Revenue Management in Video Games and with Fairness

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Abstract
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Video games represent the largest and fastest-growing segment of the entertainment industry. Despite its popularity in practice, it has received limited attention from the operations community. Managing product monetization and engagement presents unique challenges due to the characteristics of gaming platforms, where players and the gaming platform have repeated (and endogenously controlled) interactions. These practices have also led to new customer concerns and thus regulation challenges. In this thesis, we describe a body of work that provides the first analytical results for revenue management and matchmaking problems in video games, as well as the fairness issues in many e-commerce platforms. In the first part, we discuss a prevailing selling mechanism in online gaming known as a loot box. A loot box can be viewed as a random bundle of virtual items, whose contents are not revealed until after purchase. We consider how to optimally price and design loot boxes from the perspective of a revenue-maximizing video game company, and provide insights on customer surplus and protection under such selling strategies. In the second part, we consider how to manage player engagement in a game where players are repeatedly matched to compete against one another. Players have different skill levels which affect the outcomes of matches, and the win-loss record influences their willingness to remain engaged. Leveraging optimization and real data, we provide insights on how engagement may increase with optimal matchmaking policies and adding AI bots. In the third part, we consider an increasingly important concern in many e-commerce platforms: the inequality induced by price
discrimination. While the practice of discriminatory pricing is generally widespread, it can result in disparate impact against protected groups. We consider the problem of setting prices for different groups of customers under fairness regulations, which limit the differences of various metrics (such as price and demand) across the groups. We show that different types of fairness constraints may not coexist in general, and the impact of fairness levels on social welfare could be non-monotonic and non-trivial.
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To the dark days before I met operations research.
Introduction

The video game industry involves 3 billion gamers and garners $180 billion annually. Despite its popularity in practice, there has been limited research from an operations management perspective. Managing product monetization and engagement presents unique challenges due to the characteristics of gaming platforms, where players and the gaming platform have repeated (and endogenously controlled) interactions. These practices have also led to new customer concerns and thus regulation challenges. In this thesis, we describe a body of work that provides the first analytical results for revenue management and matchmaking problems in video games, as well as the fairness issues in many e-commerce platforms.

In Chapter 1, we discuss a prevailing selling mechanism in online gaming known as a loot box. Loot box is a random allocation of virtual items whose contents are not revealed until after purchase. In this work, we consider how to optimally price and design loot boxes from the perspective of a revenue-maximizing video game company, and analyze customer surplus under such selling strategies. Our paper provides the first formal treatment of loot boxes, with the aim to provide customers, companies, and regulatory bodies with insights into this popular selling strategy. We consider two types of loot boxes: a traditional one where customers can receive (unwanted) duplicates, and a unique one where customers are guaranteed to never receive duplicates. We show that as the number of virtual items grows large, the unique box strategy is asymptotically optimal among all possible strategies, while the traditional box strategy only garners 36.7% of the optimal revenue. On the other hand, the unique box strategy leaves almost zero customer surplus, while the traditional box strategy leaves positive surplus. Further, when
designing traditional and unique loot boxes, we show it is asymptotically optimal to allocate the
items uniformly, even when the item valuation distributions are heterogeneous. We also show that
when the seller purposely misrepresents the allocation probabilities, their revenue may increase
significantly and thus strict regulation is needed. Finally, we show that even if the seller allows
customers to salvage unwanted items, then the customer surplus can only increase by at most
1.4%. This work is detailed in [1].

In Chapter 2, we consider how to manage player engagement in a game where players are
repeatedly matched to compete against one another. Managing player engagement is an important
problem in the video game industry, as many games generate revenue via subscription models and
microtransactions. We consider a class of online video games whereby players are repeatedly
matched by the game to compete against one another. Players have different skill levels which
affect the outcomes of matches, and the win-loss record influence their willingness to remain
engaged. The goal is to maximize the overall player engagement over time by optimizing the
dynamic matchmaking strategy. We propose a general but tractable framework to solve this
problem, which can be formulated as an infinite linear program. We then focus on a stylized
model where there are two skill levels and players churn only when they experience a losing
streak. The optimal policy always matches as many low-skilled players who are not at risk of
churning to high-skilled players who are one loss away from churning. In some scenarios when
there are too many low-skilled players, high-skilled players are also matched to low-skilled
players that are at risk of churning. Regarding the power of the optimal policy, we compare it to
the industry status quo that matches players with the same skill level together (skill-based
matchmaking). We prove the benefit of optimizing the matchmaking system can grow linearly
with the number of skill levels. We then use our framework to analyze two common but
controversial interventions to increase engagement: adding AI bots and a pay-to-win system. We
show that optimal matchmaking may reduce the number of bots needed significantly without loss
of engagement. The pay-to-win system can influence player engagement positively when the
majority of players are low-skilled. Surprisingly, even non-paying low-skilled players may be
better off in some scenarios. Finally, we conduct a case study with real data from an online chess platform. We show the optimal policy can improve engagement by 4-6% or reducing the percentage of bot players by 15% in comparison to skill-based matchmaking. This work is detailed in [2].

In Chapter 3, we turn our attention to a new regulation concern faced by almost all e-commerce platforms nowadays: the inequality induced by price discrimination. Price discrimination strategies, which offer different prices to customers based on differences in their valuations, have become common practice. While it allows sellers to increase their profits, it also raises several concerns in terms of fairness, e.g., by charging higher prices (or denying access) to protected minorities in case they have higher (or lower) valuations than the general population. This topic has received extensive attention from media, industry, and regulatory agencies. In this paper, we consider the problem of setting prices for different groups under fairness constraints. We first propose four definitions: fairness in price, demand, consumer surplus, and no-purchase valuation. We prove that satisfying more than one of these fairness constraints is impossible even under simple settings. We then analyze the pricing strategy of a profit maximizing seller and the impact of imposing fairness on the seller’s profit, consumer surplus, and social welfare. Under a linear demand model, we find that imposing a small amount of price fairness increases social welfare, whereas too much price fairness may result in a lower welfare relative to imposing no fairness. On the other hand, imposing fairness in demand or consumer surplus always decreases social welfare. Finally, no-purchase valuation fairness always increases social welfare. We observe similar patterns under several extensions and for other common demand models numerically. Our results and insights provide a first step in understanding the impact of imposing fairness in the context of discriminatory pricing. This work is detailed in [3].
Chapter 1: Loot Box Pricing and Design

1.1 Introduction

With the recent explosion of online and mobile gaming over the last decade [4], the idea of games-as-a-service (GaaS) has been widely accepted as a way to provide video game content as a continuing revenue stream. Recently, the video game industry standard has shifted towards the freemium model, where access to a game is freely given to customers, and in-game virtual items can be acquired via microtransactions. In other words, virtual items that help players in the game are purchased with in-game or real-world currency. In many of these games, microtransactions are conducted via a randomized mechanism known in the video game industry as a loot box. A loot box is a random allocation of virtual items, the contents of which are revealed after the purchase is complete. While the concept of a loot box is not new – for instance, a pack of baseball cards is a form of loot box – modern versions of loot boxes like in Fig. 1.1 have proliferated in online video games on mobile, console, and PC platforms over the last decade. In online games such as Dota 2, FIFA 20, PlayerUnknown’s Battlegrounds, and many others, loot box sales are a core source of revenue. In these games, customers purchase loot boxes which contain a random subset of virtual items such as character costumes, cosmetic upgrades, players, virtual cards, etc. In 2018 alone, more than $30 billion dollars in sales were conducted via loot boxes [5].

Despite its popularity in the gaming industry, the use of loot boxes has invited controversy and criticism recently [6], with several development platforms enacting specific regulations in response [7, 8]. For instance, there have been issues regarding the transparency of the contents and probability of outcomes from loot boxes [9] which have led to regulatory investigations [10]. Another issue is that loot boxes have been associated with gambling, both by the media [11] and academia [12, 13]. This led the United States Congress to introduce a bill outlining new loot box
Figure 1.1: Loot Box Example.

Note. Depicted is a loot box in the popular video game Dota 2, where it is also referred to as a “treasure”. The customer may purchase the loot box for $2.49, after which they receive one of the five items depicted on the screen.

regulations [14] and in August 2019 the Federal Trade Commission held a workshop on the matter [15], which included the presentation of an early version of this paper [16].

In spite of this negative publicity, loot box selling remains as popular as ever [17]. In order to properly address the issues of transparency and gambling via regulation, we believe that it is fundamental to understand the economic motivation behind loot box selling. Why do video game companies prefer such a business strategy? How does it compare to traditional strategies such as bundling or separate selling? What are the behavioral considerations motivating customers to keep purchasing loot boxes? A rigorous framework for the operations of loot boxes would provide valuable insights for customers, companies, and regulators, which is precisely the focus of the work.

There are several salient features that distinguish the loot box mechanism and virtual items from other traditional industries. First, the virtual items have zero marginal cost, can be copied infinitely by the seller, and have no value outside the game. Second, customers engage in repeated interactions with the seller, and may potentially buy a large number of items sequentially. Third, the seller is fully aware of a customer’s current collection, and has control over the loot box allocation rules. These unique features render the models designed for other types of products inadequate and call for a new revenue management framework to specifically analyze loot box strategies.
In this work, we provide a model to analyze the optimal pricing and design of loot boxes for revenue-maximizing sellers. The model incorporates two types of commonly used loot box strategies. A *unique loot box* allocates items to customers that they do not currently own. A *traditional loot box* allocates items randomly to customers regardless of whether they already own a copy, which may result in obtaining unwanted duplicates. In both strategies, the seller may control the price, number of items allocated, allocation probabilities, and salvage value of the loot boxes. To understand the advantages and disadvantages of loot box strategies, we compare them to two traditional selling mechanisms: *separate selling*, where every virtual item is sold separately for a known price, and *grand bundle selling*, where customers pay a one-time fixed amount for access to all virtual items.

Next we provide a summary of our contributions and findings.

1. We propose the first mathematical modeling framework for selling loot boxes. Customers are endowed with i.i.d. private valuations for all items, and sequentially purchase loot boxes until their expected utility becomes negative. We show that almost no dominance relations hold between any of the four selling strategies we consider.

2. Motivated by the fact that the number of virtual items in a video game is often in the hundreds or thousands, we turn our attention to the setting where the number of items offered tends to infinity. In this asymptotic regime, both the grand bundle and unique box strategies are optimal, while the traditional box strategy only generates 36.7% as much revenue. Surprisingly, we show that the expected number of purchases from the traditional box strategy is roughly the same as the unique box strategy, although the optimal price of the traditional box is lower. However, the unique box strategy provides no customer surplus in this regime, in contrast to the traditional box strategy which leaves a positive customer surplus. We then connect our asymptotic results back to practice by conducting a numerical study to show that our findings still hold when the catalog size is finite and moderately sized.

3. Next, to accommodate the scenario that items may belong to classes with different rarities
and values in the game, we consider the case where the valuation distributions are heterogeneous across items. When the number of items is large, we show that the optimal allocation probabilities of a loot box is a random draw among all available items, independent of how customers value items in any class. Thus under an optimal price and allocation rule, a rare class is less likely to be drawn because there are less items in the class, not because customers value it highly. We also show that the seller may gain significant revenue if they can successfully deceive consumers into believing a false set of allocation probabilities, even if such allocation probabilities are accurate in expectation. We conclude that regulation is needed to protect consumers from such a practice.

4. Finally, we consider an additional design aspect where customers are allowed to salvage, or return, unwanted items. We show that traditional box strategies may earn more revenue with salvage systems and are guaranteed to dominate separate selling. Surprisingly however, we show that introducing a salvage system in a traditional box strategy can only increase customer surplus by at most 1.4%. On the other hand, as unique box strategies are already optimal, salvage systems cannot increase their revenue. However such systems do allow sellers to trade-off between revenue and customer surplus in a smooth fashion.

1.1.1 Literature Review

While loot box selling has not been previously studied in the revenue management literature to the best of our knowledge, our work draws inspiration from and is related to several areas across operations management, computer science, and economics.

In the operations management literature our work connects with the dual streams of papers on opaque selling and bundle selling. Loot boxes are an example of opaque selling, which is the practice of selling items where some features of the item are hidden from the customer until after purchase. Recent works [18, 19, 20, 21] have focused on opaque selling as a tool to manage imbalanced customer demand or induce opportunities for price discrimination. Our loot box framework diverges from the standard opaque selling models in a number of key ways. First, we consider the
performance of loot boxes in isolation, as opposed to many models in which the opaque option is sold in conjunction/in competition with traditional sales channels [22, 23, 24, 25]). Second, we model complex, repeated interactions between the loot box seller and a customer interested in obtaining a catalog of items as opposed to prior work which has focused on customers who want at most one item. Third, in our loot box model we do not have finite inventories, which diverges from the literature on using opaque products to balance inventory [26, 27, 28, 29].

Our work also resembles and references the work on bundling. As in the bundling literature, the loot box is a way to sell products to markets of customers with demand for many items. In particular, we compare our loot box selling mechanisms explicitly with the grand bundle mechanisms studied in the seminal work of [30], who show that pure bundling extracts almost all of the consumer surplus asymptotically. By leveraging results from the theory of random walks, we show that the unique box strategy (allocating one item at a time without replacement) can achieve a similar revenue to grand bundle selling without forcing the customer to choose between purchasing the whole catalog or nothing. This property allows loot boxes to circumvent many of the issues that plague bundle selling in practice (c.f. Section 1.5.2 for a detailed discussion). From a technical standpoint, this is a stark divergence from the typical techniques in the bundling literature which rely on concentration results to induce a single purchase of the entire catalog of items.

Mixed bundle strategies, strategies that allow customers to purchase the items from a menu offering both the grand bundle and the items individually, and randomized strategies, strategies that offer lotteries over possible allocations of items, have been considered recently in a stream of work on optimal mechanisms for selling items to additive buyers [31, 32, 33, 34, 35, 36]. A loot box can be thought of as a particularly simple type of randomized bundle strategy where only a single lottery over all the items is offered, and from which a single item is allocated. A similar type of mechanism is considered in [37]. They study the optimal pricing of menus of unit-demand bundles under stylized valuation assumptions, while we study repeated interactions with a customer which is the main driver of loot box revenues. Moreover, the focus of their work is on the computational hardness of computing the prices for menus of such bundles. In contrast, our
loot box mechanism is dynamic and computationally simple. [38] considers bundle selling with return options in the presence of production costs, while our paper considers loot boxes with return options and no production costs.

Closest to our paper, in the sense that an individual customer dynamically purchases multiple items from the seller, is the work of [39]. There the authors consider whether or not to offer the products in sequence or all at once, but do not consider any form of randomized selling strategies such as a loot box. The focus of their paper is on understanding the value of concealment in the context of fast fashion, whereas our loot box model does not conceal any part of the catalog, and the focus is on the choice of randomized strategy. There has also been a line of work where a customer makes decisions in multiple stages when faced with an assortment from the seller, although only at most one unit is purchased [40, 41, 42].

Finally, our work contributes to the emerging literature on operations management in video games. [43] and [44] investigate the problem of maximizing a player’s engagement in video games. [45] considers the problem of incentivizing actions in freemium games. [46] considers whether the seller should disclose an opponent’s skill level when selling in-game items that can increase the win rate. Our work is the first to investigate the popular practice of loot box selling via mathematical modeling.

1.2 Model and Preliminaries

We consider a revenue-maximizing monopolist selling a catalog of \( N \) distinct, non-perishable, virtual items. A random customer’s valuation for the items are described by non-negative i.i.d. random variables \( \{V_i\}_{i=1}^{N} \), where each \( V_i \) is drawn from a distribution \( F \). The mean and variance of \( V_i \) are denoted by \( \mu \) and \( \sigma^2 \), respectively, and are assumed to be finite. The assumption of i.i.d. valuations is reasonable when the items are cosmetic (such as character skins and customizations) or when items are of similar importance, both of which are common in many games that deploy loot boxes. In Section 1.4.2, we extend our model to address the case where items are vertically differentiated and can naturally be categorized into multiple classes based on their values or rarities.
We suppose that each customer is aware of all available items in the seller’s catalog as well as their own realized valuations for the items \( v_i \) for \( i \in [N] \), where \([N]\) is used to represent the index set \( \{1, \ldots, N\} \). Each newly obtained item \( i \) gives the customer a one-time utility of \( v_i \), which can be thought of as the lifetime value of the item in the game, and is assumed to be independent of the period in which it is received. Further, no customer values having duplicates of an item, meaning a customer’s valuation for a second unit of each item \( i \) is 0. For example, a character skin or cosmetic upgrade for the player’s avatar, once obtained, can be enjoyed for as long as the player engages with the game, and a second copy offers no additional value to the player. In some games, the seller provides a salvage mechanism through which the customer can obtain value from duplicate items by trading them in for (in-game or real-world) currency. We discuss this extension in Section 1.4.4.

A loot box can be formally defined as a random allocation of a single item to the customer, chosen according to a probability distribution over all \( N \) items. We note that the probability distribution is decided by the seller, and may or may not depend on the customer’s current inventory. There are also cases where multiple items are allocated in one loot box, which is an extension we consider in Section 1.4.1. Moreover, we assume that the customer always knows the actual allocation probabilities (that is, the probabilities of receiving each item). This is consistent with industry practice, as sellers are often forced to announce the allocation probabilities, either by government issued customer protection regulations [10] or by edict of the games distributor [7, 8]. In Section 1.4.3, we consider extensions where the seller may misrepresent the allocation probabilities.

We now describe the sequence of events in our loot box model, which capture a single customer repeatedly interacting with the seller. Before the arrival of the customer, the seller announces the price and allocation probabilities of the loot box. We consider each purchasing event to be a discrete period and emphasize that periods do not necessarily correspond to any particular unit of time, i.e. periods can be thought of as occurring in rapid succession (for a player eager to complete their collection) or occurring with long gaps between purchases (for a more judicious player). In each period \( t \), we let \( S_t \subset [N] \) denote the index set of distinct items that the customer owns before
opening the loot box in period $t$. Thus, $S_1 = \emptyset$. Based on the price, allocation probabilities, and the customer’s private valuations for items in $[N] \setminus S_t$, the customer decides whether or not to purchase the loot box. We assume customers are utility-maximizing and will purchase if their expected utility from purchasing is non-negative, otherwise the customer will not purchase further loot boxes (they may however continue to play the game). We discuss the customer behavior in greater detail in Section 1.2.1.

We now formally describe the two forms of loot box selling that we focus on as well as two benchmark strategies known as grand bundle selling and separate selling.

1) **Unique Box (UB):** In the unique box strategy, the seller offers a loot box for a fixed price $p$ in each period, with the guarantee that each purchase yields a new item that the customer does not yet own. Formally, the probability of receiving an item is 0 if $i \in S_t$, and $\frac{1}{|[N] \setminus S_t|}$ for $i \in [N] \setminus S_t$, i.e., uniform over all the items not currently owned by the customer. Fig. 1.2a shows an example of a unique box in practice. We let $R_{UB}(p)$ be the normalized revenue of a unique box strategy that uses price $p$, i.e.,

$$R_{UB}(p) := \frac{p \times \mathbb{E}[\# \text{ of Unique Box Purchases}]}{N}$$

and let $R_{UB} := \max_p R_{UB}(p)$.

2) **Traditional Box (TB):** In the traditional box strategy, the seller offers a loot box for a fixed price $p$ in each period, with the guarantee that each purchase yields an item selected uniformly at random from $[N]$, regardless of what the customer owns in $S_t$. Traditional boxes lead to the possibility of duplicate items during a customer’s purchasing process. Fig. 1.2b shows an example of a traditional box in practice. We let $R_{TB}(p)$ be the normalized revenue of a traditional box strategy that uses a fixed price $p$, i.e.,

$$R_{TB}(p) := \frac{p \times \mathbb{E}[\# \text{ of Traditional Box Purchases}]}{N}$$

and let $R_{TB} := \max_p R_{TB}(p)$.
We emphasize that, at first glance, it is not clear which loot box strategy generates more revenue. Intuitively, customers have higher valuations for unique boxes since they are guaranteed not to receive duplicates (recall we assume the utility derived from the second copy of any item is 0), which allows sellers to charge higher prices. On the other hand, although the seller may have to charge lower prices for traditional boxes, the selling volume may end up being higher because customers need to make more purchases in order to obtain new items. Indeed, for finite $N$, we provide instances where either strategy may dominate the other in Table 1.1.

Further, in both loot box strategies we assume the allocation probabilities are uniform over the remaining items/all items. We note that such allocation rules may not be optimal even though valuations for items are i.i.d. (see Example A.1). In Section 1.4.2, we provide results on the asymptotic optimality of uniform allocation rules.

We shall compare and contrast these loot box models against two classic selling models: grand bundle selling and separate selling.

3) **Grand Bundle (GB):** In the *grand bundle* strategy, the seller offers a single bundle containing *all* $N$ items for price $Np$. Customers no longer make dynamic decisions when a grand bundle is offered, but rather just make a single decision to purchase or not. The normalized revenue of a
grand bundle strategy with price $Np$ is
\[ R_{GB}(p) := \frac{Np^p \left( \sum_{i=1}^{N} V_i \geq Np \right)}{N}, \]
and the optimal normalized revenue is denoted by $R_{GB} := \max_p R_{GB}(p)$. Fig. 1.3a shows an example of a grand bundle in practice.

4) Separate Selling (SS): In the separate selling strategy, the seller offers all items individually at the same price $p$. Since we assume the valuations $V_i$ are i.i.d, the normalized revenue of a separate selling strategy with price $p$ is
\[ R_{SS}(p) := \frac{Np^p (V_i \geq p)}{N}, \]
and the optimal normalized revenue is denoted by $R_{SS} := \max_p R_{SS}(p)$. Fig. 1.3b shows an example of separate selling with uniform prices in practice.

While there are many ways to sell virtual items, we restrict our attention to these four as we believe they capture the spirit of almost all strategies observed in practice. Among them, the grand bundle and separate selling strategies provide two important benchmarks. Separate selling is the most common selling strategy in e-commerce. Grand bundle selling is also common for digital
goods such as music and television, and has been shown to be able to fully extract the maximum possible revenue when $N$ tends to infinity \[30\]. However, unlike the other three strategies, under the grand bundle strategy a customer must commit to purchasing all of the items or none of the items. In practice, this large upfront financial commitment may impair the grand bundle’s performance. In contrast, separate selling and loot box strategies are more friendly to customers that prefer smaller purchases or have a smaller budget. Although the grand bundle selling may be prohibitive in practice, it serves as a useful theoretical benchmark since it is asymptotically revenue-optimal (see Section 1.5.2 for a detailed discussion).

1.2.1 Customer Behavior

We next describe how customers value loot boxes and make purchase decisions. We assume that customers are risk-neutral, and their valuation of a loot box is simply the expectation, over the allocation probabilities, of the valuation of the random item they will receive. Let $U_t$ be the expected utility of opening a loot box in period $t$ for a price $p$. Since the allocation probabilities are uniform, $U_t$ has the following form:

$$(\text{Unique Box}) \quad U_t = \frac{\sum_{i \in [N] \setminus S_t} V_i}{N - |S_t|} - p, \quad (\text{Traditional Box}) \quad U_t = \frac{\sum_{i \in [N] \setminus S_t} V_i}{N} - p.$$  

Naturally, to maximize the expected utility, customers would purchase the $t^{th}$ loot box if $U_t \geq 0$. However, it is sometimes rational for a customer to purchase even if $U_t < 0$ for the prospect of higher utilities in future periods. The following example demonstrates that myopic behavior (purchasing if and only if $U_t \geq 0$) is not necessarily optimal for the customer.

**Example 1.1.** Let $N = 2$ and consider the unique box strategy. Let the price of each loot box be $p = 1.6$. Consider a customer whose valuations of the two products are $(v_1, v_2) = (1, 2)$. If the customer is myopic, then they will not buy a single unique box since the expected utility from the first loot box purchase is $\frac{1 \cdot 2}{2} - 1.6 < 0$. However, it can be shown by enumeration that the following purchasing strategy is optimal: buy a loot box in the first period. If the obtained item is product 2
where \( v_2 = 2 \), then stop purchasing. Otherwise purchase a second loot box, which is guaranteed to contain product 2. The expected net utility of this strategy is
\[
\frac{1}{2} (2 - 1.6) + \frac{1}{2} (1 + 2 - 1.6 \times 2) = 0.1 > 0.
\]
Thus, behaving myopically is strictly worse than the optimal strategy.

In Example A.2 we generalize Example 1.1 and show that not only is the myopic strategy suboptimal, but so is any policy that considers only a finite number of future states. Thus, in each period \( t \), a perfectly rational customer needs to solve a high-dimensional and complex optimal stopping problem to decide whether or not to purchase. However, we believe it is both impractical and unrealistic for customers to find the the optimal purchasing strategy, as the state space of the corresponding optimal stopping problem increases exponentially in the number of items. Instead, we make the natural modeling assumption that customers are indeed myopic, i.e., they purchase if and only if their expected net utility for the next loot box \( U_t \) is non-negative. Theorem 1.1 shows that myopic behavior is asymptotically optimal for a customer facing unique box selling as the catalog of items grows large. Moreover, we show in Theorem 1.1 that myopic behavior is always optimal for a customer facing traditional boxes, lending additional support to our myopic assumption for customers facing loot boxes.

**Theorem 1.1** (Myopic Purchasing Behavior is Asymptotically Optimal).

a) For unique box selling, the average net utility under the myopic strategy converges to the average net utility of the optimal strategy as \( N \to \infty \).

b) For traditional box selling, the myopic purchasing strategy is optimal for all customers.

Due to the apparent complexity and impracticality of computing the customer’s optimal purchasing policy and the fact that a myopic purchasing rule is near-optimal, we believe restricting to myopic purchasing behavior does not degrade the power of our models. Further, we note that in cases where the optimal purchasing strategy differs from the myopic strategy (i.e. unique boxes), the customer purchases strictly more loot boxes under the optimal behavior. To see this, note that a myopic customer will stop purchasing as soon a loot box gives negative utility, while an optimal customer may continue purchasing due to a positive expected future reward. Thus the revenue of a
loot box strategy under the assumption of myopic behavior is a lower bound on the revenue when customers purchase optimally.

For the remainder of this paper, we shall assume customer behavior is myopic.

1.2.2 Comparing the Strategies for Finite $N$

In this section, we aim to understand the relations between the four strategies when $N$ is finite. Specifically, we would like to establish dominance relations between the optimal revenues of (UB), (TB), (SS), and (GB). In Proposition 1.1, we show that the normalized revenue of any of the strategies is always at most $\mu$, and that the unique box strategy can never exceed the revenue of a grand bundle strategy.

**Proposition 1.1.** For any $N$ and valuation distribution $F$, the following statements hold:

(a) $\mu \geq \max\{R_{GB}, R_{SS}, R_{UB}, R_{TB}\}$, i.e., $\mu$ is a global upper bound on the normalized revenue.

(b) $R_{GB} \geq R_{UB}$, i.e., grand bundle selling weakly dominates unique box selling.

**Proof.** (a) For any strategy, the customer only makes a purchase when their expected utility is non-negative. Thus, the expected customer surplus is always non-negative. On the other hand, the total normalized expected welfare is always at most $\mathbb{E}[\sum_{i\in [N]} V_i]/N = \mu$. Together, these facts imply that the normalized revenue for any strategy is at most $\mu$.

(b) Let $p^*$ be the optimal price of the unique box strategy. An upper bound on $R_{UB}$ is then $Np^*$. For a customer to purchase the very first unique box, we must have that $\frac{\sum v_i}{N} \geq p^*$. Under the same condition, the customer would buy the grand bundle at price $Np^*$ since $\sum v_i \geq Np^*$. Similarly, if $\sum v_i < Np^*$, the customer would not buy the first unique box nor the grand bundle at price $Np^*$. Therefore, the revenue from an optimal grand bundle strategy is at least as much as the optimal unique box strategy. $\Box$

Unfortunately, outside of Proposition 1.1, there does not exist any other dominance relationships among the four selling strategies. In particular, the same argument in Proposition 1.1(b) does
not extend to the comparison of grand bundle selling and traditional box selling. Although the condition for purchasing the first traditional box remains the same, a customer may end up buying strictly more than \( N \) boxes overall due to the possibility of duplicates. In Table 1.1 we give simple examples for which all of the 11 remaining possible relationships between the four selling strategies occur.

<table>
<thead>
<tr>
<th>Relations</th>
<th>( N )</th>
<th>Valuation Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( GB &gt; UB, GB &gt; TB, GB &gt; SS )</td>
<td>3</td>
<td>( \mathbb{P}(V_i = 0.98) = 1/2, \mathbb{P}(V_i = 2.02) = 1/6, \mathbb{P}(V_i = 3.01) = 1/3 )</td>
</tr>
<tr>
<td>( UB &gt; SS, UB &gt; TB, SS &gt; TB )</td>
<td>10</td>
<td>( \mathbb{P}(V_i = 1) = 0.8, \mathbb{P}(V_i = 5) = 0.2 )</td>
</tr>
<tr>
<td>( TB &gt; UB, TB &gt; SS )</td>
<td>3</td>
<td>( \mathbb{P}(V_i = 1.01) = 1/2, \mathbb{P}(V_i = 1.98) = 1/6, \mathbb{P}(V_i = 3.03) = 1/3 )</td>
</tr>
<tr>
<td>( SS &gt; GB, TB &gt; GB )</td>
<td>2</td>
<td>( \mathbb{P}(V_i = 1) = \mathbb{P}(V_i = 100) = 1/2 )</td>
</tr>
<tr>
<td>( SS &gt; UB )</td>
<td>4</td>
<td>( \mathbb{P}(V_i = 1) = 3/10, \mathbb{P}(V_i = 10) = 7/10 )</td>
</tr>
</tbody>
</table>

From Table 1.1, we can see that it is impossible to theoretically compare the selling strategies when \( N \) is small without imposing significant additional assumptions. Therefore, in the rest of the paper we focus on an asymptotic analysis where the number of items \( N \) tends to infinity. Fortunately, this is also well-motivated in the gaming industry where \( N \), the number of items sold in a video game, is often in the hundreds or thousands. For example, in the popular online games \( Dota 2 \) and \( Overwatch \), the number of cosmetic items sold through loot boxes exceeds 3500. As we shall see, dominance relations among the four strategies naturally emerge in the asymptotic regime.

### 1.3 Asymptotic Analysis of Loot Box Strategies

In this section, we study the revenue of optimally priced loot box strategies as the number of items \( N \) in the catalog tends to infinity. Given the incomparability of the various selling strategies shown in Section 1.2.2 and the fact that \( N \) is often quite large in practice, this asymptotic analysis is quite natural. In this asymptotic regime, we shall show that an optimal unique box strategy earns a normalized revenue of \( \mu \) per item (c.f. Theorem 1.2), whereas the optimal traditional box strategy earns a normalized revenue of only \( \frac{\mu}{e} \approx 0.367\mu \) (c.f. Theorem 1.3). Since the expected
normalized revenue of any selling strategy cannot exceed the mean valuation $\mu$ by Proposition 1.1, this result proves that unique box and traditional box strategies are asymptotically optimal and sub-optimal, respectively. Additionally, we can directly compare the performance of these two loot box strategies with grand bundle selling and separate selling in this regime. Using the strong law of large numbers, it is well known that the grand bundle also obtains a normalized revenue of $\mu$ (see [30] for a detailed discussion). On the other hand, the revenue of separate selling strategies depends explicitly on the distribution of customer valuations, and can earn anywhere between 0% and 100% of the normalized revenue.

**Theorem 1.2** (Asymptotic Revenue and Convergence Rate of UB). *The unique box strategy is guaranteed to earn*

\[
R_{UB} \geq \mu \left(1 - N^{-1/5} \right) \left( 1 - \left(1 + \frac{2\sigma^2}{\mu^2} \right) N^{-1/5} - \frac{\sigma^2}{\mu^2} N^{-3/5} - \frac{\sigma^4}{\mu^4} N^{-4/5} - \left( \frac{\sigma^2}{\mu^2} + \frac{\sigma^4}{\mu^4} \right) N^{-6/5} \right).
\]

Moreover,

\[
\lim_{N \to \infty} R_{UB} = \mu.
\]

The proof of Theorem 1.2 follows by modeling the customer behavior dynamically as a random walk and explicitly constructing a sequence of prices that lead to the lower bound on the revenue. Specifically, we consider a random walk that captures the total utility a customer collects in each period and bound the number of purchases made. Unfortunately, the time a customer stops purchasing is not a stopping time since it depends on all of the customer’s valuations, which are not known to the seller. Thankfully we are able to approximate the time that the customer stops purchasing by a true stopping time, and then leverage standard machinery to bound the total number of purchases. Finally, we show that setting a price of $p = \mu \left(1 - \frac{1}{N^{1/5}} \right)$ leads to the desired result.

**Theorem 1.3** (Asymptotic Revenue and Convergence Rate of TB). *The traditional box strategy is*
guaranteed to earn

\[ \frac{\mu}{e} \log \left( \frac{1}{\frac{1}{e} + \frac{\sigma^2}{N}} \right) \leq R_{TB} \leq \frac{\mu}{e^{1-\xi_N} (1 - N^{-\frac{1}{2}})} + \frac{(1 - N^{-\frac{1}{2}}) \sigma^2 \log N}{\mu N^{\frac{1}{2}}}, \]

where \( \xi_N = \sum_{i=1}^{N} \frac{1}{i} - \log (N) - \gamma \), and \( \gamma \) is the Euler-Mascheroni constant. Moreover,

\[ \lim_{N \to \infty} R_{TB} = \frac{\mu}{e}. \]

To prove Theorem 1.3, we construct a ‘backwards’ random walk that captures the total valuation collected by a customer, starting from the very last item they would purchase. We show that in our constructed random walk, the number of unique items collected corresponds to a stopping time, and again leverage standard machinery to bound the stopping time. Since duplicates are allowed, we must also account for the number of purchases required to a collect a unique item, which depends on the number of items collected so far.

Theorems 1.2 and 1.3 highlight an important design aspect of loot boxes: the ability to monitor a customer’s current inventory and appropriately control the allocations. With full information of a customer’s inventory, a seller can implement unique boxes which are asymptotically revenue-optimal. Without this information, the seller is restricted to traditional loot boxes which garner only \( \frac{1}{e} \) fraction of the optimal revenue. Note that this is not a lower bound, but rather an exact asymptotic limit of traditional loot box selling revenue. Moreover, both of these results hold for any underlying valuation distribution (with bounded first and second moments).

Next, we investigate the properties of optimal unique and traditional box strategies and study the limiting optimal price, sales volume, and customer surplus. We present these results in Theorem 1.4 and note that it provides several seemingly counter-intuitive insights into the differences between unique and traditional box strategies. For instance, one would expect that since unique boxes never leave customers empty-handed, they are preferred by customers. In addition, one may also expect that since traditional boxes yield duplicates, customers may tend to buy strictly more
of them than unique boxes. Surprisingly, when optimally priced both of these intuitions are false.

**Theorem 1.4** (Insights into Loot Box Strategies).

(a) For the unique box strategy, as \( N \to \infty \), the optimal price converges to \( \mu \). Further, the expected fraction of unique items collected by the customer converges to 1, the expected normalized number of loot boxes purchased converges to 1, and the expected normalized customer surplus converges to 0.

(b) For the traditional box strategy, as \( N \to \infty \), the optimal price converges to \( \mu e \). Further, the expected fraction of unique items collected converges to \( 1 - \frac{1}{e} \), the expected normalized number loot boxes purchased converges to 1, and the expected normalized customer surplus converges to \( 1 - \frac{2}{e} \)\( \mu \).

Theorem 1.4(a) states that the optimal price for unique boxes as \( N \) tends to infinity is approximately \( \mu \), and that customers purchase approximately the entire catalog. Since their average valuation is \( \mu \), this leaves them with no consumer surplus. On the other hand, Theorem 1.4(b) states that the optimal price for traditional boxes as \( N \) tends to infinity is approximately \( \mu e \), and that customers purchase approximately \( N \) boxes obtaining \( 1 - \frac{1}{e} \) fraction of the catalog of items. Although the consumers do not acquire the entire catalog, they are left with a positive normalized consumer surplus of approximately \( (1 - \frac{2}{e})\mu \). To summarize, under both strategies the customer purchases approximately \( N \) loot boxes. However, the price is lower for traditional boxes, resulting in less revenue for the seller and more surplus for the consumer. Surprisingly, consumers are therefore better off with traditional boxes when duplicate allocations are allowed, since unique boxes lead to higher prices and vanishing customer surplus.

In light of Theorem 1.3, it is worthwhile to discuss why traditional boxes are popular among sellers in practice, given their substantially lower expected revenue in our model. We posit three possible explanations. First, traditional boxes have existed well before the digital age in the form of Gachapon or as packs of Pokèmon cards, and the practice may continue as a hold over from those times. Second, there is long-term value in making sure consumers are left with positive surplus (c.f.
Theorem 1.4), which is not explicitly included as an objective of the seller in our model. Third, the presence of a salvage system (resale market) may increase the revenue of a traditional box strategy since loot boxes can be sold at a higher price. We study this idea in detail in Section 1.4.4.

1.4 Loot Box Design

In this section, we extend the results of the previous section to handle various practical considerations and design aspects beyond the choice of unique versus traditional boxes. Recall that our basic model assumes that each loot box allocates one random item, that the valuations for all items are i.i.d., that the allocation distribution is uniformly random, and that customers obtain no value from duplicate items. In practice, these assumptions may sometimes be violated and thus we address them here. In Section 1.4.1, we discuss an extension where each loot box allocates multiple items. In Section 1.4.2, we discuss the case where there are multiple classes of items, and characterize the optimal allocation probabilities for potentially vertically differentiated items. In Section 1.4.3, we explore the regulatory concern when the seller deviates from the announced allocation probabilities. Finally, in Section 1.4.4, we consider the situation where the seller offers a salvage system (resale market), in which unwanted items can be salvaged by the customer for some return value.

1.4.1 Multi-item Loot Boxes

Although many games allocate one item at a time from their loot boxes, it is also a common practice to allocate multiple items at once (see Fig. A.1 for an example). A classic example of loot boxes containing multiple items are Pokémon or Baseball cards, which are sold in packs of ten or twelve. In practice, sellers may use a size-$j$ loot box when the mean valuation of a single item $\mu$ is very low (e.g., less than $0.10$. In this case, selling multiple items in one box allows the seller to set a higher price, which helps reduce the number of transactions for the customer and allows prices to conform to market norms (e.g., the phenomenon of pricing at $0.99$).

In this section, we show that Theorems 1.2 and 1.3 can be extended to the case where loot
boxes are of fixed size \( j > 1 \). We use \( R^j_{UB} \) and \( R^j_{TB} \) to denote the optimal revenue of the size-\( j \) unique box and traditional box strategies, respectively. Proposition 1.2 shows that when \( j \) is fixed and \( N \) tends to infinity, the unique box strategy is still optimal and the traditional box strategy still only earns 36.7\% as much.

**Proposition 1.2** (Multi-Item Loot Boxes). *For size-\( j \) loot boxes,*

\[
\lim_{N \to \infty} R^j_{UB} = \mu \quad \text{and} \quad \lim_{N \to \infty} R^j_{TB} = \frac{\mu}{e}.
\]

The proof uses a coupling argument. Specifically, we show that in the last period that a customer is willing to purchase a size-\( j \) box, they would have purchased a size-1 box as well. This reduces the total number of items purchased to the case studied in the previous section. In the case of the traditional box strategy, the seller needs to slightly decrease the price of the traditional size-\( j \) box as its value is now lower on average than a size-1 box, due to the possibility of more duplicates in a single box.

### 1.4.2 Optimizing Allocation Probabilities for Multiple Classes of Items

In previous sections, we assumed that valuations for all items are i.i.d., and that each item (*un-owned* item in the case of unique boxes) was equally likely to be allocated by the loot box. In practice, these assumptions may not always hold. Often in online games, the items are explicitly grouped based on rarity or effectiveness. For instance, in the popular online game *PlayerUnknown’s Battlegrounds*, customers may receive Mythic, Legendary, Epic, or Rare items from a loot box (see Fig. A.2 for an example).

To model this phenomenon, suppose there are \( M \) different classes of items, and that each item \( i \in [N] \) belongs to a specific class \( m \in [M] \). Denote \( G_m \subset [N] \) as the index set of items in class \( m \) and denote \( \beta_m \) as the proportion of the items belonging to class \( m \), i.e., \( \beta_m := |G_m|/N \) and \( \sum_{m \in [M]} \beta_m = 1 \). For each class of items \( m \), the valuations for the items in that class are sampled i.i.d. from distribution \( F_m \). We denote the mean and standard deviation of the valuations for items
in class $m$ by $\mu_m$ and $\sigma_m$. Let $\overline{\mu} := \sum_{m=1}^{M} \beta_m \mu_m$ be the expected valuation of a random item. For different classes, the distribution $F_m$ may vary arbitrarily.

For asymptotic results, we shall suppose the number of items in each class grows proportionally with $N$, i.e., there are $\beta_m N$ items belonging to class $m$ as $N$ increases. The introduction of multiple item classes allows for some items to be significantly more valuable than others, and thus it is reasonable to consider non-uniform allocation probabilities of the items. A loot box strategy is now characterized by a price $p$ and the allocation probabilities of each item, which may depend on its class. Our goal is to characterize the revenue-optimal combination of price and allocation probabilities for loot boxes over multiple classes of items.

For unique box strategies, the optimal allocation probabilities may be non-uniform, dynamic, and depend on the current set of items owned by the customer. It is difficult to explain such policies to customers, let alone characterize the optimal allocation probabilities. Thankfully, for unique boxes there is a simple allocation and pricing strategy that is asymptotically optimal for the seller. Proposition 1.3 shows that a unique box strategy that simply allocates all unowned items uniformly at random, completely ignoring the class, is asymptotically optimal. Thus, for the $t^{th}$ unique box, each unowned item is allocated with probability $\frac{1}{N-(t-1)}$.

**Proposition 1.3** ((UB) with Uniform Allocation is Asymptotically Optimal). Suppose unique boxes allocate items uniformly at random, independent of class. Then we have that

$$\lim_{N \to \infty} R_{UB} = \overline{\mu}.$$  

Surprisingly, the proof of Proposition 1.3 is almost identical to that of Theorem 1.2. Although the distribution of the unowned items is no longer i.i.d., Wald’s identity and Chebyshev’s inequality continue to hold where appropriate.

Next, we shift focus to the allocation problem for the traditional box strategy. Once again, we shall show that a simple allocation policy is asymptotically optimal. Specifically, we show that it is optimal to allocate each item uniformly at random (w.p. $\frac{1}{N}$), independent of the class. To simplify
the problem, we restrict our attention to class level allocation probabilities, i.e., allocation rules where all items in the same class have the same allocation probabilities but may differ between classes. We emphasize that class level allocation rules are common in practice (see Fig. A.2 for examples). For a class level allocation rule, we let \( d_m \) be the probability of drawing an item in class \( m \), and \( \mathbf{d} = (d_1, \ldots, d_M) \) be a probability vector. Thus, the probability of getting an item in class \( m \) is \( \frac{d_m}{\beta_m N} \). A uniform allocation rule corresponds to the case where \( \mathbf{d} = \mathbf{\beta} \).

Let \( Q^N(p, \mathbf{d}) \) denote the normalized number of loot boxes purchased by a customer (i.e. \( \frac{E[\text{# Purchases}]}{N} \)) and let \( R_{TB}(p, \mathbf{d}) \) be the normalized expected revenue, both of which are a function of the price \( p \) and class allocation probabilities \( \mathbf{d} \). In Proposition 1.4 we show that for a carefully constructed price as a function of \( \mathbf{d} \), the limit of \( Q^N(p, \mathbf{d}) \) can be characterized simply.

**Proposition 1.4.** Suppose a traditional box strategy follows a multi-class allocation rule \( \mathbf{d} \) and price \( p = \sum_{m=1}^M d_m \mu_m e^{-\frac{d_m}{\beta_m} k} \) for some \( 0 < k < 1 \). Then

\[
\lim_{N \to \infty} E\left[ Q^N(p, \mathbf{d}) \right] = k.
\]

To build intuition for Proposition 1.4, consider the case of a single class of items, and note from the proof of Theorem 1.4 that the number of unowned items after opening \( kN \) traditional boxes is roughly \( Ne^{-k} \). If a customer stops after \( kN \) purchases, then their valuation for the next box is roughly \( \mu e^{-k} \), meaning that a price of \( \mu e^{-k} \) will induce purchase up to that point but no further. The proposed price in Proposition 4 generalizes this intuition to the multi-class case.

For example, when \( d_m = \beta_m \) for all \( m \) and \( k = 0 \), the induced price in Proposition 1.4 is \( p = \bar{\mu} \).

Proposition 1.4 implies that the limiting normalized selling volume for this price with the uniform allocation strategy is 0. This agrees with our intuition, as the customer valuation drops below \( \bar{\mu} \) after opening \( \epsilon N \) boxes for any fixed \( \epsilon > 0 \). Armed with Proposition 1.4, we find in Theorem 1.5 that as \( N \) tends to infinity, the simple strategy of setting the price to be \( \frac{\bar{\mu}}{\epsilon} \) and the allocation probability vector to be \( \mathbf{\beta} \) results in asymptotically optimal normalized revenue of \( \frac{\bar{\mu}}{\epsilon} \).

**Theorem 1.5** ((TB) with Uniform Allocations are Asymptotically Optimal). For traditional box
with $M$ classes, we have

$$\lim_{N \to \infty} \max_p R_T(p, d) = \lim_{N \to \infty} R_T\left(\frac{\bar{\mu}}{e}, (\beta_1, \ldots, \beta_M)\right) = \frac{\bar{\mu}}{e}.$$ 

**Proof.** Consider a class allocation probability vector $d = (d_1, \ldots, d_M)$. For $p > \sum_{m=1}^{M} d_m \mu_m$, by the law of large numbers, the normalized selling volume will tend to 0. Thus we can focus on the case $p \leq \sum_{m=1}^{M} d_m \mu_m$. Note $\sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\bar{\mu} m}}$ equals $\bar{\mu}$ when $k = 0$, and decreases monotonically to 0 as $k \to \infty$. Thus for any price $p$, there exist a unique positive $k$ such that $p = \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\bar{\mu} m}}$. Recall by Proposition 1.4, if $p = \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\bar{\mu} m}}$, $1 > k > 0$, then

$$\lim_{N \to \infty} \mathbb{E}\left[Q^N(p, d)\right] = k.$$

Using this identity we can write the limiting revenue function in terms of $k$, i.e.,

$$\lim_{N \to \infty} R_T(p, d) = \lim_{N \to \infty} p \cdot \mathbb{E}\left[Q^N(p, d)\right] = k \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\bar{\mu} m}} := \sum_{m=1}^{M} G_m(k).$$

Consider the $m^{th}$ term of the revenue function, $G_m(k) = \mu_m d_m k e^{-\frac{d_m k}{\bar{\mu} m}}$. This function obtains its maximum at $k = \beta_m / d_m$, and the maximum value is $\beta_m \mu_m / e$, which is independent from the value of $d_m$. Hence, the limiting optimal revenue is bounded above by $\sum_{m=1}^{M} \beta_m \mu_m / e = \bar{\mu} / e$. For any $d$, we can reach the the upper bound $\bar{\mu} / e$ if every component function reaches the maximum simultaneously in the limit, i.e., $k = \beta_m / d_m$ for all $m$. Since $\sum_{m=1}^{M} \beta_m = \sum_{m=1}^{M} d_m = 1$, the only possible limiting allocation is $d_m = \beta_m$, which is the uniform allocation. In this case, $k = 1$, and the corresponding price is $p = \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\bar{\mu} m}} = \bar{\mu} / e$. Hence, the solution $p = \bar{\mu} / e$ with uniform allocation is asymptotically optimal with corresponding revenue $\bar{\mu} / e$. 

Note that the allocation $d = (\beta_1, \ldots, \beta_M)$ is simply the uniform allocation over all the items (not over all classes). Thus Theorem 1.5 is a natural generalization of Theorem 1.3 to the multi-class case. The uniform allocation strategy with price $\bar{\mu} / e$ is asymptotically optimal, for any fixed
number of classes and any set of valuation distributions. In this sense, Theorem 1.5 makes the
decisions of a seller who adopts traditional box selling simple: instead of designing complicated
allocation structures, just use uniform allocations and focus on the price. Further, Theorem 1.5
extends the asymptotic dominance of unique box strategies over traditional boxes to the case of
multiple item classes; varying the allocation probabilities cannot close the gap in revenue between
the two strategies.

We emphasize that for uniform allocations to be optimal in Theorem 1.5, the price for the loot
box must be optimally chosen. For prices other than \( \frac{\mu}{e} \), uniform allocations may not be optimal.
For example, suppose we have two classes with \( \mu_1 = 10 \), \( \mu_2 = 5 \) and \( \beta_1 = \beta_2 = 0.5 \), then the
price \( \frac{\mu}{e} = 2.76 \) with uniform allocation is asymptotically optimal. However, if the seller uses
the price $3, then by optimizing over \( d \), we find that the asymptotically optimal class allocation
probabilities are \( d = (0.514, 0.486) \).

Finally, we note that another natural selling mechanism to consider in this setting is to sell
different classes separately via loot boxes of various prices. However, this mechanism is more
complex and may create a sense of unfairness whereby wealthy players are able to obtain the high-
value items more easily. Therefore, we believe that the simple allocation rule proposed in this
section is preferable, in addition to being asymptotically optimal.

1.4.3 Transparency of the Allocation Probabilities

In previous sections, we assumed that both the customer and the seller believe and act according
to the announced allocation probabilities. In practice, the seller sometimes may lie about the
allocation probabilities by purposely using an allocation strategy different than the announced
strategy. In this section, we discuss the potential implications of such deceit.

Consider a situation where the seller deviates from the posted allocation rule, but the customers
still believe the posted allocation rule. As an example, such a situation occurred in the game
Monster Taming, where the seller claimed the chance to receive a rare item was 1% whereas the
actual odds were 0.0005% [9]. In such cases, the duped customer may end up buying many more
loot boxes due to the false announcement. In Example 1.2, we demonstrate that sellers can greatly increase their revenue by misrepresenting the allocation probabilities, and further, can do so in a way that is difficult to detect (unlike in the case of *Monster Taming*). In particular, Example 1.2 shows that a so-called *random perturbation strategy* can increase the revenue of a traditional loot box while adhering to the announced allocation rule in expectation, making such a deception hard to detect.

**Example 1.2.** Consider a traditional box with a single class of items and price $\mu/e$. Suppose the seller claims that a uniform allocation is used, but instead, the seller randomly chooses half of the items to be allocated with probability $\frac{1+\varepsilon}{N}$ and the other half to be allocated with probability $\frac{1-\varepsilon}{N}$. By Proposition 1.4, the normalized selling volume under a truly uniform allocation is asymptotically equal to 1. On the other hand, when the random perturbation is used, the traditional box can be regarded as a two-class traditional box, with $\mu_1 = \mu_2 = \mu$, and $d = (0.5(1+\varepsilon), 0.5(1-\varepsilon))$. If the customer has complete information, then by Proposition 1.4, the normalized selling volume $k$ is given by solving

$$\mu e^{-1} = \mu \left( \frac{1+\varepsilon}{2} e^{-(1+\varepsilon)k} + \frac{1-\varepsilon}{2} e^{-(1-\varepsilon)k} \right). \quad (1.1)$$

However, if the customer assumes the allocation is uniform, then the weight of two classes changes from $(\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2})$ to $(\frac{1}{2}, \frac{1}{2})$ while the exponential terms in Eq. (1.1) remain the same. In this case $k$ is given by solving

$$\mu e^{-1} = \mu \left( \frac{1}{2} e^{-(1+\varepsilon)k} + \frac{1}{2} e^{-(1-\varepsilon)k} \right),$$

and it turns out that $k$ is strictly greater than one. Thus, the selling volume (and thus revenue) increases by setting $\varepsilon > 0$. For example when $\varepsilon = 0.2$, the selling volume and revenue increases by 2.4%.

We highlight that even small deviations from uniform allocations can be profitable, while being quite difficult for consumers or regulators to discover. Notice that under the strategy in Example 1.2, when the perturbation is randomized for each customer, the total number of each item allocated among all the customers in the market is balanced. Thus a regulator examining aggre-
gate allocation data would not be able to detect the existence of such strategies. Example 1.2 demonstrates the need for regulators to focus on not only enforcing that sellers follow the stated allocation probabilities, but also to ensure that the seller follows these rules precisely for each customer. Thus, effective regulation may need to require that sellers disclose granular, customer-by-customer and transaction-level data so that the regulators may conduct statistical tests to detect unfair strategies.

1.4.4 Salvage System

In previous sections, we assumed that customers extract zero value from duplicate items received from traditional boxes, and that a customer could not resell items back to the seller. In practice however, some loot box marketplaces are equipped with salvage systems, mechanisms by which a customer can trade in unwanted items for currency. Salvage systems are a ubiquitous method for managing customer satisfaction under various sales policies, offering customers a form of recourse against unlikely or unfortunate outcomes (see Fig. A.3 for an example). In this section, we shall consider loot box selling strategies that allow customers to trade-in or salvage items for a value $c$. For simplicity, we restrict our attention to the case where the loot box allocates a single item at a time and there is only a single class of items.

The main focus in this setting is to understand the two competing effects that salvage systems have on loot box revenue. On the one hand, the presence of a salvage cost $c$ increases the minimum valuation of any item to at least $c$, increasing the expected valuation of an item (from $\mathbb{E}[V_i]$ to $\mathbb{E}[\max\{V_i, c\}]$) and thus inducing more purchases. On the other hand, salvage systems return currency to the customer which dilutes the revenue garnered from customer purchases. The results in this section characterize and extend the revenue guarantees of Theorems 1.2 and 1.3 to the case when items can be salvaged for some value $c$. Throughout this section we use the notation $R.(c)$ to denote the optimal revenue of a strategy with fixed salvage cost $c$. Note that in the presence of a salvage system, the allocation mechanism for unique box strategies is no longer well-specified. For our results we assume that customers facing a unique loot box strategy are never allocated an
item they had previously salvaged, which is the case in the example described in Fig. A.3.

We first show in Theorem 1.6 that the introduction of a salvage system by the seller makes both loot box strategies more attractive than separate selling. Specifically, by treating the salvage cost \( c \) as a parameter of a loot box strategy, the revenues of both the optimal unique box or traditional box strategies are guaranteed to dominate the revenue of separate selling.

**Theorem 1.6** (Loot Boxes with Salvage Outperform Separate Selling). For any \( N \), both the unique box and traditional box strategies with a salvage system dominate separate selling, i.e.,

\[
\max_c R_{UB}(c) \geq R_{SS} \quad \text{and} \quad \max_c R_{TB}(c) \geq R_{SS}.
\]

**Proof.** Proof of Theorem 1.6. Let \( p^* \) be the optimal price used by separate selling. Now consider a loot box strategy (either unique or traditional) with salvage cost \( p^* \) and price \( p^* \). The customer will purchase loot boxes, keeping all the items which they value at \( p^* \) or greater and returning the unwanted items for a full refund, until they obtain all items which they value above \( p^* \). Thus, such a loot box induces the same revenue as separate selling, which implies that

\[
\max_c R_{UB}(c) \geq R_{SS} \quad \text{and} \quad \max_c R_{TB}(c) \geq R_{SS}.
\]

\[\square\]

We emphasize that this result is valid for any finite \( N \). It is well known that the grand bundle is not guaranteed to outperform separate selling for finite \( N \), even though grand bundle selling is asymptotically optimal. Thus, Theorem 1.6 allows us to pin down the precise relationship between loot box strategies and separate selling and further explains the power and popularity of loot boxes in practice.

We next investigate the revenue of salvage systems in the asymptotic regime. Proposition 1.5 gives the limiting normalized revenue with respect to a fixed salvage cost \( c \).
Proposition 1.5 (Revenue and Surplus of Loot Boxes with Salvage Costs.). Let \( c \) be the salvage cost, \( \eta = \mathbb{E}[V_i | V_i > c] \), and \( \overline{F}(c) = \mathbb{P}(V_i > c) \).

a) The unique box strategy asymptotically earns

\[
\lim_{N \to \infty} R_{UB}(c) = \eta \overline{F}(c),
\]

and the limiting normalized customer surplus is \( c \overline{F}(c) \).

b) The traditional loot box strategy asymptotically earns

\[
\lim_{N \to \infty} R_{TB}(c) = \overline{F}(c) (\eta - c) \left( \frac{c}{\eta - c} + e^{-\frac{\eta}{\eta - c}} \right),
\]

and the limiting normalized customer surplus is \( \overline{F}(c) \left( \eta - c - (2\eta - c) e^{-\frac{\eta}{\eta - c}} \right) \).

This result generalizes the insights derived from Theorem 1.2 and Theorem 1.3 to the case with salvage costs. First, note that like before the (asymptotic) revenue of unique box strategies dominates the (asymptotic) revenue of traditional box strategies for any valuation distribution \( F \) and salvage cost \( c \). To see this, note since \( 0 < c \leq \eta \), we may substitute \( c \) by \( c = q\eta \), for some \( q \in [0, 1] \). Plugging in this substitution and rearranging yields:

\[
\lim_{N \to \infty} R_{TB}(c) \leq \overline{F}(c) \eta \max_{q \in (0, 1]} \left( q + (1 - q)e^{-\frac{1}{1-q}} \right) \leq \overline{F}(c) \eta = \lim_{N \to \infty} R_{UB}(c), \tag{1.2}
\]

where the final equality comes from noting \( q + (1 - q)e^{-\frac{1}{1-q}} \) is monotone increasing and tends to 1 as \( q \to 1 \). Thus \( \lim_{N \to \infty} R_{UB}(c) \geq \lim_{N \to \infty} R_{TB}(c) \). Further, the monotonicity in the maximum in Eq. (1.2) implies when \( \frac{\xi}{\eta} \) is large (close to 1), the gap in expected revenue between unique box strategies and traditional box strategies is small and generally shrinks from a factor of \( e^{\frac{\xi}{\eta}} = 0 \) monotonically down to 1 \( (\frac{\xi}{\eta} = 1) \). Thus when salvage costs are large relative to \( \eta \), the additional value of employing unique box strategies is diminished. Further, by combining Theorems 1.2 and 1.6 and Proposition 1.5, we obtain a complete ordering of the four strategies in presence of
salvage cost:
\[
\lim_{N \to \infty} R_{GB} = \lim_{N \to \infty} \max_c R_{UB}(c) \geq \lim_{N \to \infty} \max_c R_{TB}(c) \geq \lim_{N \to \infty} R_{SS}.
\]

Compared to the revenue without salvage, it is worth noting that for the unique boxes, \( \overline{F}(c) \eta \leq \mu \), with equality achieved only when \( c = 0 \). Thus, when \( N \) is large it is never optimal for a purely revenue-maximizing seller to use salvage systems with unique boxes. For traditional box, the benefit of introducing a salvage system is distribution-dependent. For example, when \( V_i \) is a uniform random variable supported on \([0, 2\mu]\), the asymptotic optimal traditional box revenue with salvage is 0.517\( \mu \), which is 40.4\% better than 0.368\( \mu \), the revenue without salvage.

Finally, salvage systems are primarily used to improve customer outcomes and overall satisfaction with the system. For unique box strategies, the expected normalized customer surplus under a revenue maximizing unique box with salvage cost \( c \) is \( c F(c) \). Note the expected normalized customer surplus is monotonically increasing in \( c \). Thus the salvage system enables the unique box seller to balance revenue and customer surplus to their desired proportion.

For traditional box strategies, we find in Theorem 1.7 that the revenue-maximizing seller may only increase the surplus by at most 1.4\%, compared to the case without salvage \( (c = 0) \).

**Theorem 1.7 (Salvage System Barely Increases Surplus for (TB)).** The limiting normalized customer surplus of the traditional box strategy with any salvage cost \( c \) is at most 1.4\% more than the customer surplus of the traditional box strategy with no salvage system \( (c = 0) \).

**Proof.** Recall from Proposition 1.5 that the customer surplus given salvage \( c \) is
\[
\overline{F}(c) \left( (\eta - c) - (2\eta - c)e^{-\frac{\eta}{\eta-c}} \right).
\]

By replacing \( c \) with \( q\eta \), one can see that the normalized customer surplus can be expressed as
\[
\overline{F}(c) \left( (\eta - c) - (2\eta - c)e^{-\frac{\eta}{\eta-c}} \right) = \overline{F}(c) \eta \left( (1 - q) - (2 - q)e^{-\frac{1}{1-q}} \right). \tag{1.3}
\]
As a special case, when $c = 0$ the normalized surplus is $(1 - \frac{2}{e})\mu = 0.264\mu$ as discussed in Theorem 1.4. When $c > 0$, note that in the right-hand side of Eq. (1.3), one can easily see that $\bar{F}(c)\eta \leq \mu$, and $\max_q (1-q) - (2-q)e^{-\frac{1}{1-q}} = 0.268$ where the maximum is reached when $q = 0.074$. So using any salvage $c$ may increase the customer surplus by at most $1.4\%$, i.e., $0.268/0.264-1$. In fact, the surplus decreases in most cases.

The following example shows that this bound is tight. Suppose valuations are constant and equal to 1 for all items. Then for any $c < \mu$, $\eta = \mu = 1$. Let $c = 0.074\eta = 0.074$. Then $\bar{F}(c)\eta = 1$, and the normalized surplus is 0.268, which is 1.4\% better than 0.264, the surplus from traditional box without salvage.

Theorem 1.7 implies that for traditional boxes, the salvage system may serve as a tool to increase revenue, but does not necessarily improve the customers satisfaction.

1.5 Numerical Experiments

In previous sections, we showed that, asymptotically, unique boxes are optimal and traditional boxes earn a constant fraction of the optimal revenue. In this section, we conduct numerical experiments to demonstrate the efficacy of unique box and traditional box selling for finite $N$. First, in Section 1.5.1 we compute and compare the optimal revenues of UB, TB, GB, and SS strategies for typical valuation distributions over a range of values for $N$ that reflect industry practices. We then numerically investigate the impact of budget constraints on the performance of UB, TB, GB and SS strategies in Section 1.5.2.

In our experiments we consider three possible valuation distributions: the uniform distribution between 0 and 2, the log-normal distribution with log-mean 0 and log-variance 1, and the exponential distribution with mean 1. These distributions are commonly used to model customer valuations, and have been previously studied in [36]. For each distribution, we let $N$ range from 1 to 3000, and consider the optimal prices and revenues of the four candidate strategies: UB, TB, GB, and SS. Computation of the revenues is done via simulation by generating 50,000 customer sample paths and using brute force to search for the optimal prices (at 1\% accuracy).
1.5.1 Finite Number of Items

In Fig. 1.4, we show how the optimal prices of the various strategies change as $N$ increases. Note that for all three distributions that $\mu = 1$ and the optimal price of the traditional box strategy quickly converges to $\frac{1}{e}$ for $N \geq 50$. The optimal price of the unique box strategy also converges to $\mu$ for each distribution, albeit at a slower rate.

Figure 1.4: Optimal Prices for Uniform (left), Log-normal (middle) and Exponential (right) Valuations.

![Optimal Prices Graph](image)

In Fig. 1.5, we plot the normalized revenues for each strategy. For traditional boxes, the revenue of the optimal policy quickly converge to $\mu e$ for all three valuation distributions. For unique boxes, the revenue of the optimal policy trends slowly towards it’s limit of 1, however it closely follows the revenue of the optimal grand bundle strategy. When $N \geq 100$, the unique box strategy garners more than 70% of the maximum possible revenue and more than 97% of the revenue of the grand bundle. The performance for finite $N$ also depends closely on the distribution. For example, under uniform valuations, the revenue of grand bundle and unique box strategies converge faster than in the case of log-normal valuations. We also note, when $N$ is small, traditional boxes may garner significantly more than $1/e$ under uniform valuations, while under log-normal valuations, it converges to $1/e$ from below.

![Normalized Revenues Graph](image)
1.5.2 Budget Constraints

Finally, we look into the impact of budgets on the relative performance of the various strategies. Recall that without budgets, as the catalog of items grows, the normalized revenue of separate selling remains fixed and distribution-dependent whereas grand bundle selling becomes asymptotically optimal [30]. However, the fact that customers are budget-constrained may hamper the performance of bundling in practice. In particular, the asymptotic optimality of bundling strategies breaks down when customers vary in their ability to spend on virtual items, i.e., when customers are budget-constrained.

In this subsection, we model the case where customers vary in their budgets to spend on virtual items. Specifically, let each customer’s budget be a realization of the random variable $B$ with distribution $F_B$. When budgets vary in this fashion, the revenue of grand bundle selling becomes highly dependent on the distribution $F_B$ and may be greatly diminished in some cases, whereas the revenues of separate selling and loot box selling are essentially unchanged.

The (unnormalized) revenue of the optimal grand bundle strategy in the presence of random budgets is:

$$\max_p \ p \ \mathbb{P} \left( \sum_{i=1}^{N} V_i \geq p, B \geq p \right)$$

which is upper bounded by the revenue of a single price strategy in the space of budgets, $\max_p \ p \ \mathbb{P} \left( B \geq p \right) = \frac{1}{2}$. 

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max_p p (1 − F_B(p)). Note that, like the normalized revenue of separate selling, max_p p (1 − F_B(p)) can be anywhere between 0% to 100% of the expected budget size E[B] and depends explicitly on F_B.

On the other hand, separate selling and both loot box strategies are quite robust to random fluctuations in customers’ budgets. To see this, first consider a customer with budget b facing separate selling. For any fixed price p such a customer simply purchases items until they either obtain all the items for which their valuation exceeds the price, or until they exhaust their budget. In the case where the customer exhausts their budget b, they will purchase \( \left\lfloor \frac{b}{p} \right\rfloor \) items, spending all of their budget except for possibly some leftover amount less than the price of one item. Thus, for each fixed price p, the revenue of separate selling garnered from each customer is either identical to the case without budgets or close to the budget up to the price of a single item.

Loot boxes, owing to the sequential nature in which they are purchased, share this property with separate selling. For each customer facing a loot box strategy and fixed price p, they either purchase the same number of boxes as they would in the case without budgets or they spend all of their budget except for possibly some leftover amount less than the price of one box. Combining this observation with Theorem 1.2 shows that unique boxes maintain the asymptotic optimality properties of grand bundle selling while remaining effective in the presence of heterogeneous customer budgets. Thus budget considerations give strong justification for the efficacy and popularity of loot boxes in practice.

In the remainder of this section we study the impact of budgets numerically. First, we specify the distribution of customer budgets. In 2016, 51% of paying customers spent less than $50 on in-game purchases, and 70% of paying customers spent less than $100 on in-game purchases [47]. To fit this data, we assume that the budget B of a random customer follows an exponential distribution with a mean of $75. Under this assumption, 49% of the customers spend less than $50, and 74% of the customers spend less than $100.

In Fig. 1.6, we plot the optimal revenues of each strategy with respect to N, facing customers with exponentially distributed budgets. We use total revenue as the performance measure because
the expected revenue now is upper bounded by the expected budget size, $75. When $N \leq 10$, we note that GB maintains its revenue dominance over the other three strategies. However, as $N$ increases, UB, TB and SS extract almost all the budget, whereas the grand bundle strategy falters, only garnering about 37% the customers budget. This experiment further shows the robustness of loot box strategies in the presence of limited budgets.

Figure 1.6: (Unnormalized) Optimal Revenue for Uniform (left), Log-normal (middle) and Exponential (right) Valuations with Budget Constraints.

1.6 Conclusions

Our work implies a host of managerial insights for sellers, customers, and regulators of loot boxes. For sellers, we give a thorough analysis of the profitability of loot boxes, yielding guidelines for how to design and price loot boxes so as to maximize revenue. We show that the unique box strategy is asymptotically optimal, whereas the traditional box can garner only 36.7% of the maximum revenue. These results hold in the cases where loot boxes allocate multiple items as well as when the items are heterogeneous. When customers are budget-constrained the implication is even clearer: unique box selling stands above the other three strategies, retaining the asymptotic efficiency properties of grand bundle selling while remaining robust to fluctuations in customers ability to spend. Finally, and perhaps surprisingly, even in the case where items come from multiple classes, designing the allocation rule for either loot box strategy is easy: a simple uniform allocation policy is effective and asymptotically optimal.
From the customer’s perspective, we show that the traditional loot box strategy is preferred and the unique box strategy does not yield any customer surplus. Further, we show that the introduction of a salvage system has surprisingly little affect on customer surplus when facing traditional loot boxes, with a potential gain of only 1.4%. We also show that customers may be at risk to seller manipulation. Specifically, if the seller deviates from the announced allocation probabilities, then they are capable of making more revenue, even when the allocation probabilities are correct in expectation. Thus, it is essential for regulators to protect consumers against such a scenario. In fact, we show that the regulator must check each customer’s allocations individually to properly ensure that the seller is being truthful. These facts together show that ex post analysis of allocations may be insufficient in detecting fraudulent behavior on behalf of the seller and suggest that regulatory bodies must inspect customer-dependent data streams or loot box mechanisms at their source implementation to ensure truthful behavior.

Finally, while loot box selling has recently gained attention in the domains of psychology and policy-making, there is a distinct lack of academic work which analyzes loot boxes from a revenue management perspective. Our work breaks ground on this topic, but leaves open several interesting avenues for future work. One particularly fruitful direction for future work may consider loot boxes under richer customer valuation models. In our work, valuations for the items are assumed to be a priori identical for each customer, and independent of the customer’s current collection. Follow-up work may consider some form of structured dependency among the customer valuations for the items, e.g., they are modulated by a common customer-specific random factor. One may also consider the case where valuations are no longer additive, but submodular or supermodular in the items currently owned. For example, customers may highly value the very last unowned item since it is necessary to complete their collection. In this case, the revenue from a traditional box strategy may increase dramatically, since it takes many purchases in expectation to collect all the items. On the other hand, if many of the items are substitutable, the customers might be less interested to open the last few boxes since their additional value is marginal relative to the current collection.
Further, our model does not specify the time scale on which the items are acquired. Customer behavior may be more complicated when the utility is proportional to how often the item is used. Within this setting, it is possible that customers have higher valuations for items acquired earlier since the player can use those items for longer. One may also consider loot box selling when the catalog of items is increased dynamically over time. For such models, the cost of introducing new items may be non-trivial and increase with the catalog size as the complexity required to maintain game balance grows. The sequence in which the new items are added to the catalog may also have a non-trivial impact on the revenue garnered. Finally, in connection with the ongoing debate in the media and governments, it would be interesting to consider loot box pricing and design problems under various legal or fairness considerations.
Chapter 2: Matchmaking Strategies for Maximizing Player Engagement in Video Games

2.1 Introduction

As of 2020, the global revenue of the gaming industry surged about 20% compared to 2019, and was estimated at $180 billion dollars [48]. One large sector in the video game industry is online competitive games, where players play against one another in matches (one-on-one or in teams). In 2020, online competitive games PUBG: Mobile and Honors of Kings earned $2.6 and $2.5 billion dollars revenue respectively, which were the most among all mobile games [49]. Both games (and many others) adopt the idea of games-as-a-service (GaaS) and provide the game content on a “continuing revenue model”. These games are usually free-to-play, and the revenue comes from in-game advertisements, microtransactions for virtual items, and subscriptions for seasonal premium passes (sometimes referred to as “battle passes”, which offers premium content for subscribers). Managing player engagement is crucial for such free-to-play games, because the total revenue is directly tied to the amount of active players. In the gaming industry, daily active users (DAU) or concurrent users (CCU) are used as key performance indicators, and a high DAU/CCU indicates good product performance in general.

Player engagement can be affected by many factors, such as game content, game mechanics, and user interfaces. Apart from these features, which are determined before the game’s release, an important factor that influences engagement in competitive video games is how the players are matched together. The matchmaking system determines with whom a player plays against in each match, which directly affects the outcome of the current match and indirectly influences the future engagement behavior of the players. [43] and [44] both documented that the outcomes of the matches, either receiving wins/losses or scores, have significant impact on engagement.
Industry practitioners have also recognized the potential of matchmaking as a tool to improve engagement.\textsuperscript{1} The status quo is called \textit{skill-based matchmaking} (SBMM), which simply matches by trying to find an opponent with the closest skill level \textsuperscript{50} and ignores the outcomes of previous matches. Since data is easily available, video game companies have begun exploration on how to leverage matchmaking system to improve player engagement (thereby increasing revenue). For example, Electronics Arts have filed a patent for an engagement-oriented matchmaking system, which largely follows \textsuperscript{43}.\textsuperscript{2} Although they recognize the dynamic nature of the problem, their patent is based on a static one-shot model. \textit{In this paper, we study the fundamental problem of dynamic matchmaking in video games and explore the value of optimal matchmaking on long-term player engagement.}\textsuperscript{3}

In addition to selecting matching partners, video game companies also influence the matchmaking system through other ways. One controversial way is to add AI-powered bots into the matchmaking pool. Different from traditional bots for tutorials or training, AI-powered bots imitate real human players in competitive games and players are not notified when they are matched with a bot. Many competitive games are suspected to have a high bot ratio \textsuperscript{51}, although only a few games such as \textit{Fortnite} publicly confirmed the existence of bots in human matches.\textsuperscript{3} Such practices are actually double-edged swords. On one hand, the outcome of a match with bots can be easily controlled, because bots can either make intentional mistakes at clutch times, or outperform human players significantly as needed (especially in games that require quick reactions). Thus, AI-powered bots may influence player engagement positively by manipulating the match outcomes properly. On the other hand, bots can be clumsy or predictable at times and often recognized by human players after repetitive encounters. Players are not happy when they intend to compete with other human players but find out that they are frequently in a match with bots. Players who have made in-game purchases and want to use (and parade) them on other players instead

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\textsuperscript{1}Josh Menke, a former lead engagement designer for the popular game \textit{Halo 5}, said that “Matchmaking guarantees gameplay experience as intended, and prevents disengagement when possible”. (https://www.gdcvault.com/play/1026588/Matchmaking-for-Engagement-Lessons-from)

\textsuperscript{2}https://www.pcgamesn.com/ea-matchmaking-microtransactions-eomm-engagement-patent

of bots can be particularly upset about this.

Another controversial practice is to introduce a pay-to-win system (PTW). For competitive games, PTW allows players to improve their competence by paying real currency. For example, the popular competitive game Dota 2 offers a subscription service called “Dota Plus”. At the start of each match, ten players pick their characters one by one, and then paying players receive analytical suggestions on which character to pick based on the existing picks of others and winrate data from recent weeks. Moreover, during the match, paying players receive data-driven suggestions on items to buy and skills to learn, as well as information that is not available to regular players. Such information may not be necessary for professional players, but it provides substantial help to most amateur players. While PTW provides direct revenue, it is not hard to imagine that PTW may also influence player engagement of both payers and non-payers, which influences the overall revenue associated with engagement. The interplay between PTW and the matchmaking system also matters, because the existence of PTW may change the optimal matchmaking policy. The benefits of PTW have not been previously studied when taking player engagement and matchmaking policies into account. In practice, the value of either AI-powered bots or PTW is hard to evaluate through field experiments since the impact is confounded with the matchmaking policies. We provide a theoretical analysis to analyze the value of such strategies in the context of matchmaking.

In this work, we propose a novel infinite-horizon dynamic program where the goal is to maximize the cumulative active players, which is our metric for engagement. In each period, the game decides who to match with one another. We assume that players can have heterogeneous skill levels, and their state depends on the win-loss outcomes of the most recent past matches. To derive sharp characterizations and managerial insights, we focus on a stylized baseline model with two skill levels and players churn (quit) only when they experience a losing streak. Below we provide a summary of our contributions and findings.

1. We fully characterize the optimal dynamic policy under the baseline model. The optimal policy always matches as many low-skilled players who are not at risk of churning to high-skilled players who are one loss away from churning, in order to keep the latter in the system.
and increase the overall engagement. In some scenarios when there are too many low-skilled players, high-skilled players can also be matched to low-skilled players that are at risk of churning.

(2) We then discuss the value of optimal matchmaking over the traditional skill-based matchmaking (SBMM). We find that under the baseline model, the value of optimal matchmaking can be up to 1.5 times of SBMM, and this ratio is tight. However, when we extend the baseline model to multiple skill levels, the advantage of the optimal policy over skill-based matchmaking grows linearly with the number of skill levels.

(3) Next, we investigate the value of optimizing the matchmaking system in presence of AI-powered bots. We show that a platform using SBMM with a relatively small bot ratio can potentially not use bots at all if they transition to an optimal matchmaking policy. On the other hand, when the bot usage is high (implying the bots have evolved to a very sophisticated state), then the gap between the optimal and SBMM policies vanishes.

(4) We then consider the interplay between PTW strategies and the matchmaking system. We find that contrary to the conventional wisdom that PTW is simply to increase revenue, it can also be an effective lever to change the distribution of player states and increase engagement. Surprisingly, even the non-paying low-skilled players may be better off in some scenarios. The potential positive externality on player engagement may transform the public perception of PTW strategies.

(5) Finally, we conduct a case study with data from Lichess (a large online chess platform) to validate our findings. After fitting a player behavior model within our framework, we show that the optimal policy may improve engagement by 4-6% over SBMM. Also, the optimal policy may reduce the bot ratio by 15% while maintaining the same level of engagement as SBMM when the bot ratio is less than 30%.
2.1.1 Literature Review

Our work contributes to the emerging literature on operations management in video games. Closest to our work, [43] and [44] also investigate the problem of maximizing player engagement in video games through matchmaking. [43] proposes a model that estimates the churn risk of every pair of possible matches through logistic regression, and myopically minimize the churn risk in the next round. Their numerical study considers one-period problems, and shows 0.7% improvement over SBMM in one period. On the other hand, [44] estimates user engagement with a hidden Markov model and propose a heuristic algorithm that assigns a selected player to one of the pre-specified candidate matches (assuming all the other players are fixed). In contrast, our paper investigates how to optimally solve the dynamic matchmaking problem through a fluid model, taking both the myopic reward as well as the long-term player engagement into account. We also explicitly solve for the optimal policy in special cases, and provide insights on how the policy looks. Our new framework also enables us to analytically investigate the value of optimal dynamic matchmaking, AI-powered bots, as well as a pay-to-win system.

Aside from matchmaking, several papers in this field consider the monetization of video games. For example, [45] considers the problem of incentivizing ad-clicking actions in freemium games. [52] considers how to optimally price and design ‘loot boxes’, which is a popular randomized selling scheme for virtual items in video games. [53] considers the introduction and pricing of premium content in freemium games. [46] considers whether the seller should disclose an opponent’s skill level when selling PTW items. Our paper also investigates PTW, but is different in several ways: first, we assume that PTW is a subscription service, which is not dependent on specific matches. Second, instead of player’s utility for a single match, we focus on the player’s lifetime engagement. Finally, our model enables us to check the joint value of PTW with the optimal matchmaking policy. Apart from monetization, there are increasing interests in the design of video games. [54] recently considers an interesting problem on how to design the loadouts in the game to maximize the diversity of the strategies. [55] considers how to maximize player utility by sequencing game elements in a level.
Our work also connects to the growing literature on managing user engagement and user lifetime value dynamically in a service system. In such systems, the revenue is usually proportional to the amount of cumulative active users, and users’ subscription behavior is based on the service history up to date. [56] considers how to maximize user engagement by dynamically adjusting the service quality. [57] considers how a fund manager should switch between risk mode and safe mode to maximize customer lifetime value. [58] and [59] considers how a content provider (e.g., video streaming services) could maximize the subscription revenue by dynamically changing their content. Our work contributes to this field by considering how to manage user retention with dynamic matchmaking in video games. While the decision variable in most of the above papers is a single variable (service quality or service mode), the decision in our paper is multi-dimensional (matchmaking flows), which brings fundamentally new technical challenges and insights.

More broadly, our work relates to dynamic decision making when players have memory of the decision history. Research in this field have considered dynamic pricing with reference effects [60, 61, 62], dynamic capacity allocation with customer memory effects [63], network revenue management with repeated customer interactions [64], improving matching rates in dating markets [65], and dynamic personalized pricing with service quality variability [66]. When customer preferences are not fully specified, [67] and [68] consider how to learn customer preferences on the fly in the context of product recommendation or promotion, with the risk that customers may leave the system permanently upon consecutive bad decisions. In our work, user churn decision depends on the outcomes of the most recent previous matches.

Finally, our work contributes to the broad literature of dynamic matching. Here we only review papers where agents may only stay in the system for certain periods before leaving. [69] provides an approximation algorithm for a setting where customers will stay a fixed number of periods before leaving. [70] propose approximate algorithms for dynamic matching over edge-weighted graphs, where the arrival and abandonment of agents are stochastic. [71] considers dynamic matching over a bipartite graph, with finite types of nodes on both sides. Unmatched supply and demand may incur waiting or holding costs, and will be partially carried over to the next period. Our paper
considers dynamic matching with finite types of players in a fluid model [see, e.g., 72] and their churn risk evolves dynamically based on outcomes of past matches.

2.2 Model and Preliminaries

We now present our model of a matchmaking system for a 1-versus-1 competitive video game. In Section 2.2.1, we describe the player behavior, and then introduce the engagement maximization problem in Section 2.2.2.

2.2.1 Player Behavior

We assume that each player has a skill level that describes their relative competence in the game. There are $K$ ordered skill levels, where level 1 is the lowest level and level $K$ is the highest. For each match, exactly one of the two players will be the winner, and the outcome of a game is either a win or a loss (no draw). The outcome of a match is a Bernoulli random variable depending on the skill levels of the two players. Let $p_{kj}$ be the winrate of a level $k$ player versus a level $j$ player, implying that $p_{kj} = 1 - p_{jk}$. Players of the same skill level are equally likely to win, i.e., $p_{kk} = 0.5$. A player with a higher skill level than their opponent has strictly larger than 0.5 probability of winning, i.e., $p_{kj} > 0.5$ if $k > j$. We assume that a player’s skill level is fixed over their lifespan in the matching system. In practice, most players are casual players, and it is reasonable to assume that they cannot significantly improve their relative competence once they are familiar with the game. In [44], players’ skill level is an evolving metric that monotonically increases as the players play more. In our setting, skill levels reflect relative competence, and is more stable because others are also getting more familiar with the game.

In practice, many factors may influence the players’ engagement behavior. The outcome of past matches has been shown to have significant impact on player engagement. For example, [43] documented that players’ churn risk varies significantly with the outcomes of the last three matches. We assume that a player’s engagement state is determined by the win-loss record of the last $m$ matches, and transitions according to a Markov chain when the player plays a new match.
We use $q$ to denote the ‘churn state’, i.e., a player quitting the game permanently. Let $\mathcal{G}$ be the set of all possible states of a player, which has cardinality at most $2^m + 1$ (history of wins/losses and the churn state). We shall assume that the probability of churning after $m + 1$ consecutive losses is strictly positive. This assumption clearly holds in practice, but is essential for ensuring finiteness of our objective (see Section 2.2.3).

A player is fully characterized by their skill level $k$ and engagement state $g \in \mathcal{G}$; we shall refer to this pair as a demographic of players. Let $P^k_{\text{win}}, P^k_{\text{lose}} \in [0, 1]^{||\mathcal{G}|| \times ||\mathcal{G}||}$ be the transition matrix of a level $k$ player’s engagement state, given that they win/lose the next match. Hence, if they are matched with a level $j$ player, their aggregate transition matrix is given by $M_{kj} = p_{kj} P^k_{\text{win}} + (1 - p_{kj}) P^k_{\text{lose}}$. For the ease of notation, we also define $\tilde{\mathcal{G}}$ to be the set of all the active states except the churn state $q$ and $\tilde{M}_{kj}$ be the reduced aggregate transition matrix without the churn state. We define an active player as one who has not churned and is thus in one of the states in $\tilde{\mathcal{G}}$. Below we use a simple example to illustrate these concepts.

**Example 2.1.** Suppose that there are two skill levels of players, either high or low (denoted by level 2 and 1, respectively). They quit with probability 0.2 if they experience two consecutive losses, and with probability 0.5 if they experience three consecutive losses. This implies that $m = 2$, as only two previous matches plus the current match outcome affects the transition state. We further assume that a high-skilled player wins against a low-skill player with probability $p_{21} = 0.8$. Hence, for each skill level, there are 4 engagement states in $\mathcal{G}$: the player may experience 0, 1 or 2 consecutive losses, or reach state $q$. We use 20, 21, 22, 2q and 10, 11, 12, 1q to denote the states of high- and low-skilled players, respectively. For $k = 1, 2$, the transition matrix $P^k_{\text{win}}$ and $P^k_{\text{lose}}$ is given by

\[
P^k_{\text{win}} = \begin{pmatrix}
  k0 & k1 & k2 & kq \\
  k0 & 1 & 0 & 0 \\
  k1 & 1 & 0 & 0 \\
  k2 & 1 & 0 & 0 \\
  kq & 0 & 0 & 1
\end{pmatrix}, \quad P^k_{\text{lose}} = \begin{pmatrix}
  k0 & k1 & k2 & kq \\
  k0 & 0 & 1 & 0 \\
  k1 & 0 & 0 & 0.8 \\
  k2 & 0 & 0 & 0.5 \\
  kq & 0 & 0 & 1
\end{pmatrix}.
\]
The aggregate transition matrix is given by

\[
M_{kk} = \begin{pmatrix}
0 \& 0.5 \& 0 \& 0 \\
0.5 \& 0 \& 0.4 \& 0.1 \\
0.5 \& 0 \& 0.25 \& 0.25 \\
0 \& 0 \& 0 \& 1
\end{pmatrix}, \quad M_{21} = \begin{pmatrix}
0 \& 0.8 \& 0 \& 0.16 \\
0.8 \& 0 \& 0.2 \& 0.2 \\
0 \& 0 \& 0 \& 1
\end{pmatrix}, \quad M_{12} = \begin{pmatrix}
0 \& 0.2 \& 0 \& 0.64 \\
0.2 \& 0 \& 0.4 \& 0.4 \\
0 \& 0 \& 0 \& 1
\end{pmatrix}
\]

and the reduced transition matrix \( \bar{M}_{kj} \) is defined as

\[
\bar{M}_{kk} = \begin{pmatrix}
0 \& 0.5 \& 0 \\
0.5 \& 0 \& 0.4 \\
0.5 \& 0 \& 0.25
\end{pmatrix}, \quad \bar{M}_{21} = \begin{pmatrix}
0 \& 0.8 \& 0 \\
0.8 \& 0 \& 0.16 \\
0 \& 0 \& 0
\end{pmatrix}, \quad \bar{M}_{12} = \begin{pmatrix}
0 \& 0.2 \& 0 \\
0.2 \& 0 \& 0.64 \\
0 \& 0 \& 0
\end{pmatrix}
\]

2.2.2 Firm’s Dynamic Optimization Problem

Now we describe the problem from the perspective of the firm/matchmaker. In practice, players request matches randomly, and the matchmaker reviews the matchmaking pool periodically and formulates matches based on specific constraints. For simplicity, we assume that for each time period, all the active players (i.e., players with engagement state that is not the churn state) request a match, and each player is assigned to an opponent by the matchmaker. The outcomes of all matches are then revealed at the same time, and players update their engagement states upon completion of their matches. Then the next time period starts and all players that have not churned request a match and so on. In Appendix B.2 we show that partial participation and random time length of a match can be naturally incorporated into our framework.

In practice, a popular competitive game usually has millions of concurrent online players.
Motivated by this, we follow the literature of fluid matching (see, e.g., [72]) and assume that players are infinitely divisible. Let \( s^t_{kg} \) be the amount of players at time \( t \) in the demographic with skill level \( k \) and engagement state \( g \). The population of level \( k \) active players in period \( t \) is given by the vector \( s^t_k \in \mathbb{R}^{\lvert \tilde{G} \rvert} \), and we use \( s^t = [s^t_1, \ldots, s^t_K] \) to denote the system state. Let \( f^t_{kg,g} \geq 0 \) be the amount of level \( k \) players in state \( g \) that are matched to a level \( j \) opponent in state \( g' \) in time \( t \). Note that it is possible to use flow variables that only consider one of \( f^t_{kg,g} \) and \( f^t_{g',kg} \). However, the current formulation allows easier presentation of the evolution of demographics’ sizes over time.

A feasible match given \( s^t \) is a set of matching flows \( f^t_{kg,jg} \) that satisfies:

\[
\begin{align*}
\sum_{j=1}^{K} \sum_{g' \in \tilde{G}} f^t_{kg,jg'} &= s^t_{kg}, \quad k = 1, \ldots, K, \forall g \in \tilde{G}, \\
\sum_{j=1}^{K} \sum_{g' \in \tilde{G}} f^t_{jg',kg} &= s^t_{kg}, \quad k = 1, \ldots, K, \forall g \in \tilde{G},
\end{align*}
\]

(FB)

Namely, (FB) are flow balance constraints that makes sure every active player is matched. The first equation makes sure that every level \( k \) player in state \( g \) is matched with some opponents, and the second equation ensures that the total amount of matches against level \( k \) players in state \( g \) equals to the amount of such players. The third equation makes sure that for every pair of demographics, a match results in an equal effect on supply and demand.

Next, we depict the evolution of the system. Using \( f^t_{kj} = \{f^t_{kg,jg} \} \in \mathbb{R}^{G \times \tilde{G}}_{\geq 0} \) to denote the flow matrix between level \( k \) and \( j \), the evolution of demographics is given by

\[
s^{t+1}_k = \sum_{j=1}^{K} \left( f^t_{kj} \mathbf{1} \right)^\top \bar{M}_{kj} \quad k = 1, \ldots, K,
\]

(ED)

where \( \mathbf{1} \) is a \( \lvert \tilde{G} \rvert \times 1 \) unit vector. Note that in (ED), \( f^t_{kj} \mathbf{1} \) is the vector describing how many
level $k$ players are matched to level $j$ players for all states in $\bar{G}$, and recall that $\bar{M}_{kj}$ is the state transition matrix for level $k$ players matched to level $j$ players. The engagement at period $t$ is given by $\sum_{k=1}^{K} \sum_{g \in \bar{G}} s^t_{kg}$, the total amount of active players. The firm’s objective is to maximize engagement, which we measure by the cumulative amount of active players across all periods, $\sum_{t=1}^{\infty} \gamma^{t-1} \sum_{k=1}^{K} \sum_{g \in \bar{G}} s^t_{kg}$, where $\gamma \in (0,1]$ is the discount factor. The engagement maximization problem can be formulated as a Markov decision process, where the states are $s^t$, the amount of active players in each demographic. Let $V^\pi$ be the value function of a feasible policy $\pi$. The value-to-go function is given by

$$V^\pi(s^t) = \sum_{k=1}^{K} \sum_{g \in \bar{G}} s^t_{kg} + \gamma V^\pi(s^{t+1})$$

subject to (FB), (ED).

Our goal is to find the optimal policy with a value function $V^*(\cdot)$, such that $V^*(s^t) \geq V^\pi(s^t)$ for any feasible policy $\pi$. Note that the system dynamics are all linear, so when the initial size of each demographic is given, maximizing the engagement is equivalent to the following infinite linear program:
\[ V^*(s^0) = \max \sum_{t=1}^{\infty} \gamma^{t-1} \sum_k \sum_{g \in \bar{G}} s_{kg}^t \] 

\[ \text{s.t.} \sum_K \sum_{j=1}^K f_{kg,jg}^t = s_{kg}^t, \forall k, \forall g \in \bar{G}, t = 0, 1, \ldots \]

\[ \sum_K \sum_{j=1}^K f_{jg',kg}^t = s_{kg}^t, \forall k, \forall g \in \bar{G}, t = 0, 1, \ldots \]

\[ f_{kg,jg}^t = f_{jg',kg}^t, j = 1, \ldots, K, k = 1, \ldots, K, \forall g \in \bar{G}, g' \in \bar{G}, t = 0, 1, \ldots \]

\[ f_{kg,jg}^t \geq 0, j = 1, \ldots, K, k = 1, \ldots, K, \forall g \in \bar{G}, g' \in \bar{G}, t = 0, 1, \ldots \]

\[ s_{k}^{t+1} = \sum_j (f_{kj}^t) \top \bar{M}_{kj}, \forall k, t = 0, 1, \ldots \]

The LP formulation makes it flexible for industry practitioners to add various practical considerations into this framework. We discuss assumptions that can be relaxed in Appendix B.2.

### 2.2.3 SBMM and Preliminary Results

As mentioned in Section 3.1, the status quo in industry is SBMM, the skill-based matchmaking policy. Hence, it is important to characterize its performance, and we will use SBMM as a benchmark in our following analysis. In our setting, SBMM refers to any policy that let players in the same skill level match with each other. Note that SBMM is always feasible by letting players in the same state match with each other, i.e., \( f_{kg,k}^t = s_{kg}^t \). Let \( V_{SBMM}^*(\cdot) \) be the value function of the SBMM. We first characterize the value of SBMM in Proposition 2.1. Because a time period in our model is short and the life cycle of a game is at most a few months or years, it is reasonable to set the discount factor to \( \gamma = 1 \) (as in [44]). Interestingly, we show that when \( \gamma = 1 \) the value of SBMM is still finite due to our mild assumption that players have a positive probability of churning when they are dealt \( m + 1 \) consecutive losses.

**Proposition 2.1** (Value of Skill-based Policy). For any \( \gamma \in (0, 1] \) and any initial state of demo-
graphics $s^0$, the value function of SBMM is

$$V^{SBMM}(s^0) = \sum_{k=1}^{K} v_k s_k^0,$$

where $v_k$ is given by

$$v_k = \gamma^{-1} \left( (I - \gamma \tilde{M}_{kk})^{-1} - I \right) 1.$$

All proofs are provided in Appendix B.1. Under SBMM, a level $k$ player starts from an active state $g$ and transitions to the next engagement state according to the Markov chain $M_{kk}$. Note that each unit of players generates one unit of engagement in each period. Thus, the total engagement a unit of player can generate, which we refer to as their shadow price, is then the average number of periods they stay active and is described by $v_k$. Note that the shadow price is finite even when $\gamma = 1$. The skill-based policy provides an important benchmark, and is trivially the only (and optimal) policy when there is only one skill level.

With Proposition 2.1, we are able to show that any policy may only induce finite engagement, even when $\gamma = 1$.

**Lemma 2.1** (Finiteness of the Value Function). For any $\gamma \in (0, 1]$, policy $\pi$, and initial state of demographics $s^0$, $V^\pi(s^0)$ is always finite.

The proof of Lemma 2.1 is based on the fact that the value of SBMM is always finite, and any policy may only be finitely better than SBMM. Finally, because (2.2) is an infinite LP, strong duality is not always guaranteed, although we prove it does hold in our setting.

**Lemma 2.2** (Strong Duality). For any $\gamma \in (0, 1]$, policy $\pi$, and initial state of demographics $s^0$, strong duality and complementary slackness hold for (2.2).

Although it is challenging and redundant to write out a dual problem in closed-form for the model presented in this section due to the generality of player behavior, we provide the dual problem of our stylized model, presented in Section 2.3, in Appendix B.1.2.
2.3 A Stylized Baseline Model

In this section, we consider a more stylized model with two types of players, which allows us to characterize the essence of the optimal matching policy explicitly and derive important insights. We first present the model of consideration and the matchmaking problem. Then we present the optimal policy and discuss its insights.

Consider the following model with only two skill levels, either high or low. Players’ churn behavior depends only on the outcomes of the last match and the current match (i.e., $m = 1$), and will churn if and only if they experience two consecutive losses (one from the previous match and one from the current match). Further, we assume a high-skilled player beats a low-skilled player with probability one. While the model is highly stylized, it does capture the essence and fundamental tradeoff of the engagement maximization problem. The assumptions of two skill levels as well as the 100% high-versus-low winrate are simplifications of the heterogeneous skill levels. The dependency on the most recent two matches captures the players’ bounded memory, which is common in literature to deliver tractable results (e.g., [61, 73]). In the context of pricing, empirical evidence also suggests that customers may only remember the current and most recent prices (see, e.g., [74] and the numerical part of [61]). It is also consistent with industry practitioners’ opinions that players have very limited memory capacity [75, 76]. The churn behavior assumption based on losing streaks is a simplification of our assumption that players have positive probability to churn after $m+1$ consecutive losses, and implies that people favor a winning outcome over a losing outcome in competitive games. This assumption follows the cognitive evolution theory in psychology that the intrinsic motivation to an activity (by the underlying need for competence and self-determination) would increase (decrease) when perceiving oneself as competent (incompetent) [77]. Winning and losing are natural signals of competence and incompetence, and the relative benefit of winning over losing on the intrinsic motivation (measured by subsequent time spent on the game) has been widely supported in the literature [78, 79]. Despite the outcome itself, in practice, game designers also offer various extrinsic incentives that are contingent with the match outcomes, such as badges
and in-game currencies, which further reinforce players’ desires to win [80]. In Section 2.5, we numerically show that looking at losing streaks over just a couple matches is sufficient for estimating churn risk and our theoretical insights hold in the case study.

Under this setting, we can classify players into 4 demographics: high (level 2) and low-skilled (level 1) players who won the past game, denoted by 2\(w\) and 1\(w\), respectively; high and low-skilled players who lost the past game, denoted by class 2\(\ell\) and 1\(\ell\), respectively. The aggregate transition matrix is given by

\[
\begin{bmatrix}
    0.5 & 0.5 \\
    0.5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
    1 & 0 \\
    k\ell & 0
\end{bmatrix}, \quad
\begin{bmatrix}
    0 & 1 \\
    0 & 0
\end{bmatrix}.
\]

For convenience, we define \(P := \{1\ell, 2\ell, 1\ell, 2\ell\}\). Let \(s' = (s'_{1w}, s'_{1\ell}, s'_{2w}, s'_{2\ell})\) denote the size of the four demographics in period \(t = 0, 1, 2, \ldots\). Furthermore, denote \(f'_{i,j}\), as the amount of players in demographic \(i\) that are assigned to match with players in demographic \(j\) in period \(t\), where \(i, j \in P\). The flow balance constraints (FB) now become

\[
\begin{align*}
\sum_{j \in P} f'_{i,j} &= s'_i, \quad \forall i \in P, \\
\sum_{i \in P} f'_{i,j} &= s'_j, \quad \forall j \in P, \\
\end{align*}
\]

\[(FBS)\]

\[
\begin{align*}
f'_{i,j} &= f'_{j,i}, \quad \forall i \neq j, i, j \in P \\
f'_{i,j} &\geq 0, \quad \forall i \neq j, i, j \in P.
\end{align*}
\]
In any period \( t = 1, 2, \ldots \), the evolution of demographics (ED) now becomes

\[
\begin{align*}
\hat{s}^{t+1}_{2w} &= \frac{1}{2} \left( f_{2w,2w}^t + f_{2w,2\ell}^t + f_{2\ell,2w}^t + f_{2\ell,2\ell}^t + f_{2w,1w}^t + f_{2w,1\ell}^t + f_{2\ell,1w}^t + f_{2\ell,1\ell}^t \right), \\
\hat{s}^{t+1}_{2\ell} &= \frac{1}{2} \left( f_{2w,2w}^t + f_{2w,2\ell}^t \right), \\
\hat{s}^{t+1}_{1w} &= \frac{1}{2} \left( f_{1w,1w}^t + f_{1w,1\ell}^t + f_{1\ell,1w}^t + f_{1\ell,1\ell}^t \right), \\
\hat{s}^{t+1}_{1\ell} &= \frac{1}{2} \left( f_{1w,1w}^t + f_{1w,1\ell}^t \right). \\
\end{align*}
\]

(EDS)

The evolution in (EDS) reflects that high-skilled players always beat low-skilled players and players from the same skill level have equal chances to win a match. Furthermore, since churn is guaranteed after two losses, only players in demographic \( 2\ell \) and \( 1\ell \) may potentially leave the system in the next period.

Consistent with our original model in Section 2.2, the matchmaker is interested in maximizing players’ engagement by designing matching flows satisfying both (FB) and (ED) constraints. We shall focus on the case where there is no discount factor, i.e., \( \gamma = 1 \). Thus, given the initial demographics \( s^0 = (s^0_{1w}, s^0_{1\ell}, s^0_{2w}, s^0_{2\ell}) \), we define the matchmaker’s problem as

\[
V^*(s^0) = \max_{\{f_{i,j}^t\}_{i,j=0}^\infty, \{s^t_i\}_{t=1}^\infty} \sum_{t=1}^\infty \sum_{i \in P} s^t_i
\]

s.t. (FB) and (EDS) \( \forall t = 0, 1, 2, \ldots \), and \( i, j \in P \),

which is an infinite dimensional linear program. Although the objective is an infinite sum, we note that Lemma 2.1 guarantees the finiteness of (P).

Although we have already stripped the matching model to its simplest form, it still possesses intricate dynamics. While traditional literature on Markov decision processes usually considers steady-state solutions, our problem (P) cannot have a steady-state solution since the total engagement is finite. We thus consider a generalization of steady-state referred to as decaying steady-state. A policy admits a decaying steady state \( s^t \), if there exists some \( c \in (0, 1] \) such that under the given policy, we have \( s^{t+1} = cs^t \). Unfortunately, Lemma 2.3 shows that no matching policy,
besides SBMM, results in its demographics reaching a decaying steady state.

**Lemma 2.3** (No Steady State Exists). Consider a fixed time period $t$.

(i) SBMM can induce a non-zero decaying steady-state, but only for $c = (1 + \sqrt{5})/4$ and $s_1^t$ a positive multiple of the vector $((1 + \sqrt{5})/2, 1, (1 + \sqrt{5})/2, 1)$.

(ii) For any matching policy that involves matching flows between high-skilled and low-skilled players, there is no non-zero decaying steady-state for any $c \in (0, 1]$.

As we show next, SBMM is actually sub-optimal as long as there are positive amounts of players in both levels. On the other hand, searching and conducting analysis for steady states is doomed to fail. Hence, even under this simplest setting, finding the optimal matching policy is highly non-trivial and needs to take the dynamics of player demographics across time into consideration.

Before presenting the optimal matching policy that maximizes the matchmaker’s problem in (2.2), we first consider a one-period matching which provides insights on the structure of the optimal matching flows. Suppose the matchmaker only needs to design the optimal matching flows for the initial period. That is, we can modify the matchmaker’s problem in (2.2) to

$$
\max \sum_{i,j} s_{ij}^1, \quad \text{s.t. (FB$_S$) and (ED$_S$) when } t = 0. \quad (P_1)
$$

In other words, by designing the matching flows in the initial period $t = 0$, the matchmaker wants to retain as many players in the next period $t = 1$ as possible. Lemma 2.4 summarizes the optimal matching flows in the one-period matching problem $(P_1)$.

**Lemma 2.4** (Myopic Policy). Consider the one-period matching problem in $(P_1)$ with demographics state $s^t = (s_{1w}^t, s_{1x}^t, s_{2w}^t, s_{2x}^t)$. The optimal matching policy maximizes flow between demographics $2\ell$ and $1w$. The rest of the players can be matched arbitrarily as long as $2w$ and $1\ell$ are not matched to each other.

Lemma 2.4 states that the myopic decision that maximizes player engagement is to maximize the matching flow between demographics $2\ell$ and $1w$. In other words, the matchmaker saves as
many high-skilled players who are about to leave the system, as possible, using low-skilled players, who are not at risk of churning. Other players can simply be matched with same-level opponents or any other ways as long as $2w$ and $1\ell$ are not matched. The intuition behind the optimal matching policy in this one-period setting is fairly intuitive. It preserves as many high-skilled players as possible without sacrificing low-skilled players at all.

The above intuition may not hold true anymore if we consider the infinite-horizon matchmaking problem, because of the trade-off between short-term and long-term reward. Compared to SBMM, the myopic policy increases the number of players in demographic $1\ell$, leading to higher churn rate in the subsequent period, and thus decreases the possible $2\ell - 1w$ matching in the future. It turns out, however, the optimal matchmaking policy in the infinite-horizon problem ($P$) resembles the one in Lemma 2.4. In the optimal matchmaking policy, the matchmaker still maximizes the flow between $2\ell$ and $1w$ demographics in order to maximize the overall player engagement, despite the forward-looking nature of the problem. However, the matchmaker also needs to balance the populations in each demographic at the same time, which is not a concern in the one-period matching problem ($P_1$). We present the formal result in the next proposition.

**Proposition 2.2** (Optimal Matchmaking Policy). The optimal matching policy is summarized in Table 2.1 with respect to different demographics. The optimal policy always maximizes matching flows between demographics $2\ell$ and $1w$. The rest of the players are matched via SBMM in most scenarios (third row in Table 2.1), except when there are less high-skilled players compared to low-skilled players and too many $1\ell$ players (first and second rows in Table 2.1). In this case, the matchmaker also matches players in demographic $2\ell$ with players in $1\ell$.

Table 2.1 summarizes the detailed matching flows by separating the four-dimensional state space into three regions. The first two regions represent special cases when there are more high-skilled players than low-skilled players in the system. The last region reflects that a myopic policy similar to the one in Lemma 2.4 can be extended to a dynamic setting and remains optimal. In our proof, we split the four-dimensional state space into 7 scenarios (presented in Appendix B.1.2), and shows the transition between scenarios under the proposed policy. Then we rely on strong
duality and complementary slackness in Lemma 2.2 to prove optimality.

According to Proposition 2.2, the optimal matching policy always maximizes the flows between demographic $2\ell$ and $1w$. Thus, just like the one-period example, matching players from different skill levels is beneficial. On one hand, the platform should always “save” as many high-skilled players ($2\ell$, who are about to leave the platform) as possible by matching them with low-skilled players ($1w$, without losing streaks). When matched with low-skilled players, high-skilled players ($2\ell$) can enjoy a free win to break the losing streak and participate in future matchings. On the other hand, since low-skilled players can potentially prevent high-skilled players from leaving when they are matched, low-skilled players are very valuable to the matchmaker so they should not be completely exhausted. Note that low-skilled players used to “save” high-skilled players are those from $1w$, so they may potentially recover from the losing streak in the next period. In most cases, the matchmaker does not match remaining players in $2\ell$ to players in $1\ell$ since players from the latter leave the matching process afterwards, resulting in no immediate improvements and losing the potential for them to come back to demographic $1w$. One exception is when there are not enough low-skilled players in $1w$ but too many players in $1\ell$ compared to the size of $2\ell$ (the first two rows in Table 2.1). Under these scenarios, after exhausting players in $1w$, the matchmaker also matches low-skilled players in $1\ell$ to high-skilled players in $2\ell$. Here, the matchmaker is essentially adjusting the overall player distribution among the demographics. Although they have the potential to increase other players’ engagement, low-skilled players are only valuable if there are enough

<table>
<thead>
<tr>
<th>Table 2.1: Optimal Matchmaking Policy, $K_1 := \frac{18}{5}s_{2w}^l + \frac{9}{5}s_{2\ell}^l + \frac{3}{5}s_{1w}^l$, $K_2 := \frac{18}{5}s_{2w}^l + \frac{23}{5}s_{2\ell}^l - \frac{11}{5}s_{1w}^l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>States of demographics</td>
</tr>
<tr>
<td>$s_{2w}^l + s_{2\ell}^l &lt; s_{1w}^l + s_{1\ell}^l, s_{2w}^l \geq s_{1w}^l, s_{2w}^l &lt; s_{1\ell}^l$, and $s_{1\ell}^l &gt; K_2$</td>
</tr>
<tr>
<td>$s_{2w}^l + s_{2\ell}^l &lt; s_{1w}^l + s_{1\ell}^l, s_{2w}^l \geq s_{1w}^l, s_{2w}^l &lt; s_{1\ell}^l$, and $K_1 &lt; s_{1\ell}^l \leq K_2$</td>
</tr>
<tr>
<td>Otherwise</td>
</tr>
</tbody>
</table>
high-skilled players that need to be saved. The adjustment preserves high-skilled players in the process who may need to be saved by low-skilled players in the future.

Another way to gain insights on the effect of player distribution on the optimal policy is to look at the shadow price of each demographic, which can be loosely interpreted as how valuable a demographic is to the matchmaker. In general, the shadow prices for players who are not on losing streaks are higher than those that just lost a game. On the other hand, the shadow prices with respect to the skills \((2w \text{ versus } 1w)\) depend on the relative population of high and low players. Interestingly, when there are many more high-skilled players compared to low-skilled players, the shadow prices of low-skilled players are always bounded. However, as the ratio between the number of high- and low-skilled players goes to 0, the shadow prices for high-skilled players increases since high-skilled players can survive “forever” if injected to the system. This also explains why the matchmaker is willing to sacrifice players in \(1\ell\) to save high-skilled players in this case. We leave the detailed descriptions on shadow prices in the proof of Proposition 2.2 in Appendix B.1.2.

Finally, we analyze the performance of the optimal matchmaking policy with the SBMM benchmark in Proposition 2.3 below.

**Proposition 2.3** (Engagement Improvement). (a) In a single period, the myopic policy from Lemma 2.4 garners at most \(4/3\) engagement as SBMM. (b) In the infinite-horizon setting, we have that the optimal policy from Proposition 2.2 garners at most \(3/2\) engagement as SBMM. In other words,

\[
\frac{V^*(s')}{V_{SBMM}(s')} \leq \frac{3}{2},
\]

for any \(s' \geq 0\). Furthermore, the upper bound is achieved for some \(s'_{2\ell} = s'_{1w} > 0\) and \(s'_{2w} = s'_{1\ell} = 0\).

Proposition 2.3 first states that the value of the myopic policy over SBMM is at most \(4/3\) in a single period. While [43] conjectured that the power of myopic matchmaking grows exponentially as the time horizon increases, we resolve this in the negative: over the whole time horizon, the
benefit of optimal policy over SBMM is at most 50%. As discussed in Section 2.3, the optimal matching policy utilizes matches between $2\ell$ and $1w$ players to improve engagement. Thus, as we can see from the second statement in Proposition 2.3, the upper bound of the performance ratio is attained when the initial demographics only have equal amount of $2\ell$ and $1w$ players. In this scenario, the matchmaker improves the most compared to SBMM by using low-skilled players who are not in danger of churning to save high-skilled players who are about to churn.

2.4 Multiple Skill Levels, AI-powered Bots, and Pay-to-win System

In this section, we use the baseline model introduced in Section 2.3 to discuss several extensions of our model. In particular, we compare the performance of the optimal policy to SBMM in Section 2.4.1 when there are multiple skill levels. Moreover, we discuss the insights of having AI-powered bots in Section 2.4.2 and implementing pay-to-win strategies in Section 2.4.3.

2.4.1 Multiple Skill Levels

Based on the discussion in Section 2.3, we know that low-skilled players are valuable for maximizing total player engagement due to the fact that they can be used to extend the lifespan of high-skilled players. Thus, it is natural to conjecture that the power of optimal matchmaking shall increases further if there are more than two skill levels in the matchmaking system. That is, when a relatively high-skilled player is about to leave, the matchmaker can match them with a relatively low-skilled player to prevent them from churning. The other player may then be saved by a player from an even lower skill level in the future, which improves the overall engagement among players. To formalize this intuition, we extend the baseline model to multiple skill levels.

Consider the baseline model introduced in Section 2.3, but with $K > 2$ skill levels. For $1 \leq k \leq K$, let $s_{k\ell}^t$ and $s_{kw}^t$ denote the amount of level $k$ players at time $t$ who just won and lost their last match, respectively. We assume that higher-skilled players always defeat lower-skilled players, i.e., for any $1 \leq j < k \leq K$ we have that $p_{jk} = 0$.

With $K > 2$ skill levels and for all $t = 1, 2, \ldots$, the state of demographics evolves according to
the following dynamics:

\[ s_{k\ell}^{t+1} = \frac{1}{2} (f_{k\ell,k\ell}^t + f_{k\ell,k\ell}^t) + \sum_{j>k} (f_{k\ell,j\ell}^t + f_{k\ell,j\ell}^t), \]  

(ED\textsubscript{K})

\[ s_{k\ell}^{t+1} = \frac{1}{2} (f_{k\ell,k\ell}^t + f_{k\ell,k\ell}^t) + \sum_{j<k} (f_{k\ell,j\ell}^t + f_{k\ell,j\ell}^t). \]

The evolution of demographics in (ED\textsubscript{K}) reflects that a player in skill level \( k \) always win a game when matched with another player who has lower skill level, i.e., \( j < k \), and only players who just lost a game are subjected to churn. Similarly, the flow balancing conditions in (FB\textsubscript{S}) needs to be modified to the following:

\[ \sum_j f_{i,j}^t = s_i^t, \forall i \in \tilde{P}, \]  

(FB\textsubscript{K})

\[ \sum_i f_{i,j}^t = s_j^t, \forall j \in \tilde{P}, \]

\[ f_{i,j}^t = f_{j,i}^t, \forall i \neq j, i, j \in \tilde{P}, \]

\[ f_{i,j}^t \geq 0, \forall i, j \in \tilde{P}, \]

where \( \tilde{P} := \{1w, 1\ell, \ldots, Kw, K\ell\} \). Therefore, with the initial demographic \( s^0 = (s_{1w}^0, s_{1\ell}^0, \ldots, s_{Kw}^0, s_{K\ell}^0) \), we define the matchmaker’s problem as

\[ \max_{\{f_{i,j}^t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{i \in \tilde{P}} s_i^t \]  

(P\textsubscript{K})

s.t. (FB\textsubscript{K}) and (ED\textsubscript{K}) \( \forall t = 0, 1, \ldots, \) and \( i, j \in \tilde{P} \),

which reduces to (P) when \( K = 2 \). Although the optimal policy can still be evaluated by solving the linear program (P\textsubscript{K}), it becomes very difficult to characterize in closed-form as a result of having multiple skill levels. Unlike the optimal policy characterized in Proposition 2.2 for the baseline model with two skill levels, the optimal policy in this case may induce matching between non-adjacent skill levels. However, we show that in this case, the matchmaker can improve the
engagement beyond 50 percent when compared to SBMM, and in fact the potential improvement is linear in $K$. The next proposition provides the upper and lower bounds on the ratio between the value functions under optimal and skill-based policies.

**Proposition 2.4** (Engagement Improvements with Multiple Classes). Denote $R_K := \max_{s'} \frac{V^*_K(s')}{V^{SBMM}(s')}$, representing the largest ratio between the value function of the optimal policy and the skill-based policy for any starting state of demographics. For $K > 2$, we have

$$\frac{3K^2 + 7K + 6}{8K + 8} \leq R_K \leq \frac{4K + 1}{5}, \quad (2.4)$$

that is, $R_K = \Theta(K)$.

To get the lower bound, we first show that in some special cases that the optimal policy is a myopic policy, and we provide the closed-form expression of the optimal value function (Lemma B.2 in Section B.1.3). We then build the lower bound by computing the ratio when the initial state falls into one of the special cases. To get the upper bound, we utilize the dual problem and the corresponding shadow prices.

The implication of Proposition 2.4 is significant. Note that the upper and lower bounds grow linearly in $K$. Thus, while the potential value of optimal matchmaking over SBMM cannot grow exponentially over the time horizon, it does grow linearly with the number of skill levels. In practice, it is reasonable to have several skill levels in a game, and thus the value of optimal matchmaking can be significant. In the extreme case, when $K$ goes to infinity, SBMM can be arbitrarily bad.

To further illustrate the intuition behind Proposition 2.4, we start with the baseline model. When there are two skill levels ($K$ and $K - 1$), note that the level $K - 1$ players can be no worse off than self-matching, because no other players can be used to prevent them from churning. Thus, their population decays exponentially fast over time. By introducing level $K - 2$ players, who have a lower skill compared to those in level $K - 1$, these players can be used to save level $K - 1$ players who would have left previously. Then the level $K - 1$ players, saved by level $K - 2$ players,
can be used to save level $K$ players. As we keep introducing new levels, players from the newly introduced lowest level can be used to save those from the second lowest level, who can further be used to save players from the third lowest level, and so on. Hence, when introducing a new skill level, it improves the engagement for all existing levels, and the marginal benefit is increasing with $K$.

2.4.2 AI-powered bots

AI-powered bots are routinely developed by game designers as an attempt to closely mimic human player behavior. Ideally, a matchmaker can use bots whenever a human player is about to churn if they experience one more loss. The bot can be designed in such a way that it is competitive but still loses to the human player, which may result in the human breaking their losing streak and remaining in the system longer. On the other hand, due to the limitations of technology, AI-powered bots can be identified by experienced players. If a human player is frequently matched with bots, they may find out that their opponents are not human and perhaps be discouraged from playing the game. In this section, we provide a framework to analyze the value of adding a percentage of AI-powered bots to the matching pool.

In this section, we focus on the impact on players’ engagement when a certain percentage of the demographics are bots. We do not consider when to match players with bots, as this may significantly alter the behavior of the system and the attitude towards bots. To be more specific, we consider a scenario where each active player can be matched to a bot with an independent and fixed probability $\alpha \in [0, 1)$ in each period. This setup can be interpreted as bots maintaining a constant percentage in the overall demographics over time. In practice, the value of $\alpha$ depends on the players’ ability to detect bots and tolerance towards bots. The more developed the AI technology is, i.e., bots are less detectable, the higher $\alpha$ can be. Consider the baseline model introduced in Section 2.3. In each period, with probability $\alpha$, a player is matched with a bot who is designed to lose with probability 1 and give the player a win. With probability $1 - \alpha$, the player is matched with another human player just like in Section 2.3. Thus, adding bots only slightly changes (FB$_S$) and
We leave the detailed formulation to the Appendix (Eq. (P_{Bot})) to avoid repetition. With slight abuse of notation, denote $V^{SBMM}(s, \alpha)$ and $V^*(s, \alpha)$ as the value functions of SBMM and the optimal matchmaking policy when the probability of matching with bots is $\alpha$, respectively.

The total engagement can be increased by improving the matchmaking policy from SBMM to optimal, but can also be improved by adding bots. To measure the value of an optimal matchmaking policy, we consider the increase in bot ratio needed under SBMM to achieve the same level of engagement as the optimal policy. It is obvious that the optimal policy should use no more bots than SBMM, and we quantify this reduction in the bot ratio. When there is only one skill level, then the optimal policy has to use the same bot ratio as SBMM as policies are the same. Thus, we are interested in the maximum power of the optimal policy. We summarize the maximum power of the optimal policy in the presence of bots in Proposition 2.5 below.

**Proposition 2.5.** Let $\alpha$ be the fractions of bots in SBMM.

(a) For any $\alpha \leq 16.9\%$, there exists some state $s$ of demographics such that $V^*(s, 0) = V^{SBMM}(s, \alpha)$, i.e., a state where the optimal policy without bots is as good as SBMM with $\alpha$ fraction of bots. For $\alpha > 16.9\%$, no such state exists.

(b) Let $a(\alpha)$ be the bot ratio such that $V^*(s, a(\alpha)) = V^{SBMM}(s, \alpha)$. For any state $s$ of demographics and bot ratio $\alpha$ we have that $\lim_{\alpha \to 1} a(\alpha) = 1$, i.e., the optimal policy requires a bot ratio that also approaches 1 in order to achieve the same engagement.

Proposition 2.5 delivers two important insights regarding matchmaking systems with bots. As of today, the moderate sophistication of AI-powered bots requires that $\alpha$ be relatively low so that players are not too frustrated. In this case, Proposition 2.5(a) shows that using the optimal policy with no bots can be just as good as SBMM with bots. In our baseline model, the optimal policy without any bots can offset a bot ratio of up to 16.9% under SBMM. Thus, for companies that are criticized for using high bot ratios [51] and that use SBMM, optimizing the matchmaking system may significantly alleviate such problems without loss of engagement. In Section 2.5, our case study based on real data also shows that the optimal policy has the power to reduce the bot ratio.
significantly when the value of $\alpha$ for SBMM is moderate. On the other hand, Proposition 2.5(b) points out that the value of using the optimal policy is negligible as $\alpha$ goes to 1. Intuitively, when $\alpha = 1$, everyone only plays with bots, and all policies are functionally the same. Thus, regardless of the matchmaking policy used, developing more sophisticated AI bots can still provide value for companies.

2.4.3 Pay-to-win system

Building upon the baseline model in Section 2.3, suppose the matchmaker can offer low-skilled players a chance to purchase items or information to gain an advantage in the next match. To be specific, we follow the practice in Dota 2 (introduced in Section 1) and consider a subscription service that provides additional information to players. Such features are largely useless for high-skilled players, but may help low-skilled players substantially. Therefore, in the model of consideration, we only consider a pay-to-win (PTW) feature for low-skilled players in every period.\(^4\) We assume that the subscription fee is $r$ per period, and low-skilled players either opt-in at time zero and keep their subscription for all matches until they churn, or never pay for it. This assumption is a reasonable reflection of reality since players that are willing to pay for the subscription feature are likely those who accept the idea behind microtransactions in video games. The majority of players opposing the idea would rarely switch to a subscription in the middle of their lifespan in the system. We use $\beta \in [0, 1)$ to denote the proportion of the low-skilled players who pay for the subscription. In practice, only about 5% players pay in a freemium game, so $\beta$ should be small [81]. We leave $r$ and $\beta$ exogenous and focus on the interplay between PTW and the matchmaker.

With a pay-to-win system such as the subscription feature mentioned above, monetary elements have been added to the matching system. We assume that there is a conversion rate between players' engagement and the seller's revenue. For simplicity, we normalize a unit of player engagement translates to one unit of revenue generated for the seller.

\(^4\)In practice, such membership is bundled with other perks, e.g., decorative staffs, so high-skilled players may also pay for it. However, our qualitative insights are consistent as long as high-skilled player do not improve their skill level any more.
Adopting the notation from Section 2.3, suppose the matchmaker faces an initial state of demographics $s^0 = \{s_{1w}^0, s_{1l}^0, s_{2w}^0, s_{2l}^0\}$. By having the subscription, a $\beta$ portion of low-skilled (level 1) players elevate their gaming skills to the high level (level 2), denoted by $\bar{2}$, so that they behave exactly like high-skilled players in Section 2.3. However, purchasing the subscription feature does not (and should not) reset a player’s losing streak. Thus, instead of having 4 demographics, with the addition of the subscription feature, there are 6 demographics in each period, denoted by $s_{2w}^t, s_{2l}^t, s_{1w}^t, s_{1l}^t, s_{1w}^0, s_{1l}^0$. Note that $s_{2w}^t, s_{2l}^t$ represent players with subscriptions who just won and lost a game, respectively. To differentiate from the original demographics, we use $s_{2w}^t, s_{2l}^t$ to denote non-paying high-skilled players, and $s_{1w}^t, s_{1l}^t$ to denote non-paying low-skilled players. In the initial period $t = 0$, given $\beta$ and $s^0$, we have $s_{2w}^0 = \beta s_{1w}^0, s_{2l}^0 = \beta s_{1l}^0, s_{2w}^t = s_{2w}^0, s_{2l}^t = s_{2l}^0, s_{1w}^0 = (1 - \beta)s_{1w}^0, s_{1l}^0 = (1 - \beta)s_{1l}^0$. Consequently, we also need to introduce new flows between demographics accordingly. In any period $t$, the flow balance constraints are now

\[
\sum_j f_{i,j}^t = s_i^t, \forall i \in \bar{P},
\]

\[
\sum_i f_{i,j}^t = s_j^t, \forall j \in \bar{P}, \quad \text{(FB}_{ptw})
\]

\[
f_{i,j}^t = f_{j,i}^t, \forall i \neq j, i, j \in \bar{P},
\]

\[
f_{i,j}^t \geq 0, \forall i \neq j, i, j \in \bar{P},
\]
where $\bar{\mathcal{P}} := \{2w, 2\ell, 2\ell, 2w, 1w, 1\ell\}$, and evolution of demographics are now

\[
\begin{align*}
    s_{2w}^{t+1} &= \frac{1}{2} (f_{2w,2w}^{t} + f_{2w,2\ell}^{t} + f_{2\ell,2w}^{t} + f_{2\ell,2\ell}^{t} + f_{2w,2w}^{t} + f_{2w,2\ell}^{t} + f_{2\ell,2w}^{t} + f_{2\ell,2\ell}^{t}) + f_{2w,1w}^{t} + f_{2w,1\ell}^{t} + f_{2\ell,1w}^{t} + f_{2\ell,1\ell}^{t}, \\
    s_{2\ell}^{t+1} &= \frac{1}{2} (f_{2w,2w}^{t} + f_{2w,2\ell}^{t} + f_{2\ell,2w}^{t} + f_{2\ell,2\ell}^{t}), \\
    s_{2w}^{t+1} &= \frac{1}{2} (f_{1w,2w}^{t} + f_{1w,2\ell}^{t} + f_{1\ell,2w}^{t} + f_{1\ell,2\ell}^{t} + f_{1w,2w}^{t} + f_{1w,2\ell}^{t} + f_{1\ell,2w}^{t} + f_{1\ell,2\ell}^{t} + f_{1w,1w}^{t} + f_{1w,1\ell}^{t} + f_{1\ell,1w}^{t} + f_{1\ell,1\ell}^{t}), \\
    s_{1w}^{t+1} &= \frac{1}{2} (f_{1w,1w}^{t} + f_{1w,1\ell}^{t} + f_{1\ell,1w}^{t} + f_{1\ell,1\ell}^{t}), \\
    s_{1\ell}^{t+1} &= \frac{1}{2} (f_{1w,1w}^{t} + f_{1w,1\ell}^{t} + f_{1\ell,1w}^{t} + f_{1\ell,1\ell}^{t}).
\end{align*}
\]

\[\text{(ED}_{ptw}\text{)}\]

From (FB$\text{ptw}$) and (ED$\text{ptw}$) conditions, we can see that players in the new demographics with skill level $\bar{\ell}$ behave the same as those with skill level $\ell$. However, the matchmaker’s objective is largely different from the one in (P). The matchmaker also needs to consider the revenue generated by paid subscriptions. Given $\beta$, $r$, and the initial demographics $s^0 = \{s_{2w}^0, s_{2\ell}^0, s_{1w}^0, s_{1\ell}^0\}$, the matchmaker’s problem is

\[
V^*(\beta, r, s^0) := \max_{(f_{i,j}^{t})_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{i=1}^{\ell} s_{i}^{t} + \sum_{t=1}^{\infty} r(s_{2w}^{t} + s_{2\ell}^{t}) = \max_{(f_{i,j}^{t})_{t=1}^{\infty}} ENG(\beta, r, s^0) + REV(\beta, r, s^0)
\]

\[\text{(P}_{ptw}\text{)}\]

s.t.

\[
\begin{align*}
    s_{2w}^0 &= \beta s_{1w}^0, \\
    s_{2\ell}^0 &= \beta s_{1\ell}^0, \\
    s_{2w}^0 &= s_{2w}^0, \\
    s_{2\ell}^0 &= s_{2\ell}^0, \\
    s_{1w}^0 &= (1 - \beta)s_{1w}^0, \\
    s_{1\ell}^0 &= (1 - \beta)s_{1\ell}^0,
\end{align*}
\]

\[\text{(FB}_{ptw}\text{)} \text{ and } \text{(ED}_{ptw}\text{)} \forall t = 0, 1, 2, \ldots, \text{ and } i, j \in \bar{\mathcal{P}},
\]

66
where the objective function is to maximize the sum of player engagement $ENG(\beta, r, s^0)$ and revenue from paid subscription $REV(\beta, r, s^0)$. Our focus is on the interplay between PTW and the matchmaker, as well as their aggregate value. We provide our main findings in Proposition 2.6 below.

**Proposition 2.6.** Consider $\beta \in [0, 1)$.

(a) With PTW under SBMM, one unit of subscribed players is as valuable as $1 + r$ units of unsubscribed players. With PTW under the optimal policy, one unit of subscribed players is more valuable than $1 + r$ units of unsubscribed players.

(b) Subscribed players have a higher priority compared to unsubscribed high-skilled players. That is, unsubscribed high-skilled players in $2w(2\ell)$ would only be matched with any unsubscribed low-skilled players after all the subscribed players in $2w(2\ell)$ have matched with unsubscribed low-skilled players.

(c) When there are more high-skilled players than low-skilled players, implementing PTW decreases engagement. However, in terms of the overall revenue (engagement plus subscription revenue) generated, implementing PTW is beneficial if and only if $r$ is greater than a specific threshold.

(d) There exists states of player demographics such that implementing PTW increases engagement and subscription revenue simultaneously.

Proposition 2.6(a) says that under SBMM, the value of a subscribed player in a PTW system equals $r + 1$ unsubscribed high-skilled players. However, under the optimal policy the value of a subscribed player in a PTW system goes beyond replacing them with $r + 1$ unsubscribed players. On the other hand, Proposition 2.6(b) depicts the impact of PTW on the matchmaking policy. We prove that the optimal policy gives subscribed players high priority, and would save an unsubscribed high-skilled player from churning only when all the paid players within the same engagement state are saved. Proposition 2.6(c) and (d) describes the influence of PTW on user
engagement and the total value function. When there are more high-skilled players than the low-skilled players, Proposition 2.6(c) claims that the corresponding engagement inevitably falls, and it is only worth to introduce PTW when the unit profit $r$ is high enough. This is because in this case, low-skilled players are already scarce resources, and a PTW system further increases their scarcity. On the other hand, Proposition 2.6(d) claims that PTW can increase engagement while providing direct revenue. This happens when there are too few high-skilled players. Intuitively, this is because now the high-skilled players are scarce resources, and a few more high-skilled players can facilitate cross-level matching significantly and thus improve the engagement. Together, Proposition 2.6(c) and (d) provide important managerial insights on the value of a PTW system, when taking matchmaking and player engagement into account: besides its direct revenue, it also works as a lever to change the skill distribution and may influence the total engagement positively. This happens when the majority of players are low-skilled, which is reasonable for most games. On the other hand, if the majority of players are high-skilled, PTW will hurt engagement and the seller should be careful to introduce it.

Intuitively, all the non-paying players are worse off in the presence of a PTW system, because high-skilled non-paying players are in low priority for cross-level matching, and the low-skilled non-paying players need to save more high-skilled players now. However, in Example 2.2 we show that it is possible that low-skilled non-paying players may also be better off due to the redistribution effect. The intuition behind such a surprising observation is that, the optimal matchmaking strategy would utilize low-skilled non-paying players in a more sustainable way as they become more scarce in the presence of PTW. Such an observation again emphasizes the potential positive externality of PTW on user engagement, in addition to the potential increase in revenue.

**Example 2.2 (PTW May Make Low-skilled Non-paying Players Better Off).** Suppose $\beta = 0.2$, $r = 0$, and the initial state is $s = (2, 2, 1, 15)$. Without PTW system, some of the players in $1\ell$ have to be matched with $2\ell$ players in the first period and leave directly, leading to an average engagement of 2.27 for all the low players over the lifespan. When the PTW system is considered, we have $s^0_{2w} = 0.2$, $s^0_{2\ell} = 3$, $s^0_{1w} = 2$, $s^0_{1\ell} = 2$, $s^0_{1w} = 0.8$, $s^0_{1\ell} = 12$. Because $r = 0$, such a problem
is equivalent to a problem with initial state \( s' = (2.2, 5, 0.8, 12) \) but without PTW system. In this case, the \( V \) players become more scarce and would not be matched to high-skilled players in the first period, and the average engagement for the low players increases to 2.31.

2.5 Case Study of Lichess

In this section, we conduct a numerical case study with the Lichess Open Database [82] to demonstrate the power of optimal matchmaking policies in a realistic setting.

2.5.1 Data Description

Lichess is a free and open-source Internet chess server, and all the match data since 2013 are available to the public. For each match, the data includes the game mode, starting time, players’ IDs, the outcome of the match, and the ratings of the players before and after the match. Each player has a rating for each game mode. The platform adopts the Glicko-2 rating, which is a generalization of Elo rating [83]. We collected all the matches in the most popular mode “Rated Blitz Game” during 2013-2014, which includes 5.41 million matches and 135,073 unique players. To simplify our problem, we remove all the matches where the outcomes are draws, which represent 3.7% of the matches, and focus on the 5.23 million matches with unique winners. We say a player is churned after a match if s/he stops playing for 14 days after the match. Hence, a player may churn more than once.

We focus on players who played for at least 5 matches for three reasons. First, players who churned in less than 5 matches may largely churn due to other factors (e.g., user interface). Second, the ratings of new players change wildly after the first few matches, which makes it difficult to estimate their skill levels. Finally, we would like to test models with a range of game memory \( m \) (up to 4), which means that the players need to play at least \( m + 1 \) matches. In our dataset, there are 60,334 qualified players. For each player, we take at most 500 matches on their record (92% of the 60,334 players play less than 500 matches). This is done to avoid underestimating the churn probability, since there are a few highly dedicated players who never quit under any circumstance.
Because the ratings oscillate at all time, we use the average of the ratings after the last 3 matches to estimate rating, which is then used to determine their skill level. A summary of the players’ engagement and ratings is in Table 2.2. Based on the estimated ratings, we divide players into 13 skill levels: for the 96% of the considered players whose rating is between 1000 and 2100, we separate them into 11 levels (level 2 to 12) based on intervals of 100; for players with rating less than 1000 and greater than 2100, we group them into two new levels, level 1 and level 13, respectively. To compute the winning probability between different skill levels, we use the realized winrate when the level difference is less than or equal to 8. When the level difference is greater than 8, we have too little data for an accurate estimation, and simply assume that the player with higher skill level wins with probability 1. We show the detailed winrate in Table B.5 in Appendix.

Table 2.2: Summary of Considered Players (N = 60334)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Matches</td>
<td>170.6</td>
<td>21</td>
<td>5</td>
<td>18764</td>
</tr>
<tr>
<td>Estimated Ratings</td>
<td>1525</td>
<td>1518</td>
<td>764</td>
<td>2663</td>
</tr>
</tbody>
</table>

2.5.2 Players’ Churn Behavior Estimation

Recall that in Section 2.2, we define a Markovian engagement model where the state $g$ is uniquely determined by the win-loss record of the last $m$ matches. We now illustrate how to estimate $P_{win}^k$ and $P_{lose}^k$ through maximum log-likelihood estimation. To simplify our problem and reduce the number of parameters, we assume that the players’ churn behavior is independent of skill level, and drop the superscript $k$. In the state transition matrix, the row that represents state $g$ in $P_{win}$ ($P_{lose}$) only has two positive entries: the probability of churning after winning (losing) the next match and the probability of moving to a certain non-churn state. Our goal is then to estimate $\rho_{win}^g$ ($\rho_{lose}^g$), the churn probability of a player in state $g \in G$ after winning (losing) the next match (Note that $\rho_{win}^q = \rho_{lose}^q = 1$ since we assume that a churned player will stay churned.) Let $g_i^t$ be the state of player $i$ reaches before playing $t$-th match, $\Psi_i^t$ be the indicator of whether the player wins the $t$-th match, $\Upsilon_i^t$ be the indicator of whether the player churns right after reaching
For a player who played $T^i$ total matches, we record his/her states and churn decisions starting from the 5th match because they only have a valid state before the 5th match if $m = 4$, and every focal player played at least 5 matches. If the state sequence is given by $(g_5^i, g_6^i, \ldots, g_{T^i}^i)$, the outcome sequence is $(\omega_5^i, \omega_6^i, \ldots, \omega_{T^i}^i)$, and churn decision sequence is $(\nu_5^i, \nu_6^i, \ldots, \nu_{T^i}^i)$, then the log-likelihood (LLH) function is given by

$$
\sum_{i=1}^{N} \sum_{t=5}^{T^i} \log \left( \rho_{g_t^i}^{\text{win}} \omega_t^i + (1 - \rho_{g_t^i}^{\text{win}}) \omega_t^i (1 - \nu_t^i) + \rho_{g_t^i}^{\text{lose}} (1 - \omega_t^i) \nu_t^i (1 - \nu_t^i) (1 - \nu_t^i) \right).
$$

Because the state transition is exogenous, the estimation is computationally simple and usually has a closed form. Below we propose two reasonable models that use $m + 2$ parameters, and shows the closed form of the parameters.

**Losing Streak Model:** The losing streak model assumes that the players’ churn probability depends on the length of losing streak they experience (after the next match). For a given $m$, players have $m + 1$ possible engagement states $0, \ldots, m$, which represents the length of losing streak before the next match. In this model, $\rho_{g}^{\text{win}}$ is the same for all $g = 0, \ldots, m$, because all such players have a 0 match losing streak upon wining the next game. On the other hand, $\rho_{g}^{\text{lose}}$ represents the churn probability upon losing the next game given that the players has already lost $g$ consecutive games (thus they experience a $(g + 1)$-match losing streak). Our baseline model in Section 2.3 can be viewed as a special case of the losing streak model, with $m = 1, \rho_{0}^{\text{win}} = 0, \rho_{0}^{\text{lose}} = 0$, and $\rho_{1}^{\text{lose}} = 1$.

For the losing streak model, the solution for the MLE is given by

$$
\rho_{g}^{\text{win}} = \frac{\text{Number of churn decisions after a win}}{\text{Number of wins}}, \quad g = 0, \ldots, m
$$

$$
\rho_{g}^{\text{lose}} = \frac{\text{Number of churn decisions after } g + 1 \text{ consecutive losses}}{\text{Number of times that players experience } g + 1 \text{ consecutive losses}}, \quad g = 0, \ldots, m.
$$

**Winrate Model:** The winrate model assumes that the players’ churn probability depends on the number of wins over the last $m$ matches plus the next match (total $m+1$ matches). Hence, the model can be described by $m + 2$ parameters, the churn probabilities when the player has 0 to
\( m + 1 \) wins. Note that although we only need to estimate \( m + 2 \) parameters, we still need \( 2^m + 1 \) states. This is because for the LP formulation, the customer transition matrix needs the full win-loss record. Hence, a state \( g \) is a length-\( m \) binary sequence denoting the win-loss record and we use \( |g| \) to denote the wins in \( g \). The solution for the MLE is

\[
\rho_g^{\text{win}} = \frac{\text{Number of churn decision when players won } |g| + 1 \text{ matches over last } m + 1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over last } m + 1 \text{ matches}}
\]

\[
\rho_g^{\text{lose}} = \frac{\text{Number of churn decision when players won } |g| \text{ matches over the } m + 1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over the } m + 1 \text{ matches}}
\]

**Estimation Results**

We use the players who played at least 5 matches to estimate the models. We randomly sample 70% of the players to train the model, and use the rest of them to validate the model. We test both the losing streak model and the winrate model, ranging \( m \) from 0 to 4. When \( m = 0 \), the churn probability only depends on the outcome of the current game, and the losing streak model coincides with the winrate model. In addition, we also estimate a null model as a benchmark, which assumes a uniform churn probability regardless of the states.

Table 2.3: Out-of-sample Negative Log-likelihood for the Candidate Models

<table>
<thead>
<tr>
<th>( m = 0 )</th>
<th>140019.4</th>
<th>140019.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>139641.5</td>
<td>139713.1</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>139488.8</td>
<td>139566.7</td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>139362.9</td>
<td>139375.9</td>
</tr>
<tr>
<td>( m = 4 )</td>
<td>139281.6</td>
<td>139255.2</td>
</tr>
</tbody>
</table>

In Table 2.3, we summarize the out-of-sample negative LLH of the candidate models. Notably, compared with the null model, even when \( m \) equals to 0 or 1, both the losing streak model and the winrate model improve the LLH by more than 4000, while increasing \( m \) from 1 to 4 only improve the LLH further by less than 500. Thus, assuming \( m = 1 \) does not result in much loss in the
goodness-of-fit of the models. When \( m = 1 \), the parameter of the losing streak model is given by \( \rho_{\text{win}}^{\text{win}} = 1.32\% \), \( \rho_{0}^{\text{lose}} = 1.62\% \), and \( \rho_{1}^{\text{lose}} = 2.53\% \). The surge in churn probability from one loss to two consecutive losses reflects the players’ negative sentiment towards a losing streak, and our stylized model in Section 2.3 reasonably captures the essence of this behavioral pattern.

2.5.3 Power of Optimal Matchmaking

In this section, we test the power of optimal matchmaking, and validate the insights in Section 2.4. We consider a scenario where in each period, all the active players join the matchmaking pool and the outcome is revealed immediately. We adopt the losing streak model since it behaves almost the same as the winrate model in terms of predictive power, and is computationally tractable for optimization. We use the realized skill levels (13 total) and the realized engagement states after the fifth match as the initial input.

We compare three policies: SBMM, the optimal policy, and the random policy where a player has a uniform chance to be matched with any other active player. To compute the total engagement under the optimal policy, we simply solve the LP formulation Eq. (2.2) with a large enough \( T \). In our case, we set \( T = 1000 \). The discount factor is 1. We use Gurobi to solve the formulation. In Table 2.4, we show the power of various policies compared to SBMM. SBMM is better than the random policy, which shows that the status quo of SBMM is better than doing nothing. Notably, the optimal policy may improve the total engagement by 4.22-6.07%, depending on the choice of \( m \).

![Table 2.4: Relative Power of Candidate Policies Compared to SBMM](image)

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal</td>
<td>4.22%</td>
<td>5.89%</td>
<td>6.07%</td>
<td>5.82%</td>
</tr>
<tr>
<td>Random</td>
<td>-2.08%</td>
<td>-3.80%</td>
<td>-5.05%</td>
<td>-5.81%</td>
</tr>
</tbody>
</table>

Next, we check the power of optimal matchmaking with a varying number of skill levels. Recall that in Proposition 2.4, the maximum power of optimal matchmaking over SBMM increases linearly with the number of skill levels. To verify this insight in a realistic setting, we now compute...
the relative power of optimal matchmaking, when the top or bottom \( K \) skill levels are considered for \( K = 2, \ldots, 13 \). In Fig. 2.1, we show a representative example when \( m = 1 \). It turns out that the power of the optimal policy monotonically increases with \( K \) regardless of how we add the levels (bottom-up or top-down), which is consistent with the insight we derive on the maximum power.

Finally, we investigate scenarios with AI-powered bots. We assume that every player has an \( \alpha \) chance to be matched with a bot, and compare the performance of the optimal policy and SBMM. In Fig. 2.2, we show how the total engagement changes with \( \alpha \) when \( m = 3 \). Notably, when \( \alpha \) is reasonably small (\( \leq 30\% \)), the optimal policy can achieve the same engagement level with SBMM but with \(~ 15\%\) less bot matches. Thus, optimizing the matchmaking system can significantly reduce the bots needed and avoid the perils of a high bot ratio. On the other hand, as \( \alpha \) keep increasing, the difference between SBMM and the optimal policy decreases. This is because when a high bot ratio can be tolerated, most players play with a bot and transition to an ideal state, making all policies similar.

2.6 Conclusions

In this paper, we present a modeling framework for matchmaking in competitive video games. Through sharp characterizations of a baseline model and a numerical case study based on real data,
we provide several novel insights to industry practitioners. The current standard is SBMM, which solely matches players with the same skill levels. Although SBMM is intuitive and easy to implement, it does not maximize the long-term engagement. For the engagement-optimal matchmaking rule, we show that in the special case, the optimal policy myopically maximize the short-term reward in the next period, but also adjust player distribution for the long-term reward. We also highlight the significant improvement in player engagement that can be achieved using an optimal matchmaking policy over SBMM. Surprisingly, the benefit increases linearly with the number of skill levels.

In addition, we provide new perspectives on controversial topics such as having AI bots and pay-to-win systems in competitive video games. Our results show that using an optimal matchmaking policy instead of SBMM may reduce the required bot ratio significantly while maintaining the same level of engagement. By investigating the interplay between pay-to-win strategies and the optimal matchmaking policy, we provide a novel viewpoint of PTW as a lever to control the distribution over the demographics of players. Importantly, we show that PTW may not necessarily hurt engagement. When most players are low-skilled, PTW can actually increase the overall engagement, and even make the low-skilled non-paying players better off. This potential positive externality on user engagement is in contrast with the negative public image of PTW strategies. Finally, using real data from an online chess platform, we show that players do indeed churn based on recent match outcomes and the optimal policy can have significant improvement over SBMM.
Chapter 3: Price Discrimination with Fairness Constraints

3.1 Introduction

The increased availability of consumer data in conjunction with the widespread use of e-commerce has led to a proliferation in discriminatory and personalized pricing strategies both in practice [84, 85] and academia [86, 87]. Specifically, companies often try to engage in first or third-degree price discrimination tactics by leveraging the available data on their consumers such as past purchase behavior, browsing history, and personal attributes to predict consumer valuations. While the practice is generally widespread, discriminatory pricing can result in disparate impact against protected groups. Protected groups may have higher (or lower) valuations for a product due to historical disadvantages or unobserved factors, and such differences can result in higher prices (or more limited access) for protected groups. [88] show that Black individuals receive higher interest rates for auto loans, while [89] show that women receive higher interest rates for small business loans even after controlling for other consumer features. [90] and [91] show that Black/Latinx and women borrowers pay higher interest rates for mortgage loans while controlling for all possible factors including risk. In fact, the study in [90] even show that this discrimination exists for FinTech lenders that make decisions based on AI algorithms. [92] show that a test-preparation provider charged Asian-Americans higher prices, even when controlling for income. In all these examples, the number of transactions is at a very large scale. Thus, it suggests that the seller’s motivation to carry forward such statistical discrimination is driven by financial gain. In contrast, the U.S. Civil Rights Act of 1964 and the Equal Credit Opportunity Act of 1974 protects against discrimination based on protected attributes such as race, color, religion, sex, national origin, and most recently, sexual orientation and gender identity.\footnote{Bostock v. Clayton County, decided on June 15, 2020} In fact, in 2020 the state of New
York banned gender-based price discrimination\(^2\) to fight against the increasing trend of large retailers selling the same product at different prices by simply changing the packaging or the product color.\(^3\) Ensuring fairness is a direct concern of the Federal Trade Commission (FTC):

“When we at the FTC evaluate an algorithm or other AI tool for illegal discrimination, we look at the inputs to the model – such as whether the model includes ethnically-based factors, or proxies for such factors, such as census tract. But, regardless of the inputs, we review the outcomes. For example, does a model, in fact, discriminate on a prohibited basis? Does a facially neutral model have an illegal disparate impact on protected classes? Our economic analysis looks at outcomes, such as the price consumers pay for credit, to determine whether a model appears to have a disparate impact on people in a protected class.”\(^4\)

Recently, there has been a surge of interest in understanding how to make discriminatory pricing practices that are fair from business \([93, 94]\), legal \([95, 96]\), and regulatory perspectives \([97, 98]\). Moral, legal, and ethical obligations are prompting sellers and regulators to ensure that pricing practices do not unfairly discriminate against protected attributes. Although this general principle is universally accepted, no formal framework prior to this work exists to properly implement or assess the impact of such fairness measures in the context of pricing decisions. In fact, in a recent discussion paper by the UK’s Financial Conduct Authority \([98]\), it clearly states the need for conducting research on fairness in pricing:

“[...] it is important that we consider the fairness of pricing in markets we regulate. It is also important to consider the harm that may be caused by particular types of pricing practice. [...] However, fairness issues can often be more complicated and the right course of action for us may be less clear.”

In light of the growing interest on fairness in the context of pricing decisions, we consider the following research questions.

1. How can we model the fairness of decisions made in the context of price discrimination? Is it possible to impose several types of fairness simultaneously?

2. What is the impact of fairness constraints on the seller, customers, and society at large?

\(^3\)https://www.huffpost.com/entry/pink-tax-examples_l_5d24da77e4b0583e482850f0
In this paper, we propose a formal framework for pricing with fairness, including several definitions of fairness and their potential impact on consumers, sellers, and society. In a first step towards the ambitious agenda of designing pricing strategies that are fair, we consider the simplest scenario of a single-product seller facing consumers who can be partitioned into two groups based on a single, binary feature observable to the seller (we then consider the extension with more than two groups in Section 3.4). For each group, we assume that the seller knows the valuation distribution, which allows us to isolate the effect of fair decision making from the machine learning task of predicting valuations. The seller’s goal is to maximize profit by optimally selecting a price for each group, potentially subject to a fairness constraint which may be self-imposed or explicitly enforced by laws and regulations. We highlight that our model assumes that price discrimination occurs due to the difference in valuation distributions of the customer groups and not due to inherent racism or biases of the seller.

In this paper, we propose four definitions of fairness based on several different contexts and motivations. More details and motivation are presented in Section 3.2.1.

- **Price fairness** enforces that the prices offered to the two groups are nearly equal, and is the common focus of the studies referenced above.

- **Demand fairness** enforces that the access to the product is as close as possible across groups, meaning that the prices should be set in a way that yields a similar market share for each group. For example, a local college may want to offer tuition loans or scholarships in such a way that each group has an equal probability of enrolling.

- **Surplus fairness** requires that the surplus of the average person in each group is similar. As is standard, surplus is defined as the consumer valuation minus the price paid, and is zero if no purchase is made.

- **No-purchase valuation fairness** imposes that the average valuation of consumers who do not purchase the product is approximately the same for each group. In other words, the
normalized value lost in each group from individuals who could not afford the product should be similar.

With our model and definitions in place, we first show that satisfying all four fairness goals simultaneously is impossible unless the mean valuations are the same for both groups. In fact, even achieving two fairness measures simultaneously cannot be done in simple settings. We then consider the impact of imposing each fairness criterion separately and identify conditions under which the consumer surplus and the social welfare increase or decrease. (Clearly, imposing fairness always results in profit loss for the seller due to the additional constraint.) Note that the impact of price discrimination on social welfare has been studied in economics [99, 100], but without explicitly considering fairness constraints. For instance, we show that when the demand for each group is linear or exponential in price, a small amount of price or no-purchase valuation fairness will increase social welfare, whereas demand or surplus fairness will decrease social welfare. We also fully characterize all scenarios under linear demand and show, for example, that imposing too much price fairness may lead to a strictly lower social welfare relative to having no fairness constraints. We first focus on the setting with two groups and a single (protected) feature. We then extend our findings to settings with more than two groups and to the case where a second, unprotected consumer feature is observed. Finally, we showcase computationally the robustness of our findings for other common demand models such as exponential, logistic, and log-log.

**Summary and implications of our research.** For industry practitioners and policy makers, our paper offers the following takeaways:

(a) We show that achieving all four fairness definitions simultaneously is impossible. In fact, even achieving two of these definitions simultaneously is impossible under standard demand models. Thus, one should focus on a single notion of fairness depending on the context.

(b) Imposing fairness constraints may not necessarily increase social welfare. The welfare change depends on both the fairness definition and the level of fairness. For price fairness, a little fairness improves social welfare, but too much fairness may lead to a lower welfare rel-
ative to imposing no fairness. For demand or surplus fairness, imposing any level of fairness will decrease social welfare. Finally, no-purchase valuation fairness always increases social welfare.

3.1.1 Related Literature

The concept of fairness has been extensively studied in economics, operations management, and computer science. Broadly speaking, fairness can be modeled either (i) as a utility term that is dependent on a reference point, or (ii) as an exogenous constraint that may be imposed by a social planner based on social justice. Our work adopts the second approach, but we still review the literature related to both approaches for completeness.

In the economics literature, fairness is typically modeled as a reference effect, which depends on either a perceived value based on historical information, or unequal outcomes across groups of individuals. In such settings, fairness is motivated in light of social comparison. Fairness with respect to perceived value refers to the situation where the price of an item should be close to its “fair” value. More precisely, customers form a reference price (based on historical information), and the demand is affected when the seller sets a price that is far from the reference price. This concept was first proposed by [101], where the authors empirically show that people perceive a price raise as unfair if the surge is driven by shifts in demand. [102] then study the pricing problem under this type of fairness. Models based on a reference price were extensively studied in the context of dynamic pricing [60, 103] and for the newsvendor problem [104]. On the other hand, several papers consider fairness with respect to unequal outcomes across groups of individuals (e.g., race, age, gender). [105] and [106] are among the first to study game-theoretic models with fairness considerations. [105] models fairness as an explicit intention and shows that a fairness equilibrium may be achieved only if the Nash equilibrium also satisfies additional fairness constraints. In [106], the need for fairness is modeled as a disutility for any unequal outcome among players. [107] consider ultimatum games with peer-induced fairness concerns. Using a similar setting, [108] consider a contract design problem and find that cooperation may be achieved when
the manufacturer and retailer are sensitive to unequal outcomes. [109] study a duopoly market with behavior-based pricing and find that incorporating fairness may increase sellers’ profit and decrease consumer surplus.

The second approach in the fairness literature models fairness as exogenous constraints in decision-making or classification problems. The fairness constraints are usually motivated by egalitarianism, where each group of people (or even each individual) should receive the same treatment, or by Rawlsian justice [110], where the social planner aims to make the least advantaged people better off. Decision making under fairness constraints have been seen in the context of stable matching [111], transportation systems [112], network design [113], advertising [114], and dynamic learning [115]. [116] investigate conditions under which uniform government subsidies are optimal. Another stream of papers considers the trade-off between fairness and efficiency in resource allocation [117, 118, 119, 120]. In our paper, we do not consider resource constraints and we investigate to what extent our fairness constraints can improve social welfare.

Research on fairness has also been increasing rapidly in the machine learning community, and fairness is also modeled as exogenous constraints. Earlier papers consider classification algorithms under various fairness constraints [121, 122, 123], or the tradeoff between different fairness metrics [124, 125]. [126] provide a framework for assessing fairness without observing the protected attribute. There has also been work on how to design fair policies using causal inference [127, 128, 129]. In fact, in the context of classification problems, several papers have tried to integrate social welfare into the loss function [130, 131, 132, 133]. While the fairness definitions in our paper resemble those in the machine learning literature, we consider the problem from a different perspective. The pricing procedure usually includes two steps: valuation prediction and pricing decisions. Machine learning models mainly focus on the first step, and the idea is to distribute the prediction error in a fair manner so that the prediction is unbiased. Our paper assumes that the seller has unbiased and accurate information on customer segmentation and valuations, and focuses on how to make fair pricing decisions given such information.

Finally, one can view uniform pricing as a revenue management problem with a (simple) fair-
ness constraint. In this view, our paper contributes to the line of research that compares social welfare under a uniform pricing strategy (i.e., perfect price fairness) versus discriminatory pricing (i.e., no fairness) [see, e.g., 99, 134, 100]. Our paper includes these two extreme cases, but also considers intermediate levels of fairness constraints as well as four different fairness definitions. This literature shows that allowing for price discrimination generally leads to a higher social welfare compared to uniform pricing (ultimately converging to the situation where the seller is able to extract the entire consumer surplus). Surprisingly, our results show that restricting price discrimination by imposing fairness constraints can sometimes increase social welfare. In fact, we identify cases where imposing intermediate levels of fairness results in a social welfare which is higher than both perfect fairness and no fairness scenarios.

3.2 Framework and Preliminary Results

We consider a single-period setting where a seller offers a single product, with marginal cost \( c \geq 0 \), to two groups of customers (we consider the extension with more than two groups in Section 3.4). The seller needs to select a price for each group with the goal of maximizing profit. Specifically, customers are categorized based on an observable binary feature \( X \in \{0, 1\} \), so that each group \( i = 0, 1 \) can be offered a different price \( p_i \). In this context, the seller may want to constrain the pricing policy to ensure fairness across the two groups, due to either a need to improve customer perception or abide by government regulations. For example, \( X \) can correspond to gender, race, operating system, age, or type of device. We let \( d_i \) denote the population size of each group \( i \). We assume that customers from group \( i \) have valuations for the product denoted by the random variable \( V_i \sim F_i(\cdot) \), where \( F_i(\cdot) \) is a given cdf. A customer in group \( i \) buys the product only if their valuation is at least the offered price \( p_i \). Thus, \( \bar{F}_i(p_i) = \mathbb{P}(V_i \geq p_i) \) represents the market share of group \( i \), and \( d_i \bar{F}_i(p_i) \) corresponds to the total demand of group \( i \). We assume that the seller has enough supply to fulfill all the demand.

The profit function for group \( i \) is then \( R_i(p_i) = (p_i - c)d_i\bar{F}_i(p_i) \). The seller’s goal is to select \( p_0 \) and \( p_1 \) to maximize \( R_0(p_0) + R_1(p_1) \), potentially subject to some fairness constraints (see
more details below). We let $p_i^* = \arg \max_p R_i(p)$ denote the optimal price offered by the seller to group $i$ under no fairness constraints, that is, the unconstrained optimal price. We capture consumer welfare by the average consumer surplus, given by $S_i(p) = \mathbb{E}[(V_i - p_i)^+]$ (note that we focus on the normalized surplus to account for possible asymmetries in population sizes). We also consider the expected no-purchase valuation, $N_i(p) = \mathbb{E}[V_i | V_i < p_i]$, that corresponds to the average valuation of non-buyers. Finally, the total welfare from group $i$, $W_i(p_i)$, can be written as the profit plus the consumer surplus, that is, $W_i(p_i) = R_i(p_i) + d_i S_i(p_i)$.

3.2.1 Fairness Definitions

In the context of pricing, we propose the four following measures of fairness, where smaller quantities imply fairer strategies:

(a) **Price fairness**, which is measured by $|p_0 - p_1|$.

(b) **Demand fairness**, which is measured by $|\bar{F}_0(p_0) - \bar{F}_1(p_1)|$.

(c) **Surplus fairness**, which is measured by $|S_0(p_0) - S_1(p_1)|$.

(d) **No-purchase valuation fairness**, which is measured by $|N_0(p_0) - N_1(p_1)|$.

We also propose a unit-less quantity, $\alpha \in [0, 1]$, to denote the fairness level. The case of $\alpha = 0$ corresponds to no fairness constraints (i.e., unconstrained discriminatory prices are used) and the case of $\alpha = 1$ corresponds to perfect fairness (i.e., the groups are treated equally with respect to the fairness measure). We emphasize that $\alpha$ is not a decision variable, but rather a parameter that is selected by the seller to meet internal goals or satisfy regulatory requirements. Formally, let $M_i(p_i)$ be the specific fairness measure of interest (price, demand, surplus, or no-purchase valuation) under price $p_i$, and let $|M_0(p_0^*) - M_1(p_1^*)|$ be the fairness gap under the optimal (unconstrained) pricing strategy. Then, a pricing strategy $p_i$ for $i = 0, 1$ is $\alpha$-fair with respect to $M_i(\cdot)$ if $|M_0(p_0) - M_1(p_1)| \leq (1 - \alpha)|M_0(p_0^*) - M_1(p_1^*)|$. Imposing a specific amount of fairness, for each measure corresponds to selecting a value for $\alpha$. Specifically, the pricing problem for the
seller becomes

\[
\mathcal{R}(\alpha) := \max_{p_0, p_1 \geq 0} R_0(p_0) + R_1(p_1)
\]

(3.1)

\[
\text{s.t. } |M_0(p_0) - M_1(p_1)| \leq (1 - \alpha)|M_0(p_0^*) - M_1(p_1^*)|,
\]

where \(\mathcal{R}(\alpha)\) denotes the optimal total profit as a function of the fairness level \(\alpha\). For convenience, we denote \(p_0(\alpha)\) and \(p_1(\alpha)\) the optimal prices obtained by solving problem (3.1) as a function of the fairness level \(\alpha\). Thus, \(\mathcal{R}(\alpha) = R_0(p_0(\alpha)) + R_1(p_1(\alpha))\). We note that \(p_i(\alpha)\) may sometimes be less than \(c\) in order to meet the fairness constraints. We define \(S(\alpha)\) as the total consumer surplus under the optimal prices with the \(\alpha\)-fairness constraint, i.e., \(S(\alpha) = d_0S_1(p_0(\alpha)) + d_1S_1(p_1(\alpha))\). Also, we let \(W(\alpha) = \mathcal{R}(\alpha) + S(\alpha)\) be the social welfare as a function of \(\alpha\).

The above fairness definitions are motivated from practical and regulatory considerations. Price fairness is directly motivated by regulations and laws that proscribe price discrimination based on specific attributes such as the U.S. Civil Rights Act and Equal Credit Opportunity Act. In fact, the U.S. Department of Housing and Urban Development makes it illegal to “impose different terms or conditions on a mortgage loan, such as different interest rates, points, or fees on the basis of race, color, national origin, religion, sex, familial status, or disability.” In October 2020, New York state banned gender-based price discrimination after observing that many products and services in brick-and-mortar locations were being sold at different sticker prices for men and women. The idea of imposing a price fairness constraint is mentioned directly by the UK Financial Conduct Authority [98] via ‘relative price caps’ that “impose limits on the differences in prices firms can charge to new and longstanding consumer groups” as an option to alleviate unfair pricing in financial services. As we mentioned in Section 3.1, several studies have found violations of price fairness, even after controlling for all relevant consumer features. In fact, [90] even show that such a price discrimination exists when decisions are made by AI algorithms, and we noted that the FTC also explicitly protects against algorithmic bias. All the aforementioned examples are occurring at a

\[5\text{https://www.hud.gov/sites/documents/FAIR_LENDING_GUIDE.PDF}\]
fairly large scale by sizeable lenders and retailers, so that these practices cannot just be explained by inherent racism or sexism. A fundamental possibility is that the groups of consumers who receive higher prices have a higher valuation on average relative to the other groups. This phenomenon can occur for several reasons. First, it is well recognized that there exist significant gender and racial differences in preferences in terms of products’ colors [135]. Similarly, [136] show that East Asian Americans are 1.5 times more likely to purchase commercial test-preparation services, which indicates that this group tends to have a higher valuation for such services. Second, this phenomenon can occur when a group of people have a higher average search cost or less bargaining intensity [91], so they are willing to accept higher prices to reduce the searching process (e.g., applying for a new loan). Third, a higher valuation can also occur when a specific group is unable or less likely to know the competitors’ prices [90], which can happen when a group is more likely to be located in a financial desert or is less likely to be eligible for a loan. Fourth, another potential reason for different valuation distributions is the difference in financial literacy across groups [95]. We note that many of these factors can potentially be connected to systemic racism and sexism, although this is beyond the scope of our paper.

Demand fairness is well motivated by applications in education and healthcare. For instance, a local college may want to charge tuition in a way such that it ensures a well-represented population of students (i.e., giving an equal opportunity to students coming from all backgrounds and income levels). In the same vein, a healthcare service provider or an insurance company may want to set prices so that every group has an equal chance of affording proper care. It is common for pharmaceutical companies to charge different prices in different countries (depending on the median income). In these types of settings, demand fairness ensures that access to essential products and needs is offered equally among all groups of customers.

Imposing surplus fairness requires the difference in normalized surplus to be small, so that individuals from different groups are similarly satisfied. Consumer surplus is perhaps the most widely-used notion in economics and operations management to measure the well-being of customers in the context of retailing [see, e.g., 137, 138, 139]. The concept of equal surplus (agents’
welfare) is one of the most fundamental principles in economics research [see, e.g., 140, 141] and has been extensively studied in resource allocation [142] and cooperative game theory [143]. Given the importance of surplus management and the popularity of equal surplus in several economics applications, it is natural to design pricing policies that ensure that the consumer surplus (which can be seen as a proxy for happiness or satisfaction) in each group is relatively similar. Our definition of surplus fairness can be seen in [144], where the author argues that “a unitary price [equal prices] affords unequal degrees of utility enhancement [unequal surplus] to buyers.”

Finally, we discuss no-purchase valuation fairness. When defining fairness measures in the context of pricing, it is important to also consider the customers who could not afford the product (because their price exceeds their valuations). Indeed, the non-buyers are directly affected by discriminatory pricing policies. For example, individuals who need a loan the most may be offered a higher interest rate from banking institutions, which further prevents these individuals from accessing the service. The non-buyers from one group may feel particularly discriminated against if their willingness-to-pay is higher than the non-buyers of the other groups, which may lead to potential complaints or lawsuits. The prices offered to each group control the number of non-buyers as well as how much the average non-buyer was willing to pay. The report by the Financial Conduct Authority [145] mentions that the significance of the harm caused by unfair pricing is not only measured by how many people are harmed but also by how much the individual level of harm is, namely: “if a small minority of consumers are affected, but we find that these consumers are a particularly vulnerable group of consumers and the level of individual harm is severe, we would likely be more concerned about the fairness of the pricing practice.” Since the utility of non-buyers in each group is zero, it is thus natural to measure the average level of individual harm among a group by looking at their valuation for the product. While demand fairness accounts for the fraction of people who cannot afford the product, no-purchase valuation fairness measures the average individual level of harm within non-buyers. No-purchase valuation fairness aims to ensure that one group of non-buyers was not more dissatisfied than the other, by measuring how much the groups were willing to spend. As we show later, no-purchase valuation fairness tends to provide
the largest increase in social welfare (see Section 3.3.4 for a detailed discussion).

We note that our fairness definitions can be also connected to Rawls’ principles of justice [146]. In particular, demand fairness can be thought of as a reflection of the equal opportunity principle, whereas price, surplus, and no-purchase valuation fairness can be seen as a reflection of the difference principle (in which any economic inequalities should benefit the least advantaged individuals).

In this paper, we characterize the pricing strategy of a profit-maximizing seller that needs to comply with such fairness constraints. We also discuss the resulting impact on consumer surplus and social welfare. Note that one can come up with alternative fairness definitions beyond the ones we proposed. However, as we show in Section 3.2.2, our four definitions do not have redundancies in the sense that it is impossible to satisfy all of them perfectly at once. In fact, satisfying any pair of fairness measures perfectly is often not possible.

3.2.2 Impossibility Results

In an ideal world, regulators would impose perfect fairness (i.e., \( \alpha = 1 \)) along all four definitions, so that customers across both groups will experience the same price, demand, surplus, and no-purchase valuation. The following theorem states that imposing 1-fairness across all four definitions simultaneously requires the necessary (and insufficient) condition that both groups have the same mean valuation. Such an assumption is very restrictive in practice, as different groups often have a different mean valuation. Impossibility results have been shown in the context of fairness for machine learning algorithms [124, 125], but under a setting related to misclassification errors rather than prescriptive pricing.

**Theorem 3.1** (Impossibility of Perfect Fairness). *If \( \mathbb{E} [V_0] \neq \mathbb{E} [V_1] \), then it is impossible to achieve 1-fairness in price, demand, surplus, and no-purchase valuation all simultaneously.*

**Proof.** Suppose for the sake of contradiction that there exists a pricing strategy that is 1-fair in price, demand, surplus, and no-purchase valuation. 1-fairness in price implies that there exists a price \( p \) such that \( p_0 = p_1 = p \). 1-fairness in demand implies that \( \mathbb{P} (V_0 \geq p) = \mathbb{P} (V_1 \geq p) \).

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Satisfying 1-fairness in surplus and no-purchase valuation implies that $E[(V_0 - p)^+] = E[(V_1 - p)^+]$ and $E[V_0 | V_0 < p] = E[V_1 | V_1 < p]$. By the law of total expectation (combined with adding and subtracting $p$ to one of the conditional expectations), $E[V_i] = E[V_i | V_i < p]P(V_i < p) + E[(V_i - p)^+] + pP(V_i \geq p)$ and thus $E[V_0] = E[V_1]$, which contradicts our assumption. □

In fact, under common demand models such as linear and exponential, even achieving 1-fairness in two metrics simultaneously is difficult. Specifically, we show that for exponential demand, any pair of 1-fairness constraints cannot coexist, unless the price is trivially set to 0. For linear demand, only 1-fairness in price and no-purchase valuation can be achieved simultaneously.

**Proposition 3.1 (Impossibility for Linear and Exponential Demand).** Assume that the demand is

(a) Exponential, that is, $V_i \sim Exp(\lambda_i)$ with $\lambda_0 \neq \lambda_1$. Then, any pair of 1-fairness in price, demand, surplus, and no-purchase valuation cannot coexist under positive prices.

(b) Linear, that is, $V_i \sim U(0, b_i)$ with $b_0 \neq b_1$. Then, only 1-fairness in price and no-purchase valuation may coexist, and any other pair of 1-fairness in price, demand, surplus, and no-purchase valuation cannot coexist under positive prices and positive demand.

Note that even when the mean valuations are equal (i.e., $E[V_0] = E[V_1]$), it is also readily possible that satisfying all the 1-fairness constraints simultaneously is impossible, unless the prices are trivially set to zero. We illustrate such a case in the following example. Specifically, in Example 3.1, we provide an example where $E[V_0] = E[V_1]$ and only 1-fairness in price and demand can be satisfied simultaneously with positive prices (any other pair of fairness constraints cannot coexist).

**Example 3.1 (Impossibility when mean valuations are equal).** Suppose that $V_0 \sim U(0, 2)$ and $V_1 \sim Exp(1)$. We find that 1-fairness in price and demand can be simultaneously satisfied when $p = 1.594$. However, for any price $p > 0$, we have $S_0(p) < S_1(p)$ and $N_0(p) > N_1(p)$, so that 1-fairness in price cannot coexist with either 1-fairness in surplus or in no-purchase valuation. Suppose that we have 1-fairness in demand, and let $q \in (0, 1)$ be the market share for each group.
(Note that $q > 0$ since group 1 follows an exponential demand, and $q < 1$ since $p > 0$.) We then have $p_0 = 2 - 2q$ and $p_1 = -\log q$. Therefore, we obtain $S_0(p_0) = q^2$ and $S_1(p_1) = q$, and thus $S_0(p_0) < S_1(p_1)$ for any $q \in (0, 1)$. Similarly, $N_0(p_0) = 1 - q$ and $N_1(p_1) = 1 + \frac{q \log q}{1 - q}$, so that $N_0(p_0) > N_1(p_1)$ for any $q \in (0, 1)$. As a result, 1-fairness in demand cannot coexist with either 1-fairness in surplus or in no-purchase valuation. Finally, under 1-fairness in surplus, we have $S_0(p_0) = (2 - p_0)^2/4 = e^{-p_1} = S_1(p_1)$ implying that $p_0 = 2 - 2e^{-p_1/2}$. Consequently, $N_0(p_0) = 1 - e^{p_1/2}$ and $N_1(p_1) = 1 - \frac{p_1 e^{-p_1}}{1 - e^{-p_1}}$. One can show that $N_0(p_0) < N_1(p_1)$ for any $p_1 > 0$, and thus 1-fairness in no-purchase valuation is not possible. Hence, only 1-fairness in price and demand can be satisfied simultaneously with positive prices in this example.

In general, the above discussion conveys that seeking fairness in multiple dimensions may not be feasible in most cases. Theorem 3.1 shows that achieving perfect fairness across all four definitions is impossible if the mean valuation of each group is different. Proposition 3.1 shows that satisfying two fairness definitions simultaneously is not possible even under simple demand models, and Example 3.1 shows the same idea can be true (except for one combination) when the mean valuations are the same. These results prompt us to focus on the case where a company or a regulator considers the impact of imposing a single fairness constraint up to a certain level of $\alpha$, which is easier to achieve. Specifically, we study the impact of fairness on the seller’s profit, consumer surplus, and social welfare.

3.2.3 Imposing a Little Fairness

In this section, we consider imposing a small amount of fairness and examine whether it increases social welfare. While it is clear that imposing fairness will decrease the seller’s profit, we are interested in the impact on social welfare. One may naturally conjecture that one of the motivations behind imposing fairness in pricing is to increase social welfare.

Recall from problem (3.1) that $R(\alpha)$ is the total seller’s profit under an $\alpha$-fairness constraint (where the measure is clear from the context). Recall also that $S(\alpha)$ is the total consumer surplus under the optimal prices with the $\alpha$-fairness constraint, i.e., $S(\alpha) = d_0S_1(p_0(\alpha)) + d_1S_1(p_1(\alpha))$, 89
and $W(\alpha) = R(\alpha) + S(\alpha)$ is the social welfare as a function of $\alpha$.\(^6\) Theorem 3.2 shows that the impact of imposing a small amount of fairness on social welfare crucially depends on the fairness definition. Mathematically, we are interested in cases where the (right) derivative of the social welfare at $\alpha = 0$, $W'(0)$, is positive.\(^7\) To gain analytical tractability, we consider two common demand models: linear and exponential.

**Theorem 3.2** (Impact of Imposing a Little Fairness on Social Welfare). Assume that the demand is either (i) linear, i.e., $V_i \sim U(0, b_i)$ with $b_0 \neq b_1$, or (ii) exponential, i.e., $V_i \sim \text{Exp}(\lambda_i)$ with $\lambda_0 \neq \lambda_1$. Then, $W'(0) > 0$ under price or no-purchase valuation fairness, whereas $W'(0) < 0$ under demand or surplus fairness.

Theorem 3.2, proved in Appendix C.1, conveys that for linear or exponential demand, imposing a small amount of fairness in price or no-purchase valuation improves social welfare, whereas imposing a small amount of fairness in demand or surplus decreases social welfare. In fact, one can identify a general necessary condition under which $W'(0) > 0$ for any demand function that leads to continuous and differentiable $R_i(\cdot)$, $S_i(\cdot)$, and $W_i(\cdot)$ at $p^*_i$ (the exact condition does not provide any further insight, and is thus omitted for conciseness; see Lemma C.1 in Appendix C.1 for more details). Our result suggests that if a seller is keen on using price discrimination tactics, then it is possible that imposing a small amount of fairness ($\alpha > 0$) can increase social welfare compared to no fairness ($\alpha = 0$). This is a surprising complement to classic economics which suggests that a seller relaxing their strategy from uniform pricing (1 group) to discriminatory pricing (2 groups with $\alpha = 0$) will increase social welfare.

To derive additional insights, we next focus on the case of uniform valuations (i.e., linear demand). We then test the robustness of our findings for three alternative demand models in Section 3.5.

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\(^6\) We highlight that the calligraphic quantities $R(\cdot)$, $S(\cdot)$, and $W(\cdot)$ denote functions of $\alpha$, whereas $R(\cdot)$, $S(\cdot)$, and $W(\cdot)$ denote functions of $p$.

\(^7\) The social welfare (right) derivative at $\alpha = 0$ is defined as $W'(0) = \lim_{\alpha \to 0^+} \frac{W(\alpha) - W(0)}{\alpha}$. 
3.3 Analysis for Linear Demand

In this section, we present a comprehensive analysis for the linear demand model. Specifically, we assume that $\bar{F}_i(p) = \max\{0, 1 - \frac{1}{b_i}p\}$, or equivalently, $V_i \sim U(0, b_i)$. Without loss of generality, we impose $c < b_0 < b_1$. The linear demand model is commonly used in various settings. Not only does linearity make our analysis tractable, it can also be viewed as a near-optimal approximation to more complex demand models (see, e.g., [147, 148]). We consider the case with two groups and study the impact of imposing each type of fairness. We then consider the case with $N$ groups in Section 3.4.1 and incorporating an unprotected feature in Section 3.4.2. We consider nonlinear demand in Section 3.5.

Under linear demand, the market share, profit, and (normalized) consumer surplus for each group $i = 0, 1$ are given by: $\bar{F}_i(p_i) = \max\{0, 1 - \frac{1}{b_i}p\}$, $R_i(p_i) = d_i(p_i - c)\bar{F}_i(p_i)$, $S_i(p_i) = \frac{(b_i - p_i)\bar{F}_i(p_i)}{2}$, and $N_i(p_i) = \frac{\min(b_i, p_i)}{2}$, respectively. It is well-known that the optimal unconstrained price for each group is given by $p_i^* = (b_i + c)/2$. At $p_i^*$, the demand, consumer surplus, and no-purchase valuation for group $i$ reduce to $\bar{F}_i(p_i^*) = (b_i - c)/2b_i$, $S_i(p_i^*) = (b_i - c)^2/8b_i$, and $N_i(p_i^*) = (b_i + c)/4$, respectively. Since $b_0 < b_1$, all of price, demand, surplus, and no-purchase valuation are lower for group 0 than for group 1. We naturally restrict the prices to be larger than 0, but they may be below the cost $c$ (this captures the situation when it is optimal for the seller to earn a negative profit for one group in order to extract a high positive profit from the other group while enforcing fairness constraints). We next discuss the optimal pricing strategy and the potential impact of imposing each type of fairness constraint for a given $\alpha$.

The price optimization problem with fairness constraints is not a straightforward extension of the nominal setting (i.e., without fairness constraints). Under linear demand, the profit function for each group is concave for $p \in [0, b_i]$. In our analysis, when imposing fairness constraints, we will show that one of the prices may reach the boundary 0 or $b_i$, thus potentially making problem (3.1) non-convex. In the left panels of Fig. 3.1, we show an example of the price dynamics, $p_i(\alpha)$, under each of the four fairness constraints. Interestingly, the four fairness constraints lead to totally
different pricing strategies. Further, for three out of the four constraints (price, surplus, and no-purchase valuation), there are nonlinear price changes. Given that the price strategies vary across different fairness constraints, the impact on profit, consumer surplus, and social welfare is also different (see the right panels of Fig. 3.1). We next provide closed-form expressions for the optimal prices as a function of $\alpha$ under each fairness measure, which allow us to assess the impact on the seller’s profit, consumer surplus, and social welfare. All the proofs can be found in Appendix C.2.

3.3.1 Price Fairness

In this section, we consider imposing price fairness. As $\alpha$ starts to increase, $p_0(\alpha)$ increases whereas $p_1(\alpha)$ decreases. Consequently, group 0 (resp. group 1) is earning a lower (resp. higher) surplus. Then, when $\alpha$ becomes large enough, it is possible that $p_0$ is set to be higher than $b_0$. This implies that it is optimal for the seller to “give up” group 0 (i.e., the demand from group 0 is zero). At this point, it is equivalent to simply set $p_0 = p_1 = p_1^*$. We formally characterize the resulting impact of imposing price fairness in Proposition 3.2.

**Proposition 3.2.** Let $\tilde{\alpha}_p = \min \left( \frac{d_0b_1 + d_1b_0}{d_1b_0}, 1 \right)$. If $0 \leq \alpha \leq \tilde{\alpha}_p$, then both the consumer surplus and the social welfare increase with $\alpha$. If $\tilde{\alpha}_p < \alpha \leq 1$, then $p_0(\alpha) = p_1(\alpha) = p_1^*$. In addition, the profit, consumer surplus, and social welfare are lower relative to the case without price fairness (i.e., $\alpha = 0$).

The impact of imposing price fairness admits two separate cases:

(a) When $\alpha \leq \tilde{\alpha}_p$, the change in consumer surplus is a quadratic function that is increasing with $\alpha$. The change in welfare is a quadratic function that is concavely increasing for any $\alpha \leq \tilde{\alpha}_p$. Thus, for any $\alpha \leq \tilde{\alpha}_p$ (i.e., before giving up group 0), imposing additional price fairness increases social welfare. Interestingly, both the gain in social welfare and the loss in profit are concave in $\alpha$. This implies that the marginal effect of imposing additional price fairness on social welfare (resp. profit) decreases (resp. increases) with $\alpha$. Consequently, imposing a small amount of price fairness yields the highest marginal benefit on social welfare coupled
Figure 3.1: Impact of fairness under linear demand.

Parameters: $d_0 = 0.35, d_1 = 0.65, b_0 = 1, b_1 = 4.5, c = 0.6$. 
with the lowest profit loss. This insight can help persuade regulators that incorporating a small amount of price fairness is worthwhile.

(b) When \( \alpha > \bar{\alpha}_p \), it becomes optimal for the seller to give up group 0 and set both prices at \( p_1^* > b_0 \). This is assuming that \( \bar{\alpha}_p < 1 \), given that if \( \bar{\alpha}_p \geq 1 \), the second case does not exist. Consequently, the profit and surplus from group 0 are lost, so that it leads to a lose-lose outcome (i.e., lower seller’s profit and lower consumer surplus). In this case, the social welfare drops below \( W(0) \) for any \( \alpha > \bar{\alpha}_p \).

On the top-right of Fig. 3.1, we consider a concrete example and show how the profit, consumer surplus, and social welfare vary as a function of \( \alpha \) under price fairness. An interesting implication of Proposition 3.2 is the fact that the social welfare reaches its maximum right before giving up group 0, that is, when \( \alpha = \bar{\alpha}_p \) (in the example of Fig. 3.1, \( \bar{\alpha}_p = 0.22 \)). The seller’s decision to give up group 0 crucially depends on the value of \( \sqrt{\frac{d_0b_1 + d_1b_0}{d_1b_0}} \frac{b_0 - c}{b_1 - b_0} \). If it is less than 1, then any \( \alpha > \bar{\alpha}_p \) leads to a lose-lose outcome. The square root term, \( \sqrt{\frac{d_0b_1 + d_1b_0}{d_1b_0}} \), depends on the relationship between \( d_0b_1 \) and \( d_1b_0 \), or equivalently, \( d_1/d_0 \) and \( b_1/b_0 \). For example, when \( d_1/d_0 = b_1/b_0 \), then \( \sqrt{\frac{d_0b_1 + d_1b_0}{d_1b_0}} = \sqrt{2} \). On the other hand, when \( d_1/d_0 = 100b_1/b_0 \), then \( \sqrt{\frac{d_0b_1 + d_1b_0}{d_1b_0}} = \sqrt{1.01} \). The higher \( d_1/d_0 \) is (for a fixed \( b_0/b_1 \)), the more likely the seller will give up group 0 when imposing fairness (i.e., it will occur for a smaller value of \( \alpha \)). This is consistent with the intuition that when the high-valuation group (group 1) dominates the market, the seller is more likely to give up the low-valuation group (group 0). Similarly, the term \( (b_0 - c)/(b_1 - b_0) \) conveys that the higher the difference between both groups’ valuations is, the more likely the seller is to give up group 0 when imposing fairness, which is also intuitive.

To summarize, imposing price fairness increases social welfare as long as \( \alpha \) remains below \( \bar{\alpha}_p \). When the differences in population size and in valuation are significant, setting \( \alpha \) to a large value may lead to a lose-lose outcome. Furthermore, the value of \( \alpha \) needs to be carefully selected given that the maximum and minimum values of \( W(\alpha) \) are right beside each other.
3.3.2 Demand Fairness

We next consider the case of demand fairness. Recall that \( \bar{F}_i(p^*_i) = (b_i - c)/2b_i \), so that group 0 has lower demand. Thus, as \( \alpha \) increases, \( p_0(\alpha) \) decreases to raise demand from group 0, while \( p_1(\alpha) \) increases to reduce demand from group 1. We characterize the impact of imposing demand fairness in Proposition 3.3.

**Proposition 3.3.** For demand fairness, the profit, consumer surplus, and social welfare all decrease with \( \alpha \).

For demand fairness, the change in surplus is always negative and reaches its minimum at \( \alpha = 1 \). Hence, the change in surplus is monotonically decreasing for \( \alpha \in [0, 1] \). Consequently, the change in social welfare is also monotonically decreasing, so that any degree of demand fairness reduces social welfare and leads to a lose-lose outcome.

3.3.3 Surplus Fairness

Recall that the surplus of group \( i \) is \( S_i(p_i) = (b_i - p_i)(1 - p_i/b_i)/2 \) and that \( S_0(p^*_0) < S_1(p^*_1) \). Thus, as \( \alpha \) starts to increase, \( p_0(\alpha) \) decreases to raise the surplus from group 0, and \( p_1(\alpha) \) increases to reduce the surplus from group 1. The closed-form expressions for surplus fairness are complicated due to the nonlinearity of the surplus function. However, as we show in in Proposition 3.4, the social welfare is always below \( \mathcal{W}(0) \) for any \( \alpha > 0 \).

**Proposition 3.4.** For surplus fairness, \( \mathcal{W}(\alpha) < \mathcal{W}(0) \) for any \( \alpha \in (0, 1] \).

Hence, regardless of the value of \( \alpha \), imposing surplus fairness always leads to lower social welfare relative to no fairness constraint.

3.3.4 No-Purchase Valuation Fairness

For no-purchase valuation fairness under linear demand, we have \( N_i(p_i) = p_i/2 \). Therefore, for small values of \( \alpha \), we obtain the same pattern as for price fairness. However, when \( \alpha \) becomes
large, the price dynamics under no-purchase valuation fairness follow a different pattern. We formalize the impact of no-purchase valuation fairness in Proposition 3.5.

**Proposition 3.5.** For no-purchase valuation fairness, both the consumer surplus and the social welfare increase with $\alpha$.

For no-purchase valuation fairness, the social welfare always increases with $\alpha$. When $\alpha$ is small, the dynamics are the same as for price fairness. As in price fairness, both the gain in social welfare and the loss in profit are concave in $\alpha$, so that imposing a small amount of fairness yields the highest marginal benefit on social welfare coupled with the lowest profit loss. When $\alpha$ is large, however, instead of setting $p_1 = p_0 = p^*_1 > b_0$ (so that the demand from group 0 is zero), the seller has to lower $p_1$ to reduce the gap in the no-purchase valuation between both groups. Indeed, for any $p_0 \geq b_0$, the expected no-purchase valuation of group 0 is equal to $b_0/2$ and cannot be raised by increasing $p_0$. Thus, the only way to reduce the difference in no-purchase valuations is to decrease $p_1$, and hence the social welfare continues to increase (since the social welfare is monotonically decreasing with price). As a result, imposing additional no-purchase valuation fairness always increases social welfare, even though group 0 may be given up (when $p_0$ is set at $b_0$). Interestingly, for a large value of $\alpha$, the only fairness definition that yields a social welfare that is greater than $W(0)$ is the no-purchase valuation fairness. As a result, no-purchase valuation fairness weakly dominates the other three fairness metrics in terms of social welfare.

3.3.5 Summary and Discussion

We next summarize the results derived so far.

(a) Under price fairness, the social welfare increases with $\alpha$ and reaches its maximum at $\alpha = \tilde{\alpha}_p$.

It then drops below $W(0)$ for any $\alpha > \tilde{\alpha}_p$.

(b) Under demand fairness, the social welfare decreases with $\alpha$—leading to a lose-lose outcome.

(c) Under surplus fairness, the social welfare is always below $W(0)$—leading to a worse outcome relative to imposing no fairness.
Under no-purchase valuation fairness, the social welfare always increases with $\alpha$, but it is possible that the demand of group 0 vanishes.

### 3.4 Extensions

In this section, we consider two extensions of our model and show the robustness of our findings from Section 3.3. In Section 3.4.1, we consider the case with $N > 2$ groups. In Section 3.4.2, we study the case where there is an additional feature $Y$ that does not need to be protected.

#### 3.4.1 Multiple Groups of Customers

We now assume that $X$ is not binary and can take on $N$ values. We index the groups by $0, \ldots, N-1$, and assume that group $i$ has population $d_i$ and parameter $b_i$. Without loss of generality, we assume that $b_0 \leq \ldots \leq b_{N-1}$. The profit maximization problem (3.1), with linear demand, can be generalized as follows:

$$
R(\alpha) := \max_{p_i \geq 0} \sum_{i=0}^{N-1} d_i (p_i - c) \max \left\{ 0, 1 - \frac{p_i}{b_i} \right\} 
$$

subject to

$$
|M_i(p_i) - M_j(p_j)| \leq (1 - \alpha) \max_{i,j \in \{0, \ldots, N-1\}} |M_i(p_i^*) - M_j(p_j^*)|, \forall i, j \in \{0, \ldots, N-1\},
$$

where $M_i$ is the fairness metric under consideration. Although problem (3.2) is easy to solve numerically (as we will show in Lemma C.2), its closed-form solution as well as the impact on social welfare are difficult to characterize. In particular, there are potentially many phase changes in the optimal solution as $\alpha$ varies. However, by leveraging the structural properties of the linear demand, we can still derive managerial insights that turn out to be similar to the two-group case studied in Section 3.3.

We start by investigating the cases of demand, surplus, and no-purchase valuation fairness. Recall that for the setting with two groups, Propositions 3.3–3.5 show that imposing fairness is either detrimental (for demand or surplus fairness) or always beneficial (for no-purchase valuation fairness) in terms of social welfare. We next extend these results to the multi-group case, as stated
in Proposition 3.6 below.

**Proposition 3.6.** Consider any $\alpha \in (0,1]$. Then, the following results hold

(a) For demand fairness, $W(\alpha)$ decreases monotonically with $\alpha$.

(b) For surplus fairness, $W'(\alpha) < W'(0)$.

(c) For no-purchase valuation fairness, $W(\alpha)$ increases monotonically with $\alpha$.

The proof for each part of Proposition 3.6 relies on different arguments and machinery (see Appendix C.3). For demand and no-purchase valuation fairness, we first show that the prices $p_i(\alpha)$ are monotonic. We then show that the problem can be reduced to an instance with two groups. For surplus fairness, we also find a reduction to an instance of the problem with two groups, but in this case, the social welfare of the two-group instance does not match the social welfare of the $N$-group problem. Instead, we leverage convexity properties of several relevant functions to arrive at our desired result. Ultimately, Proposition 3.6 shows that our findings from Section 3.3 continue to hold for settings with any finite number of customer groups.

We next consider the case of price fairness. Recall that for the setting with two groups, Proposition 3.2 shows that for small values of $\alpha$, the social welfare increases with $\alpha$. However, when $\alpha$ becomes large, it may be optimal for the seller to give up a low-value group. In a setting with more than two groups, the impact of $\alpha$ on prices is more intricate relative to the setting with two groups. In Fig. 3.2, we present an example with three groups. As $\alpha$ increases, the price changes (left panel) admit four linear pieces, and the social welfare function (right panel) includes two drops. Nevertheless, we can still partially characterize the impact on social welfare, as stated in Proposition 3.7 (the proof can be found in Appendix C.3).

**Proposition 3.7.** For price fairness, we have:

(a) $W'(0) > 0$, that is, imposing a small amount of price fairness increases social welfare.

(b) Suppose that all the groups have positive demand for all $\alpha \in [0,1]$, i.e., $\bar{F}_i(p_i(\alpha)) > 0$, then $W(\alpha)$ increases monotonically with $\alpha$. 

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Figure 3.2: Prices (left) and profit, surplus, welfare (right) under price fairness.

Note. Parameters: $d_1 = 0.1, d_2 = 0.2, d_3 = 0.7, b_0 = 1, b_1 = 1.3, b_2 = 4, c = 0.2$.

(c) If there exists an $\alpha'$ such that at least one group has zero demand, then $W(\alpha')$ may be either higher or lower than $W(0)$.

When $\alpha$ is small enough, Proposition 3.7(a) suggests that a little price fairness still increases social welfare for any finite number of groups. As illustrated in Fig. 3.2, when $\alpha$ is relatively large, some groups may be excluded by being offered high prices, thus leading to potentially complicated patterns. Nevertheless, when group exclusion does not happen, Proposition 3.7(b) conveys that the social welfare increases monotonically with $\alpha$. On the other hand, when group exclusion does happen, the social welfare could either be higher or lower than the unconstrained social welfare per Proposition 3.7(c), which is different from the two-group setting where group exclusion always leads to a lower welfare.

To summarize, for demand, surplus, or no-purchase valuation fairness, we find that our insights from the two-group setting generalize for multiple groups. For price fairness, even though the prices follow a more complex pattern, we still find that a little price fairness is always beneficial in terms of social welfare.
3.4.2 Adding an Unprotected Feature

In practice, it is possible that a subset of the features is unprotected, so that the seller is allowed to discriminate freely based on such features (e.g., loyalty status, purchase history, country). For simplicity, we consider the case of two observable features: a binary (protected) feature $X = \{0, 1\}$ on which we would like to impose fairness, and a binary unprotected feature $Y = \{0, 1\}$. This gives rise to four groups of customers. We use the subscript $xy$ to denote a specific group: for example, $d_{00}$ is the population of group $(X = 0, Y = 0)$ and $p_{10}$ is the price offered to group $(X = 1, Y = 0)$.

When adding an unprotected feature, our fairness definitions from Section 3.2 need to be revisited. We propose two refined versions for each fairness definition: conditional fairness and weighted average fairness. For example, consider the price fairness definition. The \textit{conditional $\alpha$-fairness} is defined such that for any value of $Y$, the price difference between the group with $X = 0$ and the group with $X = 1$ is small:

$$|p_{0y} - p_{1y}| \leq (1 - \alpha)|p_{0y}^* - p_{1y}^*|, \forall y = 0, 1.$$ 

The \textit{weighted average $\alpha$-fairness} is defined such that the weighted average price (with respect to population sizes) for $X = 0$ and $X = 1$ are close together, that is,

$$|\bar{p}_0 - \bar{p}_1| \leq (1 - \alpha)|\bar{p}_0^* - \bar{p}_1^*|,$$

where $\bar{p}_i = \frac{d_{0i}p_{0i} + d_{1i}p_{1i}}{d_{0i} + d_{1i}}$, for $i = 0, 1$. The same refinements extend to the other fairness definitions.

For conditional fairness, the problem separates in two parallel sub-problems for each value of $Y$. Each sub-problem has two groups, so that the results from Sections 3.3.1–3.3.4 naturally apply. On the other hand, the weighted average fairness cannot be solved using the same approach.
Specifically, the pricing problem faced by the seller becomes

\[ \mathcal{R}(\alpha) := \max_{p_{00}, p_{01}, p_{10}, p_{11}} R_{00}(p_{00}) + R_{01}(p_{01}) + R_{10}(p_{10}) + R_{11}(p_{11}) \]  
\[ \text{s.t. } |\bar{M}_0 - \bar{M}_1| \leq (1 - \alpha)|\bar{M}_0^* - \bar{M}_1^*|, \]

where \( \mathcal{R}(\alpha) \) denotes the total optimal profit as a function of \( \alpha \), \( \bar{M}_i = \frac{d_{0i}M_{i0}(p_{i0}) + d_{1i}M_{i1}(p_{i1})}{d_{0i} + d_{1i}} \) is the weighted average measure of group \( i \) with respect to \( Y \), and \( \bar{M}_i^* \) is the weighted average measure of group \( i \) under the optimal prices when the problem is unconstrained (i.e., no fairness constraints). For convenience, we denote \( p_{xy}(\alpha) \) the optimal prices to problem (3.3) as a function of \( \alpha \). Note that

\[ \mathcal{R}(\alpha) = R_{00}(p_{00}(\alpha)) + R_{01}(p_{01}(\alpha)) + R_{10}(p_{10}(\alpha)) + R_{11}(p_{11}(\alpha)). \]

For simplicity of exposition, we focus on the situation where all the groups have positive prices and demand, i.e., \( p_{xy}(\alpha) \in (0, b_{xy}) \).

Proposition 3.8 shows that our insights from Sections 3.3.1–3.3.4 still hold for weighted average fairness (the proof is in Appendix C.4).

**Proposition 3.8.** Assume that the demand is linear so that the valuations for a group \( xy \) are uniform between 0 and \( b_{xy} \), where \( x, y \in \{0, 1\} \). For all \( \alpha \) such that \( p_{xy}(\alpha) \in (0, b_{xy}) \) and for any \( x, y \in \{0, 1\} \), the following holds for weighted average \( \alpha \)-fairness.

(a) For price fairness, \( W(\alpha) \) increases with \( \alpha \).

(b) For demand fairness, \( W(\alpha) \) decreases with \( \alpha \).

(c) For surplus fairness, \( W'(0) < 0 \).

(d) For no-purchase valuation fairness, \( W(\alpha) \) increases with \( \alpha \).

Proposition 3.8 shows that all the qualitative results from Sections 3.3.1–3.3.4 still hold for weighted average fairness (with the exception of surplus fairness for which we now have a slightly weaker claim). Specifically, for small values of \( \alpha \) such that \( p_{xy}(\alpha) \in (0, b_{xy}) \), imposing additional price or no-purchase valuation fairness increases social welfare. On the other hand, imposing demand or surplus fairness has a negative impact on social welfare. Interestingly, conditional
fairness and weighted average fairness may lead to different directions of price changes (even under the same fairness metric), but the impact on social welfare is similar. For example, if $d_{00} = 0.9, d_{01} = 0.1, d_{10} = 0.1, d_{11} = 0.9$ and $b_{00} = 1, b_{01} = 2, b_{10} = 4, b_{11} = 3$, then under conditional price fairness, $p_{01}$ decreases and $p_{11}$ increases, as $b_{01} > b_{11}$. However, for weighted average price fairness, we have $\tilde{p}_{0} = 1.3$ and $\tilde{p}_{1} = 2.9$, so that $\tilde{p}_{0} < \tilde{p}_{1}$. In this case, $p_{01}$ increases and $p_{11}$ decreases with $\alpha$. Although the direction of price changes is different, surprisingly both types of price fairness increase social welfare.

Finally, we consider a scenario where the seller may not be able to (or may not want to) price discriminate based on the protected feature, implying that $p_{0y} = p_{1y}$. Meanwhile, the fairness is still measured based on the difference between $\tilde{M}_{0}$ and $\tilde{M}_{1}$, which is in general non-zero. In other words, the seller optimizes prices based only on the unprotected feature $Y$, whereas the fairness is imposed with respect to the protected feature $X$. This corresponds to solving problem (3.3) with an added price constraint. In this case, one can easily verify that our result in Proposition 3.8(a) still holds for price fairness. However, for the other fairness definitions, the social welfare can increase or decrease. In fact, Example 3.2 describes an instance where it is not possible to improve demand fairness at all in this setting.

**Example 3.2.** Let $d_{00} = 0.5, d_{01} = 0.5, d_{10} = 0.6, d_{11} = 0.4$ and $b_{00} = 1, b_{01} = 3, b_{10} = 1.2, b_{11} = 2.4$. Suppose that the seller maximizes profit subject to the constraint $p_{0y} = p_{1y}$. Then, the optimal prices are $p_{00} = p_{10} = 0.65$ and $p_{01} = p_{11} = 1.45$. The resulting weighted average demand for groups $X = 0$ and $X = 1$ are 0.417 and 0.567, respectively. We next seek to impose demand fairness across both groups. When we vary $p_{00} = p_{10}$ by any $\Delta p_{00}$, the weighted average demand for both groups will change by $-0.5\Delta p_{00}$. Similarly, when we vary $p_{01} = p_{11}$ by any $\Delta p_{01}$, the weighted average demand for both groups will change by $-0.167\Delta p_{01}$. Namely, in both cases, no matter how we vary the prices, the difference in weighted average demand will remain the same. Thus, it is impossible to reduce the demand difference across groups (unless the demands of groups are set to zero).
3.5 Computations for Nonlinear Demand Functions

In this section, we investigate computationally the impact of fairness for alternate demand functions. Specifically, we consider the following three demand models: exponential, logistic, and log-log. We report the expressions of the demand $\bar{F}_i(p)$, consumer surplus $S_i(p)$, and mean valuation $\mathbb{E}[V_i]$ in Table 3.1. Note that we made a slight adjustment to the log-log demand function to ensure that it fits into the random utility framework.$^8$

Table 3.1: Demand, surplus, and mean valuation for the different demand models.

<table>
<thead>
<tr>
<th>Demand model / Metric</th>
<th>$\bar{F}_i(p)$</th>
<th>$S_i(p)$</th>
<th>$\mathbb{E}[V_i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>$e^{-\lambda_i p}$</td>
<td>$\frac{k_i}{\lambda_i} e^{-\lambda_i p}$</td>
<td>$\frac{1}{\lambda_i}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\frac{k_i}{1+k_i e^{-\lambda_i p}}$</td>
<td>$\frac{1}{\lambda_i} \log(1+k_i e^{-\lambda_i p})$</td>
<td>$\frac{1}{\lambda_i} \log(1+k_i)$</td>
</tr>
<tr>
<td>Log-log</td>
<td>$\min\left{ \left( \frac{a_i}{p} \right)^{\beta_i}, 1 \right}$</td>
<td>$\frac{\beta_i}{\beta_i - 1} a_i (\bar{F}_i(p))^{1-\beta_i} - p \bar{F}_i(p)$</td>
<td>$\frac{\beta_i}{\beta_i - 1} a_i$</td>
</tr>
</tbody>
</table>

3.5.1 Setting with Two Groups

We first consider the problem with two groups of customers. We find the optimal pricing strategy by searching for the optimal $\bar{F}_i(p)$ between 0 and 1 using $10^{-4}$ increments. For the log-log demand, it is possible that the market share reaches 1, which corresponds to any price between 0 and $a_i$. In this case, we also search for the optimal $p_i$ between 0 and $a_i$. We report the results for one representative instance of the logistic demand in Fig. 3.3 (see Fig. C.1 and Fig. C.2 in Appendix C.6 for the exponential and log-log demand functions). Specifically, we show how the prices evolve as a function of $\alpha$ (left panels) as well as the profit, consumer surplus, and social welfare (right panels). We also list all the tested instances in Appendix C.6.1.

By conducting extensive computational tests, we find that imposing each of the four types of fairness constraints yields similar insights as in the case of linear demand. For exponential and logistic demand, we observe that under either price or no-purchase valuation fairness, the social

$^8$The common form of the log-log demand is $p = a_i q^{-1/\beta_i}$, where $q$ is the demand and $\beta_i$ is the price elasticity. Since the demand goes to infinity when the price approaches 0, we truncate the demand at 1, that is, we impose $\bar{F}_i(p) = \min\left\{ \left( \frac{a_i}{p} \right)^{\beta_i}, 1 \right\}$. We also require that $\alpha_i (\beta_i - 1) < c \beta_i$, so that $\bar{F}_i(p^*) < 1$ (otherwise all the customers are buying, hence leading to an unrealistic situation).
Figure 3.3: Impact of fairness under logistic demand (two groups).

Parameters: $d_0 = 0.5, d_1 = 0.5, \lambda_0 = 1, \lambda_1 = 0.2, k_0 = 10, k_1 = 5, c = 0.5.$
welfare first increases as a function of $\alpha$, whereas for either demand or surplus fairness, the social welfare decreases monotonically with $\alpha$.

Under price fairness, it is still possible that $p_1$ changes non-monotonically with $\alpha$ (see the top left panel of Fig. 3.3), and that group 0 is (approximately) excluded by setting both prices close to $p_{1}^{*}$ (with nearly zero demand from group 0). As we have shown in Section 3.3, such cases occur when the population of the high-valuation group is large. As a result, under price fairness, we retrieve the result that the social welfare first increases with $\alpha$ and then decreases. A major difference between the linear demand and the nonlinear models considered in this section emerge from the no-purchase valuation fairness. More specifically, the social welfare is not always increasing as it was the case for linear demand. Thus, under price or no-purchase valuation fairness, even though a small value of $\alpha$ increases social welfare, the specific value of $\alpha$ needs to be carefully selected.

3.5.2 Setting with Multiple Groups

We next test the performance of the above nonlinear demand models when there are $N > 2$ groups of customers. Since problem (3.2) is non-convex and there are $N$ decision variables, using a search heuristic can be burdensome. Interestingly, by exploiting the structure of the problem, we find that the optimal solution can be found efficiently by reducing the $N$-group pricing problem (3.2) to a one-dimensional optimization problem. We leave the detailed discussion in Appendix C.5.

We next discuss the results for 20 randomly generated instances with $N = 5$ groups. Representative figures and the details of the instances are reported in Appendix C.6.2. Although the way the prices vary with $\alpha$ is more intricate than before, most of the analytical results we derived for the case of linear demand still hold (computationally) for the nonlinear demand models we considered. For all demand models, imposing price fairness is always beneficial at first. However, increasing the level of price fairness too much may prompt the seller to exclude low-value groups via a price surge, and thus can lead to a lose-lose outcome. For demand fairness, we observe that it always
reduces social welfare under exponential and logistic demands, but for log-log demand, it can go either way as in the two-group case. Imposing a small amount of surplus fairness decreases social welfare for all demand models. Finally, imposing no-purchase valuation fairness increases social welfare when $\alpha$ initially increases from zero, under exponential and logistic demand. Under log-log demand, the social welfare may go either direction just as in the two-group case. Such findings increase our confidence that our managerial insights are robust and continue to hold for nonlinear demand models.

3.6 Conclusion

As discussed in [149], although price discrimination has become common practice, it raises important questions in terms of fairness which have been mostly unexplored. This paper offers a first step in understanding fairness in the context of pricing. We propose four possible fairness definitions—fairness in price, demand, consumer surplus, and no-purchase valuation—and investigate the impact of imposing fairness constraints on social welfare. We first show that imposing simultaneously several fairness metrics is generally impossible, hence reflecting the complexity of achieving perfect fairness in reality. We then focus on each fairness metric separately and characterize the optimal solutions in closed-form under a linear demand model. We show that imposing a small amount of price fairness increases social welfare, but imposing too much price fairness may lead to a lose-lose outcome (i.e., both the seller and the consumers are worse off). Imposing either demand or surplus fairness always reduces the social welfare. Finally, imposing no-purchase valuation fairness increases the social welfare monotonically with the fairness level. Our findings also persist for a general setting with more than two customer groups, and most of our results hold computationally for three nonlinear demand models. Our insights have the potential to inform regulatory entities who are concerned with imposing fairness constraints on pricing.

Admittedly, much more research needs to be done on this topic. First, incorporating these fairness definitions into algorithms is an interesting avenue for future research and can potentially have great practical impact. Second, one can consider the role of inventory or capacity constraints
in this setting, which may potentially evolve over time. Given the extensive research on fairness related to resource allocation, it would be interesting to develop a combined framework for fairness in both inventory allocation and pricing, which might require a different notion of surplus [150]. Another interesting extension of our model would be to consider pricing decisions that can only be made with partial information, such as the mean and variance of customer valuations [151]. We also recognize that there might be competition between multiple sellers, another dimension that is unexplored in this paper. Finally, running behavioral surveys to learn how the different fairness definitions are perceived by consumers will help better understand how to properly define a fair pricing policy.
References


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Appendix A: Loot Box Pricing and Design

A.1 Omitted Proofs

A.1.1 Omitted Proofs from Section 2.2

Proof of Theorem 1.1.

(a) Unique Box. Without loss of generality, we index the items by the order in which they are allocated, i.e., the $i^{th}$ unique box yields the $i^{th}$ item which is valued at $V_i$. Let $x_k = \sum_{i=1}^{k} (v_i - p)$, which is the net utility of buying $k$ unique boxes. Note that the myopic customer will buy the first unique box if $U_1 = x_N/N \geq 0$. The customer will continue to purchase until the first time $\tau + 1$ where $U_{\tau+1} < 0$, i.e., until

$$U_{\tau+1} = \frac{\sum_{i \in [N] \backslash S_{\tau+1}} v_i}{N - |S_{\tau+1}|} - p = \frac{\sum_{i=\tau+1}^{N} v_i}{N - \tau} - p = \frac{x_N - x_\tau}{N - \tau} < 0. \quad (A.1)$$

When this condition is met, the customer stops with $\tau$ items in total and has total utility of $x_\tau$. For clarity, we set $U_{N+1} < 0$ to include the case that $\tau = N$. From Eq. A.1, we find that $\tau$ is also the first time that $x_\tau > x_N$ (or $\tau = N$ if $x_t \leq x_N$ for all $t$) since

$$x_\tau = \sum_{i=1}^{\tau} (v_i - p) = x_N - \sum_{i=\tau+1}^{N} (v_i - p) = x_N - (N - \tau)U_{\tau+1} \geq x_N. \quad (A.2)$$

Hence, the net utility of the myopic strategy, $x_\tau$, is at least $x_N$.

Observe that an upper bound on the maximum possible utility of any purchasing strategy is the utility of the clairvoyant strategy, denoted by $M_N$, that stops when customer utility is maximized,
i.e.,

\[ M_N = \max_{k \in 0 \cup [N]} \sum_{i=1}^{k} (v_i - p), \]

where \( \sum_{i=1}^{0} (v_i - p) = 0 \). For a random customer, \( M_N \) is equivalent to the maximum value of a random walk on \( \{X_k\}_{k=0}^{N} \), where \( X_k = \sum_{i=1}^{k} (V_i - p) \). By Theorem 2.12.1 in [152], \( \lim_{N \to \infty} M_N/N \) converges to \( \max(0, \mu - p) \) almost surely. Further by the strong law of large numbers, \( \lim_{N \to \infty} X_N/N \) converges to \( \mu - p \) almost surely, which implies that the expected normalized net utility of a myopic customer converges to \( \max(0, \mu - p) \) almost surely, because a myopic customer always garners at least \( \max(0, X_N) \) utility. Therefore, the expected normalized net utility of the myopic strategy and optimal strategies must also converge to \( \max(0, \mu - p) \) almost surely.

(b) Traditional Box. Let \( \tau + 1 \) be the first period where a myopic customer decides not purchase an item, meaning the customer purchased exactly \( \tau \) traditional boxes. This means that

\[ U_{\tau+1} = \left( \sum_{i \in [N] \setminus \{\tau+1\}} v_i \right) / N - p < 0. \]  

(A.3)

However, it is clear that the the utility \( U_t \) is non-increasing in \( t \). If a duplicate is received in period \( t \), then \( U_t = U_{t+1} \). If a new item is received, then the customer now values that item as 0 in the future and their expected valuation of a traditional loot box decreases. Therefore, once \( \tau + 1 \) is reached, a customer will never see a traditional box that offers a positive utility even if they continue purchasing indefinitely. Thus, a myopic strategy is optimal. \( \Box \)

A.1.2 Omitted Proofs from Section 1.3

Proof of Theorem 1.2. The proof works by constructing a sequence of prices, \( p_N \), and showing that \( R_{UB}(p_N) \) is greater than a term that converges to \( \mu \) as \( N \) goes to \( \infty \). Since \( \mu \) is the maximum possible normalized revenue by Proposition 1.1, this then implies that \( \lim_{N \to \infty} R_{UB}(p_N) = \lim_{N \to \infty} R_{UB} = \mu \). We shall rely on the random walk from the proof of Theorem 1.1(a), \( \{X_j\}_{j=0}^{\infty} \), where \( X_j := \sum_{i=1}^{j} (V_i - p_N) \) and \( X_0 = 0 \). Without loss of generality, we assume the items are
indexed so that the \( i^{th} \) item a customer receives from the \( i^{th} \) unique box is valued at \( V_i \). Let the random variable \( \tau_N \) denote the number of boxes purchase, and recall from Eq. (A.2) that \( \tau_N \) is also the first time that \( X_{\tau_N} > X_N \), or \( \tau_N = N \) if \( X_t \leq X_N \) for all \( t \). Note that \( R_{UB}(p_N) = p_N\mathbb{E}[\tau_N]/N \). Also note that since \( X_N \) is not known to the seller, \( \tau_N \) is not a stopping time. (However, it is a stopping time from the perspective of the customer.) We shall show that for a sequence of prices that tend to \( \mu \), that \( \tau_N \) tends to \( N \) which implies our result.

For some \( \mu > \epsilon_N > 0 \) to be optimized later, let \( p_N = \mu - \epsilon_N \). We shall compare \( \tau_N \) to an actual stopping time \( \bar{\tau}_N \), which is the first time \( \{X_i\} \) crosses the threshold \((1-k_N)N\epsilon_N\), where \( 1 > k_N > 0 \) shall be optimized later. Note that if we condition on the event that \( X_N \geq (1-k_N)N\epsilon_N \), then we know that \( \tau_N \geq \bar{\tau}_N \) since \( \{X_i\} \) must hit \((1-k_N)N\epsilon_N\) before hitting \( X_N \). Therefore,

\[
\mathbb{E}[\tau_N] \geq \mathbb{E}[\tau_N1_{X_N \geq (1-k_N)N\epsilon_N}]
\geq \mathbb{E}[\bar{\tau}_N1_{X_N \geq (1-k_N)N\epsilon_N}]
= \mathbb{E}[\bar{\tau}_N] - \mathbb{E}[\bar{\tau}_N1_{X_N \in [0,(1-k_N)N\epsilon_N)}] - \mathbb{E}[\bar{\tau}_N1_{X_N < 0}]. \tag{A.4}
\]

We proceed by lower bounding Eq. (A.4) term by term, beginning with \( \mathbb{E}[\bar{\tau}_N] \). Since \( \bar{\tau}_N \) is a stopping time, by Wald’s equation we know that

\[
\mathbb{E}[X_{\bar{\tau}_N}] = \mathbb{E}\left[\sum_{i=1}^{\bar{\tau}_N} (V_i - p_N)\right] = \mathbb{E}[\bar{\tau}_N]\mathbb{E}[V_i - p_N] = \mathbb{E}[\bar{\tau}_N]\epsilon_N. \tag{A.5}
\]

Rearranging (A.5), we have

\[
\mathbb{E}[\bar{\tau}_N] = \frac{\mathbb{E}[X_{\bar{\tau}_N}]}{\epsilon_N} \geq \frac{(1-k_N)N\epsilon_N}{\epsilon_N} = (1-k_N)N, \tag{A.6}
\]

where the inequality follows from the definition of \( \bar{\tau}_N \).

Next, we provide an upper bound for the second term in (A.4), \( \mathbb{E}\left[\bar{\tau}_N1_{X_N \in [0,(1-k_N)N\epsilon_N)}\right] \). This term corresponds to the case that \( X_N \in [0, (1-k_N)N\epsilon_N) \). To derive an upper bound, we suppose that \( \{X_i\} \) has not hit \((1-k_N)N\epsilon_N\) after \( N \) steps, and further assume the worst case that \( X_N = 0 \). In
In this case, it is a fresh random walk starting from 0. We first note that for a discrete random walk crossing a threshold, by [153] Theorem 1 we have

\[
E[X_{\bar{T}_N} - (1-k_N)N\epsilon_N] \leq E[\max(V_i - \mu + \epsilon_N, 0)^2] \leq \frac{E[(V_i - \mu + \epsilon_N)^2]}{\epsilon_N} = \frac{E[V_i - \mu + \epsilon_N]^2 + \sigma^2}{\epsilon_N} \leq \frac{\mu^2 + \sigma^2}{\epsilon_N}
\]

which implies that

\[
E[X_{\bar{T}_N}] \leq (1-k_N)N\epsilon_N + \frac{\mu^2 + \sigma^2}{\epsilon_N}.
\]

Hence, by Wald’s equation (see (A.5)), in expectation it takes at most another \((1-k_N)N + \frac{\mu^2 + \sigma^2}{\epsilon_N}\) steps to hit \((1-k_N)N\epsilon_N\) if \(X_N = 0\). Thus,

\[
E[\bar{T}_N | X_N \in [0, (1-k_N)N\epsilon_N)] \leq N + (1-k_N)N + \frac{\mu^2 + \sigma^2}{\epsilon_N^2}. \quad (A.7)
\]

The probability that \(X_N \in [0, (1-k_N)N\epsilon_N]\) can be upper bounded using Chebyshev’s Inequality,

\[
P(X_N \in [0, (1-k_N)N\epsilon_N]) \leq P(X_N < (1-k_N)N\epsilon_N) \leq \frac{\sigma^2}{k^2\epsilon_N^2 N}. \quad (A.8)
\]

Combining (A.7) and (A.8), we have

\[
E[\bar{T}_N 1_{X_N \in [0, (1-k_N)N\epsilon_N]}] \leq \left(N + (1-k_N)N + \frac{\mu^2 + \sigma^2}{\epsilon_N^2}\right) \frac{\sigma^2}{k^2\epsilon_N^2 N} = \frac{(2-k_N)\sigma^2}{k^2\epsilon_N^2 N} + \frac{(\mu^2 + \sigma^2)\sigma^2}{k^2\epsilon_N^4 N}. \quad (A.9)
\]

Next, we provide an upper bound for the third term in (A.4), \(E[\bar{T}_N 1_{X_N < 0}]\). This term corresponds to the case that \(X_N < 0\). To derive an upper bound, we suppose that \(\{X_i\}\) has not hit \((1-k_N)N\epsilon_N\) after \(N\) steps, and further assume the worst case that \(X_N = -Np_N\) (since \(V_i \geq 0\)). Following the same logic as (A.7),

\[
E[\bar{T}_N | X_N < 0] \leq N + \frac{Np + (1-k_N)N\epsilon_N}{\epsilon_N} + \frac{\mu^2 + \sigma^2}{\epsilon_N^2}. \quad (A.10)
\]
As before, the probability that $X_N < 0$ can also be upper bounded using Chebyshev’s Inequality,

$$\mathbb{P}(X_N < 0) \leq \frac{\sigma^2}{\epsilon_N^2 N}.$$  \hfill (A.11)

Combining (A.10) and (A.11) yields

$$\mathbb{E}[\bar{\tau}_N I_{X_N < 0}] \leq \left( N + \frac{Np + (1 - k_N)N\epsilon_N}{\epsilon_N} + \frac{\mu^2 + \sigma^2}{\epsilon_N^2 N} \right) \frac{\sigma^2}{\epsilon_N^2 N} = \left( \frac{\mu}{\epsilon_N} + 1 - k_N \right) \frac{\sigma^2}{\epsilon_N^2 N} + \frac{(\mu^2 + \sigma^2)\sigma^2}{\epsilon_N^2 N^2} - \frac{(\mu^2 + \sigma^2)\sigma^2}{\epsilon_N^2 N^2}.$$  \hfill (A.12)

Plugging Eqs. (A.6), (A.9) and (A.12) into the right hand side of Eq. (A.4) yields

$$\mathbb{E}[\tau_N] \geq N \left( 1 - k_N - \frac{2\sigma^2}{k_N \epsilon_N^2 N} - \frac{\sigma^2 \mu}{k_N \epsilon_N^2 N} - \frac{2\sigma^2}{\epsilon_N^2 N} + \frac{\sigma^2 k_N}{\epsilon_N^2 N} - \frac{(\mu^2 + \sigma^2)\sigma^2}{k_N \epsilon_N^2 N^2} - \frac{(\mu^2 + \sigma^2)\sigma^2}{\epsilon_N^2 N^2} \right).$$

Now we can lower bound the normalized revenue of a unique box strategy,

$$R_{UB} \geq (\mu - \epsilon_N) \frac{\mathbb{E}[\tau_N]}{N} \geq \mu \left( 1 - \frac{\epsilon_N}{\mu} \right) \left( 1 - k_N - \frac{2\sigma^2}{k_N \epsilon_N^2 N} - \frac{\sigma^2 \mu}{k_N \epsilon_N^2 N} - \frac{2\sigma^2}{\epsilon_N^2 N} + \frac{\sigma^2 k_N}{\epsilon_N^2 N} - \frac{(\mu^2 + \sigma^2)\sigma^2}{k_N \epsilon_N^2 N^2} - \frac{(\mu^2 + \sigma^2)\sigma^2}{\epsilon_N^2 N^2} \right).$$

Choosing $\epsilon_N = \mu N^{-1/5}$, and $k_N = N^{-1/5}$, we have

$$R_{UB} \geq \mu \left( 1 - N^{-1/5} \right) \left( 1 - (1 + \frac{2\sigma^2}{\mu^2})N^{-1/5} - \frac{\sigma^2}{\mu^2} N^{-3/5} - \frac{\sigma^4}{\mu^4} N^{-4/5} - \frac{(\sigma^2 + \sigma^4)}{\mu^4} N^{-6/5} \right).$$

Taking the limit of both sides gives

$$\lim_{N \to \infty} R_{UB} \geq \mu.$$

Combined with the fact that $R_{UB} \leq \mu$ from Proposition 1.1, we conclude that $\lim_{N \to \infty} R_{UB} = \mu$.

□

**Proof of Theorem 1.3.** Consider a random walk for $N$ steps, $\{Y_j\}_{j=0}^N$, where $Y_j = \sum_{i=j+1}^N V_i$ for $j = 0, \ldots, N-1$ and $Y_N = 0$. Without loss of generality, we assume the items are indexed so
that the \(i^{th}\) unique item a customer receives from purchasing traditional loot boxes is valued at \(V_i\). Therefore, every time the customer receives the \(i^{th}\) unique item, their valuation for the traditional box becomes \(Y_i/N\).

Similar to our proof of Theorem 1.2, we construct a sequence of prices \(p_N\) such that \(\lim N p_N \to \frac{\mu}{\varepsilon}\) and show the expected number of traditional loot boxes purchased by a customer at price \(p_N\) tends to \(N\). Let the random variable \(\tau_N\) denote the number of unique items acquired, and recall from Eq. (A.3) that \(\tau_N\) is also the first time \(Y_{\tau_N}/N - p_N < 0\). Note that \(\tau_N\) is well defined since \(Y_N = 0\).

The number of traditional loot boxes a customer must have purchased to acquire \(\tau_N\) unique items is the sum of \(\tau_N\) independent geometric random variables, \(\text{Geo}(1) + \text{Geo}(\frac{N-1}{N}) + \ldots + \text{Geo}(\frac{N-\tau_N+1}{N})\). The revenue under price \(p_N\) is then,

\[
\mathcal{R}_{TB}(p_N) = \frac{1}{N} \mathbb{E} \left[ p_N \left( \text{Geo}(1) + \text{Geo}\left( \frac{N-1}{N} \right) + \ldots + \text{Geo}\left( \frac{N-\tau_N+1}{N} \right) \right) 1_{\tau_N \geq 1} \right]
\]

\[
= p_N \mathbb{E} \left[ \frac{1}{N} \left( 1 + \frac{N}{N-1} + \ldots + \frac{N}{N-\tau_N+1} \right) 1_{\tau_N \geq 1} \right]
\]

\[
= p_N \mathbb{E} \left[ (\log(N) + \gamma + \zeta_N - \log(N-\tau_N+1) - \gamma - \zeta_N-\tau_N+1) 1_{\tau_N \geq 1} \right]
\]

\[
= p_N \mathbb{E} \left[ \left( -\log \frac{N-\tau_N+1}{N} + \zeta_N - \zeta_{N-\tau_N+1} \right) 1_{\tau_N \geq 1} \right], \quad (A.13)
\]

where the third equality follows from the well known expression for the harmonic numbers, \(\sum_{i=1}^{k} \frac{1}{i} = \log k + \gamma + \zeta_k\), with \(\{\zeta_k\}\) converges to 0 from above, and \(\gamma\) is the Euler-Mascheroni constant.

First we bound \(\mathbb{E}[N-\tau_N+1]\). Let us define the monotonically increasing random walk \(\{C_j\}_{j=0}^{\infty}\) such that \((i)\) \(\{C_j\}_{j=0}^{N} = \{Y_{N-j}\}_{j=0}^{N}\), i.e., \(C_0 = 0\), \(C_1 = V_N\), \(C_2 = V_N + V_{N-1}\), ..., \(C_N = V_N + \ldots + V_1\) and \((ii)\) \(C_j = C_N + \sum_{k=N-j+1}^{0} V_k\) for \(j = N+1, N+2, \ldots\) where \(V_0, V_1, V_2, \ldots\) are virtual random variables that are i.i.d. samples from \(F\). Let \(r_N\) be the first time that \(\{C_j\}_{j=1}^{\infty}\) is at least \(Np_N\). By definition of \(\tau_N\), note that when \(\tau_N \geq 1\), \(r_N = N - \tau_N + 1\). Since \(r_N\) is the first passage time when \(C_j \geq Np_N\), it follows by the well known inspection paradox that \(\mathbb{E}[C_{r_N} - C_{r_N-1}] = \frac{\mathbb{E}[V_j^2]}{\mathbb{E}[V_j]} = \frac{\mu^2 + \sigma^2}{\mu}\).
Using this fact together with Wald’s equation, $\mathbb{E}[C_{r_N}] = \mathbb{E}[r_N]\mu$, we have

$$
\mathbb{E}[r_N] = \frac{\mathbb{E}[C_{r_N}]}{\mu} \in \left[ \frac{Np}{\mu}, \frac{Np}{\mu} + 1 + \frac{\sigma^2}{\mu^2} \right].
$$

(A.14)

Now we can construct a lower bound for $R_{TB}(p_N)$,

$$
R_{TB}(p_N) = p_N\mathbb{E}\left[ \left( -\log \frac{N - \tau_N + 1}{N} + \zeta_N - \zeta_{\tau_N} \right) \mathbb{1}_{\tau_N \geq 1} \right] \quad (Eq. (A.13))

\geq p_N\mathbb{E}\left[ \left( -\log \frac{N - \tau_N + 1}{N} \right) \mathbb{1}_{\tau_N \geq 1} \right] \quad (\{\zeta_k\} \text{ monotone dec.})

=p_N\mathbb{E}\left[ \left( -\log \frac{r_N}{N} \right) \mathbb{1}_{r_N \leq N} \right] \quad (r_N \leq N \iff \tau_N \geq 1) \quad (A.15)

\geq p_N\mathbb{E}\left[ \left( -\log \frac{r_N}{N} \right) \right] \quad (Jensen’s Inequality)

\geq - p_N \log \mathbb{E}\left[ \frac{r_N}{N} \right] \quad (Eq. (A.14)) \quad (A.16)

where Eq. (A.15) follows from the fact that $-\log \frac{r_N}{N} < 0$ when $r_N > N$. Setting $p_N = \frac{\mu}{e}$ yields

$$
R_{TB}(p_N) \geq \frac{\mu}{e} \log \left( \frac{1}{\frac{1 + \sigma^2}{\mu^2}} \right) \quad (A.17)
$$

which is our desired guarantee.

We now upper bound the revenue, $R_{TB}(p_N)$. Consider the event that $\frac{r_N}{N} \leq (1 - \epsilon_N)\frac{p_N}{\mu}$ for some small $1 > \epsilon_N > 0$, which is an ingredient of our proof. We can upper bound the probability
of such an event by,

\[
P \left( \frac{r_N}{N} \leq (1 - \epsilon_N) \frac{p_N}{\mu} \right) = P \left( r_N \leq (1 - \epsilon_N) \frac{Np_N}{\mu} \right) = P \left( \sum_{t=N-(1-\epsilon_N)Np_N/\mu}^{N} V_t \geq Np_N \right) = P \left( \frac{\sum_{t=N-(1-\epsilon_N)Np_N/\mu}^{N} V_t}{(1 - \epsilon_N) \frac{Np_N}{\mu}} \geq \frac{1}{1 - \epsilon_N} \mu \right) \leq P \left( \left\{ \frac{\sum_{t=N-(1-\epsilon_N)Np_N/\mu}^{N} V_t}{(1 - \epsilon_N) \frac{Np_N}{\mu}} - \mu \right\} \geq \frac{\epsilon_N}{1 - \epsilon_N} \mu \right) \leq \frac{\sigma^2}{(1 - \epsilon_N) \frac{Np_N}{\mu} \frac{\epsilon_N^2}{(1-\epsilon_N)^2} \mu^2} \quad \text{(Chebyshev’s Inequality)} = \frac{\epsilon_N^2}{(1-\epsilon_N)Np_N\mu}. \quad \text{(A.19)}
\]

Now we can upper bound the revenue from (TB) when using price \( p_N \) by

\[
\mathcal{R}_{TB}(p_N) = p_N \mathbb{E} \left[ \left( - \log \frac{N - \tau_N + 1}{N} + \zeta_N - \zeta_{\tau_N} \right) \mathbb{I}_{\tau_N \geq 1} \right] \leq p_N \mathbb{E} \left[ \left( - \log \frac{N - \tau_N + 1}{N} + \zeta_N \right) \mathbb{I}_{\tau_N \geq 1} \right] = p_N \mathbb{E} \left[ \left( - \log \frac{r_N}{N} + \zeta_N \right) \mathbb{I}_{r_N \leq N} \right] \leq p_N \zeta_N + p_N \mathbb{E} \left[ \left( - \log \frac{r_N}{N} \right) \mathbb{I}_{r_N \leq N} \right] = p_N \zeta_N + p_N \mathbb{E} \left[ \left( - \log \frac{r_N}{N} \right) \mathbb{I}_{r_N \in [N(1-\epsilon_N) p_N/\mu, N]} \right] + p_N \mathbb{E} \left[ \left( - \log \frac{r_N}{N} \right) \mathbb{I}_{r_N \in [1,N(1-\epsilon_N) p_N/\mu]} \right] \leq p_N \zeta_N + p_N \max \{- \log \left( 1 - \epsilon_N \right) \frac{p_N}{\mu} \}, 0 \} + p_N \mathbb{E} \left[ \left( - \log \frac{r_N}{N} \right) \mathbb{I}_{r_N \in [1,N(1-\epsilon_N) p_N/\mu]} \right] \leq p_N \zeta_N + p_N \max \{- \log \left( 1 - \epsilon_N \right) \frac{p_N}{\mu} \}, 0 \} + p_N \left( - \log \left( \frac{1}{N} \right) \right) \frac{\sigma^2}{\epsilon_N^2 (1-\epsilon_N)^2 Np_N \mu} \quad \text{(A.20)}
\]

\[
= p_N \zeta_N + p_N \max \{ \log \left( \frac{\mu}{(1 - \epsilon_N)p_N} \right), 0 \} + \frac{\sigma^2 \log N}{\epsilon_N^2 (1-\epsilon_N)^2 \mu N}, \quad \text{(A.21)}
\]
where the first equality follows from (A.13), the second equality follows from the facts that \( \tau_N \leq N \) and \( r_N = N - \tau_N + 1 \) when \( \tau_N \geq 1 \), and the third equality follows from the fact that \( r_N \geq 1 \). Eq. (A.20) follows by the monotonicity of \( \log(\cdot) \) and by applying Eq. (A.19).

Now setting \( \epsilon_N = N^{-\frac{1}{3}} \) and substituting in (A.21) gives

\[
R_{TB}(p_N) \leq p_N \zeta_N + p_N \max\{\log \frac{\mu}{(1 - N^{-\frac{1}{3}})p_N}, 0\} + \frac{(1 - N^{-\frac{1}{3}})\sigma^2 \log N}{\mu N^{\frac{1}{3}}}. \tag{A.22}
\]

Maximizing Eq. (A.22) over \( p_N \) gives \( \bar{p}_N \) := \( \mu / \exp(1 + \log(1 - N^{-\frac{1}{3}}) - \zeta_N) \). Plugging in \( \bar{p}_N \) into Eq. (A.22) gives the desired upper bound, and combining with Eq. (A.18) yields

\[
\frac{\mu}{e} \log \left( \frac{1}{1 + \frac{1 + \sigma^2}{\mu^2}} \right) \leq R_{TB} \leq \frac{\mu}{e^{1-\zeta_N} (1 - N^{-\frac{1}{3}})} + \frac{(1 - N^{-\frac{1}{3}})\sigma^2 \log N}{\mu N^{\frac{1}{3}}}. \tag{A.23}
\]

Taking the limit of both sides of Eq. (A.23) completes the proof. \( \square \)

Proof of Theorem 1.4.

(a) Unique Box. Suppose the optimal price \( p^*_N \) does not converge to \( \mu \), i.e., there exist \( \epsilon > 0 \) such that \( |p^*_N - \mu| > \epsilon \) infinitely often. First, consider the case where \( p^*_N < \mu - \epsilon \) infinitely often. Since the normalized selling volume is at most 1, then the revenue must be less than \( \mu - \epsilon \) infinitely often. However, this contradicts the fact that \( R_{UB} \) converges to \( \mu \).

Now consider the case that \( p^*_N > \mu + \epsilon \) infinitely often. Using Chebyshev’s inequality, the probability that a customer purchases the first loot box is at most

\[
\mathbb{P} \left( \frac{\sum_i V_i}{N} \geq p \right) \leq \frac{\sigma^2}{(p^*_N - \mu)^2 N}.
\]

Thus, \( \frac{\sigma^2}{(p^*_N - \mu)^2 N} \) is also an upper bound on the normalized sales volume, since the best case the customer buys the maximum \( N \) unique boxes. Thus, an upper bound on the the normalized revenue when \( p^*_N > \mu + \epsilon \) is \( \frac{\sigma^2 p^*_N}{(p^*_N - \mu)^2 N} \). Note that \( \frac{\sigma^2 p^*_N}{(p^*_N - \mu)^2 N} \) is decreasing \( p^*_N \) when \( p^*_N > \mu + \epsilon \), so an even greater upper bound on the revenue in this case is \( \frac{\sigma^2 (\mu + \epsilon)}{\epsilon^2 N} \). Since this upper bound tends to 0 as \( N \)
tends to \infty, then this contradicts the fact that \( \mathcal{R}_{UB} \) converges to \( \mu \) and thus \( p_N^* \) cannot be greater than \( \mu + \epsilon \) infinitely often.

Now we consider the expected fraction of unique items collected by the customer, which is also the expected normalized selling volume for the unique box strategy. Since the normalized selling volume is upper bounded by 1, if it does not converges to 1, \( \mathcal{R}_{UB} \) cannot converges to \( \mu \) given that the optimal price converges to \( \mu \). Hence the expected selling volume converges to 1.

Finally, since the expected customer valuation is \( \mu \) and \( \mathcal{R}_{UB} \) converges to \( \mu \), then no utility is left for the customer and therefore the normalized customer surplus converges to 0.

(b) Traditional Box. We first show that the optimal price converges to \( \frac{\mu}{\epsilon} \). Suppose the optimal price \( p_N^* \) does not converge to \( \frac{\mu}{\epsilon} \), i.e., there exists \( \epsilon > 0 \) such that \( |p_N^* - \frac{\mu}{\epsilon}| > \epsilon \) infinitely often. Recall from Eq. (A.22) that the revenue by using any price \( p_N \) is upper bounded by

\[
p_N \zeta_N + p_N \max\{\log \frac{\mu}{(1 - N^{-\frac{1}{3}})p_N}, 0\} + \frac{(1 - N^{-\frac{1}{3}})\sigma^2 \log N}{\mu N^{\frac{1}{3}}},
\]

and this upper bound converges to \( p_N \max\{\log \frac{\mu}{p_N}, 0\} \). Note that \( p_N \max\{\log \frac{\mu}{p_N}, 0\} < \frac{\mu}{\epsilon} \) for any \( p_N \neq \frac{\mu}{\epsilon} \). Therefore, using a price bounded away from \( \frac{\mu}{\epsilon} \) infinitely often results in a revenue that is bounded away \( \frac{\mu}{\epsilon} \) infinitely often. This contradicts the fact that \( \mathcal{R}_{TB} \) converges to \( \frac{\mu}{\epsilon} \) and thus \( p_N^* \) cannot be bounded away from \( \frac{\mu}{\epsilon} \) infinitely often.

The fraction of unique items collected by the customer is given by \( \frac{\tau_N(p)}{N} = \frac{N + 1 - r_N(p)}{N} \). By Eq. (A.14), \( \mathbb{E}[r_N(p)]/N \) converges to \( p/\mu \). When \( p < \mu \), by Theorem 7.1 in [152], \( \mathbb{E}[r_N(p)]/N \) is uniformly integrable and \( \mathbb{E}[r_N(p)I_{r_N(p) \leq N}]/N \) converges to \( p/\mu \).

Plugging in \( \lim_{N \to \infty} p_N^* = \mu/\epsilon \), we have

\[
\lim_{N \to \infty} \frac{\mathbb{E}[\tau_N(p_N^*)]}{N} = \lim_{N \to \infty} \frac{\mathbb{E}[(N - r_N(p_N^*) + 1)I_{r_N(p_N^*) \leq N}]}{N} = 1 - \lim_{N \to \infty} \frac{\mathbb{E}[r_N(p_N^*)I_{r_N(p_N^*) \leq N}]}{N} = 1 - \frac{\mu/\epsilon}{\mu} = 1 - \frac{1}{\epsilon}.
\]

For the selling volume, note that for any \( p \), Eq. (A.17) and Eq. (A.22) implies that

\[
\lim_{N \to \infty} \frac{\mathcal{R}_{TB}(p)}{p} = \max(0, \log \frac{\mu}{p}).
\]
Plugging in $p^* = \mu / e$ gives the normalized selling volume in the limit, which is 1.

Finally, the customer surplus is the total utility from the unique items $\sum_{i=1}^{TN} V_i$ minus the total price paid. Hence we have

$$
\lim_{N \to \infty} \mathbb{E} \left[ \text{Normalized Surplus} \right] = \lim_{N \to \infty} \frac{\mathbb{E} \left[ \sum_{i=1}^{TN} V_i \right]}{N} - \lim_{N \to \infty} R_{TB}
$$

$$
= \lim_{N \to \infty} \frac{\mathbb{E} \left[ \tau_N (p^* N) \right] \mathbb{E} \left[ V_i \right]}{N} - \lim_{N \to \infty} R_{TB}
$$

$$
= (1 - \frac{1}{e}) \mu - \frac{\mu}{e}
$$

$$
= (1 - \frac{2}{e}) \mu.
$$

A.1.3 Omitted Proofs from Section 2.4

Proof of Proposition 1.2. For unique boxes, we show that the revenue from the size-1 case is dominated by the revenue of the size-$j$ case with a simple coupling argument. Since the asymptotic revenue in the size-1 case is $\mu$ by Theorem 1.2, this implies that the asymptotic revenue for the size-$j$ case is also $\mu$. Now suppose $p$ is the price in the size-1 case and set $jp$ to be the price in the size-$j$ case. If a customer bought $\tau$ loot boxes in the size-1 case and would like to buy the next size-1 box given that they owned the set $S_\tau$, then we claim that the same customer would have bought a size-$j$ box. This follows from the fact that the valuation of a size-$j$ unique box in this state is exactly $j$ times the valuation of a size-1 box. Thus, since the price is also scaled by $j$, the decision of purchasing a loot box in period $\tau$ is perfectly coupled, which concludes the proof.

For traditional boxes, we use a more complex coupling argument to show that the revenue from the size-1 case is very close to the revenue of the size-$j$ case with a price slightly lower than $jp$. Let $p$ be the price of the size-1 box and let $pN (1 - (1 - 1/N)^j)$ be the price of the size-$j$ box. If a customer has purchased $\tau$ size-1 box with inventory state $S_\tau$, and would like to buy the next size-1 box, then we claim that the same customer would like to buy a size-$j$ box given the same situation. This follows from the fact that the customer may get a specific item with probability $1 - (1 - 1/N)^j$, which is close to 1 for large $N$. Therefore, the revenue is close to $\mathbb{E} \left[ \tau_N (p^* N) \right] \mathbb{E} \left[ V_i \right] / N$.
and the valuation of a size-$j$ unique box after owning $S_\tau$ is $(1 - (1 - 1/N)^j) \sum_{[N]\setminus S_\tau} V_i$, while the corresponding valuation of the size-1 box is $\frac{1}{N} \sum_{[N]\setminus S_\tau} V_i$. Therefore, a size-$j$ box is purchased in period $\tau$ if and only if a size-1 box would have been purchased:

$$\frac{1}{N} \sum_{[N]\setminus S_\tau} V_i \geq p \iff (1 - (1 - 1/N)^j) \sum_{[N]\setminus S_\tau} V_i \geq pN(1 - (1 - 1/N)^j).$$

Hence, if a customer stops after purchasing $\tau$ size-1 boxes, along with the same sampling path he will stop after purchasing $\lceil \tau/j \rceil$ size-$j$ boxes. Note that $jp \geq pN(1 - (1 - 1/N)^j)$, so the normalized revenue generated by size-$j$ box is bounded as

$$frac{pN(1 - (1 - 1/N)^j)}{jp} R_{TB}(p) \leq R_{TB}(p) \leq \frac{pN(1 - (1 - 1/N)^j)}{jp} R_{TB}(p) + \frac{pN(1 - (1 - 1/N)^j)}{N}.$$ 

Taking the limit of the above as $N \to \infty$ leads to $\lim_{N \to \infty} R_{TB}(jp) = \lim_{N \to \infty} R_{TB}(p)$. Since the optimal TB revenue is $\frac{\mu}{e}$ by Theorem 1.3, this concludes the proof. □

**Proof of Proposition 1.3.** In this proposition we show that unique box with uniform allocation is asymptotically optimal. We modify the random walk $X_t$ in the proof of Theorem 1.2 into a stochastic process $\{X'_t, t \geq 0\}$. For $t \leq N$, let $X'_t$ be the net utility of a random customer after opening $t$ unique boxes. For $t > N$, simply let $X'_t = X'_{t-1} = \bar{\mu} - p$. Note $X'_N$ has mean $N(\bar{\mu} - p)$ and variance $N\bar{\sigma}^2$, where $\bar{\sigma}^2 = \sum_{m=1}^{M} \beta_m \sigma_m^2$. Also, the expectation of $X'_t - X'_{t-1}$ is $\bar{\mu} - p$ for any $t \geq 1$. Note $\{X'_t\}_{t=1}^{\infty}$ is not a stationary random walk, since its step lengths are correlated. However, $\{X'_t\}_{t=1}^{\infty}$ satisfies the Markovian property, as for every $t$, $X'_{t+1}$ depends only on the number of items in each class that are not yet owned. Hence, following the proof of Theorem 1.2, the Wald’s equations (Eq. (A.6)) and Chebyshev’s inequalities (Eq. (A.8), Eq. (A.11)) are still valid. The only difference is the overshoot term, $\mathbb{E}[X'_{iN} - (1 - k_N)N\epsilon_N]$. By Theorem 2 in [153], it is bounded by

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\( (\sigma_{\text{max}}^2 + \mu_{\text{max}}^2) / \epsilon_N + \left((\sigma_{\text{max}}^2 + \mu_{\text{max}}^2)(1 - k_N)\epsilon_N / \epsilon_N\right)^{1/2} \), which will not influence the limit and the convergence rate. Thus, the limiting result remains the same.

□

Proof of Proposition 1.4. Fix \( k \in (0, 1) \), a probability vector \( \mathbf{d} \), and let \( p = \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\beta_m}} \).

We shall show that the normalized number of loot box purchases made by a customer under the pricing and allocation strategy \((p, \mathbf{d})\), \( \mathbb{E} \left[ Q^N(p, \mathbf{d}) \right] \), tends to \( k \) as \( N \to \infty \). For clarity, we prove the lower and upper bounds separately.

**Lower Bound**: \( \lim_{N \to \infty} \mathbb{E} \left[ Q^N(p, \mathbf{d}) \right] \geq k \)

Given \( p = \sum_{m=1}^{M} d_m \mu_m e^{-\frac{d_m k}{\beta_m}} \), we first bound the probability that \( Q^N(p, \mathbf{d}) < (1 - \epsilon)k \). Note since a customer's valuation for the next loot box decreases monotonically after each purchase, the event (a) \( Q^N(p, \mathbf{d}) \leq (1 - \epsilon)k \) is equivalent to the event (b) the customer’s valuation for the loot box is less than \( p \) after they have opened \( (1 - \epsilon)kN \) boxes. We will bound this event by applying Chebyshev’s inequality, for which we will need estimates of both the mean and variance of customers valuation after opening \( (1 - \epsilon)kN \) boxes. Let \( Z_i^m \) be an indicator random variable taking value 1 if item \( i \) from class \( m \) has not been revealed after \( (1 - \epsilon)kN \) purchases, and 0 otherwise. When the class is clear from the context we will drop the superscript. Now, after each purchase the probability that item \( i \) in class \( m \) is obtained is \( \frac{d_m}{\beta_m} \), thus the expectation of \( Z_i^m \) is,

\[
\mathbb{E}[Z_i^m] = \left(1 - \frac{d_m}{\beta_m} \right)^{(1-\epsilon)kN}.
\]

Recall that \( G_m \) denotes the set of items in class \( m \). For a random customer, since \( V_i \) and \( Z_i^m \) are independent, the valuation of the next loot box after \( (1 - \epsilon)kN \) purchases is given by \( \sum_{i \in G_m} \frac{d_m}{\beta_m} V_i Z_i^m \), and the expected valuation for a loot box after \( (1 - \epsilon)kN \) purchases is,

\[
\mathbb{E} \left[ \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m} V_i Z_i^m \right] = \sum_{m=1}^{M} \sum_{i \in G_m} \mu_m d_m \left(1 - \frac{d_m}{\beta_m} \right)^{(1-\epsilon)kN} = \sum_{m=1}^{M} \mu_m d_m \left(1 - \frac{d_m}{\beta_m} \right)^{(1-\epsilon)kN}.
\]
Moreover, observe that the set of indicators \( \{ Z_i^m | G_i \} \) is negatively correlated for all \( i \) and \( m \) since for any two different items, if one is not revealed so far then the other is more likely to have been revealed. Thus the variance can be bounded by,

\[
\text{Var} \left( \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i \right)
\]

\[
\leq \sum_{m=1}^{M} \sum_{i \in G_m} \text{Var} \left( \frac{d_m}{\beta_m N} V_i Z_i \right) \quad (\{Z_i^m\} \text{ Neg. Corr.})
\]

\[
= \sum_{m=1}^{M} \sum_{i \in G_m} \left( \frac{d_m}{\beta_m N} \right)^2 \left( \mathbb{E} \left[ V_i^2 Z_i^2 \right] - (\mathbb{E}[V_i Z_i])^2 \right)
\]

\[
= \sum_{m=1}^{M} \sum_{i \in G_m} \left( \frac{d_m}{\beta_m N} \right)^2 \left( \mathbb{E} \left[ V_i^2 Z_i \right] - (\mathbb{E}[V_i Z_i])^2 \right) \quad (Z_i^m \in \{0, 1\})
\]

\[
= \sum_{m=1}^{M} \sum_{i \in G_m} \left( \frac{d_m}{\beta_m N} \right)^2 \left( \mu_m^2 + \sigma_m^2 \right) \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN} - \mu_m^2 \left( 1 - \frac{d_m}{\beta_m N} \right)^{2(1-\epsilon)kN} \quad (V_i, Z_i \text{ independent})
\]

\[
\leq \sum_{m=1}^{M} \sum_{i \in G_m} \left( \frac{d_m}{\beta_m N} \right)^2 \left( \mu_m^2 + \sigma_m^2 \right) \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN}
\]

\[
= \sum_{m=1}^{M} \frac{d_m^2}{\beta_m N} \left( \mu_m^2 + \sigma_m^2 \right) \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN} \quad (A.25)
\]

Now applying Chebyshev’s Inequality to the event that the mean valuation is less than \( p \) after
opening \((1 - \epsilon)kN\) boxes, we have

\[
\mathbb{P}\left( Q^N(p, d) \leq (1 - \epsilon)k \right) 
\leq \mathbb{P}\left( \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i < p \right) 
\leq \mathbb{P}\left( \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i - \mathbb{E}\left[ \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i \right] < p - \mathbb{E}\left[ \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i \right] \right) 
\leq \mathbb{P}\left( \left| \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i - \mathbb{E}\left[ \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i \right] \right| > \right) 
\leq \text{Var}\left( \sum_{m=1}^{M} \sum_{i \in G_m} \frac{d_m}{\beta_m N} V_i Z_i \right) 
\leq \frac{\sum_{m=1}^{M} \frac{d_m^2}{\beta_m N} (\mu_m^2 + \sigma_m^2) \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN}}{\sum_{m=1}^{M} \mu_m d_m \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN} - p} 
\leq \frac{d_m^2 (\mu_m^2 + \sigma_m^2) \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN}}{\sum_{m=1}^{M} \beta_m N \left( \sum_{m=1}^{M} \mu_m d_m \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1-\epsilon)kN} - e^{-\frac{d_m}{\beta_m N}} \right)^2} 
\leq \mathbb{E}\left[ Q^N(p, d) \right] = \mathbb{E}\left[ Q^N(p, d) 1_{Q^N(p,d) \leq (1-\epsilon/2)k} \right] + \mathbb{E}\left[ Q^N(p, d) 1_{Q^N(p,d) > (1-\epsilon/2)k} \right] 
\geq 0 + \left( 1 - \frac{\epsilon}{2k} \right) k \left( 1 - \frac{\epsilon}{2k} \right) 
= k \left( 1 - \frac{\epsilon}{2k} \right)^2 
\geq k \left( 1 - \frac{\epsilon}{k} \right) = k - \epsilon, 
\]
which implies that the lower bound converges to $k$ as $\epsilon$ goes to 0.

**Upper Bound:** $\lim_{N \to \infty} \mathbb{E} \left[ Q^N(p, d) \right] \leq k$

Similar to the lower bound, we first control the event that $Q^N(p, d) > (1 + \epsilon)k$. Let $Z_i^m = 1$ now denote the event that after opening $(1 + \epsilon)kN$ loot boxes, item $i$ in group $m$ is still not revealed. As before we will omit the superscript when it is clear from context. Following the derivation of (A.26), we may bound the probability of this event by

$$
\mathbb{P} \left( Q^N(p, d) > (1 + \epsilon)k \right) \leq \sum_{m=1}^{M} \frac{d_m^2 \left( \mu_m^2 + c_m^2 \right) \left( 1 - \frac{d_m}{\beta_m N} \right)^{1+\epsilon}kN}{\beta_m N \left( \sum_{m=1}^{M} \mu_m d_m \left( e^{-\frac{d_m}{\beta_m} k} - \left( 1 - \frac{d_m}{\beta_m N} \right)^{1+\epsilon}kN \right) \right)^2}. \quad (A.28)
$$

Now we will choose $\epsilon = -\log(1 - N^{-1/3})/k$. Substituting our choice of $\epsilon$ into the denominator of Eq. (A.28) we may obtain a lower bound,

$$
\beta_m N \left( \sum_{m=1}^{M} \mu_m d_m \left( e^{-\frac{d_m}{\beta_m} k} - \left( 1 - \frac{d_m}{\beta_m N} \right)^{(1+\epsilon)kN} \right) \right)^{2} \geq \beta_m \left( \sum_{m=1}^{M} \mu_m d_m N^{1/2} \left( e^{-\frac{d_m}{\beta_m} k} - e^{-\frac{d_m}{\beta_m} (1+\epsilon)k} \right) \right)^{2} \quad \text{(Taylor expansion of } e^x)\]

$$
= \beta_m \left( \sum_{m=1}^{M} \mu_m d_m e^{-\frac{d_m}{\beta_m} k} N^{1/2} \left( 1 - (1 - N^{-1/3} \frac{d_m}{\beta_m}) \right) \right)^{2} \geq \beta_m \left( \sum_{m=1}^{M} \mu_m d_m e^{-\frac{d_m}{\beta_m} k} N^{1/2} \left( 1 - \frac{1}{1 + \frac{d_m}{\beta_m} N^{-1/3}} \right) \right)^{2} \quad \text{(Bernoulli’s Inequality)}
$$

$$
= \beta_m \left( \sum_{m=1}^{M} \mu_m d_m e^{-\frac{d_m}{\beta_m} k} N^{1/2} \left( \frac{d_m}{\beta_m} N^{1/3} + \frac{d_m}{\beta_m} \right) \right)^{2}.
$$

Plugging back into Eq. (A.28), the probability that customer purchases more than $(1 - \log(1 -
\(N^{-1/3}\)) \(N\) boxes is then bounded above by,

\[
\mathbb{P}\left(Q^N(p, d) > (1 - \log(1 - N^{-1/3}))\right) \leq \sum_{m=1}^{M} d_m^2 \left(\mu_m^2 + \sigma_m^2\right) \left(1 - \frac{d_m}{\beta_m N}\right)^{-(\log(1-N^{-1/3})/k)kN}.
\]

Finally, returning to \(Q^N_{d(p)}\), a trivial upper bound on \(\mathbb{E}[Q^N(p, d)]\) is given by

\[
\mathbb{E}[Q^N(p, d)] \leq \mathbb{E}[\# \text{ of purchases to collect all the items}]
\]

\[
\leq \sum_{m=1}^{M} \mathbb{E}[\# \text{ of purchases to collect all the items in class } m]
\]

\[
= \sum_{m=1}^{M} \mathbb{E}\left[\text{Geo}(d_m) + \text{Geo}\left(\frac{d_m(\beta_m N - 1)}{\beta_m N}\right) + \cdots + \text{Geo}\left(\frac{d_m}{\beta_m N}\right)\right]
\]

\[
= \sum_{m=1}^{M} \frac{\beta_m N}{d_m \beta_m N} + \frac{\beta_m N}{d_m (\beta_m N - 1)} + \cdots + \frac{\beta_m N}{d_m}
\]

\[
= \sum_{m=1}^{M} \frac{\beta_m N}{d_m} \left(\frac{1}{\beta_m N} + \frac{1}{\beta_m N - 1} + \cdots + 1\right)
\]

\[
\leq \sum_{m=1}^{M} \frac{\beta_m N}{d_m} (\log(\beta_m N) + 1)
\]

\[
\leq \sum_{m=1}^{M} \frac{\beta_m N}{d_m} (\log N + 1).
\]

Thus the expected number of purchases is upper bounded by \(\mathbb{E}[Q^N(p, d)] \leq \sum_{m=1}^{M} \frac{\beta_m N}{d_m} (\log N + 1)\) for any price \(p\). Now we can build an upper bound:

\[
\mathbb{E}[Q^N(p, d)] = \mathbb{E}[Q^N(p, d)_{Q^N(p, d) \leq (1+\epsilon)kN}] + \mathbb{E}[Q^N(p, d)_{Q^N(p, d) > (1+\epsilon)kN}]
\]

\[
\leq \left(1 - \log(1 - N^{-1/3}) / k\right) kN + N \sum_{m=1}^{M} \frac{\beta_m}{d_m} (\log N + 1) \sum_{m=1}^{M} d_m^2 \left(\mu_m^2 + \sigma_m^2\right) \left(1 - \frac{d_m}{\beta_m N}\right)^{-(\log(1-N^{-1/3})/k)kN}
\]

\[
\leq \left(1 - \log(1 - N^{-1/3}) / k\right) kN + N \sum_{m=1}^{M} \frac{\beta_m}{d_m} (\log N + 1) \sum_{m=1}^{M} d_m^2 \left(\mu_m^2 + \sigma_m^2\right) \left(1 - \frac{d_m}{\beta_m N}\right)^{-(\log(1-N^{-1/3})/k)kN}
\]

\[
\leq \beta_m \left(\sum_{m=1}^{M} \frac{d_m^2}{\mu_m^2 + \sigma_m^2} \left(1 - \frac{d_m}{\beta_m N}\right)^{-(\log(1-N^{-1/3})/k)kN}\right)^2.
\]

Taking \(N \to \infty\) on both sides, we have
Thus combining Eq. (A.27) and Eq. (A.29), we have

\[
\lim_{N \to \infty} \frac{\mathbb{E}[Q^N(p, d)]}{N} \leq \lim_{N \to \infty} \left(1 - \log(1 - N^{-1/3})/k\right) k + \lim_{N \to \infty} \sum_{m=1}^{M} \beta_m \left(\log N + 1\right) \frac{\sum_{m=1}^{M} \frac{d_m^2 \left(\mu_m^2 + \sigma_m^2\right)}{\beta_m} \left(1 - \frac{d_m}{\beta_m N}\right)^{\frac{(1 - \log(1 - N^{-1/3})/k)k N}{N^{1/3} + \frac{d_m}{\beta_m}}}}{\beta_m \left(\sum_{m=1}^{M} \mu_m d_m e^{-\frac{d_m}{\beta_m}} N^{1/2} N^{1/3} + \frac{d_m}{\beta_m}\right)^2}
\]

\[
= k + \lim_{N \to \infty} \sum_{m=1}^{M} \frac{d_m^2 \left(\mu_m^2 + \sigma_m^2\right)}{\beta_m} e^{-\frac{d_m}{\beta_m} k} \frac{\sum_{m=1}^{M} \beta_m}{\beta_m} \left(\log N + 1\right) \frac{\sum_{m=1}^{M} \beta_m}{\beta_m} \left(1 + \frac{1}{\log N}\right) \frac{\sum_{m=1}^{M} \beta_m}{\beta_m} \left(\log N^{1/2} \left(N^{1/3} + \frac{d_m}{\beta_m}\right)\right)^2
\]

\[
= k.
\]

\[
(A.29)
\]

Thus combining Eq. (A.27) and Eq. (A.29), we have

\[
\lim_{N \to \infty} \frac{n}{N} \mathbb{E}[Q^N(p, d)] = \beta_m \left(\sum_{m=1}^{M} \mu_m d_m e^{-\frac{d_m}{\beta_m}} N^{1/2} N^{1/3} + \frac{d_m}{\beta_m}\right)
\]

\[
\text{Proof of Proposition 1.5.}
\]

(a) Let \( V_i^c := \max\{V_i, c\} \) be the modified valuation of an item that has salvage value \( c \), and let 
\( F_c \) denote the distribution of \( V_i^c \). Let \( \tilde{\eta} \) be the mean of \( V_i^c \). By Theorem 1.2 and Theorem 1.4, as 
the number of items \( N \to \infty \), the optimal price tends to \( \tilde{\eta} \) and the expected proportion of items 
gained tends to 1. Since all items are obtained in expectation, the proportion of items salvaged 
tends to \( F(c) \). Thus the normalized cost of salvages by the customer is 
\( \lim_{N \to \infty} \frac{E[\# Items Salvaged]}{N} c = F(c)c \). Together, the normalized revenue is then \( \tilde{\eta} - F(c)c \). Noting \( \tilde{\eta} \) can be rewritten as 
\( \tilde{\eta} = E[\max\{V_i, c\}] = F(c)c + \bar{F}(c)E[V_i|V_i > c] = F(c)c + \bar{F}(c)\eta \), then the normalized revenue 
becomes \( E[\max\{V_i, c\}] - F(c)c = \bar{F}(c)\eta \).
For customer surplus, note that the customer all items in the limit, garnering expected utility of \( \hat{\eta} \). The cost to the customer is the revenue \( \overline{F}(c)\eta \), so the customer surplus is \( \hat{\eta} - \overline{F}(c)\eta = F(c)\eta \).

(b) Following Theorem 1.3, we consider a modified random walk for customers of a traditional box strategy with salvage cost \( c \). Let \( Y'_j = \sum_{i=j+1}^N V_i' + jc \). For a random customer, \( Y'_j/N \) is the expected valuation of the traditional box after receiving \( j \) unique items. If the new item is at value greater than \( c \), then \( Y'_j \) is decreased by \( V_j \) + 1 - \( c \), otherwise it decreases by 0. Hence, the mean step length is given by

\[
\mathbb{E}[Y'_{j+1} - Y'_j] = \overline{F}(c)(\eta - c) + F(c) \cdot 0 = \overline{F}(c)(\eta - c).
\]

Also, note that \( Y'_N = Nc \). The random walk \( Y'_j \) is still a decreasing process, which means that the valuation of the box is decreasing as customers collect more and more new items. A customer purchases until the first time such that \( \frac{Y'_N}{N} < p \iff Y'_N < Np \iff Y'_j - Nc < N(p - c) \).

Now consider the random walk \{\( Y'_j - Nc \}\}, which ends with 0, and is weakly decreasing with mean jump length \( \overline{F}(c)(\eta - c) \). Let \( \tau(p) \) be the first passage time of \{\( Y'_j - Nc \)\} hitting the line \( N(p - c) \) from above. The problem of approximating \( \tau(p) \) is exactly the same problem of approximating the expected selling volume of a vanilla traditional box in Theorem 1.3 with mean \( \overline{F}(c)(\eta - c) \) and price \( p - c \). Recall that in the proof of Theorem 1.4, we show that the limiting selling volume for a vanilla traditional box is \( \max(\log \frac{\mu}{p}, 0) \) (see (A.24)). So in the case with salvage \( c \), we know that the normalized selling volume is given by

\[
\lim_{N \to \infty} \frac{\mathbb{E}[\text{selling volume}]}{N} = \max \left( \log \frac{\overline{F}(c)(\eta - c)}{p - c}, 0 \right),
\]

and for the nontrivial case \( p - c \leq \overline{F}(c)(\eta - c) \), the selling volume is simply \( \log \frac{\overline{F}(c)(\eta - c)}{p - c} \). The net revenue is the revenue subtracted by the salvage cost. Note that only the new items with value greater than \( c \) are not salvaged. The number of unique items is \( \tau(p) \), and by the discussion in Theorem 1.4, \( \tau(p)/N \) converges to \( 1 - p/\mu \), which is \( 1 - \frac{p - c}{\overline{F}(c)(\eta - c)} \) in the new problem. Hence the
limiting revenue with price $p \leq c + \bar{F}(c)(\eta - c)$ is

$$\lim_{N \to \infty} R_{TB}(c, p) = \lim_{N \to \infty} p \frac{\mathbb{E}[\text{selling volume}]}{N} - c \frac{\mathbb{E}[\text{selling volume}]}{N} - \lim_{N \to \infty} \frac{\mathbb{E}[\# \text{ of unique item with value } > c]}{N}$$

$$= \lim_{N \to \infty} (p - c) \frac{\mathbb{E}[\text{selling volume}]}{N} + \lim_{N \to \infty} c \frac{\bar{F}(c) \mathbb{E} [\tau(p)]}{N}$$

$$= (p - c) \log \frac{\bar{F}(c)(\eta - c)}{p - c} + c(\bar{F}(c) - \frac{p - c}{\eta - c}).$$

Maximizing over the price yields $p = c + e^{-\frac{\eta}{\eta - c}} \bar{F}(c)(\eta - c)$. Plugging in $p$ gives our desired revenue $\bar{F}(c)(\eta - c) \left( \frac{c}{\eta - c} + e^{-\frac{\eta}{\eta - c}} \right)$.

Finally, customer surplus is the total utility from the unique items that the customer keeps minus the total cost (i.e. revenue of the seller). Hence we have

$$\lim_{N \to \infty} \mathbb{E} [\text{Normalized Surplus}] = \lim_{N \to \infty} \frac{\mathbb{E}[\text{Utility from unique items with value } > c]}{N} - \lim_{N \to \infty} \frac{\mathbb{E}[\# \text{ of unique item with value } > c]}{N} - \lim_{N \to \infty} R_{TB}^c$$

$$= \lim_{N \to \infty} \eta \frac{\mathbb{E}[\# \text{ of unique item with value } > c]}{N} - \lim_{N \to \infty} \frac{\mathbb{E}[\# \text{ of unique item with value } > c]}{N} - \lim_{N \to \infty} \frac{\mathbb{E}[\# \text{ of unique item with value } > c]}{N} - \lim_{N \to \infty} R_{TB}^c$$

$$= \eta \left( \bar{F}(c) - \frac{p - c}{\eta - c} \right) - \bar{F}(c)(\eta - c) \left( \frac{c}{\eta - c} + e^{-\frac{\eta}{\eta - c}} \right)$$

$$= \bar{F}(c) \left( (\eta - c) - (2\eta - c)e^{-\frac{\eta}{\eta - c}} \right).$$

\vspace{10pt}

A.2 Omitted Examples

Example A.1 (Uniform unique box may not be optimal). Consider a unique box with two items facing a customer with non-uniform allocation probabilities. At the time of the first purchase, the customer has probability $q$ to receive item 1, and $1 - q$ to receive the item 2. Since it is a unique box, upon second purchase the customer will receive the unowned item with probability 1. Now suppose items are valued as either 0 or 1, with probability 0.5.

Consider a unique box with certain allocations, $q = 1$. In this case, the first purchase of a loot box always yields item 1, and the the second purchase then gives the remaining item 2. For this box,
the optimal price is 1, and the selling volume would be \( \frac{1}{4}(1 + 2) \), i.e., customers whose valuation is \((1,0)\) will buy 1 box, and those with valuation \((1,1)\) will buy 2 boxes. The resulting revenue is \( \frac{3}{4} \).

On the other hand, if \( q = 0.5 \) then we have a uniform unique box, and the corresponding optimal price can be checked to be 0.5. The the uniform allocation and corresponding price induce selling volume \( \frac{1}{4}(1.5 + 1.5 + 2) \), i.e., customers whose valuation is \((0,1)\) or \((1,0)\) will buy 1.5 boxes on average, and customers whose valuation is \((1,1)\) will buy 2 boxes. The resulting revenue is only \( \frac{5}{8} \).

**Example A.2** (No k-step look-ahead policy is optimal for customers). Let the number of products be \( N \) and the price of each unique box be 2.5. Now consider a customer whose realized valuations for the products are \( (N, \frac{N}{N-1}, \ldots, \frac{N}{N-1}) \). If the customer is myopic, then they will not buy a single unique box since the expected utility of the first loot box is \( \frac{N+(N-1)\frac{N}{N-1}}{N} - 2.5 = -0.5 < 0 \).

Now consider the following policy: purchase unique boxes until you obtain the item which is valued at \( N \). In expectation such a strategy requires \( \frac{N}{2} \) purchases and yields utility \( N + \frac{N}{2} - 1 - \frac{2.5N}{2} = \frac{N}{4} - 1 \). It is straightforward to show this policy is optimal given these parameters. Further, the expected utility for each of the first \( \approx \frac{N}{3} \) loot box purchases before obtaining the high valued item is negative since, if a customer has acquired \( q \) percent of the catalog without obtaining the high valued item, their expected utility for the next box is \( \frac{N+(1-q)(N-1)\frac{N}{N-1}}{(1-q)N} - 2.5 \) which is less than zero when \( q \leq \frac{1}{3} \). Thus no policy that considers only a fixed number of future purchases can be optimal.

### A.3 Omitted Figures

In this section we include additional figures depicting various forms of loot box selling in practice.
Figure A.1: Multi-item Loot Boxes in Online Games.

Note. Depicted is a multi-item loot box in the game NBA 2K20. Each box contains 5 cards.

Figure A.2: Loot Box with Multiple Classes.

Note. In the game PlayerUnknown’s Battlegrounds, the traditional box contains four classes of items: Mythic, Legendary, Epic, and Rare. The allocation probability varies across classes, however items within the same class have the same probability.
Figure A.3: Salvage System in Dota 2.

Note. In the game Dota 2, players can trade in 6 unwanted items for a new loot box plus 2000 shards, a form of in-game currency.
Appendix B: Matchmaking Strategies for Maximizing Player Engagement

B.1 Omitted Proofs

B.1.1 Omitted Proofs from Section 2.2

Proof of Proposition 2.1. Denote \( v_k = \{v_{kg}\}_{g \in \tilde{G}} \) as the vector of value (engagement) that 1 unit of players in level \( k \) can create. Note that \( v_{kg} \) is the average active time starting from state \( g \) in the absorbing Markov chain \( M_{kk} \) before absorption. By Theorem 3.2.4 in [154], we have

\[
v_k = (\tilde{M}_{kk} + \gamma \tilde{M}_{kk}^2 + \gamma^2 \tilde{M}_{kk}^3 + \cdots ) \mathbf{1} \\
= \left( \frac{1}{\gamma} \left( I + \gamma \tilde{M}_{kk} + \gamma^2 \tilde{M}_{kk}^2 + \cdots \right) - \frac{1}{\gamma} I \right) \mathbf{1} \\
= \gamma^{-1} \left( (I - \gamma \tilde{M}_{kk})^{-1} - I \right) \mathbf{1}.
\]

Note that by Theorem 3.2.1 in [154], \( (I - \gamma \tilde{M}_{kk})^{-1} \) always exists. Summing up players at all levels, we have

\[
V^{SBMM}(s^0) = \sum_{k=1}^{K} v_k^T s_k^0.
\]

\[\square\]

Proof of Lemma 2.1. The claim is obviously true for \( \gamma < 1 \), since the amount of active player is non-increasing over time. Hence, we only need to focus on the case when \( \gamma = 1 \), which we prove by induction.

For the base case, consider the engagement of level 1 players. Because their winrate is at most 0.5, a positive proportion of level 1 players would experience \( m + 1 \) consecutive losses and quit, for every \( m + 1 \) rounds. Let \( \epsilon > 0 \) be the probability that a player quits after \( m + 1 \) consecutive losses. After \( m + 1 \) rounds of matches, a level 1 player starting from any state has at least \( 2^{-(m+1)} \epsilon \)
probability to quit. Hence, a player’s engagement is bounded by

\[(m + 1)(1 - 2^{-(m+1)} \epsilon) + (m + 1)(1 - 2^{-(m+1)} \epsilon)^2 + (m + 1)(1 - 2^{-(m+1)} \epsilon)^3 + \cdots = \frac{(m + 1)(1 - 2^{-(m+1)} \epsilon)}{2^{-(m+1)} \epsilon}. \]  

(B.1)

The induction hypothesis is that the engagement is finite for players of level 1 to level k. We then show that the engagement is also finite for players of level \(k + 1\). From the induction hypothesis, the engagement from matches with players from levels 1 to k must be finite. Thus, we only need to consider engagement generated from matches with players of level \(k + 1\) to \(K\), and show it is also finite. Note that level \(k + 1\) players’ win rate is at most 0.5 since they are matched with players with at least the same skill level. Thus, following the exact same argument of the base case with players of level 1, we can show that a player’s engagement is bounded by the same expression in (B.1), which is finite.

□

Proof of Lemma 2.2. Our proof relies on Theorem 2.3 in [155], which requires showing that our LP formulation (2.2) satisfies the following five hypotheses. Let \(X \subseteq \mathbb{R}^n\) be a linear subspace. For an infinite primal LP

\[
V(P) = \sup \sum_{j=1}^{\infty} c_j x_j \quad \text{(B.2)}
\]

\[
\sum_{j=1}^{\infty} a_{ij} x_j = b_i, \quad i = 1, 2, \cdots \quad \text{(B.3)}
\]

\[
x_j \geq 0, \quad j = 1, 2, \cdots \quad \text{(B.4)}
\]

\[
x \in X, \quad \text{(B.5)}
\]

we assume for any \(x \in X\)

H1. \(\sum_{j=1}^{\infty} c_j x_j < \infty\).

H2. \(\sum_{j=1}^{\infty} a_{ij} x_j < \infty, \quad j = 1, 2, \cdots\).
Further, let $Y \subseteq \mathbb{R}^\mathbb{N}$ be the subset of all $y \in \mathbb{R}^\mathbb{N}$ such that

**H3.** $\sum_{i=1}^{\infty} b_i y_i < \infty$.

**H4.** For every $x \in X$, $\sum_{j=1}^{\infty} |a_{ij} x_j y_i|$ converges to some limit $L_i(x, y_i)$ for $i = 1, 2, \cdots$, and

**H5.** The above limits $L_i(x, y_i)$ have the property that $\sum_{i=1}^{\infty} L_i(x, y_i) < \infty$.

Then consider the dual problem

$$V(D) = \inf \sum_{i=1}^{\infty} b_i y_i$$

$$\sum_{i=1}^{\infty} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots$$

$$y \in Y.$$  \hfill (B.6) \hfill (B.7) \hfill (B.8)

By Theorem 2.3 in [155], suppose $x \in X$ and $y \in Y$ are feasible to the primal and dual problems and are complementary ($x_j (c_j - \sum_{i=1}^{\infty} a_{ij} y_i) = 0$ for all $j$). Then $x$ and $y$ are optimal solutions to the primal and dual problems, and $V(P) = V(D)$.

For our problem in (2.2), let $X$ to be the $l_1$ sequence space. Because of Lemma 2.1, choosing $X$ be the $l_1$ sequence space is without of generality. We check hypotheses H1 to H5, respectively. For dual variables, we only consider $y$ from $l_\infty$ space. As we will show, any $y$ from $l_\infty$ space satisfy H3 to H5, and we only use such $y$ in the following proofs.

Hypothesis H1 is satisfied because $X$ is the $l_1$ space. Hypothesis H2 and H4 are satisfied since we have finitely many primal variables in each constraint of problem 2.2 and $X$ is the $l_1$ space. Hypothesis H3 is satisfied since only the constraints associate with the initial period $(t = 0)$ leads to nonzero values (thus finite), and $y$ is in $l_\infty$ space. Finally, for hypothesis H5, let $A_i$ be the set of nonzero columns for row $i$. Note that we have

$$\sum_{i=1}^{\infty} L_i(x, y_i) = \sum_{i=1}^{\infty} \sum_{j \in A_i} |a_{ij} x_j y_i| \leq \sum_{i=1}^{\infty} \sup_{i \in \mathbb{N}} y_i \sum_{j \in A_i} x_j = \sup_{i \in \mathbb{N}} y_i \sum_{i=1}^{\infty} \sum_{j \in A_i} x_j \leq 2|\bar{G}| \sup_{i \in \mathbb{N}} y_i \sum_{j \in \mathbb{N}} |x_j| < \infty,$$
where the first inequality follows the fact $a_{ij} \in [0, 1]$ and $Y$ is $l_\infty$ space, the second inequality follows the fact that each primal variable $x_j$ appears in two periods, so it shows up in at most $2|\bar{G}|$ constraints in problem (2.2), and the last inequality follows from the fact that $X$ is the $l_1$ space.

\[\square\]

B.1.2 Omitted Proofs from Section 2.3

Before proving results in Section 2.3, we first present a simplified formulation to the matchmaker’s problem in (P). Recall that $\mathcal{P} := \{2w, 2\ell, 1w, 1\ell\}$. To begin, in any period $t = 0, 1, 2, \ldots$, matching flows $f_{2w,2\ell}^t, f_{1w,1\ell}^t, f_{2\ell,2w}^t,$ and $f_{1\ell,1w}^t$ can be set to zero without loss of generality. Taking flows $f_{2w,2\ell}^t = f_{1w,1\ell}^t = a \in (0, \min\{s_{2w}^t, s_{1\ell}^t\})$ as an example, it can be represented by $f_{2w,2w}^t = a$ and $f_{2\ell,2\ell}^t = a$, since they induce the same evolution to players’ demographics $\{s_{2w}^{t+1}, s_{2\ell}^{t+1}\}$ in the next period. We also use the fact that $f_{i,j}^t = f_{j,i}^t, \forall i, j \in \mathcal{P}$ to reduce the problem to 8 flow variables for each period, which is half of the original described in (P). Thus, we can rewrite the
matchmaker’s problem in (P) as

$$\max_{\{f_t\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} \left( s'_{2w} + s'_{2\ell} + s'_{1w} + s'_{1\ell} \right)$$

s.t. $s^0_{2w} = f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell}$,

$s^0_{2\ell} = f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell}$,

$s^0_{1w} = f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w}$,

$s^0_{1\ell} = f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell}$,

$s^t_{2w} = f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell}, \ t = 1, 2, \ldots$ (FB)

$s^t_{2\ell} = f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell}, \ t = 1, 2, \ldots$

$s^t_{1w} = f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w}, \ t = 1, 2, \ldots$

$s^t_{1\ell} = f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell}, \ t = 1, 2, \ldots$

$s^t_{2w} = \frac{1}{2} \left( f^{t-1}_{2w,2w} + f^{t-1}_{2w,2\ell} \right) + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1\ell}, \ t = 1, 2, \ldots$ (ED)

$s^t_{2\ell} = \frac{1}{2} f^{t-1}_{2w,2\ell}, \ t = 1, 2, \ldots$

$s^t_{1w} = \frac{1}{2} \left( f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1w} \right), \ t = 1, 2, \ldots$

$s^t_{1\ell} = \frac{1}{2} f^{t-1}_{1w,1\ell} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1\ell}, \ t = 1, 2, \ldots$

$f^t_{i,j} \geq 0, \forall i, j, t \in \mathcal{P}.$

By merging the flow balance and evolution of demographic constraints, we can further remove
$s_t^j$ for $t > 1$, with only 8 decision variables per period:

$$\max_{\{F\}_{t=0}^\infty} \sum_{t=0}^\infty \left( f_{2w,2w}^t + \frac{1}{2} f_{2\ell,2\ell}^t + f_{1w,1w}^t + \frac{1}{2} f_{1\ell,1\ell}^t + 2 f_{2w,1w}^t + f_{2\ell,1\ell}^t + f_{2w,1\ell}^t + 2 f_{2\ell,1\ell}^t \right)$$

(P')

s.t.

$$s_{2w}^0 = f_{2w,2w}^0 + f_{2w,1w}^0 + f_{2w,1\ell}^0,$$
$$s_{2\ell}^0 = f_{2\ell,2\ell}^0 + f_{2\ell,1w}^0 + f_{2\ell,1\ell}^0,$$
$$s_{1w}^0 = f_{1w,1w}^0 + f_{2\ell,1w}^0 + f_{2w,1w}^0,$$
$$s_{1\ell}^0 = f_{1\ell,1\ell}^0 + f_{2w,1\ell}^0 + f_{2\ell,1\ell}^0,$$
$$f_{2w,2w}^t + f_{2w,1w}^t + f_{2w,1\ell}^t = \frac{1}{2} \left( f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1} \right) + f_{2w,1w}^{t-1} + f_{1w,1\ell}^{t-1} + f_{2\ell,1\ell}^{t-1} + f_{2w,1\ell}^{t-1}, \quad t = 1, \ldots,$$
$$f_{2\ell,2\ell}^t + f_{2\ell,1w}^t + f_{2\ell,1\ell}^t = \frac{1}{2} f_{2w,2w}^{t-1}, \quad t = 1, 2, \ldots,$$
$$f_{1w,1w}^t + f_{2\ell,1w}^t + f_{2w,1w}^t = \frac{1}{2} \left( f_{1w,1w}^{t-1} + f_{1\ell,1\ell}^{t-1} \right), \quad t = 1, 2, \ldots,$$
$$f_{1\ell,1\ell}^t + f_{2\ell,1\ell}^t + f_{2w,1\ell}^t = \frac{1}{2} \left( f_{1w,1w}^{t-1} + f_{2w,1w}^{t-1} + f_{2\ell,1\ell}^{t-1} \right), \quad t = 1, 2, \ldots,$$
$$f_{i,j}^t \geq 0, \quad \forall i, j, t \in \mathcal{P}.$$

Denote $\lambda_t^i$ for $i \in \mathcal{P}$ as the dual variables (shadow price) for each demographic in period.
\[ t = 0, 1, 2, \ldots \] Then we can write the dual problem of \((P')\) as

\[
\min_{\{\lambda^t\}} \sum_{t \in P} s^t_i \lambda^0_i \quad \text{(D')} 
\]

s.t.

\[
\begin{align*}
1 & \leq \lambda^t_{2w} - \frac{1}{2} \lambda^{t+1}_{2w} - \frac{1}{2} \lambda^{t+1}_{2\ell}, \quad t = 0, 1, 2, \ldots, \\
\frac{1}{2} & \leq \lambda^t_{2\ell} - \frac{1}{2} \lambda^{t+1}_{2w}, \quad t = 0, 1, 2, \ldots, \\
2 & \leq \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \quad t = 0, 1, 2, \ldots, \\
\frac{1}{2} & \leq \lambda^t_{1\ell} - \frac{1}{2} \lambda^{t+1}_{1w}, \quad t = 0, 1, 2, \ldots, \\
2 & \leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \quad t = 0, 1, 2, \ldots, \\
1 & \leq \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \quad t = 0, 1, 2, \ldots, \\
1 & \leq \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w}, \quad t = 0, 1, 2, \ldots, \\
1 & \leq \lambda^t_{1w} - \frac{1}{2} \lambda^{t+1}_{1\ell} - \frac{1}{2} \lambda^{t+1}_{2w}, \quad t = 0, 1, 2, \ldots.
\end{align*}
\]

**Proof of Lemma 2.3.** We work with the alternative formulation in \((P')\) which has 8 flow variables each period.

(i) Let \(f^t = (f^t_{2w,2w}, f^t_{2\ell,2\ell}, f^t_{1w,1w}, f^t_{1\ell,1\ell}, f^t_{2w,1w}, f^t_{2\ell,1\ell}, f^t_{2w,1\ell}, f^t_{2\ell,1\ell})\) be the flow vector at time \(t\). With \((FB_S)\) conditions, the state \(s^t\) can then be expressed as \(BF^t\), where

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]
Similarly, using the (EDₜ) conditions, the state \( s^{i+1} \) can be expressed as \( Af^t \), where

\[
A = \begin{pmatrix}
    f'_{2w,2w} & f'_{2\ell,2\ell} & f'_{1w,1w} & f'_{1\ell,1\ell} & f'_{2w,1\ell} & f'_{2\ell,1w} & f'_{2w,1\ell} & f'_{2\ell,1w} \\
    s'_{2w} & 0.5 & 0.5 & 0 & 1 & 1 & 1 & 1 \\
    s'_{2\ell} & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
    s'_{1w} & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\
    s'_{1\ell} & 0 & 0 & 0.5 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

If there exists a decaying steady state for some \( c \in [0, 1] \), then there exists vector \( f^t \geq 0 \) such that

\[ Af^t = cBf^t \iff (A - cB)f^t = 0. \]

Now we provide the null space of \( A - cB \). When \( c \neq (1 + \sqrt{5})/4 \), the null space is given by

\[
\begin{pmatrix}
    0.5c/g(c) & (-c^2 + 0.5c + 0.5)/g(c) & -0.5c/g(c) & (-c^2 + 0.5c)/g(c) & 0 & 0 & 0 & 1 \\
    0.5c/g(c) & (-c^2 + 0.5c + 0.5)/g(c) & (-c^2 + 0.5)/g(c) & (0.5c - 0.5)/g(c) & 0 & 1 & 0 & 0 \\
    (-c^2 + c)/g(c) & (-0.5c + 0.5)/g(c) & -0.5c/g(c) & (-c^2 + 0.5c)/g(c) & 0 & 1 & 0 & 0 \\
    (-c^2 + c)/g(c) & (-0.5c + 0.5)/g(c) & (-c^2 + 0.5)/g(c) & (0.5c - 0.5)/g(c) & 1 & 0 & 0 & 0
\end{pmatrix}
\]

where \( g(c) = c^2 - 0.5c - 0.25 \). Note that \( g(c) = 0 \) if \( c = (1 + \sqrt{5})/4 \). In that case, the null space of \( A - cB \) is given by

\[
\begin{pmatrix}
    \frac{1+\sqrt{5}}{2} & 0 & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{2}{3} & 0 & 1 \\
    \frac{1+\sqrt{5}}{2} & 0 & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{-5+\sqrt{5}}{3-\sqrt{5}} & 1 & 1 \\
    0 & 0 & \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 \\
    \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

If there exists a linear combination of rows in the null space resulting in all non-negative elements and at least one non-zero element, then we have found a valid flow vector \( f^t \) and thus a valid demographic \( s^t \) that decays steadily at a rate of \( c \).

First, consider \( c \in \left(\frac{1+\sqrt{5}}{4}, 1\right] \), which implies \( g(c) > 0 \). Then observe that elements in third
column of Eq. (B.9), representing flow $f^1_{1w,1w}$ are all negative. Hence, for any linear combination of rows in Eq. (B.9), as long as the flow $f^1_{1w,1w}$ is positive, at least one of the element representing flows $f^1_{2w,1w}, f^1_{2w,1ℓ}, f^1_{2f,1w}, f^1_{2f,1ℓ}$ is negative. Thus, no steady state exists when $c \in (\frac{1+\sqrt{5}}{4}, 1]$.

Next, consider $c \in (0, \frac{1+\sqrt{5}}{4})$, which implies $g(c) < 0$. Note that elements in the second column of Eq. (B.9), representing the flow $f^1_{2f,2ℓ}$ are all positive. Hence, for any linear combination of rows in Eq. (B.9), as long as the flow $f^1_{2f,2ℓ}$ is positive, then at least one of the element representing flows $f^1_{2w,1w}, f^1_{2w,1ℓ}, f^1_{2f,1w}, f^1_{2f,1ℓ}$ is negative. Thus, no steady state exists when $c \in (0, \frac{1+\sqrt{5}}{4})$.

Finally, consider $c = (1 + \sqrt{5})/4$. In this case, we can easily find a non-negative flow vector $f'$ by summing up the third and fourth row, which gives $\left(\frac{1+\sqrt{5}}{2}, 1, \frac{1+\sqrt{5}}{2}, 1, 0, 0, 0, 0, 0, 0\right)$, representing SBMM, and any positive multiple of $s = ((1 + \sqrt{5})/2, 1, (1 + \sqrt{5})/2, 1)$ can induce such flows. To see that no other policy can reach a steady state when $c = (1 + \sqrt{5})/4$, note that we have to find a non-zero non-negative flow vector $f'$ with the use of the first two rows. For the first row, the sign of fifth entry is opposite with the eighth entry; for the second row, the sign of fifth entry is opposite with the sixth and seventh entry. Hence, there is no way to construct a feasible flow vector $f'$ with non-negative elements. Thus, no other policy besides SBMM can induce a decaying steady state when $c = (1 + \sqrt{5})/4$. □

Proof of Lemma 2.4. For each flow that involves both high-skilled and low-skilled player, we can compare the outcome of one unit of such flow with SBMM flow that uses the same amount of players. A unit of $f^1_{2f,1w}$ uses the same amount of players as one unit of $f^1_{2f,2f}$ and $f^1_{1w,1w}$, but it leads to zero loss in the next period while SBMM loses a half unit of $2f$ players. One unit of $f^1_{2f,1f}$ or $f^1_{2w,1w}$ leads to the same losses in the next period compared to the SBMM flow of one unit of $f^1_{2f,2f}, f^1_{1f,1f}, f^1_{2w,2w}$, and $f^1_{1w,1w}$. Finally, one unit of $f^1_{2w,1f}$ leads to a one unit loss of $1f$ players in the next period, while the SBMM flow of one unit of $f^1_{1f,1f}$ and $f^1_{2w,2w}$ only leads only leads to a half unit loss of $1f$ players. Hence, to maximize the population in the next period, we should always maximize $f^1_{2f,1w}$ and set $f^1_{2w,1f} = 0$. For the rest of players, they can be matched arbitrarily as the outcome in the next period is the same as SBMM. □

Proof of Proposition 2.2. We solve the problem in (P’) by considering its dual problem (D’). In
each period, there are four constraints in the primal problem (P'). Thus, we assign dual variable $\lambda_t^i$, where $i \in \{1w, 1\ell, 2w, 2\ell\}$, to each constraint representing the evolution of a players’ demographics group in the primal problem. We will fully characterize the transition of primal and dual sequences, and show optimality by checking primal/dual feasibility and complementary slackness.

We break down the rest of this proof into 7 steps. In each step, we analyze a scenario corresponding to a parameter regime, which is mutually exclusive to parameter regimes in other scenarios and collectively exhaustive. That is, in any period $t$, we have

- **Scenario 1**: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, and $s^t_{2w} \geq s^t_{1\ell}$; The optimal matching flows are:
  \[ f^t_{2w,2w} = s^t_{2w}, f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1w}, f^t_{1w,1w} = 0, f^t_{1\ell,1\ell} = s^t_{1\ell}, \text{ and } f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}; \]

- **Scenario 2**: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} < s^t_{1w}$, and $s^t_{2w} \geq s^t_{1\ell}$; The optimal matching flows are:
  \[ f^t_{2w,2w} = s^t_{2w}, f^t_{2\ell,2\ell} = 0, f^t_{1w,1w} = s^t_{1w} - s^t_{2\ell}, f^t_{1\ell,1\ell} = s^t_{1\ell}, \text{ and } f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{2\ell}; \]

- **Scenario 3**: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, and $s^t_{2w} < s^t_{1\ell}$; The optimal matching flows are the same as those in Scenario 1;

- **Scenario 4**: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$ and $s^t_{2\ell} < s^t_{1w}$; The optimal matching flows are the same as those in Scenario 2;

- **Scenario 5**: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $s^t_{1\ell} \leq K_1 = \frac{18}{5}s^t_{2w} + \frac{9}{5}s^t_{2\ell} + \frac{3}{5}s^t_{1w};$
  The optimal matching flows are the same as those in Scenario 1;

- **Scenario 6**: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} < s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $s^t_{1\ell} > K_2 = \frac{18}{5}s^t_{2w} + \frac{23}{14}s^t_{2\ell} - \frac{11}{5}s^t_{1w};$
  The optimal matching flows are:
  \[ f^t_{2w,2w} = s^t_{2w}, f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}, f^t_{1\ell,1\ell} = f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1w}, \text{ and } f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}; \]

- **Scenario 7**: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $K_1 < s^t_{1\ell} \leq K_2$; The optimal matching flows are:
  \[ f^t_{2w,2w} = s^t_{2w}, f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}, f^t_{2\ell,2\ell} = \frac{9}{7}s^t_{2w} + \frac{23}{14}s^t_{2\ell} - \frac{11}{5}s^t_{1w} - \frac{5}{14}s^t_{1\ell}, \]
  \[ f^t_{2\ell,1\ell} = f^t_{1\ell,2\ell} = \frac{5}{14}s^t_{1\ell} - \frac{9}{14}s^t_{2w} - \frac{9}{14}s^t_{2\ell} - \frac{3}{14}s^t_{1w}, \text{ and } f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}. \]

Note that Scenarios 1 to 5 correspond to the third row of Table 2.1. In such cases, the optimal policy simply maximize the matching between $2\ell$ and $1w$, and do skill-based matching for the
remaining players. Scenario 6 to 7 corresponds to the first two rows in Table 2.1. The reason we classify the aforementioned scenarios in this way is because of how the scenarios evolve over time, as shown in Figure B.1. For example, at any period $t$, if we are in Scenario 1, then in the next period $t + 1$, one can verify that we always stay in the parameter regime of Scenario 1 under the proposed matching policy. The dual variables for any state in Scenario 1 is constant over time. Similarly, once we reach Scenario 2 at time $t$, we always transfer to Scenario 1 at $t + 1$.

For the rest of this proof, we show that the state evolves as Fig. B.1, and our proposed matching policy is optimal in each scenario and its subsequent scenarios as the state demographics evolve. To be more specific, our proof goes through each scenario and shows the corresponding optimal matching flows in Table 2.1 are optimal. Note that by construction, the proposed matching policy in Table 2.1 is primal feasible, i.e., satisfying all constraints in (FB$_S$) and (ED$_S$). We establish optimality by constructing dual variables for each scenario and show that both complementary slackness and dual feasibility conditions hold (the primal feasibility can be easily verified for our proposed policy). When possible, we write the corresponding dual variables with closed-form expressions. Our proof starts with Scenario 1 as it is at the end of all transitions (does not transition to other scenarios), then works backwards to parental scenarios. For each parental scenarios, we readily construct the dual variables for all subsequent scenarios, representing all the following time periods. For example, when the demographic state is in Scenario 2, we show it will transition to Scenario 1 in one period under the proposed matching policy. Then we only need to construct the
and $f_\ell$ as the complementary conditions corresponding to primal non-zero variables $f_\ell$. Therefore, we have

**Scenario 1**: $s_{2w}^\ell + s_{2\ell}^\ell \geq s_{1w}^\ell + s_{1\ell}^\ell$, $s_{2w}^\ell \geq s_{1w}^\ell$, and $s_{2w}^\ell \geq s_{1\ell}^\ell$; The optimal matching flows are:

\[
\begin{align*}
&f_{2w,2w}^\ell = s_{2w}^\ell, \quad f_{2\ell,2\ell}^\ell = s_{2\ell}^\ell - s_{1w}^\ell, \quad f_{1w,1\ell}^\ell = 0, \quad f_{1\ell,1\ell}^\ell = s_{1\ell}^\ell, \quad \text{and} \quad f_{2\ell,1w}^\ell = f_{1w,2\ell}^\ell = s_{1w}^\ell.
\end{align*}
\]

We first consider Scenario 1, and prove that it is the “end” of all scenarios in Figure B.1. In other words, we shall show that once the state of players’ demographics falls in Scenario 1, it will remain in Scenario 1. To see this, for some $s^\ell = \{s_{1w}^\ell, s_{1\ell}^\ell, s_{2w}^\ell, s_{2\ell}^\ell\}$ in Scenario 1, following the proposed policy, the state at $t + 1$ is given by $s_{1w}^{t+1} = \frac{1}{2}s_{1w}^t$, $s_{1\ell}^{t+1} = s_{1\ell}^t$, $s_{2w}^{t+1} = \frac{1}{2}(s_{2w}^t + s_{2\ell}^t + s_{1w}^t)$, and $s_{2\ell}^{t+1} = \frac{1}{2}s_{2\ell}^t$. Then $s_{2\ell}^{t+1} \geq s_{1w}^{t+1}$ because $s_{2w}^{t+1} \geq s_{1w}^{t+1}$, and $s_{2w}^{t+1} \geq s_{1\ell}^{t+1}$ because $s_{2w}^{t+1} - s_{1\ell}^{t+1} = \frac{1}{2}(s_{2w}^t + s_{2\ell}^t - s_{1w}^t)$ and $s_{2\ell}^t - s_{1w}^t \geq 0$. Hence, $s^{t+1}$ still belongs to Scenario 1.

Next, we show that the proposed policy in Proposition 2.2 is optimal for all subsequent periods once players’ demographics satisfy Scenario 1. The optimal solution in Proposition 2.2 suggests that in Scenario 1, only 4 variables are non-zero while all other flows are zero in each period. Therefore, we have

\[
\begin{align*}
1 &= \lambda_{2w}^\ell - \frac{1}{2}\lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{2\ell}^{t+1}, \\
\frac{1}{2} &= \lambda_{2\ell}^t - \frac{1}{2}\lambda_{2w}^{t+1}, \\
2 &= \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, \\
\frac{1}{2} &= \lambda_{1\ell}^t - \frac{1}{2}\lambda_{1w}^{t+1},
\end{align*}
\]

as the complementary conditions corresponding to primal non-zero variables $f_{2w,2w}^t, f_{2\ell,2\ell}^\ell, f_{2\ell,1w}^\ell$, and $f_{1\ell,1\ell}^\ell$, respectively, and have

\[
\begin{align*}
2 &\leq \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, \\
1 &\leq \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \\
1 &\leq \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \\
1 &\leq \lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1},
\end{align*}
\]

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as the dual feasibility conditions corresponding to variables, \( f_{2w,1w}^t, f_{2w,1\ell}^t, f_{2\ell,1\ell}^t, \) and \( f_{1w,1w}^t \), that are zero in the primal problem, respectively.

The following dual solutions:

\[
\lambda_{2w}^t = 5, \quad \lambda_{2\ell}^t = 3, \quad \lambda_{1w}^t = 9, \quad \lambda_{1\ell}^t = 5, \quad \forall t, \tag{B.10}
\]

satisfies complementary slackness in (CS1) and feasibility conditions in (DF1). Thus, the proposed policy in Proposition 2.2 is optimal once the players’ demographics fall in scenario 1.

**Scenario 2:** \( s_{2w}^t + s_{2\ell}^t \geq s_{1w}^t + s_{1\ell}^t, \) \( s_{2w}^t < s_{1w}^t, \) and \( s_{2w}^t \geq s_{1\ell}^t \). The optimal matching flows are:

\[
f_{2w,2w}^t = s_{2w}^t, \quad f_{2w,2\ell}^t = 0, \quad f_{1w,1w}^t = s_{1w}^t - s_{2w}^t, \quad f_{1\ell,1\ell}^t = s_{1\ell}^t, \quad \text{and} \quad f_{1w,2\ell}^t = f_{1w,2\ell}^t = s_{1\ell}^t.
\]

In the second step, we consider Scenario 2, which will transit to Scenario 1 after matching under the purposed policy as we have stated in Figure B.1. To see that, for some \( s^t = \{s_{1w}^t, s_{1\ell}^t, s_{2w}^t, s_{2\ell}^t\} \) in Scenario 2, following the proposed policy, the state at \( t+1 \) is given by \( s_{1w}^{t+1} = \frac{1}{2} (s_{1w}^t + s_{1\ell}^t - s_{2w}^t), \) \( s_{1\ell}^{t+1} = \frac{1}{2} (s_{1\ell}^t + s_{2w}^t), \) \( s_{2w}^{t+1} = \frac{1}{2} s_{2w}^{t+1} + s_{2\ell}^t, \) and \( s_{2\ell}^{t+1} = \frac{1}{2} s_{2\ell}^t + s_{2w}^t \). Then \( s_{1w}^{t+1} \geq s_{1w}^{t+1} \) because \( s_{2\ell}^{t+1} - s_{1w}^{t+1} = \frac{1}{2} (s_{2w}^t + s_{2\ell}^t - s_{1w}^t) \geq 0 \). Also, \( s_{2w}^{t+1} \geq s_{1\ell}^{t+1} \) because \( s_{2w}^{t+1} - s_{1\ell}^{t+1} = \frac{1}{2} (s_{2w}^t + s_{2\ell}^t - s_{1w}^t) \geq s_{1\ell}^t \geq 0. \) Hence, \( s_{1\ell}^{t+1} \) belongs to Scenario 1.

Therefore, we only need to show that in any period \( t \) such that players’ demographics satisfy Scenario 2, we can find solutions to dual variables, induced by the proposed policy in Proposition 2.2, which satisfy the complementary slackness conditions and dual feasibility conditions. Note that in period \( t \) under Scenario 2, the non-zero primal variables are \( f_{2w,2w}^t, f_{2w,1\ell}^t, f_{1w,1w}^t, \) and \( f_{1\ell,1\ell}^t \). Therefore, by taking out the condition for \( f_{2w,2\ell}^t \) and replacing it with the one for \( f_{1w,1w}^t \), the
complementary slackness conditions in (CS$_1$) change to

\[
\begin{align*}
1 &= \lambda^t_{2w} - \frac{1}{2} \lambda^{t-1}_{2w} - \frac{1}{2} \lambda^{t+1}_{2w}, \\
2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
1 &= \lambda^t_{1w} - \frac{1}{2} \lambda^{t+1}_{1w} - \frac{1}{2} \lambda^{t+1}_{1\ell}, \\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2} \lambda^{t+1}_{1w},
\end{align*}
\]

(CS$_2$)
in period $t$. Similarly, by taking out the condition for $f^t_{1w,1w}$ and replacing with the one for $f^t_{2\ell,1\ell}$, the dual feasibility conditions in (DF$_1$) turn into

\[
\begin{align*}
2 &\leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
1 &\leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\
\frac{1}{2} &\leq \lambda^t_{2\ell} - \frac{1}{2} \lambda^{t+1}_{2w}, \\
1 &\leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w},
\end{align*}
\]

(DF$_2$)
in period $t$. Note that the complementary slackness conditions and the dual feasibility conditions switch back to those in (CS$_1$) and (DF$_1$), starting period $t+1$, as player’s demographics transit into Scenario 1. Therefore, we have

\[
\lambda^t_{2w} = 5, \ \lambda^t_{2\ell} = 3, \ \lambda^t_{1w} = 9, \text{ and } \lambda^t_{1\ell} = 5, \quad \forall s = t + 1, \ldots, T - 1,
\]

from (B.10) and only need to find $\lambda^t_i$, where $i \in \{2w, 2\ell, 1w, 1\ell\}$, satisfy conditions in (CS$_2$) and (DF$_2$). Indeed, such solutions exist and one can verify that

\[
\lambda^t_{2w} = 5, \ \lambda^t_{2\ell} = 4, \ \lambda^t_{1w} = 8, \text{ and } \lambda^t_{1\ell} = 5,
\]

(B.11)
are the desired solution. Therefore, the proof for Scenario 2 is completed.
Scenario 3: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2w} \geq s^t_{1w}$, and $s^t_{2w} < s^t_{1\ell}$; The optimal matching flows are:

$$f^t_{2w,2w} = s^t_{2w}, f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1\ell}, f^t_{1w,1w} = 0, f^t_{1\ell,1\ell} = s^t_{1\ell}, \text{ and } f^t_{2\ell,1w} = f^t_{2w,1\ell} = s^t_{1w};$$

In the third step, we consider Scenario 3, which shall transit to Scenario 2 after matching is done under the purposed policy. To see this, for some $s^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 1, following the proposed policy, the state at $t + 1$ is given by $s^{t+1}_{2w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w}$, $s^{t+1}_{1w} = \frac{1}{2}s^t_{1w}$, and $s^{t+1}_{1\ell} = s^t_{1\ell}$. Then $s^{t+1}_{2\ell} < s^{t+1}_{1w}$ because $s^t_{2w} < s^t_{1\ell}$, and $s^{t+1}_{2w} \geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w})$ and $s^t_{2\ell} - s^t_{1\ell} \geq 0$. Finally, $s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell} = \frac{1}{2}(2s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) \geq 0$. Hence, $s^{t+1}$ belongs to Scenario 2.

Thus, we only need to check we can find solutions to dual variables, which satisfy the complementary slackness conditions and dual feasibility conditions. According to the policy in Proposition 2.2, in period $t$ under Scenario 3, primal variables $f^t_{2w,2w}, f^t_{2\ell,2\ell}, f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$ are non-zero. Thus, in period $t$, the complementary slackness and dual feasibility conditions are

$$\begin{align*}
1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell}, \\
\frac{1}{2} &= \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w}, \\
2 &= \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w}, \quad (CS_3)
\end{align*}$$

and

$$\begin{align*}
2 &\leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
1 &\leq \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\
1 &\leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\
1 &\leq \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}\lambda^{t+1}_{1\ell}, \quad (DF_3)
\end{align*}$$

respectively, where

$$\lambda^{t+1}_{2w} = 5, \lambda^{t+1}_{2\ell} = 4, \lambda^{t+1}_{1w} = 8, \text{ and } \lambda^{t+1}_{1\ell} = 5.$$
are from (B.11), since players’ demographic shall transit to Scenario 2 in the next period.

Finally, one can verify that

\[
\lambda'_{2w} = 5.5, \lambda'_{2f} = 3, \lambda'_{1w} = 9, \text{ and } \lambda'_{1f} = 4.5,
\]  

(B.12)

are the desired solutions we are looking for, satisfying (CS\(_3\)) and (DF\(_3\)). This completes the proof for Scenario 3.

**Scenario 4:** \(s'_{2w} + s'_{2f} < s'_{1w} + s'_{1f}\) and \(s'_{2f} < s'_{1w}:\) The optimal matching flows are: \(f'_{2w,2w} = s'_{2w},\) \(f'_{2f,2f} = 0,\) \(f'_{1w,1w} = s'_{1w} - s'_{2f},\) \(f'_{1f,1f} = s'_{1f},\) and \(f'_{2f,1w} = f'_{1w,2f} = s'_{2f}.

In the fourth step, we consider Scenario 4. For some \(s' = \{s'_{1w}, s'_{1f}, s'_{2w}, s'_{2f}\}\) in Scenario 4, following the proposed policy, the state at \(t + 1\) is given by \(s'_{1w}^{t+1} = \frac{1}{2}(s'_{1w}^t + s'_{1f}^t - s'_{2f}^t), s'_{1f}^{t+1} = \frac{1}{2}(s'_{2f}^t + s'_{1w}^t), s'_{2w}^{t+1} = \frac{1}{2}s'_{2w}^t + s'_{2f}^t,\) and \(s'_{2f}^{t+1} = \frac{1}{2}s'_{2w}^t + \frac{1}{2} s'_{1w}^t\). Then \(s'_{2w}^{t+1} < s'_{2w}^t\) because \(s'_{1w}^{t+1} - s'_{2f}^{t+1} = \frac{1}{2}(s'_{1w}^t + s'_{1f}^t - s'_{2f}^t) - s'_{2f}^t > 0.\) Hence, the state shall either transit to Scenario 2 or stay in Scenario 4 after a match. Suppose at time \(t = s,\) players’ demographic is in Scenario 4. Denote \(\tau := \min\{t \geq s \mid s'_{2w}^t + s'_{2f}^t \geq s'_{1w}^t + s'_{1f}^t\},\) representing the time period players’ demographic transit to Scenario 2. First, we argue that \(\tau < \infty.\) Suppose otherwise, we have \(\tau \to \infty,\) which implies that the players’ demographic stays in Scenario 4 forever. Note that as long as players’ demographic belongs to Scenario 4, no high-skilled players shall depart from the matching system since there are always enough low-skilled players to be matched with. Thus, we have

\[
\sum_{t=s}^{\infty} (s'_{2w}^t + s'_{2f}^t) = \sum_{t=s}^{\infty} (s'_{2w}^t + s'_{2f}^t) \to \infty,
\]

which contradicts Lemma 2.1. Therefore, we have \(\tau < \infty.\)

Next, under Scenario 2, according to the optimal policy in Proposition 2.2, we have \(\{f'_{2w,2w}\}_{t=s}^{\tau-1},\) \(\{f'_{2f,1w}\}_{t=s}^{\tau-1},\) \(\{f'_{1w,1w}\}_{t=s}^{\tau-1},\) and \(\{f'_{1f,1f}\}_{t=s}^{\tau-1}\) as the sequences that contain non-zero primal variables, whereas each elements in sequences \(\{f'_{2w,1w}\}_{t=s}^{\tau-1},\) \(\{f'_{2w,1f}\}_{t=s}^{\tau-1},\) \(\{f'_{2f,2f}\}_{t=s}^{\tau-1},\) and \(\{f'_{2f,1f}\}_{t=s}^{\tau-1}\) are zero.
\[
\begin{align*}
\{\lambda_{2w}^t\}_{t=s}^\tau, \{\lambda_{2\ell}^t\}_{t=s}^\tau, \{\lambda_{1w}^t\}_{t=s}^\tau, \text{ and } \{\lambda_{1\ell}^t\}_{t=s}^\tau, \text{ such that } & \\
\lambda_{2w}^\tau = 5, \ & \lambda_{2\ell}^\tau = 4, \ & \lambda_{1w}^\tau = 8, \ & \lambda_{1\ell}^\tau = 5, \\
\text{and for all } s \leq t \leq \tau - 1, \text{ we have } & \\
1 = \lambda_{2w}^t - \frac{1}{2}\lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{2\ell}^{t+1}, & \lambda_{2w}^t = 1 + \frac{1}{2}\lambda_{2w}^{t+1} + \frac{1}{2}\lambda_{2\ell}^{t+1} \\
2 = \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, & \lambda_{2\ell}^t = 1 + \lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{1w}^{t+1} + \frac{1}{2}\lambda_{1\ell}^{t+1} \\
1 = \lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1}, & \lambda_{1w}^t = 1 + \frac{1}{2}\lambda_{2w}^{t+1} + \frac{1}{2}\lambda_{2\ell}^{t+1} \\
\frac{1}{2} = \lambda_{1\ell}^t - \frac{1}{2}\lambda_{1\ell}^{t+1}, & \lambda_{1\ell}^t = \frac{1}{2} + \lambda_{1w}^{t+1} \\
\end{align*}
\]

as complementary slackness conditions and

\[
\begin{align*}
2 \leq \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, \\
1 \leq \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \\
\frac{1}{2} \leq \lambda_{2\ell}^t - \frac{1}{2}\lambda_{2w}^{t+1}, \\
1 \leq \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \\
\end{align*}
\]

as dual feasibility conditions.

For convenience, we also rearrange (CS\textsubscript{4}) forwardly:

\[
\begin{align*}
\lambda_{2w}^{t+1} &= \lambda_{2\ell}^t - \lambda_{1w}^t + 2\lambda_{1\ell}^t - 1, \quad \text{(B.14)} \\
\lambda_{2\ell}^{t+1} &= 2\lambda_{2w}^t - \lambda_{2\ell}^t + \lambda_{1w}^t - 2\lambda_{1\ell}^t - 1, \quad \text{(B.15)} \\
\lambda_{1w}^{t+1} &= 2\lambda_{1\ell}^t - 1, \quad \text{(B.16)}
\end{align*}
\]
\[
\lambda_{1\ell}^{t+1} = 2\lambda_{1w}^{t} - 2\lambda_{1\ell}^{t} - 1. \tag{B.17}
\]

Equation (B.16) follows the third equation of (CS4). Equation (B.17) follows the last equation in (CS4) and (B.16). Equation (B.14) follows the second equation in (CS4), (B.16), and (B.17). Finally, (B.15) follows the first equation in (CS4) and (B.15).

Taking (B.14)-(B.17) into \(DF_4\), we can rewrite the dual feasibility conditions as

\[
0 \leq \lambda_{2w}^{t} - \lambda_{2\ell}^{t}, \tag{B.18}
\]

\[
0 \leq \lambda_{2w}^{t} - \lambda_{2\ell}^{t} + \lambda_{1w}^{t} - \lambda_{1\ell}^{t}, \tag{B.19}
\]

\[
0 \leq \lambda_{2\ell}^{t} + \lambda_{1w}^{t} - 2\lambda_{1\ell}^{t}, \tag{B.20}
\]

\[
0 \leq \lambda_{1w}^{t} - \lambda_{1\ell}^{t}. \tag{B.21}
\]

Note that (B.19) is implied by summing up (B.18) and (B.21). For the rest of this step, we show that the updated dual feasibility conditions (B.18), (B.20),(B.21) are satisfied for all \(s \leq t \leq \tau\). We will prove a stronger result: (B.18), (B.20),(B.21) together with the following (B.22) are satisfied for all \(s \leq t \leq \tau\).

\[
0 \leq -\lambda_{2w}^{t} + \lambda_{2\ell}^{t} + \lambda_{1w}^{t} - \lambda_{1\ell}^{t}, \quad \text{and} \quad -1 \leq \lambda_{2w}^{t} - \lambda_{1w}^{t} + \lambda_{1\ell}^{t}. \tag{B.22}
\]

We prove this result by backwards induction with the help of the following lemma.

**Lemma B.1.** Consider the dual sequences \(\{\lambda_{2w}^{t}\}_{t=s}^{\tau}, \{\lambda_{2\ell}^{t}\}_{t=s}^{\tau}, \{\lambda_{1w}^{t}\}_{t=s}^{\tau}, \text{ and } \{\lambda_{1\ell}^{t}\}_{t=s}^{\tau}\) that is defined by Eq. (B.13) and Eq. (CS4). Then we have:
For the base step, we first verify that the conditions are satisfied for both $\tau$ and $\tau - 1$. The solution for $\tau$ is given by $\lambda_{2w}^\tau = 5$, $\lambda_{2\ell}^\tau = 4$, $\lambda_{2w}^{\tau-1} = 8$, and $\lambda_{2\ell}^{\tau-1} = 5$, and the solution for $\tau - 1$ is given by $\lambda_{2w}^{\tau-1} = 5.5$, $\lambda_{2\ell}^{\tau-1} = 4.5$, and one can easily verify that all the conditions are satisfied. As for the induction step, suppose inequalities in (B.18), (B.20), (B.21) and (B.22) hold for all periods $t = k + 1, ..., \tau$. Consider period $k \leq \tau - 2$. For (B.21), we have

$$
\lambda_{1w}^k - \lambda_{1\ell}^k = (1 + \frac{1}{2} \lambda_{1w}^{k+1} + \frac{1}{2} \lambda_{1\ell}^{k+1}) - (\frac{1}{2} + \frac{1}{2} \lambda_{1w}^{k+1}) = \frac{1}{2} (1 + \lambda_{1\ell}^{k+1}) \geq 0,
$$

where the first equality uses (CS$_4$) to expand $\lambda_{1w}^k$ and $\lambda_{1\ell}^k$, and the inequality follows Lemma B.1(1).

Next, we check the inequalities in (B.18) and (B.20). Using (CS$_4$) to expand the two conditions, we have

$$
0 \leq \lambda_{2w}^k - \lambda_{2\ell}^k \iff 0 \leq (1 + \frac{1}{2} \lambda_{2w}^{k+1} + \frac{1}{2} \lambda_{2\ell}^{k+1}) - (1 + \lambda_{2w}^{k+1} - \frac{1}{2} \lambda_{1w}^{k+1} + \frac{1}{2} \lambda_{1\ell}^{k+1}) \iff 0 \leq -\lambda_{2w}^{k+1} + \lambda_{2\ell}^{k+1} + \lambda_{1w}^{k+1} - \lambda_{1\ell}^{k+1},
$$

and

$$
0 \leq \lambda_{2\ell}^k + \lambda_{1w}^k - 2\lambda_{1\ell}^k \iff 0 \leq (1 + \lambda_{2w}^{k+1} - \frac{1}{2} \lambda_{1w}^{k+1} + \frac{1}{2} \lambda_{1\ell}^{k+1}) + (1 + \frac{1}{2} \lambda_{1w}^{k+1} + \frac{1}{2} \lambda_{1\ell}^{k+1}) - 2(\frac{1}{2} + \frac{1}{2} \lambda_{1w}^{k+1}) \iff -1 \leq \lambda_{2w}^{k+1} - \lambda_{1w}^{k+1} + \lambda_{1\ell}^{k+1}.
$$

Note that these two inequalities are exactly (B.22) in period $k + 1$, which hold by our assumption.
Finally, we check (B.22) for period $k$. Plugging in (CS$_4$) twice for period $k$ and $k + 1$, we can express $-\lambda_{2w}^k + \lambda_{2\ell}^k + \lambda_{1w}^k - \lambda_{1\ell}^k$ with variables from $k + 2$:

$$-\lambda_{2w}^k + \lambda_{2\ell}^k + \lambda_{1w}^k - \lambda_{1\ell}^k = \frac{1}{2}(1 + \lambda_{2w}^{k+1} - \lambda_{2\ell}^{k+1} - \lambda_{1w}^{k+1} + 2\lambda_{1\ell}^{k+1})$$

$$= \frac{1}{2} - \frac{1}{4}\lambda_{2w}^{k+2} + \frac{1}{4}\lambda_{2\ell}^{k+2} + \frac{1}{2}\lambda_{1w}^{k+2} - \frac{1}{2}\lambda_{1\ell}^{k+2}$$

$$= \frac{1}{2} + \frac{1}{4}(-\lambda_{2w}^{k+2} + \lambda_{2\ell}^{k+2} + \lambda_{1w}^{k+2} - \lambda_{1\ell}^{k+2}) + \frac{1}{4}(\lambda_{1w}^{k+2} - \lambda_{1\ell}^{k+2}) \geq 0$$

where the inequality follows (B.22) and (B.21) in period $k + 2$. Similarly, we also have

$$1 + \lambda_{2w}^k - \lambda_{1w}^k + \lambda_{1\ell}^k = \frac{1}{2} \left(\lambda_{2w}^{k+1} + \lambda_{2\ell}^{k+1} - \lambda_{1\ell}^{k+1}\right) + \frac{3}{2}$$

$$\geq \frac{1}{2} \left(\lambda_{2w}^{k+1} - \lambda_{2\ell}^{k+1} - \lambda_{1\ell}^{k+1}\right) + \frac{3}{2} = 4.25 > 0,$$

where the inequality uses the decreasing property of $\lambda_{2w}^t$, $\lambda_{2\ell}^t$, and the increasing property of $\lambda_{1\ell}^t$ from Lemma B.1. Thus, all dual feasible conditions in (B.18), (B.20), (B.21), and (B.22) are satisfied, which imply that conditions in (DF$_4$) also hold for all $s \leq t \leq \tau$. This completes the proof for Scenario 4.

**Scenario 5:** $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2w}^t \geq s_{1w}^t$, $s_{2\ell}^t \geq s_{1\ell}^t$, and $s_{1\ell}^t \leq K_1 = \frac{18}{5}s_{2w}^t + \frac{9}{5}s_{2\ell}^t + \frac{3}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,2\ell}^t = s_{2\ell}^t$, $f_{1w,1w}^t = s_{1w}^t$, $f_{1\ell,1\ell}^t = s_{1\ell}^t$, and $f_{2w,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$.

According to the proposed matching flows above, the state of demographics either goes to Scenario 2 or Scenario 4 in period $t + 1$. If it goes to Scenario 4, following the proved optimal solution in Scenario 4, players’ demographic eventually evolves to Scenario 2 at period $\tau \leq t + 4$. That is, we summarize players’ demographics in period $k = t + 1, \ldots, \tau$ when $\tau = t + 4$ in the next table. Note that in period $t + 4$, we have

$$s_{2w}^{t+4} + s_{2\ell}^{t+4} - s_{1w}^{t+4} - s_{1\ell}^{t+4} = \frac{1}{16}(18s_{2w}^t + 9s_{2\ell}^t + 3s_{1w}^t - 5s_{1\ell}^t) \geq 0,$$

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The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}, f^t_{2w,1w} = f^t_{1w,2w} = s^t_{1w}, f^t_{2w,1\ell} = f^t_{1\ell,2w} = s^t_{1\ell} = s^t_{2\ell}$. Therefore, we have $\tau \leq t + 4$.

In Table B.2, we list the dual variables when $\tau = t + 1, \ldots, t + 4$ respectively, and one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasible conditions.

<table>
<thead>
<tr>
<th>$\tau = t + 1$</th>
<th>$k = t$</th>
<th>$k = t + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^t_{2w}$</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td>$\lambda^t_{2\ell}$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$\lambda^t_{1w}$</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>$\lambda^t_{1\ell}$</td>
<td>4.5</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau = t + 2$</th>
<th>$k = t$</th>
<th>$k = t + 1$</th>
<th>$k = t + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^t_{2w}$</td>
<td>6</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td>$\lambda^t_{2\ell}$</td>
<td>3.25</td>
<td>4.5</td>
<td>4</td>
</tr>
<tr>
<td>$\lambda^t_{1w}$</td>
<td>8.75</td>
<td>7.5</td>
<td>8</td>
</tr>
<tr>
<td>$\lambda^t_{1\ell}$</td>
<td>4.25</td>
<td>4.5</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau = t + 3$</th>
<th>$k = t$</th>
<th>$k = t + 1$</th>
<th>$k = t + 2$</th>
<th>$k = t + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^t_{2w}$</td>
<td>6.5</td>
<td>6</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td>$\lambda^t_{2\ell}$</td>
<td>3.5</td>
<td>5</td>
<td>4.5</td>
<td>4</td>
</tr>
<tr>
<td>$\lambda^t_{1w}$</td>
<td>8.75</td>
<td>7</td>
<td>7.5</td>
<td>8</td>
</tr>
<tr>
<td>$\lambda^t_{1\ell}$</td>
<td>4</td>
<td>4.25</td>
<td>4.5</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau = t + 4$</th>
<th>$k = t$</th>
<th>$k = t + 1$</th>
<th>$k = t + 2$</th>
<th>$k = t + 3$</th>
<th>$k = t + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^t_{2w}$</td>
<td>7.0625</td>
<td>6.5</td>
<td>6</td>
<td>5.5</td>
<td>5</td>
</tr>
<tr>
<td>$\lambda^t_{2\ell}$</td>
<td>3.75</td>
<td>5.625</td>
<td>5</td>
<td>4.5</td>
<td>4</td>
</tr>
<tr>
<td>$\lambda^t_{1w}$</td>
<td>8.75</td>
<td>6.625</td>
<td>7</td>
<td>7.5</td>
<td>8</td>
</tr>
<tr>
<td>$\lambda^t_{1\ell}$</td>
<td>3.8125</td>
<td>4</td>
<td>4.25</td>
<td>4.5</td>
<td>5</td>
</tr>
</tbody>
</table>

Scenarios: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$, $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $s^t_{1\ell} > K_2 = \frac{18}{5} s^t_{2w} + \frac{23}{5} s^t_{2\ell} - \frac{11}{5} s^t_{1w}$;

The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}, f^t_{2w,1w} = f^t_{1w,2w} = s^t_{1w}, f^t_{2w,1\ell} = f^t_{1\ell,2w} = s^t_{1\ell} = s^t_{2\ell}$.
and \( f'_{1,1} = s'_{1} - f'_{2,1} \).

Following the policy above, the state of demographics in the next period is given by \( s'_{1w} = \frac{1}{2}(s'_{1l} - (s'_{2l} - s'_{1w})) \), \( s'_{1l} = s'_{1w} \), \( s'_{2w} = \frac{1}{2}s'_{2w} + s'_{2l} \), \( s'_{2l} = \frac{1}{2}s'_{2w} \). This corresponds to Scenario 4 since

\[
\begin{align*}
    s'_{1w} + s'_{1l} - s'_{2l} &= \frac{3}{2}s'_{2l} - \frac{3}{2}s'_{1w} - \frac{1}{2}s'_{1l} \\
    &< s'_{2l} + \frac{3}{2}s'_{2l} - \frac{3}{2}s'_{1w} - \frac{1}{2}K_2 \\
    &= -\frac{4}{5}s'_{2l} - \frac{4}{5}s'_{2l} - \frac{2}{5}s'_{1w} < 0.
\end{align*}
\]

Also, \( s'_{1w} - s'_{2l} = \frac{1}{2}(s'_{1l} + s'_{2l} - s'_{2l} - s'_{2l}) < 0 \).

Since \( t + 1 \), we follow the proposed policy in Scenario 4 until we reach Scenario 2 at \( \tau \). One can verify that \( \tau > t + 5 \), because following the proposed solution in Scenario 4, we have

\[
\begin{align*}
    s'_{1w} &= \frac{9}{8}s'_{2w} + \frac{23}{16}s'_{1l} - \frac{11}{16}s'_{1w} - \frac{5}{16}s'_{1l} \\
    &< \frac{9}{8}s'_{2w} + \frac{23}{16}s'_{2l} - \frac{11}{16}s'_{1w} - \frac{5}{16}K_2 = 0.
\end{align*}
\]

For period \( t + 1 \) and forward, we use the dual variables proposed in Scenario 4, 2, and 1. Hence, we only need to show that the proposed policy as well as the corresponding dual variables at period \( t \) satisfies complementary slackness and dual feasibility in order to establish optimality.

By complementary slackness, we have

\[
\lambda'_{2w} = 1 + \frac{1}{2}\lambda_{2w} + \frac{1}{2}\lambda_{1l} \tag{B.23}
\]

\[
\lambda'_{2l} = 1 + \frac{1}{2}\lambda_{2w} - \frac{1}{2}\lambda'_{1w} \tag{B.24}
\]
\[ A_{1w}^t = \frac{3}{2} + A_{1\ell}^{t+1} + \frac{1}{2} A_{1w}^{t+1}, \]  
\[ (B.25) \]

\[ A_{1\ell}^t = \frac{1}{2} + \frac{1}{2} A_{1w}^{t+1}. \]  
\[ (B.26) \]

Then, we need to validate the dual feasibility condition corresponding to \( f_{2\ell,2\ell}^t, f_{1w,1w}^t, f_{2w,1w}^t, f_{2w,1\ell}^t \), which are given by:

\[ A_{2\ell}^t - 0.5 A_{2w}^{t+1} \geq 0.5, \]  
\[ (B.27) \]

\[ A_{1w}^t - 0.5 A_{1w}^{t+1} - 0.5 A_{1\ell}^{t+1} \geq 1, \]  
\[ (B.28) \]

\[ A_{2w}^t + A_{1w}^t - A_{2w}^{t+1} - A_{1\ell}^{t+1} \geq 2, \]  
\[ (B.29) \]

\[ A_{2w}^t + A_{1\ell}^t - A_{2w}^{t+1} \geq 1. \]  
\[ (B.30) \]

Taking (B.23)-(B.26) into the above inequalities, it is equivalent to validate

\[ A_{2w}^{t+1} \geq A_{1w}^{t+1}, \]  
\[ (B.31) \]

\[ 1.5 + A_{1\ell}^{t+1} \geq 1, \]  
\[ (B.32) \]

\[ 0.5 + 0.5 A_{2\ell}^{t+1} + 0.5 A_{1w}^{t+1} \geq 0.5 A_{2w}^{t+1}, \]  
\[ (B.33) \]
1.5 + 0.5\lambda_{2\ell}^{t+1} + 0.5\lambda_{1w}^{t+1} \geq 0.5\lambda_{2w}^{t+1}. \quad (B.34)

Among them, (B.32) is trivially true, because \lambda_{1\ell}^{t+1} is in Scenario 4 and is greater than 5 by Lemma B.1. (B.33) and (B.34) are directly from (B.22). Thus, we only need to validate (B.31).

Note that in Scenario 6, we have \tau \geq t + 4. At k = \tau - 4 \geq t + 1, we have

\lambda_{2w}^{\tau-4} = 7.0625, \lambda_{2\ell}^{\tau-4} = 6.1875, \lambda_{1w}^{\tau-4} = 6.3125, and \lambda_{1\ell}^{\tau-4} = 3.8125, \quad (B.35)

by using the complementary slackness conditions recursively. By Lemma B.1, we know that \lambda_{2w}^{t+1} \geq \lambda_{2w}^{t-4} \geq \lambda_{1w}^{t-4} \geq \lambda_{1\ell}^{t+1}, which completes the proof.

Scenario 7: s_{2w}^{t} + s_{2\ell}^{t} < s_{1w}^{t}, s_{2\ell}^{t} \geq s_{1w}^{t}, s_{2w}^{t} \leq s_{1\ell}^{t}, and K_1 < s_{1\ell}^{t} \leq K_2; The optimal matching flows are: f_{2w,2w}^{t} = s_{2w}^{t}, f_{2\ell,1w}^{t} = s_{1w}^{t}, f_{2\ell,2\ell}^{t} = \frac{9}{7}s_{2w}^{t} + \frac{23}{14}s_{2\ell}^{t} - \frac{11}{14}s_{1w}^{t} - \frac{5}{14}s_{1\ell}^{t}, f_{2\ell,1\ell}^{t} = f_{1\ell,1\ell}^{t} = s_{1\ell}^{t} - f_{2\ell,1\ell}^{t}.

In this scenario, the state of demographics will transit to Scenario 4 in the second period. Further, one can verify that the system reaches Scenario 2 in period k = t + 4, with s_{2w}^{t+4} + s_{2\ell}^{t+4} = s_{1w}^{t+4} + s_{1\ell}^{t+4}. Further, in period k = t + 5, the system goes to Scenario 1 with s_{2\ell}^{t+5} = s_{1w}^{t+5}. Hence, in period k = t + 5, we reach a degenerate case with f_{2\ell,2\ell}^{t+5} = 0. From the view of the simplex method, under the solution of Scenario 5.1, the reduced cost of f_{2\ell,2\ell}^{t+5} is positive, so we take it in to the basic feasible solution, and move f_{2\ell,2\ell}^{t+5}, out of the basis. The positiveness of all the other flows remains.

To see the solution is optimal, we list out the dual variables for the first t + 6 periods in Table B.3, and for periods k > t + 6, we always have

\lambda_{2w}^{k} = 5, \lambda_{2\ell}^{k} = 3, \lambda_{1w}^{k} = 9, and \lambda_{1\ell}^{k} = 5.

Then one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasibility. \square

Proof of Lemma B.1. (1) We prove the statements by backwards induction from period \tau.

First, we show that \lambda_{1w}^{t} \geq 5, \lambda_{1\ell}^{t} \geq 3 for all t = s, \ldots, \tau. This is true for \tau according to (B.13),
Table B.3: Dual Variables for Scenario 7

<table>
<thead>
<tr>
<th>k</th>
<th>t</th>
<th>t + 1</th>
<th>t + 2</th>
<th>t + 3</th>
<th>t + 4</th>
<th>t + 5</th>
<th>t + 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ_2^k</td>
<td>50/7</td>
<td>737/112</td>
<td>85/14</td>
<td>39/7</td>
<td>71/14</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>λ_2^k</td>
<td>849/224</td>
<td>639/112</td>
<td>285/56</td>
<td>32/7</td>
<td>57/14</td>
<td>22/7</td>
<td>4</td>
</tr>
<tr>
<td>λ_2^k</td>
<td>1963/224</td>
<td>737/112</td>
<td>389/56</td>
<td>52/7</td>
<td>111/14</td>
<td>62/7</td>
<td>8</td>
</tr>
<tr>
<td>λ_1^k</td>
<td>849/224</td>
<td>445/112</td>
<td>59/14</td>
<td>125/28</td>
<td>69/14</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

which completes the base step.

The induction hypothesis is that λ_1^k ≥ 5, λ_1^k ≥ 3 for all k = t, ..., τ. Then for period t − 1, from Eq. (CS4) we have

\[ λ_{1w}^{t-1} = 1 + 0.5λ_{1w}^t + 0.5λ_{1ℓ}^t \geq 1 + 0.5 \cdot 5 + 0.5 \cdot 3 = 5, \]

\[ λ_{1ℓ}^{t-1} = 0.5 + 0.5λ_{1w}^t \geq 0.5 + 0.5 \cdot 5 = 3, \]

and the induction step is completed.

(2) We show that λ_{1w}^t ≤ λ_{1w}^{t+1}, λ_{1ℓ}^t ≤ λ_{1ℓ}^{t+1}, for t = s, ..., τ − 1. For the induction base step, we have λ_{1w}^τ = 7.5 ≤ 8 = λ_{1w}^r and λ_{1ℓ}^τ = 4.5 ≤ 5 = λ_{1ℓ}^r. Our induction hypothesis it that for all k = t, ..., τ − 1, we have λ_{1w}^k ≤ λ_{1w}^{k+1} and λ_{1ℓ}^k ≤ λ_{1ℓ}^{k+1}. Now, consider period t − 1. Take the difference between λ_{1w}^{t-1} and λ_{1w}^t, we have

\[ λ_{1w}^{t-1} - λ_{1w}^t = 1 + 0.5λ_{1w}^t + 0.5λ_{1ℓ}^t - λ_{1w}^t \]

\[ = 1 + 0.5λ_{1w}^t + 0.5λ_{1ℓ}^t - λ_{1w}^t = 0.75 - 0.25λ_{1ℓ}^{k+1} \]

\[ \leq 0.75 - 0.25 \cdot 3 = 0, \]

where the first equality follows Eq. (CS4), the third equality follows (B.17), and the last inequality follows the fact that λ_{1ℓ}^t ≥ 3. Next, we show that λ_{1ℓ}^{t-1} ≤ λ_{1ℓ}^t. Note that from the last equation in
(CS₄), we have \( \lambda'_{1\ell} = 0.5 + 0.5\lambda'_{1w} \). Since \( \lambda'_{1w} \leq \lambda'^+_{1w} \), we have \( \lambda'_{1\ell} = 0.5 + 0.5\lambda'^+_{1w} = \lambda'_{1\ell} \).

(3) Finally, we show that \( \lambda'_{2w} \geq \lambda'^+_{2w}, \lambda'_{2\ell} \geq \lambda'^+_{2\ell} \). For the base step, we have \( \lambda'^{-1}_{2w} = 5.5 \geq 5 = \lambda'^{+}_{2w} \) and \( \lambda'^{-1}_{2\ell} = 4.5 \geq 4 = \lambda'^{+}_{2\ell} \). The induction hypothesis is that for all \( k = t, \ldots, \tau - 1 \), we have \( \lambda'^{k}_{2w} \geq \lambda'^{k+1}_{2w} \) and \( \lambda'^{k}_{2\ell} \geq \lambda'^{k+1}_{2\ell} \). Now, consider period \( t - 1 \). We have

\[
\lambda'^{-1}_{2w} = 1 + 0.5\lambda'_{2w} + 0.5\lambda'_{2\ell} \geq 1 + 0.5\lambda'^{+}_{2w} + 0.5\lambda'^{+}_{2\ell} = \lambda'^{+}_{2w},
\]

where the equality on the two sides follows the first equation of (CS₄). Next, consider \( \lambda'^{-1}_{2\ell} \). We have

\[
\lambda'^{-1}_{2\ell} = 1 + \lambda'_{2w} - 0.5\lambda'_{1w} + 0.5\lambda'_{1\ell} \\
= 1 + \lambda'_{2w} - 0.5(1 + 0.5\lambda'^{+}_{1w} + 0.5\lambda'^{+}_{1\ell}) + 0.5(0.5 + 0.5\lambda'^{+}_{1w}) \\
= 1 + \lambda'_{2w} - 0.25 - 0.25\lambda'^{+}_{1\ell} \\
\geq 1 + \lambda'^{+}_{2w} - 0.25 - 0.25\lambda'^{+}_{1\ell} \\
= \lambda'^{+}_{2\ell},
\]

where the first equality follows the second equation of (CS₄), the second equality follows the third and fourth equations of (CS₄), the first inequality follows the fact that \( \lambda'^{+}_{1\ell} \leq \lambda'^{+}_{1\ell} \) and \( \lambda'^{+}_{2w} \geq \lambda'^{+}_{2w} \). Thus, the induction step is completed and this completes the proof. □

**Proof of Proposition 2.3.** (a) In order to prove the first statement, we consider the linear program for the one-shot matching problem in \((P₁)\) and use the same simplification tricks we used in Appendix B.1.2. Without loss of generality, we will set the engagement level of SBMM in the next period to be 1, which is equivalent to the constraint \( 1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell} \). Thus, the following optimization problem selects the initial state of the demographics \( s \) to maximize the ratio of the optimal policy to SBMM for the one-period problem (we drop all the superscripts, representing
time periods, since it is a one-shot problem):

$$\max_{f,s} f_{2w,2w} + \frac{1}{2} f_{2\ell,2\ell} + f_{1w,1w} + \frac{1}{2} f_{1\ell,1\ell} + 2f_{2w,1w} + f_{2\ell,1\ell} + f_{2w,1\ell} + 2f_{2,1w}$$

s.t.

$$1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}$$

$$s_{2w} = f_{2w,2w} + f_{2w,1w} + f_{2w,1\ell},$$

$$s_{2\ell} = f_{2\ell,2\ell} + f_{2\ell,1w} + f_{2\ell,1\ell},$$

$$s_{1w} = f_{1w,1w} + f_{2\ell,1w} + f_{2w,1w},$$

$$s_{1\ell} = f_{1\ell,1\ell} + f_{2w,1\ell} + f_{2\ell,1\ell}.$$  

We verify that the optimal solution to the above optimization problem is $s_{2\ell} = s_{1w} = 2/3$, $f_{2\ell,1w} = 2/3$, and all other matching flows are 0. The objective value is 4/3, which is the desired ratio. Denote $\lambda_0$ as the dual variable correspond to the constraint normalizing the engagement for SBMM to be 1. We verify the proposed solution using complementary slackness conditions:

$$0 = \frac{1}{2}\lambda_0 - \lambda_{2\ell},$$

$$0 = \lambda_0 - \lambda_{1w},$$

$$2 = \lambda_{2\ell} + \lambda_{1w},$$

where the conditions correspond to primal non-zero variables $s_{2\ell}$, $s_{1w}$, and $f_{2\ell,1w}$, respectively. There is a unique solution of dual variables solving the complementary slackness conditions: $\lambda_0 = 4/3$, $\lambda_{2\ell} = 2/3$, and $\lambda_{1w} = 4/3$. To complete the proof, we need to check dual feasibility conditions:

$$0 \leq \lambda_0 - \lambda_{2w}, \quad 0 \leq \frac{1}{2}\lambda_0 - \lambda_{1\ell}, \quad 1 \leq \lambda_{2w}, \quad \frac{1}{2} \leq \lambda_{2\ell}, \quad 1 \leq \lambda_{1w}, \quad \frac{1}{2} \leq \lambda_{1\ell},$$

$$2 \leq \lambda_{2w} + \lambda_{1w}, \quad 1 \leq \lambda_{2\ell} + \lambda_{1\ell}, \quad 1 \leq \lambda_{2w} + \lambda_{1\ell}, \quad 1 \leq \lambda_{2\ell} + \lambda_{1w},$$

representing zero state variables ($s_{2w}$, $s_{1\ell}$) and zero matching flows ($f_{2w,2w}$, $f_{2\ell,2\ell}$, $f_{1w,1w}$, $f_{1\ell,1\ell}$, $f_{2w,1\ell}$, $f_{2\ell,1w}$).
\( f_{2w,1w}, f_{2\ell,1\ell}, f_{2w,1\ell} \), respectively.

(b) Next, we turn our attention to the infinite horizon problem in (P'). Using a similar idea, we can solve an optimization problem to find the maximum ratio between the optimal matching policy and SBMM. Using Proposition 2.1, the value function of the baseline model under SBMM is

\[
V^{\text{SBMM}}(s^0) = 5(s_{2w}^2 + s_{1w}^1) + 3(s_{2\ell}^2 + s_{1\ell}^1), \quad t = 1, 2, \ldots,
\]

which we normalize to 1 without loss of generality. Thus, the following optimization problem selects the initial state of the demographics \( s^0 \) to maximize the ratio of the optimal policy to SBMM for the infinite-horizon problem (we set \( t = 0 \) without loss of generality):
\[
\max \sum_{t=0}^{\infty} \left( f'_{2w,2w} + \frac{1}{2} f'_{2\ell,2\ell} + f'_{1w,1w} + \frac{1}{2} f'_{1\ell,1\ell} + 2f^t_{2w,1w} + f^t_{2\ell,1\ell} + f^t_{2w,1\ell} + 2f^t_{2\ell,1w} \right)
\tag{B.38}
\]

s.t.
\[
\begin{align*}
1 &= 5(s^0_{2w} + s^0_{1w}) + 3(s^0_{2\ell} + s^0_{1\ell}), \\
0 &= f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell} - s^0_{2w}, \\
0 &= f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell} - s^0_{2\ell}, \\
0 &= f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w} - s^0_{1w}, \\
0 &= f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell} - s^0_{1\ell},
\end{align*}
\tag{B.39}
\]

and for all \( t = 1, 2, \ldots \),
\[
\begin{align*}
f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell} &= \frac{1}{2} \left( f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell} \right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell}, \\
f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} &= \frac{1}{2} f^{t-1}_{2w,2w}, \\
f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} &= \frac{1}{2} \left( f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell} \right), \\
f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} &= \frac{1}{2} f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w}, \\
f^t_{i,j} &\geq 0, \forall i, j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{align*}
\]

Again, we use dual complementary slackness and feasibility conditions verifying the initial state \( s^0 = \{1/8, 0, 0, 1/8\} \), along with the optimal matching flows in Proposition 2.2, is the optimal solution to the above maximization problem. The objective value is 3/2, which is the desired ratio.

With slight abuse of notation, denote the dual variable to the new constraint (B.39) as \( \lambda_0 \). We show that under the optimal initial state where \( s_{2\ell} = s_{1w} = 1/8 \), we have \( \lambda_0 = 3/2 \), representing the maximum ratio. The proposed initial state is in Scenario 1. Based on the the transition in Fig. B.1, we shall always stay in Scenario 1. We list the states in period 0, 1, and 2 in Table B.4.

Note that in period 0, we reach a degenerate period with only one positive flow \( f^0_{2\ell,1w} \), and in
period 1 we reach a degenerate case with only two positive flows $f_{1_{2w},2w}$ and $f_{1_{1l},1l}$. Thus, for these two periods, we can use only part of the equations in (CS$_1$). To be specific, given our proposed primal solution, the complementary slackness equations are given by

$$
0 = 3\lambda_0 - \lambda_{2\ell}^0,
0 = 5\lambda_0 - \lambda_{1w}^0,
2 = \lambda_{2\ell}^0 + \lambda_{1w}^0 - \lambda_{2w}^1 - \lambda_{1\ell}^1,
1 = \lambda_{2w}^1 - \frac{1}{2}\lambda_{2w}^2 - \frac{1}{2}\lambda_{2\ell}^2,
\frac{1}{2} = \lambda_{1\ell}^1 - \frac{1}{2}\lambda_{1w}^2,
$$

(CS$_1$), $t = 2, \ldots$

and the dual feasibility constraints we need to check is

$$
0 \leq 5\lambda_0 - \lambda_{2w}^0, \quad 0 \leq 3\lambda_0 - \lambda_{1w}^0, \quad 1 \leq \lambda_{2w}^0 - \frac{1}{2}\lambda_{2w}^1 - \frac{1}{2}\lambda_{2\ell}^1,
\frac{1}{2} \leq \lambda_{2\ell}^0 - \frac{1}{2}\lambda_{2w}^1, \quad \frac{1}{2} \leq \lambda_{1w}^0 - \frac{1}{2}\lambda_{1w}^1, \quad \frac{1}{2} \leq \lambda_{1\ell}^1 - \frac{1}{2}\lambda_{1w}^2,
2 \leq \lambda_{2\ell}^0 + \lambda_{1w}^0 - \lambda_{2w}^2 - \lambda_{1\ell}^2, \quad \text{(DF$_1$) for } t = 0, \ldots
$$

We then list out the dual variables in period 0, 1, 2 in Table B.4. For $t > 2$, we use $\lambda_{2w}^0 = 5$, $\lambda_{2\ell}^0 = 3$, $\lambda_{1w} = 9$, $\lambda_{1\ell} = 5$. Together with $\lambda_0 = 3/2$, one can verify that the proposed dual solution satisfies complementary slackness and dual feasibility constraints.

| Table B.4: Primal States and Dual Variables in Period 0, 1, and 2. |
|---|---|---|---|---|---|---|---|
|   | $s_{2w}^t$ | $s_{2\ell}^t$ | $s_{1w}^t$ | $s_{1\ell}^t$ | $\lambda_{2w}^t$ | $\lambda_{2\ell}^t$ | $\lambda_{1w}^t$ | $\lambda_{1\ell}^t$ |
| $t = 0$ | 0 | 1/8 | 1/8 | 0 | 11/2 | 9/2 | 15/2 | 9/2 |
| $t = 1$ | 1/8 | 0 | 0 | 1/8 | 5 | 4 | 8 | 5 |
| $t = 2$ | 1/16 | 1/16 | 1/16 | 0 | 5 | 3 | 9 | 5 |
B.1.3 Omitted Proofs from Section 2.4

Before proving Proposition 2.4, we first show that for a set of initial demographics, denoted by $S$, the optimal policy remains simple and informative. The set $S$ consists of states $s^t$ that satisfy

\begin{align}
  s^t_{k\ell} &\geq s^t_{(k-1)w}, \quad \forall k = 2, \ldots, K, \\
  s^t_{Kw} &\geq s^t_{(K-1)\ell} + s^t_{(K-2)w}, \\
  s^t_{k\ell} &\geq \frac{1}{2} \left( s^t_{(k-1)\ell} + s^t_{(k-2)w} \right), \quad \forall k = 3, \ldots, K - 1, \\
  s^t_{2w} &\geq \frac{1}{2} s^t_{1\ell}, \\
  \frac{1}{2} s^t_{Kw} &\geq s^t_{(K-1)w} \geq \ldots \geq s^t_{2w} \geq s^t_{1w}, \\
  s^t_{K\ell} &\geq s^t_{(K-1)\ell} \geq \ldots \geq s^t_{2\ell} \geq \frac{1}{2} s^t_{1\ell}, \\
  s^t_{2\ell} + s^t_{1w} &\geq s^t_{1\ell}. 
\end{align}

For initial demographics in $S$, the optimal matching policy (to be introduced in Proposition B.2 below), induces the property that $s^t_{k\ell} \leq s^t_{(k+1)w}, \forall k = 1, \ldots, K - 1$. Lemma B.2 characterizes the optimal matching policy and the corresponding value function. Lemma B.2 states that as long as the initial demographics belongs to the set $S$, then it is the best to maximize the matching flows between players in adjacent skill levels. In particular, the matchmaker wants to utilize players in the relatively low-skill level to save players in the relatively high-skill level. This optimality structure echoes the one in Section 2.3 with only two skill levels among players. Furthermore, note that the terms $4K - 4k + 5$ and $2K - 2k + 3$ in (B.40) can be interpreted as the shadow price for demographics $s_{kw}$ and $s_{k\ell}$, respectively, for all $k = 1, 2, \ldots, K$. Since the shadow prices are linearly decreasing with respect to $k$, we can conclude that the lower the skill level a player has, the more valuable they are to the matching system since they can be used to improve the engagement of players in higher skill levels. This intuition aligns with what we have observed in Section 2.3 with only two skill levels.
Lemma B.2 (Optimal Policy when Demographics are in \( S \)). If \( s^0 \in S \), then for all \( t = 0, 1, \ldots \), the optimal matching policy induces matching flows

\[
\begin{align*}
  f^t_{Kw,Kw} &= s^t_{Kw}, \\
  f^t_{(k+1)\ell,Kw} &= s^t_{k\ell}, \quad \forall k = 1, \ldots, K - 1, \\
  f^t_{k\ell,K\ell} &= s^t_{k\ell} - s^t_{(k-1)\ell}, \quad \forall k = 2, \ldots, K, \\
  f^t_{1\ell,1\ell} &= s^t_{1\ell},
\end{align*}
\]

and the optimal value function of engagement in (2.1) can be written as

\[
V^*_{K}(s^0) = \sum_{k=1}^{K} (4K - 4k + 5)s^t_{kw} + \sum_{k=1}^{K} (2K - 2k + 3)s^t_{k\ell}, \quad \forall s^0 \in S. \tag{B.40}
\]

Proof of Lemma B.2. The proof follows in two steps. First, we show that given the proposed policy, \( s^t \) always stays within \( S \), and thus the solution is primal feasible. Second, we show that \( \lambda^t_{kw} = 4(k - K + 1) + 1 \) and \( \lambda^t_{k\ell} = 2(k - K + 1) + 1 \) satisfy dual feasibility and complementary slackness, so the policy is indeed optimal.

**Step 1.** If players’ demographics satisfy conditions (a), (b), (c), (d), (e), (f), and (g), then under the proposed policy, in the next period the players’ demographics will transition to

\[
\begin{align*}
  s^{t+1}_{Kw} &= \frac{1}{2} \left( s^t_{Kw} + s^t_{(K-1)w} + s^t_{K\ell} \right), \\
  s^{t+1}_{k\ell} &= \frac{1}{2} s^t_{Kw}, \\
  s^{t+1}_{kw} &= \frac{1}{2} \left( s^t_{k\ell} + s^t_{(k-1)w} \right), \quad \forall k = 2, \ldots, K - 1, \\
  s^{t+1}_{k\ell} &= s^t_{kw}, \quad \forall k = 2, \ldots, K - 1, \\
  s^{t+1}_{1w} &= \frac{1}{2} s^t_{1\ell}, \\
  s^{t+1}_{1\ell} &= s^t_{1w},
\end{align*}
\]

since there are always less relatively “low” skilled players without losing record comparing to
adjacent “high” skill players who just lost a game. Using the expressions of players’ demographics in period $t + 1$, we can verify that the conditions $(a)$, $(b)$, $(c)$, $(d)$, $(e)$, $(f)$, and $(g)$ all still hold in period $t + 1$.

We start with condition $(a)$. When $k = K$, we need to verify that

$$s_{k\ell}^{t+1} = \frac{1}{2} s_{Kw}^t \geq \frac{1}{2} \left( s_{(K-1)\ell}^t + s_{(K-2)w}^t \right) = s_{(K-1),w}^{t+1},$$

where the inequality follows condition $(b)$ in period $t$. When $2 \leq k \leq K - 1$, we need to verify that

$$s_{k\ell}^{t+1} = s_{kw}^t \geq \frac{1}{2} \left( s_{(k-1)\ell}^t + s_{(k-2)\ell}^t \right) = s_{(k-1)w}^{t+1},$$

where the inequality follows condition $(c)$ in period $t$. Finally, when $k = 2$, we need to verify

$$s_{2\ell}^{t+1} = s_{2w}^t \geq \frac{1}{2} s_{1\ell}^t = s_{1w}^{t+1},$$

which holds according to condition $(d)$ in period $t$.

For condition $(b)$, we have

$$s_{Kw}^t = \frac{1}{2} \left( s_{K\ell}^t + s_{(K-1)\ell}^t \right) \geq \frac{1}{2} \left( s_{(K-1)\ell}^t + s_{(K-2)w}^t + 2s_{(K-1)w}^t \right)$$

$$\geq s_{(K-1)w}^t + \frac{1}{2} \left( s_{(K-2)\ell}^t + s_{(K-3)w}^t \right) = s_{(K-1)\ell}^t + s_{(K-2)w}^{t+1},$$

where the first inequality follows conditions $(a)$- $(b)$ and the second inequality follows $(e)$- $(f)$ in period $t$.

Note that condition $(c)$ in period $t + 1$ is equivalent to

$$s_{kw}^t = \frac{1}{2} \left( s_{k\ell}^t + s_{(k-1)w}^t \right) \geq \frac{1}{4} \left( s_{(k-1)w}^t + s_{(k-2)\ell}^t + s_{(k-3)w}^t \right) = \frac{1}{2} \left( s_{(k-1)\ell}^t + s_{(k-2)w}^{t+1} \right), \quad \forall k = 2, ..., K-1,$$

where the inequality holds according to conditions $(e)$ and $(f)$ in period $t$. 

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Condition (d) is equivalent to

\[ s_{2w}^{t+1} = \frac{1}{2} (s_{2\ell}^t + s_{1w}^t) \geq \frac{1}{2} s_{1w}^t = \frac{1}{2} s_{1\ell}^t , \]

were the inequality follows immediately.

Next, consider condition (e) in period \( t + 1 \). We break this condition into three parts. First, consider \( \frac{1}{2} s_{Kw}^{t+1} \geq s_{(K-1)w}^{t+1} \), which is equivalent to

\[ \frac{1}{2} s_{Kw}^{t+1} = \frac{1}{4} \left( s_{Kw}^t + s_{(K-1)w}^t + s_{K\ell}^t \right) \geq \frac{1}{4} \left( s_{(K-1)\ell}^t + s_{(K-2)w}^t + s_{(K-1)w}^t + s_{K\ell}^t \right) \geq \frac{1}{2} \left( s_{(K-1)\ell}^t + s_{(K-2)w}^t \right) = s_{(K-1)w}^{t+1}, \]

where the first inequality follows from (b) and the second inequality follows from (e)-(f) in period \( t \). Second, consider \( s_{(K-1)w}^{t+1} \geq \ldots \geq s_{2w}^{t+1} \), which is equivalent to

\[ s_{(K-1)\ell}^t + s_{(K-2)w}^t \geq \ldots \geq s_{3\ell}^t + s_{2w}^t, \]

where the inequality holds according to conditions (e) and (f) in period \( t \). Third, consider \( s_{2w}^{t+1} \geq s_{1w}^{t+1} \), which is equivalent to

\[ s_{2w}^{t+1} = \frac{1}{2} (s_{2\ell}^t + s_{1w}^t) \geq \frac{1}{2} s_{1w}^t = s_{1\ell}^{t+1} \]

which follows from condition (g).

Condition (f) in period \( t + 1 \) can be verified using similar techniques as condition (e). The details are omitted.

Finally, condition (g) in period \( t + 1 \) is equivalent to

\[ s_{2\ell}^{t+1} + s_{1w}^{t+1} = s_{2w}^t + \frac{1}{2} s_{1w}^t \geq s_{1w}^t = s_{1\ell}^{t+1}, \]

where the inequality follows from condition (e) in period \( t \).

**Step 2.** Given the proposed policy, flows \( f_{Kw,Kw}^t, f_{k\ell,(k-1)w}^t, f_{k\ell,k\ell}^t \), for all \( k = 2, \ldots, K \), and
$f^t_{\ell, \ell}$ are always non-negative, and be complementary slackness, so the associated dual constraints have to be tight. That is, for any $t$, we need

\[
1 = \lambda^t_{Kw} - \frac{1}{2} \lambda^{t+1}_{Kw} - \frac{1}{2} \lambda^{t+1}_{K\ell} \\
\frac{1}{2} = \lambda^{t+1}_{K\ell} - \frac{1}{2} \lambda^{t+1}_{kw}, \quad k = 1, \ldots, K \\
2 = \lambda^t_{K\ell} + \lambda^t_{(k-1)w} - \lambda^{t+1}_{kw} - \lambda^{t+1}_{(k-1)\ell}, \quad k = 2, \ldots, K.
\]

Then one can verify that $\lambda^{t+1}_{k0} = 4(k - K + 1) + 1$ and $\lambda^{t+1}_{K0} = 2(k - K + 1) + 1$ is the unique dual optimal solution, and satisfies the other dual feasibility constraints that correspond to $f^t_{kw,kw}$ for $k = 1, \ldots, K - 1$, $f^t_{kw,jw}$, $f^t_{kw,jl}$, and $f^t_{kl,jw}$ for $k > j$, and $f^t_{kl,jl}$ for $k > j + 1$.

\[\Box\]

**Proof of Proposition 2.4.** The lower bound of the ratio is constructed with the optimal policy in Lemma B.2. In particular, we use initial demographics within $S$ that give us the largest benefits. By Proposition 2.1, the value under SBMM is $5 \sum_{k=1}^{K} s_{kw} + 3 \sum_{k=1}^{K} s_{kl}$. Since we already know the shadow prices of each type of the player in this case, the problem can be reduced to a simple LP:

\[
\begin{align*}
\max_{s_{kw}, s_{kl}} & \sum_{k=1}^{K} (4(K - k + 1) + 1) s_{kw} + \sum_{k=1}^{K} (2(K - k + 1) + 1) s_{kl} \\
\text{s.t.} & \quad 5 \sum_{k=1}^{K} s_{kw} + 3 \sum_{k=1}^{K} s_{kl} = 1 \\
& \quad \text{conditions (a) – (g),}
\end{align*}
\]

where (B.42) requires the value under SBMM to be 1, and the coefficients in the objective comes from Lemma B.2. We now verify that $s_{Kw} = \frac{2}{8K+8}$, $s_{Kl} = \frac{1}{8K+8}$, $s_{1w} = \frac{1}{8K+8}$, $s_{1l} = \frac{2}{8K+8}$, and $s_{kw} = s_{kl} = \frac{1}{8K+8}$ for $k = 2, \ldots, K - 1$ is the optimal solution of the above LP, with objective value $\frac{3K^2+7K+6}{8K+8}$. First, note that the purposed solution is primal feasible. There are $2K$ number of primal variables, so there are $2K$ number of dual constraints. By complementary slackness, all $2K$ dual constraints are tight.

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Next, we provide a set of dual variables satisfying complementary slackness conditions, which implies that the proposed solution is optimal (there is no extra dual feasibility constraints, since all the dual constraints are tight). Let \( \lambda_0 \) be the dual variable of constraint \((B.42)\). Let the dual variables of conditions \((a)-(d)\) to be 0. Let \( \lambda_{e_k} \), \( k = 1, \ldots, K - 1 \), \( \lambda_{f_k} \), \( k = 1, \ldots, K - 1 \), and \( \lambda_g \) be the dual variables of conditions \((e),(f), (g)\), respectively. The following equality holds, which are derived from dual constraints:

\[
\begin{align*}
2K + 1 &= 3\lambda_0 + \frac{1}{2}\lambda_{f1} + \lambda_g, \\
4K + 1 &= 5\lambda_0 + \lambda_{e1} - \lambda_g, \\
2K - 1 &= 3\lambda_0 - \lambda_{f1} + \lambda_{f2} - \lambda_g, \\
4K - 3 &= 5\lambda_0 - \lambda_{e1} + \lambda_{e2}, \\
2(K - k + 1) + 1 &= 3\lambda_0 - \lambda_{f(k-1)} + \lambda_{f_k}, \forall k = 3, \ldots, K - 1, \\
4(K - k + 1) + 1 &= 5\lambda_0 - \lambda_{e(k-1)} + \lambda_{e_k}, \forall k = 3, \ldots, K - 1, \\
3 &= 3\lambda_0 - \lambda_{f(K-1)}, \\
5 &= 5\lambda_0 - \frac{1}{2}\lambda_{e(K-1)}.
\end{align*}
\]

Then one can easily verify that the following solutions solve the system of equations above:

\[
\begin{align*}
\lambda_0 &= \frac{3K^2 + 7K + 6}{8K + 8} \\
\lambda_{e_k} &= 5(K - k + 1)\lambda_0 - 5 - (2K - 2k + 3)(K - k), \; k = 1, \ldots, K - 1, \\
\lambda_{f_k} &= 3(K - k)\lambda_0 - (2 + K - k)(K - k), \; k = 2, \ldots, K - 1, \\
\lambda_{f1} &= -2(8 + 5K)\lambda_0 + 2(2K^2 + 5K + 6), \\
\lambda_g &= 5(K + 1)\lambda_0 - 5 - 3K - 2K^2,
\end{align*}
\]

which proves the optimality of the proposed solution.

The upper bound follows from constructing a feasible solution to the dual problem of the linear
program in \((P_K)\) directly. With slight abuse of notation, denote \(\lambda^t_{kw} (\lambda^t_{k\ell})\) as the dual variables of demographics with level skill level \(k\) who just won (lost) a game. One can verify that \(\lambda^t_{kw} = 4(K - k + 1) + 1\) and \(\lambda^t_{k\ell} = 2(K - k + 1) + 1\) is dual feasible. Hence, for any initial demographic, the value of optimal policy is at most \(\sum_{k=1}^{K}(4(K - k + 1) + 1)s_{kw} + \sum_{k=1}^{K}(2(K - k + 1) + 1)s_{k\ell}\), and the ratio is

\[
\frac{\sum_{k=1}^{K}(4(K - k + 1) + 1)s_{kw} + \sum_{k=1}^{K}(2(K - k + 1) + 1)s_{k\ell}}{5\sum_{k=1}^{K}s_{kw} + 3\sum_{k=1}^{K}s_{k\ell}} \leq \frac{4K + 1}{5},
\]

where the inequality follows by choosing the largest ratio among all the demographics. \(\square\)

**Proof of Proposition 2.5.** Before the proof, we first provide the primal problem in the presence of bots. The primal problem is slightly modified from Eq. \((P')\), so that every flow variable has \(\alpha\) fraction that goes to the corresponding winning state \(w\), and \((1 - \alpha)\) fraction that follows the original route:
\[
\max \sum_{t=0}^{\infty} \left( f_{2w,2w}^t + \frac{1}{2} (1 + \alpha) f_{2\ell,2\ell}^t + f_{1w,1w}^t + \frac{1}{2} (1 + \alpha) f_{1\ell,1\ell}^t + 2 f_{2w,1w}^t + (1 + \alpha) f_{2\ell,1\ell}^t + (1 + \alpha) f_{2w,1\ell}^t + 2 f_{2\ell,1\ell}^t \right)
\]

\[\text{(P}_{\text{Bot}})\]

s.t.
\[
\begin{align*}
    s_{0w}^0 &= f_{0w,2w}^0 + f_{0w,1w}^0 + f_{2w,1\ell}^0, \\
    s_{0\ell}^0 &= f_{02\ell,2\ell}^0 + f_{2\ell,1w}^0 + f_{2w,1\ell}^0, \\
    s_{0w}^0 &= f_{1w,1w}^0 + f_{0w,1w}^0 + f_{2w,1\ell}^0, \\
    s_{0\ell}^0 &= f_{1\ell,1\ell}^0 + f_{2w,1\ell}^0 + f_{2\ell,1\ell}^0,
\end{align*}
\]

and for all \( t = 1, 2, \ldots, \)
\[
\begin{align*}
    f_{2w,2w}^t + f_{2w,1w}^t + f_{2w,1\ell}^t &= \frac{1}{2} (1 + \alpha) \left( f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1} \right) + f_{2w,1w}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1\ell}^{t-1}, \\
    f_{2\ell,2\ell}^t + f_{2\ell,1\ell}^t + f_{2\ell,1\ell}^t &= \frac{1}{2} (1 - \alpha) f_{2w,2w}^{t-1}, \\
    f_{1w,1w}^t + f_{2\ell,1w}^t + f_{2w,1\ell}^t &= \frac{1}{2} (1 + \alpha) \left( f_{1w,1w}^{t-1} + f_{1\ell,1\ell}^{t-1} \right) + \alpha \left( f_{2w,1w}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1\ell}^{t-1} \right), \\
    f_{1\ell,1\ell}^t + f_{2w,1\ell}^t + f_{2\ell,1\ell}^t &= \frac{1}{2} (1 - \alpha) f_{1w,1w}^{t-1} + (1 - \alpha) \left( f_{2w,1w}^{t-1} + f_{2\ell,1w}^{t-1} \right), \\
    f_{i,j}^t &\geq 0, \forall i, j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{align*}
\]

(a) First, when having \( \alpha \) portion of bots, the SBMM value function can be derived as
\[
V^{\text{SBMM}}(s', \alpha) = \left( \frac{4}{(1 - \alpha)^2} + \frac{1 + \alpha}{1 - \alpha} \right) (s'_{2w} + s'_{1w}) + \left( \frac{4}{(1 - \alpha)^2} - 1 \right) (s'_{2\ell} + s'_{1\ell}), \tag{B.44}
\]
according to Proposition 2.1. Note that when \( \alpha = 0 \), the value function in (B.44) reduces to the one in (B.37) with shadow price 5 and 3 for players who have and lost the last game, respectively.

For a given \( \alpha \), to check whether there exists \( s \) such that \( V^*(s, 0) = V^{\text{SBMM}}(s, \alpha) \), one can solve Eq. (B.38) but change Eq. (B.39) into:
\[
\left( \frac{4}{(1 - \alpha)^2} + \frac{1 + \alpha}{1 - \alpha} \right) (s'_{2w} + s'_{1w}) + \left( \frac{4}{(1 - \alpha)^2} - 1 \right) (s'_{2\ell} + s'_{1\ell}) \leq 1, \tag{B.45}
\]
and see if the objective is greater than 1. Since the feasible region becomes smaller as $\alpha$ increases according to Eq. (B.45), the objective is monotonically decreasing with $\alpha$. Thus, using binary search, we numerically find that for $\alpha \leq 0.169$, there exists $s$ such that $V^*(s, 0) = V_{SBMM}^B(s, \alpha)$.

(b) The corresponding dual problem is given by:

$$\min_{\{\lambda^t\}} \sum_{i \in P} s_i^0 \lambda_i^0$$

s.t.

for all $t = 0, 1, 2, ...,$

$$1 \leq \lambda_{2w}^t - \frac{1}{2} (1 + \alpha) \lambda_{2w}^{t+1} - \frac{1}{2} (1 - \alpha) \lambda_{2\ell}^{t+1},$$

$$\frac{1}{2} (1 + \alpha) \leq \lambda_{2\ell}^t - \frac{1}{2} (1 + \alpha) \lambda_{2w}^{t+1},$$

$$2 \leq \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1 - \alpha) \lambda_{1\ell}^{t+1} - \alpha \lambda_{1w}^{t+1},$$

$$\frac{1}{2} (1 + \alpha) \leq \lambda_{1\ell}^t - \frac{1}{2} (1 + \alpha) \lambda_{1w}^{t+1},$$

$$2 \leq \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1 - \alpha) \lambda_{1\ell}^{t+1} - \alpha \lambda_{1w}^{t+1},$$

$$1 + \alpha \leq \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \alpha \lambda_{1w}^{t+1},$$

$$1 + \alpha \leq \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} - \alpha \lambda_{1w}^{t+1},$$

$$1 \leq \lambda_{1w}^t - \frac{1}{2} (1 + \alpha) \lambda_{1w}^{t+1} - \frac{1}{2} (1 - \alpha) \lambda_{1\ell}^{t+1}.$$

We first give an upper bound of $V^*(s, \alpha)$. Similar to the Scenario 1 in Proposition 2.2, we provide a feasible solution to Eq. (DBot):

$$\lambda_{2w}^t = \frac{4}{(1 - \alpha)^2} + \frac{1 + \alpha}{1 - \alpha},$$

$$\lambda_{2\ell}^t = \frac{4}{(1 - \alpha)^2} - 1,$$

$$\lambda_{1w}^t = \frac{4}{(1 - \alpha)^2} + \frac{4}{(1 - \alpha)^3} + \frac{1 + \alpha}{1 - \alpha},$$

$$\lambda_{1\ell}^t = \frac{2}{(1 - \alpha)^2} + \frac{4}{(1 - \alpha)^3} - 1.$$
Thus, we have

\[ V^*(s, \alpha) \leq U(s, \alpha) \]

\[
:= \left( \frac{4}{(1-\alpha)^2} + \frac{1+\alpha}{1-\alpha} \right) s_{2w} + \left( \frac{4}{(1-\alpha)^2} - 1 \right) s_{2l} \\
+ \left( \frac{4}{(1-\alpha)^2} + \frac{4}{(1-\alpha)^3} \right) s_{1w} + \left( \frac{2}{(1-\alpha)^2} + \frac{4}{(1-\alpha)^3} - 1 \right) s_{1l}.
\]

(B.46)

Let \( a = g(\alpha) \) be solution of \( U(s, a) = V_{SBMM}(s, \alpha) \), and \( a = a(\alpha) \) be solution of \( V^*(s, a) = V_{SBMM}(s, \alpha) \). Then \( a(\alpha) \geq g(\alpha) \) because \( V^*(s, \alpha) \leq U(s, \alpha) \) for any \( \alpha \in [0, 1] \) and function \( V^*(s, \alpha) \) is increasing in \( \alpha \). Since \( V_{SBMM}(s, \alpha) \) goes to infinity as \( \alpha \) goes to 1, we must have \( g(\alpha) \) as well. Hence, taking the limit on both sides of \( a(\alpha) \leq a(\alpha) \leq \alpha \), we have \( \lim_{\alpha \to 1} a(\alpha) = 1 \).

**Proof of Proposition 2.6.** We prove the four parts separately.

(a) Consider two initial states, \( s_1 = (s_{2w}, s_{2l}, s_{1w}, s_{1l}) \) and \( s_2 = (s_{2w} + (r + 1)\beta s_{1w}, s_{2l} + (r + 1)\beta s_{1l}, (1-\beta)s_{1w}, (1-\beta)s_{1l}) \). We show that \( V_{SBMM}(\beta, r, s_1) = V_{SBMM}(0, 0, s_2) \) and \( V^*(\beta, r, s_1) \geq V^*(0, 0, s_2) \).

Note that one can easily verify \( V_{SBMM}(\beta, r, s_1) = V_{SBMM}(0, 0, s_2) \) using the closed-form expression of \( V_{SBMM} \) in Proposition 2.1. Thus, we omit the details and only focused on the value functions under the optimal matching policy.

Consider the optimal trajectory when the initial state of demographics is \( s_2 = (s_{2w} + (r + 1)\beta s_{1w}, s_{2l} + (r + 1)\beta s_{1l}, (1-\beta)s_{1w}, (1-\beta)s_{1l}) \) and there is no PTW system (referred as non-PTW problem). Let \( f_{i,j}^t \) be the optimal flow between \( i, j \in \{1w, 1l, 2w, 2l\} \) at time \( t \), and \( s_i^t \) be the population at time \( t \). From Proposition 2.2, we know that it may involve cross-level matching between \( 2l \) and \( 1w \), as well as \( 2l \) and \( 1l \) (only happens when \( t = 0 \) in Scenario 6 and 7, after which the states transit to Scenario 1-4). We now show that we can collect at least the same value with demographic \( s_1 = (s_{2w}, s_{2l}, s_{1w}, s_{1l}) \) and PTW (referred as PTW problem). Note that with PTW, the reward of 1 unit flow between \( 2l \) and level 1 players are the same as \( r + 1 \) units of flow between \( 2l \) and level 1 players without PTW, due to the extra \( r \) unit of revenue. However, such a
flow use less amount of low players, which makes the level 1 players better off. This observation provides a natural way to construct a feasible flow for the PTW problem. To distinguish from the state of demographics $s^t_i$ and the matching flows $f^t_{i,g}$ in the system without PTW, where $i,j \in (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$, we denote $\sigma^t_i$ and $g^t_{i,j}$ as the state of demographics and matching flow in the PTW problem, respectively, where $i,j \in (s^t_{2w}, s^t_{2\ell}, s^t_{2w}, s^t_{2\ell}, s^t_{1w}, s^t_{1\ell})$.

Next, we construct a specific set of matching flows in the PTW problem. In each period, consider the following proposed flows: $g^t_{2\ell,1w} = \min\{f^t_{2\ell,1w}/(1+r), \sigma^t_{2\ell}\}$ and $g^t_{2\ell,1w} = f^t_{2\ell,1w} - (1+r)g^t_{2\ell,1w}$. Furthermore, at $t=0$, if the matching system without PTW has non-zero flows between players in $2\ell$ and $1\ell$, i.e., $f^0_{2\ell,1\ell} > 0$, then we set $g^0_{2\ell,1\ell} = \min\{\sigma^0_{2\ell} - g^0_{2\ell,1w}, f^0_{2\ell,1\ell}/(1+r)\}$ and $g^0_{2\ell,1\ell} = f^0_{2\ell,1\ell} - (1+r)g^0_{2\ell,1\ell}$. In other words, we prioritize cross-level matching flows between $2\ell$ players and level 1 players, and only after which is exhausted, we then use players in $2\ell$ to match with level 1 players. For all the remaining players, we simply match them with other players in the same state.

Next, we show that with the initial state of demographics $s_1$ the proposed flows $g^t_{i,j}$, where $i,j \in (s^t_{2w}, s^t_{2\ell}, s^t_{2w}, s^t_{2\ell}, s^t_{1w}, s^t_{1\ell})$ is not only feasible, but also collect at least the same rewards as the non-PTW problem with initial state of demographics $s_2$ and optimal flows $f^t_{i,g}$, where $i,j \in (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$.

To show the statement above, we prove that in the PTW problem, we must have $\sigma^t_{2w} + (1+r)\sigma^t_{2\ell} = s^t_{2w}$, $\sigma^t_{2\ell} + (1+r)\sigma^t_{1w} = s^t_{2\ell}$, $\sigma^t_{1w} + \sigma^t_{1\ell} = s^t_{1w} + s^t_{1\ell}$ for any $t \geq 0$.

It is trivially true for $t=0$ by construction. Now consider $t=1$. First, recall that one unit of subscribed (level 2) player in the PTW problem correspond to $1+r$ units of high-skilled (level 2) players in the non-PTW problem, due to the extra unit of revenue $r$ generated by subscription fee. Thus, we must have $\sigma^1_{2w} + (1+r)\sigma^1_{2\ell} = s^1_{2w}$ and $\sigma^1_{2\ell} + (1+r)\sigma^1_{1w} = s^1_{2\ell}$. Second, in the non-PTW problem, if $f^0_{2\ell,1\ell} > 0$, then $f^0_{2\ell,1\ell}$ unit of level 1 players will lose in the upcoming game and leave the system permanently. However, in a corresponding PTW problem, we have $g^0_{2\ell,1\ell} \leq f^0_{2\ell,1\ell}/(1+r)$ according to the proposed policy. Thus, instead of matched with high-skill players and leaving the system permanently in a non-PTW problem, we have $f^0_{2\ell,1\ell} - g^0_{2\ell,1\ell}$ unit
of players in state 1\(\ell\) are matched to other players in the same state in the PTW problem and half of them can survive to period \(t = 1\). Thus, we must have \(\sigma_{1w}^1 \geq s_{1w}^1\), and \(\sigma_{1\ell}^1 + \sigma_{1\ell}^{-1} \geq s_{1w}^1 + s_{1\ell}^1\).

Third, in a non-PTW problem, if we have \(f_{2,1w}^0 > 0\), then we have \(f_{2,1w}^0\) unit of players in state \(1w\) transfer to state \(1\ell\) just prior to period \(t = 1\). However, in a corresponding PTW problem, we have \(g_{2,1w}^0 \leq f_{2,1w}^0/(1 + r)\) according to the proposed policy. Thus, instead of matching with high-skilled players, we have \(f_{2,1w}^0 - g_{2,1w}^0\) unit of \(1w\) players matched with other players in the same state. As a result, just prior to period \(t = 1\), we have half of the players in the aforementioned flow with size \(f_{2,1w}^0 - g_{2,1w}^0\) remain in \(1w\) and half of them transfer to \(1\ell\). Hence, we still have \(\sigma_{1w}^1 \geq s_{1w}^1\), and \(\sigma_{1w}^1 + \sigma_{1\ell}^1 \geq s_{1w}^1 + s_{1\ell}^1\).

For \(t \geq 1\), we only need to consider the matching flow between \(1w\) and \(2\ell\) (since Scenario 5-7 in the proof of Proposition 2.2, which involves other cross-skill matching, only occurs in period \(t = 0\)). At \(t + 1\), \(\sigma_{2w}^1 + (1 + r)\sigma_{2w}^1 = s_{2w}^1\) and \(\sigma_{2\ell}^1 + (1 + r)\sigma_{2\ell}^1 = s_{2\ell}^1\) still hold for the same reason as in \(t = 1\). If in a non-PTW problem, we have \(f_{2,1w}^{t-1} > 0\). Then using the exact same third argument in the proof for period \(t = 1\), we can show that we still have \(\sigma_{1w}^1 \geq s_{1w}^1\), and \(\sigma_{1w}^1 + \sigma_{1\ell}^1 \geq s_{1w}^1 + s_{1\ell}^1\).

The reason is the same as before: some of the low-skill players who are matched with high-skilled players in a non-PTW problem shall be matched to other low-skill players in a PTW problem, which leads to less players transferring to state \(1\ell\) and more to \(1w\) instead.

Since we have \(\sigma_{2j}^1 + (1 + r)\sigma_{2j}^1 = s_{2j}^1\) and \(\sigma_{1w}^1 \geq s_{1w}^1\), the proposed flow is feasible because for any \(t \geq 1\) all matching flows are either skill-based or between \(2\ell\), \(2\ell\), and \(1w\) players, which are determined by \(f_{2,1w}^t\) in a corresponding non-PTW problem. Also, in every period we collect reward no less than the non-PTW problem because the reward we collect from high-skilled players is \(\sum_{i=\text{w,j}}(\sigma_{2i}^1 + (1 + r)\sigma_{2i}^1) = s_{2w}^1 + s_{2\ell}^1\), and the reward from low-skilled problem is \(\sigma_{1w}^1 + \sigma_{1\ell}^1\) which we have shown is no less than \(s_{1w}^1 + s_{1\ell}^1\).

\((b)\) We show that unsubscribed high-skilled players in \(2w\) \((2\ell)\) would only be matched with any unsubscribed low-skilled players after all the subscribed players in \(2w\) \((2\ell)\) have matched with unsubscribed low-skilled players.

We prove the statement above by contradiction. Suppose on the optimal trajectory, the flow
between high-skilled non-paying players in $2i$ and low-skilled unsubscribed players $1j$ are positive for some $i, j = w, \ell$, while there exists subscribed players $\bar{2}i$ who are matched by skill levels. Then by matching $\bar{2}i$ with $1j$, we can collect strictly more rewards in the current period, and a player in $\bar{2}i$ would replace a player $2i$ in all the subsequent periods. Hence, the solution cannot be optimal.

(c) We show that if $s_{2w}^0 + s_{2\ell}^0 \geq s_{1w}^0 + s_{1\ell}^0$, then $ENG(\beta, r, s^0) < V^*(0, 0, s^0)$. Furthermore, there exists a threshold $\bar{r}$ such that $V^*(\beta, r, s^0) \geq V^*(0, 0, s^0)$ if and only if $r \geq \bar{r}$.

We first show that $ENG(\beta, r, s^0) \leq V^*(0, 0, s^0)$. The engagement $ENG(\beta, r, s^0)$ is at most $V^*(\beta, 0, s^0)$, which is the optimal engagement with the same demographic but without revenue. We now show that $V^*(\beta, 0, s^0) \leq V^*(0, 0, s^0)$, and the equality only holds when we do not have low players at all.

When the high-skilled players are more than low-skilled players in a non-PTW system, we are in Scenario 1-3 in the proof of Proposition 2.2. Note that for these scenarios we have explicit shadow prices for each type of players. We now discuss the three scenarios separately.

Consider Scenario 1: $s_{2w}^0 + s_{2\ell}^0 \geq s_{1w}^0 + s_{1\ell}^0$, $s_{2\ell}^0 \geq s_{1w}^0$, and $s_{2w}^0 \geq s_{1\ell}^0$. The shadow price in this case is 5,3,9,5, respectively. The PTW system shift $1w$ ($1\ell$) player to $2w$ ($2\ell$), so with PTW the initial demographic is still in scenario 1. The total value change will be $(5-9)\beta s_{1w}^0 + (3-5) s_{1\ell}^0 \leq 0$, where the equality only holds when $s_{1w}^0 = s_{1\ell}^0 = 0$.

Consider Scenario 2: $s_{2w}^0 + s_{2\ell}^0 \geq s_{1w}^0 + s_{1\ell}^0$, $s_{2\ell}^0 < s_{1w}^0$, and $s_{2w}^0 \geq s_{1\ell}^0$; The shadow price in this case is 5,4,8,5, respectively. The PTW system shift $1w$ ($1\ell$) player to $2w$ ($2\ell$), so with PTW the initial demographic is either in Scenario 1 or Scenario 2. If it remains in Scenario 2, then the total value change will be $(5-8)\beta s_{1w}^0 + (4-5) s_{1\ell}^0 < 0$ because now we have $s_{1w}^0 > s_{2w}^0 \geq 0$. If the state
of demographics transits to Scenario 1, the total value change would be

\[
5(s_{2w}^0 + \beta s_{1w}^0) + 3(s_{2l}^0 + \beta s_{1l}^0) + 9(1 - \beta)s_{1w}^0 + 5(1 - \beta)s_{1l}^0 = 5s_{2w}^0 - 4s_{2l}^0 - 8s_{1w}^0 - 5s_{1l}^0
\]

\[
= -s_{2l}^0 + (1 - 4\beta)s_{1w}^0 - 2\beta s_{1l}^0
\]

\[
\leq -s_{2l}^0 + (1 - 4\beta)\frac{s_{2l}^0 + \beta s_{1l}^0}{1 - \beta} - 2\beta s_{1l}^0
\]

\[
= \left(-1 + \frac{1 - 4\beta}{1 - \beta}\right)s_{2l}^0 + \left(-2\beta + \frac{1 - 4\beta}{1 - \beta}\right)s_{1l}^0 < 0,
\]

(B.47)

where Eq. (B.47) comes from the fact that if the demographic transfer to Scenario 1, we must have \(s_{2l}^0 + \beta s_{1l}^0 \geq (1 - \beta)s_{1w}^0\). Eq. (B.48) comes from the fact that \(-1 + \frac{1 - 4\beta}{1 - \beta}\) and \(-2\beta + \frac{1 - 4\beta}{1 - \beta}\) are both negative when \(\beta \in (0, 1)\), and to make \(s_{2l}^0 + \beta s_{1l}^0 \geq (1 - \beta)s_{1w}^0\), one of \(s_{2l}^0\) and \(s_{1l}^0\) has to be positive.

Finally, consider Scenario 3: \(s_{2w}^0 + s_{2l}^0 \geq s_{1w}^0 + s_{1l}^0\), \(s_{2l}^0 \geq s_{1w}^0\), and \(s_{2w}^0 < s_{1l}^0\). The shadow prices in this scenario is 5.5,3,9,4.5. The PTW system shift 1w (1\(\ell\)) player to 2w (2\(\ell\)), so with PTW the initial demographic is either in Scenario 1 or Scenario 3. If it remains in Scenario 3, then the total value change will be \((5.5 - 9)\beta s_{1w}^0 + (3 - 4.5)s_{1l}^0 < 0\) we cause now \(s_{1l}^0 > s_{2w}^0 \geq 0\). If the demographic transfer to Scenario 1, to total value change would be

\[
5(s_{2w}^0 + \beta s_{1w}^0) + 3(s_{2l}^0 + \beta s_{1l}^0) + 9(1 - \beta)s_{1w}^0 + 5(1 - \beta)s_{1l}^0 = 5s_{2w}^0 - 3s_{2l}^0 - 9s_{1w}^0 - 4.5s_{1l}^0
\]

\[
= -0.5s_{2w}^0 - 4\beta s_{1w}^0 + (0.5 - 2\beta)s_{1l}^0
\]

\[
\leq -0.5s_{2w}^0 - 4\beta s_{1w}^0 + (0.5 - 2\beta)\frac{s_{2w}^0 + \beta s_{1w}^0}{1 - \beta}
\]

\[
= \left(-0.5 + \frac{0.5 - 2\beta}{1 - \beta}\right)s_{2w}^0 + \left(-4\beta + \frac{(0.5 - 2\beta)\beta}{1 - \beta}\right)s_{1w}^0 < 0,
\]

(B.49)

where Eq. (B.49) comes from the fact that if the demographic transfer to Scenario 1, we must have \(s_{2w}^0 + \beta s_{1w}^0 \geq (1 - \beta)s_{1l}^0\). The inequality in Eq. (B.48) comes from the fact that \(-0.5 + \frac{0.5 - 2\beta}{1 - \beta}\) and \(-4\beta + \frac{(0.5 - 2\beta)\beta}{1 - \beta}\) are both negative when \(\beta \in (0, 1)\), and to make \(s_{2w}^0 + \beta s_{1w}^0 \geq (1 - \beta)s_{1l}^0\), one of \(s_{2w}^0\) and \(s_{1w}^0\) has to be positive.

We have shown that \(V^*(\beta, 0, s^0) < V^*(0, 0, s^0)\) when there are positive amount of low players.
It is easy to see that $V^\ast(\beta, r, s^0)$ increases monotonically with $r$, and goes to infinity as $r$ goes to infinity. Hence, there exists a threshold $\tilde{r} > 0$ such that $V^\ast(\beta, r, s^0) > V^\ast(0, 0, s^0)$ if and only if $r > \tilde{r}$.

(d) Fix $s^0_{2w}/s^0_{2\ell}$ and $s^0_{1w}/s^0_{1\ell}$ and vary $(s^0_{2w} + s^0_{2\ell})/(s^0_{1w} + s^0_{1\ell})$. We show that if the ratio of high-over low-skilled players is sufficiently small, there is $V^\ast(\beta, r, s^0) \geq V^\ast(0, 0, s^0)$ even if $r = 0$.

Consider $s^0$ such that $(s^0_{2w} + s^0_{2\ell})/(s^0_{1w} + s^0_{1\ell}) = 0$, i.e., there are only low players. Then the optimal matching is simply SBMM. In presence of PTW system, some of the low player now becomes high player, which enables cross-level matchmaking, and we must have $V^\ast(\beta, 0, s^0) > V^\ast(0, 0, s^0)$. That said, even there is no revenue, the engagement is still higher thanks to the change in demographic distribution. For $r \geq 0$, we must have $V^\ast(\beta, r, s^0) \geq V^\ast(\beta, 0, s^0) > V^\ast(0, 0, s^0)$. Finally, note that when $r = 0$, we have $V^\ast(\beta, r, s^0) = ENG(\beta, r, s^0)$, i.e., the value of matchmaking is solely made by player engagement.

B.2 Possible Extensions

We point out that our framework is flexible enough to allow for various extensions that still result in a nice LP formulation. We discuss a few assumptions that can be easily relaxed for industry practitioners below.

1. A draw/tie outcome can be easily added, since our model only depends on the aggregate transition matrix $M_{kk}$.

2. If in each period, only $\alpha$ fraction of the idle players want to play, then we can simply multiply $\alpha$ on the right-hand-side of (FB) and add $(1-\alpha)s^t_k$ on the right-hand-side of (ED).

3. If the match duration is not one period, we can modify (ED) so that the match flow returns to the demographics after a positive and random delay.

4. New players whose amount are linear functions of past history can be introduced easily by modifying (ED).
5. Changes of player skill levels over time can be considered by modifying (ED).

B.3 Omitted Table

Table B.5: Winrate Used in Section 2.5

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<th>4</th>
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<td>0.776</td>
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<table>
<thead>
<tr>
<th>Level Difference</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.948</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Appendix C: Price Discrimination with Fairness Constraints

C.1 Proof of Proposition 3.1 and Theorem 3.2

Proof of Proposition 3.1. We refer to Table C.1 for convenient formulas used in the proof.

Table C.1: Closed-form expressions for linear and exponential demand models.

<table>
<thead>
<tr>
<th>Demand Type</th>
<th>$R_i(p)$</th>
<th>$F_i(p)$</th>
<th>$S_i(p)$</th>
<th>$N_i(p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$(p - c) \max{0, 1 - \frac{p}{b_i}}$</td>
<td>$\max{0, 1 - \frac{p}{b_i}}$</td>
<td>$\frac{(\max{0, b_i - p})^2}{2b_i}$</td>
<td>$\min{b_i, p}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$(p - c)e^{-\lambda_i p}$</td>
<td>$e^{-\lambda_i p}$</td>
<td>$\frac{1}{\lambda_i}e^{-\lambda_i p}$</td>
<td>$\frac{1}{\lambda_i} - \frac{p_1}{\lambda_i}e^{-\lambda_i p_1}$</td>
</tr>
</tbody>
</table>

(a) We first consider the case of an exponential demand, that is, $V_i \sim \Exp(\lambda_i)$ with $\lambda_0 > \lambda_1$. Suppose that we have 1-fairness in price (i.e., there exists a price $p$ such that $p = p_0 = p_1$). Then, we immediately have $\bar{F}_0(p) < \bar{F}_1(p)$, $S_0(p) < S_1(p)$, and $N_0(p) < N_1(p)$, so that 1-fairness in price cannot be satisfied along with another 1-fairness constraint. Similarly, if 1-fairness in demand is satisfied, we have $\bar{F}_0(p_0) = \bar{F}_1(p_1)$, implying that $p_0 = \frac{\lambda_1}{\lambda_0} p_1$. For such prices, we have $S_0(p_0) = \frac{\lambda_1}{\lambda_0} S_1(p_1) < S_1(p_1)$ and $N_1(p_1) - N_0(p_0) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0} + (1 - \frac{\lambda_0}{\lambda_1}) \frac{p_0 e^{-\lambda_0 p_0}}{1 - e^{-\lambda_0 p_0}}$. Note that $f(p) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0} + (1 - \frac{\lambda_0}{\lambda_1}) \frac{p e^{-\lambda p}}{1 - e^{-\lambda p}}$ is a strictly increasing function starting from $f(p) = 0$ and, hence, $N_1(p_1) - N_0(p_0) = f(p_0) > 0$. As a result, 1-fairness in demand cannot coexist with 1-fairness either in surplus or in no-purchase valuation. Finally, satisfying 1-fairness in surplus means that $S_0(p_0) = S_1(p_1)$ and, thus, $p_1 = \frac{\lambda_0}{\lambda_1} p_0 + \frac{1}{\lambda_1} \log \frac{\lambda_0}{\lambda_1}$. We have shown above that $N_1(\frac{\lambda_0}{\lambda_1} p_0) > N_0(p_0)$. Since $N_1(\cdot)$ is an increasing function and $p_1 > \frac{\lambda_0}{\lambda_1} p_0$, we then have $N_1(p_1) > N_0(p_0)$. Consequently, 1-fairness in no-purchase valuation cannot coexist with 1-fairness in surplus. In conclusion, under positive prices, any pair of 1-fairness constraints cannot be simultaneously satisfied.

(b) We next consider the case of a linear demand, that is, $V_i \sim \Unif(0, b_i)$ with $b_0 < b_1$. Suppose that we have 1-fairness in price (i.e., there exists a price $p$ such that $p = p_0 = p_1$). Then, we immediately have $\bar{F}_0(p) < \bar{F}_1(p)$ and $S_0(p) < S_1(p)$, so that 1-fairness in price cannot be satisfied...
along with another 1-fairness constraint. Similarly, if 1-fairness in demand is satisfied, we have \( \bar{F}_0(p_0) = \bar{F}_1(p_1) \). Note the surplus of group 0 can be written as \( b_i(\bar{F}_i(p_i))^2/2 \) and, hence, the surplus of group 0 is lower than group 1 under 1-fairness in demand. Finally, note that when both groups have positive demand and positive prices, the no-purchase valuation is always equal to half of the price. Thus, 1-fairness in no-purchase valuation implies 1-fairness in price, so that it cannot coexist with 1-fairness in demand or surplus. This concludes the proof. \( \square \)

Before presenting the proof of Theorem 3.2, we first state and prove Lemma C.1 which describes necessary and sufficient conditions for \( \mathcal{W}'(0) \) to be positive or negative.

**Lemma C.1.** Suppose that \( \bar{F}_i, S_i, N_i, \) and \( R_i \) are continuous and twice differentiable at \( p_i^* \). Suppose also that \( \bar{F}_i, S_i, \) and \( N_i \) are monotone and invertible.

(a) W.l.o.g. let \( p_0^* < p_1^* \). For price fairness, \( \mathcal{W}'(0) > 0 \) if and only if \( d_1 \bar{F}_1(p_1^*)R_0''(p_0^*) - d_0 \bar{F}_0(p_0^*)R_1''(p_1^*) < 0. \)

(b) W.l.o.g. let \( \bar{F}_0(p_0^*) < \bar{F}_1(p_1^*) \). For demand fairness, \( \mathcal{W}'(0) > 0 \) if and only if \( d_1 \bar{F}_1(p_1^*)\bar{F}_1'(p_1^*)R_0''(p_0^*) - d_0 \bar{F}_0(p_0^*)\bar{F}_0'(p_0^*)R_1''(p_1^*) < 0. \)

(c) W.l.o.g. let \( S_0(p_0^*) < S_1(p_1^*) \). For surplus fairness, \( \mathcal{W}'(0) > 0 \) if and only if \( d_1 \bar{F}_1(p_1^*)S_0'(p_0^*)R_0''(p_0^*) - d_0 \bar{F}_0(p_0^*)S_0'(p_0^*)R_1''(p_1^*) > 0. \)

(d) W.l.o.g. let \( N_0(p_0^*) < N_1(p_1^*) \). For no-purchase valuation fairness, \( \mathcal{W}'(0) > 0 \) if and only if \( d_1 \bar{F}_1(p_1^*)N_1'(p_1^*)R_0''(p_0^*) - d_0 \bar{F}_0(p_0^*)N_1'(p_1^*)R_1''(p_1^*) < 0. \)

**Proof of Lemma C.1.** We discuss the four problems separately.

(a) **Price Fairness.** Since \( \mathcal{W}(\alpha) = R_0(p_0(\alpha)) + R_1(p_1(\alpha)) + d_0S_0(p_0(\alpha)) + d_1S_1(p_1(\alpha)) \), the derivative of \( \mathcal{W}(\alpha) \) is given by:

\[
\mathcal{W}'(\alpha) = R_0'(p_0(\alpha))p_0'(\alpha) + R_1'(p_1(\alpha))p_1'(\alpha) + d_0S_0'(p_0(\alpha))p_0'(\alpha) + d_1S_1'(p_1(\alpha))p_1'(\alpha).
\]

By definition, at \( \alpha = 0 \) we have \( p_i(0) = p_i^* \) and \( R_i'(p_i(0)) = 0. \) Thus, we obtain \( \mathcal{W}'(0) = d_0S_0'(p_0^*)p_0'(0) + d_1S_1'(p_1^*)p_1'(0). \) By definition of the normalized surplus function \( S(\cdot), S_i'(p) = \)
\(-\tilde{F}_1(p)\) and thus we have \(W'(0) = -d_1 \tilde{F}_1(p_1^\ast)p_1'(0) - d_0 \tilde{F}_0(p_0^\ast)p_0'(0)\). The rest of the proof relies on computing \(p_i'(0)\) for each fairness definition, which we shall do in cases.

For price fairness, since we assume that \(p_0^\ast < p_1^\ast\), the seller has to increase \(p_0\) and decrease \(p_1\) in order to improve price fairness. Let \(\Delta p_0(\alpha) = p_0(\alpha) - p_0^\ast\), and \(\Delta p_1(\alpha) = p_1^\ast - p_1(\alpha)\). Hence, \(p_i'(0) = \lim_{\alpha \to 0} \Delta p_i(\alpha)/\alpha\), and \(p_i'(0) = \lim_{\alpha \to 0} -\Delta p_1(\alpha)/\alpha\). Given \(\alpha\), the profit optimization problem (3.1) for the seller can be cast as

\[
\begin{align*}
\max & R_0(p_0^\ast + \Delta p_0) + R_1(p_1^\ast - \Delta p_1) \quad \text{(C.1)} \\
\text{s.t.} & \quad \Delta p_0 + \Delta p_1 \geq (p_1^\ast - p_0^\ast)\alpha \quad \text{(C.2)} \\
& \Delta p_0, \Delta p_1 \geq 0,
\end{align*}
\]

where Eq. (C.2) requires that the total price changes is at least \((p_1^\ast - p_0^\ast)\alpha\). Further, the profit objective (C.1) can be expanded using a Taylor expansion around \((p_0^\ast, p_1^\ast)\) as

\[
R_0(p_0^\ast) + R_0'(p_0^\ast)\Delta p_0 + \frac{1}{2}R_0''(p_0^\ast)\Delta p_0^2 + g_0(\Delta p_0) + R_1(p_1^\ast) - R_1'(p_1^\ast)\Delta p_1 + \frac{1}{2}R_1''(p_1^\ast)\Delta p_1^2 + g_1(\Delta p_1),
\]

(C.3)

where \(g_i(\Delta p_i)\) corresponds to the remainder term. Since \(R_i\) is twice differentiable, \(g_i\) must be twice differentiable and \(g_i''(0) = 0\) since \(R_i''(p_i^\ast) = g_i''(0)\). Removing the constants \(R_i(p_i^\ast)\) from (C.3) and recalling that \(R_i'(p_i^\ast) = 0\), we can rewrite the profit optimization problem as

\[
\begin{align*}
\min & \quad \frac{1}{2}R_0''(p_0^\ast)\Delta p_0^2 - \frac{1}{2}R_1''(p_1^\ast)\Delta p_1^2 + g_0(\Delta p_0) + g_1(\Delta p_1) \quad \text{[Minimize the profit loss]} \quad \text{(C.4)} \\
\text{s.t.} & \quad \Delta p_0 + \Delta p_1 \geq (p_1^\ast - p_0^\ast)\alpha \\
& \Delta p_i \geq 0.
\end{align*}
\]
The KKT conditions for Eq. (C.4) are given by:

\[
\begin{bmatrix}
-R''_0(p^*_0)\Delta p_0 + g'_0(\Delta p_0) \\
-R''_1(p^*_1)\Delta p_1 + g'_1(\Delta p_1)
\end{bmatrix} = \mu \begin{bmatrix}
-1 \\
-1
\end{bmatrix},
\]

\[\Delta p_0 + \Delta p_1 \geq (p^*_1 - p^*_0)\alpha,\]

\[\Delta p_i \geq 0,\]

\[\mu \geq 0,\]

\[\mu \left[(p^*_1 - p^*_0)\alpha - \Delta p_0 - \Delta p_1\right] = 0.\]

This can be further reduced to

\[ -R''_0(p^*_0)\Delta p_0 + g'_0(\Delta p_0) = -R''_1(p^*_1)((p^*_1 - p^*_0)\alpha - \Delta p_0) + g'_1((p^*_1 - p^*_0)\alpha - \Delta p_0), \quad (C.5) \]

\[\Delta p_0 \in [0, (p^*_1 - p^*_0)\alpha]. \quad (C.6)\]

Since we assume that \(R_i\) is twice differentiable, \(R''_i(p_i) = R''_i(p^*_i) + g''_i(\Delta p_i)\) is well defined and

\[\lim_{\alpha \to 0} g'_i(\Delta p_i)/\alpha = \lim_{\alpha \to 0} \frac{g'_i(\Delta p_i)\Delta p_i}{\Delta p_i} = \lim_{\alpha \to 0} g''_i(\Delta p_i)\frac{\Delta p_i}{\alpha} = 0,\]

where the last equality comes from the facts that \(g''_i(0) = 0\) and that \(\Delta p_i/\alpha\) is bounded from Eq. (C.6). Thus, by dividing both sides of Eq. (C.5) by \(\alpha\) and taking the limit as \(\alpha\) goes to 0, we obtain:

\[-R''_0(p^*_0)p'_0(0) = -R''_1(p^*_1)[p^*_1 - p^*_0 - p'_0(0)]. \quad (C.7)\]

As a result of Eq. (C.7), as \(\alpha\) goes to 0, we have the expression of \(p'_0(0)\) and \(p'_1(0)\) (with a similar argument):

\[p'_0(0) = \frac{R''_1(p^*_1)}{R''_0(p^*_0) + R''_1(p^*_1)}(p^*_1 - p^*_0) \quad \text{and} \quad p'_1(0) = -\frac{R''_0(p^*_0)}{R''_0(p^*_0) + R''_1(p^*_1)}(p^*_1 - p^*_0). \quad (C.8)\]
Recall that we require $W'(0) = -d_1 \tilde{F}_1(p_1^*) p_1'(\alpha) - d_0 \tilde{F}_0(p_0^*) p_0'(\alpha) > 0$. By substituting Eq. (C.8) into the previous equation, we obtain our desired result $d_1 \tilde{F}_1(p_1^*) R_0''(p_0^*) - d_0 \tilde{F}_0(p_0^*) R_1''(p_1^*) < 0$.

(b) Demand Fairness. For demand fairness, since we assume that group 1 has higher demand, then $p_0$ decreases and $p_1$ increases. Note that the objective function is the same as Eq. (C.1), whereas Eq. (C.2) becomes

$$
\bar{F}_0(p_0^* - \Delta p_0) - \bar{F}_0(p_0^*) + \bar{F}_1(p_1^*) - \bar{F}_1(p_1^* + \Delta p_1) \geq \alpha [\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)].
$$

Writing the demand change into Taylor expansion, we have

$$
-\tilde{F}_0'(p_0^*) \Delta p_0 - \tilde{F}_1'(p_1^*) \Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) \geq \alpha [\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)],
$$

where $h_i(\Delta p_i)$ is the remainder term in demand. Since the demand is differentiable, we know that $h'_i(0)$ is well defined and $h'_i(0) = 0$, as $\tilde{F}_i'(p_i^*) = \tilde{F}_i'(p_i^*) + h'_i(0)$. We setup an optimization problem as in Eq. (C.4), and the KKT conditions for the new problem are given by:

$$
\begin{bmatrix}
-R_0''(p_0^*) \Delta p_0 + g_0'(\Delta p_0) \\
-R_1''(p_1^*) \Delta p_1 + g_1'(\Delta p_1)
\end{bmatrix} = \mu \begin{bmatrix}
\tilde{F}_0'(p_0^*) - h_0'(\Delta p_0) \\
\tilde{F}_1'(p_1^*) - h_1'(\Delta p_1)
\end{bmatrix},
$$

$$
-\tilde{F}_0'(p_0^*) \Delta p_0 - \tilde{F}_1'(p_1^*) \Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) \geq \alpha [\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)],
$$

$\Delta p_i \geq 0,$

$\mu \geq 0,$

$\mu [(p_1^* - p_0^*) \alpha - \Delta p_0 - \Delta p_1] = 0.$

This can be further reduced to

$$
-R_0''(p_0^*) \Delta p_0 + g_0'(\Delta p_0) = \frac{\tilde{F}_0'(p_0^*) - h_0'(\Delta p_0)}{\tilde{F}_1'(p_1^*) - h_1'(\Delta p_1)} \left[-R_1''(p_1^*) \Delta p_1 + g_1'(\Delta p_1)\right], \quad \text{(C.9)}
$$

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\[-\tilde{F}_0'(p_0^*)\Delta p_0 - \tilde{F}_1'(p_1^*)\Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) = \alpha[\tilde{F}_1(p_1^*) - \tilde{F}_0(p_0^*)]. \quad (C.10)\]

Since \(\tilde{F}_i(p)\) is twice differentiable, \(h'_i(\Delta p_i)\) is well defined and \(\lim_{\Delta p_i \to 0} h'_i(\Delta p_i) = 0\). If \(p'_1(0)\) is bounded (as we will show later), then \(\lim_{\alpha \to 0} h_1(\Delta p_1)/\alpha = \lim_{\alpha \to 0} \frac{h_1(\Delta p_1)}{\Delta p_1}/\alpha = 0\). Similarly, we have \(\lim_{\alpha \to 0} g'_i(\Delta p_i)/\alpha = 0\). Thus, dividing Eq. (C.9) and Eq. (C.10) by \(\alpha\) and taking the limit as \(\alpha\) goes to 0, leads to

\[-R''_0(p_0^*)[-p'_0(0)] = \frac{\tilde{F}'_0(p_0^*)}{\tilde{F}_1'(p_1^*)}[\tilde{F}'_0(p_0^*)]^{-1}\]

\[-\tilde{F}_0'(p_0^*)[-p'_0(0)] - \tilde{F}_1'(p_1^*)p'_1(0) = [\tilde{F}_1(p_1^*) - \tilde{F}_0(p_0^*)].\]

Note that as opposed to price fairness, we now have \(\lim_{\alpha \to 0} \Delta p_0(\alpha)/\alpha = -p'_0(0)\) and \(\lim_{\alpha \to 0} \Delta p_1(\alpha)/\alpha = p'_1(0)\) (i.e., the sign is reversed), as \(p_0\) decreases and \(p_1\) increases with \(\alpha\). Hence, we have:

\[p'_0(0) = \frac{R''_1(p_1^*) \tilde{F}'_0(p_0^*)}{R''_0(p_0^*) \tilde{F}_1'(p_1^*)^2 + R''_1(p_1^*) \tilde{F}'_1(p_0^*)^2}\Delta w\] and \(p'_1(0) = -\frac{R''_1(p_1^*) \tilde{F}'_0(p_0^*)}{R''_0(p_0^*) \tilde{F}_1'(p_1^*)^2 + R''_1(p_1^*) \tilde{F}'_1(p_0^*)^2}\Delta w,\]

where \(\Delta w = [\tilde{F}_1(p_1^*) - \tilde{F}_0(p_0^*)]\). Again, we require that \(-d_0\tilde{F}_0(p_0^*)p'_0(0) - d_1\tilde{F}_1(p_1^*)p'_1(0) > 0\). Following the same line of argument as for price fairness, we obtain:

\[W'(0) = -d_0\tilde{F}_0(p_0^*)p'_0(0) - d_1\tilde{F}_1(p_1^*)p'_1(0) > 0 \iff d_1\tilde{F}_1(p_1^*)\tilde{F}_0'(p_0^*)R''_0(p_0^*) - d_0\tilde{F}_0(p_0^*)\tilde{F}_1'(p_1^*)R''_1(p_1^*) < 0.\]

We next show that \(p'_1(0)\) is indeed bounded. Consider \(p_0\) as an example. Since the demand change in group 1 is non-negative, from Eq. (C.10) we have \(-\tilde{F}_0'(p_0^*)\Delta p_0 + h_0(\Delta p_0) \leq \alpha[\tilde{F}_1(p_1^*) - \tilde{F}_0(p_0^*)]\). Let \(\Delta \tilde{p}_0 = p_0^* - \tilde{F}_0^{-1}(\tilde{F}_0(p_0^*) + \alpha[\tilde{F}_1(p_1^*) - \tilde{F}_0(p_0^*)]),\) i.e., \(\Delta \tilde{p}_0\) increases the demand by
\[ \alpha [\bar{F}_1(p^*_1) - \bar{F}_0(p^*_0)]. \] Since the demand is monotone, \( \Delta p_0 \leq \Delta \bar{p}_0 \). We then have:

\[
|p'_0(0)| = \lim_{\alpha \to 0} \frac{\Delta p_0}{\alpha} \leq \lim_{\alpha \to 0} \frac{\Delta \bar{p}_0}{\alpha} = \lim_{\alpha \to 0} \frac{\bar{F}^{-1}_0(F_0(p^*_0)) - \bar{F}^{-1}_0(p^*_0) + \alpha [\bar{F}_1(p^*_1) - \bar{F}_0(p^*_0)]}{\alpha} = -[\bar{F}_1(p^*_1) - \bar{F}_0(p^*_0)] \cdot (\bar{F}^{-1}_0)'(\bar{F}_0(p^*_0)) = -[\bar{F}_1(p^*_1) - \bar{F}_0(p^*_0)] \frac{1}{\bar{F}'_0(p^*_0)}.
\]

Hence, showing that \( p'_i(0) \) is bounded.

The proof for (c) surplus fairness follows a similar argument as in (b). The proof for (d) no-purchase valuation fairness is also similar to (b), but note that in this case, \( p_0 \) increases and \( p_1 \) decreases, so that the sign of \( p'_i(0) \) is reversed.

\[ \square \]

**Proof of Theorem 3.2.** For linear demand, without loss of generality, we assume that \( V_i \sim U(0, b_i) \), with \( b_0 < b_1 \). For exponential demand, without loss of generality, we assume that \( V_i \sim Exp(\lambda_i) \), with \( \lambda_0 > \lambda_1 \). See Table C.1 for the closed form expressions of \( R_i, \bar{F}_i, S_i, N_i \). One can check that in both cases, we have \( p^*_0 < p^*_1, \bar{F}_0(p^*_0) < \bar{F}_1(p^*_1), S_0(p^*_0) < S_1(p^*_1) \), and \( N_0(p^*_0) < N_1(p^*_1) \).

We report the expressions of \( p^*_i, R''_i(p^*_i), \bar{F}'_i(p^*_i), \bar{F}_i(p^*_i), N'_i(p^*_i) \) in Table C.2. By substituting these expressions into the conditions in Lemma C.1, one can show that the inequalities for price and no-purchase valuation fairness are always satisfied, whereas the inequalities for demand and surplus fairness conditions are always violated.

**Table C.2: Function values for linear and exponential demand models.**

<table>
<thead>
<tr>
<th></th>
<th>( p^*_i )</th>
<th>( R''_i(p^*_i) )</th>
<th>( \bar{F}'_i(p^*_i) )</th>
<th>( \bar{F}_i(p^*_i) )</th>
<th>( N'_i(p^*_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>( \frac{b_i+c}{2} )</td>
<td>( -2d_i/b_i )</td>
<td>( -1/b_i )</td>
<td>( \frac{b_i-c}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( \frac{1}{\lambda_i} + c )</td>
<td>( -\lambda_i e^{-1} - \lambda_i c )</td>
<td>( -\lambda_i e^{-1} - \lambda_i c )</td>
<td>( e^{-1} - \lambda_i c )</td>
<td>( \frac{\lambda_i e^{\lambda_i c} - 1}{(e^{\lambda_i c} - 1)^2} )</td>
</tr>
</tbody>
</table>

\[ \square \]
C.2 Proofs of Propositions 3.2 to 3.5

In optimization problem (3.1), we may use prices which are not in $[0, b_i]$, so that the problem is not necessarily convex. To make the analysis simpler, in each proposition we discuss four cases separately: (1) $p_i \leq b_i$ for both groups; (2) $p_0 > b_0$ and $p_1 \leq b_1$; (3) $p_0 \leq b_0$ and $p_1 > b_1$; and (4) $p_i > b_i$ for both groups. We then compare the optimal solution for each case and characterize the optimal solution to the problem. We first point out that case (4) can be eliminated as it leads to zero profit, and we do not discuss this case in the subsequent proofs. Note that for case (1), for each fairness metric $M_i(p_i)$, we always have $M_0(p_0^*) \leq M_0(p_0) \leq M_1(p_1) \leq M_1(p_1^*)$. First, note that if $M_i(p_i)$ is not in $[M_0(p_0^*), M_1(p_1^*)]$, we can set the price of group $i$ to be $p_i^*$, such that the solution is still feasible, but the profit is higher because for group $i$ we use the unconstrained optimal price. Second, if $M_1(p_1) < M_0(p_0)$, then one can construct another solution $p_i'$ such that $M_0(p_0') = M_1(p_1)$ and $M_1(p_i') = M_0(p_0)$, which is also a feasible solution. However, because $M_i(p_i')$ is closer to $M_i(p_i^*)$, the price changes less when compared to using $p_i$, and hence the constructed prices have higher profit. Finally, in case (1) we have w.l.o.g. that $M_1(p_1) - M_0(p_0) = (1 - \alpha)|M_1(p_1^*) - M_0(p_0^*)|$, i.e., the solution is tight. If this is not the case, one can fix $M_0(p_0)$ and increase $M_1(p_1)$ slightly such that the solution is still feasible. However, by moving $M_1(p_1)$ closer to the unconstrained level, the profit can only increase.

Proof of Proposition 3.2. Here we first expand the proposition in the main body with closed-form solution for prices, as well as changes in profit, consumer surplus and social welfare. Let $w = p_1^* - p_0^* = \frac{b_1 - b_0}{2}$. If $0 \leq \alpha \leq \bar{\alpha}_p$, then

$$p_0(\alpha) = p_0^* + \frac{d_1 b_0}{d_0 b_1 + d_1 b_0} \alpha w \quad \text{and} \quad p_1(\alpha) = p_1^* - \frac{d_0 b_1}{d_0 b_1 + d_1 b_0} \alpha w.$$
The changes in profit, consumer surplus, and social welfare are given by:

\[
\begin{align*}
R(\alpha) - R(0) &= -\frac{d_0 d_1}{d_0 b_1 + d_1 b_0} (\alpha w)^2 \leq 0, \\
S(\alpha) - S(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} [(b_1 - b_0)\alpha w + (\alpha w)^2] \geq 0, \\
W(\alpha) - W(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} [(b_1 - b_0)\alpha w - (\alpha w)^2] \geq 0.
\end{align*}
\]

If \( p_0(\alpha) = p_1(\alpha) = p_1^* = \frac{b_1 + c}{2} > b_0 \)

and

\[
\begin{align*}
R(\alpha) - R(0) &= -R_0(p_0^*) < 0, \\
S(\alpha) - S(0) &= -d_0 S_0(p_0^*) < 0, \\
W(\alpha) - W(0) &= -R_0(p_0^*) - d_0 S_0(p_0^*) < 0.
\end{align*}
\]

We next prove these statements below.

Now we prove the above statements by analyzing the three possible cases. For case (3), the price difference is greater than \( b_1 - b_0 \). Since \( p_1^* - p_0^* = (b_1 - b_0)/2 \), then any price policy for case (3) is infeasible. We next analyze the profit from cases (1) and (2).
Case (1): Let $\Delta p_0$ and $\Delta p_1$ be the price changes, that is, $p_0 = p_0^* + \Delta p_0$ and $p_1 = p_1^* - \Delta p_1$. Let $w = (b_1 - b_0)/2$. For the seller, the profit optimization problem in Eq. (C.4) can be written as:

$$\begin{align*}
\min & \frac{d_0}{b_0} \Delta p_0^2 + \frac{d_1}{b_1} \Delta p_0^2 \\
\text{s.t.} & \Delta p_0 + \Delta p_1 = \alpha w \\
& \Delta p_0 \leq b_0 - p_0^* \\
& \Delta p_0 \leq p_1^* \\
& \Delta p_i \geq 0.
\end{align*}$$

We first relax the upper-bound constraints, and then characterize the condition under which such constraints are not tight. When the upper-bound constraints are removed, solving the above problem leads to

$$\Delta p_0 = \frac{d_1 b_0}{d_0 b_1 + d_1 b_0} \alpha w \quad \text{and} \quad \Delta p_1 = \frac{d_0 b_1}{d_0 b_1 + d_1 b_0} \alpha w.$$ (C.11)

By substituting $p_0$ and $p_1$ into the profit, consumer surplus, and social welfare functions, we obtain:

$$\begin{align*}
\mathcal{R}(\alpha) - \mathcal{R}(0) &= -\frac{d_0 d_1}{d_0 b_1 + d_1 b_0} (\alpha w)^2, \\
\mathcal{S}(\alpha) - \mathcal{S}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} \left[ (b_1 - b_0) \alpha w + (\alpha w)^2 \right], \\
\mathcal{W}(\alpha) - \mathcal{W}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} \left[ (b_1 - b_0) \alpha w - (\alpha w)^2 \right].
\end{align*}$$

Such a solution is valid as long as $\Delta p_i$ does not reach the upper bounds. Specifically, taking Eq. (C.11) into $\Delta p_0 \leq b_0 - p_0^*$, we have $\alpha \leq \frac{d_0 d_1 + d_1 b_0}{d_0 b_1 + d_1 b_0} \frac{b_0 - c}{b_1 - b_0}$. On the other hand, $\Delta p_1 \leq p_1^*$ implies that $\alpha \leq \frac{d_0 b_1 + d_1 b_0}{d_0 b_1} \frac{b_1 + c}{b_1 - b_0}$, which always holds since the right-hand side is greater than 1. We will later argue that we do not need to consider the case when $\alpha > \frac{d_0 b_1 + d_1 b_0}{d_0 b_1} \frac{b_0 - c}{b_1 - b_0}$. 

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Case (2): In this case, \( p_0 > b_0 \), so that both the profit and the consumer surplus from group 0 are zero. For group 1, the optimal price is \( p_1^* \). Therefore, the optimal solution is always \( p_0 = p_1 = p_1^* \), the profit loss is \( R_0(p_0^*) \), and the consumer surplus loss is \( S_0(p_0^*) \).

We next compare cases (1) and (2). First, the analysis of case (1) is only valid when \( p_0 + \Delta p_0 \leq b_0 \), i.e., \( \alpha \leq \frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0} \). On the other hand, by comparing the profit for cases (1) and (2), one can see that for small values of \( \alpha \), the profit in case (1) is larger (close to optimal), whereas the profit in case (2) is fixed. If \( \alpha \) is large enough, the profit loss in case (1) is greater than the profit from group 0. Formally,

\[
\frac{d_0 d_1}{d_0 b_1 + d_1 b_0} (\alpha w)^2 \geq \frac{d_0(b_0 - c)^2}{4b_0}.
\]

By rearranging terms, we obtain \( \alpha \geq \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}} \). Thus, when \( \alpha \leq \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}} \), case (1) has a higher profit. The transition from case (1) to (2) happens either when case (1) is not feasible (i.e., \( \alpha > \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}} \)), or when it has a lower profit (i.e., \( \alpha > \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}} \)). Since \( \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}} \leq \frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0} \), then the transition happens before \( p_0 \) reaches \( b_0 \). Consequently, the threshold value is \( \tilde{\alpha}_p = \max \left( \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}}, 1 \right) \). Accordingly, \( W(\alpha) \) first increases with \( \alpha \), but after \( \tilde{a} \), \( W(\alpha) \) “jumps” below \( W(0) \).

Proof of Proposition 3.3. Similar to price fairness, we first provide the closed-form solution with respect to \( \alpha \). For demand fairness, let \( w = \frac{(b_1 - b_0)c}{2b_0 b_1} \). Then, we have:

\[
p_0(\alpha) = p_0^* - \frac{d_1 b_0 b_1}{d_0 b_1 + d_1 b_0} \alpha w \quad \text{and} \quad p_1(\alpha) = p_1^* + \frac{d_0 b_0 b_1}{d_0 b_0 + d_1 b_1} \alpha w.
\]

The changes in profit, consumer surplus, and social welfare are given by:

\[
\mathcal{R}(\alpha) - \mathcal{R}(0) = -\frac{d_0 d_1 b_0 b_1}{d_0 b_0 + d_1 b_1} (\alpha w)^2 \leq 0,
\]

\[
\mathcal{S}(\alpha) - \mathcal{S}(0) = \frac{d_0 d_1}{2(d_0 b_0 + d_1 b_1)} \left[ -(b_1 - b_0) c \alpha w + b_0 b_1 (\alpha w)^2 \right] \leq 0,
\]

\[
\mathcal{W}(\alpha) - \mathcal{W}(0) = \frac{d_0 d_1}{2(d_0 b_0 + d_1 b_1)} \left[ -(b_1 - b_0) c \alpha w - b_0 b_1 (\alpha w)^2 \right] \leq 0.
\]
We next prove these statements below.

We first consider case (1), where \( p_i \leq b_i \) for both groups. Let \( \Delta p_0 \) and \( \Delta p_1 \) be the price changes, that is, \( p_0 = p_0^* - \Delta p_0 \) and \( p_1 = p_1^* + \Delta p_1 \). The initial difference in demand is \( (b_1 - b_0)c/2b_0b_1 \). Similar to Proposition 3.2, the optimization problem is given by:

\[
\min \frac{d_0}{b_0} \Delta p_0^2 + \frac{d_1}{b_1} \Delta p_1^2 \\
\text{s.t.} \quad \frac{\Delta p_0}{b_0} + \frac{\Delta p_1}{b_1} = \alpha \frac{(b_1 - b_0)c}{2b_0b_1} \\
\Delta p_0 \leq p_0^* \\
\Delta p_0 \leq b_1 - p_1^* \\
\Delta p_i \geq 0.
\]

If we ignore the upper bounds, the above problem leads to

\[
\Delta p_0 = \frac{d_1 b_0 b_1}{d_0 b_0 + d_1 b_1} \alpha w \quad \text{and} \quad \Delta p_1 = \frac{d_0 b_0 b_1}{d_0 b_0 + d_1 b_1} \alpha w,
\]

where \( w = \frac{(b_1 - b_0)c}{2b_0b_1} \). Substituting \( p_0 \) and \( p_1 \) into the profit, consumer surplus, and social welfare (defined in Section 3.3) yields the desired result. Note that the above analysis holds for any \( \alpha \), as the prices will not reach either boundary (0 and \( b_i \)). This follows from the fact that \( p_0 \) decreases and \( p_1 \) increases with \( \alpha \), and the demand can be matched before one of the prices reaches the boundary.

For case (2), one can observe that the surplus fairness and profit are not impacted whether \( p_0 > b_0 \) or is exactly \( p_0 = b_0 \). Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1).

It is easy to see that \( \mathcal{R}(\alpha) - \mathcal{R}(0) \) is always negative. To see that \( S(\alpha) - S(0) \) is always negative, note that \( S(\alpha) - S(0) = 0 \) at \( \alpha = 0 \) and decreases with \( \alpha \) on \( \alpha \in \left[0, \frac{(b_1 - b_0)c}{2b_0b_1w}\right]\), where \( \frac{(b_1 - b_0)c}{2b_0b_1w} = 1 \).

\[ \square \]

**Proof of Proposition 3.4.** For case (2), one can observe that the surplus fairness and profit are not
impacted if $p_0 > b_0$ or is exactly $p_0 = b_0$. Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1). Thus, we only need to consider case (1), where $p_i \leq b_i$ for both groups.

For case (1), the normalized consumer surplus is given by $S_i(p_i) = \frac{(b_i - p_i)(1 - p_i/b_i)}{2}$. Correspondingly, when we use $p_0 = p^*_0 - \Delta p_0$ and $p_1 = p^*_1 + \Delta p_1$, we have $S_0(p^*_0 - \Delta p_0) - S_0(p^*_0) = \frac{b_0 - c}{2b_0} \Delta p_0 + \frac{1}{2b_0} \Delta p^2_0$ and $S_1(p^*_1) - S_1(p^*_1 + \Delta p_1) = \frac{b_1 - c}{2b_1} \Delta p_1 - \frac{1}{2b_1} \Delta p^2_1$. The initial consumer surplus difference is $\frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0}$. Hence, the fairness constraint is given by

$$\frac{b_0 - c}{2b_0} \Delta p_0 + \frac{1}{2b_0} \Delta p^2_0 + \frac{b_1 - c}{2b_1} \Delta p_1 - \frac{1}{2b_1} \Delta p^2_1 = \alpha \left[ \frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0} \right].$$

The optimization problem now becomes:

$$\min \frac{d_0}{b_0} \Delta p^2_0 + \frac{d_1}{b_1} \Delta p^2_0$$

s.t. $\frac{b_0 - c}{2b_0} \Delta p_0 + \frac{1}{2b_0} \Delta p^2_0 + \frac{b_1 - c}{2b_1} \Delta p_1 - \frac{1}{2b_1} \Delta p^2_1 = \alpha \left[ \frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0} \right]$

$$\Delta p_0 \leq p^*_0$$

$$\Delta p_0 \leq b_1 - p^*_1$$

$$\Delta p_i \geq 0.$$
Consequently, we have

$$\Delta p_0 = \frac{(b_0 - c)\lambda}{4d_0 - 2\lambda} \text{ and } \Delta p_1 = \frac{(b_1 - c)\lambda}{4d_1 + 2\lambda},$$

where $\lambda$ satisfies

$$\frac{b_0 - c (b_0 - c)\lambda}{2b_0} \frac{1}{4d_0 - 2\lambda} + \frac{1}{2b_0} \left( \frac{(b_0 - c)\lambda}{4d_0 - 2\lambda} \right)^2 + \frac{1}{2} + \frac{(b_1 - c)\lambda}{4d_1 + 2\lambda} - \frac{1}{2b_1} \left( \frac{(b_1 - c)\lambda}{4d_1 + 2\lambda} \right)^2 = \alpha \left[ \frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0} \right].$$

(C.14)

While the closed-form expression of $\lambda$ can be computed by transforming Eq. (C.14) into a quartic function, it is complicated and we do not necessarily need it. On the other hand, we use several properties of $\lambda$. First, $\lambda$ must always be positive due to the fact that $2\frac{d_0}{b_0} \Delta p_0 = \lambda \frac{b_0 - c}{2b_0} + \frac{1}{\lambda} \Delta p_0$. Second, both $\Delta p_0$ and $\Delta p_1$ increase with $\lambda$, for $\lambda \in [0, 2d_0]$, and since $\Delta p_0 \geq 0$, the case with $\lambda > 2d_0$ never occurs. Third, note that the left-hand side of Eq. (C.12) increases with both $\Delta p_0$ and $\Delta p_1$, for $\Delta p_0 \in [0, p_0^*]$ and $\Delta p_1 \in [0, b_1 - p_1^*]$, while the right-hand side increases with $\alpha$. Together with the fact that $\Delta p_0$ and $\Delta p_1$ increase with $\lambda$, we know that $\lambda$ increases with $\alpha$, and one can check that $\lambda = 0$ when $\alpha = 0$.

We next show that $W(\alpha) < W(0)$ for $\alpha \in (0, 1]$. Recall that the profit loss is $d_0 \Delta p_0^2 + d_1 \Delta p_1^2$ and the surplus change is $d_0 \frac{b_0 - c}{2b_0} \Delta p_0 + \frac{d_0}{2b_0} \Delta p_0^2 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1 + \frac{d_1}{2b_1} \Delta p_1^2$. Thus, the social welfare change is $d_0 \frac{b_0 - c}{2b_0} \Delta p_0 - \frac{d_0}{2b_0} \Delta p_0^2 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1 - \frac{d_1}{2b_1} \Delta p_1^2$. Since $-\frac{d_0}{2b_0} \Delta p_0^2 - \frac{d_1}{2b_1} \Delta p_1^2$ is negative for any $\alpha$, we only need to show that $d_0 \frac{b_0 - c}{2b_0} \Delta p_0 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1$ is negative for any $\alpha$. By substituting Eq. (C.13) into $d_0 \frac{b_0 - c}{2b_0} \Delta p_0 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1$, we obtain:

$$d_0 \frac{b_0 - c}{2b_0} \Delta p_0 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1 = \left[ d_0 b_1 (b_0 - c)^2 + d_1 b_0 (b_1 - c)^2 \right] \lambda^2 + 2d_0 d_1 \left[ b_1 (b_0 - c)^2 - b_0 (b_1 - c)^2 \right] \frac{\lambda}{2b_0 b_1 (2d_0 - \lambda)(2d_1 + \lambda)}.$$

(C.15)

The denominator of Eq. (C.15) is positive since $\Delta p_i, i = 0, 1$ are positive. The numerator of Eq. (C.15) is negative for $\lambda \in (0, \bar{\lambda})$, where

$$\bar{\lambda} = \frac{2d_0 d_1 \left[ b_0 (b_1 - c)^2 - b_1 (b_0 - c)^2 \right]}{d_0 b_1 (b_0 - c)^2 + d_1 b_0 (b_1 - c)^2}.$$
We claim that the largest possible value of \( \lambda \) is below \( \bar{\lambda} \), so that \( d_0 \frac{b_0 - c}{2b_0} \Delta p_0 - d_1 \frac{b_1 - c}{2b_1} \Delta p_1 \) is negative for any \( \alpha \). To show this claim, we substitute \( \lambda \) into the left-hand side of Eq. (C.14) and obtain:

\[
\frac{b_0 - c}{2b_0} (b_0 - c) \bar{\lambda} + \frac{1}{2b_0} \left[ \frac{(b_0 - c) \bar{\lambda}}{4d_0 - 2 \bar{\lambda}} \right]^2 + \frac{b_1 - c}{2b_1} (b_1 - c) \bar{\lambda} - \frac{1}{2b_1} \left[ \frac{(b_1 - c) \bar{\lambda}}{4d_1 + 2 \bar{\lambda}} \right]^2 - \left[ \frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0} \right] = \frac{1}{8b_0 b_1 (b_0 - c)^2 (b_1 - c)^2 (d_0 + d_1)^2} \left[ b_0^2 b_1 d_0 + b_1 c^2 d_0 + b_0 \left[ b_1^2 d_1 + c^2 d_1 - 2 b_1 c (d_0 + d_1) \right] \right] > 0,
\]

where the inequality comes from the facts that \( b_1 - b_0 > 0 \) and \( b_1 b_0 - c^2 > 0 \). This indicates that if \( \lambda \) reaches \( \bar{\lambda} \), the surplus change is greater than \( \left[ \frac{(b_1 - c)^2}{8b_1} - \frac{(b_0 - c)^2}{8b_0} \right] \), which is equal to the initial difference. As a result, \( \lambda \) will never reach \( \bar{\lambda} \), and the corresponding \( \Delta W(\alpha) - \Delta W(0) \) is always negative.

The above analysis holds only for case (a), that is, before \( p_0 \) reaches zero. For case (b), if there exists \( \bar{\alpha} \) such that \( \Delta p_0(\bar{\alpha}) = p_0^* \), then for \( \alpha > \bar{\alpha} \), \( p_0 \) stays at zero and \( p_1 \) increases monotonically. In this case, \( W(\alpha) \) decreases with \( \alpha \), and \( W(\alpha) < W(\bar{\alpha}) < W(0) \). \( \square \)

**Proof of Proposition 3.5.** Similar to price fairness, we first provide closed-form solution to the problem. For no-purchase valuation fairness, let \( \bar{\alpha}_n = \min \left( \frac{d_1 b_0 + d_0 b_1}{d_1 b_0} \frac{b_0 - c}{b_1 - b_0}, 1 \right) \) and \( w = p_1^* - p_0^* = \frac{b_0}{1 - \bar{\alpha}_n} \). If \( \alpha < \bar{\alpha}_n \), then

\[
p_0(\alpha) = p_0^* + \frac{d_1 b_0}{d_0 b_1 + d_1 b_0} \alpha w \quad \text{and} \quad p_1(\alpha) = p_1^* - \frac{d_0 b_1}{d_0 b_1 + d_1 b_0} \alpha w.
\]

The changes in profit, consumer surplus, and social welfare are given by:

\[
\mathcal{R}(\alpha) - \mathcal{R}(0) = -\frac{d_0 d_1}{d_0 b_1 + d_1 b_0} (\alpha w)^2 < 0,
\]

\[
\mathcal{S}(\alpha) - \mathcal{S}(0) = \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} \left[ (b_1 - b_0) \alpha w + (\alpha w)^2 \right] \geq 0,
\]

\[
\mathcal{W}(\alpha) - \mathcal{W}(0) = \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)} \left[ (b_1 - b_0) \alpha w - (\alpha w)^2 \right] \geq 0.
\]
If $\tilde{\alpha}_n < \alpha \leq 1$, then

$$p_0(\alpha) = b_0 \quad \text{and} \quad p_1(\alpha) = b_0 + (1 - \alpha)w.$$ 

Let $\tilde{p}_1 = p_1^* - \frac{d_0b_1 - b_0 - c}{b_1 - b_0}$. Then, we have:

$$R(\alpha) - R(\tilde{\alpha}_n) = \frac{-w^2(\alpha - \tilde{\alpha}_n)^2 - (b_1 + c - 2\tilde{p}_1)w(\alpha - \tilde{\alpha}_n)}{b_1} < 0,$$

$$S(\alpha) - S(\tilde{\alpha}_n) = \frac{2(b_1 - \tilde{p}_1)w(\alpha - \tilde{\alpha}_n) + w^2(\alpha - \tilde{\alpha}_n)^2}{2b_1} > 0,$$

$$W(\alpha) - W(\tilde{\alpha}_n) = \frac{(2\tilde{p}_1 - 2c)w(\alpha - \tilde{\alpha}_n) - w^2(\alpha - \tilde{\alpha}_n)^2}{2b_1} > 0.$$

We next prove these statements below.

For case (1), since $N_i(p) = p/2$, the analysis from the proof of Proposition 3.2 holds before $p_0$ reaches $b_0$, that is, when $\alpha \leq \frac{d_0b_1 + d_1b_0}{d_1b_0}$. For $\alpha > \frac{d_0b_1 + d_1b_0}{d_1b_0} - \frac{b_0 - c}{b_1 - b_0}$, $p_0$ stays at $b_0$ and $N_0(b_0) = \frac{b_0}{2}$. The gap in no-purchase valuation is $(1 - \alpha)[N_1(p_1^*) - N_0(p_0^*)]$, and hence $N_1(p_1) = \frac{b_0}{2} + (1 - \alpha)[N_1(p_1^*) - N_0(p_0^*)]$, i.e., $\frac{p_1}{2} = \frac{b_0}{2} + (1 - \alpha)\frac{b_1 - b_0}{4}$. Rearranging terms leads to $p_1 = b_0 + (1 - \alpha)(b_1 - b_0)/2$. Substituting $p_1$ into the profit and consumer surplus functions yields our desired result.

For case (2), one can observe that the no-purchase valuation fairness and profit are not impacted if $p_0 > b_0$ or if $p_0 = b_0$. Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1).

\[\square\]

C.3 Proofs of Propositions 3.6 and 3.7

Proof of Proposition 3.6. We prove the results of each part separately. Without loss of generality, we assume that the parameters $b_i$ are indexed in increasing order.

(a) Demand fairness. Given $\alpha$, let $q_i = q_i(\alpha) = \bar{F}_i(p_i(\alpha))$ be the optimal normalized demand for group $i$. The profit of group $i$ given $q_i$ is equal to $d_iq_i(b_i - c - b_iq_i)$. Let $q_i^* = q_i(0) = (b_i - c)/2$ be the optimal unconstrained normalized demand. We define $I_{dec}(\alpha) = \{i | q_i(\alpha) < \}$
\[ q_i^* \} \text{ and } I_{\text{inc}}(\alpha) = \{ i | q_i(\alpha) > q_i^* \} \] as the sets of groups with demand that decrease and increase relative to the unconstrained optimal solution, respectively. For each specific \( \alpha \), we do not need to consider the groups whose prices remain unchanged, because these groups do not contribute to the difference in social welfare.

Consider the normalized demand for group \( i \in I_{\text{dec}}(\alpha) \). We next show that all the groups in \( I_{\text{dec}}(\alpha) \) should have the same demand. Indeed, if there exist \( i, j \in I_{\text{dec}}(\alpha) \) such that \( q_i(\alpha) > q_j(\alpha) \), one can increase \( q_j \) such that \( q_j = q_i(\alpha) \). By doing so, the fairness constraints still hold, and we arrive at a demand that is closer to \( q_j^* \), and hence corresponds to a higher profit. As a result, for all \( i \in I_{\text{dec}}(\alpha) \), the demand level must be the same. Similarly, all the groups in \( I_{\text{inc}}(\alpha) \) must have the same demand. Let \( q_{\text{dec}} \) and \( q_{\text{inc}} \) be the demand levels of decreasing and increasing groups, respectively. One can also see that w.l.o.g., \( q_{\text{dec}} - q_{\text{inc}} = (1 - \alpha)|q_{N-1}^* - q_0^*| \).

Let \( q_{\text{inc}}(\alpha) \) and \( q_{\text{dec}}(\alpha) \) be the demand levels for \( I_{\text{inc}}(\alpha) \) and \( I_{\text{dec}}(\alpha) \), respectively. We first show that \( q_{\text{inc}}(\alpha) \) (resp. \( q_{\text{dec}}(\alpha) \)) increases (resp. decreases) monotonically with \( \alpha \). First, note that given \( q_{\text{inc}} \) and \( q_{\text{dec}} \), we can construct a solution for all the \( N \) groups, by setting \( q_i = \min(\max(q_{\text{inc}}, q_i^*), q_{\text{dec}}) \). Let \( h(q_{\text{inc}}, q_{\text{dec}}) = \sum_{i=0}^{N-1} R_i(F_i^{-1}(\min(q_{\text{inc}}, q_i^*, q_{\text{dec}}))) \) be the profit with respect to \( q_{\text{inc}} \) and \( q_{\text{dec}} \). One can easily verify that \( h(q_{\text{inc}}, q_{\text{dec}}) \) is concave in the region \( 0 \leq q_{\text{inc}} \leq q_{\text{dec}} \leq 1 \). The optimization problem (3.2) can then be written as

\[
\begin{align*}
\max_{q_{\text{inc}}, q_{\text{dec}}} & \quad h(q_{\text{inc}}, q_{\text{dec}}) \\
\text{s.t.} & \quad q_{\text{dec}} - q_{\text{inc}} \leq (1 - \alpha)|q_{N-1}^* - q_0^*|, \\
& \quad q_{\text{inc}} \leq q_{\text{dec}}, \\
& \quad q_{\text{inc}}, q_{\text{dec}} \in [q_0^*, q_{N-1}^*].
\end{align*}
\]
The KKT condition is given by
\[
\begin{bmatrix}
\frac{\partial h}{\partial q_{\text{inc}}} \\
\frac{\partial h}{\partial q_{\text{dec}}}
\end{bmatrix} = \mu_1 \begin{bmatrix}
-1 \\
1
\end{bmatrix} + \mu_2 \begin{bmatrix}
1 \\
-1
\end{bmatrix},
\]
(C.16)
\[
q_{\text{dec}} - q_{\text{inc}} \leq (1 - \alpha)|q_{N-1}^* - q_0^*|,
\]
\[
q_{\text{inc}} - q_{\text{dec}} \leq 0,
\]
\[
\mu_1 \left( q_{\text{dec}} - q_{\text{inc}} - (1 - \alpha)|q_{N-1}^* - q_0^*| \right) = 0
\]
\[
\mu_2 (q_{\text{inc}} - q_{\text{dec}}) = 0
\]
\[
q_{\text{inc}}, q_{\text{dec}} \in [q_0^*, q_{N-1}^*],
\]
\[
\mu_1, \mu_2 \geq 0.
\]
Since \(q_{\text{dec}} - q_{\text{inc}} = (1 - \alpha)|q_{N-1}^* - q_0^*| > 0\), we have that \(\mu_2 = 0\) due to complementary slackness. Note that \(\frac{\partial h}{\partial q_{\text{inc}}}\) is non-positive and monotonically decreasing in the range \([q_0^*, q_{\text{dec}}]\), whereas \(\frac{\partial h}{\partial q_{\text{inc}}}\) is non-negative and monotonically decreasing in the range \([q_{\text{inc}}, q_{N-1}^*]\). With these facts in hand, we see that as \(\alpha\) increases in Eq. (C.16), \(q_{\text{inc}}\) and \(q_{\text{dec}}\) move towards one another. Since their difference is monotonically decreasing with \(\alpha\), we have \(q_{\text{inc}}(\alpha)\) monotonically increases and \(q_{\text{dec}}(\alpha)\) monotonically decreases.

Since we have shown that \(q_{\text{inc}}(\alpha)\) and \(q_{\text{dec}}(\alpha)\) are monotone and move towards one another, it follows that the functions are also continuous since \(q_{\text{dec}}(\alpha) - q_{\text{inc}}(\alpha) = (1 - \alpha)(q_{N-1}^* - q_0^*)\). Consequently, the corresponding social welfare \(W(\alpha)\) is also continuous in \(\alpha\). Since \(q_{\text{inc}}(\alpha)\) and \(q_{\text{dec}}(\alpha)\) are monotone, \(I_{\text{inc}}(\alpha)\) and \(I_{\text{dec}}(\alpha)\) are also monotone, that is, \(I_{\text{inc}}(\alpha_1) \subset I_{\text{inc}}(\alpha_2)\) and \(I_{\text{dec}}(\alpha_1) \subset I_{\text{dec}}(\alpha_2)\) for any \(\alpha_1 < \alpha_2\). We can then split \(\alpha \in [0, 1]\) into at most \(N\) non-overlapping intervals, based on the value of \(I_{\text{inc}}(\alpha)\) and \(I_{\text{dec}}(\alpha)\). For the first interval, we have \(I_{\text{inc}}(\alpha) = \{0\}\) and \(I_{\text{dec}}(\alpha) = \{N - 1\}\). As \(\alpha\) increases, we either add group 1 to \(I_{\text{inc}}\) or group \(N - 2\) to \(I_{\text{dec}}\), and so on. Since the social welfare curve is continuous, it is enough to show that for each interval such that \(I_{\text{inc}}(\alpha)\) and \(I_{\text{dec}}(\alpha)\) are fixed, the social welfare is monotonically decreasing. By the continuity of the social welfare function, this translates to the social welfare being monotonically decreasing.

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Suppose that for $\alpha \in [\alpha_1, \alpha_2]$, $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are fixed. Recall that the normalized demand for $i$ in $I_{inc}(\alpha)$ (or $I_{dec}(\alpha)$) are the same. The profit maximization problem (3.2) can be re-written as

$$\max_{q_i} \sum_{i \in I_{dec}} d_i q_i (b_i - b_i q_i - c) + \sum_{i \in I_{inc}} d_i q_i (b_i - b_i q_i - c)$$ \hspace{1cm} (C.17)

s.t. $|q_i - q_j| \leq (1 - \alpha) (q^*_N - q^*_0)$, $\forall i \in I_{dec}$, $j \in I_{inc}$,

$q_i \in [0, 1]$, $\forall i \in I_{dec} \cup I_{inc}$.

where the groups for which $p_i(\alpha) = p^*_i$ are not considered because they do not impact the optimal solution. Based on the above analysis, problem (C.17) reduces to

$$\max_{q_{dec}, q_{inc}} q_{dec} \left( \sum_{i \in I_{dec}} d_i b_i - \left( \sum_{i \in I_{dec}} d_i b_i \right) q_{dec} - \left( \sum_{i \in I_{inc}} d_i \right) c \right) + q_{inc} \left( \sum_{i \in I_{inc}} d_i b_i - \left( \sum_{i \in I_{inc}} d_i b_i \right) q_{inc} - \left( \sum_{i \in I_{inc}} d_i \right) c \right)$$

s.t. $|q_{dec} - q_{inc}| = (1 - \alpha) (q^*_N - q^*_0)$,

$q_{dec}, q_{inc} \in [0, 1]$.

Interestingly, this is exactly the problem for the setting with two groups: (i) group dec has population $\sum_{i \in I_{dec}} d_i$ and parameter $b_{dec} = \frac{\sum_{i \in I_{dec}} d_i b_i}{\sum_{i \in I_{dec}} d_i}$, and (ii) group inc has population $\sum_{i \in I_{inc}} d_i$ and parameter $b_{inc} = \frac{\sum_{i \in I_{inc}} d_i b_i}{\sum_{i \in I_{inc}} d_i}$. We further point out that the consumer surplus for multiple groups can also be represented by these two new aggregate groups (dec and inc). To see this, note that given a normalized demand $q$, the consumer surplus of this group is given by $b_i q^2 / 2$, which is linear in $b_i$. Thus, the total consumer surplus from $I_{dec}$ (resp. $I_{inc}$) is $\sum_{i \in I_{dec}} d_i b_i q_{dec}^2 / 2 = (\sum_{i \in I_{dec}} d_i) b_{dec} q_{dec}^2 / 2$. As a result, the two-group problem has exactly the same profit and consumer surplus as the multi-group problem. Following Proposition 3.3, the social welfare for the two-group problem always decreases with $\alpha$. Thus, on each piece $[\alpha_1, \alpha_2]$, the social welfare is monotonically decreasing, and the social welfare function is continuous on $[0, 1]$, which implies that $W(\alpha)$ is monotonically decreasing.
(b) Surplus fairness. Given $\alpha$, recall that $p_i(\alpha)$ is the optimal solution for group $i$. We define $I_{\text{dec}} = \{i | p_i(\alpha) > p_i^*\}$ and $I_{\text{inc}} = \{i | p_i(\alpha) < p_i^*\}$ as the sets of groups with surplus that decrease and increase with $\alpha$ relative to the unconstrained optimal solution, respectively. As before, we do not need to consider any group $i$ where $p_i(\alpha) = p_i^*$ and thus the surplus remains $S_i(p_i^*)$. These groups do not contribute to the change in social welfare, $\mathcal{W}(\alpha) - \mathcal{W}(0)$. As in part (a), all the groups in $I_{\text{dec}}$ ($I_{\text{inc}}$) share the same level of surplus, and the difference between the two sets is $(1 - \alpha)|S_{N-1}(p_{N-1}^*) - S_0(p_0^*)|$.

We consider two cases separately: $p_0(\alpha) > 0$ and $p_0(\alpha) = 0$. When $p_0(\alpha) > 0$, we note that for group $i$, if the surplus is $s_i$, then the demand is given by $\sqrt{2s_i/b_i}$, and the price is given by $b_i - \sqrt{2b_is_i}$. As a result, the profit from group $i$ is equal to $(b_i - \sqrt{2b_is_i} - c)\sqrt{2s_i/b_i} = (\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_i} - 2s_i$. Given that all the groups in $I_{\text{dec}}$ ($I_{\text{inc}}$) have the same level of surplus, we use $s_{\text{dec}}$ ($s_{\text{inc}}$) to denote the surplus for all the groups in the set. Then, the profit-maximization problem (3.2) can be re-written as

$$\max_{s_{\text{dec}},s_{\text{inc}}} \sum_{i \in I_{\text{dec}}} d_i \left[ (\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_{\text{dec}}} - 2s_{\text{dec}} \right] + \sum_{i \in I_{\text{inc}}} d_i \left[ (\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_{\text{inc}}} - 2s_{\text{inc}} \right]$$

\hspace{10cm} (C.18)

s.t. $|s_{\text{dec}} - s_{\text{inc}}| = (1 - \alpha)|S_{N-1}(p_{N-1}^*) - S_0(p_0^*)|$, where we relax the non-negativity constraints on the price as we already assume that $p_i(\alpha) > 0$.

Note that $\sqrt{2b} - c\sqrt{2/b}$ is a strictly increasing function with respect to $b$ for $b > 0$ and it ranges from negative infinity to infinity. Thus, there exists a unique $b_{\text{dec}}$ such that

$$\sqrt{2b_{\text{dec}}} - c\sqrt{2/b_{\text{dec}}} = \frac{\sum_{i \in I_{\text{dec}}} d_i(\sqrt{2b_i} - c\sqrt{2/b_i})}{\sum_{i \in I_{\text{dec}}} d_i},$$

and a unique $b_{\text{inc}}$ such that

$$\sqrt{2b_{\text{inc}}} - c\sqrt{2/b_{\text{inc}}} = \frac{\sum_{i \in I_{\text{inc}}} d_i(\sqrt{2b_i} - c\sqrt{2/b_i})}{\sum_{i \in I_{\text{inc}}} d_i}.$$
Therefore, Eq. (C.18) can be rewritten as

\[
\begin{align*}
\max_{s_{\text{dec}}, s_{\text{inc}}} & \left( \sum_{i \in I_{\text{dec}}} d_i \right) \left[ (\sqrt{2b_{\text{dec}}} - c\sqrt{2/b_{\text{dec}}})\sqrt{s_{\text{dec}}} - 2s_{\text{dec}} \right] + \left( \sum_{i \in I_{\text{inc}}} d_i \right) \left[ (\sqrt{2b_{\text{inc}}} - c\sqrt{2/b_{\text{inc}}})\sqrt{s_{\text{inc}}} - 2s_{\text{inc}} \right], \\
\text{s.t.} & \quad |s_{\text{dec}} - s_{\text{inc}}| = (1 - \alpha)|S_{N-1}(p^*_N) - S_0(p^*_0)|,
\end{align*}
\]

(C.19)

which is equivalent to a two-group problem as in (3.1). The consumer surplus of (C.19) is also the same as (C.18), both of which are \( \sum_{i \in I_{\text{dec}}} d_i s_{\text{inc}} + \sum_{i \in I_{\text{dec}}} d_i s_{\text{dec}}. \) We next note that \( \sqrt{2b - c\sqrt{2/b}} \) is strictly concave in \( b \), and thus by our definition of \( b_{\text{dec}} \) and \( b_{\text{inc}} \) we have \( b_{\text{dec}} \leq \bar{b} := \sum_{i \in I_{\text{dec}}} d_i b_i / \sum_{i \in I_{\text{dec}}} d_i \) and \( b_{\text{inc}} \leq \underline{b} := \sum_{i \in I_{\text{inc}}} d_i b_i / \sum_{i \in I_{\text{inc}}} d_i \), where \( \bar{b} \) and \( \underline{b} \) are the weighted averages of \( b_i \) in \( I_{\text{dec}} \) and \( I_{\text{inc}} \), respectively.

Recall that the surplus of a group with parameters \( d \) and \( b \) is \( d^3(b - c)^2/8b \). Thus, we have

Social Welfare from (C.19) \( < \left( \sum_{i \in I_{\text{dec}}} d_i \right) \frac{3(b_{\text{dec}} - c)^2}{8b_{\text{dec}}} + \left( \sum_{i \in I_{\text{inc}}} d_i \right) \frac{3(b_{\text{inc}} - c)^2}{8b_{\text{inc}}} \)

\( \leq \left( \sum_{i \in I_{\text{dec}}} d_i \right) \frac{3(\bar{b} - c)^2}{8\bar{b}} + \left( \sum_{i \in I_{\text{inc}}} d_i \right) \frac{3(\underline{b} - c)^2}{8\underline{b}} \)

\( \leq \sum_{i \in I_{\text{dec}}} d_i \frac{3(b_i - c)^2}{8b_i} + \sum_{i \in I_{\text{inc}}} d_i \frac{3(b_i - c)^2}{8b_i} \)

\( = \sum_{i \in I_{\text{dec}}} S_i(p^*_i) + \sum_{i \in I_{\text{inc}}} S_i(p^*_i). \)

The first inequality follows from Proposition 3.4, where we have shown that the social welfare under surplus fairness is lower than the unconstrained value in the two-group case, and the fact that (C.19) is equivalent to a two-group setting with as discussed above. The second inequality follows from the facts that \( b_{\text{dec}} \leq \bar{b} \) and \( b_{\text{inc}} \leq \underline{b} \). The third inequality follows from Jensen’s inequality, and the final equality follows by definition. Since the social welfare of (C.18) and (C.19) are equivalent, then we conclude that when \( \alpha > 0 \), the social welfare is below the social welfare in the unconstrained case (i.e., when \( \alpha = 0 \)).
When \( p_0(\alpha) = 0 \), we let \( \tilde{\alpha} \) be the smallest \( \alpha \) such that \( p_0(\alpha) = 0 \). For all \( \alpha > \tilde{\alpha} \), the lowest surplus level is fixed as \( b_0/2 \) and cannot be improved. The only way to satisfy the constraints is to increase the prices for the groups whose surpluses are still too high. As a result, the social welfare monotonically decreases for \( \alpha > \tilde{\alpha} \). Thus, we have \( W(\alpha) < W(\tilde{\alpha}) < W(0) \) for any \( \alpha \geq \tilde{\alpha} \).

(c) No-purchase valuation fairness. Given \( \alpha \), let \( p_i(\alpha) \) be the optimal solution for group \( i \). We define \( I_{dec}(\alpha) = \{ i | p_i(\alpha) < p_i^* \} \) and \( I_{inc}(\alpha) = \{ i | p_i(\alpha) > p_i^* \} \) as the sets of groups with prices that decrease and increase relative to the unconstrained optimal solution, respectively. As in demand fairness, all the groups in \( I_{dec}(\alpha) \) or \( I_{inc}(\alpha) \) share the same level of no-purchase valuation. Note that using a price higher than \( b_i \) cannot improve the no-purchase valuation, and thus \( p_i(\alpha) \) is at most \( b_i \). In this case, the no-purchase valuation is simply equal to half of \( p_i(\alpha) \) (for linear demand), i.e., \( N_i(p_i(\alpha)) = \frac{p_i(\alpha)}{2} \). As a result, all the groups in \( I_{dec}(\alpha) \) or \( I_{inc}(\alpha) \) share the same price level, and the price difference between the two sets is \( (1 - \alpha)|p_{N-1}^* - p_0^*| \).

Let \( p_{inc}(\alpha) \) and \( p_{dec}(\alpha) \) be the prices for \( I_{inc}(\alpha) \) and \( I_{dec}(\alpha) \), respectively. We first show that \( p_{inc}(\alpha) \) (resp. \( p_{dec}(\alpha) \)) increases (resp. decreases) monotonically with \( \alpha \). First, note that given \( p_{inc} \) and \( p_{dec} \), we can construct a solution for all the \( N \) groups, by setting \( p_i = \min(\max(p_{inc}, q_i^*), p_{dec}) \).

Let \( g(p_{inc}, p_{dec}) = \sum_{i=0}^{N-1} R_i(\min(\max(p_{inc}, p_i^*), p_{dec})) \) be the profit with respect to \( p_{inc} \) and \( p_{dec} \). One can easily verify that \( g(p_{inc}, p_{dec}) \) is concave in the range \( p_{inc} \in [0, \min(p_{dec}, b_0)] \) and \( p_{dec} \in [p_{inc}, b_{N-1}] \). Optimization problem (3.2) can then be written as

\[
\max_{p_{inc}, p_{dec}} g(p_{inc}, p_{dec})
\]

s.t. \( p_{dec} - p_{inc} \leq (1 - \alpha)|p_{N-1}^* - p_0^*| \),

\( p_{inc} - p_{dec} \leq 0 \)

\( p_{inc} \in [p_0^*, b_0], p_{dec} \in [p_0^*, p_{N-1}^*] \).
When \( p_{\text{inc}} \) does not reach the boundary \( b_0 \), the KKT condition is given by

\[
\begin{bmatrix}
\frac{\partial g}{\partial p_{\text{inc}}} \\
\frac{\partial g}{\partial p_{\text{dec}}} 
\end{bmatrix} = \mu_1 \begin{bmatrix}
-1 \\
1 
\end{bmatrix} + \mu_2 \begin{bmatrix}
1 \\
-1 
\end{bmatrix},
\]

(C.20)

\[
p_{\text{dec}} - p_{\text{inc}} \leq (1 - \alpha)|p^*_N - p^*_0|
\]

\[
p_{\text{inc}} - p_{\text{dec}} \leq 0
\]

\[
\mu_1 (p_{\text{dec}} - p_{\text{inc}} - (1 - \alpha)|p^*_N - p^*_0|) = 0
\]

\[
\mu_2 (p_{\text{inc}} - p_{\text{dec}}) = 0
\]

\[
\mu_1, \mu_2 \geq 0.
\]

Since the price difference between the two sets is \((1 - \alpha)|p^*_N - p^*_0|\), by complementary slackness, we have and \( \mu_2 = 0 \). Note that \( \frac{\partial g}{\partial p_{\text{inc}}} \) is non-positive and monotonically decreasing in the feasible region; similarly, \( \frac{\partial g}{\partial p_{\text{dec}}} \) is non-negative and monotonically decreasing in the feasible region. Therefore, before \( p_{\text{inc}} \) reaches \( b_0 \), when we increase \( \alpha \) to maintain Eq. (C.20), one has to move \( p_{\text{inc}} \) and \( p_{\text{dec}} \) in opposite directions. Since their difference is monotonically decreasing with \( \alpha \), then \( p_{\text{inc}}(\alpha) \) monotonically increases and \( p_{\text{dec}}(\alpha) \) monotonically decreases. When \( p_{\text{inc}}(\alpha) \) reaches the boundary \( b_0 \), to satisfy the fairness constraints, one has to decrease \( p_{\text{dec}}(\alpha) \) monotonically, while \( p_{\text{inc}}(\alpha) \) remains at \( b_0 \).

We now know that \( p_{\text{inc}}(\alpha) \) and \( p_{\text{dec}}(\alpha) \) are monotone. Since their gap is \((1 - \alpha)(p^*_N - p^*_0)\), they are both continuous. Consequently, the corresponding social welfare is also continuous. As \( \alpha \) increases from 0, \( p_0 \) is increasing and \( p_N \) is decreasing. Let \( \tilde{\alpha} \) be the smallest \( \alpha \) such that \( p_0 = b_0 \) (if it exists). Then, for any \( \alpha > \tilde{\alpha} \), since the no-purchase valuation from group 0 cannot be improved anymore, the price of group 0 (as well as all the groups in \( I_{\text{inc}} \)) remains at \( b_0 \), and the only way to decrease the differences in no-purchase valuation is to decrease the price of the remaining groups whose offered price is greater than \( b_0 \). By doing so, the social welfare must increase. We next show that for \( \alpha \leq \tilde{\alpha} \), \( W(\alpha) \) also increases monotonically, hence concluding the proof.
For $\alpha \leq \tilde{\alpha}$, since $p_{inc}(\alpha)$ and $p_{dec}(\alpha)$ are monotone, $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are also monotone, i.e., $I_{inc}(\alpha_1) \subset I_{inc}(\alpha_2)$ and $I_{dec}(\alpha_1) \subset I_{dec}(\alpha_2)$ for any $\alpha_1 < \alpha_2$. We can then split $[0, \tilde{\alpha}]$ into at most $N$ non-overlapping intervals, based on the value of $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$. For the first interval, we have $I_{inc}(\alpha) = \{1\}$ and $I_{dec}(\alpha) = \{N\}$. As $\alpha$ increases, we either add group 2 to $I_{inc}$ or group $N - 1$ to $I_{dec}$, and so on. Since the social welfare curve is continuous, it is enough to show that for each interval such that $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are fixed, the social welfare is monotonically increasing.

Suppose that $\alpha \in [\alpha_1, \alpha_2]$ and that $I_{inc}(\alpha), I_{dec}(\alpha)$ are fixed. Then, the set of tight constraints is also fixed, and we know that the prices for $i$ in $I_{inc}(\alpha)$ or $I_{dec}(\alpha)$ are the same. The profit maximization problem is thus equivalent to

$$\max_{p_{inc}, p_{dec}} \sum_{i \in I_{inc}(\alpha)} d_i(p_{inc} - c)(1 - \frac{p_{inc}}{b_i}) + \sum_{i \in I_{dec}(\alpha)} d_i(p_{dec} - c)(1 - \frac{p_{dec}}{b_i}) \quad (C.21)$$

subject to $p_{dec} - p_{inc} = (1 - \alpha)(p^*_N - p^*_0)$,

where the boundary constraints are hidden because we already assume that the prices do not hit the boundary. Rearranging the terms in (C.21) leads to

$$\max_{p_{inc}, p_{dec}} d_{inc}(p_{inc} - c)(1 - \frac{p_{inc}}{b_{inc}}) + d_{dec}(p_{dec} - c)(1 - \frac{p_{dec}}{b_{dec}}) \quad (C.22)$$

subject to $p_{dec} - p_{inc} = (1 - \alpha)(p^*_N - p^*_0)$,

where $d_{inc} = \sum_{i \in I_{inc}(\alpha)} d_i$, $d_{dec} = \sum_{i \in I_{dec}(\alpha)} d_i$, and $b_{inc}, b_{dec}$ are defined by

$$\frac{1}{b_{inc}} = \sum_{i \in I_{inc}(\alpha)} \frac{d_i}{d_{inc} b_i}, \quad \frac{1}{b_{dec}} = \sum_{i \in I_{dec}(\alpha)} \frac{d_i}{d_{dec} b_i}. \quad (C.23)$$

As a result, problem (C.21) is equivalent to a problem with two groups, $inc$ and $dec$. Using Proposition 3.5, the social welfare with respect to the aggregate groups $inc$ and $dec$ is always increasing with $\alpha$. We next show that the total social welfare of group $i \in I_{dec}(\alpha) \cup I_{inc}(\alpha)$ has a
constant difference relative to the social welfare from the two aggregate groups. The total social welfare for all the groups in $I_{\text{inc}}$ is given by

$$\sum_{i \in I_{\text{inc}}(\alpha)} d_i \left[ (p_{\text{inc}} - c)(1 - \frac{p_{\text{inc}}}{b_i}) + \frac{1}{2} (b_i - p_{\text{inc}})(1 - \frac{p_{\text{inc}}}{b_i}) \right]$$

$$= \sum_{i \in I_{\text{inc}}(\alpha)} \left[ -\frac{1}{2} \frac{d_i}{b_i} p_{\text{inc}}^2 + \frac{d_i}{b_i} p + \frac{d_i b_i}{2} - d_i c \right]$$

$$= -\frac{1}{2} \left( \sum_{i \in I_{\text{inc}}(\alpha)} \frac{d_i}{b_i} \right) p_{\text{inc}}^2 + \left( \sum_{i \in I_{\text{inc}}(\alpha)} \frac{d_i}{b_i} \right) p_{\text{inc}} + \sum_{i \in I_{\text{inc}}(\alpha)} \frac{d_i b_i}{2} - c \sum_{i \in I_{\text{inc}}(\alpha)} d_i$$

$$= -\frac{1}{2} \frac{d_{\text{inc}}}{b_{\text{inc}}} p_{\text{inc}}^2 + \frac{d_{\text{inc}}}{b_{\text{inc}}} p_{\text{inc}} + \frac{d_{\text{inc}} b_{\text{inc}}}{2} - d_{\text{inc}} c + \left( \sum_{i \in I_{\text{inc}}(\alpha)} \frac{d_i b_i}{2} - \frac{d_{\text{inc}} b_{\text{inc}}}{2} \right), \quad \text{(C.24)}$$

where the first four terms in Eq. (C.24), $-\frac{1}{2} \frac{d_{\text{inc}}}{b_{\text{inc}}} p_{\text{inc}}^2 + \frac{d_{\text{inc}}}{b_{\text{inc}}} p + \frac{d_{\text{inc}} b_{\text{inc}}}{2} - d_{\text{inc}} c$, equal to the social welfare of the aggregate group $\text{inc}$. The same result also holds for group $\text{dec}$. Hence, the total welfare of all the groups in $I_{\text{inc}}$ differs from the social welfare of the aggregate group $\text{inc}$ by a constant term. By using Proposition 3.5, the social welfare of the aggregate groups $\text{inc}$ and $\text{dec}$ are monotonically increasing on $[\alpha_1, \alpha_2]$, and the social welfare of all the groups within $I_{\text{inc}} \cup I_{\text{dec}}$ increases monotonically. Since the social welfare is a continuous function with at most $N$ pieces, and it increases with $\alpha$ on each piece, we conclude that $W(\alpha)$ is increasing for $\alpha \in [0, 1]$. \hfill \square

**Proof of Proposition 3.7.** (a) Recall that we assume $b_0 < b_1 < \cdots < b_{N-1}$. In addition, the unconstrained optimal prices, $p_i^* = (b_i + c)/2$, are also in increasing order. One can verify that $p_0^* \leq p_i(\alpha) \leq p_{N-1}^*$. For $\alpha < 1 - \frac{\max\{p_{N-2}^* - p_0^*, p_{N-1}^* - p_1^*\}}{p_{N-1}^* - p_0^*}$, we have $(1 - \alpha)(p_{N-1}^* - p_0^*) > \max\{p_{N-2}^* - p_0^*, p_{N-1}^* - p_1^*\}$, i.e., the required price range is large enough such that the prices for groups $2$ to $N - 1$ remain equal to $p_i^*$, and we only need to optimize the prices for groups $0$ and $N$. As a result, the problem reduces to a two-group problem, and the desired result follows directly from Proposition 3.2.

(b) Recall from Table C.1 that if a group $i$ has positive demand, then the no-purchase valuation metric in the case of linear demand is $N_i(p_i(\alpha)) = \frac{p_i(\alpha)}{2}$. Thus, when all groups have positive demand, ensuring price fairness is equivalent to ensuring no-purchase valuation fairness. Conse-
sequently, our result follows immediately from Proposition 3.6.

(c) We provide a proof by example. On the right panel of Fig. 3.2, when \( \alpha \in [0.47, 0.62] \), one can see that group 1 is excluded, but \( W(\alpha) > W(0) \). On the other hand, when \( \alpha > 0.62 \), both groups 1 and 2 are excluded, and \( W(\alpha) < W(0) \). \( \square \)

C.4 Proof of Proposition 3.8

Proof. (a) Price Fairness. Let \( \Delta p_{xy} \) be the absolute value of the price change for group \( xy \). Let the unconstrained weighted average price be \( \bar{p}_i^* \), \( i = 0, 1 \), where \( p_{xy}^* = \frac{b_{xy} + c}{2} \). Without loss of generality, we assume that \( \bar{p}_1 > \bar{p}_0 \). As \( \alpha \) increases, \( p_{10} \) and \( p_{11} \) decrease, whereas \( p_{00} \) and \( p_{01} \) increase. The optimization problem is given by:

\[
\min \frac{d_{00}}{b_{00}} \Delta p_{00}^2 + \frac{d_{01}}{b_{01}} \Delta p_{01}^2 + \frac{d_{10}}{b_{10}} \Delta p_{10}^2 + \frac{d_{11}}{b_{11}} \Delta p_{11}^2
\]

\[
\text{s.t.} \quad \frac{d_{00}}{d_{00} + d_{01}} \Delta p_{00} + \frac{d_{01}}{d_{00} + d_{01}} \Delta p_{01} + \frac{d_{10}}{d_{10} + d_{11}} \Delta p_{10} + \frac{d_{11}}{d_{10} + d_{11}} \Delta p_{11} = \alpha (\bar{p}_1 - \bar{p}_0)
\]

\[
\Delta p_i \geq 0.
\]

Here, we omit the upper-bound constraints as we only consider the case when \( p_{xy} \in (0, b_{xy}) \).

By solving the KKT conditions, we obtain:

\[
\frac{1}{b_{x0}} \Delta p_{x0} = \frac{1}{b_{x1}} \Delta p_{x1},
\]

\[
(d_{00} + d_{01}) \frac{1}{b_{00}} \Delta p_{00} = (d_{10} + d_{11}) \frac{1}{b_{10}} \Delta p_{10},
\]

\[
\frac{d_{00} \Delta p_{00} + d_{01} \Delta p_{01}}{d_{00} + d_{01}} + \frac{d_{10} \Delta p_{10} + d_{11} \Delta p_{11}}{d_{10} + d_{11}} = \alpha (\bar{p}_1 - \bar{p}_0).
\]

Solving the above equations leads to

\[
\Delta p_{00} = \frac{b_{00} \alpha w}{d_{00} + d_{01} d_{00} + d_{01}} (d_{10} b_{10} + d_{11} b_{11}), \quad \Delta p_{01} = \frac{b_{01} \alpha w}{d_{00} + d_{01} d_{00} + d_{01}} (d_{10} b_{10} + d_{11} b_{11}),
\]

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\[\Delta p_{10} = \frac{b_{10} \alpha \omega}{d_{10} b_{10} + d_{11} b_{11}} + \frac{b_{10} \alpha \omega}{d_{10} + d_{11}} (d_{00} b_{00} + d_{01} b_{01})\], \[\Delta p_{11} = \frac{b_{11} \alpha \omega}{d_{10} b_{10} + d_{11} b_{11}} + \frac{b_{11} \alpha \omega}{d_{10} + d_{11}} (d_{00} b_{00} + d_{01} b_{01})\],

where \(\omega = \bar{p}_1^* - \bar{p}_0^*\).

By substituting the above expressions into the profit and consumer surplus functions, we obtain:

\[R(\alpha) - R(0) = -(d_{00} + d_{01})^2 (d_{10} + d_{11})^2 \frac{2(\bar{p}_1^* - \bar{p}_0^*) \alpha \omega + (\alpha w)^2}{(b_{00} d_{00} + b_{01} d_{01}) (d_{10} + d_{11})^2 + (b_{10} d_{10} + b_{11} d_{11}) (d_{00} + d_{01})^2}\],

\[S(\alpha) - S(0) = \frac{(d_{00} + d_{01})^2 (d_{10} + d_{11})^2}{2(b_{00} d_{00} + b_{01} d_{01}) (d_{10} + d_{11})^2 + (b_{10} d_{10} + b_{11} d_{11}) (d_{00} + d_{01})^2} \left[2(\bar{p}_1^* - \bar{p}_0^*) \alpha \omega - (\alpha w)^2\right]\],

\[W(\alpha) - W(0) = \frac{(d_{00} + d_{01})^2 (d_{10} + d_{11})^2}{2(b_{00} d_{00} + b_{01} d_{01}) (d_{10} + d_{11})^2 + (b_{10} d_{10} + b_{11} d_{11}) (d_{00} + d_{01})^2} \left[2(\bar{p}_1^* - \bar{p}_0^*) \alpha \omega + (\alpha w)^2\right]\].

Note that \(\bar{p}_1^* - \bar{p}_0^* > 0\) (by assumption), so that before giving up a group, the social welfare is monotonically increasing for any \(\alpha \in [0, 1]\).

(b) Demand Fairness. For demand fairness, we assume that group 0 has a lower weighted average demand. Hence, \(p_{00}\) and \(p_{01}\) decrease, whereas \(p_{10}\) and \(p_{11}\) increase. The optimization problem is given by:

\[
\min \frac{d_{00}}{b_{00}} \Delta p_{00}^2 + \frac{d_{01}}{b_{01}} \Delta p_{01}^2 + \frac{d_{10}}{b_{10}} \Delta p_{10}^2 + \frac{d_{11}}{b_{11}} \Delta p_{11}^2
\]

s.t. \[
\frac{d_{00}}{d_{00} + d_{01}} \Delta p_{00} + \frac{d_{01}}{d_{00} + d_{01}} \Delta p_{01} + \frac{d_{10}}{d_{10} + d_{11}} \Delta p_{10} + \frac{d_{11}}{d_{10} + d_{11}} \Delta p_{11} = \alpha K
\]

\[
\Delta p_i \geq 0,
\]

where

\[
K = \frac{d_{10}}{d_{10} + d_{11}} \frac{b_{10} - c}{2b_{10}} + \frac{d_{11}}{d_{10} + d_{11}} \frac{b_{11} - c}{2b_{11}} - \frac{d_{00}}{d_{00} + d_{01}} \frac{b_{00} - c}{2b_{00}} - \frac{d_{01}}{d_{00} + d_{01}} \frac{b_{01} - c}{2b_{01}}
\]

\[
= c \left[ b_{10} b_{11} (d_{10} + d_{11}) (b_{01} d_{00} + b_{00} d_{01}) - b_{00} b_{01} (d_{00} + d_{01}) (b_{10} d_{11} + b_{11} d_{10}) \right] / 2b_{00} b_{01} b_{10} b_{11} (d_{00} + d_{01}) (d_{10} + d_{11}) > 0
\]
is the initial difference in weighted average demand. By solving the KKT conditions, we obtain:

\[
\Delta p_{00} = \Delta p_{01} = \frac{d_{10} + d_{11}}{2(d_{00} + d_{01} + d_{10} + d_{11})} \alpha K,
\]

\[
\Delta p_{10} = \Delta p_{11} = \frac{d_{00} + d_{01}}{2(d_{00} + d_{01} + d_{10} + d_{11})} \alpha K.
\]

We next consider the change in social welfare. The profit loss is

\[
d_{00} \frac{\Delta p_{00}^2}{2b_{00}} + d_{01} \frac{\Delta p_{01}^2}{2b_{10}} + d_{10} \frac{\Delta p_{10}^2}{2b_{11}} + d_{11} \frac{\Delta p_{11}^2}{2b_{11}},
\]

the consumer surplus change is

\[
d_{00} \frac{b_{00} - c}{2b_{00}} \Delta p_{00} + d_{01} \frac{b_{01} - c}{2b_{01}} \Delta p_{01} - d_{10} \frac{b_{10} - c}{2b_{10}} \Delta p_{10} - d_{11} \frac{b_{11} - c}{2b_{11}} \Delta p_{11} + \frac{d_{00}}{2b_{00}} \Delta p_{00}^2 + \frac{d_{01}}{2b_{01}} \Delta p_{01}^2 + \frac{d_{10}}{2b_{10}} \Delta p_{10}^2 + \frac{d_{11}}{2b_{11}} \Delta p_{11}^2,
\]

and the social welfare change is

\[
d_{00} \frac{b_{00} - c}{2b_{00}} \Delta p_{00} + d_{01} \frac{b_{01} - c}{2b_{01}} \Delta p_{01} - d_{10} \frac{b_{10} - c}{2b_{10}} \Delta p_{10} - d_{11} \frac{b_{11} - c}{2b_{11}} \Delta p_{11} - d_{00} \frac{b_{00}}{2b_{00}} \Delta p_{00}^2 - d_{01} \frac{b_{01}}{2b_{01}} \Delta p_{01}^2 - d_{10} \frac{b_{10}}{2b_{10}} \Delta p_{10}^2 - d_{11} \frac{b_{11}}{2b_{11}} \Delta p_{11}^2.
\]

Note that the second-order term in the social welfare is always decreasing with \( \alpha \), so that we only need to focus on the linear terms. By substituting \( \Delta p_{xy} \), we obtain:

\[
\frac{d_{00}}{2b_{00}} \frac{b_{00} - c}{2b_{00}} \Delta p_{00} + d_{01} \frac{b_{01} - c}{2b_{01}} \Delta p_{01} - d_{10} \frac{b_{10} - c}{2b_{10}} \Delta p_{10} - d_{11} \frac{b_{11} - c}{2b_{11}} \Delta p_{11}
\]

\[
= -c \left[ \frac{b_{10} b_{11}(d_{10} + d_{11})(b_{01} d_{00} + b_{00} d_{01}) - b_{00} b_{01}(d_{00} + d_{01})(b_{10} d_{11} + b_{11} d_{10})}{4b_{00} b_{01} b_{10} b_{11}(d_{00} + d_{01} + d_{10} + d_{11})} \right] \alpha K.
\] (C.25)

Note that the numerator in Eq. (C.25) equals to the numerator of \(-K\), and thus is negative by assumption. Hence, Eq. (C.25) decreases with \( \alpha \). Together with the fact that \(-\frac{d_{00}}{2b_{00}} \Delta p_{00}^2 - \frac{d_{01}}{2b_{01}} \Delta p_{01}^2 - \frac{d_{10}}{2b_{10}} \Delta p_{10}^2 - \frac{d_{11}}{2b_{11}} \Delta p_{11}^2 \) decreases with \( \alpha \), we conclude that the social welfare always decreases with \( \alpha \).

(c) *Surplus Fairness.* Finally, for surplus fairness, we follow the same idea as in Lemma C.1. We assume that group 0 has a lower weighted average surplus. Hence, \( p_{00} \) and \( p_{01} \) decrease, whereas
These conditions can be reformulated as

\[ \min \frac{d_{00}}{b_{00}} \Delta p_{00}^2 + \frac{d_{01}}{b_{01}} \Delta p_{01}^2 + \frac{d_{10}}{b_{10}} \Delta p_{10}^2 + \frac{d_{11}}{b_{11}} \Delta p_{11}^2 \]

subject to

\[ \frac{d_{00}}{d_{00} + d_{01}} \left( \frac{b_{00} - c}{2b_{00}} \Delta p_{00} + \frac{1}{2b_{00}} \Delta p_{00}^2 \right) + \frac{d_{01}}{d_{00} + d_{01}} \left( \frac{b_{01} - c}{2b_{01}} \Delta p_{01} + \frac{1}{2b_{01}} \Delta p_{01}^2 \right) \]

\[ \frac{d_{10}}{d_{10} + d_{11}} \left( \frac{b_{10} - c}{2b_{10}} \Delta p_{10} - \frac{1}{2b_{10}} \Delta p_{10}^2 \right) + \frac{d_{11}}{d_{10} + d_{11}} \left( \frac{b_{11} - c}{2b_{11}} \Delta p_{11} - \frac{1}{2b_{11}} \Delta p_{11}^2 \right) = \alpha K \]

\[ \Delta p_i \geq 0, \]

where

\[ K = \frac{d_{10}}{d_{10} + d_{11}} \frac{(b_{10} - c)^2}{8b_{10}} + \frac{d_{11}}{d_{10} + d_{11}} \frac{(b_{11} - c)^2}{8b_{11}} - \frac{d_{00}}{d_{00} + d_{01}} \frac{(b_{00} - c)^2}{8b_{00}} - \frac{d_{01}}{d_{00} + d_{01}} \frac{(b_{01} - c)^2}{8b_{01}} > 0 \]

corresponds to the initial difference. The KKT conditions are given by:

\[
\begin{bmatrix}
2 \frac{d_{00}}{b_{00}} \Delta p_{00} \\
2 \frac{d_{01}}{b_{01}} \Delta p_{01} \\
2 \frac{d_{10}}{b_{10}} \Delta p_{10} \\
2 \frac{d_{11}}{b_{11}} \Delta p_{11}
\end{bmatrix}
= \mu \begin{bmatrix}
\frac{d_{00}}{d_{00} + d_{01}} \left( \frac{b_{00} - c}{2b_{00}} - \frac{1}{b_{00}} \Delta p_{00} \right) \\
\frac{d_{01}}{d_{00} + d_{01}} \left( \frac{b_{01} - c}{2b_{01}} - \frac{1}{b_{01}} \Delta p_{01} \right) \\
\frac{d_{10}}{d_{10} + d_{11}} \left( \frac{b_{10} - c}{2b_{10}} - \frac{1}{b_{10}} \Delta p_{10} \right) \\
\frac{d_{11}}{d_{10} + d_{11}} \left( \frac{b_{11} - c}{2b_{11}} - \frac{1}{b_{11}} \Delta p_{11} \right)
\end{bmatrix}
\]

\( Eq. (C.26), \Delta p_{xy} \geq 0, \mu \geq 0. \)

These conditions can be reformulated as

\[
2 \frac{d_{00}}{b_{00}} \Delta p_{00} = \frac{d_{00}}{d_{00} + d_{01}} \left( \frac{b_{00} - c}{2b_{00}} - \frac{1}{b_{00}} \Delta p_{00} \right) 2 \frac{d_{01}}{b_{01}} \Delta p_{01} \]

\[
= \frac{d_{10}}{d_{10} + d_{11}} \left( \frac{b_{10} - c}{2b_{10}} - \frac{1}{b_{10}} \Delta p_{10} \right) 2 \frac{d_{10}}{b_{10}} \Delta p_{10} = \frac{d_{10}}{d_{10} + d_{11}} \left( \frac{b_{11} - c}{2b_{11}} - \frac{1}{b_{11}} \Delta p_{11} \right) 2 \frac{d_{11}}{b_{11}} \Delta p_{11} \]

\( Eq. (C.26), \Delta p_{xy} \geq 0. \)
Using the same argument as in Lemma C.1, we divide Eq. (C.27) and Eq. (C.26) by \( \alpha \) and take the limit as \( \alpha \) goes to 0:

\[
- \frac{d_{00}}{b_{00}} p'_{00}(0) = - \frac{d_{00}}{d_{00} + d_{01}} \frac{b_{00} - c}{2b_{00}} \frac{d_{01}}{b_{01}} p'_{01}(0) = \frac{d_{00}}{d_{00} + d_{01}} \frac{b_{00} - c}{b_{01}} \frac{d_{10}}{2b_{10}} \frac{d_{10}}{d_{10} + d_{11}} \frac{d_{11}}{b_{11}} p'_{10}(0) = \frac{d_{00}}{d_{00} + d_{01}} \frac{b_{00} - c}{b_{11}} \frac{d_{10}}{d_{10} + d_{11}} \frac{d_{11}}{2b_{11}} p'_{11}(0),
\]

\[
- \frac{d_{00}}{d_{00} + d_{01}} \frac{b_{00} - c}{2b_{00}} p'_{00}(0) - \frac{d_{01}}{d_{00} + d_{01}} \frac{b_{01} - c}{2b_{01}} p'_{01}(0) + \frac{d_{10}}{d_{00} + d_{11}} \frac{b_{10} - c}{2b_{10}} p'_{10}(0) + \frac{d_{11}}{d_{01} + d_{11}} \frac{b_{11} - c}{2b_{11}} p'_{11}(0) = K.
\]

Solving the above system of equations, we obtain:

\[
p'_{00}(0) = - \frac{b_{00} - c}{2(d_{00} + d_{01})} \frac{K}{C}, \quad p'_{01}(0) = - \frac{b_{01} - c}{2(d_{00} + d_{01})} \frac{K}{C}, \quad p'_{10}(0) = \frac{b_{10} - c}{2(d_{10} + d_{11})} \frac{K}{C}, \quad p'_{11}(0) = \frac{b_{11} - c}{2(d_{10} + d_{11})} \frac{K}{C},
\]

where \( C > 0 \) is the normalization constant. The initial social welfare derivative, \( \mathcal{W}(0)' \), becomes

\[
\mathcal{W}(0)' = - d_{00} \tilde{F}_{00}(p_{00}^*) p'_{00}(0) - d_{01} \tilde{F}_{01}(p_{01}^*) p'_{01}(0) - d_{10} \tilde{F}_{10}(p_{10}^*) p'_{10}(0) - d_{11} \tilde{F}_{11}(p_{11}^*) p'_{11}(0).
\]

By substituting \( p_{xy}'(0) \), we obtain:

\[
\mathcal{W}(0)' = - \frac{d_{00}}{d_{00} + d_{01}} \frac{(b_{00} - c)^2}{4b_{00}} + \frac{d_{01}}{d_{00} + d_{01}} \frac{(b_{01} - c)^2}{4b_{01}} - \frac{d_{10}}{d_{10} + d_{11}} \frac{(b_{10} - c)^2}{4b_{10}} - \frac{d_{11}}{d_{10} + d_{11}} \frac{(b_{11} - c)^2}{4b_{11}}.
\]

\[
\mathcal{W}(0)' = \frac{K}{C} \left( \frac{d_{00}}{d_{00} + d_{01}} \frac{(b_{00} - c)^2}{4b_{00}} + \frac{d_{01}}{d_{00} + d_{01}} \frac{(b_{01} - c)^2}{4b_{01}} - \frac{d_{10}}{d_{10} + d_{11}} \frac{(b_{10} - c)^2}{4b_{10}} - \frac{d_{11}}{d_{10} + d_{11}} \frac{(b_{11} - c)^2}{4b_{11}} \right)
\]

\[
\mathcal{W}(0)' = \frac{K}{C}(-2K) < 0.
\]

This shows that the social welfare decreases at \( \alpha = 0 \).

(d) No-Purchase Valuation Fairness. For no-purchase valuation fairness, since we only consider the case without reaching the boundary, the solutions from both price fairness and no-purchase valuation fairness are the same, just as in Proposition 3.2 and Proposition 3.5.
C.5 On the Computation Complexity of Pricing with Multiple Groups

We show in Lemma C.2 that the optimal solution can be found efficiently by reducing the N-group pricing problem (3.2) to a one-dimensional optimization problem.

**Lemma C.2.** Assume that the profit function \( R_i(p) \) is unimodal. Then, the pricing problem can be reduced to an one-dimension optimization problem.

**Proof of Lemma C.2.** Given \( \alpha \), we start by analyzing the structure of the optimal solution. We then propose an efficient way to compute the optimal solution.

**Price fairness.** Let \( p_{\min} = \min_i p_i^* \) and \( p_{\max} = \max_i p_i^* \). Given \( \alpha \), all the prices should be within \([p_{\min}, p_{\max}]\). Otherwise, if there exists \( p_i(\alpha) < p_{\min} \) for example, then setting \( p_i = p_{\min} \) will not violate the fairness constraints, but will lead to a higher profit since the profit function is unimodal. We define \( I_{\text{dec}} = \{i|p_i(\alpha) < p_i^*\} \) and \( I_{\text{inc}} = \{i|p_i(\alpha) > p_i^*\} \) as the sets of groups with prices that decrease and increase relative to the unconstrained optimal solution, respectively. It is not hard to see that all the groups in \( I_{\text{dec}} \) should have the same price. Indeed, if there exist \( i, j \in I_{\text{dec}} \) such that \( p_i(\alpha) > p_j(\alpha) \), one can increase \( p_j \) such that \( p_i = p_j \). Such a change will not violate the fairness constraints but will lead to a higher profit due to the unimodality of the profit function. As a result, we can use \( p_{\text{inc}} \) and \( p_{\text{dec}} \) to denote the prices of the groups in \( I_{\text{inc}} \) and \( I_{\text{dec}} \), respectively. In addition, the constraint should be tight, i.e., \( p_{\text{dec}} - p_{\text{inc}} = (1 - \alpha)(p_{\max} - p_{\min}) \), as otherwise, we can decrease \( p_{\text{inc}} \) such that the fairness constraint is not violated but the profit for the groups in \( I_{\text{inc}} \) increases.

From the discussion above, the decision space reduces to a single decision variable, \( p_{\text{inc}} \). Indeed, given the optimal value of \( p_{\text{inc}} \), \( p_{\text{dec}} = p_{\text{inc}} + (1 - \alpha)(p_{\max} - p_{\min}) \). For each group, if \( p_i^* < p_{\text{inc}} \), then \( p_i(\alpha) = p_{\text{inc}} \); else if \( p_i^* > p_{\text{dec}} \), then \( p_i(\alpha) = p_{\text{dec}} \); else \( p_i(\alpha) = p_i^* \).

**Demand fairness.** We follow a similar argument as for price fairness, but now search on the demand space. Let \( q_i^* = \bar{F}_i(p_i^*) \) be the demand at the unconstrained optimal solution. Let \( q_{\min} = \min_i q_i^* \) and \( q_{\max} = \max_i q_i^* \). Given \( \alpha \), all the demand values should be in \([q_{\min}, q_{\max}]\). Otherwise, if there exists \( q_i(\alpha) < q_{\min} \) for example, then setting \( q_i = q_{\min} \) would not violate the fairness
constraints, but would lead to a higher profit due to the unimodality of $R_i(\cdot)$. We define $I_{\text{dec}} = \{i|q_i(\alpha) < q_i^*\}$ and $I_{\text{inc}} = \{i|q_i(\alpha) > q_i^*\}$ as the sets of groups with demands that decrease and increase relative to the unconstrained optimal solution, respectively. As before, it is not hard to see that all the groups in $I_{\text{dec}}$ should have the same demand. Indeed, if there exist $i, j \in I_{\text{dec}}$ such that $q_i(\alpha) > q_j(\alpha)$, one can increase $q_j$ such that $q_i = q_j$. Such a change would not violate the fairness constraints but will lead to a higher profit due to the unimodality of the profit function. As a result, we can use $q_{\text{inc}}$ and $q_{\text{dec}}$ to denote the prices of the groups in $I_{\text{inc}}$ and $I_{\text{dec}}$, respectively. In addition, the constraint should be tight, i.e., $q_{\text{dec}} - q_{\text{inc}} = (1 - \alpha)(q_{\text{max}} - q_{\text{min}})$, as otherwise, we can decrease $q_{\text{inc}}$ such that the fairness constraint is not violated but the profit for the groups in $q_{\text{inc}}$ increases.

From the discussion above, the decision space reduces to a single decision variable, $q_{\text{inc}}$. Indeed, given the optimal value of $q_{\text{inc}}$, $q_{\text{dec}} = q_{\text{inc}} + (1 - \alpha)(q_{\text{max}} - q_{\text{min}})$. For each group, if $q_i^* < q_{\text{inc}}$, then $q_i(\alpha) = q_{\text{inc}}$; else if $q_i^* > q_{\text{dec}}$, then $q_i(\alpha) = q_{\text{dec}}$; else $q_i(\alpha) = q_i^*$. The corresponding prices can then be computed by inverting the demand function $F_i(\cdot)$.

**Surplus fairness and no-purchase valuation fairness.** The argument and the way of computing the optimal solution are essentially the same as for demand fairness, except that the decision variable becomes the surplus and the no-purchase valuation, respectively.

\[\square\]

### C.6 Tested Instances in Section 3.5 and Additional Figures

#### C.6.1 Instances and Figures for Two-Group Experiments

For two-group cases, we test the following instances:

**Exponential demand:** For $(d_0, d_1)$, we use $(0.1, 0.9), (0.5, 0.5), \text{ and } (0.9, 0.1)$. For $(\lambda_0, \lambda_1)$, we use $(1, 0.2) \text{ and } (1, 2)$. For $c$, we use $0.1 \text{ and } 2$. We then test all the combinations.

**Logistic demand:** For $(d_0, d_1)$, we use $(0.1, 0.9), (0.5, 0.5), \text{ and } (0.9, 0.1)$. For $(k_0, k_1)$, we use $(5, 10), (10, 5), \text{ and } (5, 5)$. For $(\lambda_0, \lambda_1)$, we use $(1, 0.2) \text{ and } (1, 0.5)$. For $c$, we use $0.5 \text{ and } 2$. We
then test all the combinations.

**Log-log demand:** For \((d_0, d_1)\), we use \((0.1, 0.9)\), \((0.5, 0.5)\), and \((0.9, 0.1)\). For \((a_0, a_1)\), we use \((2,1)\) and \((1,2)\). For \((\beta_0, \beta_1)\), we use \((3, 1.8)\) and \((3, 2.5)\). For \(c\), we use 1 and 2. We then test all the combinations.

In Fig. C.1, Fig. 3.3, and Fig. C.2, we present the results for a representative example of each demand model.
Figure C.1: Impact of fairness under exponential demand (two groups).

Parameters: $d_0 = 0.5$, $d_1 = 0.5$, $\lambda_0 = 1$, $\lambda_1 = 0.2$, $c = 0.1$. 
Figure C.2: Impact of fairness under log-log demand (two groups).

Parameters: $d_0 = 0.1, d_1 = 0.9, a_0 = 1, a_1 = 2, \beta_0 = 3, \beta_1 = 1.8, c = 2$. Note that the plot of no-purchase valuation fairness ends at $\alpha = 0.64$, since any larger $\alpha$ will result in an infeasible solution (because the demand of group 1 has reached 1 so that the no-purchase valuation is not well defined).
C.6.2 Instances and Figures for Five-Group Experiments

For five-group cases, we test the following instances:

**Exponential demand:** We sample $d_i$ uniformly between 0 and 1, and $\lambda_i$ uniformly between 0.1 and 1. The value of $c$ is set at 0.4

**Logistic demand:** We sample $d_i$ uniformly between 0 and 1, $\lambda_i$ uniformly between 0.1 and 1, and $k_i$ uniformly between 3 and 10. The value of $c$ is set at 2.

**Log-log demand:** We sample $d_i$ uniformly between 0 and 1, $\beta_i$ uniformly between 1.5 and 5. The value of $c$ is set at 2. To make sure that $a_i(\beta_i - 1) < c\beta_i$, we sample $a_i$ uniformly between $0.3c\beta_i/(\beta_i - 1)$ and $0.9c\beta_i/(\beta_i - 1)$.

In Fig. C.3, Fig. C.4, and Fig. C.5, we present the results for a representative example of each demand model.
Figure C.3: Impact of fairness under exponential demand (five groups).

Parameters: $\lambda = (0.35, 0.1, 0.56, 0.11, 0.03), \mu = (0.98, 0.63, 0.49, 0.94, 0.87), c = 0.4$. 
Figure C.4: Impact of fairness under logistic demand (five groups).

Parameters: \( \lambda = (0.99, 0.45, 0.2, 0.16, 0.32), \) \( k = (8.28, 7.1, 9.48, 7.72, 6.32), \) \( d = (0.23, 0.41, 0.17, 0.21, 0.63), \) \( c = 2. \)
Figure C.5: Impact of fairness under log-log demand (five groups).

Parameters: $\alpha = (1.89, 2.11, 2, 1.13, 2.25)$, $\beta = (1.77268, 2.81, 3.44, 4.34)$, $d = (0.32, 0.47, 0.09, 0.82, 0.12)$, $c = 2$. Note that the plot of no-purchase valuation fairness ends at $\alpha = 0.46$, since any larger $\alpha$ will result in an infeasible solution (because the demand of group 5 has reached 1 so that the no-purchase valuation is not well defined).