On some semi-linear equations related to phase transitions: Rigidity of global solutions and regularity of free boundaries

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Abstract

In this thesis, we study minimizers of the energy functional

$$J(u,\Omega)=\int_{\Omega}\frac{|\nabla u|^2}{2}+W(u)dx$$

for two different potentials $W(u)$.

In the first part we consider the Allen-Cahn energy, where $W(u)=(1-u^2)^2$ is a double well potential which is relevant in the theory of phase transitions and minimal interfaces. We investigate the rigidity properties of global minimizers in low dimensions. In particular we extend a result of Savin on the De Giorgi's conjecture to include minimizers that are not necessarily bounded, and that can have subquadratic growth at infinity.

In the second part we consider potentials of the type $W(u)=u^+$ which appear in obstacle type free boundary problems. We establish higher order estimates and the analyticity of the regular part of the free boundary. Our method relies on developing higher order boundary Harnack estimates iteratively and deducing them from Schauder estimates for certain elliptic equations with degenerate weights.

Finally we consider similar regularity questions of the free boundary in the Signorini problem which also known as the thin obstacle problem. We develop $C^{2,\alpha}$ estimates of the free boundary under sharp assumptions on the coefficients and the data.

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Chapter 1: Introduction and Background

We study minimizers of the energy functional

$$J(u, \Omega) = \int_{\Omega} \frac{|\nabla u|^2}{2} + W(u) dx \tag{1.0.1}$$

with some different examples of potentials $W(u)$. A minimizer satisfies the Euler-Lagrange equation

$$\Delta u = W'(u)$$

in the classical sense provided that W is of class $C^{1,\alpha}$.

First, we consider the Allen-Cahn equation, where $W(s) = (1 - s^2)^2$ is a positive double well potential which is extended to zero outside the interval $[-1, 1]$. We investigate the rigidity properties of global minimizers in low dimensions. In particular we extend a result of Savin on the De Giorgi's conjecture to include minimizers that are not necessarily bounded, and that can have subquadratic growth at infinity.

Next, we consider the classical obstacle problem, where $u \geq 0$ and the potential $W(s) = s^+$. We use the higher order boundary Harnack principle to show the analyticity of the free boundary $\{u > 0\}$ near a regular boundary point.

Finally, we use a similar $C^{1,\alpha}$ boundary Harnack principle as in the classical obstacle problem, in order to study the $C^{2,\alpha}$ regularity of the free boundary in a similar obstacle problem called the Signorini problem. The difference in the Signorini problem is that the obstacle is only defined on a hypersurface.

1.1 The Allen-Cahn equation

First we consider the Ginzburg-Landau energy

$$J_{GL}(u,\Omega)=\int_{\Omega}\frac{|\nabla u|^2}{2}+W(u)dx,\quad W(s)=(1-s^2)^2. \tag{1.1.1}$$

which is used to model phase transitions [1, 2]. Here, u is a scalar function that represents some density, and its stable states are ± 1.

We briefly explain the physical motivation of the Ginzburg-Landau theory. Imagine that we have a sufficiently large lattice (for example $\mathbb{Z}^2$) of particles. Each particle could be viewed as a magnetic dipole due to its self spin. Let $u(x)$ be the average magnetic moment density averaged over atomic distance. The Gibbs free energy can be approximately written as an integral:

$$G\approx\int\frac{|\nabla u|^2}{2}+W(u)dx.$$

Here the potential $W(u)=a-bu^2+cu^4$ is assumed to be a polynomial, whose coefficients are depending on some physical parameters like the temperature. In particular, when the temperature T is below the material's critical temperature T_c, then $b,c>0$.

After a rescaling W has the form as in (1.1.1). The phase field region, where $|u|\leq 1-\epsilon$, describes the phase transition region between steady states.

Let $u:\mathbb{R}^n\to[-1,1]$ be a global minimizer of (1.1.1) (i.e. u is a minimizer of each ball B_R). We denote the zero-homogeneous blow-down of u that rescales B_R to B_1 as

$$u_R(x):=u(Rx),$$

which minimizes the corresponding rescaled functional

$$\int\frac{|\nabla u|^2}{2R}+RW(u)dx. \tag{1.1.2}$$

The well-known Γ-convergence result of Modica and Mortola [3] states that as $R \to \infty$, u_R has a subsequence that L^1_{loc}-converges to a limiting function of the form

$$u_\infty = \chi_{E^c} - \chi_E,$$

where E is a set with minimal perimeter. The connection between the level sets of u at large scales and minimal surfaces is a subject of great interest, which was intensively studied in the literature, see [4, 5, 6, 7, 8, 9]. Key differences between the two problems are the invariance under dilations of minimal surfaces, and the local smoothing properties of minimizers of J.

A natural question, which is referred to as the De Giorgi conjecture, asks whether or not a global minimizer of the functional J (or global solution to its Euler-Lagrange equation) has the same rigidity as its rescaled limit, a global minimal surface. Since when $n \leq 7$, global minimal surfaces are hyperplanes (see Simons [10]), the conjecture states that level sets of a global minimizer of (1.1.1) are hyperplanes as well in $n \leq 7$ (or in other words, the minimizer is one-dimensional).

In [11], Savin gave a positive answer to this question, that a function $u : \mathbb{R}^n \to [-1, 1]$ is one-dimensional if

- Either: $n \leq 7$ and u is a global minimizer of $\displaystyle\int \frac{|\nabla u|^2}{2} + (1 - u^2)^2 dx$,
- Or: $n \leq 8$ and u is a solution of the Euler-Lagrange equation $\Delta u = W'(u)$, so that $\partial_{x_n} u \geq 0$ and $\{u = 0\}$ is a graph in the x_n-direction over $\mathbb{R}^{n-1}$.

We remark that the original formulation of the De Giorgi conjecture in [12] was stated only for bounded monotone solutions of $\Delta u = W'(u)$ without the assumption that $\{u = 0\}$ is a graph over the whole $\mathbb{R}^{n-1}$. Under these weaker hypotheses the conjecture is known to be true only in dimensions 2 and 3 by the works of Ghoussoub-Gui [8], Ambrosio-Cabre [7], and to be false in dimension $n \geq 9$ by a counterexample due to Del Pino, Kowalczyk and Wei [13].

In this thesis, we focus on global minimizers of (1.1.1). We relax the boundedness assumption $u \in [-1, 1]$, and extend the results in Savin [11] to minimizers with sub-quadratic growth at

infinity. The double well potential is extended trivially outside the interval $[-1, 1]$. That is

$$W(s) = (1 - s^2)^2 \cdot \chi_{[-1,1]}. \tag{1.1.3}$$

Before stating the main result, we make precise the notion of global minimizers.

Definition 1.1.1. *We say $u : \mathbb{R}^n \to \mathbb{R}$ is a global minimizers of the double well energy*

$$J(u, \Omega) = \int_\Omega \frac{|\nabla u|^2}{2} + W(u)dx, \tag{1.1.4}$$

if for any B_R, u is J-minimizing subject to the fixed boundary value $u\Big|_{\partial B_R}$.

We should notice that such a potential $W(s)$ can be viewed as an approximation of the δ-function centered at the origin. To see this, let $u_M(x) = u(x)/M$, then $u_M(x)$ is a minimizer of

$$\int \frac{|\nabla u_M|^2}{2} + W_M(u_M)dx, \quad W_M(s) = M \cdot W(M \cdot s).$$

As $M \to \infty$, the phase field region corresponds to the region where $u_M \approx 0$, and the potential W_M converges to a δ-function. Heuristically speaking, sudden change are expected to occur near the zero-set of u_M. In fact, in section 2, we will relate the minimizer $u(x)$ to the solutions of a well-known free boundary problem studied by Athanasopoulos, Caffarelli, Kenig and Salsa in [14].

Our main results are the following Liouville type theorems:

Theorem 1.1.2. *Let $u(x) : \mathbb{R}^n \to \mathbb{R}$ be a global minimizer of* (1.1.4) *with potential $W(s)$ as in 1.1.3. If $\max_{B_R} |u| = o(R^2)$, then u is one dimensional if $n \leq 7$. More precisely, after a Euclidean group action on $\mathbb{R}^n$, $u(x) = u(x_n)$.*

Theorem 1.1.3. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a global solution of $\Delta u = W'(u)$ with $W(s)$ as in 1.1.3. If $\partial_{x_n} u \geq 0$ and its zero set $\{u = 0\}$ is a graph of the first $(n-1)$ coordinates, then u is one-dimensional for $n \leq 8$.*

We will show that solutions satisfying the assumptions of Theorem 1.1.3 are in fact global minimizers, and we can deduce Theorem 1.1.3 from Theorem 1.1.2.

Since u is not necessarily bounded by ± 1, we divide minimizers into three categories according to their range. In the definition below, we borrow the terminology used in free boundary problems. We view the phase field region $\{|u| \leq 1\}$ as "the free boundary", the region $\{u > 1\}$ as "the positive phase", and $\{u < -1\}$ as "the negative phase".

Definition 1.1.4. *We say that a function u is*

- *zero-phase: if $|u| \leq 1$ everywhere,*
- *one-phase: if one of $\{u > 1\}$ and $\{u < -1\}$ is empty while the other is not,*
- *two-phase: if $\{u > 1\}$ and $\{u < -1\}$ are both non-empty.*

Since from Savin [11], we know that global zero-phase minimizers must be one dimensional when $n \leq 7$, we need to focus only on one-phase and two-phase minimizers and show that:

- One-phase minimizers don't exist when $n \leq 7$,
- Two-phase sub-quadratic minimizers are one-dimensional ODE solutions (no constrain on the dimension n).

The rescaling problems in the one-phase and the two-phase case are different. In the one-phase case we study rescaled functions

$$u_R(y) = \frac{u(Ry)}{\sqrt{R}},$$

while in the two-phase case we study rescaled functions

$$u_R(y) = \frac{u(Ry)}{\max_{B_R} |u|}.$$

We will perform the method of Γ-convergence as in Modica and Mortola [3], and show that rescaled functions u_R converge up to a subsequence as $R \to \infty$ to

- for one-phase case: a minimizer of the Dirichlet/perimeter functional

$$I(u,\Omega)=\int_{\Omega}\frac{|\nabla u|^2}{2}dx+\Lambda\cdot Per(u>0,\Omega)$$

  studied in Athanasopoulos-Caffarelli-Kenig-Salsa [14],

- for two-phase case: a harmonic function.

In what follows we refer to the functional $I(u,\Omega)$ as the ACKS functional.

It is yet unknown whether or not global minimizers of J with higher growth exist, especially in low dimensions. The difficulty lies on that global minimizers with at least quadratic growth, if exist, will have complicated (like self-intersection) zero-sets.

In Chapter 2 we study properties of one-phase global minimizers and show that they do not exist in $n\leq 7$. In Chapter 3 we study two-phase global minimizers and show that they are one-dimensional. At the end we give a proof of Theorem 1.1.3 concerning monotone solutions.

De Giorgi type conjectures can be posed for other energy minimizing problems. The question is whether or not a global minimizer of a generic functional J has the same rigidity as the limit of its rescaled function (such as $u(Ry)$, $\frac{u(Ry)}{\max_{B_R}|u|}$, $\frac{u(Ry)}{R^{\alpha}}$ etc.). For example in [15] Audrito and Serra studied minimizers of (1.1.4), with potential $W(s)$ being a smooth approximation of the heaviside function $\chi_{\{s>0\}}$. Its corresponding limiting problem is the Bernoulli free boundary problem (also called the one-phase problem), associated with the Alt-Caffarelli functional (see [7]):

$$\int\frac{|\nabla u|^2}{2}+\chi_{\{u>0\}}dx.$$

A natural follow-up problem related to this thesis is to extend Theorem 1.1.2 and 1.1.3 to potentials $W(s)$ with “power-decay tails", for example

$$W(s)=(1+s^2)^{-\gamma/2}.$$

Formally, the limiting problems depend on the parameter γ. If $\gamma\geq 2$ (fast decay), the limiting

problem is of ACKS type and the situation is similar to the one we consider in this thesis. If $\gamma < 2$ (slow decay), the limiting problem is of Alt-Philips type with negative powers, which was studied recently by De Silva and Savin in [16, 17, 18].

1.2 Classical obstacle problems

Next we discuss the classical obstacle problem, which in the simplest setting corresponds to the potential $W(s) = s^+$ in (1.0.1).

The obstacle problem describes the behavior of an elastic membrane stretched above a solid obstacle. The membrane and the obstacle are represented by the graphs of two functions u and ψ defined on a domain $\Omega \subseteq \mathbb{R}^n$. The function u is fixed on $\partial\Omega$, and minimizes the Dirichlet energy under the constrain that $u \geq \psi$. Then $U = u - \psi$ is non-negative and minimizes the energy functional

$$J(U, \Omega) = \int_\Omega \frac{|\nabla U|^2}{2} - U^+ \Delta\psi \, dx, \tag{1.2.1}$$

and hence satisfies the Euler-Lagrange equation $\Delta U = -\Delta\psi \chi_{\{U>0\}}$.

It is customary to simplify the model further and assume that $\Delta\psi = -1$, so the problem corresponds to the minimization problem (1.0.1) with the potential

$$W(s) = s^+.$$

It follows that $U \geq 0$ is a minimizer of the functional

$$J(U, \Omega) = \int_\Omega \frac{|\nabla U|^2}{2} + U^+ dx, \tag{1.2.2}$$

and therefore satisfies the Euler-Lagrange equation $\Delta U = \chi_{\{U>0\}}$. In [19], Frehse obtained the optimal regularity $U \in C^{1,1}_{loc}$.

Just like in the Allen-Cahn equation, where we are interested in the phase field region. In the

classical obstacle problem, we study the free boundary of U defined as

$$\Gamma := \partial\{U > 0\}. \tag{1.2.3}$$

The free boundary points are classified by their blow-up limits. Among them one type of the free boundary points is called regular:

Definition 1.2.1. *We say $x_0 \in \Gamma$ is a regular free boundary point, if the rescaled function*

$$U_r(x) := \frac{U(x_0 + rx)}{r^2} \tag{1.2.4}$$

converges as $r \to 0$ to $\frac{1}{2}((x_n)^+)^2$, after a possible rotation.

Caffarelli in the ground-breaking work [20] proved that the free boundary is a $C^{1,\alpha}$ graph near a regular free boundary point, say $y_n = \Gamma(y')$. He first proved that Γ is Lipschitz and then applied the boundary Harnack principle (see Kemper [21]) to the quotient U_{y_i}/U_{y_n} and showed that $\Gamma \in C^{1,\alpha}$ for some α, since

$$\partial_{y_i}\Gamma = -\lim_{y_n \to \Gamma(y')} \frac{U_{y_i}}{U_{y_n}} \in C^{\alpha}.$$

The higher regularity and analyticity of the regular part of the free boundary Γ was obtained by Kinderlehrer and Nirenberg in [22] using a hodograph-Legendre transform.

A second approach to obtain higher regularity of Γ is to inductively apply a higher order version of the boundary Harnack principle established by De Silva and Savin in [23]. The idea is quite simple: assume that $\Gamma \in C^{k,\alpha}$, then we apply the higher order boundary Harnack principle to $w_i = \frac{U_{x_i}}{U_{x_n}}$, and obtain that $\partial_i\Gamma = \lim_{x \to \Gamma} w_i \in C^{k,\alpha}$, hence $\Gamma \in C^{k+1,\alpha}$. This proves the smoothness of the free boundary Γ near a regular point.

More generally, the classical obstacle problem can be formulated for variable coefficients. We study the uniformly elliptic equation with $C^{k,\alpha}$ or analytic coefficients:

$$div(A \cdot \nabla U) = \chi_{\{U>0\}}, \quad U \geq 0. \tag{1.2.5}$$

The discussion above applies in variable coefficients setting, that $U \in C^{1,1}_{loc}$ and the free boundary near a regular boundary point is $C^{1,\alpha}$.

In this thesis, we investigate the higher order boundary Harnack principle in detail, and use it to study the higher regularity and analyticity of the free boundary in (1.2.5).

We recall the simplest version of the boundary Harnack principle due originally to Kemper [21], which states that if

$$u_1, u_2 \geq 0, \quad \Delta u_i = 0 \tag{1.2.6}$$

on a Lipschitz domain $\{x_n \geq \Gamma(x')\}$ and both vanish at Γ, then their ratio

$$w(x) := \frac{u_1}{u_2}(x) \tag{1.2.7}$$

is C^α near the boundary Γ. The boundary Harnack principle was extended to general elliptic operators with measureable coefficients, and to more general domains like NTA or Hölder domain, see for example [24, 25, 26, 27, 28].

De Silva and Savin established more recently in [23] a higher order boundary Harnack principle, which states that if $\Gamma \in C^{k,\alpha}$, then the ratio w is $C^{k,\alpha}$ near Γ. The idea in [23] is to use higher order polynomials to approximate the boundary Γ and two functions u_1, u_2.

In Chapter 4, we prove the following higher order boundary Harnack principle for divergence equations by a different method based on directly studying the degenerate equation satisfied by the ratio.

Theorem 1.2.2. *Assume that $u_1, u_2 > 0$ defined on $\Omega_1 = \{x_n > \Gamma(x')\} \cap B_1$ satisfy*

$$div(A \cdot \nabla u_i) = 0, \quad u_i\Big|_{x_n=\Gamma(x')} = 0 \tag{1.2.8}$$

with A symmetric and uniformly elliptic. If $\Gamma \in C^{k,\alpha}$, $A \in C^{k-1,\alpha}$ for $k \geq 1$, then the ratio $w = \dfrac{u_1}{u_2}$ is $C^{k,\alpha}$ in $\Omega_{1/2}$.

The strategy to obtain Theorem 1.2.2 is to straighten the boundary Γ by the coordinate change

$$(x', x_n) \to (x', x_n - \Gamma(x')), \tag{1.2.9}$$

and investigate the degenerate equation satisfied by w. By a Schauder type estimate of w (will be given later in Theorem 4.1.1), we can prove that if $\Gamma \in C^{1,\alpha}$, then $w \in C^{1,\alpha}$. The general result can be deduced inductively from the $C^{1,\alpha}$ estimate by taking tangential derivatives.

In Chapter 6, we use this result to establish smoothness or analyticity of the free boundary, given that the coefficient matrix A is smooth or analytic, respectively. We first straighten the boundary Γ using (1.2.9), and inductively apply Theorem 4.1.1 to $D_T^k w = D_T^k(\frac{U_{x_i}}{U_{x_n}})$. We provide an alternative proof with respect to Kinderlehrer and Nirenberg [22] of the analyticity of the free boundary.

Theorem 1.2.3. *Let U be a solution of* (1.2.5) *with $\lambda I \le A \le \Lambda I$ being analytic. If $0 \in \Gamma$ is a regular free boundary point, then Γ is an analytic surface near* 0.

We prove Theorem 1.2.3 by following Blatt's method in [29], and construct power series encoding the higher regularity of $u_i = U_{x_i}$ and $w_i = \dfrac{U_{x_i}}{U_{x_n}}$. The elliptic estimates give inductive inequalities of the higher regularity, which could be turned into an ODE inequality system about the two power series.

1.3 The Signorini problem

In Chapter 5, we use a similar higher order boundary Harnack principle to study the free boundary regularity in the Signorini problem. We briefly introduce the problem below.

Assume that $\Omega \subseteq \mathbb{R}^{n+1}$ is divided into two regions Ω_1, Ω_2 by a hypersurface $M \subseteq \Omega$ and the obstacle ψ is defined only on M. One can imagine a scenario of a carpet hanged over a string.

Such an obstacle problem is called the thin obstacle problem, or the Signorini problem. Math-

ematically it consists of minimizing the energy

$$J(U,\Omega) = \int_{\Omega} (\nabla U)^t A(\nabla U) + U(x)F(x)dx, \quad U\Big|_{\partial\Omega} = g$$

under the constrain that $U \geq \psi$ on a M.

Unlike the Allen-Cahn equation or the classical obstacle problem, the potential W is depending also on the space parameter, that $W(U(x), x) = F(x)U(x)$ with $F(x)$ varying in Ω.

The minimizing solution satisfies

$$div(A\nabla U) = F(x) \quad \text{when } U > \psi \text{ or } x \notin M.$$

The free boundary is a submanifold of M defined as

$$\Gamma := \partial\Big|_M \{U > \psi\}. \tag{1.3.1}$$

The simplest model to consider is that of constant coefficients, where we assume $A = \delta^{ij}$, $F, \psi = 0$, $M = \mathbb{R}^n$, and U is an even function in x_{n+1}-direction. The optimal regularity that $U \in C^{1,1/2}_{loc}$ in the half space $\{x_{n+1} > 0\}$ was obtained in [30] by Athanasopoulos and Caffarelli.

In this setting, the Almgren's frequency function

$$N_{x_0}(U,r) := r\frac{\int_{B_r(x_0)} |\nabla U|^2}{\int_{\partial B_r(x_0)} U^2}, \quad x_0 \in \Gamma$$

is used to classify different types of free boundary points. It is an increasing function of r and its limit as $r \to 0$ is called the frequency of U at x_0, denoted by $N_{x_0}(U)$.

In Athanasopoulos-Caffarelli-Salsa [31], it was shown that for $n \geq 1$, either $N_{x_0}(U) = 3/2$, in which case we say that x_0 is a regular free boundary point, or $N_{x_0}(U) \geq 2$, then x_0 is called a singular point. The classification of singular points is a difficult open problem when $n > 1$.

For a regular free boundary point x_0, the rescaled function

$$U_r(x) := \frac{U(rx + x_0)}{(\fint_{\partial B_r(x_0)} U^2)^{1/2}} \tag{1.3.2}$$

as $r \to 0$ has a subsequence U_{r_i} converging to a multiple of the $3/2$-homogeneous solution $Re((x_n + ix_{n+1})^{3/2})$ near 0, after a possible coordinate rotation.

Near a regular point x_0, the free boundary Γ is locally a $C^{1,\alpha}$ graph (an $(n-1)$-dimensional submanifold) for some small α.

These results were extended to variable coefficient Signorini problem in [32, 33, 34, 35], under various regularity assumptions on coefficients. The free boundary Γ is still a $C^{1,\alpha}$ graph near a regular boundary point.

In Koch-Petrosyan-Shi [36] (for constant coefficient model) and Koch-Ruland-Shi [37] (for variable coefficients), the authors showed that the set of regular boundary point in the Signorini problem is analytic, using the hodograph-Legendre transform.

Similar to the classical obstacle problem, in [38] De Silva and Savin derived a higher order boundary Harnack in a $C^{k,\alpha}$ slit domain like $B_1 \setminus S$ where

$$S = \{x = (x_1, ..., x_{n+1}) : x_n < \Gamma(x_1, \cdots, x_{n-1}), x_{n+1} = 0\}, \tag{1.3.3}$$

and showed that the free boundary Γ is a smooth $(n-1)$-dimensional graph near a regular boundary point.

In this thesis, we assume $D = B_1$, $M = \mathbb{R}^n = \{x_{n+1} = 0\}$, and the obstacle $\psi = 0$ on $\{x_{n+1} = 0\}$. Besides, U is assumed to be x_{n+1}-even, meaning

$$U(x_1, \cdots, x_n, x_{n+1}) = U(x_1, \cdots, x_n, -x_{n+1}),$$

and the symmetric matrix A, accordingly, needs to be an x_{n+1}-odd function in entries A^{ij}, A^{ji} for $i \le n$, $j = n + 1$, and be an x_{n+1}-even function in all other entries.

More precisely, the assumption on A is that $A = A_D + x_{n+1}A_O$, so that

(a) $\lambda I \le A \le \Lambda I$ and $[A]_{C^\alpha} \le \varepsilon_0(\alpha < 1/2)$ for a fixed small ϵ_0,

(b) $A_D^{ij} = A_D^{ji} = 0$ for $i \le n, j = (n+1)$, while A_O is non-zero only at these entries,

(c) $\partial_i A_D$ and $\partial_i A_O$ are $C^\alpha(\mathbb{R}^{n+1})$ for all $i \le n$.

(d) A_D and A_O are both x_{n+1}-even functions.

Assumption (d) is somehow optional, because it is not needed once we assume the solution U is x_{n+1}-even. However, if the matrix A satisfies assumption (d), then we can always assume the solution U is x_{n+1}-even by replacing U with

$$\bar{U}(x_1, \cdots, x_{n+1}) := \frac{1}{2}U(x_1, \cdots, x_n, x_{n+1}) + \frac{1}{2}U(x_1, \cdots, x_n, -x_{n+1}),$$

without affecting the free boundary Γ.

We can perform a scaling, replacing $U(x)$ by $R^{-3/2}U(Rx)$ near the origin. This keeps the blow-up of $U(x)$ at the regular boundary invariant, but we can assume that $[D^i A]_{C^\alpha}(i \le n)$ and $[F]_{W^{1,\infty}}$ are small by sending $R \to 0$.

The following theorem (as a special case of the result in [37]) shows the $C^{2,\alpha}$ regularity of the free boundary near a regular point.

Theorem 1.3.1. *Assume that the matrix A satisfies the even conditions (a)-(d), $F(x) \in W^{1,\infty}$, and an x_{n+1}-even solution U solves the thin obstacle problem*

$$div(A\nabla U) = F(x), \quad \text{when } U > 0 \text{ or } x \notin \mathbb{R}^n. \tag{1.3.4}$$

If 0 *is a regular boundary point, then the free boundary $\Gamma \subseteq \mathbb{R}^n$ is a $C^{2,\alpha}$ graph near* 0.

In this thesis we prove the theorem above using a different approach other than that in [37]. The discussion starts from knowing the free boundary $\Gamma \in C^{1,\alpha}$ near a regular point. Let's assume

that the free boundary Γ is a graph in $(x_1, \cdots, x_{n-1})$ variables near the origin, meaning that there exists $\gamma(x_1, \cdots, x_{n-1}) \in C^{1,\alpha}$ so that

$$\Gamma = \{(x_1, \cdots, x_{n-1}, \gamma(x_1, \cdots, x_{n-1}), 0) : (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \cap B_1\}.$$

The strategy is to show $w_i = U_{x_i}/U_{x_n} (i < n)$ are $C^{1,\alpha}$ on Γ. If $w_i\Big|_\Gamma \in C^{1,\alpha}$, then so do $\partial_i \gamma$'s, meaning that Γ is a $C^{2,\alpha}$ graph. Generally speaking, the methods we use in Chapter 5 for the Signorini problem are similar to those in Chapter 4 for the classical obstacle problem.

The main difference is that the blow-up limits of (1.3.2) differ along the free boundary Γ because of the changing coefficient matrix and the changing slope of Γ. Due to such a difference, we need to carry out the analysis more carefully in the Signorini problem.

Chapter 2: Global minimizers of the double well energy: one-phase case

In this and the next chapter, we will discuss the global minimizers of (1.1.4), which we write again as

$$J(u,\Omega)=\int_{\Omega}\frac{|\nabla u|^2}{2}+W(u)dx. \tag{2.0.1}$$

The potential $W(s)$ satisfies:

(W1) $W(s)\geq 0$ and $supp(W)=[-1,1]$,

(W2) $W(s)\in C^{1,1}(\mathbb{R})$, i.e. $\max\limits_{s\in\mathbb{R}}|W''(s)|<\infty$.

Clearly, $W(s)\leq C(1-s^2)^2$ in $[-1,1]$ for some $C>0$. For example, the potential $W(s)=(1-s^2)^2\chi_{[-1,1]}$ in the introduction satisfies both requirements.

In this chapter, we show that there is no one-phase minimizers when $n\leq 7$. In the next chapter, we will discuss the two phase minimizers.

Before discussing the one-phase minimizers (to be non-existing for low dimensions), we need to look at the simplest minimizers in two-phase setting, which is important in the discussion.

2.1 The one-dimensional two-phase minimizers

Let's first understand one-dimensional solutions of the Euler-Lagrange equation:

$$\Delta u=W'(u). \tag{2.1.1}$$

Definition 2.1.1. *For any $K>0$, we define $U_K(t)$ as a function such that $U_K(0)=0$, $\frac{d^2}{dt^2}U_K=W'(U_K)$ and $\frac{d}{dt}U_K(t)=K$ for all t such that $|U_K(t)|\geq 1$.*

If $W(s)$ satisfies conditions (W1), then we can multiply $\frac{d}{dt}U_K$ on both sides of the ODE and obtain a first order ODE:

$$\frac{d}{dt}U_K(t) = \sqrt{2W(U_K(t)) + K^2}. \tag{2.1.2}$$

In fact the zero-phase one-dimensional minimizers in Savin [11] are also constructed using (2.1.2) by setting $K = 0$.

We provide a few properties of $U_K(t)$.

Lemma 2.1.1. *Assume that $W(s)$ satisfying assumption (W1) is fixed, then:*

(1) Let $L = L(K)$ be defined such that $U_K(\pm L) = \pm 1$, then we always have $\frac{d}{dt}U_K \geq K$ and $L \leq 1/K$,

(2) Let $\underline{K} > 0$ be fixed, then for any $K \geq \underline{K}$, $U_K' \leq CK$ everywhere for some $C = C(\underline{K})$,

(3) Let $\underline{K} > 0$ be fixed, then for any $\underline{K} \leq K_1 < K_2$ and any $h \in \mathbb{R}$, we have

$$\frac{U_{K_2}'(U_{K_2}^{-1}(h))}{U_{K_1}'(U_{K_1}^{-1}(h))} - 1 \geq C(\underline{K})(\frac{K_2}{K_1} - 1)$$

(4) $|U_K(x) - Kx| \leq 1$ for all $x \in \mathbb{R}$.

Proof. From (2.1.2) we can easily get (1)-(3). To show (4), we let $f(x) = U_K(x) - Kx$ for $x \geq 0$. Clearly $f' \geq 0$ for $x \in [0, L]$ and $f' = 0$ for $x \geq L$. Besides, $f(0) = 0$ and $f(L) = 1 - KL \leq 1$, so $0 \leq f(x) \leq 1$ for $x \geq 0$. □

The solution $U_K(x)$ is essential in constructing a barrier function in the next section.

2.2 A volcano-shape barrier function

Assume that u is a one-phase minimizer of (2.0.1). Without loss of generality, we assume $u \geq -1$ and $\{u \geq 1\}$ is non empty. Otherwise we just need to replace $u(x)$ with $-u(x)$.

We construct a barrier function, in order to later show the $C^{1/2}$ growth rate of a one-phase minimizer u.

Lemma 2.2.1 (barrier function A)**.** *There exists universal constants $\underline{R}$ and C so that for any $R \geq \underline{R}$, there exists a radial symmetric "volcano" $V_R(x)$ defined in $B^c_{R/10}$ satisfying*

(1) $V_R(x) < -1$ *as long as* $|x| \geq 2R$,

(2) $1 < V_R(x) \leq 1 + C\sqrt{R}$ *for* $0.1R \leq |x| < 0.9R$,

(3) $\Delta V_R(x) > W'(V_R(x))$ *everywhere.*

The construction of such radial barrier uses the ODE solutions $U_K(t)$. We first look at the deformation of $\mathbb{R}$.

Lemma 2.2.2. *For any $R_{in} > 0$, there exists an increasing function $\tau(r) = \tau_{R_{in}}(r)$ vanishing at $r = 0$ satisfying*

(1) $[\tau'(r) - 1] \cdot r \leq 0$,

(2) $\tau''(r) + \dfrac{n-1}{R_{in}}\tau'(r) > 0$,

(3) For any $\epsilon \leq 0.1$, if $R_{in} \geq 100n^2/\epsilon^2$, then wherever $|\tau(r)| \leq \sqrt{R_{in}}$, we have $|r| < (1+\epsilon)\sqrt{R_{in}}$ and $|\tau'(r) - 1| < \epsilon$,

(4) For any $L > 0$ and $\epsilon \leq 0.1$, if $R_{in} > (n-1)L/\epsilon$, we have that $|\tau(r)| \leq L$ implies $|r| < (1+\epsilon)L$ and $|\tau'(r) - 1| < \epsilon$.

Proof. Let $\lambda = 0.9 \cdot \dfrac{n-1}{R_{in}}$, and we directly construct $\tau(r) = \dfrac{1 - e^{-\lambda r}}{\lambda}$, then $\tau(r)$ satisfies condition (1) and (2). Next we only prove (3), while (4) follows from the same idea. When $|\tau(r)| \leq \sqrt{R_{in}}$, we have that

$$|r| \leq \frac{-1}{\lambda}\log(1 - \lambda\sqrt{R_{in}}) < (1+\epsilon)\sqrt{R_{in}}.$$

It follows that $|\lambda r| \leq \epsilon/5$ when $|\tau(r)| \leq \sqrt{R_{in}}$ and thus $\tau'(r) = e^{-\lambda r} \in (1-\epsilon, 1+\epsilon)$. □

Now we use the deformation above to construct the "volcano" function.

Definition 2.2.1 (a "volcano" function)**.** *Given $K > 0$, $R_{in} > 0$ and $\epsilon \le 0.1$, we let $L = L(K)$ such that $U_K(\pm L) = 1$. Assume that for $\tau_{R_{in}}(r)$ constructed in Lemma 2.2.2, $\tau_{R_{in}}(r) \in [-L, L]$ is equivalent to $r \in [r_{-L}, r_L]$. We denote*

$$R = R_{in} + r_L, \quad R_{out} = R - r_{-L} = R_{in} + r_L - r_{-L}$$

and construct a radial symmetric "volcano"

$$V_{K,R,\epsilon}(x) = \begin{cases} 1 + (1+\epsilon)\dfrac{KR_{in}}{n-1.9}\left\{\left(\dfrac{|x|}{R_{in}}\right)^{1.9-n} - 1\right\}, \text{for } R/10 \le |x| \le R_{in} \\ U_K\Big(\tau_{R_{in}}(R - |x|)\Big), \text{for } R_{in} \le |x| \le R_{out} \\ -1 + (1-\epsilon)\dfrac{KR_{out}}{n-1.9}\left\{\left(\dfrac{|x|}{R_{out}}\right)^{1.9-n} - 1\right\}, \text{for } |x| \ge R_{out} \end{cases} . \tag{2.2.1}$$

Now Lemma 2.2.1 is proven by choosing $K = R_{in}^{-1/2}$ and $\epsilon = 0.1$. The construction (2.2.1) will also be used in the next chapter by choosing some other parameters (K, R, ϵ).

Proof of Lemma 2.2.1. Let $V_R(x) = V_{K,R,\epsilon}(x)$ be defined in (2.2.1) with $\epsilon = 0.1$. We set $K = R_{in}^{-1/2}$, then $L(K) \le 1/K = \sqrt{R_{in}}$, so $|r_{\pm L}| \le (1+\epsilon)\sqrt{R_{in}}$ and $\tau_R'(r) \in (1-\epsilon, 1+\epsilon)$ in $[r_{-L}, r_L]$ provided that R_{in} is large enough. As a result, when $R > \underline{R}$ (i.e. R_{in} large), $R_{in} \ge 0.9R$. Let's verify (1)-(3).

(1) When $R \ge \underline{R}$, we have $2R \ge R_{out}$ because $-r_{-L} \le 2\sqrt{R_{in}} \le R$.

(2) Obvious, $C = \dfrac{1+\epsilon}{n-1.9}\Big(10^{n-1.9} - 1\Big)$.

(3) We can easily verify that when $R/10 \le |x| < R_{in}$ or $|x| > R_{out}$, $\Delta V_R(x) > 0 = W'(V_R)$ since $|V_R| \ge 1$. When $|x| = R_{in}$, we also have $\Delta V_R(x) > 0$ because

$$\begin{aligned} \frac{d}{d|x|}\Big|_{|x|=R_{in}+} V_R(x) &= -U_K'(L) \cdot \tau_{R_{in}}'(r_L) \\ > -(1+\epsilon)K &= \frac{d}{d|x|}\Big|_{|x|=R_{in}-} V_R(x). \end{aligned}$$

Here "$\frac{d}{d|x|}$" means the derivative in radial direction. When $|x| = R_{out}$, for the same reason we also have $\Delta V_R(x) > 0$.

Finally we need to check $\Delta V_R(x) > W'(V_R(x))$ when $R_{in} < |x| < R_{out}$. Denote $r = R - |x|$, then by the chain rule, we have

$$\frac{d}{d|x|}V_R = -U_K'(\tau_{R_{in}}) \cdot \tau_{R_{in}}'(r),$$
$$\frac{d^2}{d|x|^2}V_R = U_K''(\tau_{R_{in}}) \cdot (\tau_{R_{in}}'(r))^2 + U_K'(\tau_{R_{in}}) \cdot \tau_{R_{in}}''(r).$$

As V_R is radial symmetric, we have $\Delta V_R = \frac{d^2}{d|x|^2}V_R + \frac{n-1}{|x|}\frac{d}{d|x|}V_R$. Since $\frac{d}{d|x|}V_R < 0$, when $R_{in} < |x| < R_{out}$,

$$\begin{aligned}\Delta V_R - W'(V_R) \geq & \frac{d^2}{d|x|^2}V_R + \frac{n-1}{R_{in}}\frac{d}{d|x|}V_R - W'(U_K) \\ =& W'(U_K(\tau_{R_{in}}))\left[(\tau_{R_{in}}'(r))^2 - 1\right] + U_K'(\tau_{R_{in}})\left[\tau_{R_{in}}''(r) - \frac{n-1}{R_{in}}\tau_{R_{in}}'(r)\right].\end{aligned}$$

Notice that $U_K'(\tau_{R_{in}}) > 0$ and $W'(U_K(\tau_{R_{in}}))$ has opposite sign with $\tau_{R_{in}}$, so by Lemma 2.2.2, we have $\Delta V_R - W'(V_R) > 0$ also for $R_{in} < |x| < R_{out}$.

□

2.3 Density estimate

In this section we show that if $u(x)$ is a one-phase minimizer of (2.0.1), then $\{u > 0\}$ has neither too large nor too small percentage in each B_R (unless $u \equiv c \geq 1$). More precisely, we will show the following density estimate:

Lemma 2.3.1. *Assume that $W(s)$ satisfied (W1),(W2). Assume that u is a one-phase global minimizer of* (2.0.1)*, such that $u(0) = 0$ and $u(x) = O(1 + \sqrt{|x|})$, then there exists universal constants*

$\delta > 0$ and $\underline{R}$, so that for any $R \geq \underline{R}$ we have

$$|B_R \cap \{u \geq 0\}| \geq \delta R^n, \quad |B_R \cap \{u \leq 0\}| \geq \delta R^n.$$

Remark 1. *In particular, if u is a global one-phase minimizer of* (2.0.1)*, then by Lemma 2.3.2 below, $u(x) = O(1 + \sqrt{|x|})$, so the assumption of the lemma holds.*

Before proving the density estimate, we first need to obtain a $C^{1/2}$ estimate and an energy estimate for a one-phase minimizers.

Lemma 2.3.2 ($C^{1/2}$ estimate)**.** *Assume that $W(s)$ satisfies (W1). Assume that $u \geq -1$ is a one-phase global minimizer of* (2.0.1)*. Let $\Gamma = \partial\{u > 1\}$. There exists a constant C depending only on $W(s)$ such that*

(1) For any point $x \in \mathbb{R}^n$ so that $u(x) > 1$, let $\rho = dist(x, \Gamma)$, then $u(x) \leq 1 + C\sqrt{\rho}$,

(2) If $u(0) = 1$, then $\| \max\{u, 1\}\|_{L^\infty(B_R)} \leq 1 + C\sqrt{R}$. Besides, $\max\{u, 1\}$ is globally $C^{1/2}$-continuous in the sense that $[\max\{u, 1\}]_{C^{1/2}(\mathbb{R}^n)} \leq C$.

Proof. We first show that for $\rho_0 = 2\underline{R}$ where $\underline{R}$ is the same as in Lemma 2.2.1, (1) holds for all $\rho \geq \rho_0$. After that we prove the whole lemma.

- Step 1: Let $u(x) > 1$ for some x and $\rho = dist(x, \Gamma) \geq \rho_0$. After a Euclidean group action, we assume that $x = \rho\vec{e}_n$, and $u \geq 1$ in $B_\rho(x)$ with $u(0) = 1$, then $\Delta(u - 1) = 0$ in $B_\rho(x)$. By Harnack inequality we have

$$u(y) - 1 \geq \epsilon_0[u(x) - 1], \quad \forall y \in B_{\frac{9}{10}\rho}(x).$$

  Consider a "volcano" $V(y) = V_{\rho/2}(y - x_0)$ defined in $B_{\rho/20}(x_0)^c$, where $V_R(x)$ is constructed in Lemma 2.2.1. As $\Delta V_{\rho/2}(y - x_0) > W'(V_{\rho/2})$ and $\limsup_{y\to\infty} V_{\rho/2}(y - x_0) < -1$, we have a maximal principle:

$$u(y) > V_{\rho/2}(y - x_0) \text{ in } \partial B_{\rho/20}(x_0) \Rightarrow u(y) > V_{\rho/2}(y - x_0) \text{ in } B_{\rho/20}(x_0)^c.$$

Let C_0 be the constant C in Lemma 2.2.1, then $V_{\rho/2}(y - x_0)$ obtains its maximal value $1 + C_0\sqrt{\rho/2}$ at $\partial B_{\rho/20}(x_0)$. We claim that

$$\exists y \in B_{\frac{9}{10}\rho}(x), \text{ so that } u(y) \leq 1 + C_0\sqrt{\rho/2}. \tag{2.3.1}$$

If this is true, then we have $u(x) \leq 1 + C\sqrt{\rho}$ with $C = C_0/(\sqrt{2}\epsilon_0)$. Now we prove (2.3.1) by contradiction.

We move the center x_0 along the ray $x_0(t) = x - t\vec{e_n}$ with increasing $t \geq 0$. At $t = 0$, $u(y) > V_{\rho/2}(y - x)$ in $B_{\rho/20}(x)^c$ by the contradiction assumption and the maximal principle. At $t = \rho/2$, we have $u(0) = V_{\rho/2}(0 - \frac{\rho}{2}\vec{e_n})$. Therefore, there exists some $t_0 \in (0, \rho/2)$, so that $V_{\rho/2}(y - x_0(t_0))$ touches $u(y)$ from below at a point $y_0 \in B_{\rho/20}(x_0(t_0))^c$. By maximal principle, we have

$$y_0 \in \partial B_{\rho/20}(x_0(t_0)) \subseteq B_{\frac{9}{10}\rho}(x),$$

which contradicts the assumption that (2.3.1) is not true.

- Step 2: Now we prove the whole lemma. In fact it suffices to show $[\max\{u, 1\}]_{C^{1/2}(\mathbb{R}^n)} \leq C$. Since $\Delta(u + 1) = W'(u) \in L^\infty(\mathbb{R}^n)$, for every $x_0 \in \Gamma$, that is $u(x_0) = 1$, we apply the Harnack inequality to $(u - 1)$ in $B_{8\rho_0}(x_0)$ and get that $u \leq C_1$ in $B_{4\rho_0}(x_0)$ where $C_1 = C_1(\rho_0, \max|W'|)$. By Schauder estimate we have

$$[u]_{C^{1/2}(B_{2\rho_0})} \leq C\left\{\rho_0^{-1/2}\|u - 1\|_{L^\infty(B_{4\rho_0})} + \rho_0^{3/2}\max|W'|\right\} \leq C_2. \tag{2.3.2}$$

We can also apply Schauder estimate to $(u - 1)$ in $B_\rho(x)$ where $\rho = dist(x, \Gamma)$ for any $x \in \{u > 1\}$ with $\rho \geq \rho_0$, and obtain that for some C_3 independent of x,

$$[u]_{C^{1/2}(B_{\rho/2}(x))} \leq C_3. \tag{2.3.3}$$

We combine (2.3.2) with (2.3.3) and obtain that $[\max\{u, 1\}]_{C^{1/2}(\mathbb{R}^n)} \leq C$.

□

Lemma 2.3.3 (energy estimate)**.** *Assume that $u \geq -1$ is a one-phase global minimizer of* (2.0.1)*. Let $\Gamma = \partial\{u > 1\}$ and assume $0 \in \Gamma$. There exists a uniform constant C such that*

(1) $J(u, B_R) \leq CR^{n-1}$ for all $R \geq 1$,

(2) $|\{|u| \leq s\} \cap B_R| \leq C\dfrac{R^{n-1}}{W(s)}$ for all $R \geq 1$ and any $s < 1$.

Proof. Let C_0 be defined as in Lemma 2.3.2. Let $v(x)$ be defined as

$$v(x) = \begin{cases} -1, & if\ |x| \leq R \\ |x| - R - 1, & if\ R \leq |x| \leq R + 2 \\ 1 + C_0\dfrac{|x| - R - 2}{\sqrt{R+2}}, & if\ R + 2 \leq |x| \leq 2R + 4 \end{cases}.$$

Since $v(x) \geq u(x)$ for all $|x| = 2R + 4$, we have $B_R \subseteq E \subseteq B_{2R+4}$ for $E = \{x : u(x) \geq v(x)\}$. we take v as a competitor in E and have $J(u, E) \leq J(v, E)$, so we obtain the bound

$$J(u, B_R) \leq J(v, B_{2R+4}) \leq CR^{n-1} \tag{2.3.4}$$

for all $R \geq 1$. (2) is just a direct consequence of (1). □

Now let's prove Lemma 2.3.1. Without loss of generality, we just prove the first inequality

$$|B_R \cap \{u \geq 0\}| \geq \delta R^n.$$

Proof of Lemma 2.3.1. Denote

$$A(R) = |B_R \cap \{u \geq -1/2\}| + \int_{B_R} W(u)dx =: V(R) + \Omega(R).$$

Clearly, $\Omega(R) \leq J(u, R)$. Besides, by Lemma 2.3.3 we have

$$|\{u \geq 0\} \cap B_R| \geq A(R) - CR^{n-1}$$

for R sufficiently large, so it suffices to show that there exists a $\delta > 0$, so that $A(R) \geq \delta R^n$ for an increasing sequence of $R_k \to \infty$ satisfying $R_{k+1}/R_k \leq 100$.

By Schauder estimate, $|\nabla u| \leq C$ in B_1, which means that $A(R) \geq c_1$ for $R \geq 1$. Let $R_0 = 100T$ where $T > 1$ is a fixed large constant to be decided later, then we have $A(R_0) \geq \delta R_0^n$ as long as

$$\delta \leq c_1 (100T)^{-n}. \tag{2.3.5}$$

Suppose that for some $k \geq 0$, $A(R_k) \geq \delta R_k^n$, we now try to find a $R_{k+1} \leq 100R_k$ so that $A(R_{k+1}) \geq \delta R_{k+1}^n$.

- Case 1: If in $\mathbb{A}_k = B_{3R_k} \setminus B_{R_k}$, it happens that $|\mathbb{A}_k \cap \{u \geq 1\}| \geq \delta |\mathbb{A}_k|$, then we naturally choose $R_{k+1} = 3R_k$. Clearly we still have $A(R_{k+1}) \geq \delta R_{k+1}^n$, and we go back to the next iteration.

- Case 2: Otherwise, we choose $R_{k+1} = R_k + T < 100R_k$. The remaining content in the proof is to show $A(R_{k+1}) \geq \delta R_{k+1}^n$ for such a choice.

Assume that we fall into Case 2 and $R_{k+1} = R_k + T < 100R_k$, we write $r = R_{k+1}$. Since $(u-1)^+ \in C^{1/2}$ is harmonic on its positive part, we apply the De Giorgi - Nash - Moser L^∞ estimate to the annulus $\mathbb{A}_k$, and get that

$$\|u\|_{L^\infty(\partial B_{2r})} \leq \frac{C}{R_k^{n/2}} \|u\|_{L^2(\mathbb{A}_k)} \leq C\sqrt{\delta} \|u\|_{L^\infty(\mathbb{A}_k)} \leq C_2 \sqrt{\delta r}.$$

Let $v(x)$ defined in B_{2r} so that $v \geq u$ on ∂B_{2r} be

$$v(x) = \begin{cases} -1 + 2e^{|x|-r}, \text{ if } |x| \leq r \\ 1 + C_2\sqrt{\delta}\dfrac{|x|-r}{\sqrt{r}}, \text{ if } r \leq |x| \leq 2r \end{cases}.$$

One can verify that (recall that $R_k = r - T$)

$$J(v, B_{2r} \setminus B_r) \leq C\delta r^{n-1}, \quad J(v, B_{R_k}) \leq Ce^{-2T} r^{n-1}. \tag{2.3.6}$$

Denote $E_s = B_{2r} \cap \{u > s > v\}$ which has positive distance from ∂B_{2r}, and $E = B_{2r} \cap \{u > v\} \supseteq \bigcup_{s\in[-1,1]} E_s$. For each $s \in [-1, 1]$, we denote

$$\partial E_s = \partial_u E_s \cup \partial_v E_s,$$

where $u = s$ on $\partial_u E_s$ and $v = s$ on $\partial_v E_s$. In fact, $\partial_u E_s = E \cap \{u = s\}$ and $\partial_v E_s = E \cap \{v = s\}$. By isoperimetric inequality

$$|E_s|^{\frac{n-1}{n}} \leq C[\mathcal{H}^{n-1}(\partial_u E_s) + \mathcal{H}^{n-1}(\partial_v E_s)]$$

We integrate for $s \in [-1, 1]$, and obtain that

$$\begin{aligned}\int_{-1}^{1} |E_s|^{\frac{n-1}{n}} \sqrt{2W(s)} ds \leq & C\Big\{ \int_{\partial_u E_s} \frac{|\nabla u|}{2} + \frac{W(s)}{|\nabla u|} d\sigma + \int_{\partial_v E_s} \frac{|\nabla v|}{2} + \frac{W(s)}{|\nabla v|} d\sigma \Big\} \\ \leq & C[J(u, E) + J(v, E)] \leq CJ(v, E).\end{aligned}$$

The last inequality is because we take $v(x)$ as a competitor in E and have $J(u, E) \leq J(v, E)$. We first estimate the left-hand side.

Let $h = -1 + 2e^{-T}$, then for all $s \in [h, -1/2]$, we have $E_s \supseteq B_{R_k} \cap \{u \geq -1/2\}$, so

$$\int_{-1}^{1} |E_s|^{\frac{n-1}{n}} \sqrt{2W(s)} ds \geq C^{-1} V(R_k)^{\frac{n-1}{n}} = C^{-1}[A(R_k) - \Omega(R_k)]^{\frac{n-1}{n}}.$$

Since $A(R_k) \geq A(R_0) \geq \delta_1$, we write the last term as

$$C^{-1} V(R_k)^{\frac{n-1}{n}} \geq C^{-1} \Big\{ A(R_k)^{\frac{n-1}{n}} - c_1^{-1/n} \Omega(R_k) \Big\}.$$

The term $\Omega(R_k)$ is estimated as:

$$\Omega(R_k) \leq J(u, E) + \int_{B_{R_k} \setminus E} W(v) dx \leq J(v, E) + Ce^{-2T} r^{n-1}.$$

Consequently, we have

$$C^{-1}A(R_k)^{\frac{n-1}{n}} \leq C[J(v,E) + e^{-2T}r^{n-1}].$$

Now we estimate $J(v,E)$, by (2.3.6) we have

$$J(v,E) \leq J(v, E \cap (B_r \setminus B_{R_k})) + Cr^{n-1}(\delta + e^{-2T}),$$

In the annulus $B_r \setminus B_{R_k}$, $|\nabla v|^2 \leq C(W(v) + \chi_{\{v\geq 0\}})$, while in E where $u > v$, we have $\chi_{\{v\geq 0\}} \leq \chi_{\{u\geq 0\}}$ and $W(v) \leq W(u) + C\chi_{\{u\geq 0\}}$, so

$$J(v, E \cap (B_r \setminus B_{R_k})) \leq C[A(r) - A(R_k)].$$

In total we have (recall $r = R_{k+1} = R_k + T$)

$$A(R_{k+1}) \geq A(R_k) + C^{-1}A(R_k)^{\frac{n-1}{n}} - CR_{k+1}^{n-1}(\delta + e^{-2T})$$

If $A(R_k) \geq \delta R_k^n$ can obtain $A(R_{k+1}) \geq \delta R_{k+1}^n$ if

$$(T - C)\delta \leq C^{-1}\delta^{\frac{n-1}{n}} - Ce^{-2T}. \tag{2.3.7}$$

The two requirements (2.3.5)(2.3.7) can be satisfied simultaneously, meaning that there exists $\delta > 0$ such that $A(R) \geq \delta R^k$ for $R \geq R_0$. □

2.4 The rescaled problem and the ACKS minimization problem

In this section, we rescale the function $u(x)$ to $u_R(y) = R^{-1/2}u(Ry)$. We need to understand the limit of $u_R(y)$. It turns out that $u_R(y)$ is converging to a minimizer of the ACKS functional (2.4.1) studied in [14]. Let's relate the rescaled function u_R to the minimizers of the ACKS functional.

Definition 2.4.1. *Let $s \in [-1, 1]$, we define*

$$\lambda(s) = \int_0^s \sqrt{2W(t)}dt.$$

We also define the value

$$\Lambda = \int_{-1}^{1} \sqrt{2W(t)}dt = \lambda(1) - \lambda(-1).$$

Definition 2.4.2. *We say a pair of functions $(\mu, \mathcal{E})$ is admissible in an open set Ω, denoted as $(\mu, \mathcal{E}) \in \mathcal{A}(\Omega)$, if*

(1) $\mu \geq 0$ and $-1 \leq \mathcal{E} \leq 1$ in Ω,

(2) $\mu \in H^1(\Omega)$ and $\lambda(\mathcal{E}) \in BV(\Omega)$,

(3) $\{\mu > 0\} \cap \Omega$ is contained in $\{\mathcal{E} = 1\} \cap \Omega$.

Definition 2.4.3. *If $(\mu, \mathcal{E}) \in \mathcal{A}(\Omega)$ and $\mathcal{E} \in H^1(\Omega)$, we define a functional*

$$J_R(\mu, \mathcal{E}, \Omega) := \int_\Omega \frac{|\nabla \mu|^2}{2} dx + \int_\Omega \Big[\frac{|\nabla \mathcal{E}|^2}{2R} + RW(\mathcal{E}) \Big] dx.$$

If $(\mu, \mathcal{E}) \in \mathcal{A}(\Omega)$ and $\mathcal{E} = \chi_{E^c} - \chi_E$ almost everywhere for some Caccioppoli set E, we introduce the ACKS functional studied in [14]:

$$I(\mu, \mathcal{E}, \Omega) := \int_\Omega \frac{|\nabla \mu|^2}{2} dx + \Lambda \cdot Per(E, \Omega). \tag{2.4.1}$$

In [14], a minimizer of (2.4.1) in B_1 is known to be Lipschitz in $B_{1/2}$. With this interior estimate, we obtain the global continuity if the boundary data is positive and continuous.

Lemma 2.4.1 (continuity of ACKS minimizers)**.** *Let $(\nu, \mathcal{F})$ be a minimizer of the functional $I(\nu, \mathcal{F}, B_1)$ such that the boundary data $\nu_\partial \geq 0$ is continuous on ∂B_1, then ν is uniformly continuous in $\overline{B_1}$.*

Proof. The interior pointwise continuity of v is obtained in [14], so it suffices to show the pointwise boundary continuity. Let $p \in \partial B_1$. Let μ be a harmonic function with the same continuous boundary data v_∂, then μ is continuous in $\overline{B_1}$ and we obtain that v is upper semi-continuous at p, i.e.

$$\limsup_{x\to p} v(x) \leq \limsup_{x\to p} \mu(x) = v_\partial(p).$$

In particular, if $v_\partial(p) = 0$, then clearly v is continuous at p. On the other hand, we consider the case $v_\partial(p) > 0$ and show that $\liminf_{x\to p} v(x) \geq v_\partial(p)$.

Let's assume that there exists a $\delta > 0$ such that for ϵ sufficiently small, $v_\partial \geq \delta$ in $\partial B_1 \cap B_\epsilon(q)$ where $q = (1+\epsilon/2)p$. Let $\alpha = \sqrt{\frac{2(n-1)}{(n-2)^2}\Lambda\epsilon^{2n-3}}$ and set the comparison function

$$V(x) = \max\{\alpha|x-q|^{2-n} - \alpha\epsilon^{2-n}, 0\}. \tag{2.4.2}$$

$V(x)$ is in fact an ACKS minimizer in $B_{\epsilon/2}(q)^c$ satisfying the boundary condition that is constant on $\partial B_{\epsilon/2}(q)$ and is compactly supported (we will briefly explain this after the proof).

We have that $V(x) \leq C\sqrt{\epsilon}$ outside $B_{\epsilon/2}(q)$. Let ϵ be so small that $C\sqrt{\epsilon} \leq \delta$. Since $V(x)$ is the minimizer of $I(\cdot,\cdot,B_{\epsilon/2}(q)^c)$ with the fixed Dirichlet boundary data, we have that $m(x) = \min\{v, V\}$ and $M(x) = \max\{v, V\}$ cannot simultaneously be the minimizers of $I(\cdot,\cdot,B_{\epsilon/2}(q)^c)$ and $I(\cdot,\cdot,B_1)$ respectively, unless $v \geq V > 0$ in $B_1 \cap (B_\epsilon(q) \setminus B_{\epsilon/2}(q))$.

Therefore, we have $v > 0$ and hence harmonic in a neighbourhood of p in $\overline{B_1}$. By comparing with a lower barrier function, we see v is also lower semi-continuous at p. □

Now we explain why $V(x)$ in (2.4.2) is an ACKS minimizer. First, we fix the area of all upper-level sets $A(h) = |B_{\epsilon/2}(q)^c \cap \{V > h\}|$, and obtain that the ACKS functional is minimized if $V(x)$ is rotationally symmetric about q. Next, we show $V(x)^+$ must be a fundamental solution if $A(0)$ is fixed. Finally, we just need to find the best parameter $A(0)$ to minimize the ACKS functional.

2.5 Γ convergence

In this section we prove that the minimizers of $J_R(\mu, \mathcal{E}, \Omega)$ converges up to a subsequence to the ACKS minimizers.

Theorem 2.5.1 (Γ-convergence)**.** *Assume that $(\mu_R, \mathcal{E}_R) \in \mathcal{A}(B_1)$ are minimizers of $J_R(\cdot, \cdot, B_1)$, such that $J_R(\mu_R, \mathcal{E}_R, B_1)$ and $\|\mu_R\|_{C^{1/2}(B_1)}$ are uniformly bounded. Then there exists a pair $(\mu_\infty, \mathcal{E}_\infty) \in \mathcal{A}(B_1)$ such that:*

(1) Up to a subsequence, μ_R converges to μ_∞ in $L^2(B_1) \cap C^\alpha(B_1)$ sense for any $\alpha < 1/2$, $\lambda(\mathcal{E}_R)$ converges to $\lambda(\mathcal{E}_\infty)$ in $L^1(B_1)$ sense,

(2) $\mathcal{E}_\infty \in \{-1, 1\}$ almost everywhere, that is $\mathcal{E}_\infty = \chi_{E^c} - \chi_E$ for some Caccioppoli set E,

(3) $(\mu_\infty, \mathcal{E}_\infty)$ is a minimizer of the ACKS functional $I(\cdot, \cdot, B_1)$ with respect to its boundary value.

Part (1) and (2) are easier, which can be proven right now.

Proof of Theorem 2.5.1(1)(2). Since $J_R(\mu_R, \mathcal{E}_R, B_1) \le C$, we have $\int_{B_1} |\nabla \mu_R|^2 dx \le C$ and

$$\int_{B_1} |\nabla \lambda(\mathcal{E}_R)| dx = \int_{B_1} |\nabla \mathcal{E}_R| \sqrt{2W(\mathcal{E}_R)} dx \le J_R(\mu_R, \mathcal{E}_R, B_1) \le C.$$

by Cauchy-Schwartz inequality. Therefore, by the standard compactness argument we obtain (1).

Suppose that $\mu_\infty(x) > 0$ for some $x \in B_1$, then $\mu_R > 0$ in some neighbourhood $B_\epsilon(x)$ due to C^α convergence of μ_R. As $(\mu_R, \mathcal{E}_R) \in \mathcal{A}(B_1)$, we have $\mathcal{E}_R = 1$ in $B_\epsilon(x)$, so $\mathcal{E}_\infty(x) = 1$. This proves $(\mu_\infty, \mathcal{E}_\infty) \in \mathcal{A}(B_1)$.

Since $\int_{B_1} W(\mathcal{E}_R) dx \le J_R(\mu_R, \mathcal{E}_R, B_1)/R \to 0$, for any $|h| < 1$,

$$|B_1 \cap \{|\mathcal{E}_R| \le h\}| \le C_0(h)/R,$$

so $\mathcal{E}_\infty \in \{-1, 1\}$ almost everywhere. This proves (2).

Besides, by the weak $L^2 \oplus BV$ convergence of $(\mu_R, \lambda(\mathcal{E}_R))$, we have the following lower semi-continuity

$$I(\mu_\infty, \mathcal{E}_\infty, B_1) \leq \liminf_{R\to\infty} J_R(\mu_R, \mathcal{E}_R, B_1). \tag{2.5.1}$$

□

The main difficulty lies on the proof of (3). The strategy is to prove by contradiction, that if the limiting pair $(\mu_\infty, \mathcal{E}_\infty)$ is not a minimizer of ACKS functional, then based on a minimizer of ACKS functional, say for example $(\widetilde{\mu_\infty}, \widetilde{\mathcal{E}_\infty})$, we construct a competitor, which performs better than $(\mu_R, \mathcal{E}_R)$ in minimizing J_R.

We need a few lemmas. First, for a minimizer $(\mu, \mathcal{E}) \in \mathcal{A}(B_1)$ of the ACKS functional (2.4.1), meaning $\mathcal{E} = \chi_{E^c} - \chi_E$ is characterizing a set, the functional $I(\cdot, \cdot, B_1)$ changes slightly if we smoothify the boundary ∂E wisely.

Lemma 2.5.1 (approximation by smooth set)**.** *Assume that $(\mu, \mathcal{E}) \in \mathcal{A}(B_1)$ with μ being uniformly continuous in $\overline{B_1}$ and $\mathcal{E} = \chi_{E^c} - \chi_E$. Let $a < 1$ be a fixed Lebesgue point of $\mathcal{H}^{n-1}(B_a \cap \partial E)$, then for any small $\delta > 0$, there exists an approximation $(\nu, \mathcal{F}) \in \mathcal{A}(B_a)$ such that*

(1) $\mathcal{F} = \chi_{F^c} - \chi_F$ where $\partial F \cap B_a$ is a smooth hypersurface,

(2) $I(\nu, \mathcal{F}, B_a) \leq I(\mu, \mathcal{E}, B_a) + \delta$,

(3) $\|\nu - \mu\|_{L^2(B_a)} + \|\mathcal{F} - \mathcal{E}\|_{L^1(B_a)} \leq \delta$.

Proof. As μ is continuous in $\overline{B_1}$, let's assume that there exists $\sigma(\epsilon) \to 0$, such that for any $x, y \in B_1$, if $\mu(x) = 0$ and $\mu(y) \geq \sigma(\epsilon)$, then $|x - y| \geq \epsilon$.

We first define $\nu = (\mu - \sigma(\epsilon))^+$, then $I(\nu, 0, B_a) \leq I(\mu, 0, B_a)$ and

$$\|\nu - \mu\|_{L^2(B_a)} \leq C\sigma(\epsilon) \leq \delta/2$$

when ϵ is small.

Let $\eta_\epsilon \geq 0$ be a smooth mollifier supported in B_ϵ. We define $\mathcal{F} = \chi_{F^c} - \chi_F$ where F is a lower-level set chosen among

$$F_h = \{E * \eta_\epsilon \leq h\}, \quad h \in (-1, 1).$$

Before we choose a suitable h, we should notice that ∂F is contained in the ϵ-thickening of ∂E, so $(\nu, \mathcal{F}) \in \mathcal{A}(B_a)$ and $\|\mathcal{F} - \mathcal{E}\|_{L^1(B_a)} \leq \delta/2$ for small ϵ.

As a is a Lebesgue point of $\mathcal{H}^{n-1}(B_a \cap \partial E)$, we have

$$\limsup_{\epsilon \to 0} \int_{B_a} |\nabla(E * \eta_\epsilon)| dx \leq \limsup_{\epsilon \to 0} \int_{B_{a+\epsilon}} |\nabla E| dx = 2Per(E, B_a).$$

By Sard's theorem, almost all level sets of $E * \eta_\epsilon$ are smooth hypersurfaces, and hence

$$\int_{-1}^{1} Per(F_h, B_a) = \int_{B_a} |\nabla(E * \eta_\epsilon)| dx \leq 2Per(E, B_a) + o(1).$$

Therefore, we can choose some $h \in (-1, 1)$ such that

$$Per(F_h, B_a) \leq Per(E, B_a) + o(1).$$

When ϵ is small, then (2) is also satisfied. □

Next, for $\mathcal{F} = \chi_{F^c} - \chi_F$ (cannot be inserted into J_R), we need to approximate it with H^1-integrable functions, so that $J_R(0, \mathcal{F}_R, B_a) \approx I(0, \mathcal{F}, B_a)$ for large R.

Lemma 2.5.2 (H^1-approximation)**.** *Assume that $\mathcal{F} = \chi_{F^c} - \chi_F$ is defined in B_1, where $\partial F \cap B_1$ is a compact smooth hypersurface. Let $a < 1$ be a fixed Lebesgue point of $\mathcal{H}^{n-1}(B_a \cap \partial F)$, there exists a sequence $\mathcal{F}_R$ such that*

(1) $\mathcal{F}_R = 1$ in F^c, $\mathcal{F}_R \in [-1, 1]$ in F,

(2) $\|\mathcal{F}_R - \mathcal{F}\|_{L^1(B_a)} \to 0$ as $R \to \infty$,

(3) $\limsup_{R\to\infty} J_R(0, \mathcal{F}_R, B_a) \le I(0, \mathcal{F}, B_a)$,

(4) Therefore, if $(\nu, \mathcal{F}) \in \mathcal{A}(B_a)$*, then so is the pair* $(\nu, \mathcal{F}_R)$ *for large* R.

Proof. • Step 1: Let $\phi_{R,\delta}(s)$ for $s \in [0, s_R]$ be a solution of the ODE

$$\phi_R(0) = 1, \quad \frac{d}{ds}\phi_R(s) = -R\sqrt{2W(\phi_R)} - 2\sqrt{R}, \tag{2.5.2}$$

where $s_R \le R^{-1/2}$ is the time when $\phi_R(s_R) = -1$. Such a construction ensures that for all $s \in (0, s_R)$,

$$D(\phi_R(s)) = \frac{|\phi_R(s)'|}{2R} + R\frac{W(\phi_R(s))}{|\phi_R(s)'|} \le \sqrt{2W(\phi_R(s))} + R^{-1/2}.$$

• Step 2: Define $\rho(x) = dist(x, F^c)$, then $|\nabla\rho(x)| = 1$ almost everywhere in F. Let $a < 1$ and assume that $B_{\frac{1+a}{2}} \cap \partial F$ has curvature bounded by $K = K(a)$. For every $s > 0$, denote

$$\partial^s F = B_a \cap \{x : \rho(x) = s\}. \tag{2.5.3}$$

If $s < \min\{1/K, (1-a)/2\}$, then $\partial^s F$ is a smooth hypersurface in B_a and is independent of the set $F \setminus B_{a+s}$. As $s \to 0$, we have

$$\mathcal{H}^{n-1}(\partial^s F) \le \mathcal{H}^{n-1}(B_{a+s} \cap \partial F) + o(1).$$

If $a < 1$ is a Lebesgue point of $\mathcal{H}^{n-1}(B_a \cap \partial F)$, we have

$$\limsup_{s\to 0} \mathcal{H}^{n-1}(\partial^s F) \le \mathcal{H}^{n-1}(B_a \cap \partial F).$$

Clearly, the range of valid a's is almost everywhere in $(0, 1)$.

- Step 3: We construct $\mathcal{F}_R(x)$ satisfying condition (1) as

$$\mathcal{F}_R(x) = \begin{cases} 1, & \text{if } \rho(x) \le 0 \\ \phi_R(\rho(x)), & \text{if } 0 < \rho(x) < s_R \\ -1, & \text{if } \rho(x) \ge s_R \end{cases}. \tag{2.5.4}$$

Let $R > \max\{K^2, \frac{4}{(1-a)^2}\}$ and $R \to \infty$, then $s_R \to 0$ and we have

$$\|\mathcal{F}_R - \mathcal{F}\|_{L^1(B_a)} \le 2\int_0^{s_R} \mathcal{H}^{n-1}(\partial^s F)ds \to 0.$$

This proves (2). (3) follows from the observation that

$$\begin{aligned} J_R(0, \mathcal{F}_R, B_a) =& \int_{-1}^{1} D(\phi_R(s))\mathcal{H}^{n-1}(\partial^s F)d\phi_R(s) \\ \le& \Big[\mathcal{H}^{n-1}(B_a \cap \partial F) + o(1)\Big]\int_{-1}^{1} D(\phi_R)d\phi_R \\ \le& I(0, \mathcal{F}, B_a) + o(1). \end{aligned}$$

□

Traditionally, Lemma 2.5.1 and Lemma 2.5.2 together already have proven the Γ-convergence. However, these two procedure are both changing the boundary data of the admissible pair. Therefore, we need a way to glue the admissible pair in the interior with some "unchangeable" values near the boundary.

Lemma 2.5.3 (a gluing method)**.** *Let $\Omega = B_{1-\sigma} \setminus B_{1-4\sigma}$ with σ being fixed. Assume that $(\mu_R, \mathcal{E}_R) \in \mathcal{A}(B_1)$ and $(\nu_R, \mathcal{F}_R) \in \mathcal{A}(B_{1-\sigma})$. If there exists $\delta > 0$ such that*

$$\limsup_{R\to 0}\left\{\|\mu_R - \nu_R\|_{L^2(\Omega)} + \|\mathcal{E}_R - \mathcal{F}_R\|_{L^1(\Omega)}\right\} \le \delta,$$

then there exist $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B_1)$ for some subsequence $R \to \infty$ and some $\epsilon(\delta) \to 0$ such that

(1) $(\omega_R, \mathcal{G}_R) = (\nu_R, \mathcal{F}_R)$ *in* $B_{1-4\sigma}$ *and* $(\omega_R, \mathcal{G}_R) = (\mu_R, \mathcal{E}_R)$ *in* $B^c_{1-\sigma}$,

(2) $\limsup_{R\to\infty} J_R(\omega_R, \mathcal{G}_R, \Omega) \le 3J_R(\nu_R, \mathcal{F}_R, \Omega) + 3J_R(\mu_R, \mathcal{E}_R, \Omega) + \epsilon(\delta)$.

Proof. The key point is to construct $(\omega_R, \mathcal{G}_R)$ in Ω. We first construct ω_R and then construct $\mathcal{G}_R$.

- Step 1: Let η_1, η_2, η_3 be a partition of unity in B_1 so that:

  (1) $supp(\eta_1) = B_{1-3\sigma}$, and $\eta_1 = 1$ in $B_{1-4\sigma}$,

  (2) $supp(\eta_2) = B_{1-\sigma} \setminus B_{1-4\sigma}$, and $\eta_1 = 1$ in $B_{1-2\sigma} \setminus B_{1-3\sigma}$,

  (3) $supp(\eta_1) = B^c_{1-2\sigma}$, and $\eta_1 = 1$ in $B^c_{1-\sigma}$,

  (4) $\|\nabla\eta_i\|_{L^\infty} \le C\sigma^{-1}$.

  We define $\omega_R(x)$ in B_1 such that

$$\omega_R(x) = \eta_1\nu_R + \eta_2 \min\{\mu_R, \nu_R\} + \eta_3\mu_R, \tag{2.5.5}$$

  then clearly $\omega_R = \nu_R$ in $B_{1-4\sigma}$ and $\omega_R = \mu_R$ in $B^c_{1-\sigma}$. We have that in Ω,

$$|\nabla\omega_R|(x) \le C\sigma^{-1}|u - v|(x) + |\nabla\mu_R| + |\nabla\nu_R|,$$

  so $J_R(\omega_R, 0, \Omega) \le 3J_R(\nu_R, 0, \Omega) + 3J_R(\mu_R, 0, \Omega) + C\dfrac{\delta^2}{\sigma^2} + o(1)$.

- Step 2: Let R be sufficiently large, and for simplicity, we assume that $\sigma = N/R$ for some integer N. Let $a_i = 1 - 3\sigma + i/R$ with $1 \le i \le N$, and denote $\mathbb{A}_i = B_{a_i} \setminus B_{a_i - 1/R}$. Let $\psi_{i,R}$ be supported in B_{a_i} so that

$$\psi_{i,R}\Big|_{B_{a_i-1/R}} = 1, \quad |\nabla\psi_{i,R}| \le CR\chi_{\mathbb{A}_i}.$$

  Let $\mathcal{G}_{i,R}$'s be candidates of $\mathcal{G}_R$ defined as

$$\mathcal{G}_{i,R} = \psi_{i,R}\mathcal{F}_R + (1 - \psi_{i,R})\mathcal{E}_R, \tag{2.5.6}$$

then in $\mathbb{A}_i$, $|\mathcal{G}_{i,R} - \mathcal{E}_R| \le |\mathcal{F}_R - \mathcal{E}_R| \le 2$, and

$$|\nabla \mathcal{G}_{i,R}| \le |\nabla \mathcal{E}_R| + |\nabla \mathcal{F}_R| + CR|\mathcal{F}_R - \mathcal{E}_R|.$$

This gives an estimate that for C_1 depending on $\max |W'|$,

$$J_R(0, \mathcal{G}_{i,R}, \mathbb{A}_i) \le 3J_R(0, \mathcal{E}_R, \mathbb{A}_i) + 3J_R(0, \mathcal{F}_R, \mathbb{A}_i) + C_1 R\|\mathcal{F}_R - \mathcal{E}_R\|_{L^1(\mathbb{A}_i)}.$$

There must exists an $1 \le i^* \le N$, so that the last term

$$C_1 R\|\mathcal{F}_R - \mathcal{E}_R\|_{L^1(\mathbb{A}_{i^*})} \le C\frac{\delta}{\sigma} + o(1).$$

For such an i^*, we define $\mathcal{G}_R = \mathcal{G}_{i^*,R}$ have

$$\begin{aligned} J_R(0, \mathcal{G}_R, \Omega) =& J_R(0, \mathcal{F}_R, \Omega \cap B_{a_i - 1/R}) + J_R(0, \mathcal{E}_R, \Omega \setminus B_{a_i}) \\ &+ J_R(0, \mathcal{G}_{i^*,R}, \mathbb{A}_{i^*}) \\ \le& 3J_R(0, \mathcal{F}_R, \Omega) + 3J_R(0, \mathcal{E}_R, \Omega) + C\frac{\delta}{\sigma} + o(1). \end{aligned}$$

- Step 3: We finally show $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B_1)$, so that J_R is "additive". In fact, in $B_{1-3\sigma}$, $\omega_R \le \nu_R$ and $\mathcal{G}_R = \mathcal{F}_R$, so $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B_{1-3\sigma})$. For the similar reason we get $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B^c_{1-2\sigma})$. In $B_{1-2\sigma} \setminus B_{1-3\sigma}$, $\omega_R = \min\{\mu_R, \nu_R\}$ and $\mathcal{G}_R \ge \min\{\mathcal{F}_R, \mathcal{F}_R\}$, so $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B_{1-2\sigma} \setminus B_{1-3\sigma})$.

□

With these three lemmas, we can finish the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1(3). We show by contradiction. Suppose that for some $r_0 < 1$, $(\mu_\infty, \mathcal{E}_\infty)$ is not the minimizer of I in B_{r_0}, but $(\widetilde{\mu_\infty}, \widetilde{\mathcal{E}_\infty}) \in \mathcal{A}(B_{r_0})$ is. Assume that

$$I(\mu_\infty, \mathcal{E}_\infty, B_{r_0}) \ge I(\widetilde{\mu_\infty}, \widetilde{\mathcal{E}_\infty}, B_{r_0}) + 2\epsilon.$$

By the Hölder continuity of μ_∞ and Lemma 2.4.1, we can apply Lemma 2.5.1 to the pair

$$(\mu_\infty \chi_{B_{r_0^c}} + \widetilde{\mu_\infty}\chi_{B_{r_0}}, \mathcal{E}_\infty \chi_{B_{r_0^c}} + \widetilde{\mathcal{E}_\infty}\chi_{B_{r_0}}),$$

so for arbitrarily small δ, there exists a pair $(\nu, \mathcal{F}) \in \mathcal{A}(B_a)$ satisfying

(1) $\dfrac{r_0 + 99}{100} < a < 1$,

(2) $\mathcal{F} = \chi_{F^c} - \chi_F$ and $B_a \cap \partial F$ is a smooth hypersurface,

(3) $\|\mu_\infty - \nu\|_{L^2(B_a \setminus B_{r_0})} + \|\mathcal{E}_\infty - \mathcal{F}\|_{L^1(B_a \setminus B_{r_0})} \leq \delta$,

(4) $I(\mu_\infty, \mathcal{E}_\infty, B_a) \geq I(\nu, \mathcal{F}, B_a) + \epsilon$.

Let $b = 1 - \sigma$ for some small σ be a Lebesgue point of $\mathcal{H}^{n-1}(B_b \cap \partial F)$, where we require σ to fulfill the following requirements:

(1) $1 - a < \sigma < (1 - r_0)/4$, that is $B_{r_0} \subseteq B_{1-4\sigma}$ and $B_{1-\sigma} \subseteq B_a$,

(2) $I(\mu_\infty, \mathcal{E}_\infty, B_a \setminus B_{1-4\sigma}) + I(\nu, \mathcal{F}, B_a \setminus B_{1-4\sigma}) \leq \epsilon/100$.

By Lemma 2.5.2 we have a sequence of smooth functions $\mathcal{F}_R$ so that $(\nu, \mathcal{F}_R) \in \mathcal{A}(B_{1-\sigma})$ for large R. Together with (2.5.1) we have $(\mu_R, \mathcal{E}_R)$ with $(\nu, \mathcal{F}_R)$ satisfies all assumptions in Lemma 2.5.3. We glue them and obtain a pair $(\omega_R, \mathcal{G}_R) \in \mathcal{A}(B_1)$ satisfying

$$\liminf_{R\to\infty} \left\{ J_R(\mu_R, \mathcal{E}_R, B_1) - J_R(\omega_R, \mathcal{G}_R, B_1) \right\} \geq \epsilon/2 - \epsilon(\delta).$$

When δ is so small, then $(\mu_R, \mathcal{E}_R, B_1)$ is not a minimizer of J_R, a contradiction. □

2.6 Non-existing of one-phase minimizers

Now we prove the one-phase case for Theorem 1.1.2. Let's state it in the following more general way.

Theorem 2.6.1. *Assume that the double well potential $W(s)$ satisfies (W1),(W2). Then there does not exist a one-phase global minimizer of* (2.0.1) *if* $n \leq 7$.

Proof. Suppose that there does exist $u \geq -1$ so that $\{u > 1\}$ is non-empty. For simplicity, we assume $u(0) = 1$. We denote

$$\mu_R(y) = \max\{\frac{u(Ry) - 1}{R^{1/2}}, 0\}, \quad \mathcal{E}_R(y) = \min\{u(Ry), 1\}.$$

By Lemma 2.3.2 and 2.3.3, we have that $\|\mu_R\|_{C^{1/2}(B_1)} \leq C$ and $J_R(\mu_R, \mathcal{E}_R, B_1) \leq C$. By Theorem 2.5.1, up to a subsequence, $(\mu_R, \mathcal{E}_R)$ converges to $(\mu_\infty, \mathcal{E}_\infty)$, a minimizer of $I(\cdot, \cdot, B_1)$, in L^1 sense. When $n \leq 7$, the set $\mathcal{E}_\infty$ has $C^{1,\alpha}$ boundary near the origin. For example, we assume that at the origin the exterior normal vector of $\mathcal{E}_\infty$ is $\vec{e_n}$. This means, if we define

$$\Sigma(R) = \{x \in \partial B_1 : u(Rx) > 1\}, \quad e(R) = \mathcal{H}^{n-1}(\Sigma(R) \setminus \partial B_1^+)$$

like in (3.1.3), then there exists a sequence of $R_k \to \infty$, such that

$$\frac{1}{R_k^n} \int_0^{R_k} e(r) r^{n-1} dr =: \delta(R_k) \to 0. \tag{2.6.1}$$

Let $\lambda(R)$ be the first Laplace eigenvalue of $\Sigma(R)$ and $\alpha(R)$ the same as (3.1.2), then $\alpha(R) \geq 1 - C \cdot e(R)$. Let

$$I(R) = \int_{B_R} \frac{|\nabla (u-1)^+|^2}{|x|^{n-2}} dx,$$

and assume for simplicity that $I(R_0) > 0$. We integrate the lower bound

$$\frac{I'}{I}(R) \geq \frac{2\alpha(R)}{R} \geq \frac{2}{R} - 2C\frac{e(R)}{R}$$

in $[R_0, R_k]$ and obtain that

$$\log \frac{I(R_k)}{I(R_0)} \geq 2 \log \frac{R_k}{R_0} - \int_{R_0}^{R_k} \frac{2C \cdot e(r)}{r} dr.$$

Using the restriction (2.6.1), we have that (notice that $e(R) \leq \frac{n\omega_n}{2}$)

$$\int_{R_0}^{R_k} \frac{2C \cdot e(R)}{R} \leq \int_{R_0}^{R^*} \frac{1}{r} dr, \quad R^* = O(\sqrt[n]{\delta(R)} R_k),$$

so $\frac{I(R_k)}{R_k} \to \infty$. However, this violates the $C^{1/2}$ estimate (Lemma 2.3.2) if we apply the Caccioppoli inequality to $(u-1)^+$. In conclusion, we reach a contradiction, so there is no one-phase minimizer of (2.0.1) if $n \leq 7$. □

Chapter 3: Global minimizers of the double well energy: two-phase case

In this chapter, we assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a two-phase minimizer of (2.0.1). That is, both $\{u > 1\}$ and $\{u < -1\}$ are non-empty. Like in the previous chapter, we assume throughout this chapter that the potential $W(s)$ satisfies (W1) (no need to satisfy (W2)).

We will show that a two-phase global minimizer is an one-dimensional solution for any dimension n.

3.1 The ACF monotonicity formula

We start from showing

$$M(R) = \max_{B_R} |u| \tag{3.1.1}$$

has at least linear growth, meaning $M(R) \geq 2\underline{K}R$ for some $\underline{K} > 0$ and any large R.

The method is to use the ACF monotonicity formula given in [39].

Theorem 3.1.1. *Let u_+, u_- be continuous function defined in $\mathbb{R}^n$ such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0.$$

For any $R > 0$, if we define

$$I_\pm(R) = \int_{B_R} \frac{|\nabla u_\pm|^2}{|x|^{n-2}} dx, \quad J(R) = \frac{1}{R^4} I_+(R) \cdot I_-(R),$$

then $J(R)$ is non-decreasing about R.

Since we will somehow simulate its proof later in this paper, it is necessary to sketch the main idea.

Sketch proof of Theorem 3.1.1. Use the fact that $|\nabla u_\pm|^2 \le \Delta u_\pm^2/2$ and $\Delta|x|^{2-n} \le 0$, we integrate by part and have

$$I_\pm(R) = \frac{1}{R^{n-2}} \int_{\partial B_R} u_\pm \partial_r u_\pm + \frac{n-2}{2R} u_\pm^2 dx.$$

Besides, $I_\pm(R)' = \dfrac{1}{R^{n-2}} \displaystyle\int_{\partial B_R} |\nabla u_\pm|^2 dx$. Denote $\Sigma_\pm(R) = \{x \in \partial B_1 : u_\pm(Rx) > 0\}$ and $\lambda_\pm(R)$ be the first Laplace eigenvalue of $\Sigma_\pm(R)$. It follows that if $\alpha_\pm(R) \ge 0$ are defined such that

$$\alpha_\pm(R)\Big(\alpha_\pm(R) + n - 2\Big) = \lambda_\pm(R), \tag{3.1.2}$$

then $\dfrac{I_\pm'}{I_\pm}(R) \ge \dfrac{2\alpha_\pm(R)}{R}$ and thus

$$\frac{J'}{J}(R) = \frac{I_+'}{I_+}(R) + \frac{I_+'}{I_+}(R) - \frac{4}{R} \ge \frac{2\alpha_+(R) + 2\alpha_-(R) - 4}{R} \ge 0.$$

This proves that $J(R)$ is non-decreasing. The last step $\alpha_+(R) + \alpha_-(R) \ge 2$ is called the Friedland-Hayman inequality, which holds if $\Sigma_+(R) \cap \Sigma_-(R) = \emptyset$. □

Remark 2. *One can verify that if $\Sigma_\pm(R)$ happens to be a hemisphere for some R, then the corresponding $\alpha_\pm(R) = 1$.*

Remark 3. *If $\Sigma_\pm(R)$ (or simply $\Sigma(R)$) "looks like" the upper hemisphere ∂B_1^+, we denote the area of error as*

$$e(R) = \mathcal{H}^{n-1}(\Sigma(R) \setminus \partial B_1^+). \tag{3.1.3}$$

It turns out that for some $C > 0$, $\alpha(R) \ge 1 - C \cdot e(R)$.

Now we can easily show $M(R)$ has at least linear growth.

Corollary 3.1.1. *Assume that u is a two-phase solution of* (2.1.1)*, and denote $M(R) = \max\limits_{B_R} |u|$, then $M(R)$ is at least linear for $R \ge \underline{R}$. Say for example, $M(R) \ge 2\underline{K}R$ for some $\underline{K} > 0$.*

Proof. We apply the ACF monotonicity formula (Theorem 3.1.1) to the pair

$$u_+ = (u-1)^+, \quad u_- = (-1-u)^+.$$

Since u is a two-phase function, $J(\underline{R}) > 0$ for some $\underline{R}$, so $J(R) \geq J(\underline{R})$ for any $R \geq \underline{R}$. It follows that

$$\max\{I_+(R), I_-(R)\} \geq CR^2.$$

By the Caccioppoli inequality, we have $M(R) \geq 2\underline{K}R$ for $R \geq \underline{R}$. □

3.2 Asymptotic linearity

In this section, we show that u is asymptotic linear.

Proposition 3.2.1 (asymptotic linearity)**.** *Let $\underline{K}, \epsilon > 0$ be fixed. Assume that $M(R) \geq 2\underline{K}R$ and $M(R) = o(R^2)$ for R sufficiently large. Then after a Euclidean group action, there exist subsequences $R_k \to \infty$, $\delta_k \to 0$ and $K = K(R_k) \geq \underline{K}$, such that in B_{R_k},*

$$U_{K(R_k)}(x_n - \delta_k R_k) \leq u(x) \leq U_{K(R_k)}(x_n + \delta_k R_k). \tag{3.2.1}$$

The key observation is that, as a minimizer of the double well energy 2.0.1, the rescaling of u looks harmonic as $R \to \infty$, so we can use the Schauder estimate for harmonic functions to show the asymptotic behavior.

We first show the following Hölder estimate, which guarantees the rescaling $\dfrac{u(Ry)}{M(R)}$ have a uniformly convergence limit.

Lemma 3.2.1 (Hölder estimate)**.** *Assume that $|\widetilde{W}(u)| \leq 1$ everywhere ($\widetilde{W}$ is not necessarily equal to W). If u is a minimizer in B_1 of the functional*

$$J(u, \Omega) = \int_\Omega \frac{|\nabla u|^2}{2} + \widetilde{W}(u)dx$$

and $\|u\|_{L^\infty(B_1)} \leq 1$, then $\|u\|_{C^\alpha(B_{1/2})} \leq C$ for some $\alpha > 0$ and $\|u\|_{H^1(B_{1/2})} \leq C$.

Proof. It suffices to show $osc(u, B_r) \leq Cr^\alpha$ in L^2_{avg} sense, meaning that there exists $A(r)$ satisfy-

ing

$$\int_{B_r} |u - A(r)|^2 dx \leq Cr^{n+2\alpha}.$$

Let r_0 be small enough, then still $|u| \leq 1$ in B_{r_0}. Let v_k be the harmonic replacement of u with Dirichlet boundary condition in B_R ($R = 2^{-k}r_0$), then $osc(v_k, B_{R/2}) \leq (1-\delta)osc(u, B_R)$. As $J(u, B_R) \leq J(v, B_R)$, we have

$$0 \leq \int_{B_R} |\nabla u|^2 - |\nabla v_k|^2 dx \leq 2|B_R|.$$

This already implies $\|u\|_{H^1(B_{1/2})} \leq C$. In fact, as $u = v_k$ on ∂B_R and $\Delta v_k = 0$, we have $\int_{B_R} |\nabla u|^2 - |\nabla v_k|^2 dx = \int_{B_R} |\nabla(u - v_k)|^2 dx$. By Poincare inequality,

$$\int_{B_R} |u - v_k|^2 \leq CR^{n+2}, \quad \text{i.e. } osc_{avg}(u - v_k, B_R) \leq CR.$$

Therefore, if we denote $O_k = osc_{avg}(u, B_{2^{-k}r_0})$, then

$$O_{k+1} \leq (1-\delta)O_k + C \cdot 2^{-k} r_0.$$

As long as r_0 is small enough, $O_k \leq C(1-\delta/2)^k$. □

We use the Hölder regularity to show that the rescale of $u(x)$ is harmonic. In fact, the method is quite similar to the proof of Γ-convergence in the previous chapter.

Lemma 3.2.2. *Let the rescaling function of $u(x)$ be*

$$v_R(y) = \frac{u(Ry)}{M(R)}, \quad y \in B_1.$$

Then for some subsequence $R_k \to \infty$, $v_{R_k}(y)$ converges uniformly in $B_{1/2}$ to a harmonic function.

Proof. It follows that $|v_{R_k}(y)| \leq 1$ and satisfies the equation $\Delta_y v_{R_k}(y) =$ then $v_R(x)$ is a minimizer

in B_1 of

$$\widetilde{J}_{R_k}(u,\Omega)=\int_\Omega \frac{|\nabla_y u|^2}{2}+\widetilde{W}_{R_k}(u)dy,\quad \widetilde{W}_{R_k}(u)=\frac{R^2}{M(R)^2}W(M(R_k)u).$$

Since $M(R)\geq 2\underline{K}R$, by Lemma 3.2.1, for a subsequence of R_k (still written as R_k), $v_{R_k}(y)\to v(y)$ in $C^\alpha_{loc}(B_1)\cap L^2_{loc}(B_1)$ for some $v(y)\in H^1_{loc}(B_1)$.

Now we show that $v(y)$ is harmonic. Assume that $v(y)$ is non-constant near a point y_0. We can then find some $r_0\in(0,\frac{1-|y_0|}{2})$ so that $v(y)$ is not everywhere zero in $\partial B_r(y_0)$ for any $r\in(r_0,2r_0)$. Let $\bar{v}(y)$ be a harmonic replacement in $B_r(y_0)$, still equal to $v(y)$ outside $B_r(y_0)$. Suppose that v is not harmonic at y_0, then there exists some $c_0>0$ so that

$$\int_{B_r(y_0)}\frac{|\nabla v|^2}{2}dy\geq\int_{B_r(y_0)}\frac{|\nabla\bar{v}|^2}{2}dy+c_0.$$

Since $\bar{v}$ is almost everywhere non-zero in $B_r(y_0)$, for k sufficiently large,

$$\int_{B_r(y_0)}\widetilde{W}_{R_k}(\bar{v})dy\leq C\Big|B_r(y_0)\cap\{|\bar{v}|\leq M(R_k)^{-1}\}\Big|\leq\frac{c_0}{100}.$$

Let N be sufficiently large and $r_m=\frac{N+m}{N}r_0$. Since $\|\bar{v}_{R_k}\|_{H^1(B_{2r_0}(y_0))}\leq C$, there must exists some $0\leq m<N$ such that

$$\int_{B_{r_{m+1}}(y_0)\setminus B_{r_m}(y_0)}\frac{R_k^2}{M(R_k)^2}+\frac{|\nabla v_{R_k}|^2+|\nabla v|^2}{2}dy\leq\frac{c_0}{100}.$$

Let $\eta\geq 0$ be a cut-off function supported in $B_{r_{m+1}}(y_0)$ such that $\eta\Big|_{B_{r_m}(y_0)}=1$ and $|\nabla\eta|\leq 2N/r_0$. We construct a competitor

$$w_k(y)=\eta(y)\bar{v}(y)+(1-\eta(y))v_{R_k}(y).$$

Since $\widetilde{J}_{R_k}(w, B_{r_{m+1}}(y_0)) \geq \widetilde{J}_{R_k}(v_{R_k}, B_{r_{m+1}}(y_0))$, we have

$$\int_{B_{r_{m+1}}(y_0)\setminus B_{r_m}(y_0)} \frac{|\nabla w|^2}{2} dy \geq \widetilde{J}_{R_k}(w, B_{r_{m+1}}(y_0) \setminus B_{r_m}(y_0)) - \frac{c_0}{100} \geq \frac{49}{50} c_0.$$

On the other hand,

$$\frac{|\nabla w|^2}{2} \leq 4|\nabla v_{R_k}|^2 + 2|\nabla v|^2 + 8\frac{N^2}{r_0^2}|v - v_k|^2,$$

and for k sufficiently large, $|v - v_k| \to 0$ in L^2 sense, so

$$\int_{B_{r_{m+1}}(y_0)\setminus B_{r_m}(y_0)} \frac{|\nabla w|^2}{2} dy \leq \frac{c_0}{2}$$

for k sufficiently large, a contradiction. □

Harmonic functions have good regularity. Since the limit v above is harmonic in B_1 and $\|v\|_{L^\infty(B_1)} \leq 1$, it follows that $v \in C^2(B_{1/2})$ with $\|D^2 v\|_{L^\infty(B_{1/2})} \leq C$. Yet we still need to avoid harmonic functions with vanishing gradient at the origin. This is where the sub-quadratic growth rate assumption plays a role.

Lemma 3.2.3. *Let $C, \epsilon > 0$ be fixed. Assume that*

$$\liminf_{R\to\infty} \frac{M(R)}{R^{2-\epsilon}} < \infty,$$

then for any $\delta < \delta_0 = \min\{C^{-1/\epsilon}, 1/2\}$, there is a subsequence $R_k \to \infty$ such that

$$M(\delta R_k) \geq C\delta^2 M(R_k). \tag{3.2.2}$$

Proof. We argue by contradiction, suppose $M(\delta R) < C\delta^2 M(R)$ for all $R \geq \underline{R}$, then we set $r_k = \delta^{-k}\underline{R}$ and $m_k = M(r_k)$, and obtain that

$$m_{k-1}/m_k < C\delta^2 < \delta^{2-\epsilon}, \forall k \geq 1.$$

Therefore, $M(r_k) \gg r_k^{2-\epsilon}$ for $r_k = \delta^{-k}\underline{R}$, a contradiction. □

This results in non-zero gradient of the limit v at the origin, since the harmonic function v has a $C^{1,1}$ estimate near the origin. We have:

Corollary 3.2.1. *Assume $M(R) \leq R^{2-\epsilon}$ for R sufficiently large. Then the limit v obtained as the limit of $v_R(y) = \dfrac{u(Ry)}{M(R)}$ has non-zero gradient at the origin.*

Now we prove Proposition 3.2.1, under the assumption that $M(R) \leq R^{2-\epsilon}$ for some subsequence $R_k \to \infty$. Without loss of generality, we assume $u(0) = 0$.

Proof of Proposition 3.2.1, $R^{2-\epsilon}$ case. By the previous corollary, for a subsequence $R_k \to \infty$, the limit $v(y)$ is harmonic with $\nabla v(0) \neq 0$. After a coordinate change, we might assume that

$$v(y) = K_0 y_n + O(|y|^2), \quad K_0 > 0.$$

Finally we show that for every $\delta > 0$, there exists an $R > 0$ and $K(R) \geq \underline{K}$ such that

$$U_{K(R)}(x_n - \delta R) \leq u(x) \leq U_{K(R)}(x_n + \delta R).$$

First, let $\lambda = \lambda(\delta) < 1$ satisfies that for any $y \in B_\lambda$,

$$K_0(y_n - \delta\lambda/4) \leq v(y) \leq K_0(y_n + \delta\lambda/4),$$

then by $\|v_{R_k} - v\|_{C^\alpha(B_{1/2})} \to 0$, for k sufficiently large,

$$K_0(y_n - \delta\lambda/2) \leq v_{R_k}(y) \leq K_0(y_n + \delta\lambda/2).$$

We just set $R = \lambda R_k$, then for $x \in B_R$, we have

$$K_0\frac{M(R_k)}{R_k}(x_n - \delta R/2) \leq u(x) \leq K_0\frac{M(R_k)}{R_k}(x_n + \delta R/2).$$

Set $K(R) = K_0 \dfrac{M(R_k)}{R_k}$. Notice that $M(R) \geq 2\underline{K}R$ for large R, we have $K(R) \geq \underline{K}$. For $K(R) \geq \underline{K}$, $U_{K(R)}(x_n - 2\overline{L}) \leq K(R)x_n \leq U_{K(R)}(x_n + 2\overline{L})$ for $\overline{L} = 1/\underline{K}$, so we obtain (3.2.1). □

Finally, we prove Proposition 3.2.1 in another setting that $M(R) = o(R^2)$ but $M(R) \geq R^{1.5}$ for large R. This case, together with the previous one where $\liminf \dfrac{M(R)}{R^{2-\epsilon}} < \infty$, cover all possibilities of $M(R) = o(R^2)$.

Proof of Proposition 3.2.1, $M(R) = o(R^2)$ and $M(R) \geq R^{1.5}$ case. Without loss of generality, we assume $u(0) = 0$. We compare $v_R(y)$ with its harmonic replacement $\widetilde{v_R}(y)$ in B_1, like what is done in Lemma 3.2.1. It follows that as $\dfrac{R^2}{M(R)^2} \leq \dfrac{1}{R}$, we have $\displaystyle\int_{B_1} |v_R - \widetilde{v_R}|^2 \leq C/R$ with $\|\widetilde{v_R}\|_{C^{2,1}(B_{1/2})} \leq C$. After a rotation action, let's write

$$\widetilde{v_R}(y) = \widetilde{v_R}(0) + k_R y_n + y^T A_R y + O(|y|^3).$$

Here, k_R and A_R are bounded, regardless of R. Recall that from Lemma 3.2.1 we have $\|v_R\|_{C^\alpha(B_{1/2})} \leq C$, so $|v_R - \widetilde{v_R}| \leq R^{-3\delta}$ in $B_{1/2}$ for some small $\delta > 0$. It then follows that, after a rotation action (recall that $v_R(0) = 0$),

$$|v_R(y) - k_R y_n - y^T A_R y| \leq CR^{-3\delta} \text{ in } B_{R^{-\delta}}. \tag{3.2.3}$$

The slope k_R is just known as non-negative. Based on its value, we consider the following two possibilities:

- Case 1: $k_R\sqrt{R} \geq c_0 > 0$ at least for a subsequence of $R = R_i \to \infty$. In this case, (3.2.3) written in the $x = R_i y$ coordinate, implies that

$$|u(x) - k_{R_i} \frac{M(R_i)}{R_i} x_n| \leq CM(R_i)R_i^{-2\delta} \text{ in } B_{R_i^{1-\delta}}.$$

  Let $K = k_{R_i} \dfrac{M(R_i)}{R_i}$, which is greater than c_0, and write $r_i = R_i^{1-\delta}$, then

$$\frac{1}{K}\|u_x - Kx_n\|_{L^\infty(B_{r_i})} \leq CR_i^{0.5-2\delta} = o(r_i).$$

This again implies the expression (3.2.1) since $r_k \to \infty$.

- Case 2: Otherwise, $k_R\sqrt{R} \to 0$, which means we can pretend that $k_R = 0$ in (3.2.3), that is, for any R large enough,

$$|v_R(y) - y^T A_R y| \le CR^{-3\delta} \text{ in } B_{R^{-\delta}}.$$

The rest of the proof will focus on Case 2. We will see that Case 2 violates the assumption that $M(R) = o(R^2)$.

If for a certain R, $\|A_R\|$ is uniformly non-degenerate, say $\|A_R\| \ge H$ for some small constant $H > 0$, then in this situation, we have

$$\frac{M(R^{1-\delta})}{M(R^{1-\delta}/2)} \ge \frac{H - CR^{-\delta}}{H/4 + CR^{-\delta}} \ge 4(1 - CR^{-\delta}).$$

On the other hand, if for a certain R, $\|A_R\| \le H$, then when H is small, by $\max_{B_1} |v_R(y)| = 1$ we get

$$\frac{M(R)}{M(R^{1-\delta}/2)} \ge (\frac{R}{R^{1-\delta}/2})^2$$

With this in mind, let's construct a sequence $r_k \to \infty$, that

$$N(r_k) = M(r_k)/r_k^2$$

is bounded from below. We start with a sufficiently large r_0. For each chosen $r_k (k \ge 0)$, we write $R = \sqrt[1-\delta]{2r_k}$. We choose

$$r_{k+1} = \begin{cases} 2r_k, \text{ if } \|A_R\| \ge H \\ R = \sqrt[1-\delta]{2r_k}, \text{ if } \|A_R\| < H \end{cases}.$$

We see that $N(r_{k+1}) \ge (1 - Cr_k^{-\delta})N(r_k)$ in the first situation and $N(r_{k+1}) \ge N(r_k)$ in the second one. Therefore, $N(r_k)$ is bounded from below under such a construction. This violates the assumption that $M(R) = o(R^2)$. □

3.3 A Harnack type estimate

In this section, we will prove the following Harnack type estimate.

Proposition 3.3.1 (a Harnack type estimate)**.** *For any fixed $\underline{K} > 0$, there exist constants $c, \underline{R}, \sigma > 0$. Let $R > \underline{R}$, $K \geq 2\underline{K}$, $t \leq \sigma R$. Assume that*

$$U_K(x_n - t) \leq u(x) \leq U_K(x_n + t) \text{ in } B_{2R},$$

then there exist t_1, t_2 with $|t_1 - t_2| \leq 2(1-c)t$ such that

$$U_K(x_n + t_1) \leq u(x) \leq U_K(x_n + t_2) \text{ in } B_{cR}.$$

Assuming that Proposition 3.3.1 is correct, we consider a baby case version of (3.2.1).

Corollary 3.3.1. *Assume that u satisfies* (2.1.1) *and there exist $K, t > 0$ such that after a Euclidean group action,*

$$U_K(x_n - t) \leq u(x) \leq U_K(x_n + t), \quad \forall x \in \mathbb{R}^n, \tag{3.3.1}$$

or "almost equivalently",

$$K(x_n - t) \leq u(x) \leq K(x_n + t), \quad \forall x \in \mathbb{R}^n,$$

then u is one-dimensional, meaning that $u(x) = U_K(x_n - s)$ everywhere in $\mathbb{R}^n$ for some constant $s \in [-t, t]$.

Proof. Assume that there exists some λ and $\underline{R}$, so that for any $R \geq \underline{R}$,

$$U_K(x_n + t_1(R)) \leq u(x) \leq U_K(x_n + t_2(R)) \tag{3.3.2}$$

in $B_{\lambda R}$, where $|t_1(R) - t_2(R)| \leq t/10$. Let

$$t_1^* = \inf_{R \geq \underline{R}} t_1(R), \quad t_2^* = \sup_{R \geq \underline{R}} t_2(R),$$

then for any $x \in \mathbb{R}^n$ we always have

$$U_K(x_n + t_1^*) \leq u(x) \leq U_K(x_n + t_2^*).$$

Let's plug in $x = 0$ in (3.3.2), to avoid "conflict" among different R's, we have

$$\bigcap_{R \geq \underline{R}} [t_1(R), t_2(R)] \neq \emptyset.$$

As $|t_1(R) - t_2(R)| \leq t/10$, we conclude that $|t_1^* - t_2^*| \leq t/5$. Now it suffices to repeatedly iterate the process above. □

In the rest of this section, we prove Proposition 3.3.1.

Recall that we have constructed a volcano-shape barrier function (2.2.1) in Chapter 2. Here we use it again in a different way to construct the following barrier function.

Lemma 3.3.1 (barrier function B)**.** *Given any $\underline{K} > 0$, there exist constants $\underline{R}, c_0 > 0$. For any $R \geq \underline{R}$ and $\underline{K} < K_1 < K_2$ satisfying*

$$\delta := \frac{K_2}{K_1} - 1 \leq 0.1,$$

there exists a barrier function $\Phi(x)$ defined in B_R satisfying:

(1) $\Delta\Phi(x) > W'(\Phi(x))$ in B_R,

(2) $\Phi(x) \leq U_{K_1}(x_n)$ in $\partial B_R \cap \{x_n \leq c_0 \delta R\}$,

(3) $\Phi(x) \leq U_{K_2}(x_n)$ in $\partial B_R \cap \{x_n \geq c_0 \delta R\}$,

(4) $\Phi(x) \geq U_{K_1}(x_n + c_0\delta R)$ *in* $B_{c_0 R}$. *Consequently,* $\Phi(x) \geq U_{K_2}(x_n + c_0'\delta R)$ *in* $B_{c_0' R}$ *for* $c_0' = c_0/100$.

Proof. Let $N = \max\{1000, 1000M(\underline{K}), N_0\}$. $M(\underline{K}), N_0$ will be given later.

- Step 1: We set $K = \dfrac{K_1 + K_2}{2}$, $\epsilon = \dfrac{\delta}{N}$, $c_0 = \dfrac{1}{100N}$, and construct the barrier function

$$\Phi(x) = V_{K, \frac{NR}{\delta}, \epsilon}(x - \vec{p}), \text{ where } \vec{p} = (\frac{NR}{\delta} - \frac{\delta R}{4N})\vec{e_n}$$

  as a local "view" of the "volcano" in (2.2.1). By Lemma 2.2.2, (1) is satisfied as long as we choose $\underline{R} = 1000\dfrac{(n-1)}{\underline{K}}$ (one can repeat the proof in Lemma 2.2.1).

- Step 2.1: Now we prove (2) and (3). First we notice that when $N \geq N_0$ for some uniform N_0, (2) holds for $x = -R\vec{e_n}$ and (3) holds for $x = R\vec{e_n}$.

- Step 2.2: Next, on ∂B_R, the distance $\rho = dist(x, \vec{p})$ takes the form

$$\rho = \sqrt{(\frac{NR}{\delta} + \frac{\delta R}{4N})^2 - 2(\frac{NR}{\delta} - \frac{\delta R}{4N})x_n}, \text{ so } |\frac{d\rho}{dx_n}\Big|_{\partial B_R} + 1| \leq 10\frac{\delta}{N}$$

  as long as $N \geq 1000$. By Lemma 2.2.2 we know

$$|\frac{\frac{d}{d\rho}\Phi}{U_K'(U_K^{-1}(\Phi))} + 1| \leq \epsilon = \frac{\delta}{N} \text{ when } |\Phi| \leq 1.$$

  As $K_2/K, K/K_1 \geq 1 + \delta/3$ if $\delta \leq 0.1$, by Lemma 2.1.1, there exists a large $M(\underline{K})$ such that

$$|\frac{U_{K_i}'(U_{K_i}^{-1}(h))}{U_K'(U_K^{-1}(h))} - 1| \geq \frac{\delta}{M(\underline{K})}, \forall h \in \mathbb{R}.$$

  We let $N \geq 1000M(\underline{K})$, then by chain rule

$$\frac{d\Phi}{dx_n}\Big|_{\partial B_R} = \frac{d\Phi}{d\rho}\frac{d\rho}{dx_n} \begin{cases} \geq U_{K_1}'(U_{K_1}^{-1}(\Phi)), \text{ if } \Phi \in [-1, 0] \\ \leq U_{K_2}'(U_{K_2}^{-1}(\Phi)), \text{ if } \Phi \in [0, 1] \end{cases}.$$

In other words, on ∂B_R, $\Phi(x) \le U_{K_1}(x_n)$ when $\Phi \in [-1, 0]$ and $\Phi(x) \le U_{K_2}(x_n)$ when $\Phi \in [0, 1]$.

- Step 2.3: Denote by $h_i (i = 0, \pm 1)$ so that x_n-component of $\vec{q}_i \in \partial B_R$ is h_i, where at q_i holds $\Phi(\vec{q}_i) = i$. Then we have $h_0 > c_0 \delta R$, meaning that (2) holds if $x_n \in [h_{-1}, h_0]$ and (3) holds if $x_n \in [h_0, h_1]$. From the expression of $\rho(x_n)$, $\Phi\Big|_{\partial B_R}$ is convex about x_n when $|\Phi(x)| \ge 1$, while $U_{K_i}(x_n)$ is linear about x_n, so we finally obtain (2) and (3) holds for generic points on ∂B_R.

- Step 3: Since Φ is a radial function centered at $\vec{p}$, it suffices to show (4) on $\partial B_{c_0 R}$. On $\partial B_{c_0 R}$, the distance function $\rho = dist(x, \vec{p})$ also has the form $\sqrt{A - Bx_n}$, so we can use the chain rule to check that on $\partial B_{c_0 R}$.

$$\frac{d}{dx_n}\Phi(x)\Big|_{\partial B_{c_0 R}} \ge U'_{K_1}(U^{-1}_{K_1}(\Phi)).$$

  Besides, one can check that (4) holds at the point $x = -c_0 R\vec{e_n}$, so (4) holds everywhere on $\partial B_{c_0 R}$.

□

Remark 4. *For the $\Phi(x)$ constructed above, (2)-(4) are automatically satisfied if we replace c_0 with any other $c_0' < c_0$. It also means we can simply write $\Phi(x) \ge U_{K_2}(x_n + c_0 \delta R)$ in $B_{c_0 R}$ in (4) by choosing a smaller c_0.*

Proposition 3.3.1 is obtained from the following lemma (sometimes we need to replace $u(x)$ with $-u(-x)$, $W(s)$ with $W(-s)$).

Lemma 3.3.2. *For any fixed $\underline{K} > 0$, there exist constants $c, \underline{R}, \sigma > 0$. Let $R > \underline{R}$, $K \ge 2\underline{K}$, $t \le \sigma R$, and denote $L = L(K)$. Assume that $u(x) \ge U_K(x_n)$ in B_{2R} and $u(x) \le -1$ when $x_n \le -L - 2t$. If*

$$u(R\vec{e_n}) \ge U_K(R) + Kt \quad or \quad u(-R\vec{e_n}) \ge U_K(-R) + Kt,$$

then $u(x) \geq U_K(x_n + ct)$ *in* B_{cR}.

Proof. We first copy from the last lemma the constants c_0 and $\underline{R}$. Consider two possibilities: $t \leq L$, or $t \geq L$.

- If $t \leq L$. We apply the Hopf lemma to $u(x) - U_K(x_n)$ in the region $B_{2R} \cap \{x_n \geq 3L\}$ or $B_{2R} \cap \{x_n \leq -3L\}$, and obtain (recall that R is way greater than L, t)

$$\begin{aligned} u(x) &\geq U_K(x_n) + c_1 \frac{Kt}{R}(x_n - 3L) \quad \text{in} \quad B_R \cap \{x_n \geq 3L\}, \\ \text{or} \quad u(x) &\geq U_K(x_n) + c_1 \frac{Kt}{R}(x_n + 3L) \quad \text{in} \quad B_R \cap \{x_n \leq -3L\}. \end{aligned}$$

  In particular, if $x \in B_R \cap \{x_n = \pm 4L\}$, then $u(x) \geq U_K(x_n) + c_1 \frac{Kt}{R} L$. By applying the Harnack inequality to $v(x) = u(x) - U_K(x_n)$, which satisfies

$$|\Delta v(x)| \leq \max\{|W''|\}|v(x)|$$

  in all balls $B_{10\overline{L}}(p)$ centered at $p \in B_R \cap \{x_n = 0\}$, we obtain that

$$u(x) \geq U_K(x_n) + c_2 \frac{Kt}{R} L \text{ in } B_R \cap \{|x_n| \leq 5L\}.$$

  Consequently, by setting $K_\pm = (1 \pm \delta)K$ where $\delta = c_3 t/R$, we have

$$u(x) \geq U_{K_+}(x_n) \text{ in } B_R^+, \quad \text{or} \quad u(x) \geq U_{K_-}(x_n) \text{ in } B_R^-.$$

  By Lemma 3.3.1 and the maximal principle, $u(x) \geq U_K(x_n + c_0 \delta R)$ in $B_{c_0 R}$ and we choose $c = c_0 c_3$.

- If $t \geq L$. We again apply the Hopf lemma to $u(x) - U_K(x_n)$ in $B_{2R} \cap \{x_n \geq 3t\}$ or $B_{2R} \cap \{x_n \leq$

$-3t\}$, and obtain

$$u(x) \geq U_K(x_n) + c_1 \frac{Kt}{R}(x_n - 3t) \quad \text{in} \quad B_R \cap \{x_n \geq 3t\},$$
$$\text{or} \quad u(x) \geq U_K(x_n) + c_1 \frac{Kt}{R}(x_n + 3t) \quad \text{in} \quad B_R \cap \{x_n \leq -3t\}.$$

Denote $K_\pm = (1 \pm \delta)K$, where $\delta = c_1 t/R$. It then follows that $u(x) \geq U_{K_\pm}(x_n - t_\pm)$ in B_R, where $0 \leq t_\pm \leq C_4 t^2/R$. By Lemma 3.3.1 and the maximal principle,

$$u(x) \geq U_K(x_n + c_0\delta R - t_\pm) \text{ in } B_{c_0 R}(-t_\pm \vec{e_n}).$$

One can set $\sigma = \min\{\frac{1}{100}, \frac{c_0 c_1}{100 C_4}\}$ so that $c_0\delta R \geq 100 t_\pm$, and then can set $c = c_0 c_1/100$, resulting in $u(x) \geq U_K(x_n + ct)$ in B_{cR}.

□

3.4 Improvement of flatness

In this section, we reduce the asymptotic linearity (3.2.1) given in Proposition 3.2.1 to its baby case version (3.3.1).

Without loss of generality, by choosing δ_k in (3.2.1) slightly larger, we assume that $\delta_k R_k \to \infty$, and $u(0) = 0$.

Our goal in this section is to show the following improvement of flatness result.

Lemma 3.4.1 (Improvement of flatness)**.** *Let $\underline{K}$ be given, then there exists $\underline{T}, \overline{\delta}, c_0 > 0$. For any $K \geq \underline{K}$, $\delta \leq \overline{\delta}$ and $R > 0$ satisfying $\delta R \geq \underline{T}$, if $u(x)$ as a solution of* (2.1.1) *satisfies*

$$K(x_n - \delta R) \leq u(x) \leq K(x_n + \delta R) \text{ in } B_R$$

then up to a Euclidean group action, there is another K^* such that

$$K^*(x_n - c_0\delta R) \leq u(x) \leq K^*(x_n + c_0\delta R) \text{ in } B_{c_0 R}.$$

As a preparation, we use the volcano-shape function (2.2.1) one more time to construct the third barrier function.

Lemma 3.4.2 (barrier function C)**.** *Let $\underline{K} > 0$ be fixed, then there exist constants $\underline{R}, \underline{T}, c_0 > 0$. For any $R > \underline{R}$ and any $\underline{K} \le K_1 < K_2$ such that*

$$\underline{T}/R \le \delta := \frac{K_2}{K_1} - 1 \le 0.1,$$

there is a barrier function $\Psi(x)$ defined in B_R satisfying

(1) $\Delta\Psi(x) > W'(\Psi(x))$ *in* B_R,

(2) $\Psi(x) \le K_1(x_n - c_0 R)$ *in* $\partial B_R \cap \{x_n \le c_0 R\}$,

(3) $\Phi(x) \le K_2(x_n - c_0 R)$ *in* $\partial B_R \cap \{x_n \ge c_0 R\}$,

(4) $\Phi(0) \ge K_1 c_0 R$.

Proof. Let $K = \dfrac{K_1 + K_2}{2}$ and $N > 100$ to be decided later. We construct the barrier function

$$\Psi(x) = V_{K,NR,\frac{\delta}{100}}(x - \vec{p}), \text{ where } \vec{p} = (NR - \frac{R}{4N})R\vec{e}_n.$$

When $\underline{T} \ge 1000(n-1)/\underline{K}$, we have (1) is satisfied. The remaining proof is similar to Lemma 3.3.1. □

We will prove Lemma 3.4.1 through a compactness argument. Therefore, we develop the following Lemma, in order to show that the scaling

$$\frac{u_i(R_i y) - K_i R_i y_n}{K_i \delta_i R_i}$$

converges up to a subsequence $R_i \to \infty$ to a harmonic function.

Lemma 3.4.3. *Assume that $\delta_i \to 0$, $\delta_i R_i \to \infty$, $K_i \geq \underline{K} > 0$. Assume that u_i satisfies* (2.1.1) *and $U_{K_i}(x_n - \delta_i R_i) \leq u_i(x) \leq U_{K_i}(x_n + \delta_i R_i)$ in B_{R_i}. Define $v_i(y)$ in B_1 as*

$$v_i(y) = \frac{u_i(R_i y) - U_{K_i}(R_i y_n)}{K_i \delta_i R_i},$$

Then there exists some $v(y) \in C^\alpha(B_{1/2})$ satisfying

(1) $v_i(y)$ has a subsequence converging uniformly to $v(y)$ in $B_{1/2}$ and $|v(y)| \leq 1$ everywhere,

(2) $v(y)$ is also the uniform convergence limit (up to a subsequence) of

$$\widetilde{v}_i(y) = \frac{u_i(R_i y) - K_i R_i y_n}{K_i \delta_i R_i},$$

(3) $v(y)$ is harmonic in viscosity sense.

Proof. Let $\overline{L} = 1/\underline{K}$. Recall that $U'_K \leq CK$ for some $C = C(\underline{K})$, we have $|v(y)| \leq C(\underline{K})$ everywhere. We also notice that $v_i(y)$ is harmonic and

$$|v_i(y)| \leq 1 \text{ in } B_1 \cap \{|y_n| \geq \delta_i + \overline{L}/R_i\}.$$

Therefore, if $v(y)$ is the limit of $v_i(y)$, then $|v(y)| \leq 1$ everywhere in $B_{1/2}$ and $v(y)$ is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$ in viscosity sense.

(1) Assume that $u_i(p) = 0$ for some $p \in B_{1/2}$, for simplicity suppose $p = 0$. For every $r > 0$ we denote $t_1(r) = \max\{t_1\}$, $t_2(r) = \min\{t_2\}$ such that

$$U_K(x_n + t_1) \leq u(x) \leq U_K(x_n + t_2) \text{ in } B_r(p).$$

Write $r_k = (c/100)^k R_i$ and $O_k = t_2(r_k) - t_1(r_k)$, then $O_0 = 2\delta_i R_i$. By Proposition 3.3.1, we have $O_{k+1} \leq (1 - c)O_k$ as long as $r_k \leq \sigma O_k$ and $r_k \geq \underline{R}$, or in other words when

$$k \leq \overline{k_i} = C^{-1} \min\{-\log(\delta_i), \log(R_i)\}.$$

As $U'_K \le C(\underline{K})K$, it then follows that for $q = p/R_i$,

$$\underset{B_{(c/100)^k}(q)}{osc} v_i(y) \le C(1-c)^k \text{ for } k \le \overline{k_i}.$$

By applying the Schauder estimate to v_i in $B_1 \cap \{|y_n| \ge \delta_i + \overline{L}/R_i\}$, we conclude that there is a sequence $\underline{d_i} \to 0$ such that

$$\frac{|v_i(y_1) - v_i(y_2)|}{|y_1 - y_2|^\alpha} \le C, \quad \forall y_1, y_2 \in B_{1/2} \text{ with } |y_1 - y_2| \ge \underline{d_i}.$$

Recall that $|v_i| \le C(\underline{K})$ everywhere, so up to a subsequence, v_i Hölder convergences to some $v \in C^\alpha(B_{1/2})$.

(2) Since $\delta_i R_i \to \infty$, $|\frac{U_{K_i}(R_i y_n) - K_i R_i y_n}{K_i \delta_i R_i}| \le \frac{1}{\underline{K}\delta_i R_i} \to 0$ uniformly.

(3) We only need to show v is harmonic at any point $q \in B_{1/2} \cap \{y_n = 0\}$. We argue by contradiction, suppose that $\Delta v(q) > 0$ (or < 0) in viscosity sense, meaning that there exists a sub-harmonic quadratic polynomial

$$P(y) = v(q) + (y-q)\cdot \vec{g} + (y-q)^T M(y-q) + 2\epsilon_1 |y-q|^2,$$

such that M is a quadratic form with zero trace, and $P(y) \le v(y)$ in a small neighbourhood of q. As $v(y)$ is harmonic if $y_n \ne 0$, by Hopf lemma, we may just assume $v(y) \ge Q(y) + \epsilon_1 |y-q|^2$ in $B_s(q)$ with

$$Q(y) = v(q) + (y-q)\cdot \vec{g} + (y-q)^T M(y-q) + 5\epsilon_2 |y_n|.$$

Let c_0 be the same as in Lemma 3.4.2, we can manually reset s small so that

$$s \le 0.001 \frac{c_0 \epsilon_2}{1 + \|M\|}.$$

Since $\widetilde{v}_i(y) \to v(y)$ uniformly, there is some small $h \in (0,1)$, so that when i is sufficiently large, $\widetilde{v}_i(y) \geq Q(y) + 2h$ on $\partial B_s(q)$ and $\widetilde{v}_i(q) \leq Q(q) + h$.

Let's now fix a large i. Denote

$$\mu(z) = u_i(z + R_i q),$$

then μ is still a solution of (2.1.1). Denote

$$\vec{K}^* = K_i \cdot (\vec{e_n} + \delta_i \vec{g}), \quad K^* = |\vec{K}^*|, \quad \vec{\nu} = \vec{K}^*/K^*.$$

With these, $\widetilde{v}_i(q) \leq Q(q) + h$ implies that (recall that $|v|, h \leq 1$)

$$\mu(0) \leq 2\delta_i K_i R_i,$$

while $\widetilde{v}_i(y) \geq Q(y) + 2h$ on $\partial B_s(q)$ implies that on ∂B_{sR_i},

$$\mu(z) \geq -\delta_i K_i R_i + z \cdot \vec{K}^* + \delta_i K_i (\frac{1}{R_i} z^T M z + 5\epsilon_2 |z_n|). \tag{3.4.1}$$

If i is large enough, then $\dfrac{K^*}{K_i} \in (\dfrac{9}{10}, \dfrac{11}{10})$, and

$$\frac{1}{R_i} z^T M z \geq -\epsilon_2 R_i s \text{ in } B_{sR_i}.$$

Besides, if $z \in \partial B_{sR_i}$ and $z \cdot \vec{\nu} \geq c_0 s R_i$, then

$$|z_n| \geq \frac{1}{2}|z \cdot \vec{\nu}| \geq c_0 \frac{sR_i}{2}, \quad \text{so } \frac{1}{R_i} z^T M z + 5\epsilon_2 |z_n| \geq \frac{K^*}{K_i} \epsilon_2 |z \cdot \vec{\nu}|.$$

Therefore, for i large, (3.4.1) implies

$$\mu(z)\Big|_{\partial B_{sR_i}} \begin{cases} \geq K^*(z \cdot \vec{\nu} - c_0 s R_i), \text{ if } z \cdot \vec{\nu} \leq c_0 s R_i \\ \geq (1 + \epsilon_2 \delta_i) K^*(z \cdot \vec{\nu} - c_0 s R_i), \text{ if } z \cdot \vec{\nu} \geq c_0 s R_i \end{cases}.$$

By Lemma 3.4.2 and the maximal principle, we expect $\mu(0) \geq c_0 K^* s R_i$ if i is large, but this violates $\mu(0) \leq 2\delta_i K_i R_i$.

□

Now let's prove Lemma 3.4.1.

Proof of Lemma 3.4.1. If such $\underline{T}, \overline{\delta}$ does not exist, then we have a sequence of counter examples u_i defined in B_{R_i}, and $\delta_i \to 0$ satisfying $\delta_i R_i \to \infty$, so that

$$K_i(x_n - \delta_i R_i) \leq u_i(x) \leq K_i(x_n + \delta_i R_i) \text{ in } B_{R_i}.$$

By Lemma 3.4.3, we construct a sequence

$$\widetilde{v}_i(y) = \frac{u_i(R_i y) - K_i R_i y_n}{K_i \delta_i R_i},$$

whose limit is a harmonic function v so that $\|v\|_{B_{1/2}} \leq 1$. By Schauder estimate,

$$|v(y) - v(0) - y \cdot \nabla v(0)| \leq C_1 |y|^2 \text{ in } B_{1/4}$$

Denote $\max_{B_{1/4}} |\widetilde{v}_i(y) - v(y)| = \epsilon_i \to 0$. We then approximate u_i with

$$L_i(x) = \delta_i K_i R_i v(0) + K_i x_n + \delta_i K_i x \cdot \nabla v(0),$$

then in $B_{\lambda R}$ ($\lambda \leq 0.1$)

$$|u_i(x) - L_i(x)| \leq \delta_i K_i R_i (\epsilon_i + C_1 \lambda^2) \leq 100 K_i \lambda \delta_i R_i,$$

as long as $c_0 = \lambda$ is chosen small and i is large. □

3.5 Rigidity of two-phase minimizers

We now prove the two-phase case of Theorem 1.1.2, which we write here more explicitly:

Theorem 3.5.1. *Assume that the double well potential $W(s)$ satisfies (W1). If $u(x) : \mathbb{R}^n \to \mathbb{R}$ (no restriction on n) is a global two-phase minimizer of* (2.0.1)*, such that $\max_{B_R} |u| = o(R^2)$, then it must be one-dimensional.*

Proof. As is discussed in Proposition 3.2.1, u must be asymptotically linear, satisfying (3.2.1) with δR sufficiently large. Then "almost equivalently" we have

$$K(R)(x_n - \delta(R)R) \leq u(x) \leq K(R)(x_n + \delta(R)R) \text{ in } B_R.$$

We then have (up to a Euclidean group action)

$$K'(c_0R)(x_n - c_0\delta(R)R) \leq u(x) \leq K'(c_0R)(x_n + c_0\delta(R)R) \text{ in } B_{c_0R}.$$

We keep iterating, until escaping at k-th step, when $c_0^k\delta(R)R \leq \underline{T}$. At this moment we have (up to a Euclidean group action)

$$K'(c_0^kR)(x_n - \underline{T}) \leq u(x) \leq K'(c_0^kR)(x_n + \underline{T}) \text{ in } B_{c_0^kR} \supseteq B_{c_0\underline{T}/\delta(R)}.$$

Notice that $c_0\underline{T}/\delta(R) \to \infty$, so for another subsequence $R_i' \to \infty$ and slopes $\vec{K}_i$

$$\vec{K}_i \cdot x_n - \underline{T}|\vec{K}_i| \leq u(x) \leq \vec{K}_i \cdot x_n - \underline{T}|\vec{K}_i| \text{ in } B_{R_i'}.$$

We then have that $\lim_{i\to\infty} \vec{K}_i$ exists (say for example equal to $K\vec{e_n}$) and we reach the baby case version (Corollary 3.3.1). □

3.6 Solutions with a graphical level set

We finally make a note about monotonic solutions of the Euler-Lagrange equation (2.1.1). We not only require the assumption (W1)(W2) on the potential $W(s)$, but also require an additional (W3) condition that:

(W3) $W'(s) > 0$ when $-1 < s < 0$, $W'(s) < 0$ when $0 < s < 1$, $W''(0) < 0$.

Definition 3.6.1. *Let the potential $W(s)$ satisfy (W1)-(W3) and u is a solution of the Euler-Lagrange equation* (2.1.1)*. We say u has a graphical level set if $\partial_{x_n} u \geq 0$, and $\{u = 0\}$ is a graph $x_n = \Gamma(x')$ on $\mathbb{R}^{n-1}$. Let $v(x')$ be its pointwise limit as $x_n \to +\infty$, that is*

$$v(x') = \lim_{x_n \to +\infty} u(x', x_n) \in (0, +\infty].$$

In this section we extend Theorem 1.1.2 to the graphical level set setting and prove Theorem 1.1.3. We write it more precisely as:

Theorem 3.6.2. *Assume that the potential $W(s)$ satisfy (W1)-(W3). Let $u : \mathbb{R}^n \to \mathbb{R} (n \leq 8)$ be a solution of the Euler-Lagrange equation* (2.1.1) *so that it has a graphical level set. Then $u(x)$ must be one-dimensional.*

The goal will be to show that if u has a graphical level set, then it must be a global minimizer of the functional (2.0.1). The key is to show

$$v(x') = constant \in [1, +\infty].$$

First we deal with the case that $v(x') = +\infty$.

Lemma 3.6.1. *Let $u(x)$ and $v(x')$ be described in Definition 3.6.1. If v is somewhere equal to $+\infty$, then $v(x') = +\infty$ everywhere.*

Proof. Let's just assume $v(0) = \lim_{x_n \to +\infty} u(0, x_n) = +\infty$. It then follows from continuity that there exists some $H, r > 0$ such that

$$u(x) \geq 1 \text{ in } B'_{2r} \times [H/2, +\infty).$$

Since $v(0) = +\infty$, it follows from Harnack inequality that

$$v(x') = \lim_{x_n \to \infty} u(x', x_n) = +\infty, \text{ for } x' \in B'_r$$

and the convergence to $+\infty$ is uniform.

Suppose there exists some $v(p') < +\infty$, then we set $R = 2|p'| + r$, and let $h > H$ be large enough so that the whole ball $B_R(p', h)$ is above the level set $\{u = 0\}$. Since the Euler-Lagrange equation (2.1.1) yields that $\Delta u \leq 0$ in $B_R(p', h)$, by the weak Harnack inequality we conclude that there exists a constant M such that

$$|B_R(p', h) \cap \{u \geq M \cdot u(p', h)\}| \leq \omega_n r^n.$$

However, u can be uniformly and arbitrarily large in the half cylinder $B'_r \times [H, +\infty)$, whose intersection with $B_R(p', h)$ has a measure larger than $\omega_n r^n$. This results in a contradiction since $M \cdot u(p', h) \leq M \cdot v(p')$ is just bounded. □

Next, we assume that $v(x') < +\infty$ everywhere.

Lemma 3.6.2. *Let $u(x)$ and $v(x')$ be described in Definition 3.6.1 so that $v(x') < +\infty$. Then $v(x')$ is also a solution to the Euler-Lagrange equation* (2.1.1) *in the lower dimensional space R^{n-1}.*

Proof. For example, we show that $\Delta v(x') = W'(v(x'))$ for $x' \in B'_1$. By the boundedness of $|W'|$, we have that $|\Delta u(x)| \leq C$. Let h be large enough so that the whole ball $B_4(0, h)$ is above the level

set $\{u = 0\}$. By Harnack inequality, we have

$$\max_{B_2(0,h)} u \leq C(u(0,h)+1) \leq C(v(0,h)+1)$$

and therefore $\|u\|_{C^\alpha(B_2(0,h))} \leq C$ for large h. It then follows that $\Delta u = W'(u) \in C^\alpha$, so

$$\|u\|_{C^{2,\alpha}(B_1(0,h))} \leq C, \text{ for large } h.$$

Therefore, if we abuse the notation $v(x',h) = v(x')$, then $v(x',h)$ also satisfies (2.1.1) and is constant in x_n-direction, so $\Delta v(x') = W'(v(x'))$ for $x' \in B_1'$. □

Now we show that $v(x') \geq 1$ everywhere. If that is true, then by Liouville theorem (since $v(x')$ is harmonic and bounded from below), we know $v(x')$ is constant (including the special case that $v(x') = +\infty$ everywhere).

Lemma 3.6.3. *Let $u(x)$ and $v(x')$ be described in Definition 3.6.1 so that $v(x') < +\infty$. Then $v(x') \geq 1$ everywhere.*

Proof. Clearly we have $v(x') > 0$ for all x'. Suppose that for example $v(0) < 1$, then by condition (W3) there exists a huge number L such that

$$W'(s) < -\frac{\pi^2}{4(n-1)L^2}s, \quad \forall 0 < s \leq v(0).$$

We contact $v(x')$ from below at $p' \in (-L,L)^{n-1}$ by some test function

$$\phi(x') = h \cdot \prod_{i=1}^{n-1} \cos\left(\frac{\pi}{2L}x_i\right), \quad x' \in [-L,L]^{n-1}.$$

Here, we have h exists and $0 < h \leq v_0$ since $v(x') > 0$ everywhere. At p' we have

$$\Delta\phi(p') = -\frac{\pi^2}{4(n-1)L^2}\phi(p'), \quad \Delta v(p') = W'(v(p')) < -\frac{\pi^2}{4(n-1)L^2}v(p').$$

If p' is the contact point, then $v(p') = \phi(p') \leq v(0)$, so $\Delta\phi(p') > \Delta v(p')$, a contradiction. □

Now we prove Theorem 3.6.2. Recall that the conclusion in [14] can be extended to monotonic functions, that the critical point $u(x) : B_1 \to \mathbb{R}$ of the ACKS functional (2.4.1), if is monotonic in x_n-direction, then must have a $C^{1,\alpha}$ free boundary if the dimension is $n \leq 8$.

Besides, we don't need to assume $M(R) = \max_{B_R} |u| = o(R^2)$. This is because $\dfrac{u(Ry)}{M(R)}$ must converge to some x_n-increasing harmonic function, whose gradient at the origin cannot be zero.

Proof of Theorem 3.6.2. It suffices to show that if u has a graphical level set, then it must be a global minimizer of (2.0.1). Suppose that $\widetilde{u}(x)$ is a different competitor that $J(\widetilde{u}, B_R)$ is minimized with $\widetilde{u} = u$ outside B_R, then $\widetilde{u}$ is also a solution of the Euler-Lagrange equation (2.1.1). By maximal principle we have

$$\max_{B_R} \widetilde{u} \leq \max\{\max_{B_R} u, 1\}, \quad \min_{B_R} \widetilde{u} \geq \max\{\min_{B_R} u, -1\}.$$

Suppose $\widetilde{u}(x_0) > u(x_0)$ for some $x_0 \in B_R$. We slide the function $u(x + t\vec{e_n})$ starting from $t = +\infty$ until it first touches v from above at x_1 for $t = t_1$. This is possible because

$$v(x') = \lim_{x_n \to +\infty} u(x', x_n) \geq \max\{\max_{B_R} u, 1\}.$$

Since $w = u(x + t_1\vec{e_n}) - \widetilde{u}(x)$ is non-negative, $w(x_1) = 0$ and satisfies $|\Delta w| \leq \max |W''| \cdot |w|$, we have $w = 0$ everywhere by the strong maximal principle. This violates the assumption that $\widetilde{u}(x_0) > u(x_0)$. The argument above also works if $\widetilde{u}(x_0) < u(x_0)$. □

Chapter 4: Higher order boundary Harnack on the regular set

In this chapter, we will prove the higher order boundary Harnack principle on the regular domain, which is Theorem 1.2.2. The method is to directly study the equation satisfied by the ratio of two solutions.

4.1 Relation to a degenerate equation

Let $L = div(A \cdot \nabla)$ be a uniformly-elliptic divergence operator. We first find the equation of the ratio of two solutions of a same elliptic PDE.

Lemma 4.1.1. *If u_1, u_2 satisfy $div(A \cdot \nabla u_i) = div(\vec{f_i}) + \phi_i$, then the ratio $w = \frac{u_1}{u_2}$ satisfies*

$$div(u_2^2 A \nabla w) = div(u_2 \vec{f_1} - u_1 \vec{f_2}) + (\vec{f_2} \cdot \nabla u_1 - \vec{f_1} \cdot \nabla u_2) + (u_2 \phi_1 - u_1 \phi_2) \tag{4.1.1}$$

as long as A is symmetric.

Proof. As $u_1 = u_2 w$, by expanding $u_2 div\Big(A \nabla (u_2 w)\Big) = u_2 div(\vec{f_1}) + u_2 \phi_1$, we have

$$u_2^2 div(A \nabla w) + 2 u_2 A(\nabla u_2, \nabla w) = u_2 [div(\vec{f_1}) + \phi_1] - u_2 w \cdot div(A \nabla u_2).$$

Its left-hand side is

$$u_2^2 div(A \nabla w) + A(\nabla u_2^2, \nabla w) = div(u_2^2 A \nabla w),$$

and right-hand-side is

$$\begin{aligned} & u_2 [div(\vec{f_1}) + \phi_1] - u_1 [div(\vec{f_2}) + \phi_2] \\ =& div(u_2 \vec{f_1} - u_1 \vec{f_2}) + (\vec{f_2} \cdot \nabla u_1 - \vec{f_1} \cdot \nabla u_2) + u_2 \phi_1 - u_1 \phi_2. \end{aligned}$$

□

This equation can be turned into a model degenerate equation

$$div(x_n^2 A \cdot \nabla w) = div(x_n^2 \vec{f}) \quad \text{in } B_1^+. \tag{4.1.2}$$

The equation (4.1.2) belongs to the class of general x_n^s-weight equations

$$div(x_n^s A \cdot \nabla w) = div(x_n^s \vec{f}). \tag{4.1.3}$$

When $s \in (-1, 1)$, x_n^s is an A_2-weight, which were studied extensively in the literature, see [40, 41, 42]. There are also results outside this range. For example, in [43, 44] Dong and Phan gave a weighted $W^{2,p}$ estimate for the Neumann problem when $s > -1$ and the Dirichlet problem when $s < 1$.

In our setting $s = 2$, and we prove the following Schauder estimate:

Theorem 4.1.1. *Assume A is a C^α matrix with $\lambda I \le A \le \Lambda I$. If $w \in L^2(B_1^+) \cap H^1_{loc}(B_1^+)$ satisfies* (4.1.2)*, then there exists $C = C(n, \alpha, \lambda, \Lambda, \|A\|_{C^\alpha(B_1^+)})$, so that*

$$\|w\|_{C^{1,\alpha}(B_{1/2}^+)} \le C(\|\vec{f}\|_{C^\alpha(B_1^+)} + \|w\|_{L^2(B_1^+)}). \tag{4.1.4}$$

We prove Theorem 4.1.1 through the standard Campanato iteration, together with the weighted Poincare and Caccioppoli inequality. The method can be easily generalized to equation (4.1.3) with $s > 1$.

We remark that we cannot pose a Dirichlet boundary problem to (4.1.2) at the planar boundary $B_1' = B_1 \cap \{x_n = 0\}$. Instead, $w \in L^2(B_1^+)$ can be viewed as a Neumann boundary data of (4.1.2) at B_1'. More generally, the Neumann boundary data of (4.1.3) takes the form $\lim_{x_n \to 0} x_n^{s-1} w = 0$ in trace sense.

4.2 A weighted Sobolev space in regular domain

Let's first establish a Poincare type inequality (Proposition 4.2.1) which fits the degenerate equation.

Lemma 4.2.1. *If $w \in C^1_{loc}(\mathbb{R}^n_+)$ is compactly supported, which means there exists $R > 0$ such that $w\Big|_{|x|>R} = 0$, then we have the inequality*

$$\int_{\mathbb{R}^n_+} |w| dx \leq \int_{\mathbb{R}^n_+} x_n |\partial_n w| dx. \tag{4.2.1}$$

Proof. Fix $x' = (x_1, ..., x_{n-1})$, and it suffices to show that for each x',

$$\int_0^\infty |w(x', x_n)| dx_n \leq \int_0^\infty x_n |\partial_n w(x', x_n)| dx_n.$$

To show this, we notice that since w is compactly supported,

$$|w(x', x_n)| \leq \int_{x_n}^\infty |\partial_n w(x', y_n)| dy_n,$$

and hence by changing the order of integration, we have

$$\int_0^\infty |w| \leq \int_0^\infty \int_0^{y_n} |\partial_n w(x', y_n)| dx_n dy_n = \int_0^\infty x_n |\partial_n w(x', x_n)| dx_n.$$

□

This lemma is also true for $u \in H^1_{loc}(\mathbb{R}^n_+)$ compactly supported. By replacing w with w^2 in (4.2.1), we obtain the following Poincare inequality.

Proposition 4.2.1. *If $w \in H^1_{loc}(\mathbb{R}^n_+)$ is compactly supported, i.e. $w\Big|_{|x|>R} = 0$, then*

$$\int_{\mathbb{R}^n_+} w^2 dx \leq 4 \int_{\mathbb{R}^n_+} x_n^2 |\nabla w|^2 dx. \tag{4.2.2}$$

It's thus quite natural to define a weighted Sobolev space:

Definition 4.2.1. *The space $H^1(B_r^+, x_n^2 dx)$ is the set of H^1_{loc} functions w such that*

$$\int_{B_r^+} x_n^2 |\nabla w|^2 dx + \int_{B_r^+} w^2 dx < \infty.$$

The space $H^1(B_r^+, x_n^2 dx)$ is a Hilbert space by defining the square norm as

$$\|w\|^2_{H^1(B_r^+, x_n^2 dx)} := \int_{B_r^+} x_n^2 |\nabla w|^2 dx + \int_{B_r^+} w^2 dx.$$

For each $w \in H^1(B_r^+, x_n^2 dx)$, there exists $w_i \in H^1(B_r^+, x_n^2 dx) \cap C^\infty(\overline{B_r^+})$, so that

$$\int_{B_r^+} x_n^2 |\nabla (w - w_i)|^2 dx + \int_{B_r^+} (w - w_i)^2 dx \to 0.$$

We explain the smooth approximation below:

Proof of smooth approximation. For simplicity we let $r = 1$. Let

$$F_\delta(\vec{x}) = (1 - 2\delta)\vec{x} + \delta \vec{e_n} : B_1^+ \to B_{1-\delta}^+.$$

Let $\eta_\delta \geq 0$ be a smooth mollifier supported on B_δ so that $\int \eta_\delta dx = 1$. For any $w \in H^1(B_1^+, x_n^2 dx)$, we consider the following smooth approximating function

$$w_\delta(x) = (w * \eta_\delta) \circ F_{2\delta}(x) \in H^1(B_1^+, x_n^2 dx) \cap C^\infty(\overline{B_1^+}).$$

It then suffices to show the $H^1(B_r^+, x_n^2 dx)$-convergence, especially that $\int_{B_1^+} x_n^2 |\nabla (w - w_\epsilon)|^2 dx \to 0$. Let $\vec{y} = F_{2\delta}(\vec{x})$, then correspondingly $x_n \leq 2y_n$ if $\delta \leq 1/100$, so we have

$$\int_{B_1^+} x_n^2 |\nabla (w - w_\epsilon)|^2 dx \leq C \int_{B_1^+} x_n^2 |\nabla (w - w \circ F_{2\delta})|^2 dx + C \int_{F_{2\delta}(B_1^+)} y_n^2 |\nabla (w - w * \eta_\epsilon)|^2 dy.$$

The first term is small because a continuous family of affine maps $F_{2\delta} \approx id$ is continuous in L^2

norm. The second term is small by the mollifier. □

4.3 Energy estimates and existence of weak solutions

Now, let's also develop two Caccioppoli inequalities for the degenerate model (4.1.2). First we establish an interior estimate.

Proposition 4.3.1. *Assume that $\lambda I \le A \le \Lambda I$. If $w \in H^1_{loc}(B_r^+)$ is a solution of* (4.1.2)*, then there exists $C = C(n, \lambda, \Lambda)$ so that*

$$\int_{B^+_{r/2}} x_n^2 |\nabla w|^2 dx \le C\Big\{ \int_{B_r^+} x_n^2 |\vec{f}|^2 dx + \int_{B_r^+} |w|^2 dx \Big\}. \tag{4.3.1}$$

Proof. Let $\varphi(x), \eta_h(x)$ be two positive smooth functions satisfying

$$\varphi\Big|_{|x|\le r/2} = 1, \quad \varphi\Big|_{|x|\ge r} = 0, \quad |\nabla\varphi| \le \frac{C}{r}, \tag{4.3.2}$$

$$\eta_h\Big|_{x_n \ge h} = 1, \quad \eta_h\Big|_{x_n \le 0} = 0, \quad |\nabla\eta_h| \le \frac{C}{h}. \tag{4.3.3}$$

We also denote $\varphi_h = \varphi \cdot \eta_h$ and it's not hard to show that $x_n^2 |\nabla\varphi_h|^2 \le C$ inside B_r^+. Now multiplying $\varphi_h^2 w$ on both sides of (4.1.2), integration by parts implies

$$\int_{B_r^+} x_n^2 A\Big(\nabla(\varphi_h w), \nabla(\varphi_h w)\Big) - \int_{B_r^+} x_n^2 w^2 A(\nabla\varphi_h, \nabla\varphi_h) = \int_{B_r^+} x_n^2 \vec{f} \cdot \nabla(\varphi_h^2 w).$$

As $\lambda I \le A \le \Lambda I$ and $x_n^2 |\nabla\varphi_h|^2 \le C$, we have

$$\begin{aligned} \int_{B_r^+} x_n^2 A\Big(\nabla(\varphi_h w), \nabla(\varphi_h w)\Big) \ge& \lambda \int_{B_r^+} x_n^2 |\nabla(\varphi_h w)|^2, \\ 0 \le \int_{B_r^+} x_n^2 w^2 A(\nabla\varphi_h, \nabla\varphi_h) \le& C \cdot \Lambda \int_{B_r^+} w^2. \end{aligned}$$

Besides, as $0 \le \varphi_h \le 1$,

$$|\int_{B_r^+} x_n^2 \vec{f} \cdot \nabla(\varphi_h^2 w)| \le \frac{\varepsilon}{2} \int_{B_r^+} x_n^2 |\nabla(\varphi_h w)|^2 + C\varepsilon \int_{B_r^+} w^2 + \frac{1}{\varepsilon} \int_{B_r^+} x_n^2 |\vec{f}|^2.$$

Putting those estimates together yields that when $\varepsilon = \varepsilon(n, \lambda, \Lambda)$,

$$\int_{B_{r/2}^+ \cap \{x_n \ge h\}} x_n^2 |\nabla w|^2 \le \int_{B_r^+} x_n^2 |\nabla(\varphi_h w)|^2 \le C\Big\{ \int_{B_r^+} w^2 + \int_{B_r^+} x_n^2 |\vec{f}|^2 \Big\}.$$

Finally, we just need to apply Fatou lemma to $x_n^2 |\nabla w|^2 = \lim_{h \to 0} x_n^2 |\nabla w|^2 \chi_{\{x_n \ge h\}}$. □

Remark 5. *A more careful computation yields* $x_n^2 |\nabla \varphi_h|^2 \le C\Big(\frac{x_n^2}{r^2} + \chi_{\{x_n \le h\}}\Big)$, *so if* $\int_{B_r^+} w^2 dx < \infty$, *we can get a more accurate Caccioppoli inequality:*

$$\int_{B_{r/2}^+} x_n^2 |\nabla w|^2 dx \le C\Big\{ \int_{B_r^+} x_n^2 |\vec{f}|^2 dx + \frac{1}{r^2} \int_{B_r^+} x_n^2 |w|^2 dx \Big\}. \tag{4.3.4}$$

Next, here is a global inequality for solutions of (4.1.2) with zero boundary data.

Proposition 4.3.2. *Assume that* $\lambda I \le A \le \Lambda I$. *If* $w \in L^2(B_r^+) \cap H^1_{loc}(B_r^+)$ *is a solution of* (4.1.2) *so that* $w\Big|_{|x| \ge r} = 0$, *then there exists* $C = C(n, \lambda, \Lambda)$ *so that*

$$\int_{B_r^+} x_n^2 |\nabla w|^2 dx \le C \int_{B_r^+} x_n^2 |\vec{f}|^2 dx. \tag{4.3.5}$$

Proof. The proof is similar to Proposition 4.3.1 and we adopt its cut-off function η_h, which satisfies $x_n^2 |\nabla \eta_h|^2 \le C\chi_{\{x_n \le h\}}$. We multiply by $\eta_h^2 w$ on both sides of (4.1.2) and do integration by parts. Finally, we let $h \to 0$ and apply the Fatou lemma. □

Assume that $\lambda I \le A \le \Lambda I$. To find a solution of (4.1.2) satisfying some boundary condition, we can use the Lax-Milgram lemma. Let the bilinear form and linear functional

$$B(w, \varphi) := \int_{B_r^+} x_n^2 \nabla \varphi^T A \nabla w dx, \quad F(\varphi) = \int_{B_r^+} x_n^2 \vec{f} \cdot \nabla \varphi dx \tag{4.3.6}$$

be defined on $H^1(B_r^+, x_n^2 dx)$. If $w = 0$ when $|x| > r$, then by Proposition 4.2.1,

$$B(w,w) \geq \frac{\lambda}{5}\|w\|^2_{H^1(B_r^+, x_n^2 dx)}, \quad B(u,v) \leq \Lambda \|u\|_{H^1(B_r^+, x_n^2 dx)} \|v\|_{H^1(B_r^+, x_n^2 dx)}.$$

In the special case $\vec{f} = 0$, we have the following existence result:

Proposition 4.3.3. *Assume that $\lambda I \leq A \leq \Lambda I$. If $w_0 \in H^1(B_r^+, x_n^2 dx)$, then there exists a unique weak solution $w \in H^1(B_r^+, x_n^2 dx)$ of*

$$div(x_n^2 A \nabla w) = 0$$

with boundary data w_0. Besides, there exists $C = C(n, \lambda, \Lambda)$ so that

$$\int_{B_r^+} x_n^2 |\nabla w|^2 dx \leq C \int_{B_r^+} x_n^2 |\nabla w_0|^2 dx. \tag{4.3.7}$$

Proof. It suffices to find a solution $w_1 \in H^1(B_r^+, x_n^2 dx)$ so that $w_1\Big|_{|x|>r} = 0$ and

$$div(x_n^2 A \nabla w_1) = -div(x_n^2 A \nabla w_0).$$

Let $\vec{f} = -A\nabla w_0$, then as $w_0 \in H^1(B_r^+, x_n^2 dx)$, the linear functional $F(\varphi)$ is bounded. By Lax-Milgram lemma, w_1 exists and

$$\int_{B_r^+} x_n^2 |\nabla w_1|^2 \leq C \int_{B_r^+} x_n^2 |\vec{f}|^2 \leq C \int_{B_r^+} x_n^2 |\nabla w_0|^2.$$

If $w = w_0 + w_1$, then $div(x_n^2 A \nabla w) = 0$ with boundary data w_0. □

Remark 6. *We should notice that w_0 is the Dirichlet boundary data of w only on the spherical part of the hemisphere. Even if $\|w_0\|_{L^\infty} < \infty$, what we know on the planar boundary B_r' is at most $|w| < \infty$. Therefore, we may call w_0 a half-Dirichlet-half-Neumann boundary data.*

4.4 Schauder estimate

In this section, we will give a $C^{1,\alpha}$ interior estimate of model equation (4.1.2). First, Proposition 4.4.1 gives a Schauder estimate at the planar boundary $B'_{1/2}$, then Theorem 4.1.1 follows easily by interior Schauder estimate.

We first give two approximation lemmas.

Lemma 4.4.1. *If $h \in H^1(B_1^+, x_n^2 dx)$ solves the equation*

$$div(x_n^2 \nabla h) = 0,$$

then $h \in C^2(B_{1/2}^+)$. In particular, there exists a linear polynomial

$$l(x) = h(0) + x \cdot \nabla h(0) \tag{4.4.1}$$

such that for all $\rho \leq 1/2$,

$$\|l\|^2_{C^{0,1}(B_1^+)} + \frac{1}{\rho^{n+4}} \int_{B_\rho^+} |h - l|^2 dx \leq C(n) \|h\|^2_{L^2(B_1^+)}. \tag{4.4.2}$$

Proof. Let $h_k \in H^1(B_1^+, x_n^2 dx) \cap C^\infty(\overline{B_1^+})$ be a $H^1(B_1^+, x_n^2 dx)$ approximation of h, then $H_k = x_n h_k$ is a smooth $H^1(B_1^+, dx)$ approximation of $H = x_n h$ and each H_k vanishes at $\{x_n = 0\}$. Therefore, $H\Big|_{B'_1} = 0$ in trace sense. Notice that $\Delta H = 0$, and the desired inequality follows from the C^3 boundary estimate of H. □

Remark 7. *One can also show that $l(x) = l(x')$ is vertically constant. In fact, for linear functions, $l(x) = l(x')$ is equivalent to $div(x_n^2 \nabla l) = 0$.*

Remark 8. *You can refer to [43] for a more general proof of Lemma 4.4.1. The method is to establish a De Giorgi - Nash - Moser estimate in the weighted setting.*

Later, we will use the following rescaled version of the previous lemma.

Corollary 4.4.1. *Let $r > 0$. If $h \in H^1(B_r^+, x_n^2 dx)$ solves the equation*

$$div(x_n^2 \nabla h) = 0,$$

then $h \in C^2(B_{r/2}^+)$. In particular, there exists a linear polynomial

$$l(x) = h(0) + x \cdot \nabla h(0) \tag{4.4.3}$$

such that for all $\rho \leq 1/2$,

$$|l(0)|^2 + r^2 |\nabla l|^2 + \frac{1}{\rho^{n+4} r^n} \int_{B_{\rho r}^+} |h - l|^2 dx \leq \frac{C(n)}{r^n} \|h\|^2_{L^2(B_r^+)}. \tag{4.4.4}$$

Lemma 4.4.2. *Assume that $a^{ij}(0) = \delta^{ij}$ and $[a^{ij}]_{C^\alpha(B_r^+)} \leq \varepsilon_0$. Let $w \in H^1_{loc}(B_r^+)$ be a solution of* (4.1.2)*, then there is a weak solution*

$$div\Big(x_n^2 \nabla h\Big) = 0, \quad h\Big|_{(\partial B_{r/2})^+} = w \tag{4.4.5}$$

in the space $H^1(B_{r/2}^+, x_n^2 dx)$, and there exists $C = C(n)$ so that

$$\int_{B_{r/2}^+} h^2 dx \leq C\Big\{ \int_{B_r^+} x_n^2 |\vec{f}|^2 dx + \int_{B_r^+} w^2 dx \Big\}, \tag{4.4.6}$$

$$\int_{B_{r/2}^+} |h - w|^2 dx \leq C\Big\{ \int_{B_r^+} x_n^2 |\vec{f}|^2 dx + \varepsilon_0^2 r^{2\alpha} \int_{B_r^+} w^2 dx \Big\}. \tag{4.4.7}$$

Proof. We first use Proposition 4.3.1 and 4.3.3 to get the existence of h with

$$\int_{B_{r/2}^+} x_n^2 |\nabla h|^2 \leq C \int_{B_{r/2}^+} x_n^2 |\nabla w|^2 \leq C\Big\{ \int_{B_r^+} x_n^2 |\vec{f}|^2 + \int_{B_r^+} w^2 \Big\}.$$

If we denote $\vec{f}' = \Big(a^{ij}(x) - \delta^{ij}\Big)\partial_j h \cdot e_i$, then

$$\int_{B^+_{r/2}} x_n^2 |\vec{f}'|^2 \le C\varepsilon_0^2 r^{2\alpha} \int_{B^+_{r/2}} x_n^2 |\nabla h|^2 \le C\varepsilon_0^2 r^{2\alpha} \Big\{ \int_{B^+_r} x_n^2 |\vec{f}|^2 + \int_{B^+_r} w^2 \Big\} \tag{4.4.8}$$

and $(h-w)$ satisfies $div\Big(x_n^2 A(x)\cdot\nabla(h-w)\Big) = div\Big(x_n^2(\vec{f}'-\vec{f})\Big)$ with zero boundary data at $(\partial B_{r/2})^+$. Proposition 4.2.1 and 4.3.2, applied to $(h-w)$, give

$$\int_{B^+_{r/2}} |h-w|^2 \le C \int_{B^+_{r/2}} x_n^2 |\nabla(h-w)|^2 \le C \int_{B^+_{r/2}} x_n^2 \big(|\vec{f}|^2 + |\vec{f}'|^2\big).$$

By using (4.4.8) we prove (4.4.7). Now (4.4.6) follows from triangle inequality. □

Now let's give a Schauder estimate of w at $\mathbb{R}^{n-1}$.

Proposition 4.4.1. *Assume that $a^{ij}(0) = \delta^{ij}$, $[a^{ij}]_{C^\alpha(B_1^+)} \le \varepsilon_0$ and $w \in L^2(B_1^+) \cap H^1_{loc}(B_1^+)$ is a solution of* (4.1.2). *Given that ε_0 is small, there exist a linear polynomial P and $C = C(n,\alpha)$ such that*

$$\|P\|^2_{C^{0,1}(B_1^+)} + \frac{1}{r^{n+2+2\alpha}} \int_{B_r^+} |w-P|^2 dx \le C(\|\vec{f}\|^2_{C^\alpha(B_1)} + \|w\|^2_{L^2(B_1)}). \tag{4.4.9}$$

Proof. Let's first deal with the case $\vec{f}(0) = 0$. Let's inductively define

$$w_0 = w, \quad w_{k+1} = w_k - l_k (k \ge 0), \quad P_k = w - w_k = \sum_{i=0}^{k} l_i, \tag{4.4.10}$$

where the linear polynomials $l_k(x) = l_k(x')$ will be chosen in (4.4.13). We also define

$$\vec{f}_0 = \vec{f}, \quad \vec{f}_{k+1} = \vec{f}_k + \Big(\delta^{ij} - a^{ij}(x)\Big)\Big(\partial_j l_k\Big)e_i, (k \ge 0). \tag{4.4.11}$$

By induction, we know $\vec{f}_k(0) = 0$ and $div(x_n^2 A\cdot\nabla w_k) = div(x_n^2 \vec{f}_k)$ for all $k \ge 0$. Let $\varepsilon_0^{\frac{2}{n+4}} \le S \le \frac{1}{4}$ be a shrinking rate, and we define two quantities

$$\sigma_k^2 := \frac{1}{S^{k(n+2+2\alpha)}} \int_{B^+_{S^k}} w_k^2 dx, \quad \chi_k := [\vec{f}_k]_{C^\alpha(B^+_{S^k})}. \tag{4.4.12}$$

Here, σ_k measures how w and P_k differ in a $C^{1,\alpha}$ sense. Let $h_k \in H^1(B^+_{S^k/2}, x_n^2 dx)$ be a replacement of w_k in $B^+_{S^k/2}$, and l_k be the linearization of h_k, i.e.

$$\begin{cases} div\left(x_n^2 \nabla h_k\right) = 0 \\ h\Big|_{(\partial B_{S^k/2})^+} = w_k \end{cases}, \quad l_k = h_k(0) + x \cdot \nabla h_k(0). \tag{4.4.13}$$

By applying Lemma 4.4.2 and Corollary 4.4.1 to $w_{k+1} = (w_k - h_k) + (h_k - l_k)$, we have

$$\begin{cases} \chi_{k+1} \le C(n)(\chi_k + \varepsilon_0 S^{k\alpha}\sigma_k), \\ \sigma_{k+1}^2 \le C(n) S^{2-2\alpha}\sigma_k^2 + \dfrac{C(n)}{S^{n+2+2\alpha}}\chi_k^2, \\ |l_k(0)| + |\nabla l_k(0)| \le C(n) S^{k\alpha}(\chi_k + \sigma_k). \end{cases} \tag{4.4.14}$$

Clearly, when ε_0 is very small, then there is room to choose a $S = S(n, \alpha)$ satisfying

$$\varepsilon_0^{\frac{2}{n+4}} \le S \le \frac{1}{4}, \quad C(n) S^{2-2\alpha} \le \frac{1}{2}.$$

With this the iteration inequality (4.4.14) implies

$$\sigma_k, \chi_k \le C(n, \alpha)(\chi_0 + \sigma_0) = C(\|\vec{f}\|_{C^\alpha(B_1)} + \|w\|_{L^2(B_1)}).$$

Besides, it follows that the sequence of linear polynomials P_k converges to P with

$$\|P\|_{C^{0,1}(B_1^+)} \le C(\|\vec{f}\|_{C^\alpha(B_1)} + \|w\|_{L^2(B_1)}).$$

P is a $C^{1,\alpha}$ approximation of w because σ_k's are bounded.

Finally, if $\vec{f}(0) \neq 0$, then we can subtract a linear function from w so that the remainder w' satisfies $div(x_n^2 A \cdot \nabla w') = div(x_n^2 \vec{f}')$ with $\vec{f}'(0) = 0$. Now we are reduced to the first case. □

Now, Theorem 4.1.1 is almost obvious by an interior estimate.

Proof of Theorem 4.1.1. After rescaling, we can assume that $a^{ij}(0) = \delta^{ij}$ $[a^{ij}]_{C^\alpha} \leq \varepsilon_0$. Let $P(x)$ be the linear approximation of w, then it suffices to prove the regularity of $(w - P)$, which satisfies

$$div\Big(A \cdot \nabla(w - P)\Big) = div(x_n^2 \vec{f}_\infty) \tag{4.4.15}$$

with $\vec{f}_\infty = \lim_{k\to\infty} \vec{f}_k$. It's not hard to check that

$$\vec{f}_\infty(0) = 0, \quad [\vec{f}_\infty]_{C^\alpha(B_1^+)} \leq C(\|\vec{f}\|_{C^\alpha(B_1^+)} + \|w\|_{L^2(B_1^+)}).$$

Besides, Proposition 4.4.1 gives a L^2 bound of the error $(w - P)$:

$$\frac{1}{r^{n+2+2\alpha}} \int_{B_r^+} |w - P|^2 \leq C(\|\vec{f}\|^2_{C^\alpha(B_1^+)} + \|w\|^2_{L^2(B_1^+)}).$$

This holds not only when B_r^+ is centered at 0, but its center can also move on $\mathbb{R}^{n-1}$. Therefore, we can apply the uniformly-elliptic Schauder estimate to (4.4.15) in small interior balls to get $\|\nabla(w - P)\|_{C^\alpha(B_{1/2})} \leq C(\|\vec{f}\|_{C^\alpha(B_1^+)} + \|w\|_{L^2(B_1^+)})$. □

Remark 9. *The right hand side of* (4.1.2) *can also have a non-divergence term, like*

$$div(x_n^2 A \cdot \nabla w) = div(x_n^2 \vec{f}) + x_n g. \tag{4.4.16}$$

If $g \in C^\alpha$, then $x_n g$ is absorbed by $div(x_n^2 \vec{f})$, so Theorem 4.1.1 immediately implies

$$\|w\|_{C^{1,\alpha}(B_{1/2}^+)} \leq C(\|\vec{f}\|_{C^\alpha(B_1^+)} + \|g\|_{C^\alpha(B_1^+)} + \|w\|_{L^2(B_1^+)}). \tag{4.4.17}$$

Remark 10. *In Theorem 4.1.1, the assumption that A is symmetric can be removed. In fact, Lemma 4.4.1 also works for non-symmetric matrix.*

4.5 Higher order boundary Harnack

We first prove Theorem 1.2.2 when $\Gamma = B_1'$ is straight, we allow some right-hand side.

Theorem 4.5.1. *Assume that $\lambda I \le A \le \Lambda I$ and $u_1, u_2 > 0$ defined on B_1 satisfy*

$$div(A \cdot \nabla u_i) = div(x_n \vec{f_i}), \quad u_i\Big|_{B_1'} = 0. \tag{4.5.1}$$

(a) If $D_T^{k-1}\vec{f_i}, D_T^{k-1}A \in C^\alpha$ for some $k \ge 1$, then the ratio $w = \dfrac{u_1}{u_2}$ is tangentially $C^{k,\alpha}$. More precisely, $D_T^{k-1}w \in C^{1,\alpha}(B_{1/2})$.

(b) If $\vec{f_i}, A \in C^{k-1,\alpha}$ for some $k \ge 1$, then the ratio $w = \dfrac{u_1}{u_2}$ is $C^{k,\alpha}(B_{1/2})$.

Before proving Theorem 4.5.1, let's recall that for a non-degenerate equation

$$\partial_i(a^{ij}\partial_j u) = div(\vec{f}), \quad u\Big|_{B_1'} = 0, \tag{4.5.2}$$

there is a standard $C^{1,\alpha}$ boundary estimate.

Theorem 4.5.2. *Assume that $A \in C^\alpha$ and $\lambda|\xi|^2 \le \xi^T A\xi$. If u be a solution of* (4.5.2)*, then there is $C = C(n, \alpha, \lambda, \|A\|_{C^\alpha(B_1^+)})$ so that*

$$\|u\|_{C^{1,\alpha}(B_{1/2}^+)} \le C([\vec{f}]_{C^\alpha(B_1^+)} + \|u\|_{L^2(B_1^+)}). \tag{4.5.3}$$

Notice that when u vanishes at B_1', then $u \in C^{1,\alpha}$ implies $\dfrac{u}{x_n} \in C^\alpha$. With this let's prove Theorem 4.5.1.

Proof of Theorem 4.5.1. We first show part (a). When $k = 1$, then first we know $\nabla u_i, \dfrac{u_i}{x_n} \in C^\alpha(B_{3/4}^+)$ by Theorem 4.5.2 and $\dfrac{u_i}{x_n} \ge \varepsilon > 0$ by Hopf lemma. By Lemma 4.1.1, we know $w = \dfrac{u_1}{u_2}$ satisfies

$$div(u_2^2 A\nabla w) = div(x_n u_2 \vec{f_1} - x_n u_1 \vec{f_2}) + x_n(\vec{f_2} \cdot \nabla u_1 - \vec{f_1} \cdot \nabla u_2).$$

By writing

$$\tilde{A} = \frac{u_2^2}{x_n^2}A, \quad \vec{f} = \frac{u_2}{x_n}\vec{f_1} - \frac{u_1}{x_n}\vec{f_2}, \quad g = \vec{f_2} \cdot \nabla u_1 - \vec{f_1} \cdot \nabla u_2,$$

we simplify the equation of w to the form (4.4.16). The fact $\dfrac{u_2}{x_n} \ge \varepsilon > 0$ ensures that $\tilde{A}$ is still a

uniformly elliptic matrix. Besides, $\tilde{A}, \vec{f}, g \in C^{\alpha}(B^{+}_{3/4})$. Therefore, $w \in C^{1,\alpha}(B^{+}_{1/2})$ by applying Theorem 4.1.1.

When $k \geq 2$, we first apply D_T^{k-1} to (4.5.1) and obtain $D_T^{k-1}u_i \in C^{1,\alpha}(B^{+}_{3/4})$. Therefore, $D_T^{k-1}\tilde{A}, D_T^{k-1}\vec{f}, D_T^{k-1}g \in C^{\alpha}(B^{+}_{3/4})$. By induction hypothesis, we can also assume $D^{k-2}w \in C^{1,\alpha}(B^{+}_{3/4})$. Now we just apply D_T^{k-1} to (4.4.16) and obtain $D^{k-1}w \in C^{1,\alpha}(B^{+}_{1/2})$.

For part (b), we can inductively prove that for all $0 \leq l \leq k$,

$$D_T^{k-l}\partial_n^l w \in C^{\alpha}(B^{+}_{1/2}).$$

The base case $k = 0, 1$ is shown in part (a). For larger l, we just need to inductively take normal derivatives to (4.4.16). □

Now we deal with the curvy boundary case. The strategy is to straighten the boundary under a coordinate change.

To avoid abuse of coordinate, from now on it's necessary to introduce two variables x and y to represent different coordinate systems. We call the original coordinate y-coordinate, and the new coordinate x-coordinate. That is, the coordinate change (1.2.9), is better written as

$$x' = y', \quad x_n = y_n - \Gamma(y'). \tag{4.5.4}$$

We can also treat Γ as a function in x-coordinate, i.e. $\Gamma(x) = \Gamma(x') = \Gamma(y')$. The inverse coordinate change will be

$$y(x) = x + \Gamma(x)e_n. \tag{4.5.5}$$

If u satisfies an equality in y-coordinate, then the following coordinate-change lemma gives the corresponding equality of u in x-coordinate:

Lemma 4.5.1. *Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be two coordinate charts. If $\partial_{y_p}(b^{pq}\partial_{y_q}u) =$*

$\partial_{y_p}(f_p)$, *then*

$$\partial_{x_i}(a^{ij}\partial_{x_j}u) = \partial_{x_i}\Big(\det(\frac{\partial y}{\partial x})\frac{\partial x_i}{\partial y_p}f_p\Big),\ with\ a^{ij} = \det(\frac{\partial y}{\partial x})b^{pq}\frac{\partial x_i}{\partial y_p}\frac{\partial x_j}{\partial y_q}. \tag{4.5.6}$$

Proof. Let $\varphi \in C_0^\infty$ be a test function, integration by parts in y-coordinate gives

$$\int b^{pq}\partial_{y_p}\varphi\partial_{y_q}u dy = \int f_p\partial_{y_p}\varphi dy.$$

We turn both sides into integration in x-coordinate, then

$$LHS = \int b^{pq}\Big(\frac{\partial x_i}{\partial y_p}\partial_{x_i}\varphi\Big)\Big(\frac{\partial x_j}{\partial y_q}\partial_{x_j}u\Big)\Big(\det(\frac{\partial y}{\partial x})dx\Big) = \int a^{ij}\partial_{x_i}\varphi\partial_{x_j}u dx,$$

and similarly

$$RHS = \int f_p\Big(\frac{\partial x_i}{\partial y_p}\partial_{x_i}\varphi\Big)\det(\frac{\partial y}{\partial x})dx.$$

The desired identity now follows from integration by parts using x-coordinate. □

Now Theorem 1.2.2 is easily proven by combining Theorem 4.5.1 and Lemma 4.5.1:

Proof of Theorem 1.2.2. If $\Gamma \in C^{k,\alpha}$ then $\dfrac{\partial x_i}{\partial y_p}$ is $C^{k-1,\alpha}$ in y-coordinate. By assumption, $b^{pq} \in C^{k-1,\alpha}$ in y-coordinate, and thus so does a^{ij}. Besides, $\Gamma \in C^{k,\alpha}$ also implies that the $x-y$ coordinate change is $C^{k,\alpha}$. More precisely, $y(x)$ is $C^{k,\alpha}$ in x-coordinate, so $a^{ij}(y(x))$ is also $C^{k-1,\alpha}$ in x-coordinate by chain rule. By applying Theorem 4.5.1 to the x-coordinate equation obtained by Lemma 4.5.1, we arrive at the desired higher order boundary Harnack inequality. □

We should have directly turn to the proof of Theorem 1.2.3 about the analyticity of free boundaries. However, this requires more careful estimates using the majorant power series. Instead, let's first look at how a similar process works in the Signorini problem in the next chapter, and after that we go back to the classical obstacle problem and study the analyticity of free boundaries.

Chapter 5: Higher order boundary Harnack on the slit domain

In this chapter, we use a similar method as in the previous chapter to study the $C^{2,\alpha}$-regularity of the free boundary near a regular point in the Signorini problem.

5.1 The $C^{1,\alpha}$ boundary Harnack in slit domain

We study the Signorini problem (1.3.4), which is

$$div(A\nabla U) = F(x), \quad \text{when } U > 0 \text{ or } x \notin \mathbb{R}^n.$$

Assume that the free boundary of the Signorini problem near the origin (a regular boundary point) is a graph

$$\Gamma = \{(x_1, \cdots, x_{n-1}, \gamma(x_1, \cdots, x_{n-1}), 0) : (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \cap B_1\}.$$

We perform a coordinate change

$$x \rightsquigarrow x - \gamma(x_1, \cdots, x_{n-1})e_n \tag{5.1.1}$$

which straighten the boundary Γ to $\mathbb{R}^{n-1}$ (it will be made more clear in (5.7.1)), and then establish a $C^{1,\alpha}$ boundary Harnack principle on straight boundary.

We denote the "straightened-slit" as S, more precisely,

$$S = \{x = (x_1, ..., x_{n+1}) : x_n < 0, x_{n+1} = 0\}. \tag{5.1.2}$$

It corresponds to the region $\{x_n \leq \gamma(x_1, ..., x_{n-1})\} \cap \mathbb{R}^n = \{U = 0\} \cap \mathbb{R}^n$ before the coordinate

change (5.1.1).

For each point $(x_1, \cdots, x_{n+1})$ after the coordinate change, we abbreviate its first $(n-1)$ indices as x^T, and last two indices $(x_n, x_{n+1}) = x^\perp$.

As U converges in $C^{1,\alpha}$ sense to a $3/2$-homogeneous solution, we see U_{x_i}'s converge to a multiple of $1/2$-homogeneous solution in C^α sense near the regular point. Let's first get ourselves familiar with the homogeneous solution.

Assume that after straightening the boundary Γ, which is turned in to $\mathbb{R}^{n-1}$, we have a corresponding matrix A (will be explicit in (5.7.3)). Let $\bar{A} = A(0)$ so that $\bar{A}^{i,n+1} = \bar{A}^{n+1,i} = 0$ for all $i \leq n$, then the positive $1/2$-homogeneous solution of

$$div(\bar{A}\nabla\bar{u}) = 0 \text{ in } \mathbb{R}^{n+1} \setminus S, \quad \bar{u}\Big|_S = 0 \tag{5.1.3}$$

is given up to a constant multiple by

$$\bar{u}_{\bar{A}} = \sqrt{\frac{x_n + \sqrt{x_n^2 + \kappa^2 x_{n+1}^2}}{2}}, \quad \kappa = \kappa(\bar{A}) = (\frac{\bar{A}^{nn}}{\bar{A}^{n+1,n+1}})^{1/2}. \tag{5.1.4}$$

In particular when $\bar{A} = \delta^{ij}$, then $\kappa = 1$ and we set

$$\rho := \sqrt{x_n^2 + x_{n+1}^2} = |x^\perp|, \quad \xi := \bar{u}_I = \sqrt{\frac{x_n + \rho}{2}} = Re\Big((x_n + ix_{n+1})^{1/2}\Big). \tag{5.1.5}$$

Simple computation yields that $0 \leq \xi \leq \sqrt{\rho}$ and $|\nabla\xi| = \dfrac{1}{2\sqrt{\rho}}$. We will use these new coordinates quite often.

It suffices to study the special case $\bar{A} = \delta^{ij}$ since homogeneous solutions differ only by a stretching

$$\bar{u}_{\bar{A}}(x_n, x_{n+1}) = \xi(x_n, \kappa x_{n+1}).$$

It's worthwhile to mention that when A is uniformly elliptic, then $\kappa(A)$ is bounded from above and below when we move the center point elsewhere, and there is a uniform constant $C(\lambda, \Lambda)$ so that

for any x,

$$C(\lambda,\Lambda)^{-1} \le \frac{\bar{u}_{\bar{A}_1}}{\bar{u}_{\bar{A}_2}}(x) \le C(\lambda,\Lambda).$$

However, the ratio $\frac{\bar{u}_{\bar{A}_1}}{\bar{u}_{\bar{A}_2}}(x)$ is not continuous when $\kappa(A_1) \neq \kappa(A_2)$. It is a degree 0 homogeneous function depending only on the angle $arg(x_n + ix_{n+1})$, so when $|x^{\perp}| \to 0$, the oscillation of $\frac{\bar{u}_{\bar{A}_1}}{\bar{u}_{\bar{A}_2}}(x)$ remains large. Besides, one can check that the ratio $\frac{\bar{u}_{\bar{A}_1}}{\bar{u}_{\bar{A}_2}}(x)$ is Lipschitz with respect to the angle $arg(x_n + ix_{n+1})$.

In this thesis, we will sometimes use a path distance to describe the metric space in $\mathbb{R}^{n+1} \setminus S$, that is

$$dist(x,y) := \inf\{|\gamma| : \gamma(0) = x, \gamma(1) = y, \gamma(t) \in \mathbb{R}^{n+1} \setminus S\}. \tag{5.1.6}$$

We describe a kind of Hölder continuity which will be used quite often in this paper. We say a scalar/vector value function $f(x)$ satisfies property $(\mathcal{F}_A)$ in $B_R \setminus S$ if

$(\mathcal{F}_A)$ There exists a constant C and $h(x)$ so that $f(x) = \bar{u}_{A(x^T)}(x)h(x)$ and

$$\|h(x)\|_{C^{\alpha}(B_R \setminus S)} \le C, \quad \text{w.r.t. path distance } dist(\cdot,\cdot).$$

We can also simply write $(\mathcal{F}_A)$ as $(\mathcal{F})$, provided that the matrix A is known on $\mathbb{R}^{n-1}$. Besides, if $f(x)$ is x_{n+1}-even, then there's no distinction between using the path distance $dist(\cdot,\cdot)$ and using $|x-y|$ to describe regularity.

The $C^{1,\alpha}$ boundary Harnack principle is the following.

Theorem 5.1.1. *Assume that $\lambda I \le A \le \Lambda I$ and $A \in C^{\alpha}(\alpha < 1/2)$. ϕ_1, ϕ_2 are L^{∞} functions. u_1, u_2 are defined on B_1, and they satisfy*

$$div(A \cdot \nabla u_i) = div(\vec{f_i}) + \phi_i \text{ in } B_1 \setminus S, \quad u_i\Big|_{B_1' \cap S} = 0. \tag{5.1.7}$$

We also assume $u_2/\xi \ge c_0 > 0$ in B_1. If $\vec{f_i} \cdot e_j$ for $j \le n$ and $\frac{\vec{f_i} \cdot e_{n+1}}{x_{n+1}}$ all have property $(\mathcal{F}_A)$ in

B_1, then the ratio $w = \frac{u_1}{u_2}$, when restricted on $\mathbb{R}^{n-1}$, is $C^{1,\alpha}$. Besides, u_1, u_2 have property $(\mathcal{F}_A)$ in $B_{1/2}$.

5.2 A degenerate Schauder estimate in the slit domain

Assume that $A(0) = \delta^{ij}$ for simplicity, meaning that the blow-up of u_2 at the origin is ξ (defined in (5.1.5)). We consider a model degenerate equation

$$div(\xi^2 A \cdot \nabla w) = div(\xi^2 \vec{f}) + \xi^2 g \quad \text{in } B_1 \setminus S. \tag{5.2.1}$$

Other terms can also appear on the right-hand-side, and we give some explanation in Remark 11&12.

To describe its solution, we realize that if w is an x_{n+1}-even, the linearization of w is a non-smooth "polynomial" in the variable $(x_1, ..., x_n, \rho)$, where ρ is defined in (5.1.5). We make the following definition.

Definition 5.2.1. *We refer to "polynomials" as elements in the ring $\mathbb{R}[x_1, ..., x_n, \rho]$. A "polynomial" is called linear, if it is in the form*

$$L = c_0 + c_1 x_1 + \cdots + c_n x_n + c_\rho \rho.$$

We also define "point-wise" C^α vector fields, which are the gradients of some linear "polynomial" with $O(|x|^\alpha)$ error.

Definition 5.2.2. *A vector field $\vec{f}$ is called "point-wise" C^α at the origin, if it can be written as $\vec{f} = \vec{f_0} + \vec{f_\alpha}$, where*

$$\vec{f_0} = c_1 e_1 +, \cdots + c_n e_n + c_\rho \nabla \rho, \quad |\vec{f_\alpha}| \le c_\alpha |x|^\alpha,$$

and we denote its norm as $\|\vec{f}\|_{C^\alpha(B_1,0)} := |c_1| + \cdots + |c_n| + |c_\rho| + c_\alpha$.

We prove the following point-wise Schauder estimate:

Proposition 5.2.1. *Assume that* $A^{ij}(0) = \delta^{ij}$, $[A^{ij}]_{C^\alpha(B_1^+)} \le \varepsilon_0$ *and* $w \in L^2(B_1 \setminus S, \rho^{-1}dx) \cap H^1_{loc}$ *is an* x_{n+1}*-even solution of* (5.2.1). *Let* $p = \dfrac{n+3}{1-\alpha}$. *Given that* ε_0 *is small, there exist a linear "polynomial" L and* $C = C(n, \alpha)$ *such that*

$$\begin{aligned} &\|L\|^2_{C^{0,1}(B_1\setminus S)} + \frac{1}{r^{n+2+2\alpha}} \int_{B_r\setminus S} \frac{|w-L|^2}{\rho} dx \\ \le& C\left\{ \|\vec{f}\|^2_{C^\alpha(B_1,0)} + \|g\|^2_{L^p(B_1\setminus S, \rho\xi^2 dx)} + \|w\|^2_{L^2(B_1\setminus S,\rho^{-1}dx)} \right\}. \end{aligned} \tag{5.2.2}$$

We can call such a description of w as $C^{1,\alpha}$ in average sense. If w is $C^{1,\alpha}$ in average sense everywhere on $\mathbb{R}^{n-1}$, then its restriction on $\mathbb{R}^{n-1}$ is $C^{1,\alpha}$ in the classical sense.

The L^p assumption on g can be weakened, for example we just need to assume

$$\frac{1}{r^{(n+2+2\alpha)}} \int_{B_r\setminus S} \rho\xi^4 g^2 dx \le C, \quad \forall r \le 1. \tag{5.2.3}$$

In order to apply Proposition 5.2.1 to the equation in Lemma 4.1.1, we need to show that u_2/ξ is C^α at the origin, and has a positive lower bound. Therefore, we use a similar method to study a uniform equation

$$div(A\nabla u) = div(\frac{\vec{f}}{\sqrt{\rho}}) \text{ in } B_1 \setminus S, \quad u\Big|_S = 0 \tag{5.2.4}$$

in trace sense and $\vec{f} = 0$ on $\mathbb{R}^{n-1}$. In Theorem 5.4.1 we will show that u will have property $(\mathcal{F}_A)$ if $\vec{f} \in C^\alpha$. We also use it to prove a Hopf-type result in Proposition 5.5.1.

5.3 Proof of the Schauder estimate in average sense: a sketch

In this section we briefly list an outline of the proof Proposition 5.2.1. Since the details are very similar those in Proposition 4.4.1, we omit most of the proof(details can be seen in [45]).

Let's establish a suitable Poincare inequality Proposition 5.3.1. In this sub-section, we temporarily introduce a complex coordinate. For each $x = (x^T, x^\perp)$ with $x^\perp = (x_n, x_{n+1})$, we write

$$\xi + i\eta = (x_n + ix_{n+1})^{1/2}, \tag{5.3.1}$$

where ξ here is the same as (5.1.5).

Proposition 5.3.1. *If a function $w \in H^1_{loc}(\mathbb{R}^{n+1} \setminus S)$ is compactly supported, i.e. $w\Big|_{|x|>r} = 0$, then*

$$\int_{B_r \setminus S} \frac{w^2}{\rho} dx \leq 4 \int_{B_r \setminus S} \xi^2 |\nabla w|^2 dx. \tag{5.3.2}$$

Proof. It suffices to show that for each fixed x^T,

$$\int_{\mathbb{C}\setminus\mathbb{R}_-} \frac{w^2}{\rho} dx_n dx_{n+1} \leq 4 \int_{\mathbb{C}\setminus\mathbb{R}_-} \xi^2 |\nabla^\perp w|^2 dx_n dx_{n+1}.$$

Between two coordinates (x_n, x_{n+1}) and (ξ, η), we have

$$d\xi d\eta = \frac{1}{4\rho} dx_n dx_{n+1}, \quad |\nabla_{\xi,\eta} w|^2 d\xi d\eta = |\nabla^\perp w|^2 dx_n dx_{n+1}.$$

Now, by writing (4.2.1) in the (x_n, x_{n+1})-coordinate, we have

$$\begin{aligned} &\int_{\mathbb{C}\setminus\mathbb{R}_-} \frac{w^2}{\rho} dx_n dx_{n+1} = 4 \int_{Re(\alpha)>0} w^2 d\xi d\eta \\ \leq &4 \int_{Re(\alpha)>0} \xi^2 |\nabla_{\xi,\eta} w|^2 d\xi d\eta = 4 \int_{\mathbb{C}\setminus\mathbb{R}_-} \xi^2 |\nabla^\perp w|^2 dx_n dx_{n+1}. \end{aligned}$$

□

Let's define a weighted Sobolev space $H^1(B_r \setminus S, \xi^2 dx)$.

Definition 5.3.1. *$H^1(B_r \setminus S, \xi^2 dx)$ is the space of H^1_{loc} functions w such that*

$$\|w\|^2_{H^1(B_r\setminus S, \xi^2 dx)} := \int_{B_r \setminus S} \xi^2 |\nabla w|^2 dx + \int_{B_r \setminus S} \frac{w^2}{\rho} dx < \infty. \tag{5.3.3}$$

Its subspace $H^1_0(B_r \setminus S, \xi^2 dx)$, is the set of all $w \in H^1(B_r \setminus S, \xi^2 dx)$ which vanish in B_r^c.

For each $w \in H^1(B_r \setminus S, \xi^2 dx)$, there exists $w_i \in H^1(B_r \setminus S, \xi^2 dx)$, so that $w_k \in C^\infty(B_r \setminus S) \cap$

$C^0(\overline{B_r \setminus S})$ as well and

$$\int_{B_r \setminus S} \xi^2 |\nabla(w - w_i)|^2 dx + \int_{B_r \setminus S} \frac{(w - w_i)^2}{\rho} dx \to 0.$$

The reason is quite simple. Although S is not a C^1 boundary for $B_r \setminus S$, we can use the complex coordinate (x^T, ξ, η) mentioned in (5.3.1). It turns $B_r \setminus S$ to $\Omega = \{(x^T, \xi, \eta) : \xi > 0, |x^T|^2 + (\xi^2 + \eta^2)^2 \leq r^2\}$, and the slit S becomes the plane $\{\xi = 0\}$, which is now a C^1 boundary. We can then use the global approximation theorem in (x^T, ξ, η)-coordinate.

Now, let's also develop two Caccioppoli inequalities for the degenerate model (5.2.1). We write it down again.

$$div(\xi^2 A \cdot \nabla w) = div(\xi^2 \vec{f}) + \xi^2 g \quad \text{in } B_1 \setminus S. \tag{5.3.4}$$

First we establish an interior estimate.

Proposition 5.3.2. *Assume that $\lambda I \leq A \leq \Lambda I$. If $w \in H^1_{loc}$ is a solution of* (5.3.4)*, then there exists $C = C(\lambda, \Lambda)$ so that*

$$\int_{B_{r/2} \setminus S} \xi^2 |\nabla w|^2 dx \leq C \Big\{ \int_{B_r \setminus S} \frac{w^2}{\rho} dx + \int_{B_r \setminus S} (\xi^2 |\vec{f}|^2 + \rho \xi^4 g^2) dx \Big\}. \tag{5.3.5}$$

Next, here is a global inequality for solutions of (5.3.4) with zero boundary data.

Proposition 5.3.3. *Assume that $\lambda I \leq A \leq \Lambda I$. If $w \in H^1_{loc}$ is a solution of* (5.3.4) *so that $w\Big|_{|x| \geq r} = 0$, then there exists $C = C(\lambda, \Lambda)$ so that*

$$\int_{B_r \setminus S} \xi^2 |\nabla w|^2 dx \leq C \int_{B_r \setminus S} (\xi^2 |\vec{f}|^2 + \rho \xi^4 g^2) dx. \tag{5.3.6}$$

Assume that $\lambda I \leq A \leq \Lambda I$. To find a solution of (5.3.4) satisfying some boundary condition, we can use the Lax-Milgram lemma. Let the bi-linear form and linear functional

$$B(w, \varphi) := \int_{B_r \setminus S} \xi^2 \nabla \varphi^T A \nabla w dx, \quad F(\varphi) = \int_{B_r \setminus S} (\xi^2 \vec{f} \cdot \nabla \varphi - \xi^2 g \varphi) dx$$

be defined on $H_0^1(B_r \setminus S, \xi^2 dx)$. If $w = 0$ when $|x| > r$, then by Proposition 5.3.1,

$$B(w,w) \geq \frac{\lambda}{5}\|w\|^2_{H_0^1(B_r\setminus S,\xi^2 dx)}, \quad B(w,v) \leq \Lambda \|w\|_{H_0^1(B_r\setminus S,\xi^2 dx)}\|v\|_{H_0^1(B_r\setminus S,\xi^2 dx)}.$$

Besides, the norm of $F(\varphi)$ is

$$|F(\varphi)| \leq (\|\vec{f}\|_{L^2(B_r\setminus S,\xi^2 dx)} + \|g\|_{L^2(B_r\setminus S,\rho\xi^4 dx)})\|\varphi\|_{H_0^1(B_r\setminus S,\xi^2 dx)}.$$

In the special case $\vec{f} = 0$ and $g = 0$, we have the following existence result:

Proposition 5.3.4. *Assume that $\lambda I \leq A \leq \Lambda I$. If $w_0 \in H^1(B_r \setminus S, \xi^2 dx)$, then there exists a unique weak solution $w \in H^1(B_r \setminus S, \xi^2 dx)$ of*

$$div(\xi^2 A \nabla w) = 0$$

with boundary data w_0. Besides, there exists $C = C(\lambda, \Lambda)$ so that

$$\int_{B_r\setminus S} \xi^2 |\nabla w|^2 dx \leq C \int_{B_r\setminus S} \xi^2 |\nabla w_0|^2 dx. \tag{5.3.7}$$

In this section, we should give a $C^{1,\alpha}$ estimate in average sense of model equation (5.3.4) at the origin. Before we move into the proof, we should mention that in Theorem 5.2.1, the assumption $A(0) = \delta^{ij}$ can be weakened, so that $\lambda I \leq A(0) \leq \Lambda I$, $A^{i,n+1}(0) = A^{n+1,i}(0) = 0$ for $i \leq n$, and $A^{n,n} = A^{n+1,n+1}$. This can be reduced to the case $A(0) = \delta^{ij}$ by taking a linear coordinate change.

We first give two approximation lemmas.

Lemma 5.3.1. *If $h \in H^1(B_1 \setminus S, \xi^2 dx)$ is an x_{n+1}-even function solving the equation*

$$div(\xi^2 \nabla h) = 0,$$

then there exists a linear "polynomial" l such that for all $r \leq 1/2$,

$$\|l\|^2_{C^{0,1}(B_1\setminus S)} + \frac{1}{r^{n+4}}\int_{B_r\setminus S}\frac{|h-l|^2}{\rho}dx \leq C(\lambda,\Lambda)\int_{B_1\setminus S}\frac{h^2}{\rho}dx. \tag{5.3.8}$$

Lemma 5.3.2. *Assume that $\Delta H = 0$ in $B_1 \setminus S$ and $H\Big|_S = 0$ in trace sense. If $H \in H^1(B_1, dx)$, then $H \in C^6(B_{1/2})$ when written in (x^T, ξ, η) coordinate.*

The proof can be seen in [23] (Theorem 4.5). We briefly sketch the main idea below.

Proof. We can apply the weak Harnack principle to H_+ and H_- on the slit S, showing that H is C^δ at $B_{3/4} \cap S$, hence $H \in C^\delta(B_{1/2})$ by an interior estimate. Now we can inductively show that H is differentiable in (ξ, η).

Step 1 We take discrete quotients in tangential directions to show that tangential derivatives $(D^T)^k H$ are all C^δ for arbitrarily large k. This implies that $\Delta^T H + H_{\xi\xi} + H_{\eta\eta} = (1 - 4\xi^2 - 4\eta^2)f$, where $f = \Delta^T H \in C^\delta$.

Step 2 As $H\Big|_S = 0$, it also vanishes on $\{\xi = 0\}$, so the boundary Schauder estimate applied to H in (ξ, η)-coordinate implies that $H_\xi, H_\eta \in C^\delta$.

Step 3 Let $\tilde{H} = \Delta^T H$, then it also satisfies $\Delta^T H + \tilde{H}_{\xi\xi} + \tilde{H}_{\eta\eta} = (1 - 4\xi^2 - 4\eta^2)\tilde{f}$ for some $\tilde{f}$, we repeat Step 2 and obtain that $\tilde{H}_\xi, \tilde{H}_\eta \in C^\delta$.

Step 4 Since $f_\xi = \tilde{H}_\xi$, $f_\eta = \tilde{H}_\eta \in C^\delta$, we get that $H_{\xi\xi}, H_{\xi\eta}, H_{\eta\eta} \in C^\delta$ in (ξ, η)-coordinate.

Step 5 We repeat Step 2-4 infinitely many times, and finally obtain that H is smooth in (x^T, ξ, η)-coordinate.

□

Lemma 5.3.3. *Assume that $A^{ij}(0) = \delta^{ij}$ and $[A^{ij}]_{C^\alpha(B_r\setminus S)} \leq \varepsilon_0$. Let $w \in H^1_{loc}(B_r \setminus S, \xi^2 dx)$ be a solution of* (5.3.4)*, then there is a weak solution*

$$div\Big(\xi^2 \nabla h\Big) = 0, \quad h\Big|_{(\partial B_{r/2})\setminus S} = w \tag{5.3.9}$$

in the space $H^1(B_{r/2} \setminus S, \xi^2 dx)$*, and there exists* C *so that*

$$\int_{B_{r/2}\setminus S} \frac{h^2}{\rho} dx \leq C\Big\{ \int_{B_r\setminus S} (\xi^2 |\vec{f}|^2 + \rho\xi^4 g^2) dx + \int_{B_r\setminus S} \frac{w^2}{\rho} dx \Big\}, \tag{5.3.10}$$

$$\int_{B_{r/2}\setminus S} \frac{|h-w|^2}{\rho} dx \leq C\Big\{ \int_{B_r\setminus S} (\xi^2 |\vec{f}|^2 + \rho\xi^4 g^2) dx + \varepsilon_0^2 r^{2\alpha} \int_{B_r\setminus S} \frac{w^2}{\rho} dx \Big\}. \tag{5.3.11}$$

Now let's prove the Schauder estimate of w in an average sense.

Remark 11. *If the right-hand-side of* (5.3.4) *has a term* $\frac{\xi^2}{\rho} h$*, where* $h \in C^\alpha$ *at the origin, then it could be absorbed into* $div(\xi^2 \vec{f})$ *by setting*

$$\vec{f} := \int_0^1 s \cdot h(sx + (1-s)x^T) ds \nabla\rho = \frac{h(0)}{2} \nabla\rho + O(|x|^\alpha). \tag{5.3.12}$$

To check $div(\xi^2 \vec{f}) = \frac{\xi^2}{\rho} h$*, we write* $y = \gamma(s) := sx + (1-s)x^T$*, then we have*

$$\Psi(x) := \int_0^1 s \cdot h(sx + (1-s)x^T) ds = \rho(x)^{-2} \int_\gamma \rho(y) h(y) dy,$$

so $\partial_\rho(\rho^2\Psi) = \rho h$*. Since* $div(\frac{\nabla\rho}{\rho}) = \Delta(\log\rho) = 0$*, we have*

$$div(\xi^2 \vec{f}) = div(\rho\xi^2 \Psi(x) \frac{\nabla\rho}{\rho}) = \frac{\nabla\rho}{\rho} \cdot \nabla\Big(\frac{\xi^2}{\rho}\rho^2\Psi\Big) = \frac{\xi^2}{\rho^2}\partial_\rho(\rho^2\Psi) = \frac{\xi^2}{\rho} h.$$

Remark 12. *If* $\alpha < 1/2$*, then the right-hand-side of* (5.3.4) *can also have a term* $\xi\phi$*, where* $\phi \in L^\infty$*. This is because when* $g = \phi/\xi$*, then* $\xi\phi = \xi^2 g$ *and* g *satisfies the weakened assumption* (5.2.3)*.*

5.4 Hölder estimate of ratio $v = u/\xi$

In this section we describe a somehow weird regularity estimate of u:

Theorem 5.4.1. *Under the same assumption of Lemma 5.4.1, plus that* $\vec{f}$ *is* C^α *inside* B_1 *and* $\vec{f}(x) = 0$ *for all* $x \in \mathbb{R}^{n-1}$*, then* $u(x)$ *and* $\rho\partial_j u$ $(j \leq n)$ *have property* $(\mathcal{F})$ *in* $B_{1/2}$*. Besides,*

$x_{n+1}\rho\partial_{n+1}u$ also has property $(\mathcal{F})$ in $B_{1/2}$.

The property $(\mathcal{F}_A)$, or $(\mathcal{F})$ if the matrix A is known, given in the introduction is easier for the readers to understand, but with it only it's hard to do estimate, so we provide a few parallel properties.

In $\mathbb{R}^{n+1} \setminus S$, we define a cone of opening 1, radius $r > 0$, centered at $(x^T, 0)$ as $Cone_r(x^T, 0)$, or simply $Cone_r(x^T)$. More explicitly,

$$Cone_r(x^T) := \{y = (y^T, y^\perp) : |y^\perp| \le r, |y^T - x^T| \le |y^\perp|\} \setminus S. \tag{5.4.1}$$

We say $f(x)$ defined in $B_R \setminus S$ satisfies properties $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ or $(\mathcal{F}_3)$ if:

$(\mathcal{F}_1)$ There exists a constant C, so that for every $Cone_r(x^T) \in B_R \setminus S$,

$$\left\| \frac{f(y)}{\bar{u}_{A(x^T)}(y)} \right\|_{C^\alpha(Cone_r(x^T)\setminus Cone_{r/2}(x^T))} \le C, \quad \text{w.r.t. path distance } dist(\cdot,\cdot).$$

$(\mathcal{F}_2)$ There exists a constant C, so that for every $Cone_r(x^T) \in B_R \setminus S$,

$$\left\| \frac{f(y)}{\bar{u}_{A(x^T)}(y)} \right\|_{C^\alpha(Cone_r(x^T))} \le C, \quad \text{w.r.t. path distance } dist(\cdot,\cdot).$$

$(\mathcal{F}_3)$ There exists a constant C, so that for every $B_{2r}(x^T) \subseteq B_R$, $\left|\frac{f(y)}{\bar{u}_{A(x^T)}(y)}\right| \le C$ in $B_r(x^T)$, and for every pair $y \in B_r(x^T)$, $z \in Cone_r(x^T)$, we have

$$\left|\frac{f(y)}{\bar{u}_{A(x^T)}(y)} - \frac{f(z)}{\bar{u}_{A(x^T)}(z)}\right| \le C \cdot dist(y,z)^\alpha.$$

Again, $\bar{u}_{x^T}$ is an abbreviation of $\bar{u}_{A(x^T)}$. Just like $(\mathcal{F})$, these three properties are also defined when assuming A is known on $\mathbb{R}^{n-1}$.

If $\lambda I \le A \le \Lambda I$ and $[A]_{C^\alpha}$ is small, then the properties $(\mathcal{F})$, $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ are equivalent up to a shrinking of radius, meaning that for example, if $f(x)$ has property $(\mathcal{F})$ in B_R, then it also

has property $(\mathcal{F}_1)$ in $B_{R/100}$. The proof uses some similar technique like in [46] and is postponed to section 5.8.

Now we study the Hölder continuity of (5.2.4). We state the equation again,

$$div(A\nabla u) = div(\frac{\vec{f}}{\sqrt{\rho}}) \text{ in } B_1 \setminus S, \quad u\Big|_S = 0. \tag{5.4.2}$$

Lemma 5.4.1. *Assume that $A^{ij}(0) = \delta^{ij}$, $[A^{ij}]_{C^\alpha(B_1^+)} \le \varepsilon_0$ and $u \in L^2(B_1 \setminus S, dx) \cap H^1_{loc}$ vanishing at S is an x_{n+1}-even solution of* (5.4.2) *with $\vec{f}(0) = 0$. Given that ε_0 is small, there exist a constant $\bar{c}$ and $C = C(n, \alpha)$ such that*

$$|\bar{c}|^2 + \frac{1}{r^{n+2\alpha}} \int_{B_r \setminus S} \frac{|u/\xi - \bar{c}|^2}{\rho} dx \le C \Big\{ \|\vec{f}\|^2_{C^\alpha(B_1,0)} + \|u\|^2_{L^2(B_1 \setminus S, dx)} \Big\}. \tag{5.4.3}$$

Here, the norm $\|\vec{f}\|_{C^\alpha(B_1,0)}$ is the same as that in Proposition 5.2.1.

Proof. We write $v = u/\xi$, and also denote sequences $u_k = u - c_k\xi$ and $v_k = u_k/\xi$, where c_k's are constants yet to be decided. It follows that u_k satisfies the equation

$$div(A\nabla u_k) = div(\frac{\vec{f}_k}{\sqrt{\rho}}), \quad \vec{f}_k = \vec{f} - c_k\sqrt{\rho}(A - \delta^{ij})\nabla\xi.$$

As $|\nabla\xi| = \dfrac{1}{2\sqrt{\rho}}$, we see that $|f_k(x)| \le \phi_k |x|^\alpha$, where

$$\phi_k \le C(\|\vec{f}\|_{C^\alpha(B_1,0)} + \varepsilon_0 |c_k|). \tag{5.4.4}$$

As u_k vanishes at S, there is a Hardy-like inequality (will be explained after the proof)

$$\int_{B_r \setminus S} \frac{v_k^2}{\rho} dx = \int_{B_r \setminus S} \frac{u_k^2}{\xi^2 \rho} dx \le C \int_{B_r \setminus S} |\nabla u_k|^2 dx. \tag{5.4.5}$$

Besides, the Caccioppoli inequality for uniform elliptic equation implies that

$$\int_{B_r\setminus S}|\nabla u_k|^2dx\le C\int_{B_{2r}\setminus S}(\frac{u_k^2}{r^2}+\frac{|\vec{f_k}|^2}{\rho})dx. \tag{5.4.6}$$

Therefore, if there exists a converging sequence c_k and some $\lambda<1$, so that

$$\int_{B_{\lambda^k}\setminus S}u_k^2dx\le C\lambda^{k(n+2+2\alpha)}, \tag{5.4.7}$$

then we can use (5.4.5) and (5.4.6) to get (5.4.3) for $\bar{c}=\lim\limits_{k\to\infty}c_k$. □

We now explain (5.4.5) and (5.4.7).

Proof of (5.4.5). Instead of using the x-coordinate, we use the (x^T,ξ,η)-coordinate mentioned in (5.3.1). It follows that

$$\begin{aligned}&\int_{B_r\setminus S}|\nabla u|^2dx\ge\int_{B_r\setminus S}|\nabla^\perp u|^2dx\ge\int_\Omega|\partial_\xi u|^2dx^Td\xi d\eta,\\&\int_{B_r\setminus S}\frac{u^2}{\xi^2\rho}dx=\int_\Omega\frac{u^2}{\xi^2}dx^Td\xi d\eta,\end{aligned}$$

where $\Omega=\{(x^T,\xi,\eta):\xi>0,|x^T|^2+(\xi^2+\eta^2)^2\le r^2\}$ corresponds to $B_r\setminus S$ region in (x^T,ξ,η)-coordinate. We fix x^T and η, and write $u(\xi)=u(x^T,\xi,\eta)$, then it suffices to find a constant C independent of (x^T,η) and the bound A, so that for all $u(\xi)\in H^1_{loc}(\mathbb{R}_+)$ vanishing at 0, we have

$$C\int_0^A|u'|^2d\xi\ge\int_0^A\frac{u^2}{\xi^2}d\xi.$$

The strategy is to show a $W^{1,1}$ Hardy inequality $\int_0^A\frac{|u'|}{\xi}d\xi\ge\int_0^A\frac{|u|}{\xi^2}d\xi$ and then replace u with u^2, just like the method in (4.2.1). For simplicity, we assume $w\in C^\infty$, then the $W^{1,1}$ Hardy inequality follows from changing the order of integration:

$$\int_0^A\frac{|u(\xi)|}{\xi^2}d\xi\le\int_0^A\frac{1}{\xi^2}\int_0^\xi|u'(h)|dhd\xi=\int_0^A\int_h^A\frac{1}{\xi^2}|u'(h)|d\xi dh\le\int_0^A\frac{|u'(h)|}{h}dh.$$

□

Proof of (5.4.7). We follow a similar Campanato iteration like in Proposition 5.2.1. Without loss of generality, we assume that $\|\vec{f}\|_{C^\alpha(B_1,0)} = \|u\|_{L^2(B_1\setminus S,dx)} = 1$. We let $c_0 = 0$, $\varepsilon_0^{\frac{2}{n+4}} \le \lambda \le 1/4$, and define

$$\sigma_k^2 = \frac{1}{\lambda^{k(n+2+2\alpha)}} \int_{B_{\lambda^k}\setminus S} u_k^2 dx.$$

Clearly we have $\sigma_0 = 1$. By (5.4.6) we have that

$$\int_{B_{\lambda^k/2}\setminus S} |\nabla u_k|^2 \le C\lambda^{k(n+2\alpha)}(\sigma_k^2 + \phi_k^2),$$

where $\phi_k \le C(1+\varepsilon_0|c_k|)$. Let H_k be a harmonic replacement obtained using Lax-Milgram lemma or energy minimizing method, so that it vanishes on S in trace sense and

$$\Delta H_k = 0 \text{ in } B_{\lambda^k/2} \setminus S, \quad H_k\Big|_{\partial B_{\lambda^k/2}\setminus S} = u_k.$$

By Lemma 5.3.2 plus that H_k vanishes on S, we can set $c_{k+1} = c_k + \lim_{x\to 0} \frac{H_k}{\xi}$, and

$$|c_k - c_{k+1}|^2 + \frac{1}{\lambda^{n+4} r^{n+4}} \int_{B_{\lambda r}\setminus S} |H_k + c_k\xi - c_{k+1}\xi|^2 \le \frac{C}{r^{n+2}} \int_{B_r\setminus S} |H_k|^2$$

for all $\lambda < 1$ if H_k is x_{n+1}-even. If follows that

$$|c_{k+1}| \le |c_k| + C\lambda^{k\alpha}(\sigma_k + \phi_k) \tag{5.4.8}$$

As $u_{k+1} = (u_k - H_k) + (H_k - c_{k+1}\xi + c_k\xi)$, we have

$$\sigma_{k+1}^2 \le C(n)\lambda^{2-2\alpha}\sigma_k^2 + \frac{C(n)}{\lambda^{n+2+2\alpha}}\phi_k^2. \tag{5.4.9}$$

The iterative system (5.4.4)(5.4.8)(5.4.9), if we further assume $C(n)\lambda^{2-2\alpha} \le 1/2$, will imply the boundedness of σ_k and convergence of c_k. Such a λ which simultaneously satisfies $\varepsilon_0^{\frac{2}{n+4}} \le \lambda \le 1/4$

mentioned before exists, if ε_0 is small enough. □

Now let's prove Theorem 5.4.1.

Proof of Theorem 5.4.1. In each conic annulus $Cone_r \setminus Cone_{r/2}$, whose closure is contained in $B_{2r} \setminus B_{r/4}$, (5.2.4) is a non-degenerate equation, so a boundary Schauder estimate yields that under the distance function $dist(\cdot,\cdot)$, we have

$$[u]_{C^{1,\alpha}(Cone_r \setminus Cone_{r/2})} \leq Cr^{-1/2}$$

for some C independent of r. As $u\Big|_S = 0$ and $\xi \approx \dfrac{|x_{n+1}|}{\sqrt{\rho}}$ near the slit, we conclude that $[u/\xi]_{C^\alpha(Cone_r \setminus Cone_{r/2})} \leq C$ with respect to $dist(\cdot,\cdot)$. After all, we have

$$[v]_{C^\alpha(Cone_r \setminus Cone_{r/2})}, [\rho\nabla v]_{C^\alpha(Cone_r \setminus Cone_{r/2})} \leq C,$$

which means u has property $(\mathcal{F}_1)$. The $(\mathcal{F}_1)$ regularity of $\rho\nabla u$ is obtained from

$$\frac{\rho\nabla u}{\xi} = \rho\nabla v + v \cdot (\frac{e_n}{2} + \frac{\rho - x_n}{2x_{n+1}} e_{n+1}).$$

□

Remark 13. *In fact, to get $[u]_{C^{1,\alpha}(Cone_r \setminus Cone_{r/2})} \leq Cr^{-1/2}$, we don't really require $\vec{f}$ to be $C^\alpha(B_1)$. We just need to assume $|\vec{f}(y) - \vec{f}(z)| \leq C \cdot dist(y,z)^\alpha$ for every $y \in Cone_{1/8}(x^T)$ and $z \in B_{1/8}(x^T)$.*

5.5 Non-degeneracy of u: a Hopf type estimate

We can use a similar method to prove that the growth rate of u is exactly proportional to ξ as long as u is close to ξ in L^2 sense.

Proposition 5.5.1. *Assume that $A(0) = \delta^{ij}$ and $[A]_{C^\alpha(B_1 \setminus S)} \leq \varepsilon_0$. $\vec{f}\Big|_{\mathbb{R}^{n-1}} = 0$ and $[\vec{f}]_{C^\alpha(B_1 \setminus S)} \leq \varepsilon_0$. u is an x_{n+1}-even solution of* (5.4.2) *with zero trace at S and satisfies $\|u - \xi\|_{L^2(B_1 \setminus S)} \leq \varepsilon_0$.*

Given that ε_0 is small enough, then we have

$$\inf_{B_{1/8}\setminus S} \frac{u}{\xi} \geq \frac{1}{2}. \tag{5.5.1}$$

Proof. For every point $x^T \in \mathbb{R}^{n-1} \cap B_{1/2}$, as $|A(x^T) - \delta^{ij}| \leq \varepsilon_0$, the corresponding homogeneous solution $\bar{u}_{x^T} = \bar{u}_{A(x^T)}$ is closed to ξ in L^2 sense, so

$$\|u - \bar{u}_{x^T}\|_{L^2(B_{1/2}(x^T)\setminus S)} \leq \|u - \xi\|_{L^2(B_{1/2}(x^T)\setminus S)} + \|\xi - \bar{u}_{x^T}\|_{L^2(B_{1/2}(x^T)\setminus S)} \leq C\varepsilon_0.$$

At each center x^T, the function $\tilde{u} = (u - \bar{u}_{x^T})$ satisfies

$$div(A\nabla\tilde{u}) = div\Big(\frac{\vec{f}}{\sqrt{\rho}} + (\bar{A} - A)\nabla\bar{u}_\kappa\Big) =: div(\frac{\vec{f}'}{\sqrt{\rho}}),$$

where $\vec{f}'$ vanishes at x^T with $[\vec{f}']_{C^\alpha(B_{1/2}(x^T)\setminus S)} \leq C\varepsilon_0$. We repeat the method in Theorem 5.4.1 and get that

$$\|\frac{u - \bar{u}_{x^T}}{\bar{u}_{x^T}}\|_{C^\alpha(Cone_{1/4}(x^T)\setminus S)} \leq C\varepsilon_0.$$

For every $y \in B_{1/8}$, we construct z like in (5.8.1) and get

$$\begin{aligned}
|\frac{u-\xi}{\xi}(y)| \leq & |\frac{u-\xi}{\xi}(z)| + |\frac{u}{\xi}(z) - \frac{u}{\xi}(y)| \\
= & |\frac{u-\xi}{\xi}(z)| + \frac{\bar{u}_{x^T}}{\xi}(y)|\frac{u}{\bar{u}_{x^T}}(z) - \frac{u}{\bar{u}_{x^T}}(y)| \\
\leq & \|\frac{u-\xi}{\xi}\|_{C^\alpha(Cone_{1/4}\setminus S)} + C(A(x^T))\|\frac{u - \bar{u}_{x^T}}{\bar{u}_{x^T}}\|_{C^\alpha(Cone_{1/4}(x^T)\setminus S)} \\
\leq & C\varepsilon_0.
\end{aligned}$$

When ε_0 is small, then $|\frac{u-\xi}{\xi}(y)| \leq \frac{1}{2}$ for all $y \in B_{1/8}$, which proves the non-degeneracy assertion.

□

5.6 $C^{1,\alpha}$ boundary Harnack for straight boundary

With Lemma 4.1.1, plus the Theorem 5.4.1 previously obtained, we can prove the $C^{1,\alpha}$ boundary Harnack principle for straight boundary.

Proof of Theorem 5.1.1. After a linear coordinate change, let's assume $A(0) = \delta^{ij}$ and $\bar{u}_{A(0)} = \xi$. In fact, to say $u_2/u_{A(0)} \geq c_0 > 0$, this is independent of the homogeneous solution $\bar{u}_{A(0)}$, since the ratio of $\bar{u}_{A(0)}$'s between different $A(0)$'s are bounded from above and below.

- Step 1: We first absorb ϕ_i into the divergence term by

$$div(\vec{f_i}) + \phi_i(x) = div\Big(\vec{f_i} + \int_0^{x_{n+1}} \phi_i(x_1, \cdots, x_n, h)dh \cdot e_{n+1}\Big) =: div(\vec{f_i'}). \tag{5.6.1}$$

  We write $\vec{f_i'} = \dfrac{\sqrt{\rho}\vec{f_i'}}{\sqrt{\rho}}$ so that $\sqrt{\rho}\vec{f_i'} \in C^\alpha$ and vanishes at $\mathbb{R}^{n-1}$, then by Theorem 5.4.1 we have that u_i, $\rho\partial_j u_i (i \leq n)$ and $x_{n+1}\rho\partial_{n+1}u_i$ are $(\mathcal{F})$ in $B_{3/4}$.

- Step 2: By Lemma 4.1.1, we know $w = \dfrac{u_1}{u_2}$ satisfies (4.1.1). Now we write

$$\tilde{A} = \frac{u_2^2}{\xi^2}A, \quad \vec{f} = \frac{1}{\xi^2}(u_2\vec{f_1} - u_1\vec{f_2}),$$

  and add two additional term mentioned in Remark 11.

$$h = \frac{\rho}{\xi^2}(\vec{f_2}\cdot\nabla u_1 - \vec{f_1}\cdot\nabla u_2), \quad \phi = \frac{1}{\xi}(u_2\phi_1 - u_1\phi_2).$$

  then by the assumption of $\vec{f_i}$ we know $\tilde{A}$, $\vec{f}$ and h are all C^α at the origin and $\phi \in L^\infty$, meaning that we simplify the equation of w to the form (5.2.1), but with some additional term in Remark 11. The assumption $u_2/\xi \geq c_0 > 0$ ensures that $\tilde{A}$ is still a uniformly elliptic matrix. Therefore, w at the origin is $C^{1,\alpha}$ in average sense by applying Proposition 5.2.1.

- Step 3: Such an argument works not only at the origin, but also for all points on $\mathbb{R}^{n-1}$. More

precisely, assume that at $x^T \in \mathbb{R}^{n-1}$, $\kappa = (\frac{\bar{a}^{nn}}{\bar{a}^{n+1,n+1}}(x^T))^{1/2}$, then we set

$$\rho_\kappa = \sqrt{x_n^2 + \kappa^2 x_{n+1}^2}. \tag{5.6.2}$$

Each w_m is approximated by a linear polynomial L_m in the variables $(x_1, \cdots, x_n, \rho_\kappa)$, so that

$$\|L_m\|^2_{C^{0,1}(B_1(x^T)\setminus S)} + \frac{1}{r^{n+2+2\alpha}} \int_{B_r(x^T)\setminus S} \frac{|w_m - L_m|^2}{\rho} dx \le C. \tag{5.6.3}$$

By overlapping $B_r(x^T)$ between different $x^T \in \mathbb{R}^{n-1}$, we obtain that $D^T w \in C^\alpha(B_{1/2} \cap \mathbb{R}^{n-1})$, so restriction of w on $\mathbb{R}^{n-1}$ is $C^{1,\alpha}$ in the classical sense.

□

5.7 Proof of Theorem 1.3.1

In this section, to avoid ambiguity, we use y-coordinate to represent the coordinate before straightening the boundary, and x-coordinate to be the new coordinate. We re-write (5.1.1) in a more clear way as

$$x_i = y_i, \forall i \le n-1, \quad x_n = y_n - \gamma(y^T), \quad x_{n+1} = y_{n+1}. \tag{5.7.1}$$

The equation (1.3.4) will be better written as

$$\partial_{y_p}(B^{pq}\partial_{y_q} U) = F(y). \tag{5.7.2}$$

Here, we use the matrix $B(y)$ as the original matrix in y-coordinate satisfying assumption (a)-(d) mentioned in the introduction.

Assume that $U(y) \ge 0$ solves the thin obstacle problem (5.7.2) whose blow-up is $C^{1,\alpha}$-converging to a $3/2$-homogeneous solution near the regular point 0. After the coordinate change

(5.7.1), we have that $u_m = U_{y_m}$ with $1 \le m \le n$ satisfy the equations

$$\partial_{x_i}(a^{ij}\partial_{x_j}u_m) + \partial_{x_i}(\frac{\partial x_i}{\partial y_p}\partial_{y_m}b^{pq}\cdot u_q) = \partial_{y_m}F(y), \quad a^{ij} = b^{pq}\frac{\partial x_i}{\partial y_p}\frac{\partial x_j}{\partial y_q}. \tag{5.7.3}$$

One can check that matrix A also satisfies assumptions (a)-(d). We write

$$\vec{f}_m = -\frac{\partial x_i}{\partial y_p}\partial_{y_m}b^{pq}\cdot u_q e_i, \quad \phi_m = \partial_{y_m}F(y(x)), \tag{5.7.4}$$

then

$$div(A\nabla u_m) = div(\vec{f}_m) + \phi_m. \tag{5.7.5}$$

Since the coordinate change (5.7.1) is $C^{1,\alpha}$ and $F(y) \in W^{1,\infty}$, we know $\phi_m \in L^\infty$ in x-coordinate.

It is clear that once we know all u_m's have property $(\mathcal{F})$, then so do $\vec{f}_m \cdot e_i$ for all $i \le n$. If $e_i = e_{n+1}$, then as $\dfrac{b^{pq}}{x_{n+1}}$ is $C^{1,\alpha}$ in directions $e_1, \cdots, e_n$ when $p = n+1, q \le n$, and as $\dfrac{\partial x_{n+1}}{\partial y_p} = 0$ for all $p \le n$, we realize that $\dfrac{\vec{f}_m \cdot e_{n+1}}{x_{n+1}}$ will also have property $(\mathcal{F})$ once all u_m's do.

Like in (5.6.1), we again absorb ϕ_m into the divergence term in Step 1&2.

- Step 1: By Theorem 5.4.1, as $\sqrt{\rho}\vec{f}'_m$ are C^α and vanish at $\mathbb{R}^{n-1}$, we have that u_m are all $(\mathcal{F})$ in $B_{7/8}$.

- Step 2: We now show the non-degeneracy of u_m. Without loss of generality, we assume that $B(0) = \delta^{ij}$ and $\nabla\Gamma(0) = 0$, then $a^{ij}(0) = \delta^{ij}$. As the blow-up of U converges in $C^{1,\alpha}$ to the positive $3/2$ harmonic homogeneous solution $Re\Big((x_n + ix_{n+1})^{3/2}\Big)$, we have that the blow-up of $u_n = \partial_{y_n}U$ converges to $\xi = Re\Big((x_n + ix_{n+1})^{1/2}\Big)$ in C^α sense. When $\alpha < 1/2$, then by zooming in at the origin, we have that $\sqrt{\rho}\vec{f}'_n$ vanish at S and

$$[A]_{C^\alpha}, \|u_n - \xi\|_{L^2}, [\sqrt{\rho}\vec{f}'_n]_{C^\alpha} \le \varepsilon_0$$

  as long as $\|\phi_m\|_{L^\infty}, [D^iA]_{C^\alpha} \le \varepsilon_0 (i \le n)$. By Proposition 5.5.1, we see that $u_n/\xi \ge c_0 > 0$ in $B_{3/4}$.

- Step 3: Finally, as discussed before, u_m's being $(\mathcal{F})$ implies that $\vec{f}_m \cdot e_i (i \leq n)$ and $\dfrac{\vec{f}_m \cdot e_{n+1}}{x_{n+1}}$ all have property $(\mathcal{F})$, so by Theorem 5.1.1, we see that $w_m = \dfrac{u_m}{u_n}$ is $C^{1,\alpha}$ in average sense at the origin for $m \leq n$, and in particular, w_m is $C^{1,\alpha}$ when restricted to $\mathbb{R}^{n-1}$ or Γ.

5.8 Equivalence of regularity

Let's prove the "equivalence" of properties $(\mathcal{F})$, $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$. The most obvious relation is clearly $(\mathcal{F}_3) \Rightarrow (\mathcal{F}_2) \Leftrightarrow (\mathcal{F}_1)$ up to a shrinking of radius.

The following lemmas prove the rest of "equivalence". In this section, when $\bar{u}_{\bar{A}}$ corresponds to the matrix $A(x^T)$ where $x^T \in \mathbb{R}^{n-1}$, we can also simply write the homogeneous solution $\bar{u}_{A(x^T)}$ as $\bar{u}_{x^T}$.

Lemma 5.8.1. *If $\lambda I \leq A \leq \Lambda I$, then $(\mathcal{F}_2) \Rightarrow (\mathcal{F}_3)$ up to a shrinking of radius.*

Proof. For simplicity, let's assume $x^T = 0$, $A(0) = \delta^{ij}$, meaning that $\bar{u}_0 = \xi$. If $y \in B_r(0) \setminus (S \cup Cone_r)$, $z \in Cone_r(0)$, then $|y^\perp| \leq |y^T|$ and we let

$$w = y^T + \frac{|y^T|}{|y^\perp|} y^\perp \in Cone_r(0) \cap Cone_r(y^T). \tag{5.8.1}$$

It's not hard to verify that

$$|y - w| \leq 2|y - z|, \quad |z - w| \leq 3|y - z|.$$

In the cone $Cone_r(y^T)$ we also have for $\bar{u}_{y^T} = \bar{u}_{A(y^T)}$,

$$C(\lambda, \Lambda)^{-1} \leq \bar{u}_{y^T}/\xi \leq C^{-1}(\lambda, \Lambda), \quad [f/\bar{u}_{y^T}]_{C^\alpha(Cone_r)(y^T)} \leq C.$$

Since $0, y^T, y, w$ are co-planar, we have $\frac{\bar{u}_{y^T}}{\xi}(w) = \frac{\bar{u}_{y^T}}{\xi}(y)$. Therefore,

$$\begin{aligned}|\frac{f(y)}{\xi(y)} - \frac{f(z)}{\xi(z)}| \leq & |\frac{f(y)}{\bar{u}_{y^T}(y)}\frac{\bar{u}_{y^T}(y)}{\xi(y)} - \frac{f(w)}{\bar{u}_{y^T}(w)}\frac{\bar{u}_{y^T}(w)}{\xi(w)}| + |\frac{f(w)}{\xi(w)} - \frac{f(z)}{\xi(z)}| \\ \leq & \frac{\bar{u}_{y^T}}{\xi}(y)|\frac{f}{\bar{u}_{y^T}}(y) - \frac{f}{\bar{u}_{y^T}}(w)| + |\frac{f}{\xi}(w) - \frac{f}{\xi}(z)| \\ \leq & C(|y-w|^\alpha + |w-z|^\alpha) \leq (2^\alpha + 3^\alpha)C|y-z|^\alpha.\end{aligned}$$

□

Lemma 5.8.2. *If $\lambda I \leq A \leq \Lambda I$ and $[A]_{C^\alpha}$ is small, then $(\mathcal{F}) \Rightarrow (\mathcal{F}_1)$ up to a shrinking of radius.*

Proof. For simplicity, let's assume $x^T = 0$, $A(0) = \delta^{ij}$, meaning that $\bar{u}_0 = \xi$. It suffices to show that for every $y, z \in Cone_r(0) \setminus Cone_{r/2}(0)$, we have

$$|\frac{\bar{u}_{y^T}(y)}{\xi(y)}h(y) - \frac{\bar{u}_{z^T}(z)}{\xi(z)}h(z)| \leq C \cdot dist(y,z)^\alpha.$$

The left-hand-side is a sum of $D_1 = \frac{\bar{u}_{z^T}(z)}{\xi(z)} \cdot |h(y) - h(z)|$ and

$$D_2 = |h(y)| \cdot |\frac{\bar{u}_{y^T}(z)}{\xi(z)} - \frac{\bar{u}_{z^T}(z)}{\xi(z)}|, \quad D_3 = |h(y)| \cdot |\frac{\bar{u}_{y^T}(y)}{\xi(y)} - \frac{\bar{u}_{y^T}(z)}{\xi(z)}|.$$

As $h(x)$ is C^α, we have $D_1 \leq C \cdot dist(y,z)^\alpha$. When z is fixed, $\frac{\bar{u}_{\bar{A}}(z)}{\xi(z)}$ is Lipschitz about $\bar{A}$, which is a C^α function as $[A]_{C^\alpha}$ is small. Therefore, we also have

$$D_2 \leq C \cdot dist(y^T, z^T)^\alpha \leq C \cdot dist(y,z)^\alpha.$$

To estimate D_3, we first use complex coordinate to write

$$y_n + iy_{n+1} = r_y e^{i\theta_y}, \quad z_n + iz_{n+1} = r_z e^{i\theta_z}, \quad \theta_y, \theta_z \in (-\pi, \pi).$$

It follows that $\bar{u}_{y^T}/\xi$ is determined and Lipschitz by the argument angle θ_y, θ_z. Besides, as $A \in C^\alpha$,

the Lipschitz norm about θ is in fact bounded by

$$\left|\partial_\theta \frac{\bar{u}_{y^T}(r,\theta)}{\xi(r,\theta)}\right| \le C|y^T|^\alpha$$

The angle between $y^\perp, z^\perp$ without crossing the slit S, say $\angle(y^\perp, z^\perp) = |\theta_y - \theta_z|$, which takes value in $(0, 2\pi)$, satisfies

$$\angle(y^\perp, z^\perp) \le \begin{cases} \sin^{-1}\dfrac{dist(y^\perp, z^\perp)}{|y^\perp|}, & \text{if } dist(y^\perp, z^\perp) < |y^\perp| \\ 2\pi, & \text{if } dist(y^\perp, z^\perp) \ge |y^\perp| \end{cases} \le C\frac{dist(y^\perp, z^\perp)}{|y^\perp|}.$$

Notice that when $y \in Cone_r$, $|y^\perp| \ge |y^T|$, so

$$\begin{aligned} D_3 \le & C \cdot \left\|\partial_\theta \frac{\bar{u}_{y^T}(r,\theta)}{\xi(r,\theta)}\right\|_{L^\infty} \cdot |\theta_y - \theta_z| \le C|y^T|^\alpha \cdot \frac{dist(y^\perp, z^\perp)}{|y^\perp|} \\ \le & C\Big(\frac{dist(y^\perp, z^\perp)}{|y^\perp|}\Big)^{1-\alpha} dist(y^\perp, z^\perp)^\alpha \le C3^{1-\alpha} dist(y,z)^\alpha. \end{aligned}$$

Here we used $2|y^\perp| \ge |z^\perp|$ in the last step, since $y, z \in Cone_r(0) \setminus Cone_{r/2}(0)$. □

Lemma 5.8.3. *If $\lambda I \le A \le \Lambda I$ and $[A]_{C^\alpha}$ is small, then $(\mathcal{F}_3) \Rightarrow (\mathcal{F})$ up to a shrinking of radius.*

Proof. We write $h(x) = \dfrac{f(x)}{\bar{u}_{x^T}(x)}$ in B_R, then it suffices to show $h(x) \in C^\alpha(B_{R/100})$. Let $x, y \in B_{R/100}$, then as $x \in Cone_{R/10}(x^T)$, we have that

$$|h(x) - \frac{f(y)}{\bar{u}_{x^T}(y)}| = |\frac{f(x)}{\bar{u}_{x^T}(x)} - \frac{f(y)}{\bar{u}_{x^T}(y)}| \le C|x-y|^\alpha.$$

Therefore, as it is obvious that $h \in L^\infty(B_{R/100})$, we have

$$\begin{aligned} |h(x) - h(y)| \le & C|x-y|^\alpha + |h(y)| \cdot |\frac{\bar{u}_{y^T}(y)}{\bar{u}_{x^T}(y)} - 1| \\ \le & C|x-y|^\alpha + C|x^T - y^T|^\alpha. \end{aligned}$$

□

5.9 More comments

As different matrices A correspond to different homogeneous solutions $\bar{u}_A$, the vertical coordinate change (5.1.1) is not suitable in proving $\Gamma \in C^{3,\alpha}, \cdots, C^{\omega}$.

If there is a coordinate change different from (5.1.1), so that $\kappa(A)$ is constant along $\mathbb{R}^{n-1}$, then we can take tangential derivatives of w_i and obtain higher order regularity of w_i, perhaps also obtain the analyticity of Γ like in [36][37] by a bootstrap argument (see the next chapter for regular obstacle problem).

We consider that if $A = A_D$ or equivalently $A_O = 0$, and if $A^{n+1,n+1}$ is a constant, then a geodesic flow that starts from Γ and its normal vector $\nu \in \mathbb{R}^n$ will be the desired coordinate change.

Besides, in order to prove $\Gamma \in C^{\omega}$, we also need to improve Proposition 5.2.1, so that $C^{1,\alpha}$ estimate is in the classical sense rather than in the average sense.

The coordinate change obtained by geodesic flow will be $C^{1,\alpha}$ only when we assume $\Gamma \in C^{2,\alpha}$, so in this paper we still have to use the vertical coordinate change. It seems to be a common phenomenon (like [38] and [23]) that the proof of $C^{2,\alpha}$ regularity of the free boundary is different from the proof of higher regularity.

Chapter 6: The bootstrap method and the analyticity of free boundaries in the classical obstacle problem

We now go back to the classical obstacle problem and prove the analyticity of the free boundary near a regular point. Let's first recall the notations we used in section 4.5 concerning the coordinate change:

"We call the original coordinate y-coordinate, and the new coordinate x-coordinate. That is, the coordinate change (1.2.9), is better written as $x' = y'$, $x_n = y_n - \Gamma(y')$. We can also treat Γ as a function in x-coordinate, i.e. $\Gamma(x) = \Gamma(x') = \Gamma(y')$. The inverse coordinate change will be $y(x) = x + \Gamma(x)e_n$."

6.1 A global norm concerning the growth rate at boundary

To prove analyticity, we first develop global versions of Theorem 4.1.1 and 4.5.2 by controlling growth rate of the Hölder regularity near the spherical boundary $(\partial B_1)^+$. For $X \in \mathbb{R}^n_+$, let's denote

$$B_r^+(X) := B_r(X) \cap \mathbb{R}^n_+.$$

Definition 6.1.1. *For any function f defined in B_1^+ in x-coordinate, its global $C^{k,\alpha}$ norm with b-many normal derivatives is defined as*

$$[f]^{*,l}_{C_b^{k,\alpha}} := \sup_{\substack{X\in B_1^+ \\ \Delta=1-|X|}} \max_{\substack{|\beta|=k \\ \beta_n\le b}} \Delta^l([D^\beta f]_{L^\infty(B^+_{\frac{\Delta}{l+1}}(X))} + \Delta^\alpha [D^\beta f]_{C^\alpha(B^+_{\frac{\Delta}{l+1}}(X))}). \tag{6.1.1}$$

In this paper, $b = 0, 1$, meaning that all (but at most one) derivatives are tangential. In our case we will choose $l = k$, see (6.3.1) later.

This norm has a "scaling invariance". If it's known that $[f]^{*,l}_{C_b^{k,\alpha}} \leq 1$, then for any $X \in B_1^+$, we let $|z| < 1$ such that $(X + \Delta_X \cdot z) \in B_1^+$, where $\Delta_X = 1 - |X|$, and set

$$f_{X,k,l}(z) := \Delta_X^{l-k} f(X + \Delta_X \cdot z), \tag{6.1.2}$$

then we also have $[f_{X,k,l}]^{*,l}_{C_b^{k,\alpha}} \leq 1$.

6.2 Global degenerate Schauder estimate: degenerate and non-degenerate

By using Theorem 4.1.1 at $\mathbb{R}^{n-1}$ and using the non-degenerate Schauder estimates away from $\mathbb{R}^{n-1}$, Theorem 4.1.1 holds for all $B_1^+(X)$. More precisely, we have the following lemma.

Lemma 6.2.1. *Let $X \in \mathbb{R}^n_+$ and $w \in L^2(B_1^+(X))$ satisfies* (4.1.2) *in $B_1^+(X)$. If $A \in C^\alpha$ with $\lambda I \leq A \leq \Lambda I$, then there is C independent of the center X, so that*

$$\|w\|_{C^{1,\alpha}(B^+_{1/2}(X))} \leq C(\|\vec{f}\|_{C^\alpha(B_1^+(X))} + \|w\|_{L^2(B_1^+(X))}). \tag{6.2.1}$$

Now let's give a global version of Theorem 4.1.1. In this proposition, we describe the boundary growth rate of the $C^{1,\alpha}$ estimate, and the shrinking rate of interior balls is generalized to $\dfrac{1}{l+2}$.

Proposition 6.2.1. *Let $l \geq 0$ be an arbitrary integer. Assume that $A \in C^\alpha$ with $\lambda I \leq A \leq \Lambda I$ and w satisfies* (4.1.2) *in B_1^+, which is*

$$div(x_n^2 A \nabla w) = div(x_n^2 \vec{f}),$$

then there exists C independent of l, such that

$$[w]^{*,l+1}_{C_1^{1,\alpha}} \leq C\{[f]^{*,l+1}_{C_0^{0,\alpha}} + (l+2)[w]^{*,l}_{C_0^{0,\alpha}}\}. \tag{6.2.2}$$

Proof. Let's write $\sigma = [w]^{*,l}_{C_0^{0,\alpha}}$ and $\chi = [f]^{*,l+1}_{C_0^{0,\alpha}}$. Besides, because of the "scaling invariance",

let's only consider the special case $X = 0$, which is

$$[\nabla w]_{C^0(B^+_{\frac{1}{l+2}})} + [\nabla w]_{C^\alpha(B^+_{\frac{1}{l+2}})} \le C\{\chi + (l+2)\sigma\}.$$

Other $X \in B_1^+$'s will be the same, except that we need to care about its scaling. For every $x \in B^+_{\frac{1}{l+2}}$, its $\Delta \ge \dfrac{l+1}{l+2}$. Now let's set $r = \dfrac{1}{l+2}$. From Lemma 6.2.2 which will be stated after the proof, it then suffices to show for every $x \in B^+_{\frac{1}{l+2}}$,

$$[\nabla w]_{C^0(B^+_{\frac{1}{2l+4}}(x))} + [\nabla w]_{C^\alpha(B^+_{\frac{1}{2l+4}}(x))} \le C\{\chi + (l+2)\sigma\}.$$

By using a rescaled version of Lemma 6.2.1, we know

$$\begin{aligned}[\nabla w]_{C^0(B^+_{\frac{1}{2l+4}}(x))} \le& C(n,\alpha)\{\Delta^{-l-1}[f]^{*,l+1}_{C_0^{0,\alpha}} + r^{-1}\Delta^{-l}[w]^{*,l}_{C_0^{0,\alpha}}\}\\ \le& C\cdot e\{\chi + (l+2)\sigma\}.\end{aligned}$$

The bound of $[\nabla w]_{C^\alpha(B^+_{\frac{1}{2l+4}}(x))}$ can be argued similarly. □

Lemma 6.2.2. *For any function $f \in C^\alpha(B_r^+(X))$ with $X \in \mathbb{R}^n_+$ and $r > 0$, we have*

$$[f]_{L^\infty(B_r^+(X))} \le \sup_{x\in B_r^+(X)} [f]_{L^\infty(B^+_{r/2}(x))} \tag{6.2.3}$$

$$[f]_{C^\alpha(B_r^+(X))} \le 2^{1-\alpha} \sup_{x\in B_r^+(X)} [f]_{C^\alpha(B^+_{r/2}(x))}. \tag{6.2.4}$$

Proof. The first inequality is obvious. To prove the second inequality, we pick any $x_0, x_4 \in B_r^+(X)$. We divide the line segment $\overline{x_0x_4}$ into four equal sub-segments using the points $x_i = \dfrac{i}{4}x_4 + \dfrac{4-i}{4}x_0$ with $i \in \{1,2,3\}$. We just need to combine the Hölder bounds in $B^+_{r/2}(x_1)$ and $B^+_{r/2}(x_3)$. □

Following the same method, we have a parallel result for the uniformly-elliptic equation (4.5.2).

Proposition 6.2.2. *Let $l \ge 0$ be an arbitrary integer. Assume that $A \in C^\alpha$ with $\lambda|\xi|^2 \le \xi^T A\xi$. If u*

satisfies (4.5.2) *in* B_1^+*, which is*

$$div(A\nabla u) = div(\vec{f}),$$

then there exists C *independent of* l*, such that*

$$[u]^{*,l+1}_{C_1^{1,\alpha}} \leq C\{[f]^{*,l+1}_{C_0^{0,\alpha}} + (l+2)[u]^{*,l}_{C_0^{0,\alpha}}\}. \tag{6.2.5}$$

6.3 Majorant power series

Definition 6.3.1. *For any function* f *defined in* B_1^+ *in* x*-coordinate, we define its majorant coefficient and power series as*

$$P_b[f]^{(k)} := [f]^{*,k}_{C_b^{k,\alpha}},\ P_b[f](t) := \sum_{k=0}^{\infty} P_b[f]^{(k)} \frac{t^k}{k!}. \tag{6.3.1}$$

Coefficients are allowed to be ∞ *and convergence of the power series is not required.*

Remark 14. *Here,* $P_b[f]^{(k)}$ *happens to be the* k*-th derivative of* $P_b[f](t)$*, evaluated at* $t = 0$*. Therefore, the abuse of notation* $P_b[f]^{(k)}$ *as* k*-th derivative does not cause confusion.*

Remark 15. *As* $\Gamma(x) = \Gamma(x') = \Gamma(y')$*, we can also define the majorant power series of* $\Gamma(x)$ *as* $P_b[\Gamma](t)$*.*

Here, we only consider $b = 0, 1$. In fact, $P_0[f](t) \ll P_1[f](t)$ for any f. If $\partial_n f = 0$, then $P_0[f](t) = P_1[f](t)$.

Definition 6.3.2. *Given two power series* $f(t) = \sum_{n\geq 0} a_n t^n$ *and* $g(t) = \sum_{n\geq 0} b_n t^n$ *with* $b_n \in \mathbb{R}\bigcup\{\pm\infty\}$*. We say* f *is majorized by* g*, denoted as* $f \ll g$*, if* $|a_n| \leq b_n$ *for all* $n \geq 0$*.*

The power series $P_b[f](t)$ encodes the regularity of f in tangential direction. In fact, if there exists $\delta > 0$ such that $P_b[f](t)$ is convergent when $|t| \leq \delta$, then f is analytic in tangential direction at 0, with convergence radius at least δ.

The following Lemma 6.3.1 and 6.3.2 provide basic computational rules of the majorant power series. The C^α product rule

$$[f \cdot g]_{C^\alpha(S)} \leq [f]_{L^\infty(S)}[g]_{C^\alpha(S)} + [f]_{C^\alpha(S)}[g]_{L^\infty(S)}$$

will be used several times during the proof.

Lemma 6.3.1. *Here are some basic properties of majorant power series.*

(a) (linear) For f and g defined in B_1^+, $P_b[f \pm g](t) \ll P_b[f](t) + P_b[g](t)$.

(b) (product) For f and g defined in B_1^+, $P_b[f \cdot g](t) \ll P_b[f](t) \cdot P_b[g](t)$.

(c) (integration) Let f be defined on B_1^+ and g be a power series. If $P_b[\nabla f](t) \ll g(t)$, then $P_b[f](t) \ll t \cdot g(t) + C$ for some $C = C(f, g)$. If in particular $f(0) = 0$, then $C \leq g(0)$.

Proof. Part (a) is obvious.

(b) For every $X \in B_1^+$, we bound its C^k semi-norm by

$$\begin{aligned}\Delta^k [fg]_{C^k(B_{\frac{\Delta}{k+1}}(X))} \leq & \Delta^k \sum_{l=0}^{k} \binom{k}{l} [f]_{C^l(B_{\frac{\Delta}{k+1}})}[g]_{C^{k-l}(B_{\frac{\Delta}{k+1}})} \\ \leq & \sum_{l=0}^{k} \binom{k}{l} \Delta^l [f]_{C^l(B_{\frac{\Delta}{l+1}})} \Delta^{k-l}[g]_{C^{k-l}(B_{\frac{\Delta}{k-l+1}})}\end{aligned}$$

and bound its $C^{k,\alpha}$ semi-norm by

$$\begin{aligned}\Delta^{k+\alpha} [fg]_{C^{k,\alpha}(B_{\frac{\Delta}{k+1}}(X))} \leq & \Delta^{k+\alpha} \sum_{l=0}^{k} \binom{k}{l} \Big\{ [f]_{C^{l,\alpha}(B_{\frac{\Delta}{k+1}})}[g]_{C^{k-l}(B_{\frac{\Delta}{k+1}})} \\ & + [f]_{C^l(B_{\frac{\Delta}{k+1}})}[g]_{C^{k-l,\alpha}(B_{\frac{\Delta}{k+1}})} \Big\} \\ \leq & \sum_{l=0}^{k} \binom{k}{l} \Big\{ \Delta^{l+\alpha} [f]_{C^{l,\alpha}(B_{\frac{\Delta}{l+1}})} \Delta^{k-l}[g]_{C^{k-l}(B_{\frac{\Delta}{k-l+1}})} \\ & + \Delta^l [f]_{C^l(B_{\frac{\Delta}{l+1}})} \Delta^{k-l+\alpha}[g]_{C^{k-l,\alpha}(B_{\frac{\Delta}{k-l+1}})} \Big\}.\end{aligned}$$

By adding these two inequalities together, we have

$$\begin{aligned}
&\Delta^{k+\alpha}[fg]_{C^{k,\alpha}(B_{\frac{\Delta}{k+1}}(X))} + \Delta^{k+\alpha}[fg]_{C^{k,\alpha}(B_{\frac{\Delta}{k+1}}(X))} \\
\leq &\sum_{l=0}^{k} \binom{k}{l} \left\{ \Delta^{l}[f]_{C^{l}(B_{\frac{\Delta}{l+1}})} + \Delta^{l+\alpha}[f]_{C^{l,\alpha}(B_{\frac{\Delta}{l+1}})} \right\} \\
&\cdot \left\{ \Delta^{k-l}[g]_{C^{k-l}(B_{\frac{\Delta}{k-l+1}})} + \Delta^{k-l+\alpha}[g]_{C^{k-l,\alpha}(B_{\frac{\Delta}{k-l+1}})} \right\} \\
\leq &\sum_{l=0}^{k} \binom{k}{l} P[f]^{(l)} P[g]^{(k-l)} = (P[f] \cdot P[g])^{(k)}.
\end{aligned}$$

(c) Let $X \in B_1^+$, then $\Delta^k[D^k f]_{C^0(B^+_{\frac{\Delta}{k+1}}(X))} \leq \Delta^{k-1}[D^{k-1}\nabla f]_{C^0(B^+_{\frac{\Delta}{k}}(X))}$, and similar inequality holds for $C^{0,\alpha}$-global norm. This means for $k \geq 1$ we have $P_b[f]^{(k)} \leq P_b[\nabla f]^{(k-1)}$, so $P_b[f](t) \ll \int g + C$. Besides, it's easy to check $\int g \ll t \cdot g(t)$ when $g(t)$ is a positive power series.

□

Lemma 6.3.2. *Let f be defined on B_1^+ and g is a tensor-value function defined on the range $\mathcal{R}$ of f. If $[f]_{C^1(B_1)} \leq 2$, then their composition satisfies $P_b[g \circ f] \ll 2\bar{g} \circ P_b[f]$, where $\bar{g} \gg \sum_{k=0}^{\infty} \|D^k g\|_{C^\alpha(\mathcal{R})} \frac{t^k}{k!}$ is some scalar-value power series.*

Proof. For any $\varepsilon > 0$ (we'll finally let $\varepsilon \searrow 0$), there is $X \in B_1^+$ so that

$$\frac{P_b[g \circ f]}{1+\varepsilon} \leq \Delta^k[D^k(g \circ f)]_{L^\infty(B^+_{\frac{\Delta}{k+1}}(X))} + \Delta^{k+\alpha}[D^k(g \circ f)]_{C^\alpha(B^+_{\frac{\Delta}{k+1}}(X))}.$$

We first consider the first derivative case.

$$\begin{aligned}
\frac{P_b[g \circ f]}{1+\varepsilon} \leq &\Delta[\nabla g(f) \cdot \nabla f]_{L^\infty(B^+_{\Delta/2}(X))} + \Delta^{1+\alpha}[\nabla g(f) \cdot \nabla f]_{C^\alpha(B^+_{\Delta/2}(X))} \\
\leq &\Delta[g]_{C^1}[f]_{C^1} + \Delta^{1+\alpha}\{[g]_{C^1}[f]_{C^{1,\alpha}} + [g]_{C^{1,\alpha}}[f]_{C^1}[f]_{C^1}\} \\
\leq &[g]_{C^1}\Delta\{[f]_{C^1} + \Delta^\alpha[f]_{C^{1,\alpha}}\} + ([g]_{C^{1,\alpha}}\Delta[f]_{C^1}) \cdot [f]_{C^1} \\
\leq &[g]_{C^1}P_b[f]^{(1)} + ([g]_{C^{1,\alpha}}P_b[f]^{(1)}) \cdot 2 \leq 2\bar{g}' P_b[f]^{(1)}.
\end{aligned}$$

More generally, higher order chain rule says that there are polynomials p^i generated by higher derivatives $Df, D^2f, \cdots$ so that

$$D^k(g \circ f) = \sum_{i=1}^{k} g^{(i)}(f) \cdot p^i.$$

From this, we have in $B^+_{\Delta/2}(X)$,

$$\begin{aligned}
\Delta^k [D^k(g \circ f)]_{C^0} &\leq \sum_{i=1}^{k} \|g^{(i)}\|_{C^0} \cdot \Delta^{k-i} \|p^i\|_{C^0} \\
&\leq \sum_{i=1}^{k} \|g^{(i)}\|_{C^0} \cdot |p^{(i)}(\Delta \|Df\|_{C^0}, \Delta^2 \|D^2 f\|_{C^0}, \cdots)| \\
&\leq \sum_{i=1}^{k} \|g^{(i)}\|_{C^0} \cdot [p^{(i)}(P[f]^{(1)}, P[f]^{(2)}, \cdots)]_{C^0},
\end{aligned}$$

and

$$\begin{aligned}
\Delta^{k+\alpha} [D^k(g \circ f)]_{C^\alpha} \leq & \sum_{i=1}^{k} \|g^{(i)}\|_{C^0} \cdot \Delta^{k-i+\alpha} [p^i(P[f]^{(1)}, P[f]^{(2)}, \cdots)]_{C^\alpha} \\
& + \sum_{i=1}^{k} \Delta^\alpha [g^{(i)}(f)]_{C^\alpha} \cdot \Delta^{k-i} [p^i(P[f]^{(1)}, P[f]^{(2)}, \cdots)]_{C^0}.
\end{aligned}$$

We similarly bound $\Delta^\alpha [g^{(i)}(f)]_{C^\alpha}$ by $[g^{(i)}]_{C^\alpha} [f]_{C^1} \leq 2[g^{(i)}]_{C^\alpha}$ and computation afterwards is the same as first derivative case. □

6.4 Application to regular obstacle problem: relation to boundary Harnack

Let $U \geq 0$ be a solution of (1.2.5) which is better written as

$$\partial_{y_p}(b^{pq} \partial_{y_q} U) = \chi_{\{U>0\}}, \tag{6.4.1}$$

and let Γ be the free boundary passing through the origin. As mentioned in the introduction, if 0 is a regular boundary point, then $y_n = \Gamma(y')$ is a Lipschitz graph and boundary points on Γ are

all regular near 0. In fact, one can also show that Γ is locally C^1, meaning that after a rotation, $[\Gamma]_{C^{0,1}} \leq \delta$ near the origin for some small δ.

If $b^{pq} = \delta^{pq}$, or equivalently $\Delta U = \chi_{\{U>0\}}$, then classical boundary Harnack ensures that $\frac{U_{y_i}}{U_{y_n}}, \nabla_{y'}\Gamma \in C^\alpha$ in y-coordinate. More generally, if $b^{pq} \in C^1$, then

$$\partial_{y_p}(b^{pq}\partial_{y_q}U_{y_k}) = -\partial_{y_p}(U_{y_q}\partial_{y_k}b^{pq}) =: div(\vec{f_k}).$$

in $\{y_n > \Gamma(y')\}$. Here, $\vec{f_k}$ vanishes at Γ and $[\vec{f}]_{C^\alpha}$ is small. For such an equation with right-hand-side, one can use the method given by Ros-Oton and Torres-Latorre [47] to show that if $[\Gamma]_{C^{0,1}}, [\vec{f_k}]_{C^\alpha} \leq \delta$ for some small δ, then $\frac{U_{y_i}}{U_{y_n}} \in C^\alpha$.

Now let's assume Γ is a $C^{1,\alpha}$ graph and $|\nabla_{y'}\Gamma| \leq 1$ near 0. Let's denote

$$u_i := U_{y_i}, \quad w_i := \frac{U_{y_i}}{U_{y_n}}, \quad u := (u_1, \cdots, u_n), \quad w := (w_1, \cdots, w_n). \tag{6.4.2}$$

In the region $\{y_n > \Gamma(y')\}$, what u_k satisfies in y-coordinate is

$$\partial_{y_p}(b^{pq}\partial_{y_q}u_k) = -\partial_{y_p}(u_q\partial_{y_k}b^{pq}),$$

so by Lemma 4.5.1 with $f_p = -u_q\partial_{y_k}b^{pq}$, we know that u_k satisfies

$$\partial_{x_i}(a^{ij}\partial_{x_j}u_k) + \partial_{x_i}(\frac{\partial x_i}{\partial y_p}\partial_{y_k}b^{pq} \cdot u_q) = 0, \quad a^{ij} = b^{pq}\frac{\partial x_i}{\partial y_p}\frac{\partial x_j}{\partial y_q}. \tag{6.4.3}$$

Here, $\det(\frac{\partial y}{\partial x}) = 1$ because vertical coordinate change is volume-preserving. The ratio $w_k = \frac{u_k}{u_n}$, by Lemma 4.1.1, satisfies

$$\begin{aligned}\partial_{x_i}(u_n^2 a^{ij}\partial_{x_j}w_k) =& \partial_{x_i}(u_k u_q \frac{\partial x_i}{\partial y_p}\partial_{y_n}b^{pq}) - \partial_{x_i}(u_n u_q \frac{\partial x_i}{\partial y_p}\partial_{y_k}b^{pq}) \\ &- u_q\frac{\partial x_i}{\partial y_p}\partial_{y_n}b^{pq}\partial_{x_i}u_k + u_q\frac{\partial x_i}{\partial y_p}\partial_{y_k}b^{pq}\partial_{x_i}u_n.\end{aligned} \tag{6.4.4}$$

By implicit function theorem, the ratio $w_i = \dfrac{U_{y_i}}{U_{y_n}}$ represents the slope of U's level set, so $\partial_{y_k}\Gamma = \lim\limits_{y_n \searrow \Gamma(y')} w_k$. A naive application of Theorem 4.5.1 can inductively imply $\Gamma \in C^\infty(B_{1/2})$. Now let's be more careful in order to prove $\Gamma \in C^\omega$.

6.5 A PDE system

The expression (6.4.3) and (6.4.4) depend on u, w, a^{ij}, $\dfrac{\partial x}{\partial y}$, $\nabla_y B$, which in turn depend on the free boundary Γ. Two important quantities are ∇u and w, and we define their corresponding power series:

$$\Pi(t) = P_0[\nabla u](t), \quad \Omega(t) = P_1[w](t). \tag{6.5.1}$$

Next we write (6.4.3) and (6.4.4) in a more convenient form. Let's write

$$\tilde{A} = (\frac{u_n}{x_n})^2 A, \tag{6.5.2}$$

which is uniformly elliptic by Hopf lemma, and

$$\vec{F} = (\frac{u_k}{x_n}\frac{u_q}{x_n}\frac{\partial x_i}{\partial y_p}\partial_{y_n} b^{pq} - \frac{u_n}{x_n}\frac{u_q}{x_n}\frac{\partial x_i}{\partial y_p}\partial_{y_k} b^{pq})e_i, \tag{6.5.3}$$

$$G = \frac{u_q}{x_n}\frac{\partial x_i}{\partial y_p}\partial_{y_k} b^{pq}\partial_{x_i} u_n - \frac{u_q}{x_n}\frac{\partial x_i}{\partial y_p}\partial_{y_n} b^{pq}\partial_{x_i} u_k, \tag{6.5.4}$$

$$\vec{H} = \frac{\partial x_i}{\partial y_p}\partial_{y_k} b^{pq} \cdot u_q e_i. \tag{6.5.5}$$

With these notations, $A, \tilde{A}, \vec{F}, G, \vec{H}$ are polynomials in the variables

$$\Big(B, \nabla_y B, \frac{\partial x}{\partial y}, \nabla u, \frac{u}{x_n}\Big),$$

and (6.4.4) and (6.4.3) become the following PDE system:

$$div(x_n^2 \widetilde{A} \cdot \nabla w) = div(x_n^2 \vec{F}) + x_n G, \tag{6.5.6}$$

$$div(A \cdot \nabla u) = div(\vec{H}). \tag{6.5.7}$$

The following lemma describes how the coefficients $A, \tilde{A}, \vec{F}, G, \vec{H}$ are dominated by $\Pi(t)$ and $\Omega(t)$.

Lemma 6.5.1. *Assume that* $|\nabla\Gamma| \leq 1$ *and the matrix* B *is analytic in* y*-coordinate with large convergence radius. There exist analytic functions*

$$\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{F}, \mathcal{G}, \mathcal{H}$$

in the variables (t, Π, Ω) *with large convergence radius so that*

$$P_0[A] \ll \mathcal{A}(t, \Pi(t), \Omega(t)), \quad P_0[\widetilde{A}] \ll \widetilde{\mathcal{A}}(t, \Pi(t), \Omega(t)), \quad etc.$$

Proof. The majorant inequalities follow from Lemma 6.5.2 and 6.5.3 below. Lemma 6.5.2 yields that $A, \tilde{A}, \vec{F}, G, \vec{H}$ are analytic functions in the variables

$$\Big(x, \Gamma, \nabla\Gamma, \nabla u, \frac{u}{x_n}\Big).$$

Lemma 6.5.3 implies that $P_0[\Gamma]$,$P_0[\nabla\Gamma]$ are controlled by $\Omega(t)$, and $P_0[\frac{u}{x_n}]$ is controlled by $\Pi(t)$. □

Lemma 6.5.2. *Assume that* $B = (b^{pq})$ *is analytic in* y*-coordinate with large convergence radius, and that* Γ *is passing through* 0 *with* $|\nabla_{y'}\Gamma| \leq 1$*, then after the* $x - y$ *coordinate change,*

(a) $P_1[y(x)] \ll 2 + t + P_0[\Gamma]$ *and* $P_1[\frac{\partial y}{\partial x}], P_1[\frac{\partial x}{\partial y}] \ll 1 + P_0[\nabla\Gamma]$.

(b) If we treat $B(x) = B(y(x))$ *and* $\nabla_y B(x) = \nabla_y B(y(x))$ *as functions in* x*-coordinate, then*

there is some large R so that

$$P_1[B], P_1[\nabla_y B] \ll \frac{C}{R - t - P_0[\Gamma]}. \tag{6.5.8}$$

(c) For $a^{ij} = b^{pq} \dfrac{\partial x_i}{\partial y_p} \dfrac{\partial x_j}{\partial y_q}$, there is some large R so that

$$P_0[A] \ll C \frac{(1 + P_0[\nabla\Gamma])^2}{R - t - P_0[\Gamma]}. \tag{6.5.9}$$

Proof. We only prove (a),(b), and (c) is just obtained from (a),(b).

(a) This is because $y = x + \Gamma(x)e_n$ (see (4.5.5)), and $\dfrac{\partial y}{\partial x} = I_{n\times n} + \nabla\Gamma \otimes e_n$.

(b) As we assume $|\nabla_{y'}\Gamma| \le 1$, it follows that

$$[y(x)]_{C^1} \le [x]_{C^1} + [\Gamma(x')e_n]_{C^1} \le 2.$$

This means we can apply the composition rule Lemma 6.3.2.

□

Lemma 6.5.3. *Assume that $\Gamma(0) = 0$ and $u\Big|_{B_1'} = 0$, then*

(a) $P_0[\nabla\Gamma] = P_1[\nabla\Gamma] \ll P_1[w]$,

(b) $P_0[\Gamma](t) \ll P_1[w]^{(0)} + t \cdot P_1[w](t)$,

(c) $P_0[\dfrac{u}{x_n}] \ll P_0[\nabla u]$

(d) $P_0[u] \ll (1 + t) \cdot P_0[\nabla u](t)$.

Proof. (a) By implicit function theorem, $\partial_k\Gamma = \lim\limits_{x_n\to 0} \dfrac{U_{y_k}}{U_{y_n}} = \lim\limits_{x_n\to 0} w_k$, so $P_1[\nabla\Gamma] \ll P_1[w]$. An application of integration rule in Lemma 6.3.1 hence proves (b).

(c) When $u\Big|_{B_1'} = 0$, then $\dfrac{u}{x_n} = \displaystyle\int_0^1 e_n \cdot \nabla u(x', tx_n)dt$. By taking higher order derivatives to this integral, we obtain (c). We can then use the integration or product rule in Lemma 6.3.1 to prove (d).

□

6.6 Analyticity of free boundary

Now let's inductively estimate the higher regularity of w and u using an ODE system. For convenience, we will treat analytic expressions like $\mathcal{F}(t, \Pi(t), \Omega(t))$ as power series of t and denote

$$\mathcal{F}^{(k)} := \frac{d^k}{dt^k}\Big|_{t=0} \mathcal{F}(t, \Pi(t), \Omega(t)).$$

With this notation, we have $P_0[\vec{F}]^{(k)} \le \mathcal{F}^{(k)}$, $P_0[\widetilde{A}]^{(k)} \le \widetilde{\mathcal{A}}^{(k)}$, e.t.c..

First, we have an estimate of higher derivatives of w.

Lemma 6.6.1. *Assume that* $div(x_n^2 \tilde{A} \cdot \nabla w) = div(x_n^2 \vec{F}) + x_n G$ *where* $\tilde{A}$ *is uniformly elliptic, and*

$$P_0[\widetilde{A}] \ll \widetilde{\mathcal{A}}(t, \Pi, \Omega), \quad P_0[\vec{F}] \ll \mathcal{F}(t, \Pi, \Omega), \quad P_0[G] \ll \mathcal{G}(t, \Pi, \Omega),$$

then there is analytic and positive power series $\mathcal{M}$ *so that*

$$\frac{d}{dt}\Omega \ll \mathcal{M}(t, \Pi, \Omega). \tag{6.6.1}$$

Proof. Let β be any tangential multi-index with $|\beta| = k \ge 1$, applying D^β to (6.5.6) gives

$$\begin{aligned} div(x_n^2 \widetilde{A} \cdot \nabla D^\beta w) =& div(x_n^2 D^\beta \vec{F}) + x_n D^\beta G \\ &- \sum_{0<\gamma\le\beta} \prod_{m=1}^{n} \tbinom{\beta_m}{\gamma_m} \partial_{x_i}[x_n^2 (D^\gamma \widetilde{a}^{ij})(\partial_{x_j} D^{\beta-\gamma} w_k)]. \end{aligned}$$

The global Schauder estimate proposition 6.2.1 yields that

$$\begin{aligned}\Omega^{(k+1)} \leq & C\{(k+2)[w]^{*,k}_{C_0^{k,\alpha}} + [\vec{F}]^{*,k+1}_{C_0^{k,\alpha}} + [G]^{*,k+1}_{C_0^{k,\alpha}} + \sum_{l=1}^{k} \binom{k}{l} [\widetilde{A}]^{*,l}_{C_0^{l,\alpha}} [w]^{*,k-l+1}_{C_1^{l,\alpha}}\} \\ \leq & C\{(k+2)\Omega^{(k)} + \mathcal{F}^{(k)} + \mathcal{G}^{(k)} + \sum_{l=1}^{k} \binom{k}{l} \widetilde{\mathcal{A}}^{(l)} \Omega^{(k-l+1)}\} \\ \leq & C\{(t\frac{d}{dt}\Omega)^{(k)} + (\Omega + \mathcal{F} + \mathcal{G})^{(k)} + [(\widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}^{(0)})\frac{d}{dt}\Omega]^{(k)}\}.\end{aligned}$$

Here, we have used a combinatoric identity

$$\sum_{\substack{\gamma \leq \beta \\ |\gamma| = l}} \prod_{m=1}^{n} \binom{\beta_m}{\gamma_m} = \binom{|\beta|}{l}.$$

By setting $k \geq 0$ to be arbitrary, we have

$$\frac{d}{dt}\Omega \ll C\{t\frac{d}{dt}\Omega + \Omega + \mathcal{F} + \mathcal{G} + (\widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}^{(0)})\frac{d}{dt}\Omega\}.$$

Therefore, we obtain

$$\frac{d}{dt}\Omega \ll C\frac{\Omega + \mathcal{F} + \mathcal{G}}{1 - t - (\widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}^{(0)})} =: \mathcal{M}(t, \Pi, \Omega). \tag{6.6.2}$$

We remark that $(\widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}^{(0)})$ is a positive power series starting from linear term. □

Similarly, we have an estimate of higher derivatives of u.

Lemma 6.6.2. *Assume that $div(A \cdot \nabla u) = div(\vec{H})$ where A is uniformly elliptic, and*

$$P_0[A] \ll \mathcal{A}(t, \Pi, \Omega), \quad P_0[\vec{H}] \ll \mathcal{H}(t, \Pi, \Omega),$$

then there is analytic and positive power series $\mathcal{N}$ so that

$$\frac{d}{dt}\Pi \ll \mathcal{N}(t, \Pi, \Omega). \tag{6.6.3}$$

With these the standard Cauchy-Kovalevskaya theorem for ODEs proves the analyticity.

Corollary 6.6.1. *If* $\Gamma(0) = 0$, $\|\nabla_{y'}\Gamma\|_{C^\alpha} \leq 1$, *and* $\Pi^{(0)}$ *and* $\Omega^{(0)}$ *are bounded, then* $\Pi(t)$ *and* $\Omega(t)$ *are analytic near* $t = 0$, *with converging radius uniformly bounded from below.*

Proof. As Π and Ω satisfy two ODE inequalities, they are majorized by ODE solutions of the same right hand side and the same initial data, because the right hand sides are positive power series. Analyticity of first order ODE system gives a uniform lower bound of the convergence radius of $\Pi(t)$ and $\Omega(t)$. □

Proof of Theorem 1.2.3. Analyticity of w in x-coordinate implies the analyticity of $\Gamma(x)$ in x-coordinate. As $\Gamma(x) = \Gamma(x')$, and $x' = y'$ in the $x - y$ coordinate change, $\Gamma(y')$ is also analytic in y-coordinate. □

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