The Seiberg—Witten Equations and Asymptotically Hyperbolic Einstein Metrics

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Abstract

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In this thesis we study the Seiberg–Witten equations and Einstein metrics on finite volume noncompact 4-manifolds with asymptotically hyperbolic cusps. The problem of obstructing the existence of Einstein metrics for closed and oriented 4-manifolds with nonzero Seiberg–Witten invariant was pioneered by LeBrun. Various extensions of this problem in the noncompact setting were subsequently studied by Biquard, di Cerbo, and Rollin in the asymptotically complex hyperbolic case. This dissertation extends this story in the asymptotically real hyperbolic setting where the ends of the manifolds are diffeomorphic to $T^3 \times [0, \infty)$. The main result of this dissertation is the construction of the first examples of noncompact 4-manifolds that does not admit any asymptotically hyperbolic Einstein metrics. Along the way, we also extend these techniques to the setting of the $\mathrm{Pin}^-(2)$ monopole equations developed by Nakamura.

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Chapter 1: Introduction

The Seiberg–Witten equations were originally introduced in the early 1990s and have since become a cornerstone of mathematical gauge theory. When $b^+ \geq 2$, solutions to the Seiberg–Witten equations define versatile and powerful smooth invariants of 4-manifolds and allow us to construct many examples of exotic smooth structures on 4-manifolds. Additionally, existence (or nonexistence) of solutions to the equations gives rich and subtle information about the geometry of the underlying manifold. For example, when a manifold with $b^+ \geq 2$ admits a nonzero Seiberg–Witten invariant, it cannot admit a metric of positive scalar curvature.

One of the striking applications of Seiberg-Witten theory is to the theory of Einstein metrics in dimension 4. Classically, obstructions to existence of Einstein metrics in dimension 4 use Chern–Weil theory. Berger [7] showed that all closed oriented Einstein manifolds in dimension 4 must have $\chi \geq 0$ and Hitchin–Thorpe extended this to prove the celebrated Hitchin–Thorpe inequality. Namely that if a closed oriented 4-manifold X admits an Einstein metric, then

$$2\chi(X) - 3|\sigma(X)| \geq 0 \tag{1.1}$$

where equality holds if and only if X is either flat or a finite quotient of a K3 surface. Due to the presence of exotic smooth structures in dimension 4, one does not expect a purely topological bound to be sharp, and indeed it is not. LeBrun [27] used techniques from Seiberg-Witten theory to construct the first examples of closed 4-manifolds that satisfy the strict Hitchin-Thorpe inequality yet nonetheless do not admit Einstein metrics. More concretely, he showed that when X is a minimal complex surface of general type and $k \geq \frac{2}{3}c_1^2(X)$, the k-fold blow up $X\#k\overline{\mathbb{C}P^2}$ does not admit any Einstein metric. Recall that a minimal surface of general type satisfies

$$c_1^2(X) = 2\chi(X) + 3\sigma(X) \tag{1.2}$$

so that if we additionally choose $k < c_1^2(X)$ the final manifold satisfies the strict Hitchin–Thorpe inequality.

A natural question to ask is how this story extends to the noncompact case. The manifolds that we work with in this thesis are oriented Riemannian 4-manifolds with cylindrical ends all of the form $T^3 \times [0, \infty)$. The ends can be endowed with a hyperbolic cusp metric

$$g_{hyp} = dt^2 + e^{-2t} g_{T^3}$$

where $e^{-2t} g_{T^3}$ is a flat metric on the 3-torus. Note that under change of coordinates $z = e^{-t}t$ this is equivalent to the standard upper half plane hyperbolic metric. While this definition is valid for any flat 3-manifold Y, we restrict ourselves to only the flat 3-torus.

Definition 1.0.1. *We say that a complete finite volume metric g on X is asymptotically hyperbolic if it is asymptotically C^2 close to g_{hyp} on the cusps of X.*

In this setting there is a generalization of the Hitchin–Thorpe inequality due to Dai–Wei [12]. If (X, g) is a finite volume asymptotically hyperbolic Einstein 4-manifold with cylindrical ends diffeomorphic $T^3 \times [0, \infty)$, then

$$2\chi(X) - 3|\sigma(X)| > 0. \tag{1.3}$$

The analogous question for this setting is whether this bound is sharp; i.e. whether there exists a smooth noncompact 4-manifold X with $T^3 \times [0, \infty)$ ends that satisfies $2\chi(X) - 3|\sigma(X)| > 0$ yet does not admit any asymptotically hyperbolic Einstein metric. The main result of this thesis is an affirmative answer to this question.

Theorem 1.0.2. *Let X_1 be a closed symplectic manifold with $b^+(X_1) \geq 2$, and let*

$$X_2 = \#_i^k (S^1 \times Y_i) \#_j^\ell (S^2 \times \Sigma_{g_j}) \tag{1.4}$$

Let $\overline{X} = X_1 \# X_2$ and let L be a collection of smoothly embedded 2-tori each with 0 self-intersection. If $-\chi(X_2) + 2 \geq \frac{1}{3}(2\chi(X_1) + 3\sigma(X_1))$, then it follows that $X = \overline{X} - L$ does not admit any asymptotically hyperbolic Einstein metrics.

We can construct X to satisfy the inequality in 3.3.1 by choosing X_2 to additionally satisfy the constraint

$$\frac{1}{2}(2\chi(X_1) + 3\sigma(X_1)) > -\chi(X_2) + 2. \tag{1.5}$$

Requiring components of L to have 0 self-intersection is necessary for the ends of the complement to be diffeomorphic to $T^3 \times [0, \infty)$. This theorem allows us to construct infinite families of 4-manifolds that satisfy the Dai–Wei inequality yet do not admit any asymptotically hyperbolic Einstein metrics.

Another interesting follow-up question is whether this can be used to detect exotic behavior in noncompact manifolds. In the compact setting, Kotschick [23] applied Lerun's constructions and showed that the existence of Einstein metrics depends on the smooth structure of the underlying manifold. He constructed exotic pairs of simply connected manifolds (X, Z) (so that X and Z are homeomorphic as manifolds) such that X admits a Kähler-Einstein metric yet Z does not admit any Einstein metric. See also [11] for an extension of this work to constructions of topological manifolds with k distinct smooth structures admitting Einstein metrics and infinitely many smooth structures without Einstein metrics. The construction relies on explicit examples of simply connected general type surfaces with prescribed Euler characteristic and signature, and invokes Freedman's theorem on the topological classification of simply connected 4-manifolds to complete the proof.

We are interested in seeing whether Kotschick's results can be extended to asymptotically hyperbolic manifolds. Suppose that X and Z are a pair of noncompact homeomorphic 4-manifolds with toral ends. If X admits an asymptotically hyperbolic Einstein metric yet Z does not, then clearly X and Z must be an exotic pair of 4-manifolds. In the compact setting, the exotic pairs that Kotschick constructed can already be detected by standard computations for the Seiberg–Witten invariants. In contrast, the Seiberg–Witten invariants are not defined in the noncompact setting, so the process of showing that a pair of noncompact homeomorphic 4-manifolds forms an exotic pair is not so straight forward. This program requires 3 main ingredients:

1. Methods to construct asymptotically hyperbolic Einstein manifolds.
2. A method to obstruct existence of asymptotically hyperbolic Einstein metrics on manifolds with prescribed topological invariants.
3. A method to construct a homeomorphism between the two constructions.

There are various ways one can construct 4-manifolds with asymptotically hyperbolic Einstien metrics. The simplest examples are finite volume hyperbolic 4-manifolds with T^3 cusps. For example, there are various hyperbolic 4-manifolds that are (topologically) the complement of a link of 2-tori in connect sums of $S^2 \times S^2$ constructed by Saratchandran [40]. . Anderson[1] also has a construction of asymptotically hyperbolic Einstein metrics via generalized Dehn filling of hyperbolic 4-manifolds. Note that currently all examples of asymptotically hyperbolic Einstein metrics in the literature are signature 0; therefore finding obstructions of asymptotically hyperbolic Einstein metrics for signature 0 manifolds will be of particular importance.

Our main theorem allows us to construct many candidate manifolds for the nonexistence side of this question. Given $n \geq 5, m \geq 4$, Baykur and Hamada [6] have constructions for minimal symplectic manifolds X_n that are homeomorphic to $\#^{2n+1}(S^2 \times S^2)$ and $\#^{2m+1}(\mathbb{C}P^2 \# \mathbb{C}P^2)$. Since X_2 in 1.0.2 is a signature 0 manifold, we can construct many examples of noncompact signature 0 manifolds that do not admit asymptotically hyperbolic Einstien metrics.

We can also construct many examples of 4-manifolds that do not admit any asymptotically hyperbolic Einstein metrics with nonzero signature by using results on the geography of symplectic 4-manifolds and general type surfaces. Let (x, c) be a pair of positive integers such that $c \equiv x$

mod 8. For all but finitely many such pairs satisfying $c < 9x$, Park [34] has constructed symplectic 4-manifolds X with $\chi_h(X) = x$ and $c_1^2(X) = c$ with π_1 isomorphic to any finitely generated group. Here χ_h is the holomorphic Euler characteristic, defined by

$$\chi_h(X) = \frac{\sigma(X) + \chi(X)}{4}.$$

Because for a symplectic 4-manifold $c_1^2(X) = 2\chi(X) + 3\sigma(X)$, it follows that for all but finitely many pairs (χ, σ) of integers satisfying $\chi > 0$, $-2\chi/3 \leq \sigma < \chi/3$, and $\chi + \sigma \equiv 0 \mod 4$ there exists a simply connected spin symplectic 4-manifold X with $(\chi(X), \sigma(X)) = (\chi, \sigma)$.

Unfortunately the question of whether Freedman's theorem generalizes to general π_1 is still open, so it seems difficult to finish the proof of this result at present.

1.1 Outline of the proofs

The strategy of the proof is based off of a construction due to Biquard [10]. The first step generalize LeBrun's scalar curvature estimate into the noncompact setting, so that if a spin^c structure $\mathfrak{s}$ admits an irreducible L^2 solution to the Seiberg–Witten equations (A, ϕ), then

$$\frac{1}{96\pi^2} \int_X s_g^2 \geq c_1^2(\mathfrak{s}) \tag{1.6}$$

where $c_1(\mathfrak{s})$ is the L^2 Chern class of the spin^c structure. For more details on this, see chapter 3. To find a suitable solution to the Seiberg–Witten equations on X, we follow a construction due to Biquard [10]. See also [37][14] for other versions of this construction.

Biquard originally constructed solutions to the Seiberg–Witten equations on cusped complex hyperbolic manifolds that admit a Kähler orbifold compactification. His key insight was to solve the Seiberg–Witten equations on the noncompact manifold is via approximating metrics on a closed ambient manifold with nonvanishing Seiberg–Witten invariant. More concretely, let $X = \overline{X} - L$, and suppose that L is a finite disjoint union of smoothly embedded 2-tori, each with 0 self-intersection, so that X has T^3 ends, and suppose that g is an asymptotically hyperbolic metric on X. Note that any noncompact 4-manifold with T^3 ends arises this way (although the compactification may not have nontrivial Seiberg–Witten invariant). We construct a sequence of metrics g_j on $\overline{X}$ that uniformly approximates g for any compact subset of $X \subset \overline{X}$.

The key here is then to prove a uniform compactness theorem so that we can extract a limiting subsequence. This requires a careful construction of the approximating metrics g_j as well as uniform estimates on 1-forms and run the Seiberg–Witten bootstrap.

Proposition 1.1.1. *Suppose that (A_j, ϕ_j) are solutions to the Seiberg-Witten equations on $(\overline{X}, g_j)$*

$$\begin{cases} D_{A_j}\phi_j = 0 \\ F^+_{A_j} = 2(\phi_j\phi_j^*)_0 \end{cases}$$

Then there exists a subsequence that converges, up to gauge transformation, in the C^∞_c topology to a solution (A, ϕ) of the Seiberg Witten equations, with $A - A_0 \in L^2_1\Omega(X, g)$. In particular, it follows that

$$\int_X F_A \wedge F_A = \int_X F_{A_0} \wedge F_{A_0} = c_1^2(\mathfrak{s})$$

By combining this with the L^2 estimates, we get the following scalar curvature estimate.

Theorem 1.1.2. *Suppose that $\overline{X}$ is a closed oriented 4-manifold, and suppose that α is a monopole class on $\overline{X}$. Suppose that $L \subset \overline{X}$ is a disjoint collection of smoothly embedded 2-tori, with each component having self-intersection 0. Then for any asymptotically hyperbolic metric g on X,*

$$\frac{1}{96\pi^2}\int_X s_g^2 \geq \alpha^2.$$

Remark 1.1.3. *The Dai–Wei inequality gives us an upper bound on the scalar curvature*

$$2\chi(X) - 3|\sigma(X)| \geq \frac{1}{32\pi^2}\int_X s_g^2 \geq \alpha^2. \tag{1.7}$$

so that if we additionally have

$$2\chi(\overline{X}) - 3|\sigma(\overline{X})| \leq \frac{1}{3}\alpha^2 \tag{1.8}$$

then X does not admit any asymptotically hyperbolic Einstein metrics.

The final step now is to construct a closed oriented manifold with an appropriate monopole class. Unfortunately, gluing formulas for the standard Seiberg–Witten equations are generally not suitable for constructing signature 0 examples. The recipe that LeBrun [27] developed and used in the closed case cruicially used the blow-up formua for Seiberg–Witten invariants. He showed that given X a minimal surface of general type and k satisfying $\frac{2}{3}c_1^2(X) \leq k \leq c_1^2(X)$, the k-fold blowup $X\#k\mathbb{C}P^2$ does not admit any Einstein metrics. Unfortunately, blowing up will decrease the signature of the manifold, and we need to blow up at least $2\sigma(X)$ times. So the final manifold will necessarily have negative signature.

Another approach one could try is to use the Bauer–Furuta connect sum formula [5]. Unfortunately we can only connect up to 3 manifolds with $b^+ \geq 1$. Since all signature zero manifolds with nonzero b^2 will have nonzero b^+, the derived lower bound is not large enough to obstruct existence of Einstein metrics using 1.1.2.

The $\mathrm{Pin}^-(2)$ equations are a variation of the Seiberg-Witten equations defined for a double cover $\tilde{X} \to X$ by Nakamura in [32]. The corresponding blow-up formula ofr the $Pin^-(2)$ equations are for the connect sums

$$X\#_i^k(S^1 \times Y_i)\#_j^\ell(S^2 \times \Sigma_{g_j}) \tag{1.9}$$

where Σ_{g_j} is a Riemann surface of genus at least 1 and Y_i is a closed oriented 3-manifold. This gluing formula attractive for our purposes since these components have signature equal to 0, and therefore we only need to find a suitable signature 0 manifold X. The key idea here is that given α a $\mathrm{Pin}^-(2)$ monopole class on X the lift $\tilde{\alpha}$ will be a Seiberg–Witten monopole class on $\tilde{X}$. We can then run the entire argument with approximating metrics on the double cover $\tilde{X}$, and note that any asymptotically hyperbolic Einstein metric on $X - L$ pulls pack to an asymptotically hyperbolic Einstein metric on $\tilde{X} - \tilde{L}$.

This allows us to obtain the nonexistence result 1.0.2.

1.2 Organization

This present thesis is based off of the submitted work [45]. We extend the constructions of Seiberg–Witten monopoles from Biquard [10], Rollin [37], and di Cerbo [14][15] into the setting of finite volume asymptotically hyperbolic 4-manifolds. This allows us to construct the first examples of noncompact 4-manifolds without asymptotically hyperbolic Einstein metrics. Along the way, we also extend the estimates from their techniques into the setting of the $\mathrm{Pin}^{-}(2)$ equations of Nakamura [32].

In chapter 2, we discuss the background and the motivation for this thesis and establish notation. We recall classical theorems for closed oriented 4-manifolds, such as the Hitchin–Thorpe inequality, as well as the Seiberg–Witten equations and some key existence results and estimates they give.

Chapter 3 sets the stage to generalize these results into the noncompact setting. We will set the stage for upgrading results from Chern–Weil theory and estimates from Seiberg–Witten theory into the setting for asymptotically hyperbolic metrics. Moreover, we discuss a construction due to Anderson for asymptotically hyperbolic Einstein metrics that are not themselves hyperbolic.

Chapter 4 will focus on constructing L^2 solutions to the Seiberg–Witten equations on asymptotically hyperbolic manifolds following the approach of Biquard. We derive key estimates and existence results which allow us to derive scalar curvature estimates on asymptotically hyperbolic 4-manifolds.

In chapter 5, we construct various examples of 4-manifolds that satisfy the Dai–Wei inequality yet nevertheless do not admit asymptotically hyperbolic Einstein metrics. Particular emphasis will be placed on signature 0 examples since all known examples of asymptotically hyperbolic Einstein metrics are signature 0.

Chapter 2: Background

2.1 The Hitchin–Thorpe Inequality

A Riemannian manifold (X, g) is said to be Einstein if its Ricci curvature is constant multiple of its metric tensor

$$Ric_g = \lambda g. \tag{2.1}$$

The question of which manifolds admit such metrics has been studied extensively [8] [2] [28]. In dimensions 2 and 3 the Einstein condition is equivalent to constant sectional curvature and are generally well understood. In dimension 2, this follows classically from the uniformization theorem and the classification of compact oriented Riemann surfaces, and in dimension 3 from the resolution of Thurston's geometrization conjecture, due to Hamilton and Perelman. The natural progression would be to carry out a similar program in dimension 4, and the natural beginning this program would be by identifying canonical metrics. When Einstein metrics exist, they make natural candidates for canonical metrics.

In contrast to dimensions 2 and 3, starting in dimension 4 there are no canonical local descriptions of Einstein metrics, and for dimensions 5 and above, there are no known nonexistence or uniqueness theorems. Indeed, starting in dimension 4, one begins to encounter problems with exotic smooth structures, complications with understanding the fundamental group, etc. Dimension 4 is special in this story due to deep gauge theoretic results specific to this dimension and also due to special properties of Chern-Weil theory. These allow one to prove various nonexistence theorems, such as the Hitchin–Thorpe inequality [20], and uniqueness theorems, such as those of LeBrun [26] for complex-hyperbolic metrics. See also the work of Besson–Courtois–Gallot [9] on uniqueness of hyperbolic metrics.

Theorem 2.1.1 (Hitchin–Thorpe)**.** *Suppose that X is a closed orientable 4-manifold admitting an Einstein metric. Then*

$$2\chi(X) - 3|\sigma(X)| \geq 0 \tag{2.2}$$

where equality holds if and only if X is flat or Calabi-Yau with universal cover a K3 surface. Here χ and σ denote the Euler characteristic and signature, respectively.

To see this, we recall that the Chern–Gauss–Bonnet formula states that for any Riemannian metric g on X,

$$\chi(X) = \frac{1}{8\pi^2}\int_X \frac{s^2}{24} + |W|^2 - \frac{|\mathrm{ric}_0|^2}{2} d\mu_g. \tag{2.3}$$

Here s is the scalar curvature, W is the Weyl curvature, and ric_0 is the trace-free Ricci curvature. Note that the Einstein condition is equivalent to vanishing of the trace-free Ricci curvature, so that if g is an Einstein metric, then $\chi(X) \geq 0$, with equality if and only if g is a flat metric. We also note in dimension 4, the Hirzebruch signature formula takes the form

$$\sigma(X) = \frac{1}{12\pi^2}\int_X |W^+|^2 - |W^-|^2 d\mu_g \tag{2.4}$$

where W^+ and W^- denote the self-dual and anti-self-dual components of the Weyl curvature, respectively. Combining the two equations, we see that

$$2\chi(X) \pm 3\sigma(X) = \frac{1}{4\pi^2}\int_X \frac{s^2}{24} + 2|W^\pm|^2 - \frac{|\mathrm{ric}_0|^2}{2} d\mu_g. \tag{2.5}$$

When g is Einstein, $\mathrm{ric}_0 = 0$ and therefore $2\chi(X) \pm 3\sigma(X) \geq 0$. In the equality case, one observes that $s = 0$ and assumes without loss of generality that $W^+ = 0$. In particular, the curvature of the Levi-Civita connection on the bundle of self-dual 2-forms $\Lambda^+X \subset \Lambda^2X$ must be flat. It follows that the holonomy of g is in the kernel of the representation $SO(4) \to SO(3)$ induced by self-dual 2-forms, and therefore must lie in $SU(2)$. This implies that g must either be flat or Calabi-Yau. Observe in particular, that when g is Einstein this implies an a priori upper bound on the scalar curvature functional

$$2\chi(X) - 3|\sigma(X)| \geq \frac{1}{4\pi^2}\int_X \frac{s^2}{24}$$

There have been extensions of this by Gromov [19] using simplicial volume, see also [24]. Sambussetti [39] also has an extension in this direction using the volume entropy developed by Besson, Courtois, and Gallot [9]. The general strategy of these techniques is to establish an a priori upper bound on some curvature functional for Einstein metrics, and then to observe that the functional must always be positive for all metrics.

2.2 The Seiberg–Witten Equations

The Seiberg–Witten invariants are smooth invariants of closed oriented 4-manifolds with $b^+(X) \geq 1$ first introduced in 1994 by Witten [44]. In this section, we provide a quick overview for the setup of the Seiberg–Witten equations. For more details see [31].

Let (X^4, g) be a closed, oriented Riemannian manifold. A spin^c structure $\mathfrak{s}$ on X consists of a rank 4 hermitian vector bundle $S \to X$ together with a splitting $S \cong S^+ \oplus S^-$ into rank 2 subbundles and Clifford multiplication $\rho : TX \to Hom(S^\pm, S^\mp)$ that swaps the summands and satisfies

$$\rho(v)^2 = -\|v\|^2 \cdot \mathrm{Id}_S. \tag{2.6}$$

We can extend Clifford multiplication from TX to T^*X by indentifying them via the Riemannian metric, and from T^*X to the exterior algebra Λ^*X via

$$\rho(\alpha \wedge \beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{deg(\alpha)deg(\beta)}\rho(\beta)\rho(\alpha)). \tag{2.7}$$

Implicitly, we will always use this identification. Note that Clifford multiplication by even forms preserves the summands and Clifford multiplication by odd forms swaps the summands.

The first Chern class of the spin^c structure $\mathfrak{s}$ is defined to be the first Chern class for the bundle of positive spinors, i.e. $c_1(\mathfrak{s}) = c_1(S^+)$. Note that the mod 2 reduction of the first Chern class must equal the second Stiefel-Whitney class of X; i.e.

$$w_2(X) = c_1(\mathfrak{s}) \mod 2. \tag{2.8}$$

A basic fact from spin geometry is the fact that every 4-manifold admits a spin^c structure. Furthermore, the set of spin^c structures over X is an affine space over $H^2(X,\mathbb{Z})$. To see the latter claim, recall that $H^2(X,\mathbb{Z})$ corresponds to the set of complex line bundles over X. Given a spin^c structure $\mathfrak{s}$ and a line bundle L, we can construct a new spin^c structure $\mathfrak{s} \otimes L$ given by the bundle $S \otimes L$ with Clifford multiplication given by $\rho \otimes \mathrm{Id}_L$. The converse direction follows by noting that a spin^c structure is the associated vector bundle of a principal $\mathrm{Spin}^c(4)$ bundle and an irreducible representation (the complex spin representation). Schur's lemma implies that given two spin^c bundles $\mathfrak{s}_1, \mathfrak{s}_2$ the bundle of maps $L = \mathrm{End}_X(\mathfrak{s}_1, \mathfrak{s}_2)$ intertwining the Clifford multiplications (and hence the representations) must be a complex line bundle. It follows that $\mathfrak{s}_1 \otimes L = \mathfrak{s}_2$.

A connection A on S is said to be a spin^c connection if it is compatible with Clifford multiplication; i.e.

$$\nabla^A \rho(V)\sigma = \rho(\nabla^{LC} V)\sigma + \rho(V)\nabla^A \sigma. \tag{2.9}$$

It can be checked that any two spin^c connections differ by an element of $i\Omega^1(X,\mathbb{R})$. Given a spin^c connection, we can define the corresponding Dirac operator D_A as the composition of the following maps:

$$\Gamma(S^{\pm}) \xrightarrow{\nabla^A} \Gamma(T^*X \oplus S^{\pm}) \xrightarrow{\rho} \Gamma(S^{\mp}) \tag{2.10}$$

Explicitly in local coordinates, this is given by

$$D_A = \sum_i \rho(e_i) \cdot \nabla^A_{e_i} \tag{2.11}$$

where e_i is a local orthonormal frame for TX.

The configuration space for the Seiberg–Witten equations are given by $\mathcal{C}(X,\mathfrak{s}) = \{(A,\phi)\}$ where A is a spin^c connection and $\phi \in \Gamma(S^+)$. A configuration is reducible if $\phi = 0$, and irreducible if $\phi \neq 0$. Let $\mathcal{C}^*(X, \mathfrak{s}) \subset \mathcal{C}(X, \mathfrak{s})$ denote the set of irreducible configurations. The Seiberg–Witten equations are the following system of PDEs on the configuration space:

$$\begin{cases} \frac{1}{2}\rho(F^+_{A^t}) = (\phi\phi^*)_0 \\ D_A\phi = 0 \end{cases} \tag{2.12}$$

Here A^t is the connection induced by A on the determinant bundle $det(S^+)$ (so that it is the trace of F_A) and the superscript + denotes that we take the self dual part of the curvature tensor. The term $(\phi\phi^*)_0$ on the is the trace-free part of the endomorphism of S^+ defined by $\phi \otimes \phi^*$, so

$$(\phi\phi^*)_0 = \phi\phi^* - \frac{1}{2}tr(\phi\phi^*)\mathrm{Id}_{S^+} \tag{2.13}$$

The Seiberg–Witten equations are a second order nonlinear elliptic PDE over X.

The gauge group for the Seiberg–Witten equations is $\mathcal{G}(X, \mathfrak{s}) = \{u : X \to S^1\}$ where u is a smooth map. The action of u on $(A, \phi) \in \mathcal{C}(X, \mathfrak{s})$ is by

$$u \cdot (A, \phi) = (A - u^{-1}du, u \cdot \phi) \tag{2.14}$$

and it is straightforward to verify that the stabilizer of (A, ϕ) is trivial if and only if it is irreducible. In the case (A, ϕ) is reducible, the stabilizer will consist of constant maps $X \to S^1$.

From this point out we need to extend the spaces we are working in to appropriate L^2_k Sobolev completions. Elements of the gauge group should live in L^2_3 and the spinors and connections in the configuration space should be class L^2_2. This allows us to endow all the spaces with the structure of a Hilbert manifold and access to the inverse function theorem. The moduli space of configurations is defined to be

$$\mathcal{B}(X, \mathfrak{s}) = \mathcal{C}(X, \mathfrak{s})/\mathcal{G}(X, \mathfrak{s}). \tag{2.15}$$

The irreducible part of the moduli space of configurations $\mathcal{B}^*(X, \mathfrak{s})$ is analogously defined by

$$\mathcal{B}^*(X, \mathfrak{s}) = \mathcal{C}^*(X, \mathfrak{s})/\mathcal{G}(X, \mathfrak{s}). \tag{2.16}$$

and is a Hilbert manifold homotopy equivalent to $\mathbb{C}P^\infty \times T^k$, where $k = \text{rank}(H^1(X, \mathbb{R}))$. We also define based gauge group $\mathcal{G}_0(X, \mathfrak{s})$ which consist of continuous maps $X \to S^1$ that are constant at a fixed basepoint $x_0 \in X$, and analogously define based moduli space of configurations $\mathcal{B}_0(X, \mathfrak{s})$ and based moduli space of irreducible configurations $\mathcal{B}_0^*(X, \mathfrak{s})$. Note that $\mathcal{B}_0^*(X, \mathfrak{s})$ naturally takes the form of an S^1 bundle over $\mathcal{B}^*(X, \mathfrak{s})$.

Observe that the Seiberg–Witten equations are invariant under gauge transformation. Indeed, since $u^{-1}du$ is locally closed, $F_{u \cdot A^t} = F_{A^t}$ and it is straightforward to verify that

$$\begin{aligned} D_{u \cdot A}(u \cdot \phi) &= -u^{-1}du \cdot (u \cdot \phi) + D_A(u \cdot \phi) \\ &= -du \cdot \phi + du \cdot \phi + u \cdot D_A \phi \\ &= u \cdot D_A \phi. \end{aligned} \tag{2.17}$$

This proves the claim. Indeed, if we consider the Seiberg–Witten equations as a map

$$SW : \mathcal{C}(X, \mathfrak{s}) \to \Lambda^+ T^*(X, i\mathbb{R}) \oplus \Gamma(S^-) \tag{2.18}$$

then this map is $\mathcal{G}(X, \mathfrak{s}$-equivariant (where the action on $\Lambda^+ T^*(X, i\mathbb{R})$ is trivial and the action on $\Gamma(S^+)$ is by multiplication). Therefore, solutions to Seiberg–Witten equations pass to the quotient, and we can define the Seiberg–Witten moduli space

$$\mathcal{M}(X, \mathfrak{s}) \subset \mathcal{B}(X, \mathfrak{s}) \tag{2.19}$$

To have well defined Seiberg–Witten invariants, we need the moduli space of solutions to be a smooth submanifold of $\mathcal{B}^*(X, \mathfrak{s})$. Unfortunately, this is not necessarily true since we could have reducible solutions and irreducible solutions may not necessarily be transversely cut out. To fix this, we will work with the perturbed Seiberg–Witten equations

$$\begin{cases} \frac{1}{2}\rho(F_{A^t}^+ + ih) = (\phi\phi^*)_0 \\ D_A\phi = 0 \end{cases} \tag{2.20}$$

where $h \in \Omega^+(X,\mathbb{R})$ is a self-dual 2-form. Observe that the harmonic part of $F_{A^t}^+$ is a fixed cohomology class in $H^2(X)$. Therefore when $b^+(X) \geq 1$, for a generic perturbation h, the perturbed Seiberg–Witten equations admit only irreducible solutions.

Suppose now that (A,ϕ) is a solution to the Seiberg–Witten equations. Linearizing the action of the gauge group gives an elliptic complex

$$L_3^2(X, i\mathbb{R}) \to L_2^2(\Lambda^1(X, i\mathbb{R}) \oplus S^+) \to L_1^2(\Lambda^+(X, i\mathbb{R}) \oplus S^-) \tag{2.21}$$

and when this complex has trivial second homology (i.e. the second map is surjective), the moduli space of solutions $\mathcal{M}(X,\mathfrak{s},h)$ to the perturbed Seiberg–Witten equations is transversely cut out. When $b^+(X) \geq 1$, this is true for a generic perturbation h. The dimension of the moduli space is equal to the index of the elliptic complex and can be computed to be

$$\dim \mathcal{M}(X,\mathfrak{s}) = \frac{1}{4}(c_1^2(\mathfrak{s}) - 2\chi(X) - 3\sigma(X)) \tag{2.22}$$

via the Atiyah-Singer index theorem. An orientation of the homology groups $H^0(X,\mathbb{R}), H^1(X,\mathbb{R}), H^+(X,\mathbb{R})$ determines an orientation of $\mathcal{M}(X,\mathfrak{s},h)$. To see this, note that the tangent space of $\mathcal{M}(X,\mathfrak{s},h)$ at a solution (A,ϕ) is the kernel of the second map of 2.21. Note that up to zeroth order, the elliptic complex 2.21 is equivalent to the direct sum of the following two elliptic complexes

$$\begin{gathered} L_3^2(X, i\mathbb{R}) \xrightarrow{d} L_2^2(\Lambda^1(X, i\mathbb{R})) \xrightarrow{d^+} L_1^2(\Lambda^+(X, i\mathbb{R})) \\ 0 \to L_2^2(S^+) \xrightarrow{D_A} L_1^2(S^-) \end{gathered} \tag{2.23}$$

and the homology of the former complex is exactly $H^0(X,\mathbb{R}), H^1(X,\mathbb{R}), H^+(X,\mathbb{R})$ while the second complex is complex and thus naturally oriented. This proves the claim.

One of the most striking features of Seiberg–Witten theory in comparison to Donaldson theory and Yang–Mills theory is that the moduli spaces are always compact. This is a consequence of the Lichnerowiz-Witzenböck formula. When the moduli space is zero-dimensional, the Seiberg–Witten moduli space is just a finite set of points, and the Seiberg–Witten invariants are defined to be the signed count

$$\mathcal{SW}(X, \mathfrak{s}) = \#\mathcal{M}(X, \mathfrak{s}, h) \tag{2.24}$$

of points in the moduli space. When the moduli space is $2k$-dimensional, let c_1 be the first Chern class of the S^1 bundle $\mathcal{B}_0^*(X, \mathfrak{s}) \to \mathcal{B}^*(X, \mathfrak{s})$. Then we define

$$\mathcal{SW}(X, \mathfrak{s}) = \int_{\mathcal{M}(X,\mathfrak{s},h)} c_1^k. \tag{2.25}$$

When the moduli space is $2k + 1$-dimensional, we set the invariant to be 0.

To show invariance of the Seiberg–Witten invariants, it therefore suffices to show that for any two choices of Riemannian metric g_0 and g_1 on X and suitable perturbations h_0 and h_1, the corresponding Seiberg–Witten moduli spaces are cobordant as smooth submanifolds of $\mathcal{B}^*(X, \mathfrak{s})$. To do this, we take paths g_t and h_t from (g_0, h_0) to (g_1, h_1), and consider the following cobordism:

$$\bigsqcup_t \mathcal{M}(X, \mathfrak{s}, h_t, g_t) \subset \mathcal{B}^*(X, \mathfrak{s}) \times [0, 1] \tag{2.26}$$

When $b^+(X) \geq 2$, one can show that this is a smooth cobordism for a generic path (g_t, h_t). As with the case for smoothness of the moduli space, this comes down to showing that all solutions are irreducible for a generic path and surjectivity of a certain elliptic operator. It follows from this that the Seiberg–Witten invariant $\mathcal{SW}(X, \mathfrak{s})$ is a well defined invariant for the spin^c manifold $(X, \mathfrak{s})$.

One of the key computations for Seiberg–Witten theory is the computation of the invariant for minimal surfaces of general type and $b^+ \geq 2$, due to Morgan [31], and the subsequent generalizations to symplectic surfaces with $b^+ \geq 2$ due to Taubes [41][42]. Suppose that X is a closed orented

4-manifold, and suppose that J is an almost complex structure. There is a canonical spin^c structure defined by J, where the spinor bundles are given by

$$\begin{cases} S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \\ S^- = \Lambda^{0,1} \end{cases}$$

and Clifford multiplication by a 1-form v is given by

$$\rho(v) \cdot \sigma = \sqrt{2}(\sigma \wedge \pi^{0,1} v + \sigma \lrcorner \pi^{0,1} v).$$

This defines a canonical spin^c structure $\mathfrak{s}_0$. Clearly, $c_1(\mathfrak{s}_0) = c_1(X)$. Since all other spin^c structures arise as $\mathfrak{s}_0 \otimes L$ for complex line bundles L, this allows one to explicitly write out the Seiberg–Witten equations and compute its solutions. For more details, see chapter 7 of [31].

Theorem 2.2.1 (Morgan)**.** *Suppose that X is a minimal surface of general type. Then the Seiberg–Witten invariants are*

$$SW(X, \mathfrak{s}) = \begin{cases} 1 & \mathfrak{s} = \mathfrak{s}_0 \\ \pm 1 & \mathfrak{s} = \overline{\mathfrak{s}_0} \\ 0 & \textit{otherwise} \end{cases} \tag{2.27}$$

where $\overline{\mathfrak{s}_0}$ denotes the conjugate spin^c *structure.*

In general when the surface is not a minimal model, we can pass to the blow-up formula[16].

Theorem 2.2.2 (Friedman-Morgan)**.** *Suppose that X is a minimal surface of general type and let $Z = X \# k\overline{\mathbb{C}P^2}$. Let $E_1, \cdots, E_k$ denote the exceptional classes on Z and let $\mathfrak{s}_0$ denote the pullback of the canonical class of X along the blowdown map $Z \to X$. Then*

$$SW(Z, \mathfrak{s}) = \begin{cases} \pm 1 & \mathfrak{s} = \mathfrak{s}_0 \otimes \pm E_1 \cdots \otimes \pm E_k \\ 0 & \textit{otherwise} \end{cases} \tag{2.28}$$

In the symplectic case, Taubes proved an analogous result using pseudoholomorphic curves[41][42].

Theorem 2.2.3 (Taubes)**.** *Suppose that X is a symplectic 4-manifold. Then*

$$SW(X, \mathfrak{s}) = \begin{cases} 1 & \mathfrak{s} = \mathfrak{s}_0 \\ \pm 1 & \mathfrak{s} = \overline{\mathfrak{s}_0} \end{cases} \tag{2.29}$$

2.3 Obstructions to Einstein Metrics from Seiberg–Witten Theory

One of the key features of Seiberg–Witten theory that makes it much more tractable than other gauge theories is the Lichnerowicz–Weitzenböck formula:

$$D_A^* D_A \phi = \nabla^{A,*} \nabla^A \phi + \frac{s}{4}\phi + \rho(F_A^+)\phi. \tag{2.30}$$

This allows one to obtain a priori C^0 bounds on solutions to the Seiberg–Witten equations and start the elliptic bootstrap via Sobolev multiplication and easily prove compactness of the moduli space. Observe that when (A, ϕ) is an irreducible solution to the (unperturbed) Seiberg–Witten equations, we have the following identity:

$$\begin{aligned} 0 &= \int_X \langle D_A^* D_A \phi, \phi \rangle \\ &= \int_X \langle \nabla^{A,*} \nabla^A \phi, \phi \rangle + \frac{s}{4}\|\phi\|^2 + \frac{\|\phi\|^4}{4} \\ &\geq \int_X \frac{s}{4}\|\phi\|^2 + \frac{\|\phi\|^4}{4}. \end{aligned} \tag{2.31}$$

From this it follows that the scalar curvature cannot be everywhere positive, and from Cauchy–Schwartz that

$$\int_X s^2 \geq \int_X \|\phi\|^4. \tag{2.32}$$

Similarly, we can also compute that [1]

$$\|F_{A^t}^+\|^2 = \frac{1}{4}\|(\phi\phi^*)_0\|^2 = \frac{1}{8}\|\phi\|^4. \tag{2.33}$$

Combining these two observations, we can see that

$$\int_X s^2 \geq \int_X \|\phi\|^4 = 8\int_X |F_{A^t}^+|^2. \tag{2.34}$$

Let $\eta \in \Omega^2(X, i\mathbb{R})$ be the harmonic representative of $2\pi c_1(\mathfrak{s})$; i.e. the harmonic representative of the class $[F_{A^t}] \in H^2(X, i\mathbb{R})$. Since harmonic forms minimize norm within their cohomology class:

$$\begin{aligned}\int_X |\eta^+|^2 &= \frac{1}{2}\int_X \eta\wedge\eta + |\eta|^2 \\ &= 2\pi^2 c_1^2(\mathfrak{s}) + \frac{1}{2}\int_X |\eta|^2 \\ &\leq \frac{1}{2}\int_X F_{A^t}\wedge F_{A^t} + |F_{A^t}|^2 \\ &= \int_X |F_{A^t}^+|^2\end{aligned} \tag{2.35}$$

In particular, this implies that the existence of an irreducible solution to the Seiberg–Witten equations gives an a priori lower bound on the total scalar curvature of X. LeBrun [26] used these observations to prove the following theorem.

Theorem 2.3.1 (LeBrun). *Let (X, g) be a closed oriented Riemannina 4-manifold, and let $\mathfrak{s}$ be a* spin^c *structure. If there exists an irreducible solution (A, ϕ) to the Seiberg–Witten equations, then*

$$\int_X s^2 \geq 8\int_X |F_{A^t}^+|^2 \geq 32\pi^2(c_1^+(\mathfrak{s})^2 \tag{2.36}$$

where here $2\pi c_1^+(\mathfrak{s}) = [\eta^+]$. Furthermore, if equality holds, then g has constant scalar curvature and is Kähler with respect to a complex structure that induces $\mathfrak{s}$.

[1] Note that when we consider the Frobenius norm on $End(S)$, Clifford multiplication scales up norms by a factor of 4

Proof. We simply combine the above inequalities to see that

$$\begin{aligned} \int_X s^2 &\geq 8 \int_X |F_{A^t}^+|^2 \\ &\geq 8 \int_X |\eta^+|^2 = 32\pi^2 (c_1^+(\mathfrak{s})^2 \end{aligned} \tag{2.37}$$

In the equality case, we notice that we must have equality

$$0 = \int_X \frac{s}{4} \|\phi\|^2 + \frac{\|\phi\|^4}{4} \tag{2.38}$$

and it follows that

$$\nabla^A \phi = 0.$$

So ϕ is a parallel spinor and $|\phi|$ must be constant and nonzero. Therefore both s and $|F_A^+|$ must be constant. Since constant length self-dual 2-forms correspond to g-compatible almost complex structures, this gives us an almost complex structure J. Now we note that

$$\nabla J = \nabla F_A^+ = \nabla^A (\phi\phi^*)_0 = 0$$

so J must be integrable and g must be Kähler, proving the claim. □

One can also use $c_1^2(\mathfrak{s})$ rather than $\int_X |\eta^+|^2$ for the lower bound since the latter depends on the metric. However in the setting where there are multiple spin^c structures that admit irreducible solutions to the Seiberg–Witten equations, one can sometimes obtain better metric independent lower bounds than $32\pi^2 c_1^2(\mathfrak{s})$.

The critical observation here is from the remarkable fact that a nonvanishing Seiberg–Witten invariant implies that any metric admits solutions to the unperturbed Seiberg–Witten equations. This is from the observation that if there are no solutions to the equations, then trivially, the Seiberg–Witten moduli space is transversely cut out and thus the Seiberg–Witten invariant is 0. This observation combined with the above inequality gives an a priori lower bound to the scalar curvature functional. This was first observed by LeBrun [26][27] and he leveraged this to prove uniqueness of the Einstein metric on closed oriented quotients of $\mathbb{CH}^2$ as well as construct infinitely many examples of smooth 4-manifolds that satisfy the Hitchin–Thorpe inequality yet do not admit any Einstein metrics.

In order to relate this to the Hitchin–Thorpe inequality, recall that Hitchin–Thorpe gives us an upper bound to the total scalar curvature functional, so that when X admits an Einstein metric and a spin^c structure $\mathfrak{s}$ with $SW(X, \mathfrak{s}) \neq \bar{0}$, we have the following chain of inequalities

$$\begin{aligned} 2\chi(X) - 3|\sigma(X)| &\geq \frac{1}{4\pi^2} \int_X \frac{s^2}{24} + 2|W^{\pm}|^2 \\ &\geq \frac{1}{96\pi^2} \int_X s^2 \\ &\geq \frac{1}{12\pi^2} \int_X |F^+_{A^t}|^2 \\ &\geq \frac{1}{3}(c_1^+(\mathfrak{s}))^2 \\ &\geq \frac{1}{3}c_1^2(\mathfrak{s}) \end{aligned} \tag{2.39}$$

Here $c_1^+(\mathfrak{s})$ is the self-dual part of the harmonic representative of $c_1(\mathfrak{s})$; note that this is generally metric dependent. However, in the case where there are multiple spin^c structures with nonzero Seiberg–Witten invariant the supremum of all such bounds usually will give us a better metric independent lower bound that just using $c_1^2(\mathfrak{s})$ will. The way forward from here is to find a smooth manifold X with nontrivial Seiberg–Witten invariants so that the a priori lower bound for $\int_X s^2$ we derived above is larger than the upper bound from the Hitchin–Thorpe inequality.

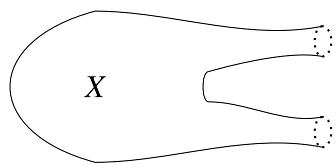
Notice that what we really needed for the scalar curvature estimate was a priori existence of solutions to the unperturbed Seiberg–Witten equations. Indeed, this motivates the notion of a monopole class class [25] :

Definition 2.3.2. *Let $\overline{X}$ be a closed orientable manifold, and let $\alpha \in H^2(\overline{X})/torsion$. If there exists a* spin^c *structure $\mathfrak{s}$ such that for any metric g the Seiberg Witten equations on $\mathfrak{s}$ admits an irreducible solution with $\alpha = c_1(\mathfrak{s})$, say that α is a monopole class.*

For example, this is true for manifolds with nonzero Bauer-Furuta invariant [17][5][4], and is the basis for some extensions of LeBrun's original work [21].

Chapter 3: Asymptotically Hyperbolic 4-Manifolds

Suppose that X is a noncompact oriented Riemannian 4-manifold of finite topological type with cylindrical ends all of the form $T^3 \times [0, \infty)$. Such manifolds arise as link complements $\overline{X} - L$, where $\overline{X}$ is ambient closed oriented 4-manifold and the link L is a disjoint union of smoothly embedded 2-tori with each component T_i having self-intersection 0. Note that in general, the end corresponding to the complement of a 2-torus with self intersection k will take the form $Y \times [0, \infty)$, where Y is the S^1-bundle over T^2 with Euler number k. Conversely, we can realize all noncompact



oriented Riemannian 4-manifolds X of finite topological type with T^3 ends as link complements in some ambient closed oriented 4-manifold via generalized Dehn filling. For each end $T_i \times [0, \infty)$ choose a simple closed geodesic $\sigma_i \subset T_i$ and attach a solid torus $D^2 \times T^2$ to T_i by a diffeomorphism that sends ∂D^2 to σ_i to obtain a closed 4-manifold $\overline{X}$. It follows that the relevant link L will consist of the collection of core tori $\{0\} \times T_i^2 \subset \overline{X}$.

Given a flat metric g_{T^3} on T^3, the corresponding hyperbolic cusp metric on $T^3 \times [0, \infty)$ is

$$g_{hyp} = dt^2 + e^{-2t} g_{T^3} \tag{3.1}$$

where t is the coordinate on $[0, \infty)$–note that this is precisely the hyperbolic upper half space metric with $z = e^{-t}$. Say that a metric g on X is asymptotically hyperbolic if on each end there is some flat metric g_{T^3} so that g is asymptotically C^2-close to g_{hyp}. Here, by asymptotically C^2-close we mean that g approaches g_{hyp} in the C^2-topology on the interval $T^3 \times [k, \infty)$ as $k \to \infty$.

3.1 L^2 Cohomology

Suppose that (X, g) is an oriented, noncompact n-manifold. Denote by $L^2\Omega^k(X, g)$ the L^2 completion of smooth k-forms with bounded L^2 norm taken with respect to g. If we restrict the de Rham differential d to a suitable dense linear subspace, we have a Hilbert complex

$$\cdots \to L^2\Omega^{k-1}(X, g) \to L^2\Omega^k(X, g) \to L^2\Omega^{k+1}(X, g) \to \cdots \tag{3.2}$$

The maximal domain of existence for d is

$$Dom^k(d) = \{\alpha \in L^2\Omega^k(X, g) | d\alpha \in L^2\Omega^{k+1}(X, g)\} \tag{3.3}$$

Note that when g is a complete metric, $Dom^k(d)$ is essentially the L^2_1-closure of the space of smooth k-forms with compact support Ω^k_c. Let

$$Z^k(X, g) = \{\alpha \in L^2\Omega^{k-1}(X, g) | d\alpha = 0\} \tag{3.4}$$

and define the (reduced) L^2 cohomology to be

$$H^*_{(2)}(X, g) = Z^k(X, g) / \overline{dDom^{k-1}(d)} \tag{3.5}$$

As defined, the L^2 cohomology is not a topological invariant and it is not hard to see that it is quite sensitive to the geometry of X at infinity. Indeed, the reduced L^2 cohomology is in general only an invariant for a quasi-isometry class of metrics, that is for metrics g_1 and g_2 such that there exist constants $c, C > 0$ such that

$$cg_1 \leq g_2 \leq Cg_1. \tag{3.6}$$

In particular, since asymptotically hyperbolic metrics are C^2-close to hyperbolic metrics, they must also be quasi-isometric. Therefore computations for L^2 cohomology for hyperbolic metrics also hold for asymptotically hyperbolic metrics.

The natural inclusions of $\Omega_c^* \subset L^2\Omega^* \subset \Omega^*$ induce natural maps on the level of de Rham cohomology

$$H_c^*(X) \to H_{(2)}^*(X,g) \to H^*(X) \tag{3.7}$$

where H_c^* is de Rham cohomology with compact support and H^* is standard de Rham cohomology. In general, these maps are neither injectve nor surjective. – indeed, in the case that X has cusped / cylindrical ends, recall that $H_c^*(X)$ is just the relative cohomology group $H^*(X, \partial X)$, where ∂X is the geometric boundary of X consisting of the cusp cross-sections.

Let $\mathcal{H}_{(2)}^k(X,g)$ denote the space of g-harmonic k-forms, i.e. kernels of

$$\Delta = dd^* + d^*d \tag{3.8}$$

with finite L^2 norm. A natural question to ask here is whether there exists a Hodge-Kodaira decomposition; i.e.

$$L^2\Omega^k(X,g) = \mathcal{H}_{(2)}^k(X,g) \oplus \overline{d\Omega_c^{k-1}} \oplus \overline{d^*\Omega_c^{k+1}} \tag{3.9}$$

where here the subscript c denotes forms with compact support. When g is a complete metric, Gaffney [18] showed that this exists, and this implies that the natural inclusion map

$$\mathcal{H}_{(2)}^*(X,g) \xrightarrow{\cong} H_{(2)}^*(X,g). \tag{3.10}$$

is in fact an isomorphism. In certain cases when we have an explicit description of the metric at infinity (such as in the case of hyperbolic cusps), this allows us to get an explicit description of the space of harmonic forms and relate the L^2 cohomology to the ordinary de Rham cohomologies.

When X is orientable, since the Poincaré duality isomorphism is an isometry, we also have Poincaré duality via the Hodge star

$$H_{(2)}^k(X,g) \cong H_{(2)}^{n-k}(X,g). \tag{3.11}$$

Therefore, when $n = 4k$ it makes sense to talk about L^2 self-dual and anti-self-dual forms. For hyperbolic cusps on $2n$ dimensional manifolds of maximal rank and finite volume, we have a computation due to Zucker [46]. See also [30].

Theorem 3.1.1 (Zucker)**.** *Suppose that (X^{2n}, g) is a finite volume complete orientable $2n$ dimensional manifold with asymptotically hyperbolic cusps. Then the L^2 cohomology is isomorphic to the various homology groups as follows:*

$$H^k_{(2)}(X,g) \cong \begin{cases} H^k_c(X) & k > n \\ Im(H^k_c(X) \to H^k(X)) & k = n \\ H^k(X) & k < n \end{cases} \tag{3.12}$$

Since L^2 cohomology is invariant under quasi-isometry, it follows that the same is true for asymptotically hyperbolic cusps. The idea behind the proof is as follows. Given a harmonic form $\omega \in \mathcal{H}^k_{(2)}(X,g)$, on the cusp $T^3 \times [0,\infty)$, we take a Fourier decomposition

$$\omega = \alpha(t) + dt \wedge \beta(t) \tag{3.13}$$

where $\alpha(t) \in C^k_{dR}(T^3)$ and $\beta(t) \in C^{k-1}_{dR}(T^3)$. Then it follows that the L^2 norm of ω with the metric given by g_{hyp} is

$$\|\omega\|^2_{L^2(T^3\times[0,\infty))} = \int_0^\infty e^{(2k-3)t}\|\alpha(t)\|^2_{L^2(T^3)} + e^{(2k-5)t}\|\beta(t)\|^2_{L^2(T^3)}dt \tag{3.14}$$

where the norms $\|\alpha(t)\|_{L^2(T^3)}$ and $\|\beta(t)\|_{L^2(T^3)}$ are taken with respect to a fixed flat metric on T^3. One can then decompose d, d^* and $\Delta = dd^* + d^*d$ with respect to this decomposition and note that

α and β satisfy a coupled first order ODE:

$$\begin{cases} d_{T^3}\alpha = d^*_{T^3}\beta = 0 \\ d_{T^3}\beta = \frac{\partial}{\partial t}\alpha \\ d^*_{T^3}\alpha = e^{-2t}\frac{\partial}{\partial t}\beta - (5-2k)e^{-2t}\beta \end{cases} \tag{3.15}$$

To proceed, we can take a decomposition of α and β into eigenforms on T^3 and analyze the decay rates for solutions of the ODE. One can show that for $k \geq 2$ there is a a suitable retraction map $\mathcal{H}^k(X,g) \to H^k_c(X)$ which is an isomorphism when $k > 2$.

When $k = 2$, it follows that

$$H^2_{(2)}(X,g) = Im(H^2_c(X) \to H^2(X)). \tag{3.16}$$

When $X = \overline{X} - L$, there are two relative long exact sequences of pairs given by $(\overline{X}, L)$ and $(X, \partial X)$:

$$\cdots \longrightarrow H^2(\overline{X}, L) \longrightarrow H^2(\overline{X}) \longrightarrow H^2(L) \longrightarrow \cdots \tag{3.17}$$

and

$$\cdots \longrightarrow H^2(X, \partial X) \longrightarrow H^2(X) \longrightarrow H^2(\partial X) \longrightarrow \cdots \tag{3.18}$$

These long exact sequences are further related by the long exact sequences induced by he pairs $(\overline{X}, X)$ and $(L, \partial X)$, where by L we really mean a suitable tubular neighborhood so that we can identify $\partial L = \partial X$. But by the excision theorem, we see that $H^*(\overline{X}, X) \cong H^*(L, \partial X)$. This gives the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & H^2(\overline{X}, X) & \xrightarrow{\cong} & H^2(L, \partial X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^2(\overline{X}, L) & \longrightarrow & H^2(\overline{X}) & \longrightarrow & H^2(L) & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^2(X, \partial X) & \longrightarrow & H^2(X) & \longrightarrow & H^2(\partial X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & H^3(\overline{X}, X) & \xrightarrow{\cong} & H^3(L, \partial X) & \longrightarrow & \cdots
\end{array} \tag{3.19}$$

where the maps in the above diagram are induced by inclusion. Note that $H^2(\overline{X}, L) \to H^2(X, \partial X)$ is an isomorphism by the Thom isomorphism. A diagram chase shows that $\omega \in Im(H^2(X, \partial X) \to H^2(X))$ if and only if $\omega \in Im(H^2(\overline{X}, L) \to H^2(\overline{X})$. But this is true if and only if $\omega \in Ker(H^2(\overline{X} \to H^2(L))$ with is equivalent to $\langle \omega, [T_i] \rangle = 0$ for all 2-tori $T_i \subset L$. Summarizing, this shows that:

Proposition 3.1.2. *Suppose that $\overline{X}$ is a closed, oriented 4-manifold and suppose that $L \subset \overline{X}$ is a disjoint collection of smoothly embedded 2-tori, with each component having self-intersection 0. Let $X = \overline{X} - L$ and let g be an asymptotically hyperbolic metric on X. Then the reduced L^2 cohomology is*

$$H^k_{(2)}(X, g) \cong \begin{cases} H^k(\overline{X}, L) & k > 2 \\ \{\omega \in H^2(X) | \langle \omega, [T_i] \rangle = 0\} & k = 2 \\ H^k(X) & k < 2 \end{cases} \tag{3.20}$$

with isomorphism determined by pullback of the inclusion $X \hookrightarrow \overline{X}$. In other words, this theorem gives necessary and sufficient conditions for a k-form in $H^*(\overline{X})$ to have a suitable representative in $H^2_{(2)}(X, g)$.

3.2 L^2 Chern-Weil theory

In this section, we extend Chern-Weil theory into the setting L^2 forms to define the L^2-Chern class of complex vector bundles in the case of asymptotically hyperbolic manifolds. Suppose that (X, g) is a complete, finite volume 4-manifold. Suppose that $L \subset \overline{X}$ is a disjoint collection of smoothly embedded 2-tori, with each component having self-intersection 0 and let $X = \overline{X} - L$.

Suppose that $\mathcal{L}$ is a complex line bundle over X, and A is a connection on $\mathcal{L}$ so that $F_A \in L^2\Omega^2(X, g)$. Then, we can define the L^2 Chern class of L to be given by

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F_A]_{L^2} \tag{3.21}$$

On complete manifolds, the L^2 Chern class is invariant under L^2_1 perturbations of the underlying connection. More precisely, if $\alpha \in L^2_1\Omega^2(X, g)$, then it follows that

$$[F_A]_{L^2} = [F_{A+a}]_{L^2}. \tag{3.22}$$

In particular, we have the equality

$$\begin{aligned} \int_X F_{A+a} \wedge F_{A+a} &= \int_X F_A \wedge F_A + da \wedge (2F_A + da) \\ &= \int_X F_A \wedge F_A \end{aligned} \tag{3.23}$$

where the second equality follows by considering a sequence $a_n \in C^\infty_c\Omega^1(X)$ with $a_n \xrightarrow{L^2_1} a$. By applying Stokes's theorem to a finite truncation of X, we see that

$$\int_X da \wedge (2F_A + da) = \lim_{n\to\infty} \int_X da_n \wedge (2F_A + da) = 0 \tag{3.24}$$

proving the claim. Indeed, by the Hodge-Kodaira decomposition, we have the following lemma.

Lemma 3.2.1. *Suppose that $\omega \in L^2\Omega^2(X, g)$ is a closed form, and let $\gamma \in \mathcal{H}^2(X, g)$ be the har-*

monic representative for the class $[\omega]$. Then

$$\int_X |\omega|^2 \geq \int_X |\gamma|^2. \tag{3.25}$$

Let ω^+ and γ^+ denote the self-dual components of ω and γ, respectively. Then we also have that

$$\int_X |\omega^+|^2 \geq \int_X |\gamma^+|^2. \tag{3.26}$$

Note that this lemma holds for all complete manifolds, since $\omega \in L^2\Omega^2(X, g)$ being closed implies that there exists an L^2 harmonic representative.

In the case where $\mathcal{L}$ extends to a line bundle over $\overline{X}$ with $c_1(\mathcal{L})[T_i] = 0$, we can do this more explicitly. Our assumption allows us to choose a base connection A_0 so that $F_{A_0}|_{\nu\Sigma} = 0$ on some tubular neighborhood of Σ. It then follows that for all spin^c connections A on $\mathfrak{s}$ that differ from A_0 by an L^2_1 form,

$$\int_X F_A \wedge F_A = \int_X F_{A_0} \wedge F_{A_0} = 4\pi^2 c_1^2(\mathcal{L}) \tag{3.27}$$

so the self pairing of the L^2 Chern class of $\mathfrak{s}$ on X agrees with the self pairing of the standard Chern class of $\mathfrak{s}$ on $\overline{X}$.

We now have the ingredients necessary to extend the story of the scalar curvature lower bound from Seiberg–Witten monopoles to the asymptotically hyperbolic setting.

Lemma 3.2.2. *Let $\mathfrak{s}$ be a* spin^c *structure on X and suppose that (A, ϕ) is a solution to the Seiberg–Witten equations so that the curvature $F_A \in H^2_{(2)}(X, g)$. Then,*

$$\int_X s_g^2 \geq 8 \int_X |F_A^+| \geq 32\pi^2 (c_1^+(\mathfrak{s}))^2$$

where $c_1^+(\mathfrak{s})$ is the g-self-dual component of $c_1(\mathfrak{s})$. Equality holds if and only if g has constant scalar curvature and $\mathfrak{s}$ is the spin^c *structure induced by a Kähler metric.*

Proof. The proof of the lemma is essentially running the proof of LeBrun's original estimate through sequences of compact approximations. By the Lichnerowicz–Weiztenböck formula,

$$0 = \int_X \langle \nabla^{A,*}\nabla^A \phi, \phi\rangle + \frac{s}{4}\|\phi\|^2 + \frac{\|\phi\|^4}{4}.$$

Since g is complete, we can find a sequence $\phi_n \in \Omega^2_c(X, g)$ such that $\phi_n \xrightarrow{L^2_1} \phi$. Since ϕ_n has compact support, we can integrate by parts

$$\int_X \langle \nabla^{A,*}\nabla^A \phi_n, \phi_n\rangle = \int_X \langle \nabla^A \phi_n, \nabla^A \phi_n\rangle \geq 0$$

and therefore it follows that

$$\int_X \langle \nabla^{A,*}\nabla^A \phi, \phi\rangle$$

so that

$$\int_X \frac{s}{4}\|\phi\|^2 + \frac{\|\phi\|^4}{4} \leq 0$$

and the rest of the proof follows identically to the original case. □

3.3 The Dai–Wei inequality

A natural question to ask at this point is whether the Hitchin–Thorpe inequality extends to the noncompact setting. Dai-Wei [12] have proved a generalization of the Hitchin-Thorpe inequality for complete finite volume manifolds with with a specified asymptotic geometry at infinity. They consider noncompact manifolds X with fibered cusps; i.e. ends taking the form $Y \times [0, \infty)$ where Y is a fiber bundle $F \to Y \to B$ and the metric is

$$g \approx dt^2 + \pi^* g_B + e^{2t} g_F$$

The version of their theorem adapted to complete, asymptotically hyperbolic Einstein metrics with T^3 ends is the following.

Theorem 3.3.1 (Dai-Wei)**.** *Suppose that (X, g) is a complete noncompact Einstein 4-manifold* *with cylindrical ends all of the form $T^3 \times [0, \infty)$. Furthermore, suppose that g is an asymptotically hyperbolic metric. Then X satisfies*

$$2\chi(X) \geq 3\,|\sigma(X)|\,. \tag{3.28}$$

The strategy of their proof is as follows. When the metric is a cylindrical product metric near the boundary, the APS index theorem allows us to establish he Chern-Gauss-Bonnet theorem and Hirzebruch signature formula in the noncompact setting. However, the fibered cusp metrics cylindrical near the boundary of X, and there will be Chern–Simons terms coming out of the various curvature functionals. In the general case, this turns out to be the adiabatic limit of the eta invariant of Y, but in the setting of symptotically hyperbolic metrics, this will be 0.

The rest of the proof follows the Hitchin–Thorpe inequality. In the process, Dai–Wei derive a scalar curvature lower bound that implies that on asymptotically hyperbolic Einstein metrics takes the following form.

$$2\chi(X) - 3|\sigma(X)| > \frac{1}{4\pi^2}\int_X \frac{s_g^2}{24} dvol_g. \tag{3.29}$$

Similar to the original Hitchin–Thorpe inequality, the equality case occurs if and only if g is Ricci flat, but since asymptotically hyperbolic metrics have negative curvature in the limit, the equality case cannot occur.

3.4 Examples of Asymptotically Hyperbolic Einstein Metrics

The most obvious examples of asymptotically hyperbolic Einstein manifolds are the hyperbolic ones. There are many ways to construct them, such as gluing sides of ideal polyhedra (e.g. [13], [22]), constructing suitable lattices in $SO(4,1) \cong \mathrm{Isom}(\mathbb{H}^4)$, or constructing suitable covers of such manifolds (e.g. [40]). See also [36] for a census of some hyperbolic 4-manifolds arising from gluing the sides of a regular ideal 24-cell.

Due to the restrictive nature of hyperbolic manifolds, there are quite a few restrictions on the topology of these examples. For example, by Preissman's theorem every abelian subgroup of the fundamental group must either be trivial or isomorphic to $\mathbb{Z}$. In [12], Dai–Wei show that the Chern-Gauss-Bonnet theorem extends to the case of fibered cusp manifolds, so when X is a finite volume hyperbolic 4-manifold

$$\chi(X) = \frac{1}{12\pi^2}\int_X \frac{s^2}{24} = \frac{1}{2\pi^2}Vol(X) \tag{3.30}$$

since $s = 12$, $W = 0$. Long–Reid [29] show that the Hirzebruch signature formula extends into the cusped setting, but there is a correction coming from the η-invariant of the cusp cross-section Y, so that for a finite volume hyperbolic 4-manifold

$$\sigma(X) = -\eta(Y) \tag{3.31}$$

This follows from the observation by Dai–Wei that the corresponding extension for the Hirzebruch signature formula in the fibered cusp case will have a correction term coming from the adiabatic limit of the η-invariant of the boundary. In the asymptotically hyperbolic case this is literally just $-\eta(Y)$. Note that $\eta(T^3) = 0$, so the signature of all hyperbolic manifolds with toral cusps will be 0.

Another construction of asymptotically hyperbolic Einstein Metrics comes from a generalized Dehn filling construction due to Anderson [1]. See also [3]. The strategy proceeds as follows. Let X be a finite volume hyperbolic manifold with q cusps $C_1, \cdots, C_q$ and truncate the cusps so that they have boundary T^3_k. For $1 \leq k \leq q$, choose a simple closed geodesic $\sigma_k \subset T^3_k$, and glue in $D^2 \times T^2$ to the truncated cusp by a diffeomorphism of $\partial(D^2 \times T^2)$ that sends ∂D^2 to σ_k. Write

$$\sigma = (\sigma_1, \cdots, \sigma_q) \tag{3.32}$$

and denote the glued (closed) manifold by X_σ. There is an almost Einstein metric on X_σ by putting the AdS toral black hole metric

$$g_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^2} \tag{3.33}$$

on each $D^2 \times T^2$, where $V = r^2 - 2mr$.

What Anderson shows is that if the geodesics σ_k are long enough, then this approximately Eisntein metric can be perturbed to an actual Einstein metric via an inverse function theorem on an appropriate gauge fixing of the Einstein operator. Furthermore, for any cusp C_k, if we take a sequence of geodesics $\sigma_{k,i}$ with $\ell(\sigma_{k,i}) \to \infty$ as $i \to \infty$, then there is a subsequence such that the metric on the cusp will be converging smoothly and uniformly on compact sets to an Einstein metric with bounded curvature. Since the initial model metrics converge smoothly and uniformly on compact sets to the hyperbolic cusp metric, it follows that the limiting Einstein metrics must be asymptotically hyperbolic. Doing this procedure over any subcollection of cusps gives us examples of asymptotically hyperbolic Eisntein manifolds.

Note that the inclusion of the core T^2 in each cusp necessarily induces an injection on π_1 by Van-Kampen's theorem. Since this will be an abelian subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, it follows as a consequence of Preissman's theorem that X_σ will never admit a metric of negative sectional curvature and in particular cannot be a hyperbolic manifold. Indeed, the manifolds constructed will not be locally isometric.

Notice also that since this story started with hyperbolic manifolds with toral cusps, the signature of the final cusped manifolds must necessarily be 0. At the present moment it is unknown to the author whether there exist examples of asymptotically hyperbolic Einstein manifolds with nonzero signature. Therefore, it will be of particular interest to find obstructions to asymptotically hyperbolic metrics in the signature 0 case.

Chapter 4: Monopoles on Asymptotically Hyperbolic Manifolds

We now approach the problem of constructing monopoles on asymptotically hyperbolic 4-manifolds. Let $X = \overline{X} - L$, and suppose that L is a finite disjoint union of smoothly embedded 2-tori, each with 0 self-intersection, so that X has T^3 ends, and suppose that g is an asymptotically hyperbolic metric on X. For a given end $T^3 \times [0, \infty)$, suppose that with respect to a given flat metric g_0 on T^3, g is asymptotically C^2 close to the model hyperbolic metric $dt^2 + e^{-2t} g_0$. Decompose $T^3 = S^1 \times T^2$, where the T^2 component comes from embedded 2-torus and the S^1 comes from the unit normal bundle, and write $g_0 = g_{T^2} + g_{S^1}$.

Biquard's approach [10] to solving the Seiberg–Witten equations on noncompact manifolds is elegant and geometric. The idea is that one takes a sequence of approximating metrics g_j on $\overline{X}$ so that $g_j \to g$ over any compact subset of X. If $\overline{X}$ has nonvanishing Seiberg–Witten invariant, then it follows that there exists an irreducible solution (A_j, ϕ_j) to the unperturbed Seiberg–Witten equations on $(\overline{X}, g_j)$. By carefully constructing the metrics g_j, one can obtain suitable a priori estimates and show that a suitable subseqeunce limits to solution (A, ϕ) to the unperturbed Seiberg–Witten equations on (X, g).

This goal of this section is to derive existence results needed to prove 1.1.2.

Proof. Let g be any asymptotically hyperbolic metric on X. Let $\mathfrak{s}$ be a spin^c structure with $c_1(\mathfrak{s}) = \alpha$. From proposition 1.1.1, it follows that there exists a solution (A, ϕ) to the Seiberg-Witten equations on X. Thus, it follows from theorem **??** that

$$\frac{1}{32\pi^2} \int_X s_g^2 dvol_g \geq \alpha^2. \tag{4.1}$$

If g were to be Einstein, then by 3.29,

$$2\chi(X) - 3|\sigma| \geq \frac{1}{96\pi^2} \int_X s_g^2 dvol_g \tag{4.2}$$

with equality if and only if g is Calabi-Yau with respect to a complex structure J. Since asymptotically hyperbolic metrics cannot be Ricci flat, it follows that when $2\chi(X) - 3|\sigma| \leq \frac{1}{3}\alpha^2$, g cannot possibly be Einstein. □

4.1 Approximating Metrics

In this section we construct the approximating metrics g_j on $\overline{X}$. We define warping functions ϕ_j and ψ_j as follows. Let $\phi_j(t) = e^{-t}$ for $t \in [0, j]$. For ε_j small we let ϕ_j' very rapidly decrease to -1 on the interval $[j, j + \varepsilon_j]$ by making ϕ_j'' very negative, and for an appropriate choice of T_J, let $\phi_j(t) = T_j - t$ for $t \in [j + \varepsilon_j, T_j]$. Similarly, define ψ_j with $\psi_j(t) = e^{-t}$ for $t \in [0, j]$ and $\psi_j(t) = C_j$ for an appropriately chosen constant C_j on the interval $t \in [j + \varepsilon_j, T_j)$. In the intermediate interval $[j, j + \varepsilon_j]$, we decrease ψ_j'' to 0 such that $|\psi''| \leq |\psi'| \leq |\psi|$. This is always possible for ε_j small enough and C_j close enough to e^{-j}. We furthermore also impose the conditions that

$$\begin{aligned} \frac{\phi_j'}{\phi_j} + 2\frac{\psi_j'}{\psi_j} &\leq -3 \\ \frac{-2}{1 - \frac{\phi_j'}{\phi_j}} &\leq \frac{\psi_j'}{\psi_j} \end{aligned} \tag{4.3}$$

This can be done by allowing ϕ_j to decrease at a faster rate than ψ_j.

These inequalities will be important for curvature estimates. On the interval $[0, T_j) \times T^3$ define the metric

$$\tilde{g}_j = dt^2 + \psi_j^2 g_{T^2} + \phi_j^2 g_{S^1}. \tag{4.4}$$

Since $\phi_j(T_j) = 0$ follows that the S_1 factor is completely pinched off, so $\tilde{g}_j$ extends to the compactification $D^2 \times T^2$. The metric $\tilde{g}_j$ is a standard hyperbolic cusp on the interval $[0, j]$ and a standard flat metric on the interval $[j+\varepsilon_j, T_j]$. The condition on ψ_j'/ψ_j is to ensure that the mean curvature of the slices in $\tilde{g}_j$ is at least $3/2$. Indeed, we note that the following inequalities are true for all t and j.

$$\begin{aligned} \frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t^2\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2} &\geq -2 \\ \frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} &\geq -2 \\ \frac{-\partial_t^2\psi_j}{\psi_j} - \frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} &\geq -2 \end{aligned} \tag{4.5}$$

These will be used in section 4.3 to study the convergence of solutions of Seiberg-Witten equations. To see that

$$\frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t^2\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2} \geq -2 \tag{4.6}$$

note that on the interval $[0, j]$, $\phi_j(t) = \psi_j(t) = e^{-t}$, so that

$$\frac{-\partial_t^2\phi_j}{\phi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} = \partial_t(\frac{-\partial_t\phi_j}{\phi_j}) = 0 \tag{4.7}$$

since $\frac{-\partial_t\phi_j}{\phi_j} = 1$ is the log derivative of ϕ_j. Similarly,

$$-2\frac{\partial_t^2\psi_j}{\psi_j} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2} = 2\partial_t(\frac{-\partial_t\psi_j}{\psi_j}) = 0 \tag{4.8}$$

On the interval $[j, j+\varepsilon)$ since ϕ_j'' is very negative and $-1 \leq \phi_j' \leq -e^{-j}$, it follows that

$$\frac{-\partial_t^2\phi_j}{\phi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} \geq 0 \tag{4.9}$$

Note that $2\frac{(\partial_t\psi_j)^2}{\psi_j^2} \geq 0$ and since $|\psi''| < |\psi|$ it follows that $-2\frac{\partial_t^2\psi_j}{\psi_j} \geq -2$.

On the interval $[j+\varepsilon_j, T_j)$, since $\phi_j(t) = T_j - t$ it follows that $\frac{(\partial_t\phi_j)^2}{\phi_j^2} = \frac{1}{(T_j-t)^2} \geq e^{2j}$. Since ψ_j

is constant on this interval, again

$$-2\frac{\partial_t^2\psi_j}{\psi_j} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2} = 2\partial_t(\frac{-\partial_t\psi_j}{\psi_j}) = 0. \tag{4.10}$$

For the inequality

$$\frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} \geq -2 \tag{4.11}$$

note that as before, on the interval $[0, j)$, we can directly evaluate

$$\frac{-\partial_t^2\phi_j}{\phi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} \geq 0 \tag{4.12}$$

and

$$-2\frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} = -2 \tag{4.13}$$

to verify the inequality. On the interval $[j, j+\varepsilon_j)$, as before we have that

$$\frac{-\partial_t^2\phi_j}{\phi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} \geq 0. \tag{4.14}$$

The assumption

$$\frac{-2}{1-\frac{\phi_j'}{\phi_j}} \leq \frac{\psi_j'}{\psi_j} \tag{4.15}$$

allows us to substitute $\frac{\psi_j'}{\psi_j}$ for $\frac{-2}{1-\frac{\phi_j'}{\phi_j}}$ so that

$$-2\frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} \geq -4\frac{1}{1-\frac{\partial_t\phi_j}{\phi_j}} \geq -2 \tag{4.16}$$

Where the final inequality follows since

$$1-\frac{\partial_t\phi_j}{\phi_j} \geq 2.$$

Finally, on the interval $[j+\varepsilon_j, T_j)$, note taht $\frac{\partial_t \psi_j}{\psi_j} = 0$ and as in the previous case, $\frac{(\partial_t \phi_j)^2}{\phi_j^2} = \frac{1}{(T_j - t)^2} \geq e^{2j}$.

For the last inequality

$$\frac{-\partial_t^2 \psi_j}{\psi_j} - \frac{\partial_t \phi_j}{\phi_j} \cdot \frac{\partial_t \psi_j}{\psi_j} \geq -2 \tag{4.17}$$

we note that $\frac{\partial_t \phi_j}{\phi_j} \cdot \frac{\partial_t \psi_j}{\psi_j} \geq 0$ and $\frac{-\partial_t^2 \psi_j}{\psi_j} \geq -1$.

Fix a smooth cutoff function χ with $\chi(t) = 0$ for $t \geq 1$ and $\chi(t) = 1$ for $t \leq 0$. Set $\chi_j(t) = \chi(t - j)$. It follows that the metric

$$g_j = \chi_j g + (1 - \chi_j) \tilde{g}_j \tag{4.18}$$

agrees with the original metric g on the compact pieces $[0, j] \times T^3$ on each end, and extends to a metric on the compact manifold $\overline{X}$.

We now compute the various curvature tensors of the metric $\tilde{g}_j$, on the ends $T^3 \times [0, \infty)$. The Ricci curvature of $\tilde{g}_j$ can be explicitly computed to be

$$\begin{aligned} Ric_{\tilde{g}_j} = & \left(\frac{-\partial_t^2 \phi_j}{\phi_j} - 2\frac{\partial_t^2 \psi_j}{\psi_j}\right) dt^2 + \left(\frac{-\partial_t^2 \phi_j}{\phi_j} - 2\frac{\partial_t \phi_j}{\phi_j} \cdot \frac{\partial_t \psi_j}{\psi_j}\right)(\phi_j d\theta)^2 \\ & + \left(\frac{-\partial_t^2 \psi_j}{\psi_j} - \frac{(\partial_t \psi_j)^2}{\psi_j^2} - \frac{\partial_t \phi_j}{\phi_j} \cdot \frac{\partial_t \psi_j}{\psi_j}\right) \psi_j^2 g_{T^2} \end{aligned} \tag{4.19}$$

By taking the trace, we may also compute the scalar curvature to be

$$s_{\tilde{g}_j} = 2\frac{-\partial_t^2 \phi_j}{\phi_j} - 4\frac{\partial_t^2 \psi_j}{\psi_j} - 4\frac{\partial_t \phi_j}{\phi_j} \cdot \frac{\partial_t \psi_j}{\psi_j} - 2\frac{(\partial_t \psi_j)^2}{\psi_j^2} \tag{4.20}$$

Note here that by construction, $s_{\tilde{g}_j}$ is uniformly bounded from below. The second fundamental form of metrics taking the form $dt^2 + g_t$ is given by $-\partial_t g_t / 2$. For us, we can explicitly see it to be

$$\mathrm{II}_{\tilde{g}_j} = \frac{(\partial_t \phi_j)^2}{\phi_j^2} \phi_j^2 d\theta^2 + \frac{(\partial_t \psi_j)^2}{\psi_j^2} \psi_j^2 g_{T^2} \tag{4.21}$$

And the mean curvature can be computed as

$$h_{\tilde{g}_j} = \frac{-1}{2}\left(\frac{(\partial_t \phi_j)^2}{\phi_j^2} + 2\frac{(\partial_t \psi_j)^2}{\psi_j^2}\right) \geq \frac{3}{2} \tag{4.22}$$

, where the lower bound follows from 4.3. By the hypothesis of C^2 closeness, for all j, the scalar curvatures of $(\overline{X}, g_j)$ will be bounded from below by $s_{g_j} \geq C$ where C is a uniform constant. Similarly, it follows that there is a uniform constant so that $Vol(\overline{X}, g_j) \leq C \cdot Vol(X, g)$.

4.2 Apriori Estimates on 1-forms

In this section we derive various L^2 estimates on 1-forms satisfying gauge fixing conditions that will be important later in performing the Seiberg-Witten bootstrap. We follow the approach of Biquard and di Cerbo. The first step is a uniform Poincaré inequality for functions over the family of metrics g_j. We begin with the lemma

Lemma 4.2.1. *Suppose that $g = dt^2 + g_t$ on the half cylinder $[0, \infty) \times Y$, and suppose that for some constant $h_0 > 0$, the mean curvature h satisfies*

$$h = \frac{-\partial_t dvol_{g_t}}{2dvol_{g_t}} \geq h_0 \tag{4.23}$$

Then, for any function f, we have

$$\frac{1}{h_0}\int_{[0,T]\times Y} |\partial_t f|^2 \geq h_0 \int_{[0,T]\times Y} |f|^2 + \int_{t=T} |f|^2 - \int_{t=0} |f|^2 \tag{4.24}$$

for all $T \in [0, \infty)$.

Proof. We integrate by parts to obtain

$$\int_{t=T} |f|^2 dvol_{g_0} - \int_{t=0} |f|^2 dvol_{g_T} = \int_0^T 2f\partial_t f + f^2 \frac{\partial_t dvol_{g_t}}{dvol_{g_t}} dvol_{g_t} \tag{4.25}$$

By Cauchy-Schwarz, we have the pointwise inequality $2|f\partial_t f| \leq h_0 f^2 + h_0^{-1}(\partial_t f|^2$, and the result is immediate. ◻

For the model metrics $\tilde{g_j}$, we can more explicitly set $h_0 = \frac{3}{2}$ to get a uniform constant. And the family of metrics g_j are quasi-isometric to the model metrics $\tilde{g_j}$, so we can get a similar estimate by weakening the constant factor.

Corollary 4.2.2. *For the family of metrics $(\overline{X}, g_j)$, there exists a constant c, uniform in j so that for all functions f with $\int_{\overline{X}} f dvol_{g_j} = 0$*

$$\int_{\overline{X}} |df|^2_{g_j} dvol_{g_j} \geq c \int_{\overline{X}} f^2 dvol_{g_j} \tag{4.26}$$

Proof. We proceed by contradiction. Suppose that there exists a sequence of functions f_j with

$$\int_{\overline{X}} f_j dvol_{g_j} = 0, \|f_j\|_{g_j} = 0, \tag{4.27}$$

and so that $\|df_j\|_{g_j} \to 0$. Then it follows that f_j is bounded in L^2_1 and thus has a weak limit f. Note that $\|df\|_g = 0$, so f must be the constant function.

Let $K \subset X$ be a compact subset. Then we note that

$$\int_K f dvol_g = \lim_{j\to\infty} \int_K f_j dvol_{g_j} = \lim_{j\to\infty} \int_{\overline{X}-K} f_j dvol_{g_j} \tag{4.28}$$

But

$$\left| \int_{\overline{X}-K} f_j \right| \leq \|f_j\|_{L^2(g_j)} \sqrt{Vol(\overline{X}-K)} \tag{4.29}$$

It follows that

$$\left| \int_K f dvol_g \right| \leq \sqrt{Vol(\overline{X}-K)} \tag{4.30}$$

Since for all ε, we can find a K so that $Vol(\overline{X}-K) \leq \varepsilon$, it follows that $f = 0$. Now choose $K \subset X$ to be compact with $\{0\} \times Y \subset K$, and let χ be a bump function with compact support in X, with $\chi|_K = 1$. Since χf_j have compact support and are bounded in L^2_1, it follows by Rellich embedding that $\|\chi f_j\|^2_{L^2(g)} \to 0$. On the other hand, the lemma tells us that there is a constant C so that

$$\begin{aligned} C\int_X |(1-\chi)f_j|^2 dvol_{g_j} &\leq \int_X |d(1-\chi)f_j|^2 dvol_{g_j} \\ &\leq \int_X |df_j|^2 + |(d\chi_j)f_j|^2 dvol_{g_j} \end{aligned} \tag{4.31}$$

By assumption $\|df_j\|^2_{L^2(g_j)} \to 0$ and $\|(d\chi_j)f_j\|^2_{L^2(g_j)} \to 0$ again by the Rellich lemma. In light of the decomposition $f_j = \chi f_j + (1-\chi)f_j$, this contradicts $\|f_j\|_{L^2(g_j)} = 1$. □

Our next goal is to prove a similar uniform Poincaré inequality for 1-forms. We first need to understand convergence of harmonic 1-forms.

Proposition 4.2.3. *Let $[a] \in H^1_{dR}(\overline{X})$ be a cohomology class and let $\alpha_j \in \mathcal{H}^1(X, g_j)$ be the corresponding harmonic representatives. Then α_j converges in $C^\infty_C(X)$ to a harmonic form $\alpha \in L^2\Omega^1(X)$.*

This proposition ensures that for any compact subset $K \subset X$, $L \subset H^1(\overline{X})$, there exists a uniform L^2_k bound on g_j-harmonic representatives of cohomology classes lying in L, depending only on k, K, and L.

Proof. Fix a representative β for $[a]$ on $\overline{X}$, and write

$$\alpha_j = \beta + df_j \tag{4.32}$$

for f_j satisfying

$$\begin{cases} \int_{\overline{X}} f_j = 0 \\ \Delta^{g_j} f_j = -d^{*_j}\beta \end{cases} \tag{4.33}$$

Via integration by parts, we have

$$\|df_j\|^2_{L^2(g_j)} = -\int_{\overline{X}} f_j d^{*_j}\beta dvol_{g_j} \leq \|f_j\|_{L^2(g_j}\|d^{*_j}\beta\|_{L^2(g_j)} \tag{4.34}$$

By construction of the metrics g_j, the norms of $\|\beta\|_{L^2(g_j)}$ and $\|d^{*_j}\beta\|_{L^2(g_j)}$ are both uniformly bounded in j. By lemma 3.2, we have that

$$c\|f_j\|^2_{L^2(g_j} \leq \|df_j\|^2_{L^2(g_j)} \leq \|f_j\|_{L^2(g_j}\|d^{*_j}\beta\|_{L^2(g_j)} \tag{4.35}$$

Thus, it follows that we have bounds $\|f_j\|_{L^2(g_j} \leq c^{-1}\|d^{*_j}\beta\|_{L^2(g_j)}$, and so by elliptic bootstrapping, we can find a C^∞_c convergent subsequence of f_j with limit f satisfying

$$\begin{cases} \int_{\overline{X}} f = 0 \\ \Delta^g f = -d^*\beta \end{cases} \tag{4.36}$$

And it follows that $\alpha = \beta + df$ is our desired harmonic representative. □

We also need to show a uniform Poincaré inequality on the cusp ends of the approximating metrics.

Lemma 4.2.4. *For the model metrics $\tilde{g_j}$ on $T^3 \times [0, T_j)$, we have the estimate for all 1-forms α with support in $[t_1, t_2] \subset [0, T_j)$,*

$$\int_{[t_1,t_2]\times Y} |d\alpha|^2 + |d^{*_j}\alpha|^2 dvol_{\tilde{g}} \geq c\int_{[t_1,t_2]\times Y} |\alpha|^2 dvol_{\tilde{g}} \tag{4.37}$$

Proof. We recall that by the Bochner formula for 1-forms we have

$$\int_{[t_1,t_2]\times Y} |d\alpha|^2 + |d^{*_j}\alpha|^2 dvol_{\tilde{g}} = \int_{[t_1,t_2]\times Y} |\nabla\alpha|^2 + Ric(\alpha,\alpha) dvol_{\tilde{g}} \tag{4.38}$$

Writing $\alpha = \beta + f dt$, with $\beta \in \Omega^1(Y)$, we can decompose the term

$$\begin{aligned} \nabla\alpha &= \nabla_{\partial_t}\alpha + d_Y f \otimes dt + f\nabla|_Y dt + \nabla|_Y dt\beta \\ &= \nabla_{\partial_t}\alpha + d_Y f \otimes dt - f\mathrm{I\!I}(\cdot,\cdot) + \mathrm{I\!I}(\beta,\cdot)\otimes dt \end{aligned} \tag{4.39}$$

Here II denotes the second fundamental form of the slices $T^3 \times \{t\}$. It follows that

$$\begin{aligned}\int |\nabla\alpha|^2 &= \int |\nabla_{\partial_t}\alpha|^2 + |d_Y f + \mathrm{II}(\beta,\cdot)|^2 + |-f\mathrm{II}(\cdot,\cdot) + \nabla^Y\beta|^2 \\ &= \int |\nabla_{\partial_t}\alpha|^2 + |d_Y f|^2 + |\mathrm{II}(\beta,\cdot)|^2 + |f\mathrm{II}(\cdot,\cdot)|^2 + |\nabla^Y\beta|^2 + 2\langle d_Y f, \mathrm{II}(\beta,\cdot)\rangle - 2\langle f\mathrm{II}(\cdot,\cdot), \nabla^Y\beta\rangle \\ &.\end{aligned} \tag{4.40}$$

From 4.22, we can write down the mean curvature computation as

$$\frac{-1}{2}\partial_t g_t = \frac{-\partial_t\phi_j}{\phi_j}(\phi_j d\theta)^2 + \frac{-\partial_t\psi_j}{\psi_j}\psi_j^2 g_{T^2}. \tag{4.41}$$

II commutes with ∇^Y because it is constant on the slices, so we can deduce that

$$\begin{aligned}\langle\nabla^Y f, \mathrm{II}(\beta,\cdot)\rangle &= \langle f, (\nabla^Y)^*\mathrm{II}(\beta,\cdot)\rangle \\ &= \langle f, \mathrm{II}(\nabla^Y\beta)\rangle\end{aligned} \tag{4.42}$$

and the 2 cross terms cancel out. By further decomposing $\alpha = f\,dt + f_1 d\theta + \gamma$, we note that

$$\begin{aligned}|\mathrm{II}(\beta,\cdot)|^2 &= \frac{(\partial_t\phi_j)^2}{\phi_j^2}|f_1 d\theta|^2 + \frac{(\partial_t\psi_j)^2}{\psi_j^2}|\gamma|^2 \\ |f\mathrm{II}|^2 &= \left(\frac{(\partial_t\phi_j)^2}{\phi_j^2} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2}\right)|f\,dt|^2\end{aligned} \tag{4.43}$$

By also plugging in the formula for the Ricci curvature 4.19 from section 2, we can finally deduce that

$$\begin{aligned}Ric(\alpha,\alpha) + |\mathrm{II}(\beta,\cdot)|^2 + |f\mathrm{II}|^2 = &\left(\frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t^2\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2} + 2\frac{(\partial_t\psi_j)^2}{\psi_j^2}\right)|f\,dt|^2 \\ &+ \left(\frac{-\partial_t^2\phi_j}{\phi_j} - 2\frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j} + \frac{(\partial_t\phi_j)^2}{\phi_j^2}\right)|f_1 d\theta|^2 \\ &+ \left(\frac{-\partial_t^2\psi_j}{\psi_j} - \frac{\partial_t\phi_j}{\phi_j}\cdot\frac{\partial_t\psi_j}{\psi_j}\right)|\gamma|^2\end{aligned} \tag{4.44}$$

But by the metric approximations 4.5, we know that each of the coefficients is bounded from below by -2. Thus, we have

$$Ric(\alpha,\alpha) + |\mathrm{II}(\beta,\cdot)|^2 + |f\mathrm{II}|^2 \geq -2|\alpha|^2. \tag{4.45}$$

But now, because the mean curvature of g_j is bounded from below by $\frac{3}{2}$, it follows that

$$\int_{[t_1,t_2]\times Y} |\nabla_{\partial_t}\alpha|^2 \geq \frac{9}{4}\int_{[t_1,t_2]\times Y} |\alpha|^2 \tag{4.46}$$

Thus,

$$\int_{[t_1,t_2]\times Y} |d\alpha|^2 + |d^{*j}\alpha|^2 dvol_{\tilde{g}} \geq \frac{1}{4}\int_{[t_1,t_2]\times Y} |\alpha|^2 \tag{4.47}$$

□

Note that the situation here holds also for metrics that are C^2-close to such metrics, such as the metrics g_j that we consider. This is because in local coordinates d^* depends only on the metric g and its first derivatives.

Corollary 4.2.5. *There exists $T > 0$ and $c > 0$ so that for all $j \geq T$ and $[t_1, t_2] \subset [T, T_j)$, we have the uniform estimate for the metrics g_j*

$$\int_{[t_1,t_2]\times Y} |d\alpha|^2 + |d^{*j}\alpha|^2 dvol_{\tilde{g}_j} \geq c\int_{[t_1,t_2]\times Y} |\alpha|^2 dvol_{\tilde{g}_j} \tag{4.48}$$

For all 1-forms α with support in $[t_1, t_2] \times Y$.

This allows us to prove the estimates:

Proposition 4.2.6. *For all 1-forms a orthogonal to the space of g_j harmonic 1-forms, there exists a c independent of j so that*

$$\int_{\overline{X}} |da|^2 + |d^{*g_j}a|^2 dvol_{g_j} \geq c\int_{\overline{X}} |a|^2 \tag{4.49}$$

Proof. Suppose to the contrary that there is a sequence α_j of 1-forms so that $|\alpha_j|_{L^2(g_j)} = 1$ and

$\int_{\overline{X}} |d\alpha_j|^2 + |d^{*_j}\alpha_j|^2 dvol_{g_j} \to 0$.Then, we may choose a diagonal subsequence, so that in the C_c^∞ topology, $\alpha_j \to \alpha$. By construction, $d\alpha = d^{*_g}\alpha = 0$, so $\alpha \in \mathcal{H}_g^1(X)$. But by our computation of the L^2 cohomology of X and proposition 3.3, it follows that $\alpha = 0$.

By using corollary 3.6, it now follows from the same proof as lemma 3.2 that we have a contradiction. □

4.3 The Seiberg-Witten Equations on Asymptotically Hyperbolic Manifolds

The goal of this section is the existence theorem for monopoles on asymptotically hyperbolic manifolds 1.1.1. This implies the following existence theorem:

Proposition 4.3.1. *Suppose that $\alpha \in H^2(\overline{X})/torsion$ is a monopole class, and suppose that $\langle \alpha, [T_i] \rangle = 0$ for all 2-tori $T_i \subset L$. Then it follows that there exists L^2 solutions (A, ϕ) to the Seiberg–Witten equations on (X, g) with*

$$[F_A] = 2\pi\alpha \in H^2_{(2)}(X, g)/torsion \tag{4.50}$$

We first derive apriori estimates that we need for the Seiberg-Witten bootstrap. The strategy is entirely analogous to the standard bootstrap in [31]. Let α be a monopole class, and let $\mathfrak{s}$ be the corresponding spin^c structure. Note that by the adjunction inequality [25], $\langle c_1(\mathfrak{s}), [T_i] \rangle = 0$ for each embedded torus T_i. Fix a base connection A_0 so that $F_{A_0} = 0$ on a tubular neighborhood of each of the 2-tori $\nu(T_i)$.

Proposition 4.3.2. *Suppose that we have a sequence of solutions (A_j, ϕ_j) of the unperturbed Seiberg Witten equations on $(\overline{X}, g_j)$*

$$\begin{cases} D_{A_j}\phi_j = 0 \\ F^+_{A_j} = 2(\phi_j\phi_j^*)_0 \end{cases} \tag{4.51}$$

Let $a_j = A_0 - A_j$*. By gauge fixing, we may assume that* $d^{*g_j} a_j = 0$ *and*

$$\int_{\overline{X}} a_j \wedge \beta_i \in [0, 2\pi] \tag{4.52}$$

where $\beta_i \in H^3_{dR}(\overline{X})$ *forms a fixed orthonormal basis. Then there exist uniform constants* K_1, K_2, K_3*, independent of* j *so that*

$$\|a_j\|_{L^2_2(g_j)} \leq K_1, \|\psi_j\|_{L^2_1(g_j)} \leq K_2, \|\psi_j\|_{L^\infty} \leq K_3 \tag{4.53}$$

Proof. These bounds follow from the usual approach to apriori estimates for the Seiberg-Witten equations. We first derive the L^∞ bound on ϕ_j. Suppose that $x \in \overline{X}$ is a maximal point for $|\phi_j|^2$. Then $Re\langle \nabla^*_{A_j} \nabla_{A_j} \phi_j, \phi_j \rangle \geq 0$ at x. By the Weiztenböck formula,

$$0 = Re\langle \nabla^*_{A_j} \nabla_{A_j} \phi_j, \phi_j \rangle + \frac{1}{4}|\phi_j|^4 + \frac{1}{4} s_{g_j} |\phi_j|^2 \tag{4.54}$$

we see that at the point x,

$$|\phi_j|^2 \leq -s_{g_j} \tag{4.55}$$

By construction, s_{g_j} is uniformly bounded. So, it follows that $|\phi_j|$ is uniformly bounded, independent of j. Since $vol(\overline{X}, g_j)$ is uniformly bounded in j as well, it also follows that $|\phi_j|_{L^2}$ is uniformly bounded. By integrating the Weitzenböck formula for the Seiberg Witten equations, we have that

$$0 = \int_{\overline{X}} |\nabla_{A_j} \phi_j|^2 + \frac{1}{4} s_{g_j} |\phi_j|^2 + \frac{1}{4}|\phi_j|^4 \tag{4.56}$$

Since the terms $s_{g_j}|\phi_j|^2$ and $|\phi_j|^4$ are both bounded, it follows that $|\nabla_{A_j}\phi_j|^2$ is also uniformly bounded independent of j. Therefore, we have a uniform bound on $|\phi_j|_{L^2_1}$.

We now proceed to show the L^2_2 bounds on a_j. Write $a_j = b_j + c_j$, where $b_j \in (\mathcal{H}^1_{g_j})^\perp$ and $c_j \in \mathcal{H}^1_{g_j}$. Proposition 3.3 gives us a uniform L^2_2 bound on c_j since c_j is contained inside a compact set in $H^1(\overline{X})$. Since $\int_{\overline{X}} |\nabla_{A_j}\phi_j|^2 \geq 0$, by the Weitzenböck formula, we have the inequality

$$\int_{\overline{X}} |F^+_{A_j}|^2 = \frac{1}{8}\int_{\overline{X}} |\phi_j|^4 \leq \frac{1}{8}\int_{\overline{X}} s^2_{g_j} \tag{4.57}$$

The first equality is from the curvature equation in the Seiberg Witten equations. This gives us a bound for $|F^+_{A_j}|_{L^2(g_j}$. Since $d^{+j}b_j = \frac{1}{2}(F^+_{A_j} - F^+_{A_0})$ and since $|F^+_{A_0}|_{L^2(g_j)}$ is uniformly bounded, it follows that we have a bound for $|d^{+j}b_j|_{L^2(g_j)}$. Note that $\|db_j\|_{L^2} = 2\|d^{+j}b_j\|_{L^2}$ and so it follows from corollary 3.7 that we have L^2_1 bounds on b_j.

We note from taking the covariant derivative on the curvature equation that

$$\nabla F^+_{A_j} = (\nabla\phi_j) \otimes \phi^*_j + \phi_j \otimes (\nabla\phi^*_j) - Re\langle\nabla\phi_j, \phi_j\rangle Id \tag{4.58}$$

Note that the L^∞ bound and the L^2_1 bound on ϕ_j combine to give us an L^2_1 bound on $F^+_{A_j}$. We use the curvature Seiberg-Witten equation $d^+b_j = F^+_{A_j} - F^+_{A_0}$ to see that this then gives us an L^2_2 bound on b_j. Combining the L^2_2 uniform bounds for b_j and c_j, we get a uniform bound for a_j, as desired. □

By using these estimates, we are ready to use the Seiberg-Witten bootstrap to construct a solution to the Seiberg-Witten equations on (X, g) and prove 1.1.1.

Proof. Suppose that (A_n, ϕ_n) are solutions to the Seiberg-Witten equations on $(\overline{X}, g_n)$ and let $a_n = A_0 - A_n$. We proceed via the Seiberg Witten bootstrap argument on the compact sets $X_n \subset X$ (where $g_n = g$), and finish by extracting a diagonal subsequence of the solutions. To run the bootstrap, we need:

1. A uniform L^∞ bound and a uniform L^2_2 bound on $|\phi_n|$
2. A uniform L^2_1 bound $\|a_n\|_{L^2_1(g_n)}$

These uniform bounds come from proposition 1.1.1. To start the bootstrap, by the first equation we note that

$$D_{A_n}\phi_n = -a_n \cdot \phi_n \tag{4.59}$$

By the uniform L^2_2 bound on a_n and the uniform L^∞ bound on ϕ_n, it follows that we have an L^4_0 bound on $D_{A_n}\phi_n$ by Sobolev multiplication. By elliptic regularity, this implies an L^4_1 bound on ϕ_n. Repeating the previous step with the Sobolev multiplication $L^2_2 \otimes L^4_1 \to L^3_1$, it follows that we get uniform L^3_2 bounds on ϕ_n. And repeating again with $L^2_2 \otimes L^3_2 \to L^2_2$, we get a uniform L^2_3 bound on ϕ_n.

The uniform part of the bootstrapping now follows from $L^2_k \otimes L^2_k \to L^2_k$ for $k \geq 3$. We use the curvature Seiberg-Witten equation

$$d^+ a_n = \frac{1}{2}(\phi_n \phi_n^*)_0 - F^+_{A_0} \tag{4.60}$$

and the L^2_k bounds on ϕ_n to acquire uniform L^2_{k+1} bounds on a_n (here we use the assumption of Coulomb gauge). And we use the Dirac equation to acquire L^2_{k+1} bounds for ϕ_n. The Sobolev embedding theorem then tells us that a diagonal subsequence converges in C^∞_c. □

Chapter 5: The $\mathrm{Pin}^{-}(2)$ Monopole Invariants

In the first part of this section, we reveiw the $Pin^{-}(2)$ equations and discuss how to extend the estimates from the Seiberg–Witten equations in the previous chapter into this setting. This will allow us to deduce 1.0.2.

5.1 The $\mathrm{Pin}^{-}(2)$ Monopole Equations

Let $\ell = \tilde{X} \times_{\pm 1} \mathbb{Z}$ be the local system corresponding to the double cover. Note that local systems on X with coefficient group $\mathbb{Z}$ are in bijective correspondence with double covers of X, so we sill sometimes use the two notions interchangeably. The main goal of this section is to prove an analogous version of the nonexistence theorem from the previous section.

Theorem 5.1.1. *Let X be a closed oriented manifold and let $\tilde{X} \to X$ be a double cover with $b^{+}(X,\ell) \geq 2$. Suppose that $\omega \in H^2(\overline{X},\ell)$ is a $Pin^{-}(2)$ monopole basic class, and let $L \subset \overline{X}$ be a disjoint collection of smoothly embedded 2-tori, with each component having self-intersection 0. Then if*

$$2\chi(\overline{X}) - 3|\sigma(\overline{X})| \leq \frac{1}{3}\omega^2 \tag{5.1}$$

the noncompact manifold $X = \overline{X} - L$ does not admit any asymptotically hyperbolic Einstein metrics.

We briefly recall the setup of $Pin^{-}(2)$ monopoles. For a more thorough exposition, see section 3 of [32]. Define the Lie group $Pin^{-}(2) = U(1) \sqcup jU(1) \subset Sp(1)$, with a double covering map $\phi : Pin^{-}(2) \to O(2)$ that maps

$$\phi(z) = z^2, \phi(j) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

The Lie group $Spin^{c-}(4)$ is defined by $Spin^{c-}(4) = Spin(4) \times_{\pm 1} Pin^{-}(2)$, so that there is an exact sequence of groups

$$1 \to \{\pm 1\} \to Spin^{c-}(4) \to SO(4) \times O(2) \to 1 \tag{5.3}$$

The double covering $Pin^{-}(2) \to U(1)$ also induces a double covering $Spin^{c-}(4) \to \mathrm{spin}^c(4)$. The spinor representations $\Delta^{\pm} : Spin(4) \to M_4(\mathbb{C})$ extend to representations of $Spin^{c-}(4)$ where $U(1)$ acts by complex multiplication and j acts by conjugation.

Let X be a closed oriented 4-manifold and let $\pi : \tilde{X} \to X$ be an unbranched double cover, and let g be any Riemannian metric on X. A $Spin^{c-}$ structure $\mathfrak{s} = (P, s, t)$ on X, where P is a principal $Spin^{c-}$ bundle over X, $s : \tilde{X} \to P/\mathrm{spin}^c(4)$ is an isomorphism of double covers of X and $t : Fr(X) \to P/Pin^{-}(2)$ is an isomorphism of principal $SO(4)$ bundles. The $O(2)$ bundle $E = P/Spin(4)$ is the characteristic bundle of the $spin^{c-}$ structure. By fixing an ℓ-orientation on E, we can define a corresponding ℓ-coefficient Euler class of E

$$c_1(\mathfrak{s}) = c_1(E) \in H^2(X, \ell). \tag{5.4}$$

Define the spinor bundles

$$S^{\pm} = P \times_{Spin^{c-}} \Delta^{\pm} \tag{5.5}$$

to be the rank 2 complex vector bundles associated to the spinor representations. We can now write down the $Pin^{-}(2)$ monopole equations. Suppose that a $spin^{c-}$ structure $\mathfrak{s}$ is given. Let E be the corresponding principal $O(2)$ bundle, and let $\lambda = \tilde{X} \times_{\pm 1} \mathbb{R}$, and $\ell = \tilde{X} \times_{\pm 1} \mathbb{Z}$ be the corresponding $\mathbb{R}$ and $\mathbb{Z}$ coefficient systems associated to the double covering. An $O(2)$ connection A on E induces a Dirac operator $D_A : \Gamma(S^+) \to \Gamma(S^+)$ and there is a standard quadratic map $q : S^+ \to \Omega^+(X, i\lambda)$. Note that $F_A \in \Omega^2(X, i\lambda)$. The $Pin^{-}(2)$-monopole equations are

$$\begin{cases} D_A \phi &= 0 \\ F_A^+ &= \frac{1}{2}\rho(\phi) + i\mu \end{cases} \tag{5.6}$$

here $\mu \in \Omega^+(X, i\lambda)$ is a perturbation; when $\mu = 0$, the equations are called the unperturbed $Pin^-(2)$-monopole equations. Let $\mathcal{A}$ be the space of $O(2)$ connections on E and define the configuration space $\mathcal{C} = \mathcal{A} \times \Gamma(S^+)$. We also define the space of irreducible configurations $\mathcal{C}^* = \mathcal{A} \times (\Gamma(S^+) \setminus \{0\})$. The gauge group $\mathcal{G} = \Gamma(\tilde{X} \times_{\pm 1} U(1))$ acts on $\mathcal{C}$ and preserves solutions to the $Pin^-(2)$-monopole equations. Let$\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ be the moduli space of configurations and let $\mathcal{M}_{Pin^-(2)}(X, \mathfrak{s}) \subset \mathcal{B}^*$ denote the moduli space of irreducible solutions.

For a given $k \geq 3$, we work with the L^2_k completion of $\mathcal{C}$, and the L^2_{k+1} completion of the gauge group. Fix a reference $O(2)$ connection A_0. As with Seiberg-Witten theory when $b_+(X, \ell) \geq 2$, for a generic choice of perturbation the $Pin^-(2)$ moduli space will be a smooth compact d-dimensional submanifold of $\mathcal{B}^*$, wehre the dimension d is given by

$$d = \frac{1}{4}(c_1^2(E) - \sigma(X)) - (b_0(X, \ell) - b_1(X, \ell) + b_+(X, \ell)). \tag{5.7}$$

Note that in contrast to the ordinary Seiberg-Witten equations, the moduli space is not always orientable. The $Pin^-(2)$ invariant of $(X, \mathfrak{s})$ is defined is defined as the fundamental class

$$[\mathcal{M}_{Pin^-(2)}(X, \mathfrak{s})] \in H_d(\mathcal{B}^*). \tag{5.8}$$

In the case where $d = 0$, this amounts to a mod 2 count of solutions up to guage of the $Pin^-(2)$ equations under a generic pertubation. In this case, let $SW_{Pin^-(2)} \in \mathbb{Z}/2\mathbb{Z}$ denote the mod 2 count.

Nakamura proved a gluing formula for $Pin^-(2)$ monopoles [33]. It can be viewed as an analog of the blow-up formula from Seiberg-Witten theory in the $Pin^-(2)$ setting. For any 3-manifold Y, let ℓ be the $\mathbb{Z}$ local system on $S^1 \times Y$ coming from the nontrivial local system on S^1. For $g \geq 1$, construct a $\mathbb{Z}$ local system ℓ on $S^2 \times \Sigma_g$ as follows. Let ℓ_{T^2} be any nontrivial $\mathbb{Z}$ local system on the 2-torus, and recall that $\Sigma_g = \#^g T^2$. Define $\ell_{\Sigma_g} = \#^g \ell_{T^2}$, and define ℓ on $S^2 \times \Sigma_g$ by taking the pullback of ℓ_{Σ_g} through the pullback map. In [33], Nakamura proved the following gluing result:

Theorem 5.1.2 (Nakamura)**.** *Let X_1 be a closed oriented connected 4-manifold with $b_+(X_1) \geq 2$ and a* spin^c *structure $\mathfrak{s}_1$ so that the formal dimension of the Seiberg-Witten moduli space is 0 and*

$SW(X_1, \mathfrak{s}_1) = 1 \pmod 2$. *Let* $X_2 = \#_i(S^2 \times \Sigma_{g_i})\#_j(S^1 \times Y_j)$ *with the a local system* ℓ *defined by connect summing the local systems on* $S^2 \times \Sigma_{g_i}$ *and* $S^1 \times Y_j$ *defined about. Then for any spin*$^{c-}$ *structure* $\mathfrak{s}_2$ *on* X_2*, we have*

$$SW_{Pin^-(2)}(X_1\#X_2, \mathfrak{s}_1\#\mathfrak{s}_2) = 1 \pmod 2 \tag{5.9}$$

The condition for X_1 is satisfied for symplectic manifolds when $\mathfrak{s}_1$ is taken to be the spin^c structure induced by the symplectic form. The reason that it should be considered a $Pin^-(2)$ analog of the blow-up formula is because $b_+(X_2, \ell_{X_2}) = 0$. Note also that

$$c_1^2(\mathfrak{s}_1) = c_1^2(\mathfrak{s}_1\#\mathfrak{s}_2). \tag{5.10}$$

The proof of theorem 1.0.2 now follows from 5.2.1 by combining 5.1.1 with 5.1.2 for $Pin^-(2)$ monopole invariants in [33].

Let X_1 be a closed symplectic manifold with $b^+(X_1) \geq 2$, and let $\mathfrak{s}_1$ be the corresponding canonical spin^c structure. Then it follows that $SW(X_1, \mathfrak{s}_1) = 1$. Pick X_2 as in 5.2.1 so that

$$-\chi(X_2) + 2 \geq \frac{1}{3}(2\chi(X_1) + 3\sigma(X_1)). \tag{5.11}$$

Since $c_1^2(\mathfrak{s}_1) = 2\chi(X_1) + 3\sigma(X_1)$, and $\sigma(X_2) = 0$, it follows that

$$2\chi(X_1\#X_2) - 3|\sigma(X_1\#X_2)| \geq c_1^2(\mathfrak{s}_1) + 2\chi(X_2) - 4 \geq \frac{1}{3}c_1^2(\mathfrak{s}_1) \tag{5.12}$$

Since $c_1^2(\mathfrak{s}_1\#\mathfrak{s}_2) = c_1^2(\mathfrak{s}_1)$, it follows that we can apply 5.1.1 and deduce 1.0.2.

5.2 Relation with the Seiberg-Witten equations

The $Pin^-(2)$ equations on X are related to the standard $U(1)$ Seiberg-Witten equations on $\tilde{X}$. As with the previous section, for a more detailed exposition, see section 3(iii) of [32]. Given a

$spin^{c-}$ structure $\mathfrak{s}$ on X, there exists a corresponding spin^c structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$ defined as follows.

$$\begin{array}{ccc} P & \xrightarrow{\tilde{s}} & \tilde{X} \\ & \searrow & \downarrow \pi \\ & & X \end{array} \tag{5.13}$$

The isomorphism $s : P/\mathrm{spin}^c \to \tilde{X}$ gives us a map $\tilde{s} : P \to \tilde{X}$ that takes the structure of a principal spin^c bundle over $\tilde{X}$. Since the isomorphism t lifts to an isomorphism $\tilde{t} : P/U(1) \to Fr(\tilde{X})$, it follows that this defines a spin^c structure $\tilde{\mathfrak{s}}$ over $\tilde{X}$. The characteristic $O(2)$ bundle E over X also lifts into the determinant $U(1)$ bundle $\tilde{E}$ of $\tilde{\mathfrak{s}}$ over $\tilde{X}$. So it also follows that $2c_1^2(\mathfrak{s}) = c_1^2(\tilde{\mathfrak{s}})$. The spinor bundles $\tilde{S}^{\pm}$ of $\tilde{\mathfrak{s}}$ have a canonical identification $\pi^* S^{\pm}$, and a $Spin^{c-}$ connection A has a canonical lift to a spin^c connection $\tilde{A}$.

Let $\iota : \tilde{X} \to \tilde{X}$ be the deck transformation of the double covering. Then, the spin^c structure defined by $\iota \circ \tilde{s}$ will be the conjugate spin^c structure, and therefore ι is naturally covered by a principal spin^c-bundle map $\tilde{\iota} : \tilde{\mathfrak{s}} \to \overline{\tilde{\mathfrak{s}}}$. Define

$$J = [1, j^{-1}] \in Spin^{c-} = Spin(4) \times_{\pm 1} Pin^-(2). \tag{5.14}$$

Since J is on the nonidentity component of $Spin^{c-}$, the right action of J on $P \to X$ covers ι, and it is not hard to see that J is $\tilde{\iota}$ composed with the complex conjugation map $\tilde{\mathfrak{s}} \to \overline{\tilde{\mathfrak{s}}}$.

The J-action will induce actions on the spinor bundles $\tilde{S}^{\pm} = P \times_{\mathrm{spin}^c} \Delta^{\pm}$ by $I(p, \phi) = [pJ, J^{-1}\phi]$. This I-action is an antilinear involution of the spinor bundles, and

$$S^{\pm} \cong \tilde{S}^{\pm}/I. \tag{5.15}$$

Under this identifications, it follows that $\Gamma(S^{\pm}) \cong \Gamma(\tilde{S}^{\pm})^I$. Similarly, there is also an I-action on the space of $U(1)$ connections on the determinant line bundle L of $\tilde{\mathfrak{s}}$. This is because J passes to an antilinear involution of the determinant line of $\tilde{\mathfrak{s}}$. It follows that

$$\mathcal{A}(E) = \mathcal{A}(L)^I, \tag{5.16}$$

where $\mathcal{A}(E)$ are the principal $O(2)$-connections on E and $\mathcal{A}(L)$ are the principal $U(1)$-connections on L. Nakamura showed in [32] that solutions to the $Pin^{-}(2)$-monopole equations on X are precisely the I-invariant solutions of the usual Seiberg-Witten solutions on $\tilde{X}$ that are invariant under J.

Proposition 5.2.1 (Nakamura)**.** *Fix a Riemannian metric g on X and let $\tilde{g}$ be the covering metric on $\tilde{X}$. There is a bijective correspondence between the set of $Pin^{-}(2)$ monopoles on $(X, \mathfrak{s})$ and the set of I-invariant Seiberg-Witten monopoles on $(\tilde{X}, \tilde{\mathfrak{s}})$. Furthermore, there is a canonical identification of moduli spaces*

$$\mathcal{M}_{Pin^{-}(2)}(X, \mathfrak{s}) \cong \mathcal{M}_{U(1)}(\tilde{X}, \tilde{\mathfrak{s}})^I \tag{5.17}$$

In particular, this implies that if $(X, \mathfrak{s})$ has nonvanishing $Pin^{-}(2)$ monopole invariant then given any metric g on X, the pullback metric $\pi^* g$ on $\tilde{X}$ will admit solutions to the unperturbed Seiberg-Witten equations for the spin^c structure $\tilde{\mathfrak{s}}$.

We are now ready to prove theorem 5.1.1.

Proof. Let $\tilde{L} = \pi^{-1} L$, and let g be any asymptotically hyperbolic metric on $\overline{X} - L$. Let g_j be a sequence of approximating metrics on $\overline{X}$ constructed in section **??**. Then the pullback metric $\tilde{g} = \pi^* g$ will be an asymptotically hyperbolic metric on $\tilde{X} - \tilde{L}$, and the metrics $\tilde{g}_j = \pi^* g_j$ will be approximating metrics on $\tilde{X}$ for $\tilde{X} - \tilde{L}$. In particular, they will satisfy the same estimates 4.19 4.20 4.22 in section **??**. Our goal is now to show that there exists a solution (A, ϕ) to the unperturbed Seiberg-Witten equations on $\tilde{X} - \tilde{L}$ with $[F_A] = c_1(\tilde{\mathfrak{s}}) \in H^2_{(2)}(\tilde{X} - \tilde{L})$.

Since $\mathfrak{s}$ is a $Pin^{-}(2)$ basic class, it follows that there exists irreducible solutions (A_j, ϕ_j) to the unperturbed $Pin^{-}(2)$ monopole equations on (X, g_j). As with the Seiberg-Witten case, fix a base connection A_0 with F_{A_0} vanishing in a neighborhood of L; we can do this by the $Pin^-(2)$ adjunction inequality in [33]. By theorem 5.2.1, these solutions (A_j, ϕ_j) lift to solutions $(\tilde{A}_j, \tilde{\phi}_j)$ of the Seiberg-Witten equations on $(\tilde{X}, \tilde{g}_j)$ on the spin^c structure $\tilde{\mathfrak{s}}$. It follows that we can apply theorem 1.1.1 to $(\tilde{A}_j, \tilde{\phi}_j)$ to obtain a solution to the usual Seiberg-Witten equations on $(\tilde{X} - \tilde{L}, \tilde{g})$. In particular, we have the estimate:

$$\int_{\tilde{X}-\tilde{L}} s_{\tilde{g}}^2 dvol_{\tilde{g}} \geq \frac{1}{3} c_1^2(\tilde{\mathfrak{s}}). \tag{5.18}$$

Since both these quantities are multiplicative under covers, we get corresponding estimates

$$\int_X s_g^2 dvol_g \geq \frac{1}{3} c_1^2(\mathfrak{s}). \tag{5.19}$$

It follows as in the proof of theorem 1.1.2 that if

$$2\chi(\overline{X}) - 3|\sigma(\overline{X})| \leq \frac{1}{3} c_1^2(\mathfrak{s}) \tag{5.20}$$

then X will not admit any asymptotically hyperbolic Einstein metrics. □

5.3 Examples

We now briefly discuss the strategy for constructing examples in 1.0.2. The main goal of this section is to survey the literature on constructions of manifolds that can play the role of X_1 in the main theorem.

There is a thriving industry in constructing various exotic symplectic surfaces. Given $n \geq 5$, Baykur and Hamada [6] have recently constructed minimal symplectic manifolds X_n that are homeomorphic to $\#^{2n+1}(S^2 \times S^2)$. It follows that for $g \geq (4n+7)/6$, the manifold

$$\overline{X} = X_n \#(S^2 \times \Sigma_g) \tag{5.21}$$

satisfies the hypothesis of theorem 1.0.2.

We can also construct many examples of 4-manifolds that do not admit any asymptotically hyperbolic Einstein metrics by using results on the geography of symplectic 4-manifolds and general type surfaces. Let (x, c) be a pair of positive integers such that $c \equiv x \mod 8$. For all but finitely many such pairs satisfying $c < 9x$, Park [34] has constructed symplectic 4-manifolds X with $\chi_h(X) = x$ and $c_1^2(X) = c$. The fundamental group of these manifolds can be chosen to be any finitely generated group. Here χ_h is the holomorphic Euler characteristic, defined by

$$\chi_h(X) = \frac{\sigma(X) + \chi(X)}{4}.$$

Because for a symplectic 4-manifold $c_1^2(X) = 2\chi(X) + 3\sigma(X)$, it follows that for all but finitely many pairs (χ, σ) of integers satisfying $\chi > 0$, $-2\chi/3 \leq \sigma < \chi/3$, and $\chi + \sigma \equiv 0 \mod 4$ there exists a simply connected spin symplectic 4-manifold X with $(\chi(X), \sigma(X)) = (\chi, \sigma)$.

Now fix a pair of integers (χ, σ) with $\chi + \sigma \equiv 0 \mod 2$. We construct construct a closed oriented 4-manifold $\overline{X} = X_1 \# X_2$ satisfying the hypothesis of theorem 1.0.2 with $\chi(\overline{X}) = \chi$ and $\sigma(\overline{X}) = \sigma$ as follows. Choose X_1 following Park's construction with $\sigma(X_1) = \sigma$ and $\chi(X_1) \geq 3\chi + 9|\sigma|$, and let X_2 be a connect sum of components of the form $Y \times S^1$ and $\Sigma_g \times S^2$ so that $\chi(X_2) = \chi + 2 - \chi(X_1)$. Since $-\chi \leq \chi(X_1)/3$, it follows that

$$\begin{aligned} -\chi(X_2) + 2 &= \chi(X_1) - \chi \\ &\geq \frac{1}{3}(2\chi(X_1) + 3|\sigma(X_1)|) \end{aligned}$$

and therefore $\overline{X}$ satisfies the hypothesis of theorem 1.0.2. So any complement of smoothly embedded 2-tori in $\overline{X}$, each with 0 self-intersection, will not admit any cusped asymptotically hyperbolic Einstein metrics.

We can also find many examples coming from general type complex surfaces. Given X a surface of general type, define the Chern slope to be $c_1^2(X)/c_2(X)$. Persson [35] showed that every rational number $r \in [1/5, 2]$ arises as the Chern slope of some simply connected general type surface. Rolleau and Urzúa [38] showed that the Chern slopes of simply connected complex surfaces of general type are dense in the interval $[2,3]$. When G is the fundamental group of nonsingular complex projective surface, Troncoso and Urzúa [43] show that Chern slopes of complex surfaces of general type with fundamental group G are dense in $[1,3]$.

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