (potentially) multiple updates performed between times t and $t + 1$. Therefore, proving that L_t is dominated by Geo(q) for the refined sequence of updates will give the desired result for times t.

How can L_t change when updating a vertex? It will only change when the incoming particles are exactly a second- and a third-class particle. In this case, the two incoming particles must be in positions $Z_t(k)$ and $Z_t(k + 1)$ for some $0 \leq k \leq M - 1$ and L_t is either k or $k + 1$. Then L_t changes in the following way:

- If $L_t = k$, then $L_{t+1} = k + 1$ with probability b_1 and $L_{t+1} = k$ with probability $1 - b_1$.
- If $L_t = k + 1$, then $L_{t+1} = k$ with probability b_2 and $L_{t+1} = k + 1$ with probability $1 - b_2$.

Now let us see how the law of L_t evolves. We will identify laws on $\{0, 1, \ldots, M\}$ with vectors in $\mathbb{R}^{M+1}$ and write $(e_i)_{0 \leq i \leq M}$ for the standard coordinate basis of this space. The law of L_0 is given by e_0 since L_0 is deterministically 0. Let $S(t)$ denote the collection of updates at which the two incoming particles are both either second- or third-class particles, which is given by $\boldsymbol{Z}$. By the above observation, the law of L_t is given by

$$\left(\prod_{k \in S(t)} P_k \right) e_0 \,, \tag{3.16}$$

where the matrices P_k are given by the transition rates above, i.e.

$$P_k = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1-b_1 & b_2 & & \\ & & b_1 & 1-b_2 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}. \tag{3.17}$$

To understand this product we introduce a new basis $(\mathbf{v}_i)_{i=0}^M$. Let $\mathbf{v}_i$ be the vector corresponding to the law of the random variable $\min(i, G)$ where $G \sim \text{Geo}(q)$, i.e.

$$(\mathbf{v}_i)_k := \mathbb{P}\left[\min(i, G) = k\right] = \begin{cases} (1-q)q^k & \text{if } k < i \\ q^i & \text{if } k = i \\ 0, & \text{if } k > i. \end{cases} \tag{3.18}$$

This basis satisfies the following relation with the matrices P_k for all $0 \leq k \leq M-1$ and $0 \leq j \leq M$:

$$P_k \mathbf{v}_j = \begin{cases} \mathbf{v}_j & \text{if } j \neq k, k+1 \\ (1-b_2)\mathbf{v}_k + b_2 \mathbf{v}_{k+1} & \text{if } j = k \\ b_1 \mathbf{v}_k + (1-b_1)\mathbf{v}_{k+1} & \text{if } j = k+1\,. \end{cases} \tag{3.19}$$

To see this, first recall that $q = \frac{b_1}{b_2}$ so that $qb_2 = b_1$. We now check each of the three cases in (3.19):

1. $j \neq k, k+1$: Since P_k is equal to the identity matrix in all rows except k and $k+1$ $(P_k \mathbf{v}_j)_i = (\mathbf{v}_j)_i$ for $i \neq k, k+1$. For $j < k$, we have $(\mathbf{v}_j)_k = (\mathbf{v}_j)_{k+1} = 0$ and therefore also

$(P_k \mathbf{v}_j)_i = (\mathbf{v}_j)_i$ for $i = k, k+1$. For $j > k+1$, we have $(\mathbf{v}_j)_{k+1} = q(\mathbf{v}_j)_k$ and therefore:

$$(P_k v_j)_k = (1 - b_1)(\mathbf{v}_j)_k + b_2(\mathbf{v}_j)_{k+1} = (1 - b_1 + qb_2)(\mathbf{v}_j)_k = (\mathbf{v}_j)_k \tag{3.20}$$

and

$$(P_k \mathbf{v}_j)_{k+1} = b_1(\mathbf{v}_j)_k + (1 - b_2)(\mathbf{v}_j)_{k+1} = \left(\frac{b_1}{q} + 1 - b_2\right)(\mathbf{v}_j)_{k+1} = (\mathbf{v}_j)_{k+1}. \tag{3.21}$$

2. $j = k$*:* We have $(\mathbf{v}_j)_k = q^k$ and $(\mathbf{v}_j)_{k+1} = 0$. Therefore,

$$\begin{pmatrix} (P_k \mathbf{v}_j)_k \\ (P_k \mathbf{v}_j)_{k+1} \end{pmatrix} = \begin{pmatrix} 1 - b_1 & b_2 \\ b_1 & 1 - b_2 \end{pmatrix} \begin{pmatrix} q^k \\ 0 \end{pmatrix} \tag{3.22}$$

$$= \frac{b_1}{q} \begin{pmatrix} (1-q)q^k \\ q^{k+1} \end{pmatrix} + \left(1 - b_1 - \frac{b_1(1-q)}{q}\right) \begin{pmatrix} q^k \\ 0 \end{pmatrix} \tag{3.23}$$

$$= b_2 \begin{pmatrix} (1-q)q^k \\ q^{k+1} \end{pmatrix} + (1 - b_2) \begin{pmatrix} q^k \\ 0 \end{pmatrix}. \tag{3.24}$$

This suffices since $(\mathbf{v}_k)_j = (\mathbf{v}_{k+1})_j$ for j different from $k, k+1$.

3. $j = k + 1$*:* The calculation is similar to the one above and we omit the details.

Returning to the law of L_t, we can write it as

$$\sum_{i=0}^{M} \lambda_t(i) \mathbf{v}_i \tag{3.25}$$

for some random coefficients $\lambda_t(i)$. Using (3.16) together with (3.19), we see that for any time t, the vector λ_t is the law of a random variable on $\{0, 1, \ldots, N\}$. Letting X be a random variable with this law independent of $G \sim \text{Geo}(q)$, we see that L_t is equal in distribution to $\min(X, G)$, and therefore is dominated stochastically by G. This proves the statement.

Note that M being finite was only used to define the vertex by vertex updates. This assumption can easily be removed. □

Remark 3.3.1. *The proof shows that the law of L_t is equal in distribution to the law of the minimum between a geometric random variable and a process X_t, which behaves in the same way as L_t, except that b_1 and b_2 are reversed. This seems to be some kind of duality statement. It would be interesting to see if this is a specific case of some more general duality.*

We can also obtain a dual statement to Theorem 3.1.6:

Corollary 3.3.2. *Let $(\eta_t)_t$ be a multi-class stochastic six-vertex process with the following initial conditions:*

- *There are some first-class particles (finitely or infinitely many).*
- *There are M second-class particles.*
- *There is a single third-class particle, to the left of all second-class particles.*

Let L_t be the number of second-class particles to the left of the third-class particle. Then conditioned on the paths of the first-class particles, and the joint paths of the second- and third-class particles, for any $t \geq 0$ the random variable L_t is stochastically dominated by Geo(q).

Proof. In the initial configuration, there are four classes of particles: $\{1, 2, 3, \infty\}$, (recall that holes are considered particles of class ∞). We invert the order of classes so that particles of class 1 become holes, holes become particles of class 1, and the second and third-class particles swap class. Doing this and swapping the x and t coordinates, we obtain a stochastic six-vertex model with the same parameters b_1 and b_2 by Propositions 3.2.11 and 3.2.12. This is now a stochastic six-vertex process on the domain $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$, i.e. the right half plane, which can be defined in the same way as the stochastic six-vertex model on the line. The boundary conditions obtained after these transformations satisfy the hypothesis of Theorem 3.1.6, with the third-class particles being above the second-class particle. The proof then goes through without any changes. □

Remark 3.3.3. *While we stated Theorem 3.1.6 and Corollary 3.3.2 for the stochastic six-vertex process on the line, they can also be stated for the stochastic six-vertex model on domains whose boundary is a down-right path. Since the proof takes a vertex-by-vertex approach, it will carry through with minimal changes.*

3.4 Effective hydrodynamic estimates

The purpose of this section is to prove Propositions 3.1.7 and 3.1.8. Before doing that, we combine them to prove the following theorem:

Recall that $H(X,T)$ refers to the height function of a stochastic six-vertex model on the quadrant with step initial conditions and that $g(x) = g(x,1)$ is the limit shape of the height function (see (3.5)).

Theorem 3.4.1. *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that the following holds. For any $\mu_1, \mu_2 \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$, and for any $T \geq 1$, $s \in [0,T]$,*

$$\mathbb{P}\left[|H(T\mu_1,T) - H(T\mu_2,T) - (g(\mu_1) - g(\mu_2))\,T| \geq sT^{1/3}\right] \leq c^{-1}e^{-cs}\,. \tag{3.26}$$

Furthermore, the constant c can be chosen to weakly decrease in ε.

Proof of Theorem 3.4.1. For any $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ we have the following two bounds from Propositions 3.1.7 and 3.1.8, respectively. There exists a c (that will change from line to line) such that

$$\mathbb{P}\left[H(T\mu,T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{2}{3}}}$$
$$\mathbb{P}\left[H(T\mu,T) \leq g(\mu)T - sT^{1/3}\right] \leq c^{-1}(e^{-cs} + e^{-cT}) \leq 2c^{-1}e^{-cs}.$$

Combining these two bounds, we obtain

$$\mathbb{P}\left[|H(T\mu,T) - g(\mu)T| \geq sT^{1/3}\right] \leq c^{-1}e^{-cs}\,.$$

It follows from a union bound that

$$\begin{aligned}
&\mathbb{P}\left[|H(T\mu_1, T) - H(T\mu_1, T) - (g(\mu_1) - g(\mu_2))\, T| \geq sT^{1/3}\right] \\
&\leq \mathbb{P}\left[|H(T\mu_1, T) - g(\mu_1)T| \geq \frac{s}{2}T^{1/3}\right] + \mathbb{P}\left[|H(T\mu_2, T) - g(\mu_2)T| \geq \frac{s}{2}T^{1/3}\right] \\
&\leq c^{-1}e^{-cs}.
\end{aligned}$$

This finishes the proof of Theorem 3.4.1. The constant c can be chosen to be weakly decreasing in ε since this is the case for both Proposition 3.1.7 and Proposition 3.1.8. □

We immediately obtain the following corollary of Theorem 3.4.1:

Corollary 3.4.2. *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that the following holds. For any $T \geq 1$ and for any $s \in [0, T]$,*

$$\mathbb{P}\left[\max_{\mu_1, \mu_2 \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]} |H(T\mu_1, T) - H(T\mu_2, T) - (g(\mu_1) - g(\mu_2))\, T| \geq sT^{1/3}\right] \leq c^{-1}T^2 e^{-cs}, \tag{3.27}$$

and c can be chosen to weakly decrease in ε.

Proof. Notice that there are only finitely many μ_i satisfying $\kappa^{-1} + \varepsilon \leq \mu_i \leq \kappa - \varepsilon$ and such that $T\mu_i$ is an integer. In fact, there are at most κT of them, giving at most $O(T^2)$ possible pairs (μ_1, μ_2). Taking a union bound of (3.26) over all such pairs yields the result. □

Finally, we can quickly extend Proposition 3.1.7 to the case of step Bernoulli boundary conditions, i.e. the incoming arrows from the left are given by i.i.d. Bernoulli(ρ) random variables, while the incoming positions from the bottom are all still empty. Denote these boundary conditions as $(\rho, 0)$-Bernoulli boundary conditions. Even though we don't need this result to prove our main theorem, we state it as a corollary for completeness.

Corollary 3.4.3. *Fix $\varepsilon > 0$. There exists a constant $c = c(\varepsilon) > 0$ such that the following holds: Let $\rho \in [0, 1]$ and let $H^\rho(x, y)$ be the height function for the stochastic six-vertex model on the*

quadrant with $(\rho;0)$-Bernoulli boundary conditions. For any $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and for any $T \geq 1$, $s \geq 0$,

$$\mathbb{P}\left[H^{\rho}(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq c^{-1}e^{-cs^{\frac{3}{2}}}. \tag{3.28}$$

Proof. This is a straightforward consequence of the attractivity of the stochastic six-vertex model, by which we can couple the model with $(\rho;0)$-Bernoulli initial data with the model with step initial data. In this coupling the height function of the model with $(\rho;0)$-Bernoulli initial data is smaller at every point, and thus the statement follows from Proposition 3.1.7. □

Remark 3.4.4. *Proposition 3.1.8 can also be extended to the case of $(\rho,0)$-Bernoulli boundary conditions by restricting $\mu \in \left[\frac{\kappa}{(\kappa\rho-\rho+1)^2}+\varepsilon, \kappa-\varepsilon\right]$, since $\left[\frac{\kappa}{(\kappa\rho-\rho+1)^2}, \kappa\right]$ is the rarefaction fan for $(\rho,0)$-Bernoulli boundary conditions. In fact, we actually prove this more general version in the proof of Proposition 3.1.8.*

3.4.1 Proof of Proposition 3.1.7

To prove Proposition 3.1.7, we will make use of a remarkable exact identity that relates the height function of the stochastic six-vertex model to the holes of the Meixner ensemble. We can then reduce the question of studying the tail of the height function to studying the tail for the position of the smallest hole in this determinantal point process. We study this tail by taking asymptotics of the associated kernel.

In this subsection, we define the Meixner ensemble, which is a determinantal point process on $\mathbb{Z}$. We will then relate the q-Laplace transform of the stochastic six-vertex model height function to an expectation with respect to the Meixner ensemble.

We give a brief introduction to the theory of discrete determinantal point processes. Let $\mathfrak{X}$ denote the state space of a single particle, which we will take to be a countable set (for the Meixner ensemble, we will take $\mathfrak{X} = \mathbb{Z}_{\geq 0}$). A subset $X \subseteq \mathfrak{X}$ is called a *point configuration*, and we define $\mathrm{Conf}(\mathfrak{X}) = 2^{\mathfrak{X}}$ to be the set of all possible point configurations.

We define the following Borel sigma algebra for $\text{Conf}(\mathfrak{X})$:

$$\mathcal{B} := \sigma\left(\{X \in \text{Conf}(\mathfrak{X}) : |A \cap X| = n\} : n \in \mathbb{N}, A \subseteq \mathfrak{X} \text{ compact }\right).$$

A probability measure $\mathbb{P}$ on $(\text{Conf}(\mathfrak{X}), \mathcal{B})$ is called a *random point process*. From now on, we will use X to denote this random point process by setting $X : \text{Conf}(\mathfrak{X}) \to \text{Conf}(\mathfrak{X})$, $X(\omega) = \omega$.

We define the *n-point correlation function* as follows: for $A = \{x_1, ..., x_n\} \subseteq \mathfrak{X}$, let

$$\rho_n(A) = \rho_n(x_1, ..., x_n) := \mathbb{P}\left[A \subseteq X\right].$$

Definition 3.4.5 (Determinantal Point Process)**.** *A random point process X is determinantal if there exists a kernel $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ such that for all $n \geq 1$ and for all $x_1, \ldots, x_n$ with $x_i \neq x_j$ for $i \neq j$, we have*

$$\rho_n(x_1, ..., x_n) = \det\left(K(x_i, x_j)_{i,j=1}^n\right) \tag{3.29}$$

Let $W(x) : \mathfrak{X} \to \mathbb{R}$ be a weight function, and let $P_0, P_1, \ldots$ be the family of orthonormal polynomials with respect to W, i.e.,

$$\int_{\mathfrak{X}} P_i(x) P_j(x) W(x) dx = \mathbb{1}_{i=j}.$$

The corresponding *N-point ensemble* (a random point process where $\mathbb{P}$ is supported on configurations with exactly N particles) is given by

$$\mathbb{P}(x_1, \ldots, x_N) \propto \det(V(x_1, \ldots, x_N))^2 \prod_{i=1}^N W(x_i),$$

where $V(x_1, \ldots, x_N) = (x_i^{j-1})_{i,j=1}^N$ is the Vandermonde matrix, and $\det(V(x_1, \ldots, x_N)) = \prod_{i<j}(x_j - x_i)$ is the Vandermonde determinant. An N-point ensemble generated in this way is determinantal

with the *Christoffel-Darboux* kernel

$$K_N(x,y) = (W(x)W(y))^{\frac{1}{2}} \sum_{n=0}^{N-1} P_n(x)P_n(y). \tag{3.30}$$

The Meixner polynomials are a family of orthogonal polynomials on $\mathbb{Z}_{\geq 0}$. We fix two parameters: $\beta > 0$ and $\xi \in (0,1)$, and then define the weight function $W : \mathbb{Z}_{\geq 0} \to \mathbb{R}$:

$$W(x) = \frac{\Gamma(\beta + x)}{\Gamma(\beta)x!}\xi^x. \tag{3.31}$$

We can then define the Meixner polynomials to be the family of orthogonal polynomials with respect to the weight function W. Using these orthogonal polynomials, we can define the Meixner ensemble Meixner(N, β, ξ) to be the corresponding N-point ensemble.

We use the term *particles* to refer to the elements of a point process X and use the term *holes* to refer to elements of $\mathfrak{X} \setminus X$. *Particle-hole involution* is an involution from Conf$(\mathfrak{X}) \to$ Conf$(\mathfrak{X})$ that exchanges particles with holes. In other words, $X \mapsto X^\circ := \mathfrak{X} \setminus X$. If we start with an N-point ensemble, then particle-hole involution yields a point process with infinitely many particles. Suppose that X is a determinantal point process with kernel K. Then X° is a determinantal point process with kernel $1 - K$.

Next, we give a brief overview of Schur measures. An integer partition is denoted as $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $l(\lambda)$ denotes the number of nonzero λ_i in the partition λ. Let $\mathbb{Y}$ denote the set of all integer partitions. Let $\mathbf{x} = (x_1, x_2 \ldots), \mathbf{y} = (y_1, y_2, \ldots)$ be two sets of nonnegative variables. For fixed $\mathbf{x}$ and $\mathbf{y}$, we define the Schur measure $\mathbf{SM}(\mathbf{x}; \mathbf{y})(\lambda)$ as a measure on partitions λ as follows:

$$\mathbf{SM}(\mathbf{x}; \mathbf{y})(\lambda) := \frac{s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})}{\Pi(\mathbf{x}; \mathbf{y})} \tag{3.32}$$

where s_λ is the Schur symmetric function indexed by λ and $\Pi(\mathbf{x}; \mathbf{y}) = \sum_\lambda s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})$ is the partition function. We need to assume that $\Pi(\mathbf{x}; \mathbf{y}) < \infty$ for our choice of $\mathbf{x}$ and $\mathbf{y}$ for this to define a valid probability measure.

The Meixner ensemble can be obtained as a pushforward of the Schur measure as follows: Consider the Schur measure of the form $\mathbf{SM}(x,\dots,x;y\dots,y)$ where we take n copies of x and m copies of y. Using standard properties of Schur functions, it follows that this measure is supported on $\mathbb{Y}^{(\min\{m,n\})}$, which is the set of partitions with $l(\lambda)\leq \min\{m,n\}$. Finally, consider the map from $\mathbb{Y}^{(\min\{m,n\})}\to \text{Conf}(\mathbb{Z}_{\geq 0})$ such that $\lambda\mapsto\{\min\{m,n\}+\lambda_i-i\}_{i=1}^{\min\{m,n\}}$. Then the pushforward of $\mathbf{SM}(x,\dots,x;y\dots,y)$ to a measure on $\text{Conf}(\mathbb{Z}_{\geq 0})$ gives us the Meixner ensemble $\text{Meixner}(\min\{m,n\},|m-n|+1,xy)$. This can be checked directly, see Proposition 8.2 in [70].

The following identity originates from [67], although we state a version written in [70]: Let $\mathbb{E}_{\mathbf{6v}}$ refer to the expectation with respect to the stochastic six-vertex model and let $\mathbb{E}_{\mathbf{SM}}$ denote the expectation with respect to a specified Schur measure.

Proposition 3.4.6 (Proposition 8.4 in [70])**.** *Take any $0<q<1$ and $\kappa>1$ and consider the stochastic six-vertex model on the quadrant parameterized by q and κ. Consider any integers $M,N\geq 1$. Then for any $\xi\notin -q^{\mathbb{Z}_{\leq 0}}$ we have*

$$\mathbb{E}_{\mathbf{6v}}\prod_{i\geq 0}\frac{1}{1+\xi q^{H(M,N)+i}}=\mathbb{E}_{\mathbf{SM}}\prod_{j\geq 0}\frac{1+\xi q^{\lambda_{N-j}+j}}{1+\xi q^{j}} \tag{3.33}$$

where in the right-hand side we assume that $q^{\lambda_{-m}}=0$ for $m\geq 0$, and the expectation is with respect to the Schur measure $\mathbf{SM}(\underbrace{\kappa^{-1}q^{1/2},\dots,\kappa^{-1}q^{1/2}}_{N};\underbrace{q^{-1/2},\dots,q^{-1/2}}_{M-1})$.

If $M>N$, then the Schur measure in (3.33) is supported on $\mathbb{Y}^{(N)}$. We can obtain the N particles of the $\text{Meixner}(N,M-N,\kappa^{-1})$ ensemble by taking the above-mentioned pushforward of the Schur measure so that the particles in the Meixner ensemble are given by $\{\lambda_i+N-i\}_{i=1}^{N}=\{\lambda_{N-j}+j\}_{j=0}^{N-1}$. On the other hand, if $M\leq N$, then the Schur measure is supported on $\mathbb{Y}^{(M-1)}$. We now have that

$$\{\lambda_{N-j}+j\}_{j=0}^{N-1}=\{0,\cdots,N-M\}\sqcup\{\lambda_i+(N-M)+M-i\}_{i=1}^{M-1}.$$

This gives us the $M-1$ particles in $\text{Meixner}(M-1,N-M+2,\kappa^{-1})$ each shifted over determin-

istically by $N-(M-1)$ along with the addition of particles packed from 0 to $N-M$. In either case, we can obtain the following identity:

Proposition 3.4.7. *Take any $0<q<1$ and $\kappa>1$ and consider the stochastic six-vertex model on the quadrant parameterized by q and κ. Consider any integers $M,N\geq 1$. Then for any $\xi\notin -q^{\mathbb{Z}_{\leq 0}}$ we have*

$$\mathbb{E}_{\mathbf{SM}}\prod_{j\geq 0}\frac{1+\xi q^{\lambda_{N-j}+j}}{1+\xi q^{j}}=\mathbb{E}_{X}\prod_{x\in X}\frac{1}{1+\xi q^{x}}. \tag{3.34}$$

where in the left-hand side we assume that $q^{\lambda_{-m}}=0$ for $m\geq 0$ and the right-hand expectation is with respect to the point process

$$X\sim\begin{cases}\text{Meixner}^{\circ}(N,M-N,\kappa^{-1}) & \text{if } M>N\\ N-(M-1)+\text{Meixner}^{\circ}(M-1,N-M+2,\kappa^{-1}) & \text{if } M\leq N\,,\end{cases} \tag{3.35}$$

where for a point process X, $n+X$ denotes the point process obtained by deterministically shifting over each particle in X by n, and as mentioned above, X° denotes the holes of X.

Proof. The proof of this follows from crossing out each term in the denominator that equals one of the nontrivial terms in the numerator (i.e., a term corresponding to one of the particles in the Meixner ensemble). All terms that remain in the denominator will correspond to holes of the associated Meixner ensemble. □

We now explain how we go from Propositions 3.4.6 and 3.4.7 to proving Proposition 3.1.7. We will first need the following definition and lemma:

Definition 3.4.8 (q-Pochhammer symbol)**.** *For any complex numbers q and a such that $|q|<1$, we define $(a;q)_{\infty}=\prod_{j=0}^{\infty}(1-aq^{j})$.*

The following Lemma is taken from [46] and it allows us to connect the q-Laplace transform of H (the left-hand side of (3.33)) with the tail probability of H.

Lemma 3.4.9 (Lemma B.7 in [46]). *Let* $\mathbf{A}$ *be a real-valued random variable,* $q \in [0,1)$ *and* $b \in \mathbb{R}$. *Then,*

$$\mathbb{P}[\mathbf{A} \leq 0] \leq 2 \cdot \left(1 - \mathbb{E}\left[\left(-q^{\mathbf{A}}; q\right)_{\infty}^{-1}\right]\right), \tag{3.36}$$

$$\mathbb{E}\left[\left(-q^{\mathbf{A}}; q\right)_{\infty}^{-1}\right] \geq e^{q^b/(q-1)} \cdot \mathbb{P}[\mathbf{A} \geq b], \tag{3.37}$$

$$\mathbb{E}\left[\left(1 + q^{\mathbf{A}}\right)^{-1}\right] \leq \mathbb{P}[\mathbf{A} > -b] + q^b \cdot \mathbb{P}[\mathbf{A} \leq -b]. \tag{3.38}$$

Remark 3.4.10. *While in the statement of Proposition 3.1.7 we consider a height function of the form* $H(T\mu, T)$*, for the remainder of this section we will work with the more general form* $H(T\mu, T\nu)$ *in order to highlight that many of the formulas that we will use in our analysis will have some symmetries in* μ *and* ν*. In the end, we will simply take* $\nu = 1$*. This does not actually reduce generality, since any appropriate* $H(M, N)$ *can be obtained by taking* $T = N$ *and* $\mu = M/T$.

The constants in this section are allowed to depend on κ *freely, but can be chosen to be uniform in* μ *and* ν *as long as* $\kappa^{-1} + \varepsilon \leq \mu \leq \kappa - \varepsilon$ *and* $\nu = 1$ *(Any other compact set bounded away from the two lines* $\frac{\mu}{\nu} = \kappa^{-1}$ *and* $\frac{\mu}{\nu} = \kappa$ *would also work). In particular, this will also be true for all implicit constants hidden in big* O *notation terms.*

Take $M = T\mu, N = T\nu$, and $\xi = q^{g(\mu,\nu)T - sT^{1/3}}$. Then using (3.37), with $\mathbf{A} = H(T\mu, T\nu) - g(\mu,\nu)T - sT^{1/3}$ and $b = 0$, we obtain

$$\mathbb{P}\left[H(T\mu, T\nu) \geq g(\mu,\nu)T + sT^{1/3}\right] \leq e^{-1/(q-1)} \cdot \mathbb{E}_{\mathbf{6v}} \prod_{i \geq 0} \frac{1}{1 + \xi q^{H(M,N)+i}} \tag{3.39}$$

$$= e^{-1/(q-1)} \cdot \mathbb{E}_X \prod_{x \in X} \frac{1}{1 + \xi q^x} \tag{3.40}$$

where the point process X is defined as in Proposition 3.4.7. We can estimate the last product by dropping all terms in the product except for that corresponding to the smallest hole x_1. More precisely, since all the terms in the product are at most 1, we have

$$\mathbb{E}_X \prod_{x \in X} \frac{1}{1+\xi q^x} \leq \frac{1}{1+\xi q^{x_1}}. \tag{3.41}$$

Using (3.38) with $\mathbf{A} = x_1 - g(\mu,\nu)T - sT^{1/3}$ and $b = \frac{sT^{1/3}}{2}$, we see that

$$\frac{1}{1+\xi q^{x_1}} \leq \mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}} \mathbb{P}\left[x_1 \leq g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] \tag{3.42}$$

$$\leq \mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}}. \tag{3.43}$$

So in order to obtain an upper bound on $\mathbb{P}\left[H(T\mu, T\nu) \geq g(\mu,\nu)T + sT^{1/3}\right]$, it will suffice to obtain an upper bound on $\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right]$. Let us denote the holes of the Meixner ensemble by $x_1, x_2, ...$ We know that λ has at most N nonzero parts. If there are only k nonzero parts, then $\lambda_{k+1}, ..., \lambda_N = 0$, so there are $N-k$ Meixner particles at positions $0, ..., N-k-1$. Therefore, the smallest hole x_1 will occur at position $N-k$. It follows that $\ell(\lambda) = N - x_1$. Equivalently, we have

$$x_1 = N - \ell(\lambda). \tag{3.44}$$

It follows from (3.44) that

$$\mathbb{P}\left[x_1 > g(\mu,\nu)T + \frac{sT^{1/3}}{2}\right] = \mathbb{P}\left[-\ell(\lambda) > (g(\mu,\nu) - \nu)\, T + \frac{sT^{1/3}}{2}\right]. \tag{3.45}$$

According to [67, In the proof of Theorem 6.1] we can represent the tail probability $\mathbb{P}\left[-\ell(\lambda) > h\right]$ as a Fredholm determinant. We first recall the definition of a Fredholm determinant:

Definition 3.4.11 (Fredholm Determinant)**.** *We define the Fredholm determinant of a kernel* K :

$\mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ *as*

$$\det(\mathrm{Id} + K) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{x_1,\dots,x_k \in \mathbb{Z}} \det\left[K\left(x_i, x_j\right)\right]_{i,j=1}^{k}.$$

We then have

$$\mathbb{P}[-\ell(\lambda) > h] = \det(1 - \widetilde{K})_{\ell^2(h,h-1,\dots)} := \det(1 - \Pi_h \widetilde{K} \Pi_h). \tag{3.46}$$

where $\widetilde{K}$ is a correlation kernel obtained as a dual of the kernel for the Meixner ensemble and Π_h is the projection from $\ell^2(\mathbb{Z})$ to $\ell^2(h, h-1, \dots)$.

We can write out $\widetilde{K}$ explicitly as (see [67, Equation (6.1) and the subsequent paragraph in the reference])

$$\widetilde{K}(x,y) = \frac{1}{(2\pi i)^2} \oint \oint \frac{(\sqrt{\kappa} - z^{-1})^N}{(\sqrt{\kappa} - z)^{M-1}} \frac{(\sqrt{\kappa} - w)^{M-1}}{(\sqrt{\kappa} - w^{-1})^N} \frac{dz dw}{(w - z) z^{x+1} w^{-y}} \tag{3.47}$$

where $x, y \in \mathbb{Z}$ and the integrals are taken over positively oriented circular contours with $1/\sqrt{\kappa} < |z| < 1 < |w| < \sqrt{\kappa}$. Note that our integrand has poles at $0, 1/\sqrt{\kappa}$ and $\sqrt{\kappa}$, so the contours are chosen so that they do not pass through the poles.

To estimate the Fredholm determinant in (3.46) we will use a technique known as Widom's trick first used in [68, Lemma 1]. It consists of the observation that for a kernel K with eigenvalues in $[0, 1]$ it holds that

$$\det(1 - K) \le \exp(-\mathrm{Tr}(K)). \tag{3.48}$$

The following lemma checks that the operator $\Pi_h \widetilde{K} \Pi_h$ satisfies this condition.

Lemma 3.4.12. *The operator $\Pi_h \widetilde{K} \Pi_h$ has real eigenvalues $(\mu_j)_{j \ge 0}$ all of which are in $[0, 1]$ and hence*

$$\det(1 - \Pi_h \widetilde{K} \Pi_h) \le \exp(-\mathit{Tr}(\Pi_h \widetilde{K} \Pi_h)). \tag{3.49}$$

Proof. Let $I(x, y) = \mathbf{1}_{x=y}$ be the identity operator and K as in [67, Equation (6.1)]. As noted in [67, Below Equation (6.1)] the operator $\widetilde{K}$ satisfies $\widetilde{K} = I - K$. The operator K is related to

the Christoffel-Darboux kernel K_N (see (3.30)) associated to the Meixner ensemble via a gauge transformation, see Theorem 3.3 and Lemma 3.5 in [171]. Since I is invariant under gauge transformations, this also means that $\widetilde{K}$ is related to $I - K_N$ via a gauge transformation. The operator K_N is a projection operator since it is a Christoffel-Darboux kernel. Therefore, $I - K_N$ is also a projection operator and finally $\widetilde{K}$ as well, since a gauge transform of a projection operator is a projection operator. After the gauge transformation, $\Pi_h \widetilde{K} \Pi_h$ becomes self-adjoint, so the eigenvalues are real and non-negative. Since we have now also seen that this operator is a composition of projections, the eigenvalues are at most 1. Since $1 - x \leq \exp(-x)$ for $x \in [0, 1]$ this implies (3.49). □

Therefore, obtaining an upper bound on $\mathbb{P}[-\ell(\lambda) > h]$ reduces to obtaining a lower bound for $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h)$. Similar kinds of bounds were obtained for other kernels in e.g. [68, 169]. We first compute this trace in the following lemma.

Lemma 3.4.13. *We have*

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \frac{1}{(2\pi i)^2} \oint \oint \exp(T(G_x(z) - G_x(w)) \frac{dzdw}{(w-z)^2}, \tag{3.50}$$

where

$$G_x(z) = \nu \ln\left(\sqrt{\kappa} - z^{-1}\right) - \mu \ln\left(\sqrt{\kappa} - z\right) - x \ln(z) \tag{3.51}$$

and we have reparameterized M, N and h as

$$\mu = \frac{M-1}{T}; \quad \nu = \frac{N}{T}; \quad x = \frac{h}{T}.$$

Proof. The trace is given by

$$\begin{aligned} \mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) &= \sum_{j=-\infty}^{h} \widetilde{K}(j,j) \\ &= \sum_{j=-\infty}^{h} \frac{1}{(2\pi i)^2} \oint \oint \frac{(\sqrt{\kappa} - z^{-1})^N}{(\sqrt{\kappa} - z)^{M-1}} \frac{(\sqrt{\kappa} - w)^{M-1}}{(\sqrt{\kappa} - w^{-1})^N} \left(\frac{w}{z}\right)^j \frac{dzdw}{(w-z)z}. \end{aligned}$$

Since $|w/z| > 1$ by our choice of contours, we can sum w/z from $-\infty$ to h which yields

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \frac{1}{(2\pi i)^2} \oint \oint \frac{(\sqrt{\kappa} - z^{-1})^N}{(\sqrt{\kappa} - z)^{M-1}} \frac{(\sqrt{\kappa} - w)^{M-1}}{(\sqrt{\kappa} - w^{-1})^N} \left(\frac{w}{z}\right)^h \frac{dzdw}{(w-z)^2}.$$

Finally, we can rewrite the integrand in exponential form to obtain (3.50). □

The function G has the following two critical points:

$$z_c^{\pm} = \frac{\mu + \nu + (\kappa+1)x \pm \sqrt{-4\kappa(\mu+x)(\nu+x) + (\mu+\nu+(\kappa+1)x)^2}}{2\sqrt{\kappa}(\mu+x)} \tag{3.52}$$

We can see that these two critical points are equal if we choose $x = x_c^{\pm}$ where

$$x_c^{\pm} = \frac{(\sqrt{\mu} \pm \sqrt{\kappa\nu})^2}{\kappa - 1} - \nu. \tag{3.53}$$

Note that $x_c^- = g(\mu, \nu) - \nu$. For $x = x_c^-$, we have

$$z_c^+ = z_c^- = \frac{\sqrt{\mu} - \sqrt{\kappa\nu}}{\sqrt{\kappa\mu} - \sqrt{\nu}}.$$

Denote this value as z_c. For general x, we can rewrite the formula for $z_c^{\pm}$ as

$$z_c^{\pm} = \frac{\mu + \nu + (\kappa+1)x \pm \sqrt{(\kappa-1)^2(x - x_c^+)(x - x_c^-)}}{2\sqrt{\kappa}(\mu+x)}. \tag{3.54}$$

If $x_c^- < x < x_c^+$, then the two critical points $z_c^{\pm}$ are not real. Then it holds that

$$|z_c^{\pm}| = \sqrt{\frac{\nu+x}{\mu+x}}. \tag{3.55}$$

The following lemma describes how this function behaves around (x_c^-, z_c).

Lemma 3.4.14. *The function $G_x(z)$ satisfies:*

$$G'''_{x_c^-}(z_c) = 2\frac{\sqrt{\kappa\mu\nu}(\sqrt{\kappa} - \sqrt{\mu/\nu})^2(\sqrt{\kappa} - \sqrt{\nu/\mu})^2}{z_c^3(\kappa-1)^3}.$$

As $x \to x_c^-$ from above we have the following:

$$z_c^+ = z_c + \frac{i\,(1-\kappa^{-1})^{\frac{1}{2}}\,(\kappa^{-1}\mu\nu)^{\frac{1}{4}}\sqrt{x-x_c^-}}{\sqrt{\kappa}(\kappa\mu+\nu+2\sqrt{\kappa\mu\nu})} + O(x-x_c^-) \text{ and} \tag{3.56}$$

$$G_x''(z_c^+) = \frac{iG'''_{x_c^-}(z_c)(1-\kappa^{-1})^{\frac{1}{2}}(\kappa^{-1}\mu\nu)^{\frac{1}{4}}\sqrt{x-x_c^-}}{\sqrt{\kappa}(\kappa\mu+\nu+2\sqrt{\kappa\mu\nu})} + O(x-x_c^-)\,, \tag{3.57}$$

where the implicit constant in the big O term can be chosen independently of $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $x \in [x_c^-, x_c^+]$, i.e. it depends only on κ and ε.

Proof. The first two equalities are calculations, the third one is the Taylor expansion of G'' in x and z around (x_c^-, z_c). □

Now that we have established all the variables at play we can state an estimate on the trace.

Proposition 3.4.15. *Define $s = 2(x-x_c^-)T^{\frac{2}{3}}$. For any ε there exist s_0, T_0 and C such that for any $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $\nu = 1$, any $x \in \left[x_c^- + \frac{s_0}{2}T^{-2/3}, 0\right]$ and any $T > T_0$ it holds that:*

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) \geq C s^{3/2}.$$

Here s is seen as function of h via the two equations $x = \frac{h}{T}$ and $s = (x-x_c^-)T^{\frac{2}{3}}$.

To prove Proposition 3.4.15, we will deform the contours in (3.50) to make the asymptotic analysis simpler. To do so we need to understand the level lines of $\mathrm{Re}(G_x(z))$ which pass through the critical points $z_c^{\pm}$. The following proposition describes the properties of these level lines, which are depicted in the left panel in Figure 3.6:

Proposition 3.4.16. *Let x be such that $x_c^- < x < x_c^+$. Then there are two smooth curves $\mathcal{L}_1$ and $\mathcal{L}_2$ such that:*

1. *The two curves only intersect at the critical points, i.e. $\mathcal{L}_1 \cap \mathcal{L}_2 = \{z_c^{\pm}\}$.*
2. *The two curves are the level lines through the critical points, i.e. $\mathrm{Re}(G_x(z)) = \mathrm{Re}(G_x(z_c^+))$ if and only if $z \in \mathcal{L}_1 \cup \mathcal{L}_2$.*

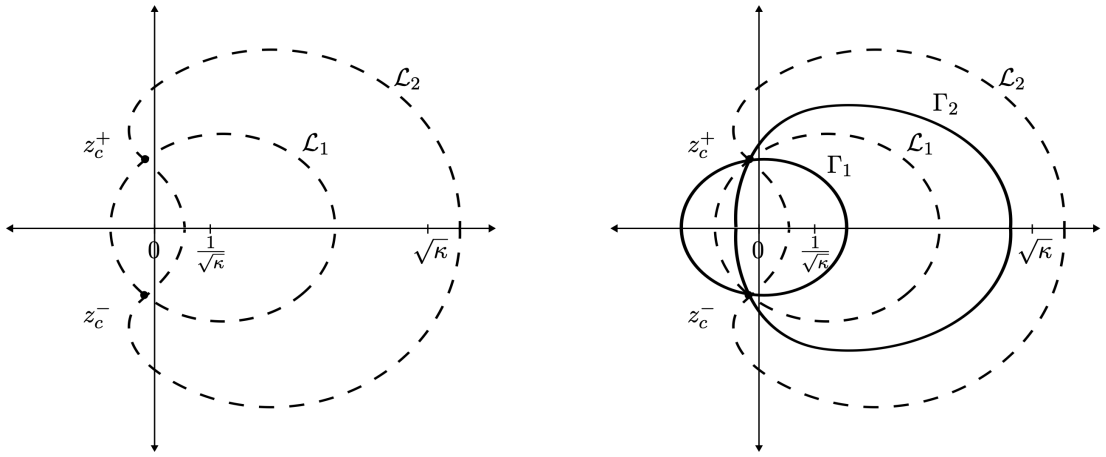

Figure 3.6: Left panel: the level lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Right panel: the contours Γ_1 and Γ_2.

3. *Both curves are bounded simple loops.*

4. *$\mathcal{L}_1$ contains 0 and $1/\sqrt{\kappa}$ but not $\sqrt{\kappa}$, while $\mathcal{L}_2$ contains $\sqrt{\kappa}$ and $1/\sqrt{\kappa}$ but not 0.*

Proof. Note that while the logarithms in the definition of $G(z)$ require a choice of branch cut, $\mathrm{Re}(\log(z))$ is defined and smooth everywhere except 0 and does not depend on the choice of branch cut. Let us consider the level lines through the critical points. Since the function is critical at these points and the second derivative does not vanish, there are exactly two level lines emerging, which intersect each other at those points. These cannot intersect at any other points, since the intersection points would again be critical points of G_x. For z with $|z|$ large it holds that

$$\mathrm{Re}(G_x(z)) = -(\mu + x)\ln(|z|) + \nu \ln\left(\sqrt{\kappa}\right) + o(1). \tag{3.58}$$

Since for $x \geq x_c^-$ we have $\mu + x > 0$, the level lines must be bounded.

Considering $\mathrm{Re}(G_x(z))$ on the real line, we see poles at 0, $1/\sqrt{\kappa}$, and $\sqrt{\kappa}$, where this function converges to $+$, $-$, and $+\infty$ respectively. Between two consecutive poles, the level lines can only cross once, since otherwise between two crossings there would be another critical point of G. This means there are exactly four points $d_1, \ldots, d_4$ along the real line such that $\mathrm{Re}(G_x(d_i)) = \mathrm{Re}(G(z_c^{\pm}))$ which satisfy $d_1 < 0 < d_2 < 1/\sqrt{\kappa} < d_3 < \sqrt{\kappa} < d_4$. Each of the four half-

lines emanating from one of the critical points will intersect the real line at exactly one of those four points. Indeed the only other option would be for two of these lines to meet, but that would create a closed level-line loop containing no pole, which would force the function to be constant by harmonicity. A brief consideration shows that the only way to connect the half-lines gives the description in the fourth point.

Finally, there cannot be any other points z for which $\mathrm{Re}(G_x(z)) = \mathrm{Re}(G(z_c^{\pm}))$ since each of those would need to lie on a closed level-line, and such a level line would need to surround a pole and therefore also intersect the real line. But all points on the real line with value $\mathrm{Re}(G(z_c^{\pm}))$ already lie on the two level lines through the critical points.

□

Using these properties of the level lines we can choose contours Γ_1 and Γ_2, as depicted in the right panel in Figure 3.6:

Proposition 3.4.17. *Let x be such that $x_c^- < x < x_c^+$. Then there are two simple curves Γ_1 and Γ_2 such that:*

1. *The two curves only intersect at the critical points, i.e. $\Gamma_1 \cap \Gamma_2 = \{z_c^+, z_c^-\}$.*
2. *At the critical points the two curves intersect perpendicularly and in the direction of steepest ascent and descent respectively.*
3. *The two curves only intersect the level lines $\mathcal{L}_1$ and $\mathcal{L}_2$ at the critical points.*
4. *Both curves contain 0 and $1/\sqrt{\kappa}$ but not $\sqrt{\kappa}$.*
5. *On Γ_1, the function G_x is always larger than $G_x(z_c^+(x))$. On Γ_2 it is always smaller.*
6. *There exists an $r = r(\kappa)$ such that for $w \in \Gamma_1$ and $z \in \Gamma_2$, the inequality $|w - z| \leq r(x - x_c^-)^{\frac{1}{2}}$ implies that either*

$$|w - z_c^+| < 2r(x - x_c^-)^{\frac{1}{2}} \text{ and } |z - z_c^+| < 2r(x - x_c^-)^{\frac{1}{2}}$$

or

$$|w - z_c^-| < 2r(x - x_c^-)^{\frac{1}{2}} \text{ and } |z - z_c^-| < 2r(x - x_c^-)^{\frac{1}{2}}.$$

Furthermore r can be chosen such that $4r(x - x_c^-)^{\frac{1}{2}} < |z_c^+ - z_c^-|$ for all $x \in [x_c^-, 0]$.

Proof. Let us first consider the steepest descent/ascent curves through the critical points. These are given by the level lines of $\mathrm{Im}(G_x(z))$. They cannot cross $\mathcal{L}_1$ or $\mathcal{L}_2$ at points other than the critical points $z_c^{\pm}$. Along these curves, the real part is strictly increasing/decreasing, therefore these curves must end at the poles of $G_x(z)$ which are at $0, \kappa^{-\frac{1}{2}}$, and $\kappa^{\frac{1}{2}}$. By considering the signs of the poles one can see that the steepest descent curve (which is in the region where $\mathrm{Re}(G_x)$ is positive) connects the pole at 0 to the pole at $\kappa^{\frac{1}{2}}$. The steepest ascent curve connects $\kappa^{-\frac{1}{2}}$ to ∞.

By considering small circles $K_0, K_{\kappa^{-\frac{1}{2}}}, K_{\kappa^{\frac{1}{2}}}$ around each pole and a large circle K_∞ around the origin, we can construct the contours as follows: The curve Γ_1 is given by the steepest descent curves through the critical points until those hit the circles K_0 and $K_{\kappa^{\frac{1}{2}}}$. Then it follows those circles such that it contains 0 but not $\kappa^{\frac{1}{2}}$. The curve Γ_2 is given by the steepest ascent curve until it hits K_∞ and $K_{\kappa^{-\frac{1}{2}}}$, where it similarly follows the circles such that it includes 0 and $\kappa^{-\frac{1}{2}}$.

By considering (3.58) and (3.55), one can see that the choice of circle can be made independently of μ and x. Indeed one can see that $\mathrm{Re}(G(z_c))$ depends continuously on $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and $x \in [x_c^-, x_c^+]$ and is therefore bounded uniformly in absolute value, with the bound depending only on κ. Around each of the poles, one can also find a uniform lower or upper bound depending on the sign of the pole. For example, around 0 one can bound:

$$\mathrm{Re}(G_x(z)) = \nu \ln(|\sqrt{\kappa} - z^{-1}|) - \mu \ln(|\sqrt{\kappa} - z|) - x \ln(|z|) \gtrsim (\nu + x) \ln(|z|^{-1}), \tag{3.59}$$

where for $|z|$ small enough the implicit constant depends only on κ. The prefactor $\nu + x = \frac{(\sqrt{\mu} \pm \sqrt{\kappa\nu})^2}{\kappa - 1}$ is bounded below by a constant which only depends on κ and ε. Therefore one can find a radius small enough, depending only on κ and ε such that for K_0, a circle of this radius, and $z \in K_0$, we have $\mathrm{Re}(G_x(z)) > \mathrm{Re}(G(z_c)) + 1$ for all $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and $x \in [x_c^-, x_c^+]$. In particular, this ensures that z_c^+ is outside this ball around 0. With very similar arguments one can

determine the radii of $K_{\kappa^{-\frac{1}{2}}}, K_{\kappa^{\frac{1}{2}}}$ and K_∞, such that for all x and μ, the values of $\mathrm{Re}(G_x(z))$ on these circles is respectively larger, larger and smaller than the value of $\mathbb{R}(G_x(z_c^+))$.

For the last point, we will actually show the following stronger statement: There exists an $r_0 = r_0(\kappa)$ such that for all $r < r_0$, $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$, and $x \in [x_c^-, 0]$ it holds that for $w \in \Gamma_1$ and $z \in \Gamma_2$, the inequality $|w - z| \leq r$ implies that either

$$(|w - z_c^+| < 2r \text{ and } |z - z_c^+| < 2r) \text{ or } (|w - z_c^-| < 2r \text{ and } |z - z_c^-| < 2r)\,.$$

This implies the desired statement.

To prove this, first consider fixed $x \in (x_c^-, 0]$ and $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$. Since the two curves only intersect at z_c^+ and z_c^- and intersect there perpendicularly, there exists an $\widetilde{r}_0 = \widetilde{r}_0(x, \mu, \kappa)$ such that for all $r < \widetilde{r}_0$ the statement holds. For $x = x_c^-$, the two critical points merge into a double critical point, and Γ_1 and Γ_2 deform in the following way. The part of Γ_1 that connects the critical points to $K_{\kappa^{\frac{1}{2}}}$ deforms into a piecewise continuous curve, which has a $2\pi/3$ angle at the z_c and leaves this point in the directions $e^{\pi i/3}$ and $e^{-\pi i/3}$ The part of Γ_1 which connects the critical points to K_0 becomes straight lines connecting z_c to K_0, parallel to the horizontal axis. Similarly Γ_2 deforms into a straight line segment connecting z_c to $K_{\kappa^{-1/2}}$ and a piece-wise continuous curve which goes through z_c at a $2\pi/3$ angle, in the directions $e^{2\pi i/3}$ and $e^{4\pi i/3}$. See Figure 3.8 for the level lines of a double critical point that appears in the proof of Proposition 3.1.8 for an illustration of what this looks like.

Since these curves still only meet at z_c at a $\pi/3$ angle, there is an $\widetilde{r}_0$ such that the statement holds for $x = x_c$. Since Γ_1 and Γ_2 deform continuously in $x \in [x_c^-, 0]$ and $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$, one can find r by taking a minimum over all r_0. Decreasing r further one can obtain $4r(x - x_c^-)^{\frac{1}{2}} < |z_c^+ - z_c^-|$, which guarantees that w and z are indeed close to the same critical point when $|w - z| \leq r(x - x_c^-)^{\frac{1}{2}}$. □

Proof of Proposition 3.4.15. For clarity, we divide the proof into several steps:

Step 1. Decomposing the trace into two parts: Recall that in our original definition of

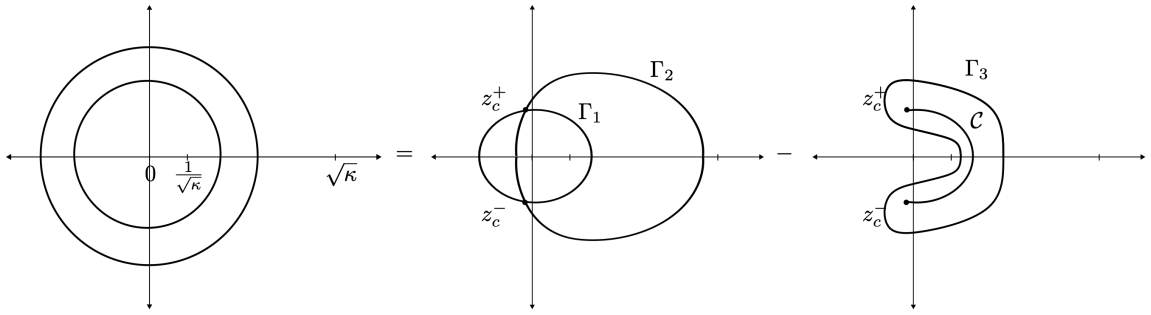

Figure 3.7: An illustration of how we pick up residues when deforming the original circular contours into Γ_1 and Γ_2.

$\widetilde{K}$ in (3.47), we started off with two positively oriented circular contours for w and z such that $1/\sqrt{\kappa} < |z| < 1 < |w| < \sqrt{\kappa}$. We will now deform these two circular contours into our new choice of contours Γ_1 and Γ_2, respectively.

Originally, the z contour is nested inside of the w contour. When we deform the z contour into Γ_2, part of it will cross through the w contour, see Figure 3.7. Therefore we will pick up some residues since our integrand has a pole of order 2 at $z = w$ due to the term $\frac{1}{(w-z)^2}$. After doing this deformation, we can decompose $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h)$ into two parts as follows:

$$\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = \left[\frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \exp(T(G_x(z) - G_x(w)) \frac{dzdw}{(w-z)^2} \right] + \left[-\frac{1}{(2\pi i)^2} \oint_{\mathcal{C}} \oint_{\Gamma_3} \exp(T(G_x(z) - G_x(w)) \frac{dzdw}{(w-z)^2} \right], \quad (3.60)$$

where Γ_1 and Γ_2 are the contours given in Proposition 3.4.17, $\mathcal{C}$ is an arc connecting z_c^+ and z_c^- and intersecting the real line between $\kappa^{-\frac{1}{2}}$ and $\kappa^{\frac{1}{2}}$, and Γ_3 is a contour around this arc, intersecting the real line only between $\kappa^{-\frac{1}{2}}$ and $\kappa^{\frac{1}{2}}$ (i.e. not including any pole other than $z = w$). Denote the first expression in (3.60) as I_1 and the second one as I_2, such that $\mathrm{Tr}(\Pi_h \widetilde{K} \Pi_h) = I_1 + I_2$. Here I_1 needs to be interpreted as a principal value integral due to the quadratic singularity at the intersection points. I_2 accounts for the residues picked up in the above-described deformation.

Step 2: Estimating I_2: In this step we show that there exists a constant $C = C(\varepsilon)$, such that $I_2 > C s_0^{3/2}$. Let $f_w(z) := \frac{\exp(T(G_x(z) - G_x(w))}{(w-z)^2}$. Using Cauchy's residue theorem for a pole of order

2, we can compute

$$I_2 = -\frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\oint_{\Gamma_3} f_w(z)dzdw = -\frac{1}{2\pi i}\int_{\mathcal{C}} \text{Res}(f_w, w)dw \tag{3.61}$$

$$= -\frac{1}{2\pi i}\int_{\mathcal{C}} TG'_x(w)dw \tag{3.62}$$

$$= -\frac{T(G_x(z_c^+) - G_x(z_c^-))}{2\pi i}. \tag{3.63}$$

Since $z_c^{\pm}(y)$ are critical points, and since only one term in G_y depends explicitly on y, we can compute

$$\frac{d}{dy}G_y(z_c^{\pm}(y)) = \ln(z_c^{\pm}(y)).$$

At x_c^-, we have $z_c^+ = z_c^-$, so we can add and subtract $G_{x_c^-}(z_c^+) = G_{x_c^-}(z_c^-)$ to get

$$G_x(z_c^+) - G_x(z_c^-) = (G_x(z_c^+) - G_{x_c^-}(z_c^+)) - (G_x(z_c^-) - G_{x_c^-}(z_c^-)) \tag{3.64}$$

$$= \int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) dy. \tag{3.65}$$

This integrand is purely imaginary since we are taking the log of the ratio of complex conjugates (which has modulus 1). Therefore, when we divide by $2\pi i$ we will get something real. We now estimate the integrand. Since the modulus is 1, the integrand is just the argument of $\frac{z_c^+(y)}{z_c^-(y)}$, which varies along the unit circle clockwise starting at 0.

Let us define the new variable $v = \sqrt{(y - x_c^-)}$ and also define

$$Z(v) := \frac{z_c^+(y)}{z_c^-(y)} = \frac{\mu + \nu + (\kappa+1)y + \sqrt{(\kappa-1)^2(y-x_c^+)(y-x_c^-)}}{\mu + \nu + (\kappa+1)y - \sqrt{(\kappa-1)^2(y-x_c^+)(y-x_c^-)}}$$

$$= \frac{\mu + \nu + (\kappa+1)(v^2+x_c^-) + v\sqrt{(\kappa-1)^2(v^2+x_c^- - x_c^+)}}{\mu + \nu + (\kappa+1)(v^2+x_c^-) - v\sqrt{(\kappa-1)^2(v^2+x_c^- - x_c^+)}}.$$

Note that $Z(0) = 1$, so that

$$Z(v) = 1 + Z'(0)v + O(v^2)$$

where

$$Z'(0) = \frac{-i2(\kappa-1)^{3/2}\kappa^{1/4}(\frac{\mu}{\nu})^{1/4}}{\nu^{1/2}\left[\kappa^{1/2}(\frac{\mu}{\nu})^{1/2}(\kappa+1) - (1+\frac{\mu}{\nu})\kappa\right]}.$$

We can see that the numerator above is bounded for $\frac{\mu}{\nu} \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$. We can also see that the denominator is zero precisely when $\frac{\mu}{\nu} \to \kappa^{-1}$ or $\frac{\mu}{\nu} \to \kappa$ and is positive between those two values. Finally, note that $Z'(0)$ is purely imaginary and $iZ'(0) > 0$. It follows that there exists $C = C(\varepsilon)$ such that $iZ'(0) > C$ for all $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$ and $\nu = 1$.

It follows that

$$\begin{aligned}\ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) &= Z'(0)v + O(v^2) \\ &= Z'(0)\sqrt{(y - x_c^-)} + O(y - x_c^-)\end{aligned}$$

and that

$$\int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) dy = Z'(0)(x - x_c^-)^{3/2} + O((x - x_c^-)^2)\,,$$

where the $O(x - x_c^-)^2$ is uniform in μ. Plugging this back into (3.63) and (3.65), we conclude that

$$\begin{aligned}I_2 = -\frac{T}{2\pi i}\int_{x_c^-}^{x} \ln\left(\frac{z_c^+(y)}{z_c^-(y)}\right) dy &> CT(x - x_c^-)^{3/2} \\ &= CT(sT^{-2/3})^{3/2} \\ &= Cs^{3/2}\end{aligned} \tag{3.66}$$

for some positive real constant C depending on ε, which changes from line to line.

Step 3: Estimating I_1. In this section, we show that the integral I_1 defined above is bounded uniformly in x, i.e., there exists constants C, T_0, s_0 depending on ε, such that $|I_1| < C$ for all x in $[x_c^- + s_0 T^{-\frac{2}{3}}, 0]$ and $T > T_0$.

Consider first the part of the integral I_1 where $|z-w| \geq r(x-x_c)^{\frac{1}{2}}$, where r is defined as in Proposition 3.4.17:

$$\frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \exp(T(G_x(z) - G_x(w))) \frac{\mathbf{1}_{|z-w| \geq r(x-x_c)^{\frac{1}{2}}} dzdw}{(w-z)^2}.$$

This integral we can bound by taking absolute values and the triangle inequality to obtain

$$\frac{1}{(x-x_c) r^2 (2\pi)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \exp(T(\mathrm{Re}(G_x(z) - G_x(w)))) dzdw.$$

This integral has no singularities and can be split into the product of two integrals, each of which can be treated using the method of steepest descent. Each of them gives a contribution $\frac{C}{\sqrt{TG_x''(z_c^+)}}$ for T large enough. Combined with the prefactor $\frac{1}{(x-x_c)r^2(2\pi)^2}$, we obtain the upper bound

$$T^{-1} |G_x''(z_c^+)|^{-1} r^{-2} (x-x_c)^{-1}.$$

By Lemma 3.4.14, this is of order $\frac{(x-x_c)^{-\frac{3}{2}}}{T} \lesssim s_0^{-3/2}$ and therefore $O(1)$.

Proposition 3.4.17 states the following. For small but fixed r, all $w \in \Gamma_1$ and $z \in \Gamma_2$ such that $|w-z| \leq r(x-x_c)^{\frac{1}{2}}$, satisfy either

$$|w - z_c^+| < 2r(x-x_c)^{\frac{1}{2}} \text{ and } |z - z_c^+| < 2r(x-x_c)^{\frac{1}{2}}$$

or

$$|w - z_c^-| < 2r(x-x_c)^{\frac{1}{2}} \text{ and } |z - z_c^-| < 2r(x-x_c)^{\frac{1}{2}},$$

i.e. they are both close to the same critical point. By symmetry it suffices to consider both w and

z in the ball around z_c^+ of radius $2r(x-x_c)^{\frac{1}{2}}$. Denote this ball by B. Let us Taylor expand around our integrand.

$$\exp(T(G_x(z))-G_x(w)) = \exp\left(\frac{1}{2}TG_x''(z_c^+)((z-z_c^+)^2-(w-z_c^+)^2)\right)(1+O(T|z-z_c^+|^3+|w-z_c^+|^3)), \tag{3.67}$$

where the big O constant depends on ε, but not on x or T. Let us first consider the contribution of the big O term on the right. After taking absolute values we have to bound

$$\oint_{\Gamma_1\cap B}\oint_{\Gamma_2\cap B} T(|z-z_c^+|^3+|w-z_c^+|^3)\exp\left(\frac{1}{2}T\mathrm{Re}(G_x''(z_c^+)((z-z_c^+)^2-(w-z_c^+)^2))\right)\frac{dzdw}{|w-z|^2}\,.$$

Since our contours meet at a right angle at the critical point, we have $\frac{1}{|z-w|^2} = O\left(\frac{1}{|z-z_c^+|^2+|w-z_c^+|^2}\right)$ and the integral is bounded by

$$T\int_{\mathbb{R}}\int_{\mathbb{R}}(|z|+|w|)\exp\left(-\frac{1}{2}T|G_x''(z_c^+)|(-z^2-w^2)\right)dzdw \lesssim \frac{T}{(TG_x''(z_c^+))^{\frac{3}{2}}} = O(s_0^{-\frac{3}{4}}).$$

For the other part of (3.67) we use a change of variables $z = z_c^+ + \xi_1/\sqrt{TG_x''(z_c^+)}$ and $w = z_c^+ + \xi_2/\sqrt{TG_x''(z_c^+)}$ which yields the principal value integral:

$$\int_{\widetilde{\Gamma}_1}\int_{\widetilde{\Gamma}_2}\frac{\exp(\xi_1^2-\xi_2^2)}{(\xi_1-\xi_2)^2}d\xi_1 d\xi_2\,,$$

where $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are contours crossing at the origin with $\widetilde{\Gamma}_1$ vertical and $\widetilde{\Gamma}_2$ horizontal there. This is clearly bounded away from the origin. Close to the origin the exponential can be estimated by 1 up to an $O(1)$ error, and the resulting principal value integral is also of order $O(1)$.

Combining the above estimates, one obtains that I_1 is $O(1)$. Combining this with (3.66) one obtains that

$$\mathrm{Tr}(\Pi_h\widetilde{K}\Pi_h) = I_1 + I_2 > Cs^{\frac{3}{2}}$$

for a different constant C, using that $s \geq s_0$ to absorb the $O(1)$ term. □

We now prove Proposition 3.1.7:

Proof of Proposition 3.1.7. Combining Equations (3.40)-(3.46) and (3.49) and setting $\nu = 1$ and h as

$$h = (g(\mu) - 1)T + \frac{sT^{1/3}}{2} = x_c^- T + \frac{sT^{1/3}}{2}\,.$$

we have that

$$\mathbb{P}\left[H(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq e^{-1/(q-1)}\left(\mathbb{P}\left[-\ell(\lambda) > (g(\mu) - 1)\,T + \frac{sT^{1/3}}{2}\right] + q^{\frac{sT^{1/3}}{2}}\right) \tag{3.68}$$

$$= e^{-1/(q-1)}\left(\det(1 - \Pi_h \widetilde{K}\Pi_h) + q^{\frac{sT^{1/3}}{2}}\right) \tag{3.69}$$

$$\leq e^{-1/(q-1)}\left(\exp(-\mathrm{Tr}(\Pi_h \widetilde{K}\Pi_h)) + q^{\frac{sT^{1/3}}{2}}\right)\,, \tag{3.70}$$

By Proposition 3.4.15 there exist constants C, T_0, and s_0 depending on ε such that for $s > s_0, T > T_0$ and $x \in [x_c^- + \frac{s_0}{2}T^{-2/3}, 0]$:

$$\mathrm{Tr}(\Pi_h \widetilde{K}\Pi_h) \geq Cs^{3/2}\,.$$

Noting that x is given by

$$x = \frac{h}{T} = x_c^- + \frac{sT^{-2/3}}{2}\,,$$

the restriction $x \in [x_c^- + \frac{s_0}{2}T^{-2/3}, 0]$ is equivalent to $s \in [s_0, -2T^{\frac{2}{3}}x_c^-]$. For s in this range, applying Proposition 3.4.15 to (3.70) yields

$$\mathbb{P}\left[H(T\mu, T) \geq g(\mu)T + sT^{1/3}\right] \leq e^{-1/(q-1)}\left(\exp(-Cs^{\frac{3}{2}}) + q^{\frac{sT^{1/3}}{2}}\right) \leq c^{-1}\exp(-cs^{\frac{3}{2}})\,. \tag{3.71}$$

For $s > -2T^{\frac{2}{3}}x_c^-$ we have

$$g(\mu)T + sT^{\frac{1}{3}} > (x_c^- + 1)T - 2Tx_c^- = T(1 - x_c^-) > T\,.$$

Since $H(\mu T, T)$ can be at most T, for such s the left-hand side of (3.71) is 0 and therefore (3.7) is trivially satisfied.

In summary, we have proved Proposition 3.1.7 for all $s \geq s_0$ and all $T \geq T_0$. For fixed $T < T_0$, the statement is trivial for s large enough as the left-hand side is 0 if $g(\mu)T + sT^{\frac{1}{3}} > T$. Thus by increasing s_0, the statement holds for all $T \geq 1$ and $s > s_0$. By decreasing c such that $c^{-1}e^{-cs} \leq 1$ for $s < s_0$ the statement holds for all $T \geq 1$ and $s \geq 0$. That c can be chosen weakly decreasing in ε is easily checked by checking that all constants in the above estimates depend continuously on ε. □

3.4.2 Proof of Proposition 3.1.8

The goal of this section is to prove Proposition 3.1.8.

Remark 3.4.18. *Again, it suffices to prove the statement for $T \geq T_0$ and $s \geq s_0$ for some T_0 and s_0 large enough. For fixed T, the left-hand side becomes 0 for s large enough since the height function is always non-negative. We can also alter the constant c to be small enough so that the right-hand side becomes greater than 1 for all $s < s_0$.*

In this section, we will closely follow the results in [69]. We start with an identity that relates the q-Laplace transform of the stochastic six-vertex model under step Bernoulli initial data, to a Fredholm determinant of some kernel. Recall that $(\rho, 0)$-Bernoulli boundary conditions denotes the boundary condition in which the incoming arrows from the left are given by i.i.d. Bernoulli(ρ) random variables, while the incoming positions from the bottom are all empty.

Proposition 3.4.19 (Prop 5.1 in [69])**.** *Fix $b_1, b_2 \in (0,1); \rho \in (0,1]; x \in \mathbb{Z}$; and $p \in \mathbb{R}$. Denote $\beta = \rho/(1-\rho)$.*

Let $\Gamma \subset \mathbb{C}$ be a positively oriented, star-shaped contour in the complex plane containing 0, but leaving outside $-q\kappa$ and $q\beta$. Let $\mathcal{C} \subset \mathbb{C}$ be a positively oriented, star-shaped contour contained inside $q^{-1}\Gamma$; that contains $0, -q$, and Γ; but that leaves outside $q\beta$.

Let $\mathbb{E}_{\mathbf{6v}}$ denote the expectation with respect to the stochastic six-vertex model with left jump probability b_1, right jump probability b_2, and $(\rho, 0)$-Bernoulli initial data. Then, we have that

$$\mathbb{E}_{\mathbf{6v}}\left[\frac{1}{(-q^{H^\rho(X,T)+p};q)_\infty}\right] = \det\left(\mathrm{Id} + K^{(p)}\right)_{L^2(\mathcal{C})} \tag{3.72}$$

where

$$K^{(p)}(w,w') = \frac{1}{2\mathbf{i}\log q}\sum_{j=-\infty}^{\infty}\oint_\Gamma \frac{(\kappa^{-1}w+q)^{X-1}}{(\kappa^{-1}v+q)^{X-1}}\frac{(v+q)^T}{(w+q)^T}\frac{(q^{-1}\beta^{-1}v;q)_\infty}{(q^{-1}\beta^{-1}v;q)_\infty} \cdot \frac{v^{p-1}w^{-p}}{\sin\left(\frac{\pi}{\log q}(\log v - \log w + 2\pi\mathbf{i}j)\right)}\frac{dv}{w'-v}. \tag{3.73}$$

While the above is stated for $\rho \in (0,1)$, we will ultimately need statements for $\rho = 1$, which is the case of step initial conditions. For ease of presentation, we will first obtain the tail bounds for $\rho < 1$ (see Remark 3.4.4) and extend to $\rho = 1$ by attractivity. However, the above proposition and the estimates that follow do also extend directly to the case $\rho = 1$, see [30, Theorem 4.16].

Once we choose p appropriately, (3.72) implies a bound on the tail probability $\mathbb{P}[H^\rho(X,T) \leq -p]$ by applying (3.36). Let $\mu = \frac{X-1}{T}$ and for $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ we define

$$f_\mu = \frac{(\sqrt{\kappa\mu}-1)^{2/3}(\kappa-\sqrt{\kappa\mu})^{2/3}}{(\kappa-1)\kappa^{1/6}\mu^{1/6}}.$$

The function f_μ appears as the scaling factor in the convergence to Tracy-Widom GUE fluctuations, see [30, Theorem 1.2]. We then define for $s \geq 0$:

$$p_T = sf_\mu T^{1/3} - g(\mu)T.$$

We will now study asymptotics of the kernel K^{p_T}, closely following section 6 of [69], but

adding in control of the decay in s as well, since in [69] they treat s as a constant. Our first step is to rewrite the formula for the kernel $K^{(p_T)}$ in an exponential form that utilizes the explicit form we chose for p_T. Plugging $p = p_T$ into (3.73), we obtain

$$\begin{aligned} K^{(p_T)}(w, w') =& \frac{1}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \oint_\Gamma \frac{\exp\left(T\left(G(w) - G(v)\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log v - \log w)\right)} \\ & \times \frac{\left(q^{-1}\beta^{-1}v; q\right)_\infty}{\left(q^{-1}\beta^{-1}w; q\right)_\infty} \\ & \times \left(\frac{v}{w}\right)^{sf_\mu T^{1/3}} \frac{dv}{v\left(w' - v\right)}, \end{aligned}$$

where we define

$$G(z) = \mu \log\left(\kappa^{-1}z + q\right) - \log(z + q) + g(\mu)\log z.$$

Next, we Taylor expand G around its critical point: We compute its derivative

$$G'(z) = \frac{(\sqrt{\kappa\mu} - 1)^2}{\kappa - 1} \frac{(z - \psi)^2}{z(z + q\kappa)(z + q)}$$

with

$$\psi = \frac{q(\kappa - \sqrt{\kappa\mu})}{\sqrt{\kappa\mu} - 1}.$$

Therefore, ψ is a critical point of G, and we have $G''(\psi) = 0$. We also have

$$G'''(\psi) = \frac{2(\sqrt{\kappa\mu} - 1)^5}{q^3(\kappa - 1)^3(\kappa - \sqrt{\kappa\mu})\sqrt{\kappa\mu}} = 2\left(\frac{f_\mu}{\psi}\right)^3.$$

Putting this all together, the Taylor expansion of G can be written as

$$G(z) - G(\psi) = \frac{1}{3}\left(\frac{f_\mu(z - \psi)}{\psi}\right)^3 + R\left(\frac{f_\mu(z - \psi)}{\psi}\right), \tag{3.74}$$

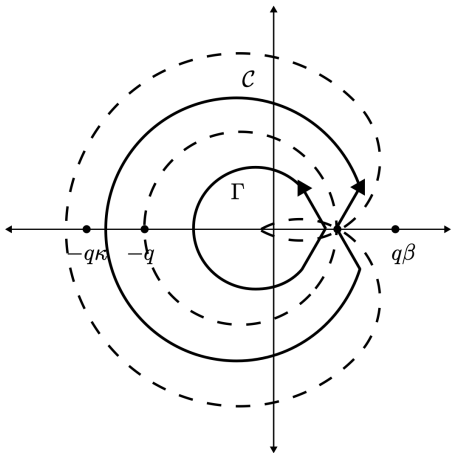

Figure 3.8: The solid lines represent the contours Γ and $\mathcal{C}$, and the dashed curves represent the level lines of $\mathrm{Re}(G(z))$.

where R is the remainder. By Taylor's remainder theorem, we have

$$R\left(\frac{f_\mu\left(z-\psi\right)}{\psi}\right) = O\left(|z-\psi|^4\right) \qquad \text{as } |z-\psi| \to 0.$$

This remainder can be bounded uniformly for $\kappa^{-1}+\varepsilon \le \mu \le \kappa - \varepsilon$. Note that $G'''(\psi) = 0$ at $\mu = \kappa^{-1}$ and $G'''(\psi)) = \infty$ at $\mu = \kappa^{-1}$, so μ being bounded away from κ^{-1} and κ is really necessary for uniformity.

We now need to choose contours Γ and $\mathcal{C}$. We use the contours defined in [69, Definitions 6.2-6.5], which will take the following shape. The contour Γ will consist of two parts: a piecewise linear part $\Gamma^{(1)}$ and a round part $\Gamma^{(2)}$ that connects the endpoints of $\Gamma^{(1)}$. Similarly $\mathcal{C}$ will consist of two parts $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, where $\mathcal{C}^{(1)}$ is piecewise linear and $\mathcal{C}^{(2)}$ is a round part connecting the end points of $\mathcal{C}^{(1)}$.

Definition 3.4.20. *For a real number $r \in \mathbb{R}$ and a positive real number $\varpi > 0$ (possibly infinite), let $\mathfrak{W}_{r,\varpi}$ denote the piecewise linear curve in the complex plane that connects $r + \varpi e^{-\pi \mathbf{i}/3}$ to r to $r + \varpi e^{\pi \mathbf{i}/3}$. Similarly, let $\mathfrak{V}_{r,\varpi}$ denote the piecewise linear curve in the complex plane that connects $r + \varpi e^{-2\pi \mathbf{i}/3}$ to r to $r + \varpi^{2\pi \mathbf{i}/3}$.*

The contours $\mathcal{C}$ and Γ look as follows:

- $\mathcal{C}^{(1)} = \mathfrak{W}_{\psi,\varpi}$ and $\Gamma^{(1)} = \mathfrak{V}_{\psi-\psi f_\mu^{-1}T^{-1/3},\varpi}$, for some sufficiently small ϖ (independently of T).

- $\mathcal{C}^{(2)}$ is a positively oriented contour from the top endpoint $\psi + \varpi e^{\pi \mathbf{i}/3}$ of $\mathcal{C}^{(1)}$ to the bottom endpoint $\psi + \varpi e^{-\pi \mathbf{i}/3}$ of $\mathcal{C}^{(1)}$, and $\Gamma^{(2)}$ is a positively oriented contour from the top endpoint $\psi - \psi f_\mu^{-1} T^{-1/3} + \varpi e^{2\pi \mathbf{i}/3}$ of $\Gamma^{(1)}$ to the bottom endpoint $\psi + \psi f_\mu^{-1} T^{-1/3} + \varpi e^{-2\pi \mathbf{i}/3}$ of $\Gamma^{(1)}$.

- We take $\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ and $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$.

See Figure 3.8 for a depiction of these contours.

Proposition 3.4.21. *The contours Γ and $\mathcal{C}$ satisfy the following properties: The contour Γ is positively oriented and star-shaped; it contains 0, but leaves outside $-q\kappa$ and $q\beta$. Furthermore, $\mathcal{C}$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma$; that contains $0, -q$ and Γ; but that leaves outside $q\beta$. Furthermore, there exists some positive real number $c_1 > 0$, independent of T, such that*

$$\max \left\{ \sup_{\substack{w \in \mathcal{C} \\ v \in \Gamma^{(2)}}} \operatorname{Re}(G(w) - G(v)), \sup_{\substack{w \in \mathcal{C}^{(2)} \\ v \in \Gamma}} \operatorname{Re}(G(w) - G(v)) \right\} < -c_1 ,$$

where c_1 depends on ε, but is uniform in μ.

Proof. These properties are all stated in [69, Definition 6.3, Lemma 6.6, and Lemma 6.13] except for the uniformity of c_1. This uniformity follows from the uniformity of G and ψ in $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$. □

By Proposition 3.4.21, Γ and $\mathcal{C}$ satisfy the necessary conditions stated in Proposition 3.4.19. We can now analyze the kernel $K := K^{(p_T)}$ for the different cases where v, w belong to the different components of these contours. The first case is where $w \in \mathcal{C}^{(1)}$ and $v \in \Gamma^{(1)}$. In this case, both w and v are close to ψ. The second case is where either $w \in \mathcal{C}^{(2)}$ or $v \in \Gamma^{(2)}$. Let $\widetilde{K}(w, w')$ be the same kernel as $K(w, w')$, but where we replace the contour Γ with $\Gamma^{(1)}$.

We perform a change of variables to zoom in around ψ: Let $\sigma = \psi f_\mu^{-1} T^{-1/3}$ and set

$$w = \psi + \sigma\widehat{w}; \qquad w' = \sigma + \sigma\widehat{w}';$$
$$v = \psi + \sigma\widehat{v}; \qquad \widehat{K}(\widehat{w}, \widehat{w}') = \sigma\widetilde{K}(w, w')$$
$$\overline{K}(\widehat{w}, \widehat{w}') = \sigma K(w, w').$$

For any contour $\mathcal{D}$, set $\widehat{\mathcal{D}} = \sigma^{-1}(\mathcal{D} - \psi)$. In particular, we have

$$\widehat{\mathcal{C}^{(1)}} = \mathfrak{W}_{0,\varpi/\sigma}, \quad \widehat{\Gamma^{(1)}} = \mathfrak{V}_{-1,\varpi/\sigma}.$$

Notice that as $T \to \infty$, we have

$$\widehat{\mathcal{C}^{(1)}} \to \mathfrak{W}_{0,\infty}, \quad \widehat{\Gamma^{(1)}} \to \mathfrak{V}_{-1,\infty/\sigma},$$

each of which consists of a pair of rays emanating from 0 and -1 respectively.

The only difference between $\widehat{K}$ and $\overline{K}$ is that for $\widehat{K}$ we are integrating over the contour $\widehat{\Gamma}^{(1)}$ and for $\overline{K}$ we are integrating over the contour $\widehat{\Gamma}$. The following lemma will deal with estimating these two kernels.

Lemma 3.4.22. *There exist positive constants c, C and T_0 all depending on ε such that for $T \geq T_0$ we have*

1. $$\left|\overline{K}(\widehat{w}, \widehat{w}') - \widehat{K}(\widehat{w}, \widehat{w}')\right| < c^{-1}\exp\left(-c\left(T + |\widehat{w}|^3\right)\right)$$
   for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$ and $\widehat{w}' \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$

2. $$\left|\overline{K}(\widehat{w}, \widehat{w}')\right| < c^{-1}\exp\left(-c\left(T + |\widehat{w}|^3\right)\right)$$
   for all $\widehat{w} \in \widehat{\mathcal{C}^{(2)}}$ and $\widehat{w}' \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$.

3.

$$|\widehat{K}(\widehat{w},\widehat{w}')| \leq \frac{C}{1+|\widehat{w}'|}\exp(-c|\widehat{w}|^3 - cs)$$

for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}}$.

Combining the above items, we can conclude that

$$\left|\overline{K}(\widehat{w},\widehat{w}')\right| \leq c^{-1}\exp\left(-c|\widehat{w}|^3 - cs\right) + c^{-1}\exp\left(-c|\widehat{w}|^3 - cT\right) \tag{3.75}$$

for all $\widehat{w} \in \widehat{\mathcal{C}}$ *and* $\widehat{w}' \in \widehat{\mathcal{C}}$.

Proof. The proof of the first two items is the content of Corollary 6.14 in [69]. For the proof of Item 3 we write out the formula for kernel $\widehat{K}(\widehat{w},\widehat{w}')$ in terms of the variables $\widehat{w}, \widehat{w}'$, and $\widehat{v}$, using the fact that $\sigma = \psi f_\mu^{-1} T^{-1/3}$ as well as the Taylor estimates in (3.74). We have

$$\begin{aligned}
\widehat{K}(\widehat{w},\widehat{w}') &= \sigma\widetilde{K}(w,w') \\
&= \frac{\sigma}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \oint_{\widehat{\Gamma^{(1)}}} \frac{\exp\left(T\left(G(\psi+\sigma\widehat{w}) - G(\psi+\sigma\widehat{v})\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log(\psi+\sigma\widehat{v}) - \log(\psi+\sigma\widehat{w}))\right)} \\
&\times \frac{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q)_\infty}{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q)_\infty} \\
&\times \left(\frac{\psi+\sigma\widehat{v}}{\psi+\sigma\widehat{w}}\right)^{sf_\mu T^{1/3}} \frac{\sigma d\widehat{v}}{(\psi+\sigma\widehat{v})(\sigma\widehat{w}'-\sigma\widehat{v})} \\
&= \oint_{\widehat{\Gamma^{(1)}}} \frac{\sigma\psi^{-1}}{2\mathbf{i}\log q} \times \sum_{j\in\mathbb{Z}} \frac{\exp\left(\frac{\widehat{w}^3-\widehat{v}^3}{3} + T\left(R\left(T^{-1/3}\widehat{w}\right) - R\left(T^{-1/3}\widehat{v}\right)\right)\right)}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log(1+\sigma\psi^{-1}\widehat{v}) - \log(1+\sigma\psi^{-1}\widehat{w}))\right)} \\
&\times \frac{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q)_\infty}{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q)_\infty} \\
&\times \left(\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right)^{s\psi\sigma^{-1}} \frac{d\widehat{v}}{(1+\sigma\psi^{-1}\widehat{v})(\widehat{w}'-\widehat{v})}\,.
\end{aligned}$$

Next, we estimate each of the terms in the integrand, using Proposition 3.4.23. Multiplying the seven bounds in Proposition 3.4.23 together, we obtain that the integrand is bounded in absolute value by $\frac{c^{-1}}{1+|\widehat{w}'|}\exp\left(cs\mathrm{Re}\widehat{v} - c(|\widehat{w}|^3 + |\widehat{v}|^3)\right)$. Noting that $\mathrm{Re}\widehat{v} \leq -1$ for $\widehat{v} \in \widehat{\Gamma^{(1)}}$, we obtain

$$
\begin{aligned}
|\widehat{K}(\widehat{w},\widehat{w}')| &\leq \oint_{\widehat{\Gamma^{(1)}}} \frac{c^{-1}}{1+|\widehat{w}'|} \exp\left(cs\mathrm{Re}\widehat{v} - c(|\widehat{w}|^3+|\widehat{v}|^3)\right) d\widehat{v} \\
&\leq \oint_{\widehat{\Gamma^{(1)}}} \frac{c^{-1}}{1+|\widehat{w}'|} \exp\left(-cs - c(|\widehat{w}|^3+|\widehat{v}|^3)\right) d\widehat{v} \\
&\leq \frac{C}{1+|\widehat{w}'|} \exp(-c|\widehat{w}|^3 - cs).
\end{aligned}
$$

□

Proposition 3.4.23. *In this proposition, we prove estimates for the terms in the integrand of the kernel $\widehat{K}$. There exists $c = c(\varepsilon) > 0$ such that for T large enough, the following seven bounds hold for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}} = \mathfrak{W}_{0,\varpi/\sigma}$, $\widehat{w}' \in \widehat{\mathcal{C}}$, and $\widehat{v} \in \widehat{\Gamma^{(1)}} = \mathfrak{V}_{-1,\varpi/\sigma}$:*

$$\left|\frac{1}{1+\sigma\psi^{-1}\widehat{v}}\right| \leq c^{-1}; \tag{3.76}$$

$$\left|\frac{1}{\widehat{w}'-\widehat{v}}\right| \leq \frac{c^{-1}}{1+|\widehat{w}'|}; \tag{3.77}$$

$$\left|\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right|^{s\psi\sigma^{-1}} \leq c^{-1}\exp\left(cs\mathrm{Re}(\widehat{v})\right); \tag{3.78}$$

$$\frac{\sigma\psi^{-1}}{|\log q|}\sum_{j\neq 0}\left|\frac{1}{\sin\left(\pi(\log q)^{-1}(2\pi\mathbf{i}j+\log(1+\sigma\psi^{-1}\widehat{v})-\log(1+\sigma\psi^{-1}\widehat{w}))\right)}\right| \leq c^{-1}T^{-1/3}; \tag{3.79}$$

$$\left|\frac{\sigma\psi^{-1}}{\log q\sin\left(\pi(\log q)^{-1}(\log(1+\sigma\psi^{-1}\widehat{v})-\log(1+\sigma\psi^{-1}\widehat{w}))\right)}\right| \leq c^{-1}; \tag{3.80}$$

$$\left|\frac{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{v});q)_\infty}{(q^{-1}\beta^{-1}(\psi+\sigma\widehat{w});q)_\infty}\right| \leq c^{-1}\exp\left(c^{-1}\left(|\widehat{w}|+|\widehat{v}|\right)\right) \tag{3.81}$$

$$\left|\exp\left(\frac{\widehat{w}^3-\widehat{v}^3}{3}+T\left(R\left(T^{-1/3}\widehat{w}\right)-R\left(T^{-1/3}\widehat{v}\right)\right)\right)\right| \leq c^{-1}\exp\left(-\frac{1}{5}(|\widehat{w}|^3+|\widehat{v}|^3)\right) \tag{3.82}$$

Proof. The proof of all of these inequalities except for (3.78) can be found in [69, Proof of Lemma 6.12] without uniformity in μ. Uniformity in μ is checked in [46, Proof of Lemma C.9]. In [69], s is fixed, and therefore they do not need estimates that depend on s. The proof of (3.78) is as follows:

First note that $|1+\sigma\psi^{-1}\widehat{w}| \geq 1$ for all $\widehat{w} \in \widehat{\mathcal{C}^{(1)}}$, so that we have

$$\left|\frac{1+\sigma\psi^{-1}\widehat{v}}{1+\sigma\psi^{-1}\widehat{w}}\right|^{s\psi\sigma^{-1}} \leq \left|1+\sigma\psi^{-1}\widehat{v}\right|^{s\psi\sigma^{-1}}.$$

For fixed $\widehat{v}$, we have $\lim_{T\to\infty}|1+\sigma\psi^{-1}\widehat{v}|^{s\psi\sigma^{-1}} = \exp(s\mathrm{Re}\widehat{v})$. Without loss of generality let us suppose that we choose $\widehat{v}$ to be on the upper half plane so that $\widehat{v} = -1+\frac{r\varpi}{\sigma}e^{2\pi i/3} = -1-\frac{r\varpi}{2\sigma}+\frac{r\sqrt{3}\varpi i}{2\sigma}$ for some $0 \leq r \leq 1$. Then we have

$$1+\sigma\psi^{-1}\widehat{v} = 1-\sigma\psi^{-1}-\frac{r\varpi\psi^{-1}}{2}+\frac{r\sqrt{3}\varpi\psi^{-1}i}{2}.$$

If we choose ϖ small enough then there exists a T_0 such that for $T \geq T_0$, $|1+\sigma\psi^{-1}\widehat{v}| < 1$. Therefore, $|1+\sigma\psi^{-1}\widehat{v}|^{s\psi\sigma^{-1}}$ is increasing monotonically in T for $T \geq T_0$ and is therefore bounded by $\exp(s\mathrm{Re}\widehat{v})$. □

We will need the following definition of a Fredholm determinant on a contour, see e.g. [69, Definition A.1]

Definition 3.4.24 (Fredholm Determinant)**.** *Fix a contour $\mathcal{C} \subset \mathbb{C}$ in the complex plane. Let $K : \mathcal{C}\times\mathcal{C} \to \mathbb{C}$ be a meromorphic function with no poles on $\mathcal{C}\times\mathcal{C}$. We define the Fredholm determinant*

$$\det(\mathrm{Id}+K)_{L^2(\mathcal{C})} = 1+\sum_{k=1}^{\infty}\frac{1}{(2\pi\mathrm{i})^k k!}\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\det\left[K(x_i,x_j)\right]_{i,j=1}^{k}\prod_{j=1}^{k}dx_j.$$

We will also need the following lemma from [69]:

Lemma 3.4.25 (Lemma A.4 in [69])**.** *We have*

$$\left|\det(\mathrm{Id}+K)_{L^2(\mathcal{C})}-1\right| \leq \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\prod_{i=1}^{k}\left|\frac{1}{k}\sum_{j=1}^{k}|K(x_i,x_j)|^2\right|^{1/2}\prod_{i=1}^{k}dx_i. \tag{3.83}$$

Proof of Proposition 3.1.8. Combining Lemma 3.4.25 with Lemma 3.4.22, we obtain the follow-

ing (allowing the constant c to change between lines):

$$
\begin{aligned}
&\left|\det(\mathrm{Id}+K^{(p_T)})_{L^2(\mathcal{C})}-1\right| = \left|\det(\mathrm{Id}+\overline{K})_{L^2(\widehat{\mathcal{C}})}-1\right| \\
&\leq \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\int_{\widehat{\mathcal{C}}}\cdots\int_{\widehat{\mathcal{C}}}\prod_{i=1}^{k}\left|\frac{1}{k}\sum_{j=1}^{k}|\overline{K}(x_i,x_j)|^2\right|^{1/2}\prod_{i=1}^{k}dx_i \\
&\leq \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\int_{\widehat{\mathcal{C}}}\cdots\int_{\widehat{\mathcal{C}}}\prod_{i=1}^{k}\left|\frac{1}{k}\sum_{j=1}^{k}\left(c^{-1}\exp\left(-c|x_i|^3-cs\right)+c^{-1}\exp\left(-c|x_i|^3-cT\right)\right)^2\right|^{1/2}\prod_{i=1}^{k}dx_i \\
&= \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\left(\int_{\widehat{\mathcal{C}}}\left(c^{-1}\exp\left(-c|x|^3-cs\right)+c^{-1}\exp\left(-c|x|^3-cT\right)\right)dx\right)^k \\
&= \sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{(k-1)!}\left(c^{-1}e^{-cs}+c^{-1}e^{-cT}\right)^k\left(\int_{\widehat{\mathcal{C}}}c^{-1}\exp\left(-c|x|^3\right)dx\right)^k.
\end{aligned}
$$

This last integral is bounded above by a constant, and we can also bound $(k-1)! \geq 8^{-k}k^k$. We then obtain

$$
\left|\det(\mathrm{Id}+K^{(p_T)})_{L^2(\mathcal{C})}-1\right| \leq \sum_{k=1}^{\infty}\frac{16^k}{k^{k/2}}\left(c^{-1}e^{-cs}+c^{-1}e^{-cT}\right)^k \leq c^{-1}(e^{-cs}+e^{-cT}). \tag{3.84}
$$

Combining (3.36), (3.72), and (3.84) we have the following: There exists $c = c(\varepsilon)$ such that for all T large enough and for all $s \geq 0$ we have that for $\mu \in \left[\frac{\kappa}{(\kappa\rho-\rho+1)^2}+\varepsilon, \kappa-\varepsilon\right]$:

$$
\begin{aligned}
\mathbb{P}[H^\rho(T\mu,T) \leq g(\mu)T - sf_\mu T^{1/3}] &\leq 2\left(1-\det(\mathrm{Id}+K^{(p_T)})_{L^2(\mathcal{C})}\right) \\
&\leq c^{-1}(e^{-cs}+e^{-cT}).
\end{aligned}
$$

Due to the fact that f_μ is bounded uniformly for $\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$, we can absorb the constant f_μ into s by simply substituting $s \to f_\mu^{-1}s$. We then get the desired bound on $\mathbb{P}[H^\rho(T\mu,T) \leq g(\mu)T - sT^{1/3}]$ for all $\rho \in (0,1)$. Finally, we use attractivity to obtain the same bound for step initial data (i.e. $\rho = 1$). Chose ρ such that $[\kappa^{-1}+\varepsilon, \kappa-\varepsilon] \subset \left[\frac{\kappa}{(\kappa\rho-\rho+1)^2}+\frac{\varepsilon}{2}, \kappa-\frac{\varepsilon}{2}\right]$. Then we

have that for all $\mu \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon]$:

$$\mathbb{P}[H(T\mu, T) \leq g(\mu)T - sT^{1/3}] \leq \mathbb{P}[H^{\rho}(T\mu, T) \leq g(\mu)T - sT^{1/3}] \leq c^{-1}(e^{-cs} + e^{-cT}).$$

Finally, the term e^{-cT} can be absorbed into the term e^{-cs} because the left-hand side becomes 0 if $s > \max_{\mu \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]} g(\mu)T^{2/3}$. We can choose c to be weakly decreasing in ε since all constants in the above estimates depend continuously on ε. □

3.5 From linear trajectories to the proof of the main theorem

We can now begin to prove the main theorem. Let us recall the setup of Theorem 3.1.1. We start a stochastic six-vertex process from step initial conditions with a second-class particle at the origin. In this section, and in Sections 3.6 and 3.7, we always view the stochastic six-vertex model as a particle system on the line (see Definition 3.1.4). On the line, the above initial conditions are given by a first-class particle at every position $x < 0$, a second-class particle at position 0, and holes at positions $x > 0$. Denote by $(\mathcal{A}_t)_{t \geq 0}$ the single-class stochastic six-vertex process given by the first-class particles in this process and by $\boldsymbol{X}_t$ the position of the second-class particle at time t. These processes are started from the initial conditions

$$\mathcal{A}_0(x) = \mathbf{1}_{x<0} \quad \text{and} \quad X_0 = 0\,,$$

and $(\mathcal{A}, \boldsymbol{X})$ contains all the information of the multi-class process. Let $\mathcal{F}_s$ denote the σ-algebra generated by $(\mathcal{A}, \boldsymbol{X})$ until time s, for $s \in \mathbb{Z}_{\geq 0}$.

Let us now define some events, which will be vital to the proof of the main theorem.

Definition 3.5.1. *Fix positive integers S and T. We define the following $\mathcal{F}_S$-measurable event, which depends on some $\varepsilon > 0$:*

$$P_S = \left\{ \frac{\boldsymbol{X}_S}{S} \in [\kappa^{-1} + \varepsilon, \kappa - \varepsilon] \right\}. \tag{3.85}$$

We also define the following $\mathcal{F}_{S+T}$-measurable events, which depend on some $\gamma \in [0,1]$:

$$E_S^{\geq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \geq \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma} \right\} \tag{3.86}$$
$$E_S^{\leq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \leq \frac{\boldsymbol{X}_S}{S} T + S^{1-\gamma} \right\}.$$

Finally, we let $E_S := E_S^{\geq} \cap E_S^{\leq}$.

On the event P_S, the speed $\frac{\boldsymbol{X}_S}{S}$ is bounded strictly away from the edge of the rarefaction fan so that the effective hydrodynamic bounds in Corollary 3.4.2 will apply. On event E_S, we control how much the speed of the second-class particle at time $S+T$ deviates from the speed at time S. In the following proposition, we choose appropriate time steps S and T and show the existence of a high-probability $\mathcal{F}_S$-measurable *hydrodynamic event* H_S upon which E_S will hold with high probability at time $S+T$. The precise definition of this event will be given in Proposition 3.7.2. We call H_s the hydrodynamic event since it is the event upon which at time S the height function of $\mathcal{A}$ has not deviated too much from the hydrodynamic limit. We will show that the same is true at time $S+T$ with high probability, for any possible configuration in H_S.

Proposition 3.5.2. *For any integer $S \geq 1$, let $T = S^\beta$ for some $\beta \in (\frac{2}{3}, 1)$. For any positive $\alpha < \frac{\beta}{2} - 1/3$ and for any $\varepsilon \in (0, \frac{1}{4})$, there is a $c = c(\varepsilon, \alpha) > 0$ and an $\mathcal{F}_S$-measurable event H_S such that for all $S \geq 1$ and for $\gamma = \frac{5}{6} - \frac{\beta}{2} - \alpha$ we have*

$$\mathbb{P}[H_S] \geq 1 - c^{-1} e^{-cS^\alpha} \quad \mathbb{P}[E_S | \mathcal{F}_S] \geq (1 - c^{-1} e^{-cT^\alpha}) \mathbf{1}_{P_s \cap H_s}. \tag{3.87}$$

Note that since $T < S$, the bound on $\mathbb{P}[H_S]$ remains true if we replace S with T.

Remark 3.5.3. *The parameters α, β, and γ have the following meaning. The parameter β controls the size of the time steps, with β closer to 1 giving larger time steps. The parameter γ determines our control on the trajectory of the particle, with larger γ giving tighter bounds. For each β we can prove the statement of the proposition for each $\gamma < \frac{5}{6} - \frac{\beta}{2}$ and the difference between the two is α. Therefore, small α are of interest. There is a trade-off for the value of β. Bigger β gives*

better control along the sequence of time steps. To see this note that $S^{1-\gamma} = T^{\frac{1}{6}\beta^{-1}+\frac{1}{2}+\alpha\beta^{-1}}$*, and this exponent is minimized for* $\beta = 1$*. However, between the time steps, bigger* β *leads to less control, since we use a rough bound based on monotonicity between time steps. We will ultimately set* $\beta = \frac{7}{9}$*, where the value* $\beta = \frac{7}{9}$ *is obtained by an optimization balancing these two effects.*

Before proving this proposition, let us see how it implies Theorems 3.1.1 and 3.1.2. This largely follows [46, Section 4], especially Proposition 3.5.8, except that there a different choice of time sequence was made, see Remark 3.5.9.

Definition 3.5.4. *For* $S_0 \geq 2$ *define* $(S_m)_{m\geq 1}$ *and* $(T_m)_{m\geq 0}$ *as follows. Let* $T(S) = S^\beta$*. Let* $T_m = T(S_m)$ *and* $S_{m+1} = S_m + T_m$*. Note that* $T(S)$ *is strictly increasing for all* $S \geq 1$*.*

Lemma 3.5.5. *There exists constants* $z_- = z_-(\beta) > 0$ *and* $z_+ = z_+(\beta) > 0$*, such that* $z_- m^{\frac{1}{1-\beta}} \leq S_m \leq z_+ m^{\frac{1}{1-\beta}}$ *for all* $m \geq 1$*.*

Proof. This follows from a straightforward induction argument, using the fact that $z_- x^{\frac{1}{1-\beta}} + (z_- x^{\frac{1}{1-\beta}})^\beta - z_-(x+1)^{\frac{1}{1-\beta}}$ is both positive at $x = 2$ and strictly increasing on $[2, \infty]$ for z_- small enough (but still positive). The upper bound also follows by induction, for z^+ bigger than 1 and such that the inequality is true for S_1. □

Lemma 3.5.6. *For each* ε *there exists a* $D = D(\varepsilon, \beta, \alpha)$ *such that for* $S_0 \geq D$ *the following holds*

$$\sum_{m\geq 0} S_m^{-\gamma} \leq \varepsilon/9; \quad \sum_{m\geq 0} c^{-1} e^{-cT_m^\alpha} < \varepsilon/2; \quad \kappa \frac{T_m}{S_m} < \varepsilon/9, \text{ for all } m \geq 0\,. \tag{3.88}$$

Proof. By Lemma 3.5.5 and the monotonicity mentioned in Definition 3.5.4, $S_m \geq \max(S_0, z_- S^{\frac{1}{1-\beta}})$. The term $(z_- m^{\frac{1}{1-\beta}})^{-\gamma} = z_-^{-\gamma} m^{\frac{-\frac{5}{6}+\frac{\beta}{2}+\alpha}{1-\beta}}$ is summable since the exponent is less than -1 by the assumption $\alpha < \frac{\beta}{2} - \frac{1}{3}$. Therefore by the dominated convergence theorem as $D \to \infty$ this sum goes to 0. The second and third sums are convergent since S_m and T_m grow faster than linear in m, and therefore the sums can be bounded by geometric series. By the same dominated convergence argument, the statement follows. Finally $\kappa \frac{T_m}{S_m} < \varepsilon/4$ simply follows from the fact that $\frac{T_m}{S_m}$ goes to 0 as S_0 goes to ∞. □

Definition 3.5.7. *Define ε_m by setting $\varepsilon_0 = \varepsilon \in (0, \frac{1}{4})$ and*

$$\varepsilon_{m+1} = \varepsilon_m - S_m^{-\gamma}$$

and note that for $S_0 > D(\varepsilon, \beta, \alpha)$ all ε_m are positive (in fact they are greater than $\frac{8\varepsilon}{9}$) by Lemma 3.5.6. Define further the event

$$L^{\varepsilon}_{S_0}(k) = \bigcap_{m=0}^{k-1} P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap E_{S_m} \tag{3.89}$$

and let $L^{\varepsilon}_{S_0} = L^{\varepsilon}_{S_0}(\infty)$.

Since the constant c in Proposition 3.5.2 can be taken to be weakly decreasing in ε, we can assume that the statement of Proposition 3.5.2 holds with the same c for all ε_m in the definition above.

Proposition 3.5.8. *There exists a constant d such that for all $\varepsilon \in (0, 1/4)$ there is a constant D such that for all $S_0 \geq D$ the probability of $L^{\varepsilon}_{S_0}$ is at least $1 - (d+1)\varepsilon$.*

Proof. Note first that the event $P^{\varepsilon_m}_{S_m} \cap E_{S_m}$ is contained in the event $P^{\varepsilon_{m+1}}_{S_{m+1}}$, since on the event E_{S_m} it holds that $\left| \frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m} \right| \leq S_m^{-\gamma}$. Using this we obtain

$$\begin{aligned}
\mathbb{P}[L^{\varepsilon}_{S_0}(k)] &= \mathbb{P}[P^{\varepsilon_0}_{S_0}] - \sum_{m=0}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c] && (3.90)\\
&\quad - \sum_{m=0}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap (E^{\varepsilon_m}_{S_m})^c] - \sum_{m=1}^{k-1} \mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap (P^{\varepsilon_m}_{S_m})^c] && (3.91)\\
&\geq \mathbb{P}[P^{\varepsilon_0}_{S_0}] - \sum_{m \geq 0} \mathbb{P}[P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c] - \mathbb{P}[P^{\varepsilon_m}_{S_m} \cap H^{\varepsilon_m}_{S_m} \cap (E_{S_m})^c]\,, && (3.92)
\end{aligned}$$

where we used the fact noted above to observe that $\mathbb{P}[L^{\varepsilon_0}_{S_0}(m) \cap (P^{\varepsilon_m}_{S_m})^c] = 0$. By Proposition 3.9.4, $\frac{\boldsymbol{X}_t}{t}$ converges in law to a continuous random variable on $[\kappa^{-1}, \kappa]$, with density $\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}}$. This density is bounded, and therefore there is a constant d such that for all S_0 large enough $\mathbb{P}[P^{\varepsilon_0}_{S_0}] \geq 1 - d\varepsilon$, for all ε_0. Note further that by Proposition 3.5.2, both $\mathbb{P}[P^{\varepsilon_m}_{S_m} \cap (H^{\varepsilon_m}_{S_m})^c]$ and

$\mathbb{P}[P_{S_m}^{\varepsilon_m} \cap H_{S_m}^{\varepsilon_m} \cap (E_{S_m})^c])$ are less than $c^{-1}e^{-cT_m^\alpha}$, and thus by Lemma 3.5.6 the right-hand side of (3.90) is at least $1-(d+1)\varepsilon$. □

Proof of Theorem 3.1.1. We already have weak convergence to the desired distribution from Proposition 3.9.4, so it remains to prove a.s. convergence. We will show that with probability at least $1-(d+1)\varepsilon$ the difference between the limit superior and the limit inferior of $\frac{\boldsymbol{X}_t}{t}$ is less than ε. This immediately implies that this probability is indeed 1 since these events form a decreasing family as ε goes to 0. Since this holds for any ε, the conclusion follows as the limit superior and the limit inferior must then be equal with probability 1.

Claim: Fix $\varepsilon > 0$. There exists D such that for $s, s' > D$, with probability at least $1-(d+1)\varepsilon$:

$$\left| \frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{s'}}{s'} \right| \leq \varepsilon \,. \tag{3.93}$$

Proof of Claim: Let D be large enough such that both Lemma 3.5.6 and Proposition 3.5.8 hold. Let $S_0 = D$. For $S_0 < s < s'$ let m be the largest integer such that $S_m < s$ and m' be the smallest integer such that $s' < S_{m'}$. By Proposition 3.5.8 the event $L_{S_0}^{\varepsilon}$ holds with probability at least $1-(d+1)\varepsilon$. Assume now that the event $L_{S_0}^{\varepsilon}$ takes place, so that in particular E_{S_m} takes place for all $m \geq 0$. Then

$$\begin{aligned}
\left| \frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{S_m}}{S_m} \right| &\leq \left| \frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{S_{m+1}}}{s} \right| + \left| \frac{\boldsymbol{X}_{S_{m+1}}}{s} - \frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} \right| + \left| \frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m} \right| && (3.94)\\
&\leq \frac{|\boldsymbol{X}_{S_m} - \boldsymbol{X}_{S_{m+1}}|}{S_m} + \frac{|\boldsymbol{X}_{S_{m+1}}|}{S_{m+1}} \frac{S_{m+1} - S_m}{S_m} + S_m^{-\gamma} && (3.95)\\
&\leq 2\left(\kappa \frac{T_m}{S_m} + S_m^{-\gamma} \right) \leq \tfrac{4}{9}\varepsilon \,, && (3.96)
\end{aligned}$$

where in the second inequality we used monotonicity of $\boldsymbol{X}_s$ for the first two terms, and for the third term we used that on E_{S_m} it holds that $|\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}| \leq S_m^{-\gamma}$. For the third inequality we used the event E_{S_m} to control $|\boldsymbol{X}_{S_m} - \boldsymbol{X}_{S_{m+1}}|$ and that $\frac{\boldsymbol{X}_{S_m}}{S_m} < \kappa$, since $P_{S_m}^{\varepsilon_m}$ holds. The final inequality

then follows from Lemma 3.5.6. By the same argument $\left|\frac{\boldsymbol{X}_{s'}}{s'} - \frac{\boldsymbol{X}_{S_{m'}}}{S_{m'}}\right| \leq \frac{4}{9}\varepsilon$. Finally, it holds that

$$\left|\frac{\boldsymbol{X}_{S_m}}{S_m} - \frac{\boldsymbol{X}_{S_{m'}}}{S_{m'}}\right| \leq \sum_{n=m}^{m'-1} \left|\frac{\boldsymbol{X}_{S_n}}{S_n} - \frac{\boldsymbol{X}_{S_{n+1}}}{S_{n+1}}\right| \leq \sum_{n=m}^{m'-1} S_n^{-\gamma} \leq \tfrac{1}{9}\varepsilon \tag{3.97}$$

since E_{S_n} holds for all $m \leq n \leq m'$ and Lemma 3.5.6. It follows that with probability at least $1-(d+1)\varepsilon$,

$$\left|\frac{\boldsymbol{X}_s}{s} - \frac{\boldsymbol{X}_{s'}}{s'}\right| \leq \varepsilon\,, \tag{3.98}$$

for all $s, s' > S_0$ which was the claim.

This now implies that with probability at least $1-(d+1)\varepsilon$,

$$\left|\limsup_{t\to\infty} \frac{\boldsymbol{X}_t}{t} - \liminf_{t\to\infty} \frac{\boldsymbol{X}_t}{t}\right| < \varepsilon, \tag{3.99}$$

and the proof of Theorem 3.1.1 follows.

□

Note that the conditions on β and α together with the definition of γ imply

$$\beta + \gamma > 1 \tag{3.100}$$

Proof of Theorem 3.1.2. For time steps given by $T(S) = S^\beta$, again for each ε there exists a $D(\varepsilon, \beta, \alpha)$ such that the event $L^\varepsilon_{S_0}$ occurs with probability at least $1-(d+1)\varepsilon$ for all $S_0 > D$. Letting C be a constant that can depend on β and α and that can change from line to line, we note

that on this event we have

$$|\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| = \left|\boldsymbol{X}_{S_n} - S_n\left(\frac{\boldsymbol{X}_{S_n}}{S_n} + \sum_{m\geq n}\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right)\right| \tag{3.101}$$

$$= S_n\left|\sum_{m\geq n}\frac{\boldsymbol{X}_{S_{m+1}}}{S_{m+1}} - \frac{\boldsymbol{X}_{S_m}}{S_m}\right| \tag{3.102}$$

$$\leq S_n\sum_{m\geq n} S_m^{-\gamma} \tag{3.103}$$

$$\leq CS_n\sum_{m\geq n} m^{-\gamma\frac{1}{1-\beta}} \tag{3.104}$$

$$\leq CS_n n^{-\gamma\frac{1}{1-\beta}+1} \tag{3.105}$$

$$\leq CS_n^{2-\gamma-\beta}. \tag{3.106}$$

In (3.104) and (3.106), we used Lemma 3.5.5 to bound S_n from above and below by constants times $n^{\frac{1}{1-\beta}}$ and we used the fact that the $-\gamma\frac{1}{1-\beta} \leq -1$ by (3.100) to ensure that p-series in (3.104) converges and can be bounded. It remains to check the behavior times s that are not of the form S_n. For s between S_n and S_{n+1} we get

$$|\boldsymbol{X}_s - s\boldsymbol{U}| \leq |\boldsymbol{X}_s - \boldsymbol{X}_{S_n}| + |\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| + |S_n\boldsymbol{U} - s\boldsymbol{U}| \tag{3.107}$$

$$\leq |\boldsymbol{X}_{S_{n+1}} - \boldsymbol{X}_{S_n}| + |\boldsymbol{X}_{S_n} - S_n\boldsymbol{U}| + \kappa|S_{n+1} - S_n| \tag{3.108}$$

$$\leq C(S_n^{1-\gamma} + S_n^{2-\gamma-\beta} + S_n^{\beta}) \leq C(s^{2-\gamma-\beta} + s^{\beta}),, \tag{3.109}$$

where in the second inequality, we used monotonicity of $\boldsymbol{X}_s$ in the first term and that U is bounded by κ in the third term. For the third inequality, we used that the event E_n holds for the first term, (3.106) for the second term, and the definition of S_{n+1} for the third term. Setting $\beta = \frac{7}{9}$ gives $\gamma = \frac{4}{9} - \alpha$, which gives $2 - \gamma - \beta = \frac{7}{9} + \alpha$, therefore showing that on an event of probability $1 - (d+1)\varepsilon$ the limit of $|\boldsymbol{X}_S - s\boldsymbol{U}|s^{-\frac{7}{9}-2\alpha}$ is 0, which concludes the proof. □

Remark 3.5.9. *While we used the sequence $S_{n+1} = S_n + S_n^{\beta}$, which grows polynomially, in [46], they used the sequence $S_{n+1} = S_n + \frac{S_n}{\log S_n}$, which grows like $e^{\sqrt{n}}$. The latter sequence would have*

sufficed to prove Theorem 3.1.1. Furthermore, this time scale gives

$$|X_{S_n} - S_n U| \leq S_n^{\frac{2}{3}+}$$

which is the expected order of fluctuations. However, it grows too quickly to prove the finer statement in Theorem 3.1.2 as in (3.109)*, we crucially used the fact that* $|S_{n+1} - S_n| \leq s^\beta$ *to bound the fluctuations between the times* S_n*.*

3.6 From hydrodynamic events to linear trajectories

The purpose of this section is to prove Proposition 3.5.2. We write the full proof only for $E_S^{\geq}$, since $E_S^{\leq}$ can be treated similarly. We will outline the proof for $E^{\leq}$ at the end of Section 3.7.

Let us now proceed with the proof for the process $E^{\geq}$. We couple the process $(\mathcal{A}, \boldsymbol{X})$ to a new multi-class stochastic six-vertex process $\boldsymbol{\mathcal{B}}$ by filling in every position to the left of $\boldsymbol{X}_S$ with particles. Then Theorem 3.1.6 allows us to control the position of $\boldsymbol{X}_{S+T}$ by controlling a large number of these additional particles.

Definition 3.6.1. *Define a new multi-class process* $(\boldsymbol{\mathcal{B}}_t)_{t\geq 0}$*, with first-, second- and third-class particles, depending on* $\mathcal{A}_S$ *in the following way:*

- *It has the same parameters* b_1 *and* b_2*.*
- *At time* 0 *each site* $j \in \mathbb{Z}$ *in* $\boldsymbol{\mathcal{B}}_0$ *is occupied by a first-class particle if it is occupied by a particle in* $\mathcal{A}_S$*,*
- *the site* $\boldsymbol{X}_S$ *is occupied by a second-class particle,*
- *and each site to the left of* $\boldsymbol{X}_S$ *not occupied by a particle in* $\mathcal{A}_S$ *is also occupied by a third-class particle.*

Further let $\boldsymbol{M}$ *be the number of third-class particles in* $\boldsymbol{\mathcal{B}}$*, which is finite, since it is at most* $\boldsymbol{X}_S$*. Further, let* $\boldsymbol{Z}_t(0) > \boldsymbol{Z}_t(1) > \cdots > \boldsymbol{Z}_t(\boldsymbol{M})$ *denote the ordered positions of the second-class*

particle and the third-class particles in $\boldsymbol{\mathcal{B}}_t$*. At time* 0 *we have* $\boldsymbol{Z}_0(0) = \boldsymbol{X}_S$*, but at later times,* $\boldsymbol{X}_{S+t}$ *can be any of the positions* $\boldsymbol{Z}_t$*.*

We will also consider the two single-class processes obtained by merging the second-class and third-class particles in $\boldsymbol{\mathcal{B}}$ with either the holes or the first-class particles. That is, let $(\boldsymbol{\mathcal{B}}_t^{(1)})_{t\geq 0}$ be the single-class process given by just the first-class particles in $\boldsymbol{\mathcal{B}}$ and let $(\boldsymbol{\mathcal{B}}_t^{(1,2,3)})_{t\geq 0}$ be the single-class process of the first-, second- and third-class particles in $\boldsymbol{\mathcal{B}}$ forgetting their classes. The triplet $(\boldsymbol{\mathcal{B}}_t^{(1)}, \boldsymbol{\mathcal{B}}_t^{(1,2,3)}, \boldsymbol{X}_{S+t})_{t\geq 0}$ contains all the information of $\boldsymbol{\mathcal{B}}$.

The process $\boldsymbol{\mathcal{B}}$ depends both on $\mathcal{A}_S$ and $\boldsymbol{X}_S$. We will often use a union bound over possible values of $\boldsymbol{X}_S$. To do so it will be convenient to introduce a version of $\boldsymbol{\mathcal{B}}$ in which the position of the second-class particle is replaced by a deterministic position X. For X in the interval $[S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \cap \mathbb{Z}$, define $\boldsymbol{\mathcal{B}}^X$ as the process obtained from $\mathcal{A}_S$ by adding a second-class particle at X if that position is empty and filling all empty positions $k < X$ with third-class particles (note that $\boldsymbol{\mathcal{B}}^X$ may not have a second-class particle if there is a particle at position X in $\mathcal{A}_S$). Let further $\boldsymbol{\mathcal{B}}^{(1,2,3),X}$ be the single-class process of the first- and second-class particles in $\boldsymbol{\mathcal{B}}^X$. Note that it possible to couple all the processes $\boldsymbol{\mathcal{B}}^X$, $\boldsymbol{\mathcal{B}}$, $\mathcal{A}$ and $\boldsymbol{X}$ such that the first-class particles in $\boldsymbol{\mathcal{B}}_t^X$ and $\boldsymbol{\mathcal{B}}_t$ are given by $\mathcal{A}_{S+t}$, and such that substituting $\boldsymbol{X_S}$ for X it holds that

$$\boldsymbol{\mathcal{B}}^{\boldsymbol{X}_S} = \boldsymbol{\mathcal{B}}\,. \tag{3.110}$$

Note however that the law of $\boldsymbol{\mathcal{B}}$ conditioned on $\boldsymbol{X}_S = X$ is not given by $\boldsymbol{\mathcal{B}}^X$, since $\boldsymbol{X}_S$ and the first-class particles in $\boldsymbol{\mathcal{B}}$ are non-trivially correlated. However, by showing that certain events happen for all $\boldsymbol{\mathcal{B}}^X$ with exponentially small probability, one can use (3.110) to show that they also happen for $\boldsymbol{\mathcal{B}}$ with exponentially small probability.

Lemma 3.6.2. *Let* $\boldsymbol{G} \sim \mathrm{Geo}(q)$ *be independent of* $\mathcal{F}_S$ *and* $\boldsymbol{Z}$*. Then for any* $y \in \mathbb{Z}$ *and any* $S, T \geq 1$ *it holds that*

$$\mathbb{P}[\boldsymbol{X}_{S+T} \geq \boldsymbol{X}_S + y | \mathcal{F}_S] \geq \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{G} \wedge \boldsymbol{M}) \geq \boldsymbol{X}_S + y | \mathcal{F}_S]\,. \tag{3.111}$$

Proof. Denote the number of third-class particles that are to the right of the second-class particle in $\boldsymbol{\mathcal{B}}_T$ by $\boldsymbol{K}$, such that $\boldsymbol{X}_{S+T} = \boldsymbol{Z}_T(\boldsymbol{K})$. By Theorem 3.1.6, the law of $\boldsymbol{K}$ conditioned on $\mathcal{F}_S$ and $\boldsymbol{Z}_T$ is dominated by $\mathrm{Geo}(q)$. Therfore $\boldsymbol{K}$ can be coupled to a random variable $\boldsymbol{G}^* \sim \mathrm{Geo}(q)$ independent of $\mathcal{F}_S$ and $\boldsymbol{Z}_T$ such that $\boldsymbol{K} \leq \boldsymbol{G}^*$ almost surely. Thus we obtain

$$\mathbb{P}[\boldsymbol{X}_{S+T} \geq \boldsymbol{X}_S + y | \mathcal{F}_S] = \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{K}) \geq \boldsymbol{X}_S + y | \mathcal{F}_S] \tag{3.112}$$

$$\geq \mathbb{P}[\boldsymbol{Z}_T(\boldsymbol{G}^* \wedge \boldsymbol{M}) \geq \boldsymbol{X}_S + y | \mathcal{F}_S], \tag{3.113}$$

where in (3.113) we used $\boldsymbol{K} \leq \boldsymbol{G}^* \wedge \boldsymbol{M}$ and the ordering of $\boldsymbol{Z}_T$. Note that the right-hand side does not depend on the coupling between $\boldsymbol{K}$ and $\boldsymbol{G}^*$ since $\boldsymbol{G}^*$ is independent of $\mathcal{F}_S$ and $\boldsymbol{Z}$. Therefore, we can replace $\boldsymbol{G}^*$ by $\boldsymbol{G}$ in (3.113). □

Let $\boldsymbol{L}$ be defined as

$$\boldsymbol{L} = \#\left\{\text{second- and third-class class particles in } \boldsymbol{\mathcal{B}}_T \text{ to the right of } \frac{\boldsymbol{X}_S}{S}(S+T) - S^{1-\gamma}\right\}. \tag{3.114}$$

Using Lemma 3.6.2, we can reduce the proof of Proposition 3.5.2 to the following lemma which states that $\boldsymbol{L}$ is of order at least $S^{\frac{1}{3}}$ with high probability.

Lemma 3.6.3. *For any positive $\varepsilon < \frac{1}{4}$, and for T, α and γ as in Proposition 3.5.2, there is a constant $c > 0$ and an $\mathcal{F}_S$-measurable event H_S such that for all $S \geq 1$*

$$\mathbb{P}(H_S) \geq 1 - c^{-1}e^{-cS^{\alpha}} \tag{3.115}$$

and

$$\mathbb{P}[\boldsymbol{L} \geq S^{\frac{1}{3}} | \mathcal{F}_S] \geq (1 - c^{-1}e^{-cT^{\alpha}})\mathbf{1}_{P_S \cap H_S}\,. \tag{3.116}$$

Before proving this proposition let us see how it implies Proposition 3.5.2.

Proof of Proposition 3.5.2. As said above, we only prove the statement for $E_S^{\geq}$. Condition on $\mathcal{F}_S$ and assume that $H_S \cap P_S$ holds. Let $\boldsymbol{G} \sim \mathrm{Geo}(q)$ be independent of $\boldsymbol{Z}$ and $\mathcal{F}_S$, as above. Define

the events

$$N_S = \left\{ \boldsymbol{Z}_T(\boldsymbol{G} \wedge \boldsymbol{M}) \geq \frac{\boldsymbol{X}_S}{S}(S+T) + S^{1-\gamma} \right\} \quad \text{and} \quad G_S = \left\{ \boldsymbol{L} > S^{\frac{1}{3}} \right\} \tag{3.117}$$

and recall the definition of the event $E_S^{\geq}$ is

$$E_S^{\geq} = \left\{ \boldsymbol{X}_{S+T} - \boldsymbol{X}_S \geq \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma} \right\}.$$

Setting $y = \frac{\boldsymbol{X}_S}{S} T - S^{1-\gamma}$, it follows from Lemma 3.6.2 that $\mathbb{P}[E_S^{\geq} | \mathcal{F}_S] \geq \mathbb{P}[N_S | \mathcal{F}_S]$. By the distribution of $\boldsymbol{G}$,

$$\mathbb{P}[N_S | G_S, \mathcal{F}_S] \geq \mathbb{P}[\boldsymbol{G} \leq S^{1/3}] \geq 1 - q^{S^{\frac{1}{3}}}.$$

Combining this with the statement of Lemma 3.6.3 gives

$$\begin{aligned}
\mathbb{P}[E_S^{\geq} | \mathcal{F}_S] &\geq \mathbb{P}[F_S | \mathcal{F}_S] \\
&\geq \mathbb{P}[N_S | G_S, \mathcal{F}_S] \mathbb{P}[G_S | \mathcal{F}_S] \\
&\geq \left(1 - q^{S^{\frac{1}{3}}}\right) \left(1 - c^{-1} e^{-cT^{\alpha}}\right) \mathbf{1}_{H_S \cap P_S} \\
&\geq (1 - c^{-1} e^{-cT^{\alpha}}) \mathbf{1}_{H_S \cap P_S}.
\end{aligned}$$

The final inequality is obtained by decreasing c so that we can absorb the term $q^{S^{\frac{1}{3}}}$ (since $\alpha \leq 1/3$ and $S \geq T$). □

3.7 From effective hydrodynamics to hydrodynamic events

The purpose of this section is to prove Lemma 3.6.3. To this end let $(\boldsymbol{\mathcal{B}}^{\text{step},X})_{t \geq 0}$ be a stochastic six-vertex process started from step initial conditions shifted by X, i.e. $\boldsymbol{\mathcal{B}}_0^{\text{step},X}(x) = \mathbf{1}_{x \leq X}$. Clearly, at time 0 we have that $\boldsymbol{\mathcal{B}}_0^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_0^{\text{step},X}(x)$ for all $x \in \mathbb{Z}$, so by attractivity (Proposition 3.2.3) we can couple them so that this holds for any later time as well. Note that $\boldsymbol{\mathcal{B}}^{(1,2,3),X}$ and $\boldsymbol{\mathcal{B}}^{(1)}$ are already coupled such that $\boldsymbol{\mathcal{B}}_t^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_t^{(1)}(x)$ for any time t by their relation to the

multi-class process $\boldsymbol{\mathcal{B}}^X$. This gives a coupling of the three processes $\boldsymbol{\mathcal{B}}^{(1,2,3),X}, \boldsymbol{\mathcal{B}}^{(1)}, \boldsymbol{\mathcal{B}}^{\text{step},X}$ such that at all times t it holds that

$$\boldsymbol{\mathcal{B}}_t^{(1,2,3),X}(x) \geq \max(\boldsymbol{\mathcal{B}}_t^{(1)}(x), \boldsymbol{\mathcal{B}}_t^{\text{step},X}(x)) \text{ for all } x \in \mathbb{Z}.$$

The process $\boldsymbol{\mathcal{B}}^{\text{step},X}$ is a stochastic six-vertex process started from step initial conditions, thus we can use Corollary 3.4.2 to control its height function. Understanding $\boldsymbol{\mathcal{B}}^{(1)}$ is more intricate since its initial conditions are given by $\mathcal{A}_S$. We will not be able to control $\boldsymbol{\mathcal{B}}^{(1)}$ for all values of $\mathcal{A}_S$, and instead find an $\mathcal{F}_S$-measurable event H_S, which we call the *hydrodynamic event*, that holds with high probability. On this event, with high probability, $\boldsymbol{\mathcal{B}}_T^{(1)}$ is close to the hydrodynamic profile of a stochastic six-vertex process started from step initial conditions and evaluated at time $S+T$.

To simplify notation throughout this section, we will define $h_t([X,Y];\eta) := h_t(X;\eta) - h_t(Y;\eta)$. Let us now define the following shorthand notation for the event that a process is close to its hydrodynamic limit.

Definition 3.7.1. *For a single-class stochastic six-vertex process $(\mathcal{A}_t)_{t\geq 0}$ started from step initial conditions, a time t and $\alpha, \varepsilon > 0$ define the event $C_t^{\alpha,\varepsilon}(\mathcal{A})$ as*

$$\left\{ \left| h_t([X,Y];\mathcal{A}) - t(g(\tfrac{X}{t}) - g(\tfrac{Y}{t})) \right| \leq t^{\frac{1}{3}+\alpha}, \text{ for all } X \text{ and } Y \text{ in } [t(\kappa^{-1}+\varepsilon), t(\kappa-\varepsilon)] \right\}. \tag{3.118}$$

Note that this event does not depend on the choice of height function $h(x,\mathcal{A})$ since it only concerns height function differences.

Using this definition, Corollary 3.4.2 with $s = t^\alpha$ gives $\mathbb{P}[C_t^{\alpha,\varepsilon}(\mathcal{A})] \geq 1 - c^{-1}t^2 e^{-ct^\alpha}$, with the constant $c > 0$ depending on ε.

Proposition 3.7.2. *For $T = T(S)$ as in Proposition 3.5.2, for $\alpha, \varepsilon > 0$, and for $S \geq 1$, there is a constant $c = c(\varepsilon)$ and an $\mathcal{F}_S$-measurable event H_S that holds with probability $1 - c^{-1}e^{-cS^\alpha}$ such that*

$$\mathbb{P}\left[C_{S+T}^{\alpha,\varepsilon}(\mathcal{A}) | \mathcal{F}_S\right] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha}). \tag{3.119}$$

Proof. By making c smaller the right-hand side of (3.118) can be made non-positive for S small, so it suffices to consider S large enough. Let $(\widetilde{\mathcal{A}}_t)_{0\le t\le S}$ be an independent copy of $(\mathcal{A}_t)_{0\le t\le S}$, i.e. a stochastic six-vertex process started from step initial conditions and run until time S. After time S we will couple these two processes, such that they are no longer independent.

Define $H_S = C_S^{\frac{a}{2},\frac{\varepsilon}{2}}(\mathcal{A})$. This event has the desired probability by Corollary 3.4.2 and is $\mathcal{F}_S$-measurable. Further, let $\widetilde{H}_S = C_S^{\frac{a}{2},\frac{\varepsilon}{2}}(\widetilde{\mathcal{A}})$. On the intersection of H_S and $\widetilde{H}_S$ it holds that

$$\left|h_S([X,Y];\mathcal{A}) - S\left(g\left(\tfrac{X}{S}\right) - g\left(\tfrac{Y}{S}\right)\right)\right| \le S^{\frac{1}{3}+\frac{\alpha}{2}}, \text{ for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)] \tag{3.120}$$

and

$$\left|h_S([X,Y];\widetilde{\mathcal{A}}) - S\left(g\left(\tfrac{X}{S}\right) - g\left(\tfrac{Y}{S}\right)\right)\right| \le S^{\frac{1}{3}+\frac{\alpha}{2}}, \text{ for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)]. \tag{3.121}$$

These events do not depend on the choice of height function and therefore we can choose the height functions which satisfy $h_S(S(\kappa-\frac{\varepsilon}{2}),\mathcal{A}) = h_S(S(\kappa-\frac{\varepsilon}{2}),\widetilde{\mathcal{A}}) = 0$. With this choice of height function and setting $Y = S(\kappa-\frac{\varepsilon}{2})$ in (3.120) and (3.121), it follows that on the intersection $H_S \cap \widetilde{H}_S$ we have

$$|h_S(X;\mathcal{A}) - h_S(X;\widetilde{\mathcal{A}})| \le 2S^{\frac{1}{3}+\frac{\alpha}{2}} \text{ for } X \in [S(\kappa^{-1}+\varepsilon/2), S(\kappa-\varepsilon/2)]\,.$$

Thus we can couple $(\widetilde{\mathcal{A}}_t)_{t\ge S}$ with $(\mathcal{A}_t)_{t\ge S}$ via the coupling given in Lemma 3.2.9 with $M = S^{\frac{1}{3}}$ and $K = 2S^{\frac{1}{3}+\frac{\alpha}{2}}$. Define the event that this coupling succeeds as

$$D = \left\{\left|h_{S+T}(X;\widetilde{\mathcal{A}}) - h_{S+T}(X;\mathcal{A})\right| < 6S^{\frac{1}{3}+\frac{\alpha}{2}} + S^{\frac{1}{3}} \text{ for } X \in [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa-\tfrac{\varepsilon}{2})]\right\} \tag{3.122}$$

Then the statement of Lemma 3.2.9 gives

$$\mathbb{P}[D|H_S \cap \widetilde{H}_S, \mathcal{F}_S] \ge 1 - c^{-1}e^{-cT} - c^{-1}e^{-cS^{\frac{1}{3}}}\,,$$

which implies that

$$\mathbb{P}[D|\mathcal{F}_S] \geq \mathbb{P}[D|H_S \cap \widetilde{H}_S, \mathcal{F}_S]\mathbb{P}[H_S \cap \widetilde{H}_S|\mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha}) \,, \tag{3.123}$$

where we used that $\widetilde{H}_S$ is independent of $\mathcal{F}_S$ and both $c^{-1}e^{-cT}$ and $c^{-1}e^{-cS^{\frac{1}{3}}}$ were absorbed into $c^{-1}e^{-cS^\alpha}$ by decreasing c.

Consider now the event

$$D \cap C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\mathcal{A}}) \,. \tag{3.124}$$

On this event, it holds that

$$|h_{S+T}([X,Y];\mathcal{A}) - (S+T)(g(\tfrac{X}{S+T}) - g(\tfrac{Y}{S+T}))| \leq 13S^{\frac{1}{3}+\frac{\alpha}{2}} + 2S^{\frac{1}{3}}$$
$$\text{for all } X \text{ and } Y \text{ in } [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa - \tfrac{\varepsilon}{2})] \,, \tag{3.125}$$

by repeated use of the triangle inequality. For large enough S we have

$$[S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \subset [S(\kappa^{-1}+\tfrac{\varepsilon}{2}) + \tfrac{2T}{1-b_2} + 1, S(\kappa - \tfrac{\varepsilon}{2})]$$

since $T = S^\beta$ for some $\beta < 1$. Furthermore, for large enough S

$$13S^{\frac{1}{3}+\frac{\alpha}{2}} + 2S^{\frac{1}{3}} \leq S^{\frac{1}{3}+\alpha} \,.$$

Therefore, for large enough S, the event $C^{\alpha,\varepsilon}_{S+T}(\mathcal{A})$ contains $D \cap C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\mathcal{A}})$. Using (3.123) and

noting that $C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\mathcal{A}})$ is independent of $\mathcal{F}_S$ we obtain that

$$\mathbb{P}[C^{\alpha,\varepsilon}_{S+T}(\mathcal{A})|\mathcal{F}_S] \geq \mathbb{P}[D \cap C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\mathcal{A}})|\mathcal{F}_S] \tag{3.126}$$

$$\geq (\mathbb{P}[D|\mathcal{F}_S] - \mathbb{P}[C^{\frac{a}{2},\frac{\varepsilon}{2}}_{S+T}(\widetilde{\mathcal{A}})^c|\mathcal{F}_S]) \vee 0 \tag{3.127}$$

$$\geq (1 - c^{-1}e^{-cS^\alpha} - c^{-1}(S+T)^2 e^{-c(S+T)^\alpha})\mathbf{1}_{H_S} \tag{3.128}$$

$$\geq (1 - c^{-1}e^{-cS^\alpha})\mathbf{1}_{H_S}\,, \tag{3.129}$$

where $(c^{-1}(S+T)^2 e^{-c(S+T)^\alpha})\mathbf{1}_{H_S}$ is absorbed by increasing c. □

We can now prove Lemma 3.6.3.

Proof of Lemma 3.6.3. We can decrease c such that (3.116) is trivial for S small and therefore it suffices to consider large enough S. First fix some $X \in [S(\kappa^{-1}+\varepsilon), S(\kappa-\varepsilon)] \cap \mathbb{Z}$. By Proposition 3.7.2 we have

$$\mathbb{P}[C^{\alpha,\varepsilon/2}_{S+T}(\mathcal{A})|\mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cS^\alpha})\,.$$

The event $C^{\alpha,\varepsilon/2}_{S+T}(\mathcal{A})$ states a bound for all pairs of points in $[(S+T)(\kappa^{-1}+\varepsilon/2), (S+T)(\kappa-\varepsilon/2)]$. For S large enough both $\frac{X}{S}(S+T)-S^{1-\gamma}$ and $\frac{X}{S}(S+T)$ are in this interval since $\frac{X}{S} \in [\kappa^{-1}+\varepsilon, \kappa-\varepsilon]$ and $\frac{S^{1-\gamma}}{S+T} \leq \varepsilon/2$ for S large enough. Since the law of $\boldsymbol{\mathcal{B}}^{(1)}_T$ is equal to the law of $\mathcal{A}_{S+T}$, we have

$$\mathbb{P}\left[\left|h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{(1)}) - (S+T)(g(\tfrac{X}{S} - \tfrac{S^{1-\gamma}}{S+T}) - g(\tfrac{X}{S}))\right| \leq (S+T)^{\frac{1}{3}+\alpha}\middle|\mathcal{F}_S\right]$$
$$\geq (1 - c^{-1}e^{-cS^\alpha})\mathbf{1}_{H_S}\,. \tag{3.130}$$

The process $\boldsymbol{\mathcal{B}}^{\text{step},X}$ is also started from step initial data translated by X. For large enough S both $\frac{X}{S}(S+T) - S^{1-\gamma} - X$ and $\frac{X}{S}(S+T) - X$ are in $[T(\kappa^{-1}+\varepsilon/2), T(\kappa-\varepsilon/2)]$. Indeed $\frac{X}{S}(S+T) - X = \frac{X}{S}T$, and $\frac{S^{1-\gamma}}{T} < \varepsilon/2$ since $T = S^\beta$ and $\beta + \gamma > 1$ (see (3.100)). Thus we can

apply Theorem 3.4.1 to obtain:

$$\mathbb{P}\left[\left|h_T([\tfrac{X}{S}(S+T)-S^{1-\gamma},\tfrac{X}{S}(S+T)];\boldsymbol{\mathcal{B}}^{\text{step},X})-T(g(\tfrac{X}{S}-\tfrac{S^{1-\gamma}}{T})-g(\tfrac{X}{S}))\right|\le T^{\frac{1}{3}+\alpha}\right] \geq 1-c^{-1}e^{-cT^{\alpha}}. \quad (3.131)$$

Let us compare the limit shape terms in (3.130) and (3.131) by bounding their difference

$$\Delta = T\left(g\left(\frac{X}{S}-\frac{S^{1-\gamma}}{T}\right)-g\left(\frac{X}{S}\right)\right)-(S+T)\left(g\left(\frac{X}{S}-\frac{S^{1-\gamma}}{S+T}\right)-g\left(\frac{X}{S}\right)\right), \quad (3.132)$$

By the explicit form of $g(x)=\frac{(\sqrt{x}-\sqrt{\kappa})^2}{\kappa-1}$, one can easily see that on the interval $[\kappa^{-1},\kappa]$ both the first and second derivative of g are uniformly bounded and in particular $g''(x)\geq C$ for some C depending only on κ, for all $x\in[\kappa^{-1},\kappa]$. Considering the second-order Taylor expansion of g at $\frac{X}{S}$, we have

$$\begin{aligned}T(g(\tfrac{X}{S}-\tfrac{S^{1-\gamma}}{T})-g(\tfrac{X}{S})) &= -T(g'(\tfrac{X}{S})\tfrac{S^{1-\gamma}}{T}+\tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{1-\gamma}}{T})^2+O(\tfrac{S^{1-\gamma}}{T}))\\ &= -g'(\tfrac{X}{S})S^{1-\gamma}+\tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{2-2\gamma}}{T})+O(\tfrac{S^{3-3\gamma}}{T^2}).\end{aligned}$$

Similarly for the other term in (3.132) we obtain

$$(S+T)(g(\tfrac{X}{S}-\tfrac{S^{1-\gamma}}{S+T})-g(\tfrac{X}{S}))=-g'(\tfrac{X}{S})S^{1-\gamma}+\tfrac{g''(\frac{X}{S})}{2}(\tfrac{S^{2-2\gamma}}{S+T})+O(\tfrac{S^{3-3\gamma}}{(S+T)^2})$$

For large enough S the error terms are smaller than the second order terms, since $\frac{S^{1-\gamma}}{T}\to 0$ and we obtain

$$\Delta \geq CS^{2-2\gamma}\left(\frac{1}{T}-\frac{1}{S+T}\right)\geq\frac{C}{2}S^{2-2\gamma}T^{-1}\geq\frac{C}{2}S^{1/3+2\alpha} \quad (3.133)$$

where the linear terms cancel each other out and we use that $T=S^{\beta}$ and $\gamma=\frac{5}{6}-\frac{\beta}{2}-\alpha$.

Let $\boldsymbol{L}^X$ be defined as

$$\boldsymbol{L}^X = \#\{\text{second- and third-class particles in } \boldsymbol{\mathcal{B}}_T^X \text{ to the right of } \frac{X}{S}(S+T) - S^{1-\gamma}\}$$

and note that

$$\boldsymbol{L}^X = \sum_{x \geq \frac{\boldsymbol{X}_S}{S}(S+T) - S^{1-\gamma}} \boldsymbol{\mathcal{B}}_T^{(1,2,3),X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x) \geq \sum_{x = \frac{\boldsymbol{X}_S}{S}(S+T) - S^{1-\gamma}}^{\frac{X}{S}(S+T)} \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x)\,,$$

since $\boldsymbol{\mathcal{B}}_T^{(1,2,3),X}(x) \geq \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x)$ for every $x \in \mathbb{Z}$. If the events in (3.130) and (3.131) take place this sum can be bounded from below by

$$\begin{aligned}
&\sum_{x = \frac{\boldsymbol{X}_S}{S}(S+T) - S^{1-\gamma}}^{\frac{X}{S}(S+T)} \boldsymbol{\mathcal{B}}_T^{\text{step},X}(x) - \boldsymbol{\mathcal{B}}_T^{(1)}(x) \\
&= h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{\text{step},X}) - h_T([\tfrac{X}{S}(S+T) - S^{1-\gamma}, \tfrac{X}{S}(S+T)]; \boldsymbol{\mathcal{B}}^{(1)})) \\
&\geq C S^{\frac{1}{3}+2\alpha} - (S+T)^{\frac{1}{3}+\alpha} - T^{\frac{1}{3}+\alpha} \geq S^{\frac{1}{3}}\,, \quad (3.134)
\end{aligned}$$

for S large enough. Therefore

$$\mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} | \mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - c^{-1}e^{-cT^\alpha})\,,$$

where we use that the event (3.131) is independent of $\mathcal{F}_S$, since it only depends on $\boldsymbol{\mathcal{B}}^{\text{step},X}$, which is only coupled to $\boldsymbol{\mathcal{B}}$ after time S. Using a union bound we obtain

$$\mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} \text{ for all } X \in [S(\kappa^{-1}+\varepsilon), S(\kappa+\varepsilon)] \cap \mathbb{Z} | \mathcal{F}_S] \geq \mathbf{1}_{H_S}(1 - Sc^{-1}e^{-cT^\alpha})$$

and we further can absorb S into $c^{-1}e^{-cT^\alpha}$ by decreasing c. By the definition of $\boldsymbol{L}$ in (3.114), the

definition of P_S in (3.85), and the observation (3.110) we have that

$$\begin{aligned}\mathbb{P}[\boldsymbol{L} \geq S^{\frac{1}{3}} | \mathcal{F}_S] &\geq \mathbb{P}[\boldsymbol{L}^X \geq S^{\frac{1}{3}} \text{ for all } X \in [S(\kappa^{-1}+\varepsilon), S(\kappa+\varepsilon)] \cap \mathbb{Z} | \mathcal{F}_S] \mathbf{1}_{P_S} \\ &\geq \mathbf{1}_{H_S \cap P_S}(1 - c^{-1}e^{-cT^\alpha})\end{aligned}$$

as desired. □

Let us now sketch what needs to be changed for $E^{\leq}$. We need to show that on the hydrodynamic event (which is the same) the second-class particle does not deviate too much to the right. To do so we will delete all particles to the right of the second-class particle, which corresponds to a multi-class particle system $(\boldsymbol{\mathcal{B}}_t)_{t\geq 0}$ with the following initial conditions

- A first-class particle in every position that is occupied by a first-class particle in $\mathcal{A}_S$ and is to the left of $\boldsymbol{X}_S$,
- A second-class particle in every position that is occupied by a first-class particle in $\mathcal{A}_S$ that is to the right of $\boldsymbol{X}_S$ and
- A third-class particle in the position $\boldsymbol{X}_S$.

This process satisfies the conditions of Corollary 3.3.2 and by an argument analogous to Lemma 3.6.3 it suffices to show that there are a large number of second-class particles in $\boldsymbol{\mathcal{B}}_T$ to the left of $\frac{\boldsymbol{X}_S}{S}(S+T) + S^{1-\gamma}$. Denoting by $\boldsymbol{\mathcal{B}}^{(1)}$ the process of the first-class particles in $\boldsymbol{\mathcal{B}}$, by $\boldsymbol{\mathcal{B}}^{(1,2,3)}$ the process of the first, second- and third-class particles, and by $\boldsymbol{\mathcal{B}}^{\text{step}}$ a stochastic six-vertex process with step initial conditions translated to position $\boldsymbol{X}_t$. At time 0 we have

$$\boldsymbol{\mathcal{B}}_0^{(1)}(x) \leq \min(\boldsymbol{\mathcal{B}}_0^{(1,2,3)}(x), \boldsymbol{\mathcal{B}}_0^{\text{step}}(x)),$$

so $\boldsymbol{\mathcal{B}}^{\text{step}}$ can be coupled to $\boldsymbol{\mathcal{B}}^{(1)}$ such that $\boldsymbol{\mathcal{B}}_t^{(1)}(x) \leq \boldsymbol{\mathcal{B}}_t^{\text{step}}(x)$ at all later times t as well. Note that $\boldsymbol{\mathcal{B}}_t^{(1)}(x) \leq \boldsymbol{\mathcal{B}}_t^{(1,2,3)}(x)$ already holds by definition. By Proposition 3.7.2, $\boldsymbol{\mathcal{B}}_T^{(1,2,3)} = \mathcal{A}_{S+T}$ is close to the hydrodynamic limit at time $S+T$ with high probability. Since $\boldsymbol{\mathcal{B}}^{\text{step}}$ is also a stochastic

six-vertex model started from step initial conditions we can use Theorem 3.4.1 to say that it is also close to the hydrodynamic limit at time T translated by $\boldsymbol{X}_T$ with high probability. These two results, together with a union bound over all possible values of $\boldsymbol{X}_t$ and a calculation similar to (3.133), yield the desired result.

Remark 3.7.3. *For ASEP one could have simply used particle-hole duality to obtain the proof for $E^{\leq}$ as a corollary of the proof for $E^{\geq}$. After exchanging first-class particles with holes, and reversing space, one again obtains a multi-class ASEP, and the events $E^{\geq}$ and $E^{\leq}$ are exchanged. For the stochastic six-vertex model, this is not the case. Applying the particle-hole duality for the stochastic six-vertex model exchanges the two axes and therefore maps the event $E^{\geq}$ into an event that concerns the times at which the second-class particle hits positions S and $S+T$. This is clearly not the same as the event $E^{\leq}$. A different choice of $E^{\geq}$ and $E^{\leq}$ such that they are symmetric with respect to the particle-hole symmetry of the stochastic six-vertex model could be considered.*

3.8 Symmetry and stationarity of the speed process

In this section, we will prove Corollary 3.1.5 and discuss various properties of the stochastic six-vertex model speed process.

Proof of Corollary 3.1.5. By the color merging property, the law of $(\boldsymbol{X}_t(x))_{t\geq 0}$ is equal to that of $(x + \boldsymbol{X}_t(0))_{t\geq 0}$. By Theorem 3.1.1 the speed $\frac{\boldsymbol{X}_t(0)}{t}$ converges almost surely, and therefore so does each speed $\frac{\boldsymbol{X}_t(x)}{t}$. Since there are countably many particles, this also implies that almost surely all of the speeds converge. □

An immediate consequence of the construction is the ergodicity of the speed process. In this section **ergodic** always refers to ergodicity with respect to translations of $\mathbb{Z}$, i.e. of space.

Proposition 3.8.1 (Ergodicity for the Speed Process)**.** *The stochastic six-vertex speed process is ergodic.*

Proof. This is immediately inherited from the fact that the process $(\boldsymbol{X}_t(x))_{x\in\mathbb{Z},t\geq 0}$ can be constructed by sampling i.i.d. pairs of Bernoulli(b_1) and Bernoulli(b_2) random variables at every vertex, which are ergodic under the shift. □

To obtain stationarity of the speed process we need the following symmetry, which is a special case of [52, Corollary 7.1.]

Proposition 3.8.2 (Color Position Symmetry in Finite Domains)**.** *Consider the stochastic six-vertex model on an $M \times N$ box, with particles of class 1 to $M + N$ coming in on the left and lower boundaries such that from the top left to the bottom right the classes are in increasing order. Enumerate the outgoing positions along the top and right boundary with $\{1, \dots, M + N\}$ in descending order, first from left to right along the top and then from top to bottom along the right edge. Denote by π the (random) permutation of $\{1, \dots, M + N\}$ obtained by letting $\pi(x)$ equal the class of the particle at position x. Then π and π^{-1} are equal in law.*

Proof. This follows from [52, Corollary 7.1] by specializing the Ferrer diagram S to a rectangle and using the fact that rectangles are invariant under point reflections. □

We now extend this to the stochastic six-vertex model on the line.

Proposition 3.8.3 (Color Position Symmetry on the Line)**.** *Consider the random bijection $\pi_N : \mathbb{Z} \to \mathbb{Z}$ obtained by running the stochastic six-vertex model from packed initial conditions until time N (i.e. on a box of infinite width and height $N + 1$) and letting $\pi_N(x)$ be the class of the particle exiting at the vertex (x, N). Then $(\pi_N(x))_{x\in\mathbb{Z}}$ and $(-\pi_N^{-1}(-x))_{x\in\mathbb{Z}}$ are equal in law.*

Proof. Consider the box $[\![-M, M]\!] \times [\![0, N]\!]$. Consider the boundary conditions consisting of the incoming arrows from the left with $\{-M - N - 1, \dots, -M - 1\}$, in increasing order from top to bottom, and incoming arrows from the bottom with classes $\{-M, M\}$ from the bottom, again in increasing order. Enumerate the outgoing positions on the top and right boundary with $\{-M - N - 1, \dots, M\}$, again in clockwise order, i.e. starting with $-M - N - 1$ in the bottom right corner and ending with M in the top left corner. Again let $\pi_{M,N}(x)$ be the class of the

outgoing particle at position x. Note that there is the following relation between $\pi_{M,N}$ and π_N for all $x, y \in [-M, M]$

$$\mathbb{P}(\pi_N(x) = y) = \mathbb{P}(\pi_{M,N}(x) = -y)\,. \tag{3.135}$$

Indeed, by using the merging property (Proposition 3.2.5), one can see that the trace of the particle of class y inside the box $[\![-M, M]\!] \times [\![0, N]\!]$ is the same for both models since all particles coming from the left have a smaller class than y. Thus the probability to exit the box through a specific vertex along the top edge is the same in both models. The negative sign on the right-hand side is due to the outgoing boundary positions being enumerated in descending order. Using this twice along with Proposition 3.8.2, we obtain

$$\mathbb{P}(\pi_N(x) = y) = \mathbb{P}(\pi_{M,N}(x) = -y) = \mathbb{P}(\pi_{M,N}(-y) = x) = \mathbb{P}(\pi_N(-y) = -x)\,, \tag{3.136}$$

which proves the statement. □

Now note that $\pi_N^{-1}(x) = \boldsymbol{X}_N(x)$, since it is the position of the particle of class x at time N. We can use this to prove that the speed process is stationary with respect to the dynamics of the multi-class stochastic six-vertex process.

Proposition 3.8.4 (Stationarity of the Speed Process)**.** *Let U be sampled from the stochastic six-vertex speed process. Consider the multi-class stochastic six-vertex model with initial conditions given by $(-U(-x))_{x\in\mathbb{Z}}$. This process is stationary.*

Proof. Start with packed initial conditions and run the process until times N and $N+1$. Since $\pi_N(x)$ equals in distribution $-\pi_N^{-1}(-x)$, which equals $-\boldsymbol{X}_N(-x)$, we know that both $(\pi_N(x)/N)_{x\in\mathbb{Z}}$ and $(\pi_{N+1}(x)/N)_{x\in\mathbb{Z}}$ converge in law to $(-U(-x))_{x\in\mathbb{Z}}$ by Corollary 3.1.5. Let μ_N be the law of $(\pi_N(x)/N)_{x\in\mathbb{Z}}$ on the space $\mathbb{R}^{\mathbb{Z}}$ and ν_N the law of $(\pi_{N+1}(x)/N)_{x\in\mathbb{Z}}$. The laws μ_{N+1} and ν_N only differ by multiplying the corresponding random variables with a factor $\frac{N}{N+1}$. Since the dynamics do not change under monotone relabeling of classes, and division by N is such a monotone

relabeling, we have for any bounded function f on $\mathbb{R}^{\mathbb{Z}}$

$$\int f(\eta) d\nu_N(\eta) = \int P_1 f(\eta) d\mu_N(\eta) \tag{3.137}$$

where P_1 is the one-step evolution operator for the process. Since both μ_N and ν_N converge to the law of $(-U(-x))_{x\in\mathbb{Z}}$, this proves the statement. □

The following result is known for multi-class TASEP and ASEP respectively proven in [39] and [157] respectively. However, for the stochastic six-vertex model, no proof of this seems to be in the literature, see also [158, Remark 7.9]. We prove this in Section 3.10.

Proposition 3.8.5 (Uniqueness of Stationary Translation-invariant Measures)**.** *For $\lambda_k \in (0,1)$ with $\sum_{k=0}^{n} \lambda_k = 1$, there is a unique ergodic stationary measure for the n-class stochastic six-vertex process on the line with $\mathbb{P}(\eta_0(x) = k) = \lambda_k$.*

The existence can be derived abstractly from a compactness argument. Recently in [158] such measures have also been constructed in a way that is amenable to calculating marginals.

One can conclude the following:

Proposition 3.8.6. *The ergodic stationary measures for the multi-class stochastic six-vertex model (on the line) are the ergodic stationary measures for the multi-class ASEP.*

Proof. This follows from Proposition 3.8.5 together with the observation that the stationary measures constructed for both ASEP and the stochastic six-vertex model in [158] are identical. This can be seen by noticing that the stationary measures in [158, Theorem 3.3] when specialized to ASEP, as done in [158, Section 4.2] and considered on the line instead of the cylinder, are exactly the stationary measures in [158, Section 7.3] for the stochastic six-vertex model. □

This result implies the following connection between the stochastic six-vertex speed process and the ASEP speed process.

Proposition 3.8.7. *Let f be the unique increasing map from $[-\kappa, -\kappa^{-1}]$ to $[-1, 1]$, which maps a random variable with density $\frac{\sqrt{\kappa}}{2(\kappa-1)}|x|^{-\frac{3}{2}}\mathbf{1}_{x\in[-\kappa,-\kappa^{-1}]}$ to a uniform random variable in $[-1, 1]$. If U is sampled according to the stochastic six-vertex speed process. Then*

$$(f(-U(-x)))_{x\in\mathbb{Z}}$$

has the same law as the ASEP speed process.

Proof. Note first that since f is increasing, and the dynamics only considers the relative ordering of the labels, $(f(-U(-x)))_{x\in\mathbb{Z}}$ is still stationary for the stochastic six-vertex process. Given any $(\lambda_k)_{k=1,\dots,n}$ which satisfy $\sum_{k=1}^{n}\lambda_k = 1$, consider the increasing map ϕ_λ from $[-1, 1]$ to $\{1, \dots, n\}$ such that $\phi_\lambda^{-1}(k)$ is an interval of length $2\lambda_k$. By the merging property (Lemma 3.2.5), $(\phi_\lambda \circ f(-U(-x)))_{x\in\mathbb{Z}}$ is a stationary measure for the n-class stochastic six-vertex process. By the definition of ϕ_λ and f it also satisfies

$$\mathbb{P}[\phi_\lambda \circ f(-U(0)) = k] = \lambda_k\,.$$

Ergodicity is also inherited from U. By Proposition 3.8.5 there is only one such measure, and by Proposition 3.8.6 this is the same as the unique ASEP stationary measure with the same densities. If one denotes by $\widetilde{U}$ a sample of the ASEP speed process, this implies the equality in law

$$(\phi_\lambda \circ f(-U(-x)))_{x\in\mathbb{Z}} \overset{d}{=} (\phi_\lambda \circ \widetilde{U}(x))_{x\in\mathbb{Z}}$$

This equality is satisfied for all ϕ_λ, and this set of functions is sufficiently large to determine all the marginals of the speed processes, as was outlined in [49]. It follows that

$$(f(-U(-x)))_{x\in\mathbb{Z}} \overset{d}{=} (\widetilde{U}(x))_{x\in\mathbb{Z}}$$

as desired. □

3.9 Hydrodynamic limit and weak convergence

In this section we will give a summary of the hydrodynamic limit and local statistics for the stochastic six-vertex model proved in [31], specialized to the step initial conditions on the corner. We will then use these results to show the weak convergence of $\frac{\boldsymbol{X}_t}{t}$, which mirrors the arguments for ASEP from [42].

We will work with the single-class stochastic six-vertex model on the quadrant, i.e. as process $(\eta_t(x))_{x,t\in\mathbb{Z}_{\geq 0}}$ taking values in $\{0,1\}$. To state the hydrodynamic limit, we define the function φ as

$$\varphi(\rho) := \frac{\kappa\rho}{(\kappa-1)\rho+1}\,,$$

where we recall that $\kappa = \frac{1-b_1}{1-b_2}$. This function encodes the "slope relation" of the stochastic six-vertex model (see [172]). In particular, if the stochastic six-vertex model is run from i.i.d. Bernoulli(ρ) initial conditions on the bottom and i.i.d. Bernoulli$(\phi(\rho))$ from the left, the process is stationary under space-time shifts. Furthermore, the asymptotic speed of a single second-class particle added to such initial conditions will be given by $\varphi'(\rho)$.

We can now state the general hydrodynamic limit in [31, Theorem 1.1]. This is stated for the stochastic six-vertex model on the Torus. Let $\mathbb{T}^N$ be the discrete torus $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{T}$ the torus $\mathbb{R}/\mathbb{Z}$.

Theorem 3.9.1 (Theorem 1.1 from [31])**.** *Consider initial conditions $\eta_0^N(x):\mathbb{T}^N\to\{0,1\}$ which approximate a profile $\rho_0:\mathbb{T}\to[0,1]$, in the sense that*

$$\lim_{N\to\infty}\sup_{x,y}\left|\frac{1}{N}\sum_{i=\lfloor Nx\rfloor}^{\lfloor Ny\rfloor}\eta_0^N(i)-\int_x^y\rho_0(x)\right|=0\,.$$

Let $(\rho_t(x))_{t\geq 0,x\in\mathbb{R}}$ be the entropy solution of the partial differential equation

$$\frac{\partial}{\partial_t}\rho_t(x)+\frac{\partial}{\partial_x}\varphi(\rho_t(x))=0 \tag{3.138}$$

with initial condition given by ρ_0. Then uniformly on compact sets, we have the following conver-

gence in probability:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{t=0}^{NT} \sum_{x=0}^{NX} \eta_t^N(x) = \int_0^T \int_0^X \rho_t(x) dx dt \,.$$

This theorem gives the expected density of particles at large times and scales. It is supplemented by the following local statistics result, given in [31, Theorem 1.3], which states that whenever $\rho_t(x)$ is continuous at (x,t), the system will be approximately at equilibrium. In other words, the microscopic behavior around $\eta_{Nt}(Nx)$ will approach a stationary measure.

Theorem 3.9.2 (Theorem 1.3 from [31])**.** *In the setting of Theorem 3.9.1, consider a point (x,t) such that $\rho_t(x)$ is continuous at (x,t), and fix an integer $k \geq 1$. Then the law of*

$$[\eta^N_{\lfloor Nt \rfloor + s}(\lfloor Nx \rfloor + y)]_{y \in [\![-k,k]\!], s \in [\![0,k]\!]}$$

converges in law to the stationary process started from i.i.d. Bernoulli$(\rho_t(x))$ random variables and restricted to the rectangle $[\![-k,k]\!] \times [\![0,k]\!]$.

To prove the weak convergence we will now need the following consequence of these theorems.

Corollary 3.9.3. *Let $\eta_t : \mathbb{Z} \to \{0,1\}$ be the stochastic six-vertex process on the line started from $\eta_0(x) = \mathbf{1}_{x<0}$, and let α be a positive real number. Then*

$$\lim_{t\to\infty} \mathbb{P}[\eta_t(\alpha t) = 1] = \rho_1(\alpha)\,,$$

where

$$\rho_t(x) = \begin{cases} 1 & \text{if } \frac{x}{t} < \kappa^{-1} \\ \frac{\sqrt{\kappa t/x}-1}{\kappa - 1} & \text{if } \kappa^{-1} \leq \frac{x}{t} \leq \kappa \\ 0 & \text{if } \frac{x}{t} > \kappa \end{cases} \tag{3.139}$$

is the unique weak solution to (3.138) *from $\rho_0(x) = \mathbf{1}_{x<0}$.*

Proof. Since Theorems 3.9.1 and 3.9.2 are stated on the torus, we need to connect the process on the line with the process on a torus. Let B be some integer, which will later be chosen to be large. Let η_t be the stochastic six-vertex model on the line started from step initial conditions and let $\eta_t^N : \mathbb{T}^{2B^3N+1} \to \{0,1\}$ be the stochastic six-vertex process on $\mathbb{T}^{2B^3N+1}$ started from the initial conditions

$$\eta_t^{N,\mathbb{T}}(x) = \begin{cases} 1, \text{ for } -B^2N \leq x \leq 0 \\ 0 \text{ else,} \end{cases}$$

where we identify the torus $\mathbb{T}^{2B^3N+1}$ with the set $[\![-B^3N, B^3N]\!]$. Then by [31, Proposition 5.7] the processes η and η^N can be coupled to agree on the interval $[\![-BN, BN]\!]$ until time BN with probability $1 - \exp -BN$, if B is large enough such that $B^2N - \frac{4BN}{1-b_2} > BN$.

Assuming further that B is large than α, we obtain

$$|\mathbb{P}[\eta_N(\alpha N)] - \mathbb{P}[\eta_N^{N,\mathbb{T}}(\alpha N)]| \leq \exp -BN\,.$$

Therefore it remains to apply Theorem 3.9.2 to η^N. Note that the PDE (3.138) is invariant under scaling space and time by the same factor, and we can therefore also consider ρ_t defined on the torus of size $2B^3$, which simplifies the notation. The initial condition of η^N approximate initial conditions $\rho_0^{\mathbb{T}}(x) = \mathbf{1}_{x\in[-B^2,0]}$ on the torus of side-length $2B^3$ identified with $[-B^3, B^3]$. By Proposition 5.3 and Remark 5.4 of [31], the solution $\rho_t^{\mathbb{T}}$ of (3.138) for these initial conditions and ρ_t (the solution for step initial conditions on $\mathbb{R}$) agree at time t on $[-B^2 + ct, B^2 - ct]$, where $c = \max_{x\in[\kappa^{-1},\kappa]} |\phi'(x)|$. Given B large enough such that $B^2 - c > \alpha$, this implies that

$$\rho_1^{\mathbb{T}}(\alpha) = \rho_1(\alpha)\,.$$

Applying Theorem 3.9.2 with $k = 0, t = 1$ and $x = \alpha$ gives the desired result. □

Proposition 3.9.4 (Weak Convergence of the Speed of the Second-Class Particle)**.** *Let $\boldsymbol{X}_t$ be the position of the second-class particle under step initial conditions with a single second-class parti-*

cle at the origin, as in Theorem 3.1.1 and let $\rho_t(x)$ be given by (3.139)*. Then the asymptotic speed of the second-class particle $\frac{\boldsymbol{X}_t}{t}$ converges weakly to a random variable with density*

$$\frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}} \mathbf{1}_{\kappa^{-1} \le x \le \kappa}\,. \tag{3.140}$$

Proof. Consider step initial conditions (for the single-class model), i.e. $\eta_0(x) = \mathbf{1}_{x<0}$, and let $_0$ be the same initial conditions shifted by 1 to the right i.e. $\widetilde{\eta}_0(x) = \mathbf{1}_{x\le 0}$. Let $(\eta_t)_{t\ge 0}$ and $(\widetilde{\eta}_t)_{t\ge 0}$ be the two stochastic six-vertex processes started from these initial conditions, with the height functions made unique by the choice that $h_0(1;\eta) = h_0(1;\widetilde{\eta}) = 0$. In particular $h_0(0;\eta) = 0$, but $h_0(0;\widetilde{\eta}) = 1$. We will couple them in two different ways. The first coupling π_1 is given by the multi-class stochastic six-vertex model with step initial conditions and a single second-class particle at the origin, i.e. the setup of Theorem 3.1.1 and of this proposition. Denote the position of the second-class particle at time t with $\boldsymbol{X}_t$, as above. The second coupling π_2 is given by the deterministic shift i.e.

$$\mathbb{P}_{\pi_2}[\widetilde{\eta}_t(x) = \eta_t(x-1)] = 1.$$

We will now calculate $\mathbb{E}[h_t(x;\widetilde{\eta}) - h_t(x;\eta)]$ under both of these couplings. Under π_1 the configurations are identical except for the second-class particle and so the height functions agree for all (x,t) such that $x > \boldsymbol{X}_t$ and $h_t(x;\eta) = h_t(x;\widetilde{\eta}) - 1$ for (x,t) such that $x \le \boldsymbol{X}_t$. This gives

$$\mathbb{E}_{\pi_1}[h_t(x;\widetilde{\eta}) - h_t(x;\eta)] = \mathbb{P}[x \le \boldsymbol{X}_t]\,.$$

Under π_2 the height functions are related by deterministic shift $h_t(x;\widetilde{\eta}) = h_t(x-1;\eta)$ and therefore

$$\mathbb{E}_{\pi_2}[h_t(x;\widetilde{\eta}) - h_t(x;\eta)] = \mathbb{E}_{\pi_2}[h_t(x-1;\eta) - h_t(x;\eta)] = \mathbb{E}[\eta_t(x-1)]\,.$$

By the linearity of expectations, $\mathbb{E}[h_t(x;\widetilde{\eta}) - h_t(x;\eta)]$ does not depend on the coupling we take

and therefore we obtain the following identity:

$$\mathbb{P}[x \leq \boldsymbol{X}_t] = \mathbb{E}[\eta_t(x-1)]\,.$$

Choosing $x = \lfloor \alpha t \rfloor$ we obtain

$$\mathbb{P}\left[\frac{\boldsymbol{X}_t}{t} \geq \alpha\right] = \mathbb{E}[\eta_t(\lfloor \alpha t \rfloor - 1)].$$

By Corollary 3.9.3 the right hand side converges to $\rho_1(\alpha)$. Therefore $\frac{\boldsymbol{X}_t}{t}$ converges weakly to the random variable with density

$$-\rho_1'(x) = \frac{\sqrt{\kappa}}{2(\kappa-1)} x^{-\frac{3}{2}} \mathbf{1}_{\kappa^{-1} \leq x \leq \kappa}\,,$$

which proves the proposition. □

3.10 Uniqueness of stationary measures

The goal of this section is to prove Proposition 3.8.5. The proof builds on ideas in [39] and [157]. First, we define some sets of measures that we will be working with:

Definition 3.10.1.

(a) Denote by $\mathcal{P}_n$ the set of measures on $\{1, 2, \ldots, n-1, \infty\}^{\mathbb{Z}}$.

(b) Denote by $\mathfrak{M}_t$ the map from $\mathcal{P}_n$to $\mathcal{P}_n$ given by the dynamics of the multi-class stochastic six-vertex model, i.e. if the initial conditions η_0 are sampled according to μ, then $\mathfrak{M}_t$ is the law of η_t.

(c) Denote by $\mathcal{S}_n \subset \mathcal{P}_n$ the set of stationary measures with respect to the dynamics of the multi-class stochastic six-vertex model.

(d) Denote by $\mathcal{T}_n \subset \mathcal{P}_n$ the set of translation invariant measures with respect to shifts of $\mathbb{Z}$.

(e) Denote by $\mathcal{P}_n^{\otimes 2}$ the set of measures on $\{1,2,\dots,n-1,\infty\}^{\mathbb{Z}} \times \{1,2,\dots,n-1,\infty\}^{\mathbb{Z}}$, i.e. each $\pi \in \mathcal{P}_n^{\otimes 2}$ is a coupling of two measures in $\mathcal{P}_n$. Similarly, let $\mathcal{T}_n^{\otimes 2}$ be the set of translation invariant measures in $\mathcal{P}_n^{\otimes 2}$ and let $\mathcal{S}_n^{\otimes 2}$ be the set of measures which are stationary under the dynamics given by the basic coupling.

Definition 3.10.2. *For a set of measures $\mathcal{A}$, denote by $\mathcal{A}_e$ the set of extremal measures in $\mathcal{A}$. In otherwords, if $\mu \in \mathcal{A}_e$, then $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in \mathcal{A}$ implies that λ is either 0 or 1. Measures in $(\mathcal{T}_n)_e$ are called ergodic.*

Proposition 3.10.3. *For any n it holds that*

$$\mathcal{S}_n \cap (\mathcal{T}_n)_e = (\mathcal{S}_n \cap \mathcal{T}_n)_e \,,$$

i.e. the extremal stationary and translation-invariant measures are exactly the stationary ergodic measures.

Proof. Note first that

$$\mathcal{S}_n \cap (\mathcal{T}_n)_e \subset (\mathcal{S}_n \cap \mathcal{T}_n)_e \,,$$

since any measure that cannot be decomposed into translation-invariant measures cannot either be decomposed into translation-invariant stationary measures.

For the other inclusion, assume that $\mu \in (\mathcal{S}_n \cap \mathcal{T}_n)_e$ but $\mu \notin \mathcal{S}_n \cap (\mathcal{T}_n)_e$. This means it must be possible to decompose μ as

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2 \,,$$

with $\lambda \in (0,1)$ and $\mu_1, \mu_2 \in \mathcal{T}_n$ but $\mu_1, \mu_2 \notin \mathcal{S}_n$. Denote by $\mu_1(t)$ and $\mu_2(t)$ the measures obtained by running the dynamics of the stochastic six-vertex model for t steps starting from $\mu_1(0) = \mu_1$ and $\mu_2(0) = \mu_2$ respectively. By the construction of the dynamics, these are still translation invariant measures, and by linearity and stationarity of μ they also satisfy

$$\mu = \lambda\mu_1(t) + (1-\lambda)\mu_1(t).$$

By the compactness of $\{1, 2, \ldots, n-1, \infty\}^{\mathbb{Z}}$, there is a subsequence t_n such that $\frac{1}{t_n}\sum_{t=0}^{t_n}\mu_1(t)$ has a limit $\tilde{\mu}_1 \in \mathcal{S}_n \cap \mathcal{T}_n$ as $n \to \infty$. Taking a further subsequence there is a sequence t_n along which both $\frac{1}{t_n}\sum_{t=0}^{t_n}\mu_1(t)$ and $\frac{1}{t_n}\sum_{t=0}^{t_n}\mu_2(t)$ have limits $\tilde{\mu}_1$ and $\tilde{\mu}_2$ in $\mathcal{S}_n \cap \mathcal{T}_n$. These limits then satisfy

$$\mu = \lambda\tilde{\mu}_1 + (1-\lambda)\tilde{\mu}_1\,,$$

which contradicts the assumption that $\mu \in (\mathcal{S}_n \cap \mathcal{T}_n)_e$. □

We will use the following two results from [31], the second of which we need to modify slightly:

Theorem 3.10.4 (Theorem 3.6 from [31])**.** *If $\mu \in (\mathcal{T}_1 \cap \mathcal{S}_1)_e$, it is given by the law of i.i.d. Bernoulli(ρ) random variables for some $\rho \in [0,1]$.*

Proposition 3.10.5. *Let π be in $\mathcal{S}_1^{\otimes 2} \cap (\mathcal{T}_1^{\otimes 2})_e$ and (η, ξ) be sampled according to π. Then $\mathbb{P}[\eta \leq \xi$ or $\xi \leq \eta] = 1$.*

Proof. In [31, Corollary 3.9] this is stated for a different coupling, namely the "higher rank coupling". However, the proof generalizes to the basic coupling in a straightforward manner. Most of the proof only uses attractivity, the merging property, and finite speed of propagation, all of which hold for both the higher rank and the basic coupling (Lemmas 3.2.3, 3.2.5, and Proposition 3.2.8, respectively). The only point where specific properties of the higher rank coupling are used is at the end of the proof of [31, Proposition 3.7] to obtain the following fact.

Let

$$R(I;\eta,\xi) := \begin{cases} 1 & \text{if there are } x, y \in I \subset \mathbb{Z}, \text{ such that } \eta(x) = \xi(y) = 1 \text{ and } \eta(y) = \xi(x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words $R(I;\eta,\xi) = 1$ if there are uncoupled arrows in both η and ξ and is 0 otherwise. Given two coupled stochastic six-vertex processes η and ξ and an interval I of length k, let $F(I;t)$ be the event that there are uncoupled arrows in both η and ξ in the interval I at time $t-1$ which

couple at time t. Then

$$\mathbb{P}[F(I;t)|R(I;\eta_{t-1},\xi_{t-1})=1] \geq c^k\,, \tag{3.141}$$

for some $c = c(b_1, b_2) > 0$ independent of k, η and ξ.

The inequality (3.141) also holds for the basic coupling, and can be seen as follows: Let x and y be the positions of the two uncoupled arrows i.e. $\eta_t(x) = \xi_t(y) = 1$ and $\eta_t(y) = \xi_t(x) = 0$. Without loss of generality we assume that $x < y$ and $\eta_{t-1}(z) = \xi_{t-1}(z)$ for all $z \in [x+1, y-1]$. Using the Bernoulli random variables from Definition 3.2.1, the two arrows will couple if

- $\chi^2(t,x) = 0$,
- $\chi^1(t,x) = 0$,
- $\chi^2(t,z) = 1$ for $z \in [x+1, y-1]$ and
- $\chi^2(t,y) = 0$ and $\chi^1(t,y) = 1$ (alternatively $\chi^2(t,y) = 1$ and $\chi^1(t,y) = 0$).

Indeed the first two conditions guarantee that the arrow started at x will not couple with an arrow coming from the left, the second and third conditions guarantee that it will move to y, and the last condition guarantees that it will couple with the other uncoupled arrow at y. Since the χ are all independent, this implies that $\mathbb{P}[F(I;t)|R(I;\eta_{t-1},\xi_{t-1}) = 1]$ is at least $(1-b_2)^2 b_2^{k-2}(1-b_1)b_1$, which clearly can be bounded below by c^k for some c depending only on b_1 and b_2. □

To prove Proposition 3.8.5 we will consider two types of projections from $\mathcal{P}_{n+1}$ to $\mathcal{P}_n$:

Definition 3.10.6. *Let* $\phi : \{1,\dots,n,\infty\} \to \{1,\dots,n-1,\infty\}$ *and* $\psi : \{1,\dots,n,\infty\} \to \{1,\infty\}$ *be given by*

$$\phi(x) = \begin{cases} x, \text{ if } x < n \\ \infty, \text{ if } x = n, \infty \end{cases} \qquad \text{and} \quad \psi(x) = \begin{cases} 1, \text{ if } x \leq n \\ \infty, \text{ if } x = \infty \end{cases}.$$

For $\mu \in \mathcal{P}_{n+1}$ *and* η *distributed according to* μ, *let* $\Phi(\mu) \in \mathcal{P}_n$ *be the law of* $\phi \circ \eta$, *and* $\Psi(\mu)$ *be the law of* $\psi \circ \eta$. *Similarly define* $\Phi : \mathcal{P}_{n+1}^{\otimes 2} \to \mathcal{P}_n^{\otimes 2}$ *by setting* $\Phi(\pi)$ *to be the law of* $(\phi \circ \eta, \phi \circ \xi)$

where (η, ξ) is sampled according to π.

Both of these maps correspond to merging classes of particles. The map Φ merges class n with class ∞, i.e. it turns all particles of class n into holes. The map Ψ merges the classes 1 to n into a single class to obtain a single-class stochastic six-vertex model.

Lemma 3.10.7. *The maps $\Phi : \mathcal{P}_{n+1} \to \mathcal{P}_n$ and $\Psi : \mathcal{P}_{n+1} \to \mathcal{P}_1$ satisfy*

$$\Phi(\mathcal{S}_{n+1}) \subset \mathcal{S}_n \,,$$
$$\Phi(\mathcal{T}_{n+1}) \subset \mathcal{T}_n \,,$$
$$\Phi((\mathcal{T}_{n+1})_e) \subset (\mathcal{T}_n)_e \,,$$

and the same holds for Ψ, with the sets on the right-hand side replaced with their single-class counterparts.

Proof. The first statement is a consequence of the merging property, which implies that Φ commutes with the Markov kernel for the multi-class dynamics. The second one follows since Φ commutes with shifts. Finally note for the last one that any translation-invariant event of non-trivial probability for the measure $\Phi(\mu)$ also defines such an event for μ, since $\phi \circ \eta$ is a deterministic function of η. □

Proof of Proposition 3.8.5. The existence of such measures is shown in [158, Theorem 7.6], see in particular Section 7.3 therein, where the specialization to the stochastic six-vertex model is discussed, and [158, Equation (7.8)] where the densities of vertical and horizontal arrows are calculated. From this equation, one can see that all possible densities $(\lambda_k)_{k\in\{0,\dots,n\}}$ satisfying $\sum_{k=0}^{n} \lambda_k = 1$ can be obtained by correctly choosing the parameters α_i. Note that one could also prove the existence of such measures abstractly by a compactness argument. However, the above construction allows us to calculate marginals for these measures.

As remarked in [158, Remark 7.9], the uniqueness of such measures does not seem to be present in the literature. We will now prove this by induction on n. For $n = 1$ this is a consequence of [31,

Theorem 3.6] (restated above as Theorem 3.10.4), which states that the only stationary ergodic measures for the single-class stochastic six-vertex process are the i.i.d. Bernoulli(ρ) measures.

Let us assume that the statement has already been proven for some $n \geq 1$. Consider two measures $\mu, \nu \in (\mathcal{T}_{n+1} \cap \mathcal{S}_{n+1})_e$, which satisfy

$$\mathbb{P}_\mu[\eta(0) = k] = \mathbb{P}_\nu[\eta(0) = k] \text{ for } k = 1, \dots, n\,. \tag{3.142}$$

By Lemma 3.10.7 both $\Phi(\mu)$ and $\Phi(\nu)$ are in $\mathcal{S}_n \cap (\mathcal{T}_n)_e$. By (3.142) they have the same densities and hence by the induction hypothesis, they are equal.

We will now construct a coupling $\pi \in \mathcal{S}_{n+1}^{\otimes 2} \cap (\mathcal{T}_{n+1}^{\otimes 2})_e$ of the measures μ and ν such that

$$\text{if } (\eta, \xi) \text{ are sampled according to } \pi, \text{ then } \phi \circ \eta = \phi \circ \xi \text{ almost surely.} \tag{3.143}$$

To do so, first let π_0 be the coupling obtained by first sampling $\phi \circ \eta = \phi \circ \xi$ according to $\Phi(\mu) = \Phi(\nu)$ and then η according to μ conditioned on $\phi \circ \eta$ and ξ according to ν conditioned on $\phi \circ \xi$. Since μ and ν are translation invariant, so is π_0.

Next, let π_t be the law given by running the basic coupling started from initial conditions (η, ξ) sampled according to π_0. By the construction of the basic coupling and the stationarity of μ and ν, π_t will be translation invariant and satisfy the condition (3.143).

Since $\{1, 2, \dots, n, \infty\}^{\mathbb{Z}} \times \{1, 2, \dots, n, \infty\}^{\mathbb{Z}}$ is compact, there is a subsequence t_n such that $\frac{1}{t_n} \sum_{i=0}^{t_n} \pi_i$ converges weakly to a measure $\tilde{\pi} \in \mathcal{S}_{n+1}^{\otimes 2} \cap \mathcal{T}_{n+1}^{\otimes 2}$, which satisfies (3.143). The set of measures in $\mathcal{S}_{n+1}^{\otimes 2} \cap \mathcal{T}_{n+1}^{\otimes 2}$ that satisfy (3.143) is convex and by the above argument non-empty. Taking an extremal measure in this set we obtain the desired measure π.

To see that $\pi \in (\mathcal{T}_{n+1}^{\otimes 2})_e$ we first note that $\pi \in (\mathcal{S}_{n+1}^{\otimes 2} \cap T_{n+1}^{\otimes 2})_e$. If π were not in $(\mathcal{S}_{n+1}^{\otimes 2} \cap T_{n+1}^{\otimes 2})_e$, that would mean that there are $\mu_1 \neq \mu_2 \in \mathcal{S}_{n+1}^{\otimes 2} \cap T_{n+1}^{\otimes 2}$ and $\lambda \in (0, 1)$ such that $\pi = \lambda\mu_1 + (1-\lambda)\mu_2$. However since π is extremal in the set of measures in $\mathcal{S}_{n+1}^{\otimes 2} \cap T_{n+1}^{\otimes 2}$ satisfying condition (3.143), this means that either μ_1 or μ_2 does not satisfy (3.143). This cannot be, since for any event A the equality $\pi(A) = 1$ implies $\mu_1(A) = 1$ and $\mu_2(A) = 1$. Therefore $\pi \in (\mathcal{S}_{n+1}^{\otimes 2} \cap T_{n+1}^{\otimes 2})_e =$

$\mathcal{S}_{n+1}^{\otimes 2} \cap (T_{n+1}^{\otimes 2})_e$ and in particular it is in $(T_{n+1}^{\otimes 2})_e$.

Consider now $\Psi(\pi)$, which is a coupling of $\Psi(\mu)$ and $\Psi(\nu)$. Both $\Psi(\mu)$ and $\Psi(\nu)$ are in $\mathcal{S}_1 \cap (\mathcal{T}_1)_e$ and have the same particle density, therefore they are both given by the law of i.i.d. Bernoulli(ρ) random variables with the same ρ, by Theorem 3.10.4. Furthermore, $\Psi(\pi)$ satisfies the conditions of Proposition 3.10.5 and therefore if $(\tilde{\eta}, \tilde{\xi})$ is sampled according to $\Psi(\pi)$, either almost surely $\tilde{\eta} \leq \tilde{\xi}$ or $\tilde{\xi} \leq \tilde{\eta}$. This implies that under $\Psi(\pi)$ it holds that $\tilde{\eta} = \tilde{\xi}$ almost surely. This means that for (η, ξ) sampled according to π, $\psi \circ \eta = \psi \circ \xi$ almost surely. By the definition of π, we also have that $\phi \circ \eta = \phi \circ \xi$ almost surely. Note that any two configurations that are equal under both projections ψ and ϕ are themselves equal. Therefore, we conclude that almost surely $\eta = \xi$, and hence $\mu = \nu$. $\square$

Chapter 4: KPZ equation limit of sticky Brownian motion

This chapter is adapted from the paper [71] and is coauthored with Sayan Das and Shalin Parekh.

4.1 Introduction

4.1.1 Preface

Diffusion in time-dependent random environments has been a subject of intense investigation recently due to its connection with the KPZ universality class [73]. It is well known that the quenched density of the position of a particle in the diffusive regime (when its location $=$ $O(\sqrt{\text{time}})$) converges to the Gaussian distribution with a second-order correction given by the fixed point of the Edwards-Wilkinson (EW) universality class [79, Theorem 3.2], [80, Theorem 1.6], see also [173, 174, 78, 175, 176, 177, 178]. Meanwhile in the large deviation (LD) regime (when location $= O(\text{time})$), the quenched density is expected to admit a large deviation principle with linear speed, and the second-order correction is given by the Tracy-Widom (TW) distribution, the one-point marginal of the fixed point in the Kardar-Parisi-Zhang (KPZ) universality class (proven rigorously only for a few special models in [73, 82]). The goal of this paper is to show that for a large class of such diffusions in the moderate deviation (MD) regime (when location $= O(\text{time}^{3/4})$), the crossover distribution between these two classes, namely the *KPZ equation*, arises as a second-order correction.

The (1+1) dimensional KPZ equation is a stochastic partial differential equation (SPDE) given by

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \sigma^{1/2}\cdot\xi, \qquad \mathcal{H} = \mathcal{H}_t(x), \tag{KPZ}$$

where $\sigma > 0$ and $\xi = \xi(t, x)$ is a space-time white noise. The KPZ equation was first introduced in [2] as a prototypical model for interfaces of random growth. Since then, the model has been studied intensively in both the mathematics and physics literature. We refer to [179, 132, 133, 180, 181, 182] for some surveys of the mathematical studies of the KPZ equation.

As an SPDE, (KPZ) is ill-posed due to the presence of the non-linear term $(\partial_x \mathcal{H})^2$. One way to make sense of the equation is to consider $\mathcal{Z} := e^{\mathcal{H}}$ which formally solves the stochastic heat equation (SHE) with multiplicative noise:

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \sigma^{1/2} \cdot \mathcal{Z}\xi, \qquad \mathcal{Z} = \mathcal{Z}_t(x). \tag{SHE}$$

The SHE is known to be well-posed and has a well-developed solution theory based on the Itô integral and chaos expansions [11, 92, 132, 133]. In this paper, we will consider the solution of the (SHE) started with Dirac delta initial data $\mathcal{Z}_0(x) = \delta_0(x)$. For this initial data, [10] established that $\mathcal{Z}_t(x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ almost surely (see also [9]). Thus $\mathcal{H} = \log \mathcal{Z}$ is well-defined and is called the Cole-Hopf solution of the KPZ equation. This is the notion of solution that we will work with in this paper, and it coincides with other existing notions of solutions [3, 4, 5, 6, 7, 8], under suitable assumptions.

The work of [14, 183, 184, 185] demonstrates that the one-point distribution of the rescaled KPZ equation as $t \to \infty$ goes to the TW distribution, whereas as $t \to 0$ then under a different rescaling, the KPZ equation converges to the Gaussian distribution (the one-point distribution of the EW fixed point). Thus, the KPZ equation serves as a mechanism for crossing over between the EW and the KPZ universality classes. Going back to the diffusion models in time-dependent random environments, in [83] it was argued that these diffusion models are rich enough to admit the KPZ equation as limiting statistics. By physical arguments (explained briefly in Section 4.1.5), [83] derived that in the moderate deviation (MD) regime (when location $= O(\text{time}^{3/4})$), (KPZ) arises as a second-order correction for the quenched density (see Figure 4.1). Their heuristic arguments were later supported by [85] via rigorous moment-level computations for certain integrable discrete

and continuous diffusion models. More recently, using high-precision numerical simulations, [94] provided strong numerical evidence for this KPZ equation limiting behavior.

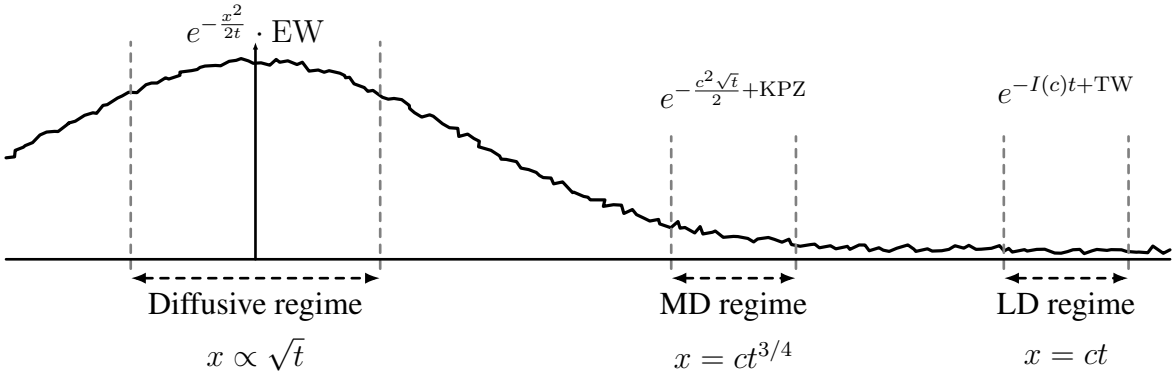

Figure 4.1: Schematic diagram of the quenched density $p(x)$ of the position of a particle at time t. The results in the diffusive regime need to be interpreted appropriately and may be found in [80]. The fluctuations in the Tracy-Widom regime are known only in some exactly solvable cases [82], but are conjectured to hold generally. We expect EW fluctuations whenever $x \ll t^{3/4}$ and TW fluctuations whenever $x \gg t^{3/4}$. The exponent $3/4$ is expected to be the unique exponent where the KPZ equation fluctuations appear.

In this paper, we work with diffusion in continuum random environments. We consider the stochastic flow of kernels whose k-point motions solve the Howitt-Warren martingale problem [186]. Such stochastic flows of kernels are called Howitt-Warren flows and give rise to diffusions in continuum random environments (see Section 4.1.2 for more details). We show that the logarithm of the quenched density of the motion of a particle under the Howitt-Warren flow, upon appropriate centering and scaling, converges weakly to the KPZ equation.

Our work is the first rigorous instance of weak convergence to the KPZ equation for diffusion in time-dependent random media under the moderate deviation regime. We mention that such weak convergence to the KPZ equation has been shown in the large deviation regime under weak random environment settings [90, 187]. Our proof techniques rely on a certain Girsanov transform related to sticky Brownian motions (see Section 4.1.5 for details). In particular, we do not rely on tools from integrable probability, and our results hold for all Howitt-Warren flows with finite and positive characteristic measures.

4.1.2 The model: sticky Brownian motion

In order to define random motions in a continuum random environment, we need to introduce the notion of *stochastic flow of kernels*. For $s \leq t$, a random probability kernel, denoted $K_{s,t}(x, A)$, is a measurable function defined on some underlying probability space Ω, such that it defines a probability measure on $\mathbb{R}$ for each $(x, \omega) \in \mathbb{R} \times \Omega$. $K_{s,t}(x, A)$ is interpreted as the random probability to arrive in A at time t starting at x at time s.

Definition 4.1.1. *A family of random probability kernels $(K_{s,t})_{s \leq t}$ on $\mathbb{R}$ is called a stochastic flow of kernels if*

(a) For any $s \leq t \leq u$ and $x \in \mathbb{R}$, almost surely $K_{s,s}(x, A) = \delta_x(A)$, and

$$\int_{\mathbb{R}} K_{t,u}(y, A) K_{s,t}(x, dy) = K_{s,u}(x, A)$$

for all A in the Borel σ-algebra of $\mathbb{R}$.

(b) For any $t_1 \leq t_2 \leq \cdots \leq t_k$, the $(K_{t_i,t_{i+1}})_{i=1}^{k-1}$ are independent.

(c) For any $s \leq u$ and $t \in \mathbb{R}$, $K_{s,u}$ and $K_{s+t,u+t}$ have the same finite dimensional distributions.

The general theory of stochastic flow was developed by Le Jan and Raimond in [188]; see also Tsirelson [189]. For the stochastic flow of kernels that we consider in this text, one can ensure that the random set of probability 1 on which (a) holds is independent of $x \in \mathbb{R}$ and $s \leq t \leq u$. This allows us to interpret $(K_{s,t})_{s \leq t}$ as bona fide transition kernels of a random motion in a continuum random environment. The averaged law of such a motion is called the 1-point motion associated to $(K_{s,t})_{s \leq t}$. More generally, the k-point motion of a stochastic flow of kernels is defined as the $\mathbb{R}^k$ valued stochastic process $\mathbf{X} = (X^1, \ldots, X^k)$ with transition probabilities given by

$$P_t(\vec{x}, d\vec{y}) = \mathbb{E}\left[\prod_{i=1}^{k} K_{0,t}(x_i, dy_i)\right].$$

We will be interested in a particular random motion in a continuum random environment originating from the *Howitt-Warren* flow of kernels. Its corresponding k-point motion solves a well-posed martingale problem that was first studied by Howitt and Warren in [186]. Below, we introduce the k-point motion by stating the martingale problem formulated in [190].

Definition 4.1.2. *Let ν be any finite measure on $[0,1]$. We say an $\mathbb{R}^k$-valued process $\mathbf{X}_t = (X_t^1, \ldots, X_t^k)$ solves the Howitt-Warren martingale problem with characteristic measure ν if $\mathbf{X}$ is a continuous, square-integrable martingale with the covariance process between X^i and X^j given by*

$$\langle X^i, X^j \rangle_t = \int_0^t \mathbb{1}_{\{X_s^i = X_s^j\}} ds, \tag{4.1}$$

and furthermore it satisfies the following condition:

Consider any nonempty $\Delta \subset \{1, 2, \ldots, k\}$. For $\mathbf{x} \in \mathbb{R}^k$, let

$$f_\Delta(\mathbf{x}) := \max\{x_i : i \in \Delta\}, \qquad \textit{and} \qquad g_\Delta(\mathbf{x}) := \big|\{i \in \Delta \mid x_i = f_\Delta(\mathbf{x})\}\big|.$$

Then the process $f_\Delta(\mathbf{X}_t) - \int_0^t \beta_+\big(g_\Delta(\mathbf{X}_s)\big) ds$ is a martingale with respect to the filtration generated by $\mathbf{X}$, where $\beta_+(1) := 0$ and

$$\beta_+(m) := 2 \int \sum_{k=0}^{m-2} (1-y)^k \nu(dy), \quad m \geq 2.$$

Note that from the covariance process formula above, we see that each X^i is marginally a Brownian motion. Focusing on the $k = 2$ case, one can check that the last condition in Definition 4.1.2 is equivalent to

$$|X_t^1 - X_t^2| - 4\nu([0,1]) \int_0^t \mathbb{1}_{\{X_s^1 = X_s^2\}} ds \tag{4.2}$$

being a martingale.

We define the local time at a of a continuous semimartingale X as

$$L_a^X(t) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon < X_s < a+\varepsilon\}} d\langle X, X \rangle_s. \tag{4.3}$$

Then, using Tanaka's formula we see that (4.2) implies that

$$L_0^{X^1 - X^2}(t) = 4\nu([0,1]) \int_0^t \mathbb{1}_{\{X_s^1 = X_s^2\}} ds.$$

Thus, from the above formula, we see that X^1 and X^2 can be interpreted as Brownian motions evolving independently of each other when apart, but when they meet there is some stickiness. Due to this stickiness, the two motions momentarily move together in the sense that they are equal on a nowhere-dense set of positive measure. The k-point motion defined above is a generalization of this stickiness phenomenon and is thus referred to as *sticky Brownian motion* (SBM) in the literature (Figure 4.2).

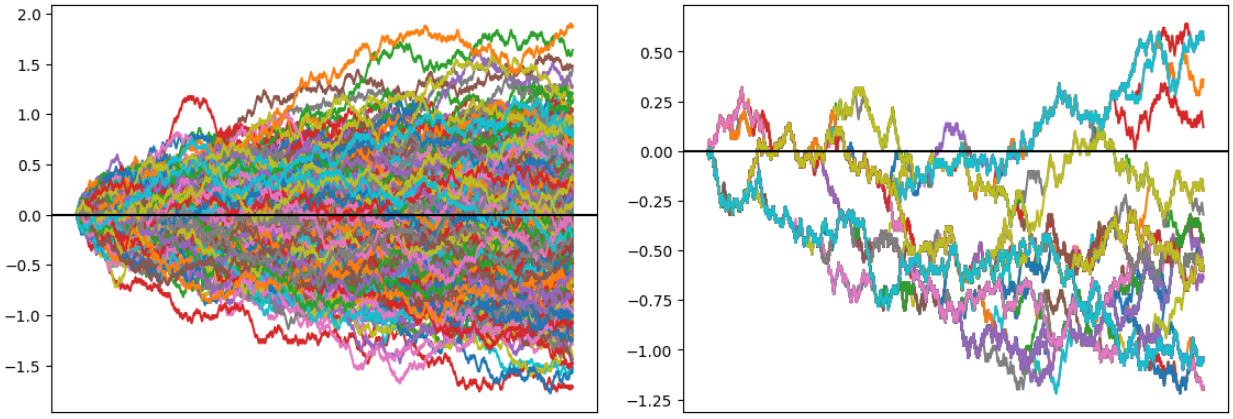

Figure 4.2: 500-point SBM with weak and strong stickiness simulated from discrete random walks in random environments.

The study of Brownian motions with stickiness goes back to the work of Feller [191], where he studied general boundary conditions for diffusions on the half line. Since then, sticky Brownian motion has been observed to arise as a diffusive scaling limit of various models: storage processes [192], discrete random walks in random environments [193, 194], and certain families of exclusion processes with a tunable interaction [195]. An SBM with a uniform characteristic measure inherits

integrability from the beta random walk in random environment model studied in [73]. This was exploited in [82, 85, 196] to extract various exact formulas and asymptotics. SBM models also bear connections to the Kraichnan model of turbulent flow [197]. Indeed, in the works [198, 199], the sticky behavior of particles was observed under certain fine-tuning in the Kraichnan model. We refer to the series of the physics works [200, 201, 202, 203, 204] and expository notes [205, 206] on the Kraichnan model for more details in this direction.

Howitt and Warren [186, Proposition 8.1] proved that the martingale problem in Definition 4.1.2 is well-posed, and its solutions form a consistent family of Feller processes. By a remarkable result of Le Jan and Raimond [188, Theorem 2.1], any consistent family of Feller processes can be viewed as a k-point motion of some stochastic flow of kernels, unique in finite-dimensional distributions. Thus, in particular, the solution of the Howitt-Warren martingale problem can be viewed as the k-point motion of some stochastic flow of kernels. This stochastic flow of kernels is called the Howitt-Warren flow. When referring to the k-point motions, we will continue to use SBM. We refer to [207, 190, 208] for more background on how SBM can be viewed as random motions in continuum random environments, and how to give a concrete construction of such a flow in a space of measure-valued flows using a coupling with the Brownian web and net.

4.1.3 Main results

Let $K_{0,t}(dx)$ denote the Howitt-Warren flow started from a Dirac mass at 0 whose characteristic measure ν is non-degenerate in the sense that $\nu([0,1]) > 0$. As mentioned before, $K_{0,t}(x)$ can be interpreted as the random probability of a particle hitting x at time t. The goal of this paper is to study the density of this quenched probability in the moderate deviation regime where we take $(t,x) \mapsto (Nt, tN^{3/4} + xN^{1/2})$. Formally speaking, we show that the log of the quenched density after appropriate centering:

$$\log K_{0,Nt}\big(tN^{3/4} + xN^{1/2}\big) + \tfrac{1}{2}\log N + \tfrac{t}{2}N^{1/2} + xN^{1/4}$$

converges in law to the solution $\mathcal{H}_t(x)$ of the KPZ equation defined in (KPZ).

It is actually the case that $K_{0,t}$ is *singular* with respect to the Lebesgue measure. In fact, by [190, Theorem 2.8] it is almost surely purely atomic at deterministic times, and may thus be viewed (formally) as a large system of interacting Brownian particles of different masses that dynamically split and recombine according to a time-homogeneous rule determined by the characteristic measure ν. Since K only exists as a measure and not as a function in general, some care must be taken in order to make sense of the convergence statement written above. To do this, we introduce the fields $\{\mathscr{X}^N\}_{N\geq 1}$ below.

For $t \geq 0$ and a bounded test function $\phi : \mathbb{R} \to \mathbb{R}$ we first define

$$\mathscr{X}_t^N(\phi) := \int_{\mathbb{R}} e^{-\frac{t}{2}N^{1/2}+uN^{-1/4}}\phi(N^{-1/2}(u-N^{3/4}t))K_{0,Nt}(du) \tag{4.4}$$

so that on a purely formal level, one has (via the substitution $x = N^{-1/2}(u - N^{3/4}t)$)

$$\mathscr{X}_t^N(\phi) = \int_{\mathbb{R}} \phi(x)N^{1/2}e^{\frac{t}{2}N^{1/2}+xN^{1/4}}K_{0,Nt}(tN^{3/4}+xN^{1/2})dx. \tag{4.5}$$

The above formally defines a *spatial* pairing of $\mathscr{X}_t^N$ with ϕ in $L^2(\mathbb{R})$, and we can also define *space-time* pairings of smooth compactly supported test functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$(\mathscr{X}^N, \varphi)_{L^2(\mathbb{R}^2)} := \int_0^\infty \mathscr{X}_t^N(\varphi(t,\cdot))dt, \tag{4.6}$$

though again we emphasize that $\mathscr{X}^N$ is not an element of $L^2(\mathbb{R}^2)$ and the subscript here is merely suggestive. Our first result shows that for a fixed $t > 0$ and a spatial test function ϕ, the moments of $\mathscr{X}_t^N(\phi)$ converge to the moments of the stochastic heat equation paired with ϕ.

Proposition 4.1.3 (Convergence of moments)**.** *Fix $t > 0$, and $k \in \mathbb{N}$. For all $\phi \in \mathcal{S}(\mathbb{R})$ (the*

Schwartz space on $\mathbb{R}$), one has the following limit:

$$\lim_{N\to\infty} \mathbb{E}[\mathscr{X}_t^N(\phi)^k] = \mathbf{E}_{B^{\otimes k}}\left[\prod_{j=1}^{k} \phi(B_t^j) e^{\frac{\sigma}{2}\sum_{i<j} L_0^{B^i - B^j}(t)}\right] = \mathbf{E}[\mathcal{Z}_t(\phi)^k]. \tag{4.7}$$

Here $\mathbb{E}$ denotes the annealed expectation over the environment ω and $\mathcal{Z}_t(\phi) := \int_{\mathbb{R}} \mathcal{Z}_t(x)\phi(x)dx$, where $(t,x) \mapsto \mathcal{Z}_t(x)$ solves (SHE) *with $\sigma = \frac{1}{2\nu([0,1])}$ under Dirac delta initial condition. The expectation of the middle term is with respect to k independent Brownian motions, and $L_0^Y(t)$ denotes the local time accrued by Y at zero by time t.*

Using different methods, a similar multipoint moment convergence result is established in [85] for the case where the characteristic measure ν is a uniform measure. However, in contrast to the field (4.4) that we use in this paper, [85] used a slight variant which we refer to as the "quenched tail field." We refer the reader to Section 4.1.3 where we define the quenched tail field and discuss our results related to it.

We will now describe our weak convergence result for the above field under the appropriate topology. Note that $\mathcal{Z}_t(x)$ is not uniquely characterized by its moments, since they grow too fast. Therefore, the convergence of moments alone will not be enough to establish weak convergence of $\mathscr{X}^N$. However, Proposition 4.1.3 will still be relevant and will help us to identify the limit points of $\mathscr{X}^N$ once we show tightness in an appropriate Banach space.

We next introduce these suitable topologies for $\mathscr{X}^N$. Fix any $T > 0$ and set $\Lambda_T := [0,T] \times \mathbb{R}$. We denote by $C_c^\infty(\Lambda_T)$ the set of functions $\Lambda_T \to \mathbb{R}$ that are restrictions to Λ_T of some function in $C_c^\infty(\mathbb{R}^2)$. For $\alpha < 0$ we let $r = -\lfloor \alpha \rfloor$ and define the weighted parabolic Hölder space $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ to be the closure of $C_c^\infty(\Lambda_T)$ with respect to the norm given by

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)} := \sup_{(t,x)\in\Lambda_T} \sup_{\lambda\in(0,1]} \sup_{\varphi\in B_r} \frac{(f, S^\lambda_{(t,x)}\varphi)_{L^2(\Lambda_T)}}{(1+x^2)^\tau \lambda^\alpha}$$

where the scaling operators are defined by $S^\lambda_{(t,x)}\varphi(s,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(x-y))$, and where B_r is the set of all smooth functions of C^r norm (see (4.103)) less than 1 with support contained

in the unit ball of $\mathbb{R}^2$. These spaces are separable and embed naturally into $\mathcal{S}'(\mathbb{R}^2)$ (the space of tempered distributions); see Sections 2 and 3 of [209].

Similarly, for $\alpha < 0, r = -\lfloor \alpha \rfloor$, we define the weighted elliptic Hölder space $C^{\alpha,\tau}(\mathbb{R})$ to be the closure of $C_c^\infty(\mathbb{R})$ with respect to the norm given by

$$\|f\|_{C^{\alpha,\tau}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \sup_{\lambda\in(0,1]} \sup_{\phi\in B_r} \frac{(f, S_x^\lambda \phi)_{L^2(\mathbb{R})}}{(1+x^2)^\tau \lambda^\alpha}$$

where the scaling operators are defined by $S_x^\lambda \phi(s,y) = \lambda^{-1}\phi(\lambda^{-1}(x-y))$, and where B_r is now the set of all smooth functions of C^r-norm (see (4.103)) less than 1 with support contained in the unit ball of $\mathbb{R}$. As before, these spaces are separable and embed naturally into $\mathcal{S}'(\mathbb{R})$; see Sections 2 and 3 of [209]. Finally, for a Banach space Ξ we define the function space $C([0,T],\Xi)$ containing all continuous paths $[0,T] \to \Xi$, equipped with a norm given by $\|v\| := \sup_{t\in[0,T]} \|v(t)\|_\Xi$. In particular, we will consider the spaces $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$.

Our main result, stated below, shows that the collection $\{\mathscr{X}^N\}_{N\geq 1}$ converges weakly to the stochastic heat equation when viewed as elements of certain weighted parabolic Hölder spaces or certain function spaces.

Theorem 4.1.4 (Weak Convergence)**.** *Fix any $T > 0$, and $\tau > 1$.*

(a) Fix any $\alpha < -1$. For each $N \geq 1$, $\mathscr{X}^N$ as defined by (4.4) *and* (4.6) *can be viewed as an element of $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$. Furthermore, the collection $\{\mathscr{X}^N\}_{N\geq 1}$ is tight with respect to the above topology of $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$. Additionally, any limit point as $N \to \infty$ is concentrated on $C((0,T], C(\mathbb{R}))$ and coincides with the law of* (SHE) *with $\sigma = \frac{1}{2\nu([0,1])}$, started from Dirac delta initial condition.*

(b) Fix any $\gamma < -2$. For each $N \geq 1$, $(\mathscr{X}_t^N)_{t\in[0,T]}$ as defined by (4.4) *can be viewed as an element of the space $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$. Furthermore, the collection $\{\mathscr{X}^N\}_{N\geq 1}$ is tight with respect to that topology. Any limit point as $N \to \infty$ is concentrated on $C((0,T], C(\mathbb{R}))$ and coincides with the law of* (SHE) *with $\sigma = \frac{1}{2\nu([0,1])}$, started from Dirac delta initial condition.*

Remark 4.1.5. *A few remarks related to the above theorem are now in order.*

(a) We explain how to interpret the above results in terms of test functions. For any $\varphi_1, \dots, \varphi_m \in C_c^\infty(\mathbb{R}^2)$*, either part of Theorem 4.1.4 implies the joint convergence of* $\big((\mathscr{X}^N, \varphi_j)_{L^2(\mathbb{R}^2)}\big)_{1\le j\le m}$ *as defined in* (4.6) *to* $\big(\int_{(0,\infty)\times\mathbb{R}} \mathcal{Z}_t(x)\varphi_j(t,x)dtdx\big)_{1\le j\le m}$.

(b) Theorem 4.1.4(b) implies convergence of $(\mathscr{X}_t^N(\phi))_{t\in[0,T]}$ *to* $\big(\int_{\mathbb{R}} \mathcal{Z}_t(x)\phi(x)dx\big)_{t\in[0,T]}$*, viewed as random variables in* $C[0,T]$*, for any* $\phi \in C_c^\infty(\mathbb{R})$*. In particular, one may evaluate the field at some finite collection of fixed times to obtain multi-time convergence.*

(c) There is a tradeoff between the two parts of the theorem. Theorem 4.1.4(a) is a statement about the convergence of the field $\mathscr{X}^N$ *when tested against smooth functions in both space and time, and it does not imply convergence in law of* $\mathscr{X}_t^N(\phi)$ *for fixed* $t > 0$*. However,* $\alpha < -1$ *is indeed the optimal Hölder exponent that one could hope to obtain for convergence of the fields* $\mathscr{X}^N$ *in the parabolic spaces (the heat kernel itself does not have better regularity). On the other hand, Theorem 4.1.4(b) implies convergence of the spatial field for fixed* $t > 0$*, but we believe* $\gamma < -2$ *is no longer the optimal Hölder exponent for the function space.*

(d) The weights $\tau > 1$ *are not optimal. It should be possible to get rid of the weights altogether, since* (SHE) *started from Dirac initial condition is known to have nice decay properties in both space and time, but some technical aspects of the paper are simplified by using weights.*

Theorem 4.1.4 is part of a series of efforts that have sought to show the weak KPZ universality conjecture, which postulates that a large class of weakly asymmetric models rescale to the KPZ equation (see the introduction of [210] for a brief background). For instance, convergence to the KPZ equation has been established in a variety of models: directed polymers in the intermediate disorder regime [86], exclusion processes [88, 14, 211, 126, 212, 213, 138, 214, 215], and a large class of stochastic growth models [210, 216, 217, 218]. In the context of diffusion in time-dependent random environments, [90] studied nearest neighbor random walk on $\mathbb{Z}$ in random environments. They showed that under the weak scaling of the environment, the rescaled

quenched transition probability evaluated in the large deviation regime converges to the solution of the stochastic heat equation. The work [82] considered the analogous setting for SBM with a uniform characteristic measure. They showed that by weakly scaling the environment, which in this case corresponds to rescaling the characteristic measure, the moments of the Howitt-Warren flow evaluated in the large deviation regime converge to the moments of the stochastic heat equation. In a similar spirit, [187] considered a continuous SPDE model that models the trajectory of a particle in a turbulent fluid. They showed that under a weak environment setting, the limiting fluctuations of the quenched law of the underlying process are given by the KPZ equation.

We emphasize that we do not deliberately introduce any weak asymmetry into our model, i.e., the environment is independent of N and there are no parameters of the model that are being tuned. Rather, the KPZ fluctuations suggest that the weak asymmetry is somehow introduced naturally as a consequence of the moderate deviation scaling. In fact, by the scaling property of SBM [190, Proposition 2.4], our result can be converted to a large deviation regime result under weak stickiness, which would confirm the moment-based predictions for the case of a uniform characteristic measure in [82].

Quenched tail field and connection to extreme value theory

In this subsection, we describe how KPZ equation convergence can be established for the quenched tail probability from our results on the density field stated in the previous section.

Definition 4.1.6. *We define the quenched tail field by*

$$F_N(t,x) := N^{1/4} e^{\frac{t}{2}\sqrt{N}+N^{1/4}x} K_{0,Nt}[N^{3/4}t + N^{1/2}x, \infty).$$

The prefactor $N^{1/4}$, as opposed to the $N^{1/2}$ observed in (4.5), essentially comes from integration by parts which absorbs an $N^{1/4}$ factor from the $N^{1/2}$ (see the proof of Proposition 4.7.2). We remark that although the F_N are function-valued, they are *discontinuous* functions due to the atomic nature of $K_{0,t}$ [190, Theorem 2.8]. Our next theorem states that the family $\{\log F_N\}_{N\geq 1}$ of

space-time processes converges to the KPZ equation, in the sense of finite-dimensional distributions of pointwise values (t, x).

Theorem 4.1.7. *For any finite collection of space-time points* $\{(t_i, x_i)\}_{i=1}^m \in ((0,\infty)\times\mathbb{R})^m$ *one has the joint convergence*

$$\big(\log F_N(t_i, x_i)\big)_{i=1}^m \xrightarrow{d} \big(\mathcal{H}_{t_i}(x_i)\big)_{i=1}^m,$$

where $\mathcal{H}_t(x)$ *solves* (KPZ) *with* $\sigma = \frac{1}{2\nu([0,1])}$.

The above theorem was conjectured in [85] where the authors established a multipoint moment convergence of the field $F_N(t, x)$ to that of the stochastic heat equation for the case where the characteristic measure ν is a uniform measure. Again, since the moments do not determine the distribution of the stochastic heat equation, the results in [85] do not yield Theorem 4.1.7 even for the uniform case.

To prove Theorem 4.1.7, we rely on Theorem 4.1.4(b) and an integration by parts argument to first obtain the joint convergence

$$\left(\int_{\mathbb{R}} \phi_i(x) F_N(t_i, x)dx\right)_{i=1}^m \xrightarrow{d} \left(\int_{\mathbb{R}} \phi_i(x)\mathcal{Z}_{t_i}(x)dx\right)_{i=1}^m. \tag{4.8}$$

for $\phi_1, \ldots, \phi_m \in C_c^\infty(\mathbb{R})$ and $t_1, \ldots, t_m > 0$. We then establish regularity bounds for the two-point spatial difference of the quenched tail field. This essentially follows from the work of [190], [80], and [82]. Given this regularity bound, the finite-dimensional convergence in (4.8) can be upgraded to finite-dimensional convergence of pointwise space-time values by an application of Fatou's lemma. The full details of the proof of Theorem 4.1.7 are presented in Section 4.7.

One could go even further and ask about the convergence of F_N in a stronger topology such as the Skorohod topology (recall the F_N are discontinuous), which implies the multipoint result. We do not pursue this in the present paper and leave this as a future work.

As a consequence of Theorem 4.1.7, we obtain the limiting distribution for the maximum particle of a k-point sticky Brownian motion $(X_t^1, \ldots, X_t^k)$ in the regime $t = O((\log k)^2)$.

Theorem 4.1.8. *Fix $c, t > 0$ and $d \in \mathbb{R}$. Let $(X_t^1, \dots, X_t^k)$ be a k-point sticky Brownian motion with characteristic measure ν. Set the number of particles $k = k(N) := \lfloor \exp(\frac{1}{2}cN^{1/2} + dN^{1/4} + r_N) \rfloor$ where r_N can be any sequence satisfying $r_N = o(N^{1/4})$. Then*

$$\max_{1 \le i \le k(N)} \left\{ N^{-\frac{1}{4}} X^i(Nt) \right\} - a_N(c, d, t) \xrightarrow{d} \sqrt{\tfrac{t}{c}} \big(G + \mathcal{H}_c(d) \big),$$

where

$$a_N(c, d, t) := \sqrt{ctN} - dN^{\frac{1}{4}} \sqrt{\tfrac{t}{c}} - \sqrt{\tfrac{c}{t}} \big(r_N - \tfrac{1}{4} \log N \big).$$

Here G is a Gumbel random variable (i.e., $P(G \le a) = e^{-e^{-a}}$) which is independent of $\mathcal{H}$, the solution to (KPZ) *with $\sigma = \frac{\sqrt{c}}{2\nu([0,1])\sqrt{t}}$.*

We remark that k-point sticky Brownian motion refers to the *annealed* law of the X^i: we are not making a pathwise statement about the maximum for each realization of the kernels $K_{s,t}$, which is consistent with the fact that Theorem 4.1.4 is a weak convergence statement, not an almost sure convergence statement. We also remark that rather than allowing σ to depend on t and c, one may instead take $\sigma = \frac{1}{2\nu([0,1])}$ but then $\mathcal{H}_c(d)$ must be changed accordingly to $\mathcal{H}_{\frac{t^2}{c}}\left(\frac{td}{c}\right) + \log\left(\frac{t}{c}\right)$.

Taking $d = r_N = 0$ and $t = 1$, we see that the above statement is a result of the maximum of $e^{\frac{1}{2}c\sqrt{N}}$ many sticky Brownian motion particles at time N. This is the same as understanding the maximum of N particles when time is of the order $(\log N)^2$. This leads to the question of what happens when (for a fixed characteristic measure ν) one looks at the maximum of N particles at timescales different from $(\log N)^2$. At timescales of order 1, we do not expect universality, as the answer may depend on the characteristic measure. If the characteristic measure satisfies $\int_{[0,1]} q^{-1} \nu(dq) < \infty$, then the support result of [190, Theorem 2.5a] implies that the maximum of N particles at time $t = 1$ converges in law (without any centering or scaling) to a Gaussian of mean $2 \int_{[0,1]} q^{-1} \nu(dq)$. If $\int_{[0,1]} q^{-1} \nu(dq) = \infty$, we have no conjecture what happens.

At timescales of order $\log N$, the maximum of N particles fluctuates like $(\log N)^{1/3}$ times a Tracy-Widom distribution. This is conjectured to be universal, but it is currently only provable

in certain exactly solvable cases (see [73, Corollary 5.8] or [82]). Finally, on timescales greater than $(\log N)^2$ we believe that the Gumbel term will dominate, since in this regime the motions closely resemble i.i.d. Brownian motions. We conjecture that $(\log N)^2$ is the unique timescale at which one sees a mix of Gumbel with the KPZ equation. It remains to explore what happens on timescales between 1 and $\log N$, or between $\log N$ and $(\log N)^2$.

The physics paper [94] contains numerical simulations which explore these regimes and seem to support some of our conjectures, although they consider the random walk in random environment which is a discrete analogue of our model. The physics paper [219] also contains interesting conjectures related to timescales slightly larger than $(\log N)^2$.

4.1.4 Issues with the chaos expansion technique

Before explaining the core ideas and novel techniques of the proof, it is important to highlight the constraints of traditional methods used in showing convergence to the (SHE). Among the existing methods, the polynomial chaos method is a widely used approach in establishing weak convergence to the (SHE). In this method, the prelimiting object is first identified as a sequence of multi-linear polynomials of independent random variables (called polynomial chaos expansions). Then each term of the chaos series is shown to converge in L^2 to that of the Wiener chaos series of (SHE). This idea was first implemented by Alberts, Khanin, and Quastel [86] for directed polymers. Later, [87] set up a general framework, formulating general conditions under which a polynomial chaos series converges in law to a Wiener chaos expansion. This framework has since been utilized extensively to show that (SHE) arises as a limit from several models of interest. In particular, Corwin and Gu [90] used the framework of [87] to obtain KPZ equation convergence for the random walk in a random environment (RWRE) model in the large deviation regime under a weak environment scaling. Although sticky Brownian motion bears a strong resemblance to the RWRE model and can be realized as its diffusive limit [208], there are two serious obstacles in carrying out the polynomial chaos approach in our setting.

1. Firstly, it is not clear how to set up the polynomial chaos for the quenched density in the

context of a continuum random environment. Indeed, as shown in [220] the noise generated by the Howitt-Warren flow is a *black noise* in the sense of Tsirelson [189] (see also [220]). Black noises arise as a scaling limit in various discrete models, such as systems of coalescing random walks [221, 222] and 2D critical percolation [223]. These non-classical noises are a much more subtle and less understood subject than white noise. A stochastic calculus for black noise is not known, and in particular, there is no notion of iterated stochastic integrals with respect to black noise.

2. Secondly, even for the discrete RWRE model, it is not straightforward to replicate the ideas of Corwin and Gu [90] to prove KPZ equation convergence under the moderate deviation regime. Although a polynomial chaos expansion for the quenched density is available in this regime, taking a naive limit of this discrete chaos expansion interestingly gives a noise coefficient in the limiting stochastic heat equation which is *strictly smaller* than the physics prediction from [85]. We refer to Section 1.1 of our sequel paper on discrete random walks in dynamical random environments [72] for more details on this. This suggests that this polynomial chaos does not satisfy the conditions in [87, Theorem 2.3] needed to apply their framework. In this particular scenario, a nonzero proportion of the L^2-mass of the polynomial chaos series escapes into the tails of the series in the $N \to \infty$ limit, suggesting that additional independent noise is generated in the limit.

As far as we know the latter phenomenon has not been observed previously. We study this phenomenon in our upcoming work [72] where we prove a similar KPZ equation universality result for the quenched transition kernel of the RWRE using a similar strategy. Like this paper and in contrast with [90], the environment law will be *fixed* under the scalings, and we will focus on the moderate deviation setting.

4.1.5 Proof idea

In this section, we describe the broad ideas of the proof of our main theorem: Theorem 4.1.4. We focus on the proof of Theorem 4.1.4(a). The proof of Theorem 4.1.4(b) will follow readily

from Theorem 4.1.4(a), together with a short embedding lemma about these Hölder spaces under the action of the heat kernel.

Girsanov's formula

The main technique in our analysis will be a Girsanov-type formula for sticky Brownian motion. For simplicity, we illustrate here the 1-point case. Using the definition of $\mathscr{X}_t^N(\phi)$ from (4.4) we may write

$$\mathscr{X}_t^N(\phi) = \mathbb{E}_\omega^{(1)}\left[e^{-\frac{1}{2}t\sqrt{N}+N^{-1/4}X_{Nt}} \cdot \phi\left(N^{-1/2}(X_{Nt} - N^{3/4}t)\right)\right], \tag{4.9}$$

where $\mathbb{E}_\omega^{(1)}$ denotes the quenched expectation with respect to a single motion X sampled from the environment $\omega = \{K_{s,t} : -\infty \le s \le t < \infty\}$. Note that in the annealed sense $K_{0,Nt}$ is simply the law of B_{Nt} for a standard Brownian motion B. Thus, taking the annealed expectation on both sides of the above equation, and by the tower property for conditional expectation, we obtain

$$\begin{aligned}\mathbb{E}[\mathscr{X}_t^N(\phi)] &= \mathbf{E}_{BM}\left[e^{-\frac{1}{2}t\sqrt{N}+N^{-1/4}X_{Nt}} \cdot \phi\left(N^{-1/2}(X_{Nt} - N^{3/4}t)\right)\right] \\ &= \mathbf{E}_{BM}\left[e^{-\frac{1}{2}t\sqrt{N}+N^{1/4}X_t} \cdot \phi\left(X_t - N^{1/4}t\right)\right].\end{aligned}$$

Here the expectations are taken with respect to a standard Brownian motion B. In the last line, we used the scale invariance of Brownian motion to say that $X \overset{d}{=} (N^{-1/2}X_{Nt})_{t\ge 0}$. Note that $Z_t := e^{-\frac{t}{2}\sqrt{N}+N^{1/4}X_t}$ is the stochastic exponential of $N^{1/4}X_t$. By Girsanov's theorem for Brownian motion, under the changed measure $Q(A) := \mathbb{E}_{BM}(Z_T \mathbb{1}_A)$, the process $(X_t - N^{1/4}t)_{t\in[0,T]}$ is again a Brownian motion. Thus, the last expression in the above equation is precisely equal to $\mathbf{E}_{BM}[\phi(B_t)]$ which no longer depends on N. This matches the first moment of $\int_{\mathbb{R}} \mathscr{Z}_t(x)\phi(x)dx$ where $(t,x) \mapsto \mathscr{Z}_t(x)$ is defined in Proposition 4.1.3.

In the case of higher moments, $\mathscr{X}_t^N(\phi)^k$ can be viewed as the quenched expectation with respect to k-point motion sample from the environment ω. Then taking the annealed expectation over the quenched expectation will lead to expressions in terms of a k-point sticky Brownian motion.

Then the key idea is to use a Girsanov-type formula for sticky Brownian motion (see Lemma 4.2.2) to get rid of divergent terms appearing in the annealed expectation expression. The resulting higher moment formulas appear in Lemma 4.3.1. Unlike the first-moment computation, the resulting expressions for higher moments still depend on N. However, the expressions are amenable to taking the large N limit. The expressions obtained in Lemma 4.3.1 are essentially annealed expectations with respect to a k-point sticky Brownian motion with characteristic measure $N^{1/2}\nu$. As $N \to \infty$, the stickiness disappears, and we are left with expectations with respect to a standard Brownian motion on $\mathbb{R}^k$. In Theorem 4.3.2 we compute these limits and show that they indeed match with the moments of (SHE) defined in Proposition 4.1.3.

Through our method, the $N^{3/4}$ term appearing in the scaling is seen to be the unique and natural choice of exponent that universally gives KPZ fluctuations. Indeed, following (4.9), one could potentially consider a model with more general exponents:

$$\mathscr{Z}_t^{N,c_N}(\phi) := \mathbb{E}_\omega^{(1)}\left[e^{-\frac{1}{2}c_N^2 Nt + c_N X_{Nt}} \cdot \phi\left(N^{-1/2}(X_{Nt} - c_N Nt)\right)\right],$$

and certain aspects of the paper would still go through. Indeed, following the arguments in the proof of Lemma 4.3.1, one can check that the second moment is given by

$$\mathbb{E}\big[\mathscr{Z}_t^{N,c_N}(\phi)^2\big] = \mathbf{E}\left[\phi(X_t)\phi(Y_t)\exp\left(c_N^2 N\int_0^t \mathbb{1}_{\{X_s=Y_s\}}ds\right)\right],$$

where (X, Y) is a certain tilted version of 2-point SBM with characteristic measure $N^{1/2}\nu$. The key observation here is that $c_N = N^{-1/4}$ is the unique choice for which local times appear in the limiting expressions of the intersection times (see Theorem 4.3.2). When $c_N \ll N^{-1/4}$, the contribution of the intersection times would vanish in the limit, whereas for $c_N \gg N^{-1/4}$ the expressions blow up.

Tightness

We now explain the main idea used in proving the tightness of the field $\mathscr{X}^N$. Roughly speaking, the original conjecture made in [85] interpreted the Howitt-Warren flows $K_{0,t}$ as a Kolmogorov forward equation associated to an SDE with drift coefficient formally given by space-time white noise. They then apply a shear transform of space-time given by $(t, x) \mapsto (Nt, N^{3/4}t + N^{1/2}x)$ and note that (at least formally) this transforms the Kolmogorov forward equation into an SPDE which is essentially (SHE) plus some term that should vanish in the limit. Their derivation is non-rigorous because such a Kolmogorov SPDE is ill-posed due to the roughness of the noise. The main idea in our proof is to use a rigorous variant of this idea.

More precisely, in Lemma 4.4.3 we will show that the fields $\mathscr{X}^N$ satisfy a forced heat equation of the form

$$(\partial_t - \tfrac{1}{2}\partial_x^2)\mathscr{X}^N = dM^N \tag{4.10}$$

in the sense of space-time Schwartz distributions, where M^N is a *martingale* forcing that is constructed in Section 4.4 below. We do not aim to explicitly describe M^N but simply work with it as though (4.10) is the definition. Despite this non-explicit description of M^N, we can nonetheless show that the quadratic variations of M^N admit the following nice decomposition:

$$\langle M^N(\phi)\rangle_t = Q_t^N(\phi^2) + \mathcal{E}_t^N(\phi), \tag{4.11}$$

where Q^N is the quadratic variation field introduced in (4.52) and $\mathcal{E}_t^N(\phi)$ is an error term defined in (4.53) that goes to zero in L^2 norm. Our tightness proof then proceeds in two steps:

- In the first step, we obtain various moment estimates for Q^N. This is done by the same method outlined in Section 4.1.5. Indeed, the Girsanov approach allows us to get precise expressions for other relevant observables related to the field $\mathscr{X}^N$, not just the moments. In particular, it gives us access to moment estimates for Q^N as well (Proposition 4.5.6).

- The next step is to use the moment estimates for Q^N to obtain tightness estimates on the fields $\mathscr{X}^N$. Indeed, since the fields M^N have a martingale structure, the Burkholder-Davis-Gundy inequality yields moment estimates for M^N from moment estimates for Q^N. By Schauder estimates for the heat equation, we may, in turn, translate moment estimates for M^N into tightness estimates for the fields $\mathscr{X}^N$ using (4.10).

From the above steps, we obtain tightness for the fields $\mathscr{X}^N$, Q^N, and M^N in an appropriate topology (see Propositions 4.6.12 and 4.6.13). This roundabout method turns out to be much more tractable than trying to obtain tightness for the fields $\mathscr{X}^N$ directly, see Proposition 4.5.6 below. This type of method is somewhat similar to that used in [88] where the authors proved KPZ fluctuations for WASEP.

Identification of the limit points

After tightness is obtained, it remains to identify the limit points. To do this, we will use the martingale characterization of the solution of the multiplicative noise stochastic heat equation. Specifically, consider a measure μ on $C([0,T], C(\mathbb{R}))$, and let $(X_t)_{t\in[0,T]}$ denote the canonical process on that space. The canonical filtration $\mathcal{F}_t$ on $C([0,T], C(\mathbb{R}))$ is the one generated by $\{X_s : s \leq t\}$. A result of [88, Proposition 4.11] inspired by the work of [224] says that if for all $\phi \in C_c^\infty(\mathbb{R})$ the processes

$$M_t(\phi) := (X_t, \phi)_{L^2(\mathbb{R})} - \frac{1}{2}\int_0^t (X_s, \phi'')_{L^2(\mathbb{R})} ds \tag{4.12}$$

are $(\mathcal{F}_t, \mu)$-martingales with quadratic variation given by

$$\langle M(\phi)\rangle_t = \sigma \int_0^t (X_t^2, \phi^2)_{L^2(\mathbb{R})} ds, \tag{4.13}$$

then (under reasonable assumptions on the spatial growth of X_t at infinity) the measure μ necessarily coincides with the law of (SHE) started from an initial condition that is distributed as X_0 under μ.

Let $(X^\infty, Q^\infty, M^\infty)$ be a limit point of $(\mathscr{X}^N, Q^N, M^N)$. Since in the prelimit the observables satisfy (4.10), from that equation it is not hard to deduce that (X^∞, M^∞) satisfies (4.12) with $(X, M) = (X^\infty, M^\infty)$. To show (4.13), we again rely on the Girsanov approach to extract moment formulas for certain observables in the prelimit. Using these formulas, loosely speaking, we shall show in Proposition 4.5.3 that as $N \to \infty$

$$Q_t^N(a) - \sigma \int_0^t \left(\mathscr{X}_s^N(a)\right)^2 ds \overset{L^2}{\to} 0$$

for each $a \in \mathbb{R} \setminus \{0\}$. The precise formulation of the above equation requires more care, as Q_t^N and $\mathscr{X}_t^N$ exist only as distributions. Assuming this, thanks to the decomposition in (4.11) and the fact that $\mathcal{E}_t^N(\phi)$ vanishes in the limit, we get (4.13) with $(X, M) = (X^\infty, M^\infty)$.

The proof strategy outlined above has the potential to generalize to the random walk in random environment (RWRE) setting. In an upcoming work, we plan to prove a similar KPZ equation universality result for the quenched transition kernel of the RWRE in the moderate deviation regime using the same strategy.

Remark 4.1.9 (Universality)**.** *From the proof outlined above, we see that only the 1-point and 2-point motions associated with the kernels $K_{s,t}$ are consequential in the limit. The 1-point motion is simply Brownian motion which is why $M_t(\phi)$ as defined above is a martingale, while the 2-point motions appear in the expressions for the quadratic variations of those martingales. Indeed the kernels $K_{s,t}$ and their "squares" $K_{s,t}^{Sq}$ (see* (4.45)*) which appear in the expressions for the martingale M^N and the quadratic martingale field Q^N are purely in terms of the quenched expectations of at most two-point motions of* SBM *(see for instance* (4.54)*). Since the 1-point and 2-point motions of* SBM *are completely determined in law by just the total mass $\nu([0,1])$ of the characteristic measure (this can be seen for the 2-point motion by looking at* (4.2)*), the limiting law in Theorem 4.1.4 only depends on* SBM *via $\nu([0,1])$. In other words, our result is universal in the sense that it yields the same limit for all characteristic measures ν with the same total mass.*

Organization

The rest of the article is organized as follows. In Section 4.2 we describe the Girsanov transform and collect estimates related to sticky Brownian motion. In Section 4.3 we prove the moment convergence (Proposition 4.1.3). In Section 4.4 we identify the martingale M^N and show that the field $\mathscr{X}^N$ satisfies a forced heat equation with forcing M^N. Section 4.5 is devoted to analyzing the quadratic variation of M^N. Finally, in Section 4.6 we prove Theorem 4.1.4 by utilizing the estimates from the previous sections to obtain tightness estimates and to identify the limit points for the fields $\mathscr{X}^N$. In Section 4.7, we prove results related to the quenched tail field discussed in Section 4.1.3. In Section 4.8 we prove a few technical estimates related to Brownian bridges which are used in proving our main theorems.

Notation and Conventions

Throughout this paper we use $\mathrm{C} = \mathrm{C}(a, b, c, \ldots) > 0$ to denote a generic deterministic positive finite constant depending on $a, b, c, \ldots$ that may change from line to line. We write $\mathcal{S}(\mathbb{R}^d)$ to denote the space of all Schwartz functions on $\mathbb{R}^d$ and use $\mathcal{S}'(\mathbb{R}^d)$ to denote it's dual: the space of all tempered distributions. We write $\mathbb{E}$ and $\mathbb{E}_\omega$ for annealed and quenched expectations in the context of random motions in random environments. We use $\mathbf{E}$ to denote expectation under path measures such as Brownian motion, sticky Brownian motion, etc.

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4.2 Girsanov transform and sticky Brownian motion estimates

In this section, we develop the basic framework of our proof. As mentioned in the introduction, our proof relies on a certain Girsanov-type transform for sticky Brownian motion. In Section 4.2.1 we describe this transform that we will use repeatedly in our later analysis. In Section 4.2.2, we collect several estimates related to sticky Brownian motion.

4.2.1 Girsanov transform

We begin with some necessary notation and definitions. Throughout this paper we assume ν is a fixed finite measure on $[0,1]$ with $\nu([0,1]) > 0$. Fix $T > 0$. For each $k \in \mathbb{N}$, we denote

- $\mathbf{P}_{B^{\otimes k}}$: the law on the canonical space $C([0,T],\mathbb{R}^k)$ of k independent Brownian motions.
- $\mathbf{P}_{SB_\nu^{(k)}}$: the law on the canonical space $C([0,T],\mathbb{R}^k)$ of the k-point motion of a sticky Brownian motion with characteristic measure ν.

Given $(X^1,\dots,X^k)$ distributed as $\mathbf{P}_{SB_\nu^{(k)}}$, using (4.1) we have

$$\left\langle \sum_{j=1}^k X^j \right\rangle(t) = kt + \sum_{i\neq j} \int_0^t \mathbb{1}_{\{X_s^i = X_s^j\}} ds \leq k^2 t. \tag{4.14}$$

Thus by Novikov's criterion for each $\lambda \in \mathbb{R}$

$$Z(t) := \exp\left[\lambda \sum_{j=1}^k X^j(s) - \frac{k\lambda^2 t}{2} - \lambda^2 \sum_{i<j} \int_0^t \mathbb{1}_{\{X_s^i = X_s^j\}} ds\right]. \tag{4.15}$$

is a martingale. With this information in hand, we now introduce two more measures in the following definition.

Definition 4.2.1. *We denote by* $\mathbf{P}_{T_\lambda SB_\nu^{(k)}}$ *the measure which is absolutely continuous with respect to* $\mathbf{P}_{SB_\nu^{(k)}}$ *with density* $\mathbf{P}_{T_\lambda SB_\nu^{(k)}} = Z(t) \cdot \mathbf{P}_{SB_\nu^{(k)}}$ *on* $\mathcal{F}_t$ *where* $(\mathcal{F}_t)_{t\in[0,T]}$ *is the filtration generated by* $\mathbf{X}$*. We define the measure* $\mathbf{P}_{D_\lambda SB_\nu^{(k)}}$ *to be the law on the canonical space* $C([0,T],\mathbb{R}^k)$ *of the process* $t \mapsto (X_t^i - \lambda t)_{i=1}^k$ *where* $(X^i)_{i=1}^k$ *is distributed as* $\mathbf{P}_{T_\lambda SB_\nu^{(k)}}$*. We shall often refer to them as the* $T_\lambda SB_\nu^{(k)}$ *and* $D_\lambda SB_\nu^{(k)}$ *measures.*

The following lemma shows that $\mathbf{P}_{D_\lambda SB_\nu^{(k)}}$ is also absolutely continuous with respect to $\mathbf{P}_{SB_\nu^{(k)}}$ on the interval $[0,T]$ with an explicit Radon-Nikodym derivative. To state the lemma, we introduce a few more pieces of notation. For $\mathbf{x} \in \mathbb{R}^k$, define $m_i(\mathbf{x})$ to be the cardinality of the set $\{j \leq k : x_i = x_j\}$. Define

$$G(t) := \sum_{j=1}^k \int_0^t \left[1 - \frac{1}{m_j(\mathbf{X}(s))}\right] dX^j(s), \quad \mathcal{M}(t) := \exp\left(\lambda G(t) - \frac{\lambda^2}{2}\langle G\rangle(t)\right), \tag{4.16}$$

where $\mathbf{X} = (X^1, \ldots, X^k)$ is distributed as $\mathbf{P}_{SB_\nu^{(k)}}$.

Lemma 4.2.2. *The measure* $\mathbf{P}_{D_\lambda SB_\nu^{(k)}}$ *defined in Definition 4.2.1 is absolutely continuous with respect to* $\mathbf{P}_{SB_\nu^{(k)}}$ *with density* $\mathbf{P}_{D_\lambda SB_\nu^{(k)}} = \mathcal{M}(t) \cdot \mathbf{P}_{SB_\nu^{(k)}}$ *on* $\mathcal{F}_t$ *where* $(\mathcal{F}_t)_{t\in[0,T]}$ *is the filtration generated by* $\mathbf{X}$*.*

Proof. Define the measure $\mathbf{P}_{Q_\lambda SB_\nu^{(k)}}$ to be the law on the canonical space $C([0,T],\mathbb{R}^k)$ of the process $t \mapsto (X_t^i - \lambda t)_{i=1}^k$ where $(X^i)_{i=1}^k$ is distributed as $\mathbf{P}_{SB_\nu^{(k)}}$. By [186, Lemma 5.4], we know that $\mathbf{P}_{Q_\lambda SB_\nu^{(k)}}$ is absolutely continuous with respect to $\mathbf{P}_{SB_\nu^{(k)}}$ on the interval $[0,T]$ with Radon-Nikodym derivative given by

$$\left.\frac{d\mathbf{P}_{Q_\lambda SB_\nu^{(k)}}}{d\mathbf{P}_{SB_\nu^{(k)}}}\right|_{\mathcal{F}_t}(\mathbf{X}) = \exp\left(-\lambda G_1(t) - \frac{\lambda^2}{2}\langle G_1\rangle(t)\right), \tag{4.17}$$

where

$$G_1(t) := \sum_{j=1}^{k} \int_0^t \frac{1}{m_i(\mathbf{X}(s))} dX^i(s).$$

Take any bounded $\mathcal{F}_t$-measurable functional $F : C([0,T], \mathbb{R}^k) \to \mathbb{R}$. We have

$$\begin{aligned}
\mathbf{E}_{D_\lambda SB_\nu^{(k)}}[F(\mathbf{X})] &= \mathbf{E}_{T_\lambda SB_\nu^{(k)}}\left[F\big((\mathbf{X}_u - \boldsymbol{\lambda} u)_{0\le u\le t}\big)\right] \\
&= \mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(\lambda \sum_{j=1}^{k} X_t^j - \frac{\lambda^2}{2}\left\langle \sum_{j=1}^{k} X^j \right\rangle(t)\right) F\big((\mathbf{X}_u - \boldsymbol{\lambda} u)_{0\le u\le t}\big)\right] \\
&= \mathbf{E}_{Q_\lambda SB_\nu^{(k)}}\left[\exp\left(\lambda \sum_{j=1}^{k} \big(X_t^j + \lambda t\big) - \frac{\lambda^2}{2}\left\langle \sum_{j=1}^{k} X^j \right\rangle(t)\right) F(\mathbf{X})\right] \\
&= \mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(\lambda \sum_{j=1}^{k} X_t^j - \lambda G_1(t) + \lambda^2 kt - \frac{\lambda^2}{2}\left\langle \sum_{j=1}^{k} X^j \right\rangle(t) - \frac{\lambda^2}{2}\langle G_1\rangle(t)\right) F(\mathbf{X})\right].
\end{aligned}$$

Here $\boldsymbol{\lambda} u$ denotes the vector $(\lambda u, \ldots, \lambda u) \in \mathbb{R}^k$ so that $(\mathbf{X}_u - \boldsymbol{\lambda} u) = (X_u^i - \lambda u)_{i=1}^k \in \mathbb{R}^k$. The first three equalities in the above equation follow directly from Definition 4.2.1 and the definition of $\mathbf{P}_{Q_\lambda SB_\nu^{(k)}}$. The last equality uses (4.17). Note that $G(t)$, defined in (4.16), equals to $\sum_{j=1}^k X^j(t) - G_1(t)$. Using (4.14) and the fact that $\langle G_1, X^j\rangle(t) = t$ for each j, we observe that the last expression in the above equation is precisely equal to $\mathbf{E}_{SB_\nu^{(k)}}[\mathcal{M}(t) \cdot F(\mathbf{X})]$ where $\mathcal{M}(t)$ is defined in (4.16). This completes the proof. □

Remark 4.2.3. *In view of Lemma 4.2.2, a bit of definition chasing shows that for all bounded $\mathcal{F}_t$-measurable F*

$$\mathbf{E}_{SB_\nu^{(k)}}\left[Z(t) F\big(\,(\mathbf{X}_u - \boldsymbol{\lambda} u)_{0\le u\le t}\,\big)\right] = \mathbf{E}_{SB_\nu^{(k)}}\left[\mathcal{M}(t) F\big((\mathbf{X}_u)_{0\le u\le t}\big)\right],$$

where Z and $\mathcal{M}$ are defined in (4.15) *and* (4.16) *respectively. Using the same argument, one also has*

$$\mathbf{E}_{SB_\nu^{(k)}}\left[\frac{Z(t)}{Z(s)} F\big(\,(\mathbf{X}_u - \boldsymbol{\lambda}(u-s))_{s\le u\le t}\,\big) \mid \mathcal{F}_s\right] = \mathbf{E}_{SB_\nu^{(k)}}\left[\frac{\mathcal{M}(t)}{\mathcal{M}(s)} F\big(\,(\mathbf{X}_u)_{s\le u\le t}\,\big) \mid \mathcal{F}_s\right]$$

almost surely. This implies that for all bounded $\mathcal{F}_t$-measurable H

$$\begin{aligned}&\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(\lambda\sum_{j=1}^{k}(X_t^j-X_s^j)-\tfrac{k}{2}\lambda^2(t-s)\right)H\big(\,(\mathbf{X}_u-\boldsymbol{\lambda}(u-s))_{s\le u\le t}\,\big)\mid\mathcal{F}_s\right]\\&=\mathbf{E}_{SB_\nu^{(k)}}\left[e^{\lambda(G(t)-G(s))-\frac{\lambda^2}{2}[\langle G\rangle(t)-\langle G\rangle(s)]}H\big(\,(\mathbf{X}_u)_{s\le u\le t}\,\big)\exp\left(\lambda^2\sum_{i<j}\int_s^t\mathbb{1}_{\{X_u^i=X_u^j\}}du\right)\mid\mathcal{F}_s\right].\end{aligned}\tag{4.18}$$

4.2.2 Sticky Brownian motion estimates

In this section, we gather various estimates related to sticky Brownian motion that are necessary for our later proofs. In our analysis, we will often encounter intersection times of a k-point sticky Brownian motion. For ease of notation, for each $k\in\mathbb{N}$ and $1\le i\ne j\le k$ we define the functional $V^{ij}:C([0,T],\mathbb{R}^k)\to C([0,T],[0,\infty))$ by

$$V_t^{ij}(\mathbf{X}):=\int_0^t\mathbb{1}_{\{X_s^i=X_s^j\}}ds.\tag{4.19}$$

We remark that the processes V^{ij} are Borel-measurable and adapted to the canonical filtration on $C([0,T],\mathbb{R}^k)$. The following two lemmas record moment estimates related to these functionals.

Lemma 4.2.4. *There exists an absolute constant $\mathrm{C}>0$ such that for all $k>1$, $\theta,t>0$ and for all characteristic measures ν we have*

$$\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(\theta\sum_{i=1}^{k}\left[\big(\sup_{s\le t}|X_s^i|\big)+4\nu([0,1])\sum_{j\ne i}V_t^{ij}(\mathbf{X})\right]\right)\right]\le\mathrm{C}\cdot\exp(\mathrm{C}\cdot k^4\theta^2t).$$

Proof. Applying Hölder's inequality we get

$$\begin{aligned}&\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(\theta\sum_{i=1}^{k}\big(\sup_{s\le t}|X_s^i|\big)+4\nu([0,1])\sum_{j\ne i}V_t^{ij}(\mathbf{X})]\right)\right]\\&\quad\le\prod_{i=1}^{k}\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(2k\theta\sup_{s\le t}|X_s^i|\right)\right]^{\frac{1}{2k}}\cdot\prod_{i=1}^{k}\mathbf{E}_{SB_\nu^{(k)}}\left[e^{8k\theta\nu([0,1])\sum_{j\ne i}V_t^{ij}(\mathbf{X})}\right]^{\frac{1}{2k}}.\end{aligned}\tag{4.20}$$

We now proceed to bound the above two product terms individually. Note that marginally each X^i is a Brownian motion under $\mathbf{P}_{SB_\nu^{(k)}}$. For a Brownian motion, one has exponential tail estimates of the form

$$\mathbf{P}_B\left[\sup_{s\leq t}|X_s|>a\right]\leq 2e^{-a^2/2t}. \tag{4.21}$$

We thus have

$$\prod_{i=1}^k \mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(2k\theta\sup_{s\leq t}|X_s^i|\right)\right]^{\frac{1}{2k}} = \mathbf{E}_B\left[\exp\left(2k\theta\sup_{s\leq t}|X_s|\right)\right]^{\frac{1}{2}} \leq \mathrm{C}e^{\mathrm{C}k^2\theta^2 t}, \tag{4.22}$$

for some absolute constant $\mathrm{C}>0$.

To deal with the second product in (4.20), another application of the Hölder inequality on each of the expectation terms in the product shows that for each $i=1,\ldots,k$ we have

$$\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(8k\theta\nu([0,1])\sum_{j\neq i}V_t^{ij}(\mathbf{X})\right)\right]\leq\prod_{j\neq i}\mathbf{E}_{SB_\nu^{(k)}}\left[\exp\left(8k(k-1)\theta\nu([0,1])\int_0^t \mathbb{1}_{\{X_s^i=X_s^j\}}ds\right)\right]^{\frac{1}{k-1}}.$$

Notice that for all $i\neq j$ the pair (X^i,X^j) is a 2-point sticky Brownian motion under $\mathbf{P}_{SB_\nu^{(k)}}$ and therefore the latter product is the same as

$$\mathbf{E}_{SB_\nu^{(2)}}\left[\exp\left(8k(k-1)\theta\nu([0,1])\int_0^t \mathbb{1}_{\{X_s^1=X_s^2\}}ds\right)\right].$$

Summarizing, we find that

$$\prod_{i=1}^k \mathbf{E}_{SB_\nu^{(k)}}\left[e^{8k\theta\nu([0,1])\sum_{j\neq i}V_t^{ij}(\mathbf{X})}\right]^{\frac{1}{2k}}\leq\mathbf{E}_{SB_\nu^{(2)}}\left[\exp\left(8k(k-1)\theta\nu([0,1])\int_0^t \mathbb{1}_{\{X_s^1=X_s^2\}}ds\right)\right]^{\frac{1}{2}}.$$

Note that for a 2-point sticky Brownian motion (X,Y), recall from (4.2) that the process $4\nu([0,1])\int_0^t \mathbb{1}_{\{X_s^1=Y_s\}}ds$ agrees with the local time $L_0^{X-Y}(t)$. By [225], $X-Y$ can be viewed

as a time changed Brownian motion. Indeed, given a Brownian motion B, if we define

$$T(t) = t + \frac{1}{4\nu([0,1])} L_0^B(t), \tag{4.23}$$

the process $\sqrt{2}B_{T^{-1}(\cdot)}$ has the same distribution as $X - Y$. Note that for $0 \le s \le t$ we have

$$\begin{aligned} T(T^{-1}(s) + t - s) &= T^{-1}(s) + t - s + \tfrac{1}{4\nu([0,1])} L_0^{X-Y}(T^{-1}(s) + t - s)) \\ &\ge T^{-1}(s) + t - s + \tfrac{1}{4\nu([0,1])} L_0^{X-Y}(T^{-1}(s))) \ge s + t - s = t. \end{aligned}$$

Writing $t = T(T^{-1}(t))$, we see that

$$T^{-1}(t) - T^{-1}(s) \le t - s. \tag{4.24}$$

This means that the time increments of the time-changed process are slower than that of the standard time. In particular, $T^{-1}(t) \le t$, hence $L_0^{X-Y}(t)$ is stochastically dominated by $L_0^{\sqrt{2}B}(t) = \sqrt{2}L_0^B(t)$. By Levy's identity for Brownian local time, we have $L_0^B(t) \stackrel{d}{=} \max_{s \le t} B(t)$ for each fixed $t > 0$. Using (4.21) we thus have

$$\begin{aligned} \prod_{i=1}^{k} \mathbf{E}_{SB_\nu^{(k)}} \left[e^{8k\theta\nu([0,1]) \sum_{j \ne i} V_t^{ij}(\mathbf{X})} \right]^{\frac{1}{2k}} &\le \mathbf{E}_{SB_\nu^{(2)}} \left[\exp\left(8k(k-1)\theta\nu([0,1]) \int_0^t \mathbb{1}_{\{X_s^1 = X_s^2\}} ds \right) \right]^{\frac{1}{2}} \\ &\le \mathrm{C} e^{\mathrm{C}k^2(k-1)^2\theta^2 t} \end{aligned}$$

for some absolute constant $\mathrm{C} > 0$. Inserting the above bound and the bound in (4.22) back in (4.20) we get the desired bound. This completes the proof. □

Lemma 4.2.5. *Fix $k \in \mathbb{N}$ and $p \ge 1$. There exists a constant $\mathrm{C} = \mathrm{C}(p,k) > 0$ such that for all $t > s \ge 0$*

$$\sup_{\nu \in \mathcal{M}([0,1])} \mathbf{E}_{SB_\nu^{(k)}} \left[\left(4\nu([0,1]) \sum_{i<j} (V_t^{ij}(\mathbf{X}) - V_s^{ij}(\mathbf{X})) \right)^p \right] \le \mathrm{C}|t-s|^{p/2},$$

where $\mathcal{M}([0,1])$ is the space of finite and non-negative Borel measures on $[0,1]$.

Proof. Use Minkowski's inequality and the fact that the 2-point motions of $\mathbf{P}_{SB_\nu^{(k)}}$ are distributed as $\mathbf{P}_{SB_\nu^{(2)}}$ to write

$$\begin{aligned}\mathbf{E}_{SB_\nu^{(k)}}\left[\left(4\nu([0,1])\sum_{i<j}(V_t^{ij}(\mathbf{X})-V_s^{ij}(\mathbf{X}))\right)^p\right]^{1/p} &\leq \sum_{i<j}\mathbf{E}_{SB_\nu^{(k)}}\left[\left(4\nu([0,1])(V_t^{ij}(\mathbf{X})-V_s^{ij}(\mathbf{X}))\right)^p\right]^{1/p}\\ &= \sum_{i<j}\mathbf{E}_{SB_\nu^{(2)}}\left[\left(4\nu([0,1])\int_s^t \mathbb{1}_{\{X_u=Y_u\}}du\right)^p\right]^{1/p}\\ &= \frac{k(k-1)}{2}\mathbf{E}_{SB_\nu^{(2)}}\left[\left(L_0^{X-Y}(t)-L_0^{X-Y}(s)\right)^p\right]^{1/p}.\end{aligned} \tag{4.25}$$

The last identity follows from the fact that for a 2-point sticky Brownian motion (X,Y) the process $4\nu([0,1])\int_0^t \mathbb{1}_{\{X_s^1=Y_s\}}ds$ agrees with the local time $L_0^{X-Y}(t)$. Recall that $X-Y \overset{d}{=} \sqrt{2}B_{T^{-1}(\cdot)}$ where B is a standard Brownian motion and T is defined in (4.23). It follows from (4.24) that $L_0^{X-Y}(t)-L_0^{X-Y}(s)$ is stochastically dominated by $\sqrt{2}(L_0^B(t)-L_0^B(s))$. By Levy's identity for Brownian local time, we have $L_0^B(t)-L_0^B(s) \overset{d}{=} M(t)-M(s)$ where $M(u) := \max_{v\leq u} B(v)$. Since $\mathbb{E}[|M(t)-M(s)|^p] \leq \mathbb{E}[\sup_{s\leq u\leq t}|B(u)-B(s)|^p] = \mathrm{C}|t-s|^{p/2}$ for some constant $\mathrm{C}>0$ depending on p, we have (4.25) $\leq \mathrm{C}|t-s|^{1/2}$ for some constant $\mathrm{C}>0$ depending on p,k. This completes the proof. □

We end this section by recording a uniform exponential moment estimate for a certain class of $\mathbf{P}_{SB_{N^{1/2}\nu}^{(k)}}$-martingales that will be very important.

Proposition 4.2.6. *Fix $k\in\mathbb{N}$. Let $\mathcal{G}_N=\mathcal{G}_N(\mathbf{X})$ be any family of continuous $\mathbf{P}_{SB_{N^{1/2}\nu}^{(k)}}$-martingales satisfying*

$$\langle \mathcal{G}_N\rangle(t) \leq \mathrm{C}N^{1/2}\sum_{i<j}V_t^{ij}(\mathbf{X}) \tag{4.26}$$

for some deterministic constant $\mathrm{C} > 0$ *independent of* N. *For all* $p, t > 0$ *we have*

$$\sup_{N\geq 1} \mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}} \left[e^{p\mathcal{G}_N(t) - \frac{p}{2}\langle \mathcal{G}_N\rangle(t)} \right] < \infty. \tag{4.27}$$

Remark 4.2.7. *Suppose* $\mathbf{X} = (X^1, \ldots, X^k)$ *is distributed as* $\mathbf{P}_{SB^{(k)}_{N^{1/2}\nu}}$. *Consider* G *defined in* (4.16). *Clearly* $\mathcal{G}_N := N^{1/4}G$ *satisfies* (4.26). *Let us write*

$$\mathcal{M}_N(t) := \exp\left(N^{1/4}G(t) - \tfrac{1}{2}N^{1/2}\langle G\rangle(t) \right). \tag{4.28}$$

Note that $\mathcal{M}_N$ *is the same as* $\mathcal{M}$ *defined in* (4.16) *with* $\lambda \mapsto N^{1/4}$ *and* $\nu \mapsto N^{1/2}\nu$. *The above proposition shows that the* L^p *norms of* $\mathcal{M}_N(t)$ *are uniformly bounded.*

Proof of Proposition 4.2.6. For convenience, we write $\mathbf{E}_N$ for $\mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}}$. Due to the hypothesis (4.26), for every $\lambda > 0$ we have

$$\sup_{N\geq 1} \mathbf{E}_N \left[e^{\lambda\langle \mathcal{G}_N\rangle(t)} \right] \leq \sup_{N\geq 1} \mathbf{E}_N \left[\exp\left(\lambda \mathrm{C} N^{1/2} \sum_{i<j} V_t^{ij}(\mathbf{X}) \right) \right] < \infty. \tag{4.29}$$

where the last inequality is due to Lemma 4.2.4 (applied with $\nu \mapsto N^{1/2}\nu$). Thus,

$$\mathbf{E}_N \left[e^{p\mathcal{G}_N(t) - \frac{p}{2}\langle \mathcal{G}_N\rangle(t)} \right] \leq \mathbf{E}_N \left[e^{p\mathcal{G}_N(t)} \right] \leq \mathbf{E}_N \left[e^{2p^2\langle \mathcal{G}_N\rangle(t)} \right]^{1/2}. \tag{4.30}$$

The last inequality will be explained shortly. Assuming it is true, taking supremum over $N \geq 1$ on both sides of (4.30), in view of (4.29), we get (4.27).

To prove the last inequality in (4.30), we note more generally that uniformly over all continuous real-valued martingales M defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $M_0 = 0$, one has the bound

$$\mathbb{E}[e^{M_t}] \leq \mathbb{E}[e^{2\langle M\rangle_t}]^{1/2}.$$

Indeed, if the right side is infinite then the statement is trivial. If the right-hand side is finite, then

by Novikov's condition $e^{2M_t-2\langle M\rangle_t}$ is a martingale. Consequently we have the bounds

$$\mathbb{E}[e^{M_t}] = \mathbb{E}[e^{M_t-\langle M\rangle_t}e^{\langle M\rangle_t}] \leq \mathbb{E}[e^{2M_t-2\langle M\rangle_t}]^{1/2}\mathbb{E}[e^{2\langle M\rangle_t}]^{1/2} = 1 \cdot \mathbb{E}[e^{2\langle M\rangle_t}]^{1/2}.$$

□

4.3 Proof of Proposition 4.1.3

The goal of this section is to prove Proposition 4.1.3, the moment convergence of the field $\mathscr{X}_t^N$ (defined in (4.4)) to that of the stochastic heat equation. The key idea is to observe that the moments of $\mathscr{X}_t^N$ can be written as expectations of certain functionals of a tilted version of a sticky Brownian motion (see Lemma 4.3.1). We then carefully analyze sticky Brownian motions and the associated tilted measures to obtain a weak convergence result for the tilted measure. This eventually leads to the moment convergence for the field $\mathscr{X}_t^N$.

4.3.1 Moment computations and moment convergence

We first identify moments of $\mathscr{X}_t^N(\phi)$ as expectations of certain functions under the $D_{N^{1/4}}SB^{(k)}_{N^{1/2}\nu}$ measure introduced in Definition 4.2.1.

Lemma 4.3.1. *For all bounded functions ϕ on $\mathbb{R}$, $t > s \geq 0$, and $k \in \mathbb{N}$ one has that*

$$\mathbb{E}[\mathscr{X}_t^N(\phi)^k] = \mathbf{E}_{D_{N^{1/4}}SB^{(k)}_{N^{1/2}\nu}}\left[\prod_{j=1}^{k}\phi(X_t^j)\exp\left(N^{1/2}\sum_{1\leq i<j\leq k}V_t^{ij}(\mathbf{X})\right)\right], \tag{4.31}$$

where the functionals V^{ij} are defined in (4.19).

Proof. The proof essentially follows by keeping track of all the definitions. Indeed, using the definition of $\mathscr{X}_t^N(\phi)$ from (4.4) we may write

$$\mathscr{X}_t^N(\phi) = \mathbb{E}_\omega^{(1)}\left[e^{-\frac{1}{2}t\sqrt{N}+N^{-1/4}X_{Nt}}\cdot\phi\left(N^{-1/2}(X_{Nt}-N^{3/4}t)\right)\right], \tag{4.32}$$

where $\mathbb{E}_\omega^{(1)}$ denotes the quenched expectation with respect to a single motion X sampled from the environment $\omega = \{K_{s,t} : -\infty \leq s \leq t < \infty\}$. Taking the k^{th} power we find that

$$\mathscr{X}_t^N(\phi)^k = \mathbb{E}_\omega^{(k)} \left[e^{-\frac{k}{2}t\sqrt{N} + N^{-1/4}(X_{Nt}^1 + \cdots + X_{Nt}^k)} \prod_{j=1}^k \phi\big(N^{-1/2}(X_{Nt}^j - N^{3/4}t)\big) \right],$$

where $\mathbb{E}_\omega^{(k)}$ denotes the quenched expectation with respect to a k-point motion $(X^1, \ldots, X^k)$ sampled from the environment $\omega = \{K_{s,t} : -\infty \leq s \leq t < \infty\}$. Taking an annealed expectation over the above expression and then using the fact that the averaged law of $(X^1, \ldots, X^k)$ is a k-point sticky Brownian motion with characteristic measure ν we obtain that

$$\mathbb{E}\big[\mathscr{X}_t^N(\phi)^k\big] = \mathbf{E}_{SB_\nu^{(k)}} \left[e^{-\frac{k}{2}t\sqrt{N} + N^{-1/4}(X_{Nt}^1 + \cdots + X_{Nt}^k)} \prod_{j=1}^k \phi\big(N^{-1/2}(X_{Nt}^j - N^{3/4}t)\big) \right].$$

A standard fact about sticky Brownian motion is that $N^{-1/2}(X_{Ns}^1, \ldots, X_{Ns}^k)$ is distributed as $SB_{N^{1/2}\nu}^{(k)}$ whenever $(X_s^1, \ldots, X_s^k)$ is distributed as $SB_\nu^{(k)}$ (see [190, Proposition 2.4]). Therefore the expectation in the last math display can be rewritten as

$$\mathbf{E}_{SB_{N^{1/2}\nu}^{(k)}} \left[e^{-\frac{k}{2}t\sqrt{N} + N^{1/4}(X_t^1 + \cdots + X_t^k)} \prod_{j=1}^k \phi(X_t^j - N^{1/4}t) \right].$$

Using the definition of $D_{N^{1/4}} SB_{N^{1/2}\nu}^{(k)}$ measure from Definition 4.2.1 and the definition of V^{ij} from (4.19), it is not hard to check that the above expression is precisely equal to the right-hand side of (4.31). This establishes the lemma. □

Thus, in view of the above lemma, it suffices to study weak convergence of $D_{N^{1/4}} SB_{N^{1/2}\nu}^{(k)}$ measures to extract moment convergence for $\mathscr{X}_t^N(\phi)$. Our next theorem is the main technical result of this section. It shows that if we consider the path $\mathbf{X}$ to be distributed according to the $D_{N^{1/4}} SB_{N^{1/2}\nu}^{(k)}$ measure, as $N \to \infty$,

- $\mathbf{X}$ converges to k independent Brownian motions.

- On the $N^{1/2}$ scale, pairwise intersection times converge to pairwise local times of the corresponding Brownian motions.

Throughout the theorem and its proof, we will diverge from our usual notational conventions by describing the random variables instead of referring to the path measures on the canonical space explicitly. In particular, we will write the associated expectations generically as $\mathbf{E}$, rather than $\mathbf{E}_{SB_\nu^{(k)}}$ and $\mathbf{E}_{B^{\otimes k}}$. This will avoid heavy notation, and hopefully does not cause confusion.

Theorem 4.3.2. *Fix any $k \in \mathbb{N}$ and constants $A, b, C, T > 0$. Fix any continuous function $f : C([0,T], \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2}) \to \mathbb{R}$ such that $|f(p)| \leq Ae^{b\|p\|_\infty}$ for all paths $p : [0,T] \to \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2}$. Suppose $\mathbf{X}^N$ is distributed according to the $SB^{(k)}_{N^{1/2}\nu}$ measure from Definition 4.2.1. Set $V^{ij} = V^{ij}(\mathbf{X}^N)$ as defined in* (4.19)*. The following holds.*

(a) We have the following convergence

$$\lim_{N\to\infty} \mathbf{E}\big[f\big(\mathbf{X}^N\ ,\ 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\leq i<j\leq k}\big)\big] = \mathbf{E}\big[f\big(\mathbf{U}\ ,\ \big(L_0^{ij}\big)_{1\leq i<j\leq k}\big)\big],$$

where $\mathbf{U}$ is a standard k-dimensional Brownian motion on $[0,T]$, and $L_0^{ij}(t) := L_0^{U^i - U^j}(t)$ is the local time accrued by $U_i - U_j$ at zero by time t.

(b) Let $\mathcal{G}_N = \mathcal{G}_N(\mathbf{X}^N)$ be any family of continuous $\mathbf{P}_{SB^{(k)}_{N^{1/2}\nu}}$-martingales with $\mathcal{G}_N(0) = 0$ satisfying

$$\int_s^t d\langle \mathcal{G}_N\rangle(u) \leq CN^{1/2} \sum_{i<j} \big(V_t^{ij}(\mathbf{X}^N) - V_s^{ij}(\mathbf{X}^N)\big), \tag{4.33}$$

for all $0 \leq s < t \leq T$. Then $\{\mathcal{G}_N\}_N$ is a tight family of random variables in $C[0,T]$. Furthermore, any limit point of the triple $\big(\mathcal{G}_N,\ \mathbf{X}^N,\ 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\leq i<j\leq k}\big)$ is of the form $(\mathcal{G}, \mathbf{U}, \big(L_0^{ij}\big)_{1\leq i<j\leq k})$, where $\mathbf{U}, L_0^{ij}$ are as in part (a), and $\mathcal{G}$ is a $C[0,T]$-valued random variable satisfying $\mathbf{E}[e^{p\mathcal{G}(T) - \frac{p}{2}\langle\mathcal{G}\rangle(T)}] < \infty$ for all $p > 0$ as well as

$$\mathbf{E}[\exp(\mathcal{G}(T) - \tfrac{1}{2}\langle\mathcal{G}\rangle(T)) \mid \mathcal{F}_T(\mathbf{U})] = 1, \tag{4.34}$$

where $\mathcal{F}_T(\mathbf{U})$ denotes the σ-algebra generated by $\mathbf{U}$. In particular, we have

$$\lim_{N\to\infty} \mathbf{E}\left[e^{\mathcal{G}_N(T)-\frac{1}{2}\langle\mathcal{G}_N\rangle(T)} f\left(\mathbf{X}^N\ ,\ 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\le i<j\le k}\right)\right] = \mathbf{E}\big[f\big(\mathbf{U},\big(L_0^{ij}\big)_{1\le i<j\le k}\big)\big]. \tag{4.35}$$

(c) Suppose $k = 2m$ is even. Let us write

$$\Delta_m(s,t) := \{(s_1,s_2,\dots,s_m)\in[s,t]^m \mid s\le s_1\le s_2\le\cdots\le s_m\le t\} \tag{4.36}$$

for the set of all m ordered points in $[s,t]$. We define $\mathcal{M}(\Delta_m(0,T))$ to be the space of finite and non-negative Borel measures on that simplex, equipped with the topology of weak convergence. Consider the following sequence of $\mathcal{M}(\Delta_m(0,T))$-valued random variables

$$\gamma_N(du_1,\dots,du_m) := N^{m/2}\prod_{j=1}^{m} \mathbb{1}_{\{X_N^{2j-1}(u_j)=X_N^{2j}(u_j)\}}\, du_1\dots\, du_m, \tag{4.37}$$

where $\mathbf{X}^N = (X_N^1,...,X_N^k)$. The random variables $\{\gamma_N\}_{N\ge1}$ are tight in $\mathcal{M}(\Delta_m(0,T))$. Moreover, any limit point of the triple $(\mathbf{X}^N,\ 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\le i<j\le k},\ \gamma_N)$ is of the form

$$\left(\mathbf{U},\ \big(L_0^{ij}\big)_{1\le i<j\le k},\ \Big(\frac{\sigma}{2}\Big)^m\prod_{j=1}^{m} dL_0^{U^{2j-1}-U^{2j}}(u_j)\right)$$

where $\mathbf{U}, L_0^{ij}$ are as in part (a), where $\sigma := \frac{1}{2\nu([0,1])}$, and $dL(t)$ denotes the Lebesgue-Stiltjes measure induced by the increasing function $t\mapsto L(t)$.

By Remark 4.2.7, taking $\mathcal{G}_N := N^{1/4}G$ with G defined as in (4.16), we see that equation (4.35) in part (b) of the above theorem deals with the weak convergence of $D_{N^{1/4}}SB^{(k)}_{N^{1/2}\nu}$ measures. Let us first show how it implies Proposition 4.1.3.

Proof of Proposition 4.1.3. Fix any $\phi \in \mathcal{S}(\mathbb{R})$. Given $\mathbf{X}$ distributed as $\mathbf{P}_{SB^{(k)}_{N^{1/2}\nu}}$, the martingales

$$\mathcal{G}_N(t) = N^{1/4} \sum_{j=1}^{k} \int_0^t \left[1 - \frac{1}{m_j(\mathbf{X}(s))}\right] dX^j(s)$$

clearly satisfy (4.33). Thus, Theorem 4.3.2 part (b) holds for this choice of $\mathcal{G}_N$. In view of Lemma 4.2.2, equation (4.35) yields that

$$\lim_{N\to\infty} \mathbf{E}_{D_{N^{1/4}} SB^{(k)}_{N^{1/2}\nu}} \left[f\left(\mathbf{X} \, , \; 4N^{1/2}\nu([0,1]) \big(V^{ij}\big)_{1\le i<j\le k} \right) \right] = \mathbf{E}_{B^{\otimes k}} \left[f\left(\mathbf{U} \, , \; \big(L_0^{ij}\big)_{1\le i<j\le k} \right) \right],$$

for all continuous functions $f : C([0,T], \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2}) \to \mathbb{R}$ with $|f(p)| \le Ae^{b\|p\|_\infty}$ for some $A, b > 0$. Specializing f to a particular choice given by a product of ϕ's multiplied by the exponential of the intersection times, we get

$$\lim_{N\to\infty} \mathbf{E}_{D_{N^{1/4}} SB^{(k)}_{N^{1/2}\nu}} \left[\prod_{j=1}^{k} \phi(X_t^j) e^{N^{1/2} \sum_{i<j} V_t^{ij}(\mathbf{X})} \right] = \mathbf{E}_{B^{\otimes k}} \left[\prod_{j=1}^{k} \phi(B_t^j) e^{\frac{\sigma}{2} \sum_{i<j} L_0^{B^i - B^j}(t)} \right].$$

where $\sigma = \frac{1}{2\nu([0,1])}$. Thanks to the moment formula from Lemma 4.3.1, we thus have the first equality in (4.7) from the above equation.

By the Feynman-Kac formula, the stochastic heat equation (SHE) admits well-known moment formulas in terms of local times of Brownian bridges (see [92] for example). In particular, from [92] we have

$$\mathbf{E}\left[\prod_{i=1}^{k} \mathcal{Z}_t(x_i) \right] = p_t(x_1) \cdots p_t(x_k) \mathbf{E}_{\vec{x}} \left[e^{\frac{\sigma}{2} \sum_{i<j} L_0^{B^i - B^j}(t)} \right], \tag{4.38}$$

where p is the standard heat kernel, and the B^i are independent Brownian bridges on $[0,t]$ from 0 to x_i respectively. Note that in [92], local time is defined as an integral against Lebesgue measure: $L_a^X(t) := \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon<X_s<a+\varepsilon\}} ds$, in contrast with our definition of local time (4.3) which is as an integral against $d\langle X, X\rangle_s$. Hence, when taking the local time of $B^i - B^j$, our definition of local time differs from theirs by a factor of 2. This explains why we have $\frac{\sigma}{2}$ in the exponent on the

right-hand side of (4.38) instead of just σ, while this extra factor of $1/2$ does not appear in [92].

From here one arrives at the same moment formula for $\mathcal{Z}_t(\phi) := \int_{\mathbb{R}} \mathcal{Z}_t(x)\phi(x)dx$, yielding the second equality in (4.7). This completes the proof. □

4.3.2 Proof of Theorem 4.3.2

Throughout the proof, we will write $X^{N,i}$ and U^i for the i^{th} coordinate of the $\mathbb{R}^k$-valued processes $\mathbf{X}^N$ and $\mathbf{U}$ respectively.

Proof of (a). We prove part (a) for any *bounded* continuous function f. In view of Lemma 4.2.4, a uniform integrability argument then extends the result to all continuous functions f with exponential growth. Recall that the stickiness of the sticky Brownian motion has an inverse relationship to the characteristic measure, so intuitively, as we take $N \to \infty$, the Brownian motions will no longer stick together, resulting in k independent Brownian motions in the limit.

We now flesh out the technical details of the above-claimed Brownian motion convergence. We assume for convenience that all $(\mathbf{X}^N)_{N\geq 1}$ are coupled onto the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For each N, we partition $[0,T]$ into the two random sets A_N, B_N as

$$\begin{aligned} A_N &:= \left\{ s \in [0,T] \mid \#\{X_s^{N,1}, X_s^{N,2}, \ldots, X_s^{N,k}\} = k \right\}, \\ B_N &:= \left\{ s \in [0,T] \mid \#\{X_s^{N,1}, X_s^{N,2}, \ldots, X_s^{N,k}\} \leq k-1 \right\}. \end{aligned} \tag{4.39}$$

Only for this proof, we will use $|S|$ to denote the Lebesgue measure of a set S. Clearly by the definition of V_T^{ij} (see (4.19)), we have $|B_N| \leq \sum_{i<j} V_T^{ij}(\mathbf{X}^N)$. Using $a \leq e^a$ for $a > 0$, we see that

$$\mathbf{E}\left[|B_N|\right] \leq N^{-1/2} \cdot \mathbf{E}\left[\exp\left(N^{1/2} \sum_{i<j} V_T^{ij}(\mathbf{X}^N) \right) \right] \leq \mathrm{C} \cdot N^{-1/2}, \tag{4.40}$$

where $\mathrm{C} = \mathrm{C}(k,T) > 0$ may be chosen independent of N. The last inequality above is a consequence of Lemma 4.2.4 with $\nu \mapsto N^{1/2}\nu$. Let $\mathbf{W}$ be a Brownian motion independent of $\mathbf{X}^N$

defined on the same probability space. Let us set

$$\mathbf{Y}_t^N := \int_0^t \mathbb{1}_{\{s\in A_N\}} d\mathbf{X}_s^N + \int_0^t \mathbb{1}_{\{s\in B_N\}} d\mathbf{W}_s.$$

As $X_s^{N,i} \neq X_s^{N,j}$ for all $s \in A_N$, we get that

$$\langle Y^{N,i}, Y^{N,j}\rangle(t) = \int_0^t \mathbb{1}_{\{s\in A_N | X_s^{N,i}=X_s^{N,j}\}} ds + \mathbb{1}_{\{i=j\}} \int_0^t \mathbb{1}_{\{s\in B_N\}} ds = \mathbb{1}_{\{i=j\}} \cdot t.$$

for all $t \in [0,T]$. Thus, by Levy's criterion, we find that the $\mathbf{Y}^N$ is a k-dimensional standard Brownian motion in the combined filtration of $(\mathbf{X}^N, \mathbf{W})$. We claim that

$$\mathbf{P}\left[\sup_{t\leq T} \left\|\mathbf{Y}_t^N - \mathbf{X}_t^N\right\| > N^{-1/6}\right] \leq 2k \cdot \mathrm{C}N^{-1/6}. \tag{4.41}$$

where the C is the same as in (4.40). An application of Doob's submartingale inequality followed by Itô's isometry yields

$$\begin{aligned}
\mathbf{P}\left[\sup_{t\leq T} \left\|\mathbf{Y}_t^N - \mathbf{X}_t^N\right\| > N^{-1/6}\right] &\leq N^{1/3} \cdot \mathbf{E}\left[\left\|\mathbf{Y}_T^N - \mathbf{X}_T^N\right\|^2\right] \\
&= N^{1/3} \cdot \mathbf{E}\left[\left\| \int_0^T \mathbb{1}_{\{s\in B_N\}} d\big(\mathbf{W}_s - \mathbf{X}_s^N\big)\right\|^2\right] \\
&= 2kN^{1/3} \cdot \mathbf{E}\left[\int_0^T \mathbb{1}_{\{s\in B_N\}} ds\right] = 2kN^{1/3} \cdot \mathbf{E}[|B_N|].
\end{aligned}$$

Inserting the bound from (4.40) leads to (4.41). As $\mathbf{Y}^N$ is a standard k-dimensional Brownian motion, (4.41) implies $\mathbf{X}^N$ converges weakly to a standard k-dimensional Brownian motion.

Let us now settle the weak convergence of pairwise intersection times. Observe that by Lemma 4.2.5, $4N^{1/2}\nu([0,1])V^{ij}(\mathbf{X}^N)$ is a tight sequence in $C[0,T]$ for each $i<j$. Let us consider any limit point $(\mathbf{U}, (K^{ij})_{1\leq i<j\leq k})$ of

$$(\mathbf{X}^N\ ,\ (4N^{1/2}\nu([0,1])V^{ij}(\mathbf{X}^N))_{1\leq i<j\leq k}).$$

By the property (4.2) of sticky Brownian motion, for each $i < j$ we have that

$$|X^{N,i}(t) - X^{N,j}(t)| - 4N^{1/2}\nu([0,1])V^{ij}(\mathbf{X}^N)$$

is a martingale for each N. Thanks to the estimates in Lemma 4.2.4, the above expression is uniformly integrable. Thus in the limit point, the process $|U^i - U^j| - K^{ij}$ is a martingale as well. However, we have already established that $\mathbf{U}$ is a standard k-dimensional Brownian motion. Thus, by Tanaka's formula, $K^{ij}(\cdot)$ has to be $L_0^{U^i - U^j}(\cdot)$.

Thus, summarizing, we have identified the limit point as $(\mathbf{U}, (L_0^{ij})_{1\le i<j\le k})$ where $\mathbf{U}$ is a standard k-dimensional Brownian motion and $L_0^{ij}(t) := L_0^{U^i - U^j}(t)$. This verifies part (a).

Proof of (b). Let us take any $\mathcal{G}_N$ satisfying the assumption of the theorem. We first prove that $\mathcal{G}_N$ is tight in $C[0,T]$. Indeed, by Burkholder-Davis-Gundy inequality, we have that

$$\mathbf{E}\big[|\mathcal{G}_N(t) - \mathcal{G}_N(s)|^p\big] \le \mathbf{E}\bigg[\bigg(\int_s^t d\langle \mathcal{G}_N\rangle(u)\bigg)^{\frac{p}{2}}\bigg] \le C\cdot \mathbf{E}\bigg[\bigg(N^{1/2}\sum_{i<j}\big(V_t^{ij}(\mathbf{X}^N) - V_s^{ij}(\mathbf{X}^N)\big)\bigg)^{\frac{p}{2}}\bigg].$$

The second inequality above is due to the assumption (4.33). Lemma 4.2.5 implies that the term on the right-hand side of the above equation is bounded by $\mathrm{C}|t-s|^{p/4}$ for some constant $\mathrm{C} > 0$ independent of N. This verifies the tightness of $\mathcal{G}_N$.

Note that (4.35) would be immediate from (4.34), because the latter would imply that any subsequence of the left side of (4.35) has a further subsequence which converges to the right side of (4.35). Let us therefore prove (4.34). Consider any joint limit point $(\mathcal{G}, \mathbf{U}, (K^{ij})_{1\le i<j\le k})$ of

$$(\mathcal{G}_N\ ,\ \mathbf{X}^N\ ,\ (4N^{1/2}\nu([0,1])V^{ij}(\mathbf{X}^N))_{1\le i<j\le k}).$$

By part (a), $\mathbf{U}$ is a standard k-dimensional Brownian motion and $K^{ij} = L_0^{U^i - U^j}$. By our moment estimates, we have uniform integrability of both prelimiting martingales $\mathcal{G}_N$ and $\mathbf{X}^N$. Thus in the limit, $\mathcal{G}$ and $\mathbf{U}$ are martingales in their *joint* filtrations. By Proposition 4.2.6 and the assumption (4.33), we have boundedness of p^{th} moments (hence uniform integrability) of $\{e^{\mathcal{G}_N(T) - \frac{1}{2}\langle \mathcal{G}_N\rangle_T}\}_{N\ge 1}$.

Thus, in the limit, $\exp(\mathcal{G}(t) - \frac{1}{2}\langle\mathcal{G}\rangle(t))$ is a martingale as well.

To complete the proof of (4.34), the main idea is to use (4.33) to prove that the martingales $\mathcal{G}_N$ are asymptotically decoupled from the sticky Brownian motions. To be precise, we shall prove that for each i and for all $0 \leq t \leq T$

$$\lim_{N\to\infty} \mathbf{E}\bigg[\int_0^t d|\langle X^{N,i}, \mathcal{G}_N\rangle(s)|\bigg] = 0. \tag{4.42}$$

Let us assume (4.42) for the moment and complete the proof of (4.34). Note that in the prelimit $X^{N,i}\cdot\mathcal{G}_N - \langle X^{N,i}, \mathcal{G}_N\rangle$ are martingales and (4.42) shows that the covariations vanish in probability. Therefore, in the limit, we can conclude that $\mathcal{G}\cdot U^i$ is a martingale so that $\langle U^i, \mathcal{G}\rangle = 0$.

Take any bounded $\mathcal{F}_T(\mathbf{U})$-measurable functional $H : C([0,T],\mathbb{R}^k) \to \mathbb{R}$. Note that $t \mapsto \mathbf{E}[H(\mathbf{U})|\mathcal{F}_t(\mathbf{U})]$ is a martingale in the filtration of $\mathbf{U}$. As $\mathbf{U}$ is a standard k-dimensional Brownian motion, by the martingale representation theorem we have

$$H(\mathbf{U}) = \mathbf{E}[H(\mathbf{U})] + \sum_{i=1}^k \int_0^T h_s^i dU_s^i$$

for some adapted $\mathbb{R}$-valued processes $h^1, \ldots, h^k$. For the stochastic exponential we have

$$\exp\left(\mathcal{G}(T) - \frac{1}{2}\langle\mathcal{G}\rangle(T)\right) = 1 + \int_0^T \exp\left(\mathcal{G}(s) - \frac{1}{2}\langle\mathcal{G}\rangle(s)\right) d\mathcal{G}(s).$$

Using this we find that

$$\begin{aligned}\mathbf{E}\Big[\big(e^{\mathcal{G}(T)-\frac{1}{2}\langle\mathcal{G}\rangle(T)} - 1\big)\big(H(\mathbf{U}) - \mathbf{E}[H(\mathbf{U})]\big)\Big] &= \sum_{i=1}^k \mathbf{E}\bigg[\bigg(\int_0^T e^{\mathcal{G}(s)-\frac{1}{2}\langle\mathcal{G}\rangle(s)} d\mathcal{G}(s)\bigg)\bigg(\int_0^T h_s^i dU_s^i\bigg)\bigg] \\ &= \sum_{i=1}^k \mathbf{E}\bigg[\int_0^T h_s^i e^{\mathcal{G}(s)-\frac{1}{2}\langle\mathcal{G}\rangle(s)} d\langle\mathcal{G}, U^i\rangle(s)\bigg] = 0,\end{aligned}$$

where the last equality is due to the fact $\langle U^i, \mathcal{G}\rangle \equiv 0$ almost surely. This proves that for all bounded measurable H we have $\mathbf{E}[e^{\mathcal{G}(T)-\frac{1}{2}\langle\mathcal{G}\rangle(T)} H(\mathbf{U})] = \mathbf{E}[H(\mathbf{U})]$, establishing (4.34) modulo (4.42).

Let us now explain why (4.42) holds. Note that by assumption (4.33)

$$\mathcal{G}_N(t) = \int_0^t \mathbb{1}_{\{s\in B_N\}} d\mathcal{G}_N(s),$$

where B_N is defined in (4.39). By the Kunita-Watanabe inequality (see Proposition 2.14 in Chapter 3 of [226]), we have that

$$\begin{aligned}\mathbf{E}\Big[\int_0^t d|\langle X^{N,i}, \mathcal{G}_N\rangle(s)|\Big] &= \mathbf{E}\Big[\int_0^t \mathbb{1}_{\{s\in B_N\}} d|\langle X^{N,i}, \mathcal{G}_N\rangle(s)|\Big] \\ &\leq \mathbf{E}\Big[\Big(\int_0^t \mathbb{1}_{\{s\in B_N\}} ds\Big)^{\frac{1}{2}} \sqrt{\langle \mathcal{G}_N\rangle(t)}\Big]. \end{aligned} \tag{4.43}$$

By definition, $\int_0^t \mathbb{1}_{\{s\in B_N\}} ds \leq \sum_{i<j} V_t^{ij}$. Using assumption (4.33) we get that

$$\text{r.h.s. of (4.43)} \leq N^{1/4} \cdot \mathbf{E}\Big[\sum_{i<j} V_t^{ij}\Big] \leq N^{-1/4} \cdot \mathbf{E}\Big[\exp\Big(N^{1/2}\sum_{i<j} V_t^{ij}\Big)\Big] \leq \mathrm{C}\cdot N^{-1/4},$$

where we used $a \leq e^a$ and then applied Lemma 4.2.4 in the last two bounds. This verifies (4.42), completing the proof of the theorem.

Proof of (c). Note that

$$\gamma_N(\Delta_m(0,T)) \leq \prod_{j=1}^m \Big[N^{\frac{1}{2}} \int_0^T \mathbb{1}_{\{X_u^{2j-1}=X_u^{2j}\}} du\Big].$$

From Lemma 4.2.4 we know the exponential moments for each of the terms in the product are uniformly bounded under $SB^{(k)}_{N^{1/2}\nu}$ measure. Thus for all $p \geq 1$,

$$\sup_{N\geq 1} \mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}}[\gamma_N(\Delta_m(0,T))^p] < \infty. \tag{4.44}$$

Hence the laws of $\{\gamma_N\}_{N\geq 1}$ are tight, because the total mass of γ_N is a tight family of random variables and because $\Delta_m(0,T)$ as defined in (4.36) is a compact space. Let $\Big(\mathbf{U}, \big(K^{ij}\big)_{1\leq i<j\leq k}, \gamma\Big)$ be any limit point of the sequence $(\mathbf{X}^N, 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\leq i<j\leq k}, \gamma_N)$. By part (a), we know

$\mathbf{U}$ is a standard k-dimensional Brownian motion and $K^{ij} = L_0^{ij}$. Note that the joint cumulative distribution function of the measure γ is necessarily given by $\left(\frac{\sigma}{2}\right)^m$ times a product of the K^{ij}, where $\sigma = \frac{1}{2\nu([0,1])}$. Since we know that $K^{ij} = L_0^{ij}$, this immediately implies that $\gamma = \left(\frac{\sigma}{2}\right)^m \prod_{j=1}^m dL_0^{U^{2j-1}-U^{2j}}(u_j)$. This establishes part (c).

4.4 Martingales associated to the sticky kernels

In this section, we begin our analysis of the field $\mathscr{X}_t^N$ by identifying some useful martingale observables which will later be used to identify the limit in Section 4.6.

Definition 4.4.1. *For a finite measure $\nu = \lambda + \sum_{k\in\mathbb{N}} p_k \delta_{x_k}$ on the real number line with λ atomless, we define its "square"*

$$\nu^{Sq} := \sum_k p_k^2 \delta_{x_k}. \tag{4.45}$$

For a continuous function ϕ we also denote by $(\nu, \phi) := \int_{\mathbb{R}} \phi \, d\nu$ the natural (spatial) pairing. When ϕ is smooth, we will use ϕ', ϕ'' to denote the first and second (spatial) derivatives of ϕ respectively. We then have the following lemma.

Lemma 4.4.2. *Let $K_{0,t}$ denote the Howitt-Warren flow. Then for all $\phi \in C_c^\infty(\mathbb{R})$, the process*

$$S_t(\phi) := (K_{0,t}, \phi) - \frac{1}{2}\int_0^t (K_{0,s}, \phi'')ds$$

is a continuous martingale in the filtration $\mathcal{F}_t := \sigma(\{K_{a,b}(x, A) : 0 \le a < b \le t, x \in \mathbb{R}, A \text{ Borel}\})$. Furthermore its quadratic variation is given by

$$\langle S(\phi)\rangle_t = \int_0^t (K_{0,s}^{Sq}, (\phi')^2)ds.$$

From [190, Theorem 2.8], we know for each fixed $t > 0$, $K_{0,t}$ is atomic almost surely. Thus $K_{0,s}^{Sq}$ is nonzero for almost every $s > 0$.

Proof. For the moment being, we consider a fixed realization of the kernels $\{K_{s,t} : 0 \le s \le t\}$. By [188, Section 2.6], we can sample paths $B^1, B^2, B^3, \ldots$ starting from 0 in such a way that:

- given $\{K_{s,t} : 0 \le s \le t\}$, the paths $\{B^i\}_{i \ge 1}$ are all independent.
- given $\{K_{s,t} : 0 \le s \le t\}$, the law of B^i_t given $(B^i_u)_{0 \le u \le s}$ is $K_{s,t}(B_s, \cdot)$ for all $0 \le s \le t$.

It is clear that the joint quenched law of these paths $\{B^i\}_{i \ge 1}$ is a deterministic function of the collection $\{K_{s,t} : 0 \le s \le t\}$, because the two bullet points automatically determine the multi-time marginal laws of any finite subcollection $\{B^i\}_{i=1}^r$ in terms of the kernels $K_{s,t}$. It is also clear that the *annealed* law of $(B^1, \ldots, B^r)$ is just r-point sticky Brownian motion (see the discussion after Definition 4.1.1). We define $\mu^r_t := \frac{1}{r} \sum_{i=1}^r \delta_{B^i_t}$. By the law of large numbers, as $r \to \infty$ we have

$$\mu^r_t \to K_{0,t} \qquad \text{and} \qquad \int_0^t \mu^r_s ds \to \int_0^t K_{0,s} ds$$

almost surely in the topology of weak convergence of Borel measures, for each $t > 0$. Notice that

$$|(\mu^r_t, \phi) - (\mu^r_s, \phi)| \le \frac{1}{r} \sum_{i=1}^r |\phi(B^i_t) - \phi(B^i_s)| \le \frac{\|\phi'\|_{L^\infty(\mathbb{R})}}{r} \sum_{i=1}^r |B^i_t - B^i_s|$$

for all $s, t \ge 0$, all $r \in \mathbb{N}$, and all $\phi \in C^\infty_c(\mathbb{R})$. By Minkowski's inequality and the fact that the B^i are individually Brownian motions in the annealed sense, we obtain

$$\mathbb{E}[|(\mu^r_t, \phi) - (\mu^r_s, \phi)|^p]^{1/p} \le \frac{\|\phi'\|_{L^\infty(\mathbb{R})}}{r} \sum_{i=1}^r \mathbb{E}[|B^i_t - B^i_s|^p]^{1/p} = C_p \|\phi'\|_{L^\infty} |t - s|^{1/2}, \tag{4.46}$$

for any $p \ge 1$. Define

$$\widetilde{S}^r_t(\phi) := (\mu^r_t, \phi) - \frac{1}{2} \int_0^t (\mu^r_s, \phi'') ds = \frac{1}{r} \sum_{i=1}^r \int_0^t \phi'(B^i_s) dB^i_s. \tag{4.47}$$

where the second equality above is due to Itô's formula. Hence, $\widetilde{S}^r_t(\phi)$ is a martingale. Clearly

$|(\mu_t^r, \phi)| \leq \|\phi\|_{L^\infty(\mathbb{R})}$. Combining this with (4.46) and the first equality in (4.47), one has

$$\mathbb{E}[|\widetilde{S}_t^r(\phi) - \widetilde{S}_s^r(\phi)|^p]^{1/p} \leq C_p \|\phi'\|_{L^\infty} |t-s|^{1/2} + \frac{1}{2}\|\phi''\|_{L^\infty}|t-s|,$$

uniformly over $s, t \geq 0$. On any compact interval $[0, T]$ the right side can be bounded above by $\mathrm{C}|t-s|^{1/2}$ for some constant $\mathrm{C} = \mathrm{C}(p, T, \phi)$ independent of $s, t \in [0, T]$. Since $S_t(\phi) = \lim_{r\to\infty} \widetilde{S}_t^r(\phi)$ almost surely, this L^p bound implies that the limit $S_t(\phi)$ is a continuous L^p-martingale for each $\phi \in C_c^\infty(\mathbb{R})$. The quadratic variations of $\widetilde{S}^r(\phi)$ can be computed explicitly as

$$\langle \widetilde{S}^r(\phi)\rangle_t = \frac{1}{r^2}\sum_{i,j}\int_0^t \phi'(B_s^i)\phi'(B_s^j) d\langle B^i, B^j\rangle_s = \frac{1}{r^2}\sum_{i,j}\int_0^t \phi'(B_s^i)^2 \mathbf{1}_{\{B_s^i = B_s^j\}} ds = \int_0^t ((\mu_s^r)^{Sq}, (\phi')^2) ds.$$

We consider what happens to the quadratic variation as we take $r \to \infty$. By [190], we have that for each $t > 0$, $K_{0,t}$ is a.s. atomic. Using this, it is easy to show that for each $s > 0$, the quantity $((\mu_s^r)^{Sq}, (\phi')^2)$ will converge almost surely as $r \to \infty$ to $(K_{0,s}^{Sq}, (\phi')^2)$. Since μ_t^r is a probability measure, we have $((\mu_s^r)^{Sq}, (\phi')^2) \leq (\mu_s^r, (\phi')^2) \leq \|\phi'\|_{L^\infty}^2$ deterministically, and likewise $(K_{0,s}^{Sq}, (\phi')^2) \leq (K_{0,s}, (\phi')^2) \leq \|\phi'\|_{L^\infty}^2$. Therefore the dominated convergence theorem applied on the product space $\mathbb{P} \otimes ds$ gives

$$\lim_{r\to\infty} \mathbb{E}\bigg[\int_0^t \big|((\mu_s^r)^{Sq}, (\phi')^2) - (K_{0,s}^{Sq}, (\phi')^2)\big| ds\bigg] = 0.$$

Here as always, $\mathbb{E}$ is annealed expectation. In other words, the expression for $\langle \widetilde{S}^r(\phi)\rangle_t$ will converge in $L^1(\mathbb{P})$ as $r \to \infty$ to the quantity $\int_0^t (K_{0,s}^{Sq}, (\phi')^2) ds$. Putting all of this together, we obtain that $S_t(\phi) = \lim_{r\to\infty} \widetilde{S}_t^r(\phi)$ is a martingale with quadratic variation

$$\langle S(\phi)\rangle_t = \lim_{r\to\infty} \langle \widetilde{S}^r(\phi)\rangle_t = \int_0^t (K_{0,s}^{Sq}, (\phi')^2) ds,$$

where the limit is interpreted in $L^1(\mathbb{P})$. This completes the proof. □

Given the above lemma, our next goal is to derive a (local) martingale that is related to the

rescaled field $\mathscr{X}^N$ from (4.4) instead of $K_{0,t}$. To motivate the choice of our next martingale, we first perform some martingale calculations on a formal level. The process S from Lemma 4.4.2 resembles an *orthomartingale* in the sense that the cross-variation vanishes whenever ϕ and ψ have disjoint support, and the above lemma states that $K_{0,t}$ solves the SPDE

$$dK = \partial_x^2 K dt + dS.$$

Note that if u solves the forced heat equation $\partial_t u = \frac{1}{2}\partial_x^2 u + F$ and if $a, b, c \in \mathbb{R}$ then $e^{ax+bt}u(t, x + ct)$ solves the equation $\partial_t v = Lv + \overline{F}$ where

$$L = \frac{1}{2}\big(\partial_x - (a-c)I\big)^2 + \big(b - \frac{1}{2}c^2\big)I,$$

$$\overline{F}(t,x) = e^{ax+bt}F(t, x+ct).$$

This makes sense even when F is a Schwartz distribution. Note that when $a = c$ and $b = \frac{1}{2}c^2$ then $L = \frac{1}{2}\partial_x^2$ is unchanged.

We now formally go from $K_{0,t}$ to $\mathscr{X}_t^N$ defined in (4.4) in two steps. We first apply the above argument for $u = K$ and $F = dS$, with $a = c = N^{-1/4}$ and $b = \frac{1}{2}N^{-1/2}$. This gives us that the measure-valued process

$$\widetilde{K}^N(t,x) := e^{ax+bt}K_{0,t}(x+ct) = e^{\frac{t}{2}N^{-1/2} + xN^{-1/4}}K_{0,t}(x + N^{-1/4}t)$$

solves the SPDE

$$\partial_t \widetilde{K}^N = \partial_x^2 \tilde{K}^N dt + dS^N$$

where S^N is the orthomartingale measure defined by

$$dS^N(t,x) = e^{\frac{t}{2}N^{-1/2} + xN^{-1/4}} \cdot dS(t, x + N^{-1/4}t),$$

to be interpreted by integration against test functions. Next, we apply a diffusive scaling to $\widetilde{K}^N$.

Let us set

$$\mathscr{X}_t^N(x) := N^{1/2}\widetilde{K}_N(Nt, N^{1/2}x) = N^{1/2}e^{\frac{t}{2}N^{1/2}+xN^{1/4}}K_{0,Nt}(tN^{3/4}+xN^{1/2}).$$

Note that $\partial_t \mathscr{X}_t^N = N^{3/2}\partial_t \widetilde{K}^N(Nt, N^{1/2}x)$ and $\partial_x^2 \mathscr{X}^N dt = N^{3/2}\partial_x^2 \widetilde{K}^N(Nt, N^{1/2}x)dt$. Thus, $\mathscr{X}_t^N$ solves the SPDE

$$\partial_t \mathscr{X}^N = \partial_x^2 \mathscr{X}^N dt + dM^N$$

where M^N is the orthomartingale measure defined by

$$dM^N(t,x) = N^{3/2}dS^N(Nt, N^{1/2}x) = N^{3/2}e^{\frac{t}{2}N^{1/2}+xN^{1/4}} \cdot dS(Nt, tN^{3/4}+xN^{1/2}).$$

The above formal calculations suggest the following lemma.

Lemma 4.4.3. *With $\mathscr{X}^N$ as defined in* (4.4)*, for all $\phi \in C_c^\infty(\mathbb{R})$ the processes*

$$M_t^N(\phi) := \mathscr{X}_t^N(\phi) - \phi(0) - \frac{1}{2}\int_0^t \mathscr{X}_s^N(\phi'')ds \tag{4.48}$$

are continuous martingales with

$$\langle M^N(\phi)\rangle_t = \int_0^t \big((\mathscr{X}_s^N)^{Sq}, (\phi' + N^{1/4}\phi)^2\big)ds$$

where $(\mathscr{X}_s^N)^{Sq}$ is defined via (4.45)*.*

We remark that the $M^N(\phi)$ are actually tight in $C[0,T]$ as $N \to \infty$. This will follow from the Burkholder-Davis-Gundy inequality as well as moment bounds on the quadratic variation that are proved in Proposition 4.5.6. Tightness will be proved in Proposition 4.6.13, using moment formulas and bounds that we will derive in Section 4.5. This will be crucial in solving the martingale problem for (SHE).

Proof. The proof follows in a similar manner to the proof of Lemma 4.4.2. Note that ϕ' from

Lemma 4.4.2 has changed to $\phi' + N^{1/4}\phi$ due to the transformation of space-time $(t, x) \mapsto (Nt, N^{3/4}t + N^{1/2}x)$ which is used to obtain $\mathscr{X}_t^N$ from $K_{0,t}$. We omit the details. □

Lemma 4.4.3 can be extended to include time-dependent test functions as well.

Lemma 4.4.4. *Let $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Consider $M_t^N(\phi)$ as defined in* (4.48)*. The process*

$$R_t^N(\varphi) := M_t^N\big(\varphi(t,\cdot)\big) - \int_0^t M_s^N\big(\partial_s\varphi(s,\cdot)\big)ds \tag{4.49}$$

is a continuous martingale, and its quadratic variation up to time t is given by

$$\langle R^N(\varphi)\rangle_t = \int_0^t \big((\mathscr{X}_s^N)^{Sq}, N^{1/2}(\varphi(s,\cdot) + N^{-1/4}\partial_x\varphi(s,\cdot))^2\big)ds. \tag{4.50}$$

Intuitively, one should think of $R_t^N(\varphi)$ as being equal to $\int_0^t (\varphi(s,\cdot), dM_s^N)_{L^2(\mathbb{R})}$, which yields (4.49) by a formal integration-by-parts.

Proof. Let $\mathcal{F}_t := \sigma(\{K_{Na,Nb}(x, A) : 0 \le a < b \le t, x \in \mathbb{R}, A \text{ Borel}\})$. Fix $0 \le u \le t$. By Lemma 4.4.3 we have

$$\mathbb{E}[R_t^N(\varphi) \mid \mathcal{F}_u] = M_u^N\big(\varphi(t,\cdot)\big) - \int_0^u M_s^N\big(\partial_s\varphi(s,\cdot)\big)ds - \int_u^t M_u^N\big(\partial_s\varphi(s,\cdot)\big)ds. \tag{4.51}$$

Note that

$$\int_u^t M_u^N\big(\partial_s\varphi(s,\cdot)\big)ds = M_u^N\big(\varphi(t,\cdot)\big) - M_u^N\big(\varphi(u,\cdot)\big).$$

Inserting this identity back in (4.51), we get $\mathbb{E}[R_t^N(\varphi) \mid \mathcal{F}_u] = R_u^N(\varphi)$. This verifies that $R_t^N(\varphi)$ is a martingale with respect to $\mathcal{F}_t$. The claimed quadratic variation for $R_t^N(\varphi)$ in (4.50) can be verified similarly. □

4.5 Analysis of the quadratic martingale field

Recall the martingale M_t^N from (4.48). As explained in the introduction, in order to identify limit points of the field $M_t^N(\phi)$, one must study the quadratic variation of this martingale. In this section, we identify the leading order contribution of $\langle M^N(\phi)\rangle$ as a functional which we call the *quadratic martingale field*—introduced below.

Definition 4.5.1. *Recall the field $\mathscr{X}_t^N$ from* (4.4)*, as well as the squaring operation from Definition 4.4.5. For a bounded test function ϕ on $\mathbb{R}$, we define the quadratic martingale field (QMF) as*

$$\begin{aligned} Q_t^N(\phi) &:= N^{1/2}\int_0^t \big((\mathscr{X}_s^N)^{Sq},\phi\big)ds \\ &= N^{1/2}\int_0^t\int_{\mathbb{R}} e^{-s\sqrt{N}+2uN^{-1/4}}\phi(N^{-1/2}(u-N^{3/4}s))K_{0,Ns}^{Sq}(du)ds, \end{aligned} \tag{4.52}$$

From Lemma 4.4.3, in view of the above definition, one has that

$$\langle M^N(\phi)\rangle_t = Q_t^N(\phi^2) + \mathcal{E}_t^N(\phi),$$

where $\mathcal{E}_t^N(\phi)$ is the "error term" given by

$$\mathcal{E}_t^N(\phi) := \int_0^t \big((\mathscr{X}_s^N)^{Sq}, 2N^{1/4}\phi\phi' + (\phi')^2\big)ds = Q_t^N\big(N^{-1/2}(\phi')^2 + 2N^{-1/4}\phi\phi'\big). \tag{4.53}$$

The main goal of this section is to show that only Q_t^N contributes to the limit (in a very specific way, see Proposition 4.5.3) and that $\mathcal{E}_t^N$ vanishes in the limit (Proposition 4.5.5). The starting point of our analysis is a probabilistic interpretation of $Q_t^N(\phi)$. Indeed, similar to the representation (4.32) for $\mathscr{X}_t^N(\phi)$, the field $Q_t^N(\phi)$ also admits a formula in terms of a quenched expectation of 2-point motion sampled from the environment $\omega := \{K_{s,t} : -\infty < s < t < \infty\}$. Given a 2-point

motion sampled from the environment $\omega := \{K_{s,t} : -\infty < s < t < \infty\}$, we may view $Q_t^N(\phi)$ as

$$\begin{aligned} Q_t^N(\phi) &= N^{1/2} \int_0^t \mathbb{E}_\omega^{(2)} \left[e^{-s\sqrt{N}+2N^{-1/4}X_{Ns}} \phi(N^{-1/2}(X_{Ns} - N^{3/4}s)) \mathbb{1}_{\{X_{Ns}=Y_{Ns}\}} \right] ds \\ &= N^{1/2} \int_0^t \mathbb{E}_\omega^{(2)} \left[e^{-s\sqrt{N}+N^{-1/4}(X_{Ns}+Y_{Ns})} \phi(N^{-1/2}(X_{Ns} - N^{3/4}s)) \mathbb{1}_{\{X_{Ns}=Y_{Ns}\}} \right] ds \end{aligned} \quad (4.54)$$

where the second equality follows by replacing $2N^{-1/4}X_{Ns}$ in the exponential with $N^{-1/4}(X_{Ns} + Y_{Ns})$ due to the presence of the indicator $\mathbb{1}_{\{X_{Ns}=Y_{Ns}\}}$. The representation (4.54) will be used as a starting point in the next section, Section 4.5.1, in extracting formulas and estimates for the QMF.

4.5.1 Formulas and limits for the quadratic martingale field

Recall the field $\mathscr{X}_t^N$ from (4.4). The goal of this section is to show that the limit of Q_t^N can be related to that of $\mathscr{X}_t^N$ in a precise way (see Proposition 4.5.3). To do this, we first extract moment formulas of certain observables of $(\mathscr{X}_t^N, Q_t^N)$ in terms of sticky Brownian motion in the same spirit as Lemma 4.3.1.

Lemma 4.5.2 (Moment formulas)**.** *Fix any bounded functions ψ, ϕ on $\mathbb{R}$ and $N \geq 1$. Suppose that $(X^1, \dots, X^{2k})$ denotes a $2k$-point sticky Brownian motion with characteristic measure $N^{1/2}\nu$. Recall $\Delta_m(s,t)$ from* (4.36) *and $\sigma = \frac{1}{2\nu([0,1])}$. We have the following moment formulas.*

(a) For all $t > 0$, we have

$$\begin{aligned} &\mathbb{E}\left[\left(Q_t^N(\psi) - \sigma \int_0^t \mathscr{X}_s^N(\phi)^2 ds \right)^k \right] \\ &\quad = k! \int_{\Delta_k(0,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\prod_{i=1}^k e^{-s_i\sqrt{N}+N^{1/4}(X^{2i-1}_{s_i}+X^{2i}_{s_i})} \Upsilon_{s_i}^N(X^{2i-1}, X^{2i}) \right] \prod_{i=1}^k ds_i, \end{aligned} \quad (4.55)$$

where

$$\Upsilon_u^N(X,Y) := N^{1/2}\psi\big(X_u - N^{1/4}u\big)\mathbb{1}_{\{X_u=Y_u\}} - \sigma\phi\big(X_u - N^{1/4}u\big)\phi\big(Y_u - N^{1/4}u\big).$$

(b) For all $0 \leq s < t$ we have the following moment formula for the increment of the QMF

$$\begin{aligned}&\mathbb{E}\left[\left(Q_t^N(\psi) - Q_s^N(\psi)\right)^k\right]\\&= N^{\frac{k}{2}} k! \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[\prod_{i=1}^k e^{-s_i\sqrt{N}+N^{\frac{1}{4}}(X^{2i-1}_{s_i}+X^{2i}_{s_i})}\psi(X^{2i}_s - N^{1/4}s_i)\mathbb{1}_{\{X^{2i-1}_{s_i}=X^{2i}_{s_i}\}}\right]\prod_{i=1}^k ds_i.\end{aligned} \tag{4.56}$$

Proof. Recall the probabilistic interpretation of $\mathscr{X}_t^N(\phi)$ and $Q_t^N(\psi)$ from (4.32) and (4.54) respectively. Combining them we get that

$$Q_t^N(\psi) - \sigma\int_0^t \mathscr{X}_s^N(\phi)^2 ds = \int_0^t \mathbb{E}_\omega^{(2)}\left[e^{-s\sqrt{N}+N^{-1/4}(X^1_{Ns}+X^2_{Ns})}\widetilde{\Upsilon}_s^N(X^1,X^2)\right]ds \tag{4.57}$$

where $\mathbb{E}_\omega^{(2)}$ denotes the quenched expectation of a 2-point motion sampled from the environment $\omega := \{K_{s,t} : -\infty < s < t < \infty\}$ and

$$\widetilde{\Upsilon}_s^N(X^1,X^2) := N^{1/2}\psi\left(N^{-1/2}(X^1_{Ns} - N^{3/4}s)\right)\mathbb{1}_{\{X^1_{Ns}=X^2_{Ns}\}} - \sigma\prod_{i=1}^2\phi\left(N^{-1/2}(X^i_{Ns} - N^{3/4}s)\right).$$

Taking the k-th power of (4.57) we see that

$$\begin{aligned}&\left(Q_t^N(\psi) - \sigma\int_0^t \mathscr{X}_s^N(\phi)^2 ds\right)^k\\&\qquad = k!\int_{\Delta_k(0,t)} \mathbb{E}_\omega^{(2k)}\left[\prod_{i=1}^k e^{-s_i\sqrt{N}+N^{-1/4}(X^{2i-1}_{Ns_i}+X^{2i}_{Ns_i})}\widetilde{\Upsilon}_{s_i}^N(X^{2i-1},X^{2i})\right]\prod_{i=1}^k ds_i.\end{aligned} \tag{4.58}$$

In the above line, we have also used the fact that $\int_{[0,t]^k}\prod_{i=1}^k F(x_i)dx_i = k!\int_{\Delta_k(0,t)}\prod_{i=1}^k F(x_i)dx_i$ (recall Δ_k from (4.36)). Here $\mathbb{E}_\omega^{(2k)}$ denotes the quenched expectation of a $2k$-point motion sampled from the environment $\omega := \{K_{s,t} : -\infty < s < t < \infty\}$. Taking annealed expectation on both sides of the above equation, interchanging the order of integration and expectation, and then using the fact that the averaged law of $(X^1,\ldots,X^{2k})$ is a $2k$-point sticky Brownian motion with

characteristic measure ν, we get

$$\mathbb{E}\left[\left(Q_t^N(\psi) - \sigma\int_0^t \mathscr{X}_s^N(\phi)^2 ds\right)^k\right]$$
$$= k!\int_{\Delta_k(0,t)} \mathbf{E}_{SB_\nu^{(2k)}}\left[\prod_{i=1}^k e^{-s_i\sqrt{N}+N^{-1/4}(X^{2i-1}_{Ns_i}+X^{2i}_{Ns_i})}\widetilde{\Upsilon}^N_{s_i}(X^{2i-1},X^{2i})\right]\prod_{i=1}^k ds_i.$$

The interchanging of the order of integration and expectation is permissible by Fubini's theorem. Indeed, Fubini's theorem is applicable as applying Lemma 4.2.4, one can check that for each fixed N, the expectation of the absolute value of the integrand on the right-hand side of (4.58) is uniformly bounded as $\vec{s}$ varies in $\Delta_k(0,t)$. Using the fact that $N^{-1/2}(X^1_{Ns},\dots,X^{2k}_{Ns})$ is distributed as $SB^{(2k)}_{N^{1/2}\nu}$ measure, (4.55) follows from the above formula. Relying on (4.54) alone, one can derive the formula in (4.56) by the exact same argument. This completes the proof. □

We now come to the main result of this section, which shows that the quadratic martingale field Q_t^N is well approximated by the integrated version of the square of $\mathscr{X}_t^N$. Loosely speaking, for each $a \in \mathbb{R}\setminus\{0\}$ we shall show as $N\to\infty$

$$Q_t^N(\delta_a) - \sigma\int_0^t \left(\mathscr{X}_s^N(\delta_a)\right)^2 ds \xrightarrow{L^2} 0.$$

In other words, mollifying the squared field is comparable to squaring the mollified field. Since Q_t^N and $\mathscr{X}_s^N$ exist as distributions, to make sense of the above display, we work with a sequence of Gaussian test functions converging to δ_a.

Proposition 4.5.3 (Limiting behavior of the QMF)**.** *Let $a\in\mathbb{R}$ and let $\xi(x) := \frac{1}{\sqrt{\pi}}e^{-x^2}$ be the Gaussian test function and let $\xi^a_\varepsilon(x) := \varepsilon^{-1}\xi(\varepsilon^{-1}(x-a))$. Then for all $t>0$ and $a\in\mathbb{R}\setminus\{0\}$,*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}\left[\left(Q_t^N(\xi^a_\varepsilon) - \sigma\int_0^t \mathscr{X}_s^N\left(\xi^a_{\varepsilon\sqrt{2}}\right)^2 ds\right)^2\right] = 0, \tag{4.59}$$

where as usual, $\sigma = \frac{1}{2\nu([0,1])}$. Furthermore, we have the bound

$$\sup_{\substack{\varepsilon>0 \\ a\in\mathbb{R}\setminus\{0\}}} \limsup_{N\to\infty} \left[1 \wedge (|\log a|^{-2})\right] \cdot \mathbb{E}\left[\left(Q_t^N(\xi_\varepsilon^a) - \sigma \int_0^t \mathscr{X}_s^N\left(\xi_{\varepsilon\sqrt{2}}^a\right)^2 ds\right)^2\right] < \infty. \tag{4.60}$$

We remark that the relevance of the second bound (4.60) is that even though we may not have convergence to 0 in (4.59) when $a = 0$, we can still control the behavior near $a = 0$ by a blow-up that is polylogarithmic at worst. Note that the $|\log a|^{-2}$ factor appears above only when $a \leq e$. In the proof of (4.117) in Theorem 4.6.16 below, we will see that such blow-ups are irrelevant in the limit.

The starting point of the proof of Proposition 4.5.3 is the moment formula in Lemma 4.5.2 (a). As noted in (4.55), the moments can be expressed as integrals of the expectation of certain observables under sticky Brownian motion. Recall that in Theorem 4.3.2 we saw that the expectation of a certain class of observables under sticky Brownian motion converges to the expectation of those observables under standard k dimensional Brownian motion. That theorem will be used to prove a similar convergence result for the expectations of the type of observables that appear in (4.55).

Proposition 4.5.4. *Fix any $k \in \mathbb{N}$ and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty$. Set $s_0 = t_0$ and $s_{2i-1} = s_{2i} = t_i$ for $1 \leq i \leq k$. Suppose $(X^1, \ldots, X^{2k})$ is distributed according $SB^{(2k)}_{N^{1/2}\nu}$. Let $(\phi_i, \psi_i)_{i=1}^{2k}$ be bounded continuous functions on $\mathbb{R}$. Define*

$$\mathcal{L}_k(\vec{t}) := \sum_{r=1}^{k} \sum_{2r-1\leq p<q\leq 2k} \left[L_0^{U^p - U^q}(t_r) - L_0^{U^p - U^q}(t_{r-1})\right]. \tag{4.61}$$

We have the following two limits:

(a)

$$\lim_{N\to\infty} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[\prod_{i=1}^{2k} e^{-\frac{1}{2}s_i\sqrt{N} + N^{\frac{1}{4}}X^i_{s_i}} \phi_i(X^i_{s_i} - N^{\frac{1}{4}}s_i)\right] = \mathbf{E}_{B^{\otimes 2k}}\left[e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})} \prod_{i=1}^{2k} \phi_i(U^i_{s_i})\right]. \tag{4.62}$$

(b) Let A be any subset of $\{1, 2, \ldots, k\}$. Set $B = A^c$ and define

$$E_1(\vec{t}) := \prod_{i\in A} e^{-\frac{1}{2}t_i\sqrt{N}+N^{\frac{1}{4}}X_{t_i}^{2i-1}}\phi_i(X_{t_i}^{2i-1} - N^{\frac{1}{4}}t_i)\prod_{i\in A} e^{-\frac{1}{2}t_i\sqrt{N}+N^{\frac{1}{4}}X_{t_i}^{2i}}\phi_i(X_{t_i}^{2i} - N^{\frac{1}{4}}t_i), \quad (4.63)$$

$$E_2(\vec{t}) := \prod_{i\in B} e^{-t_i\sqrt{N}+N^{\frac{1}{4}}(X_{t_i}^{2i-1}+X_{t_i}^{2i})}\psi_i(X_{t_i}^{2i} - N^{\frac{1}{4}}t_i)\mathbb{1}_{\{X_{t_i}^{2i-1}=X_{t_i}^{2i}\}}. \quad (4.64)$$

Recall $\Delta_k(s,t)$ from (4.36)*. For each $0 \le s < t \le T < \infty$ we have*

$$\begin{aligned}&\lim_{N\to\infty} N^{\frac{|B|}{2}} \cdot \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\bigg[\int_{\Delta_k(s,t)} E_1(\vec{t})E_2(\vec{t})\prod_{i=1}^{k} dt_i\bigg]\\ &= \Big(\frac{\sigma}{2}\Big)^{|B|}\mathbf{E}_{B^{\otimes 2k}}\bigg[\int_{\Delta_k(s,t)} e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})}\prod_{i\in A}\phi_i(U_{t_i}^{2i-1})\phi_i(U_{t_i}^{2i})dt_i\prod_{i\in B}\psi_i(U_{t_i}^{2i})dL_0^{U^{2i-1}-U^{2i}}(t_i)\bigg].\end{aligned} \quad (4.65)$$

Here $\int_0^t f(s)dL_0^{U^i-U^j}(s)$ denotes the integration of the continuous function $f : [0,t] \to \mathbb{R}$ against the random Lebesgue-Stiltjes measure $dL_0^{U^i-U^j}$ induced from the increasing function $t \mapsto L_0^{U^i-U^j}(t)$.

The proof of Proposition 4.5.4 builds up on the estimates proved in Section 4.2 and uses Theorem 4.3.2. We postpone the proof of Proposition 4.5.4 to Section 4.5.2 and complete the proof of Proposition 4.5.3 assuming it.

Proof of Proposition 4.5.3. For clarity, we split the proof into three steps. We first provide a brief outline of the steps below. Note that, given the moment formulas from Lemma 4.5.2 (a) and the limit of those formulas from Proposition 4.5.4, one can compute the $N \to \infty$ limit of the expectation in (4.60) in terms of expectation of certain observables under a 4D Brownian motion measure (X^1, X^2, Y^1, Y^2). We then proceed to carefully massage this expectation formula to obtain the desired result.

- In **Step 1**, we use a simple transformation to express the limit in terms of an expectation under

a different 4D Brownian motion measure:

$$(X^1 - Y^1, X^2 - Y^2, X^1 + Y^1, X^2 + Y^2).$$

- In **Step 2**, we show how local time heuristics (which are rigorously shown in Section 4.8) can be used to rewrite the expectation formula in terms of a certain concatenation of Brownian bridge and Brownian motion measures.
- In **Step 3**, we use local time estimates for concatenated processes (Lemma 4.8.6) and heat kernel calculations to bound the formula obtained in the previous step. The bound obtained in this step is sharp enough to conclude the proof of Proposition 4.5.3.

Step 1. Applying Lemma 4.5.2 (a) and Proposition 4.5.4 (b) with $k = 2$ we get

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\left(Q_t^N(\psi) - \sigma\int_0^t \mathscr{X}_s^N(\phi)^2 ds\right)^2\right] \\
&= 2\sigma^2 \cdot \mathbf{E}_{B^{\otimes 4}}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\left(\psi(X_{s_i}^i)\frac{1}{2}dL_0^{X^i-Y^i}(s_i) - \phi(X^i)\phi(Y^i)ds_i\right)\right] \\
&= 2\sigma^2 \cdot \mathbf{E}_{B^{\otimes 4}}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\left(\psi\big(\tfrac{1}{2}(X_{s_i}^i + Y_{s_i}^i)\big)\frac{1}{2}dL_0^{X^i-Y^i}(s_i) - \phi(X^i)\phi(Y^i)ds_i\right)\right]
\end{aligned} \tag{4.66}$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{L}_2$ is defined in (4.61). The second equality in the above equation follows by observing that $X_u^i = Y_u^i$ for u in the support of $L_0^{X^i-Y^i}(du)$.

We shall now write $\mathbf{E}$ instead of $\mathbf{E}_{B^{\otimes 4}}$ for convenience. Let us now take

$$\psi(x) := \xi_\varepsilon^a(x) = \frac{1}{\sqrt{\pi\varepsilon^2}} e^{-(x-a)^2/\varepsilon^2}, \quad \phi(x) := \xi_{\varepsilon\sqrt{2}}^a(x) = \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-(x-a)^2/2\varepsilon^2},$$

in (4.66). Using the identity $\xi_{\varepsilon\sqrt{2}}^a(x)\xi_{\varepsilon\sqrt{2}}^a(y) = \xi_\varepsilon^a((x+y)/2)\xi_{2\varepsilon}^0(x-y)$, we may now write (4.66)

as

$$2\sigma^2 \cdot \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\left(\xi_\varepsilon^a\big(\tfrac{1}{2}(X^i_{s_i}+Y^i_{s_i})\big)\cdot\left(\frac{1}{2}dL_0^{X^i-Y^i}(s_i) - \xi_{2\varepsilon}^0(X^i_{s_i}-Y^i_{s_i})ds_i\right)\right)\right]. \tag{4.67}$$

Let us write $U^{i,-} := X^i - Y^i$ and $U^{i,+} := X^i + Y^i$. Note that under $\mathbf{P}_{B^{\otimes 4}}$ the four processes $U^{1,-}, U^{1,+}, U^{2,-}, U^{2,+}$ are independent Brownian motions with diffusion coefficient 2. This enables us to view (4.67) as

$$\begin{aligned}&2\sigma^2 \cdot \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\left(\xi_\varepsilon^a\big(\tfrac{1}{2}U^{i,+}_{s_i}\big)\big)\cdot\left(\frac{1}{2}dL_0^{U^{i,-}}(s_i) - \xi_{2\varepsilon}^0(U^{i,-}_{s_i})ds_i\right)\right)\right]\\ &=: 2\sigma^2[A_1(\varepsilon) - A_2(\varepsilon) - A_3(\varepsilon) + A_4(\varepsilon)],\end{aligned} \tag{4.68}$$

where

$$A_1(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\,\frac{1}{2}dL_0^{U^{1,-}}(s_1)\,\frac{1}{2}dL_0^{U^{2,-}}(s_2)\right] \tag{4.69}$$

$$A_2(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\xi_{2\varepsilon}^0(U^{2,-}_{s_2})\,\frac{1}{2}dL_0^{U^{1,-}}(s_1)\,ds_2\right] \tag{4.70}$$

$$A_3(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\xi_{2\varepsilon}^0(U^{1,-}_{s_1})\,\frac{1}{2}dL_0^{U^{2,-}}(s_2)\,ds_1\right] \tag{4.71}$$

$$A_4(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\xi_{2\varepsilon}^0(U^{1,-}_{s_1})\xi_{2\varepsilon}^0(U^{2,-}_{s_2})\,ds_1\,ds_2\right]. \tag{4.72}$$

Step 2. In this step we focus on each of the $A_i(\varepsilon)$ terms separately. We drop the ε and write A_i for simplicity. Note that informally $\frac{1}{2}dL_0^{U^{i,-}}(s_i)$ may be written as $\delta_0(U^{i,-})ds_i$ which suggests that each of the A_i may be written in terms of Brownian bridge expectations. To this end, we introduce the function

$$H_{s_1,s_2}(\vec{x},\vec{y}) := \mathbf{E}\left[e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)} \mid U^{i,-}_{s_i} = x_i, U^{i,+}_{s_i} = y_i, i=1,2\right] \tag{4.73}$$

where the above conditional expectation is interpreted as taking expectation under the measure where $U^{i,-}$ ($U^{i,+}$ resp.) is a concatenation of a Brownian bridge from 0 to x_i (0 to y_i resp.) over the interval $[0, s_i]$ and an independent Brownian motion started from x_i (y_i resp.) on $[s_i, t]$. All Brownian objects considered are independent with diffusion coefficient 2.

A_1 term: Let us consider the A_1 term from (4.69). Let $\mathcal{F}_t^-$ be the σ-field generated by $\{U_s^{1,-}, U_s^{2,-} \mid 0 \le s \le t\}$. By the tower property of conditional expectation, we have

$$A_1 := \mathbf{E}\left[\int_{\Delta_2(0,t)} \mathbf{E}\left[e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_1}^{1,+}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_2}^{2,+}\big) \mid \mathcal{F}_t^-\right] \frac{1}{2}dL_0^{U^{1,-}}(s_1)\,\frac{1}{2}dL_0^{U^{2,-}}(s_2)\right].$$

By Lemma 4.8.2, the right-hand side of the above equation simplifies to

$$A_1 = \int_{\Delta_2(0,t)} \mathbf{E}\left[e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_1}^{1,+}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_2}^{2,+}\big) \mid U_{s_i}^{i,-} = 0, i = 1,2\right] p_{2s_1}(0)p_{2s_2}(0)\,ds_1\,ds_2 \tag{4.74}$$

where again the conditional expectation is interpreted via concatenation of Brownian bridges and Brownian motions. Note that via Brownian motion decomposition, we have

$$\begin{aligned}&\mathbf{E}\left[e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\cdot\xi_\varepsilon^a(\tfrac{1}{2}U_{s_1}^{1,+})\xi_\varepsilon^a(\tfrac{1}{2}U_{s_2}^{2,+}) \mid U_{s_i}^{i,-} = x_i.i = 1,2\right]\\ &\qquad = \int_{\mathbb{R}^2} H_{s_1,s_2}(\vec{x},\vec{y})\prod_{i=1}^{2}\xi_\varepsilon^a(\tfrac{1}{2}y_i)p_{2s_i}(y_i)dy_i\end{aligned} \tag{4.75}$$

where H is defined in (4.73). Inserting the above formula back in (4.74) we get

$$A_1 = \int_{\Delta_2(0,t)}\int_{\mathbb{R}^2} H_{s_1,s_2}((0,0),(y_1,y_2))\prod_{i=1}^{2}\xi_\varepsilon^a(\tfrac{1}{2}y_i)p_{2s_i}(y_i)p_{2s_i}(0)\,dy_i\,ds_i. \tag{4.76}$$

A_2 and A_3 terms: Recall A_2 from (4.70). We condition only on $\mathcal{F}_t^{1,-} := \sigma\{U_s^{1,-} \mid 0 \le s \le t\}$ to get

$$A_2 := \mathbf{E}\left[\int_0^t F(U^{1,-})\frac{1}{2}dL_0^{U^{1,-}}(s_1)\right]$$

where

$$F(U^{1,-}) := \int_0^{s_1} \mathbf{E}\left[e^{\frac{\sigma}{2}\mathcal{L}_2(s_1,s_2)}\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_1}^{1,+}\big)\xi_\varepsilon^a\big(\tfrac{1}{2}U_{s_2}^{2,+}\big)\xi_{2\varepsilon}^0(U_{s_2}^{2,-}) \mid \mathcal{F}_t^{1,-}\right] ds_2$$

Applying Lemma 4.8.2 again, we see that

$$A_2 = \int_0^t \mathbb{E}[F(U^{1,-}) \mid U_{s_1}^{1,-} = 0]p_{2s_1}(0)ds_2.$$

where again the conditional expectation is interpreted in a similar manner. Performing a similar trick as in (4.75), we see that

$$A_2 = \int_{\Delta_2(0,t)} \int_{\mathbb{R}^3} H_{s_1,s_2}((0,x_2);\vec{y}) \left[\prod_{i=1}^2 \xi_\varepsilon^a\big(\tfrac{1}{2}y_i\big)p_{2s_i}(y_i)\right] \xi_{2\varepsilon}^0(x_2)p_{2s_2}(x_2)p_{2s_1}(0)d\vec{y}\,dx_2\,d\vec{s}. \tag{4.77}$$

Similarly for A_3 defined in (4.71) one has

$$A_3 = \int_{\Delta_2(0,t)} \int_{\mathbb{R}^3} H_{s_1,s_2}((x_1,0);\vec{y}) \left[\prod_{i=1}^2 \xi_\varepsilon^a\big(\tfrac{1}{2}y_i\big)p_{2s_i}(y_i)\right] \xi_{2\varepsilon}^0(x_1)p_{2s_1}(x_1)p_{2s_2}(0)d\vec{y}\,dx_1\,d\vec{s}. \tag{4.78}$$

A_4 term: Recall A_4 from (4.72). Note that A_4 does not involve any integration with respect to local times. Thus we may use H from (4.73) directly and tricks similar to (4.75) to get

$$A_4 := \int_{\Delta_2(0,t)} \int_{\mathbb{R}^4} H_{s_1,s_2}(\vec{x},\vec{y}) \prod_{i=1}^2 \xi_\varepsilon^a\big(\tfrac{1}{2}y_i\big)\xi_{2\varepsilon}^0(x_i)p_{2s_i}(x_i)p_{2s_i}(y_i)dx_i\,dy_i\,ds_i. \tag{4.79}$$

Step 3. In this step, we determine convergence and provide bounds for each of the A_i terms defined at the end of **Step 1**. We claim that for each i,

$$\lim_{\varepsilon\to 0} A_i(\varepsilon) = 4\int_{\Delta_2(0,t)} H_{s_1,s_2}((0,0),(2a,2a)) \prod_{i=1}^2 p_{2s_i}(0)p_{2s_i}(2a)ds_i, \tag{4.80}$$

where H is defined in (4.73). Inserting this limit back in (4.68) verifies (4.59). Let us consider $A_1(\varepsilon)$ from (4.69). Let us consider the integrand on the right-hand side of (4.76):

$$p_{2s_i}(0) \int_{\mathbb{R}^2} H_{s_1,s_2}((0,0),(y_1,y_2)) \prod_{i=1}^{2} \xi_\varepsilon^a(\tfrac{1}{2}y_i) p_{2s_i}(y_i)\, dy_i.$$

Clearly as $\varepsilon \to 0$, it converges to $4p_{2s_i}(0)H_{s_1,s_2}((0,0),(2a,2a))p_{2s_1}(2a)p_{2s_2}(2a)$. Thus, to show (4.80) holds for $A_1(\varepsilon)$, it suffices by the dominated convergence theorem to show that for all $a \neq 0$,

$$\sup_{\varepsilon\in(0,1]} p_{2s_i}(0) \int_{\mathbb{R}^2} H_{s_1,s_2}((0,0),(y_1,y_2)) \prod_{i=1}^{2} \xi_\varepsilon^a(\tfrac{1}{2}y_i) p_{2s_i}(y_i)\, dy_i \tag{4.81}$$

is dominated by some integrable function of s_1, s_2. Towards this end, note that H is defined as the expectation of exponentials of local times of certain linear functions of concatenated processes introduced at the beginning of **Step 2**. In Lemma 4.8.6, we study these expectations and in particular show that for all $s_1, s_2 \in [0,t]$ and for all $\vec{x}, \vec{y} \in \mathbb{R}^2$, $H_{s_1,s_2}(\vec{x},\vec{y}) \leq \mathrm{C}$ for some constant $\mathrm{C} > 0$ that depends only on t. As $\xi_\varepsilon^a(\frac{1}{2}y) = 2p_{2\varepsilon^2}(y-2a)$, using $H_{s_1,s_2}(\vec{x},\vec{y}) \leq \mathrm{C}$ and then using the semigroup property of the heat kernel (i.e., $\int_{\mathbb{R}} p_u(x-b)p_t(x)dx = p_{u+t}(b)$) we have from (4.76) that

$$(4.81) \leq \mathrm{C} \sup_{\varepsilon\in(0,1]} \prod_{i=1}^{2} p_{2s_i+2\varepsilon^2}(2a) p_{2s_i}(0). \tag{4.82}$$

Note that the heat kernel globally satisfies the bound $p_t(x) \leq \mathrm{C}(t+x^2)^{-1/2}\mathbb{1}_{\{t>0\}}$ for some large enough constant $\mathrm{C} > 0$. Consequently we have

$$\begin{aligned}\int_{\Delta_2(0,t)} \sup_{\varepsilon\in(0,1]} \prod_{i=1}^{2} p_{2s_i+2\varepsilon^2}(2a) p_{2s_i}(0) ds_i &\leq \mathrm{C} \int_{\Delta_2(0,t)} \prod_{i=1}^{2} s_i^{-1/2}(s_i+a^2)^{-1/2}\, ds_i \\ &= \mathrm{C} \int_{\Delta_2(0,a^{-2}t)} \prod_{i=1}^{2} v_i^{-1/2}(v_i+1)^{-1/2}\, dv_i.\end{aligned}$$

where the equality above follows by making the substitution $s_i = a^2 v_i$. Extending the range of

integration we have

$$\int_{\Delta_2(0,a^{-2}t)} \prod_{i=1}^{2} v_i^{-1/2}(v_i+1)^{-1/2}\,dv_i \le \int_{[0,a^{-2}t]^2} \prod_{i=1}^{2} v_i^{-1/2}(v_i+1)^{-1/2}\,dv_i \le \mathrm{C}\max\{1,|\log a|^2\}. \tag{4.83}$$

This verifies the integrability of the bound in (4.82) and consequently proves (4.80) for A_1. Starting from (4.77), (4.78), and (4.79), an analogous computation verifies (4.80) for A_2, A_3 and A_4 respectively. This establishes (4.59). From the polylogarithmic bound in (4.83) (and its analogous counterparts for A_2, A_3, A_4) one arrives at (4.60). This completes the proof. □

Proposition 4.5.5 (Error term limit)**.** *For any $t \ge 0$ and $\phi \in \mathcal{S}(\mathbb{R})$, we have that*

$$\limsup_{N\to\infty} \mathbb{E}[\mathcal{E}_t^N(\phi)^2] = 0. \tag{4.84}$$

Proof. Note that $\phi\phi' \le \frac{1}{2}(\phi^2 + (\phi')^2)$ thus

$$|\mathcal{E}_t^N(\phi)| \le 2N^{1/4}\left[\int_0^t ((\mathscr{X}_s^N)^{Sq}, \phi^2)ds + \int_0^t ((\mathscr{X}_s^N)^{Sq}, (\phi')^2)ds\right] = 2N^{-1/4}\left[Q_t^N(\phi^2) + Q_t^N(\phi'^2)\right].$$

Now by Lemma 4.5.2 (b) and Proposition 4.5.4 (a), we see that for every $\psi \in \mathcal{S}(\mathbb{R})$, L^2 moments of $Q_t^N(\psi)$ are uniformly bounded in N. Since ϕ^2 and ϕ'^2 are also Schwartz functions, in view of the above inequality, we readily have (4.84). □

4.5.2 Supporting estimates

In this section we prove Proposition 4.5.4 and establish an estimate for the moments of the increments of the quadratic martingale field which will be useful in Section 4.6.2.

Proof of Proposition 4.5.4. **Proof of** (4.62)**.** We continue with the same notation as in the state-

ment of the Proposition 4.5.4. Let us set

$$\mathcal{G}_N^r(t) = \sum_{j=2r-1}^{2k} \int_0^t \left[1 - \frac{1}{m_j^r(\mathbf{X}(u))}\right] dX^j(u)$$

where $m_j^r(\mathbf{x}) := \#\{i \in \{2r-1, \ldots, 2k\} : x_i = x_j\}$. Note that $\mathcal{G}_N^r$ are martingales of the form (4.16) where only the last $2k - 2r$ of the $2k$ particles are taken into account. Define the martingale

$$d\overline{\mathcal{G}}_N(t) := N^{1/4} \sum_{r=1}^{k} \mathbb{1}_{\{t \in [t_{r-1}, t_r)\}} d\mathcal{G}_N^r(t). \tag{4.85}$$

It turns out that the stochastic exponential of $\overline{\mathcal{G}}_N$ is precisely the tilt that gets rid of the divergent term appearing in the expectation of the left-hand side of (4.62). To be precise, we have that

$$\begin{aligned} &\mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\prod_{i=1}^{2k} e^{-\frac{1}{2} s_i \sqrt{N} + N^{\frac{1}{4}} X^i_{s_i}} \phi_i(X^i_{s_i} - N^{\frac{1}{4}} s_i) \right] \\ &= \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[e^{\overline{\mathcal{G}}_N(T) - \frac{1}{2}\langle \overline{\mathcal{G}}_N \rangle(T)} \exp\left(N^{\frac{1}{2}} \sum_{r=1}^{k} \sum_{2r-1 \le i < j \le 2k} \int_{t_{r-1}}^{t_r} \mathbb{1}_{\{X^i_u = X^j_u\}} du \right) \prod_{r=1}^{2k} \phi_r(X^r_{s_r}) \right]. \end{aligned} \tag{4.86}$$

Note that $N^{1/4}\mathcal{G}_N^r$ satisfies the assumption (4.33). Consequently, $\overline{\mathcal{G}}_N$ satisfies the assumption (4.33) as well. From Theorem 4.3.2 (b), it is immediate that, as $N \to \infty$, the right-hand side of (4.86) converges to the right-hand side of (4.62). This proves (4.62) modulo (4.86).

We now turn towards the proof of (4.86) which is done by iteratively applying (4.18). We illustrate the proof for the $k = 2$ case only; the general case follows in the exact same manner. Let us set $k = 2$ and write $\mathbf{E}$ for the expectation with respect to $SB^{(2k)}_{N^{1/2}\nu}$ measure. Recall that $s_{2i-1} = s_{2i} = t_i$ for $i = 1, 2$. Let $(\mathcal{F}_t)_{t \le T}$ denote the filtration of the $2k$-point sticky Brownian motion. Note that the expression inside the first expectation in (4.86) can be written as $\mathbf{A} \cdot \mathbf{B}$ where

$$\begin{aligned} \mathbf{A} &:= e^{-2t_1\sqrt{N} + N^{1/4} \sum_{j=1}^{4} X^j_{t_1}} \prod_{i=1}^{2} \phi_i(X^i_{s_i} - N^{\frac{1}{4}} s_i) \\ \mathbf{B} &:= e^{-(t_2 - t_1)\sqrt{N} + N^{1/4}(X^3_{t_2} + X^4_{t_2} - X^3_{t_1} - X^4_{t_1})} \cdot \prod_{i=3}^{4} \phi_i(X^i_{s_i} - N^{\frac{1}{4}} s_i). \end{aligned}$$

$\mathbf{A}$ is measurable with respect to $\mathcal{F}_{t_1}$. By the Markov property of sticky Brownian motion $\mathbf{E}[\mathbf{B} \mid \mathcal{F}_{t_1}]$ is measurable with respect to $\sigma\big(X^i_{t_1})_{i=1}^4\big)$. Let us write

$$H\big(\mathbf{X}_{t_1}\big) := \mathbf{E}[\mathbf{B} \mid \mathcal{F}_{t_1}].$$

By the tower property of conditional expectation followed by an application of (4.18) with $\lambda = N^{\frac{1}{4}}$, $s = 0$, $t = t_1$, $\nu \mapsto N^{\frac{1}{2}}\nu$, and $k = 4$ we get

$$\begin{aligned} \text{l.h.s. of (4.86)} &= \mathbf{E}\left[\mathbf{A} \cdot H\big(\mathbf{X}_{t_1}\big)\right] \\ &= \mathbf{E}\left[e^{N^{\frac{1}{4}}\mathcal{G}^1_N(t_1) - N^{\frac{1}{2}}\langle \mathcal{G}^1_N\rangle(t_1)} \exp\left(N^{\frac{1}{2}} \sum_{1\le i<j\le 4} \int_{t_1}^{t_2} \mathbb{1}_{\{X^i_u = X^j_u\}} du\right)\right. \\ &\qquad \left. \cdot \prod_{i=1}^{2} \phi_i(X^i_{s_i}) \cdot H\big((X^i_{t_1} + N^{\frac{1}{4}}t_1)_{i=1}^k\big)\right]. \end{aligned} \tag{4.87}$$

Observe that by the Markov property of the sticky Brownian motion, we have

$$H\big((X^i_{t_1} + N^{\frac{1}{4}}t_1)_{i=1}^k\big) = \mathbf{E}\left[e^{-(t_2-t_1)\sqrt{N} + N^{\frac{1}{4}}(X^3_{t_2} + X^4_{t_2} - X^3_{t_1} - X^4_{t_1})} \prod_{i=3}^{4} \phi_i(X^i_{s_i} - N^{\frac{1}{4}}(t_2 - t_1)) \,|\, \mathcal{F}_{t_1}\right]. \tag{4.88}$$

We use (4.18) with $\lambda = N^{\frac{1}{4}}$, $s = t_1$, $t = t_2$, $\nu \mapsto N^{\frac{1}{2}}\nu$, and $k = 2$ to get that

$$\text{r.h.s. of (4.88)} = \mathbf{E}\left[e^{N^{\frac{1}{4}}\mathcal{G}^2_N(t_1,t_2) - N^{\frac{1}{2}}\langle \mathcal{G}^2_N\rangle(t_1,t_2)} \cdot \prod_{i=3}^{4} \phi_i(X^i_{s_i}) \exp\left(N^{\frac{1}{2}} \int_{t_1}^{t_2} \mathbb{1}_{\{X^3_u = X^4_u\}} du\right) \mid \mathcal{F}_{t_1}\right],$$

where $g(s,t) := g(t) - g(s)$. Inserting the above expression back in (4.87) and again using the tower property of the conditional expectation, we derive the identity in (4.86).

Proof of (4.65)**.** Following the same argument as in the proof of (4.62), in the same spirit as (4.86),

one has that

$$\begin{aligned}
&\mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[E_1(\vec{t})E_2(\vec{t})\right] \\
&= \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[e^{\overline{\mathcal{G}}_N(T)-\frac{1}{2}\langle\overline{\mathcal{G}}_N\rangle(T)}\exp\left(N^{\frac{1}{2}}\sum_{r=1}^{k}\sum_{2r-1\le i<j\le 2k}\int_{s_{r-1}}^{s_r}\mathbb{1}_{\{X^i_u=X^j_u\}}du\right)\right. \\
&\qquad\left.\cdot\prod_{i\in A}\phi_i(X^{2i-1}_{t_i})\phi_i(X^{2i}_{t_i})\prod_{i\in B}\psi_i(X^{2i}_{t_i})\mathbb{1}_{\{X^{2i-1}_{t_i}=X^{2i}_{t_i}\}}\right].
\end{aligned} \tag{4.89}$$

Set

$$\mathbf{A}_N(\vec{t}) := e^{\overline{\mathcal{G}}_N(T)-\frac{1}{2}\langle\overline{\mathcal{G}}_N\rangle(T)}, \quad \mathbf{B}_N(\vec{t}) := \exp\left(N^{\frac{1}{2}}\sum_{r=1}^{k}\sum_{2r-1\le i<j\le 2k}\int_{t_{r-1}}^{t_r}\mathbb{1}_{\{X^i_u=X^j_u\}}du\right). \tag{4.90}$$

Recall γ_N from (4.37). By Theorem 4.3.2 (b) and (c), any limit point of the sequence

$$\left(\overline{\mathcal{G}}_N, \mathbf{X}, 4N^{1/2}\nu([0,1])\big(V^{ij}\big)_{1\le i<j\le 2k}, N^{|B|/2}\prod_{i\in B}\mathbb{1}_{\{X^{2i-1}_{t_i}=X^{2i}_{t_i}\}}dt_i\right)$$

is of the form

$$\left(\overline{\mathcal{G}}, \mathbf{U}, \big(L_0^{U_i-U_j}\big)_{1\le i<j\le k}, \left(\frac{\sigma}{2}\right)^{|B|}\prod_{i\in B}dL_0^{U^{2i-1}-U^{2i}}(u_i)\right)$$

for some $\overline{\mathcal{G}}$ with $\mathbf{E}[e^{p\overline{\mathcal{G}}(T)-\frac{p}{2}\langle\overline{\mathcal{G}}\rangle(T)}] < \infty$ for all $p > 0$ and satisfying (4.34). Here U is standard $2k$-dimensional Brownian motion. By the continuous mapping theorem, it follows that along a subsequence we have

$$\begin{aligned}
&N^{\frac{|B|}{2}}\int_{\Delta_k(s,t)}D_N(\vec{t})\cdot\prod_{i\in B}\mathbb{1}_{\{X^{2i-1}_{t_i}=X^{2i}_{t_i}\}}\prod_{i=1}^{k}dt_i \\
&\xrightarrow{d}\left(\frac{\sigma}{2}\right)^{|B|}\int_{\Delta_k(s,t)}e^{\overline{\mathcal{G}}(T)-\frac{1}{2}\langle\overline{\mathcal{G}}\rangle(T)+\frac{\sigma}{2}\mathcal{L}_k(\vec{t})}\prod_{i\in A}\phi_i(U^{2i-1}_{t_i})\phi_i(U^{2i}_{t_i})dt_i\prod_{i\in B}\psi_i(U^{2i}_{t_i})dL_0^{U^{2i-1}-U^{2i}}(t_i),
\end{aligned} \tag{4.91}$$

where

$$D_N(\vec{t}) := \mathbf{A}_N(\vec{t}) \cdot \mathbf{B}_N(\vec{t}) \prod_{i \in A} \phi_i(X^{2i-1}_{t_i}) \phi_i(X^{2i}_{t_i}) \prod_{i \in B} \psi_i(X^{2i}_{t_i}). \tag{4.92}$$

We now upgrade (4.91) to imply convergence of first moments. To do this, we shall show the sequence of random variables in the left-hand side of (4.91) is uniformly integrable. Repeatedly using Proposition 4.2.6 and Doob's martingale inequality to the stochastic exponentials of each of the martingales $\mathcal{G}^r_N$ appearing in the expression for $\overline{\mathcal{G}}_N$ we find that for all $p \geq 1$, we have

$$\sup_{N \geq 1} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\sup_{\vec{t} \in \Delta_k(0,T)} \left(\mathbf{A}_N(\vec{t}) \right)^p \right] < \infty, \quad \sup_{N \geq 1} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\sup_{\vec{t} \in \Delta_k(0,T)} \left(\mathbf{B}_N(\vec{t}) \right)^p \right] < \infty, \tag{4.93}$$

and thus via the definition of $D_N(\vec{t})$ from (4.92) we get

$$\sup_{N \geq 1} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\sup_{\vec{t} \in \Delta_k(0,T)} \left(D_N(\vec{t}) \right)^p \right] < \infty.$$

In view of the above bound and (4.44), we obtain uniform integrability for the sequence of random variables in (4.91). This implies that along the same subsequence,

$$\begin{aligned}
&\lim_{N \to \infty} N^{\frac{|B|}{2}} \mathbf{E} \left[\int_{\Delta_k(s,t)} D_N(\vec{t}) \cdot \prod_{i \in B} \mathbb{1}_{\{X^{2i-1}_{t_i} = X^{2i}_{t_i}\}} \prod_{i=1}^{k} dt_i \right] \\
&\quad = \left(\frac{\sigma}{2} \right)^{|B|} \mathbf{E} \left[\int_{\Delta_k(s,t)} e^{\overline{\mathcal{G}}(T) - \frac{1}{2}\langle \overline{\mathcal{G}} \rangle(T) + \frac{\sigma}{2} \mathcal{L}_k(\vec{t})} \prod_{i \in A} \phi_i(U^{2i-1}_{t_i}) \phi_i(U^{2i}_{t_i}) dt_i \prod_{i \in B} \psi_i(U^{2i}_{t_i}) dL_0^{U^{2i-1} - U^{2i}}(t_i) \right] \\
&\quad = \left(\frac{\sigma}{2} \right)^{|B|} \mathbf{E} \left[\int_{\Delta_k(s,t)} e^{\frac{\sigma}{2} \mathcal{L}_k(\vec{t})} \mathbf{E} \left[e^{\overline{\mathcal{G}}(T) - \frac{1}{2}\langle \overline{\mathcal{G}} \rangle(T)} \mid \mathcal{F}_T(\mathbf{U}) \right] \right. \\
&\qquad\qquad \left. \cdot \prod_{i \in A} \phi_i(U^{2i-1}_{t_i}) \phi_i(U^{2i}_{t_i}) dt_i \prod_{i \in B} \psi_i(U^{2i}_{t_i}) dL_0^{U^{2i-1} - U^{2i}}(t_i) \right],
\end{aligned}$$

where the last equality follows from the tower property of the conditional expectation. By (4.34), the above inner expectation is 1. Since the limit is free of $\overline{\mathcal{G}}$, we have the same above limit along every subsequence. Thus we arrive at the (4.65) formula. This completes the proof. □

We end this section by recording a couple of useful estimates for moments of the increments

of the quadratic martingale field.

Proposition 4.5.6 (Estimates for moments of the increments of QMF)**.** *Fix $k \in \mathbb{N}$ and $T > 0$. Then there exists a constant $\mathrm{C} = \mathrm{C}(k,T) > 0$ such that for all bounded measurable functions ϕ on $\mathbb{R}$ and all $0 \leq s < t \leq T$ one has that*

$$\sup_{N\geq 1} \mathbb{E}\big[(Q_t^N(\phi) - Q_s^N(\phi))^k\big] \leq \mathrm{C}\|\phi\|_{L^\infty(\mathbb{R})}^k (t-s)^{k/4}. \tag{4.94}$$

Furthermore fix $p > 1$ and $\varepsilon > 0$. Then there exists $\mathrm{C} = \mathrm{C}(p,\varepsilon,k,T) > 0$ such that for all functions $\phi \in L^p(\mathbb{R})$ and all $\varepsilon \leq s < t \leq T$ one has

$$\lim_{N\to\infty} \mathbb{E}\big[(Q_t^N(\phi) - Q_s^N(\phi))^k\big] \leq \mathrm{C}\|\phi\|_{L^p(\mathbb{R})}^k (t-s)^{k}. \tag{4.95}$$

We remark that the first bound is crude and nowhere near optimality. However, the latter bound is more important, and it will be most powerful when p is very close to 1, as this will allow us to obtain optimal tightness bounds for limit points in the next section.

Proof. **Proof of** (4.94)**.** Using (4.56) together with the trivial bound $|\phi(N^{-1/2}X_{t_j}^j - N^{1/4}t_j)| \leq \|\phi\|_{L^\infty}$, we obtain that

$$\begin{aligned}
&\mathbb{E}[\big(Q_t^N(\phi) - Q_s^N(\phi)\big)^k] \\
&\leq N^{k/2} k! \|\phi\|_{L^\infty}^k \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[\prod_{j=1}^k e^{-t_j\sqrt{N}+N^{1/4}(X_{t_j}^{2j-1}+X_{t_j}^{2j})} \mathbb{1}_{\{X_{t_j}^{2j-1}=X_{t_j}^{2j}\}}\right] dt_1\cdots dt_k \\
&= k!\|\phi\|_{L^\infty}^k \cdot N^{k/2} \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[\mathbf{A}_N(\vec{t})\cdot\mathbf{B}_N(\vec{t})\cdot\prod_{j=1}^k \mathbb{1}_{\{X_{t_j}^{2j-1}=X_{t_j}^{2j}\}}\right] dt_1\cdots dt_k,
\end{aligned} \tag{4.96}$$

where in above we use the same notation from (4.90):

$$\mathbf{A}_N(\vec{t}) := e^{\overline{\mathcal{G}}_N - \frac{1}{2}\langle\overline{\mathcal{G}}_N\rangle}, \quad \mathbf{B}_N(\vec{t}) := \exp\left(N^{\frac{1}{2}}\sum_{r=1}^k \sum_{2r-1\leq i<j\leq 2k} \int_{t_{r-1}}^{t_r} \mathbb{1}_{\{X_u^i = X_u^j\}} du\right),$$

and where $\overline{\mathcal{G}}_N = \overline{\mathcal{G}}_N(t)$ defined in (4.85). The equality in (4.96) is due to (4.89). Note that by

Cauchy-Schwarz inequality and using the fact that $\mathbf{A}^2 \cdot \mathbf{B}^2 \leq \mathbf{A}^4 + \mathbf{B}^4$, we have

$$(4.96) \leq k! \|\phi\|_{L^\infty}^k \sqrt{N^{\frac{k}{2}} \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[(\mathbf{A}_N(\vec{t})^4 + \mathbf{B}_N(\vec{t})^4) \cdot \prod_{j=1}^k \mathbb{1}_{\{X^j_{t_j} = Y^j_{t_j}\}} \right] dt_1 \cdots dt_k} \tag{4.97}$$

$$\cdot \sqrt{N^{\frac{k}{2}} \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\prod_{j=1}^k \mathbb{1}_{\{X^j_{t_j} = Y^j_{t_j}\}} \right] dt_1 \cdots dt_k} \tag{4.98}$$

For the factor in the square root of (4.98), note that

$$\begin{aligned} N^{\frac{k}{2}} \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\prod_{j=1}^k \mathbb{1}_{\{X^j_{t_j} = Y^j_{t_j}\}} \right] dt_1 \cdots dt_k &\leq \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\prod_{j=1}^k N^{\frac{1}{2}} \int_s^t \mathbb{1}_{\{X^j_t = Y^j_t\}} dt \right] \\ &\leq \prod_{j=1}^k \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\left(N^{\frac{1}{2}} \int_s^t \mathbb{1}_{\{X^j_t = Y^j_t\}} dt \right)^k \right]^{\frac{1}{k}} \\ &\leq \mathrm{C}|t-s|^{k/2}. \end{aligned} \tag{4.99}$$

where the last inequality follows by applying Lemma 4.2.5 on each of the expectations. Here the constant $\mathrm{C} > 0$ depends only on k. We now focus on the factor in the square root of (4.97). We shall show that this factor can be bounded uniformly in $N \geq 1$ and $s, t \in [0, T]$. We split it into two parts: one containing $\mathbf{A}(\vec{t})$ and $\mathbf{B}(\vec{t})$. Recall the γ_N measure from (4.37). Note that

$$\begin{aligned} &\sup_{N \geq 1} N^{k/2} \int_{\Delta_k(s,t)} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\mathbf{A}_N(\vec{t})^4 \prod_{j=1}^k \mathbb{1}_{\{X^j_{t_j} = Y^j_{t_j}\}} \right] dt_1 \cdots dt_k \\ &\quad \leq \sup_{N \geq 1} \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\sup_{\vec{t} \in \Delta_k(0,T)} \mathbf{A}_N(\vec{t})^4 \int_{\Delta_k(s,t)} d\gamma_N \right] \\ &\quad \leq \sup_{N \geq 1} \sqrt{\mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\sup_{\vec{t} \in \Delta_k(0,T)} \mathbf{A}_N(\vec{t})^8 \right] \cdot \mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}} \left[\gamma_N(\Delta_k(0,T))^2 \right]}. \end{aligned}$$

where the last inequality is due to Cauchy-Schwarz inequality. By (4.44) and by (4.93), we find that the above expression is finite. On the other hand, for the $\mathbf{B}(\vec{t})$ term it is clear from Theorem

4.3.2(a) and (c) that

$$N^{k/2}\int_{\Delta_k(0,T)}\mathbf{E}_{SB^{(2k)}_{N^{1/2}\nu}}\left[\mathbf{B}_N(\vec{t})^4\prod_{j=1}^{k}\mathbb{1}_{\{X^j_{t_j}=Y^j_{t_j}\}}\right]dt_1\cdots dt_k$$

converges to some finite quantity as $N\to\infty$, which implies that it is bounded independently of N. Thus the last two math displays imply that the term in (4.97) is bounded uniformly in $N\geq 1$ and $s,t\in[0,T]$. From (4.99), we see that (4.98) $\leq \mathrm{C}|t-s|^{k/4}$. Inserting these two bounds back in (4.97) and (4.98), we arrive at (4.94).

Proof of (4.95)**.** Appealing to the moment formula for the increment of QMF from (4.56) and the convergence from (4.65) we have

$$\lim_{N\to\infty}\mathbb{E}\left[\left(Q^N_t(\phi)-Q^N_s(\phi)\right)^k\right]=k!\left(\frac{\sigma}{2}\right)^k\mathbf{E}_{B^{\otimes 2k}}\left[\int_{\Delta_k(s,t)}e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})}\prod_{j=1}^{k}\phi(U^{2j}_{t_j})dL_0^{U^{2j-1}-U^{2j}}(t_j)\right], \tag{4.100}$$

where $\mathcal{L}_k(\vec{t})$ is defined in (4.61). Lemma 4.8.2 formalizes the intuition that $\frac{1}{2}dL_0^{U^{2j-1}-U^{2j}}(t_j)=\delta_0(U^{2j-1}_{t_j}-U^{2j}_{t_j})$. Thus by appealing to that lemma, we can write the above expectation in terms of a certain family of concatenated bridge processes whose law we will write as $\mathbf{P}_c$. Specifically, we have that

$$\text{r.h.s. of (4.100)}=k!\sigma^k\int_{\Delta_k(s,t)}\mathbf{E}_c\left[e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})}\prod_{j=1}^{k}\phi(U^{2j}_{t_j})\right]\prod_{j=1}^{k}p_{2t_j}(0)dt_1\cdots dt_k, \tag{4.101}$$

where the expectation is taken over a collection of paths U^j such that

- $(U^{2i-1}-U^{2i},U^{2i-1}+U^{2i})_{i=1}^k$ are $2k$ many independent processes.
- $U^{2i-1}+U^{2i}$ is a Brownian motion of diffusion rate 2 for $i=1,2,\ldots,k$.
- $U^{2i-1}-U^{2i}$ is a Brownian bridge (from 0 to 0) of diffusion rate 2 from $[0,t_i]$ and an independent Brownian motion of diffusion rate 2 from $[t_i,\infty)$ for $i=1,2,\ldots,k$.

We have used a different notation for the expectation operator $\mathbf{E}_c$ in (4.101) (c for concatenation) just to stress that the law is different from the standard Brownian motion. We claim that for all $p > 1$ we have

$$\sup_{\vec{t}\in[0,T]} \mathbf{E}_c\left[e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})} \prod_{j=1}^{k} |\phi(U_{t_j}^{2j})| \right] \leq \mathrm{C} \cdot \|\phi\|_{L^p}^k. \tag{4.102}$$

where the $\mathrm{C} > 0$ depends on p, k, ε, T. Let us assume (4.102) for the moment. By hypothesis, $t_j \geq \varepsilon > 0$. Thus, $p_{2t_j}(0) \leq \mathrm{C}$ for some constant $\mathrm{C} > 0$ depending on ε. Thus, in view of (4.102), to get an upper bound for the right-hand side of (4.101), we may take the supremum of the integrand in the right-hand side of (4.101) and pull it outside of the integration. As the Lebesgue measure of $\Delta_k(s,t)$ is $\frac{(t-s)^k}{k!}$, we thus have the desired estimate in (4.95).

Let us now establish (4.102). Fix $p > 1$ and take $q > 1$ so that $p^{-1} + q^{-1} = 1$. Use Hölder's inequality to write

$$\mathbf{E}_c\left[e^{\frac{\sigma}{2}\mathcal{L}_k(\vec{t})} \prod_{j=1}^{k} |\phi(U_{t_j}^{2j})| \right] \leq \mathbf{E}_c\left[e^{q\frac{\sigma}{2}\mathcal{L}_k(\vec{t})} \right]^{1/q} \mathbf{E}_c\left[\prod_{j=1}^{k} |\phi(U_{t_j}^{2j})|^p \right]^{1/p}.$$

For the first expectation above, observe that by Lemma 4.8.6, $\mathbf{E}_c\left[e^{q\frac{\sigma}{2}\mathcal{L}_k(\vec{t})} \right]$ is uniformly bounded over $\vec{t} \in \Delta_k(0,T)$. For the second expectation above, note that under $\mathbf{E}_c$, we have

$$U_{t_j}^{2j} = \tfrac{1}{2}(U_{t_j}^{2j-1} + U_{t_j}^{2j}) - \tfrac{1}{2}(U_{t_j}^{2j-1} - U_{t_j}^{2j}) = \tfrac{1}{2}(U_{t_j}^{2j-1} + U_{t_j}^{2j}) - 0.$$

Thus under $\mathbf{E}_c$, $U_{t_j}^{2j}$ are independent Gaussian random variables with variance $t_j/2$. Hence,

$$\mathbf{E}_c\left[\prod_{j=1}^{k} |\phi(U_{t_j}^{2j})|^p \right]^{1/p} = \prod_{j=1}^{k} \mathbf{E}_c\left[|\phi(U_{t_j}^{2j})|^p \right]^{1/p} = \prod_{j=1}^{k} \left(\int_{\mathbb{R}} p_{t_j/2}(y) |\phi(y)|^p \, dy \right)^{1/p} \leq \mathrm{C} \cdot \|\phi\|_{L^p}^k.$$

where the last inequality follows by again using the fact that $t_j \geq \varepsilon > 0$, together with the uniform bound $\sup_y p_t(y) \leq (2\pi t)^{-1/2}$, and noting that the constant C is allowed to depend on ε. This

establishes (4.102) completing the proof of (4.95). □

4.6 Solving the martingale problem for the SHE

The goal of this section is to establish the tightness of the field $\mathscr{X}^N$ defined in (4.4) and identify its limit points as the solution of the stochastic heat equation (SHE). As the field $\mathscr{X}^N$ does not exist in a functional sense, we first introduce in Section 4.6.1 appropriate spaces and topologies that work well with the bounds derived in the previous section. In Section 4.6.2 and Section 4.6.3 we deal with the tightness and identification of the limit points of $\mathscr{X}^N$ respectively.

4.6.1 Weighted Hölder spaces and Schauder estimates

In this subsection, we introduce various natural topologies for our field $\mathscr{X}^N$ and its limit points. We then discuss how the heat flow affects these topologies and record Kolmogorov-type lemmas that will be key in showing tightness under these topologies. We begin by recalling many familiar and useful spaces of continuous and differentiable functions that have natural metric structures. For $d \geq 1$, we denote by $C_c^\infty(\mathbb{R}^d)$ the space of all compactly supported smooth functions on $\mathbb{R}^d$. For a smooth function on $\mathbb{R}^d$, we define its C^r norm as

$$\|f\|_{C^r} := \sum_{\vec{\alpha}} \sup_{\mathbf{x}\in\mathbb{R}^d} |D^{\vec{\alpha}} f(\mathbf{x})| \tag{4.103}$$

where the sum is over all $\vec{\alpha} \in \mathbb{Z}^d_{\geq 0}$ with $\sum \alpha_i \leq r$ and $D^{\vec{\alpha}} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ denotes the mixed partial derivative.

We now recall the definition of weighted Hölder spaces from [227, Definitions 2.2 and 2.3]. For the remainder of this paper, we shall work with *elliptic and parabolic* weighted Hölder spaces with polynomial weight function

$$w(x) := (1 + x^2)^\tau \tag{4.104}$$

for some fixed $\tau > 1$. We introduce these weights because weighted spaces will be more convenient to obtain tightness estimates. Since the solution of (SHE) started from Dirac initial condition is known to be globally bounded away from $(0,0)$, we expect that it is possible to remove the weights throughout this section, but this would require more precise moment estimates than the ones we derived in previous sections, which take into account spatial decay of the fields.

Definition 4.6.1 (Elliptic Hölder spaces)**.** *For $\alpha \in (0,1)$ we define the space $C^{\alpha,\tau}(\mathbb{R})$ to be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm given by*

$$\|f\|_{C^{\alpha,\tau}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{w(x)} + \sup_{|x-y|\leq 1} \frac{|f(x)-f(y)|}{w(x)|x-y|^\alpha}.$$

For $\alpha < 0$ we let $r = -\lfloor \alpha \rfloor$ and we define $C^{\alpha,\tau}(\mathbb{R})$ to be the closure of $C_c^\infty(\mathbb{R})$ with respect to the norm given by

$$\|f\|_{C^{\alpha,\tau}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \sup_{\lambda\in(0,1]} \sup_{\phi\in B_r} \frac{(f, S_x^\lambda \phi)_{L^2(\mathbb{R})}}{w(x)\lambda^\alpha}$$

where the scaling operators S_x^λ are defined by

$$S_x^\lambda \phi(y) = \lambda^{-1}\phi(\lambda^{-1}(x-y)), \tag{4.105}$$

and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}$.

Definition 4.6.2 (Function spaces)**.** *Let $C^{\alpha,\tau}(\mathbb{R})$ be as in Definition 4.6.1. We define $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ to be the space of continuous maps $g : [0,T] \to C^{\alpha,\tau}(\mathbb{R})$, equipped with the norm*

$$\|g\|_{C([0,T],C^{\alpha,\tau}(\mathbb{R}))} := \sup_{t\in[0,T]} \|g(t)\|_{C^{\alpha,\tau}(\mathbb{R})}.$$

Here and henceforth we will define $\Lambda_{[a,b]} := [a,b] \times \mathbb{R}$ and we will define $\Lambda_T := \Lambda_{[0,T]}$.

So far we have used ϕ, ψ for test functions on $\mathbb{R}$. To make the distinction clear between test functions on $\mathbb{R}$ and $\mathbb{R}^2$, we shall use variant Greek letters such as $\varphi, \vartheta, \varrho$ for test functions on $\mathbb{R}^2$.

In some instances, we will explicitly write $(f, \varphi)_{\mathbb{R}^2}$ or $(g, \phi)_{\mathbb{R}}$ when we want to be clear about the space in which we are pairing.

Definition 4.6.3 (Parabolic Hölder spaces)**.** *We define $C_c^\infty(\Lambda_T)$ to be the set of functions on Λ_T that are restrictions to Λ_T of some function in $C_c^\infty(\mathbb{R}^2)$ (in particular we do not impose that elements of $C_c^\infty(\Lambda_T)$ vanish at the boundaries of Λ_T).*

For $\alpha \in (0,1)$ we define the space $C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)$ to be the completion of $C_c^\infty(\Lambda_T)$ with respect to the norm given by

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)} := \sup_{(t,x)\in\Lambda_T} \frac{|f(t,x)|}{w(x)} + \sup_{|s-t|^{1/2}+|x-y|\leq 1} \frac{|f(t,x)-f(s,y)|}{w(x)(|t-s|^{1/2}+|x-y|)^\alpha}.$$

For $\alpha < 0$ we let $r = -\lfloor \alpha \rfloor$ and we define $C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)$ to be the closure of $C_c^\infty(\Lambda_T)$ with respect to the norm given by

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)} := \sup_{(t,x)\in\Lambda_T} \sup_{\lambda\in(0,1]} \sup_{\varphi\in B_r} \frac{(f, S^\lambda_{(t,x)}\varphi)_{L^2(\Lambda_T)}}{w(x)\lambda^\alpha}$$

where the scaling operators are defined by

$$S^\lambda_{(t,x)}\varphi(s,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(x-y)), \tag{4.106}$$

and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}^2$.

Remark 4.6.4 (Derivatives of distributions)**.** *By definition, any element $f \in C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)$ admits an L^2-pairing with any smooth function $\varphi : \Lambda_T \to \mathbb{R}$ of rapid decay, which we can write as $(f, \varphi)_{\Lambda_T}$. Consequently there is a natural embedding $C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ which is defined by formally setting f to be zero outside of $[0,T]\times\mathbb{R}$. More rigorously, this means that the $L^2(\mathbb{R}^2)$-pairing of f with any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ is defined to be equal to $(f, \varphi|_{\Lambda_T})_{\Lambda_T}$.*

The image of this embedding consists of some specific collection of tempered distributions that are necessarily supported on $[0,T]\times\mathbb{R}$. Consequently we can sensibly define $\partial_t f$ and $\partial_x f$ as elements of $\mathcal{S}'(\mathbb{R}^2)$ whenever $f \in C_{\mathfrak{s}}^{\alpha,\mathfrak{r}}(\Lambda_T)$. Explicitly these derivatives are defined by the

formulas

$$(\partial_t f, \varphi)_{\Lambda_T} := -(f, \partial_t \varphi)_{\Lambda_T}, \qquad (\partial_x f, \varphi)_{\Lambda_T} := -(f, \partial_x \varphi)_{\Lambda_T}$$

for all smooth $\varphi : \Lambda_T \to \mathbb{R}$ of rapid decay. This convention on derivatives will be useful for certain computations later. From the definitions, when $\alpha < 0$ one can check that for such f one necessarily has $\partial_t f \in C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$ and $\partial_x f \in C_{\mathfrak{s}}^{\alpha-1,\tau}(\Lambda_T)$.

We remark that this fails for $\alpha > 0$. Indeed, by our convention of derivatives, $\partial_t f$ may no longer be a smooth function (or even a function) even if $f \in C_c^\infty(\Lambda_T)$. This is because such an f gets extended to all of $\mathbb{R}^2$ by setting it to be zero outside Λ_T. In particular, if f does not vanish on the boundary of Λ_T, then it may become a discontinuous function under our convention of extension to $\mathbb{R}^2$. Due to these discontinuities, the distributional derivative $\partial_t f$ can be a tempered distribution with singular parts along the boundaries (one may verify that $\partial_t f$ can be at best an element of $C_{\mathfrak{s}}^{-2,\tau}(\Lambda_T)$ for generic $f \in C_c^\infty(\Lambda_T)$). In our later computations, we never take derivatives of functions in $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ with $\alpha > 0$.

We now discuss the smoothing effect of heat flow on these weighted Hölder spaces.

Proposition 4.6.5 (Smoothing effect on elliptic spaces)**.** *For $f \in C_c^\infty(\mathbb{R})$ and $t > 0$ define*

$$P_t f(x) := \int_{\mathbb{R}} p_t(x - y) f(y) dy.$$

Then for all $\alpha \leq \beta < 1$, there exists $\mathrm{C} = \mathrm{C}(\alpha, \beta, T) > 0$ such that

$$\|P_t f\|_{C^{\beta,\tau}(\mathbb{R})} \leq \mathrm{C} t^{-(\beta-\alpha)/2} \|f\|_{C^{\alpha,\tau}(\mathbb{R})}$$

uniformly over $f \in C_c^\infty$ and $t \in [0, T]$. In particular, P_t extends to a globally defined linear operator on $C^{\alpha,\tau}(\mathbb{R})$ which maps boundedly into $C^{\beta,\tau}(\mathbb{R})$.

A proof may be found in [227, Lemma 2.8] in the case of an exponential weight. The proof for polynomial weights is identical.

For the parabolic Hölder spaces, the following lemma states that the heat flow improves the regularity by a factor of 2 and provides a Schauder-type estimate.

Proposition 4.6.6 (Schauder estimate)**.** *For $f \in C_c^\infty(\Lambda_T)$ let us define*

$$Kf(t,x) := \int_{\Lambda_T} p_{t-s}(x-y) f(s,y) ds dy, \tag{4.107}$$

where p_t is the standard heat kernel for $t > 0, x \in \mathbb{R}$ and $p_t(x) := 0$ for $t < 0$. Then for all $\alpha < -1$ with $\alpha \notin \mathbb{Z}$ there exists $\mathrm{C} = \mathrm{C}(\alpha) > 0$ independent of f such that

$$\|Kf\|_{C_{\mathfrak{s}}^{\alpha+2,\tau}(\Lambda_T)} \leq \mathrm{C} \cdot \|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)}.$$

In particular K extends to a globally defined linear operator on $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ that maps boundedly into $C_{\mathfrak{s}}^{\alpha+2,\tau}(\Lambda_T)$. Furthermore, if $f \in C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$, then $K(\partial_t - \frac{1}{2}\partial_x^2) f = f$.

We remark that the last statement is only true because of our convention on distributional derivatives that we have explained in Remark 4.6.4. Without that convention, that statement would be false even for a smooth function f that does not vanish on the line $\{t = 0\}$.

Proof. Let φ be any smooth function that equals 1 in a ball of radius 1 about the origin in $\mathbb{R}^2$ and is supported on a ball of radius 2 about the origin in $\mathbb{R}^2$. For $(t,x) \in \mathbb{R}^2$ let us define $P^+(t,x) := p_t(x)\varphi(t,x)$ and $P^-(t,x) := p_t(x) - P^+(t,x)$ so that

$$p_t(x) = P^+(t,x) + P^-(t,x).$$

We have that P^- is a globally smooth function on $\mathbb{R}^2$, and without any loss of generality, we may assume that it is supported on Λ_{T+1} since values of the heat kernel outside Λ_T do not matter when convolving with distributions supported on Λ_T. Then we get a corresponding decomposition $K = K^+ + K^-$ of the operator defined in the proposition statement.

Since P^+ is supported on a ball of radius 2, the proof of the weighted Schauder estimate for K^+ is sketched in [227, Page 8, proof of Corollary 1.2], and we do not repeat it here. That proof

is given in an elliptic setting, but the proof of the parabolic version is no different because the singularity of the 1+1 dimensional heat satisfies a bound of the form

$$|\partial_t^{k_1}\partial_x^{k_2} p_t(x)| \leq \mathrm{C}(|t|^{1/2} + |x|)^{-1-2k_1-k_2},$$

where C may depend on $k_1, k_2 \geq 0$ but not on $t, x \in \mathbb{R}$.

Thus it only remains to prove the weighted Schauder estimate for K^-. As P^- is globally smooth on Λ_T and has a Gaussian decay at infinity, we may write

$$P^-(t,x) = \sum_{n=1}^{\infty} \psi_n(t,x)$$

where ψ_n can be taken to be smooth functions supported on $[0, T+1] \times \big([n-1, n+1] \cup [-n-1, 1-n]\big)$ and $\|\psi_n\|_{L^\infty} \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}n^2}$, for some constant $\mathrm{C} > 0$ independent of n. Consequently, if we write $K^- = \sum_n K_n^-$ where K_n^- corresponds to convolution with ψ_n, then the infinite sum of the operator norms $\sum_n \|K_n\|_{C_{\mathfrak{s}}^{\alpha,\tau} \to C_{\mathfrak{s}}^{\alpha+2,\tau}}$ may be shown to be finite using the fact that Gaussian decay of the ψ_n dominates the growth of the polynomial weights.

Finally, it remains to prove $K(\partial_t - \frac{1}{2}\partial_x^2) f = f$ for $f \in C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ with $\alpha < 0$. To prove this, we remark that Kf may actually be defined for *any* $f \in \mathcal{S}'(\mathbb{R}^2)$ with support contained in $[0,T] \times \mathbb{R}$, via the formula

$$(Kf, \varphi) := (f, K^\dagger * \varphi),$$

where $K^\dagger(t,x) = \frac{1}{\sqrt{2\pi|t|}} e^{-x^2/2|t|} \mathbb{1}_{\{t \leq 0\}}$ and $*$ denotes convolution in both space and time. The right side is well-defined because f is supported on $[0,T] \times \mathbb{R}$, and because one can ensure that if $\varphi \in \mathcal{S}(\mathbb{R}^2)$ then $K^\dagger * \varphi$ agrees on that strip with an element of $\mathcal{S}(\mathbb{R}^2)$ thanks to the Gaussian decay properties of $K^\dagger$ in the spatial variable (this is our reason for working on Λ_T rather than all of $[0,\infty) \times \mathbb{R}$).

Then Kf as defined above will be an element of $\mathcal{S}'(\mathbb{R}^2)$ that is supported on $[0,\infty) \times \mathbb{R}$, and this definition of Kf agrees with the one in the proposition statement because it agrees on smooth test

functions using integration by parts. Now we claim (more generally than stated in the proposition) that $K(\partial_t - \frac{1}{2}\partial_x^2)f = f$ for *any* $f \in \mathcal{S}'(\mathbb{R}^2)$ that is supported on $[0,T] \times \mathbb{R}$. But this just amounts to showing that $((\partial_t - \frac{1}{2}\partial_x^2)f, K^\dagger * \varphi) = (f, \varphi)$ for all such f and all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. By definition of $(\partial_t - \frac{1}{2}\partial_x^2)f$, this is equivalent to showing $(f, (-\partial_t - \frac{1}{2}\partial_x^2)K^\dagger * \varphi) = (f, \varphi)$ for all such f and all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. As f is supported on $[0,T] \times \mathbb{R}$, this reduces to showing that $(-\partial_t - \frac{1}{2}\partial_x^2)K^\dagger * \varphi = \varphi$ for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ that are supported on $[-1, T+1] \times \mathbb{R}$ (for instance), which is clear. □

Corollary 4.6.7. *Define* $J : C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ *by*

$$Jf(t,x) := \big(f, p_t(x - \cdot)\big)_{\mathbb{R}},$$

J *is a bounded linear operator for any* $\alpha < 1$ *and any* $\tau > 0$. *Consider the operator* $\widehat{P}_t := JP_t$ *with* P_t *defined in Proposition 4.6.5. By the semigroup property of the heat kernel, we have that*

$$\widehat{P}_s f(t,x) = (Jf)(t+s, x). \tag{4.108}$$

Furthermore, the operators satisfy

$$\|\widehat{P}_t f\|_{C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)} \leq \mathrm{C} \cdot t^{-(\beta-\alpha)/2} \|f\|_{C^{\alpha,\tau}(\mathbb{R})}, \tag{4.109}$$

where $\mathrm{C} = \mathrm{C}(\alpha, \beta, T) > 0$ *is independent of* t *and* f.

Proof. The first part follows by using the Schauder estimate in Proposition 4.6.6, noting that $Jf = K(\delta_0 \otimes f)$ where for $f \in C^{\alpha,\tau}(\mathbb{R})$ the latter distribution is defined by $(\delta_0 \otimes f, \varphi)_{\mathbb{R}^2} := (f, \varphi(0,\cdot))_{\mathbb{R}}$. Directly from the definitions one can check that $f \mapsto \delta_0 \otimes f$ is bounded from $C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$. Finally, (4.109) follows from Proposition 4.6.5. □

Remark 4.6.8. *Although we have only defined it for* $0 < \alpha < 1$, *one may actually define* $C_{\mathfrak{s}}^{\alpha,\tau}$ *for*

all $\alpha > 0$, as the closure of $C_c^\infty(\Lambda_T)$ with respect to

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)} := \sup_{(t,x)\in\Lambda_T} \frac{|f(t,x)|}{w(x)} + \sup_{(t,x)\in\Lambda_T} \sup_{\lambda\in(0,1]} \sup_{\varphi\in B_0} \frac{(f, S^\lambda_{(t,x)}\varphi)}{w(x)\lambda^\alpha}$$

where the scaling operators are defined in (4.106) *and where B_0 is the set of all smooth functions of C^0 norm less than 1, supported on the unit ball of $\mathbb{R}^2$ that are orthogonal in $L^2(\mathbb{R}^2)$ to all polynomials of parabolic degree less than or equal to $\lceil\alpha\rceil$, see for instance equation (3.8) and (3.9) in [181]. One may verify that this norm is equivalent to the one in Definition 4.6.3 for $\alpha \in (0,1)$. Stated in this way, the Schauder estimate actually holds for all $\alpha \in \mathbb{R}\backslash\mathbb{Z}$, though we will not need this.*

We next define a space-time distribution that is supported on a single temporal cross-section:

Definition 4.6.9. *Let $\alpha < 0$. Given some $f \in C^{\alpha,\tau}(\mathbb{R})$ and $b \in [0,T]$ we define $\delta_b \otimes f \in C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$ by the formula $(\delta_b \otimes f, \varphi)_{\mathbb{R}^2} := (f, \varphi(b,\cdot))_{\mathbb{R}}$.*

Directly from the definitions of the scaling operators in (4.105) and (4.106), it is clear that for fixed $b \in [0,T]$, the linear map $f \mapsto \delta_b \otimes f$ is bounded from $C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$ as long as $\alpha < 0$. It is also clear that $\delta_b \otimes f$ is necessarily supported on the line $\{b\}\times\mathbb{R}$. The following lemma shows that under mild conditions, $\delta_T \otimes f$ vanishes upon the action of the heat flow K defined in (4.107).

Lemma 4.6.10. *Let $f \in C^{\alpha,\tau}(\mathbb{R})$, where $\alpha > -2$. Then viewing K as an operator from $C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T) \to C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$, we have that $K(\delta_T \otimes f) = 0$.*

Proof. Note that $\delta_T \otimes f \in C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ where $\alpha > -4$. Thus by Proposition 4.6.6, we know that $K(\delta_T \otimes f) \in C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)$ for some $\beta > -2$. Now since $\delta_T \otimes f$ is supported on $\{T\}\times\mathbb{R}$, it immediately follows that $K(\delta_T \otimes f)$ is also supported on $\{T\}\times\mathbb{R}$.

Thus it suffices to show that if $g \in C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)$ for some $\beta \in (-2,0)$, and if g is supported on $\{T\}\times\mathbb{R}$ then $g = 0$. We can prove this directly: fix some smooth nonnegative compactly supported function φ on $\mathbb{R}^2$, and define $g^\lambda(t,x) := (g, S^\lambda_{(t,x)}\varphi)$, where the scaling operators are given by

(4.106). Using $g \in C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)$ for some $\beta \in (-2,0)$, one has that $g^\lambda(t,x) \leq Cw(x)|T-t|^{\frac{\beta}{2}}$ where C is independent of λ, x, t, and where w is the weight function given by (4.104). Now if ψ is any compactly supported function on $\mathbb{R}^2$, then by integrability of $t \mapsto t^{\beta/2}$ on $[0,1]$ and the dominated convergence theorem, we find that

$$(g,\psi) = \lim_{\lambda \to 0} \int_{\mathbb{R}^2} g^\lambda(t,x)\psi(t,x)dtdx = \int_{\mathbb{R}^2} \lim_{\lambda \to 0} g^\lambda(t,x)\psi(t,x)dtdx = 0,$$

where we used the fact that $\lim_{\lambda \to 0} g^\lambda(t,x) = 0$ at all points (t,x) outside of the measure-zero subset given by $\{T\} \times \mathbb{R}$. □

We remark that the last paragraph of the above proof has an intuitive analogue in the elliptic setting as well: if $f \in C^{\alpha,\tau}(\mathbb{R})$ where $\alpha > -1$, and if f is supported on a single point, then $f = 0$. The proof proceeds in a similar manner.

We end this subsection by recording a Kolmogorov-type lemma for the three spaces introduced at the beginning of this subsection. It will be crucial in proving tightness in those respective spaces.

Lemma 4.6.11 (Kolmogorov lemma)**.** *Let $L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moment. We have the following:*

(a) (Elliptic Hölder Space) Let $\phi \mapsto V(\phi)$ be a bounded linear map from $\mathcal{S}(\mathbb{R})$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Recall S_x^λ from (4.105)*. Assume there exists some $p > 1$ and $\alpha < 0$ and $\mathrm{C} = \mathrm{C}(\alpha, p) > 0$ such that one has*

$$\mathbf{E}[|V(S_x^\lambda \phi)|^p] \leq \mathrm{C}\lambda^{\alpha p},$$

uniformly over all smooth functions ϕ on $\mathbb{R}$ supported on the unit ball of $\mathbb{R}$ with $\|\phi\|_{L^\infty} \leq 1$, and uniformly over $\lambda \in (0,1]$ and $x \in \mathbb{R}$. Then for any $\tau > 1$ and any $\beta < \alpha - 1/p$ there exists a random variable $\mathscr{V}$ taking values in $C^{\beta,\tau}(\mathbb{R})$ such that $(\mathscr{V}, \phi) = V(\phi)$ almost surely for all ϕ. Furthermore one has that

$$\mathbf{E}[\|\mathscr{V}\|_{C^{\beta,\tau}(\mathbb{R})}^p] \leq \mathrm{C}',$$

where C' *depends on the choice of* α, p*, and the constant* C *appearing in the moment bound above but not on* $V, \Omega, \mathcal{F}, \mathbb{P}$*.*

(b) (Function space) Let $(t, \phi) \mapsto V(t, \phi)$ *be a map from* $[0, T] \times \mathcal{S}(\mathbb{R})$ *into* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *which is linear and continuous in* ϕ*. Fix a non-negative integer* r*. Assume there exists some* $\kappa > 0, p > 1/\kappa$ *and* $\alpha < -r$ *and* $\mathrm{C} = \mathrm{C}(\kappa, \alpha, p, T) > 0$ *such that one has*

$$\mathbf{E}[|V(t, S_x^\lambda \phi)|^p] \leq \mathrm{C}\lambda^{\alpha p},$$
$$\mathbf{E}[|V(t, S_x^\lambda \phi) - V(s, S_x^\lambda \phi)|^p] \leq \mathrm{C}\lambda^{(\alpha-\kappa)p}|t - s|^{\kappa p},$$

uniformly over all smooth functions ϕ *on* $\mathbb{R}$ *supported on the unit ball of* $\mathbb{R}$ *with* $\|\phi\|_{C^r} \leq 1$*, and uniformly over* $\lambda \in (0, 1]$ *and* $0 \leq s, t \leq T$*. Then for any* $\tau > 1$ *and any* $\beta < \alpha - \kappa$ *there exists a random variable* $\big(\mathscr{V}(t)\big)_{t\in[0,T]}$ *taking values in* $C([0, T], C^{\beta,\tau}(\mathbb{R}))$ *such that* $(\mathscr{V}(t), \phi) = V(t, \phi)$ *almost surely for all* ϕ *and* t*. Furthermore, one has that*

$$\mathbf{E}[\|\mathscr{V}\|^p_{C([0,T],C^{\beta,\tau}(\mathbb{R}))}] \leq \mathrm{C}',$$

where C' *depends on the choice of* α, β, p, κ*, and the constant* C *appearing in the moment bound above but not on* $V, \Omega, \mathcal{F}, \mathbb{P}$*.*

(c) (Parabolic Hölder Space) Let $\varphi \mapsto V(\varphi)$ *be a bounded linear map from* $\mathcal{S}(\mathbb{R}^2)$ *to* $L^2(\Omega, \mathcal{F}, \mathbb{P})$*. Assume* $V(\varphi) = 0$ *for all* φ *with support contained in the complement of* Λ_T*. Recall* $S^\lambda_{(t,x)}$ *from* (4.106)*. Fix a non-negative integer* r*. Assume there exists some* $p > 1$ *and* $\alpha < 0$ *and* $\mathrm{C} = \mathrm{C}(\alpha, p) > 0$ *such that one has*

$$\mathbf{E}[|V(S^\lambda_{(t,x)}\varphi)|^p] \leq \mathrm{C}\lambda^{\alpha p},$$

uniformly over all smooth functions φ *on* $\mathbb{R}^2$ *supported on the unit ball of* $\mathbb{R}^2$ *with* $\|\varphi\|_{C^r} \leq 1$*, and uniformly over* $\lambda \in (0, 1]$ *and* $(t, x) \in \Lambda_T$*. Then for any* $\tau > 1$ *and any* $\beta < \alpha - 3/p$ *there*

exists a random variable $\mathscr{V}$ taking values in $C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)$ such that $(\mathscr{V},\varphi) = V(\varphi)$ almost surely for all φ. Furthermore one has that

$$\mathbf{E}[\|\mathscr{V}\|^p_{C_{\mathfrak{s}}^{\beta,\tau}(\Lambda_T)}] \leq \mathrm{C}',$$

where C' depends on the choice of α, p, and the constant C appearing in the moment bound above but not on $V, \Omega, \mathcal{F}, \mathbb{P}$.

A proof of the above results may be adapted from the proof of Lemma 9 in Section 5 of [228]. We remark that we do not actually need uniformity over a large class of test functions as we have written above, just a single well-chosen test function would suffice (e.g. the Littlewood-Paley blocks as used in [228] or the Daubechies wavelets in [227]). Part (a) of the above lemma can be stated with C^r functions just like part (c), but we will not need this general version.

4.6.2 Tightness

In this section, we prove tightness of the field $\mathscr{X}^N$ and establish regularity estimates for the limit points. At this stage, readers may find it helpful to revisit Section 4.1.5 for a proof sketch of the tightness argument.

Proposition 4.6.12. *For each $t \geq 0$, the fields $\{\mathscr{X}_t^N\}_{N\geq 1}$ defined in* (4.4) *may be realized as random elements of the elliptic Hölder space $C^{\alpha,\tau}(\mathbb{R})$ for any $\tau > 1$ and any $\alpha < -1$. Moreover they are tight with respect to that topology as $N \to \infty$, in fact, we have that $\sup_N \mathbb{E}[\|\mathscr{X}_t^N\|^p_{C^{-\alpha,\tau}(\mathbb{R})}] < \infty$ for all $p \geq 1$. Furthermore, the $\mathscr{X}^N$ defined in* (4.6) *themselves may be viewed as elements of the parabolic Hölder space $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ for $\alpha < -1$ and $\tau > 1$. Moreover they are tight with respect to that topology as $N \to \infty$.*

Proof. Using Lemma 4.2.2 and Lemma 4.3.1 we obtain that

$$\begin{aligned}\mathbb{E}[\mathscr{X}_t^N(\phi)^{2k}] &= \mathbf{E}_{D_{N^{1/4}}SB^{(k)}_{N^{1/2}\nu}}\left[\prod_{j=1}^{2k}\phi(X_t^j)e^{\frac{1}{2}N^{1/2}\sum_{i<j}V_t^{ij}(\mathbf{X})}\right]\\ &= \mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}}\left[\mathcal{M}_N(t)\prod_{j=1}^{2k}\phi(X_t^j)e^{\frac{1}{2}N^{1/2}\sum_{i<j}V_t^{ij}(\mathbf{X})}\right]\\ &\leq \|\phi\|^{2k}_{L^\infty(\mathbb{R})}\mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}}[\mathcal{M}_N(t)^2]^{1/2}\mathbf{E}_{SB^{(k)}_{N^{1/2}\nu}}\left[e^{N^{1/2}\sum_{i<j}V_t^{ij}(\mathbf{X})}\right]^{1/2},\end{aligned}$$

where $\mathcal{M}_N(t)$ is defined in (4.28) and the last inequality above is due to the Cauchy-Schwarz inequality. From Remark 4.2.7 and Lemma 4.2.4 (with $\nu \mapsto N^{1/2}\nu$), we know that the above two expectations are uniformly bounded in N. Therefore the last expression is bounded by some universal constant times $\|\phi\|^{2k}_{L^\infty}$. In particular we get that

$$\mathbb{E}[\mathscr{X}_t^N(S_x^\lambda\phi)^{2k}] \leq \mathrm{C}\|S_x^\lambda\phi\|^{2k}_{L^\infty} \leq C\lambda^{-2k},$$

uniformly over all $\phi \in C_c^\infty(\mathbb{R})$ with $\|\phi\|_{L^\infty} \leq 1$, where S_x^λ is defined in (4.105). Appealing to Lemma 4.6.11 (a) we get the desired result for fixed t.

To prove the second statement about tightness in the parabolic Hölder space, we will as in (4.6) denote space-time pairings as $(\mathscr{X}^N, \varphi)_{L^2(\mathbb{R}^2)}$. Note by Minkowski's inequality that

$$\mathbb{E}[(\mathscr{X}^N,\varphi)^{2k}_{L^2(\mathbb{R}^2)}]^{1/2k} = \mathbb{E}\left[\int_{\mathbb{R}}\left(\mathscr{X}_s^N(\varphi(s,\cdot))ds\right)^{2k}\right]^{1/2k} \leq \int_{\mathbb{R}}\mathbb{E}[\mathscr{X}_s^N(\varphi(s,\cdot))^{2k}]^{1/2k}ds.$$

Now with $S^\lambda_{(t,x)}$ defined in (4.106), we have

$$\mathbb{E}[(\mathscr{X}^N, S^\lambda_{(t,x)}\varphi)^{2k}_{L^2(\mathbb{R}^2)}]^{1/2k} \leq \int_{\mathbb{R}}\mathbb{E}\big[\mathscr{X}_s^N\big((S^\lambda_{(t,x)}\varphi)(s,\cdot)\big)^{2k}\big]^{1/2k}ds \leq \mathrm{C}\int_{t-\lambda^2}^{t+\lambda^2}\lambda^{-3}ds \leq \mathrm{C}\lambda^{-1},$$

uniformly over all $\varphi \in C_c^\infty(\mathbb{R}^2)$ supported on the unit ball, with $\|\varphi\|_{L^\infty} \leq 1$. Here the λ^{-3} comes from the fact that the parabolic scaling operator differs from the elliptic one by exactly a factor of λ^{-2}. In the second bound, we used the same bounds appearing in the proof for the elliptic case

above, noting that the constants there are independent of $t \in [0,T]$. Appealing to Lemma 4.6.11 (c) we get the desired result. □

Proposition 4.6.13. *The fields M^N defined in* (4.48) *may be realized as an element of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ for any $\alpha < -2$ and $\tau > 1$. Moreover, they are tight with respect to that topology. Q^N defined in* (4.52) *may be realized as an element of $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$ for any $\gamma < -1$ and $\tau > 1$. Moreover, they are tight with respect to that topology. Let (M^∞, Q^∞) be a joint limit point of (M^N, Q^N) in $C([0,T], C^{\alpha,\tau}(\mathbb{R}) \times C^{\gamma,\tau}(\mathbb{R}))$. For all $\phi \in C_c^\infty(\mathbb{R})$ the process $(M_t^\infty(\phi))_{t\in[0,T]}$ is a martingale with respect to the canonical filtration on that space, and moreover one has*

$$\langle M^\infty(\phi)\rangle_t = Q_t^\infty(\phi^2). \tag{4.110}$$

We remark that the canonical filtration $\mathscr{F}_t$ on $C([0,T],\Xi)$ for any Polish space Ξ is the one generated by the random variables $X(s)$ for $s \le t$ as X varies through all elements of $C([0,T],\Xi)$.

Proof. Take any $\phi \in C_c^\infty(\mathbb{R})$ with $\|\phi\|_{L^\infty} \le 1$. Recall $S^\lambda_{(t,x)}$ from (4.106). Using the first bound in Proposition 4.5.6, we have

$$\mathbb{E}[(Q_t^N(S_x^\lambda\phi) - Q_s^N(S_x^\lambda\phi))^k] \le \mathrm{C}|t-s|^{k/4}\lambda^{-k},$$

uniformly over $x \in \mathbb{R}, \lambda \in (0,1], 0 \le s,t \le T$. Since $Q_0^N(\phi) = 0$ by definition, therefore the assumptions of Lemma 4.6.11 (b) are satisfied for any $\kappa \le 1/4$, any $p > 1/\kappa$, and any $\alpha \le -1$, and we conclude the desired tightness for Q^N.

Now we address the tightness of the M^N. By Lemma 4.4.3 and the Burkholder-Davis-Gundy inequality, we have that

$$\mathbb{E}[(M_t^N(\phi) - M_s^N(\phi))^{2k}] \le \mathrm{C}\cdot\mathbb{E}\left[\left((Q_t^N\big((\phi + N^{-1/4}\phi')^2\big) - Q_s^N\big((\phi+N^{-1/4}\phi')^2\big)\right)^k\right],$$

where $\mathrm{C} = \mathrm{C}(k) > 0$ is free of ϕ, s, t, N. Using a crude bound

$$(\phi + N^{-1/4}\phi')^2 \leq 2\phi^2 + 2N^{-1/2}(\phi')^2 \leq 2\phi^2 + 2(\phi')^2,$$

we find from the first bound in Proposition 4.5.6 that

$$\mathbb{E}[(M_t^N(\phi) - M_s^N(\phi))^{2k}] \leq \mathrm{C}(t-s)^{k/4}(\|\phi\|_{L^\infty}^2 + \|\phi'\|_{L^\infty}^2)^k \leq \mathrm{C}(t-s)^{k/4}\|\phi\|_{C^1}^{2k}. \tag{4.111}$$

This gives

$$\mathbb{E}[(M_t^N(S_x^\lambda\phi) - M_s^N(S_x^\lambda\phi))^{2k}] \leq \mathrm{C}(t-s)^{k/4}\lambda^{-4k}$$

uniformly over $x \in \mathbb{R}, \lambda \in (0,1], 0 \leq s, t \leq T$, and $\phi \in C_c^\infty(\mathbb{R})$ with $\|\phi\|_{C^1} \leq 1$. Moreover $M_0(\phi) = 0$ by definition, therefore the assumptions of Lemma 4.6.11 (b) are satisfied for any $\kappa \leq 1/8$, any $p > 1/\kappa$, and any $\alpha \leq -2$. Hence, we conclude the desired tightness for M^N.

We next show that the limit point $M^\infty(\phi)$ is a martingale. Since $M_0^N(\phi) = 0$, from (4.111), we see that $\sup_N \mathbb{E}[M_t^N(\phi)^{2k}] < \infty$. Thus by uniform integrability, $M^\infty(\phi)$ is a martingale. In the prelimit we know that

$$(M_t^N(\phi))^2 - Q_t^N\big((\phi + N^{-1/4}\phi')^2\big) = (M_t^N(\phi))^2 - Q_t^N(\phi^2) - \mathcal{E}_t^N(\phi)$$

is a martingale, where the error term $\mathcal{E}_t^N$ is defined in (4.53). By Proposition 4.5.5 we know that $\mathcal{E}_t^N$ vanishes in probability, so we conclude (again by uniform L^p boundedness guaranteed by Proposition 4.5.6) that $(M_t^\infty(\phi))^2 - Q_t^\infty(\phi^2)$ is a martingale. This verifies (4.110) completing the proof. □

Proposition 4.6.14. *For $\alpha < 0$, the derivative operator $\partial_s : C([0,T], C^{\alpha,\tau}(\mathbb{R})) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$ which is defined by sending $(v_t)_{t\in[0,T]}$ to the distribution*

$$(\partial_s v, \varphi)_{L^2(\Lambda_T)} := v_T(\varphi(T,\cdot)) - \int_0^T v_t(\partial_t\varphi(t,\cdot))dt - v_0(\varphi(0,\cdot)),$$

whenever $\varphi \in C_c^\infty(\Lambda_T)$, is a bounded linear map. Let M^N be as in Proposition 4.6.13, viewed as elements of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ for some $\alpha < -2$. Also let $\mathscr{X}_t^N$ and $\mathscr{X}^N$ and be as in Proposition 4.6.12, viewed as elements of $C^{\gamma,\tau}(\mathbb{R})$ and $C_{\mathfrak{s}}^{\gamma,\tau}(\Lambda_T)$ respectively for some $\gamma < -1$. Then one has that

$$\partial_s M^N = -\delta_{(0,0)} + (\partial_t - \tfrac{1}{2}\partial_x^2)\mathscr{X}^N + \delta_T \otimes \mathscr{X}_T^N \tag{4.112}$$

in the sense of distributions.

Proof. Recall $S^\lambda_{(t,x)}$ from (4.106). From the definition of the space $C^{\alpha,\tau}(\mathbb{R})$ one verifies directly that if $v = (v_t)_{t\in[0,T]} \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ then

$$\int_0^T |v_t\big(\partial_t(S^\lambda_{(s,y)}\varphi)(t,\cdot)\big)|dt \le w(y)\int_{s-\lambda^2}^{s+\lambda^2} \lambda^{\alpha-4}dt \le w(y)\lambda^{\alpha-2}$$

uniformly over $(s,y) \in \Lambda_T$ and $\varphi \in C_c^\infty(\mathbb{R}^2)$ supported on the unit ball of $\mathbb{R}^2$ with $\|\varphi\|_{C^r} \le 1$ where $r = \lceil -\alpha \rceil$. This proves the first part of the proposition that ∂_s is bounded from $C([0,T], C^{\alpha,\tau}(\mathbb{R})) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$.

For the second part, let us abbreviate $\mathscr{M}^N := \partial_s M^N$. We first note by definition of ∂_s that for any smooth φ of compact support on $\mathbb{R}^2$, $\mathscr{M}^N(\varphi)$ is explicitly given by

$$\mathscr{M}^N(\varphi) = M_T^N(\varphi(T,\cdot)) - \int_0^T M_s^N\big(\partial_s\varphi(s,\cdot)\big)ds,$$

where we used the fact that $M_0^N \equiv 0$. Note that the above expression is the same as $R_T^N(\varphi)$ defined

in (4.49). Now observe that using (4.48) we have

$$\begin{aligned}\int_0^T M_s^N\big(\partial_s\varphi(s,\cdot)\big)ds &= \int_0^T \mathscr{X}_s^N\big(\partial_s\varphi(s,\cdot)\big)ds - \int_0^T \big(\partial_s\varphi(s,0)\big)ds \\ &\qquad - \int_0^T \frac{1}{2}\int_0^s \mathscr{X}_u^N\big(\partial_{xx}\partial_s\varphi(s,\cdot)\big)duds \\ &= \int_0^T \mathscr{X}_s^N\big(\partial_s\varphi(s,\cdot)\big)ds + \varphi(0,0) - \varphi(T,0) \\ &\qquad - \int_0^T \frac{1}{2}\int_0^s \mathscr{X}_u^N\big(\partial_{xx}\partial_s\varphi(s,\cdot)\big)duds.\end{aligned}$$

The last term on the right-hand side of the above equation can be simplified as follows.

$$\begin{aligned}\int_0^T \frac{1}{2}\int_0^s \mathscr{X}_u^N\big(\partial_{xx}\partial_s\varphi(s,\cdot)\big)duds &= \int_0^T \frac{1}{2}\int_u^T \mathscr{X}_u^N\big(\partial_{xx}\partial_s\varphi(s,\cdot)\big)dsdu \\ &= \frac{1}{2}\int_0^T \big[\mathscr{X}_u^N\big(\partial_{xx}\varphi(T,\cdot)\big) - \mathscr{X}_u^N\big(\partial_{xx}\varphi(u,\cdot)\big)\big]du \\ &= -M_T^N(\varphi(T,\cdot)) + \mathscr{X}_T^N(\varphi(T,\cdot)) - \varphi(T,0) \\ &\qquad - \frac{1}{2}\int_0^T \mathscr{X}_u^N(\partial_{xx}\varphi(u,\cdot)\big)du.\end{aligned}$$

We applied (4.48) with $\phi = \varphi(T,\cdot)$ in the last line. Combining these expressions, we derive that

$$\mathscr{M}^N(\varphi) = -\varphi(0,0) - \int_0^T \mathscr{X}_s^N(\partial_s\varphi(s,\cdot)ds - \frac{1}{2}\int_0^T \mathscr{X}_s^N(\partial_{xx}\varphi(s,\cdot)\big)ds + \mathscr{X}_T^N(\varphi(T,\cdot)).$$

This verifies (4.112) completing the proof. □

Since we expect to obtain a Dirac initial data in the limiting SPDE, there is an additional singularity at the origin that we have not yet taken into account. To fix this issue, we now formulate a tightness result taking into account this singularity, by starting our martingale-forced heat equation from some positive time ε (which should be thought of as being close to 0).

Proposition 4.6.15 (Tightness). *Fix $\varepsilon > 0$, and consider the fields $\mathscr{X}^{N,\varepsilon}(t,\cdot) := \mathscr{X}_{t+\varepsilon}^N(\cdot)$, for $t \geq 0$. Set $\mathscr{X}^{N,\varepsilon}(t,\cdot) := 0$ for $t < 0$. The fields $\mathscr{X}^{N,\varepsilon}$ may be realized as random variables taking values in $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ for any $\tau > 1$ and $\alpha < -1$. Furthermore, they are tight with respect to that*

topology, and any limit point necessarily lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_T)$ *for any* $\kappa > 0$.

Proof. The proof that the fields $\mathscr{X}^{N,\varepsilon}$ are tight in $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ for any $\tau > 1$ and $\alpha < -1$ follows by using the exact same strategy followed in the proof of Proposition 4.6.12. Let us set

$$\mathscr{M}^{N,\varepsilon} := \big(\partial_t - \frac{1}{2}\partial_x^2\big)\mathscr{X}^{N,\varepsilon} + (\delta_T \otimes \mathscr{X}_{T+\varepsilon}^N - \delta_0 \otimes \mathscr{X}_\varepsilon^N).$$

We first claim that for fixed $\varepsilon > 0$, the fields $\{\mathscr{M}^{N,\varepsilon}\}_{N>1}$ are tight in $C_{\mathfrak{s}}^{-3-\kappa,\tau}(\Lambda_T)$ for any $\tau > 1$ and $\kappa > 0$. As $\mathscr{X}^{N,\varepsilon}$ is tight in $C_{\mathfrak{s}}^{-1-\kappa,\tau}(\Lambda_T)$, the tightness of the $(\partial_t - \frac{1}{2}\partial_x^2)\mathscr{X}^{N,\varepsilon}$ in $C_{\mathfrak{s}}^{-3-\kappa,\tau}(\Lambda_T)$ is immediate since $\partial_t - \frac{1}{2}\partial_x^2$ is a bounded linear map from $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ into $C_{\mathfrak{s}}^{\alpha-2,\tau}(\Lambda_T)$ as explained in Remark 4.6.4. From Proposition 4.6.12 we know that both $\mathscr{X}_{T+\varepsilon}^N$ and $\mathscr{X}_\varepsilon^N$ are tight in $C^{-1-\kappa,\tau}(\mathbb{R})$. As we remarked after Definition 4.6.9, the map $f \mapsto \delta_a \otimes f$ is continuous from $C^{-1-\kappa,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{-3-\kappa,\tau}(\Lambda_T)$, therefore we can immediately conclude that $(\delta_T \otimes \mathscr{X}_{T+\varepsilon}^N - \delta_0 \otimes \mathscr{X}_\varepsilon^N)$ are tight in $C_{\mathfrak{s}}^{-3-\kappa,\tau}(\Lambda_T)$. Hence, the sum given by $\mathscr{M}^{N,\varepsilon}$ is also tight in the latter space as $N \to \infty$ (for fixed ε).

Next, we address the regularity of the limit points of $\mathscr{X}^{N,\varepsilon}$. By the last statement in Proposition 4.6.6, we can apply the linear operator K to both sides to obtain

$$\mathscr{X}^{N,\varepsilon} = K(\delta_0 \otimes \mathscr{X}_\varepsilon^N) + K\mathscr{M}^{N,\varepsilon} - K(\delta_T \otimes \mathscr{X}_{T+\varepsilon}^N).$$

By Proposition 4.6.12 we know that $\mathscr{X}_{T+\varepsilon}^N$ is tight in $C^{\gamma,\tau}(\mathbb{R})$ for $\gamma < -1$, and from Lemma 4.6.10 we can conclude that $K(\delta_T \otimes \mathscr{X}_{T+\varepsilon}^N) = 0$. On the other hand, $K(\delta_0 \otimes \mathscr{X}_\varepsilon^N) = J\mathscr{X}_\varepsilon^N$, where J is defined in Corollary 4.6.7. We thus have the Duhamel equation

$$\mathscr{X}^{N,\varepsilon} = J\mathscr{X}_\varepsilon^N + K\mathscr{M}^{N,\varepsilon}.$$

For technical reasons that will be made clear below, we now replace ε by $\varepsilon/2$ and T by $T+1$. If we take any joint limit point $(\mathscr{X}^{\infty,\varepsilon/2}, \mathscr{M}^{\infty,\varepsilon/2}, h)$ in $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_{T+1}) \times C_{\mathfrak{s}}^{\alpha+2,\tau}(\Lambda_{T+1}) \times C^{\gamma,\tau}(\mathbb{R})$ as $N \to \infty$ of $(\mathscr{X}^{N,\varepsilon/2}, \mathscr{M}^{N,\varepsilon/2}, \mathscr{X}_{\varepsilon/2}^N)$, then (by continuity of J and K on the relevant spaces) the

Duhamel relation still holds in the limit, i.e.,

$$\mathscr{X}^{\infty,\varepsilon/2} = Jh + K\mathscr{M}^{\infty,\varepsilon/2}. \tag{4.113}$$

We now claim that

$$\mathbb{E}[\|\mathscr{M}^{\infty,\varepsilon/2}\|^p_{C_{\mathfrak{s}}^{-\kappa-3/2,\tau}(\Lambda_{T+1})}] < \infty. \tag{4.114}$$

Let us now complete the proof of regularity assuming (4.114).

- Using (4.114) and Proposition 4.6.6, it follows that

$$\mathbb{E}\left[\left\|K\mathscr{M}^{\infty,\varepsilon/2}\right\|^p_{C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_{T+1})}\right] < \infty.$$

  This implies that the restriction of $K\mathscr{M}^{\infty,\varepsilon/2}$ to $[\varepsilon/2, T+\varepsilon/2]\times\mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_{[\varepsilon/2,T+\varepsilon/2]})$.

- By Proposition 4.6.12 we have $\mathbb{E}[\|h\|^p_{C^{-1-\kappa,\tau}}] < \infty$. So from (4.109) with $\alpha := -1-\kappa$ and $\beta := \frac{1}{2} - \kappa$. It follows that

$$\mathbb{E}[\|\widehat{P}_{\varepsilon/2}h\|^p_{C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_{T+1})}] < \infty.$$

  Since from (4.108), we know $Jh(t+\varepsilon/2, x) = \widehat{P}_{\varepsilon/2}h(t,x)$, we thus have that the restriction of Jh to $[\varepsilon/2, T+\varepsilon/2]\times\mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_{[\varepsilon/2,T+\varepsilon/2]})$.

Thanks to the above two bullet points and the relation (4.113), we have showed that the restriction of $\mathscr{X}^{\infty,\varepsilon/2}$ to $[\varepsilon/2, T+\varepsilon/2]\times\mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_{[\varepsilon/2,T+\varepsilon/2]})$. This is equivalent to the fact that $\mathscr{X}^{\infty,\varepsilon}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_T)$. This completes the proof modulo (4.114).

Proof of (4.114). We shall show (4.114) with $\varepsilon/2$ and $T+1$ replaced by ε and T respectively. Following the same proof as in Lemma 4.6.14, one can check that

$$\mathscr{M}^{N,\varepsilon}(\varphi) = R^N_{T+\varepsilon}(\varphi) - R^N_{\varepsilon}(\varphi),$$

where R^N are the same martingales defined in (4.49). We will denote the right side as $R^{N,\varepsilon}(\varphi)$ and the quadratic variation up to time T as $\langle R^{N,\varepsilon}(\varphi)\rangle_T$. Let us fix any $\alpha < -3$ and take any joint limit point $(\mathscr{M}^{\infty,\varepsilon}, \mathscr{X}^{\infty,\varepsilon})$ in $C^{\alpha,\tau}_{\mathfrak{s}}(\Lambda_T) \times C^{\alpha+2,\tau}_{\mathfrak{s}}(\Lambda_T)$ of $(\mathscr{M}^{N,\varepsilon}, \mathscr{X}^{N,\varepsilon})$ as $N \to \infty$. Let us consider any smooth function φ supported on the unit ball of $\mathbb{R}^2$ such that $\|\varphi\|_{C^1} \leq 1$. Recall $S^\lambda_{(t,x)}$ from (4.106). Using Skorohod's lemma, Fatou's lemma, and the Burkholder-Davis-Gundy inequality, one has that

$$\mathbb{E}[|\mathscr{M}^{\infty,\varepsilon}(S^\lambda_{(t,x)}\varphi)|^p] \leq \liminf_{N\to\infty} \mathbb{E}[|\mathscr{M}^{N,\varepsilon}(\varphi)|^p] \leq \mathrm{C} \cdot \liminf_{N\to\infty} \mathbb{E}\big[\langle R^{N,\varepsilon}(S^\lambda_{(t,x)}\varphi)\rangle_T^{p/2}\big],$$

where $\mathrm{C} = \mathrm{C}(p) > 0$. From Lemma 4.4.4, the quadratic variation is given by

$$\langle R^{N,\varepsilon}(S^\lambda_{(t,x)}\varphi)\rangle_T = N^{1/2} \int_0^T \big((\mathscr{X}^N_{s+\varepsilon})^{Sq}, ((1+N^{-1/4}\partial_x)S^\lambda_{(t,x)}\varphi(s,\cdot))^2\big) ds.$$

Using $(a+b)^2 \leq 2(a^2+b^2)$, it is clear that $\big((1+N^{-1/4}\partial_x)\varphi\big)^2 \leq 2\varphi^2 + 2N^{-1/2}(\partial_x\varphi)^2$. Furthermore, the term with a factor $N^{-1/2}$ will not contribute when taking the $\liminf$ as $N \to \infty$ (by e.g. Proposition 4.5.5). Therefore, we can obtain that

$$\liminf_{N\to\infty} \mathbb{E}[\langle R^{N,\varepsilon}(S^\lambda_{(t,x)}\varphi)\rangle_T] \leq \liminf_{N\to\infty} N^{1/2}\mathbb{E}\bigg[\int_0^T \big((\mathscr{X}^N_{s+\varepsilon})^{Sq}, (S^\lambda_{(t,x)}\varphi)^2\big) ds\bigg].$$

Recall the fields Q^N defined in (4.52). Notice that $\big(S^\lambda_{(t,x)}\varphi\big)^2 \leq \lambda^{-6}\mathbb{1}_{[t-\lambda^2,t+\lambda^2]\times[x-\lambda,x+\lambda]}$. Therefore, taking any $\delta > 0$, by the above bound and by the second bound in Proposition 4.5.6 we find that

$$\begin{aligned}\mathbb{E}[|\mathscr{M}^{\infty,\varepsilon}(S^\lambda_{(t,x)}\varphi)|^p] &\leq 2\mathrm{C}\lambda^{-3p} \liminf_{N\to\infty} \mathbb{E}\big[\big(Q^N_{t+\varepsilon+\lambda^2}(\mathbb{1}_{[x-\lambda,x+\lambda]}) - Q^N_{t+\varepsilon-\lambda^2}(\mathbb{1}_{[x-\lambda,x+\lambda]})\big)^{p/2}\big] \\ &\leq 2\mathrm{C}\lambda^{-3p}(\lambda^2)^{p/2}\|\mathbb{1}_{[x-\lambda,x+\lambda]}\|^{p/2}_{L^{1+\delta}(\mathbb{R})} = \mathrm{C}\lambda^{-\frac{3}{2}p-\left[\frac{\delta}{1+\delta}\right]p}.\end{aligned}$$

Given any $\kappa > 0$, by making $\delta = \delta(\kappa)$ close to 0 and making $p = p(\kappa)$ large enough, we may then apply Lemma 4.6.11(c) to conclude (4.114). □

4.6.3 Identification of the limit points

In this subsection, we identify the limit points as the solution of the stochastic heat equation (SHE) and thereby prove our weak convergence theorem: Theorem 4.1.4.

Theorem 4.6.16 (Solving the martingale problem)**.** *Consider the triplet of processes* $(\mathscr{X}^N, M^N, Q^N)_{t\geq 0}$*, where* $\mathscr{X}^N$*,* M^N*, and* Q^N *are defined in* (4.4)*,* (4.48)*, and* (4.52) *respectively. Fix* $\alpha < -1, \beta < -2, \gamma < -1$*, and* $\tau > 1$*. These triples are jointly tight in the space*

$$C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T) \times C([0,T], C^{\beta,\tau}(\mathbb{R})) \times C([0,T], C^{\gamma,\tau}(\mathbb{R})).$$

Consider any joint limit point $(X^\infty, M^\infty, Q^\infty)$*. Then for any* $s > 0$*, the process* $(t,x) \mapsto X_{s+t}^\infty(x)$ *is necessarily supported on the space* $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_T)$*. Furthermore,* $(M_t^\infty(\phi))_{t\geq 0}$ *is a continuous martingale for all* $\phi \in C_c^\infty(\mathbb{R})$*, and moreover for all* $0 < s \leq t < T$ *one has the almost sure identities*

$$M_t^\infty(\phi) - M_s^\infty(\phi) = \int_{\mathbb{R}} (X_t^\infty(x) - X_s^\infty(x))\phi(x)dx - \frac{1}{2}\int_s^t \int_{\mathbb{R}} X_u^\infty(x)\phi''(x)dxdu \tag{4.115}$$

$$\langle M^\infty(\phi)\rangle_t = Q_t^\infty(\phi^2) \tag{4.116}$$

$$Q_t^\infty(\phi) - Q_s^\infty(\phi) = \sigma \int_{\mathbb{R}} \int_s^t (X_u^\infty(x))^2 \phi(x)\, du\, dx. \tag{4.117}$$

Before going into the proof, we remark that with some inspection it may be verified that all quantities make sense given the spaces they lie in, as long as we choose $\kappa < 1/2$ to ensure that X^∞ is a continuous function in space-time away from $t = 0$.

Proof. Most parts of the following theorem are already established in previous propositions and lemmas. Note that the tightness of $\mathscr{X}^N$ was proved in Proposition 4.6.12. The tightness of M^N and Q^N was shown in Proposition 4.6.13. Thus the joint tightness follows from individual tightness. Consider any limit point $(\mathscr{X}^\infty, M^\infty, Q^\infty)$. The fact that for any $s > 0$, the process $(t,x) \mapsto X^\infty(s+t,x)$ is necessarily supported on the space $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_T)$ was proved in Proposition

4.6.15. From Proposition 4.6.13 we have that for all $\phi \in C_c^\infty(\mathbb{R})$ the process $(M_t^\infty(\phi))_{t\geq 0}$ is a martingale. (4.116) is already proven in Proposition 4.6.13 as (4.110). All we are left to show is (4.115) and (4.117).

Proof of (4.115)**.** In Lemma 4.6.14 we obtained that $\partial_s M^N = -\delta_{(0,0)} + (\partial_t - \frac{1}{2}\partial_x^2)\mathscr{X}^N + \delta_T \otimes \mathscr{X}_T^N$ in the sense of distributions, which by disregarding the boundary terms implies that

$$(\partial_s M^N, \varphi) = ((\partial_t - \tfrac{1}{2}\partial_x^2)\mathscr{X}^N, \varphi)$$

for all φ of compact support contained in $\Lambda_{[\varepsilon, T-\varepsilon]}$ for some $\varepsilon > 0$. Note that this relation is still respected by any limit points and hence remains true when $N = \infty$. We thus have

$$(\partial_s M^\infty, \varphi) = ((\partial_t - \tfrac{1}{2}\partial_x^2)\mathscr{X}^\infty, \varphi)$$

Since both $\mathscr{X}^\infty$ and M^∞ lie in spaces with strong enough topologies ($C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Lambda_T)$ and $C([0,T], C^{\beta,\tau}(\mathbb{R}))$ respectively), taking $\varphi(u,x)$ to approach the function $(u,x) \mapsto \mathbb{1}_{[s,t]}(u)\phi(x)$ in the above equation leads to (4.115).

Proof of (4.117)**.** Fix any $0 \leq t \leq T$ and let μ be a $\mathcal{S}'(\mathbb{R})$ valued random variable defined as

$$(\mu, \phi) := Q_t^\infty(\phi) - \sigma \int_{\mathbb{R}} \int_0^t (\mathscr{X}_s^\infty(x))^2 \phi(x)\, ds\, dx.$$

We claim that $(\mu, \phi) = 0$ a.s. for all $\phi \in \mathcal{S}(\mathbb{R})$. This will validate (4.117). To verify this, let us define a sequence of function-valued random variables:

$$\mu^\varepsilon(a) := Q_t^\infty(\xi_\varepsilon^a) - \sigma \int_0^t (\mathscr{X}_s^\infty(\xi_{\varepsilon\sqrt{2}}^a))^2 ds$$

where $\xi_\varepsilon^a(x) := \varepsilon^{-1}\xi(\varepsilon^{-1}(x-a))$ with $\xi(x) := \frac{1}{\sqrt{\pi}} e^{-x^2}$. Observe that for any $\phi \in C_c^\infty(\mathbb{R})$

supported on $[-S, S]$, by Cauchy-Schwarz and Jensen's inequality we have that

$$\mathbb{E}\big[|(\mu^\varepsilon, \phi)|\big] \leq \|\phi\|_{L^2(\mathbb{R})} \cdot \mathbb{E}\bigg[\sqrt{\int_{[-S,S]} \mu^\varepsilon(a)^2 da}\bigg] \leq \|\phi\|_{L^2} \bigg[\int_{[-S,S]} \mathbb{E}[\mu^\varepsilon(a)^2] da\bigg]^{1/2}.$$

We now make use of Proposition 4.5.3. From (4.60), we see that the integral is uniformly bounded in $\varepsilon > 0$. On the other hand, (4.59) tells us that the integrand goes to zero as $\varepsilon \to 0$. Thus, by the dominated convergence theorem $(\mu^\varepsilon, \phi) \to 0$ in probability. But $(\mu^\varepsilon, \phi) \to (\mu, \phi)$ almost surely as $\varepsilon \to 0$. Hence, $(\mu, \phi) = 0$ almost surely. This proves the claim for compactly supported ϕ. For general $\phi \in \mathcal{S}(\mathbb{R})$, we may find a sequence of functions $\phi_n \in C_c^\infty(\mathbb{R})$ such that $\phi_n \to \phi$ in the topology of $\mathcal{S}(\mathbb{R})$. Then $0 = (\mu, \phi_n) \to (\mu, \phi)$ almost surely as $n \to \infty$. Hence, $(\mu, \phi) = 0$ almost surely for all $\phi \in \mathcal{S}(\mathbb{R})$. This verifies the claim, completing the proof. □

We now complete the proof of our main theorem, Theorem 4.1.4.

Proof of Theorem 4.1.4(a). We continue with the notation and setup of Theorem 4.6.16. We have already established the tightness of $\mathscr{X}^N$ in the topology of $C_{\mathfrak{s}}^{\alpha,\tau}(\Lambda_T)$ in Proposition 4.6.12. Consider any limit point X^∞ of $\mathscr{X}^N$. From the previous theorem, we already know that $(t, x) \mapsto X_{t+\varepsilon}^\infty(x)$ is a continuous function in space and time. From the three equations (4.115), (4.116), and (4.117) in Theorem 4.6.16 it follows that the martingale problem for (SHE) is satisfied by any limit point X^∞. We refer the reader to [88, Proposition 4.11] for the characterization of the law of (SHE) as the solution to this martingale problem.

The result there is only stated for continuous initial conditions, so what this really shows is that for any $\varepsilon > 0$ the law of the continuous field $(t, x) \mapsto X_{t+\varepsilon}^\infty(x)$ is that of the solution of (SHE) with initial condition $X_\varepsilon^\infty(\cdot)$. Thus we still need to pin down the initial data as δ_0, by showing that we can let $\varepsilon \to 0$ and see that the limit of $X_\varepsilon^\infty(\cdot)$ is equal to δ_0 in some sense.

In [229, Section 6] there is a general approach to do exactly this. Specifically, it suffices to

show as in Lemma 6.6 of that paper the two bounds

$$\mathbb{E}[|X_t^\infty(x)|^r]^{2/r} \leq \mathrm{C} \cdot t^{-1/2} p_t(x), \tag{4.118}$$

$$\mathbb{E}[|X_t^\infty(x) - p_t(x)|^r]^{2/r} \leq \mathrm{C} \cdot p_t(x), \tag{4.119}$$

where $r > 1$ is arbitrary, p_t is the standard heat kernel, and C is independent of $t > 0$ and $x \in \mathbb{R}$. Clearly, it suffices to show this when r is an even integer. Using L^2 bounds for each term in the chaos expansion of (SHE) with δ_0 initial condition (see Lemma 2.4 in [230]), it follows that the solution of (SHE) with δ_0 initial condition satisfies the above two bounds for $r = 2$. The bounds for general r then follow from the $r = 2$ case by the hypercontractivity inequality [231, Theorem 1.4.1]. By the moment convergence result (Proposition 4.1.3), we know any limit point X^∞ must satisfy $\mathbb{E}[(\int_{\mathbb{R}} X_t^\infty(x)\phi(x)dx)^k] = \mathbf{E}[(\int_{\mathbb{R}} \mathcal{Z}_t(x)\phi(x)dx)^k]$ for all $k \in \mathbb{N}$ and $\phi \in C_c^\infty(\mathbb{R})$, where $(t, x) \mapsto \mathcal{Z}_t(x)$ solves (SHE) with δ_0 initial data and $\sigma = \frac{1}{2\nu([0,1])}$. From here we can conclude by letting $\phi \to \delta_x$ that $\mathbb{E}[X_t^\infty(x)^k] = \mathbf{E}[\mathcal{Z}_t(x)^k]$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}$. Consequently, we may immediately deduce (4.118) and (4.119) by the corresponding bounds for $\mathcal{Z}_t$. This completes the proof. □

Finally, we conclude this section by proving Theorem 4.1.4(b) which follows from the existence of a certain bounded linear operator.

Proof of Theorem 4.1.4(b). Fix $\alpha < 0$. We claim that there exists a bounded linear map $\overline{K\partial_s} : C([0,T], C^{\alpha,\tau}(\mathbb{R})) \to C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ such that for all $v = (v_t)_{t\in[0,T]} \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ one has that $\overline{K\partial_s}v = K(\partial_s(v))$ where K and ∂_s are the same operators defined in Propositions 4.6.6 and 4.6.14 respectively (where the equality may be interpreted in the sense that both sides are viewed as elements of $\mathcal{S}'(\mathbb{R}^2)$).

Note that the existence of $\overline{K\partial_s}$ would immediately imply the desired result. Indeed by applying K to both sides of (4.112) and using Proposition 4.6.6 and Lemma 4.6.10 we get that

$$\mathscr{X}^N = p + K(\partial_s(M^N)),$$

where p is the standard heat kernel. We also know from Proposition 4.6.13 that the M^N are tight in $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$ for $\gamma < -2$, therefore tightness of $\mathscr{X}^N$ in $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$ for $\gamma < -2$ would be immediate by continuity of $\overline{K\partial_s}$. The fact that any limit point coincides with the law of (SHE) follows from Theorem 4.6.16, (4.118) and (4.119) by taking a joint limit point with the triple appearing there, then using the fact that $K(\partial_s(M^N)) = \overline{K\partial_s}M^N$.

We now turn toward the existence of $\overline{K\partial_s}$. To construct $\overline{K\partial_s}$ we first remark that $C_c^\infty(\Lambda_T)$ embeds into $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ in the natural way: a function $v \in C_c^\infty(\Lambda_T)$ may be identified with $(v(t))_{t\in[0,T]}$ given by $(v(t),\phi)_{L^2(\mathbb{R})} := \int_{\mathbb{R}} v(t,x)\phi(x)dx$. Then $\partial_s v =: v'$ is given by $(v'(t),\phi) := \int_{\mathbb{R}}(\partial_t v)(t,x)\phi(x)dx$. In this case we may integrate by parts to obtain

$$\begin{aligned}(K(\partial_s v)(t),\phi) &= \left(\int_0^t P_{t-s}v'(s)ds\ ,\ \phi\right)\\ &= \left(v(t) - P_t v(0) + \int_0^t \partial_s P_{t-s}v(s)ds\ ,\ \phi\right)\\ &= (v(t),\phi) - (v(0),P_t\phi) - \int_0^t (v(s), P_{t-s}\partial_x^2\phi)ds,\end{aligned}$$

where all pairings are in $L^2(\mathbb{R})$ and we use that $\partial_s P_{t-s} = -\partial_x^2 P_{t-s}$, and ∂_x^2 and P_t are self-adjoint operators on $L^2(\mathbb{R})$.

Based on this calculation, we consider a general path $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ and we finally define $\overline{K\partial_s}v$ suggestively by the formula

$$\big(\big[\ \overline{K\partial_s}v\ \big](t),\phi) := (v(t),\phi) - (v(0),P_t\phi) - \int_0^t (v(s),P_{t-s}\partial_x^2\phi)ds. \tag{4.120}$$

First note that the pairings on the right-hand side are indeed well-defined for all $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. What remains to be seen is that the integral is convergent and that the right-hand side is indeed an element of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ whose norm may be controlled by $\|v\|_{C([0,T],C^{\alpha,\tau}(\mathbb{R}))}$. By the definition of these spaces, we must replace ϕ by $S_x^\lambda\phi$ (where the scaling operators are given in Definition 4.6.1) and then study the growth as λ becomes close to 0.

The first two terms on the right side of (4.120) are completely straightforward to deal with: the

growth is at worst $\lambda^\alpha(1+x^2)^\tau$ uniformly over λ, ϕ, x, T, since we assumed $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. To deal with the third term, we claim that one has

$$|(v(s), P_{t-s}\partial_x^2 S_x^\lambda \phi)| \lesssim \big(\lambda^{\alpha-2} \wedge (t-s)^{\frac{\alpha}{2}-1}\big)(1+x^2)^\tau \tag{4.121}$$

uniformly over $s < t \in [0,T]$, as well as λ, ϕ, x. Indeed the bound of the form $\lambda^{\alpha-2}$ follows by noting that when $t-s$ is much smaller than λ, P_{t-s} is essentially the identity operator, so we can effectively disregard the heat kernel and note that $\partial_x^2 S_x^\lambda \phi$ paired with $v(s)$ satisfies a bound of order $\lambda^{\alpha-2}$. Likewise the bound of the form $(t-s)^{\frac{\alpha}{2}-1}$ is obtained by noting that when λ is very small compared to $t-s$, $P_{t-s}S_x^\lambda\phi$ behaves like $S_x^{\sqrt{t-s}}\phi$, giving a bound of order $(\sqrt{t-s})^{\alpha-2}$ after applying ∂_x^2 and pairing with $v(s)$. This proves the bound (4.121).

Now the fact that $\|\overline{K\partial_s}v\|$ can be controlled by $\|v\|$ follows simply by noting that uniformly over $0 < \lambda^2 \leq t \leq T$ one has

$$\int_0^t \big(\lambda^{\alpha-2} \wedge (t-s)^{\frac{\alpha}{2}-1}\big)ds = \int_0^{t-\lambda^2} (t-s)^{\frac{\alpha}{2}-1}ds + \int_{t-\lambda^2}^t \lambda^{\alpha-2}ds \leq \mathrm{C}(\alpha)\cdot\lambda^\alpha.$$

It remains to verify that $\overline{K\partial_s}v$ is indeed a *continuous* path. So far, what we have effectively shown is that $\overline{K\partial_s}$ is a bounded operator from $L^\infty([0,T], C^{\alpha,\tau}(\mathbb{R}))$ to itself. Note that $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ is precisely the closure of $\mathcal{S}(\Lambda_T)$ inside of $L^\infty([0,T], C^{\alpha,\tau}(\mathbb{R}))$, where $\mathcal{S}(\Lambda_T)$ denotes restrictions to Λ_T of those functions in $\mathcal{S}(\mathbb{R}^2)$. From here, path continuity of $\overline{K\partial_s}v$ for all $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ is immediate because the bounded operator $\overline{K\partial_s} : L^\infty([0,T], C^{\alpha,\tau}(\mathbb{R})) \to L^\infty([0,T], C^{\alpha,\tau}(\mathbb{R}))$ maps the linear subspace $\mathcal{S}(\Lambda_T)$ into itself.

Since $\overline{K\partial_s}$ is indeed a bounded operator, and since $\overline{K\partial_s}v = K(\partial_s(v))$ for smooth paths $v \in C_c^\infty(\Lambda_T)$ as shown above, this equality remains true for general v by the denseness of the linear subspace $C_c^\infty(\Lambda_T)$ in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. □

We remark that the above proof is essentially sharp, in the sense that K improves parabolic regularity by 2, while ∂_s destroys parabolic regularity by 2. Hence, we expect their composition to preserve regularity, which is what we showed. Nonetheless, the Hölder exponent appearing in

Theorem 4.1.4(b) may not be sharp, because we do not know if we obtained tightness of M^N in the best possible space in Proposition 4.6.13.

4.7 Quenched tail field

The goal of this section is to prove the results from Section 4.1.3, namely Theorem 4.1.7 and Theorem 4.1.8. Recall the quenched tail field $F_N(t,x)$ from Definition 4.1.6. We begin with a preliminary estimate for the first moment of $F_N(t,x)$.

Lemma 4.7.1. *Let B be a standard Brownian motion. For all $t > 0, x \in \mathbb{R}$, we have*

$$\mathbb{E}[F_N(t,x)] = N^{1/4}\mathbf{E}[e^{-N^{1/4}(B_t - x)}\mathbb{1}_{\{B_t \geq x\}}] \leq (2\pi t)^{-1/2}.$$

Proof. The proof proceeds by writing F_N as a quenched expectation, exactly as we did in Section 4.1.5 or Section 4.2. Then one takes the annealed expectation over the quenched expectation and uses the scale invariance of Brownian motion to obtain

$$\mathbb{E}[F_N(t,x)] = N^{1/4}e^{\frac{t}{2}\sqrt{N}+N^{1/4}x}\mathbf{P}(B_t - N^{1/4}t \geq x).$$

Now applying Girsanov with the stochastic exponential of $-N^{1/4}B$ will give the claimed equality. To prove the inequality, we note that

$$\begin{aligned} N^{1/4}\mathbf{E}[e^{-N^{1/4}(B_t - x)}\mathbb{1}_{\{B_t \geq x\}}] &= N^{1/4}\int_x^\infty e^{-N^{1/4}(u-x)}p_t(u)du \\ &\leq (2\pi t)^{-1/2}\int_x^\infty N^{1/4}e^{-N^{1/4}(u-x)}du = (2\pi t)^{-1/2}. \end{aligned}$$

□

We first prove a variant of Theorem 4.1.7 where the quenched tail field is integrated against test functions.

Proposition 4.7.2. *Fix $m \in \mathbb{N}$. For all $\phi_1, \dots, \phi_m \in C_c^\infty(\mathbb{R})$ and $t_1, \dots, t_m > 0$ we have*

$$\left(\int_{\mathbb{R}} \phi_i(x) F_N(t_i, x) dx \right)_{i=1}^m \xrightarrow{d} \left(\int_{\mathbb{R}} \phi_i(x) \mathcal{Z}_{t_i}(x) dx \right)_{i=1}^m. \tag{4.122}$$

Proof. Let us formally define

$$\mu_t^N(dx) := N^{1/2} e^{\frac{t}{2}\sqrt{N}} K_{0,Nt}(N^{3/4}t + N^{1/2}x) dx.$$

The above definition can be interpreted rigorously as a random Borel measure by pushing forward $K_{0,Nt}$ under a linear change of coordinates, just as we did in (4.4). Then we notice that Definition 4.1.6 is equivalent to

$$F_N(t, x) := N^{1/4} e^{N^{1/4}x} \mu_t^N[x, \infty).$$

Integrating by parts we have

$$\int_{-\infty}^x N^{1/4} e^{N^{1/4}u} \mu_t^N[u, \infty) du = \int_{-\infty}^x e^{N^{1/4}u} \mu_t^N(du) + e^{N^{1/4}x} \mu_t^N[x, \infty).$$

In the sense of distributions, it is clear that

$$F_N(t, x) = \partial_x \left[\int_{-\infty}^x N^{1/4} e^{N^{1/4}u} \mu_t^N[u, \infty) du \right], \quad e^{N^{1/4}x} \mu_t^N(dx) = \partial_x \left[\int_{-\infty}^x e^{N^{1/4}u} \mu_t^N(du) \right].$$

Consequently, for each $1 \le i \le m$, we can repeatedly integrate by parts to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} \phi_i(x) F_N(t_i, x) dx &= -\int_{\mathbb{R}} \phi_i'(x) \left[\int_{-\infty}^x N^{1/4} e^{N^{1/4}u} \mu_{t_i}^N[u, \infty) du \right] dx \\
&= -\int_{\mathbb{R}} \phi_i'(x) \left[\int_{-\infty}^x e^{N^{1/4}u} \mu_{t_i}^N(du) + e^{N^{1/4}x} \mu_{t_i}^N[x, \infty) \right] dx \\
&= \int_{\mathbb{R}} \phi_i(x) e^{N^{1/4}x} \mu_{t_i}^N(dx) - \int_{\mathbb{R}} \phi_i'(x) e^{N^{1/4}x} \mu_{t_i}^N[x, \infty) dx.
\end{aligned}$$

Thanks to Theorem 4.1.4(b) (see Remark 4.1.5(b)) we know that as $N \to \infty$

$$\left(\int_{\mathbb{R}} \phi_i(x) e^{N^{1/4}x} \mu^N_{t_i}(dx) \right)_{i=1}^m \xrightarrow{d} \left(\int_{\mathbb{R}} \phi_i(x) \mathcal{Z}_{t_i}(x) dx \right)_{i=1}^m .$$

Thus, to show (4.122) it suffices to show that as $N \to \infty$, $\int_{\mathbb{R}} \phi_i'(x) e^{N^{1/4}x} \mu^N_{t_i}[x, \infty) dx$ goes to zero in $L^1(\mathbb{P})$ for each $1 \le i \le m$. Indeed, observe that

$$\mathbb{E}\left| \int_{\mathbb{R}} \phi_i'(x) e^{N^{1/4}x} \mu^N_{t_i}[x, \infty) dx \right| \le \int_{\mathbb{R}} |\phi_i'(x)| e^{N^{1/4}x} \mathbb{E}[\mu^N_{t_i}[x, \infty)] dx.$$

By Lemma 4.7.1 we get

$$e^{N^{1/4}x} \mathbb{E}[\mu^N_{t_i}[x, \infty)] = \mathbf{E}[e^{-N^{1/4}(B_{t_i} - x)} \mathbb{1}_{\{B_{t_i} \ge x\}}] \le N^{-1/4} (2\pi t_i)^{-1/2}$$

for all $x \in \mathbb{R}$, which implies the desired L^1 convergence. This completes the proof. □

Note that composition with the logarithm function is not a continuous operation in the topology of integration against test functions, hence the convergence result stated above cannot be directly translated to any convergence result for the KPZ equation. Our next lemma demonstrates uniform convergence for the two-point correlation function of the quenched tail field. This is the crucial result that allows us to improve Proposition 4.7.2 to obtain multipoint convergence in law of $F_N(t, x)$ to $\mathcal{Z}_t(x)$ for *individual* values of $(t, x) \in (0, \infty) \times \mathbb{R}$, after which taking a logarithm becomes sensible.

Lemma 4.7.3. *As $N \to \infty$, we have that*

$$\mathbb{E}[F_N(t, x) F_N(t, y)] \to \mathbf{E}[\mathcal{Z}_t(x) \mathcal{Z}_t(y)]$$

where $\mathcal{Z}$ as in Proposition 4.1.3. Furthermore, the above convergence is uniform over compact sets of $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$.

Proof. By adapting the methods in Section 4.2, the two-point correlation function given by $(t, x, y) \mapsto$

$\mathbb{E}[F_N(t,x)F_N(t,y)]$ can be written as an expression involving only the 2-point motion associated to the kernels $K_{0,t}$. More specifically, in the notation of Section 4.2, it equals

$$Ne^{t\sqrt{N}}\mathbf{P}_{SB^{(2)}_{N^{1/2}\nu}}(X_t - N^{1/4}t \geq x, Y_t - N^{1/4}t \geq y).$$

Consequently the two-point correlation function only depends on the total mass of the characteristic measure. We shall thus assume ν is a multiple of uniform measure. Without loss of generality, we will assume $\sigma = 1$ (i.e., ν is precisely half times the uniform measure).

We shall now invoke results from [82] which are for the case where the characteristic measure is a multiple of the uniform measure on $[0,1]$. In [82], the authors consider a slightly different field $\widehat{F}_N(t,x)$ which is "dual" to ours in a certain sense. More specifically, they vary the starting point x and fix the tail probability as $[0,\infty)$, whereas we fix the starting point 0 and vary the tail probability as $[x,\infty)$. One may show that the distributions of both fields are the same as space-time processes, modulo a reflection of the characteristic measure. More precisely, we have $F_N^{\nu} \stackrel{d}{=} \widehat{F}_N^{\mu}$, where μ is the pushforward of ν by $x \mapsto 1-x$, and the superscript highlights the dependence of each field on the underlying characteristic measure.[1] This is proved using an explicit coupling of the two fields via the Brownian web. See [80, Equation (1.8)] and the subsequent discussion for a short proof, which in turn is based on a construction of [190, Section 3].

Note that for the case where the characteristic measure is a multiple of the uniform measure, we have $\mu = \nu$. Proposition 1.22 in [82] provides exact moment formulas for the unscaled version of $F_N(t,x)$ in this case. Taking the scalings into account we have that

$$\mathbb{E}[F_N(t,x)F_N(t,y)] = \oint_{r_1+i\mathbb{R}} \oint_{r_2+i\mathbb{R}} \frac{N^{-1/2}(z_2-z_1)}{N^{-1/2}(z_2-z_1) - N^{-1/2} - N^{-3/4}(z_1+z_2) - N^{-1}z_1z_2} \cdot e^{\frac{t}{2}(z_1^2+z_2^2)+(xz_1+yz_2)} \frac{N^{-1/2}}{(N^{-1/4}+N^{-1/2}z_1)(N^{-1/4}+N^{-1/2}z_2)} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i},$$

[1] In particular, our results imply multipoint convergence of their field $\widehat{F}_N$ to (SHE) as well.

where r_1, r_2 are fixed numbers not depending on t, x, y and N (large enough), and such that $r_2 > r_1 + 1$.

Now assume that we have a sequence (t^N, x^N, y^N) converging to $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$. Note that for $z_1 \in r_1 + i\mathbb{R}$ and $z_2 \in r_2 + i\mathbb{R}$ we have that $|e^{\frac{t^N}{2}(z_1^2+z_2^2)+(x^N z_1 + y^N z_2)}| \leq \mathrm{C} e^{-\frac{1}{\mathrm{C}}(|z_1|^2+|z_2|^2)}$ for some constant $\mathrm{C} > 0$ which is independent of N, z_1, z_2. Also, note that the ratios appearing in the integral may be bounded independently of N thanks to the fact that $z_1 \in r_1 + i\mathbb{R}$ and $z_2 \in r_2 + i\mathbb{R}$ with $r_2 > r_1 + 1$. Therefore, by the dominated convergence theorem (with dominating function given by $e^{-\frac{1}{\mathrm{C}}(|z_1|^2+|z_2|^2)}$) we may conclude that the integral expression for $\mathbb{E}[F_N(t^N, x^N) F_N(t^N, y^N)]$ converges as $N \to \infty$ to

$$\oint_{r_1+i\mathbb{R}} \oint_{r_2+i\mathbb{R}} \frac{z_2 - z_1}{z_2 - z_1 - 1} e^{\frac{t}{2}(z_1^2+z_2^2)+(xz_1+yz_2)} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i}.$$

This is known to agree with $\mathbf{E}[\mathcal{Z}_t(x)\mathcal{Z}_t(y)]$, see [232, Section 6.2], thus proving the uniform convergence. □

With the above lemma in place, we now prove a stronger version of Theorem 4.1.7.

Proposition 4.7.4. *Suppose that the deterministic sequence of vectors $(x_1^N, \ldots, x_m^N) \in \mathbb{R}^m$ converges as $N \to \infty$ to $(x_1, \ldots, x_m) \in \mathbb{R}^m$. Fix $t_1, \ldots, t_m > 0$. Then we have the joint convergence*

$$\big(F_N(t_i, x_i^N)\big)_{i=1}^m \xrightarrow{d} \big(\mathcal{Z}_{t_i}(x_i)\big)_{i=1}^m.$$

Proof. We first prove the case when $x_i^N = x_i$ for $1 \leq i \leq m$. We give a proof for $m = 1$ to simplify the notation, but the generalization to larger m is straightforward. We will simply write (t_1, x_1) as (t, x), which will be fixed throughout the proof. By Lemma 4.7.1 we get that $\mathbb{E}[F_N(t, x)] \leq (2\pi t)^{-1/2}$ which is independent of N, thus it follows that the $\{F_N\}_{N \geq 1}$ are tight. Consider any limit point μ_0 of the laws of $\{F_N(t, x)\}_{N \geq 1}$. Fix a smooth compactly supported nonnegative function $\phi : \mathbb{R} \to \mathbb{R}$ which integrates to 1, and define $\phi_x^\lambda(y) = \lambda^{-1}\phi(\lambda^{-1}(y - x))$. By Proposition 4.7.2 we know that for each $\lambda, x \in \mathbb{R}$, $\big\{ \int_{\mathbb{R}} \phi_x^\lambda(y) F_N(t, y) dy \big\}_{N \geq 1}$ is a tight sequence.

For $r \geq 1$, consider any measure μ_r on $\mathbb{R}^{r+1}$ which is a joint limit point as $N \to \infty$ of

$$\Big(F_N(t,x) \ , \ \big(\int_{\mathbb{R}} \phi_x^{2^{-k}}(y) F_N(t,y) dy \big)_{k=1}^r \Big),$$

such that the first marginal of μ_r is μ_0. Using a diagonalization argument, these measures μ_r may be chosen so that they form a projective family, and therefore by the Kolmogorov extension theorem, we may consider any projective limit μ which will be a measure on the space of sequences $(a_k)_{k\geq 0} \in \mathbb{R}^{\mathbb{N}_0}$, equipped with the σ-algebra generated by the projection maps. By Proposition 4.7.2, we find that for such a measure μ the marginal distribution of $(a_k)_{k\geq 1}$ is simply equal to the law of $\big(\int_{\mathbb{R}} \mathcal{Z}_t(y) \phi_x^{2^{-k}}(y) dy \big)_{k\geq 1}$. We now claim that

$$\limsup_{\lambda \to 0} \limsup_{N\to\infty} \mathbb{E} \Big| F_N(t,x) - \int_{\mathbb{R}} F_N(t,y) \phi_x^{\lambda}(y) dy \Big| = 0. \tag{4.123}$$

Assuming this fact, one finds that such a measure μ is necessarily supported on those sequences $(a_k)_{k\geq 0}$ which satisfy $a_0 = \lim_{k\to\infty} a_k$ in $L^1(\mu)$, which means that a_0 must have the law of $\mathcal{Z}_t(x)$ under μ. We are thus left to check (4.123). By Lemma 4.7.3, we have that

$$\limsup_{N\to\infty} \mathbb{E}[(F_N(t,x) - F_N(t,y))^2] = \mathbf{E}[(\mathcal{Z}_t(x) - \mathcal{Z}_t(y))^2] \leq \mathrm{C}|x-y|$$

uniformly over compacts sets of $(t,x,y) \in (0,\infty) \times \mathbb{R}^2$ where the above inequality is a known estimate for (SHE) (see [233, Proposition 2.4-nw] for example). Thus, by Fatou's lemma, we note that

$$\begin{aligned}
\limsup_{N\to\infty} \mathbb{E} \Big| F_N(t,x) - \int_{\mathbb{R}} F_N(t,y) \phi_x^{\lambda}(y) dy \Big| &\leq \limsup_{N\to\infty} \int_{\mathbb{R}} \mathbb{E}|F_N(t,x) - F_N(t,y)| \phi_x^{\lambda}(y) dy \\
&\leq \int_{\mathbb{R}} \limsup_{N\to\infty} \mathbb{E}|F_N(t,x) - F_N(t,y)| \phi_x^{\lambda}(y) dy \\
&\leq \int_{\mathbb{R}} \limsup_{N\to\infty} \mathbb{E}[(F_N(t,x) - F_N(t,y))^2]^{1/2} \phi_x^{\lambda}(y) dy \\
&\leq \mathrm{C} \int_{\mathbb{R}} |x-y|^{1/2} \phi_x^{\lambda}(y) dy = \mathrm{C}\lambda^{1/2} \int_{\mathbb{R}} |u|^{1/2} \phi(u) du.
\end{aligned}$$

Taking $\lambda \to 0$ leads to (4.123). To justify the above application of Fatou's lemma with $\limsup$, one needs a bound on the integrand which is independent of N and is in $L^1(\phi_x^\lambda(y)dy)$, for which we may use Lemma 4.7.1 to get

$$\mathbb{E}|F_N(t,x) - F_N(t,y)| \leq \mathbb{E}[F_N(t,x) + F_N(t,y)] \leq 2(2\pi t)^{-1/2}.$$

Finally, the case when x_i^N varies with N follows from the fact that as $N \to \infty$ we have

$$\mathbb{E}\big[\big(F_N(t_i, x_i^N) - F_N(t_i, x_i)\big)^2\big] \to 0$$

due to the uniform convergence in Lemma 4.7.3. This completes the proof. □

Remark 4.7.5. *We remark that one can allow $t_1, \ldots, t_m$ to depend on N as well, and the multi-point convergence result analogous to Proposition 4.7.4 would still be true as long as the t_i^N converge to some positive real numbers t_i as $N \to \infty$. These are the precise modifications needed: first one modifies the proof of Proposition 4.7.2 to allow the t_i to depend on N. This is fairly immediate since Theorem 4.1.4(b) gives convergence in a topology that is uniform in time. Then notice that in Lemma 4.7.3 we have already allowed the t_i to depend on N. Thus we can replace t_i by t_i^N in Proposition 4.7.4 as well (the estimate using Fatou's Lemma will still work), which is enough to establish the claim. After the claim is established, some interesting effects can be observed by setting $t_i^N := t_i + \alpha N^{-1/4}$ for $\alpha \in \mathbb{R}$. Using Definition 4.1.6 one sees in this case that α gets reinterpreted as a shift of the x variable, but with different scaling values.*

Proof of Theorem 4.1.8. We continue with the same notations as in the statement of the Theorem 4.1.8. Fix any $a \in \mathbb{R}$. Set $v := \sqrt{ct}$ and $x := d\sqrt{\frac{t}{c}}$ so that $c = \frac{v^2}{t}$ and $d = vx/t$. It suffices to show that

$$\lim_{N\to\infty} \mathbf{P}_{SB_\nu^{(k)}}\left(\max_{1\leq i\leq k(N)} \big\{X^i(Nt)\big\} - vN^{3/4} - x\sqrt{N} - \tfrac{tN^{1/4}}{v}\big(r_N - \tfrac{1}{4}\log N\big) \leq aN^{1/4}\right)$$
$$= \mathbb{P}(t/v[G + \log \mathcal{Z}_{v^2/t}(vx/t)] \leq a), \tag{4.124}$$

where $\mathcal{Z}$ solves (SHE) with $\sigma = \frac{v}{2t\nu([0,1])}$ and G is an independent Gumbel random variable.

For now, fix any $k \in \mathbb{N}$ which may be arbitrary. Fix a realization of the kernels $\omega = \{K_{s,t} : -\infty < s < t < \infty\}$. As before, we let $\mathbb{P}_\omega^{(k)}$ be the quenched probability of k random motions $(X^i(t))_{i=1}^k$ sampled from these kernels. By the scale invariance of SBM kernels, the random variable

$$\mathbb{P}_\omega^{(k)}\left(\max_{1\le i\le k} \{X^i(Nt)\} \le vN^{3/4} + x\sqrt{N} + \frac{t}{v}N^{1/4}(r_N - \frac{1}{4}\log N) + aN^{1/4} \right)$$

has the same law for any $y > 0$ as the random variable

$$\mathbb{P}_\omega^{(k)}\left(\max_{1\le i\le k} \{X^i(Ny^2t)\} \le y\big[vN^{3/4} + x\sqrt{N} + \frac{t}{v}N^{1/4}(r_N - \frac{1}{4}\log N) + aN^{1/4}\big] \right)$$

except that the characteristic measure in the latter expression has changed to $y \cdot \nu$. Now we make a specific choice of $y := v/t$ and we get by Definition 4.1.6 that it is equal (pathwise, not just in distribution) to the quantity

$$\left(1 - \exp\left[-\frac{v^2}{2t}\sqrt{N} - N^{\frac{1}{4}}\frac{vx}{t} - \frac{av}{t} - r_N \right] \cdot F_N\big(v^2/t,\ \tfrac{vx}{t} + N^{-\frac{1}{4}}(\tfrac{av}{t} + r_N - \tfrac{1}{4}\log N)\big)\right)^k. \tag{4.125}$$

where it is implicit that the characteristic measure associated to F_N is $(v/t)\cdot\nu$. Then we may choose

$$k = k(N) = \left\lfloor \exp\left[\frac{v^2}{2t}\sqrt{N} + \frac{vx}{t}N^{1/4} + r_N \right] \right\rfloor$$

as stated in the theorem to obtain that (4.125) is a quantity of the form $(1 - \frac{u(N)}{k(N)+O_N(1)})^{k(N)}$, where the $O_N(1)$ term is deterministic and bounded between 0 and 1, and $u(N)$ is a sequence of random variables which by Proposition 4.7.4 converges in law to the strictly positive random variable $e^{-\frac{av}{t}}\mathcal{Z}_{v^2/t}(vx/t)$. Since the functions $u \mapsto (1 - \frac{u}{k(N)+O_N(1)})^{k(N)}$ converge uniformly to $u \mapsto e^{-u}$ on compact subsets of $[0,\infty)$, we can conclude that the expression in (4.125), therefore

converges in law to the random variable

$$\exp\big[-e^{-\frac{av}{t}}\mathcal{Z}_{v^2/t}(vx/t)\big],$$

where $\mathcal{Z}$ solves (SHE) with $\sigma = \frac{v}{2t\nu([0,1])}$. Now this convergence in law also implies convergence of the associated annealed expectations thanks to the boundedness of all quantities involved, which implies that the limit of the left side of (4.124) equals

$$\mathbb{E}\bigg[\exp\big[-e^{-\frac{av}{t}}\mathcal{Z}_{v^2/t}(vx/t)\big]\bigg],$$

which agrees with the right-hand side of (4.124) by the explicit form of the Gumbel distribution.

□

Using Remark 4.7.5, one could also consider replacing time Nt by time Nt plus a correction of order N^α for some $\alpha < 1$, and this would change the constants appearing in the proof in an explicit manner and introduce even more terms in the recentering if $\alpha \in [1/2, 1)$. For brevity, we will not explore this.

4.8 Technical lemmas on Brownian bridges

Definition 4.8.1 (Brownian concatenated process)**.** *Fix any $x \in \mathbb{R}$, and $s \in [0,1]$. Let B_{br} be a Brownian bridge on $[0,s]$ from 0 to x. Let W be a standard Brownian motion independent of B_{br}. We define the Brownian concatenated process $B_{s,x}$ with anchor at (s,x) as*

$$B_{s,x}(y) := \begin{cases} B_{\mathrm{br}}(y) & y \in [0,s] \\ W(y-s)+x & y \in [s,1]. \end{cases}$$

Lemma 4.8.2. *Let $W = (W_s)_{s\in[0,1]}$ be a Brownian motion on $[0,1]$. Let $F = (F_s)_{s\in[0,1]} : C[0,1] \to C[0,1]$ be a Borel-measurable functional. Suppose that $\mathbf{E}[\sup_{s\le 1} F_s(W)^2] < \infty$. Suppose further that the function $g(s,x) := \mathbf{E}\left[F_s(B_{s,x})\right]$ is bounded on $[0,1]\times\mathbb{R}$ and continuous in x*

for each $s > 0$. Then for all compactly supported smooth functions ϕ we have the identity

$$\mathbf{E}\Big[\int_0^1 F_s(W)\phi(W_s)ds\Big] = \int_0^1 \int_{\mathbb{R}} g(s,x)p_s(x)\phi(x)\,dx\,ds,$$

and moreover for all $x \in \mathbb{R}$ we have

$$\mathbf{E}\Big[\int_0^1 F_s(W)dL_x^W(s)\Big] = \int_0^1 g(s,x)p_s(x)ds,$$

where $dL_x^W(s)$ is the Lebesgue-Stiltjes measure associated to the nondecreasing function $s \mapsto L_x^W(s)$.

We remark that all expectations appearing above are finite. Let $Z := \sup_{s\le 1}|F_s(W)|$. Note that $\big|\int_0^1 F_s(W)\phi(W_s)ds\big| \le Z \cdot \int_0^1 |\phi(W_s)|ds$ and $\big|\int_0^1 F_s(W)dL_x^W(s)\big| \le Z \cdot L_x^W(1)$. Therefore a simple application of Cauchy-Schwarz (noting that $\mathbf{E}[Z^2] < \infty$ by assumption) shows that all expectations above are finite.

Proof. Note that for all ϕ continuous and compactly supported, by the tower property of conditional expectation we have

$$\mathbf{E}\Big[\int_0^1 F_s(W)\phi(W_s)ds\Big] = \int_0^1 \mathbf{E}[F_s(W)\phi(W_s)]ds = \int_0^1 \mathbf{E}\big[\mathbf{E}[F_s(W)|W_s]\phi(W_s)\big]ds.$$

Now the law of W conditioned on W_s is precisely given by the Brownian concatenated process with anchor at (s, W_s). Thus, $\mathbf{E}[F_s(W)|W_s] = g(s, W_s)$. Therefore, the above expectation equals

$$\int_0^1 \mathbf{E}[g(s,W_s)\phi(W_s)]ds = \int_0^1 \int_{\mathbb{R}} g(s,x)p_s(x)\phi(x)dxds,$$

proving the first identity. To prove the second identity, we will let ϕ converge weakly to a Dirac mass and use the continuity in x of g. Let ϕ be smooth, nonnegative, even, supported in $[-1,1]$, and suppose that it integrates to 1. Let $\phi^\varepsilon(u) := \varepsilon^{-1}\phi(\varepsilon^{-1}u)$. It is standard that the random measure on $[0,1]$ given by $\phi^\varepsilon(x - W_s)ds$ converges almost surely as $\varepsilon \to 0$ to $L_x^W(ds)$, with respect to the

topology of weak convergence on the space of finite measures on $[0,1]$. Consequently, since we assume continuity of $s \mapsto F_s(W)$ we have the almost sure convergence

$$\int_0^1 F_s(W)\phi^\varepsilon(x - W_s)ds \xrightarrow{\varepsilon \to 0} \int_0^1 F_s(W)L_x^W(ds). \tag{4.126}$$

Now by the first identity, the expectation of the left-hand side equals

$$\int_0^1 H_x^\varepsilon(s)ds, \quad \text{where } H_x^\varepsilon(s) := \int_{\mathbb{R}} g(s,u)p_s(u)\phi^\varepsilon(x-u)du.$$

It is clear from the continuity of g in x that $H_x^\varepsilon(s) \to g(s,x)p_s(x)$ as $\varepsilon \to 0$, and moreover since g is assumed to be globally bounded and ϕ is supported in $[-1,1]$, we have that

$$|H_x^\varepsilon(s)| \leq \max_{u \in [x-1,x+1]} |g(s,u)|p_s(u) \leq \|g\|_{L^\infty} s^{-1/2}.$$

Hence by the dominated convergence theorem we find that

$$\mathbf{E}\bigg[\int_0^1 F_s(W)\phi^\varepsilon(x-W_s)ds\bigg] = \int_0^1 H^\varepsilon(s)ds \to \int_0^1 g(s,x)p_s(x)ds.$$

Thus it suffices to prove that (4.126) also holds in L^1. To prove that, we note that

$$\bigg|\int_0^1 F_s(W)\phi^\varepsilon(x-W_s)ds\bigg| \leq \int_0^1 |F_s(W)|\phi^\varepsilon(x-W_s)ds \leq Z\int_0^1 \phi^\varepsilon(x-W_s)ds,$$

where $Z := \sup_{s\in[0,1]}|F_s(W)|$. By the occupation time formula for the local time, we have

$$\sup_{\varepsilon\in(0,1]}\int_0^1 \phi^\varepsilon(x-W_s)ds = \sup_{\varepsilon\in(0,1]}\int_{\mathbb{R}} \phi^\varepsilon(x-u)L_u^W(1)du \leq \sup_{u\in\mathbb{R}} L_u^W(1).$$

Summarizing the last two expressions and applying the Cauchy-Schwarz inequality, we have that

$$\mathbf{E}_{BM}\bigg[\sup_{\varepsilon\in(0,1]}\bigg|\int_0^1 F_s(W)\phi^\varepsilon(x-W_s)ds\bigg|\bigg] \leq \mathbf{E}[Z^2]^{1/2}\mathbf{E}\big[\sup_{u\in\mathbb{R}} L_u^W(1)^2\big]^{1/2}.$$

The expected square of the sup over local times is finite, e.g., by [234, Chapter XI, Theorem (2.4)]. Therefore, we have uniform integrability and thus L^1 convergence in (4.126) as desired. □

Lemma 4.8.3. *Let $B^{s,x}(t)$ be a Brownian bridge on $[0,s]$ from 0 to x. Denote by $L_0^{B^{s,x}}(t)$ the local time at 0 of $B^{s,x}$ at time t. Then the map $(s,x) \mapsto L_0^{B^{s,x}}(s)$ has a modification which is almost surely continuous in both variables on $(0,1] \times \mathbb{R}$.*

Proof. Let $G(s,x) := L_0^{B^{s,x}}(s)$. Note that the zero set of $B^{s,xs^{1/2}}$ is just a reparametrization of that of $B^{x,1}$, in fact $B^{x,xs^{1/2}}$ is just $s^{1/2}$ multiplied by a reparametrization of $B^{x,1}$, and so we have that $G(s, xs^{1/2}) = G(1,x)$. Thus $G(s,x) = G(1, xs^{-1/2})$ and therefore it suffices to show a.s. Hölder continuity of $x \mapsto G(1,x)$.

To prove continuity of $x \mapsto G(1,x)$, let us simplify notation and henceforth write $B^{1,x} = B^x$. Now write $L_0^{B^x}(1) = L_0^{B^x}(1/2) + L_0^{W^x}(1/2)$, where $W^x(t) = B^x(1-t)$. Note that on the time interval $[0,1/2]$, B^x is absolutely continuous with respect to a Brownian motion of drift x started from 0, and moreover the Radon-Nikodym derivative is *independent* of x. Likewise, W^x is absolutely continuous with respect to a Brownian motion of drift $-x$ started from x, and similarly, the Radon-Nikodym derivative is *independent* of x. Consequently, for a standard Brownian motion Z if we define $Z^{a,\lambda}(t) := a + \lambda t + Z(t)$, then it suffices to show that $(a,\lambda) \mapsto L_0^{Z^{a,\lambda}}(1/2)$ is a.s. jointly Hölder continuous as this would imply the desired result for both B^x and W^x. Note that $L_0^{Z^{a,\lambda}}(1/2) = L_{-a}^{Z^{0,\lambda}}(1/2)$. For Brownian motion, joint continuity of local times in the drift and the spatial variable is now a standard result. □

Lemma 4.8.4. *Fix $a,b,c,d,x \in \mathbb{R}$ with $c^2 + d^2 = 1$. Let B be a Brownian bridge from 0 to x on $[0,1]$ and let W be an independent standard Brownian motion. Consider the process $X := a + cB + dW$. For all $\theta > 0$, we have*

$$\mathbf{E}[e^{\theta L_0^X(1)}] \leq \theta\sqrt{2\pi}e^{\theta^2/2}. \tag{4.127}$$

Remark 4.8.5. *Since the above bound is uniform in a,b,c,d,x, the same bound continues to hold when a,b,c,d,x are random variables (with $c^2+d^2=1$) independent of B and W.*

Proof. Note that conditioned on $W(1)$, X is a standard Brownian bridge from a to $a+cx+dW(1)$. In [235, (3)] it is shown that for a standard Brownian bridge Y from a to $a+b$ one has for all $v>0$ that

$$\mathbb{P}(L_0^Y(1)>v)=e^{b^2/2}e^{-(|a|+|a+b|+v)^2/2}$$

and therefore if $\theta>0$ then

$$\mathbf{E}[e^{\theta L_0^Y(1)}]=\int_0^\infty \theta e^{\theta y}\mathbb{P}(L_0^Y(1)>y)dy\le \theta e^{b^2/2}\int_{\mathbb{R}} e^{\theta y}e^{-(|a|+|a+b|+y)^2/2}dy=\theta\sqrt{2\pi}e^{\theta^2/2}e^{-(|a|+|a+b|)\theta}.$$

Thus returning to the process X we simply take $b=cx+dW(1)$, and we find that

$$\mathbf{E}[e^{\theta L_0^X(1)}|W(1)]\le \theta\sqrt{2\pi}e^{\theta^2/2}e^{-(|a|+|a+cx+dW(1)|)\theta}.$$

Taking the expectation again and noting that $e^{-(|a|+|a+cx+dW(1)|)\theta}\le 1$ we arrive at (4.127), proving the claim. □

Lemma 4.8.6. *Fix $r,m\in\mathbb{Z}_{>0}$. Fix $\mathbf{x}=(x_1,x_2,\dots,x_r)\in\mathbb{R}^r$, $\mathbf{t}=(t_1,t_2,\dots,t_m)\in[0,1]^m$, and $\mathbf{s}=(s_1,s_2,\dots,s_r)\in[0,1]^r$. Let $\mathbf{B}^{\mathbf{s},\mathbf{x}}=(B^i)_{i=1}^r$ be r-many independent processes with each B^i being distributed as a Brownian concatenated process with the anchor at (s_i,x_i) (recall Definition 4.8.1). Let $\ell_k:\mathbb{R}^n\to\mathbb{R}$ be linear functions for $1\le k\le m$. Define*

$$f(\mathbf{x},\mathbf{s},\mathbf{t}):=\mathbf{E}\left[\exp\left(\sum_{k=1}^m L_0^{\ell_k(\mathbf{B}^{\mathbf{s},\mathbf{x}})}(t_k)\right)\right].$$

Then f is globally bounded and continuous on $\mathbb{R}^n\times(0,1]^r\times(0,1]^m$.

Proof. We use a coupling argument based on Lemma 4.8.3. Take $W^1,\dots,W^n$ to be i.i.d. Brownian motions on $[0,1]$ and then define

$$B^{j,s,x}(t):=W^j(t)-(\tfrac{t}{s}\wedge 1)(W^j(s)-x).$$

Note that for all r-tuples of pairs (s_j,x_j) the processes $(B^{j,s_j,x_j})_{j=1}^r$ are distributed as $(B^i)_{i=1}^r$.

Applying Lemma 4.8.3 we see that if $(\mathbf{x}_n, \mathbf{s}_n, \mathbf{t}_n)_{n\in\mathbb{N}}$ converges to $(\mathbf{x}, \mathbf{s}, \mathbf{t}) \in \mathbb{R}^n \times (0,1]^r \times (0,1]^m$, we have

$$L_0^{\ell_k(\mathbf{B}^{\mathbf{s}_n,\mathbf{x}_n})}(t_{n,k}) \to L_0^{\ell_k(\mathbf{B}^{\mathbf{s},\mathbf{x}})}(t_k)$$

for each $1 \le k \le n$ almost surely. Consequently, the exponential of the sum over k also converges almost surely. To show convergence of the expectations of those exponentials we need uniform integrability of the exponentials. We will prove this in a way that also gives global boundedness of the function f. Since local time is an increasing process we have the automatic bound $f(\mathbf{x}, \mathbf{s}, \mathbf{t}) \le f(\mathbf{x}, \mathbf{s}, \mathbb{1})$. To prove uniform integrability we will upper-bound $f(\mathbf{x}, \mathbf{s}, \mathbb{1})$ in L^2 independently of s_j, x_j. First, apply Hölder's inequality to obtain

$$\mathbf{E}\left[\exp\left(\sum_{k=1}^m L_0^{\ell_k(\mathbf{B}^{\mathbf{s},\mathbf{x}})}(1)\right)\right] \le \prod_{k=1}^m \mathbf{E}\left[\exp\left(mL_0^{\ell_k(\mathbf{B}^{\mathbf{s},\mathbf{x}})}(1)\right)\right]^{1/m}.$$

Thus, it suffices to show each term in the product above is finite. Let us take any $\ell(\mathbf{v}) := a_1 v_1 + \cdots + a_r v_r$. Assume without loss of generality that $s_1 \le \cdots \le s_r$. Then on each subinterval, $[s_j, s_{j+1}]$, we have

$$\ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(t) = \sum_{i=1}^j a_i[W^i(t) - (W^i(s_i) - x_i)] + \sum_{i=j+1}^r a_i[W^i(t) - \tfrac{t}{s_i}(W^i(s_i) - x_i)].$$

Let us set

$$D_{1,j} := \sum_{i=1}^j a_i^2, \qquad D_{2,j} := \sum_{i=j+1}^r a_i^2, \qquad \gamma := D_{2,j} \cdot \big[\sum_{i=j+1}^r a_i^2/s_i\big]^{-1}. \tag{4.128}$$

Note that $\ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(t + s_j) - \ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(s_j)$ has mean $t\sum_{i=j+1}^r \frac{a_i x_i}{s_i}$, and for $s, t \in [0, s_{j+1} - s_j]$ its covariance function is given by

$$C(s,t) := (s \wedge t) \cdot D_{1,j} + \sum_{i=j+1}^r a_i^2\big[(s \wedge t) - \tfrac{ts}{s_i}\big] = (s \wedge t) \cdot D_{1,j} + \big[(s \wedge t) - \tfrac{ts}{\gamma}\big] \cdot D_{2,j}.$$

As $s_1 \leq \cdots \leq s_r$, one can check that $\gamma \geq (s_{j+1} - s_j)$. Thus,

$$\ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(t + s_j) - \ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(s_j) \stackrel{d}{=} U_1(t) + U_2(t)$$

as processes on $[0, s_{j+1} - s_j]$ where U_1 is a Brownian bridge with diffusion rate $D_{2,j}$ from 0 to $\gamma \sum_{i=j+1}^{r} a_i x_i / s_i$ on $[0, \gamma]$ and U_2 is an independent Brownian motion with diffusion rate $D_{1,j}$.

On $[0, s_{j+1} - s_j]$, we have shown that $\ell(\mathbf{B}^{\mathbf{s},\mathbf{x}}) - \ell(\mathbf{B}^{\mathbf{s},\mathbf{x}})(s_j)$ is distributed as the independent sum of a Brownian bridge (not necessarily ending at zero) and a Brownian motion whose diffusivities sum to $\|\ell\|_2^2 = a_1^2 + \cdots + a_r^2$. Consequently, by breaking up the expectation on each of these subintervals (by again applying Hölder's inequality) and using the fact that each of these subintervals has size at most 1, it suffices to bound $\mathbf{E}[e^{\theta L_0^X(1)}]$ for all $\theta > 0$ where $X := a + cB + dW$, where B is a standard Brownian bridge from 0 to x on $[0, 1]$ and W is an independent standard Brownian motion on $[0, 1]$. More precisely, we need a bound which is uniform over all $a \in \mathbb{R}$ and $c^2 + d^2 = 1$. This is precisely done in Lemma 4.8.4. This completes the proof. □

We remark that Definition 4.8.1 can be naturally extended to Brownian concatenated processes defined on $[0, T]$ for any fixed $T > 0$. Both Lemma 4.8.2 and Lemma 4.8.6 continue to hold under this setting.

Chapter 5: Multiplicative SHE limit of random walks in space-time random environments

This chapter is adapted from the paper [72] and is coauthored with Sayan Das and Shalin Parekh.

5.1 Introduction

The (1+1) dimensional Kardar–Parisi–Zhang (KPZ) equation [2] is a stochastic partial differential equation (SPDE) given by

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \gamma \cdot \xi, \qquad \mathcal{H} = \mathcal{H}_t(x), \tag{KPZ}$$

where $\gamma > 0$ and $\xi = \xi(t,x)$ is a space-time white noise. The KPZ equation has been studied intensively over the past few decades in both the mathematics and physics literature. We refer to [179, 132, 133, 180, 181, 182] for some surveys of the mathematical studies of the KPZ equation.

As an SPDE, (KPZ) is ill-posed due to the presence of the non-linear term $(\partial_x \mathcal{H})^2$. One way to make sense of the equation is to consider $\mathcal{U} := e^{\mathcal{H}}$ which formally solves the stochastic heat equation (SHE) with multiplicative noise:

$$\partial_t \mathcal{U} = \tfrac{1}{2}\partial_{xx}\mathcal{U} + \gamma \cdot \mathcal{U}\xi, \qquad \mathcal{U} = \mathcal{U}_t(x). \tag{SHE}$$

The SHE is known to be well-posed and has a well-developed solution theory based on the Itô integral and chaos expansions [11, 92, 132, 133]. In this paper, we will consider the solution to (SHE) started with Dirac delta initial data $\mathcal{U}_0(x) = \delta_0(x)$. For this initial data, [10] established that $\mathcal{U}_t(x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ almost surely (see also [9]). Thus $\mathcal{H} = \log \mathcal{U}$ is well-defined

and is called the Cole-Hopf solution of the KPZ equation. This is the notion of solution that we will work with in this paper, and it coincides with other existing notions of solutions [3, 4, 5, 6, 7, 8], under suitable assumptions.

The subject of the present work will be to show that (SHE) arises as the fluctuations limit of a simple model of random walks in random environments (RWRE), with a nontrivial noise coefficient γ that has an interesting story behind it. Consider a family of iid $[0,1]$-valued random variables $\omega_{r,y}$ with $r \in \mathbb{Z}_{\geq 0}$ and $y \in \mathbb{Z}$, drawn from a common law ν. A random walk $(R(r))_{r\geq 0}$ in the environment ω starts from $(r,y) = (0,0)$ and goes from $(r, R(r)) \to (r+1, R(r)+1)$ with probability $\omega_{r,R(r)}$ and from $(r, R(r)) \to (r+1, R(r)-1)$ with probability $1 - \omega_{r,R(r)}$.

We will consider the (quenched) random transition probabilities

$$\mathsf{P}^{\omega}(r,y) := \mathsf{P}^{\omega}(R(r) = y). \tag{5.1}$$

We will show that in a window of size $r^{1/2}$ around the spatial location $y \sim r^{3/4}$, which we call the moderate deviation regime in this paper, these transition kernels have fluctuations that are described by the KPZ equation. We define for $t \geq 0$ and $x \in \mathbb{R}$ and $N \in \mathbb{N}$ the constant

$$C_{N,t,x} := e^{N^{1/4}x + (N^{1/2} - N\log\cosh(N^{-1/4}))t} \tag{5.2}$$

and then we define for each $N \geq 1$ and $t \in N^{-1}\mathbb{Z}_{\geq 0}$ a random measure $\mathscr{U}_N(t,\cdot)$ on $\mathbb{R}$, given by a superposition of Dirac masses as follows

$$\mathscr{U}_N(t,\cdot) := \sum_{x \in N^{-1/2}\mathbb{Z} - N^{1/4}t} C_{N,t,x} \cdot \mathsf{P}^{\omega}(Nt, N^{3/4}t + N^{1/2}x) \cdot \delta_x. \tag{5.3}$$

Here δ_x denotes a Dirac mass at spatial position x, and $a\mathbb{Z} + b := \{ax + b : x \in \mathbb{Z}\}$. The definition of $\mathscr{U}_N(t,\cdot)$ is extended to $t \in \mathbb{R}_+$ by linearly interpolating, i.e., taking an appropriate convex combination of the two measures at the two nearest points of $N^{-1}\mathbb{Z}_{\geq 0}$. We call $\mathscr{U}_N$ the **quenched density field**. In all of our results below, we fix a terminal time $T > 0$ which is arbitrary. Our main

result is as follows.

Theorem 5.1.1. *Let ω denote an environment as above with $\mathbb{E}[\omega_{t,x}] = \frac{1}{2}$ and $\mathrm{Var}(\omega_{t,x}) = \sigma^2 \in [0, 1/4)$. There is an explicit Banach space X of distributions on $\mathbb{R}$, which is continuously embedded in $\mathcal{S}'(\mathbb{R})$, such that the collection $\{\mathscr{U}_N\}_{N\geq 1}$ is tight with respect to the topology of $C([0,T], X)$. Furthermore any limit point as $N \to \infty$ lies in $C((0,T], C(\mathbb{R}))$ and coincides with the law of the Itô solution of the SPDE*

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\partial_x^2 \mathcal{U}(t,x) + \sqrt{\frac{8\sigma^2}{1-4\sigma^2}} \cdot \mathcal{U}(t,x)\xi(t,x), \qquad t \geq 0, x \in \mathbb{R} \tag{5.4}$$

started with δ_0 initial condition at $t = 0$. Here ξ is a standard (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$.

Note that tightness and uniqueness of the limit point means that $\mathscr{U}_N$ *converges in distribution* to $\mathcal{U}$ in the aforementioned topology, as $N \to \infty$. The explicit Banach space X is given by a weighted Hölder space $C^{\alpha,\tau}(\mathbb{R})$ with negative exponent $\alpha < -3$ and polynomial weight of degree $\tau > 1$, see Definition 5.6.8. These exponents are non-optimal but the result is strong enough to imply e.g. convergence in law of $(\mathscr{U}_N(t,\phi))_{t\in[0,T]}$ to $\big(\int_{\mathbb{R}} \mathcal{U}_t(x)\phi(x)dx\big)_{t\in[0,T]}$ in the topology of $C[0,T]$ for any fixed $\phi \in C_c^\infty(\mathbb{R})$, where $\mathscr{U}_N(t,\phi) := \int_{\mathbb{R}} \phi(x)\mathscr{U}_N(t,dx)$.

While our main result above is formulated in terms of the quenched density field, there is another (somewhat weaker) formulation in terms of the **quenched tail field** which is concerned with the behavior of the tail probability $\mathsf{P}^\omega\big(R(r) \geq y\big)$. We obtain the following:

Theorem 5.1.2 (Multi-point convergence of the quenched tail field to the SHE)**.** *Suppose that the deterministic sequence of vectors $(x_{N,1}, \ldots, x_{N,m}) \in \mathbb{R}^m$ converges as $N \to \infty$ to $(x_1, \ldots, x_m) \in \mathbb{R}^m$. Fix $t_1, \ldots, t_m > 0$ and suppose we have sequences $t_{N,i} \in N^{-1}\mathbb{Z}_{\geq 0}$ such that $t_{N,i} \to t_i$ as $N \to \infty$. Then we have the joint convergence*

$$\big(N^{1/4} C_{N,t,x} \mathsf{P}^\omega\big(R(Nt) \geq N^{3/4}t_{N,i} + N^{1/2}x_{N,i}\big)\big)_{i=1}^m \xrightarrow{d} \big(\mathcal{U}_{t_i}(x_i)\big)_{i=1}^m,$$

with $\mathcal{U}$ as in (5.4).

This will be proved in Section 7.

Remark 5.1.3. *Besides the fact that the limit is given by the multiplicative-noise stochastic heat equation* (5.4)*, two other aspects of the above result are striking.*

1. *As* (5.3) *suggests, these KPZ fluctuations are obtained by probing the tail of the probability distribution $\mathsf{P}^{\omega}(r, y)$ near spatial location $y = N^{3/4}t + N^{1/2}x$ at time $r = Nt$, where N is very large. It is not so surprising that the spatial variable x needs a factor $N^{1/2}$ at time of order N, since this scaling respects the parabolic structure of the limiting equation. However the factor $N^{3/4}$ in front of t is more mysterious and eluded a rigorous understanding until recently. It is now known that $3/4$ is the unique exponent at which one expects the KPZ equation to arise. For exponents strictly smaller than $3/4$ one instead obtains a Gaussian field as the limit, as evidenced by works [236, 80, 177]. On the other hand, when the exponent is equal to 1 (i.e., at the large deviation scale) it was shown in [73] that under a Beta-distributed random environment, one obtains the Tracy-Widom GUE distribution as the one-point fluctuations. The Tracy-Widom GUE distribution is the one-point marginal of the directed landscape [237, 238], a recently constructed universal scaling limit of models in the KPZ universality class. We expect that for all exponents strictly larger than $3/4$, we have the directed landscape as the scaling limit. In this sense $3/4$ is the* ***crossover exponent*** *of this model, that is, the unique exponent where one observes the crossover from Gaussian to non-Gaussian behavior.*

2. *There is a nontrivial coefficient $\sqrt{\frac{8\sigma^2}{1-4\sigma^2}}$ appearing in front of the noise, which depends on the weights only through their variance σ^2. Note that this coefficient can be factored as $\sqrt{2}\sqrt{\frac{4\sigma^2}{1-4\sigma^2}}$ where $\sqrt{2}$ can be viewed as a consequence of the periodicity of the random walk. We call this coefficient the* ***environmental variance coefficient****. We shall see later in our proofs that this coefficient arises in a nontrivial way through intersection processes of the two-point motion for the RWRE model. To give some idea for why this is the correct coefficient, note that $1 - 4\sigma^2$ is the long-term quenched diffusion rate of the random walk given the realization of the*

environment ω. Consequently, if one samples two independent walks in a fixed realization of the environment, then their pairwise intersection process converges in law (upon diffusive scaling) to the local time of Brownian motion divided by a factor of $1 - 4\sigma^2$. This is precisely why we repeatedly see this factor arise in our analysis and in the limiting SPDE.

One may ask why we have used Dirac masses to define the field in (5.3), instead of linear interpolation. Indeed since the limiting field in (5.4) is a continuous function of space and time, one could ask about convergence in a space of continuous functions, rather than tempered distributions. While Dirac masses are computationally easier to work with, this is *not* the only reason for our choice. Heuristically, the $\mathscr{U}_N$ density field has a tendency to concentrate more heavily on certain favored sites microscopically, such that macroscopically these favored sites have a Lebesgue density equal to $1 - 4\sigma^2 > 0$ (see the discussions at the end of Section 5.6 for a more detailed explanation of this fact). Therefore, it is hopeless to ask for convergence in a topology of continuous functions. Indeed, we can rigorously show that it is impossible for convergence to occur in any space of continuous functions, as it would contradict one of the key estimates of the paper. See the discussion just after Proposition 5.5.3 below.

Given these discontinuity phenomena, a natural question to ask is what happens when one keeps track of the rescaled field $\mathscr{U}_N$ *together* with the noise variables $\omega_{t,x}$. This leads us to define the prelimiting noise field, given by a superposition of Dirac masses on space-time:

$$\Xi_N(\cdot,\cdot) := (2N^{3/2}\sigma^2)^{-1/2} \sum_{t \in N^{-1}\mathbb{Z} \cap [0,T]} \sum_{\substack{x \in N^{-1/2}\mathbb{Z} - N^{1/4}t \\ \text{same parity}}} \left(\omega_{Nt, N^{1/2}x + N^{3/4}t} - \tfrac{1}{2}\right) \cdot \delta_{(t,x)}. \tag{5.5}$$

where the sum is understood as being over those points (t, x) such that Nt and $N^{1/2}x + N^{3/4}t$ are both integers of the same parity. We now address the question of taking a **joint** limit of $(\Xi_N, \mathscr{U}_N)$. In principle Ξ_N just tracks the driving noise of the discrete system, so one might be led to believe that it will converge to the driving noise ξ of $\mathcal{U}$. This is indeed what happens for other systems such as directed polymers when one takes the limit to the stochastic heat equation. This can be deduced from the methods used in [86, 87] (those papers prove term-by-term convergence of the respective

chaos series, and in particular the first term of the series which encodes the full information of the noise, will converge jointly with the rest of the series). In fact, this is not the case in our model, and we have the following refinement of Theorem 5.1.1.

Theorem 5.1.4 (Creation of independent noise in the limit)**.** *Let $X \subset \mathcal{S}'(\mathbb{R})$ be as in Theorem 5.1.1. There exists an explicit Banach space $Y \subset \mathcal{S}'(\mathbb{R}^2)$ of distributions supported on $[0,T] \times \mathbb{R}$, such that $(\Xi_N, \mathscr{U}_N)$ are tight with respect to the topology of $Y \times C([0,T], X)$. Any limit point is of the form $(\xi_1\ ,\ \mathcal{U})$, where ξ_1 is a standard space-time white noise, and*

$$\partial_t \mathcal{U}(t,x) = \frac{1}{2}\partial_x^2 \mathcal{U}(t,x) + \left(\sqrt{8\sigma^2}\,\xi_1(t,x) + \sqrt{\frac{32\sigma^4}{1-4\sigma^2}}\,\xi_2(t,x)\right)\mathcal{U}(t,x), \qquad t \geq 0, x \in \mathbb{R},$$

*with $\mathcal{U}(0,x) = \delta_0(x)$. Here ξ_2 is another space-time white noise, **independent** of ξ_1.*

This will be proved as Theorem 5.6.26 in the main body of the paper. The proof of the theorem will be non-constructive, and it remains an open problem to identify at the microscopic scale precisely where the independent part ξ_2 comes from. Note that this recovers the earlier result since the sum of squares of the two noise coefficients is exactly the squared noise coefficient from Theorem 5.1.1. As far as we know, this phenomenon of creating independent noise in the limit is quite new. In Section 5.1.1, we give an intuitive interpretation of the independent part ξ_2 in terms of the failure of a naive chaos expansion.

Our main results confirm and generalize a physics prediction of [85]. They conjecture the convergence of the RWRE to the KPZ equation in the moderate deviations regime, but only for the specific choice of Beta-distributed weights $\omega_{t,x}$ which makes the model integrable. Their field is defined slightly differently, but it agrees with the "quenched tail field" that we study in Section 5.7. Note that when $\omega \sim \text{Beta}(\alpha,\alpha)$ one has $8\sigma^2/(1-4\sigma^2) = 1/\alpha$ which agrees precisely with the prediction of [85, Section 4.3]. Their predictions were based on moment computations using contour integral formulas that are only available when the weights are Beta-distributed. As moments of (SHE) grow too fast to determine the distribution uniquely, the convergence of moments in this case does not imply weak convergence as in Theorem 5.1.1. Our proof of Theorem 5.1.1

will instead be based purely on stochastic analysis and will not involve any exact solvability. The denominator $1 - 4\sigma^2$ of the noise coefficient will instead be seen to arise naturally from local time considerations coming from pairwise interactions in the 2-point motions, see Theorem 5.3.1.

Note that the extreme case $\sigma^2 = 0$, which occurs when all $\omega_{t,x}$ are deterministically $1/2$, corresponds to deterministic P^ω satisfying a discrete heat equation. The other extreme case $\sigma^2 = 1/4$ occurs when the $\omega_{t,x}$ are Bernoulli$(1/2)$ which gives coalescing random walks. Consequently, the limit of the $\mathscr{U}_N$ will be zero *almost surely*, since the probability of a simple random walk path being at location of order $N^{3/4}$ at time of order N is superpolynomially small. Both of these extreme cases make sense at the level of the limiting SPDE. In the $\sigma^2 = 0$ case, (5.4) becomes the heat equation. The $\sigma^2 \to 1/4$ case is slightly more difficult to see: in this case the renormalization effect inherent in the Ito equation (5.4) becomes so strong that $\mathcal{U}$ becomes zero everywhere at positive times, and this is indeed the correct limit for the coalescing walks under the above scaling regime.

Let us now give an overview of the literature related to the fluctuations in RWRE and related models. The model of random walks in dynamical random media gained attention in the context of stochastic flows and their discrete counterparts, see e.g. [188, 236, 186, 194, 207, 190, 208, 177], which studied existence, uniqueness, continuity, and construction of invariant measures for these types of stochastic processes. The question arose naturally to study the fluctuations of the quenched density and related quantities in such models. Gaussian fluctuations in the bulk region were proved in works such as [236, 80, 177]. Tracy-Widom fluctuations for the large deviation regime were proved in [73, 82] for certain exactly solvable models (the Beta-distributed RWRE and a continuous counterpart), though it still remains wide-open for general weights. The physics papers [84, 83, 85] conjectured that an explicit crossover from Gaussian to Tracy-Widom fluctuations occurs at spatial location $N^{3/4}$ in these types of stochastic flows and that the KPZ equation should, in fact, appear at this crossover location. In [71] and the present work, we develop general methods to tackle the problem of studying the quenched density at this crossover exponent, for the continuous and discrete models of random walks in random environments respectively. Forthcom-

ing work will explore how to rigorously find the crossover exponent and environmental variance coefficient for more general models of stochastic flows.

In Theorem 5.1.1 we did not aim for the greatest possible generality, but let us briefly describe the sort of generalizations that can be easily proved by adapting our methods, while also commenting on results that are out of reach with our methods. First, rather than assuming that the weights have mean $1/2$, we can assume they have mean $\mu \in (0,1)$. In this case, one needs to change the definition of the constant $C_{N,t,x}$, but one still has weak convergence to (SHE) in the moderate deviation regime. However in the scalings, one will observe an additional shearing of space-time $(t,x) \mapsto (t, x+Vt)$ for appropriate $V \in \mathbb{R}$, and the stochastic PDE limit will now be given by the solution of

$$\partial_t \mathcal{U}(t,x) = \sqrt{\mu(1-\mu)}\partial_x^2 \mathcal{U}(t,x) + \sqrt{\tfrac{2\sigma^2}{\mu(1-\mu)-\sigma^2}}\, \mathcal{U}(t,x)\xi(t,x), \qquad t \geq 0, x \in \mathbb{R}.$$

A further generalization is if the weight distribution is not fixed with N, but varies in such a way that the mean stays within a window of size $N^{-1/4}$ of some fixed value μ and the variance converges to some $\sigma^2 \in [0, \mu(1-\mu))$. Then our methods can still show weak convergence to (SHE) with a different shearing coefficient V of space-time.

Still another generalization is to look at $\mathsf{P}^\omega(Nt, \beta N^{3/4} t + N^{1/2} x)$ where $\beta > 0$, and then look at how the limit depends on β. In this case our methods still work, and in fact this is actually equivalent to our result by substituting $(N,t,x) \to (\beta^{-4}N, \beta^4 t, \beta^2 x)$. The limit will thus be

$$\partial_t \mathcal{U}_\beta(t,x) = \frac{1}{2}\partial_x^2 \mathcal{U}_\beta(t,x) + \sqrt{\frac{8\beta^2\sigma^2}{1-4\sigma^2}}\, \mathcal{U}_\beta(t,x)\xi(t,x), \qquad t \geq 0, x \in \mathbb{R}.$$

This indicates that one should observe a “smooth crossover" from Gaussian to Tracy-Widom fluctuations as β goes from 0 to $+\infty$, see Remark 5.1.3 just above.

Finally, we can also allow jumps of size larger than one. This seems like a more interesting problem because the correct noise coefficient would now depend on more than just the mean and variance of the weights. The latter is *not* something we can easily do by adapting the methods

of our paper. In particular, when the jumps have unbounded range, our methods break down completely as one no longer has the power of Azuma's inequality which we repeatedly use. This is something we plan to explore in some depth in a future work.

5.1.1 A failure of the chaos expansion

In this section we discuss interpretations of Theorem 5.1.4. One interesting feature of Theorem 5.1.1 is the noise coefficient $\sqrt{\frac{8\sigma^2}{1-4\sigma^2}}$. This coefficient is quite different from the situation of directed polymers, in which one also sees the KPZ equation appear in the fluctuations of a physically complex system driven by iid weights on a discrete space-time lattice. The present result is different for two reasons. Firstly, our weights are **fixed** with N, they are not being tuned so that their variance approaches zero as in the directed polymer setting. Secondly, in the polymer setting if the iid weights have variance σ^2 before rescaling, then the noise coefficient in the limiting KPZ equation is simply σ, as opposed to something more complicated as we see in the present work.

In the polymer setting, the method used to prove convergence to (SHE) is the method of *chaos expansion* [86, 87]. One expands the partition function as a multilinear polynomial of the weights $\omega_{t,x}$ and then proves term-by-term joint convergence of the resultant expression to the term-by-term expansion of the solution of (SHE) given by the Gaussian Fock space structure generated by the noise ξ. The natural question to ask for our model is if a chaos expansion exists and if it can be used to prove the convergence stated in Theorem 5.1.1. It turns out that a polynomial chaos expansion is indeed available for the model considered herein, but it is **not** useful in proving the above result if applied naively. We will now illustrate its failure.

Let $p_{\mathsf{RW}}(n,y)$ denote the transition density of a simple symmetric random walk on $\mathbb{Z}$. Let $\widehat{\omega}_{t,x} = 2\omega_{t,x} - 1$. By Lemma 2.1 in [90] for n, y of the same parity we have

$$\mathsf{P}^{\omega}(n,y) = p_{\mathsf{RW}}(n,y) + \sum_{k=1}^{n} \sum_{(z_1,z_2,\ldots,z_k)} \phi_k^{(n,y)}(z_1,z_2,\ldots,z_k)\widehat{\omega}_{z_1}\widehat{\omega}_{z_2}\cdots\widehat{\omega}_{z_k}$$

where the summation $\sum_{(z_1,z_2,\ldots,z_k)}$ is over all possible $z_\ell = (i_\ell, j_\ell)$ with $0 \le i_1 < i_2 < \cdots < i_k \le n-1$ and $j_1, j_2, \ldots, j_k \in \mathbb{Z}$, and the expansion coefficients are given by

$$\begin{aligned}\phi_k^{(n,y)}(z_1,\ldots,z_k) = \tfrac{1}{2^k} p_{\mathsf{RW}}(i_1,j_1) &\times [p_{\mathsf{RW}}(i_2-i_1-1, j_2-j_1-1) - p_{\mathsf{RW}}(i_2-i_1-1, j_2-j_1+1)] \\ &\times [p_{\mathsf{RW}}(i_3-i_2-1, j_3-j_2-1) - p_{\mathsf{RW}}(i_3-i_2-1, j_3-j_2+1)] \\ &\times \cdots \\ &\times [p_{\mathsf{RW}}(n-i_k-1, y-j_k-1) - p_{\mathsf{RW}}(n-i_k-1, y-j_k+1)]\,.\end{aligned}$$

Now fix $t > 0, x \in \mathbb{R}$, and $k \in \mathbb{N}$. Set $t_0 = 0, t_{k+1} = t, x_0 = 0$, and $x_{k+1} = x$. Then for **fixed** $k \in \mathbb{N}$ one has the convergence in law as $N \to \infty$,

$$\frac{C_{N,t,x}}{N^{k/2}} \sum_{(z_1,z_2,\ldots,z_k)} \phi_k^{(Nt, N^{1/2}x + N^{3/4}t)}(z_1,\ldots,z_k)\widehat{\omega}_{z_1}\cdots\widehat{\omega}_{z_k} \xrightarrow{d} (8\sigma^2)^{k/2} \int_{\mathbb{R}^k \times \Delta_k(t)} \prod_{\ell=1}^{k+1} p(\overline{t}_\ell, \overline{x}_\ell)\xi^{\otimes k}(d\mathbf{t}, d\mathbf{x})$$

where $\overline{t}_\ell := t_\ell - t_{\ell-1}$, $\overline{x}_\ell := x_\ell - x_{\ell-1}$, and $p(t,x)$ is the continuum heat kernel. The right-hand side is a k-fold iterated Wiener-Ito integral where ξ is a standard Gaussian space-time white noise and $\Delta_k(0,t)$ is the simplex of ordered times $0 \le t_1 \le \cdots \le t_k \le t$.

The above convergence is fairly straightforward to verify since the weights $\widehat{\omega}$ have variance $4\sigma^2$ and one picks up an extra factor of 2 from the parity considerations of the discrete lattice on which the ϕ_k are supported. In particular the above convergence would suggest that if the limit (in distribution as $N \to \infty$) of the individual terms commuted with the infinite sum (over $k \in \mathbb{N}$) of those terms, then the coefficient $\sqrt{8\sigma^2/(1-4\sigma^2)}$ in Theorem 5.1.1 is incorrect, and that the more naive guess of $\sqrt{8\sigma^2}$ should be the correct one.

Since Theorem 5.1.1 states that this is *not* the case, it suggests that taking the limit of the above chaos expansion does not work. More precisely, it formally converges to the correct equation but with an incorrect noise coefficient that is strictly smaller than the correct one. The reason that this happens is that a nonzero proportion of the L^2-mass in the terms of the above chaos expansion

escapes into the tails of the series (i.e., terms in the expansion indexed by large values of k that are of unbounded order as $N \to \infty$), which causes a failure of the infinite sum over k to commute with the limit in distribution of each term of the chaos series as $N \to \infty$. In the context of the generalized polynomial chaos convergence result of [87], this exactly means that condition (iii) fails when trying to apply their main result Theorem 2.3. Note that the naive limit of the expansion is exactly the limiting object in Theorem 5.1.4 *without* the ξ_2 term. Thus a heuristic interpretation of that theorem is that the amount of extra noise that has escaped into the tails of the chaos expansion is exactly the ξ_2 part in Theorem 5.1.4.

As a reality check for the above calculation, note that in the extreme case $\sigma^2 = 1/4$, the weights are Bernoulli distributed. This means that independent random walkers in the environment will coalesce forever upon collision. In particular, the quenched transition densities $\mathsf{P}^\omega(n, \cdot)$ have all of their mass concentrated at a single point for all n almost surely. Consequently, there will *not* be any nontrivial fluctuation behavior. Nonetheless, the above discrete chaos expansion is valid and formally converges to the stochastic heat equation with noise coefficient $\sqrt{2}$ if one believes that the sum over k commutes with the limit $N \to \infty$. This is already a clear contradiction.

The reason why the above chaos expansion fails is that the variance of the weights is fixed and positive, rather than tending to zero. In other words, we are not in the "weak noise regime" of [86, 87]. Note however that when σ is very close to 0, the correct coefficient of $\sqrt{8\sigma^2/(1-4\sigma^2)}$ agrees to first order with the more naive guess of $\sqrt{8\sigma^2}$, which makes sense at a non-rigorous level since $\sigma \to 0$ is a "weak noise limit."

There are results in higher spatial dimensions where we suspect that independent noise is also created in the limit, see for example [89, 239, 240, 241, 242, 243, 244, 245, 246]. However, Theorem 5.1.4 seems to be the first precise result along these lines. To the best of our knowledge, this also seems to be the first result where this phenomenon was observed for a model with just one spatial dimension. A related question is that given the failure of Theorem 5.1.1 to hold in a space of continuous functions, it is natural to ask about probing additional *pointwise* fluctuations of the field $\mathscr{U}_N(t,x)$ as $N \to \infty$, similar to the quenched local limit theorem proved in [178]. These

pointwise fluctuations would give some clue as to where exactly the additional noise is generated. We leave this as a future work.

The fact that the distribution of weights is fixed as a function of N is in stark contrast with the results of [90] where the authors use a weak noise scaling and consider the large deviation regime. In contrast to the present work, their rescaled field does not concentrate on certain favorite sites. In fact, that regime is more similar to the polymers, because the weights converge as $N \to \infty$ to $\delta_{1/2}$ in a certain precise way, and the variance therefore tends to zero under the diffusive scaling. Consequently, the chaos expansion method works to prove convergence, and the convergence occurs in the space of continuous functions. Our method of proof is completely different, as the regime considered here seems not to be conducive to such chaos expansion methods as shown above. The work of [187] studies the weak-noise regime of a related continuum model of transport SPDEs, and Figure 1 therein illustrates an interesting phase transition. This transition clearly distinguishes the weak and strong noise regimes. Our results fall into the upper right quadrant of the figure, which corresponds to strong noise, whereas the results of [187, 90] fall into the upper left quadrant, which corresponds to weak noise. Independent noise does not appear in the latter regime when taking the limit to (SHE).

5.1.2 Extrema of sticky random walks

Results about the behavior of a probability distribution at its extreme ends can often be translated into information about the extremal behavior of a large number of independent particles sampled from that distribution. Theorem 5.1.1 is a result about the tails of the probability distribution $\mathsf{P}^{\omega}(Nt, \cdot)$ at a specific location in the tail near $N^{3/4}t$. Theorem 5.1.1 combined with some additional calculations thus leads to the following result.

Theorem 5.1.5 (Extremal particle limit theorem)**.** *Fix $c, T > 0$ and $d \in \mathbb{R}$. Let $(R^1(r), \dots, R^k(r))_{r \geq 0}$ be sampled according to the ω-averaged law of k independent random walk particles in the environment ω. Set the number of particles $k = k(N) := \lfloor \exp(\frac{1}{2}cN^{1/2} + dN^{1/4} + r_N) \rfloor$ where r_N can be any sequence satisfying $r_N = o(N^{1/4})$. Consider any sequence $t_N \in N^{-1}\mathbb{Z}_{\geq 0}$ such that*

$t_N \to t > 0$ *as* $N \to \infty$. *Then we have convergence in distribution as* $N \to \infty$

$$\max_{1 \le i \le k(N)} \left\{ N^{-\frac{1}{4}} R^i(N t_N) \right\} - a_N(c, d, t_N) \xrightarrow{d} \sqrt{\tfrac{t}{c}} \big(G + \log \mathcal{U}_c(d) \big),$$

where

$$a_N(c, d, s) := \sqrt{csN} - dN^{\frac{1}{4}} \sqrt{\tfrac{s}{c}} - \sqrt{\tfrac{c}{s}} \big(r_N - \tfrac{1}{4} \log N \big) - \frac{c^{3/2}}{6\sqrt{s}}.$$

Here G *is a standard Gumbel random variable which is independent of* $\mathcal{U}$, *and* $(t, x) \mapsto \mathcal{U}_t(x)$ *solves* (5.4) *but with noise coefficient replaced by* $\sqrt{\frac{8\sigma^2 c}{(1-4\sigma^2)t}}$.

This will be proved as Theorem 5.7.5. The Gumbel distribution appearing in the maximum is not that surprising, since it often arises in extreme value theory. But the question of how the solution $\mathcal{U}$ of (5.4) appears is more mysterious. Unlike in the case of *independent* random walkers, the k separate particles in the random environment model may see a nontrivial *interaction* upon collision, and this nontrivial interaction is precisely what gives rise to the non-Gumbel term in the above limit theorem. The nontrivial interaction is sometimes referred to as "stickiness," since particles that are together tend to stay together for a while before separating.

Taking $d = r_N = 0$ and $t = 1$, we see that the above statement is a result of the maximum of $e^{\frac{1}{2}c\sqrt{N}}$ many sticky random walk particles at time N, where N is very large. This is the same as understanding the maximum of M sticky particles when time is of the order $(\log M)^2$, which then leads to the question of what happens when one looks at the maximum of M particles at timescales different from $(\log M)^2$. At timescales smaller than $(\log M)^2$, one expects Tracy-Widom fluctuations, with no Gumbel term at all as $M \to \infty$. However this has only been proved when the timescale is $\log M$ in certain exactly solvable cases, see [73, 82]. At timescales larger than $(\log M)^2$, one expects *purely* Gumbel fluctuations in the $M \to \infty$ limit, as the particles begin to behave independently. This is evidenced by the Gaussian fluctuation behavior below the crossover exponent, which was explained above. There are numerous open questions related to obtaining a more complete picture in this area, see the physics papers [85, 94, 247, 248, 249].

From a physical standpoint, the sticky model of particles can be used to describe a certain turbulent flow of particles caused by a nontrivial viscosity of the fluid in which they evolve, see e.g. [198]. The physics works [94, 248] demonstrate through numerics that the edge of the cloud of particles will exhibit a certain behavior that is different from that of the classical Einstein's model of diffusion. Our main results (Theorems 5.1.1 and 5.1.5) thus serve to make some rigorous progress in this area by showing the correct crossover exponent, but there are still a number of open conjectures.

5.1.3 Main ideas of the proof

The main technique used in our analysis will be a discrete Girsanov's formula. We illustrate this idea here by computing the first moment of $\mathscr{U}_N$ as defined in (5.3) and showing that it converges to the first moment of the SHE. This computation contains the core idea that will be used in many of the later proofs. For a simple symmetric random walk $R(r)$ on $\mathbb{Z}$ starting from the origin, the process

$$e^{\lambda R(r) - r\log\cosh(\lambda)} = \frac{e^{\lambda R(r)}}{\mathbb{E}[e^{\lambda R(r)}]}$$

is a martingale in r, simply because it is a product of r iid strictly positive mean-one random variables $e^{\lambda\xi_i}/\mathbb{E}[e^{\lambda\xi_i}]$ where the ξ_i are the increments of the process $(R(r))_{r\geq 0}$. Given this fact, let us now fix some Schwartz function ϕ and compute the first moment of the pairing $(\mathscr{U}_N(t,\cdot),\phi)_{L^2(\mathbb{R})} =: \mathscr{U}_N(t,\phi)$. Note that for $t \in \mathbb{Z}/N$

$$\begin{aligned}
\mathscr{U}_N(t,\phi) &= \sum_{x\in N^{-1/2}\mathbb{Z}-N^{1/4}t} C_{N,t,x}\cdot \mathsf{P}^\omega(Nt, N^{3/4}t + N^{1/2}x)\cdot\phi(x)\\
&= \sum_{x\in\mathbb{Z}} C_{N,t,N^{-1/2}(x-N^{3/4}t)}\mathsf{P}^\omega(Nt,x)\phi\big(N^{-1/2}(x-N^{3/4}t)\big)\\
&= \mathsf{E}^\omega[C_{N,t,N^{-1/2}(R(Nt)-N^{3/4}t)}\phi\big(N^{-1/2}(R(Nt)-N^{3/4}t)\big)], \qquad (5.6)
\end{aligned}$$

where E^ω denotes a quenched expectation operator given the realization of the environment ω. Thus after taking the *annealed* expectation (that is, averaging over all possible environments ω)

we have

$$\mathbb{E}[\mathscr{U}_N(t,\phi)] = \mathbf{E}[C_{N,t,N^{-1/2}(R(Nt)-N^{3/4}t)}\phi\big(N^{-1/2}(R(Nt)-N^{3/4}t)\big)],$$

where now the expectation on the right is with respect to a simple symmetric random walk path on $\mathbb{Z}$ (recall $\mathbb{E}[\omega] = \frac{1}{2}$).

Notice that

$$C_{N,t,N^{-1/2}(R(Nt)-N^{3/4}t)} = e^{N^{-1/4}R(Nt)-Nt\log\cosh(N^{-1/4})}.$$

We already know that $e^{N^{-1/4}R(Nt)-Nt\log\cosh(N^{-1/4})}$ is a martingale in the Nt variable and thus has mean 1, so it can be interpreted as a change of measure where the annealed law of the increments of $R(r)$ has changed from the usual symmetric law $\frac{1}{2}(\delta_1+\delta_{-1})$ to the new law given by $\frac{1}{2\cosh(N^{-1/4})}(e^{N^{-1/4}}\delta_1 + e^{-N^{-1/4}}\delta_{-1})$.

This new law has mean asymptotically given by $N^{-1/4} + O(N^{-3/4})$. Consequently, under the new law the process $N^{-1/2}(R(Nt) - N^{3/4}t)$ is centered up to a vanishing error term, and by Donsker's principle will converge in law to a standard Brownian motion. In summary, denoting $\widetilde{\mathbf{E}}$ as the tilted expectation on the path space of the random walk, we have that

$$\mathbb{E}[\mathscr{U}_N(t,\phi)] = \widetilde{\mathbf{E}}[\phi(N^{-1/2}(R(Nt)-N^{3/4}t))] \overset{N\to\infty}{\longrightarrow} \mathbf{E}_{BM}[\phi(B_t)] = \int_{\mathbb{R}} p(t,x)\phi(x)dx$$

for a standard Brownian motion B. Here $p(t,x) = (2\pi t)^{-1/2}e^{-x^2/2t}$ is the standard heat kernel. The right-hand side is indeed equal to $\mathbb{E}\big[\int_{\mathbb{R}}\mathcal{U}_t(x)\phi(x)dx\big]$, where $(t,x)\mapsto\mathcal{U}_t(x)$ solves (5.4) with initial condition δ_0. This shows that the first moment of $\mathscr{U}_N(t,\cdot)$ converges to the first moment of the SHE as desired. While the calculation for higher moments is more complicated, the core idea of using a "discrete Girsanov transform" is rather similar, and these higher moments will be computed in Section 5.3.1.

Once the moments are computed using this method, in Section 5.4 we derive a discrete Hopf-Cole transform for the field in (5.3). The martingale observables arising from the Hopf-Cole transform can also be analyzed using the above discrete Girsanov transforms. In particular, this Girsanov trick yields moments and regularity estimates for our observables which eventually leads to

tightness estimates for our field. Sufficiently strong estimates using this method will eventually lead us to the proof of Theorem 5.1.1, by showing that any limit point must satisfy the martingale problem for (5.4). This martingale-based approach circumvents all of the issues with the chaos expansion, as the quadratic variations arising in the martingales automatically take into account the singular behavior of the model on small scales. Remarkably, this model has the property that the error terms appearing in the discrete martingale equation behave very nicely in relation to the original object itself, which is very rare among KPZ-related models where martingale characterizations have been used, see e.g. [88, 211, 7, 215] where extremely careful analysis was needed to show vanishing of error terms for exclusion-type models.

Remark 5.1.6 (Comparison to sticky Brownian motion)**.** *Sticky Brownian motion is obtained by a diffusive scaling of sticky random walks where the law of the weights converges as $N \to \infty$ to $\frac{1}{2}(\delta_0 + \delta_1)$ in a certain precise way [190, Theorem 2.10], which is in a sense the opposite of the weak-noise regime considered in [90]. Since in the present work the weights are not being scaled, the discrete kernel P^ω is not converging to a sticky Brownian motion kernel. Therefore, our result is not simply a "commutation of limits" with our other work [71] on sticky Brownian motion (just as it is not a corollary of [90] as explained earlier in Section 5.1.1). Indeed, the noise amplitude of $\frac{8\sigma^2}{1-4\sigma^2}$ does not correspond in any natural way to the noise amplitude of $2\mu([0,1])^{-1}$ appearing in that paper. However, the overall approach we use here is similar to the one that we developed in the sticky Brownian motion case [71]. Nonetheless, a number of new challenges arise, and several ideas need to be introduced that were not present in that paper. For instance, in Section 5.2.1 the Radon-Nikodym derivatives on the path space are substantially more complicated than that of the continuous case, involving the stochastic exponential of entropy-type observables of the k-point motion rather than just simple linear functions of the process. In Section 5.4 we find a discrete Hopf-Cole transform, which is more difficult than the continuous case where the usual heat operator applied to the field immediately yielded a martingale. In Section 5.6, we need to develop several estimates that bound the difference between discrete and continuum heat operators, which was not needed in the previous work.*

Organization

The rest of the paper is organized as follows. In Section 5.2 we describe the discrete Girsanov transform and prove estimates related to the k-point motion of sticky random walks. In Section 5.3, we prove a general weak convergence theorem under the tilted measure. In Section 5.4, we show that our prelimiting field satisfies a discrete version of the SHE. Section 5.5 is devoted to the analysis of the quadratic variation of the martingale observables in the prelimiting field. Theorems 5.1.1 and 5.1.4 are proven in Section 5.6 using the estimates from the previous sections. Finally, in Section 5.7, we prove the results related to the extrema of sticky random walks discussed in Section 5.1.2. There is a detailed glossary at the end of this paper that recalls and points to the definitions of much of the notation introduced elsewhere.

Notations and conventions

Throughout this paper we use $\mathrm{C} = \mathrm{C}(a, b, c, \ldots) > 0$ to denote a generic deterministic positive finite constant depending on $a, b, c, \ldots$ that may change from line to line. We respectively write $\mathbb{E}$ and E^{ω} for annealed and quenched expectations in the context of random motions in random environments. We use $\mathbf{E}$ to denote expectation under path measures such as Brownian motion.

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5.2 Transformations and estimates for the k-point motion

In this section, we develop the basic framework of our proof. As mentioned in the introduction, our proof relies on a certain Girsanov-type transform for random walks. In Section 5.2.1 we describe this transform that we will use repeatedly in our later analysis. In Section 5.2.2, we collect several estimates related to the random walks.

5.2.1 Girsanov transforms

We begin with some necessary notation and definitions. Throughout this paper, we assume ν is a probability measure on $[0,1]$. We define $\mu := \int_{[0,1]} x\nu(dx)$. We set $\sigma^2 = \int_{[0,1]}(x-\mu)^2\nu(dx)$, and sometimes we may write μ_ν, σ_ν^2 to emphasize the dependence on ν. While we take $\mu_\nu = 1/2$ in the setting of Theorem 5.1.1, we will need this more general framework for our proofs. In our analysis, we shall often consider a special sequence of measures obtained by small perturbation of ν. We introduce this below.

Definition 5.2.1 (Skewed sequence of measures)**.** *Fix any probability measure ν on $[0,1]$ of mean 1/2. Assume ω has law ν. Let ν_N^* be the law of $\min\{1, \omega + d_N\}$ where $d_N = \frac{1}{2}N^{-1/4} + o(N^{-1/4})$ are constants chosen so that the expectation $\mu_{\nu_N^*}$ is exactly equal to ρ_N where*

$$\rho_N := \frac{e^{N^{-1/4}}}{2\cosh(N^{-1/4})} = \frac{1}{2} + \frac{1}{2}N^{-1/4} + O(N^{-3/4}). \tag{5.7}$$

The choice of d_N implies that $\lim_{N\to\infty}\sigma^2_{\nu_N^*} = \sigma_\nu^2$. We shall call $\{\nu_N^*\}_{N\geq 1}$ the skewed sequence of measures corresponding to ν.

Definition 5.2.2. *For each $k \in \mathbb{N}$ we denote $\mathbf{P}_{RW_\nu^{(k)}}$ to be the annealed law on the canonical space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$ of k independent walks sampled from the environment ω whose weights are distributed*

according to ν, all started from 0.

- We denote by $\mathbf{R} = (R^1, \dots, R^k)$ the canonical process on $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$. We will usually index such discrete-time processes by a time-variable $r \in \mathbb{N}$, e.g., $\mathbf{R} = (\mathbf{R}(r))_{r \geq 0}$. We further define

$$\Gamma^{(v)} := \big\{ r \in \mathbb{Z}_{\geq 0} \mid \#\{R^1(r), \dots, R^k(r)\} = v \big\}, \qquad \Gamma^{(\leq v)} := \bigcup_{u=1}^{v} \Gamma^{(u)} \tag{5.8}$$

  to be the sets of times that we have exactly v and no more than v distinct particles, respectively.

- For each $k \in \mathbb{N}$ and $1 \leq i \neq j \leq k$ we define the functional $V^{ij} : (\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}} \to \mathbb{Z}^{\mathbb{Z}_{\geq 0}}$ by

$$V^{ij}(r) := \sum_{s=0}^{r-1} \mathbb{1}_{\{R^i(s) = R^j(s)\}}. \tag{5.9}$$

  We remark that the processes V^{ij} are predictable with respect to the canonical filtration on $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$.

- For a martingale $(M(r))_{r \geq 0}$ adapted to the canonical filtration of $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$, we will denote by $[M](r) := \sum_{s=1}^{r} (M(s) - M(s-1))^2$ its optional quadratic variation.

- All objects in this section should implicitly be understood as being adapted path functionals of $\mathbf{R}$, though we will not write this out explicitly, e.g. $V^{ij}(r) = V^{ij}(\mathbf{R}; r)$, $\mathcal{M}^{\lambda}(r) = \mathcal{M}^{\lambda}(\mathbf{R}; r)$.

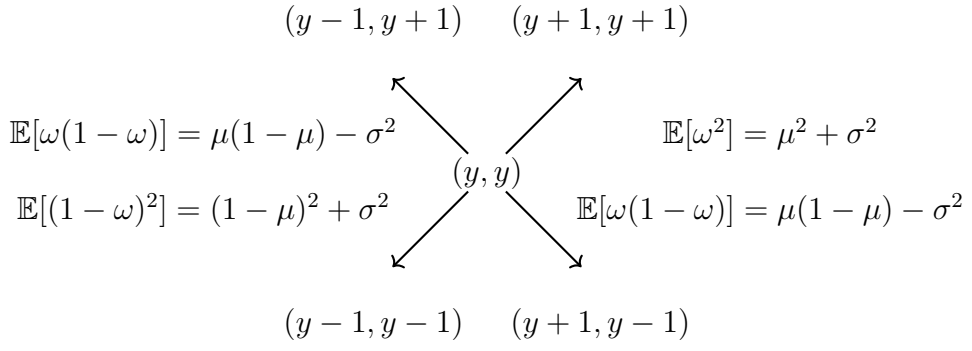

Figure 5.1: Markov chain transition diagram of the two-point motion (R^i, R^j) at points along the diagonal of $\mathbb{Z}^2$, where ω denotes a random variable sampled from ν.

Note that when $\nu = \delta_a$ for some $a \in [0,1]$, the path measure $\mathbf{P}_{RW_\nu^{(k)}}$ is just the law of k independent random walks on $\mathbb{Z}$ where the increments have mean $2a-1$. However, for $\nu \neq \delta_a$, the law is more complicated: while the k particles still behave independently when they are apart, there can be a nontrivial interaction between them when they collide. As an extreme case, consider $\nu = p\delta_0 + (1-p)\delta_1$ with $p \in [0,1]$. In this setting, the particles coalesce forever upon collision.

If we only look at a single particle, then its trajectory is just that of a nearest-neighbor random walk with right jump rate given by $\mathbb{E}[\omega] = \mu_\nu$ and left jump rate given by $\mathbb{E}[1-\omega] = 1-\mu_\nu$. So for all $1 \leq i \leq k$, the one-point motion $R^i(r)$ is a random walk with mean $(2\mu_\nu - 1)$ under $\mathbf{P}_{RW_\nu^{(k)}}$. It follows that the process $R^i(r) - (2\mu_\nu - 1)r$ is a mean-zero random walk.

Next, we examine the interactions between two particles. For $1 \leq i < j \leq k$, the two-point motion (R^i, R^j) is a random walk whose transition probabilities at time t differ depending on whether $R^i(t) = R^j(t)$ or not. If $R^i(t) \neq R^j(t)$, then R^i and R^j behave as two independent random walks. If $R^i(t) = R^j(t) = y$, then the two walks are no longer independent since they are sampled using the same environment variable $\omega_{t,y}$. See Figure 5.1 for the transition probabilities in this case.

Using these transition probabilities, one can check that the following process

$$\mathcal{M}(r) := 4\big[\mu_\nu(1-\mu_\nu) - \sigma_\nu^2\big]V^{ij}(r) - |R^i(r) - R^j(r)| \tag{5.10}$$

is a $\mathbf{P}_{RW_\nu^{(k)}}$-martingale. Rewriting this as

$$|R^i(r) - R^j(r)| = 4\big[\mu_\nu(1-\mu_\nu) - \sigma_\nu^2\big]V^{ij}(r) - \mathcal{M}(r), \tag{5.11}$$

we can view this as a discrete version of Tanaka's formula, where the term $4\big[\mu_\nu(1-\mu_\nu)-\sigma_\nu^2\big]V^{ij}(r)$ describes the intersection times of the two walks and corresponds to the local time term in the continuous version of Tanaka's formula. If we take $\mu_\nu = 1/2$, as we will in the setting of Theorem 5.1.1, the coefficient in front of $V^{ij}(r)$ becomes $1-4\sigma^2$, which as discussed in the introduction will show up in the denominator of the noise coefficient in (5.4).

Note that if $(X(r))_{r\geq 0}$ is any sequence of random variables with finite exponential moments defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and adapted to a filtration $(\mathcal{F}_r)_{r\geq 0}$ on this space, then the process

$$\exp\left(X(r) - \sum_{s=1}^{r} \log \mathbf{E}[e^{X(s)-X(s-1)}|\mathcal{F}_{s-1}]\right) \tag{5.12}$$

is a martingale in $r \geq 0$, with respect to the same filtration. Henceforth, our probability space will always be $\Omega = (\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$, equipped with its canonical filtration. For $\lambda > 0$, if we apply (5.12) to the process $X := \lambda(R^1 + \cdots + R^k)$, we see that for each $\lambda > 0$

$$\mathcal{M}^{\lambda}(r) := \exp\left(\lambda \sum_{j=1}^{k} R^j(r) - \sum_{\ell=0}^{r-1} f_{\ell}^{\lambda,k}\right) \tag{5.13}$$

is a $\mathbf{P}_{RW_{\nu}^{(k)}}$-martingale for the k-point motion where

$$f_{\ell}^{\lambda,k} := f^{\lambda,k,\nu}\big(R^1(\ell), \ldots, R^k(\ell)\big), \text{ and } f^{\lambda,k,\nu}(x^1, \ldots, x^k) := \log \mathbf{E}_{RW_{\nu}^{(k)}}^{(x^1,\ldots,x^k)}[e^{\lambda \sum_{j=1}^{k}(R^j(1)-x^j)}]. \tag{5.14}$$

Here $\mathbf{E}_{RW_{\nu}^{(k)}}^{(x^1,\ldots,x^k)}$ denotes expectation of the Markov chain when started from $(x^1, \ldots, x^k)$. $f_{\ell}^{\lambda,k}$ admits a simple expression for $\ell \in \Gamma^{(k)}, \Gamma^{(k-1)}$ where $\Gamma^{(v)}$ is defined in (5.8). Indeed, for $\ell \in \Gamma^{(k)}$, $f_{\ell}^{\lambda,k} = k \log \cosh(\lambda)$, whereas for $\ell \in \Gamma^{(k-1)}$ we have

$$f_{\ell}^{\lambda,k} = k \log \cosh(\lambda) - g(\lambda, \sigma^2).$$

where $g(\lambda, \sigma^2) := \log(\frac{1+4\sigma^2}{2} \cosh(2\lambda) + \frac{1-4\sigma^2}{2}) - 2 \log \cosh(\lambda)$, as may be verified by an inspection of Figure 5.1. This fact will be very useful in many calculations below.

Before discussing the importance of the martingale defined in (5.13), we record a useful estimate for the $f^{\lambda,\kappa,\nu}$.

Lemma 5.2.3. *Assume $\mu_\nu = 1/2$. For any $k \in \mathbb{N}$, we have the bound*

$$\sup_{(x^1,\dots,x^k)\in\mathbb{Z}^k} \sup_{|\lambda|<1} \lambda^{-2}|f^{\lambda,k,\nu}(x^1,\dots,x^k)| < \infty.$$

Proof. Note that the function $f^{\lambda,k,\nu} : \mathbb{Z}^k \to \mathbb{R}$ is defined using only the first step of the Markov chain, so that $f^{\lambda,k,\nu}(x^1,\dots,x^k)$ depends only on the first k moments of the measure ν and the number of particle clusters in the set $\{x^1,\dots,x^k\}$, as well as the exact number of particles present in each cluster. This means that the supremum over $\mathbb{Z}^k$ is really a supremum over some **finite** set of partitions of k. Thus it suffices to show for each *individual* $(x^1,\dots,x^k) \in \mathbb{Z}^k$ that the sup over λ is finite. Since the $R^j - x^j$ are centered if $\mu_\nu = 1/2$, we have that $\lambda^{-2}\log \mathbf{E}_{RW_\nu^{(k)}}^{(x^1,\dots,x^k)}[e^{\lambda\sum_{j=1}^k (R_1^j - x^j)}]$ actually converges as $\lambda \to 0$ to the quantity $\frac{1}{2}\mathbf{E}_{RW_\nu^{(k)}}^{(r^1,\dots,r^k)}\big[\big(\sum_{j=1}^k (R^j(1) - r^j)\big)^2\big]$. □

We use the martingale $\mathcal{M}^\lambda(r)$ in (5.13) to define the following measure.

Definition 5.2.4. *Define the measure $\mathbf{Q}^\lambda_{RW_\nu^{(k)}}$ on the canonical space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$ to be given by*

$$\frac{d\mathbf{Q}^\lambda_{RW_\nu^{(k)}}}{d\mathbf{P}_{RW_\nu^{(k)}}}\bigg|_{\mathcal{F}_r} = \mathcal{M}^\lambda(r),$$

where $\mathcal{F}_r$ is the σ-algebra on $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$ generated by the first r coordinate maps, and $\mathcal{M}^\lambda(r)$ is the martingale given by (5.13)*.*

It turns out that proving KPZ convergence for the quenched kernels will amount to studying the canonical process under the measures $\mathbf{Q}^\lambda_{RW_\nu^{(k)}}$, with $\lambda = N^{-1/4}$. The following proposition is the main result of this section. It essentially tells us that $\mathbf{Q}^{N^{-1/4}}_{RW_\nu^{(k)}}$ is similar to the measure $\mathbf{P}_{RW_{\nu_N}^{(k)}}$, albeit with a different underlying measure ν_N that has a different mean and variance than the original measure ν, but which is close enough for our purposes in studying the diffusive limits and the intersection times.

Proposition 5.2.5. *Consider any probability measure ν on $[0,1]$ with $\mu_\nu = 1/2$ and $\sigma_\nu^2 \in [0, 1/4)$. Fix $k \in \mathbb{N}$, and let $\mathbf{Q}^\lambda_{RW_\nu^{(k)}}$ be as in Definition 5.2.4. Let $\{\nu_N^*\}_{N\geq 1}$ be the skewed sequence of*

measures corresponding to ν defined in Definition 5.2.1. For all $r \geq 0$, one has

$$\frac{d\mathbf{Q}^{N^{-1/4}}_{RW^{(k)}_{\nu}}}{d\mathbf{P}_{RW^{(k)}_{\nu^*_N}}}\Big|_{\mathcal{F}_r}(\mathbf{R}) = \exp\Big(\widetilde{\mathcal{G}_N}(r)\Big) := \exp\Bigg(\mathcal{G}_N(r) - \sum_{s\leq r}\log \mathbf{E}_{RW^{(k)}_{\nu^*_N}}[e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)}|\mathcal{F}_{s-1}]\Bigg), \tag{5.15}$$

where $\mathcal{G}_N$ is a family of $\mathbf{P}_{RW^{(k)}_{\nu^*_N}}$-martingales satisfying the optional variation bound

$$[\mathcal{G}_N](r+1) - [\mathcal{G}_N](r) \leq CN^{-1/2}\sum_{i<j}\big(V^{ij}(r+1) - V^{ij}(r)\big), \tag{5.16}$$

for all $r \geq 0$, where V^{ij} are defined in (5.9)*, and C is a constant independent of N, r.*

Proof. For clarity the proof is divided into two steps.

Step 1. In this step, we will construct a family of $\mathbf{P}_{RW^{(k)}_{\nu_N}}$ martingales $\mathcal{G}_N$ satisfying (5.15). Let $\{\nu_N\}_N$ be any arbitrary sequence of probability measures on $[0,1]$. Write the Radon-Nikodym derivative in (5.15) as a product of two terms:

$$\frac{d\mathbf{Q}^{N^{-1/4}}_{RW^{(k)}_{\nu}}}{d\mathbf{P}_{RW^{(k)}_{\nu}}}\Big|_{\mathcal{F}_r}(\mathbf{R}) \cdot \frac{d\mathbf{P}_{RW^{(k)}_{\nu}}}{d\mathbf{P}_{RW^{(k)}_{\nu_N}}}\Big|_{\mathcal{F}_r}(\mathbf{R}). \tag{5.17}$$

The second term in (5.17) is the quotient of two Markov chain transition probabilities. This can be written in the form $\exp\big(\mathcal{D}_N(r)\big)$ where $\mathcal{D}_N(0) = 0$ and where $\mathcal{D}_N(r+1) - \mathcal{D}_N(r)$ can be described deterministically from the path $\mathbf{R}$ as follows:

Assume that at time r, the k particles of $\mathbf{R}(r) \in \mathbb{Z}^k$ form $v = v(r)$ disjoint groups, with all particles in each distinct group at the same site in $\mathbb{Z}$. Assume that the v respective groups contain $n_1, \ldots, n_v$ respective particles, where $n_1 + \cdots + n_v = k$. Suppose that from time r to $r+1$, b_j of the n_j particles in the j^{th} group go up one step and $n_j - b_j$ particles go down one step for each

$1 \leq j \leq v$. Then we have the entropy formula:

$$\mathcal{D}_N(r+1) - \mathcal{D}_N(r) = \sum_{j=1}^{v} \log\left[m_{b_j, n_j - b_j}\right], \quad \text{where } m_{b,n-b} := \frac{\int_{[0,1]} x^b (1-x)^{n-b} \nu(dx)}{\int_{[0,1]} x^b (1-x)^{n-b} \nu_N(dx)}. \tag{5.18}$$

The exponential of the r.h.s. of the above equation is precisely the quotient of the two relevant Markov chain transition probabilities from $\mathbf{R}(r) \to \mathbf{R}(r+1)$. Although $\mathcal{D}_N$ is not a $\mathbf{P}_{RW_{\nu_N}^{(k)}}$-martingale, we can define another process $\widetilde{\mathcal{D}}_N$ with $\widetilde{\mathcal{D}}_N(0) = 0$, and

$$\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r) := \mathcal{D}_N(r+1) - \mathcal{D}_N(r) - \mathbf{E}_{RW_{\nu_N}^{(k)}}[\mathcal{D}_N(r+1) - \mathcal{D}_N(r) | \mathcal{F}_r]. \tag{5.19}$$

By construction, $\widetilde{\mathcal{D}}_N$ is a $\mathbf{P}_{RW_{\nu_N}^{(k)}}$-martingale. Utilizing the above relation, we observe that

$$\begin{aligned}
\log \mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r)} | \mathcal{F}_r\right] &= \log \mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\mathcal{D}_N(r+1) - \mathcal{D}_N(r) - \mathbf{E}_{RW_{\nu_N}^{(k)}}[\mathcal{D}_N(r+1) - \mathcal{D}_N(r) | \mathcal{F}_r]} | \mathcal{F}_r\right] \\
&= \log \mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\mathcal{D}_N(r+1) - \mathcal{D}_N(r)} | \mathcal{F}_r\right] - \mathbf{E}_{RW_{\nu_N}^{(k)}}[\mathcal{D}_N(r+1) - \mathcal{D}_N(r) | \mathcal{F}_r] \\
&= -\mathbf{E}_{RW_{\nu_N}^{(k)}}[\mathcal{D}_N(r+1) - \mathcal{D}_N(r) | \mathcal{F}_r],
\end{aligned}$$

where in the last line we used that $e^{\mathcal{D}_N}$ is a $\mathbf{P}_{RW_{\nu_N}^{(k)}}$-martingale. This implies

$$\left.\frac{d\mathbf{P}_{RW_{\nu}^{(k)}}}{d\mathbf{P}_{RW_{\nu_N}^{(k)}}}\right|_{\mathcal{F}_r}(\mathbf{R}) = \exp\left(\mathcal{D}_N(r)\right) = \exp\left(\widetilde{\mathcal{D}}_N(r) - \sum_{s \leq r} \log \mathbf{E}_{RW_{\nu_N}^{(k)}}[e^{\widetilde{\mathcal{D}}_N(s) - \widetilde{\mathcal{D}}_N(s-1)} | \mathcal{F}_{s-1}]\right). \tag{5.20}$$

On the other hand, by Definition 5.2.4, we have

$$\left.\frac{d\mathbf{Q}^{N^{-1/4}}_{RW_{\nu}^{(k)}}}{d\mathbf{P}_{RW_{\nu}^{(k)}}}\right|_{\mathcal{F}_r}(\mathbf{R}) = \exp\left(\mathcal{H}_N(r) - \sum_{s \leq r} \log \mathbf{E}_{RW_{\nu}^{(k)}}[e^{\mathcal{H}_N(s) - \mathcal{H}_N(s-1)} | \mathcal{F}_{s-1}]\right) \tag{5.21}$$

where $\mathcal{H}_N(r) = N^{-1/4}(R_r^1 + \cdots + R_r^k)$. Note that the process $\widetilde{\mathcal{H}}_N(r) := \mathcal{H}_N(r) - rkN^{-1/4}(2\mu_{\nu_N} -$

1) is a $\mathbf{P}_{RW_{\nu_N}^{(k)}}$-martingale since each of the processes $R^i(r) - r(2\mu_{\nu_N} - 1)$ for $1 \le i \le k$ are mean-zero random walks under $\mathbf{P}_{RW_{\nu_N}^{(k)}}$ and replacing $\mathcal{H}$ by $\widetilde{\mathcal{H}}$ leaves the above expression unchanged. Let us set $\mathcal{G}_N := \widetilde{\mathcal{H}}_N + \widetilde{\mathcal{D}}_N$. Clearly $\mathcal{G}_N$ is a $\mathbf{P}_{RW_{\nu_N}^{(k)}}$-martingale. Multiplying both sides of (5.20) and (5.21) we get

$$\begin{aligned}
&\frac{d\mathbf{Q}^{N^{-1/4}}_{RW_{\nu}^{(k)}}}{d\mathbf{P}_{RW_{\nu_N}^{(k)}}}\Big|_{\mathcal{F}_r}(\mathbf{R}) \\
&= \exp\left(\mathcal{G}_N(r) - \sum_{s\le r} \log\left[\mathbf{E}_{RW_{\nu}^{(k)}}\left[e^{\widetilde{\mathcal{H}}_N(s)-\widetilde{\mathcal{H}}_N(s-1)} \mid \mathcal{F}_{s-1}\right]\mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\widetilde{\mathcal{D}}_N(s)-\widetilde{\mathcal{D}}_N(s-1)} \mid \mathcal{F}_{s-1}\right]\right)\right) \\
&= \exp\left(\mathcal{G}_N(r) - \sum_{s\le r} \log \mathbf{E}_{RW_{\nu_N}^{(k)}}[e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)}|\mathcal{F}_{s-1}]\right),
\end{aligned}$$

where the last equality follows by observing that for each $s \le r$, by tilting we have

$$\begin{aligned}
&\mathbf{E}_{RW_{\nu}^{(k)}}\left[e^{\widetilde{\mathcal{H}}_N(s)-\widetilde{\mathcal{H}}_N(s-1)} \mid \mathcal{F}_{s-1}\right] \\
&= \mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\widetilde{\mathcal{D}}_N(s)-\widetilde{\mathcal{D}}_N(s-1)-\log \mathbf{E}_{RW_{\nu_N}^{(k)}}[e^{\widetilde{\mathcal{D}}_N(s)-\widetilde{\mathcal{D}}_N(s-1)}|\mathcal{F}_{s-1}]+\widetilde{\mathcal{H}}_N(s)-\widetilde{\mathcal{H}}_N(s-1)} \mid \mathcal{F}_{s-1}\right] \\
&= \frac{\mathbf{E}_{RW_{\nu_N}^{(k)}}\left[e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)} \mid \mathcal{F}_{s-1}\right]}{\mathbf{E}_{RW_{\nu_N}^{(k)}}[e^{\widetilde{\mathcal{D}}_N(s)-\widetilde{\mathcal{D}}_N(s-1)}|\mathcal{F}_{s-1}]}.
\end{aligned}$$

This verifies (5.15).

Step 2. In this step, we will show (5.16). This is where we set $\nu_N = \nu_N^*$, the skewed sequence of probability measures corresponding to ν (see Definition 5.2.1). With this choice, we claim that for all r

$$[\mathcal{G}_N](r+1) - [\mathcal{G}_N](r) = \mathbb{1}_{r\in\Gamma^{(\le k-1)}}\big(\mathcal{G}_N(r+1) - \mathcal{G}_N(r)\big)^2, \tag{5.22}$$

$$\big(\mathcal{G}_N(r+1) - \mathcal{G}_N(r)\big)^2 \le CN^{-1/2}, \tag{5.23}$$

where $\Gamma^{(\le k-1)}$ is defined in (5.8). From the definition of the V^{ij} functionals from (5.9), we have $\mathbb{1}_{r\in\Gamma^{(\le k-1)}} \le \sum_{i<j}\big(V^{ij}(r+1) - V^{ij}(r)\big)$. Inserting this bound and the bound in (5.23) in the

right-hand side of (5.22) leads to (5.16).

Recall the sets $\Gamma^{(v)}$ from (5.8). We claim that

$$\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r) = -\frac{N^{1/4}}{2} \log \frac{\mu_{\nu_N^*}}{1-\mu_{\nu_N^*}} (\widetilde{\mathcal{H}}_N(r+1) - \widetilde{\mathcal{H}}_N(r)), \tag{5.24}$$

for all $r \in \Gamma^{(k)}$. We will prove (5.24) below, but we first complete the rest of the proof, assuming (5.24) to be true. Since the ν_N^* have mean exactly equal to $\mu_{\nu_N^*} = \rho_N = e^{N^{-1/4}}(2\cosh(N^{-1/4}))^{-1}$, we get that $\frac{N^{1/4}}{2} \log \frac{\mu_{\nu_N^*}}{1-\mu_{\nu_N^*}} = 1$ which in turn forces

$$\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r) = -(\widetilde{\mathcal{H}}_N(r+1) - \widetilde{\mathcal{H}}_N(r)).$$

Since $\mathcal{G}_N = \widetilde{\mathcal{H}}_N + \widetilde{\mathcal{D}}_N$, this implies that $\mathcal{G}_N(r+1) - \mathcal{G}_N(r) = 0$ for all $r \in \Gamma^{(k)}$, in turn implying (5.22).

We now turn to the proof of (5.23). By the Cauchy-Schwarz inequality, we have

$$\big(\mathcal{G}_N(r+1) - \mathcal{G}_N(r)\big)^2 \le 2\big(\widetilde{\mathcal{H}}_N(r+1) - \widetilde{\mathcal{H}}_N(r)\big)^2 + 2\big(\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r)\big)^2. \tag{5.25}$$

By definition

$$\big(\widetilde{\mathcal{H}}_N(r+1) - \widetilde{\mathcal{H}}_N(r)\big)^2 = N^{-1/2}\bigg(\sum_{j=1}^{k} (R^j(r+1) - R^j(r) - (2\mu_{\nu_N^*} - 1))\bigg)^2 \le 4k^2 N^{-1/2} \tag{5.26}$$

as $|R^i(r+1) - R^i(r) - (2\mu_{\nu_N^*} - 1)| \le 2$. Recall the definition of $\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r)$ from (5.18) and (5.19). We have

$$\begin{aligned}\big(\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r)\big)^2 &= \bigg(\sum_{j=1}^{v} \big(\log(m_{b_j,n_j-b_j}) - \mathbf{E}_{RW^{(k)}_{\nu_N^*}}[\log(m_{b_j,n_j-b_j})|\mathcal{F}_r]\big)\bigg)^2 \\ &\le 2k \sum_{j=1}^{v} \bigg[\big(\log(m_{b_j,n_j-b_j})\big)^2 + \mathbf{E}_{RW^{(k)}_{\nu_N^*}}\big[\big(\log(m_{b_j,n_j-b_j})\big)^2 \big| \mathcal{F}_r\big]\bigg],\end{aligned} \tag{5.27}$$

where we used Cauchy-Schwarz and $v \leq k$. Our particular choice of ν_N implies that

$$\left| \int_{[0,1]} x^j \nu_N^*(dx) - \int_{[0,1]} x^j \nu(dx) \right| \leq CN^{-1/4} \text{ for all } 1 \leq j \leq k.$$

This implies

$$\left| \int_{[0,1]} x^b (1-x)^{n-b} \nu_N^*(dx) - \int_{[0,1]} x^b (1-x)^{n-b} \nu(dx) \right| \leq CN^{-1/4} \text{ for all } 0 \leq b \leq n \leq k.$$

Since $\sigma_\nu^2 < \frac{1}{4}$, we have that $\int_{[0,1]} x^b (1-x)^{n-b} \nu(dx) > 0$ for all $0 \leq b \leq n \leq k$ and hence $|m_{b,n-b}^{-1} - 1| \leq CN^{-1/4}$ for $0 \leq b \leq n \leq k$, where $C = C(\nu, k)$ is independent of N, b, n. In turn, this is enough to imply that $|\log(m_{b,n-b})| \leq CN^{-1/4}$, for a deterministic constant C, so that $(\log(m_{b,n-b}))^2 \leq CN^{-1/2}$. Inserting this bound back in (5.27) and combining it with the bound in (5.26), in view of (5.25), we have the desired bound of (5.23).

Proof of (5.24)**.** Recall the sets $\Gamma^{(v)}$ from (5.8) and the notation in (5.18). Note that for $r \in \Gamma^{(k)}$, we have $v = k$, $n_j = 1$, and $b_j = \frac{1}{2}(R_{r+1}^j - R^j(r) + 1)$ for $1 \leq j \leq k$. Note that b_j is either one or zero depending on whether the particle jumps right or left. We can therefore compute

$$\begin{aligned} m_{b_j, 1-b_j} &= \frac{\int_{[0,1]} x^{b_j} (1-x)^{1-b_j} \nu(dx)}{\int_{[0,1]} x^{b_j} (1-x)^{1-b_j} \nu_N(dx)} \\ &= \frac{\mathbb{1}_{R_{r+1}^j - R^j(r) = 1} \mu_\nu + \mathbb{1}_{R_{r+1}^j - R^j(r) = -1}(1-\mu_\nu)}{\mathbb{1}_{R_{r+1}^j - R^j(r) = 1} \mu_{\nu_N} + \mathbb{1}_{R_{r+1}^j - R^j(r) = -1}(1-\mu_{\nu_N})} = \frac{1/2}{1/2 + (\mu_{\nu_N} - 1/2)(R_{r+1}^j - R^j(r))}. \end{aligned}$$

We observe that for $x \in \{-1, 1\}$ the following relation holds true:

$$\log(1 + Ax) = \frac{1}{2}\left[x \log\left(\frac{1+A}{1-A} \right) + \log(1 - A^2) \right].$$

Since $R^j_{r+1} - R^j(r) \in \{-1, 1\}$, utilizing the above identity we have

$$\begin{aligned}\log[m_{b_j,1-b_j}] &= -\log\left[1 + (2\mu_{\nu_N} - 1)(R^j_{r+1} - R^j(r))\right] \\ &= -\frac{1}{2}\left[(R^j_{r+1} - R^j(r))\log\left(\frac{\mu_{\nu_N}}{1-\mu_{\nu_N}}\right) + \log(1 - (2\mu_{\nu_N} - 1)^2)\right].\end{aligned}$$

Thus for $r \in \Gamma^{(k)}$, (5.18) boils down to

$$\mathcal{D}_N(r+1) - \mathcal{D}_N(r) = -\frac{1}{2}\log\left(\frac{\mu_{\nu_N}}{1-\mu_{\nu_N}}\right)\sum_{j=1}^{k}(R^j_{r+1} - R^j(r)) - \frac{k}{2}\log(1 - (2\mu_{\nu_N} - 1)^2).$$

This implies for $r \in \Gamma^{(k)}$

$$\mathbf{E}_{RW^{(k)}_{\nu_N}}[\mathcal{D}_N(r+1) - \mathcal{D}_N(r)|\mathcal{F}_r] = -\frac{1}{2}\log\left(\frac{\mu_{\nu_N}}{1-\mu_{\nu_N}}\right)\sum_{j=1}^{k}(2\mu_{\nu_N} - 1) - \frac{k}{2}\log(1 - (2\mu_{\nu_N} - 1)^2).$$

Plugging these identities back in (5.19) we get

$$\begin{aligned}\widetilde{\mathcal{D}}_N(r+1) - \widetilde{\mathcal{D}}_N(r) &= -\frac{1}{2}\log\left(\frac{\mu_{\nu_N}}{1-\mu_{\nu_N}}\right)\sum_{j=1}^{k}(R^j_{r+1} - R^j(r) - (2\mu_{\nu_N} - 1)) \\ &= -\frac{N^{1/4}}{2}\log\left(\frac{\mu_{\nu_N}}{1-\mu_{\nu_N}}\right)(\widetilde{\mathcal{H}}_N(r+1) - \widetilde{\mathcal{H}}_N(r)),\end{aligned}$$

where the second equality follows from the definition of $\widetilde{\mathcal{H}}_N(r)$. This verifies (5.24). □

5.2.2 k-point motion moment estimates

In this subsection, we collect various moment estimates that will be used in our later analysis. Recall the functionals V^{ij} from (5.9).

Lemma 5.2.6. *There exists an absolute constant* $\mathrm{C} > 0$ *such that uniformly over all* $k, N > 1$,

$\theta > 0$ *and all probability measures* ν *on* $[0,1]$*, we have*

$$\mathbf{E}_{RW_\nu^{(k)}}\left[\exp\left(\theta\sum_{i=1}^{k}\left[\big(\max_{r\le N}|R^i(r)-(2\mu_\nu-1)r|\big)+(\mu_\nu(1-\mu_\nu)-\sigma_\nu^2)\sum_{j\neq i}V^{ij}(N)\right]\right)\right]\le \mathrm{C}\cdot e^{\mathrm{C}\cdot k^4\theta^2 N}. \tag{5.28}$$

We remark that this bound will be most useful after replacing $N \to Nt$ and $\theta \to \theta N^{-1/2}$. The bound is optimal in the sense that the constant $\mu_\nu(1-\mu_\nu)-\sigma_\nu^2$ cannot be improved in a manner that is uniform over all probability laws on $[0,1]$. Indeed this factor vanishes precisely if ν is the law of Bernoulli(p) for some $p \in [0,1]$, which corresponds to coalescing walks $(R^1,\dots,R^k)$.

Proof. By the Cauchy-Schwarz inequality, the expectation on the left-hand side of (5.28) is bounded above by $\sqrt{E_1 \cdot E_2}$ where

$$\begin{aligned}
E_1 &:= \mathbf{E}_{RW_\nu^{(k)}}\left[\exp\left(2\theta\sum_{i=1}^{k}\max_{r\le N}|R^i(r)-(2\mu_\nu-1)r|\right)\right]\\
&\le \prod_{i=1}^{k}\mathbf{E}_{RW_\nu^{(k)}}\left[\exp\left(2k\theta\max_{r\le N}|R^i(r)-(2\mu_\nu-1)r|\right)\right]^{1/k}\\
&= \mathbf{E}_{RW_\nu^{(1)}}\left[\exp\left(2k\theta\max_{r\le N}|R(r)-(2\mu_\nu-1)r|\right)\right],
\end{aligned}$$

$$\begin{aligned}
E_2 &:= \mathbf{E}_{RW_\nu^{(k)}}\left[\exp\left(4\theta\big[\mu_\nu(1-\mu_\nu)-\sigma_\nu^2\big]\sum_{1\le i<j\le k}V^{ij}(N)\right)\right]\\
&\le \prod_{1\le i<j\le k}\mathbf{E}_{RW_\nu^{(k)}}\left[\exp\left(2\theta k(k-1)\big[\mu_\nu(1-\mu_\nu)-\sigma_\nu^2\big]V^{ij}(N)\right)\right]^{2/k(k-1)}\\
&= \mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(2\theta k(k-1)\big[\mu_\nu(1-\mu_\nu)-\sigma_\nu^2\big]V^{12}(N)\right)\right].
\end{aligned}$$

Above, in obtaining the subsequent bounds, we used Hölder's inequality and the fact that the laws $\mathbf{P}_{RW_\nu^{(k)}}$ form a projective family of k-point motions. Now the law of $(R(r)-(2\mu_\nu-1)r)_{r\ge 0}$ under $\mathbf{P}_{RW_\nu^{(1)}}$ is simply that of a mean-zero random walk. Therefore $X(r) := \exp\big(k\theta|R(r)-(2\mu_\nu-$

$1)r|\big)$ is a positive $\mathbf{P}_{RW_\nu^{(1)}}$-submartingale, thus by Doob's L^2 inequality we find that

$$\begin{aligned}\mathbf{E}_{RW_\nu^{(1)}}\left[\exp\left(2k\theta\max_{r\le N}|R(r)-(2\mu_\nu-1)r|\right)\right] &= \mathbf{E}_{RW_\nu^{(1)}}\big[\max_{r\le N}X(r)^2\big]\\ &\le 4\mathbf{E}_{RW_\nu^{(1)}}[X(N)^2]\\ &= 4\mathbf{E}_{RW_\nu^{(1)}}\left[\exp\left(2k\theta|R(N)-(2\mu_\nu-1)N|\right)\right].\end{aligned}$$

Then by Azuma's martingale inequality, we obtain that the last expression is bounded above by $Ce^{Ck^2\theta^2N}$, where C is independent of k,θ,N,ν.

Now we need to bound the term with the exponential of the intersection times. To do this, we recall $\mathcal{M}(r)$ from (5.10). Note that $\mathcal{M}(r)$ is a $\mathbf{P}_{RW_\nu^{(2)}}$-martingale with increments bounded above by 3, which may be verified by a direct calculation using the Markov chain transition diagram of (R^1,R^2) given in Figure 5.1. Therefore by the Cauchy-Schwarz inequality followed by Azuma's martingale inequality, we see that

$$\begin{aligned}&\mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(8\theta k(k-1)[\mu_\nu(1-\mu_\nu)-\sigma_\nu^2]V^{ij}(N)\right)\right]\\ &= \mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(2\theta k(k-1)\big[\mathcal{M}(N)+|R^1(N)-R^2(N)|\big]\right)\right]\\ &\le \sqrt{\mathbf{E}_{RW_\nu^{(2)}}\left[e^{4\theta k^2\mathcal{M}(N)}\right]\mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(4\theta k^2\big|(R^1(N)-N(2\mu_\nu-1))-(R^2(N)-N(2\mu_\nu-1))\big|\right)\right]}\\ &\le Ce^{C\theta^2k^4N}\sqrt{\mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(4\theta k^2(|R^1(N)-N(2\mu_\nu-1)|+|R^2(N)-N(2\mu_\nu-1)|)\right)\right]}.\end{aligned}$$

To bound the last term, another application of Cauchy-Schwarz, the projective property of the random walk measures, and then Azuma's inequality gives

$$\mathbf{E}_{RW_\nu^{(2)}}\left[\exp\left(4\theta k^2(|R^1(N)-(2\mu_\nu-1)N|+|R^2(N)-(2\mu_\nu-1)N|)\right]\right. \tag{5.29}$$

$$\left.\le \mathbf{E}_{RW_\nu^{(1)}}\left[\exp\left(8\theta k^2|R(N)-(2\mu_\nu-1)N|\right)\right]\le Ce^{C\theta^2k^4N},\right. \tag{5.30}$$

where we again used that R minus its mean drift is a mean-zero random walk (with increments

bounded absolutely by 2) under $\mathbf{P}_{RW_\nu^{(1)}}$. $\square$

Lemma 5.2.7. *Fix $k \in \mathbb{N}$ and $p \geq 1$. There exists a constant $C = C(p,k) > 0$ such that uniformly over all $N \geq M \geq 0$ and all probability measures ν on $[0,1]$, we have*

$$\mathbf{E}_{RW_\nu^{(k)}}\left[\left(\left[\mu_\nu(1-\mu_\nu) - \sigma_\nu^2\right]\sum_{i<j}(V^{ij}(N) - V^{ij}(M))\right)^p\right]^{1/p} \leq C|M-N|^{1/2}.$$

Proof. By Minkowski's inequality, the left side is bounded above by

$$\sum_{1\leq i<j\leq k}\mathbf{E}_{RW_\nu^{(k)}}\left[\left((\mu_\nu(1-\mu_\nu) - \sigma_\nu^2)(V^{ij}(N) - V^{ij}(M))\right)^p\right]^{1/p}$$

which by the projective property of the Markovian laws is the same as

$$\frac{1}{2}k(k-1)\mathbf{E}_{RW_\nu^{(2)}}\left[\left((\mu_\nu(1-\mu_\nu) - \sigma_\nu^2)(V^{12}(N) - V^{12}(M))\right)^p\right]^{1/p}.$$

Using (5.10), we may write $4(\mu_\nu(1-\mu_\nu) - \sigma^2)V^{12}(N) = \mathcal{M}(N) + |R^1(N) - R^2(N)|$, where $\mathcal{M}(N)$ is a martingale with increments bounded absolutely by 3. Thus using $||x| - |y|| \leq |x-y|$ and Minkowski's inequality again, we find that the last expression is bounded above by $\frac{1}{8}k(k-1)$ times

$$\mathbf{E}_{RW_\nu^{(2)}}\left[\left|\mathcal{M}(N) - \mathcal{M}(M)\right|^p\right]^{1/p} + 2\mathbf{E}_{RW_\nu^{(1)}}\left[\left|R(N) - R(M) - (N-M)(2\mu_\nu - 1)\right|^p\right]^{1/p}.$$

From here, an application of Azuma's inequality allows us to bound both terms by $C|M-N|^{1/2}$, where we again use that R minus its mean drift is a mean-zero random walk (with increments bounded absolutely by 2) under $\mathbf{P}_{RW_\nu^{(1)}}$. $\square$

Proposition 5.2.8. *Fix $k \in \mathbb{N}$. Suppose $\{\nu_N\}_N$ is a family of probability measures on $[0,1]$ satisfying*

$$\liminf_{N\to\infty}\int_{[0,1]} x(1-x)\nu_N(dx) > 0. \tag{5.31}$$

Let $\mathcal{G}_N = \big(\mathcal{G}_N(r)\big)_{r\geq 0}$ *be any family of* $\mathbf{P}_{RW_{\nu_N}^{(k)}}$*-martingales, adapted to the canonical filtration* $(\mathcal{F}_r)_{r\geq 0}$ *on* $(\mathbb{Z}^k)^{\mathbb{N}}$*, satisfying*

$$[\mathcal{G}_N](r) - [\mathcal{G}_N](s) \leq CN^{-1/2}\sum_{i<j}(V^{ij}(r) - V^{ij}(s)) \tag{5.32}$$

for some deterministic constant $C > 0$ *independent of* N*. For all* $p, T > 0$ *we have*

$$\sup_{N\geq 1}\sup_{1\leq r\leq NT}\mathbf{E}_{RW_{\nu_N}^{(k)}}\big[\exp\big(p\mathcal{G}_N(r)\big)\big] < \infty. \tag{5.33}$$

Proof. For simplicity of notation, we write $\mathbf{E}_N$ for $\mathbf{E}_{RW_{\nu_N}^{(k)}}$. We further assume $T = 1$ for simplicity of notation. The general case is analogous. Since $\exp\big(p\mathcal{G}_N(r)\big)$ for $r \geq 0$ is a positive submartingale, by Doob's L^p inequality it suffices to show $\sup_{N\geq 1}\mathbf{E}_N\big[\exp\big(p\mathcal{G}_N(N)\big)\big] < \infty$, i.e., we may ignore the inner supremum after replacing r by the terminal time N. Define

$$W_N(r) := \frac{1}{2}\sum_{s\leq r}\log\mathbf{E}_N[e^{2p(\mathcal{G}_N(s)-\mathcal{G}_N(s-1))}|\mathcal{F}_{s-1}].$$

By (5.12), the process $\exp\big(2p\mathcal{G}_N(r) - 2W_N(r)\big)$ for $r \geq 0$ is a martingale, therefore

$$\mathbf{E}_N[e^{p\mathcal{G}_N(N)}] = \mathbf{E}_N[e^{p\mathcal{G}_N(N)-W_N(N)}e^{W_N(N)}] \tag{5.34}$$

$$\leq \mathbf{E}_N[e^{2p\mathcal{G}_N(N)-2W_N(N)}]^{1/2}\mathbf{E}_N[e^{2W_N(N)}]^{1/2} = \mathbf{E}_N[e^{2W_N(N)}]^{1/2}. \tag{5.35}$$

By the assumption (5.32), we have $|\mathcal{G}_N(r+1) - \mathcal{G}_N(r)| \leq C^{1/2}N^{-1/4} \leq 1$ deterministically for large enough N. For x in the interval $[-2p, 2p]$ we have $e^x \leq 1 + x + Dx^2$ for a large constant $D = D(p) > 0$, therefore we find that

$$e^{2p(\mathcal{G}_N(s)-\mathcal{G}_N(s-1))} \leq 1 + 2p(\mathcal{G}_N(s) - \mathcal{G}_N(s-1)) + 4p^2D\cdot(\mathcal{G}_N(s) - \mathcal{G}_N(s-1))^2,$$

so that by martingality of $\mathcal{G}_N$ and $\log(1+u) \le u$ we have

$$\log \mathbf{E}_N[e^{2p(\mathcal{G}_N(s)-\mathcal{G}_N(s-1))}|\mathcal{F}_{s-1}] \le 4p^2 D \cdot \mathbf{E}_N[(\mathcal{G}_N(s)-\mathcal{G}_N(s-1))^2|\mathcal{F}_{s-1}].$$

Hence we find that

$$\mathbf{E}_N[e^{2W_N(N)}] \le \mathbf{E}_N[e^{4p^2 D\sum_{s\le N}\mathbf{E}_N[(\mathcal{G}_N(s)-\mathcal{G}_N(s-1))^2|\mathcal{F}_{s-1}]}] \le \mathbf{E}_N[e^{C'N^{-1/2}\sum_{i<j}V^{ij}(N)}],$$

where C' is a larger constant that has now absorbed D, and we used (5.32) in the last bound. The bound is clearly finite for each fixed N. We thus have to justify that the last expression remains bounded as $N \to \infty$. Now, (5.31) guarantees that $\mu_{\nu_N}(1-\mu_{\nu_N}) - \sigma^2_{\nu_N}$ is strictly positive for large enough N. A direct application of Lemma 5.2.6 with $\theta = C'N^{-1/2}/(\mu_{\nu_N}(1-\mu_{\nu_N}) - \sigma^2_{\nu_N})$ shows that the last expression remains bounded as $N \to \infty$. □

5.3 A general theorem for calculating limits via tilting

The goal of this section is to establish a general weak convergence theorem that takes the Girsanov tilting into account. As a consequence of this general theorem, we will prove moment convergence in Section 5.3.1. Recall V^{ij} from (5.9) and $\Gamma^{(\le v)}$ from (5.8). Then we have the following.

Theorem 5.3.1. *Fix any $k \in \mathbb{N}$ and constants $C, T > 0$. Assume that ν_N is a family of probability measures on $[0,1]$ satisfying $\mu_{\nu_N} \to \frac{1}{2}$ and $\sigma^2_{\nu_N} \to \sigma^2 \in [0, \frac{1}{4})$. Let $\mathbf{R} = (\mathbf{R}(r))_{r\ge 0}$ be the canonical process on $(\mathbb{Z}^k)^{\mathbb{Z}_{\ge 0}}$, and define the rescaled processes*

$$\mathbf{X}_N(t) := \frac{\big[\mathbf{R}(Nt) - (2\mu_{\nu_N}-1)Nt\big]}{\sqrt{N}}, \qquad \mathscr{V}^{ij}_N(t) := \frac{V^{ij}(Nt)}{\sqrt{N}}, \qquad \mathscr{T}_N(t) := \frac{\#\{[Nt]\cap\Gamma^{\le(k-2)}\}}{\sqrt{N}}. \tag{5.36}$$

for $t \in N^{-1}\mathbb{Z}_{\ge 0}$, and linearly interpolated for $t \notin N^{-1}\mathbb{Z}_{\ge 0}$. We will use $\mathbf{E}_N$ to abbreviate path measures $\mathbf{E}_{RW^{(k)}_{\nu_N}}$, as defined in Section 5.2.1. Let $\mathcal{G}_N = \big(\mathcal{G}_N(r)\big)_{r\ge 0}$ be any family of $\mathbf{P}_N$-

martingales, adapted to the canonical filtration $(\mathcal{F}_r)_{r\geq 0}$ *on* $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$*, satisfying*

$$[\mathcal{G}_N](r) - [\mathcal{G}_N](s) \leq CN^{-1/2} \sum_{i<j} \big(V^{ij}(r) - V^{ij}(s)\big), \tag{5.37}$$

for all $r \geq s \geq 0$*. Define:*

$$W_N(r) := \sum_{s=1}^{r} \log \mathbf{E}_N[e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)} | \mathcal{F}_{s-1}]. \tag{5.38}$$

Set $\widetilde{\mathcal{G}}_N(t) = \mathcal{G}_N(Nt)$ *and* $\widetilde{W}_N(t) = W_N(Nt)$ *for* $t \in N^{-1}\mathbb{Z}$ *and linearly interpolated for* $t \notin N^{-1}\mathbb{Z}$*. Under the* $\mathbf{P}_N$ *law,*

$$\Big(\mathbf{X}_N, ((1-4\sigma^2)\mathscr{V}_N^{ij})_{1\leq i<j\leq k}, \mathscr{T}_N, \widetilde{\mathcal{G}}_N, \widetilde{\mathcal{W}}_N\Big)$$

is a tight family of random variables in the canonical space of 5-tuples $C([0,T], \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2} \times \mathbb{R}\times\mathbb{R}\times\mathbb{R})$*. Consider any (joint) limit point* $\mathbf{P}_\infty$ *of that 5-tuple of processes, which is a probability measure on the canonical space of 5-tuples. Let* $(\mathbf{U}, (L^{ij})_{1\leq i<j\leq k}, \mathbf{0}, \mathcal{G}, \mathcal{W})$ *denote the coordinate process on the canonical space of 5-tuples. Then*

(i) $\mathbf{U}$ *is a standard* k*-dimensional Brownian motion on* $[0,T]$ *under* $\mathbf{P}_\infty$*.*

(ii) $L^{ij}(t) := L_0^{U^i-U^j}(t)$ *is the local time at zero of* $U^i - U^j$ *accrued by time* t*. Our convention for the local time at* $a \in \mathbb{R}$ *of a continuous semimartingale* X *is*

$$L_a^X(t) := \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon<X_s<a+\varepsilon\}} d\langle X, X\rangle_s. \tag{5.39}$$

(iii) $\mathbf{0}$ *is* $\mathbf{P}_\infty$*-almost surely equal to the function from* $[0,T] \to \mathbb{R}$ *which is identically zero.*

(iv) $\mathcal{G}$ *is a continuous* $\mathbf{P}_\infty$*-martingale satisfying* $\mathbf{E}_\infty[e^{p\mathcal{G}(T)-\frac{p}{2}\langle\mathcal{G}\rangle(T)}] < \infty$ *for all* $p > 0$ *as well as*

$$\mathbf{E}_\infty[\exp(\mathcal{G}(T) - \tfrac{1}{2}\langle\mathcal{G}\rangle(T)) \mid \mathcal{F}_T(\mathbf{U})] = 1, \tag{5.40}$$

where $\mathcal{F}_T(\mathbf{U})$ denotes the σ-algebra generated by $\mathbf{U}$.

(v) $\mathcal{W} = \frac{1}{2}\langle \mathcal{G} \rangle$, where the latter denotes quadratic variation of the continuous martingale $\mathcal{G}$.

We remark that the above five conditions do not uniquely characterize the limit points $\mathbf{P}_\infty$, however, they do uniquely characterize expectations with respect to the particular types of observables that we will be interested in, thus the non-uniqueness will not be an issue. We will abuse notation and use $\mathbf{P}_{\text{lim}}$ to mean different marginal measures of $\mathbf{P}_\infty$ throughout the proof, clarifying when needed.

Proof. Write out the vector components as $\mathbf{X}_N = (\mathbf{X}_N^1, \ldots, \mathbf{X}_N^k)$. Note that each of the processes $(\mathbf{X}_N^i(t))_{t \in N^{-1}\mathbb{Z}_{\geq 0}}$ is a $\mathbf{P}_N$-martingale indexed by $N^{-1}\mathbb{Z}_{\geq 0}$, with increments bounded above by $2N^{-1/2}$. Therefore Azuma's inequality implies the bound $\sup_N \mathbf{E}_N[|\mathbf{X}_N^i(t) - \mathbf{X}_N^i(s)|^p]^{1/p} \leq C|t-s|^{1/2}$, for the linearly interpolated processes $\mathbf{X}_N^i$. Thus each coordinate of $\mathbf{X}_N^i$ is tight in $C[0,T]$. Furthermore, the process $\mathscr{V}_N$ is tight via Lemmas 5.2.6 and 5.2.7.

Let us thus consider any *joint* limit point of $(\mathbf{X}_N, \mathscr{V}_N)$, say $(\mathbf{U}, (\mathscr{L}^{ij})_{i<j})$, whose law we view as a measure $\mathbf{P}_{\text{lim}}$ on the canonical space $C([0,T], \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2})$ equipped with its canonical filtration. Without loss of generality, we will assume $(\mathbf{U}, (\mathscr{L}^{ij})_{i<j})$ is simply the canonical process on this space. We first show that $\mathbf{U}$ is a standard Brownian motion in $\mathbb{R}^k$ and that the processes $(1 - 4\sigma^2)\mathscr{L}^{ij}$ necessarily agree with the local time at zero of $U^i - U^j$. We will use the Levy characterization and Tanaka's formula after identifying useful martingales in the prelimit.

For $1 \leq i \leq k$, define the process $S^i(r) := R^i(r) - (2\mu_{\nu_N} - 1)r$ which are $\mathbf{P}_N$-martingales. In the prelimit, the processes for $1 \leq i \leq k$ given by

$$Y_N^i(r) = S^i(r)^2 - \sum_{s=1}^{r} \mathbf{E}_N\big[\big(S^i(s) - S^i(s-1)\big)^2 \big| \mathcal{F}_{s-1}\big] = S^i(r)^2 - 4\mu_{\nu_N}(1 - \mu_{\nu_N})r$$

are $\mathbf{P}_N$-martingales. Additionally, the processes for $1 \leq i < j \leq k$ given by

$$\begin{aligned} K_N^{ij}(r) &= S^i(r)S^j(r) - \sum_{s=1}^{r} \mathbf{E}_N\big[\big(S^i(s) - S^i(s-1)\big)\big(S^j(s) - S^j(s-1)\big)\big|\mathcal{F}_{s-1}\big] \\ &= S^i(r)S^j(r) - 4\sigma_{\nu_N}^2 V_r^{ij} \end{aligned}$$

are $\mathbf{P}_N$-martingales. Rescaling, we find that the processes

$$N^{-1}Y_N^i(Nt) = \mathbf{X}_N^i(t)^2 - 4\mu_{\nu_N}(1-\mu_{\nu_N})t \tag{5.41}$$

$$N^{-1}K_N^{ij}(Nt) = \mathbf{X}_N^i(t)\mathbf{X}_N^j(t) - N^{-1/2}4\sigma_{\nu_N}^2 \mathscr{V}_N^{ij}(t) \tag{5.42}$$

are $\mathbf{P}_N$ martingales indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$. Using Lemma 5.2.6 we find that these martingales are uniformly integrable as well.

Proof of (i). Now we pass this information to the limit points. Martingality is preserved by limit points as long as one has uniform integrability, thus it is clear that $\mathbf{U}$ is necessarily a $\mathbf{P}_{\text{lim}}$-martingale (with respect to the canonical filtration on $C([0,T], \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2})$). Next, note that $\lim_{N\to\infty} 4\mu_{\nu_N}(1-\mu_{\nu_N}) = 1$ by our assumption on μ_{ν_N}. Again using that martingality is preserved by limit points, we can thus apply (5.41) to conclude that $\mathbf{U}^i(t)^2 - t$ is a $\mathbf{P}_{\text{lim}}$-martingale, i.e., $\mathbf{U}^i$ necessarily has quadratic variation t. Yet again using the fact that martingality is preserved by limit points, we can use (5.42) and tightness in $C[0,T]$ of the processes $\mathscr{V}_N^{ij}$ to conclude that for all $i < j$, the process $\mathbf{U}^i\mathbf{U}^j$ is a $\mathbf{P}_{\text{lim}}$-martingale, i.e., $\langle \mathbf{U}^i, \mathbf{U}^j \rangle = 0$ for $i \neq j$. By Levy's characterization, we may conclude that $\mathbf{U}$ is therefore a standard Brownian motion in $\mathbb{R}^k$. This proves (i).

Proof of (ii). We will again use a martingale problem. Recall the martingales $\mathcal{M}(r)$ from (5.10). Rescaling, we find that

$$|\mathbf{X}_N^i(t) - \mathbf{X}_N^j(t)| - 4\big[(\mu_{\nu_N}(1-\mu_{\nu_N}) - \sigma_{\nu_N}^2)\big]\mathscr{V}_N^{ij}(t)$$

are $\mathbf{P}_N$-martingales indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$. All quantities are uniformly integrable and tight in $C[0,T]$ as $N \to \infty$ (as explained earlier). By our hypothesis on ν_N, we have $\mu_{\nu_N}(1-\mu_{\nu_N}) \to 1/4$. Thus for the limit points, we conclude that the processes $|\mathbf{U}^i - \mathbf{U}^j| - (1-4\sigma^2)\mathscr{L}^{ij}$ are $\mathbf{P}_{\text{lim}}$-martingales. As $\mathbf{U}$ is a standard k-dimensional Brownian motion, and the processes $\mathscr{L}^{ij}$ are $\mathbf{P}_{\text{lim}}$-a.s. nondecreasing, we conclude by Tanaka's formula that $(1-4\sigma^2)\mathscr{L}^{ij}$ is the local time at zero of $U^i - U^j$. This uniquely identifies the law of the joint limit point $(\mathbf{U}, (\mathscr{L}^{ij})_{i<j})$.

Proof of (iii) We now show that as $N \to \infty$ the processes $\mathscr{T}_N$ converge in probability to the zero process with respect to the topology of $C[0,T]$. Indeed these processes are tight because for $\delta > 0$,

$$\sup_{|t-s|<\delta} |\mathscr{T}_N(t) - \mathscr{T}_N(s)| \leq \sup_{|t-s|<\delta} N^{-1/2} \sum_{1\leq i<j\leq k} V^{ij}(Nt) - V^{ij}(Ns)$$

and we already know that the intersection time process on the right side is tight. Now if we take any *joint* limit point $(F_t, \mathbf{U}_t)_{t\in[0,T]}$ of the pair of processes $(\mathscr{T}_N, \mathbf{X}_N)$, then F_t is a nondecreasing process with $F_0 = 0$. Furthermore by definition of the set $\Gamma^{(\leq k-2)}$ (see (5.8)), this limit point F is constant on every connected component of the complement of the random closed set $K \cup L$ where

$$K := \{t \in [0,T] : \mathbf{U}^i_t = \mathbf{U}^j_t = \mathbf{U}^p_t \text{ for some } 1 \leq i < j < p \leq k\},$$
$$L := \{t \in [0,T] : \mathbf{U}^i_t = \mathbf{U}^j_t \text{ and } \mathbf{U}^p_t = \mathbf{U}^q_t \text{ for some } i < j \text{ and } p < q \text{ with } \{i,j\} \cap \{p,q\} = \emptyset\}.$$

We already know that $\mathbf{U}$ is a standard Brownian motion in $\mathbb{R}^k$. But 3d Brownian motion never hits the diagonal $\{x = y = z\}$ of $\mathbb{R}^3$ after time 0, and likewise, a 4d Brownian motion never hits the proper 2d linear subspace of $\mathbb{R}^4$ given by $\{x = y\} \cap \{z = w\}$ after time 0. Consequently $K \cup L = \{0\}$, and so $F_t = 0$ identically.

Proof of (iv) Let us now take any $\mathcal{G}_N$ satisfying the assumption of the theorem. We first prove that the linearly interpolated processes $\big(\mathcal{G}_N(Nt)\big)_{t\in[0,T]\cap N^{-1}\mathbb{Z}_{\geq 0}}$ are tight in $C[0,T]$. Indeed, by

Burkholder-Davis-Gundy inequality, we have for all $s, t \in N^{-1}\mathbb{Z}_{\geq 0}$ that

$$\mathbf{E}\big[|\mathcal{G}_N(Nt) - \mathcal{G}_N(Ns)|^p\big] \leq \mathbf{E}\bigg[\Big([\mathcal{G}_N](Nt) - [\mathcal{G}_N](Ns)\Big)^{\frac{p}{2}}\bigg] \leq C \cdot \mathbf{E}\bigg[\Big(N^{-\frac{1}{2}} \sum_{i<j} \big(V^{ij}(Nt) - V^{ij}(Ns)\big)\Big)^{\frac{p}{2}}\bigg].$$

The second inequality above is due to the assumption (5.37). Lemma 5.2.7 implies that the term on the right-hand side of the above equation is bounded by $C|t-s|^{p/4}$ for some constant $C > 0$ free of N. This verifies the tightness of $t \mapsto \mathcal{G}_N(Nt)$.

Consider any limit point of the triple $\big(\widetilde{\mathcal{G}}_N,\ \mathbf{X}_N,\ (1-4\sigma^2)\big(\mathscr{V}_N^{ij}\big)_{1\leq i<j\leq k}\big)$, say $(\mathcal{G}, \mathbf{U}, \big(L^{ij}\big)_{1\leq i<j\leq k})$ whose law $\mathbf{P}_{\text{lim}}$ (now a different object from the previous parts of the proof) we view as a measure on the canonical space $C([0,T], \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2})$. By the tightness and L^p bounds above, the process $\mathcal{G}$ is still a $\mathbf{P}_{\text{lim}}$-martingale with respect to the canonical filtration on this space. We claim that the quadratic covariations satisfy $\langle G, \mathbf{U}^i \rangle_t = 0$ for all $t \in [0,T]$. To prove this, note that in the prelimit, $\mathbf{X}_N^i(t)\mathcal{G}_N(Nt) - \mathscr{P}_N(t)$ for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ is a $\mathbf{P}_N$-martingale where

$$\mathscr{P}_N(t) := N^{-1/2} \sum_{s=1}^{Nt} \mathbf{E}_N[(\mathcal{G}_N(s) - \mathcal{G}_N(s-1))(R^i(s) - R^i(s-1) - (2\mu_{\nu_N} - 1))|\mathcal{F}_{s-1}].$$

Note that

$$\begin{aligned}
|\mathscr{P}_N(t)| &= N^{-\frac{1}{2}} \sum_{s \in [Nt] \cap \Gamma^{(\leq k-1)}} \big|\mathbf{E}_N[(\mathcal{G}_N(s) - \mathcal{G}_N(s-1))(R^i(s) - R^i(s-1) - (2\mu_{\nu_N} - 1)|\mathcal{F}_{s-1}]\big| \\
&\leq N^{-\frac{1}{2}} \sum_{s \in [Nt] \cap \Gamma^{(\leq k-1)}} \bigg[\sqrt{\mathbf{E}_N[(\mathcal{G}_N(s) - \mathcal{G}_N(s-1))^2|\mathcal{F}_{s-1}]} \\
&\qquad\qquad \cdot \sqrt{\mathbf{E}_N[(R^i(s) - R^i(s-1) - (2\mu_{\nu_N} - 1))^2|\mathcal{F}_{s-1}]}\bigg] \\
&\leq \sum_{s \in [Nt] \cap \Gamma^{(\leq k-1)}} C^{1/2} N^{-3/4} \leq C^{1/2} N^{-1/4} \sum_{i<j} \mathscr{V}_N^{ij}.
\end{aligned}$$

Let us briefly explain the above deductions. The first equality is due to the fact that $\mathcal{G}_N(s) - \mathcal{G}_N(s-1) = 0$ for $s \in \Gamma^{(k)}$ via assumption (5.37). The inequality in the second line follows from

Cauchy-Schwarz inequality. Note that $R^i(s) - R^i(s-1) - (2\mu_{\nu_N} - 1)$ is uniformly bounded by 2. Thus, employing the bound in (5.37), we get the inequality in the third line. The final inequality follows from the definition of $\Gamma^{(k-1)}$ and $\mathscr{V}^{ij}$.

Now, since we already know $\mathscr{V}_N$ are tight, we conclude that $|\mathscr{P}_N(t)| \to 0$. Invoking Lemmas 5.2.6 and 5.2.8, we see that $\mathbf{X}_N^i \mathcal{G}_N - \mathscr{P}_N$ is uniformly integrable. Hence $\mathbf{U}^i\mathcal{G}$, being a limit point of $\mathbf{X}_N^i \mathcal{G}_N - \mathscr{P}_N$, is a $\mathbf{P}_{\text{lim}}$-martingale for all $1 \le i \le k$, so that $\langle \mathcal{G}, \mathbf{U}^i \rangle_t = 0$ for all $t \in [0,T]$, as claimed. In fact, Proposition 5.2.8 guarantees that the process $\exp(\mathcal{G} - \frac{1}{2}\langle \mathcal{G} \rangle)$ is a $\mathbf{P}_{\text{lim}}$-martingale. Now a standard argument that we used in our previous paper [71, Proof of Theorem 3.2] ensures that $\mathbf{E}_{\text{lim}}[\exp(\mathcal{G}(T) - \frac{1}{2}\langle \mathcal{G} \rangle(T)) \mid \mathcal{F}_T(\mathbf{U})] = 1$. For completeness, we reproduce the argument in the next paragraph.

Take any bounded $\mathcal{F}_T(\mathbf{U})$-measurable functional $H : C([0,T], \mathbb{R}^k) \to \mathbb{R}$. Note that $t \mapsto \mathbf{E}_{\text{lim}}[H(\mathbf{U})|\mathcal{F}_t(\mathbf{U})]$ is a martingale in the filtration of $\mathbf{U}$. Under $\mathbf{P}_{\text{lim}}$, the marginal law of $\mathbf{U}$ is a standard k-dimensional Brownian motion by the result of part (i), thus by the martingale representation theorem we have

$$H(\mathbf{U}) = \mathbf{E}_{\text{lim}}[H(\mathbf{U})] + \sum_{i=1}^{k} \int_0^T h_s^i dU_s^i$$

for some adapted $\mathbb{R}$-valued processes $h^1, \ldots, h^k$. For stochastic exponential we have

$$\exp\left(\mathcal{G}(T) - \frac{1}{2}\langle \mathcal{G} \rangle(T)\right) = 1 + \int_0^T \exp\left(\mathcal{G}(s) - \frac{1}{2}\langle \mathcal{G} \rangle(s)\right) d\mathcal{G}(s).$$

Using this we find that

$$\begin{aligned}
&\mathbf{E}_{\text{lim}}\left[\left(e^{\mathcal{G}(T) - \frac{1}{2}\langle \mathcal{G} \rangle(T)} - 1\right)\left(H(\mathbf{U}) - \mathbf{E}_{\text{lim}}[H(\mathbf{U})]\right)\right] \\
&= \sum_{i=1}^{k} \mathbf{E}_{\text{lim}}\left[\left(\int_0^T e^{\mathcal{G}(s) - \frac{1}{2}\langle \mathcal{G} \rangle(s)} d\mathcal{G}(s)\right)\left(\int_0^T h_s^i dU_s^i\right)\right] \\
&= \sum_{i=1}^{k} \mathbf{E}_{\text{lim}}\left[\int_0^T h_s^i e^{\mathcal{G}(s) - \frac{1}{2}\langle \mathcal{G} \rangle(s)} d\langle \mathcal{G}, U^i \rangle(s)\right] = 0,
\end{aligned}$$

where the last equality is due to the fact $\langle U^i, \mathcal{G}\rangle \equiv 0$ almost surely. This proves that for all bounded measurable H we have $\mathbf{E}_{\text{lim}}[e^{\mathcal{G}(T)-\frac{1}{2}\langle \mathcal{G}\rangle(T)}H(\mathbf{U})] = \mathbf{E}_{\text{lim}}[H(\mathbf{U})]$, establishing (5.40).

Proof of (v) By assumption (5.37), we have $|\mathcal{G}_N(r+1) - \mathcal{G}_N(r)| \leq C^{1/2}N^{-1/4} \leq 1$ deterministically for large enough N. For $x \in [-1,1]$ we have $e^x \leq 1 + x + x^2$, therefore we find that

$$e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)} \leq 1 + (\mathcal{G}_N(s) - \mathcal{G}_N(s-1)) + (\mathcal{G}_N(s) - \mathcal{G}_N(s-1))^2,$$

consequently using martingality of $\mathcal{G}_N$ and $\log(1+u) \leq u$ we find that

$$\begin{aligned} W_N(r) - W_N(r-1) &\leq \log \mathbf{E}_N[e^{\mathcal{G}_N(r)-\mathcal{G}_N(r-1)}|\mathcal{F}_{r-1}] \\ &\leq \mathbf{E}_N[(\mathcal{G}_N(r) - \mathcal{G}_N(r-1))^2|\mathcal{F}_{r-1}] \leq CN^{-1/2}\sum_{i<j}(V^{ij}(r) - V^{ij}(r-1)), \end{aligned}$$

where we used (5.37). Using this bound together with the nondecreasing property of W_N, and Lemmas 5.2.6 and 5.2.7, it is clear that for all $p \geq 0$

$$\sup_{N\geq 1}\mathbf{E}_N[|W_N(Nt) - W_N(Ns)|^p]^{1/p} \leq C|t-s|^{p/2}, \qquad \sup_{N\geq 1}\mathbf{E}_N[e^{pW_N(Nt)}] < \infty.$$

Thus the linearly interpolated processes $t \mapsto W_N(Nt)$ for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ are tight in $C[0,T]$, so we may consider any limit point $\mathcal{W}$ which is taken *jointly* with the earlier triple of processes $(\mathcal{G}, \mathbf{U}, \left(L^{ij}\right)_{1\leq i<j\leq k})$. Note that since $\exp(\mathcal{G}_N - W_N)$ are martingales in the prelimit, and the above exponential bound on W_N guarantees uniform integrability of its exponential, it follows that $\exp(\mathcal{G} - \mathcal{W})$ is a martingale for any limit point of the joint 4-tuple. Since $\mathcal{G}$ is a continuous martingale, this then forces that $\mathcal{W} = \frac{1}{2}\langle \mathcal{G}\rangle$. □

We record an extension of the above theorem which also takes certain types of measures into account. The following corollary will be useful in our later analysis.

Corollary 5.3.2. *Assume the same notations and assumptions of Theorem 5.3.1. Suppose* $k = 2m$

is even. For $t > s > 0$ define

$$\Delta_m(s,t) := \{(s_1, s_2, \ldots, s_m) \in [s,t]^m \mid s \leq s_1 \leq s_2 \leq \cdots \leq s_m \leq t\} \tag{5.43}$$

to be the set of all m ordered points in $[s,t]$. We define $\mathcal{M}(\Delta_m(0,T))$ to be the space of finite and non-negative Borel measures on that simplex, equipped with the topology of weak convergence. Consider the following sequence of $\mathcal{M}(\Delta_m(0,T))$-valued random variables

$$\gamma_N := N^{-m/2}(1-4\sigma^2)^m \sum_{u_1 \leq \ldots \leq u_m \in (N^{-1}\mathbb{Z}_{\geq 0}) \cap [0,T]} \prod_{j=1}^{m} \mathbb{1}_{\{\mathbf{X}_N^{2j-1}(u_j) = \mathbf{X}_N^{2j}(u_j)\}} \delta_{(u_1,\ldots,u_m)} \tag{5.44}$$

where $\mathbf{X}_N = (\mathbf{X}_N^1, \ldots, \mathbf{X}_N^k)$. The random variables $\{\gamma_N\}_{N\geq 1}$ are tight in $\mathcal{M}(\Delta_m(0,T))$. Moreover, any limit point of the triple $(\mathbf{X}_N,\ (1-4\sigma^2)\big(\mathscr{V}_N^{ij}\big)_{1\leq i<j\leq k},\ \gamma_N)$ is of the form

$$\left(\mathbf{U},\ \big(L^{ij}\big)_{1\leq i<j\leq k},\ \prod_{j=1}^{m} dL^{2j-1,2j}(u_j)\right)$$

where $\mathbf{U}, L^{ij}$ are as in Theorem 5.3.1, and $dL(t)$ denotes the Lebesgue-Stieltjes measure induced by the increasing function $t \mapsto L(t)$.

Proof. Note that

$$\gamma_N(\Delta_m(0,T)) \leq \prod_{j=1}^{m} \left[N^{-\frac{1}{2}} \sum_{u \in [0,T] \cap (N^{-1}\mathbb{Z}_{\geq 0})} \mathbb{1}_{\{\mathbf{X}_N^{2j-1}(u) = \mathbf{X}_N^{2j}(u)\}} \right].$$

From Lemma 5.2.6 we know the exponential moments for each of the terms in the product are uniformly bounded under the $\mathbf{E}_N$ measure. Thus for all $p \geq 1$,

$$\sup_{N\geq 1} \mathbf{E}_N[\gamma_N(\Delta_m(0,T))^p] < \infty. \tag{5.45}$$

Hence the laws of $\{\gamma_N\}_{N\geq 1}$ are tight, because the total mass of γ_N is a tight family of random variables and because $\Delta_m(0,T)$ as defined in (5.43) is a compact space. Let $\Big(\mathbf{U}, \big(K^{ij}\big)_{1\leq i<j\leq k}, \gamma\Big)$

be any limit point of the sequence $(\mathbf{X}_N, (1-4\sigma^2)\big(\mathscr{V}_N^{ij}\big)_{1\le i<j\le k}, \gamma_N)$. Note that the joint cumulative distribution function of the measure γ is necessarily given by a product of the K^{ij}. Since we know that $K^{ij} = L_0^{U^i - U^j}$ by Theorem 5.3.1(ii), this implies that $\gamma = \prod_{j=1}^m dL_0^{U^{2j-1}-U^{2j}}(u_j)$. □

5.3.1 Convergence of moments

We will now prove that moments of the field (5.3) converge to the moments of (5.4). It will be a consequence of Theorem 5.3.1, illustrating the applicability of the convergence theorem. While the moment convergence is not enough to prove the weak convergence of Theorem 5.1.1, it gives a strong indication of it, and more importantly, it implies some useful estimates that will be used later.

Proposition 5.3.3 (Moment convergence)**.** *Fix $\phi \in C_c^\infty(\mathbb{R})$, $t > 0$, and $k \in \mathbb{N}$. With $\mathscr{U}_N$ as defined in* (5.3)*, we have that*

$$\lim_{N\to\infty} \mathbb{E}[\mathscr{U}_N(t,\phi)^k] = \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}} \mathcal{U}_t(x)\phi(x)dx\bigg)^k\bigg],$$

where $(t,x) \mapsto \mathcal{U}_t(x)$ solves (5.4) *with initial condition $\delta_0(x)$.*

Proof. Fix any $\phi \in \mathcal{S}(\mathbb{R})$. With $\mathscr{U}_N$ defined in (5.3), we wish to compute the annealed expectation $\mathbb{E}[\mathscr{U}_N(t,\phi)^k]$. Just like (5.6) we can write

$$\mathscr{U}_N(t,\phi)^k = \mathsf{E}_{(k)}^\omega\bigg[\prod_{j=1}^k C_{N,t,N^{-1/2}(R^j(Nt)-N^{3/4}t)}\phi\big(N^{-1/2}(R^j(Nt)-N^{3/4}t)\big)\bigg],$$

where $\mathsf{E}_{(k)}^\omega$ is a *quenched* expectation of k independent particles sampled from a fixed realization of the environment $(\omega_{t,x})_{t\ge 0, x\in\mathbb{Z}}$. Taking the annealed expectation over the quenched one we get

$$\mathbb{E}[\mathscr{U}_N(t,\phi)^k] = \mathbf{E}_{RW_\nu^{(k)}}\bigg[\prod_{j=1}^k C_{N,t,N^{-1/2}(R^j(Nt)-N^{3/4}t)}\phi\big(N^{-1/2}(R^j(Nt)-N^{3/4}t)\big)\bigg]. \tag{5.46}$$

Let $\{\nu_N^*\}_{N\ge 1}$ be the skewed sequence of measures corresponding to ν. Recall the measure $\mathbf{Q}^\lambda_{RW_\nu^{(k)}}$,

$\mathcal{M}^{\lambda}$ and $\widetilde{\mathcal{G}}_N$ from Definition 5.2.4, (5.13) and (5.15) respectively. The key idea from here is to get rid of divergent terms appearing in the above expectation through the following change of variable:

$$\mathbf{E}_{RW_{\nu}^{(k)}}[\mathbf{F}] = \mathbf{Q}^{N^{-1/4}}_{RW_{\nu}^{(k)}}\left[\frac{1}{\mathcal{M}^{N^{-\frac{1}{4}}}(Nt)}\mathbf{F}\right] = \mathbf{E}_{RW^{(k)}_{\nu^*_N}}\left[\frac{1}{\mathcal{M}^{N^{-\frac{1}{4}}}(Nt)}e^{\widetilde{\mathcal{G}_N}(Nt)}\mathbf{F}\right]. \tag{5.47}$$

where $\mathbf{F} = \mathbf{F}(\mathbf{R})$ is any measurable functional on the path space. Then we appeal to Theorem 5.3.1 to deduce convergence under the measure $\mathbf{P}_{RW^{(k)}_{\nu^*_N}}$.

Indeed, applying the tilts in (5.47) we get

$$\begin{aligned}\text{r.h.s. of (5.46)} = \mathbf{E}_{RW^{(k)}_{\nu^*_N}}\Bigg[&\exp\left(\mathcal{G}_N(Nt) - \sum_{s\le Nt}\mathbf{E}_{RW^{(k)}_{\nu_N}}[e^{\mathcal{G}_N(s)-\mathcal{G}_N(s-1)} \mid \mathcal{F}_{s-1}]\right)\\ &\cdot\frac{1}{\mathcal{M}^{N^{-\frac{1}{4}}}(NT)}\prod_{j=1}^{k}C_{N,t,N^{-\frac{1}{2}}(R^j(Nt)-N^{\frac{3}{4}}t)}\phi\big(N^{-\frac{1}{2}}(R^j(Nt)-N^{\frac{3}{4}}t)\big)\Bigg].\end{aligned} \tag{5.48}$$

Here $\mathcal{G}_N$ is a family of $\mathbf{P}_{RW_{\nu}^{(k)}}$-martingales satisfying (5.16). We now use the notations of rescaled processes: $\mathbf{X}_N$, $\mathscr{V}_N^{ij}$, and $\mathscr{T}_N$, considered in (5.36). We claim that

$$\frac{1}{\mathcal{M}^{N^{-1/4}}(NT)}\prod_{j=1}^{k}C_{N,t,N^{-1/2}(R^j(Nt)-N^{3/4}t)} = \exp\left(4\sigma^2\sum_{1\le i<j\le k}\mathscr{V}_N^{ij}(t) - \mathsf{Err}_N(t)\right) \tag{5.49}$$

where

$$|\mathsf{Err}_N(t)| \le \mathrm{C}\bigg(\mathscr{T}_N(t) + N^{-1/2}\sum_{1\le i<j\le k}\mathscr{V}_N^{ij}(t)\bigg). \tag{5.50}$$

The proofs of (5.49) and (5.50) follow by some algebraic calculations and definition chasing. We shall prove them in a moment. Note that ν_N and $\mathcal{G}_N$ satisfy the hypothesis of Theorem 5.3.1. Thus applying Theorem 5.3.1, in view of the bound in (5.50), we see that any limit point of the random variable inside the expectation on the r.h.s. of (5.48) is of the form

$$\exp\big(\mathcal{G}(t) - \tfrac{1}{2}\langle\mathcal{G}\rangle(t)\big)\exp\left(\frac{4\sigma^2}{1-4\sigma^2}\sum_{1\le i<j\le k}L^{ij}(t)\right)\prod_{j=1}^{k}\phi(U^i(t)),$$

where $\mathbf{U}$ is a standard k-dimensional Brownian motion on $[0,T]$, defined on a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $L^{ij}(t) := L_0^{U^i - U^j}(t)$ is the local time at zero of $U_i - U_j$ accrued by time t and $\mathcal{G}$ is a $C[0,T]$-valued random variable satisfying $\mathbf{E}[e^{p\mathcal{G}(T) - \frac{p}{2}\langle \mathcal{G} \rangle(T)}] < \infty$ for all $p > 0$ as well as (5.40).

Thanks to the estimates in Lemma 5.2.6 and Proposition 5.2.8, we see that the random variables in the prelimit are uniformly integrable. Thus every subsequence of the expectation on r.h.s. of (5.48) has a further subsequence which as $N \to \infty$ converges to

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\mathcal{G}(t) - \tfrac{1}{2}\langle \mathcal{G} \rangle(t)\right) \exp\left(\frac{4\sigma^2}{1-4\sigma^2} \sum_{1 \le i < j \le k} L^{ij}(t)\right) \prod_{j=1}^{k} \phi(\mathbf{U}^i(t))\right] \\
&= \mathbf{E}\left[\exp\left(\frac{4\sigma^2}{1-4\sigma^2} \sum_{1 \le i < j \le k} L^{ij}(t)\right) \prod_{j=1}^{k} \phi(\mathbf{U}^i(t))\right].
\end{aligned}$$

The last equality is due to (5.40). Since this expression is free $\mathcal{G}$ (which may depend on the subsequence), we thus get that

$$\lim_{N \to \infty} \mathbb{E}[\mathscr{U}_N(t,\phi)^k] = \mathbf{E}\left[\exp\left(\frac{4\sigma^2}{1-4\sigma^2} \sum_{1 \le i < j \le k} L^{ij}(t)\right) \prod_{j=1}^{k} \phi(\mathbf{U}^i(t))\right]. \tag{5.51}$$

By the Feynman-Kac formula, the solution $\mathcal{U}$ of the stochastic heat equation (5.4) admits well-known moment formulas in terms of local times of Brownian bridges. In particular, from [92, Lemma 2.3] we have

$$\mathbf{E}\left[\prod_{i=1}^{k} \mathcal{U}_t(x_i)\right] = p(t,x_1) \cdots p(t,x_k) \mathbf{E}_{\vec{x}}\left[\prod_{j=1}^{k} e^{\frac{4\sigma^2}{1-4\sigma^2} \sum_{i<j} L_0^{B^i - B^j}(t)}\right],$$

where p is the standard heat kernel, and the B^i are independent Brownian bridges on $[0,t]$ from 0 to x_i respectively. Note that in [92], local time is defined as an integral against Lebesgue measure: $L_a^X(t) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon < X_s < a+\varepsilon\}} ds$, in contrast with our definition of local time in (5.39) which is an integral against $d\langle X, X \rangle_s$. Hence, when taking the local time of $B^i - B^j$, our definition of local time differs from theirs by a factor of 2.

We can integrate this formula to arrive at the moment formula for $\mathcal{U}_t(\phi) := \int_{\mathbb{R}} \mathcal{U}_t(x)\phi(x)dx$,

with $(t,x) \mapsto \mathcal{U}_t(x)$ solving (5.4), and we see that it matches the right-hand side of (5.51), thus completing the proof of moment convergence for $\mathscr{U}_N$ modulo (5.49) and (5.50).

Let us now prove (5.49)-(5.50). Using the definition of martingale $\mathcal{M}^\lambda$ from (5.13) we see that

$$\frac{1}{\mathcal{M}^{N^{-\frac{1}{4}}}(Nt)} \prod_{j=1}^{k} C_{N,t,N^{-\frac{1}{2}}(R^j(Nt)-N^{3/4}t)} = \exp\Big(\sum_{\ell=0}^{Nt-1} f_\ell^{N^{-\frac{1}{4}},k} - kNt \log\cosh(N^{-\frac{1}{4}}) \Big), \quad (5.52)$$

where $f_\ell^{N^{-1/4},k}$ is defined in (5.14). Recall $\Gamma^{(v)}$ from (5.8). Using the expressions for $f_\ell^{\lambda,k}$ for $\ell \in \Gamma^{(k)}, \Gamma^{(k-1)}$ (obtained after (5.14)) we have

$$\begin{aligned}\sum_{\ell=0}^{Nt-1} f_\ell^{N^{-\frac{1}{4}},k} - kNt \log\cosh(N^{-\frac{1}{4}}) = g(N^{-1/4},\sigma^2) \cdot \#\{[Nt] \cap \Gamma^{(k-1)}\} + \sum_{\ell \in [Nt] \cap \Gamma^{(\le k-2)}} f_\ell^{N^{-1/4},k} \\ - k \log\cosh(N^{-1/4}) \cdot \#\{[Nt] \cap \Gamma^{(\le k-2)}\}. \end{aligned} \quad (5.53)$$

We note that

$$\#\{[Nt] \cap \Gamma^{(k-1)}\} = N^{1/2} \sum_{1 \le i < j \le k} \mathscr{V}_N^{ij}(t) - \sum_{\ell \in [Nt] \cap \Gamma^{(\le k-2)}} \sum_{1 \le i < j \le k} \mathbb{1}_{\{R^i(\ell) = R^j(\ell)\}}, \quad (5.54)$$

where $\mathscr{V}_N^{ij}$ is defined in (5.36). Let us now define

$$\begin{aligned}\mathsf{Err}_N^{(1)}(t) &:= g(N^{-1/4},\sigma^2) \sum_{\ell \in [Nt] \cap \Gamma^{(\le k-2)}} \sum_{1 \le i < j \le k} \mathbb{1}_{\{R^i(\ell) = R^j(\ell)\}} \\ &\quad - \sum_{\ell \in [Nt] \cap \Gamma^{(\le k-2)}} f_\ell^{N^{-1/4},k} + k \log\cosh(N^{-1/4}) \cdot \#\{[Nt] \cap \Gamma^{(\le k-2)}\}, \\ \mathsf{Err}_N^{(2)}(t) &:= \big(g(N^{-1/4},\sigma^2) - 4\sigma^2 N^{-1/2} \big) N^{1/2} \sum_{1 \le i < j \le k} \mathscr{V}_N^{ij}(t).\end{aligned}$$

In view of the relation in (5.54) and above definitions, we obtain from (5.53) that

$$\sum_{\ell=0}^{Nt-1} f_\ell^{N^{-\frac{1}{4}},k} - kNt \log\cosh(N^{-\frac{1}{4}}) = 4\sigma^2 \sum_{1 \le i < j \le k} \mathscr{V}_N^{ij}(t) + \mathsf{Err}_N^{(2)}(t) - \mathsf{Err}_N^{(1)}(t).$$

Setting $\mathsf{Err}_N(t) := \mathsf{Err}_N^{(2)}(t) - \mathsf{Err}_N^{(1)}(t)$ leads to (5.49) from (5.52) and the above relation. To prove the bound in (5.50), note that $g(N^{-1/4}, \sigma^2) = 4\sigma^2 N^{-1/2} + O(N^{-1})$. Thus we have

$$|\mathsf{Err}_N^{(2)}(t)| \leq \mathrm{C} N^{-1/2} \sum_{1 \leq i < j \leq k} \mathscr{V}_N^{ij}(t).$$

Furthermore $\log\cosh(N^{-1/4}) = O(N^{-1/2})$ and $f_\ell^{N^{-1/4},k}$ is uniformly bounded by $\mathrm{C}N^{-1/2}$ by Lemma 5.2.3. Using these bounds and recalling the definition of $\mathscr{T}_N$ from (5.36) we see that

$$|\mathsf{Err}_N^{(1)}(t)| \leq \mathrm{C} \cdot \mathscr{T}_N(t).$$

Combining the bounds in the last two displays verifies (5.50). □

5.4 A discrete Hopf-Cole transform: Proof overview of Theorem 5.1.1

Objects	Discrete Version		Continuous Version
Heat Operator	$\mathcal{L}_N$ (5.58)		$\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2$
	Microscopic version	Rescaled field	Continuum limit
Process of interest	Z_N^ω (5.59)	$\mathscr{U}_N$ (5.3)	$\mathcal{U}$ (5.4)
Relevant martingale	v_N (5.60)	M_N (5.63)	$\mathcal{Y}$ (5.55)
Quadratic Martingale field		Q_N (5.68)	Integrated $\mathcal{U}^2$

Table 5.1: Objects that will appear in the proof of weak convergence

Having established the moment convergence, in this section, we present a technical roadmap to the proof of the main result Theorem 5.1.1. We will rely on the martingale characterization of the solution of the multiplicative noise stochastic heat equation. Specifically, consider a measure μ on $C([0,T], C(\mathbb{R}))$, and let $(\mathcal{U}(t))_{t\in[0,T]}$ denote the canonical process on that space. The canonical filtration $\mathcal{F}_t$ on $C([0,T], C(\mathbb{R}))$ is the one generated by $\{\mathcal{U}(s) : s \leq t\}$. Define the heat operator

$\mathcal{L} := \partial_t - \frac{1}{2}\partial_x^2$, which acts on tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ by the formula $((\partial_t - \frac{1}{2}\partial_x^2)f, \varphi) = (f, (-\partial_t - \frac{1}{2}\partial_x^2)\varphi)$.

We recall a result of [88, Proposition 4.11] inspired by the work of [224]. If for all $\phi \in C_c^\infty(\mathbb{R})$ the processes

$$\mathcal{Y}(t, \phi) := \mathcal{U}(t, \phi) - \frac{1}{2}\int_0^t (\mathcal{U}(s), \phi'')ds \tag{5.55}$$

(which is formally the same as $\int_0^t \int_{\mathbb{R}} (\mathcal{L}\mathcal{U})(s, y)\phi(y)dyds$) are $(\mathcal{F}_t, \mu)$-martingales with quadratic variation

$$\langle \mathcal{Y}(\cdot, \phi)\rangle(t) = \frac{8\sigma^2}{1 - 4\sigma^2}\int_0^t (\mathcal{U}(s)^2, \phi^2)_{L^2(\mathbb{R})}ds, \tag{5.56}$$

then (under reasonable assumptions on the spatial growth of $\mathcal{U}(t)$ at infinity) the measure μ necessarily coincides with the law of (5.4) started from an initial condition that is distributed as $\mathcal{U}(0)$ under μ.

We shall eventually show our prelimiting field from (5.3) is tight and any limit point satisfies the above martingale characterization. The key observation that drives our proof is that the prelimiting field itself satisfies a lattice stochastic heat equation. This will then allow us to define some observables that will be crucial in proving tightness and identifying the limit points in later sections. We now end the heuristic discussions and rigorously derive the lattice SPDE, then write out the quadratic variations of the resultant martingales.

Definition 5.4.1. *Fix $N \in \mathbb{N}$. Define a discrete lattice*

$$\Lambda_N := \{(t, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R} : x + tN^{-1/4} \in \mathbb{Z}\}, \tag{5.57}$$

*and a **discrete heat operator** $\mathcal{L}_N$, which acts on real-valued functions $f : \Lambda_N \to \mathbb{R}$ by*

$$(\mathcal{L}_N f)(t, x) := f(t + 1, x - N^{-1/4}) - \big(\rho_N f(t, x - 1) + (1 - \rho_N)f(t, x + 1)\big), \tag{5.58}$$

with ρ_N given by (5.7)*.*

Recall $\mathsf{P}^\omega(t,x)$ from (5.1). Assume $\mathbb{E}[\omega_{ij}] = \frac{1}{2}$. For each realization of the environment ω, define the random function $Z_N^\omega : \Lambda_N \to \mathbb{R}$:

$$Z_N^\omega(t,x) := C_{N,N^{-1}t,N^{-1/2}x}\mathsf{P}^\omega(t, x + N^{-1/4}t), \tag{5.59}$$

where $C_{N,T,x}$ is defined in (5.2). Note that Z_N^ω is the pointwise version of the rescaled field $\mathscr{U}_N$ defined in (5.3). Roughly speaking, Theorem 5.1.1 suggests that we wish to prove convergence to SHE for the fields $N^{1/2}Z_N^\omega(Nt, N^{1/2}x)$. The following lemma suggests $Z_N^\omega(t,x)$ satisfies a discrete version of (5.55) on Λ_N.

Lemma 5.4.2. *For $(t,x) \in \Lambda_N$, define*

$$v_N(t,x) := \big(\mathcal{L}_N Z_N^\omega\big)(t,x). \tag{5.60}$$

Then v_N is a martingale-difference field in the filtration $\mathcal{F}_t^\omega = \sigma(\{\omega_{s,x} : x \in \mathbb{Z}, 0 \le s \le t\})$, i.e.

$$\mathbb{E}\big[v_N(t,x)\big|\mathcal{F}_{t-1}^\omega\big] = 0. \tag{5.61}$$

Proof. First note that $v_N(t,x)$ is indeed adapted to $\mathcal{F}_t^\omega$ because $Z_N^\omega(t+1,\cdot)$ is measurable with respect to $\mathcal{F}_t^\omega$. Note that the quenched probabilities satisfy the relation

$$\mathsf{P}^\omega(t+1,x) = (1-\omega_{t,x+1})\mathsf{P}^\omega(t,x+1) + \omega_{t,x-1}\mathsf{P}^\omega(t,x-1),$$

which can be derived from conditioning on the previous step. Using the definition of Z_N^ω and $C_{N,t,x}$

from (5.59) and (5.2) respectively, we thus have

$$\begin{aligned} Z_N^\omega(t+1, x-N^{-1/4}) &= e^{N^{-\frac{1}{2}} - \log\cosh(N^{-\frac{1}{4}})}\big[(1-\omega_{t,x+N^{-1/4}t+1})e^{N^{-1/4}(-1-N^{-1/4})}Z_N^\omega(t,x+1) \\ &\qquad + \omega_{t,x+N^{-1/4}t-1}e^{N^{-1/4}(1-N^{-1/4})}Z_N^\omega(t,x-1)\big] \\ &= 2(1-\omega_{t,x+N^{-1/4}t+1})(1-\rho_N)Z_N^\omega(t,x+1) + 2\omega_{t,x+N^{-1/4}t-1}\rho_N Z_N^\omega(t,x-1), \end{aligned}$$

for $(t,x) \in \Lambda_N$. Consequently we have that

$$v_N(t,x) = (1-2\omega_{t,x+N^{-1/4}t+1})(1-\rho_N)Z_N^\omega(t,x+1)+(2\omega_{t,x+N^{-1/4}t-1}-1)\rho_N Z_N^\omega(t,x-1). \quad (5.62)$$

Since the $\omega_{t,x}$ have mean 1/2 and are independent of $\mathcal{F}_{t-1}^\omega$, and since $Z_N^\omega(t,\cdot)$ is $\mathcal{F}_{t-1}^\omega$ measurable, (5.61) follows. □

Definition 5.4.3. *Define the rescaled martingale field as follows. For $\phi \in C_c^\infty(\mathbb{R})$ and $t \in N^{-1}\mathbb{Z}_{\geq 0}$ let*

$$M_N(t,\phi) := \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z}-rN^{-1/4}} \phi(N^{-1/2}x)v_N(r,x). \quad (5.63)$$

$M_N(t,\phi)$ is the macroscopic field corresponding to the microscopic variables $v_N(t,x)$. $M_N(t,\phi)$ is the discrete analog of $\mathcal{Y}(t,\phi)$ defined in (5.55). Note by (5.61) that $M_N(t,\phi)$ is a martingale indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$. A substantial amount of effort in this paper will be spent studying the quadratic variations of $M_N(t,\phi)$, as we need to verify that (5.56) holds for any limit point. By defining $\eta_N(r,x) := 1 - 2\omega_{r,x+N^{-1/4}r}$, we can use (5.62) to collect neighboring terms and rewrite (5.63) as follows for $t \in N^{-1}\mathbb{Z}_{\geq 0}$:

$$M_N(t,\phi) = \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z}-rN^{-1/4}} (\nabla_N\phi)\big(N^{-1/2}x\big)Z_N^\omega(r,x)\eta_N(r,x), \quad (5.64)$$

where

$$(\nabla_N\phi)(x) := (1-\rho_N)\phi(x-N^{-1/2}) - \rho_N\phi(x+N^{-1/2}).$$

Consequently the optional quadratic variation is given for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ by

$$[M_N(\phi)]_t = \sum_{r=0}^{Nt} \Big(\sum_{x \in \mathbb{Z} - rN^{-1/4}} (\nabla_N \phi)\big(N^{-1/2}x\big) Z_N^{\omega}(r,x) \eta_N(r,x) \Big)^2. \tag{5.65}$$

Since $\mathbb{E}[\eta_N(r,x)\eta_N(r,y)|\mathcal{F}_{r-1}^{\omega}] = 4\sigma^2 \mathbb{1}_{\{x=y\}}$, the predictable quadratic variation of $M_N(\phi)$ is given for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} \langle M_N(\phi) \rangle_t &= \sum_{r=0}^{Nt} \mathbb{E}\Big[\Big(\sum_{x \in \mathbb{Z} - rN^{-1/4}} (\nabla_N \phi)\big(N^{-1/2}x\big) Z_N^{\omega}(r,x) \eta_N(r,x) \Big)^2 \Big| \mathcal{F}_{r-1}^{\omega} \Big] \\ &= 4\sigma^2 \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \big[(\nabla_N \phi)\big(N^{-1/2}x\big) Z_N^{\omega}(r,x) \big]^2. \end{aligned} \tag{5.66}$$

Note that $M_N(t,\phi)^2 - [M_N(\phi)](t)$ and $M_N(t,\phi)^2 - \langle M_N(\phi) \rangle(t)$ are both martingales indexed by $N^{-1}\mathbb{Z}_{\geq 0}$, which will be relevant in later sections. We decompose (5.66) as

$$\langle M_N(\phi) \rangle_t = \mathcal{E}_N(t,\phi) + \big((2\rho_N - 1)^2 \sqrt{N} \big) Q_N(t,\phi^2) \tag{5.67}$$

where for $t \in N^{-1}\mathbb{Z}_{\geq 0}$

$$\begin{aligned} Q_N(t,\phi) &:= \frac{4\sigma^2}{\sqrt{N}} \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \phi(N^{-1/2}x) Z_N^{\omega}(r,x)^2 \\ &= \frac{4\sigma^2}{\sqrt{N}} \sum_{r=0}^{Nt} \mathsf{E}_{(2)}^{\omega} \Big[\phi(N^{-1/2}(S(r) - N^{-1/4}r)) \mathbb{1}_{\{S(r)=R(r)\}} C^2_{N, N^{-1}r, N^{-1/2}(S(r) - N^{-1/4}r)} \Big] \end{aligned} \tag{5.68}$$

where $\mathsf{E}_{(2)}^{\omega}$ denotes quenched expectation for two independent motions $(R(r), S(r))_{r \geq 0}$ in the fixed realization of the environment ω, and where $\mathcal{E}_N(T,\phi)$ is an "error term" given by

$$\mathcal{E}_N(t,\phi) := 4\sigma^2 \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \big[(\nabla_N \phi)\big(N^{-1/2}x\big)^2 - (2\rho_N - 1)^2 \phi(N^{-1/2}x)^2 \big] Z_N^{\omega}(r,x)^2. \tag{5.69}$$

Definition 5.4.4. *We shall call $Q_N(t,\phi)$ the **Quadratic martingale field** (QMF), and $\mathcal{E}_N(t,\phi)$ the **error term**.*

It will be shown that Q_N contributes meaningfully in the limit, while the error term $\mathcal{E}_N$ vanishes in the limit. Let us see why $\mathcal{E}_N$ should vanish. Notice that a first order Taylor expansion gives $\nabla_N\phi(x) = (1-2\rho_N)\phi(x) - O(N^{-1/2})$, and $1-2\rho_N$ is of order $N^{-1/4}$. Denote by $\|\phi\|_{L^\infty}$ the supremum of ϕ on $\mathbb{R}$, and denote by $\|\phi\|_{C^k} := \sum_{j=0}^k \|\phi^{(j)}\|_{L^\infty}$. Using the Taylor expansion, one verifies that the term in the square brackets of (5.69) is bounded above absolutely by $N^{-3/4}$ multiplied by $2\|\phi\|_{L^\infty}\|\phi'\|_{L^\infty} \le \|\phi\|_{L^\infty}^2 + \|\phi'\|_{L^\infty}^2 \le \|\phi\|_{C^1}^2$. For $\phi \in C_c^\infty(\mathbb{R})$ let $A_\phi := \sup\{|x| : x \in supp(\phi)\}$, and note that the summands in (5.69) vanish whenever $|N^{-1/2}x| > A_\phi + 1$. We have thus shown that the error term satisfies a pathwise bound written in (5.70) just below.

Lemma 5.4.5. *For all $\phi \in C_c^\infty(\mathbb{R})$ and $t \in N^{-1}\mathbb{Z}_{\ge 0}$ the predictable quadratic variation of $M_N(\phi)$ may be written as in* (5.67)*, where Q_N and $\mathcal{E}_N$ are defined by* (5.68) *and* (5.69) *respectively. Furthermore, we have uniformly over all $s,t \in N^{-1}\mathbb{Z}_{\ge 0}$, $\phi \in C_c^\infty(\mathbb{R})$ and $N \ge 1$ the bound*

$$\begin{aligned}|\mathcal{E}_N(t,\phi) - \mathcal{E}_N(s,\phi)| &\le 8N^{-3/4}\sigma^2\|\phi\|_{C^1}^2 \sum_{r=Ns}^{Nt} \sum_{x\in\mathbb{Z}-rN^{-1/4}} Z_N^\omega(r,x)^2 \mathbb{1}_{[-A_\phi-1,A_\phi+1]}(N^{-1/2}x) \\ &\le 8N^{-1/4}\sigma^2\|\phi\|_{C^1}^2 \big(Q_N(t,\mathbb{1}_{[-A_\phi-1,A_\phi+1]}) - Q_N(s,\mathbb{1}_{[-A_\phi-1,A_\phi+1]})\big),\end{aligned} \tag{5.70}$$

where $A_\phi := \sup\{|x| : x \in supp(\phi)\}$.

What this estimate shows is that the error term $\mathcal{E}_N$ is of the same form as Q_N, but with an extra factor of $N^{-1/4}$ in front. The field Q_N will be shown to be tight in later sections, and therefore the irrelevance of $\mathcal{E}_N$ will then be immediate from (5.70) without any further work needed. This error calculation illustrates a remarkable property of the model under consideration, which is that the error terms behave very nicely in relation to the original object itself, which is rare among KPZ-related models where a martingale characterization has been used, see e.g. [88, 211, 214] where extremely careful analysis was needed to show vanishing of error terms.

Now that the important objects M_N and Q_N have been introduced, we give a brief discussion on how the proof of Theorem 5.1.1 will be completed. The next section is heavily devoted to studying the quadratic martingale field $Q_N(t,\phi)$. In Proposition 5.5.4, we will establish tightness-related bounds of the form

$$\mathbf{E}[|Q_N(t,\phi) - Q_N(s,\phi)|^p]^{1/p} \leq C\|\phi\|_{L^\infty}^2 |t-s|^{1/2}.$$

This will then lead to a similar regularity bound for $[M_N(\cdot,\phi)]$ and $\mathscr{U}_N(\cdot,\phi)$ as well.

Together with these bounds, a Kolmogorov-type lemma will tell us that the triple $(\mathscr{U}_N, M_N, Q_N)$ is tight. This will be the subject of Section 5.6. Since the prelimiting object satisfies a lattice stochastic heat equation, one can obtain that any limit point of the triple satisfies (5.55) by showing that the discrete and continuous heat operators are close.

To justify why the limit point satisfies the other part (5.56) of the martingale problem, we study the QMF extensively in Section 5. In particular, informally speaking, we shall show the "key estimate" that

$$Q_N(t,\phi) - \frac{8\sigma^2}{1-4\sigma^2}\frac{1}{N}\sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]} \mathscr{U}_N(s,\phi)^2 \quad \overset{L^2(\mathbb{P})}{\longrightarrow} \quad 0$$

if we take $N \to \infty$ and then $\phi \to \delta_a$ in that order, for any $a \in \mathbb{R}\backslash\{0\}$. The precise statement is interpreted in terms of well-chosen Gaussian bump functions (see Proposition 5.5.3). This type of estimate will allow us to conclude that the limit point satisfies (5.56).

5.5 Formulas and estimates for the martingales

This section will heavily focus on obtaining crucial formulas and estimates for the quadratic martingale field (QMF) Q_N defined in (5.68). Later, these formulas and estimates will allow us to show the tightness of (5.3) and also identify the limit points. We first have two lemmas before stating the key estimate of this section.

Lemma 5.5.1 (Moment formulas). *Fix any bounded functions ψ, ϕ on $\mathbb{R}$ and $N \geq 1$. Suppose that $(R^1, \dots, R^{2k})$ denotes the canonical process on $(\mathbb{Z}^{2k})^{\mathbb{Z}_{\geq 0}}$. Recall $\Delta_k(s,t)$ from* (5.43)*, and define $\Delta_k^N(s,t) := (N^{-1}\mathbb{Z}_{\geq 0})^k \cap \Delta_k(s,t)$. For any $\vec{s} = (s_1, s_2, \dots, s_k) \in \Delta_k(0,t)$, we define*

$$\overline{g}_N(\vec{s}) = \prod_{r \in N^{-1}\mathbb{Z}_{\geq 0}} \frac{1}{(\#\{i : s_i = r\})!}. \tag{5.71}$$

Note that $\#\{i : s_i = r\} = 0$ for all but finitely many $r \in N^{-1}\mathbb{Z}_{\geq 0}$. We have the following moment formulas.

(a) For all $t \in N^{-1}\mathbb{Z}_{\geq 0}$ and $\gamma > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(Q_N(t,\psi) - \frac{\gamma}{N} \sum_{s \in (N^{-1}\mathbb{Z}_{\geq 0}) \cap [0,t]} \mathscr{U}_N(s,\phi)^2\right)^k\right] \\
&= k! N^{-k} \sum_{(s_1,\dots,s_k) \in \Delta_k^N(0,t)} \overline{g}_N(\vec{s}) \mathbf{E}_{RW_\nu^{(2k)}}\Bigg[\prod_{i=1}^k C_{N,s_i,N^{-1/2}(R^{2i-1}(Ns_i) - N^{3/4}s_i)} \\
&\qquad \cdot C_{N,s_i,N^{-1/2}(R^{2i}(Ns_i) - N^{3/4}s_i)} \Upsilon_N(Ns_i; R^{2i-1}, R^{2i})\Bigg],
\end{aligned} \tag{5.72}$$

where

$$\begin{aligned}
\Upsilon_N(r; X, Y) := {} & 4N^{1/2}\sigma^2 \psi\big(N^{-1/2}(X(r) - N^{-1/4}r)\big) \mathbb{1}_{\{X(r) = Y(r)\}} \\
& - \gamma \phi\big(N^{-1/2}(X(r) - N^{-1/4}r)\big) \phi\big(N^{-1/2}(Y(r) - N^{-1/4}r)\big).
\end{aligned}$$

(b) For all $0 \leq s < t$ we have the following moment formula for the increment of the QMF

$$\begin{aligned}
&\mathbb{E}\left[\Big(Q_N(t,\psi) - Q_N(s,\psi)\Big)^k\right] \\
&= (4\sigma^2)^k N^{-k/2} k! \sum_{(s_1,\dots,s_k) \in \Delta_k^N(s,t)} \overline{g}_N(\vec{s}) \mathbf{E}_{RW_\nu^{(2k)}}\Bigg[\prod_{i=1}^k C^2_{N,s_i,N^{-1/2}(R^{2i}(Ns_i) - N^{3/4}s_i)} \\
&\qquad \psi(N^{-1/2}(R^{2i}(Ns_i) - N^{3/4}s_i)) \mathbb{1}_{\{R^{2i-1}(Ns_i) = R^{2i}(Ns_i)\}}\Bigg].
\end{aligned} \tag{5.73}$$

Proof. Note by (5.3) that

$$\begin{aligned}\mathscr{U}_N(t,\phi)^2 &= \sum_{(x,y)\in\mathbb{Z}^2} C_{N,t,N^{-1/2}(x-N^{3/4}t)}C_{N,t,N^{-1/2}(y-N^{3/4}t)} \\ &\qquad \cdot \mathsf{P}^\omega(Nt,x)\mathsf{P}^\omega(Nt,y)\phi\big(N^{-1/2}(x-N^{3/4}t)\big)\phi\big(N^{-1/2}(y-N^{3/4}t)\big) \\ &= \mathsf{E}^\omega_{(2)}\big[C_{N,t,N^{-1/2}(S(Nt)-N^{3/4}t)}C_{N,t,N^{-1/2}(R(Nt)-N^{3/4}t)} \\ &\qquad \cdot \phi\big(N^{-1/2}(S(Nt)-N^{3/4}t)\big)\phi\big(N^{-1/2}(R(Nt)-N^{3/4}t)\big)\big].\end{aligned}$$

Here $\mathsf{E}^\omega_{(2)}$ denotes quenched expectation for two independent motions $(R(r),S(r))_{r\geq 0}$ in the fixed realization of the environment ω. Note that if we apply $\frac{1}{N}\sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]}$ to the last expression, then it is of a similar form to the definition (5.68) of Q_N, hence we have that

$$\begin{aligned}&Q_N(t,\psi) - \frac{\gamma}{N}\sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]}\mathscr{U}_N(s,\phi)^2 \\ &\quad = \frac{1}{N}\sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]}\mathsf{E}^\omega_{(2)}[C_{N,s,N^{-1/2}(R(Ns)-N^{3/4}s)}C_{N,s,N^{-1/2}(S(Ns)-N^{3/4}s)}\Upsilon_N(Ns;R,S)].\end{aligned}$$

From here one expands out the k^{th} power of both sides of this equation, and then one applies the annealed expectation over the quenched expectation to deduce (5.72), noting in general that by the multinomial theorem one has the following expansion for a function f:

$$\left(\sum_{s=0}^{r-1} f(s)\right)^k = k!\sum_{0\leq s_1\leq\ldots\leq s_k<r}\overline{g}_N(\vec{s})f(s_1)\cdots f(s_k).$$

The proof of (5.73) is similar. □

The following proposition will allow us to further analyze the moment formulas obtained in Lemma 5.5.1.

Proposition 5.5.2. *Assume that ν is a probability measure on $[0,1]$ of mean $1/2$. Fix any $k\in\mathbb{N}$. Let $\{\psi_i,\phi_i\}_{i=1}^k$ be bounded continuous functions on $\mathbb{R}$.*

Let A be a subset of $\{1,2,\ldots,k\}$. Let $B = \{1,2\ldots,k\} \cap A^c$. Define

$$E_1(\vec{t}) := \prod_{i\in A} C_{N,t_i,N^{-1/2}(R^{2i-1}(Nt_i)-N^{3/4}t_i)}\phi_i\big(N^{-1/2}(R^{2i-1}(Nt_i)-N^{3/4}t_i)\big) \tag{5.74}$$

$$\cdot \prod_{i\in A} C_{N,t_i,N^{-1/2}(R^{2i}(Nt_i)-N^{3/4}t_i)}\phi_i\big(N^{-1/2}(R^{2i}(Nt_i)-N^{3/4}t_i)\big) \tag{5.75}$$

$$E_2(\vec{t}) := \prod_{i\in B} C^2_{N,t_i,N^{-1/2}(R^{2i}(Nt_i)-N^{3/4}t_i)}\psi_i(N^{-1/2}(R^{2i}(Nt_i)-N^{3/4}t_i))\mathbb{1}_{\{R^{2i-1}(Nt_i)=R^{2i}(Nt_i)\}} \tag{5.76}$$

Consider any uniformly bounded sequence of deterministic functions $g_N : \Delta_k^N(s,t) \to [0,\infty)$ that converge uniformly to 1 on the interior of the simplex (consisting of those points $(t_1,...,t_k)$ with all t_i distinct) as $N \to \infty$. For each $0 \le s < t \le T < \infty$ we have

$$\begin{aligned}&\lim_{N\to\infty} N^{-|A|-\frac{1}{2}|B|}(1-4\sigma^2)^{|B|}\cdot \mathbf{E}_{RW_\nu^{(2k)}}\Bigg[\sum_{(t_1,\ldots,t_k)\in\Delta_k^N(s,t)} g_N(\vec{t})E_1(\vec{t})E_2(\vec{t})\Bigg]\\ &= \mathbf{E}_{B^{\otimes(2k)}}\Bigg[\int_{\Delta_k(s,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_k(\vec{t})}\prod_{i\in A}\phi_i(U^{2i-1}_{t_i})\phi_i(U^{2i}_{t_i})dt_i\cdot\prod_{i\in B}\psi_i(U^{2i}_{t_i})dL_0^{U^{2i-1}-U^{2i}}(t_i)\Bigg],\end{aligned} \tag{5.77}$$

where the expectation on the right is with respect to a $2k$-dimensional standard Brownian motion $(U^1,\ldots,U^{2k})$, and

$$\mathcal{L}_k(\vec{t}) := \sum_{i=1}^{k}\sum_{2i-1\le p<q\le 2k}\left[L_0^{U^p-U^q}(t_i) - L_0^{U^p-U^q}(t_{i-1})\right]. \tag{5.78}$$

Here $\int_0^t f(s)dL_0^{U^i-U^j}(s)$ denotes the integration of the continuous function $f:[0,t]\to\mathbb{R}$ against the random Lebesgue-Stieltjes measure $dL_0^{U^i-U^j}$ induced from the increasing function $t \mapsto L_0^{U^i-U^j}(t)$.

We will be most interested in the case where $g_N(\vec{t}) = \overline{g}_N(\vec{t})$ defined in (5.71).

Proof. We are going to assume that $g_N \equiv 1$; the general case follows by slight modifications. Let ν_N^* be the skewed sequence of measures corresponding to ν, as in Definition 5.2.1. For simplicity we write $\mathbf{E}$ and $\mathbf{E}_N$ for $\mathbf{E}_{RW_\nu^{(2k)}}$ and $\mathbf{E}_{RW_{\nu_N^*}^{(2k)}}$ respectively. Let us fix any $\vec{t}\in\Delta_k^N(s,t)$ and a subset A of $\{1,2,\ldots,k\}$. Just as in the proof of moment convergence, the main idea of the proof is to first

apply a change of measure to get rid of the divergent terms inside the expectation of $\mathbf{E}[E_1(\vec{t})E_2(\vec{t})]$ and then use the weak convergence result from Theorem 5.3.1. The change of measure here will be based on a glorified version of (5.47).

Step 1. Post-processing the expectation via tilting. To carry out the tilting procedure, we first note that

$$\mathbf{E}[E_1(\vec{t})E_2(\vec{t})] = \mathbf{E}\left[\prod_{i=1}^{k} \widetilde{C}_N^i \Lambda_N^i\right] = \mathbf{E}\left[\widetilde{C}_N^1 \Lambda_N^1 \mathbf{E}\left[\widetilde{C}_N^2 \Lambda_N^2 \cdots \mathbf{E}\left[\widetilde{C}_N^k \Lambda_N^k \mid \mathcal{F}_{Nt_{k-1}}\right] \cdots \mid \mathcal{F}_{Nt_1}\right]\right]. \tag{5.79}$$

where

$$\Lambda_N^i := \begin{cases} \phi_i\big(N^{-1/2}(R^{2i-1}(Nt_i) - N^{3/4}t_i)\big)\phi_i\big(N^{-1/2}(R^{2i}(Nt_i) - N^{3/4}t_i)\big) & i \in A \\ \psi_i(N^{-1/2}(R^{2i}(Nt_i) - N^{3/4}t_i))\mathbb{1}_{\{R^{2i-1}(Nt_i)=R^{2i}(Nt_i)\}} & i \in B, \end{cases}$$

and

$$\widetilde{C}_N^i := \prod_{j=2i-1}^{2k} C_{N,t_i-t_{i-1},N^{-1/2}(R^j(Nt_i)-R^j(Nt_{i-1})-N^{3/4}(t_i-t_{i-1}))\cdot} \tag{5.80}$$

We shall now apply tilting to each of the expectations on the r.h.s. of (5.79) and go from $\mathbf{P}$ to $\mathbf{P}_N$. To perform the tilting for conditional expectations we need generalized versions of $\mathcal{M}^\lambda(\cdot)$ (defined in (5.13)) and $\widetilde{\mathcal{G}}_N$ (defined in (5.15)):

Let us define

$$\mathcal{M}^{[k_1:k_2],\lambda}(r) = \exp\left(\lambda \sum_{j=k_1}^{k_2} R^j(r) - \sum_{\ell=0}^{r-1} f_\ell^{\lambda,k_2-k_1+1,\nu}(R^{k_1}(\ell),\ldots,R^{k_2}(\ell))\right) \tag{5.81}$$

The above martingale is obtained in the same spirit as $\mathcal{M}^\lambda$ by considering only the motion of $(k_2 - k_1 + 1)$ particles: $(R^j(\cdot))_{j=k_1}^{k_2}$.

Let us also define $\mathbf{P}_N$-martingales $\mathcal{G}_N^1, \ldots, \mathcal{G}_N^k$ by $\mathcal{G}_N^i = \widetilde{\mathcal{H}}_N^i + \widetilde{\mathcal{D}}_N^i$, where the $\mathbf{P}_N$-martingales

$\widetilde{\mathcal{D}}_N^i$ and $\widetilde{\mathcal{H}}_N^i$ are defined by $\widetilde{\mathcal{H}}_N^i(0) = \widetilde{\mathcal{D}}_N^i(0) = 0$, and

$$\widetilde{\mathcal{H}}_N^i(r+1) - \widetilde{\mathcal{H}}_N^i(r) = \begin{cases} N^{-1/4} \sum_{j=2i-1}^{2k} (R^j(r+1) - R^j(r) - (2\mu_{\nu_N^*} - 1)), & Nt_i \le r < Nt_{i+1} \\ 0 & \text{otherwise} \end{cases}.$$

$$\widetilde{\mathcal{D}}_N^i(r+1) - \widetilde{\mathcal{D}}_N^i(r) = \begin{cases} \sum_{q=1}^{v} \big(\log(m_{b_q, n_q - b_q}) - \mathbf{E}_N[\log(m_{b_q, n_q - b_q}) | \mathcal{F}_r] \big), & Nt_i \le r < Nt_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

where $m_{b,n-b}$ is defined in (5.18) and the parameters $v = v(i,r), n_q = n_q(i,r)$, and $b_q = b_q(i,r)$ are obtained deterministically from the path $(R^{2i-1}, \ldots, R^{2k})$ as follows. Assume that at time r, the $2k - 2i + 2$ "particles" of $(R^{2i-1}(r), \ldots, R^{2k}(r)) \in \mathbb{Z}^{2k-2i+2}$ may be grouped into $v = v(i,r)$ disjoint groups with all particles in each distinct group at the same site in $\mathbb{Z}$. Assume that the v respective groups contain $n_1, \ldots, n_v$ respective particles, where $n_q = n_q(i,r)$ are positive integers such that $n_1 + \cdots + n_v = 2k - 2i$. From time r to $r+1$ in the path $(R^{2i-1}, \ldots, R^{2k})$, assume (for each $1 \le q \le v$) in the q^{th} group, that $b_q = b_q(i,r)$ of the n_j particles go up one step, so that $n_j - b_j$ particles go down one step.

The processes $\mathcal{G}_N^i$ are martingales of the same form as those in the proof of Proposition 5.2.5, but where only the last $2k - 2i + 2$ of the $2k$ particles are taken into account on the time interval $[Nt_i, Nt_{i+1}) \cap \mathbb{Z}$. In particular, by the exact same argument as in that proof, each of the martingales $\mathcal{G}_N^i$ satisfies the bound (5.37) for some absolute constant $C > 0$ independent of N. Define $W_N^i(r) = \sum_{s \le r} \log \mathbf{E}_N[e^{\mathcal{G}_N^i(s) - \mathcal{G}_N^i(s-1)} \mid \mathcal{F}_{s-1}]$. Set $\widetilde{\mathcal{G}_N^i} := \mathcal{G}_N^i - W_N^i$. We will now see that the exponentials of $\widetilde{\mathcal{G}_N^i}$ are the "correct" martingales to tilt the expressions in the lemma statement in such a way that Theorem 5.3.1 is applicable.

Just like (5.47), one can check that for each i the conditional expectation $\mathbf{E}[\mathbf{F} \mid \mathcal{F}_{Nt_i}]$ can be tilted as follows:

$$\mathbf{E}[\mathbf{F} \mid \mathcal{F}_{Nt_i}] = \mathbf{E}_N \left[\frac{\mathcal{M}^{[2i-1:2k]}(Nt_{i-1})}{\mathcal{M}^{[2i-1:2k]}(Nt_i)} \exp\left(\widetilde{\mathcal{G}_N^i}(Nt_i) \right) \mathbf{F} \mid \mathcal{F}_{Nt_i} \right], \tag{5.82}$$

where $\mathbf{F}$ is measurable with respect to $\sigma(\{(R^{2i-1}(r),\dots,R^{2k}(r))_{r\in[0,Nt]}\})$.

Using this, we tilt each of the conditional expectations in (5.79) to get

$$\mathbf{E}\left[E_1(\vec{t})E_2(\vec{t})\right]=\mathbf{E}_N\left[\prod_{i=1}^{k}\frac{\mathcal{M}^{[2i-1:2k]}(Nt_{i-1})}{\mathcal{M}^{[2i-1:2k]}(Nt_i)}\exp\left(\widetilde{\mathcal{G}_N^i}(Nt_i)\right)\widetilde{C}_N^i\Lambda_N^i\right] \tag{5.83}$$

Step 2. Convergence. We shall now study the weak convergence of the sum of random variables inside the expectation on the r.h.s. of (5.83). We use the notations of rescaled processes from (5.36). Recalling the identity (5.49) and the bound in (5.50) we see that

$$\prod_{i=1}^{k}\frac{\mathcal{M}^{[2i-1:2k],N^{-1/4}}(Nt_{i-1})}{\mathcal{M}^{[2i-1:2k],N^{-1/4}}(Nt_i)}\widetilde{C}_N^i=\exp\left(4\sigma^2\sum_{i=1}^{k}\sum_{2i-1<p<q\le 2k}[\mathscr{V}_N^{pq}(t_i)-\mathscr{V}_N^{pq}(t_{i-1})]-\widetilde{\mathsf{Err}}_N(\vec{t})\right).$$

where

$$|\widetilde{\mathsf{Err}}_N(\vec{t})|\le \mathrm{C}\left(\mathscr{T}_N(t)+N^{-1/2}\sum_{1\le p<q\le 2k}\mathscr{V}_N^{pq}(t)\right). \tag{5.84}$$

Let us define $W_N^{\vec{t}}(s):=W_N^1(Ns)+\cdots+W_N^k(Ns)$ and $\mathcal{G}_N^{\vec{t}}(s):=\mathcal{G}_N^1(Ns)+\cdots+\mathcal{G}_N^k(Ns)$ for $s\in N^{-1}\mathbb{Z}$ and linearly interpolated for $s\notin N^{-1}\mathbb{Z}$. Let us view W_N and $\mathcal{G}_N$ as random continuous functions from $\Delta_k(0,t)\times[0,t]\to\mathbb{R}$. Using the same arguments from the proof of Theorem 5.3.1, it follows that W_N and $\{\mathcal{G}_N\}_{N\ge1}$ are tight in the space of $C(\Delta_k(0,t)\times[0,t])$ equipped with uniform topology, and furthermore $\mathcal{G}_N^{\vec{t}}$ satisfies (5.37) with constant C independent of $\vec{t}$. By Theorem 5.3.1 and Corollary 5.3.2 we know that any limit point as $N\to\infty$ of the sequence

$$\left(\mathcal{G}_N^{\vec{t}},W_N^{\vec{t}},\mathbf{X}_N,\mathscr{V}_N,\mathscr{T}_N,N^{-|B|/2}\sum_{u_1\le\cdots\le u_{|B|}\in N^{-1}\mathbb{Z}}\prod_{i\in B}\mathbb{1}_{\{\mathbf{X}_N^{2i-1}(u_i)=\mathbf{X}_N^{2i}(u_i)\}}\delta_{(u_1,\dots,u_{|B|})}\right) \tag{5.85}$$

(considered under the measure $\mathbf{E}_N$) is of the form

$$\left(\mathcal{G}^{\vec{t}},\tfrac{1}{2}\langle\mathcal{G}^{\vec{t}}\rangle,\mathbf{U},(L^{ij})_{1\le i<j\le 2k},\mathbf{0},(1-4\sigma^2)^{-|B|}\prod_{i\in B}dL^{2i-1,2i}(t_i)\right), \tag{5.86}$$

where $\mathbf{U}$ is a standard Brownian motion in $\mathbb{R}^k$, where $L^{i,j}$ are its pairwise local times, and $\mathcal{G}^{\vec{t}}$ are martingales satisfying (5.40). Here the last coordinate has a topology of $\mathcal{M}(\Delta_k(0,T))$ which was defined in the statement of Corollary 5.3.2, while all other coordinates have a uniform topology with respect to all relevant variables. Notice that the map from $\mathcal{M}(\Delta_k(0,T)) \times C(\Delta_k(0,T)) \to \mathbb{R}$ given by $(f,\mu) \mapsto \int_{\Delta_k(0,T)} f \, d\mu$ is a continuous map. Using the continuous mapping theorem, we therefore see that for the limit point (5.86) of (5.85) one has

$$N^{-|A|-\frac{1}{2}|B|}(1-4\sigma^2)^{|B|} \sum_{\vec{t}\in\Delta_k^N(s,t)} \prod_{i=1}^{k} \frac{\mathcal{M}^{[2i-1:2k]}(Nt_{i-1})}{\mathcal{M}^{[2i-1:2k]}(Nt_i)} \exp\left(\widetilde{\mathcal{G}_N^i}(Nt_i)\right) \widetilde{C}_N^i \Lambda_N^i$$
$$\xrightarrow{d} \int_{\Delta_k(s,t)} e^{\mathcal{G}^{\vec{t}}(t)-\frac{1}{2}\langle \mathcal{G}^{\vec{t}}\rangle(t)+\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_k(\vec{t})} \prod_{i\in A} \phi_i(U_{t_i}^{2i-1})\phi_i(U_{t_i}^{2i}) dt_i \cdot \prod_{i\in B} \psi_i(U_{t_i}^{2i}) dL^{2i-1,2i}(t_i),$$

where we again emphasize that the objects in the prelimit are viewed as observables under $\mathbf{E}_N$. Thanks to the estimates in Lemma 5.2.7, Proposition 5.2.8, and (5.45), we see that the prelimiting sum above is uniformly integrable. Now we finally prove the proposition. Taking into account the identity in (5.83), the preceding observations imply that for every subsequence of indices $N \to \infty$, there is a further subsequence along which we have

$$N^{-|A|-\frac{1}{2}|B|}(1-4\sigma^2)^{|B|} \sum_{\vec{t}\in\Delta_k^N(s,t)} \mathbf{E}[E_1(\vec{t})E_2(\vec{t})]$$
$$\longrightarrow \mathbf{E}_{\text{lim}}\left[\int_{\Delta_k(s,t)} e^{\mathcal{G}^{\vec{t}}(t)-\frac{1}{2}\langle \mathcal{G}^{\vec{t}}\rangle(t)+\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_k(\vec{t})} \prod_{i\in A} \phi_i(U_{t_i}^{2i-1})\phi_i(U_{t_i}^{2i}) dt_i \cdot \prod_{i\in B} \psi_i(U_{t_i}^{2i}) dL^{2i-1,2i}(t_i)\right]$$
$$= \mathbf{E}_{\text{lim}}\Bigg[\int_{\Delta_k(s,t)} \mathbf{E}_{\text{lim}}[e^{\mathcal{G}^{\vec{t}}(t)-\frac{1}{2}\langle \mathcal{G}^{\vec{t}}\rangle(t)} \mid \mathcal{F}_t(\mathbf{U})]$$
$$\cdot e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_k(\vec{t})} \prod_{i\in A} \phi_i(U_{t_i}^{2i-1})\phi_i(U_{t_i}^{2i}) dt_i \cdot \prod_{i\in B} \psi_i(U_{t_i}^{2i}) dL^{2i-1,2i}(t_i)\Bigg].$$

Here $\mathbf{E}_{\text{lim}}$ denotes a possible limit point on the canonical space, of the entire tuple of processes given by (5.85), and we are viewing (5.86) as the canonical process on that space. The inner conditional expectation in the last expression equals 1 due to (5.40). Thus the last expression must

be equal to the right side of (5.77), completing the proof. □

Proposition 5.5.3 (Key estimate for the QMF)**.** *Let $a \in \mathbb{R}$ let $\xi(x) := \frac{1}{\sqrt{\pi}} e^{-x^2}$, and let $\xi_\varepsilon^a(x) := \varepsilon^{-1}\xi(\varepsilon^{-1}(x-a))$. Then for all $t > 0$ and $a \in \mathbb{R} \setminus \{0\}$,*

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{E}\left[\left(Q_N(t,\xi_\varepsilon^a) - \frac{8\sigma^2}{1-4\sigma^2}\frac{1}{N} \sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]} \mathscr{U}_N(s,\xi_{\varepsilon\sqrt{2}}^a)^2\right)^2\right] = 0. \tag{5.87}$$

Furthermore, we have the bound

$$\sup_{\substack{\varepsilon>0\\ a\in\mathbb{R}\setminus\{0\}}} \limsup_{N\to\infty} \left[1 \wedge \frac{1}{|\log a|^2}\right] \cdot \mathbb{E}\left[\left(Q_N(t,\xi_\varepsilon^a) - \frac{8\sigma^2}{1-4\sigma^2}\frac{1}{N} \sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]} \mathscr{U}_N(s,\xi_{\varepsilon\sqrt{2}}^a)^2\right)^2\right] < \infty. \tag{5.88}$$

The above proposition is the key estimate that will allow us to identify limit points. This proposition also illustrates why we cannot hope to obtain convergence in a space of continuous functions. Indeed if this were possible, then by first equality in the definition (5.68) of the field Q_N it is easy to see that in (5.87), the correct coefficient would have to be $8\sigma^2 = 2 \cdot 4\sigma^2$ rather than $\frac{8\sigma^2}{1-4\sigma^2}$ (where the extra factor of 2 comes from the fact that $Z_N^\omega(t,x)$ as defined in (5.59) are nonzero only at those points such that $t - x$ is even). Clearly, both coefficients cannot be correct unless $\mathscr{U}_N$ vanishes in the limit which is certainly not the case, as we have already shown that its moments of all orders converge to a nontrivial limit.

Proof. Applying Lemma 5.5.1 (a) with $\gamma = \frac{8\sigma^2}{1-4\sigma^2}$, and Proposition 5.5.2 with $k = 2$, we get

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\left(Q_N(t,\psi) - \frac{8\sigma^2}{1-4\sigma^2}\frac{1}{N} \sum_{s\in(N^{-1}\mathbb{Z}_{\geq 0})\cap[0,t]} \mathscr{U}_N(s,\phi)^2\right)^2\right] \\
&= 2\left(\frac{8\sigma^2}{1-4\sigma^2}\right)^2 \cdot \mathbf{E}_{B^{\otimes 4}}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\left(\psi(X_{s_i}^i)\tfrac{1}{2}dL_0^{X^i-Y^i}(s_i) - \phi(X^i)\phi(Y^i)ds_i\right)\right] \\
&= 2\left(\frac{8\sigma^2}{1-4\sigma^2}\right)^2 \cdot \mathbf{E}_{B^{\otimes 4}}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2}\Big(\psi\big(\tfrac{1}{2}(X_{s_i}^i + Y_{s_i}^i)\big)\tfrac{1}{2}dL_0^{X^i-Y^i}(s_i) - \phi(X^i)\phi(Y^i)ds_i\Big)\right]
\end{aligned} \tag{5.89}$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{L}_2$ is defined in (5.78), and (X^1, X^2, Y^1, Y^2) is a 4d standard BM under $\mathbf{P}_{B^{\otimes 4}}$. The second equality in the above equation follows by observing that $X^i_u = Y^i_u$ for u in the support of $L_0^{X^i - Y^i}(du)$.

We shall now write $\mathbf{E}$ instead of $\mathbf{E}_{B^{\otimes 4}}$ for convenience. Let us now take

$$\psi(x) := \xi^a_\varepsilon(x) = \frac{1}{\sqrt{\pi\varepsilon^2}} e^{-(x-a)^2/\varepsilon^2}, \quad \phi(x) := \xi^a_{\varepsilon\sqrt{2}}(x) = \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-(x-a)^2/2\varepsilon^2},$$

in (5.89). Using the identity $\xi^a_{\varepsilon\sqrt{2}}(x)\xi^a_{\varepsilon\sqrt{2}}(y) = \xi^a_\varepsilon((x+y)/2)\xi^0_{2\varepsilon}(x-y)$, we may now write (5.89) as

$$2\left(\frac{8\sigma^2}{1-4\sigma^2}\right)^2 \cdot \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2} \left(\xi^a_\varepsilon\left(\tfrac{1}{2}(X^i_{s_i} + Y^i_{s_i})\right) \cdot (\tfrac{1}{2} dL_0^{X^i - Y^i}(s_i) - \xi^0_{2\varepsilon}(X^i_{s_i} - Y^i_{s_i}) ds_i)\right)\right]. \tag{5.90}$$

Let us write $U^{i,-} := X^i - Y^i$ and $U^{i,+} := X^i + Y^i$. Note that under $\mathbf{P}_{B^{\otimes 4}}$ the four processes $U^{1,-}, U^{1,+}, U^{2,-}, U^{2,+}$ are independent Brownian motions with diffusion coefficient 2. This enables us to view (5.90) as

$$\begin{aligned} &2\left(\frac{8\sigma^2}{1-4\sigma^2}\right)^2 \cdot \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \prod_{i=1}^{2} \left(\xi^a_\varepsilon\left(\tfrac{1}{2}U^{i,+}_{s_i}\right)\right) \cdot (\tfrac{1}{2} dL_0^{U^{i,-}}(s_i) - \xi^0_{2\varepsilon}(U^{i,-}_{s_i}) ds_i)\right)\right] \\ &=: 2\left(\frac{8\sigma^2}{1-4\sigma^2}\right)^2 [A_1(\varepsilon) - A_2(\varepsilon) - A_3(\varepsilon) + A_4(\varepsilon)], \end{aligned} \tag{5.91}$$

where

$$A_1(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \xi^a_\varepsilon\left(\tfrac{1}{2}U^{1,+}_{s_1}\right)\xi^a_\varepsilon\left(\tfrac{1}{2}U^{2,+}_{s_2}\right) \tfrac{1}{2} dL_0^{U^{1,-}}(s_1) \tfrac{1}{2} dL_0^{U^{2,-}}(s_2)\right]$$

$$A_2(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)} \xi^a_\varepsilon\left(\tfrac{1}{2}U^{1,+}_{s_1}\right)\xi^a_\varepsilon\left(\tfrac{1}{2}U^{2,+}_{s_2}\right)\xi^0_{2\varepsilon}(U^{2,-}_{s_2}) \tfrac{1}{2} dL_0^{U^{1,-}}(s_1)\, ds_2\right]$$

$$A_3(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)}\xi^a_\varepsilon\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi^a_\varepsilon\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\xi^0_{2\varepsilon}(U^{1,-}_{s_1})\,\tfrac{1}{2}dL_0^{U^{2,-}}(s_2)\,ds_1\right]$$

$$A_4(\varepsilon) := \mathbf{E}\left[\int_{\Delta_2(0,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_2(s_1,s_2)}\xi^a_\varepsilon\big(\tfrac{1}{2}U^{1,+}_{s_1}\big)\xi^a_\varepsilon\big(\tfrac{1}{2}U^{2,+}_{s_2}\big)\xi^0_{2\varepsilon}(U^{1,-}_{s_1})\xi^0_{2\varepsilon}(U^{2,-}_{s_2})\,ds_1\,ds_2\right].$$

From here, the goal is to show that (5.91) vanishes as $\varepsilon \to 0$ as long as $a \neq 0$, as well as establish a bound given by the right side of (5.88) for each of the four terms $A_i(\varepsilon)$.

Note that informally $\frac{1}{2}dL_0^{U^{i,\pm}}(s_i)$ may be written as $\delta_0(U^{i,\pm})ds_i$ which suggests that each of the A_i may be written in terms of Brownian bridge expectations. Indeed this is the case, and consequently the proofs of the desired convergence statements and bounds for the terms $A_i(\varepsilon)$ rely purely on elementary (albeit lengthy) disintegration formulas for Brownian motion at its endpoint, and these proofs can be copied verbatim from [71, Proof of Proposition 5.3: Steps 2 and 3], replacing the coefficient σ appearing there with our coefficient $\frac{8\sigma^2}{1-4\sigma^2}$ throughout the proof. For brevity, we do not reproduce the details here. □

With the "key estimate" proved, next we focus on obtaining bounds that will be useful for proving the tightness of the rescaled field (5.3).

Proposition 5.5.4 (Estimates for moments of the increments of QMF)**.** *Fix $k \in \mathbb{N}$ and $T > 0$. Then there exists a constant $\mathrm{C} = \mathrm{C}(k,T) > 0$ such that for all bounded measurable functions ϕ on $\mathbb{R}$ and all $0 \leq s < t \leq T$ with $s,t \in N^{-1}\mathbb{Z}_{\geq 0}$ one has that*

$$\sup_{N\geq 1}\mathbb{E}\big[(Q_N(t,\phi) - Q_N(s,\phi))^k\big] \leq \mathrm{C}\|\phi\|^k_{L^\infty(\mathbb{R})}(t-s)^{k/2}. \tag{5.92}$$

Furthermore fix $p > 1$ and $\varepsilon > 0$. Then there exists $\mathrm{C} = \mathrm{C}(p,\varepsilon,k,T) > 0$ such that for all functions $\phi \in L^p(\mathbb{R})$ and all $\varepsilon \leq s < t \leq T$ one has

$$\lim_{N\to\infty}\mathbb{E}\big[(Q_N(t,\phi) - Q_N(s,\phi))^k\big] \leq \mathrm{C}\|\phi\|^k_{L^p(\mathbb{R})}(t-s)^k. \tag{5.93}$$

Proof. We are going to use the same notation and the same family of martingales from the proof of Proposition 5.5.2. Using (5.73) together with the trivial bound $|\phi(N^{-1/2}(R^j(Nt_j) - N^{3/4}t_j))| \leq \|\phi\|_{L^\infty}$, we obtain that

$$\begin{aligned}
&\mathbb{E}[\big(Q_N(t,\phi) - Q_N(s,\phi)\big)^k] \\
&\leq N^{-k/2} k! \|\phi\|_{L^\infty}^k \sum_{(t_1,\ldots,t_k)\in\Delta_k^N(s,t)} \mathbf{E}_{RW_\nu^{(2k)}}\left[\prod_{j=1}^k C^2_{N,t_j,N^{-1/2}(R^{2j}(Nt_j)-N^{3/4}t_j)} \mathbb{1}_{\{R^{2j-1}(Nt_j)=R^{2j}(Nt_j)\}}\right] \\
&= k!\|\phi\|_{L^\infty}^k \cdot N^{-k/2} \sum_{(t_1,\ldots,t_k)\in\Delta_k^N(s,t)} \mathbf{E}_{RW_{\nu_N^*}^{(2k)}}\left[\mathbf{A}_N(\vec{t}) \cdot \mathbf{B}_N(\vec{t}) \cdot \prod_{j=1}^k \mathbb{1}_{\{R^{2j-1}(Nt_j)=R^{2j}(Nt_j)\}}\right], \quad (5.94)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A}_N(\vec{t}) &:= \exp\left(\mathcal{G}_N^{\vec{t}}(t) - W_N^{\vec{t}}(t)\right), \\
\mathbf{B}_N(\vec{t}) &:= \exp\left(4\sigma^2 \sum_{i=1}^k \sum_{2i-1<p<q\leq 2k} [\mathscr{V}_N^{pq}(t_i) - \mathscr{V}_N^{pq}(t_{i-1})] - \widetilde{\mathsf{Err}}_N(\vec{t})\right),
\end{aligned}$$

and where all of these objects are exactly the same as those introduced in the proof of Proposition 5.5.2. The equality in (5.94) is due to the argument given in the proof of Proposition 5.5.2. As before, let us abbreviate $\mathbf{E}_N := \mathbf{E}_{RW_{\nu_N^*}^{(2k)}}$. Recalling the measure γ_N introduced in (5.44), let us rewrite (5.94) as

$$k!\|\phi\|_{L^\infty}^k \mathbf{E}_N\left[\int_{\Delta_k^N(s,t)} \mathbf{A}_N(\vec{t}) \cdot \mathbf{B}_N(\vec{t}) \gamma_N(d\vec{t})\right].$$

Now using a trivial bound

$$\int_{\Delta_k^N(s,t)} \mathbf{A}_N(\vec{t}) \cdot \mathbf{B}_N(\vec{t}) \gamma_N(d\vec{t}) \leq \|\mathbf{A}_N\|_{L^\infty(\Delta_k^N(s,t))} \|\mathbf{B}_N\|_{L^\infty(\Delta_k^N(s,t))} \gamma_N(\Delta_k^N(s,t)),$$

we have by Holder's inequality that

$$\mathbf{E}_N\Big[\int_{\Delta_k^N(s,t)} \mathbf{A}_N(\vec{t}) \cdot \mathbf{B}_N(\vec{t})\gamma_N(d\vec{t})\Big] \tag{5.95}$$

$$\leq \mathbf{E}_N\big[\|\mathbf{A}_N\|^3_{L^\infty(\Delta_k^N(s,t))}\big]^{1/3}\mathbf{E}_N\big[\|\mathbf{B}_N\|^3_{L^\infty(\Delta_k^N(s,t))}\big]^{1/3}\mathbf{E}_N\big[\gamma_N(\Delta_k^N(s,t))^3\big]^{1/3}. \tag{5.96}$$

Recalling $\mathscr{V}^{ij}$ from Theorem (5.36), note that $\gamma_N(\Delta_k^N(s,t)) \leq \big(\sum_{1\leq i<j\leq k} \mathscr{V}_N^{ij}(t) - \mathscr{V}_N^{ij}(s)\big)^k$. By Lemma 5.2.7, we can bound $\mathbf{E}_N\big[\gamma_N(\Delta_k^N(s,t))^3\big]^{1/3} \leq C(t-s)^{k/2}$ where C does not depend on N. This is because of the fact that $\mu_{\nu_N^*}(1-\mu_{\nu_N^*}) - \sigma^2_{\nu_N^*}$ is positive for large enough N. Thus it suffices to bound I_1, I_2 independently of N.

Note that the martingales $\mathcal{G}_N^{\vec{t}}(t)$ satisfy (5.37), by their construction in the proof of Proposition 5.5.2. Consequently an application of Doob's L^p inequality combined with Proposition 5.2.8 allows us to bound the expected maximum $\mathbf{E}_N\big[\|\mathbf{A}_N\|^3_{L^\infty(\Delta_k^N(s,t))}\big]^{1/3}$ independently of N. Finally using Lemma 5.2.6 we can bound $\mathbf{E}_N\big[\|\mathbf{B}_N\|^3_{L^\infty(\Delta_k^N(s,t))}\big]^{1/3}$ for large enough N as well (as again $\mu_{\nu_N^*}(1-\mu_{\nu_N^*}) - \sigma^2_{\nu_N^*}$ is positive for large enough N). This verifies the bound in (5.92).

For (5.93). appealing to the moment formula for the increment of QMF from (5.73) and the convergence from (5.77) we have

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(Q_N(t,\phi) - Q_N(s,\phi)\big)^k\Big] \tag{5.97}$$

$$= k!\bigg(\frac{4\sigma^2}{1-4\sigma^2}\bigg)^k \cdot \mathbf{E}_{B^{\otimes 2k}}\bigg[\int_{\Delta_k(s,t)} e^{\frac{4\sigma^2}{1-4\sigma^2}\mathcal{L}_k(\vec{t})} \prod_{j=1}^k \phi(U_{t_j}^{2j}) dL_0^{U^{2j-1}-U^{2j}}(t_j)\bigg], \tag{5.98}$$

where $\mathcal{L}_k(\vec{t})$ is defined in (5.78). The rest of the proof is analogous to the proof of [71, Eq. (5.43)].

□

Recall the martingale field M_N from (5.63). The next estimate will obtain a L^p bound on its optional quadratic variation from (5.65), which will be the main tool in obtaining tightness for the field (5.3).

Proposition 5.5.5 (Optional quadratic variation bound)**.** *Fix $p \geq 1$. For all $s, t \in N^{-1}\mathbb{Z}_{\geq 0}$ and all*

$\phi \in C_c^\infty(\mathbb{R})$ *we have that*

$$\mathbb{E}\big[\big|[M_N(\phi)]_t - [M_N(\phi)]_s\big|^p\big]^{1/p} \leq C\|\phi\|_{C^1}^2|t-s|^{1/2}. \tag{5.99}$$

Here C is independent of N, ϕ, s, t.

Proof. The main idea of the proof will be to split the quadratic variation $[M_N(\phi)]$ into a "predictable part" (denoted A_N below) which is easy to control, and a "discontinuity part" (denoted B_N below) which we expect to vanish as $N \to \infty$. Then we bound each of these separately.

It suffices to prove the claim when $p = 2k$ for some positive integer k. Letting $\eta_N(r,x) := 1 - 2\omega_{r,x+N^{-1/4}r}$, recall from (5.64) that for $t \in N^{-1}\mathbb{Z}_{\geq 0}$:

$$M_N(t,\phi) = \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z}-rN^{-1/4}} (\nabla_N \phi)\big(N^{-1/2}x\big) Z_N^\omega(r,x)\eta_N(r,x).$$

where $(\nabla_N\phi)(x) := (1-\rho_N)\phi(x-N^{-1/2}) - \rho_N\phi(x+N^{-1/2})$. A first-order Taylor expansion of ϕ yields that $\|\nabla_N\phi\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{C^1}N^{-1/4}$. Then for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ we have that

$$(M_N(t+N^{-1},\phi) - M_N(t,\phi))^2 = A_N(Nt,\phi) + B_N(Nt,\phi).$$

where

$$A_N(r,\phi) := \sum_{x \in \mathbb{Z}-rN^{-1/4}} (\nabla_N\phi)\big(N^{-1/2}x\big)^2 Z_N^\omega(r,x)^2\eta_N(r,x)^2,$$

$$B_N(r,\phi) := \sum_{\substack{x,y \in \mathbb{Z}-rN^{-1/4} \\ x \neq y}} (\nabla_N\phi)\big(N^{-1/2}x\big)(\nabla_N\phi)\big(N^{-1/2}y\big) Z_N^\omega(r,x)Z_N^\omega(r,y)\eta_N(r,x)\eta_N(r,y).$$

We will separately obtain bounds of the desired form for both A_N and B_N. Using the fact that $\eta_N(r,x)^2 \leq 1$ deterministically, and using the definition (5.68) of Q_N, we obtain the pathwise bound

$$\sum_{r=Ns}^{Nt} A_N(r,\phi) \leq \sigma^{-2}\|\phi\|_{C^1}^2\big(Q_N(t,\mathbb{1}) - Q_N(s,\mathbb{1})\big).$$

In particular, by (5.92) we see that

$$\mathbb{E}\Big[\Big(\sum_{r=Ns}^{Nt} A_N(r,\phi)\Big)^{2k}\Big] \le \|\phi\|_{C^1}^4 \mathbb{E}\big[\big(Q_N(t,\mathbb{1}) - Q_N(s,\mathbb{1})\big)^{2k}\big] \le C\|\phi\|_{C^1}^{2k}|t-s|^k. \tag{5.100}$$

Now let us bound the moments of B_N. Note that

$$(B_N(r,\phi))^2 \le N^{-1}\|\phi\|_{C^1}^4 \Big(\sum_x Z_N^\omega(r,x)\Big)^4 = N^{-1}\|\phi\|_{C^1}^4 \cdot \big(\mathsf{E}^\omega\big[C_{N,N^{-1}r,N^{-1/2}(R(r)-N^{-1/4}r)}\big]\big)^4.$$

Taking the annealed expectation of a product of B_N's we get

$$\mathbb{E}\left[\prod_{j=1}^k B_N(r_j,\phi)^2\right] \le N^{-k}\|\phi\|_{C^1}^{4k} \cdot \mathbb{E}\left[\prod_{j=1}^k \Big(\mathsf{E}^\omega\big[C_{N,N^{-1}r_j,N^{-1/2}(R(r_j)-N^{-1/4}r_j)}\big]\Big)^4\right]. \tag{5.101}$$

By Hölder's inequality the right-hand side of the above equation is less than

$$N^{-k}\|\phi\|_{C^1}^{4k} \prod_{j=1}^k \left(\mathbf{E}_{RW_\nu^{(4k)}}\left[\prod_{i=1}^{4k} C_{N,N^{-1}r_j,N^{-1/2}(R_i(r_j)-N^{-1/4}r_j)}\right]\right)^{1/k}. \tag{5.102}$$

Using the same arguments as in the proof of Proposition 5.5.4, the above expectation can be bounded uniformly over r_j's in $[0,Nt]$. Using the fact that $\mathbb{E}[B_N(r,\phi)|\mathcal{F}_{r-1}^\omega] = 0$, we then obtain from Burkholder-Davis-Gundy that

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{r=Ns}^{Nt} B_N(r,\phi)\Big)^{2k}\Big] &\le C_k\mathbb{E}\Big[\Big(\sum_{r=Ns}^{Nt} B_N(r,\phi)^2\Big)^k\Big] \\ &= C_k\mathbb{E}\Big[\sum_{r_1,\dots,r_k=Ns}^{Nt} B_N(r_1,\phi)^2\cdots B_N(r_k,\phi)^2\Big] \le C_k\|\phi\|_{C^1}^{4k}|t-s|^k.\end{aligned}$$

This gives the desired claim after combining this bound with the other bound (5.100) for $A_N(r,\phi)$.

□

5.6 Tightness and identification of limit points

Throughout this section, we are going to fix a terminal time $T > 0$. To prove Theorem 5.1.1 we need to introduce a number of preliminaries.

Definition 5.6.1. *Recall the lattice Λ_N from* (5.57) *and define its rescaled version*

$$\Psi_{N,T} := \{(t,x) \in \mathbb{R}^2 : (Nt, N^{1/2}x) \in \Lambda_N, \quad 0 \le t \le T\}.$$

Definition 5.6.2. *For $(s,y) \in \Lambda_N$ define $p_N(s,y)$ to be the transition density at time s and position y of a random walker on Λ_N with increment distribution given by going from 0 to $-1+N^{-1/4}$ with probability ρ_N (as defined in* (5.7)*), and from 0 to $1+N^{-1/4}$ with probability $1-\rho_N$.*

Define linear operators D_N, L_N, K_N on $\mathcal{S}'(\mathbb{R}^2)$ by

$$\begin{aligned}
D_N f(t,x) &:= N\big[f(t+N^{-1},x) - f(t,x)\big],\\
L_N f(t,x) &:= N\big[f(t+N^{-1},x-N^{-3/4}) - \big(\rho_N f(t,x-N^{-1/2}) + (1-\rho_N) f(t,x+N^{-1/2})\big)\big],\\
K_N f(t,x) &:= N^{-1} \sum_{(s,y)\in\Psi_{N,T+1}} p_N(Ns, N^{1/2}y) f(t-s, x-y).
\end{aligned}$$

These equalities should be understood by integration against smooth functions $\varphi \in \mathcal{S}(\mathbb{R}^2)$.

The sum defining $K_N f$ is well defined because it is actually a finite sum, as p_N is finitely supported for bounded time intervals. Intuitively it is clear that L_N is a diffusively rescaled version of the discrete heat operator $\mathcal{L}_N$ from Section 5.4, but which acts on tempered distributions rather than functions on Λ_N. Indeed if $\varphi \in \mathcal{S}(\mathbb{R}^2)$ then a second-order Taylor expansion shows that $L_N\varphi$ converges in $\mathcal{S}(\mathbb{R}^2)$ as $N \to \infty$ to $(\partial_t - \frac{1}{2}\partial_x^2)f$, i.e., L_N approaches the continuum heat operator. Then K_N is the inverse operator to L_N, in the following sense.

Lemma 5.6.3. *$L_N K_N f = K_N L_N f = f$ whenever f is a tempered distribution supported on $[a,b] \times \mathbb{R}$ with $b - a < T + 1$.*

Proof. Note that for each N the operators K_N, L_N are continuous on $\mathcal{S}'(\mathbb{R}^2)$, since they are finite linear combinations of translation operators. Therefore it suffices to prove the claim for all smooth functions f that have compact support contained in $(a,b) \times \mathbb{R}$, since these are dense in the subset of distributions supported on $[a,b] \times \mathbb{R}$, with respect to the topology of $\mathcal{S}'(\mathbb{R}^2)$. The smooth claim is true by direct calculations since $p_N(s,y)$ is the kernel for the inverse operator to the discrete heat operator $\mathcal{L}_N$ introduced in Section 5.4. □

Definition 5.6.4. *With the fields M_N and Q_N as defined in* (5.63) *and* (5.68) *respectively, we will associate random elements of $C([0,T+1], \mathcal{S}'(\mathbb{R}))$ by the formulas $\widehat{M}_N(t,\phi) = 0$ for $t \in [0, N^{-1}]$ and*

$$\widehat{M}_N(t,\phi) := M_N(t - N^{-1}, \phi) - M_N(0,\phi), \qquad t \in N^{-1}\mathbb{Z}_{\geq 1}.$$
$$\widehat{Q}_N(t,\phi) := Q_N(t,\phi), \qquad t \in N^{-1}\mathbb{Z}_{\geq 0}.$$

for $\phi \in \mathcal{S}(\mathbb{R})$. We define these fields by linear interpolation for $t \notin N^{-1}\mathbb{Z}_{\geq 0}$.

For any T, note that $C([0,T], \mathcal{S}'(\mathbb{R}))$ embeds naturally into the linear subspace of $\mathcal{S}'(\mathbb{R}^2)$ consisting of distributions supported on $[0,T] \times \mathbb{R}$, thus we can make sense of $D_N f, L_N f, K_N f$ for all $f \in C([0,T], \mathcal{S}'(\mathbb{R}))$, and these will be elements of $\mathcal{S}'(\mathbb{R}^2)$ in general.

Definition 5.6.5. *We will say that two tempered distributions $f, g \in \mathcal{S}'(\mathbb{R}^2)$ agree on $[0,T]$ if there exists $\varepsilon > 0$ such that $(f,\varphi) = (g,\varphi)$ for all φ supported on $[-\varepsilon, T+\varepsilon] \times \mathbb{R}$.*

Definition 5.6.6. *Sample the environment ω and then define a collection of coefficients $a_N(s,y)$ with $(s,y) \in \Lambda_N$ as $a_N(0,y) = \mathbb{1}_{\{y=0\}}$ and for $s \geq 1$,*

$$a_N(s,y) = \omega_{0,0} p_N(s-1, y-1-N^{-1/4}) + (1-\omega_{0,0}) p_N(s-1, y+1-N^{-1/4}).$$

Then define the distribution $\mathfrak{p}_N \in C([0,T+1], \mathcal{S}'(\mathbb{R}))$ for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ by

$$\mathfrak{p}_N(t,\cdot) := N^{-1} \sum_{y:(t,y) \in \Psi_{N,T+1}} a_N(Nt, N^{1/2}y)\delta_y$$

and linearly interpolated for $t \notin N^{-1}\mathbb{Z}_{\geq 0}$.

Although its definition is tedious, $\mathfrak{p}_N$ admits the following simple description: it agrees with the density field $\mathscr{U}_N$ from (5.3) on the time interval $[0, N^{-1}]$, and it is a linearly interpolated solution of the discrete heat equation $\mathcal{L}_N a_N = 0$ at all positive times in $N^{-1}\mathbb{Z}_{\geq 0}$. The purpose of this will be to make certain technical details work later. Since $\mathfrak{p}_N$ behaves deterministically after time N^{-1}, its scaling limit will simply be the continuum heat kernel.

Lemma 5.6.7. *Let* $\widehat{M}_N$ *be as defined above. Furthermore, let* $\mathscr{U}_N$ *be as defined in* (5.3)*. Restrict* $\mathscr{U}_N$ *to* $[0,T]$ *thus viewed as an element of* $C([0,T], \mathcal{S}'(\mathbb{R}))$*. Then* $L_N(\mathscr{U}_N - \mathfrak{p}_N)$ *agrees with* $D_N\widehat{M}_N$ *on* $[0,T]$*. Consequently* $\mathscr{U}_N$ *agrees with* $\mathfrak{p}_N + K_N D_N \widehat{M}_N$ *on* $[0,T]$.

Proof. Note that $\mathscr{U}_N - \mathfrak{p}_N$ and $\widehat{M}_N$ both vanish on $[0, N^{-1}]$, consequently $D_N\widehat{M}_N(0,\phi) = L_N(\mathscr{U}_N - \mathfrak{p}_N)(0,\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{R})$. On the other hand, $L_N\mathfrak{p}_N(t,\cdot) = 0$ for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ with $t > 0$, consequently $L_N(\mathscr{U}_N - \mathfrak{p}_N)(t,\phi) = L_N\mathscr{U}_N(t,\phi)$ for such t.

Recall v_N defined in (5.60). Then it is clear from (5.63) that

$$D_N\widehat{M}_N(t,\phi) = N\big[M_N(t,\phi) - M_N(t-N^{-1},\phi)\big] = \Big(\sum_{x:(t,x)\in\Psi_{N,T}} N v_N(Nt, N^{1/2}x)\delta_x \ \ , \ \ \phi\Big),$$

for all $t \in N^{-1}\mathbb{Z}_{\geq 1}$ and $\phi \in \mathcal{S}(\mathbb{R})$. Now, with Z_N^ω defined in (5.59) we have that for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ we have

$$\mathscr{U}_N(t,\cdot) = \sum_{x:(t,x)\in\Psi_{N,T}} Z_N^\omega(Nt, N^{1/2}x)\delta_x.$$

With $\mathcal{L}_N$ as defined in Section 5.4, it is then clear that for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ the expression for $L_N\mathscr{U}_N(t,\cdot)$ can be written as

$$N\sum_{x:(t,x)\in\Psi_{N,T}} (\mathcal{L}_N Z_N^\omega)(Nt, N^{1/2}x)\delta_x = N\sum_{x:(t,x)\in\Psi_{N,T}} v_N(Nt, N^{1/2}x)\delta_x,$$

which is the same as the expression for $D_N\widehat{M}_N(t,\phi)$. Since linear operators respect linear interpolation, and since all of the fields $\mathscr{U}_N, D_N\widehat{M}_N, \mathfrak{p}_N$ are defined via linear interpolation, we have

$L_N(\mathscr{U}_N - \mathfrak{p}_N) = D_N \widehat{M}_N$ for $t \notin N^{-1}\mathbb{Z}_{\geq 0}$ as well.

Finally, if we view the restriction of $D_N \widehat{M}_N$ to $[0,T]$ as an element of $\mathcal{S}'(\mathbb{R}^2)$ supported on $[0,T] \times \mathbb{R}$, then we can apply K_N to both sides and we obtain that $(\mathscr{U}_N - \mathfrak{p}_N)(t,\cdot) = K_N D_N \widehat{M}_N(t,\cdot)$ for all $t \in [0,T]$ by Lemma 5.6.3. □

5.6.1 Weighted Hölder spaces and Schauder estimates

We now introduce various natural topologies for our field $\mathscr{U}_N$ and its limit points. We then discuss how the heat flow affects these topologies and record Kolmogorov-type lemmas that will be key in showing tightness under these topologies. We begin by recalling many familiar and useful spaces of continuous and differentiable functions that have natural metric structures. For $d \geq 1$, we denote by $C_c^\infty(\mathbb{R}^d)$ the space of all compactly supported smooth functions on $\mathbb{R}^d$. For a smooth function on $\mathbb{R}^d$, we define its C^r norm as

$$\|f\|_{C^r} := \sum_{\vec{\alpha}} \sup_{\mathbf{x} \in \mathbb{R}^d} |D^{\vec{\alpha}} f(\mathbf{x})| \tag{5.103}$$

where the sum is over all $\vec{\alpha} \in \mathbb{Z}_{\geq 0}^d$ with $\sum \alpha_i \leq r$ and $D^{\vec{\alpha}} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ denotes the mixed partial derivative.

We now recall the definition of weighted Hölder spaces from [227, Definitions 2.2 and 2.3]. For the remainder of this paper, we shall work with *elliptic and parabolic* weighted Hölder spaces with polynomial weight function

$$w(x) := (1 + x^2)^\tau \tag{5.104}$$

for some fixed $\tau > 1$. We introduce these weights because weighted spaces will be more convenient to obtain tightness estimates. Since the solution of (5.4) started from Dirac initial condition is known to be globally bounded away from $(0,0)$, we expect that it is possible to remove the weights throughout this section, but this would require more precise moment estimates than the ones we derived in previous sections, which take into account spatial decay of the fields.

Definition 5.6.8 (Elliptic Hölder spaces)**.** *For $\alpha \in (0,1)$ we define the space $C^{\alpha,\tau}(\mathbb{R})$ to be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm given by*

$$\|f\|_{C^{\alpha,\tau}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{w(x)} + \sup_{|x-y|\leq 1} \frac{|f(x)-f(y)|}{w(x)|x-y|^\alpha}.$$

For $\alpha < 0$ we let $r = -\lfloor \alpha \rfloor$ and we define $C^{\alpha,\tau}(\mathbb{R})$ to be the closure of $C_c^\infty(\mathbb{R})$ with respect to the norm given by

$$\|f\|_{C^{\alpha,\tau}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \sup_{\lambda\in(0,1]} \sup_{\phi\in B_r} \frac{(f, S_x^\lambda \phi)_{L^2(\mathbb{R})}}{w(x)\lambda^\alpha}$$

where the scaling operators S_x^λ are defined by

$$S_x^\lambda \phi(y) = \lambda^{-1}\phi(\lambda^{-1}(x-y)), \tag{5.105}$$

and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}$.

Definition 5.6.9 (Function spaces)**.** *Let $C^{\alpha,\tau}(\mathbb{R})$ be as in Definition 5.6.8. We define $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ to be the space of continuous maps $g : [0,T] \to C^{\alpha,\tau}(\mathbb{R})$, equipped with the norm*

$$\|g\|_{C([0,T],C^{\alpha,\tau}(\mathbb{R}))} := \sup_{t\in[0,T]} \|g(t)\|_{C^{\alpha,\tau}(\mathbb{R})}.$$

Here and henceforth we will define $\Psi_{[a,b]} := [a,b] \times \mathbb{R}$ and we will define $\Psi_T := \Psi_{[0,T]}$.

So far we have used ϕ, ψ for test functions on $\mathbb{R}$. To make the distinction clear between test functions on $\mathbb{R}$ and $\mathbb{R}^2$, we shall use variant Greek letters such as $\varphi, \vartheta, \varrho$ for test functions on $\mathbb{R}^2$. In some instances, we will explicitly write $(f,\varphi)_{\mathbb{R}^2}$ or $(g,\phi)_{\mathbb{R}}$ when we want to be clear about the space in which we are pairing.

Definition 5.6.10 (Parabolic Hölder spaces)**.** *We define $C_c^\infty(\Psi_T)$ to be the set of functions on Ψ_T that are restrictions to Ψ_T of some function in $C_c^\infty(\mathbb{R}^2)$ (in particular we do not impose that elements of $C_c^\infty(\Psi_T)$ vanish at the boundaries of Ψ_T).*

For $\alpha \in (0,1)$ we define the space $C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ to be the completion of $C_c^\infty(\Psi_T)$ with respect to the norm given by

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)} := \sup_{(t,x)\in\Psi_T} \frac{|f(t,x)|}{w(x)} + \sup_{|s-t|^{1/2}+|x-y|\leq 1} \frac{|f(t,x)-f(s,y)|}{w(x)(|t-s|^{1/2}+|x-y|)^\alpha}.$$

For $\alpha < 0$ we let $r = -\lfloor \alpha \rfloor$ and we define $C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ to be the closure of $C_c^\infty(\Psi_T)$ with respect to the norm given by

$$\|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)} := \sup_{(t,x)\in\Psi_T} \sup_{\lambda\in(0,1]} \sup_{\varphi\in B_r} \frac{(f, S^\lambda_{(t,x)}\varphi)_{L^2(\Psi_T)}}{w(x)\lambda^\alpha}$$

where the scaling operators are defined by

$$S^\lambda_{(t,x)}\varphi(s,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(x-y)), \tag{5.106}$$

and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}^2$.

An important property of both the parabolic and elliptic spaces is that one has a continuous embedding $C^{\alpha,\tau} \hookrightarrow C^{\beta,\tau}$ whenever $\beta < \alpha$. In fact this embedding is compact, though we will not use this. We also have the following embedding of function spaces inside parabolic spaces.

Lemma 5.6.11. *For $\alpha < 0, \tau > 0$ one has a continuous embedding $C([0,T], C^{\alpha,\tau}(\mathbb{R})) \hookrightarrow C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ given by identifying $v = (v(t))_{t\in[0,T]}$ with the tempered space-time distribution given by*

$$(v,\varphi)_{\mathbb{R}^2} = \int_0^T (v(t), \varphi(t,\cdot))_{\mathbb{R}} dt.$$

The proof is straightforward from the definitions and is omitted.

Remark 5.6.12 (Derivatives of distributions)**.** *Let $\alpha < 0$. By definition, any element $f \in C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ admits an L^2-pairing with any smooth function $\varphi : \Psi_T \to \mathbb{R}$ of rapid decay, which we can write as $(f,\varphi)_{\Psi_T}$. Consequently there is a natural embedding $C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ which is defined by*

formally setting f to be zero outside of $[0,T]\times\mathbb{R}$. More rigorously, this means that the $L^2(\mathbb{R}^2)$-pairing of f with any $\varphi\in\mathcal{S}(\mathbb{R}^2)$ is defined to be equal to $(f,\varphi|_{\Psi_T})_{\Psi_T}$.

The image of this embedding consists of some specific collection of tempered distributions that are necessarily supported on $[0,T]\times\mathbb{R}$. Consequently we can sensibly define $\partial_t f$ and $\partial_x f$ as elements of $\mathcal{S}'(\mathbb{R}^2)$ whenever $f\in C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)$. Explicitly these derivatives are defined by the formulas

$$(\partial_t f,\varphi)_{\Psi_T}:=-(f,\partial_t\varphi)_{\Psi_T},\qquad (\partial_x f,\varphi)_{\Psi_T}:=-(f,\partial_x\varphi)_{\Psi_T}$$

for all smooth $\varphi:\Psi_T\to\mathbb{R}$ of rapid decay. This convention on derivatives will be useful for certain computations later. From the definitions, when $\alpha<0$ one can check that for such f one necessarily has $\partial_t f\in C^{\alpha-2,\tau}_{\mathfrak{s}}(\Psi_T)$ and $\partial_x f\in C^{\alpha-1,\tau}_{\mathfrak{s}}(\Psi_T)$.

The latter statement fails for $\alpha>0$. Indeed, by our convention of derivatives, $\partial_t f$ may no longer be a smooth function (or even a function) even if $f\in C^\infty_c(\Psi_T)$. This is because such an f gets extended to all of $\mathbb{R}^2$ by setting it to be zero outside Ψ_T. In particular, if f does not vanish on the boundary of Ψ_T, then it may become a discontinuous function under our convention of extension to $\mathbb{R}^2$. Due to these discontinuities, the distributional derivative $\partial_t f$ can be a tempered distribution with singular parts along the boundaries (one may verify that $\partial_t f$ can be at best an element of $C^{-2,\tau}_{\mathfrak{s}}(\Psi_T)$ for generic $f\in C^\infty_c(\Psi_T)$). In our later computations, we never take derivatives of functions in $C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)$ with $\alpha>0$.

We now discuss the smoothing effect of heat flow on these weighted Hölder spaces.

Proposition 5.6.13 (Smoothing effect on elliptic spaces)**.** *For $f\in C^\infty_c(\mathbb{R})$ and $t>0$ define*

$$\mathcal{P}_t f(x):=\int_{\mathbb{R}}p(t,x-y)f(y)dy.$$

Then for all $\alpha\le\beta<1$, there exists $\mathrm{C}=\mathrm{C}(\alpha,\beta,T)>0$ such that

$$\|\mathcal{P}_t f\|_{C^{\beta,\tau}(\mathbb{R})}\le \mathrm{C}t^{-(\beta-\alpha)/2}\|f\|_{C^{\alpha,\tau}(\mathbb{R})}$$

uniformly over $f \in C_c^\infty$ *and* $t \in [0,T]$. *In particular,* $\mathcal{P}_t$ *extends to a globally defined linear operator on* $C^{\alpha,\tau}(\mathbb{R})$ *which maps boundedly into* $C^{\beta,\tau}(\mathbb{R})$.

A proof may be found in [227, Lemma 2.8] in the case of an exponential weight. The proof for polynomial weights is identical.

For the parabolic Hölder spaces, the following lemma states that the heat flow improves the regularity by a factor of 2 and provides a Schauder-type estimate.

Proposition 5.6.14 (Schauder estimate)**.** *For* $f \in C_c^\infty(\Psi_T)$ *let us define*

$$Kf(t,x) := \int_{\Psi_T} p(t-s, x-y) f(s,y) ds dy, \tag{5.107}$$

where p *is the standard heat kernel for* $t > 0, x \in \mathbb{R}$ *and* $p(t,x) := 0$ *for* $t < 0$. *Then for all* $\alpha < -1$ *with* $\alpha \notin \mathbb{Z}$ *there exists* $\mathrm{C} = \mathrm{C}(\alpha) > 0$ *independent of* f *such that*

$$\|Kf\|_{C_{\mathfrak{s}}^{\alpha+2,\tau}(\Psi_T)} \leq \mathrm{C} \cdot \|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)}.$$

In particular K *extends to a globally defined linear operator on* $C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ *that maps boundedly into* $C_{\mathfrak{s}}^{\alpha+2,\tau}(\Psi_T)$. *Furthermore, if* $f \in C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$, *then* $K(\partial_t - \frac{1}{2}\partial_x^2) f = f$.

We remark that the last statement is only true because of our convention on distributional derivatives that we have explained in Remark 5.6.12. Without that convention, that statement would be false even for a smooth function f that does not vanish on the line $\{t = 0\}$.

Proof. See [71, Proposition 6.6]. □

Corollary 5.6.15. *Define* $J : C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)$ *by*

$$Jf(t,x) := \big(f, p(t, x-\cdot)\big)_{\mathbb{R}},$$

J *is a bounded linear operator for any* $\alpha < 1$ *and any* $\tau > 0$. *Consider the operator* $\widehat{\mathcal{P}}_t := J\mathcal{P}_t$

with $\mathcal{P}_t$ defined in Proposition 5.6.13. By the semigroup property of the heat kernel, we have that

$$\widehat{\mathcal{P}}_s f(t,x) = (Jf)(t+s,x). \tag{5.108}$$

Furthermore, the operators satisfy

$$\|\widehat{\mathcal{P}}_t f\|_{C_{\mathfrak{s}}^{\beta,\tau}(\Psi_T)} \leq \mathrm{C} \cdot t^{-(\beta-\alpha)/2} \|f\|_{C^{\alpha,\tau}(\mathbb{R})}, \tag{5.109}$$

where $\mathrm{C} = \mathrm{C}(\alpha,\beta,T) > 0$ is independent of t and f.

Proof. The first part follows by using the Schauder estimate in Proposition 5.6.14, noting that $Jf = K(\delta_0 \otimes f)$ where for $f \in C^{\alpha,\tau}(\mathbb{R})$ the latter distribution is defined by $(\delta_0 \otimes f, \varphi)_{\mathbb{R}^2} := (f, \varphi(0,\cdot))_{\mathbb{R}}$. Directly from the definitions one can check that $f \mapsto \delta_0 \otimes f$ is bounded from $C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Psi_T)$. Finally, (5.109) follows from Proposition 5.6.13. □

We next define a space-time distribution that is supported on a single temporal cross-section:

Definition 5.6.16. *Let $\alpha < 0$. Given some $f \in C^{\alpha,\tau}(\mathbb{R})$ and $b \in [0,T]$ we define $\delta_b \otimes f \in C_{\mathfrak{s}}^{\alpha-2,\tau}(\Psi_T)$ by the formula $(\delta_b \otimes f, \varphi)_{\mathbb{R}^2} := (f, \varphi(b,\cdot))_{\mathbb{R}}$.*

Directly from the definitions of the scaling operators in (5.105) and (5.106), it is clear that for fixed $b \in [0,T]$, the linear map $f \mapsto \delta_b \otimes f$ is bounded from $C^{\alpha,\tau}(\mathbb{R}) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Psi_T)$ as long as $\alpha < 0$. It is also clear that $\delta_b \otimes f$ is necessarily supported on the line $\{b\} \times \mathbb{R}$. The following lemma shows that under mild conditions, $\delta_T \otimes f$ vanishes upon the action of the heat flow K defined in (5.107).

We end this subsection by recording a Kolmogorov-type lemma for the function spaces and parabolic Hölder spaces introduced at the beginning of this subsection. It will be crucial in proving tightness in those respective spaces.

Lemma 5.6.17 (Kolmogorov lemma)**.** *Let $L^2(\Omega,\mathcal{F},\mathbb{P})$ be the space of all random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with finite second moment. We have the following:*

(a) (Function space) Let $(t,\phi) \mapsto V(t,\phi)$ *be a map from* $[0,T] \times \mathcal{S}(\mathbb{R})$ *into* $L^2(\Omega,\mathcal{F},\mathbb{P})$ *which is linear and continuous in* ϕ*. Fix a non-negative integer* r*. Assume there exists some* $\kappa > 0, p > 1/\kappa$ *and* $\alpha < -r$ *and* $\mathrm{C} = \mathrm{C}(\kappa,\alpha,p,T) > 0$ *such that one has*

$$\mathbf{E}[|V(t,S_x^\lambda\phi)|^p]^{1/p} \leq \mathrm{C}\lambda^\alpha,$$
$$\mathbf{E}[|V(t,S_x^\lambda\phi) - V(s,S_x^\lambda\phi)|^p]^{1/p} \leq \mathrm{C}\lambda^{\alpha-\kappa}|t-s|^\kappa,$$

uniformly over all smooth functions ϕ *on* $\mathbb{R}$ *supported on the unit ball of* $\mathbb{R}$ *with* $\|\phi\|_{C^r} \leq 1$*, and uniformly over* $\lambda \in (0,1]$ *and* $0 \leq s,t \leq T$*. Then for any* $\tau > 1$ *and any* $\beta < \alpha - \kappa$ *there exists a random variable* $\big(\mathscr{V}(t)\big)_{t\in[0,T]}$ *taking values in* $C([0,T],C^{\beta,\tau}(\mathbb{R}))$ *such that* $(\mathscr{V}(t),\phi) = V(t,\phi)$ *almost surely for all* ϕ *and* t*. Furthermore, one has that*

$$\mathbf{E}[\|\mathscr{V}\|^p_{C([0,T],C^{\beta,\tau}(\mathbb{R}))}] \leq \mathrm{C}',$$

where C' *depends on the choice of* α,β,p,κ*, and the constant* C *appearing in the moment bound above but not on* $V,\Omega,\mathcal{F},\mathbb{P}$*.*

(b) (Parabolic Hölder Space) Let $\varphi \mapsto V(\varphi)$ *be a bounded linear map from* $\mathcal{S}(\mathbb{R}^2)$ *to* $L^2(\Omega,\mathcal{F},\mathbb{P})$*. Assume* $V(\varphi) = 0$ *for all* φ *with support contained in the complement of* Ψ_T*. Recall* $S^\lambda_{(t,x)}$ *from* (5.106)*. Fix a non-negative integer* r*. Assume there exists some* $p > 1$ *and* $\alpha < 0$ *and* $\mathrm{C} = \mathrm{C}(\alpha,p) > 0$ *such that one has*

$$\mathbf{E}[|V(S^\lambda_{(t,x)}\varphi)|^p]^{1/p} \leq \mathrm{C}\lambda^\alpha,$$

uniformly over all smooth functions φ *on* $\mathbb{R}^2$ *supported on the unit ball of* $\mathbb{R}^2$ *with* $\|\varphi\|_{C^r} \leq 1$*, and uniformly over* $\lambda \in (0,1]$ *and* $(t,x) \in \Psi_T$*. Then for any* $\tau > 1$ *and any* $\beta < \alpha - 3/p$ *there exists a random variable* $\mathscr{V}$ *taking values in* $C^{\beta,\tau}_{\mathfrak{s}}(\Psi_T)$ *such that* $(\mathscr{V},\varphi) = V(\varphi)$ *almost surely*

for all φ. Furthermore one has that

$$\mathbf{E}[\|\mathscr{V}\|^p_{C^{\beta,\tau}_{\mathfrak{s}}(\Psi_T)}] \leq \mathrm{C}',$$

where C' depends on the choice of α, p, and the constant C appearing in the moment bound above but not on $V, \Omega, \mathcal{F}, \mathbb{P}$.

A proof of the above results may be adapted from the proof of Lemma 9 in Section 5 of [228]. We remark that we do not actually need uniformity over a large class of test functions as we have written above, just a single well-chosen test function would suffice (e.g. the Littlewood-Paley blocks as used in [228] or the Daubechies wavelets in [227]).

5.6.2 Tightness

Here we will prove tightness of the rescaled field from (5.3).

Lemma 5.6.18. *Fix $\alpha < 0, \tau > 0$. Let K_N, D_N be the operators from Definition 5.6.2. Recall the function spaces from Definition 5.6.9, and let $\mathcal{X}_N^{\alpha,\tau}$ denote the closed linear subspace of $C([0, T + 1], C^{\alpha,\tau}(\mathbb{R}))$ consisting of those paths $v = (v(t))_{t\in[0,T+1]}$ such that $v(t) = 0$ for all $t \in [0, N^{-1}]$. Then we have the operator norm bound*

$$\sup_{N\geq 1} \|K_N D_N\|_{\mathcal{X}_N^{\alpha,\tau} \to C([0,T],C^{\alpha,\tau}(\mathbb{R}))} < \infty.$$

Proof. Note that $v(t) = 0$ for $t \in [0, N^{-1}]$ ensures that $D_N v$ is actually a continuous path taking values in $C([0, T + 1 - N^{-1}], C^{\alpha,\tau}(\mathbb{R}))$, consequently so is $K_N D_N v$. Let us now define four

families of linear operators on $\mathcal{S}'(\mathbb{R})$, indexed by N. For $f \in \mathcal{S}'(\mathbb{R})$ and $s \in N^{-1}\mathbb{Z}_{\geq 0}$ let

$$\begin{aligned}
P_N(s)f(x) &:= \sum_{y \in N^{-1/2}\mathbb{Z} - N^{3/4}s} p_N(Ns, N^{1/2}y) f(x-y), \\
\delta_N f(x) &:= N(P_N(N^{-1}) - I)f(x) \\
&= N\big[\rho_N f(x + N^{-\frac{1}{2}} - N^{-\frac{3}{4}}) + (1-\rho_N) f(x - N^{-\frac{1}{2}} - N^{-\frac{3}{4}}) - f(x)\big] \\
\delta_N^* f(x) &:= N\big[\rho_N f(x - N^{-1/2} + N^{-3/4}) + (1-\rho_N) f(x + N^{-1/2} + N^{-3/4}) - f(x)\big], \\
P_N^*(s)f(x) &:= \big(I + N^{-1}\delta_N^*\big)^{Ns} f(x) = \sum_{y \in N^{-1/2}\mathbb{Z} - N^{3/4}s} p_N(Ns, N^{1/2}y) f(x+y).
\end{aligned}$$

Then for $f \in \mathcal{S}'(\mathbb{R})$ each of the four expressions $P_N(s)f, \delta_N f, \delta_N^* f, P_N^*(s)f$ also make sense as elements of $\mathcal{S}'(\mathbb{R})$. Note that δ_N and δ_N^*, both of which approximate the spatial Laplacian ∂_x^2, are adjoint to each other on $L^2(\mathbb{R})$. Therefore P_N and P_N^* are also adjoint to each other. Now let $v = (v(t))_{t \in [0, T+1]} \in \mathcal{X}_N^{\alpha,\tau}$. For $\phi \in C_c^\infty(\mathbb{R})$ and $t \in [0,T]$, if we apply summation by parts and use the fact that v vanishes on $[0, N^{-1}]$ we can write

$$\begin{aligned}
\big(K_N D_N v(t), \phi\big) &= \Big(N^{-1} \sum_{s \in [0,t] \cap (N^{-1}\mathbb{Z}_{\geq 0})} N P_N(t-s)\big[v(s+N^{-1}) - v(s)\big] \ , \ \phi \Big) \\
&= \Big(v(t+N^{-1}) - 0 + \sum_{s \in [0,t] \cap (N^{-1}\mathbb{Z}_{\geq 0})} \big[P_N(t-s) - P_N(t-s-N^{-1})\big](v(s)) \ , \ \phi \Big) \\
&= (v(t+N^{-1}), \phi) + \frac{1}{N} \sum_{s \in [0,t] \cap (N^{-1}\mathbb{Z}_{\geq 0})} (\delta_N P_N(t-s-N^{-1}) v(s), \phi) \\
&= (v(t+N^{-1}), \phi) + \frac{1}{N} \sum_{s \in [0,t] \cap (N^{-1}\mathbb{Z}_{\geq 0})} (v(s), P_N^*(t-s-N^{-1}) \delta_N^* \phi).
\end{aligned}$$

Here we are using the $L^2(\mathbb{R})$-pairing between distributions and smooth functions. By the definition of the function spaces (Definition 5.6.9), we must replace ϕ by $S_x^\lambda \phi$ (where the scaling operators are given in Definition 5.6.8) in the last expression and then study the growth as λ becomes close to 0.

The first term on the right side is completely straightforward to deal with: the growth is at worst

$\lambda^\alpha(1+x^2)^\tau$ uniformly over λ, ϕ, x, T, since we assumed $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. To deal with the second term, we claim that one has

$$|(v(s), P_N^*(t-s-N^{-1})\delta_N^* S_x^\lambda \phi)| \lesssim \big(\lambda^{\alpha-2} \wedge (t-s)^{\frac{\alpha}{2}-1}\big)(1+x^2)^\tau \tag{5.110}$$

uniformly over $s < t \in [0,T]$, as well as λ, ϕ, x, N and v with $\|v\|_{C([0,T+1],C^{\alpha,\tau}(\mathbb{R}))} \leq 1$. Indeed the bound of the form $\lambda^{\alpha-2}$ follows by noting that when $t-s$ is much smaller than λ, $P_N^*(t-s)$ is essentially the identity operator, so we can effectively disregard the heat kernel and note that $\delta_N^* S_x^\lambda \phi$ paired with $v(s)$ satisfies a bound of order $\lambda^{\alpha-2}$, uniformly over N by e.g. second-order Taylor expansion. Likewise the bound of the form $(t-s)^{\frac{\alpha}{2}-1}$ is obtained by noting that when λ is very small compared to $t-s$, $P_N^*(t-s-N^{-1})\delta_N^* S_x^\lambda \phi$ behaves like $S_x^{\sqrt{t-s}}\phi$, giving a bound of order $(\sqrt{t-s})^{\alpha-2}$ after applying δ_N^* and pairing with $v(s)$. This proves the bound (5.110).

Now the fact that $\|K_N D_N v\|$ can be controlled by $\|v\|$ follows simply by noting that uniformly over $0 < \lambda^2 \leq t \leq T$ one has

$$\int_0^t \big(\lambda^{\alpha-2} \wedge (t-s)^{\frac{\alpha}{2}-1}\big) ds = \int_0^{t-\lambda^2} (t-s)^{\frac{\alpha}{2}-1} ds + \int_{t-\lambda^2}^t \lambda^{\alpha-2} ds \leq \mathrm{C}(\alpha) \cdot \lambda^\alpha.$$

This implies the uniform bound on the operator norm of $K_N D_N$. □

Proposition 5.6.19 (Tightness of all relevant processes)**.** *The following are true.*

1. *The fields $\widehat{M}_N$ from Definition 5.6.4 may be realized as an element of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ for any $\alpha < -3$ and $\tau > 1$. Moreover, they are tight with respect to that topology.*

2. *The fields $\mathscr{U}_N$ from* (5.3) *may be realized as an element of as an element of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ for any $\alpha < -3$ and $\tau > 1$. Moreover, they are tight with respect to that topology.*

3. *The fields $\widehat{Q}_N$ from Definition 5.6.4 may be realized as an element of $C([0,T], C^{\gamma,\tau}(\mathbb{R}))$ for any $\gamma < -1$ and $\tau > 1$. Moreover, they are tight with respect to that topology.*

4. *Let $\alpha < -3, \gamma < -1$, and $\tau > 1$. Let $(M^\infty, Q^\infty, U^\infty)$ be a joint limit point of $(\widehat{M}_N, \widehat{Q}_N, \mathscr{U}_N)$*

in $C([0,T], C^{\alpha,\tau}(\mathbb{R}) \times C^{\gamma,\tau}(\mathbb{R}) \times C^{\alpha,\tau}(\mathbb{R}))$. *For all* $\phi \in C_c^\infty(\mathbb{R})$ *the process* $(M_t^\infty(\phi))_{t\in[0,T]}$ *is a continuous martingale with respect to the canonical filtration on that space, and moreover*

$$\langle M^\infty(\phi)\rangle_t = Q_t^\infty(\phi^2). \tag{5.111}$$

Proof. Take any $\phi \in C_c^\infty(\mathbb{R})$ with $\|\phi\|_{L^\infty} \le 1$. Recall $S^\lambda_{(t,x)}$ from (5.106). Using the first bound in Proposition 5.5.4, we have

$$\mathbb{E}[(Q_N(t, S_x^\lambda \phi) - Q_N(s, S_x^\lambda \phi))^k] \le \mathrm{C}|t-s|^{k/2}\lambda^{-k},$$

uniformly over $x \in \mathbb{R}, \lambda \in (0,1], 0 \le s,t \le T$ with $s,t \in N^{-1}\mathbb{Z}_{\ge 0}$. Since $Q_N(0,\phi) = 0$ by definition, the assumptions of Lemma 5.6.17 (a) are therefore satisfied for any $\kappa \le 1/4$, any $p > 1/\kappa$, and any $\alpha \le -1$, and we conclude the desired tightness for $\widehat{Q}_N = Q_N$. This proves Item *(3)*.

Now we address the tightness of the $\widehat{M}_N$. Using the Burkholder-Davis-Gundy inequality and then Proposition 5.5.5, we have that

$$\mathbb{E}[(M_N(t,\phi) - M_N(s,\phi))^8]^{1/8} \le \mathrm{C}\cdot \mathbb{E}\big[\big([M_N(\phi)]_t - [M_N(\phi)]_s\big)^4\big]^{1/8} \le \mathrm{C}(t-s)^{1/4}\|\phi\|_{C^1(\mathbb{R})}, \tag{5.112}$$

where $\phi \in C_c^\infty(\mathbb{R})$, $\mathrm{C} = \mathrm{C}(k) > 0$ is free of ϕ, s, t, N. This gives

$$\mathbb{E}[(M_N(t, S_x^\lambda\phi) - M_N(s, S_x^\lambda\phi))^8]^{1/8} \le \mathrm{C}(t-s)^{1/4}\lambda^{-2}$$

uniformly over $x \in \mathbb{R}, \lambda \in (0,1], 0 \le s,t \le T$, and $\phi \in C_c^\infty(\mathbb{R})$ with $\|\phi\|_{C^1} \le 1$ with support contained in the unit interval. Moreover $\widehat{M}_N(0,\phi) = 0$ by definition, therefore the assumptions of Lemma 5.6.17 (a) are satisfied with $\kappa = 1/4$, $p = 8 > 1/\kappa$, and any $\alpha \le -2$. Hence, we conclude the desired tightness for $\widehat{M}_N$. This proves Item *(1)*.

Now tightness for the fields $\mathscr{U}_N$ is immediate from Lemma 5.6.18, since we know from Lemma

5.6.7 that $\mathscr{U}_N = \mathfrak{p}_N + K_N D_N \widehat{M}_N$ where we view $K_N D_N$ as a bounded operator from $\mathcal{X}_N^{\alpha,\tau} \to C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ and the convergence of $\mathfrak{p}_N$ in this topology is straightforward to deal with. This proves Item *(2)*.

We next show that the limit point $M^\infty(\phi)$ is a martingale indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$. Since $M_N(0,\phi) = 0$, from (5.112), we see that $\sup_N \mathbb{E}[M_N(t,\phi)^{2k}] < \infty$. Thus $M^\infty(\phi)$ is a martingale since martingality is preserved by limit points under the uniform integrability assumption. Continuity is guaranteed by the definition of the spaces in which we proved tightness. In the prelimit we know from (5.67) that

$$M_N(t,\phi)^2 - \big((2\rho_N - 1)^2\sqrt{N}\big)Q_N(t,\phi^2) - \mathcal{E}_N(t,\phi)$$

is a martingale indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$, where the error term $\mathcal{E}_N$ is defined in (5.69) and satisfies the bound (5.70). Note that $\big((2\rho_N - 1)^2\sqrt{N}\big) \to 1$ as $N \to \infty$. By (5.70) and tightness estimates (5.92) of Q_N, it follows that $\mathcal{E}_N(t,\phi)$ vanishes in probability in the topology of $C[0,T]$, so we conclude (again by uniform L^p boundedness guaranteed by Proposition 5.5.4) that $M_t^\infty(\phi)^2 - Q_t^\infty(\phi^2)$ is a martingale. This verifies (5.111) completing the proof of Item *(4)*. □

Lemma 5.6.20 (Controlling the difference between the discrete and continuum heat operators)**.** *Fix $\alpha < 0, \tau > 1$. Let K_N be as in Definition 5.6.2, and let K be as in* (5.107)*. Then we have the operator norm bound*

$$\|K_N - K\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T) \to C_{\mathfrak{s}}^{\alpha-1,\tau}(\Psi_T)} \leq C N^{-1/4}.$$

Here C is independent of N.

The above bound is crude, we do not claim optimality of the Holder exponents here.

Proof. Define $f_\varphi^\lambda(t,x) := (f, S_{(t,x)}^\lambda \varphi)_{L^2(\Psi_T)}$. It suffices to show that

$$\sup_{\|f\|_{C_{\mathfrak{s}}^{\alpha,\tau}(\Psi_T)} \leq 1} \sup_{(t,x)\in\Psi_T} \sup_{\lambda\in[(0,1]} \sup_{\varphi \in B_r} \frac{|(K_N - K) f_\varphi^\lambda(t,x)|}{(1+x^2)^\tau \lambda^{\alpha-1}} \leq C N^{-1/4},$$

where the scaling operators are defined by $S^{\lambda}_{(t,x)}\varphi(s,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(x-y))$, and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}^2$. We shall use a probabilistic interpretation of the kernels to prove this. Note that

$$(K_N - K)f^{\lambda}_{\varphi}(t,x) = \frac{1}{N}\sum_{s\in[0,t]\cap(N^{-1}\mathbb{Z}_{\geq 0})} \mathbf{E}[f^{\lambda}_{\varphi}(t-s, x-W_N(s))] - \int_0^t \mathbf{E}[f^{\lambda}_{\varphi}(t-s,x-W(s))]ds,$$

where $W_N(s) = N^{-1/2}R_N(Ns)$ for the discrete-time random walk $(R_N(r))_{r\geq 0}$ with increment distribution described in Definition 5.6.2, and W is a standard Brownian motion. Define $W_N(s)$ by linear interpolation for $s \notin N^{-1}\mathbb{Z}_{\geq 0}$.

By the KMT coupling [250], we may assume that W_N and W are all coupled onto the same probability space so that $\sup_{s\in[0,T]}|W_N(s) - W(s)| \leq GN^{-1/4}$ where G is a random variable on that same probability space independent of N such that $\mathbf{E}|G|^p < \infty$ for all p. Note that $-1/4$ is the optimal exponent here because, despite KMT giving a better exponent in principle, the increments of the original random walk R_N are not centered but rather they have a non-zero mean of order $N^{-3/4}$. Consequently, by the definition of the parabolic spaces, we have uniformly over $s,t\in[0,T]$ the bound

$$\begin{aligned}
&|f^{\lambda}_{\varphi}(t-s,x-W_N(s)) - f^{\lambda}_{\varphi}(t-s,x-W(s))| \\
&\leq \left(\sup_{u\in\left[-|x|-|W(s)|-|W_N(s)|,\, |x|+|W(s)|+|W_N(s)|\right]} |\partial_x f^{\lambda}_{\varphi}(t-s,u)|\right)\cdot|W_N(s)-W(s)| \\
&\leq \left(\|f\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)}\lambda^{\alpha-1}(1+4x^2+4W(s)^2+4(W_N(s))^2)^{\tau}\right)\cdot GN^{-1/4}.
\end{aligned}$$

Here the factor $\|f\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)}\lambda^{\alpha-1}$ can be deduced using e.g. Remark 5.6.12 which says that ∂_x boundedly reduces the parabolic regularity by 1 exponent. We also used the fact that $(|x|+|y|+|z|)^2 \leq 4x^2+4y^2+4z^2$.

We claim that

$$\mathbf{E}\big|G\cdot(1+4x^2+4W(s)^2+4(W_N(s))^2)^{\tau}\big| < C(1+x^2)^{\tau}$$

for a constant $C = C(\alpha, T, \tau)$ independent of N, x, s. To prove this, one uses $(a+b+c+d)^\tau \leq 4^\tau(a^\tau + b^\tau + c^\tau + d^\tau)$, then one notes that $\sup_N \mathbf{E}|G \cdot \sup_{s\leq T}(|W(s)|^{2\tau} + |W_N(s)|^{2\tau})| < \infty$ by e.g. Cauchy-Schwartz.

Consequently the above expression for $(K_N - K)f^\lambda_\varphi(t,x)$ can be bounded in absolute value by a universal constant times $\|f\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)}\lambda^{\alpha-1}N^{-1/4}$. □

Lemma 5.6.21 (Controlling the difference between the discrete and continuum time-derivatives)**.** *Fix $\alpha < 0, \tau > 1$. The derivative operator $\partial_s : C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T) \to C^{\alpha-2,\tau}_{\mathfrak{s}}(\Psi_T)$ which was defined in Remark 5.6.12, is a bounded linear map. Furthermore, let D_N be as in Definition 5.6.2. Then we have the operator norm bounds*

$$\sup_{N\geq 1} \|D_N\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)\to C^{\alpha-2,\tau}_{\mathfrak{s}}(\Psi_T)} < \infty.$$
$$\|D_N - \partial_s\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)\to C^{\alpha-4,\tau}_{\mathfrak{s}}(\Psi_T)} \leq \frac{1}{2}N^{-1}.$$

Proof. The proof that ∂_s is bounded is straightforward from the definitions of the spaces. Note that $(D_N f, \varphi) = (f, D_N^- \varphi)$ where $D_N^-\varphi(t,x) = N\big[\varphi(t,x) - \varphi(t - N^{-1}, x)\big]$. Recall $S^\lambda_{(t,x)}$ from (5.106). From the definition of the space $C^{\alpha,\tau}(\mathbb{R})$ one verifies directly that if $v \in C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ with norm less than or equal to 1, then for $N > \lambda^{-1}$ one has

$$\int_0^T |v_t\big(D_N^-(S^\lambda_{(s,y)}\varphi)(t,\cdot)\big)|dt \leq (1+y^2)^\tau \int_{s-\lambda^2}^{s+\lambda^2} \lambda^{\alpha-4}dt \leq (1+y^2)^\tau \lambda^{\alpha-2}$$

uniformly over $(s,y) \in \Psi_T$ and $\varphi \in C^\infty_c(\mathbb{R}^2)$ supported on the unit ball of $\mathbb{R}^2$ with $\|\varphi\|_{C^r} \leq 1$ where $r = \lceil -\alpha \rceil$. Here the bound of order $\lambda^{\alpha-4}$ may be deduced by writing $D_N^-\varphi(t,x) = N\int_{t-N^{-1}}^t \partial_s\varphi(s,y)ds$, interchanging the two integrals, and noting that $\partial_s(S^\lambda_{(s,y)}\varphi)(t,\cdot)$ satisfy such a bound. This proves the first operator norm bound.

Now we prove the second bound. Define $f^\lambda_\varphi(t,x) := (f, S^\lambda_{(t,x)}\varphi)_{L^2(\Psi_T)}$. It suffices to show that

$$\sup_{\|f\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)}\leq 1} \sup_{(t,x)\in\Psi_T} \sup_{\lambda\in(0,1]} \sup_{\varphi\in B_r} \frac{|(D_N - \partial_s)f^\lambda_\varphi(t,x)|}{(1+x^2)^\tau\lambda^{\alpha-4}} \leq \frac{1}{2}N^{-1},$$

where the scaling operators are defined by $S^\lambda_{(t,x)}\varphi(s,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(x-y))$, and where B_r is the set of all smooth functions of C^r norm less than 1 with support contained in the unit ball of $\mathbb{R}^2$. Note that

$$\begin{aligned}|(D_N - \partial_s) f^\lambda_\varphi(t,x)| &= \Big| N\big(f^\lambda_\varphi(t+N^{-1},x) - f^\lambda_\varphi(t,x)\big) - \partial_s f^\lambda_\varphi(t,x)\Big| \\ &= \left| N\int_t^{t+N^{-1}} \big(\partial_s f^\lambda_\varphi(u,x) - \partial_s f^\lambda_\varphi(t,x)\big)du\right| \\ &\le N\int_t^{t+N^{-1}} \|\partial_s^2 f^\lambda_\varphi\|_{L^\infty([t,t+N^{-1}]\times\{x\})}|t-u|du \le \frac{1}{2}N^{-1}\|\partial_s^2 f^\lambda_\varphi\|_{L^\infty([t,t+N^{-1}]\times\{x\})}.\end{aligned}$$

From the definitions one has $\|\partial_s^2 f^\lambda_\varphi\|_{L^\infty([t,t+N^{-1}]\times\{x\})} \le (1+x^2)^\tau \lambda^{\alpha-4}\|f\|_{C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)}$. □

Since we expect to obtain a Dirac initial data in the limiting SPDE, there is an additional singularity at the origin that we have not yet taken into account. To fix this issue, we now formulate a tightness result taking into account this singularity, by starting the field (5.3) from some positive time ε (which should be thought of as being close to 0).

Proposition 5.6.22 (Regularity of limit points)**.** *Fix $\varepsilon > 0$, and consider the fields $\mathscr{U}_{N,\varepsilon}(t,\cdot) := \mathscr{U}_N(t+\varepsilon,\cdot)$, for $t \ge 0$. Set $\mathscr{U}_{N,\varepsilon}(t,\cdot) := 0$ for $t < 0$. The fields $\mathscr{U}_{N,\varepsilon}$ may be realized as random variables taking values in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ for any $\tau > 1$ and $\alpha < -3$. Furthermore, they are tight with respect to that topology, and any limit point is necessarily supported on $C^{-\kappa+1/2,\tau}_{\mathfrak{s}}(\Psi_T)$ for any $\kappa > 0$.*

Proof. We already know from Item (2) of Proposition 5.6.19 that the $\mathscr{U}_N$ are tight in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. We also know from Item (1) of Proposition 5.6.19 that $\widehat{M}_N$ are tight in $C([0,T+1], C^{\alpha,\tau}(\mathbb{R}))$, which is embeds continuously into $C^{\alpha,\tau}_{\mathfrak{s}}(\Psi_T)$ by Lemma 5.6.11. Thus using the first bound in Lemma 5.6.21, we have that $D_N\widehat{M}_N$ are tight in $C^{\alpha-2}_{\mathfrak{s}}(\Psi_T)$.

Consider any joint limit point $(U^\infty, \mathscr{M}^\infty)$ as $N \to \infty$ of the pair $(\mathscr{U}_N, D_N\widehat{M}_N)$ in the product space $C([0,T], C^{\alpha,\tau}(\mathbb{R})) \times C^{\alpha-2,\tau}_{\mathfrak{s}}(\Psi_T)$. By Lemmas 5.6.7 and 5.6.20 we may conclude that

$$U^\infty = p + K\mathscr{M}^\infty, \tag{5.113}$$

where $p(t,x) = (2\pi t)^{-1/2} e^{-x^2/2t} \mathbb{1}_{\{t \geq 0\}}$ is the (deterministic) standard heat kernel, and K is the operator defined in (5.107).

Fix $\varepsilon > 0$. Now we consider the ε-translates of these statements. It is automatic from Item (2) of Proposition 5.6.19 that $\mathscr{U}_{N,\varepsilon}$ are tight in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. Likewise, it is automatic from Item (1) that the family $\widehat{M}_{N,\varepsilon} := \widehat{M}_N(\varepsilon + \cdot)$ is tight in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ so that from Lemma 5.6.21 we see that $D_N \widehat{M}_{N,\varepsilon}$ are tight in $C_{\mathfrak{s}}^{\alpha-2,\tau}(\Psi_T)$. Let us now take a joint limit point $(U^\infty, \mathscr{M}^\infty, U^{\infty,\varepsilon}, \mathscr{M}^{\infty,\varepsilon})$ of the family $(\mathscr{U}_N, D_N \widehat{M}_N, \mathscr{U}_{N,\varepsilon}, D_N \widehat{M}_{N,\varepsilon})$. On one hand we necessarily have $U^{\infty,\varepsilon}(t) = U^\infty(t+\varepsilon)$ and on the other hand we necessarily have that $U^\infty = p + K\mathscr{M}^\infty$. From the last statement in Proposition 5.6.14, this then forces the relation

$$\mathscr{U}^{\infty,\varepsilon} = K(\delta_0 \otimes U^\infty(\varepsilon, \cdot)) + K\mathscr{M}^{\infty,\varepsilon},$$

where the tensor product was defined in Definition 5.6.16. Now we notice $K(\delta_0 \otimes U^\infty(\varepsilon,\cdot)) = J\big(U^\infty(\varepsilon,\cdot)\big)$, where J is defined in Corollary 5.6.15. We thus have the Duhamel equation

$$\mathscr{U}^{\infty,\varepsilon} = J\big(U^\infty(\varepsilon,\cdot)\big) + K\mathscr{M}^{\infty,\varepsilon}. \tag{5.114}$$

For technical reasons that will be made clear below, we now replace ε by $\varepsilon/2$ and T by $T+1$. We now claim that for all $q > 0$

$$\mathbb{E}[\|\mathscr{M}^{\infty,\varepsilon/2}\|^q_{C_{\mathfrak{s}}^{-\kappa-3/2,\tau}(\Psi_{T+1})}] < \infty. \tag{5.115}$$

Let us now complete the proof of regularity assuming (5.115).

- Using (5.115) and Proposition 5.6.14, it follows that

$$\mathbb{E}\left[\left\|K\mathscr{M}^{\infty,\varepsilon/2}\right\|^q_{C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_{T+1})}\right] < \infty.$$

This implies that the restriction of $K\mathscr{M}^{\infty,\varepsilon/2}$ to $[\varepsilon/2, T+\varepsilon/2]\times\mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_{[\varepsilon/2,T+\varepsilon/2]})$.

- Let $h := \mathscr{U}_N(\varepsilon/2, \cdot)$. By Proposition 5.6.19 Item (2) we have $h \in C^{-2-\kappa,\tau}$ almost surely. So from (5.109) with $\alpha := -2-\kappa$ and $\beta := \frac{1}{2} - \kappa$, it follows that $\widehat{\mathcal{P}}_{\varepsilon/2} h \in C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_{T+1})$ almost surely. Since from (5.108), we know $Jh(t+\varepsilon/2, x) = \widehat{\mathcal{P}}_{\varepsilon/2} h(t,x)$, we thus have that the restriction of Jh to $[\varepsilon/2, T+\varepsilon/2] \times \mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_{[\varepsilon/2,T+\varepsilon/2]})$.

Thanks to the above two bullet points and the relation (5.114), we have showed that the restriction of $U^{\infty,\varepsilon/2}$ to $[\varepsilon/2, T+\varepsilon/2] \times \mathbb{R}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_{[\varepsilon/2,T+\varepsilon/2]})$. This is equivalent to the fact that $U^{\infty,\varepsilon}$ lies in $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_T)$. This completes the proof modulo (5.115).

Proof of (5.115). We shall show (5.115) with $\varepsilon/2$ and $T+1$ replaced by ε and T respectively. For $\alpha < 0$, the derivative operator $\overline{\partial}_s : C([0,T], C^{\alpha,\tau}(\mathbb{R})) \to C_{\mathfrak{s}}^{\alpha-2,\tau}(\Psi_T)$ which is defined by sending $(v_t)_{t\in[0,T]}$ to the distribution

$$(\partial_s v, \varphi)_{L^2(\Psi_T)} := v_T(\varphi(T,\cdot)) - \int_0^T v_t(\partial_t \varphi(t,\cdot))dt - v_0(\varphi(0,\cdot)), \tag{5.116}$$

whenever $\varphi \in C_c^\infty(\Psi_T)$, is a bounded linear map. Indeed this operator is just the composition of the embedding map of Lemma 5.6.11 with the ∂_s operator which we proved was bounded in Lemma 5.6.21.

Let $\widehat{M}_{N,\varepsilon} := \widehat{M}_N(\varepsilon + \cdot) - \widehat{M}_N(\varepsilon)$ and $\widehat{Q}_{N,\varepsilon} := \widehat{Q}_N(\varepsilon + \cdot) - \widehat{Q}_N(\varepsilon)$. Let us consider any joint limit point $(Q^{\infty,\varepsilon}, M^{\infty,\varepsilon}, \mathscr{M}^{\infty,\varepsilon})$ of $(\widehat{Q}_{N,\varepsilon}, \widehat{M}_{N,\varepsilon}, D_N \widehat{M}_{N,\varepsilon})$ in the space $C([0,T], C^{\alpha,\tau}(\mathbb{R})) \times C([0,T], C^{\alpha,\tau}(\mathbb{R})) \times C^{\alpha-2,\tau}(\Psi_T)$, where $\gamma < -1$. Then one may verify from the second bound in Lemma 5.6.21 that one necessarily has $\mathscr{M}^{\infty,\varepsilon} = \overline{\partial}_s M^{\infty,\varepsilon}$. Furthermore by Proposition 5.6.19 one has martingality of $M_t^{\infty,\varepsilon}(\phi)$, with $\langle M^{\infty,\varepsilon}(\phi)\rangle = Q^{\infty,\varepsilon}(\phi^2)$.

Let us consider any smooth function φ supported on the unit ball of $\mathbb{R}^2$ such that $\|\varphi\|_{C^1} \leq 1$. Recall $S^\lambda_{(t,x)}$ from (5.106), and notice that $\left(\partial_t S^\lambda_{(t,x)}\varphi\right)^2 \leq \lambda^{-10} \mathbb{1}_{[t-\lambda^2, t+\lambda^2]\times[x-\lambda, x+\lambda]}$. We will now estimate the q^{th} moments of $\mathscr{M}^{\infty,\varepsilon}(S^\lambda_{(t,x)}\varphi) = \overline{\partial}_s M^{\infty,\varepsilon}(S^\lambda_{(t,x)}\varphi)$. By making ε smaller and T larger we may simply ignore the boundary terms in (5.116). Note that for fixed s, t, x, λ the martingale $u \mapsto M_u^{\infty,\varepsilon}(\partial_t S^\lambda_{(t,x)}\varphi(s,\cdot))$ is constant outside of the interval $[t-\lambda^2, t+\lambda^2]$. Thus we may apply

the inequalities of Minkowski and then Burkholder-Davis-Gundy in (5.116) to obtain

$$\begin{aligned}
\mathbb{E}[|\mathscr{M}^{\infty,\varepsilon}(S^{\lambda}_{(t,x)}\varphi)|^q]^{1/q} &= \mathbb{E}[|\overline{\partial}_s M^{\infty,\varepsilon}(S^{\lambda}_{(t,x)}\varphi)|^q]^{1/q} \\
&\leq \int_{t-\lambda^2}^{t+\lambda^2} \mathbb{E}\big[\big|M^{\infty,\varepsilon}_s(\partial_t S^{\lambda}_{(t,x)}\varphi(s,\cdot))\big|^q\big]^{1/q} ds \\
&\leq \mathrm{C}_q \int_{t-\lambda^2}^{t+\lambda^2} \mathbb{E}\bigg[\Big(Q^{\infty,\varepsilon}_{t+\lambda^2}\big((\partial_t S^{\lambda}_{(t,x)}\varphi)^2(s,\cdot)\big) - Q^{\infty,\varepsilon}_{t-\lambda^2}\big((\partial_t S^{\lambda}_{(t,x)}\varphi)^2(s,\cdot)\big)\Big)^{q/2}\bigg]^{1/q} \\
&\leq 2\mathrm{C}_q\lambda^{-3}\mathbb{E}\bigg[\Big(Q^{\infty,\varepsilon}_{t+\lambda^2}\big(\mathbb{1}_{[x-\lambda,x+\lambda]}\big) - Q^{\infty,\varepsilon}_{t-\lambda^2}\big(\mathbb{1}_{[x-\lambda,x+\lambda]}\big)\Big)^{q/2}\bigg]^{1/q}
\end{aligned}$$

where $\mathrm{C}_q > 0$. For any $\delta > 0$, by the second bound in Proposition 5.5.4 we find that the last expression may be bounded above by

$$2\mathrm{C}\lambda^{-3}(2\lambda^2)^{1/2}\|\mathbb{1}_{[x-\lambda,x+\lambda]}\|^{1/2}_{L^{1+\delta}(\mathbb{R})} = \mathrm{C}\lambda^{-\frac{3}{2}-\left[\frac{\delta}{2(1+\delta)}\right]}.$$

Given any $\kappa > 0$, by making $\delta = \delta(\kappa)$ close to 0 and making $q = q(\kappa)$ large enough, we may then apply Lemma 5.6.17(b) to conclude (5.115). □

5.6.3 Identification of the limit points

In this subsection, we identify the limit points as the solution of the stochastic heat equation (5.4) and thereby prove our weak convergence theorem: Theorem 5.1.1.

Lemma 5.6.23. *Let μ be a random variable in $\mathcal{S}'(\mathbb{R})$ such that for some smooth even (deterministic) $\phi \in \mathcal{S}(\mathbb{R})$ and some $\delta > 0$ one has*

$$\sup_{a,\varepsilon}(1 \wedge a^{1-\delta})\mathbb{E}[(\mu,\phi^a_\varepsilon)^2] < \infty$$

$$\limsup_{\varepsilon\to 0}\mathbb{E}[(\mu,\phi^a_\varepsilon)^2] = 0, \textit{ for all } a \in \mathbb{R}\backslash\{0\},$$

where $\phi^a_\varepsilon(x) := \varepsilon^{-1}\phi(\varepsilon^{-1}(x-a))$. Then $(\mu,\psi) = 0$ almost surely for all $\psi \in \mathcal{S}(\mathbb{R})$.

Proof. Define $\mu^\varepsilon(x) := \mu * \phi_\varepsilon(x)$ so that $(\mu,\phi^a_\varepsilon) = \mu^\varepsilon(a)$. Given some smooth $\psi : \mathbb{R} \to \mathbb{R}$ of

compact support note that $(\mu^\varepsilon, \psi) \to (\mu, \psi)$ a.s. as $\varepsilon \to 0$. This is a purely deterministic statement. Thus it suffices to show that $(\mu^\varepsilon, \psi) \to 0$ in probability. To prove that, suppose that the support of ψ is contained in $[-S, S]$ and note by Cauchy-Schwarz that

$$|(\mu^\varepsilon, \psi)| = \left| \int_{\mathbb{R}} \mu^\varepsilon(a)\psi(a)da \right| \leq \left[\int_{[-S,S]} \mu^\varepsilon(a)^2 da \right]^{1/2} \|\psi\|_{L^2(\mathbb{R})},$$

so that by taking expectation and applying Jensen we find

$$\mathbb{E}[|(\mu^\varepsilon, \psi)|] \leq \|\psi\|_{L^2} \left[\int_{[-S,S]} \mathbb{E}[\mu^\varepsilon(a)^2] da \right]^{1/2}.$$

Since by assumption $\mathbb{E}[\mu^\varepsilon(a)^2] \leq Ca^{\delta-1}$ and $\mathbb{E}[\mu^\varepsilon(a)^2] \to 0$ as $\varepsilon \to 0$ the dominated convergence theorem now gives the result by letting $\varepsilon \to 0$ on the right side. This proves the claim for ψ of compact support. For general $\psi \in \mathcal{S}(\mathbb{R})$ we may find a sequence $\psi_n \to \psi$ in the topology of $\mathcal{S}(\mathbb{R})$, with each ψ_n compactly supported. Then $0 = (\mu, \psi_n) \to (\mu, \psi)$ a.s. as $n \to \infty$. □

Theorem 5.6.24 (Solving the martingale problem)**.** *Consider the triple of processes* $(\mathscr{U}_N, M_N, Q_N)_{t\geq 0}$*, where* $\mathscr{U}_N$*,* M_N*, and* Q_N *are defined in* (5.3)*,* (5.63)*, and* (5.68) *respectively. Fix* $\alpha < -3, \gamma < -1$*, and* $\tau > 1$*. These triples are jointly tight in the space*

$$C\big([0,T]\ ,\ C^{\alpha,\tau}(\mathbb{R}) \times C^{\gamma,\tau}(\mathbb{R}) \times C^{\alpha,\tau}(\mathbb{R})\big).$$

Consider any joint limit point $(U^\infty, M^\infty, Q^\infty)$*. Then for any* $s > 0$*, the process* $(t, x) \mapsto U^\infty_{s+t}(x)$ *is necessarily supported on the space* $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_T)$*. Furthermore,* $(M^\infty_t(\phi))_{t\geq 0}$ *is a continuous martingale for all* $\phi \in C^\infty_c(\mathbb{R})$*, and moreover for all* $0 < s \leq t < T$ *one has the almost sure*

identities

$$M_t^\infty(\phi) - M_s^\infty(\phi) = \int_{\mathbb{R}} (U_t^\infty(x) - U_s^\infty(x))\phi(x)dx - \frac{1}{2}\int_s^t \int_{\mathbb{R}} U_u^\infty(x)\phi''(x)dxdu \tag{5.117}$$

$$\langle M^\infty(\phi)\rangle_t = Q_t^\infty(\phi^2) \tag{5.118}$$

$$Q_t^\infty(\phi) - Q_s^\infty(\phi) = \frac{8\sigma^2}{1-4\sigma^2}\int_{\mathbb{R}}\int_s^t (U_u^\infty(x))^2\phi(x)\,du\,dx. \tag{5.119}$$

Before going into the proof, we remark that with some inspection it may be verified that all quantities make sense given the spaces they lie in, as long as we choose $\kappa < 1/2$ to ensure that U^∞ is a continuous function in space-time away from $t = 0$.

Proof. Most parts of the following theorem are already established in previous propositions and lemmas. The tightness of the triple was shown in Proposition 5.6.19. Consider any limit point $(U^\infty, M^\infty, Q^\infty)$. The fact that for any $s > 0$, the process $(t, x) \mapsto U^\infty(s + t, x)$ is necessarily supported on the space $C_{\mathfrak{s}}^{-\kappa+1/2,\tau}(\Psi_T)$ was proved in Proposition 5.6.22. From Proposition 5.6.19 we have that for all $\phi \in C_c^\infty(\mathbb{R})$ the process $(M_t^\infty(\phi))_{t\geq 0}$ is a martingale. (5.118) is already proven in Proposition 5.6.19 as (5.111). All we are left to show is (5.117) and (5.119).

Proof of (5.117)**.** In (5.113) we obtained that $U^\infty = p + K\mathscr{M}^\infty$ where (just as we observed after (5.116)) one necessarily has $\mathscr{M}^\infty = \overline{\partial}_s M^\infty$. By disregarding the boundary terms implies that

$$(\overline{\partial}_s M^\infty, \varphi) = ((\partial_t - \tfrac{1}{2}\partial_x^2)U^\infty, \varphi)$$

for all φ of compact support contained in $[\varepsilon, T - \varepsilon] \times \mathbb{R}$ for some $\varepsilon > 0$. Since both M^∞ and U^∞ lie in spaces with strong enough topologies, taking $\varphi(u, x)$ to approach the function $(u, x) \mapsto \mathbb{1}_{[s,t]}(u)\phi(x)$ in the above equation leads to (5.117).

Proof of (5.119)**.** Fix any $0 \leq t \leq T$ and let μ be a $\mathcal{S}'(\mathbb{R})$ valued random variable defined as

$$(\mu, \phi) := Q_t^\infty(\phi) - \frac{8\sigma^2}{1-4\sigma^2}\int_{\mathbb{R}}\int_0^t (U_s^\infty(x))^2\phi(x)\,ds\,dx. \tag{5.120}$$

We claim that $(\mu, \phi) = 0$ a.s. for all $\phi \in \mathcal{S}(\mathbb{R})$. This will validate (5.119). To verify this, let us define $\xi_\varepsilon^a(x) := \varepsilon^{-1}\xi(\varepsilon^{-1}(x-a))$ with $\xi(x) := \frac{1}{\sqrt{\pi}}e^{-x^2}$. Using the "key estimate" (5.87) of Proposition 5.5.3, one verifies that the assumptions of Lemma 5.6.23 hold for μ with this family of mollifiers.

To see why Lemma 5.6.23 is applicable, first note that for fixed $\phi \in C_c^\infty(\mathbb{R})$, the difference between the sum appearing in (5.87) and integral over $[0,t]$ of the same quantity tends to zero in probability with respect to the topology of $C[0,T]$, since we proved tightness of $\mathscr{U}_N$ in a topology given by the norm of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. Consequently that sum in (5.87) converges in law, jointly with $\mathscr{U}_N$, to the integral appearing in (5.120). Then an application of (5.87) and (5.88) shows that (μ, ϕ) as defined by (5.120) satisfies the conditions of Lemma 5.6.23 for any $\delta \in (0,1)$, since (5.88) gives a polylogarithmic bound which is less than $\mathrm{C}a^{-\delta}$ for arbitrary $\delta > 0$. This is enough to complete the proof. □

We now complete the proof of our main theorem, Theorem 5.1.1.

Proof of Theorem 5.1.1. We continue with the notation and setup of Theorem 5.6.24. We have already established the tightness of $\mathscr{U}_N$ in Proposition 5.6.19. Consider any limit point U^∞ of $\mathscr{U}_N$. From the previous theorem, we already know that $(t,x) \mapsto U^\infty_{t+\varepsilon}(x)$ is a continuous function in space and time. From the three equations (5.117), (5.118), and (5.119) in Theorem 5.6.24 it follows that the martingale problem for (5.4) is satisfied by any limit point U^∞. We refer the reader to [88, Proposition 4.11] for the characterization of the law of (5.4) as the solution to this martingale problem.

The result there is only stated for continuous initial conditions, so what this really shows is that for any $\varepsilon > 0$ the law of the continuous field $(t,x) \mapsto U^\infty(t+\varepsilon, x)$ is that of the solution of (5.4) with initial condition $U^\infty(\varepsilon, \cdot)$. Thus we still need to pin down the initial data as δ_0, by showing that we can let $\varepsilon \to 0$ and see that the limit of $U^\infty(\varepsilon, \cdot)$ is equal to δ_0 in some sense.

In [229, Section 6] there is a general approach to do exactly this. Specifically, it suffices to

show as in Lemma 6.6 of that paper the two bounds

$$\mathbb{E}[|U_t^\infty(x)|^r]^{2/r} \leq \mathrm{C} \cdot t^{-1/2} p(t,x), \tag{5.121}$$

$$\mathbb{E}[|U_t^\infty(x) - p(t,x)|^r]^{2/r} \leq \mathrm{C} \cdot p(t,x), \tag{5.122}$$

where $r > 1$ is arbitrary, p is the standard heat kernel, and C is independent of $t > 0$ and $x \in \mathbb{R}$. Clearly, it suffices to show this when r is an even integer. The solution of (5.4) with δ_0 initial condition certainly satisfies this bound, and by the moment convergence result in Section 5.3.1, we know any limit point U^∞ must satisfy $\mathbb{E}[(\int_{\mathbb{R}} U_t^\infty(x)\phi(x)dx)^k] = \mathbf{E}[(\int_{\mathbb{R}} \mathcal{U}_t(x)\phi(x)dx)^k]$ for all $k \in \mathbb{N}$ and $\phi \in C_c^\infty(\mathbb{R})$, where $(t,x) \mapsto \mathcal{U}_t(x)$ solves (5.4) with δ_0 initial data. From here we can conclude by letting $\phi \to \delta_x$ that $\mathbb{E}[U_t^\infty(x)^k] = \mathbf{E}[\mathcal{U}_t(x)^k]$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}$. Consequently, we may immediately deduce (5.121) and (5.122) by the corresponding bounds for $\mathcal{U}_t$. This completes the proof. □

5.6.4 Creation of independent noise in the limit

In this subsection, we prove our independent noise result: Theorem 5.1.4.

Definition 5.6.25. *Define for $t \in N^{-1}\mathbb{Z}_{\geq 0}$ and $\phi \in C_c^\infty(\mathbb{R})$ the field*

$$\mathcal{W}_N(t,\phi) := N^{-3/4} \sum_{r=0}^{Nt} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \phi(N^{-1/2}x)\eta_N(r,x) \tag{5.123}$$

where $\eta_N(r,x) := 2\omega_{r,x+rN^{-1/4}} - 1$, whenever $r-(x+rN^{-1/4})$ is even, and $\eta_N(r,x) = 0$ whenever $r-(x+rN^{-1/4})$ is odd. We also define $\mathcal{W}_N(t,\phi)$ by linear interpolation whenever $t \notin N^{-1}\mathbb{Z}_{\geq 0}$.

Recall the linear operator ∂_s from Remark 5.6.12, which was shown to be bounded in Lemma 5.6.21. We prove the following version of Theorem 5.1.4.

Theorem 5.6.26. *Fix $\alpha < -3, \tau > 1$. Then the pair $(\mathcal{W}_N, \mathscr{U}_N)$ is tight with respect to the topology of $C([0,T], C^{\alpha,\tau}(\mathbb{R}) \times C^{\alpha,\tau}(\mathbb{R}))$. Furthermore any joint limit point $(\mathcal{W},\mathcal{U})$ is uniquely characterized as follows:*

1. *If we define* $\xi_1 := (2\sigma^2)^{-1/2}\partial_s \mathcal{W} \in C_{\mathfrak{s}}^{\alpha-2}(\Psi_T)$ *then* ξ_1 *is a standard space-time white noise.*

2. $\mathcal{U}$ *is the solution to the Ito SPDE given by*

$$\partial_t \mathcal{U} = \partial_x^2 \mathcal{U} + \sqrt{8\sigma^2} \cdot \mathcal{U}\xi_1 + \sqrt{\frac{32\sigma^4}{1-4\sigma^2}} \cdot \mathcal{U}\xi_2,$$

where ξ_2 *is a standard space-time white noise independent of* ξ_1.

Note that $\mathcal{W}_N$ is simply a time-integrated and linearly interpolated version of the rescaled noise field Ξ_N from (5.5). The above theorem readily implies Theorem 5.1.4.

Proof. We are going to consider the martingales $\mathcal{W}_N(t,\phi)$ as defined in (5.123), and we are going to study the cross-variations with the martingales $M_N(t,\phi)$ from (5.63), both indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$.

First, let us note that if $\phi \in C_c^\infty(\mathbb{R})$ then the process $\mathcal{W}_N(t,\phi)$ indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$ has independent increments, where each increment has mean zero and has variance

$$4\sigma^2 N^{-3/2} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \phi(N^{-1/2}x)^2 \mathbb{1}_{\{r-x-rN^{-1/4} \equiv 0 \pmod 2\}}.$$

From here (using BDG) the tightness of $\mathcal{W}_N$ in the space $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$ will follow easily from Lemma 5.6.17(a). Since martingality is preserved by limit points as long as uniform integrability holds, one may verify that for any limit point $\mathcal{W}$, the processes $\mathcal{W}_t(\phi)$ and $\mathcal{W}_t(\phi)^2 - 2\sigma^2\|\phi\|_{L^2}^2 t$ will be martingales in the canonical filtration of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$, for all $\phi \in C_c^\infty(\mathbb{R})$. By Levy's criterion, $\mathcal{W}$ must therefore be distributed as $\sigma\sqrt{2}$ times a standard cylindrical Wiener process over $L^2(\mathbb{R})$, that is, a mean-zero Gaussian process such that $\mathbb{E}[\mathcal{W}_t(\phi)\mathcal{W}_s(\psi)] = 2\sigma^2(s \wedge t)(\phi,\psi)_{L^2(\mathbb{R})}$.

Next, with the martingales $M_N(\phi)$ from (5.63), we note that the processes

$$M_N(t,\phi)\mathcal{W}_N(t,\phi) - \langle M_N(\phi), \mathcal{W}_N(\phi)\rangle_t$$

are martingales indexed by $t \in N^{-1}\mathbb{Z}_{\geq 0}$, where the predictable quadratic variation is defined by

$$\begin{aligned}\langle M_N(\phi), \mathcal{W}_N(\phi)\rangle_t &:= \sum_{r=1}^{Nt} \mathbb{E}\big[(M_N(r,\phi) - M_N(r-1,\phi))(\mathcal{W}_N(r,\phi) - \mathcal{W}_N(r-1,\phi))|\mathcal{F}_{r-1}\big] \\ &= 4\sigma^2 N^{-3/4} \sum_{r=0}^{Nt-1} \sum_{x \in \mathbb{Z} - rN^{-1/4}} \phi(N^{-1/2}x) \cdot \big(\nabla_N \phi\big)(N^{-1/2}x) \cdot Z_N^{\omega}(r,x).\end{aligned} \tag{5.124}$$

We used (5.64) and the fact that the $\eta_N(r,x)$ are iid in the last equality.

Let us consider a joint limit point $(\mathcal{W}, \mathscr{M}, \mathcal{U})$ of the triple $(\mathcal{W}_N, \widehat{M}_N, \mathscr{U}_N)$ in the function space $C\big([0,T], C^{\alpha,\tau}(\mathbb{R})^3\big)$, where we recall that $\widehat{M}_N$ is the interpolated version of M_N from Definition 5.6.4, and tightness of $\widehat{M}_N, \mathscr{U}_N$ in this topology was proved in Proposition 5.6.19.

Since martingality is preserved by limit points as long as one has uniform integrability, we see that $\mathscr{M}_t(\phi)$ and $\mathcal{W}_t(\phi)$ are martingales in the canonical filtration of the space $C\big([0,T], C^{\alpha,\tau}(\mathbb{R})^3\big)$. Recall from Section 5.4 that $\big(\nabla_N \phi\big)(N^{-1/2}x) = N^{-1/4}\phi(N^{-1/2}x) + O(N^{-1/2})$. Thus by (5.124) we see in the limit that

$$\mathscr{M}_t(\phi)\mathcal{W}_t(\phi) - 4\sigma^2 \int_0^t \mathcal{U}_s(\phi^2) ds \tag{5.125}$$

is also a martingale with respect to the canonical filtration. Indeed the difference between the sum in (5.124) and integral in (5.125) is easily controlled, since we proved tightness of $\mathscr{U}_N$ in a topology given by the norm of $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$. On the other hand, we know by Theorem 5.1.1 that $\mathcal{U}$ solves (5.4) for some driving noise ξ which can be deterministically recovered from $\mathcal{U}$ by the following argument. By (5.117) we have that

$$\begin{aligned}\mathscr{M}_t(\phi) - \mathscr{M}_s(\phi) &= \int_{\mathbb{R}} (\mathcal{U}_t(x) - \mathcal{U}_s(x))\phi(x)dx - \frac{1}{2}\int_s^t \int_{\mathbb{R}} \mathcal{U}_u(x)\phi''(x)dxdu \\ &= \sqrt{\frac{8\sigma^2}{1-4\sigma^2}} \int_s^t \mathcal{U}_u(x)\phi(x)\xi(du,dx),\end{aligned} \tag{5.126}$$

where the latter is an Ito-Walsh stochastic integral against the noise (which a priori may be defined on an enriched probability space), and the last equality is a standard result for the stochastic heat

equation (5.4), see [11]. Now we may define

$$\mathcal{B}_t(\phi) := \sqrt{\frac{1-4\sigma^2}{8\sigma^2}} \int_0^t \mathcal{U}_u(x)^{-1}\phi(x)\mathscr{M}(du,dx)$$

where the latter should be understood as an Ito-Walsh stochastic integral against the orthomartingale $\mathscr{M}$, as defined in [11]. The above stochastic integral is well-defined by strict positivity of $\mathcal{U}$ [9]. Then from Levy's criterion, it is clear that $\mathcal{B}_t$ is a standard cylindrical Wiener process (hence can be realized in $C([0,T], C^{\alpha,\tau}(\mathbb{R}))$, such that one has $\partial_s\mathcal{B} = \xi$. Thus ξ has been deterministically recovered from $\mathcal{U}$.

Define $\overline{\mathcal{W}}_t := (\sigma\sqrt{2})^{-1}\mathcal{W}_t$, which is also a standard cylindrical Wiener process. Combining (5.126) with the fact that (5.125) is a martingale, one sees that

$$\mathcal{B}_t(\phi)\overline{\mathcal{W}}_t(\phi) - (1-4\sigma^2)^{1/2}\|\phi\|^2_{L^2(\mathbb{R})}t$$

is a martingale for all $\phi \in C_c^\infty(\mathbb{R})$. By Levy's criterion, this is enough to imply that $(\mathcal{B}_t, \overline{\mathcal{W}}_t)_{t\geq 0}$ is a *jointly* Gaussian process, such that both coordinates are standard cylindrical Wiener processes, and furthermore $\mathbb{E}[\overline{\mathcal{W}}_t(\phi)\mathcal{B}_s(\phi)] = (s\wedge t)(1-4\sigma^2)^{1/2}\|\phi\|^2_{L^2}$. By elementary algebraic manipulations of Gaussian variables, we may therefore write $\mathcal{B} = (1-4\sigma^2)^{1/2}\overline{\mathcal{W}} + 2\sigma\mathcal{X}$ where $\mathcal{X} = (\mathcal{X}_t)_{t\geq 0}$ is a standard cylindrical Wiener process independent of $\overline{\mathcal{W}}$. Now the theorem is proved, setting $\xi_1 = \partial_s\overline{\mathcal{W}}$ and $\xi_2 = \partial_s\mathcal{X}$. □

With the theorem proved, we now discuss some different interpretations of the above result, which will be more heuristic than rigorous.

The term $\sqrt{\frac{32\sigma^4}{1-4\sigma^2}}\cdot\xi_2$ appearing in the preceding theorem can be viewed as the contribution which causes the chaos expansion method to fail in the introduction, as this term precisely contains the L^2-mass which escapes into the tails of the chaos expansion. Indeed one expects that this piece which escapes will decouple and become independent of the remaining noise in the limit, just as we see in Theorem 5.6.26.

Rather than "creation of independent noise," there is an alternative interpretation of the above

theorem. In the proof we showed $\mathbb{E}[\overline{\mathcal{W}}_t(\phi)\mathcal{B}_s(\phi)] = (s \wedge t)(1 - 4\sigma^2)^{1/2}(\phi, \psi)_{L^2(\mathbb{R})}$. Thus, an equivalent way of formulating Theorem 5.6.26 is that ξ_1, ξ are jointly Gaussian such that

$$\mathbb{E}\big[(\xi_1, \varphi)_{L^2(\mathbb{R}^2)}(\xi, \varphi)_{L^2(\mathbb{R}^2)}\big] = (1 - 4\sigma^2)^{1/2}\|\varphi\|^2_{L^2(\mathbb{R}^2)},$$

where ξ is the driving noise of $\mathcal{U}$ as in (5.4). The above equality strongly suggests that the rescaled density field $\mathscr{U}_N$ from (5.3) has a tendency to concentrate on certain favored sites, such that macroscopically these favored sites have Lebesgue density $1 - 4\sigma^2$ on average. In the limit, the noise ξ is generated by only the weight variables $\omega_{t,x}$ from these favored sites, in other words, the memory of only a proportion $1 - 4\sigma^2$ of the weights is remembered in the limit. The concentration of the field $\mathscr{U}_N$ inside these favored sites is the reason why we fail to see convergence in a space of continuous functions.

Thus we see that heuristically, the three pathological phenomena that we have observed in this paper all seem to be equivalent: the failure of the naive chaos expansion, the failure of Theorem 5.1.1 to be strengthened to a topology of continuous functions, and the creation of independent noise in the limiting SPDE. It may be interesting to seek a more precise statement establishing a rigorous correspondence between these phenomena.

5.7 Results for the quenched tail field

Definition 5.7.1. *Fix a realization of the environment variables ω, as in Theorem 5.1.1. For $t \in N^{-1}\mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$ we define the quenched tail field by*

$$F_N(t, x) := N^{1/4}C_{N,t,x}\mathsf{P}^{\omega}\big(R(Nt) \geq N^{3/4}t + N^{1/2}x\big).$$

The main result of this section will be that the family $\{\log F_N\}_{N\geq 1}$ of space-time processes converges to the KPZ equation, in the sense of finite-dimensional distributions of pointwise values (t, x), thus confirming the physics prediction of [85, Section 4].

Lemma 5.7.2. *For sufficiently large N and for all $t \in N^{-1}\mathbb{Z}_{\geq 0}, x \in \mathbb{R}$, we have*

$$\mathbb{E}[F_N(t,x)] \leq 4(\pi t)^{-1/2}.$$

Proof. Let $\mathbf{P} = \mathbf{P}_{RW_\nu^{(1)}}$ denote the law of a simple symmetric random walk on $\mathbb{Z}$. One takes the annealed expectation over the quenched expectation in the above definition of F_N to obtain

$$\begin{aligned}\mathbb{E}[F_N(t,x)] &= N^{1/4} C_{N,t,x} \mathbf{P}(N^{-1/2}(S(Nt) - N^{3/4}t) \geq x) \\ &= N^{1/4} \widehat{\mathbf{E}}[e^{-N^{-1/4}(S(Nt)-N^{3/4}t-N^{1/2}x)} \mathbb{1}_{\{N^{-1/2}(S(Nt)-N^{3/4}t)\geq x\}}].\end{aligned}$$

where $\widehat{\mathbf{P}}$ is a change of measure induced by the exponential martingale $C_{N,t,S(Nt)-N^{3/4}t}$, which changes the annealed law of the increments of S_i from the usual symmetric law $\frac{1}{2}(\delta_1 + \delta_{-1})$ to the new law given by $\frac{1}{2\cosh(N^{-1/4})}(e^{N^{-1/4}}\delta_1 + e^{-N^{-1/4}}\delta_{-1})$. Denote the new expectation as $\widehat{\mathbf{E}}$. To prove the inequality, we let $\widehat{p}_N(r,y) = \widehat{\mathbf{P}}(S(r) - N^{-1/4}r = y)$. Note that under the tilted measure $\widehat{\mathbf{P}}$, the increments of the random walk have mean $N^{-1/4} + O(N^{-3/4})$. Then the local central limit theorem implies that for sufficiently large N,

$$\sup_{y \in \mathbb{Z} - N^{-1/4}r} \widehat{p}_N(r,y) \leq 2(\pi r)^{-1/2}.$$

Consequently by setting $r = Nt$ we see that

$$\begin{aligned}&N^{1/4}\widehat{\mathbf{E}}[e^{-N^{-1/4}(S(Nt)-N^{3/4}t-N^{1/2}x)} \mathbb{1}_{\{N^{-1/2}(S(Nt)-N^{3/4}t)\geq x\}}] \\ &= N^{1/4} \sum_{a \geq N^{1/2}x} \widehat{p}_N(Nt,a) e^{-N^{-1/4}(a-N^{1/2}x)} \\ &\leq 2(\pi Nt)^{-1/2} N^{1/4} \sum_{a \geq N^{1/2}x} e^{-N^{-1/4}(a-N^{1/2}x)} \\ &\leq 2(\pi t)^{-1/2} N^{-1/4} \sum_{b \geq 0} e^{-N^{-1/4}b} = 2(\pi t)^{-1/2} \frac{N^{-1/4}}{1 - e^{-N^{-1/4}}},\end{aligned}$$

where the sums over a should be understood as being over $a \in \mathbb{Z} - N^{-1/4}Nt$ with $a \geq N^{1/2}x$.

The last quotient is always bounded above by 2, thus proving the claim. □

Proposition 5.7.3. *Fix $m \in \mathbb{N}$, and $\phi_1, \dots, \phi_m \in C_c^\infty(\mathbb{R})$. Consider sequences $t_{N,1}, \dots, t_{N,m} \in N^{-1}\mathbb{Z}_{\geq 0}$ such that $t_{N,i} \to t_i > 0$ for all $1 \leq i \leq m$ as $N \to \infty$. Then*

$$\left(\int_{\mathbb{R}} \phi_i(x) F_N(t_{N,i}, x) dx \right)_{i=1}^m \xrightarrow{d} \left(\int_{\mathbb{R}} \phi_i(x) \mathcal{U}_{t_i}(x) dx \right)_{i=1}^m. \tag{5.127}$$

Here $(t, x) \mapsto \mathcal{U}_t(x)$ is the solution of (5.4)*.*

Proof. For $t \in N^{-1}\mathbb{Z}_{\geq 0}$ let us define a family of measures on $\mathbb{R}$ given by

$$\mu_{N,t} := e^{(N^{1/2} - N \log\cosh(N^{-1/4}))t} \sum_{x \in N^{-1/2}\mathbb{Z} - N^{1/4}t} \mathsf{P}^\omega(Nt, N^{3/4}t + N^{1/2}x) \cdot \delta_x,$$

Then we notice that Definition 5.7.1 is equivalent to

$$F_N(t, x) := N^{1/4} e^{N^{1/4}x} \mu_{N,t}[x, \infty).$$

An application of Fubini's theorem gives

$$\int_{-\infty}^x N^{1/4} e^{N^{1/4}u} \mu_{N,t}[u, \infty) du = \int_{-\infty}^x e^{N^{1/4}u} \mu_{N,t}(du) + e^{N^{1/4}x} \mu_{N,t}[x, \infty).$$

In the sense of distributions, it is clear that

$$F_N(t, x) = \partial_x \left[\int_{-\infty}^x N^{1/4} e^{N^{1/4}u} \mu_{N,t}[u, \infty) du \right], \quad e^{N^{1/4}x} \mu_{N,t}(dx) = \partial_x \left[\int_{-\infty}^x e^{N^{1/4}u} \mu_{N,t}(du) \right].$$

Consequently, for all $t \in N^{-1}\mathbb{Z}_{\geq 0}$ and $\phi \in C_c^\infty(\mathbb{R})$, we can repeatedly integrate by parts to obtain

that

$$\begin{aligned}\int_{\mathbb{R}} \phi(x) F_N(t,x)dx &= -\int_{\mathbb{R}} \phi'(x)\bigg[\int_{-\infty}^{x} N^{1/4} e^{N^{1/4}u}\mu_{N,t}[u,\infty)du\bigg]dx \\ &= -\int_{\mathbb{R}} \phi'(x)\bigg[\int_{-\infty}^{x} e^{N^{1/4}u}\mu_{N,t}(du) + e^{N^{1/4}x}\mu_{N,t}[x,\infty)\bigg]dx \\ &= \int_{\mathbb{R}} \phi(x) e^{N^{1/4}x}\mu_{N,t}(dx) - \int_{\mathbb{R}} \phi'(x) e^{N^{1/4}x}\mu_{N,t}[x,\infty)dx.\end{aligned}$$

Now let us prove the proposition. In the setting of the proposition statement, by Theorem 5.1.1 we know that as $N \to \infty$

$$\bigg(\int_{\mathbb{R}} \phi_i(x) e^{N^{1/4}x}\mu_{N,t_{N,i}}(dx)\bigg)_{i=1}^{m} \xrightarrow{d} \bigg(\int_{\mathbb{R}} \phi_i(x)\mathcal{U}_{t_i}(x)dx\bigg)_{i=1}^{m}.$$

Thus, to show (5.127) it suffices to show that as $N \to \infty$, $\int_{\mathbb{R}} \phi_i'(x) e^{N^{1/4}x}\mu_{N,t_{N,i}}[x,\infty)dx$ goes to zero in $L^1(\mathbb{P})$ for each $1 \le i \le m$. Indeed, observe that

$$\mathbb{E}\bigg|\int_{\mathbb{R}} \phi_i'(x) e^{N^{1/4}x}\mu_{N,t_{N,i}}[x,\infty)dx\bigg| \le \int_{\mathbb{R}} |\phi_i'(x)| e^{N^{1/4}x}\mathbb{E}[\mu_{N,t_{N,i}}[x,\infty)]dx.$$

By Lemma 5.7.2 we get that

$$e^{N^{1/4}x}\mathbb{E}[\mu_{N,t_{N,i}}[x,\infty)] \le 4N^{-1/4}(\pi t_{N,i})^{-1/2}$$

for all $x \in \mathbb{R}$. Since the $t_{N,i}$s remain bounded away from 0 as $N \to \infty$, this implies the desired L^1 convergence and completes the proof. □

Our next lemma shows uniform convergence of the two-point correlation function of the quenched tail field. This is the crucial result that allows us to improve Proposition 5.7.3 to obtain multipoint convergence in law of $F_N(t,x)$ to $\mathcal{U}_t(x)$ for *individual* values of $(t,x) \in (0,\infty) \times \mathbb{R}$.

Lemma 5.7.4. *Suppose that $t_N \in N^{-1}\mathbb{Z}_{\ge 0}$ and $x_N, y_N \in \mathbb{R}$. Assume that as $N \to \infty$ we have*

$t_N \to t > 0$ *and* $(x_N, y_N) \to (x, y) \in \mathbb{R}^2$. *Then*

$$\lim_{N\to\infty} \mathbb{E}[F_N(t_N, x_N)F_N(t_N, y_N)] = \mathbb{E}[\mathcal{U}_t(x)\mathcal{U}_t(y)]$$

where $\mathcal{U}$ *is as in* (5.4).

Proof. By taking an annealed expectation over the quenched one, notice that the two-point correlation function given by $(t, x, y) \mapsto \mathbb{E}[F_N(t, x)F_N(t, y)]$ can be written as an expression involving only the 2-point motion (R^1, R^2). More specifically, in the notation of Section 5.2.1, it equals

$$Ne^{2(N^{1/2}-N\log\cosh(N^{-1/4}))t}\mathbf{P}_{RW_\nu^{(2)}}(N^{-1/2}(R^1(Nt) - N^{3/4}t) \geq x, N^{-1/2}(R^2(Nt) - N^{3/4}t) \geq y).$$

Consequently the two-point correlation function only depends on the mean and variance of the weight measure ν. Without any loss of generality we shall henceforth assume ν is the law of Beta(α, α) where $\alpha := \frac{1-4\sigma^2}{8\sigma^2}$.

We shall now invoke results from [73] which are for the Beta measure. In [73], the authors consider a slightly different field $\widehat{F}_N(t, x)$ which is "dual" to ours in a certain sense. More specifically, they vary the starting point x and fix the tail probability as $[0, \infty)$, whereas we fix the starting point 0 and vary the tail probability as $[x, \infty)$. One may show that the distributions of both fields are the same as space-time processes, modulo a reflection of the weight measure. More precisely, we have $F_N^\nu \stackrel{d}{=} \widehat{F}_N^\mu$, where μ is the pushforward of ν by $x \mapsto 1 - x$, and the superscript highlights the dependence of each field on the underlying weight measure.[1] This is proved using an explicit coupling of the two fields via the "discrete Brownian web". See [80, Equation (1.8)] for a discussion of the coupling construction, which is in turn based on [190, Section 3].

Note that for the case of Beta(α, α) we have $\mu = \nu$. Proposition 3.4 in [73] provides exact moment formulas for the unscaled version of $F_N(t, x)$ in this case. Taking the scalings into account

[1] In particular, our results imply the space-time multi-point convergence in law of their field $\widehat{F}_N$ to (5.4) as well.

(see e.g. [85, (34) and (40)-(43)]) we then have for all $t \in N^{-1}\mathbb{Z}_{\geq 0}$ and $x_1, x_2 \in N^{-1/2}(\mathbb{Z} - N^{3/4}t)$

$$\mathbb{E}[F_N(t,x_1)F_N(t,x_2)]$$
$$= N^{1/2} \oint_{\gamma_1} \oint_{\gamma_2} \frac{z_1 - z_2}{z_1 - z_2 - 1} \prod_{j=1}^{2} C_{N,t,x_j} \left(\frac{2\alpha + z_j}{z_j}\right)^{\frac{1}{2}(Nt - N^{\frac{3}{4}}t - x_j N^{\frac{1}{2}})+1} \left(\frac{\alpha + z_j}{2\alpha + z_j}\right)^{Nt} \frac{1}{2\alpha + z_j} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i},$$

where γ_1, γ_2 are positively oriented closed curves around 0 such that γ_1 contains $\gamma_2 + 1$ and both contours exclude -2α.

Now assume that we have a sequence (t_N, x_N, y_N) converging to $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$. We claim that the integral expression for $\mathbb{E}[F_N(t_N, x_N)F_N(t_N, y_N)]$ converges as $N \to \infty$ to

$$\alpha^{-2} \oint_{r_1 + i\mathbb{R}} \oint_{r_2 + i\mathbb{R}} \frac{z_2 - z_1}{z_2 - z_1 - 1} e^{\frac{t}{2\alpha^2}(z_1^2 + z_2^2) + \frac{1}{\alpha}(xz_1 + yz_2)} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i}.$$

Here r_1, r_2 are chosen such that $r_2 > r_1 + 1$. This convergence can be justified by dominated convergence and Taylor expansions, see [232, Proposition 5.4.2] for a similar argument. This is known to agree with $\mathbf{E}[\mathcal{U}_t(x)\mathcal{U}_t(y)]$, see [232, Section 6.2], thus proving the lemma. □

With the above lemma in place, we prove our main convergence result for the quenched tail field: Theorem 5.1.2.

Proof of Theorem 5.1.2. We first prove the case when $x_{N,i} = x_i$ for $1 \leq i \leq m$. We give a proof for $m = 1$ to simplify the notation, but the generalization to larger m is straightforward. We will simply write $(t_{N,1}, x_1)$ as (t_N, x), which will be fixed for the moment being. By Lemma 5.7.2 we get that $\mathbb{E}[F_N(t_N, x)] \leq 2(\pi t_N)^{-1/2}$ which is bounded independently of N, thus it follows that the $\{F_N(t_N, x)\}_{N \geq 1}$ are tight. Consider any limit point μ_0 of the laws of $\{F_N(t_N, x)\}_{N \geq 1}$. Fix a smooth compactly supported nonnegative function $\phi : \mathbb{R} \to \mathbb{R}$ which integrates to 1, and define $\phi_x^\lambda(y) = \lambda^{-1}\phi(\lambda^{-1}(y - x))$. By Proposition 5.7.3 we know that for each $\lambda, x \in \mathbb{R}$, $\left\{ \int_{\mathbb{R}} \phi_x^\lambda(y) F_N(t_N, y) dy \right\}_{N \geq 1}$ is a tight sequence. For $r \geq 1$, consider any measure μ_r on $\mathbb{R}^{r+1}$

which is a joint limit point as $N \to \infty$ of

$$\Big(F_N(t_N, x) \ , \ \big(\int_{\mathbb{R}} \phi_x^{2^{-k}}(y) F_N(t_N, y) dy \big)_{k=1}^r \Big),$$

such that the first marginal of μ_r is μ_0. Using a diagonalization argument, these measures μ_r may be chosen so that they form a projective family, and therefore by the Kolmogorov extension theorem, we may consider any projective limit μ as $r \to \infty$, which will be a measure on the space of sequences $(a_k)_{k \geq 0} \in \mathbb{R}^{\mathbb{Z}_{\geq 0}}$, equipped with the σ-algebra generated by the projection maps. By Proposition 5.7.3, we find that for such a measure μ the marginal distribution of $(a_k)_{k \geq 1}$ is simply equal to the law of $\big(\int_{\mathbb{R}} \mathcal{U}_t(y) \phi_x^{2^{-k}}(y) dy \big)_{k \geq 1}$. We now claim that

$$\limsup_{\lambda \to 0} \limsup_{N \to \infty} \mathbb{E} \Big| F_N(t_N, x) - \int_{\mathbb{R}} F_N(t_N, y) \phi_x^{\lambda}(y) dy \Big| = 0. \tag{5.128}$$

Assuming this fact, one finds that such a measure μ is necessarily supported on those sequences $(a_k)_{k \geq 0}$ which satisfy $a_0 = \lim_{k \to \infty} a_k$ in $L^1(\mu)$, which means that a_0 must indeed have the law of $\mathcal{U}_t(x)$ under μ, proving the lemma. We are thus left to check (5.128). By Lemma 5.7.4, we have that

$$\limsup_{N \to \infty} \mathbb{E}[(F_N(t, x) - F_N(t, y))^2] = \mathbf{E}[(\mathcal{U}_t(x) - \mathcal{U}_t(y))^2] \leq \mathrm{C}|x - y|$$

uniformly over compacts sets of $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$ where the above inequality is a known estimate for (5.4) (see [233, Proposition 2.4-nw] for example). Thus, by Fatou's lemma, we note that

$$\begin{aligned}
\limsup_{N \to \infty} \mathbb{E} \Big| F_N(t_N, x) - \int_{\mathbb{R}} F_N(t_N, y) \phi_x^{\lambda}(y) dy \Big| &\leq \limsup_{N \to \infty} \int_{\mathbb{R}} \mathbb{E} |F_N(t_N, x) - F_N(t_N, y)| \phi_x^{\lambda}(y) dy \\
&\leq \int_{\mathbb{R}} \limsup_{N \to \infty} \mathbb{E} |F_N(t_N, x) - F_N(t_N, y)| \phi_x^{\lambda}(y) dy \\
&\leq \int_{\mathbb{R}} \limsup_{N \to \infty} \mathbb{E}[(F_N(t_N, x) - F_N(t_N, y))^2]^{1/2} \phi_x^{\lambda}(y) dy \\
&\leq \mathrm{C} \int_{\mathbb{R}} |x - y|^{1/2} \phi_x^{\lambda}(y) dy = \mathrm{C} \lambda^{1/2} \int_{\mathbb{R}} |u|^{1/2} \phi(u) du.
\end{aligned}$$

Taking $\lambda \to 0$ leads to (5.128). To justify the above application of Fatou's lemma with $\limsup$, one needs a bound on the y-integrand which is independent of N and is in $L^1(\phi_x^\lambda(y)dy)$. For this we may use Lemma 5.7.2 to get

$$\mathbb{E}|F_N(t_N, x) - F_N(t_N, y)| \le \mathbb{E}[F_N(t_N, x) + F_N(t_N, y)] \le 2(2\pi t_N)^{-1/2}.$$

The right side may be bounded by a constant independent of N since $t_N \to t > 0$. Finally, the case when $x_{N,i}$ varies with N follows from the fact that as $N \to \infty$ we have

$$\mathbb{E}\big[\big(F_N(t_{N,i}, x_{N,i}) - F_N(t_{N,i}, x_i)\big)^2\big] \to 0$$

due to the uniform convergence in Lemma 5.7.4. This completes the proof. □

Theorem 5.7.5 (Extremal particle limit theorem)**.** *Fix $c, t > 0$ and $d \in \mathbb{R}$. Let $(R^1(r), \dots, R^k(r))_{r\ge 0}$ denote the canonical process sampled from the measure $\mathbf{P}_{RW_\nu^{(k)}}$. Set the number of particles $k = k(N) := \lfloor \exp(\frac{1}{2}cN^{1/2} + dN^{1/4} + r_N)\rfloor$ where r_N can be any sequence satisfying $r_N = o(N^{1/4})$. Consider any sequence $t_N \in N^{-1}\mathbb{Z}_{\ge 0}$ such that $t_N \to t > 0$ as $N \to \infty$. Then*

$$\max_{1\le i\le k(N)} \big\{N^{-\frac{1}{4}} R^i(Nt_N)\big\} - a_N(c, d, t_N) \xrightarrow{d} \sqrt{\tfrac{t}{c}}\big(G + \log \mathcal{U}_c(d)\big),$$

where

$$a_N(c, d, s) := \sqrt{csN} - dN^{\frac{1}{4}}\sqrt{\tfrac{s}{c}} - \sqrt{\tfrac{c}{s}}\big(r_N - \tfrac{1}{4}\log N\big) - \frac{c^{3/2}}{6\sqrt{s}}.$$

Here G is a standard Gumbel random variable which is independent of $\mathcal{U}$, and $(t, x) \mapsto \mathcal{U}_t(x)$ solves (5.4) *but with noise coefficient replaced by $\sqrt{\frac{8\sigma^2 c}{(1-4\sigma^2)t}}$.*

Proof. Fix any $a \in \mathbb{R}$. Set $v_N := \sqrt{ct_N}$ and $x_N := d\sqrt{\frac{t_N}{c}}$ so that $c = \frac{v_N^2}{t_N}$ and $d = v_N x_N / t_N$. Also

define $v = \sqrt{ct}$ and $x = d\sqrt{\frac{t}{c}}$. It suffices to show

$$\lim_{N\to\infty} \mathbf{P}_{RW_\nu^{(k(N))}}\left(\max_{1\le i\le k(N)} \left\{R^i(Nt_N)\right\} - v_N N^{3/4} - x_N\sqrt{N} - \tfrac{tN^{1/4}}{v_N}\left(r_N - \tfrac{1}{4}\log N\right) \le aN^{1/4}\right) = \mathbb{P}\left(t/v[G + \log \mathcal{U}_{v^2/t}(vx/t)] \le a + \frac{v^3}{6t^2}\right), \tag{5.129}$$

where $\mathcal{U}$ solves (5.4) with noise coefficient replaced by $\frac{v}{t}\sqrt{\frac{8\sigma^2}{1-4\sigma^2}}$, and G is an independent Gumbel random variable.

For now, fix any $k \in \mathbb{N}$ which may be arbitrary. Fix a realization of the environment $\omega = \{\omega_{i,j} : (i,j) \in \mathbb{Z}^2\}$. As before, we let $\mathsf{P}^\omega_{(k)}$ be the quenched probability of k random motions $(R^i(r))_{i=1}^k$ sampled given the environment. Let us consider the random variable

$$\mathsf{P}^\omega_{(k)}\left(\max_{1\le i\le k} \left\{R^i(Nt_N)\right\} \le v_N N^{3/4} + x_N\sqrt{N} + \frac{t_N}{v_N} N^{1/4}(r_N - \frac{1}{4}\log N) + aN^{1/4}\right). \tag{5.130}$$

Notice that until this point, we have assumed in Theorem 5.1.1 that the parameter N is a *discrete* parameter taking values in $\mathbb{N}$. However nothing in our proof actually used this discreteness, and in fact, there is no loss of generality in assuming it is actually a *continuous* parameter indexed by $[1,\infty)$ for instance. In this context, the results of Theorem 5.1.1 and therefore Theorem 5.1.2 still hold as $N \to \infty$.

Now we define $\mathfrak{N} = \mathfrak{N}(N) := N\left(\frac{t_N}{v_N}\right)^4$, so that $\mathfrak{N} \to \infty$ is equivalent to $N \to \infty$. Notice by Definition 5.7.1 that (5.130) is pathwise equal to the quantity

$$\left(1 - \mathfrak{N}^{-1/4} C^{-1}_{\mathfrak{N}, \frac{v_N^4}{t_N^3}, \left(\frac{v_N}{t_N}\right)^2 x_N + \mathfrak{N}^{-\frac{1}{4}}\left(\frac{av_N}{t_N} + r_N - \frac{1}{4}\log N\right)} F_{\mathfrak{N}}\left(\tfrac{v_N^4}{t_N^3},\ \left(\tfrac{v_N}{t_N}\right)^2 x_N + \mathfrak{N}^{-\frac{1}{4}}\left(\tfrac{av_N}{t_N} + r_N - \tfrac{1}{4}\log N\right)\right)\right)^k. \tag{5.131}$$

Then we may choose

$$k = k(N) = \Big\lfloor \mathfrak{N}^{1/4} C_{\mathfrak{N}, \frac{v_N^4}{t_N^3}, \left(\frac{v_N}{t_N}\right)^2 x_N + \mathfrak{N}^{-\frac{1}{4}}(\frac{av_N}{t_N} + r_N - \frac{1}{4}\log N))} \exp\Big[- \frac{v_N^4}{6t_N^3} - a\frac{v_N}{t_N} - \log(t_N/v_N)\Big]\Big\rfloor$$
$$= \Big\lfloor \exp\Big[\frac{v_N^2}{2t_N}\sqrt{N} + \frac{v_N x_N}{t_N} N^{1/4} + r_N + O(N^{-1/2})\Big]\Big\rfloor$$

where we applied a fourth order Taylor expansion of $\log\cosh$ in order to deduce the $O(N^{-1/2})$ term. Note that the latter expression agrees with the choice of $k(N)$ in the theorem statement. Then (5.131) is a quantity of the form $(1 - \frac{u(N)}{k(N)+O_N(1)})^{k(N)}$, where the $O_N(1)$ term is deterministic and bounded between 0 and 1, and $u(N)$ is a sequence of random variables which by Proposition 5.1.2 converges in law to the strictly positive random variable $\frac{v}{t} e^{-\frac{av}{t} - \frac{v^4}{6t^3}} \mathcal{U}_{v^4/t^3}(v^2 x/t^2)$ with $\mathcal{U}$ solving (5.4). The latter has the same law as $e^{-\frac{av}{t} - \frac{v^4}{6t^3}} \mathcal{U}_{v^2/t}(vx/t)$ where now the noise coefficient in (5.4) has been replaced by $\frac{v}{t}\sqrt{\frac{8\sigma^2}{1-4\sigma^2}}$. Since the functions $u \mapsto (1 - \frac{u}{k(N)+O_N(1)})^{k(N)}$ converge uniformly to $u \mapsto e^{-u}$ on compact subsets of $[0, \infty)$, we can conclude that the expression in (5.131) therefore converges in law to the random variable

$$\exp\big[- e^{-\frac{av}{t} - \frac{v^4}{6t^3}} \mathcal{U}_{v^2/t}(vx/t)\big],$$

where $\mathcal{U}$ solves (5.4) with noise coefficient replaced by $\frac{v}{t}\sqrt{\frac{8\sigma^2}{1-4\sigma^2}}$. Now this convergence in law also implies convergence of the associated annealed expectations thanks to the boundedness of all quantities involved, which implies that the limit of the left side of (5.129) equals

$$\mathbb{E}\Big[- e^{-\frac{av}{t} - \frac{v^4}{6t^3}} \mathcal{U}_{v^2/t}(vx/t)\Big],$$

which agrees with the right-hand side of (5.129) by the explicit form of the Gumbel distribution.

□

Note that in the above theorem, a natural choice is $t_N := N^{-1}\lfloor Nt \rfloor$ where $t > 0$ is fixed. However, we also have the liberty to set $t_N := N^{-1}\lfloor Nt + N^\alpha \rfloor$, where $\alpha \in (0, 1)$, and this would

have a nontrivial effect in the recentering coefficients $a_N(c, d, t_N)$ if $\alpha \in [1/2, 1)$.

Glossary

Processes used in text		
$\mathcal{H}$	KPZ equation	Eq. (KPZ)
$\mathcal{U}$	Stochastic heat equation (SHE)	Eq. (SHE)
$\mathsf{P}^\omega(t, x)$	Quenched transition probability of the random walk	Eq. (5.1)
$C_{N,t,x}$	Rescaling constants for the moderate deviation scaling	Eq. (5.2)
$\mathbf{R}$	The canonical process on $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$	Sec. 5.2.1
V^{ij}	Intersection process of i^{th} and j^{th} coordinate processes	Eq. (5.9)
$\Gamma^{(v)}$	Set of times with v distinct particles	Eq. (5.8)
$\mathcal{M}$	Martingale in the Tanaka formula for 2-point motion	Eq. (5.10)
$\mathcal{M}^\lambda$	Exponential martingale used to tilt path measure $\mathbf{P}_{RW_\nu^{(k)}}$	Eq. (5.13)
$M_N(t, \phi)$	Martingales arising in discrete SHE	Eq. (5.63)
$\mathscr{U}_N(t, \phi)$	Rescaled field converging to $\mathcal{U}$ in main results	Eq. (5.3)
$Q_N(t, \phi)$	Quadratic martingale field	Eq. (5.68)
Ξ_N	Prelimiting noise field	Eq. (5.5)
$\mathcal{W}_N$	Time-integrated and linearly interpolated version of Ξ_N	Eq. (5.123)
Probability measures and filtrations		
ν	Probability measure on $[0, 1]$ representing law of each $\omega_{t,x}$	Sec. 5.2.1
μ_ν	Mean of ν	Sec. 5.2.1
σ_ν^2	Variance of ν	Sec. 5.2.1

$\mathbf{P}_{RW_\nu^{(k)}}$	Probability measure on the path space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$, used to denote the annealed law of k independent walks sampled from the environment ω whose weights are distributed as ν, or more precisely $\mathbf{P}_{RW_\nu^{(k)}} = \mathbb{P}\mathsf{P}_{(k)}^{\omega}$	Def. 5.2.2
$\mathsf{P}_{(k)}^{\omega}$	Probability measure on the path space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$, used to denote the quenched law of k independent walks sampled from the fixed realization of environment $\omega = \{\omega_{t,x}\}_{t,x}$	
$\mathbb{P}$	Probability measure on the environment space $[0,1]^{\mathbb{Z}_{\geq 0}\times\mathbb{Z}}$, used to denote the annealed probability associated to any observable on the probability space of the environment $\{\omega_{t,x}\}_{t,x}$	
$\mathbf{P}_{\text{lim}}$ and $\mathbf{P}_\infty$	Notation used for limit points of objects on the canonical space of continuous paths (or tuples of continuous paths)	
$(\mathcal{F}_r)_{r\in\mathbb{Z}_{\geq 0}}$	Canonical filtration on the path space $(\mathbb{Z}^k)^{\mathbb{Z}_{\geq 0}}$	Def. 5.2.4
$(\mathcal{F}_r^{\omega})_{r\in\mathbb{Z}_{\geq 0}}$	Canonical filtration on the environment space $[0,1]^{\mathbb{Z}_{\geq 0}\times\mathbb{Z}}$	Eq. (5.61)

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