Open Enumerative Mirror Symmetry for Lines in the Mirror Quintic

Ryuichi Sebastian Haney

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Abstract

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Mirror symmetry is a family of conjectural equivalences between the symplectic geometry (A-model) of a symplectic manifold and the algebraic geometry (B-model) of a corresponding mirror space. The earliest mathematical achievement of mirror symmetry was a prediction for the genus zero Gromov–Witten invariants of the quintic threefold in terms of period integrals on a mirror family of Calabi–Yau varieties. Using homological mirror symmetry, we deduce an analogous mirror theorem for the open Gromov–Witten invariants of a particular Lagrangian submanifold in the quintic threefold assuming the existence of a negative cyclic open-closed map. The Lagrangian we consider can be thought of as an SYZ mirror to a line in the mirror quintic and their open Gromov-Witten (OGW) invariants coincide with the Abel–Jacobi images of these lines as calculated by Walcher. In this example, the OGW invariants are irrational numbers contained in $\mathbb{Q}(\sqrt{-3})$ and admit an expression involving special values of a Dirichlet L-function. The field in which the OGW invariants lie arises as the invariant trace field of (the smooth locus of) a closely related hyperbolic Lagrangian submanifold with conical singularities in the quintic. These results explain some of the predictions on the existence of hyperbolic Lagrangian submanifolds in the quintic put forward by Jockers, Morrison, and Walcher. We achieve these results by computing the Lagrangian Floer theory of a different Lagrangian immersion in the quintic supporting a family of objects in the Fukaya category that are homologically mirror to coherent sheaves supported on line in the mirror quintic. Some of the key technical ingredients in

this proof include a definition of the wrapped Fukaya category with Lagrangian immersions, an adjunction isomorphism on wrapped Floer cohomology associated to a non-compact Lagrangian immersion, a construction of the open Gromov–Witten invariants from a strong proper Calabi–Yau structure on the Fukaya category, and a framework for relating open Gromov–Witten invariants to Abel–Jacobi images under homological mirror symmetry.

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Chapter 1

Introduction

The first mathematical manifestation of mirror symmetry was a prediction for the genus zero Gromov–Witten invariants of the quintic threefold X in terms of period integrals on a mirror family $\mathcal{X}^\vee$ of Calabi–Yau threefolds [1]. Morrison [2] later observed that these enumerative predictions can be reformulated as an isomorphism between variations of Hodge structures (VHS) associated to X and $\mathcal{X}^\vee$. One can construct a VHS from $\mathcal{X}^\vee$ using classical Hodge theory, and there is a candidate mirror VHS obtained by equipping the quantum cohomology $QH^*(X)$ with the (small) quantum connection. The genus zero Gromov–Witten invariants of the quintic were calculated by Givental [3] and Lian–Liu–Yau [4], and shown to agree with the values predicted by [1].

Kontsevich [5] proposed homological mirror symmetry as a conceptual framework for explaining enumerative predictions from mirror symmetry. Ganatra–Perutz–Sheridan [6] showed that homological mirror symmetry for Calabi–Yau varieties implies Hodge-theoretic mirror symmetry in the sense of [2], assuming the existence of a *negative cyclic open-closed map* on the Fukaya category. In this setting, Hodge-theoretic mirror symmetry is roughly recovered by comparing the natural (weak proper) Calabi–Yau structure on the Fukaya category with the Calabi–Yau structure on the derived category of the mirror.

It has long been expected [7, 8, 9, 10] that the (genus zero) *open Gromov–Witten* invariants, which count pseudoholomorphic disks with Lagrangian boundary, can also be recovered from mirror symmetry. Using classical mirror principles, Walcher [9] gave a prediction for the open Gromov–Witten invariants of the 'real quintic' (the set of real points in a quintic threefold defined over $\mathbb{R}$), which was later verified by Pandharipande–Solomon–Walcher [11] using equivariant localization and the real structure. As in the closed case, the predictions for open Gromov–Witten invariants can be rephrased as the prediction that there is an isomorphism between two extensions of variations of mixed Hodge structure (VMHS) [12]. The extension of VMHSs in the B-model arises from a classical construction from a smooth family of homologically trivial algebraic cycles (see e.g. [13]), while the candidate mirror A-model extension of VMHSs is governed by the open Gromov–Witten invariants of a nullhomologous Lagrangian brane. This extension should essentially come from the relative quantum cohomology constructed by Solomon–Tukachinsky [14].

The purpose of this paper is to prove a mirror theorem, in a specific example, involving the open Gromov–Witten invariants of an immersed Lagrangian brane in a closed Calabi–Yau threefold, and to illustrate how such results can be recovered from homological symmetry. Our results require that the Fukaya category satisfies some widely expected structural properties related to the existence of a cyclic open-closed map. These assumptions are needed in the proof of homological mirror symmetry [15, 16, 17] and to extract closed Gromov–Witten invariants from the Fukaya category [6].

On the A-side, we construct an immersed Lagrangian that supports an infinite family of objects in the Fukaya category mirror to (non-isolated) lines in the mirror quintic. Our main result recovers the open Gromov–Witten potential of one of these objects. The open Gromov–Witten potential is thought of as a formal power series in the Novikov variable Q, and the coefficients of this power series are the open Gromov–Witten invariants. In this paper, we denote by ω a primitive third root of 1, which we take to be $\omega = e^{\frac{2\pi i}{3}}$ for definiteness.

Theorem 1.0.1. *Suppose that Assumptions 4.9.1, 4.9.2, and 4.9.3 are all satisfied by the quintic threefold. Then there is an embedded Lagrangian submanifold $\widetilde{L}^5_{\mathrm{sm}}$ and two rank 1 local systems over the Novikov ring, denoted $\nabla^{\mathrm{vG}}_{\omega^2}$ and $\nabla^{\mathrm{vG}}_{\omega}$, which give rise to unobstructed Lagrangian branes $(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\omega^i}_{\Lambda})$ for $i = 1, 2$ when equipped with suitable bounding cochains. The open Gromov–Witten potentials, in the sense of Fukaya [18], of these two branes satisfy*

$$\Psi(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\omega^2}_{\Lambda}) - \Psi(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\omega}_{\Lambda}) = \sum_{d=1}^{\infty} n_d \sum_{k=1}^{\infty} \frac{\chi(k)}{k^2} Q^{kd} \tag{1.0.1}$$

where $\chi\colon \mathbb{Z} \to \{-1,0,1\}$ *is the nontrivial Dirichlet character of order* 3 *given by*

$$\chi(k) \coloneqq \frac{1}{\sqrt{-3}}(\omega^k - \omega^{2k}) = \begin{cases} 0 & k \equiv 0 \mod 3 \\ 1 & k \equiv 1 \mod 3 \\ 2 & k \equiv -1 \mod 3 \end{cases}$$

and $n_d \in \sqrt{-3}\mathbb{Z}[\frac{1}{3}]$. *The coefficients* n_d *in* (1.0.1) *are two times the corresponding numbers in [19, §6.1]. In particular, the open Gromov–Witten invariants, which are the coefficients* $\widetilde{n}_d$ *in the expansion*

$$\Psi(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\omega^2}_{\Lambda}) - \Psi(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\omega}_{\Lambda}) = \sum_{d=1}^{\infty} \widetilde{n}_d Q^d$$

of (1.0.1)*, lie in the field* $\mathbb{Q}(\sqrt{-3})$*.*

The Lagrangian submanifold $\widetilde{L}^5_{\mathrm{sm}}$ can intuitively be thought of as the lift of a tropical curve, similar the examples in mirror quintic threefolds constructed in [20]. The irrationality of the open Gromov–Witten invariants is determined by the holonomy representations of the local systems $\nabla^{\omega^i}_{\Lambda}$, which are defined over the Novikov ring with coefficients in $\mathbb{Q}(\sqrt{-3})$. The first few values of $\widetilde{n}_d$ are

$$\begin{aligned}
\widetilde{n}_1 &= 280000\sqrt{-3} \\
\widetilde{n}_2 &= \frac{22296200000}{3}\sqrt{-3} \\
\widetilde{n}_3 &= \frac{10031895589000000}{27}\sqrt{-3} \\
\widetilde{n}_4 &= \frac{660275805871745000000}{27}\sqrt{-3}\,.
\end{aligned}$$

Notice that

$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k^2}$$

can be written as $L(2;\chi)$, where $L(s;\chi)$ is the Dirichlet L-function associated to χ. This is also a special value of the D-logarithm introduced by Walcher [19]. One will notice a similarity between this formula and the Ooguri–Vafa multiple cover formula (see [21, 8, 22]). As explained more thoroughly in Remark 1.2.4, the result of Theorem 1.0.1, and the auxiliary results appearing in its proof, can be regarded as confirming a prediction on the existence of *hyperbolic Lagrangians* in the quintic threefold due to Jockers–Morrison–Walcher based partially on the results of [19].

The open Gromov–Witten invariants of Theorem 1.0.1 are obtained from homological mirror symmetry for the quintic threefold. This is achieved by studying the Floer theory of a *different* Lagrangian immersion $\widetilde{L}^5_{\mathrm{im}}$ (cf. Theorem 1.2.6) in a way that we summarize in the rest of this introduction. To the author's knowledge, these are the first examples of an enumerative invariant to be recovered from homological mirror symmetry before being calculated using classical techniques (e.g. Atiyah–Bott localization).

1.1 Background on lines in the mirror quintic

Let Δ denote the unit disk in $\mathbb{C}$ and let $\Delta^* := \Delta \setminus \{0\}$ denote the punctured unit disk. To define the mirror quintic family, we first consider the so-called Dwork family quintics with fibers given by

$$X^5_z := \left\{ \prod_{j=1}^{5} x_j - \frac{z^{1/5}}{5} \sum_{j=1}^{5} x_j^5 = 0 \right\} \subset \mathbb{CP}^4 \,;\ z \in \Delta^* \,. \tag{1.1.1}$$

There is an action of $(\mathbb{Z}/5)^3$ on each X^5_z, which is inherited from the natural action of $(\mathbb{Z}/5)^5$ on $\mathbb{CP}^4$ by restricting to the subgroup that preserves the monomial $\prod_{j=1}^5 x_j$, and taking the quotient by the diagonal subgroup. The mirror quintic family $\pi\colon \mathcal{X}^{5,\vee} \to \Delta^*$ has fibers $\pi^{-1}(z) := X^{5,\vee}_z$ which are crepant resolutions of the orbifold quotients $X^5_z/(\mathbb{Z}/5)^3$.

As was first observed by van Geemen, cf. [23], the Dwork quintics X_z all contain infinite families of lines. This can be seen by considering the special lines of the form

$$\widetilde{C}_z^{\omega} := \left\{ x_1 + \omega x_2 + \omega^2 x_3 = 0\,,\ x_4 = \frac{a}{3}(x_1 + x_2 + x_3)\,,\ x_5 = \frac{b}{3}(x_1 + x_2 + x_3) \right\} \subset X_z^5 \qquad (1.1.2)$$

where $a, b, \omega \in \mathbb{C}$ are constants subject to the relations

$$\begin{aligned} 1 + \omega + \omega^2 &= 0 \\ a^5 + b^5 &= 27 \\ ab &= 6z^{1/5}\,. \end{aligned} \qquad (1.1.3)$$

Each of the lines $\widetilde{C}_z^{\omega}$ is called a *van Geemen line*. Fix one such line, and note that the orbit of $\widetilde{C}_z^{\omega}$ under the group $(\mathbb{Z}/5)^3$ contains 125 elements. Moreover, the orbit of $\widetilde{C}_z^{\omega}$ under the action of the symmetric group S_5, which permutes the coordinates on $\mathbb{CP}^4$, has order 40, since the subgroup given by permuting the first three coordinates preserves such a line. These facts imply that there are at least 5000 lines on X_z, and since this exceeds the virtual number of lines, 2875, there must actually be infinitely many.

The lines on a generic Dwork quintic X_z^5 were described explicitly in [24]. In addition to 375 isolated lines, there are two families of lines, each of which can be identified with a 125-fold cover of a genus six curve, meaning that they are both genus 626 curves. It follows from the results of [24] that the families of lines in the mirror quintic can be identified with two genus six curves. Denote by C_z^{ω} the line in the mirror quintic $X_z^{\vee}$ corresponding to any of the lines $\widetilde{C}_z^{\omega}$, for some $a, b \in \mathbb{C}$ as in (1.1.3). Notice that all such choices of $a, b \in \mathbb{C}$ yield the same line. Abusing terminology, we also call the lines C_z^{ω} in the mirror quintic van Geemen lines.

Convention 1.1.1. Note that the mirror quintic family can be viewed as a scheme over $\mathbb{C}((z))$, where z here denotes a formal variable. We denote this scheme by $X^{5,\vee}$. Similarly, the van Geemen lines determine lines in $X^{5,\vee}$, which we denote by C^{ω} and C^{ω^2}.

1.2 Tropical Lagrangian submanifolds in the quintic

The Lagrangian $\widetilde{L}^5_{\text{sm}}$ of Theorem 1.0.1 is constructed as the lift of a tropical curve, which comes from considering the tropicalization of a van Geemen line. To explain the meaning of this statement, consider the cycles $\widetilde{C}^{\omega}_0$ in the toric boundary $X^5_0 := \left\{\prod_{j=1}^5 x_j = 0\right\}$ of $\mathbb{CP}^4$ obtained from (1.1.2) by setting $z = 0$. It follows from (1.1.3) that either $a = 0$ or $b = 0$, meaning that $\widetilde{C}^{\omega}_z$ either lies in one of the hyperplanes $\{x_4 = 0\}$ or $\{x_5 = 0\}$. For concreteness, we will consider the case where $b = 0$, so we can think of $\widetilde{C}^{\omega}_z$, for some fixed $a \in \mathbb{C}$, as a cycle in $\mathbb{CP}^3 \cong \{x_5 = 0\}$. An easy computation shows that $\widetilde{C}^{\omega}_z$ intersects the toric boundary of $\mathbb{CP}^3$ in exactly four points, meaning that its restriction to the big torus $(\mathbb{C}^*)^3 \subset \mathbb{CP}^3 \cong \{x_5 = 0\}$ is a rational curve with four punctures.

The tropicalization of a curve $C \subset (\mathbb{C}^*)^n$ can be thought of as the image $\text{Log}_t(C)$, in the limit $t \to \infty$, under the map $\text{Log}_t\colon (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by

$$\text{Log}_t(x_1, \ldots, x_n) = (\log_t |x_1|, \ldots, \log_t |x_n|)\,. \tag{1.2.1}$$

The tropicalization will be a 1-dimensional polyhedral complex whose 1-dimensional faces are affine line segments in $\mathbb{R}^n$ and whose 0-dimensional faces satisfy a balancing condition that dictates how the 1-faces meet at the 0-faces. A direct calculation shows that the tropicalization of $\widetilde{C}^{\omega}_z \cap (\mathbb{C}^*)^3$ is a 4-*valent* tropical curve with a single vertex. The tropicalization is independent of the choice of a, since all such choices have the same norm.

There is a general theory for constructing Lagrangian lifts of *trivalent* tropical subvarieties of $\mathbb{R}^3$ that should correspond, under homological mirror symmetry, to (structure sheaves on) the corresponding complex subvarieties of $(\mathbb{C}^*)^3$; see inter alia [25, 26, 27, 28]. One of the main results of this thesis, which appeared in [29, Theorem 1.1], develops a natural analogue of these constructions for 4-valent tropical curves which arise, for examples, as tropicalizations of the very affine van Geemen lines. Somewhat surprisingly, the lift of such a curve is naturally a singular

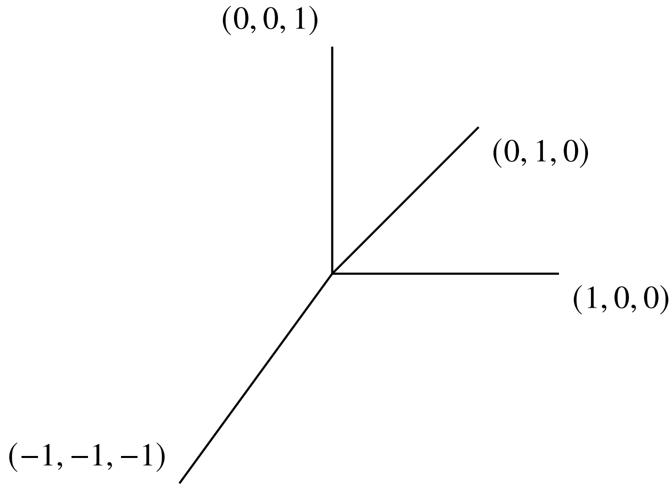

Figure 1.1: The tropical curve V.

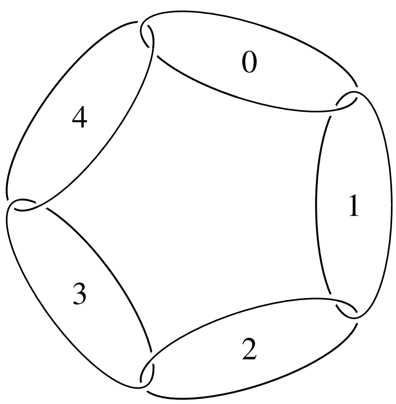

Figure 1.2: The minimally-twisted five-component chain link complement, and a labeling of its components.

Lagrangian submanifold $L_{\text{sing}} \subset T^*T^3$.

Theorem 1.2.1 (Paraphrasing of Theorem 2.3.1)**.** *The* 4*-valent tropical curve V in $\mathbb{R}^3$ depicted in Figure 1.1 has a Lagrangian lift L_{sing} with one singular point homeomorphic to the cone over a* 2*-torus. In a Darboux chart, the cone point can be identified with the Harvey–Lawson cone of [30]. Away from the singular point, L_{sing} is diffeomorphic to a hyperbolic link complement in S^3, called the minimally twisted five-component chain link, which is also drawn in Figure 1.2.*

Corollary 1.2.2. *Any tropical curve in $\mathbb{R}^n$ with vertices of valence at most* 4 *has a Lagrangian lift in T^*T^n which is immersed away from a discrete set of isolated conical singular points, which* *project to the* 4*-valent vertices under projection to the cotangent fiber.*

Proof. Given any tropical curve with a single 4-valent vertex with the property that any three of its adjacent edges are linearly independent, we can construct a lift of it with isolated conical singularities by taking a cover of the Lagrangian from Theorem 1.2.1 using the argument in [28]. By crossing these with the isotropic copies of $T^{n-3} \subset T^n \subset T^*T^n$ orthogonal to the affine space spanned by the edges of the 4-valent vertex, we obtain a Lagrangian lift of any non-planar 4-valtent tropical vertex in any dimension.

Any tropical curve W in $\mathbb{R}^2$ with a single vertex is a tropical hypersurface, and thus is the vanishing locus of a single tropical polynomial. If we choose any polynomial over $\mathbb{C}$ which tropicalizes to the tropical polynomial, it will cut out a curve C in $(\mathbb{C}^*)^2$. Applying a hyperKähler rotation to C produces an immersed Lagrangian in $(\mathbb{C}^*)^2 \cong T^*T^2$ which is, up to Hamiltonian isotopy, a Lagrangian lift of W (cf. [25]). After thickening these lifts by taking products with isotropic copies of T^{n-2}, we obtain a Lagrangian lift of any planar 4-valent tropical curve in $\mathbb{R}^n$. We can glue the local Lagrangian lifts together over the legs of any tropical curve to produce lifts of an arbitrary tropical curve. □

There are three ways of desingularizing L_{sing} to produce a smooth Lagrangian. It turns out that all three of these desingularizations are diffeomorphic to the graph manifold

$$(P \times S^1) \bigcup_{\sigma} (P \times S^1), \tag{1.2.2}$$

where $P := \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ denotes the pair of pants and σ is the self-diffeomorphism of one of the T^2-boundary components of $P \times S^1$ that interchanges the S^1-factor with the boundary circle of P.

Correspondingly, there are three trivalent tropical curves obtained by replacing the vertex of V with an edge. More precisely, consider the tropical line $V_{1,\epsilon} \subset \mathbb{R}^3$ consisting of two pairs of pants centered at the points $\left(0, -\frac{\epsilon}{2}, -\frac{\epsilon}{2}\right)$ and $\left(0, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$. The legs of the pair of pants centered at $\left(0, -\frac{\epsilon}{2}, -\frac{\epsilon}{2}\right)$

point in the directions

$$(0,1,1), \quad (0,-1,0), \quad (0,0,-1)$$

and the legs of the pair of pants centered at $\left(0, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$ point in the directions

$$(0,-1,-1), \quad (-1,0,0), \quad (1,1,1)\,.$$

One would expect stronger relationship between the tropical Lagrangian lifts of a tropical subvariety and the corresponding complex subvariety C of the mirror. Specifically, the tropical Lagrangian lift should be mirror to the structure sheaf $\mathcal{O}_C$ under the equivalence between the wrapped Fukaya category of T^*T^n and $D^b\mathrm{Coh}((\mathbb{C}^*)^n)$. For smooth tropical hypersurfaces, a mirror relation of this type is proven by Hicks [26], using the fact that tropical Lagrangian hypersurfaces are mapping cones in the Fukaya category, essentially by construction.

To prove a similar result for the Lagrangian L_{sing} of Theorem 1.2.1, one would have to make sense of Lagrangian Floer theory for Lagrangians with conical singularities. Rather than taking this approach, we construct an immersed Lagrangian in T^*T^3 by using a 'doubling' trick, similar to [31], [32], [33], in which we morally construct an immersed Lagrangian by gluing two copies of the singular Lagrangian together.

Let $\mathbb{K}$ be a field, and identify $\mathrm{Hom}(\pi_1(T^3), GL(1,\mathbb{K}))$ with $(\mathbb{K}^*)^3$. Our other main result is the following.

Theorem 1.2.3. *There is an exact Lagrangian immersion $\widetilde{L}_{\mathrm{im}} \xrightarrow{\mathrm{im}} T^*T^3$ whose image L_{im} is C^0-close to L_{sing}, and with the property that for a generic line $C \subset \mathbb{P}^3$, there is a $GL(1)$-local system ∇_C on $\widetilde{L}_{\mathrm{im}}$ such that the set*

$$\{\nabla_p \in \mathrm{Hom}(\pi_1(T^3), GL(1,\mathbb{K}))\colon HF^*((L_{\mathrm{im}}, \nabla_C), (T^3, \nabla_p)) \neq 0\}$$

is equal to the affine part $C \cap (\mathbb{K}^*)^3$.

We call the set of such local systems the Floer-theoretic support of L_{im} following [34]. By a generic line, we mean one which lies in a certain Zariski open subset of the Grassmannian $Gr(2,4)$: if we embed $Gr(2,4)$ in $\mathbb{P}^5$ via the Plücker embedding, then the lines to which our theorem applies are those in $Gr(2,4) \cap (\mathbb{K}^*)^5$. This restriction comes from the fact that the holonomy of a local system on L_{im} must always be nonzero.

Under homological mirror symmetry, the Floer-theoretic support should correspond to the usual support of the mirror sheaf. This is closely related to the support of the mirror sheaf constructed using family Floer theory, except that instead of considering families of Lagrangian tori in T^*T^3, most of which are not exact, we work with a single Lagrangian torus and equip it with non-unitary local systems. Consequently, our mirror is defined over $\mathbb{K}$, as opposed to the Novikov field. While significantly less general than family Floer theory, our technical setup does allow us to prove a statement that does not refer to the SYZ fibration or to tropical geometry.

The starting point for our construction is to identify a particular Lagrangian torus $L_{\tau,0}$ in a Dwork quintic X^5_τ, which supports a family of objects mirror to points on a quintic. By this we mean that the objects of the Fukaya category $\mathcal{F}(X_\tau)$ obtained by equipping $L_{\tau,0}$ with a rank 1 local system are all mirror to skyscraper sheaves on the mirror quintic. The Lagrangian torus in question is obtained from a moment fiber in toric boundary of $\mathbb{CP}^4$ under symplectic parallel transport. One would expect the Lagrangian tori constructed this way to be smooth SYZ fibers of an SYZ fibration on the quintic (cf. [35, 36]).

We then consider a 125-fold cover of L_{sing} contained inside a Weinstein neighborhood of this torus. Intuitively, taking this cover corresponds, under mirror symmetry, to taking a quotient by the action of $(\mathbb{Z}/5)^3$ in the construction of the mirror quintic. We obtain a compact singular Lagrangian $\widetilde{L}^5_{\mathrm{sing}}$ by attaching suitable copies of the Harvey–Lawson cone liing near the intersection of the toric boundary with X^5_τ to this (non-compact) singular Lagrangian. The approach we use to perform this gluing construction is similar to, but much easier than, the construction of tropical Lagrangians in mirror quintics in [20].

The Lagrangian submanifold $\widetilde{L}^5_{\mathrm{sm}}$ of the quintic is obtained from $\widetilde{L}^5_{\mathrm{sing}}$ using the (a topological version of) the construction of [37]. This roughly replaces the a neighborhood of each cone point of such a singular Lagrangian with a copy of an Aganagic–Vafa brane [8]. Thought of in these terms, it is unclear how to calculate the open Gromov–Witten invariants of $\widetilde{L}^5_{\text{sm}}$ directly: it has very few apparent symmetries given its construction by soft topological methods, and there are currently few available tools for counting disks with Lagrangian boundary tropically in this context. This might make one hopeful that the open Gromov–Witten invariants can be extracted from homological mirror symmetry using the framework of [38], but studying the Floer theory of $\widetilde{L}^5_{\text{sm}}$ is also difficult. Instead of studying $\widetilde{L}^5_{\text{sm}}$ directly as an object of the Fukaya category, we will study the Floer theory of a closely related immersed Lagrangian denoted $\widetilde{L}^5_{\text{im}}$.

An immersed Lagrangian lift, denoted L_{im}, of a curve with the same combinatorial type as the tropicalization of $\widetilde{C}^{\omega}_{z} \cap (\mathbb{C}^*)^3$ was constructed in [29] by 'doubling' the singular Lagrangian L_{sing}. Moreover, it was shown [29, Theorem 1.2] that equipping L_{im} with suitable local systems yields objects of the wrapped Fukaya category of T^*T^3 mirror to lines in $\mathbb{CP}^3$. This was achieved by computing the Floer cohomology of such objects with the 0-section in T^*T^3, which amounts to computing the support of the mirror sheaf. This immersed Lagrangian appears as a local piece of the immersed Lagrangian $\widetilde{L}^5_{\text{im}}$. Near the singular points of $\widetilde{L}_{\text{sing}}$, we can also think of $\widetilde{L}^5_{\text{im}}$ as being obtained by replacing the conical pieces of L_{sing} with 'doubled' immersed Lagrangian submanifolds [29, §5.3] that miss the toric boundary of $\mathbb{CP}^4$. Consequently, we can think of objects of the Fukaya category supported on $\widetilde{L}^5_{\text{im}}$ as objects of the *relative Fukaya category*, as is usually considered in proofs homological mirror symmetry for closed Calabi–Yau manifolds [15, 16, 17]. This fact makes the Floer theory of $\widetilde{L}^5_{\text{im}}$ significantly easier to work with than the Floer theory of $\widetilde{L}^5_{\text{sm}}$.

Remark 1.2.4. The smooth part of $\widetilde{L}^5_{\text{sing}}$, denoted $\widetilde{L}'$ is a cover of the minimally-twisted five-component chain link complement L', and hence it is a hyperbolic 3-manifold. The hyperbolicity of L' determines a faithful representation $\pi_1(L') \to SL(2,\mathbb{C})$. Given any subgroup Γ of $SL(2,\mathbb{C})$, one can construct an extension of $\mathbb{Q}$ called the *invariant trace field* of Γ by adjoining the traces of certain element of Γ to $\mathbb{Q}$ (see [39, Chapter 3] for a precise definition). This turns out to depend only on the commensurability class of Γ. From this we obtain a field-valued invariant of hyperbolic 3-manifolds which is preserved under taking finite degree covers.

The invariant trace field of L' is $\mathbb{Q}(\sqrt{-3})$, which can be seen from an ideal triangulation [40, §2.6], and hence it is also the invariant trace field of $\widetilde{L}'$. This is also the field in which the open Gromov–Witten invariants appearing in Theorem 1.0.1 lie. Such a relation between invariant trace fields and open Gromov–Witten invariants has been long-expected in string theory based on computations of Jockers–Morrison–Walcher. In this way, one can thin of the field $\mathbb{Q}(\sqrt{-3})$ as a *period ring* for the branes $(\widetilde{L}^5_{\mathrm{sm}}, \nabla^{\Lambda}_{\omega^i}$, which corresponds to the role that this field plays on the B-side. By [41, Corollary 1.2], there is a general relation on the A-side between the field of definition of brane structures on a Lagrangian submanifold and the field of definition of its open Gromov–Witten invariants.

Remark 1.2.5. It is remains unclear whether there should also be a general relationship between topological invariants of a Lagrangian submanifold, such as the hyperbolic volume or invariant trace field, and the values of its open Gromov–Witten invariants. In particular, we note that $\widetilde{L}^5_{\mathrm{sm}}$ is not a hyperbolic 3-manifold. The results of §4.2.4 show that $\widetilde{L}^5_{\mathrm{sm}}$ is obtained by Dehn filling (a cover of) the Lagrangian lift of a tropical curve given in (4.2.17). It follows that $\widetilde{L}^5_{\mathrm{sm}}$ contains an essential torus, which can be thought of in this context as a conormal fiber over the finite edge (cf. Figure 4.1) of this tropical curve. We also remark that one of the original motivations for the predictions of Jockers–Morrison–Walcher comes from their announced computation of a constant first approximated by Laporte–Walcher [42], which they conjectured to be the volume of a hyperbolic 3-manifold mirror to the van Geemen lines. The hyperbolic volume of $\widetilde{L}'$ is $1250\,\mathrm{Im}\,\mathrm{Li}_2(-\omega)$, where $\mathrm{Im}\,\mathrm{Li}_2(-\omega)$ denotes the imaginary part of the dilogarithm. We compute this volume using the fact that the hyperbolic volume of the minimally-twisted five=component chain link complement is $10\,\mathrm{Im}\,\mathrm{Li}_2(-\omega)$, since it can be written explicitly as the union of 10 regular ideal tetrahedra. This disagrees with the predicted volume $130\,\mathrm{Im}\,\mathrm{Li}_2(-\omega)$, which we obtain by rewriting [42, (17)] in terms of the dilogarithm. We also point out that one should not expect to obtain an *embedded* hyperbolic Lagrangian submanifold from $\widetilde{L}^5_{\mathrm{sing}}$, per the discussion in [29, §1] on Dehn fillings of the chain link complement.

Given its close relation to the Lagrangian torus in X^5_τ supporting mirrors to points, we can describe mirror sheaves, with respect to the mirror equivalence of [15] to objects supported on L^5_{im}. We use a Morse–Bott model for the Fukaya category following [43], but with some modifications that allow for immersed Lagrangians equipped with local systems and gradings [44]. A description of the Fukaya category we use is given in Appendix 4.8. The results of [15] still apply provided our model for the Fukaya category satisfies certain technical assumptions, namely Assumption 4.9.1, which are expected to hold for any reasonable version of the Fukaya category. Most crucially, these assumptions say that the Fukaya category should have all of the algebraic structure needed to mimic the proof of Abouzaid's generation criterion [45].

Theorem 1.2.6 (Paraphrasing of Theorem 4.3.21)**.** *Under Assumption 4.9.1, consider a generic non-isolated line C in the mirror quintic. Then there is an unobstructed rank one $\mathbb{C}$-local system ∇_C on $\widetilde{L}^5_{\mathrm{im}}$ such that the resulting object $(\widetilde{L}^5_{\mathrm{im}}, \nabla_C)$ of the Fukaya category is mirror, under the equivalence of [15], to a rank 2 vector bundle supported on C.*

We can, in particular, take C to be a van Geemen line C^ω, and the corresponding local system is denoted $\nabla^{\mathrm{vG}}_\omega$ as above. Here the mirror quintic is understood as a variety defined over the Novikov field. It is clear from their definition that the van Geemen lines can be naturally identified with lines in this variety. To prove Theorem 1.2.6, we will calculate its Floer cohomology with the torus $L_{\tau,0}$. This computation uses [29, Theorem 1.2] in an essential way. By computing $HF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla_C)$, we will show that this object is mirror to the pushforward of a vector bundle. This determines the *algebraic second Chern class* of the mirror sheaf, which is crucial for deducing Theorem 1.0.1 from Theorem 1.2.6.

To obtain Theorem 1.0.1 from Theorem 1.2.6, we will relate the A_∞-algebra of one of the Lagrangian branes $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ to the A_∞-algebra of $\widetilde{L}^5_{\mathrm{sm}}$ equipped with a suitable brane structure. This is carried out in §4.3.5, and involves studying the behavior of the Fukaya A_∞-algebra under clean Lagrangian surgery in a particularly simple case.

1.3 Open Gromov–Witten invariants and homological mirror symmetry

The genus zero Gromov–Witten invariants of a closed Lagrangian L in a closed symplectic manifold M should count pseudoholomorphic disks with boundary on L in a way that is independent of the almost complex structure used to write the Cauchy–Riemann equation. In contrast to the theory of closed Gromov–Witten invariants, moduli spaces of pseudoholomorpic disks have boundary, which can potentially obstruct the invariance of open Gromov–Witten invariants. One expects the boundaries of these moduli spaces to contains pseudoholomorphic disks with boundary nodes and pseudoholomorphic spheres intersecting L. To account for disk bubbles, Joyce [46] proposed that L should be equipped with a bounding cochain as defined in [47].

For a graded Lagrangian L in a Calabi–Yau threefold M, this idea was implemented by Fukaya in [18], where he constructed a generating function for the open Gromov–Witten invariants. Fukaya's open Gromov–Witten potential is a function from the space of bounding cochains on L modulo gauge equivalence to the Novikov ring. The proof that Fukaya's open Gromov–Witten potential is gauge-invariant and invariant under changes of almost complex structure, up to a count of sphere bubbles, uses the existence of a cyclically symmetric pairing on the Fukaya A_∞-algebra of L, i.e. an inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_0 \rangle = \pm \langle \mathfrak{m}_k(\alpha_0, \ldots, \alpha_{k-1}), \alpha_k \rangle$$

up to a sign determined by the gradings of the inputs.

Geometrically, this symmetry should arise by cyclically permuting boundary constraints on pseudoholomorphic disks on L and using the S^1-symmetry of the domain. In [48], Fukaya constructs an A_∞-structure on the de Rham complex $\Omega^*(L)$, and shows that the integration pairing

$$\langle \alpha, \beta \rangle = \pm \int_L \alpha \wedge \beta$$

(which is also defined up to a sign depending on the degrees of the inputs) is cyclically symmetric in this sense. Using this cyclic pairing, Fukaya defines the open Gromov–Witten potential of L to be

$$\Psi(b) := \mathfrak{m}_{-1} + \sum_{k=0}^{\infty} \frac{1}{k+1} \langle \mathfrak{m}_k(a_1, \ldots, a_k), a_0 \rangle \tag{1.3.1}$$

where b is a bounding cochain on L and $\mathfrak{m}_{-1}$ counts disks without boundary marked points.

The statement of Theorem 1.0.1 involves the difference of open Gromov–Witten potentials for two Lagrangian branes in the quintic supported on the same embedded Lagrangian submanifold $\widetilde{L}^5_{\mathrm{sm}}$. Since this Lagrangian is not nullhomologous, the open Gromov–Witten potential defined for graded Lagrangian submanifolds of Calabi–Yau threefolds by Fukaya [18] is not independent of the almost complex structure used to define it. We can, however, show that the difference of open Gromov–Witten potentials is independent of the almost complex structure by viewing it as the open Gromov–Witten potential of an *immersed Lagrangian submanifold* given by two copies of $\widetilde{L}^5_{\mathrm{sm}}$. This would require a definition of the open Gromov–Witten potential for Lagrangian immersions. The construction of the open Gromov–Witten in [18] does not immediately generalize to the immersed case, as the Fukaya A_∞-algebra of a clean Lagrangian immersion does not possess cyclic symmetry [43].

The key technical input required in [48] is the construction of a system of Kuranishi structures on the moduli spaces pseudoholomorphic disks with boundary on L, which are compatible with forgetful maps of marked points, and for which the evaluation map at one of the boundary marked points is a submersion. When L is an immersed Lagrangian, the moduli spaces of holomorphic disks with corners does not admit forgetful maps at boundary marked points corresponding to self-intersection points of L, and so one cannot mimic the arguments of [48].

A closely related technical issue also arises when one attempts to extend these constructions to fields of characteristic 0 that do not contain $\mathbb{R}$. The construction of the Fukaya A_∞-algebra of an embedded Lagrangian over $\mathbb{Q}$ in [47] involves taking fiber products over L^k, a product of k

copies of L with itself. The moudli spaces of disks with k marked points admit an evaluation map to L^k given by evaluation at all marked points simultaneously. It is not known how to construct a Kuranishi structure which is compatible with forgetful maps and for this map is a submersion [48, Remark 3.2]. This lack of submersivity means that one cannot construct a cyclic pairing on the A_∞-algebras of [47], which use smooth singular chains as a model for the cohomology of L. Consequently, it is unclear whether or not the Fukaya category generally carries a strictly cyclic structure over fields which do not contain $\mathbb{R}$. This implies that the open Gromov–Witten invariants of [18] are only real-valued.

Solomon and Tukachinsky [49] explain how to extend Fukaya's construction of the open Gromov–Witten potential to Lagrangians of any dimension with possibly non-vanishing Maslov class by working over a ground ring with nontrivial grading. Their construction proceeds under the assumption that all relevant moduli spaces of disks are smooth orbifolds with corners, and that one of the boundary evaluation maps is a submersion. Without using virtual techniques, the dimension-counting argument of [48, Remark 3.2] shows that one cannot construct an A_∞-algebra structure on the singular chain complex of a Lagrangian. Consequently, it is unknown how one would associate a cyclically symmetric A_∞-algebra to a Lagrnagian submanifold. Moreover, assuming submersivity of evaluation maps and regularity simultaneously means that the invariants of [49] are only shown to be invariant under changes of almost complex structure in a weak sense. Specifically, their proof of invariance [49, Theorem 1] requires that one has a path of *regular* almost complex structures, which usually cannot be shown to exist by standard transversality arguments.

One of the main results of this thesis presents a construction of the open Gromov–Witten potential over arbitrary fields of characteristic 0 that does not require cyclic symmetry and can be carried out without any submersivity assumptions. To address the second of these problems, we define the Fukaya A_∞-algebra using the Morse complex rather than the de Rham complex or smooth singular chains. Because the A_∞-structures on the Morse complex counts configurations of pseudoholomorphic disks joined by Morse flow lines, it manifestly lacks cyclic symmetry.

Instead of using a cyclic pairing for our construction, we use the existence of a Calabi–Yau structure on the Fukaya category. One can define a Calabi–Yau structure on $\mathcal{A}$ to be an A_∞-bimodule homomorphism from the diagonal bimdoule $\mathcal{A}_\Delta$ to the dual bimodule $\mathcal{A}^\vee$. Hence, a cyclic pairing can be thought of as a special case of a Calabi–Yau structure. It is a theorem of Kontsevich and Soibelman [50, Theorem 10.7] that, for an uncurved A_∞-algebra, (weak proper) Calabi–Yau structures on $\mathcal{A}$ correspond to strictly cyclic pairings on a minimal model of $\mathcal{A}$.

In light of Kontsevich and Soibelman's theorem, one might hope to mimic the constructions of [18] and [49] on a minimal model of the Fukaya A_∞-algebra of L. The problem with this approach is that, in the filtered case, the potential of a cyclic A_∞-algebras does not behave well under quasi-isomorphisms. In particular, it is only a quasi-isomorphism invariant up to additive constants [51]. This phenomenon arises as one studies the dependence of the open Gromov–Witten potential on the choice almost complex structure used to define it, wherein such additive constants are given explicitly as counts of pseudoholomorphic teardrops.

Instead of passing to a cyclic minimal model, we construct the open Gromov–Witten potential by incorporating the higher order terms of the A_∞-bimodule homomorphism to account for the lack of cyclic symmetry. More specifically, an A_∞-bimodule homomorphism $\mathcal{A}_\Delta \to \mathcal{A}^\vee$ consists of a family of linear maps

$$\phi_{p,q} \colon \mathcal{A}^{\otimes p} \otimes \underline{\mathcal{A}} \otimes \mathcal{A}^{\otimes q} \to \mathcal{A}^\vee$$

where the underline signifies that the corresponding factor of $\mathcal{A}$ is thought of as a bimodule over $\mathcal{A}$.

The potential Φ associated to a Lagrangian L equipped with a bounding cochain b, defined over a field $\Bbbk$ of characteristic zero, is defined by the formula

$$\Phi(b) := \mathfrak{m}_{-1} + \sum_{N=0}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \phi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k(b^{\otimes k})} \otimes b^{\otimes q})(b).$$

Here, the structure coefficients $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ arise from linear maps on the Morse complex $CM^*(L)$ defined over $\Bbbk$ and $\mathfrak{m}_{-1}$ is a count of rigid pearly trees with no inputs from the Morse cochain complex of L. The A_∞-bimodule homomorphism appearing in this definition arises from a (possibly bulk-deformed) cyclic open-closed map on the Morse complex of L, in the sense of [52].

As we will show, this potential satisfies a wall-crossing formula of the same sort as the open Gromov–Witten potential in the cyclic case.

Theorem 1.3.1 (Paraphrasing of Theorem 3.4.9)**.** *The open Gromov–Witten potential is invariant under changes of almost complex structure up to a count of closed pseudoholomorphic spheres intersecting L in a point.*

Consequently, the open Gromov–Witten potential is defined over the field of definition of Maurer–Cartan elements of L.

Corollary 1.3.2. *If $L \subset M$ is a Lagrangian submanifold which is unobstructed by a bounding cochain b defined over a field $\mathbb{k}$ of characteristic 0, then its open Gromov–Witten potential is valued in $\mathbb{k}$.*

Using the open Gromov–Witten potential, one can extract open Gromov–Witten invariants of certain Lagrangians whose spaces of bounding cochains are sufficiently well-understood. In the situations for which [49] define open Gromov–Witten invariants, one can in fact show that they are unobstruced over $\mathbb{Q}$. By Corollary 1.3.2, we can guarantee the rationality of these invariants.

Corollary 1.3.3. *If $L \subset M$ is a rational homology sphere (cf. [49, Theorem 2]) or a real locus satisfying the conditions of [49, Theorem 3], then there exist $\mathbb{Q}$-valued open Gromov–Witten invariants for L.*

To keep our exposition simple and self-contained, we have only constructed the open Gromov–Witten potential under regularity assumptions which say that all moduli spaces of pearly trees needed for our construction are cut out transversely. These assumptions are achieved for any Lagrangian considered in [49] and for any monotone Lagrangian [53], as explained in Appendix 3.6, where a complete list of all assumptions introduced throughout this paper can be found.

Because we have avoided using cyclic symmetry, our construction should also extend to cases where one defines the Fukaya A_∞-algebra of L using domain-dependent perturbations of the

Cauchy–Riemann equation satisfying certain consistency conditions. In particular, this would allow for the definition of the open Gromov–Witten potential for Lagrangian immersions with clean self-intersections. This is a desirable generalization even if one is only interested in embedded Lagrangians, since considering Lagrangian immersions with disconnected domains enables easy constructions of nullhomologous Lagrangian immersions, for which the terms involving closed curves in the wall-crossing formula can be algebraically canceled by a choice of bounding chain.

We will avoid this issue by following the strategy of [41], where it was shown that the open Gromov–Witten invariants can be defined using a strong proper Calabi–Yau structure on the Fukaya category.

Geometrically, this is obtained from a *cyclic open closed map*. We recall that this is a map from the cyclic homology of the Fukaya category to the quantum cohomology which intertwines the relevant S^1-actions. Incidentally, the (negative) cyclic open-closed map is also needed to extract genus zero Gromov–Witten invariants from the Fukaya category, as shown by Ganatra–Perutz–Sheridan [6]. Hence, we will assume (cf. Assumptions 4.9.3 and 4.9.2) that suitable versions of the cyclic open-closed map exist.

Using results of Cho and Lee [54], we can construct a homotopy cyclic ∞-inner product ψ on $\mathcal{A}$ from the trace of Assumption 4.9.2. More specifically, this is an A_∞-bimodule homomorphism $\psi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ from the diagonal bimodule to the linear dual satisfying some additional symmetries. By closely following the arguments in [41], we can use this to construct the open Gromov–Witten potential of a clean Lagrangian immersion.

Theorem 1.3.4. *Let $\mathcal{A}$ denote the curved Fukaya A_∞-algebra of a graded clean Lagrangian immersion L in a Calabi–Yau threefold equipped with a rank 1 local system, and suppose that it satisfies Assumption 4.9.2. Then there is a well-defined open Gromov–Witten potential $\Psi\colon \mathcal{M}(L) \to \Lambda$, where $\mathcal{M}(L)$ denotes the space of gauge-equivalence classes of bounding cochains on L, given by*

$$\Psi(b) \coloneqq \mathfrak{m}_{-1} + \sum_{N=0}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \psi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k(b^{\otimes k})} \otimes b^{\otimes q})(b)$$

where $\mathfrak{m}_{-1}$ is a count of pseudoholomorphic disks with boundary on L and no boundary marked points. This is invariant of the almost complex structure on M up to a wall-crossing term given by counting closed pseudoholomorphic curves intersecting L in a point.

Remark 1.3.5. Assuming that L is graded and 3-dimensional allows us to work over the Novikov ring, rather than an extension thereof as in [49, 14]. There should be no essential difficulty in defining the open Gromov–Witten potential of a general Lagrangian immersion using the techniques of this paper by considering a larger coefficient ring. It would be interesting to formulate the notion of a *point-like* bounding cochain in this situation. Such a definition might make it possible to obtain meaningful counts of disks with boundary and corners on a Lagrangian immersion.

One can compensate for the wall-crossing term mentioned in Theorem 1.3.4 when L is nullhomologous using a choice of bounding 4-chain, obtaining a potential which is truly invariant under changes of almost complex structure. In this setting, Solomon–Tukachinsky [14] construct a connection on the relative cohomology $QH^*(X, L)$, essentially constructing an extension of VHSs, but without specifying an integral structure. The connection on $QH^*(X, L)$ is defined using the open Gromov–Witten invariants, and so one would expect open enumerative mirror symmetry to be implied by an isomorphism between $QH^*(X, L)$ and an extension of VHSs defined in the B-model, as described in [55].

Let X be a Calabi–Yau 3-fold, and let $X^\vee$ be a mirror family of Calabi–Yau threefolds, thought of here as a variety over a Novikov field. We further assume that $X^\vee$ is *maximally unipotent*, in the sense of [6, §1.1], a suitable analogue over the Novikov field of the notion of a large complex structure limit point [56]. In [6] Ganatra–Perutz–Sheridan show how to recover the enumerative predictions of [1] from homological mirror symmetry using variations of *semi-infinite* Hodge structures (VSHS) defined at the categorical level. A VSHS can be thought of as a variation of Hodge structure without a choice of integral local system. If $X^\vee$ is smooth and compact, one can endow its negative cyclic homology $HC^-_\bullet(D^b \operatorname{Coh}(X^\vee))$ with the structure of a VSHS. Assuming the existence of a weak proper Calabi–Yau structure on the Fukaya category (cf. Assumption 4.9.2), and using the fact that $X^\vee$ is maximally unipotent, one can also construct a VSHS associated to

$HC^-_\bullet(\mathcal{F}(X))$ on the A-side. Homological mirror symmetry determines an isomorhpism between these two VSHSs. Ganatra–Perutz–Sheridan use the negative cyclic open-closed map (cf. Assumption 4.9.3) and an appropriate version of the HKR isomorphism (cf. [57]) to show that these VSHSs are isomorphic to the ones relevant to homological mirror symmetry, which suffices to deduce the genus zero Gromov–Witten invariants of X from homological mirror symmetry.

To prove Theorem 1.0.1, we will establish a partial analogue of the main Theorem of [6] in settings like the one considered in Theorem 1.2.6 for *extensions* of VSHS. We do so under similar assumptions about the existence and properties of the cyclic open-closed map.

Remark 1.3.6. Hugtenburg [38] proposes a construction of extensions of VSHSs at the categorical level, as well as a framework for showing that this extension is isomorphic to relative quantum cohomology. This relies on the definition of the cyclic open-closed map in [58], which was defined on the negative cyclic homology of a Fukaya A_∞-algebra, as opposed to the Fukaya category, in a way the relies on cyclic symmetry. The same issues that arise when one attempts to construct a cyclically symmetric pairing on the A_∞-algebra of an immersed Lagrangian would also arise when one attempts to construct such a pairing on the Fukaya category, which would pose a challenge in attempting to extend the chain-level arguments of [38] to the Fukaya category of a Calabi–Yau manifold. Similar difficulties would arise when attempting to adapt the results of Solomon–Tukachinsky [49, 14] to the present setting.

Instead of comparing extensions of VSHS at the categorical level, our approach transfers this difficulty to a comparison of B-model VSHSs, which admit a description in terms of normal functions that is particularly helpful in the setting of Theorem 1.2.6. This means that our arguments can be carried out at the homological level, albeit in significantly less general settings than one could hope to address with the techniques of [38].

For the purpose of relating the open Gromov–Witten invariants to *mirror symmetry*, we find it convenient to formulate the extension of VSHSs associated to a Lagrangian submanifold slightly differently than in [14]. The relative period integrals are determined by a normal function, which can be thought an element of an Ext group in the category of VSHSs. This normal function is itself determined by a choice of bounding chain for a nullhomologous algebraic cycle. Therefore, we choose a formulation of the A-model extension that privileges the normal function and a bounding smooth singular chain for an immersed Lagrangian submanifold. To carry out this approach, we work under some restrictive assumptions on the Hodge numbers of X, which are satisfied by the quintic.

The following theorem encapsulates the results of §4.7, which show how the open Gromov–Witten invariants are recovered from homological mirror symmetry in sufficiently simple geometric situations. We assume that X and $X^\vee$ are a mirror pair of Calabi–Yau 3-folds, where $X^\vee$ is defined over a certain Novikov ring. Alternatively, one can think of $X^\vee$ as a family $\mathcal{X}^\vee$ of Calabi–Yau threefolds over the punctured unit disk. We assume that X is simply connected and that its even degree Hodge numbers coincide with those of the quintic. These conditions are summarized in Assumptions 4.6.5 and 4.7.2. In this setting, we consider a pair of Lagrangian branes $L_0, L_1 \in \mathcal{F}(X)$ with the same open-closed image, and let $\mathcal{L}_0, \mathcal{L}_1 \in D^b_{dg}\operatorname{Coh}(X^\vee)$ denote the mirror objects. These should satisfy Assumption 4.7.1, meaning that $\mathcal{L}_i$ is required be the pushforward of a vector bundle on an algebraic curve C^i, and that C^0 and C^1 are homologous. Theorem 1.2.6 verifies this assumption for the pair of branes $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega^2})$ and $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega})$. Since a nontrivial algebraic curve is never nullhomologous, it is natural to phrase our results in terms of a pair of Lagrangian branes on the A-side, rather than a single Lagrangian brane.

Theorem 1.3.7. *Suppose that X and $X^\vee$ are a mirror pair of Calabi–Yau 3-folds subject to Assumptions 4.6.5 and 4.7.2, and that L_0 and L_1 are a pair of immersed Lagrangian branes subject to Assumption 4.7.1. Then there is a normal function in the quantum cohomology of X determined by the open Gromov–Witten invariants of the immersion $L_0 \cup L_1 \subset X$. This normal function coincides, up to changing coordinates by the mirror map, with a normal function determined by mC, where C is the algebraic cycle $C_0 - C_1$.*

Remark 1.3.8 (Open enumerative mirror symmetry for the real quintic)**.** Consider the set of real points in a Dwork quintic threefold X^5_τ for a small real constant τ, i.e. the *real quintic*. This is an embedded Lagrangian submanifold diffeomorphic to $\mathbb{RP}^3$, which supports two objects of the Fukaya category corresponding to its two unitary local systems. Provided that one can show that these objects supported on the real quintic are mirror to the expected B-model objects, which can be thought of as structure sheaves of conics in the mirror quintic [10], it should follow from Theorem 1.3.7 that the main theorem of [11] can be recovered from homological mirror symmetry.

This result does not establish a relationship between period integrals and and open Gromov–Witten invariants in the maximum conceivable generality, but it does apply in the setting of Theorem 1.2.6, and it is also expected to apply to the real quintic and its mirror conics [9, 11, 10]. The strategy of the proof is to define a candidate normal function in quantum cohomology by hand, using the simple form of the quantum connection on the quintic to check that we can indeed define a Hodge structure this way. For this purpose, we use an expression for the quantum connection in terms of a basis on quantum cohomology specified by Schwarz–Vologodsky–Walcher [59]. The determination of this basis is where a *splitting* of the Hodge filtration (cf. [2] or [6, Def. 2.9 and §2.2]) is used in our argument. Our construction of the open Gromov–Witten potential, combined with assumptions about the cyclic open-closed map, shows that this normal function lifts to a normal function in the negative cyclic homology of the Fukaya category. Under homological mirror symmetry, this corresponds to a normal function in the derived category of the mirror, and hence yields a normal function for the B-model VSHS associated to $X^\vee$.

We can compare this extension of VSHSs to the one derived from the cycle mC by appealing to [60], which relates the Chern character to the HKR isomorphism used in [6] to compare closed B-model Hodge structures. This, combined with the calculations of [19] then implies Theorem 1.0.1, since the extensions of B-model Hodge structures are both essentially determined by the algebraic second Chern class of an object in the derived category. For the pushforward of a vector bundle on a curve, the algebraic second Chern class is just an integer multiple of the support. The need to determine the second Chern class is the reason that merely computing the support of the mirror sheaf to $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_\omega)$ would be insufficient to prove Theorem 1.0.1.

Chapter 2

Cusped hyperbolic Lagrangians as mirrors to lines in three-space

The contents of this chapter first appeared in [29].

2.1 Immersed wrapped Floer theory

In this section, we will construct the wrapped Fukaya category of a Liouville manifold M. We will allow certain exact Lagrangian immersions which are cylindrical outside of M^{in} and which possess appropriate unobstructedness and monotonicity properties to be detailed below. Intuitively, one might expect the wrapped Floer A_∞-algebra of a Lagrangian immersion $\iota\colon \widetilde{L} \to M$ to have generators corresponding either to self-intersection points of the immersion or to Hamiltonian chords in the cylindrical end of M. In that case, the A_∞-operations would need to count disks with both corners and boundary punctures asymptotic to Hamiltonian chords. To prove that the A_∞-relations hold, one would then have to find perturbation data which is coherent, in an appropriate sense, with respect to gluing in pseudoholomorphic teardrops (i.e. disks with one corner). Since gluing in teardrops lowers the number of boundary punctures on a disk, the usual argument for the existence of perturbation data needs to be modified appropriately.

To address this issue, we will study the curvature terms that one would expect to see in the A_∞-structure first before attempting to define the full wrapped A_∞-category. More precisely, for a Lagrangian immersion $\iota\colon \widetilde{L} \to M$ equipped with a rank-one local system ∇, we will define a curvature term $\mathfrak{m}_0(\widetilde{L}, \nabla)$ which counts pseudoholomorphic teardrops (weighted by holonomy) on L of the appropriate index, and we will only take branes for which $\mathfrak{m}_0(\widetilde{L}, \nabla) = 0$ to be objects of the wrapped Fukaya category. To rule out disk bubbles of nonpositive index, we must also impose a positivity assumption, Definition 2.1.7, on the immersed Lagrangians that we consider.

Assuming that our Lagrangians are embedded outside of a compact subset of M^{in}, this first step can be carried out entirely in the interior part of M. When we construct the wrapped A_∞-operations, we will define them by counting inhomogeneous pseudoholomorphic polygons with boundary punctures asymptotic to Hamiltonian chords. The 1-dimensional moduli spaces may have boundary strata arising from bubbling of pseudoholomorphic teardrops, but standard gluing results and the vanishing of $\mathfrak{m}_0(\widetilde{L}, \nabla)$ imply that weighted counts of points in these 0-dimensional strata cancel, and so we are able to prove the uncurved A_∞-relations. It is possible to allow for deformations by nontrivial bounding cochains with similar techniques, but we do not use this.

2.1.1 Lagrangian immersions

We will start by recalling some basic notions regarding Lagrangian immersions. Let M be a Liouville manifold, meaning that it is equipped with a one-form λ such that $\omega := d\lambda$ is a symplectic form, with the property that the *Liouville vector field* Z_λ, characterized by the property that

$$\omega(Z_\lambda, \cdot) \equiv \lambda(\cdot)$$

generates a complete expanding flow. Outside of a compact subset of M, we assume that the flow is modeled on multiplication in the positive end of the symplectization of a contact manifold. We will write

$$M = M^{in} \cup_{\partial M} (\partial M \times [1, \infty)),$$

where M^{in} is a (compact) Lioville domain. In the cylindrical end $M \times [1, \infty)$, the Liouville form is given by $r\lambda|_{\partial M}$, where r is the $[1, \infty)$-factor, and hence the Liouville vector field is $\frac{\partial}{\partial r}$ in the cylindrical end. Also assume that $c_1(M) = 0$.

Definition 2.1.1. A Lagrangian immersion $\iota\colon \widetilde{L} \to M$ is said to have clean self-intersections if

(i) The fiber product

$$\widetilde{L} \times_\iota \widetilde{L} = \{(p_-, p_+) \in \widetilde{L} \times \widetilde{L} \colon \iota(p_-) = \iota(p_+)\}$$

is a smooth submanifold of $\widetilde{L} \times \widetilde{L}$.

(ii) At each point (p_-, p_+) of the fiber product, the tangent space is given by the fiber product of tangent spaces, i.e.

$$T_{(p_-,p_+)}(\widetilde{L} \times_\iota \widetilde{L}) = \{(v_-, v_+) \in T_{p_-}\widetilde{L} \times T_{p_+}\widetilde{L} \colon d\iota_{p_-}(v_-) = d\iota_{p_+}(v_+)\}.$$

We will often let $L := \iota(\widetilde{L})$ denote the image of such an immersion.

Definition 2.1.2. Suppose that $\iota \colon \widetilde{L} \to M$ is a Lagrangian immersion with clean self-intersections. We say that $\widetilde{L}$ is exact if there is a smooth function $f_L \colon \widetilde{L} \to \mathbb{R}$ such that $df_L = \iota^*\lambda$.

This definition of exactness does not prevent the existence of holomorphic teardrops on L.

Definition 2.1.3. A Lagrangian immersion $\iota \colon \widetilde{L} \to M$ is said to be cylindrical at infinity if there is a closed embedded Legendrian $\Lambda \subset \partial M$ such that L satisfies

$$L \cap (\partial M \times [1, \infty)) = \Lambda \times [1, \infty)$$

and such that the restriction of ι to $\iota^{-1}(\Lambda \times [1, \infty))$ is an embedding.

For a Lagrangian immersion with clean self-intersections, we write

$$\widetilde{L} \times_\iota \widetilde{L} = \coprod_{a \in A} L_a = \Delta_{\widetilde{L}} \sqcup \coprod_{a \in A \setminus \{0\}} L_a,$$

where A is an index set with a distinguished element $0 \in A$. Here, $L_0 = \Delta_{\widetilde{L}}$ denotes the diagonal component of the fiber product. Note that $\Delta_{\widetilde{L}}$ is disconnected if $\widetilde{L}$ is disconnected. Each other component L_a of the disjoint union is a non-diagonal connected component of the fiber product, which we will call switching components. Observe that there is a natural involution on $A \setminus \{0\}$ induced by the function on $\widetilde{L} \times_\iota \widetilde{L}$ which swaps coordinates.

2.1.2 Obstructions

In this subsection, we will define a certain count $\mathfrak{m}_0(\widetilde{L}, \nabla) \in \mathbb{K}$ of pseudoholomorphic disks on L with one corner weighted by holonomy, where $\mathbb{K}$ is a field and ∇ is a $GL(1, \mathbb{K})$-local system on $\widetilde{L}$.

Definition 2.1.4. Consider a Lagrangian immersion $\iota\colon \widetilde{L} \to M$ which is exact, cylindrical at infinity (or compact), and has at worst clean self-intersections. Additionally, we assume that all switching components of $\widetilde{L} \times_\iota \widetilde{L}$ are closed manifolds. A Lagrangian brane consists of such a Lagrangian immersion $\widetilde{L}$, together with a $GL(1, \mathbb{K})$-local system ∇, a spin structure (and induced orientation), and a grading $\alpha^\#\colon \widetilde{L} \to \mathbb{R}$. We will denote such an object by $(\widetilde{L}, \nabla)$. Sometimes we all also write this as L, which we have also used for the image of the immersion, in an abuse of notation.

To show that the moduli spaces of pseudoholomorphic disks on L are cut out transversely, we will impose a constraint on $\widetilde{L}$ which is similar to monotonicity. We will now recall the notion of index from [61] in order to state this assumption properly.

Definition 2.1.5. Let (V, ω) be a symplectic vector space with a compatible almost complex structure J, and let $\Lambda_0, \Lambda_1 \subset V$ be Lagrangian subspaces. Choose a path Λ_t, for $t \in [0, 1]$, of Lagrangian subspaces from Λ_0 to Λ_1 which satisfies the following:

- $\Lambda_0 \cap \Lambda_1 \subset \Lambda_t \subset \Lambda_0 + \Lambda_1$ for all t, and
- $\Lambda_t/(\Lambda_0 \cap \Lambda_1) \subset (\Lambda_0 + \Lambda_1)/(\Lambda_0 \cap \Lambda_1)$ is a path of positive-definite subspaces (with respect to the metric induced from ω and J) from $\Lambda_0/(\Lambda_0 \cap \Lambda_1)$ to $\Lambda_1/(\Lambda_0 \cap \Lambda_1)$.

Next consider a path α_t in $\mathbb{R}$ such that

$$\exp(2\pi i \alpha_t) = \det{}^2(\Lambda_t)$$

for all $t \in [0,1]$. Then we define the angle between Λ_0 and Λ_1 to be

$$\text{Angle}(\Lambda_0, \Lambda_1) = \alpha_1 - \alpha_0$$

This definition gives rise to the appropriate notion of index for switching components of a Lagrangian immersion.

Definition 2.1.6. Given a switching component L_{a_-} of $\widetilde{L} \times_\iota \widetilde{L}$, let L_{a_+} denote the corresponding switching component under the involution on A. Then for any points $p_- \in L_{a_-}$ and $p_+ \in L_{a_+}$, we define the index

$$\deg(p_-, p_+) \coloneqq n + \alpha^\#(p_+) - \alpha^\#(p_-) - 2 \cdot \text{Angle}(d\iota(T_{p_-}L), d\iota(T_{p_+}L)).$$

Since this is independent of $(p_-, p_+) \in L_{a_-} \times L_{a_+}$, we set

$$\deg(L_{a_-}) \coloneqq \deg(L_{a_-}, L_{a_+}) \coloneqq \deg(p_-, p_+).$$

The following definition records the positivity assumption that we will use to verify the wrapped A_∞-relations.

Definition 2.1.7 (Positivity)**.** Given an exact cylindrical Lagrangian immersion with clean self-intersections we say that it is positive if for any switching components L_{a_+} and L_{a_-} of $\widetilde{L} \times_\iota \widetilde{L}$ which are ordered such that the action satisfies

$$f_L(L_{a_+}) - f_L(L_{a_-}) > 0$$

one also has that

$$\deg(L_{a_-}) > 0.$$

For the rest of this section, we will assume that all Lagrangian immersions we consider satisfy this positivity condition. Let S be a Riemann surface with one boundary puncture, denoted ξ_0, whose compactification is a disk. Consider the space $\mathcal{J}(M)$ of ω-compatible almost complex structure on M which are of contact type when restricted to the cylindrical end $\partial M \times [1, \infty)$, meaning that $\lambda \circ J = dr$.

Definition 2.1.8. Let $a \in A \setminus \{0\}$ be a label for a switching component of $\iota\colon \widetilde{L} \to M$, and fix some $J \in \mathcal{J}(M)$. A pseudoholomorphic teardrop with corners on L consists of the data $(S, u, \widetilde{\partial u})$, where

(i) $u\colon S \to M$ is continuous and $u(\partial S) \subset L$.

(ii) $\widetilde{\partial u}\colon \partial S \to \widetilde{L}$ is a continuous map such that $\iota \circ \widetilde{\partial u} = u|_{\partial S}$ on ∂S.

(iii) The restriction of u to the interior of S is J-holomorphic, meaning that $(du)^{0,1} = 0$.

(iv) We have that
$$\left(\lim_{z \to \xi_0^-} \widetilde{\partial u}(z), \lim_{z \to \xi_0^+} \widetilde{\partial u}(z)\right) \in L_a.$$

Let $\mathcal{M}_1(L, a)$ denote the moduli space of J-holomorphic teardrops with a corner on L_a up to automorphisms of the domain.

By adapting the main results of [62] or [63] to our situation, one can prove that pseudoholomorphic teardrops are simple for generic almost complex structures.

Lemma 2.1.9. *Assume that* $\dim \widetilde{L} \geq 3$. *Then there is a second category subset of* $\mathcal{J}(M)$ *such that for any* J *in this subset, all* J*-holomorphic teardrops with boundary on L are simple.* □

The proof of this lemma in [63] is written for compact Lagrangians with transverse double-points, but it carries over to clean immersions in Liouville manifolds with no changes. Note that by our assumptions on M, all pseuodholomorphic teardrops are contained in the compact part M^{in} of M. We expect that the same result can be proven if $\dim \widetilde{L} \leq 2$, but this would require a more intricate combinatorial argument along the lines of [53]. It now follows from the standard Sard–Smale argument that we achieve regularity for teardrops.

Lemma 2.1.10. *If* $\dim \widetilde{L} \geq 3$*, then for generic* $J \in \mathcal{J}(M)$*, the moduli spaces* $\mathcal{M}_1(L, a)$*, defined with respect to the domain-independent almost complex structure* J*, are smooth, compact, oriented manifolds.* □

The proof that these moduli spaces are oriented uses the orientation local systems on the switching components and the spin structure on $\widetilde{L}$, as discussed in [47] or [43]. The moduli spaces $\mathcal{M}_1(L, a)$ admit natural evaluation maps

$$\mathrm{ev}\colon \mathcal{M}_1(L, a) \to L_a$$

which has the value $\left(\lim_{z\to\xi_0^-} \widetilde{\partial u}(z), \lim_{z\to\xi_0^+} \widetilde{\partial u}(z)\right)$ at $(u, \widetilde{\partial u})$. The spaces $\mathcal{M}_1(L, a)$ have dimension $\deg(L_a) - 2 + \dim(L_a)$, so when $\deg(L_a) = 2$ we can push the fundamental class forward to obtain

$$\mathrm{ev}_*[\mathcal{M}_1(L, a)] \in H_*(L_a)$$

of top dimension. Thus by Poincaré duality we can interpret this as the count of pseudoholomorphic teardrops with a corner on L_a. This justifies the following definition.

Definition 2.1.11. Let $(\widetilde{L}, \nabla)$ be a pair consisting of a Lagrangian immersion and rank-one local system. For any relative homology class $\beta \in H_2(M, L; \mathbb{Z})$, write $\mathcal{M}(L, a; \beta)$ for the connected component of $\mathcal{M}(L, a)$ consisting of all teardrops representing the class β. Then define the curvature term

$$\mathfrak{m}_0(\widetilde{L}, \nabla) := \sum_{a\in A\setminus\{0\}} \sum_{\beta\in H_2(M,L;\mathbb{Z})} \mathrm{hol}_\nabla(\partial\beta) PD(\mathrm{ev}_*[\mathcal{M}_1(L, a, \beta)]) \in \mathbb{K}.$$

Additionally, we say that the Lagrangian brane $(\widetilde{L}, \nabla)$ is unobstructed if $\mathfrak{m}_0(\widetilde{L}, \nabla) = 0$.

2.1.3 The wrapped Fukaya category

Consider a finite collection $\mathrm{Ob}(\mathcal{W})$ of unobstructed Lagrangian branes $(\widetilde{L}, \nabla)$ in M. Here $\widetilde{L} \to M$ is an exact Lagrangian immersion which is cylindrical at infinity and has clean self-intersections, and ∇ is a rank one local system. Each brane in $\mathrm{Ob}(\mathcal{W})$ is also equipped with a grading and spin structure. If $\dim \widetilde{L} \leq 2$, we also assume that $L = \iota(\widetilde{L})$ can be written as a union of embedded Lagrangians in M intersecting each other cleanly. Apart from the discussion of curvature, this subsection follows [64] closely. Since proofs of most results in this section are standard, we will omit many details, emphasizing the new issues that appear in the presence of Lagrangian immersions.

Let $\mathcal{H}(M)$ denote the set of all Hamiltonians which are of the form

$$H(r, y) = r^2$$

away from a compact subset of M. We choose to define the wrapped Fukaya category using Hamiltonians quadratic at infinity so that we can appeal to the main result of [65] in our discussion of wrapped Floer cohomology in products of Liouville manifolds. Fix some $H \in \mathcal{H}(M)$. Let X denote the Hamiltonian flow of H and, for a pair $L_0, L_1 \in \mathrm{Ob}(\mathcal{W})$, let $\mathcal{X}(L_0, L_1)$ denote the set of time-one flow lines of X from L_0 to L_1, which as usual denote the images of the immersions. To define a graded cochain complex, we should assume that:

Assumption 2.1.12. All chords in $\mathcal{X}(L_0, L_1)$ are nondegenerate.

Let $\deg(x)$ denote the Maslov index of the chord $x \in \mathcal{X}(L_0, L_1)$.

Definition 2.1.13. Let $Z := (-\infty, \infty) \times [0, 1]$ with coordinates (s, t). Given $x_0, x_1 \in \mathcal{X}(L_0, L_1)$, define the spaces $\widehat{\mathcal{M}}(x_0, x_1)$ to be the set of all maps $u \colon Z \to M$ which converge exponentially to x_0 at the negative end and x_1 at the positive end, satisfying the boundary conditions

$$u(s, 0) \in L_0$$
$$u(s, 1) \in L_1$$

and which satisfy Floer's equation

$$(du - X \otimes dt)^{0,1} = 0,$$

with respect to a generic family $\{J_t\}_{t\in[0,1]}$ of domain-dependent almost complex structures of contact type.

Let J denote an almost complex structure satisfying the conclusion of Lemma 2.1.10 for every Lagrangian brane in $\mathrm{Ob}(\mathcal{W})$.

Lemma 2.1.14. *For a generic choice of $\{J_t\}$ with the property that $J_0 = J_1 = J$, the moduli space $\widehat{\mathcal{M}}(x_0, x_1)$ is a smooth manifold of dimension* $\deg(x_0) - \deg(x_1)$. *Whenever* $\deg(x_0) - \deg(x_1) > 0$, *the natural $\mathbb{R}$-action induced by translations in the s-direction is smooth and free.* □

Define $\mathcal{M}(x_0, x_1)$ to be $\widehat{\mathcal{M}}(x_0, x_1)/\mathbb{R}$ whenever $\deg(x_0) - \deg(x_1) > 0$, and set $\mathcal{M}(x_0, x_1) = \emptyset$ if not.

Having fixed perturbation data for inhomogeneous strips, we can now explain how to choose perturbation data for disks with more boundary punctures consistently with the choices made thus far. In the following definition, let $Z_\pm$ denote the positive and negative half-strips of Z, respectively. To discuss disks with corners on a Lagrangian immersion, we use some terminology from [61] for boundary punctures.

Definition 2.1.15. Let $d \geq 1$ and $m \geq 0$ be integers such that $d + m \geq 2$, and let S be a stable disk with one negative strip-like end, d positive strip-like ends which are said to be of Type I, and m positive strip-like ends which are said to be of Type II. Let $\zeta_0, \zeta_1, \ldots, \zeta_d$ denote the Type I marked points and $\eta_1, \ldots, \eta_m$ denote the Type II marked points. We assume that both sets of marked points are ordered cyclically on the boundary of S. A perturbation datum for S consists of:

(i) Strip-like ends $\epsilon_i^I \colon Z_+ \to S$ and $\epsilon_j^{II} \colon Z_+$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$, and $\epsilon_0^I \colon Z_- \to S$ near the corresponding boundary punctures.

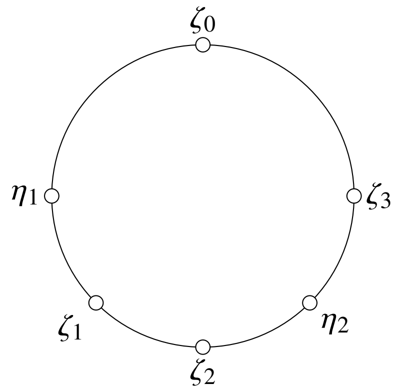

Figure 2.1: A stable disk with 3 positive punctures of Type I and 2 negative punctures of Type II.

(ii) A time-shifting function $\rho_S \colon \partial S \to [1, \infty)$ which takes the value 1 away from the Type I marked points, and is constant of value $w_{i,S}$ near the type I marked points for $i = 0, 1, \ldots, d$.

(iii) A one-form α_S which vanishes along ∂S and near the Type II marked points, and domain-dependent Hamiltonian $H_S \colon S \to \mathcal{H}(M)$ defining a Hamiltonian vector field X_S such that $X_S \otimes \alpha_S$ pulls back to $X_{H/w_{i,S} \circ \phi^{w_{i,S}}} \otimes dt$ in each strip-like end near the Type I marked points.

(iv) A domain-dependent almost-complex structure which agrees with J along the boundary, and pulls back to J_t in the strip-like ends near the Type I marked points, and pulls back to J in the ends near the Type II marked points.

Our definition of the wrapped A_∞-operations will count disks whose domains only have Type I marked points. Let $(\widetilde{L}_0, \nabla_0), \ldots, (\widetilde{L}_d, \nabla_d)$, where $d \geq 2$, be a sequence of Lagrangian branes in $\mathrm{Ob}(\mathcal{W})$. Consider a sequence of Hamiltonian chords $x_i \in \mathcal{X}(L_{i-1}, L_i)$ for all $i = 1, \ldots, d$, and a chord $x_0 \in \mathcal{X}(L_0, L_d)$. Let S be a stable disk with d positive ends of Type I and one negative end (but with no ends of Type II). The moduli spaces $\mathcal{M}_{d+1}(x_0, x_1, \ldots, x_d)$ consist of all maps $u \colon S \to M$ satisfying the boundary conditions

$$\begin{cases} u(z) \in \phi^{\rho_S(z)} L_i & \text{for } z \text{ between } \zeta_i \text{ and } \zeta_{i+1} \\ u(z) \in \phi^{\rho_S(z)} L_d & \text{for } z \text{ between } \zeta_d \text{ and } \zeta_0 \\ \lim_{s \to \pm\infty} u \circ \epsilon_i^I(s, \cdot) = \phi^{\rho_S(z)} x_i & \end{cases}$$

and satisfying the inhomogeneous Cauchy–Riemann equation

$$(du - X_S \otimes \alpha_S)^{0,1}.$$

By the same argument as in [64], we can choose perturbation data appropriately.

Lemma 2.1.16. *There is a universal and conformally consistent, in the sense of [64], choice of perturbation data. This choice of data has the property that all of the moduli spaces $\mathcal{M}(x_0, x_1, \dots, x_d)$ defined above are smooth oriented manifolds of the expected dimension.* □

Definition 2.1.17. For each pair of unobstructed Lagrangian branes $(\widetilde{L}_0, \nabla_0)$ and $(\widetilde{L}_1, \nabla_1)$ which are cylindrical at infinity, we define the wrapped Floer cochain complex to be

$$CW^*(L_0, L_1; H; J_t) := \bigoplus_{x \in \mathcal{X}(L_0, L_1)} \mathbb{K}\langle x \rangle$$

as a graded vector space, where the grading of the $\mathbb{K}\langle x \rangle$ summand is $\deg(x)$.

Let $\beta \in H_2(M, L_0 \cup L_1)$, and let $\mathcal{M}(x_0, x_1; \beta)$ denote the elements of $\mathcal{M}(x_0, x_1)$ in this relative homology class. We define the differential

$$\begin{aligned} \mathfrak{m}_1 \colon CW^*(L_0, L_1) &\to CW^*(L_0, L_1) \\ \mathfrak{m}_1(x_0) &= (-1)^{\deg(x_0)} \sum_{\substack{\deg(x_0) - \deg(x_1) = 1, \\ u \in \mathcal{M}(x_0, x_1)}} \mathrm{hol}(\partial u) \mathrm{sgn}(u) x_1 \end{aligned}$$

where the holonomy $\mathrm{hol}(\partial u) \in \mathbb{K}^*$ is taken with respect to the flat connections on L_0 and L_1 and the sign $\mathrm{sgn}(u) \in \{\pm 1\}$ is determined by the orientation of the moduli space. Similarly, we define the higher compositions

$$\mathfrak{m}_d \colon CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_d)$$

by

$$\mathfrak{m}_d(x_d \otimes \cdots \otimes x_1) = (-1)^{\sum i \deg(x_i)} \sum_{\substack{\deg(x_0)=2-d+\sum \deg(x_i) \\ u \in \mathcal{M}(x_0,\ldots,x_d)}} \mathrm{hol}(\partial u)\mathrm{sgn}(u) x_0,$$

where $\mathrm{hol}(\partial u)$ and $\mathrm{sgn}(u)$ are defined as before. Note that there is a canonical isomorphism

$$CW^*(L_i, L_j; H, J_t) \cong CW^*\left(\phi^\rho L_i, \phi^\rho L_j; \frac{H}{\rho} \circ \phi^\rho, (\phi^\rho)^* J_t\right)$$

which we use to ensure that $\mathfrak{m}_d$ has the correct domain and codomain. For more details on this point, see [64].

We must now discuss the Gromov compactifications of the moduli spaces considered above, and show that the A_∞-relations hold.

Definition 2.1.18. Let S be a disk with $d \geq 1$ positive ends of Type I and $m \geq 0$ positive ends of Type II. A disk with corners on a sequence of Lagrangian immersions $\iota_i \colon \widetilde{L}_i \to M$ for $i = 0, 1, \ldots, d$ is a tuple $(S, u, \widetilde{\partial u}, \gamma)$, where

(i) $u \colon S \to M$ is continuous.

(ii) $u(z) \in \phi^{\rho_S(z)} L_i$ for all z between ζ_i and ζ_{i+1} and $u(z) \in \phi^{\rho_S(z)} L_d$ for all z between ζ_d and ζ_0.

(iii) $\widetilde{\partial u} \colon \partial S \to \coprod_{i=0}^d \widetilde{L}_i$ is a continuous map whose composition with the Lagrangian immersion $\coprod_{i=0}^d \iota_i$ coincides with the restriction of u to ∂S.

(iv) The restriction of u to the interior of S satisfies the inhomogeneous Cauchy–Riemann equation $(du - X_S \otimes \alpha_S)^{0,1}$, which is defined with respect to the perturbation data already chosen.

(v) γ is a function whose domain is a subset $I \subset \{1, \ldots, m\}$, and whose value at j is a switching component $L_{i,\gamma(j)}$ of the immersion $\widetilde{L}_i \to M$, where the marked point η_j is between ζ_i and ζ_{i+1} (or it is a switching component of L_d if η_j is between ζ_d and ζ_0).

(vi) Near the Type I ends, u converges exponentially to $\phi^{\rho s(z)} x_i(\cdot)$, where $x_i \in \mathcal{X}(L_i, L_{i+1})$ for $i = 1, \ldots, d$ and $x_0 \in \mathcal{X}(L_0, L_d)$.

(vii) Near the Type II ends, we have that

$$\left(\lim_{z \to \eta_j^-} \widetilde{\partial u}(z), \lim_{z \to \eta_j^+} \widetilde{\partial u}(z) \right) \in L_{i,\gamma(j)}$$

if $j \in I$, and if $j \notin I$ then u extends smoothly over the puncture.

We write $\mathcal{M}(x_0, \ldots, x_d, \gamma)$ for the moduli space of all such stable disks. In the special case where $d = 1$ and $m = 0$, this definition reduces to the definition of inhomogenous strips from before, otherwise all such disks are stable.

These moduli spaces admit natural evaluation maps at each Type II marked point.

Lemma 2.1.19. *The moduli spaces* $\mathcal{M}(x_0, x_1, \ldots, x_d)$ *admit Gromov compactifications denoted* $\overline{\mathcal{M}}(x_0, x_1, \ldots, x_d)$ *which are smooth manifolds with corners. Elements of the compactified moduli space can be written as*

$$(T, \{u_i, F_i, v_1^i, \ldots, v_{m_i}^i\}_{i \in \mathrm{Vert}(T)})$$

where

(i) T *is a tree with vertex set* $\mathrm{Vert}(T)$ *and a distinguished root*

(ii) $u_i \in \mathcal{M}(x_0^i, \ldots, x_{d_i}^i; \gamma_i)$

(iii) $F_i \subset \{1, \ldots, d_i\}$ *for nonroot vertices, and* $F_r \subset \{0, \ldots, d_r\}$ *for the root vertex* $r \in \mathrm{Vert}(T)$*. The sets* F_i *label the chords which are among the original chords* $x_0, x_1, \ldots, x_d$*.*

(iv) $v_1, \ldots, v_{m_i}$ *are elements of moduli spaces* $\mathcal{M}_1(L, \gamma_i(j))$*.*

□

These data should satisfy additional constraints as detailed in [61] which say that the components u_i and $v_1^i, \ldots, v_{m_i}^i$ can be glued along their domains. The proof of this lemma is similar to the proof of Gromov compactness in [66], and hence we omit it. For a discussion of the exact case, see [61].

Lemma 2.1.20. *The operation $\mathfrak{m}_d$ defined above satisfy the A_∞-relations*

$$\sum_{\substack{d_1+d_2=d+1\\0\leq k<d_1}} (-1)^{\star_k}\mathfrak{m}_{d_1}(x_d,\ldots,x_{k+d_2+1},\mathfrak{m}_{d_2}(x_{k+d_2},\ldots,x_{k+1}),\ldots,x_k,\ldots,x_1)=0$$

where the sign is determined by setting $\star_k = \sum_{i=1}^{k=1}(\deg(x_i)+1)$.

Proof. As usual, we need to describe the boundaries of the 1-dimensional moduli spaces $\overline{\mathcal{M}}(x_0,\ldots,x_d)$. We will adapt the argument of [61] to use our weaker positivity assumption. So suppose that we are given an element

$$(T,\{u_i,F_i,v_1^i,\ldots,v_{m_i}^i\}_{i\in\mathrm{Vert}(T)})$$

in the boundary. Each of the u_i's must belong to a moduli space of nonnegative dimension, whence

$$\deg(x_{i,0})-\sum_{j=1}^{d_i}\deg(x_{i,j})-\sum_{j=1}^{m_i}\deg(\gamma_i(j))+d_i+m_i-2\geq 0.$$

Applying Definition 2.1.7, we can rewrite this inequality to obtain

$$\deg(x_{i,0})-\sum_{j=1}^{d_i}\deg(x_{i,j})\geq 2-d_i.$$

Now suppose that $x_0,\ldots,x_d$ satisfy $\deg(x_0)=2-d+\sum\deg(x_j)$. Then for any element in the boundary of the Gromov compactification $\overline{\mathcal{M}}(x_0,\ldots,x_d)$, we get the inequality

$$2-d=\deg(x_0)-\sum_{j=1}^{d}\deg(x_j)\geq 2|\mathrm{Vert}(T)|-\sum_{i=1}^{|\mathrm{Vert}(T)|}d_i.$$

Using that $|\mathrm{Vert}(T)|-1+d=\sum_{i=1}^{|\mathrm{Vert}(T)|}d_i$, we obtain

$$2-d\geq|\mathrm{Vert}(T)|+1-d$$

or equivalently

$$1 \geq |\mathrm{Vert}(T)|.$$

This implies that any element of $\overline{\mathcal{M}}(x_0, \ldots, x_d)$ consists of a single disk, and by Definition 2.1.7, it cannot have any teardrops attached, since they would need to have nonnegative index. In other words, $\mathcal{M}(x_0, \ldots, x_d)$ is compact, so the A_∞-operations are well-defined.

To check that the A_∞-relations are satisfied, we argue similarly. If $x_0. \ldots, x_d$ are such that $\mathcal{M}(x_0, \ldots, x_d$ is one-dimensional, then a similar calculation for elements of the compactified moduli space shows that

$$2 \geq |\mathrm{Vert}(T)|.$$

Hence elements of the compactified moduli space can have at most two components that are not teardrops. If there were any elements in the compactified moduli space with teardrop components $v_1^1, \ldots, v_{m_1}^1, v_1^2, \ldots, v_{m_2}^2$, then by regularity, they must all have index at least 2. In particular, there can be at most one teardrop component, say v^i. But because we have assumed that all Lagrangians under consideration are unobstructed, meaning that the count of index 2 teardrops $\mathfrak{m}_0(\widetilde{L}, \nabla)$ vanishes, a gluing argument shows that the corresponding components of the boundary $\overline{\mathcal{M}}(x_0, \ldots, x_d)$, when accounting for orientations and holonomy, must cancel. The only boundary components unaccounted for so far have $|\mathrm{Vert}(T)| = 2$ and no teardrop components. These are precisely the boundary components corresponding to the quadratic terms in the A_∞-relations. □

Definition 2.1.21. Given the choice of object $\mathrm{Ob}(\mathcal{W})$ from before, we define the wrapped Fukaya category to be the A_∞-category with this set of objects and composition maps $\{\mathfrak{m}_d\}_{d=1}^\infty$.

Using the same argument as in [65], we can construct a cohomology unit on the A_∞-algebra $HW^*(L, L)$.

Lemma 2.1.22. *There is an element $e_L \in HW^*(L, L)$ which is a unit for the composition map $\mathfrak{m}_2$, which can be represented by a Hamiltonian chord in $\mathcal{X}(L, L)$ for an appropriate choice of H.* □

This unit can be thought of as the image of the unit in $H^*(L)$ under a PSS map. More specifically, consider a disk $S = D^2 \setminus \{-1\}$ with one negative boundary puncture. Fix a strip-like end

$\epsilon_-\colon Z_- \to S$ near this boundary puncture. Choose a domain-dependent Hamiltonian H_S on S which agrees with H over the strip-like end, a domain-dependent almost complex structure J_S which agrees with J near the boundary and J_t over the strip-like end, and a closed one-form α_S which vanishes along ∂S. For any $x \in \mathcal{X}(L, L)$, consider the moduli spaces $\mathcal{M}(x)$ consisting of maps $u\colon S \to M$ satisfying

(i) $(du - X_{H_S} \otimes \alpha_S)^{0,1} = 0$

(ii) $u(\partial S) \subset L$

(iii) $\lim_{s\to-\infty} u \circ \epsilon_-(s, \cdot) = x(\cdot)$.

There is an element of $CF^0(L, L)$ defined by counting elements of the 0-dimensional moduli spaces $\mathcal{M}(x)$ with signs. Examining the compactifications of the one-dimensional moduli spaces $\mathcal{M}(x)$, and using positivity and unobstructedness, we see that the element constructed this way is closed.

The proof that this element is a cohomological unit works by gluing the elements of these 0-dimensional moduli spaces to triangles which contribute to the product on $CF^*(L, L)$, and identifying the resulting strips with the ones used to define continuation maps. This step is not different from the embedded case [65]. The existence of cohomological units means that we can use the standard arguments to show that our construction of the wrapped Fukaya category is independent of the choices used to construct it.

2.2 Background on SYZ mirror symmetry

In this section, we will review the setup of SYZ mirror symmetry for Lagrangian torus fibrations without singularities, mainly for the purpose of fixing notation. We will then collect some definitions from tropical geometry and review the construction of the Lagrangian pair-of-pants following [28], ending with a computation of its Floer-theoretic support.

2.2.1 Lagrangian torus fibrations

Let Q be an n-dimensional integral affine space, meaning that it comes with a choice of full-rank lattice. Then there is a local system $T^*_{\mathbb{Z}}Q$ of integral 1-forms. Given this data, there is an associated symplectic manifold $X := T^*Q/T^*_{\mathbb{Z}}Q$ which admits a Lagrangian T^n-fibration $\pi\colon X \to Q$ induced by the bundle projection $T^*Q \to Q$. Notice that we can identify $\check{X}$ with T^*T^n, and that under this identification $\pi\colon T^*T^n \to Q$ is the projection map to the cotangent fiber.

If $W \subset Q$ is an integral affine subspace, then one can construct a Lagrangian $L_W := N^*W/N^*_{\mathbb{Z}}W \subset X$ called the periodized conormal bundle with the property that $\pi(L_W) = W$. If $\dim W = k$, then L_W is a T^{n-k}-bundle over W. In particular, the Lagrangian lift of a point $q \in Q$ is just $L_q = \pi^{-1}(q)$.

There is a dual complex manifold $\check{X} := TQ/T_{\mathbb{Z}}Q$, were $T_{\mathbb{Z}}Q$ is the local system of integral tangent vectors. The complex structure arises by identifying TQ with $\mathbb{C}^n$ via

$$
\begin{aligned}
TQ &\to \mathbb{C}^n \\
\left(x, \sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j}\right) &\mapsto (z_j)_{j=1,\dots,n} = (x_j + i y_j)_{j=1,\dots,n}
\end{aligned}
$$

The complex structure on TQ descends to one on $\check{X}$ which respect to which the fibers $\check{\pi}(q)$ are totally real tori. Notice that $\check{X}$ comes with a natural map $\check{\pi}\colon \check{X} \to Q$ induced by the tangent bundle projection. Under the identification $TQ \cong \mathbb{C}^n$, we see that X is identified with $(\mathbb{C}^*)^n$, and the map $\check{\pi}$ coincides with $\mathrm{Log}\colon (\mathbb{C}^*)^n$, which is given by $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$.

To an integral affine subspace $W \subset Q$, there is an associated complex submanifold $C_W := TC/T_{\mathbb{Z}}C \subset \check{X}$ such that $\check{\pi}(C_W) = W$. This construction associates, to a point $q \in Q$, a point $\check{X}$, namely the zero vector in T_qQ. This is not the only complex submanifold of $\check{X}$ mapped to q under $\check{\pi}$: any other point in $\check{\pi}^{-1}(q)$ has this property.

On the other hand, $q \in Q$ only has one Lagrangian lift in X. To obtain a family of A-model objects corresponding to the points of $\check{\pi}^{-1}(q)$, we can equip L_q wit local systems. There is a bijection between pairs (L_q, ∇), where ∇ is a $U(1)$-local system on L_q, and points $z \in \check{X}$. In particular, $\check{\pi}^{-1}(q)$ is identified with $\operatorname{Hom}(\pi_1(L_q), U(1))$, so we can think of $\check{\pi}^{-1}(q)$ as the dual torus to L_q. Similarly, equipping $L_W \subset X$ with $U(1)$-local systems corresponds to taking different almost complex lifts of W in $\check{X}$.

We will need to consider exact Lagrangians equipped with local systems over a field $\mathbb{K}$ of arbitrary characteristic. With respect to the standard choice of primitive, the zero section $T^n \subset T^*T^n$ is exact. The space of $GL(1)$-local systems on T^n is $\operatorname{Hom}(\pi_1(T^n), \mathbb{K}^*) \cong (\mathbb{K}^*)^n$, which can be thought of as a copy of the mirror space $(\mathbb{K}^*)^n$. We can make this more precise as follows.

Consider the wrapped Fukaya category $\mathcal{W}(T^*T^n)$ whose objects consist of exact Lagrangians which are cylindrical at infinity equipped with $GL(1)$-local systems. In this setting, we can appeal to homological mirror symmetry.

Theorem 2.2.1. *There is an equivalence of derived categories*

$$\mathcal{F}\colon D^\pi\mathcal{W}(T^*T^n) \to D^b \operatorname{Coh}((\mathbb{K}^*)^n)$$

where the wrapped Fukaya category has coefficients in $\mathbb{K}$. □

To prove this, one checks that $\mathcal{W}(T^*T^n)$ is generated by a cotangent fiber L_Q, and that there are algebra isomorphisms

$$HW^*(L_Q, L_Q) \cong \mathbb{K}[t, t^{-1}] \cong \operatorname{Hom}_{D^b \operatorname{Coh}}(\mathcal{O}_{(\mathbb{K}^*)^n}, \mathcal{O}_{(\mathbb{K}^*)^n}).$$

Because $\mathcal{O}_{(\mathbb{K}^*)^n}$ generates the dg-enhancement of $D^b \operatorname{Coh}(\mathbb{K}^*)^n)$, the theorem follows. Under this equivalence, objects (T^n, ∇) of $\mathcal{W}(T^*T^n)$, where ∇ is a $GL(1)$-local system on T^n, correspond to skyscraper sheaves over points in $(\mathbb{K}^*)^n$. Given this, we can attempt to match objects of $D^\pi\mathcal{W}(T^*T^n)$ supported on L_W to their images under the mirror functor. We remark that L_W will only be an exact Lagrangian if $W \subset \mathbb{R}^n$ is a *linear* subspace.

If $W \subset Q \cong \mathbb{R}^n$ is the k-dimensional linear subspace spanned by the last k coordinates, then its Lagrangian lift L_W is the conormal bundle N^*T^{n-k}, where T^{n-k} is the subtorus of T^n supported on the first $n-k$ coordinates. The proof of the next lemma follows [34].

Lemma 2.2.2. *Let ∇ be a $GL(1)$ local system on N^*T^{n-k} with holonomy α_j about the circle in the θ_j-direction for $1 \le j \le n-k$. Then the mirror sheaf $\mathcal{F}(N^*T^{n-k}, \nabla)$ on $(\mathbb{K}^*)^n$ is supported on the subvariety $\{z_i = \alpha_i | 1 \le i \le n-k\}$ which is isomorphic to $(\mathbb{K}^*)^k$.*

Proof. Our strategy will be to find all local systems $\nabla_p \in \mathrm{Hom}(\pi_1(T^n), \mathbb{K}^*)$ for which

$$HW^*((N^*T^{n-k}, \nabla), (T^n, \nabla_p)) \neq 0.$$

Since $\mathcal{F}$ takes Floer cohmology algebras to Ext-algebras, this characterizes the support of the mirror sheaf.

For a small positive real constant a, consider the Hamiltonian

$$H = \sum_{i=1}^{n-k} a\cos(\pi\theta_i)$$

on T^*T^n, and let ϕ denote its time-one Hamiltonioan flow. Then the intersection of $\phi(N^*T^{n-k})$ with T^n is

$$\phi(N^*T^{n-k}) \cap T^n = \{(\epsilon_1, \ldots, \epsilon_{n-k}, 0, \ldots, 0) \mid \epsilon_i \in \{0,1\}\}$$

and the index of each intersection point is $\sum_{i=1}^{n-k} \epsilon_i$. We index these intersection points x_I by subsets $I \subset \{1, \ldots, n-k\}$. We also write $x_I <_1 x_J$ whenever $J = I \cup \{i\}$ for some $i \in \{1, \ldots, n-k\}$.

As a vector space, $CW^*(\phi(N^*T^{n-k}), T^n)$ coincides with the Morse complex $CM^*(T^{n-k})$ defined using the Morse function H restricted to T^{n-k}. In particular, it is the exterior algebra of an $(n-k)$-dimensional vector space. The holomorphic strips connecting x_I to x_J correspond to Morse flowlines between critical points of H. When $x_I <_1 x_J$, there are exactly two holomorphic strips, which we denote by u_{IJ}^+ and u_{IJ}^-, by the sign they appear with in the Floer differential.

Let ∇_p be a local system on T^n with holonomy z_i in the θ_i-direction. Suppose that $I <_1 J$ differ only in the ith factor. We can normalize ∇ and ∇_p such that the intersection of u_{IJ}^+ with N^*T^{n-k} is a path of holonomy 1 and its intersection with T^n is a path of holonomy z_i. Hence u_{IJ}^- should intersect N^*T^{n-k} in a path with holonomy α_i^{-1} and T^n in a path of holonomy 1. Therefore the Floer differential is characterized as follows.

$$\langle d(x_I), x_J\rangle = \begin{cases} z_i - \alpha_i & I <_1 J \\ 0 & I \not<_1 J \end{cases}$$

We now see that the Floer homology group will vanish unless $z_i = \alpha_i$ for all $i \in \{1, \dots, n-k\}$. $\square$

2.2.2 The Lagrangian pair of pants

To achieve a geometric description of mirror symmetry in full generality, one should consider tropical subvarieties of $Q \cong \mathbb{R}^n$ rather than just affine subspaces.

A tropical curve $W \subset Q$ is a collection of 1-dimensional rational convex polyhedral domains $\{W_s \subset Q\}$ and weights $\{w_s \in \mathbb{Z}_{>0}\}$ which are required to satisfy the following conditions.

(i) The intersection $W_s \cap W_t$ is either empty or a boundary point of both W_s and W_t.

(ii) At each boundary point $v \in W_s$, let $u_s \in T_{\mathbb{Z},v}Q$ denote the primitive integral vector tangent to W_s at v. We require that

$$\sum_{\{s \mid v \in W_s\}} w_s u_s = 0.$$

Example 2.2.3. Consider the following rays in $\mathbb{R}^2$.

$$W_1 = \{(-t, 0) \mid t \in \mathbb{R}_{\geq 0}\}$$

$$W_2 = \{(0, -t) \mid t \in \mathbb{R}_{\geq 0}\}$$

$$W_3 = \{(t, t) \mid t \in \mathbb{R}_{\geq 0}\}$$

Their union W is a tropical curve which we call the tropical pair of pants.

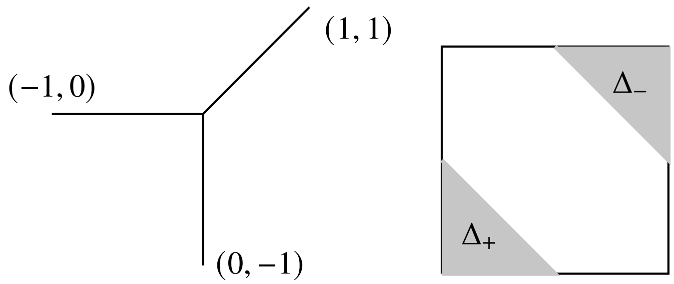

Figure 2.2: The tropical pair of pants (left) and its coamoeba (right).

On the B-side, this tropical curve arises from the standard pair of pants

$$C = \{1 + z_1 + z_2 = 0\}$$

as the limit of the amoebae $\mathrm{Log}_t(C) \subset \mathbb{R}^n$ as $t \to \infty$. There is a corresponding Lagrangian submanifold in T^*T^n. In the following, let $\pi_{SYZ}\colon T^*T^n \to Q$ denote projection to the cotangent fiber.

Theorem 2.2.4 ([28], [26])**.** *For any $\epsilon > 0$, there is a Lagrangian pair of pants $L_{\text{pants}} \subset T^*T^n$ which agrees with the conormal lifts of the cones W_i when restricted to the subset $\pi_{SYZ}^{-1}(\mathbb{R}^2 \setminus B_\epsilon(0))$.*

Up to Hamiltonian isotopy, this Lagrangian can be constructed by applying a hyperKähler rotation to C [25], or as a mapping cone in the Fukaya category corresponding to a mapping cone of vector bundles in $D^b\mathrm{Coh}((\mathbb{C}^*)^2)$ defining $\mathcal{O}_C$ [26]. We will briefly review Matessi's construction of the Lagrangian pair of pants, since this is the method of construction we will generalize later, and because the embedding $L_{\text{pants}} \hookrightarrow T^*T^2$ can be made particularly explicit in this setup. This construction starts by considering the expected image of the projection $L_{\text{pants}} \to T^2$ to the SYZ fiber.

Definition 2.2.5. Identifying $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ with the usual quotient of the unit cube $[0, 1]^2$. Consider the simplices $\overline{\Delta}_\pm \subset [0, 1]^2$ defined as

$$\overline{\Delta}_+ = \left\{(\theta_1, \theta_2) \in [0, 1]^2 \mid \theta_1 + \theta_2 \leq \frac{1}{2}\right\}$$

$$\overline{\Delta}_- = \left\{(\theta_1, \theta_2) \in [0,1]^2 \mid \theta_1 + \theta_2 \geq \frac{3}{2}\right\}$$

Let $\Delta_\pm$ denote the union of the 0-cells and 2-cells of $\overline{\Delta}_\pm$. Then the coamoeba Δ of W is the union of the images of $\Delta_\pm$ in T^2.

The Lagrangian L_{pants} can be thought of as the closure of the graph of an exact 1-form on the interior of Δ. To make this more precise, we will replace the coamoeba with a smooth pair of pants before attempting to define a Lagrangian embedding.

The real blowup $\widetilde{\Delta}$ of Δ is the space obtained by blowing up Δ at each of its vertices. More precisely, if θ_0 is a vertex of Δ and $U_{\theta_0} \subset \Delta$ is a small neighborhood of θ_0, we can form

$$\widetilde{U}_{\theta_0} = \{(\theta, \ell) \in U_{\theta_0} \times \mathbb{RP}^1 \mid \theta - \theta_0 \in \ell\},$$

which comes with a natural projection map $\widetilde{U_{\theta_0}} \to U_{\theta_0}$. We construct $\widetilde{\Delta}^n$ by gluing $\widetilde{U_{\theta_0}}$ to $\mathrm{Int}(\Delta^n)$, which is just the complement of the vertices in Δ^n. There is an obvious projection $\pi \colon \widetilde{\Delta}^n \to \Delta^n$.

Consider the function $g : \Delta \to \mathbb{R}$ defined by

$$g(\theta) = \begin{cases} -\sqrt{\cos(\theta_1 + \theta_2)\sin(\theta_1)\sin(\theta_2)}, & \theta \in \Delta_+ \\ \sqrt{\cos(\theta_1 + \theta_2)\sin(\theta_1)\cdots\sin(\theta_2)}, & \theta \in \Delta_-. \end{cases}$$

The following Lemma is proven by Matessi [28].

Lemma 2.2.6. *The function* $dg = \left(\frac{\partial g}{\partial \theta_1}, \frac{\partial g}{\partial \theta_2}\right)$ *extends to a smooth map* $\widetilde{dg} \colon \widetilde{\Delta} \to \mathbb{R}^2$.

Matessi's Lagrangian embedding is $\Phi \colon \widetilde{\Delta} \to T^*T^2 = T^2 \times \mathbb{R}^2$ defined by $\Phi(\theta) = (\pi(\theta), \widetilde{dg}(\theta))$.

Since $\Phi(\widetilde{\Delta})$ is exact and approaches the conormals to the top-dimensional cones of W asymptotically, one can deform it via Hamiltonian isotopy to lie in tropical position. This means that it coincides with the conormal lifts to the top-dimensional cones of W_n away from the $(n-2)$-skeleton of W_n. Since the vertex of W_n is placed at the origin, this means that W_n becomes an object of $\mathcal{W}(T^*T^n)$ when equipped with a $GL(1)$-local system. We can now compute the support of the mirror sheaf using the same strategy as in Lemma 2.2.2.

Observe that by construction L_{pants} is homotopy equivalent to $\widetilde{\Delta}$. Hence we can identify a set of generators for $H_1(L_{\text{pants}})$ with the generators $[\theta_1], \ldots, [\theta_n]$ of $H_1(T^n)$. Th compute Floer homology, one should also equip L_{pants} with a spin structure, but we leave this unspecified for now.

Lemma 2.2.7. *Suppose that L_{pants} is equipped with a choice of spin structure. Let ∇ be a $GL(1)$-local system on L_{pants} with holonomy ρ_i along $[\theta_i]$. If $\nabla_z \in \mathrm{Hom}(\pi_1(T^2), \mathbb{K}^*)$ has holonomy z_i along $[\theta_i]$, then $HW^*((L_{\text{pants}}, \nabla), (T^2, \nabla_z))$ is nonzero, in which case it is isomorphic to $H^*(S^1)$ as a graded vector space, if and only if*

$$\pm\rho_1^{-1} z_1 \pm \rho_2^{-1} z_2 \pm 1 = 0. \tag{2.2.1}$$

Proof. We identify T^*T^2 with $(\mathbb{C}^*)^2$ symplectically via $(q_i, \theta_i) \mapsto z_i := (\log|q_i|, \theta_i)$, and we write $r_i := \log|q_i|$. From the formula for g it is easy to see that L_{pants} intersects T^2 transversely in two points corresponding to the barycenters of Δ_+ and Δ_-. Call these two intersection points x_+ and x_-. There are 3 obvious holomorphic strips connecting x_+ to x_-, which are obtained as portions of holomorphic cylinders lying over lines in Q through the origin in the directions of $-r_1 - r_2$, $-r_1 + 2r_2$, and $2r_1 - r_2$. These holomorphic strips intersect T^2 in the three geodesic paths connecting x_+ to x_-. By the exactness of L_{pants} and T^2, it follows that these are the only strips which contribute to the Floer differential.

Thus we need only compute the holonomy of the local systems along the boundaries of these strips. We can normalize the local systems on L_{pants} and T^2 such that parallel transport along the blue arc in Figure 2.3 is multiplication by $1 \in \mathbb{K}^*$. In that case, we must have that parallel transport along the orange and green arcs act as multiplication by z_1 and ρ_1 and by z_2 and ρ_2, respectively. Thus, without loss of generality, the Floer differential is

$$x_+ \mapsto \pm(\rho_1^{-1} z_1 \pm \rho_2^{-1} z_2 \pm 1) x_-,$$

which completes the proof. □

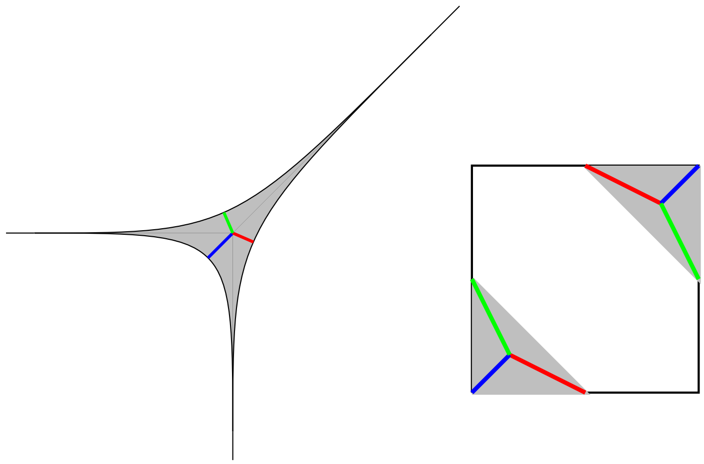

Figure 2.3: The boundary of the three holomorphic strips contributing to the Floer differential on $CW^*(L_{\text{pants}}, T^2)$, projected to the amoeba (left), and the coamoeba (right).

Remark 2.2.8. For different choices of spin structure on L_{pants}, the differential in this chain complex can be any of $\pm\rho_1^{-1}z_1 \pm \cdots \pm \rho_n^{-1}z_n \pm 1$ up to an overall sign.

Remark 2.2.9. A similar computation in two dimensions using non-exact Lagrangian tori is carried out in [34]. The Floer differential we computed above also appears in [27].

2.3 Singular Lagrangian lift of a 4-valent vertex

This section is devoted to the construction of a Lagrangian lift of a certain genus 0 tropical curve $V \subset \mathbb{R}^3$. Specifically, V is defined to be the union of the following four one-dimensional cones

$$V_1 = \{(-t, 0, 0) \mid t \in \mathbb{R}_{\geq 0}\},$$

$$V_2 = \{(0, -t, 0) \mid t \in \mathbb{R}_{\geq 0}\},$$

$$V_3 = \{(0, 0, -t) \mid t \in \mathbb{R}_{\geq 0}\},$$

$$V_4 = \{(t, t, t) \mid t \in \mathbb{R}_{\geq 0}\}.$$

This has one 4-valent vertex at the origin. We will prove that

Theorem 2.3.1. *There is a family of Hamiltonian isotopic Lagrangian subsets $L^{\epsilon}_{\text{sing}} \subset T^*T^3$ for sufficiently small $\epsilon > 0$, each with one singular point. The image of $L^{\epsilon}_{\text{sing}}$ under projection to the cotangent fiber is ϵ-close to V in the Hausdorff metric. The link of the singular point of $L^{\epsilon}_{\text{sing}}$ is Legendrian isotopic to the link of the Harvey–Lawson cone. For each $L^{\epsilon}_{\text{sing}}$, there is a small open ball $B \subset \mathbb{R}^3$ centered at the origin such that*

$$L^{\epsilon}_{\text{sing}} \mid_{T^*T^3 \setminus \pi^{-1}_{SYZ}(B)} = \bigcup_{j=1}^{4} L_{V_j} \mid_{T^*T^3 \setminus \pi^{-1}_{SYZ}(B)} .$$

Away from the cone points, $L^{\epsilon}_{\text{sing}}$ is diffeomorphic to the minimally twisted five-component chain link complement in S^3, drawn in Figure 1.2.

The cone point of L_{sing} lies at $(0,0) \in T^*T^3$. We will let v_0 denote this cone point. Consider the symplectomorphism $\phi_0\colon T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ induced by the linear map

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

acting on $\mathbb{R}^3$. Denote the image of a ball in $T^*\mathbb{R}^3 \cong \mathbb{C}^3$ of sufficiently small radius under the composition ϕ_0 with the universal covering map map by

$$B_0 \subset T^*T^3 . \tag{2.3.1}$$

This is a Darboux ball centered at v_0.

2.3.1 The minimally twisted five-component chain link

We begin by collecting some topological facts about the minimally-twisted five component chain link complement, hereafter denoted L'. The link complement L' is a hyperbolic 3-manifold, which one can see by exhibiting an ideal triangulation for it, following Dunfield–Thurston [40].

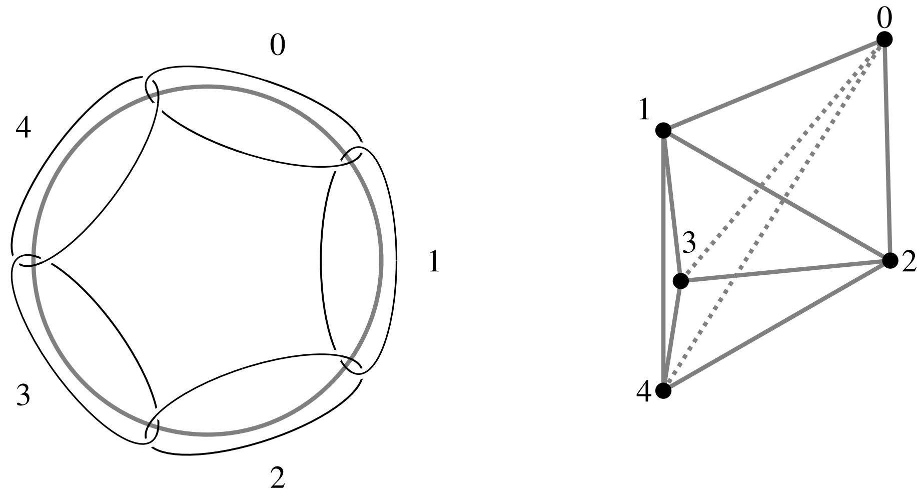

Figure 2.4: The circle of symmetry (left) and its image in the quotient orbifold (right).

Fix the labeling of the cusps of M_5 shown in Figure 2.4. The labeling we choose does not matter, since the group of hyperbolic isometries of L' acts transitively on the cusps of L'.

Observe that there is an involution of L' given by rotation about the circle depicted in Figure 2.4. Note that this circle should intersect the chain-link in S^3, so in L' the fixed locus is actually a union of 10 arcs between the cusps. Inspecting the diagram shows that any two cusps of L' are joined by one of these arcs. The quotient N of L' by this involution is an orbifold whose underlying 3-manifold is S^3 with five points removed, with each puncture in S^3 corresponding to the hyperelliptic quotient of a cusp of L'. The orbifold locus consists of an unknotted arc in S^3 connecting each pair of punctures, and all orbifold points are cone points of order 2.

Viewing N as the boundary 3-sphere of a standard 4-simplex carrying the induced triangulation with its vertices deleted gives us an ideal triangulation of N. The union of edges of the triangulation is the orbifold locus of N. It is shown in [40] that these can be thought of as regular ideal tetrahedra in hyperbolic 3-space $\mathbb{H}^3$, so this gives N the structure of a hyperbolic orbifold decomposed into five regular ideal tetrahedra. We can lift this to an ideal triangulation of L' consisting of ten regular ideal tetrahedra.

Since we will build $L^{\epsilon}_{\text{sing}}$ using a coamoeba, it will be convenient to recast this ideal triangulation as a decomposition of L' into two ideal cubes, which will correspond to the two pieces of

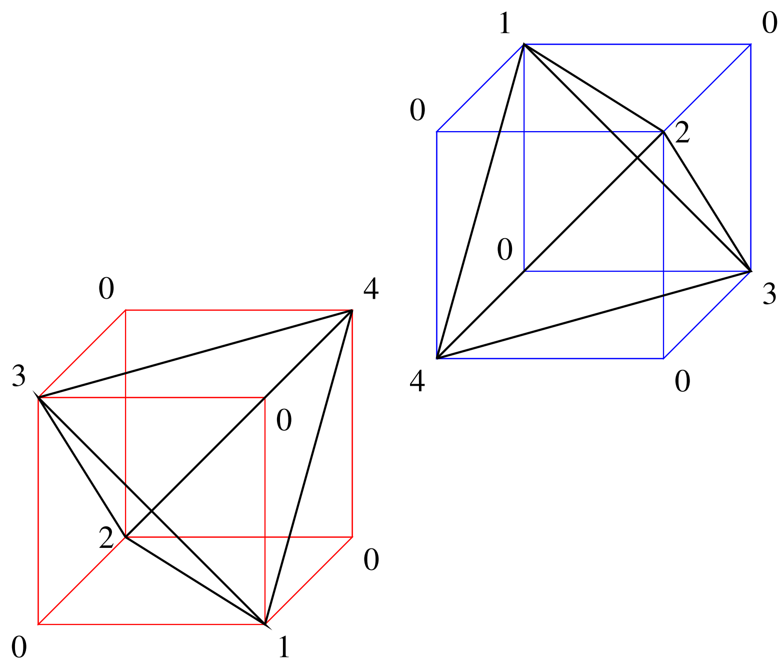

Figure 2.5: The two ideal cubes in the decomposition of L', decomposed into five ideal tetrahedra each. The edges of the tetrahedra $T_{0,\pm}$ are the black lines.

the coamoeba. From our description of the ideal triangulation of L', it is clear that exactly two of the ideal tetrahedra, which we call $T_{0,\pm}$, in the triangulation will have ideal vertices at the cusps labeled 1 through 4. At each face of $T_{0,+}$, the adjacent tetrahedron will have one ideal vertex at the 0th cusp. The union of these five ideal tetrahedra in L' is an ideal cube. Similar remarks apply to $T_{0,-}$, implying that the union of the other five ideal tetrahedra is also an ideal cube. The next lemma summarizes our discussion thus far.

Lemma 2.3.2. *L' admits a decomposition glued into two ideal cubes glued together as shown in Figure 2.5, where opposite faces identified by reflections along the dashed lines, so that vertices with the same label are identified.* □

The edges of the tetrahedra $T_{0,\pm}$ are the dashed black lines in Figure 2.5.

It will be convenient to describe the spin structures on L' that we will use in terms of the ideal triangulation, following Benedetti and Petronio [67], as we will now quickly review. Their description of ideal triangulations makes use of so-called special spines, which are certain 2-dimensional complexes dual to a (possibly ideal) triangulation of a 3-manifold. The special spine is the union

P of compact 2-dimensional polyhedra dual to each maximal cell in the ideal triangulation. The singular set $S(P)$ of P is a 4-valent graph whose edges are dual to the 2-dimensional faces of the triangulation on L'.

To specify a spin structure, we will first need to smooth P to obtain a new spine Y which has a well-defined normal direction. A choice of smoothing is determined by the following combinatorial data associated to a triangulation. A branching on an oriented tetrahedron T is an orientation of its edges so that none of its faces are cycles. In particular, each vertex of T should have a different number of incoming edges. Each face of T is given the orientation that restricts to the prevailing orientation of its edges.

Lemma 2.3.3. *The triangulation of L' admits a branching.*

Proof. Fix a branching of $T_{0,\pm}$ for which the ith vertex, where $i = 1, 2, 3, 4$, has $i - 1$ incoming edges. Any other tetrahedron in the ideal triangulation has a 0th vertex, and we declare that all edges adjacent to this vertex are incoming. With these choices, it is easy to see that all edges have been given an orientation, and that no face in this triangulation is a cycle. □

As we will see momentarily, this choice of branching determines a spin structure, and when we compute Floer homology, we will always assume that L' is equipped with this spin structure.

Remark 2.3.4. For an arbitrary 3-manifold, not every triangulation admits a branching, so in general one requires so-called weak branchings [67] to describe spin structures.

Given a branching of the triangulation, one can orient the 2-dimensional faces of P so that they are positively transverse to the dual edges of the triangulation.

Lemma 2.3.5. *The spine P can be smoothed in such a way that it gains a well-defined normal vector field ν. Let Y denote the smoothed spine.* □

This lemma tells us that there is a canonically defined nonvanishing vector field ν along Y. We can choose another vector field η defined along Y which is tangent to Y. These two vector fields are always linearly independent, so they specify a spin structure on M.

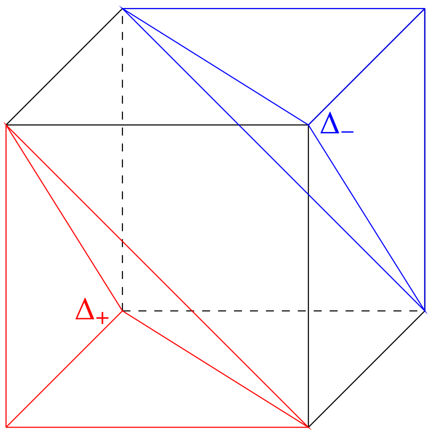

Figure 2.6: The coamoeba $\Delta \subset [0,1]^3$ associated to 4-valent tropical vertex.

2.3.2 Construction

As a starting point for the construction of $L^{\epsilon}_{\text{sing}}$, we define a coamoeba dual to V. Identify $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with a quotient of $[0,1]^3$.

Definition 2.3.6. Let $\overline{\Delta}_+ \subset [0,1]^3$ denote the 3-simplex with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let $\overline{\Delta}_- \subset [0,1]^3$ denote the 3-simplex with vertices at $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, and $(1,1,1)$. Define $\Delta_\pm$ to be the union of the 1-cells and 3-cells in $\overline{\Delta}_\pm$, and define the coamoeba to be $\Delta := \Delta_+ \cup \Delta_-$.

The images of these simplices in T^3 intersect each other along their edges, which are either of the form $E_k := \{\theta_i = \theta_j = 0\} \subset [0,1]^3$, where $\{i,j,k\} = \{1,2,3\}$ or $E_{ij} := \{\theta_i + \theta_j = 0, \text{ and } \theta_k = 0\} \subset [0,1]^3$, where $\{i,j,k\} = \{1,2,3\}$. All vertices of $\overline{\Delta}_+$ and $\overline{\Delta}_-$ project to the same point in T^3. Define E°_k and E°_{ij} to be the 1-cells of Δ (which do not include the vertices).

It will be helpful to understand the symmetries of Δ for our construction. Let $\theta_0, \dots, \theta_3$ denote the vertices of $\Delta_+ \subset [0,1]^3$ as in Figure 2.5. For $k = 1,2,3$, there is a unique affine automorphism of $\mathbb{R}^3$ which fixes Δ_+, switches θ_0 with θ_k, and leaves the other vertices of Δ_+ fixed. This descends to an affine automorphism of T^3 which we call R_k. Let G be the group generated by $\{R_1, R_2, R_3\}$. Note that these automorphisms also fix Δ_-, so G acts on Δ. The obvious action of G on $\{0,1,2,3\}$ corresponds to its action on the edges and faces of Δ.

As in the construction of the Lagrangian pair of pants, we need to find an appropriate smoothing of Δ. Choose neighborhoods U_k of E_k° and U_{ij} of E_{ij}° in Δ which are diffeomorphic to subsets of $\mathbb{R}^2$ crossed with open intervals. We can form the real blowups $\widetilde{U}_k$ and $\widetilde{U}_{ij}$ by blowing up along E_k° and E_{ij}°, respectively, in the obvious sense. These come with natural maps $\widetilde{U}_k \to U_k$ and $\widetilde{U}_{ij} \to U_{ij}$.

Definition 2.3.7. The real blowup $\widetilde{\Delta}$ is the space obtained from $\mathrm{Int}(\Delta)$ by gluing $\widetilde{U}_k$ and $\widetilde{U}_{ij}$ to $\mathrm{Int}(\Delta)$ using the projection maps. Let $\pi\colon \widetilde{\Delta} \to \Delta$ denote the blowup map.

Lemma 2.3.8. *The real blowup $\widetilde{\Delta}$ is a smooth 3-manifold diffeomorphic to L'.*

Proof. Identify $\mathrm{Int}\,\Delta_+$ and $\mathrm{Int}\,\Delta_-$ with the interiors of solid cubes in such a way that the edges of Δ_+ and Δ_- correspond to faces of the cube. Consequently, vertices of $\Delta_\pm$ will correspond to the vertices of the cubes labeled 0, and the faces of $\Delta_\pm$ will correspond to the vertices of the cubes labeled 1, 2, 3, and 4. Under this identification, the real blowup will correspond to gluing the two cubes together by reflections along the dashed lines. □

Define the function $g\colon \Delta \to \mathbb{R}$ by

$$g(\theta_1,\theta_2,\theta_3) = \begin{cases} -\sqrt{\sin(\pi\theta_1+\pi\theta_2+\pi\theta_3)\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)}, & \theta\in\Delta_+ \\ \sqrt{\sin(\pi\theta_1+\pi\theta_2+\pi\theta_3)\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)}, & \theta\in\Delta_-. \end{cases}$$

Observe that g vanishes on the boundary of $\overline{\Delta}$, and that it is invariant under the involution $\theta \mapsto -\theta$ on T^3.

Lemma 2.3.9. *The differential dg restricted to* $\mathrm{Int}\,\Delta$ *extends to a smooth function $\widetilde{dg}\colon \widetilde{\Delta} \to \mathbb{R}$.*

Proof. One easily computes the components of the differential dg on $\Delta_\pm$ to be

$$\frac{\partial g}{\partial\theta_1} = \mp\frac{\pi\sin(\pi\theta_2)\sin(\pi\theta_3)\sin(2\pi\theta_1+\pi\theta_2+\pi\theta_3)}{2\sqrt{\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)\sin(\pi\theta_1+\pi\theta_2+\pi\theta_3)}}$$
$$\frac{\partial g}{\partial\theta_2} = \mp\frac{\pi\sin(\pi\theta_1)\sin(\pi\theta_3)\sin(\pi\theta_1+2\pi\theta_2+\pi\theta_3)}{2\sqrt{\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)\sin(\pi\theta_1+\pi\theta_2+\pi\theta_3)}}$$

$$\frac{\partial g}{\partial \theta_3} = \mp \frac{\pi \sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_1 + \pi\theta_2 + 2\pi\theta_3)}{2\sqrt{\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)\sin(\pi\theta_1 + \pi\theta_2 + \pi\theta_3)}}.$$

First we will show that dg extends over the edge E_3°. In the neighborhood $\widetilde{U_3}$, the real blowup has coordinates (θ_1, t, θ_3), where $t \in \mathbb{R}$, and the map $\widetilde{U_3} \to U_3$ is $(\theta_1, t, \theta_3) \mapsto (t\theta_1, t, \theta_3)$. Rewriting dg in these coordinates, we get

$$\begin{aligned}
\frac{\partial g}{\partial \theta_1} &= \mp \frac{\pi \sin(\pi t)\sin(\pi\theta_3)\sin(2\pi t\theta_1 + \pi t + \pi\theta_3)}{2\sqrt{\sin(\pi t\theta_1)\sin(\pi t)\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}} \\
\frac{\partial g}{\partial \theta_2} &= \mp \frac{\pi \sin(\pi t\theta_1)\sin(\pi\theta_3)\sin(\pi t\theta_1 + 2\pi t + \pi\theta_3)}{2\sqrt{\sin(\pi t\theta_1)\sin(\pi t)\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}} \\
\frac{\partial g}{\partial \theta_3} &= \mp \frac{\pi \sin(\pi t\theta_1)\sin(\pi t)\sin(\pi t\theta_1 + \pi t + 2\pi\theta_3)}{2\sqrt{\sin(\pi t\theta_1)\sin(\pi t)\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}}.
\end{aligned}$$

Notice that nonzero values of t correspond to points $(t\theta_1, t, \theta_3)$ in the interior of Δ, so checking whether these functions extend smoothly across the blowup amounts to checking whether or not they extend smoothly to $t = 0$. To check whether $\frac{\partial g}{\partial \theta_1}$ extends to the blowup, we rewrite it as

$$\begin{aligned}
\frac{\partial g}{\partial \theta_1} &= \mp \frac{\pi \sin(\pi t)\sin(\pi\theta_3)\sin(2\pi t\theta_1 + \pi t + \pi\theta_3)}{2\sqrt{\sin(\pi t\theta_1)\sin(\pi t)\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}} \\
&= \mp\epsilon \frac{\pi \frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(2\pi t\theta_1 + \pi t + \pi\theta_3)}{2\sqrt{\frac{\sin(\pi t\theta_1)}{t}\frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}},
\end{aligned}$$

where ϵ is the sign $\frac{t}{\sqrt{t^2}}$, which is (-1) when $t < 0$ and 1 when $t > 0$. Because the value of θ_3 will always be nonzero, since we are considering the blowup along E_3°. Since θ_1 is close to 0 or 1, depending on whether the point lies in Δ_+ or Δ_-, all factors in these expressions involving θ_3 are never 0. Thus it is clear that $\frac{\partial g}{\partial \theta_1}$ extends smoothly across $t = 0$, with the extension given by

$$\widetilde{dg}_1 = -\frac{\pi \frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(2\pi t\theta_1 + \pi t + \pi\theta_3)}{2\sqrt{\frac{\sin(\pi t\theta_1)}{t}\frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}}.$$

The sign -1 comes from comparing the sign in the definition of g and the sign $\frac{t}{\sqrt{t^2}}$. A similar computation shows that $\frac{\partial g}{\partial \theta_2}$ extends smoothly to the blowup, and the extension has the formula

$$\widetilde{dg}_2 = -\frac{\pi \frac{\sin(\pi t\theta_1)}{t} \sin(\pi\theta_3)\sin(\pi t\theta_1 + 2\pi t + \pi\theta_3)}{2\sqrt{\frac{\sin(\pi t\theta_1)}{t}\frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}}.$$

Treating $\frac{\partial g}{\partial \theta_3}$ in the same way, we find that the third component of the extension is

$$\widetilde{dg}_3 = -t\frac{\pi \frac{\sin(\pi t\theta_1)}{t}\frac{\sin(\pi t)}{t} \sin(\pi t\theta_1 + \pi t + 2\pi\theta_3)}{2\sqrt{\frac{\sin(\pi t\theta_1)}{t}\frac{\sin(\pi t)}{t}\sin(\pi\theta_3)\sin(\pi t\theta_1 + \pi t + \pi\theta_3)}}.$$

We note that this vanishes when $t = 0$, so points in E_3 are mapped to the hyperplane $\{q_3 = 0\}$ in $\mathbb{R}^3$. Since g is symmetric under the group G of symmetries of Δ, it follows that dg extends over the blowups along all other edges of Δ. □

Given a finite subset $E = \{e_1, \dots, e_k\} \subset \mathbb{R}^3$, let

$$\mathrm{Cone}(E) := \left\{ \sum_{j=1}^{k} t_j e_j \mid t_j \in \mathbb{R}_{\geq 0} \right\}$$

denote the cone over E. The following is a consequence of the formula for $\widetilde{dg}$.

Lemma 2.3.10. *Let $\{i, j, k\} = \{1, 2, 3\}$. The map $\widetilde{dg}\colon \widetilde{\Delta} \to \mathbb{R}^3$ maps points lying over the edge E_i in Δ to points in* $\mathrm{Cone}\{u_j, u_k\}$. *It maps points lying over the edge E_{jk} to points in* $\mathrm{Cone}\{u_0, u_i\}$. □

Definition 2.3.11. Define L_{sing} to be the closure of the embedding

$$\Phi\colon \widetilde{\Delta} \to T^*T^3 = T^3 \times \mathbb{R}^3$$
$$\theta \mapsto (\pi(\theta), \widetilde{dg}(\theta))$$

Since $\lim_{\theta\to 0} dg(\theta) = 0$, this is just the union $L_{\mathrm{sing}} = \Phi(\widetilde{\Delta}) \cup \{0\} \subset T^*T^3$. Clearly L_{sing} is smooth everywhere except at $0 \in T^*T^3$, so for the rest of this subsection, we will study the behavior of L_{sing} near this point.

Remark 2.3.12. Given an arbitrary tubular neighborhood of the zero-section in T^*T^3, we can assume that the preimage of a ball B in the cotangent fiber containing the origin is contained in this neighborhood by taking B to be sufficiently small.

By the construction of the diffeomorphism between $\widetilde{\Delta}$ and the minimally twisted five component chain link complement, we know that the intersection of L_{sing} with a small sphere centered $0 \in T^*T^3$ is a (Legendrian) 2-torus. We now prove that this link is the Legendrian link of the Harvey–Lawson cone.

Definition 2.3.13. The Harvey–Lawson cone C_{HL} is defined to be the subset

$$C_{HL} := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1| = |z_2| = |z_3|,\ z_1 z_2 z_3 \in \mathbb{R}_{>0}\}.$$

Let $\Lambda_{HL} := C_{HL} \cap S^5(\epsilon)$ denote the link of C_{HL}.

The link Λ_{HL} is the image of a map $S^1 \times S^1 \to S^5(\epsilon) \subset \mathbb{C}^3$ defined to be

$$(s, t) \mapsto \left(\frac{1}{3}\epsilon e^{is}, \frac{1}{3}\epsilon e^{it}, \frac{1}{3}\epsilon e^{-is-it}\right). \tag{2.3.2}$$

We write the coordinates on $\mathbb{C}^3$ in terms of real and imaginary parts as $z = x + iy$, where $z = (z_1, z_2, z_3)$. Then if we define $n := \frac{1}{|x|}x \in S^2$ and $f := n \cdot y$, the front projection of Λ_{HL} is

$$(s, t) \mapsto e^f n \in \mathbb{R}^3 \setminus \{0\} \cong S^2 \times \mathbb{R}.$$

The caustic set of this projection is the edge graph of a tetrahedron embedded on S^2, where the vertices are at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$, and $\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$.

It will be helpful to think of T^*T^3 as $(\mathbb{C}^*)^3$ using the identification $(q_i, \theta_i) = (\log(r_i), \theta_i)$, where q_i is the coordinate on the cotangent fiber of T^*T^3. Hence, if U is a small neighborhood of $0 \in T^*T^3$, it is identified with a small neighborhood of $0 \in \mathbb{C}^3$ under the exponential map $\mathbb{C}^3 \to (\mathbb{C}^*)^3$. Under these identifications, the projection $U \to \mathbb{R}^3$ to the cotangent fiber can be thought of as a restriction of the projection $\mathbb{C}^3 \to \mathbb{R}^3$ to the real axis.

Lemma 2.3.14. *For a sufficiently small sphere $S^5(\epsilon) \subset \mathbb{C}^3$, the intersection $\Lambda_\epsilon := S^5(\epsilon) \cap L$ is Legendrian isotopic to Λ_{HL}.*

Proof. Since $\widetilde{dg}$ is invariant under the involution on T^3, the image of Λ in $\mathbb{R}^3$ under projection $\Lambda \to \mathbb{R}^3$ will be 2-to-1, except over the cones $\mathrm{Cone}\{u_i, u_j\}$ for $i, j \in \{0, 1, 2, 3\}$, where it is 1-to-1. The intersection of these regions with $S^2 \subset \mathbb{R}^3$ is precisely the edge graph of a tetrahedron. Since the map $T^*T^3 \to \mathbb{R}^3$ restricted to a link of the cone points can be thought of as the front projection of the link, it follows that Λ_ϵ is the Legendrian surface associated to the weave given by this intersection, in the sense of [68]. There is an isotopy of S^2 which carries this graph to the weave defining Λ_{HL} described in [69]. Explicitly, if one thinks of a neighborhood of the origin in $T^*\mathbb{R}^3$ as a neighborhood of the cone point of T^*T^3, this isotopy comes from restricting the symplectic automorphism automorpism of $T^*\mathbb{R}^3$ induced by the linear map

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Hence there is a Legendrian isotopy between Λ and Λ_{HL}. □

This implies the following description of L near the singular point.

Corollary 2.3.15. *There is a small ball $B \subset \mathbb{C}^3$ centered at the origin such that $L_{\mathrm{sing}} \cap (B \setminus \{0\})$ is diffeomorphic to $T^2 \times (0, \epsilon)$. There is a Hamiltonian isotopy of B taking the Lagrangian cone $L \cap B$ to the Harvey–Lawson cone.*

Extending this Hamiltonian isotopy by the identity outside of a slightly larger neighborhood of the singular point lets us identify the cone point of L_{sing} with the Harvey–Lawson cone.

We can similarly deform the Lagrangian over the legs of V. Since L_{sing} is exact in the appropriate sense, it can be put in tropical position by a Hamiltonian isotopy.

Lemma 2.3.16. *Let $\pi\colon T^*T^3 \to \mathbb{R}^3$ denote the projection. For any $\epsilon > 0$, there is a Hamiltonian isotopy which takes L_{sing} to a (singular) Lagrangian $L^\epsilon_{\mathrm{sing}}$ for which $\pi_{SYZ}(L^\epsilon_{\mathrm{sing}})$ is within Haus-*

dorff distance ϵ of V. Moreover, $L^{\epsilon}_{\text{sing}}$ agrees with the conormal lifts of the 1*-dimensional faces V_i of V outside of an arbitrarily small ball B centered at the origin in $\mathbb{R}^3$, i.e.*

$$L^{\epsilon}_{\text{sing}} \mid_{T^*T^3 \setminus \pi^{-1}_{SYZ}(B)} = \bigcup_{i=0}^{3} L_{V_i} \mid_{T^*T^3 \setminus \pi^{-1}_{SYZ}(B)} .$$ □

The argument in the proof of [27] applies in our situation with no changes, since $L \mid_{T^*T^3 \setminus \pi^{-1}(B_0(\epsilon))}$ is smooth. Therefore we can construct a Lagrangian lift which agrees with the periodized conormals to the legs of V outside of a ball in $\mathbb{R}^3$ centered at the origin. The statements about Hausdorff distance and the size of B can be proven by rescaling $\widetilde{dg}$ by an arbitrarily small real constant. From now on, we will use L_{sing} to refer interchangeably to the closure of $\Phi(M)$ constructed using $\lambda\widetilde{dg}$ for any $\lambda > 0$, or to any of the Lagrangians $L^{\epsilon}_{\text{sing}}$ in tropical position.

2.3.3 First homology

In this section, we will discuss $H_1(L')$ and describe the induced map $H_1(L') \to H_1(T^*T^3) = H_1(T^3)$ in preparation for the proof of Theorem 1.2.3. Since L' is a link complement, we know that $H_1(L')$ is generated by the meridians $m_0, \dots, m_4$ to the components of the minimally twisted five-component chain link. It will also be helpful for us to consider the longitudes of the link components, denoted by $\ell_0, \dots, \ell_4$. We choose orientations on the m_i's and ℓ_i's which are shown in Figure 2.7. Examining this figure has the following immediate consequence.

Lemma 2.3.17. *The longitudes can be expressed in terms of the meridians in $H_1(L')$ by the following formulas.*

$$\ell_0 = -m_1 - m_4,$$

$$\ell_1 = -m_0 + m_2,$$

$$\ell_2 = m_1 - m_3,$$

$$\ell_3 = -m_2 + m_4,$$

$$\ell_4 = -m_0 + m_3.$$

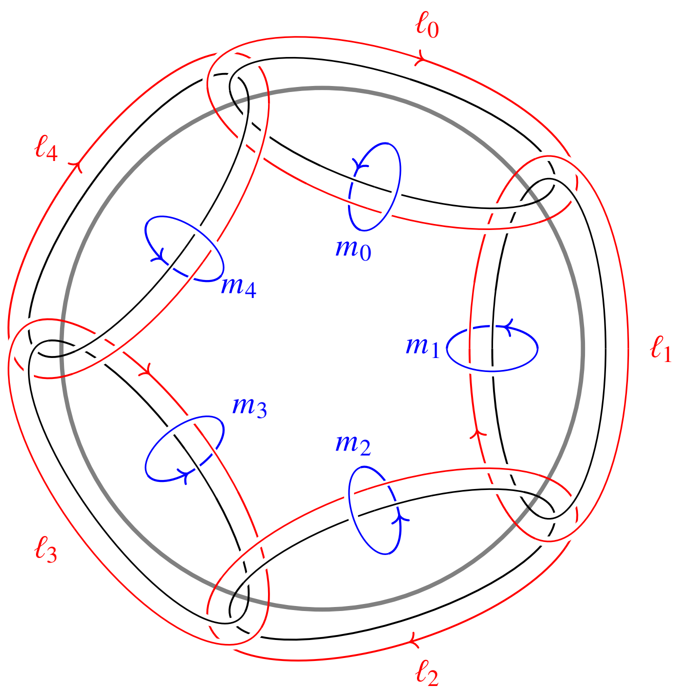

Figure 2.7: m_i's (blue) and ℓ_i's (red).

□

Let $\{e_1, e_2, e_3\} \subset H_1(T^3)$ denote the set of generators which lift to the coordinate axes in the universal cover $\mathbb{R}^3 \to T^3$.

Lemma 2.3.18. *The map $H_1(L') \to H_1(T^3)$ is determined by the following values (up to automorphisms of $H_1(T^3)$ changing the signs of e_1, e_2, and e_3).*

$$m_0 \mapsto 0,$$

$$m_1 \mapsto e_2 - e_3,$$

$$m_2 \mapsto e_3,$$

$$m_3 \mapsto -e_1 + e_2,$$

$$m_4 \mapsto -e_2 + e_3.$$

Consequently, the values of this map on the longitudes are as follows.

$$\ell_0 \mapsto 0,$$

$$\ell_1 \mapsto e_3,$$

$$\ell_2 \mapsto e_1 - e_3,$$

$$\ell_3 \mapsto -e_2,$$

$$\ell_4 \mapsto -e_1 + e_2.$$

Proof. It will be more convenient to determine the values on the longitudes first. Recall that in the ideal triangulation of L' described in subsection 2.3.1, there were two ideal tetrahedra $T_{0,\pm}$ with no vertices on the zeroth cusp. Informally, we can think of these as the dual tetrahedra to the components $\Delta_\pm$ of the coamoeba.

For $i = 1, 2, 3, 4$, the longitude ℓ_i restricts to a small arc on $T_{0,+}$ near the ith vertex which connects two edges adjacent to the ith vertex. Recall that in the ideal cubulation of L' from subsection 2.3.1, each ideal cube was written as the union of five ideal tetrahedra. We can think of these five ideal tetrahedra as representatives of the ideal tetrahedra in the quotient orbifold N of L' by the obvious $\mathbb{Z}/2$-action on L'. Note that in terms of the coamoeba, this $\mathbb{Z}/2$-action comes from the involution $\theta \mapsto -\theta$ on T^3.

From Figure 2.8, which shows the images of the ℓ_i's under the orbifold quotient, we can determine the arcs on $T_{0,+}$ to which each longitude restricts. These are also drawn on the ideal cube in Figure 2.9. Thus the values of the induced map on homology of ℓ_i for $i = 1, 2, 3, 4$ are as given in the statement of the lemma, assuming that the arcs are oriented as in Figure 2.7. We also have $\ell_0 \mapsto 0$ because the zeroth cusp of $\widetilde{L}$ is the link of the cone point in L_{sing}. The values of the map on the meridians follow from Lemma 2.3.17.

□

Although ℓ_0 and m_0 are mapped to nullhomotopic curves in T^*T^3, we can still characterize their images inside a neighborhood of the cone point of L_{sing}. Recall that the link Λ_{HL} of the Harvey–

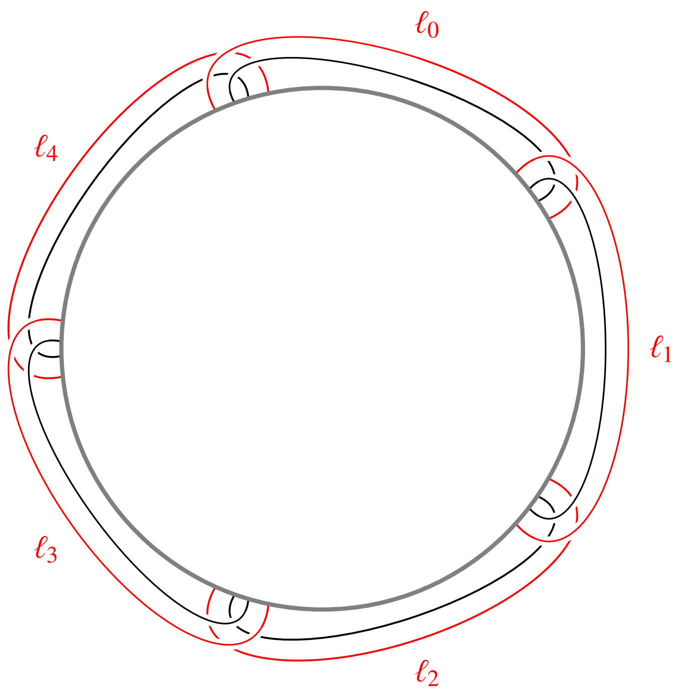

Figure 2.8: The images of the longitudes in the quotient oribfold.

Lawson double covers a sphere in $\mathbb{R}^3$ with branch points (a,a,a), $(a,-a,-a)$, $(-a,a,-a)$, and $(-a,-a,a)$ for some $a > 0$. Under our identification of the cone point of L with C_{HL}, these branch points correspond to points in $\mathbb{R}^3$ of the form (b,b,b), $(-b,0,0)$, $(0,-b,0)$, and $(0,0,-b)$, where $b > 0$. It is easy to see that both m_0 and ℓ_0 will project to arcs in $\mathbb{R}^3$ connecting pairs of branch points. From Figure 2.8, we can take ℓ_0 to be a curve which projects to an arc connecting $(0,0,-b)$ to $(0,-b,0)$. Similarly, we take m_0 to be a curve which projects to an arc connecting $(-b,0,0)$ to $(0,-b,0)$. Under the identification of C_{HL} with the cone points of L, this implies that ℓ_0^{-1} is homologous to the curve $(1, e^{it}, e^{-it})$ and m_0 is homologous to the curve $(e^{is}, e^{-is}, 1)$, where these are given in terms of the parametrization 2.3.2 of C_{HL}. The orientations on m_0 and ℓ_0 are determined by the orientations we used in Lemma 2.3.18.

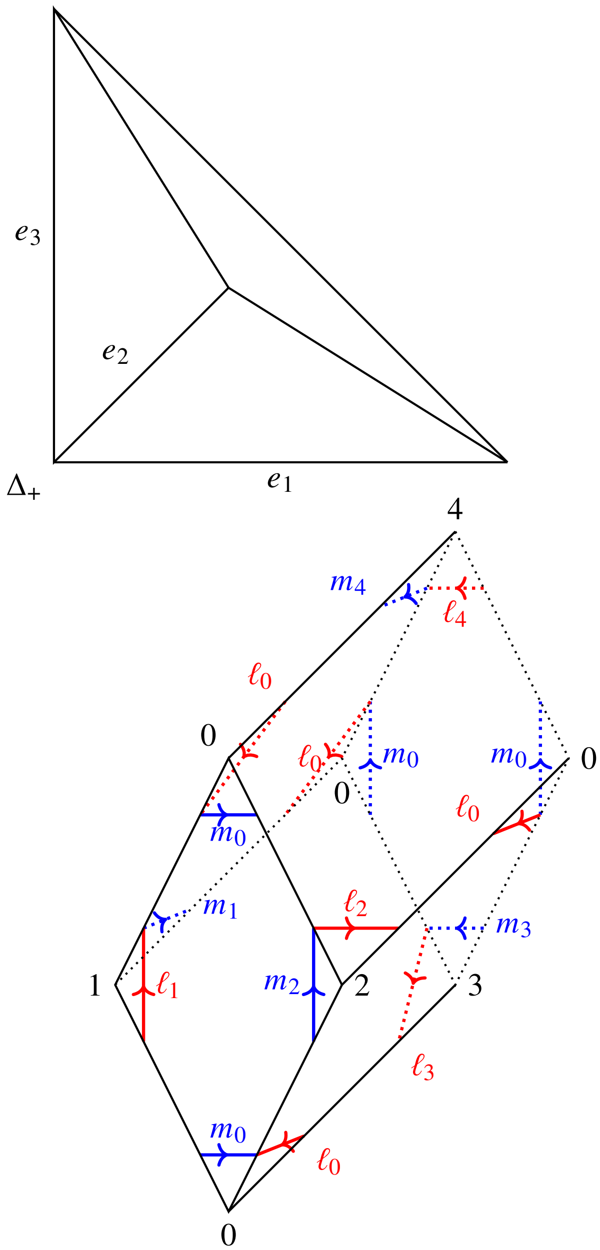

Figure 2.9: Δ_+ (top), and the corresponding ideal cube with the restrictions of m_i and ℓ_i with orientations (bottom).

2.4 An Immersed Tropical Lagrangian

In this section, we will construct a cleanly immersed Lagrangian L_{im} whose Floer theory we will later see behaves as one would expect the Floer theory of the direct sum of L_{sing} with itself to behave, if it were to be defined. We will also discuss gradings and unobstrucedness for this immersed Lagrangian.

2.4.1 Lagrangian neighborhood theorems

In this subsection, we will prove a version of the Weinstein neighborhood theorem for Lagrangians in Liouville manifolds which have conical singular points and which are cylindrical at infinity. Although the results that we use are similar to those in [37], we will state them here for the reader's convenience.

The starting point for the proof of the Lagrangian neighborhood theorem we will use is the following statement from Lagrangian foliations, which is stated in [37] and extracted from [70].

Theorem 2.4.1 (Joyce [37])**.** *Let (M, ω) be a symplectic manifold and $N \subset M$ a half-dimensional embedded submanifold. Let $\{L_x \mid x \in N\}$ be a smooth family of embedded, noncompact Lagrangians in M for which $x \in L_x$ and $T_xL_x \cap T_xN = \{0\}$ for all $x \in N$. Then there is an open neighborhood U of the zero section N in T^*N such that the fibers of the restricted bundle projection $\pi\colon U \to N$ are connected along with a unique embedding $\Phi\colon U \to M$ with $\Phi(\pi^{-1}(x)) \subset L_x$ for each $x \in N$ with the property that Φ restricts to the identity on N and that $\Phi^*(\omega) = \hat{\omega} + \pi^*(\omega\mid_N)$, where $\hat{\omega}$ is the canonical symplectic structure on T^*N.*

One also has a neighborhood theorem for Lagrangian cones in $C \subset \mathbb{C}^n$, by which we mean singular Lagrangians in $\mathbb{C}^n$ with the property that $tC = C$ for any $t \in \mathbb{R}_{>0}$. This can either be proven using Theorem 2.4.1, as in [37], or using the Legendrian neighborhood theorem. Such neighborhoods can be made dilation invariant with respect to a certain $\mathbb{R}_{>0}$-action on $T^*(\Sigma \times \mathbb{R}_{>0})$ which we will now describe.

Let C be a Lagrangian cone in $\mathbb{C}^n$ with isolated singularity at 0 and $\Sigma = C \cap S^{2n-1}$. Define $\iota\colon \Sigma \times \mathbb{R}_{>0} \to \mathbb{C}^n$ by the formula $\iota(\sigma, s) = s\sigma$. For $\sigma \in \Sigma$, $\tau \in T^*_\sigma\Sigma$, $s \in \mathbb{R}_{>0}$ and $u \in \mathbb{R}$, let (σ, s, τ, u) denote the point $\tau + u ds \in T^*_{(\sigma,r)}(\Sigma \times \mathbb{R}_{>0})$. Then there is an action of $\mathbb{R}_{\geq 0}$ on $T^*(\Sigma \times (0, \infty))$ given by

$$t\colon (\sigma, s, \tau, u) \mapsto (\sigma, ts, t^2\tau, tu)$$

for $t \in \mathbb{R}_{>0}$. If we let $\hat{\omega}$ denote the canonical symplectic form on $T^*(\Sigma \times \mathbb{R}_{>0})$, then this action has the property that $t^*(\hat{\omega})t^2\hat{\omega}$.

Lemma 2.4.2 (Joyce [37]). *There is an open neighborhood U_C of $\Sigma \times \mathbb{R}_{>0}$ in $T^*(\Sigma \times \mathbb{R}_{>0})$ invariant under the action of $\mathbb{R}_{>0}$ and an embedding $\Phi_C\colon U_C \to \mathbb{C}^n$ which restricts to ι over the zero section, for which the pullback of the usual symplectic form on $\mathbb{C}^n$ is $\hat{\omega}$, and such that $\Phi_C \circ t = t\Phi_C$.*

Now we will move on to discuss a global version of this theorem. Let M be a Liouville manifold, and let $L \subset M$ be a Lagrangian which is cylindrical at infinity and embedded, except at a discrete set of conical singular points, say $x_1, \dots, x_m \in L$, all of which are contained in M^{in} away from ∂M. Let $\Lambda_1, \dots, \Lambda_\ell$ denote the connected components of the Legendrian $\partial M \cap L$. In particular the Lagrangian $L_{M^{in}} := L \cap M^{in}$ is compact with Legendrian boundary.

Suppose that there are Darboux charts $\Psi_1, \dots, \Psi_m$ near $x_1, \dots, x_m$ such that $\Psi_i\colon B_r(0) \to M$ is defined on a ball centered at the origin in $\mathbb{C}^n$ and takes the value $\Psi_i(0) = x_i$. Denote by ψ_i the linear isomorphism $d_0\Psi_i\colon \mathbb{C}^n \to T_{x_i}M$.

Further assume that $\Psi_i^{-1}(L)$ is a (dilation-invariant) Legendrian cone in $B_\epsilon(0) \subset \mathbb{C}^n$ with Legendrian link denoted Σ_i. We will let $\iota_i\colon \Sigma_i(0, \epsilon) \to B_\epsilon(0)$ denote the map $(\sigma, s) \mapsto s\sigma$.

Define the punctured Lagrangian $L' := L \setminus \{x_1, \dots, x_m\}$. Then we can write $\Psi_i^{-1}(L')$ as the image under Φ_{Σ_i} of the graph of the zero section in $T^*(\Sigma_i \times (0, \epsilon'))$ for some $\epsilon' \in (0, \epsilon]$. Now define $\phi\colon \Sigma_i \times (0, \epsilon') \to B_\epsilon(0)$ by $\phi_i(\sigma, s) = \Phi_{\Sigma_i}(\sigma, s)$. Then $\Psi_i \circ \phi_i$ maps $\Sigma_i \times (0, \epsilon') \to L'$. Let S_i denote the image of $\Psi_i \circ \phi_i$, and define K to be

$$K := L' \setminus \left(S \cup \bigcup_{j=1}^{\ell} (\Lambda_i \times [1, \infty)) \right).$$

Lemma 2.4.3. *There is an open tubular neighborhood $U_{L'} \subset T^*L'$ of the zero section and a symplectic embedding $\Phi_{L'}\colon U_{L'} \to M$ which restricts to the inclusion $L' \hookrightarrow L$ over the zero section. Moreover, this embedding satisfies*

$$\Phi_{L'} \circ (d\Psi_i \circ \phi_i) = \Psi_i \circ \Phi_{\Sigma_i} \tag{2.4.1}$$

for all points $(\sigma, s, \tau, u) \in T^*(\Sigma_i \times (0, \epsilon'))$ in the open neighborhood from Lemma 2.4.2.

Proof. We take the equation 2.4.1 to be a definition for $\Phi_{L'}$ over the subset S, and so this determines what $U_{L'}$ should be near each of the cone points. Similarly we can define $\Phi_{L'}$ and $U_{L'}$ over the cylindrical ends using the Legendrian neighborhood theorem, meaning that we need only extend $\Phi_{L'}$ over K.

To that end, define $L_x = \Phi_{L'}(T^*_x L' \cap U_{L'})$ for all points x in S or in the cylindrical ends. We see that L_x is an open Lagrangian ball which intersects L' transversely at x, and this family depends smoothly on x. We can then extend this to a family parametrized by $x \in L'$, since the set where the family isn't already defined is the compact set K. Applying Theorem 2.4.1 to $\{L_x \mid x \in L'\}$ yields an open neighborhood U of L' in T^*L' and a symplectic embedding $\Phi\colon U \to T^*T^3$ which restricts to the identity on L'. By the local uniqueness of Theorem 2.4.1 we see that $\Phi_{C_{HL}}$ and $\Phi_{L'}$ coincide where they are both defined. □

2.4.2 Desingularizations and lifts of smooth tropical curves

In this subsection, we briefly consider the embedded Lagrangian submanifolds of T^*T^3 that we can construct by smoothing the singular point of L_{sing}. Recall that the Harvey–Lawson cone C_{HL} has three asymptotically conical smoothings, which are given in coordinates by

$$C^i_{HL}(\epsilon) = \{|z_i|^2 - \epsilon = |z_j|^2 = |z_k|^2,\ z_1 z_2 z_3 \in \mathbb{R}_{\geq 0}\} \subset \mathbb{C}^3$$

for all $\{i, j, k\} = \{1, 2, 3\}$. Let $L'_{\text{sing}} := L \setminus \{0\}$. We can construct three distinct smoothings of L_{sing} by perturbing L'_{sing} by a Lagrangian isotopy inside of a Weinstein neighborhood furnished by Lemma 2.4.3, and then filling the resulting Lagrangian using one of $C^i_{HL}(\epsilon)$ to obtain a proper Lagrangian embedding that we call $L^i_\delta(\epsilon)$, which depends on a choice of one-form δ.

By abuse of notation, let $\partial L'_0$ denote the link of the zeroth cusp of L', which is identified with Λ_{HL} under the Lagrangian embedding $L' \hookrightarrow T^*T^3$. The existence of appropriate perturbations for the purposes of constructing smoothings follows from the fact that $H^1(L'; \mathbb{R}) \to H^1(\partial L'_0; \mathbb{R})$ is a surjection. Consequently, any closed 1-form on $\partial L'_0$ extends to a closed 1-form on L'.

Now apply Lemma 2.4.2 to C_{HL} and Lemma 2.4.3 to L_{sing}. From this, we obtain Lagrangian neighborhoods $U_{L'} \subset T^*L'$ and $U_{C_{HL}} \subset \mathbb{C}^3$ of L' and C_{HL}, respectively, along wit symplectic embeddings $\Phi_{L'}$ and $\Phi_{C_{HL}}$ of these neighborhoods into T^*T^3. Choosing $\epsilon > 0$ small, we can write

$$\{|z_i|^2 - \epsilon = |z_j|^2 = |z_k|^2,\ z_1 z_2 z_3 \in \mathbb{R}_{>0}\}$$

as the graph of a closed one-form on $T^2 \times (0, \infty)$ in $\Phi_{C_{HL}}(U_{C_{HL}})$. By choosing a closed one-form δ on L' which restricts to this one-form in a neighborhood of $\partial L'_0$, we obtain the desired smoothing $L^i_\delta(\epsilon)$.

Note that the kernel of $H^1(L'; \mathbb{R}) \to H^1(\partial L'; \mathbb{R})$ is three-dimensional, and thus each $L^i_\delta(\epsilon)$ moves in a three-dimensional family of Lagrangians. It is easy to see that any $[\delta] \in H^1(L'; \mathbb{R})$ whose image in $H^1(\partial L'; \mathbb{R})$ is as described will restriction to a closed, but not exact, 1-form near the other cusps of L'.

A straightforward calculation now shows that the smoothing $L^i_\delta(\epsilon)$ constructed this way will lie over one of the three tropical smoothings of V described in the introduction in which the finite edge has length ϵ, possibly up to a translation of $\mathbb{R}^3$, which corresponds to elements of the three-dimensional kernel of $H^1(L') \to H^1(\partial L'_0)$. This should also be compared with the description of the asymptotically conical fillings of Λ_{HL} from [69].

It follows from the discussion in subsection 2.3.3 that the three families of smoothings of L

are all topologically obtained by Dehn filling along the curves m_0, ℓ_0, or $m_0\ell_0^{-1}$. As explained in [71], these three Dehn fillings of L' are all diffeomorphic to the three-manifold 1.2.2 of the introduction. The smooth Lagrangian lifts of the tropical smoothings of V are also diffeomorphic to this three-manifold.

We are currently unable to show that the smoothings $L^i_\delta(\epsilon)$ are Hamiltonian isotopic to tropical Lagrangian lifts of these smooth curves. The main difficulty lies in constructing a Hamiltonian isotopy that puts these curves in tropical position over the finite edges of these curves, since these $L^i_\delta(\epsilon)$ bounds a holomorphic disk lying over the finite edge.

2.4.3 Construction

We construct the immersed Lagrangian using a 'doubling' trick, similar to [33]. We first describe this for the Harvey–Lawson cone.

Consider the map $w\colon \mathbb{C}^3 \to \mathbb{C}$ given by $q(z_1, z_2, z_3) = z_1z_2z_3$. Under this map, the Harvey–Lawson cone C_{HL} is sent to the positive real axis in $\mathbb{C}$. Identify $\mathbb{C}^3 \setminus \{z_1z_2z_3 = 1\}$ with

$$X := \{(z_1, z_2, z_3, v) \in \mathbb{C}^3 \times \mathbb{C}^* \mid z_1z_2z_3 = 1 + v\}.$$

Note that this is one of the local models considered in [33]. Let $D := w^{-1}(0) \subset X$ be the normal crossing divisor.

Lemma 2.4.4. *The maps (z_1, z_2, w) define coordinates on $X \setminus D$ such that the following diagram commutes:*

$$\begin{array}{ccc} X \setminus D & \xrightarrow{\cong} & \mathbb{C}^2 \times (\mathbb{C} \setminus \{0,1\}) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{C}^2 \times \mathbb{C} \end{array}$$

This enables us to define a local model for the immersed Lagrangian as a product. Let L_1 denote the immersed arc in $\mathbb{C} \setminus \{0, 1\}$ shown in Figure 2.10.

Definition 2.4.5. Define the Lagrangian $L_{loc} \subset X$ to be $T^2 \times L_1$, where the T^2-factor is a Clifford

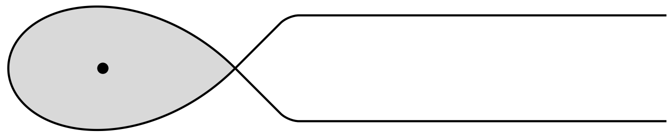

Figure 2.10: The immersed Lagrangian $L_1 \subset \mathbb{C}$, bounding a holomorphic teardrop through the origin.

torus $T^2 \subset \mathbb{C}^2$.

We can think of this immersed Lagrangian as lying inside the neighborhood $\Psi(B_{\epsilon'})$ constructed during the proof of Lemma 2.4.3. We let L' denote the minimally-twisted five-component chain link complement.

Lemma 2.4.6. *There is a Morse function $h\colon L' \to [0,\infty)$ with 10 index 0 critical points, 20 index 1-critical points, and 10 index 2 critical points. Moreover, we can arrange that in collar neighborhoods of the cusps of L' modeled on $T^2 \times (0,\epsilon)$, the gradient vector field of h points outward in the radial direction.*

Proof. This Morse function is constructed from the ideal triangulation of L', with the numbers of critical points correspond to the number of 3-cells, 2-cells, and 1-cells, respectively, in the triangulation. □

Recall that we can identify a neighborhood of the point $(0,0) \in T^*T^3$ with an open ball via $\Phi\colon \mathbb{C}^3 \cong \mathbb{R}^6 \to T^*T^3$, where the second map is the universal cover. Let $U \subset T^*T^3$ denote the image of this ball. In this Darboux chart, the Liouville form on T^*T^3 pulls back to the Liouville form on $\mathbb{C}^3$. We can also assume that U is the Darboux ball used to construct the Weinstein neighborhood of L_{sing} constructed in Lemma 2.4.3.

Now consider the graphs $\Gamma(dh)$ and $\Gamma(-dh)$ taken inside the cotangent bundle T^*L'. Since we have Hamiltonian isotoped L_{sing} near $(0,0) \in T^*T^3$ to agree with the Harvey–Lawson cone, it follows that the intersection $(\Gamma(dh) \cup \Gamma(-dh)) \cap U$ pulled back to $\mathbb{C}^3$ and projected to $\mathbb{C}$ under $q\colon \mathbb{C}^3 \to \mathbb{C}$ is the union of two open arcs

$$q \circ \Phi^{-1}((\Gamma(dh) \cup \Gamma(-dh)) \cap U) = \{re^{i\alpha}\colon r \in (0,\epsilon)\} \cup \{re^{-i\alpha}\colon r \in (0,\epsilon)\},$$

for a small (in absolute value) real number α. Next, we choose a slightly smaller Darboux ball $U' \subset U$ centered at 0, and a Hamiltonian isotopy ψ supported in a neighborhood of U' such that

$$\begin{aligned} & q \circ \Phi^{-1} \circ \psi((\Gamma(dh) \cup \Gamma(-dh)) \cap (U \setminus U')) \\ & = \{x + i\delta \colon x \in (a,b)\} \cup \{x - i\delta \colon x \in (a,b)\} \end{aligned}$$

agrees with the union of two horizontal arcs in $\mathbb{C}$. Since $\psi((\Gamma(dh) \cup \Gamma(-dh))$ has this form, we can form the union

$$L_{\text{im}} := L_{loc} \cup (\psi(\Gamma(dh) \cup \Gamma(-dh)) \setminus U'),$$

which is a smooth Lagrangian immersion.

Lemma 2.4.7. *The Lagrangian L_{im} is cylindrical at infinity, and the cylindrical ends consist of two disjoint Hamiltonian isotopic copies of the periodized conormal bundles over the infinite edges of V. Furthermore, L_{im} is exact, with a locally constant primitive over the cylindrical ends.*

Proof. The statement about cylindrical ends is clear. Recall that the smooth part L' of the Lagrangian $L \subset T^*T^3$ is exact and cylindrical at infinity by construction. Since $\psi(\Gamma(dh))$ and $\psi(\Gamma(-dh))$ are both obtained by applying a Hamiltonian isotopy to L', it follows that $(\psi(\Gamma(dh) \cup \Gamma(-dh)) \setminus U')$ has an induced primitive. By our assumption about the gradient flow of h near the cusps of L', it follows that L_{im} remains cylindrical at infinity, and hence that the primitive on $(\psi(\Gamma(dh) \cup \Gamma(-dh)) \setminus U')$ is necessarily constant on the cylindrical ends.

To finish the proof, we must show that the primitive on $(\psi(\Gamma(dh) \cup \Gamma(-dh)) \setminus U')$ extends over L_{loc}. Since we know that the projection $\Phi^{-1} \circ \psi((\Gamma(dh) \cup \Gamma(-dh) \cap (U \setminus U'))$ under q is the union of two horizontal arcs in $\mathbb{C}$, it follows the restriction of the primitive to $\psi((\Gamma(dh) \cup \Gamma(-dh) \cap (U \setminus U'))$ can be written in the collar

$$(T^2 \sqcup T^2) \times (0,1) \cong \psi((\Gamma(dh) \cup \Gamma(-dh) \cap (U \setminus U'))$$

as the product of a linear function on the $(0,1)$-factor with a constant function on the $T^2 \sqcup T^2$-factor.

This extends uniquely to a primitive defined on L_{loc} (which is determined by the area of this disk shown in Figure 2.10. □

Let $\widetilde{L}_{\text{im}}$ denote the 3-manifold obtained as follows. Choose a collar neighborhood of the zeroth cusp of L' which we identify with $T^2 \times (0,1)$, so that by removing $T^2 \times (0,\epsilon)$, we get a manifold with one T^2-boundary component. Then define $\widetilde{L}_{\text{im}} := (L' \setminus T^2 \times (0,\epsilon)) \cup_{T^2 \times \{\epsilon\}} (L' \setminus T^2 \times (0,\epsilon))$, where the gluing is by the identity on the boundary 2-torus $T^2 \times \{\epsilon\}$. Then we can think of L_{im} as the image of a Lagrangian immersion $\widetilde{L}_{\text{im}} \to T^*T^3$.

2.4.4 Grading

In the next two subsections we will identify all local systems for which $CF^*(L_{\text{im}})$ is unobstructed. We begin by determining the gradings of generators of $CF^*(L_{\text{im}})$ associated to self-intersections.

Lemma 2.4.8. *L_{im} is graded, and the grading function $\alpha^\#\colon \widetilde{L}_{\text{im}} \to \mathbb{R}$ is approximately 0 at the critical points of $-h$ and approximately 1 at the critical points of h.*

Proof. Since $c_1(T^*T^3) = 0$, it admits a quadratic volume form $\det^2 := \bigwedge_{j=1}^{3}(dq_j + id\theta_j)^{\otimes 2}$. If $\mathcal{L} \to T^*T^3$ denotes the Lagrangian Grassmannian bundle over T^*T^3, then the Lagrangian immersion L_{im} determines a section σ of $\mathcal{L}\mid_{\widetilde{L}_{\text{im}}}$. We need to show that the composition $\alpha := \det^2 \circ \sigma\colon \widetilde{L}_{\text{im}} \to S^1$ admits a lift $\alpha^\#\colon \widetilde{L}_{\text{im}} \to \mathbb{R}$.

The restriction of α to the cylindrical ends of L_{im} is constantly 1, so we can take $\alpha^\#$ to be locally constant over the ends. Because the inclusion of the cylindrical ends into L_{im} induces an isomorphism on H_1, it follows that the lift $\alpha^\#$ can be defined globally. In the local model for L_1, one can see that the tangent spaces to L_1 away from the origin will have opposite orientations, so we can take $\alpha^\#$ to have the values given in the statement of the lemma. □

The switching components of $\widetilde{L}_{\text{im}} \times_{T^*T^3} \widetilde{L}_{\text{im}}$ are either transverse double points, or the 2-torus in L_1. Letting $\iota\colon \widetilde{L}_{\text{im}} \to T^*T^3$ denote the immersion, each transverse double point is of the form $\iota(p_-) = \iota(p_+)$, where p_- is a Morse critical point of $-h$ and p_+ is the corresponding Morse critical point of h. Since the two intersecting sheets of L_{im} at these points are locally the graphs of $-dh$ and dh, it follows from [44] that the index of p_- is given by

$$\begin{aligned} \deg(p_-) &= \mu_{\mathrm{Morse}}(p_-) - \alpha^{\#}(p_-) + \alpha^{\#}(p_+) \\ &= \mu_{\mathrm{Morse}}(p_-) - \alpha^{\#}(p_-) + \alpha^{\#}(p_-) + 1 \\ &= \mu_{\mathrm{Morse}}(p_-) + 1. \end{aligned}$$

Symmetrically, one has that

$$\deg(p_+) = \mu_{\mathrm{Morse}}(p_+) - 1$$

for generators on the positive sheet. To compute gradings for the Floer generators coming from the T^2-family of self-intersection points, we consider the arc $U \subset \mathbb{C}$. A straightforward computation shows that the T^2 switching component of L_{im} contributes copies of $H^*(T^2)[-2]$ and $H^*(T^2)[1]$ to the Floer cochain space.

2.4.5 Unobstructedness

It will turn out that L_{im} will only be an object of the wrapped Fukaya category for certain choices of local system. To determine these local systems, we will need to compute $\mathfrak{m}_0$. Recall that we described a particular choice of spin structure for L' in Lemma 2.3.3, and this induces a spin structure on $\widetilde{L}_{\mathrm{im}}$. Hereafter, we assume that $\widetilde{L}_{\mathrm{im}}$ is equipped with this spin structure.

Lemma 2.4.9. *Let ∇ be a $GL(1)$-local system on L', and let μ_0 and λ_0 denote the holonomy of ∇ about the meridian m_0 and longitude ℓ_0. Then*

$$\mathfrak{m}_0 = \pm(1 + \mu_0 + \lambda_0^{-1}).$$

In particular, $(\widetilde{L}_{\text{im}}, \nabla)$ *is an object of the Fukaya category if and only if the holonomy of* ∇ *satisfies*

$$1 + \mu_0^{-1} + \mu_0^{-1}\lambda_0^{-1} = 0. \tag{2.4.2}$$

Proof. Using the product decomposition of Lemma 2.4.8, we can see that any holomorphic teardrop with boundary on L_{loc} must be a product, where one factor is the holomorphic teardrop in $\mathbb{C}$ with boundary on L_1, and the other factor is either constant or a Maslov 2 disk with boundary on the Clifford torus in $\mathbb{C}^2$. All three of these disks are regular with respect to the integrable almost complex structure on $T^*T^3 \cong (\mathbb{C}^*)^3$ by the argument in [33].

For topological reasons, L_{im} cannot bound any other holomorphic teardrops. If there another teardrop D with boundary on L_{im}, then its boundary would have to lift to a path on $\widetilde{L}_{\text{im}}$ between the two branches. Then its intersection $D \cap U$ with the Darboux ball we chose near $(0,0) \in T^*T^3$ would project to a region in $q \circ \Phi^{-1}(U) \subset \mathbb{C}$, which is an open ball by construction. Since $q \colon \mathbb{C}^3 \to \mathbb{C}$ is a proper map (because it is a homogeneous polynomial), it follows that this region must be bounded by L_1. Because the boundary of D passes through the switching component, this region is the union of the teardrop bounded by L_1 with a portion of $\mathbb{C}$ lying between the two horizontal arcs of L_1. In particular, the image of D is not open, and since q is holomorphic this contradicts the open mapping theorem.

By our choice of coordinates near the singular point of L, it follows that the two Maslov 2-disks in $\mathbb{C}^2$ will have boundary circles homologous to m_0 and ℓ_0^{-1} in $\widetilde{L}_{\text{im}}$. The spin structure we have chosen for L' induces a spin structure on $\widetilde{L}_{\text{im}}$, and using this spin structure all three of the holomorphic teardrops described here contribute with the same sign. □

2.5 Floer-theoretic support of the immersed Lagrangian

We will prove Theorem 1.2.3 in this section. In the first subsection, we will introduce some notation relating to Lagrangian correspondences. The second subsection proves the main auxiliary results regarding quilted Floer theory that we will use. Finally, the last subsection is devoted to the proof of Theorem 1.2.3.

2.5.1 Lagrangian correspondences

The Floer cohomology computations in the proof of Theorem 1.2.3 is via an adjunction isomorphism which is proved using quilted Floer theory. This subsection fixes some notation for Lagrangian correspondences.

Definition 2.5.1. A Lagrangian correspondence between two symplectic manifolds (M_0, ω_0) and (M_1, ω_1) is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$, where M_0^- denotes M_0 equipped with the symplectic form $-\omega_0$.

Given two Lagrangian correspondences $L_{01} \subset M_0 \times M_1$ and $L_{12} \subset M_1 \times M_2$, their geometric composition $L_{01} \circ L_{02}$ is defined to be the image of the fiber product $L_{01} \times_{M_1} L_{12} := (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$ under the projection $M_0 \times M_1 \times M_1 \times M_2 \to M_0 \times M_2$.

Note that a Lagrangian correspondence from $\{\mathrm{pt}\}$ to (M_1, ω_1) is just a Lagrangian $L \subset M_1$. Given a Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$, we let $L_{01}^t \subset M_1^- \times M_0$ denote the same submanifold thought of as a correspondence from M_1 to M_0. We assume that L_{01} is embedded for convenience, as this will be the case for the correspondences appearing in the proof of Theorem 1.2.3. To make sense of the Floer theory for L_{01}, we should also require that it either be cylindrical at infinity in $M_0 \times M_1$ with respect to the natural cylindrical end, or a product of Lagrangians in M_0 and M_1 which are both cylindrical at infinity (cf. Section 2.5.2).

One should think of Lagrangian correspondences as acting on Lagrangian submanifolds in the following way.

Definition 2.5.2. Given $L_0 \subset M_0$ and $L_{01} \subset M_0^- \times M_1$, their geometric composition $L_0 \circ L_{01}$ is defined to be the image under the projection $M_0^- \times M_1$ of the fiber product $L_0 \times_{M_0} L_{01}$ over the diagonal in $M_0 \times M_0^-$. Recall that the fiber product is the intersection $(L_0 \times L_{01}) \cap (\Delta_{M_0} \times M_1)$.

This is an immersed Lagrangian submanifold when the fiber product is a transverse intersection. Given local systems on L_0 and L_{01}, the fiber product $L_0 \times_{M_0} L_{01}$ carries an induced local system. If one thinks of local systems as representations of π_1, this local system is obtained by first forming the external tensor product representation of $\pi_1(L_0) \times \pi_1(L_{01})$, and precomposing this with the group homomorphism $\pi_1(L_0 \times_{M_0} L_{01}) \to \pi_1(L_0 \times L_{01}) \cong \pi_1(L_0) \times \pi_1(L_{01})$.

Spin structures on L_0 and L_{01} also induce a spin structure on $L_0 \circ L_{01}$. Since we will need to consider these induced spin structures, we sketch the proof of this fact following Fukaya [43].

Lemma 2.5.3. *Suppose that the fiber product $L_0 \times_{M_0} L_{01}$ is a transverse intersection and that L_1 is an immersed Lagrangian with clean self-intersections. Then if L_0 and L_{01} have spin structures, these determine a choice of spin structure on L_1.*

Proof. For $x = (y, z) \in L_1$, there is a canonical isomorphism

$$T_x L_1 \oplus T_y M_0 \cong T_y L_0 \oplus T_z L_{01}.$$

Since the fiber product $L_0 \times_{M_0} L_{01}$ is transverse, we can choose smooth triangulations of L_0, M_0, L_{01}, and $L_0 \times_{M_0} L_{01}$ such that:

(i) The maps $L_0 \to M_0$ and $L_{01} \to M_0$ preserve 2-skeletons.

(ii) The 2-skeleton of $L_0 \times_{M_0} L_{01}$ is contained in the fiber product of 2-skeletons $(L_0)_{[2]} \times_{(M_0)_{[2]}} (L_{01})_{[2]}$.

The trivialization on the 2-skeleton $(L_{01})_{[2]}$ of $\pi_1^*(TM_0) \oplus TL_{01}$ and the trivialization on $(L_0)_{[2]}$ of TL_0 induce a trivialization of $T_y M_0 \oplus T_z L_{01} \oplus T_y L_0$ on the fiber product $(L_0)_{[2]} \times_{(M_0)_{[2]}} (L_{01})_{[2]}$. Therefore it induces a trivialization of $T_x L_1 \oplus T_y M_0 \oplus T_y M_0$. Since TM_0 is oriented, $TM_0 \oplus TM_0$ is spin, so the existence of a trivialization of this bundle on $(L_1)_{[2]}$ implies the existence of a trivialization of TL_1 on $(L_1)_{[2]}$. □

2.5.2 Wrapped Floer theory in product manifolds

Let M_0 and M_1 be a pair of Liouville manifolds. Recall that their product $M_0 \times M_1$ is also a Liouville manifold. The product admits a natural cylindrical end constructed by Oancea [72]. To construct the end, consider the subsets $U_0, U_1, U_2 \subset M_0 \times M_1$ given by

$$U_0 = M_0 \times (\partial M_1 \times [1, \infty))$$
$$U_1 = (\partial M_0 \times [0, \infty)) \times M_1$$
$$U_2 = (\partial M_0 \times [1, \infty)) \times (\partial M_1 \times [1, \infty)).$$

Let r_0 and r_1 denote the radial coordinates on M_0 and M_1, and consider real constants $\kappa_0, \kappa_1 > 1$. Choose a hypersurface Σ transverse to the Liouville vector field for $M_0 \times M_1$, subject to the conditions

$$r_0|_{\Sigma \cap U_1} = \kappa_0$$
$$r_0|_{\Sigma \cap U_2} \in [1, \kappa_0]$$
$$r_1|_{\Sigma \cap U_0} = \kappa_1$$
$$r_1|_{\Sigma \cap U_2} \in [1, \kappa_1].$$

The cylindrical end is parametrized by the map

$$\Sigma \times [1, \infty) \to (M_0 \times M_1)$$
$$(z, r) \mapsto (\phi^r_{M_0}(\pi_{M_0}(z)), \phi^r_{M_1}(\pi_{M_1}(z))).$$

Now consider a pair of Lagrangian branes (L_i, ∇_i) in M_i, for $i = 0, 1$, both of which are unobstructed and cylindrical at infinity. We will need to consider wrapped Floer cohomology for $\widetilde{L}_0 \times \widetilde{L}_1 \to M_0 \times M_1$. It is not immediately clear that this can be done, as the standard compactness argument for wrapped Floer trajectories requires Lagrangians to be cylindrical at infinity, which will not be the case for a product of cylindrical Lagrangians. This issue is resolved in [65, §3.4-3.5], by estimating the action of any Hamiltonian chord which would arise as the generator of a Floer complex.

Lemma 2.5.4. *Consider two unobstructed Lagrangian branes* $(\widetilde{L}, \nabla_L)$ *and* $(\widetilde{K}, \nabla_K)$ *in* $M_0 \times M_1$*, each of which can either be a product of cylindrical Lagrangians, or cylindrical at infinity with respect to the product end in* $M_0 \times M_1$*. Suppose that* H *is a Hamiltonian on* $M_0 \times M_1$ *which is either a split Hamiltonian, or quadratic in the product cylindrical end. Then for any pair of Hamiltonian chords* x *and* y *joining* L *and* K*,*

(i) The moduli space $\overline{\mathcal{M}}(x, y)$ *is compact.*

(ii) For each fixed chord y*, the moduli spaces* $\overline{\mathcal{M}}(x, y)$ *are empty for all but finitely many chords* x*.*

To prove this, one follows the arguments of Gao, for all possibilities for L and K, in the embedded case. Note that there will be additional boundary strata corresponding to disk bubbles on the Lagrangian immersions, but since these are constrained to a compact subset of $M_0 \times M_1$, they do not affect the action estimates which take place in the cylindrical end.

It is similarly straightforward to adapt Gao's argument [65, §7] for the invariance of wrapped Floer cohomology in the product to the unobstructed setting.

Lemma 2.5.5. *Suppose that* $(\widetilde{L}, \nabla_L)$ *and* $(\widetilde{K}, \nabla_K)$ *are Lagrangian branes in* $M_0 \times M_1$*, each of which are either cylindrical or products of cylindrical Lagrangian immersion. Then the wrapped Floer cohomology defined with respect to the split Hamiltonian* H_{M_0,M_1} *and split almost complex structures* J_{M_0,M_1}*, denoted*

$$HW^*(L, K; H_{M_0,M_1}, J_{M_0,M_1})$$

and the wrapped Floer cohomology defined with respect to an appropriate choice of admissible Hamiltonians and almost complex structures

$$HW^*(L, K; H, J)$$

are isomorphic.

The key point of Gao's proof is to deform the split Hamiltonian H_{M_0,M_1} to an appropriate quadratic Hamiltonian H outside of a compact subset $M_0 \times M_1$. The presence of self-intersection points on a Lagrangian, which by our assumptions all lie inside a compact set, does not affect this part of the argument. Having done this, it is possible to define a map between truncated Floer complexes, which can be assembled into an action-restriction map, which is a filtration-preserving cochain map that induces the desired isomorphism on cohomology.

Definition 2.5.6. For L_0, L_{01}, and L_1 as above, define the quilted Floer cohomology to be

$$HW^*(L_0, L_{01}, L_1) \cong HW^*(L_{01}, L_0 \times L_1; H_{M_0,M_1}, J_{M_0,M_1}).$$

Since the differential on $CW^*(L_{01}, L_0 \times L_1; H_{M_0,M_1}, J_{M_0,M_1})$ only counts pseudoholomorphic strips, there is an obvious correspondence between Floer trajectories in the product and pseudo-holomorphic quilts with seam conditions on (L_0, L_{01}, L_1) (cf. [65] and [73]). For a definition of quilted surfaces, we refer the reader to [65].

Our proof of the next theorem follows the argument of [43] (see also [74]).

Theorem 2.5.7. *Suppose that L_0 and L_1 are unobstructed Lagrangian branes which are positive in the sense of Definition 2.1.7. Let $L_{01} \subset M_0 \times M_1$ be an exact embedded Lagrangian correspondence equipped with a local system, grading, and Spin structure, and suppose that the geometric composition $L_0 \circ L_{01}$ with the induced brane data is also positive, and additionally that there are no chords in $\mathcal{X}(L_0 \circ L_{01}, L_0 \circ L_{01})$ of degree 1 which have greater action than the chord representing the unit. Then there is an isomorphism*

$$HW^*(L_0, L_{01}, L_1) \cong HW^*(L_0 \circ L_{01}, L_1)$$

of wrapped Floer cohomology groups.

Proof. First observe that there is a canonical identification of Floer cochain spaces

$$CW^*(L_0, L_{01}, L_1) \cong CW^*(L_0 \circ L_{01}, L_1).$$

We will deform this to a cochain map by counting certain inhomogeneous quilts. Consider inhomogeneous pseudoholomorphic quilts whose domain consists of two patches $S_0 = \mathbb{R} \times [0,1]$ and $S_1 = (\mathbb{R} \times [1,2]) \setminus \{(0,2)\}$, for which the maps $u_0 \colon S_0 \to M_0$ and $u_1 \colon S_1 \to M_1$ satisfy the seam conditions

$$\begin{aligned} u_0(\mathbb{R} \times \{0\}) &\subset L_0 \\ u_0(\mathbb{R} \times \{1\}) \times u_1(\mathbb{R} \times \{1\}) &\subset L_{01} \\ u_1((0,\infty) \times \{2\}) &\subset L_0 \circ L_{01} \\ u_1((-\infty,0) \times \{2\}) &\subset L_1 \end{aligned}$$

and which converge exponentially at the ends to a chord $y \in \mathcal{X}(L_0 \circ L_{01}, L_1)$, a chord (x_-, x_+) for (L_0, L_{01}, L_1), and the chord e for $(L_0, L_{01}, L_0 \circ L_{01})$ corresponding to a chord representing the unit of $HW^*(L_0 \circ L_{01})$ (cf. Figure 2.11). Let $\mathcal{M}((x_-, x_+), e, y)$ denote the moduli space of all such quilts. We must choose suitable perturbation data for these quilts, specifically domain-dependent Hamiltonians, domain-dependent almost complex structures, times-shifting functions, and basic one-forms. This is identical to the discussion of perturbation data in the construction of the wrapped Fukaya category, so we will not repeat it. Consider the map

$$CW^*(L_0, L_{01}, L_1) \to CW^*(L_0 \circ L_{01}, L_1) \tag{2.5.1}$$

given by counting elements in the zero-dimensional moduli spaces $\mathcal{M}((x_-, x_+), e, y)$. Examining the boundary strata of the one-dimensional moduli spaces of this form, elements of which are depicted in Figure 2.12, shows that this is a chain map if and only if e is closed under the differential on $CW^*(L_0, L_{01}, L_0 \circ L_{01})$, but this is immediate from our choice of e, and our assumption on

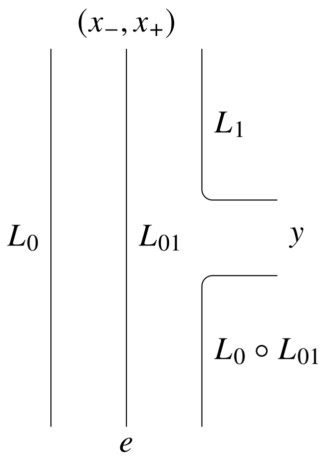

Figure 2.11: Elements of the moduli spaces $\mathcal{M}((x_-, x_+), e, y)$ used to construct 2.5.1. Strictly speaking, the boundary conditions should be translates of these Lagrangians by the Liouville flow for times specified by time-shifting functions.

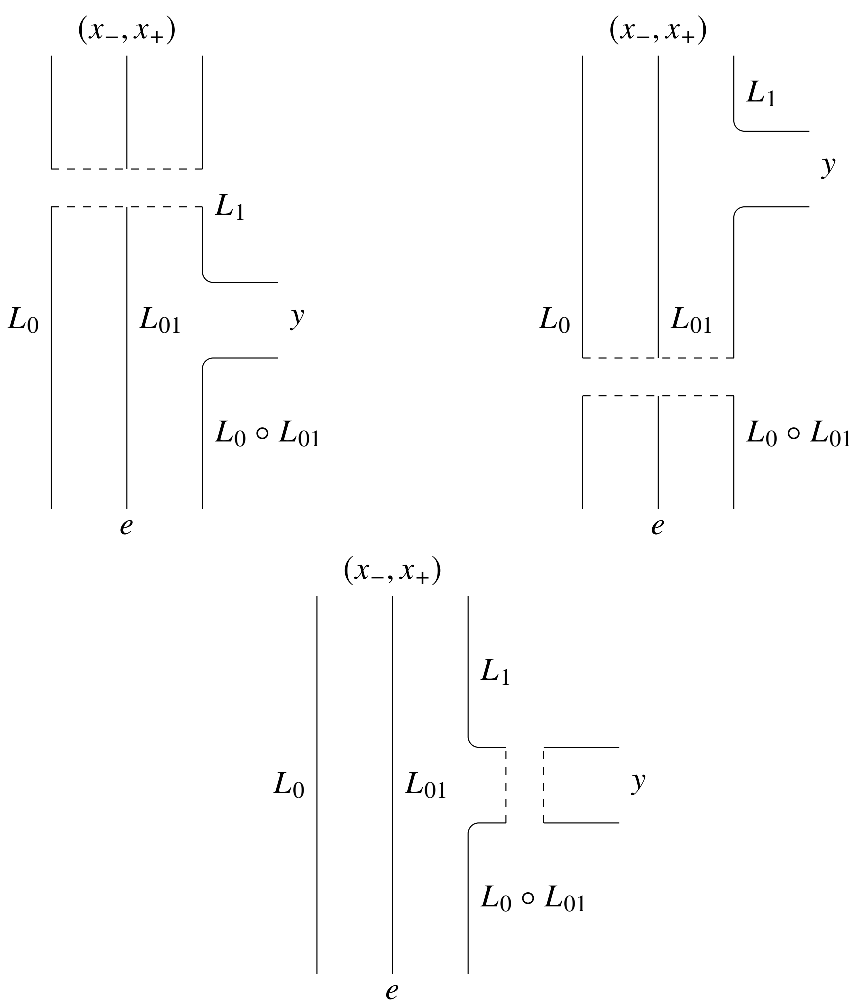

Figure 2.12: Quilts in the codimension 1 boundary strata of $\mathcal{M}((x_-, x_+), e, y)$.

the action of degree 1 chords. That this map is a quasi-isomorphism follows because it can be decomposed as a sum of linear maps which increases the action namely, the original identification of wrapped Floer cochain spaces, and a linear map which strictly increases action by a larger amount. It is therefore a vector space isomorphism at the chain level, and thus it induces the deisred isomorphism of Floer homology groups. □

Corollary 2.5.8 (Adjunction Isomorphism)**.** *Suppose we are in the situation of Theorem 2.5.7. Then there is an isomorphism of Floer cohomology groups*

$$HW^*(L_0 \circ L_{01}, L_1) \cong HW^*(L_0, L_1 \circ L_{01}^t),$$

where the geometric compositions $L_0 \circ L_{01}$ and $L_1 \circ L_{01}^t$ are equipped with the induced local systems, gradings, and spin structures.

Remark 2.5.9. When $L_0 \circ L_{01}$ is embedded, e can be chosen to have maximal action among all chords [65], but our action assumption is nonetheless somewhat unnatural. We expect that it can be removed by adapting the strip-shrinking argument of [73] to wrapped Floer cohomology. The positivity condition should be enough to rule out figure eight bubbling. One could then conclude the main result, but also note that in the language of [43] and [74], strip-shrinking would let us conclude that the chord e is closed under the (undeformed) differential. We have not, however, checked that the elliptic estimates of [73, §5.2] hold in our situation.

2.5.3 Computation of support

The strategy of the proof of Theorem 1.2.3 is to reduce the computation of $HW^*((L_{\mathrm{im}}, \nabla), (T^3, \nabla_p))$ to the support computation from Lemma 2.2.7, where we computed the Floer-theoretic support of a tropical pair of pants, using Lagrangian correspondences.

In the following definition, we will identify T^*T^n with the product $(T^*S^1)^n \cong (\mathbb{R} \times S^1)$, where the latter has coordinates $x = (q, \theta)$. Also recall that we have fixed an identification of T^n with the unit cube.

Definition 2.5.10. Define the Lagrangian correspondences $\mathcal{L}_i \subset (T^*T^3)^- \times (T^*T^2)$ to be

$$\mathcal{L}_1 = \left\{ (x_1, x_2, x_3, x_2, x_3) \,\middle|\, x_1 = \left(q_1, \frac{1}{2}\right),\ q_1 \in \mathbb{R} \right\},$$
$$\mathcal{L}_2 = \left\{ (x_1, x_2, x_3, x_1, x_3) \,\middle|\, x_2 = \left(q_2, \frac{1}{2}\right),\ q_2 \in \mathbb{R} \right\},$$
$$\mathcal{L}_3 = \left\{ (x_1, x_2, x_3, x_1, x_2) \,\middle|\, x_3 = \left(q_3, \frac{1}{2}\right),\ q_3 \in \mathbb{R} \right\}.$$

For $\{i, j, k\} = \{1, 2, 3\}$, we will use (x_j, x_k) to denote the coordinates on the codomain, denoted $(T^*T^2)_{jk}$ of $\mathcal{L}_i$.

These correspondences are Lagrangian because up to a reordering of coordinates, each $\mathcal{L}_i$ is the product of the diagonal in $T^*T^2 \times T^*T^2$ with a cotangent fiber of T^*S^1. As a set, the image of the fiber product $L \times_{T^*T^3} \mathcal{L}_i$ in T^*T^2 agrees with Matessi's pair of pants, which one easily sees by setting $\theta_i = 1/2$ in the formulae for $\widetilde{dg}$.

The geometric composition $L_{\mathrm{im}} \circ \mathcal{L}_i$ is an immersed Lagrangian submanifold, but it will turn out that the support of the geometric composition ultimately follows from Lemma 2.2.7. The next lemma describes this geometric composition as an immersed Lagrangian submanifold.

Lemma 2.5.11. *The geometric composition $L_{\mathrm{im}} \circ \mathcal{L}_i$ is the union of two embedded Lagrangian submanifolds $\Gamma(dh_i) \cup \Gamma(-dh_i)$ lying in a Weinstein neighborhood of the Lagrangian pair of pants in T^*T^2. Here h_i is a Morse function on the pair of pants with two critical points of index* 0 *and three critical points of index* 1.

Proof. Recall that away from the singular point of L, we defined L_{im} as the union $\Gamma(dh) \cup \Gamma(-dh)$, where h was the Morse function associated to the ideal triangulation of M. We can identify the Lagrangian pair of pants in $(T^*T^2)_{jk}$ with a certain submanifold of M. In particular, it is the union of the faces of $T_{0,\pm}$ which intersect the jth, kth, and 4th vertices. Hence the Morse function h on M restricts to a Morse function h_i on these three pairs of pants with critical points as described above. □

Although $T^3 \circ \mathcal{L}_i = T^2$, it will not be the case that the Floer homology groups $HF^*(L_{\mathrm{im}}, T^3)$ and

$HF^*(L_{\mathrm{im}} \circ \mathcal{L}, T^2)$ are isomorphic. Indeed, one should not expect this by mirror symmetry; if $C \subset (\mathbb{C}^*)^n$ is a curve, then for any point $p \in C$, the group $\mathrm{Ext}^*(\mathcal{O}_C, \mathcal{O}_p)$ depends on the codimension of C. What we have instead is an adjunction isomorphism involving another Lagrangian in T^*T^3 mirror to a particularly simple curve.

Lemma 2.5.12. *The geometric composition $T^2 \circ \mathcal{L}^t_{1,2}$ is the subsapce $\{\theta_3 = \frac{1}{2}\}$, and hence it is Hamiltonian isotopic to $N^*\langle q_3\rangle / N^*_{\mathbb{Z}}\langle q_3\rangle$.*

Given a $GL(1)$-local system ∇ on $\widetilde{L}_{\mathrm{im}}$, we need to determine the holonomy of the induced local systems on the geometric compositions $L_{\mathrm{im}} \circ \Gamma_i$. Before computing it, we fix some notation and conventions. Note by the Mayer–Vietoris sequence that $H_1(\widetilde{L}_{\mathrm{im}}) \cong H_1(L') \oplus H_1(L')/H_1(T^2)$, where $H_1(T^2)$ is generated by $\{m_0, \ell_0\}$, the longitude and meridian of the 0th component of the minimally twisted five-component chain link. This means that for $i = 1, 2, 3, 4$, the group $H_1(\widetilde{L}_{\mathrm{im}})$ has two distinct generators, say $m_{i,+}$ and $m_{i,-}$ corresponding to meridians of the ith link component in each of the copies of $\widetilde{L}$. From now on, we will only consider local systems ∇ for which the holonomy satisfies $\mathrm{hol}_\nabla(m_{i,+}) = \mathrm{hol}_\nabla(m_{i,-})$ for $i = 1, 2, 3, 4$. We define $\mu_i = \mathrm{hol}_\nabla(m_{i,+}) = \mathrm{hol}_\nabla(m_{i,-})$. Similarly the longitudes $\ell_{i,+}$ and $\ell_{i,-}$ near the ith cusps of the two copies of $\widetilde{L}$ are different in $H_1(\widetilde{L}_{\mathrm{im}})$, and our assumption implies that $\lambda_i := \mathrm{hol}_\nabla(\ell_{i,+}) = \mathrm{hol}_\nabla(\ell_{i,-})$ for $i = 1, 2, 3, 4$.

The Lagrangian correspondences $\mathcal{L}_i$ are always equipped with the trivial $GL(1)$ local system.

Lemma 2.5.13. *Fix ∇ on $\widetilde{L}_{\mathrm{im}}$ as above.*

(i) *The induced local system on the geometric composition $L_{\mathrm{im}} \circ \mathcal{L}_1$ has holonomy μ_2 about the two circles conormal to q_2, and holonomy λ_3^{-1} about the two circles conormal to q_3.*

(ii) *The induced local system on $L_{\mathrm{im}} \circ \mathcal{L}_2$ has holonomy $(\mu_3\lambda_3)^{-1}$ about the two circles conormal to q_3, and holonomy λ_1 about the two circles conormal to q_1.*

(iii) *The induced local system on $L_{\mathrm{im}} \circ \mathcal{L}_3$ has holonomy $\mu_2\lambda_2$ about the two circles conormal to q_2, and holonomy $\mu_1\lambda_1$ about the circles conormal to q_1.*

Proof. Fix a basepoint pt on one of the two path components of $L_{\mathrm{im}} \times T^*T^3 \mathcal{L}_i$ of maximal dimension. Recall that the holonomy representation of the induced local system is given by composing $\pi_1(\widetilde{L}_{\mathrm{im}} \times_{T^*T^3} \mathcal{L}_i, \mathrm{pt}) \to \pi_1(\widetilde{L}_{\mathrm{im}}) \times \pi_1(\mathcal{L}_i)$ with the box tensor product representation. Since $\mathcal{L}_i$ carries the trivial local system, the induced local system on the chosen component of $L_{\mathrm{im}} \times_{T^*T^3} \mathcal{L}_i$ is determined by the map $\pi_1(L_{\mathrm{im}} \times_{T^*T^3} \mathcal{L}_i, \mathrm{pt}) \to \pi_1(L_{\mathrm{im}})$, which in our situation is easily seen to be the map induced by the inclusion of a pair of pants into M, as discussed in the proof of Lemma 2.5.11. There we identified the pair of pants with the union of the two faces in the ideal triangulation of M with vertices at the jth, kth and 4th vertices, where $\{i, j, k\} = \{1, 2, 3\}$. The puncture on this pair of pants at the jth vertex is the conormal of one component of $L_{\mathrm{im}} \times_{T^*Ts} \mathcal{L}_i \subset (T^*T^2)_{jk}$ to q_k. From this, the holonomies on the induced local systems in the θ_j and θ_k-directions are determined by the holonomies of ∇ at the kth cusp in the θ_j-direction and at the kth csp in the θ_k-direction, respectively. ☐

We are now ready to compute the Floer-theoretic support of the geometric composition.

Lemma 2.5.14. *For $i = 1, 2, 3$, the wrapped Floer cohomology $HW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2)$ is nonzero in the following cases.*

- **(i)** $HW^*(L_{\mathrm{im}} \circ \mathcal{L}_1, T^2) \neq 0$ *if and only if* $\lambda_3 x_2 - \mu_2^{-1} x_3 + 1 = 0$.
- **(ii)** $HW^*(L_{\mathrm{im}} \circ \mathcal{L}_2, T^2) \neq 0$ *if and only if* $-\mu_2^{-1}\lambda_2^{-1} x_1 + \mu_1^{-1}\lambda_1^{-1} x_2 + 1 = 0$.
- **(iii)** $HW^*(L_{\mathrm{im}} \circ \mathcal{L}_3, F_0^2) \neq 0$ *if and only if* $\mu_3\lambda_3 x_1 - \lambda_1^{-1} x_3 + 1 = 0$.

Proof. Since the induced bounding cochain on $L_{\mathrm{im}} \circ \mathcal{L}_i$ is 0, it follows that the chain complex $CW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2)$ splits as a direct sum $CW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2) \cong CW^*(L_{pants}, T^2) \oplus CW^*(L_{pants}, T^2)$, since any disk which contributes nontrivially to the differential must have exactly two boundary marked points and one boundary component on (one diagonal component of) $L_{\mathrm{im}} \circ \mathcal{L}_i$. As a graded vector space, $CW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2)$ has two generators in degree 1, denoted $x_1^{\pm}$, and two generators in degree 0, denoted $x_0^{\pm}$.

By the proof of Lemma 2.2.7 and Lemma 2.5.13, the differentials on both summands of $CW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2)$ are as follows.

$$d(x_0^{\pm}) = \begin{cases} \pm(\lambda_3 x_2 - \mu_2^{-1} x_3 + 1)x_1^{\mp} & i = 1 \\ \pm(-\mu_2^{-1}\lambda_2^{-1} x_1 + \mu_1^{-1}\lambda_1^{-1} x_2 + 1)x_1^{\mp} & i = 2 \\ \pm(\mu_3 \lambda_3 x_1 - \lambda_1^{-1} x_3 + 1)x_1^{\pm} & i = 3. \end{cases}$$

The signs are as above because, each of the induced spin structures on $L_{\mathrm{im}} \times_{T^*T^3} \mathcal{L}_i$ is only symmetric under one possible transposition of the vertices in the induced ideal triangulation of the pair of pants. □

Remark 2.5.15 (Signs)**.** Although the proof of Lemma 2.5.14 referenced a specific choice of spin structure, any other spin structure we could have chosen on $\widetilde{L}_{\mathrm{im}}$ would have had similar symmetries with respect to different pairs of coordinates. In the proof of Theorem 1.2.3, we will identify the space of unobstructed $GL(1)$-local systems on $\widetilde{L}_{\mathrm{im}}$ with a subspace of the Grassmannian $Gr(2,4) \subset \mathbb{P}^5$. There are 32 automorphisms of $\mathbb{P}^5$ induced by scaling coordinates by -1. These correspond to the 32 local systems on M, since $H_1(M;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^5$. Hence a change of local system on $\widetilde{L}_{\mathrm{im}}$ corresponds to acting on $Gr(2,4)$ by an ambient automorphism.

We can now prove Theorem 1.2.3.

Proof of Theorem 1.2.3. For a fixed local system ∇ on $\widetilde{L}_{\mathrm{im}}$ and spin structure as above, the adjunction isomorphism gives isomorphisms

$$HW^*(L_{\mathrm{im}}, N^*\langle q_i \rangle / N^*_{\mathbb{Z}}\langle q_i \rangle) \cong HW^*(L_{\mathrm{im}} \circ \mathcal{L}_i, T^2),$$

with the induced local systems on $N^*\langle q_i \rangle / N^*_{\mathbb{Z}}\langle q_i \rangle$ and $L_{\mathrm{im}} \circ \mathcal{L}_i$.

Under homological mirror symmetry, the periodized conormal $N^*\langle q_i \rangle / N^*_{\mathbb{Z}}\langle q_i \rangle$ corresponds to a sheaf supported on the curve $\{z_j = x_j,\ z_k = x_k\}$, where x_j and x_k refer to the holonomies of the local system on the periodized conormal. Lemma 2.5.14 tells us exactly when the vector space on the left is nonzero.

By homological mirror symmetry this occurs precisely when the supports of the mirror sheaves to L_{im} and $N^*\langle q_i\rangle/N^*_{\mathbb{Z}}\langle q_i\rangle$ intersect. If this supports intersect nontrivially, it follows that the support of the mirror sheaf to L_{im} contains a point on the curve $\{z_j = x_j,\ z_k = x_k\}$. Applying homological mirror symmetry again shows that there must be some local system on T^3 such that $HW^*(L_{\mathrm{im}}, T^3)$ is nontrivial whose holonomy in the jth and kth directions is subject to the constraints of Lemma 2.5.14. Thus in particular the support of the mirror sheaf is nonempty (in particular L_{im} is not the trivial object in the Fukaya category), and at least one-dimensional. Therefore, if we let x_1, x_2, x_3 denote the holonomies of a local system on T^3 for which $HW^*(L_{\mathrm{im}}, T^3)$ is nontrivial, it follows that it must satisfy the equations

$$\begin{aligned} \lambda_3 x_2 - \mu_2^{-1} x_3 + 1 &= 0, \\ -\mu_2^{-1}\lambda_2^{-1} x_1 + \mu_1^{-1}\lambda_1^{-1} x_2 + 1 &= 0, \\ \mu_3\lambda_3 x_1 - \lambda_1^{-1} x_3 + 1 &= 0. \end{aligned}$$

It now remains to show that these three equations determine the affine part of a line in $\mathbb{P}^3$. If we scale the first of these equations by μ_1^{-1}, the second by μ_3^{-1}, and the third by $\mu_1^{-1}\mu_3^{-1}$, and apply Lemma 2.3.17, we are left with the following relations.

$$\begin{aligned} \mu_1^{-1}\mu_2^{-1}\mu_4 x_2 - \mu_1^{-1}\mu_2^{-1} x_3 + \mu_1^{-1} &= 0, \\ -\mu_1\mu_2 x_1 + \mu_0^{-1}\mu_1\mu_2\mu_3 x_2 + \mu_3 &= 0, \\ \mu_1\mu_2\mu_4^{-1} x_1 - \mu_0^{-1}\mu_1\mu_2\mu_3 x_3 + \mu_1\mu_3 &= 0, \end{aligned}$$

which we can rewrite more compactly as

$$\phi_{12} x_2 - \phi_{13} x_3 + \phi_{14} = 0,$$

$$-\phi_{12}x_1 + \phi_{23}x_3 + \phi_{24} = 0,$$
$$\phi_{13}x_1 - \phi_{23}x_2 + \phi_{34} = 0.$$

To check that these three linear forms define a line in $\mathbb{P}^3$, it suffices to check that the Grassmann–Plücker relation

$$\phi_{12}\phi_{34} + \phi_{13}\phi_{24} + \phi_{14}\phi_{23} = 0,$$

is satisfied. Note that replacing ϕ_{13} with $-\phi_{13}$ yields the usual sign conventions for the Plücker relations. One verifies this relation by computing

$$\begin{aligned}
&\phi_{23}^{-1}\phi_{14}^{-1}\phi_{12}\phi_{34} \\
&= (\mu_0^{-1}\mu_1\mu_2\mu_3)\mu_1(\mu_1^{-1}\mu_2^{-1}\mu_4)\mu_3^{-1} \\
&= \mu_0^{-1}\mu_1\mu_4 \\
&= \mu_0^{-1}\lambda_0^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&\phi_{23}^{-1}\phi_{14}^{-1}\phi_{13}\phi_{24} \\
&= (\mu_0^{-1}\mu_1\mu_2\mu_3)\mu_1(\mu_1^{-1}\mu_2^{-1})(\mu_1^{-1}\mu_3^{-1}) \\
&= \mu_0^{-1}.
\end{aligned}$$

But by Lemma 2.4.9, the relation $\mu_0^{-1}\lambda_0^{-1} + \mu_0^{-1} + 1$ says precisely that $(L_{\mathrm{im}}, \nabla)$ is unobstructed. □

Remark 2.5.16. The proof of Theorem 1.2.3 furnishes a map from the space of unobstructed local systems on $\widetilde{L}_{\mathrm{im}}$ to $Gr(2,4)$ given by

$$\begin{aligned}
(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) \mapsto &[\phi_{12} : \phi_{13} : \phi_{14} : \phi_{23} : \phi_{24} : \phi_{34}] \\
&= [\mu_1^{-1}\mu_2^{-1}\mu_4 : \mu_1^{-1}\mu_2^{-1} : \mu_1^{-1} : \mu_0\mu_1^{-1}\mu_2^{-1}\mu_3^{-1} : \mu_1^{-1}\mu_3^{-1} : \mu_3^{-1}].
\end{aligned}$$

It is easy to see that this map is injective. Moreover, since the values of $\mu_0, \dots, \mu_3$ can be chosen independently, it follows that the image of this map contains all points of $Gr(2,4) \subset \mathbb{P}^5$ which do not lie on the coordinate hyperplanes. Hence, all lines in $\mathbb{P}^3$ represented by such a point of $Gr(2,4)$ are mirror to A-model objects supported on L_{im}.

We end by observing that the computations carried out in this course of this proof also determine the rank of the mirror sheaf to L_{im}.

Corollary 2.5.17. *For every local system $\nabla_p \in \text{Hom}(\pi_1(T^3), \mathbb{K}^*)$ in the Floer-theoretic support of a Lagrangian brane $(\widetilde{L_{\text{im}}}, \nabla_C)$, the degree 0 Floer cohomology group $HF^0((L_{\text{im}}, \nabla_C), (T^3, \nabla_p))$ has rank 2.*

Proof. Recall that (T^3, ∇_p) is mirror to a skyscraper sheaf on $(\mathbb{K}^*)^3$, which we denote O_p, where $p \in (\mathbb{K}^*)^3$. The point p lies in the intersection of the support of the mirror to $(\widetilde{L_{\text{im}}}, \nabla_C)$ and a curve in $(\mathbb{K}^*)$ of the form $\{z_j = x_j, z_k = x_k\}$, where $x_j, x_k \in \mathbb{K}^*$ and $j, k \in \{1, 2, 3\}$. The defining relations for the Floer-theoretic imply that p is the unique point in this intersection. Moreover, we know from the adjunction isomorphism that $HW^*(L_{\text{im}}, N^*\langle q_i \rangle / N^*_{\mathbb{Z}}\langle q_i \rangle) \cong H^*(S^1) \oplus H^*(S^1)$, where both Lagrangians are equipped with the appropriate local systems. The statement now follows from an easy argument using the local-to-global spectral sequence. $\square$

Chapter 3

Infinity inner products and open Gromov–Witten invariants

The contents of this chapter first appeared in [41].

3.1 Hochschild invariants

In this section, we will collect some definitions pertaining to A_∞-algebras, mainly for the purposes of fixing our notation and conventions, and review the definition of Hochschild and cyclic homology in the curved case. These invariants behave somewhat differently than for uncurved A_∞-algebras, since the bar complex is no longer acyclic. Our primary use for this theory is to eventually extract a particular A_∞-bimodule homomorphism $\mathcal{A}_\Delta \to \mathcal{A}^\vee$ from the diagonal bimodule over an A_∞-algebra $\mathcal{A}$ to its dual over the ground field, so this does not pose a serious problem for us.

3.1.1 A_∞-algebras

All A_∞-algebras we consider will be thought of as modules of a certain extension of the Novikov ring.

Definition 3.1.1. Let $\Bbbk$ be a field of characteristic 0. For the formal variables T of degree 0 and e of degree 2, we denote by Λ_{nov} the universal Novikov ring

$$\Lambda_{\text{nov}} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} : a_i \in \Bbbk\,,\ \lambda_i \in \mathbb{R}_{\geq 0}\,,\ \lim_{i\to\infty} \lambda_i = 0 \right\}. \tag{3.1.1}$$

When we define Lagrangian Floer cohomology, it will actually be a module over a certain $\mathbb{Z}$-graded Λ_{nov}-algebra, following [49].

Definition 3.1.2. Let $s, t_0, \ldots, t_N$ be formal variables with integer gradings $|s|$ and $|t_i|$. Define the following $\mathbb{Z}$-graded and graded-commutative rings

$$R \coloneqq \Lambda_{\text{nov}}[[s, t_0, \ldots, t_N]] \tag{3.1.2}$$

$$Q \coloneqq \Lambda_{\text{nov}}[[t_0, \ldots, t_N]]. \tag{3.1.3}$$

The ring R will be the coefficient ring for all A_∞-algebras we consider. The ring Q will be the ring of coefficients for bulk deformation classes, and thus does not appear explicitly in this section. There is a valuation on R defined by

$$\nu \colon R \to \mathbb{R}$$
$$\nu\left(\sum_{j=0}^{\infty} a_j T^{\lambda_j} e^{\mu_j} s^k \prod_{i=0}^{N} t_i^{\ell_{ij}}\right) = \min_{\{j \,:\, a_j \neq 0\}} \left(\lambda_j + k + \sum_{i=0}^{N} \ell_{ij}\right).$$

The extra variables in R and Q enable us to, for example, define the notion of a point-like bounding cochain.

Let $\mathcal{A}$ be a free graded R-module. It follows that there is a free graded $\Bbbk[[s, t_0, \ldots, t_N]]$-module $\overline{\mathcal{A}}$ such that

$$\mathcal{A} = \overline{\mathcal{A}} \otimes_{\Bbbk[[s, t_0, \ldots, t_N]]} R.$$

We denote by $|x|$ the grading of an element $x \in A$, and set $|x|' = |x| - 1$. Note that the grading $|x|$ also incorporates the grading of coefficients in R.

In Lagrangian Floer theory, Gromov compactness implies that the A_∞-algebras we will construct are gapped and filtered. To explain what this means, let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a monoid such that

- the image of G in $\mathbb{R}_{\geq 0}$ is discrete;

- $G \cap (\{0\} \times 2\mathbb{Z}) = \{(0,0)\}$;
- for any $\lambda \in \mathbb{R}_{\geq 0}$, the set $G \cap (\{\lambda\} \times 2\mathbb{Z})$ is finite.

For each $\beta = (\lambda(\beta), \mu(\beta)) \in G$, suppose that we have a collection of linear maps

$$\mathfrak{m}_{k,\beta} \colon (\overline{\mathcal{A}}[1])^{\otimes k} \to \overline{\mathcal{A}}[1] \tag{3.1.4}$$

for which $\mathfrak{m}_{0,(0,0)} = 0$. These induce Λ_{nov}-linear maps

$$\mathfrak{m}_k \colon (\mathcal{A}[1])^{\otimes k} \to \mathcal{A}[1] \tag{3.1.5}$$

$$\mathfrak{m}_k \coloneqq \sum T^{\lambda(\beta)} e^{\mu(\beta)/2} \mathfrak{m}_{k,\beta}. \tag{3.1.6}$$

The operations $\mathfrak{m}_k$ induce coderivations $\widehat{\mathfrak{m}}_k$ given by

$$\widehat{\mathfrak{m}}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-k} (-1)^{\maltese_i} x_n \otimes \cdots \otimes \mathfrak{m}_k(x_{i+1} \otimes \cdots \otimes x_{i+k}) \otimes \cdots \otimes x_n$$

for all $n \geq k$, where

$$\maltese_i \coloneqq \sum_{j=1}^{i} |x_j|' \tag{3.1.7}$$

and setting $\widehat{\mathfrak{m}}_k(x_1 \otimes \cdots \otimes x_n) = 0$ for $n < k$.

Definition 3.1.3. We say that $(\mathcal{A}, \{\mathfrak{m}_{k,\beta}\}_{k=0}^{\infty})$ form a *gapped filtered A_∞-algebra* if the coderivation

$$\widehat{d} = \sum_{k=1}^{\infty} \widehat{\mathfrak{m}}_k$$

satisfies

$$\widehat{d} \circ \widehat{d} = 0.$$

Equivalently, the operations $\mathfrak{m}_k$ are required to satisfy the curved A_∞-relations

$$\sum_{i,\ell} (-1)^{\maltese_i} \mathfrak{m}_{k-\ell+1}(x_1 \otimes \cdots \otimes \mathfrak{m}_\ell(x_{i+1} \otimes \cdots \otimes x_{i+\ell}) \otimes \cdots \otimes x_k) = 0. \tag{3.1.8}$$

Remark 3.1.4 (Sign conventions)**.** The sign (3.1.7) can be thought of as a Koszul sign arising when $\mathcal{A}$ acts on itself on the right. These are the sign conventions used b [51], [75], [76], [54], and [47]. One can translate these signs to those of [52] by replacing $\mathcal{A}$ with the opposite A_∞-algebra.

Additionally, $\mathcal{A}$ is said to be *strictly unital* if there is an element $1 \in \mathcal{A}$ with $|1| = 0$ such that

- $\mathfrak{m}_2(1, x) = x = (-1)^{|x|}\mathfrak{m}_2(x, 1)$ and
- $\mathfrak{m}_k(x_k, \ldots, 1, \ldots, x_1) = 0$ whenever $k \neq 2$.

The Fukaya A_∞-algebras we construct will possess strict units, but all of our arguments can be reworked in the homotopy unital setting [47, §3.3.1]. To compensate for the nonvanishing of $\mathfrak{m}_0$, we use weak bounding cochains, defined using the strict unit.

Definition 3.1.5. Let $\mathcal{A}$ be a strictly unital gapped filtered A_∞-algebra and denote by 1 its strict unit. For any $b \in \mathcal{A}$ with $|b| = 1$ and $\mathrm{val}(b) > 0$, we say that it is a *weak bounding cochain* if it satisfies the weak Maurer–Cartan equation

$$\mathfrak{m}_0^b := \sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = c \cdot 1.$$

Let $\widehat{\mathcal{M}}_{weak}(\mathcal{A})$ denote the set of bounding cochains on $\mathcal{A}$. If b satisfies the Maurer–Cartan equation

$$\mathfrak{m}_0^b = 0 \tag{3.1.9}$$

it is said to be a *bounding cochain*. Let $\widehat{\mathcal{M}}(\mathcal{A})$ denote the set of all bounding cochains on $\mathcal{A}$.

We end this subsection by reviewing some basic notions related to filtered A_∞-bimodules. Let $\mathcal{B}$ be a graded free filtered R-module, and fix a gapped filtered A_∞-algebra $\mathcal{A}$ as above. An A_∞-bimodule structure on $\mathcal{B}$ consists of a family of operations

$$\mathfrak{n}_{p,q} \colon B_p(\mathcal{A}[1]) \otimes \mathcal{B}[1] \otimes B_q(\mathcal{A}[1]) \to \mathcal{B}[1].$$

These maps induce

$$\widehat{\delta}\colon \widehat{B}(\mathcal{A}[1])\widehat{\otimes}\mathcal{B}[1]\widehat{\otimes}\widehat{B}(\mathcal{A}[1]) \to \widehat{B}(\mathcal{A}[1]) \otimes \mathcal{B}[1]\widehat{\otimes}\widehat{B}(\mathcal{A}[1])$$

defined by

$$\widehat{\delta}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes z_1 \otimes \cdots \otimes z_\ell) = \widehat{d}(x_1 \otimes \cdots \otimes x_k) \otimes y \otimes z_1 \otimes \cdots \otimes z_\ell$$
$$+ \sum (-1)^{\sum_{i=1}^{k-p} |x_i|'} x_1 \otimes \cdots \otimes x_{k-p} \otimes \mathfrak{n}_{p,q}(x_{k-p+1} \otimes \cdots \otimes y \otimes \cdots \otimes z_q) \otimes \cdots \otimes z_\ell$$

The maps $\{\mathfrak{n}_{p,q}\}_{p,q\geq 0}$ give $\mathcal{B}$ the structure of an A_∞-bimodule if

$$\widehat{\delta} \circ \widehat{\delta} = 0.$$

The notion of an A_∞-bimodule homomorphism (over the pair of A_∞-algebra homomorphisms $(\mathrm{id}, \mathrm{id})$ on $\mathcal{A}$) consists of a family of linear maps

$$\phi_{p,q}\colon B_p(\mathcal{A}[1])\widehat{\otimes}\mathcal{B}[1]\widehat{\otimes}B_q(\mathcal{A}[1]) \to \mathcal{B}'[1]$$

which respect the filtration on $\mathcal{B}$. We form

$$\widehat{\phi}\colon B(\mathcal{A}[1])\widehat{\otimes}\mathcal{B}[1]\widehat{\otimes}B\mathcal{A} \to B(\mathcal{A}[1])\widehat{\otimes}\mathcal{B}'[1]\widehat{\otimes}B(\mathcal{A}[1])$$

by setting

$$\widehat{\phi}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes z_1 \otimes z_\ell) =$$
$$\sum x_1 \otimes \cdots \otimes x_{k-p} \otimes \phi_{p,q}(x_{k-p+1} \otimes \cdots \otimes y \otimes \cdots \otimes z_q) \otimes z_{q+1} \otimes \cdots \otimes z_\ell.$$

The defining condition for an A_∞-bimodule homomorphism is

$$\widehat{\phi} \circ \widehat{\delta} = \widehat{\delta}' \circ \widehat{\phi} \tag{3.1.10}$$

where δ' is induced from the A_∞-bimodule structure maps on $\mathcal{B}'$.

The two main examples of A_∞-bimodule we will use are the diagonal bimodule $\mathcal{A}_\Delta$, and the dual bimodule $\mathcal{A}^\vee$. To define the latter, let $\mathcal{A}^\vee$ denote the R-dual of $\mathcal{A}$, and equip it with structure maps $\{\mathfrak{m}^\vee_{k,\ell}\}_{k,\ell\geq 0}$ given by

$$\begin{aligned}&\mathfrak{m}^\vee_{k,\ell}(x_1 \otimes \cdots \otimes x_k \otimes v^\vee \otimes x_{k+1} \otimes \cdots \otimes x_\ell)(w) = \\ &(-1)^\epsilon v^\vee(\mathfrak{m}_{k+\ell+1}(x_{k+1} \otimes \cdots \otimes x_{k+\ell} \otimes w \otimes x_1 \otimes \cdots \otimes x_k))\end{aligned}$$

where the sign is determined by

$$\epsilon = |v^*|' + \left(\sum_{i=1}^{k} |x_i|'\right)\left(|v^*|' + \sum_{i=k+1}^{k+\ell} |x_i|' + |w|'\right).$$

3.1.2 The Hochschild complex

Let $\mathcal{A}$ be a strictly unital gapped filtered A_∞-algebra, and consider an A_∞-bimodule $\mathcal{B}$ over $\mathcal{A}$. Denote by

$$CH^k_*(\mathcal{A},\mathcal{B}) := \underline{\mathcal{B}[1]} \otimes (\mathcal{A}[1])^{\otimes k}$$

the space of Hochschild chains of length k, with degree given by

$$|\underline{b} \otimes a_1 \otimes \cdots \otimes a_k| = |b| + \sum_{i=1}^{k} |a_i|'.$$

We have underlined the bimodule factor for readability. Since $\mathcal{A}$ can be curved, it is usually more convenient to consider reduced Hochschild chains, which are given by

$$CH_*^{\mathrm{red},k}(\mathcal{A},\mathcal{B}) \coloneqq \underline{\mathcal{B}[1]} \otimes (\mathcal{A}[1]/R\cdot 1)^{\otimes k}.$$

The Hochschild chain complex is the completed direct sum

$$CH_*(\mathcal{A},\mathcal{B}) \coloneqq \widehat{\bigoplus}_{k\geq 0} CH_*^k(\mathcal{A},\mathcal{B}).$$

Similarly, the reduced Hochschild chain space is the completed direct sum

$$CH_*^{\mathrm{red}}(\mathcal{A},\mathcal{B}) \coloneqq \widehat{\bigoplus}_{k\geq 0} CH_*^{\mathrm{red},k}(\mathcal{A},\mathcal{B}).$$

For $v \in \mathcal{B}$ and $a_i \in \mathcal{A}$, the Hochschild differential is defined by

$$\begin{aligned}
&b(\underline{v} \otimes a_1 \otimes \cdots \otimes a_k) \coloneqq \\
&\sum (-1)^{\#^i_j} \underline{\mathfrak{m}_{i+j+1}(a_{k-i+1} \otimes \cdots \otimes a_k \otimes \underline{v} \otimes a_1 \otimes \cdots \otimes a_j)} \otimes a_{j+1} \otimes \cdots \otimes a_{k-i} \\
&+ \sum (-1)^{\maltese'_i} \underline{v} \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \mathfrak{m}_j(a_i \otimes \cdots \otimes a_{i+j-1}) \otimes \cdots \otimes a_k.
\end{aligned}$$

The signs above are determined by

$$\begin{aligned}
\#^i_j &\coloneqq \left(\sum_{s=1}^{i} |a_{k-i+s}|'\right)\left(|v|' + \sum_{t=1}^{j} |a_t|'\right) \\
\maltese'_i &\coloneqq |v|' + \sum_{s=1}^{i-1} |a_s|'.
\end{aligned}$$

Notice that $\mathfrak{m}_0$ can appear in terms of the second sum in the definition of the Hochschild differential, but we still have that $b \circ b = 0$. The main examples we will consider are the cases where $\mathcal{B} = \mathcal{A}_\Delta$ is the diagonal bimodule, or where $\mathcal{B} = \mathcal{A}^\vee$ is the Λ_{nov}-dual bimodule.

We use the strict unit on $\mathcal{A}$ to construct a homological S^1-action on

$$CH_*(\mathcal{A}) := CH_*(\mathcal{A}, \mathcal{A}_\Delta)$$

in the following sense. Let $t \in \mathbb{Z}/k\mathbb{Z}$ denote the generator, and define its action on $\mathcal{A}^{\otimes k}$ by

$$t(a_1 \otimes \cdots \otimes a_k) = (-1)^{\maltese_{k-1}\cdot|a_k|'} a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

Define an operator $N = 1 + t + \cdots + t^{k-1}$ on

$$CH_*^k(\mathcal{A}, \mathcal{A}_\Delta).$$

We abuse notation and write t for the generator of any cyclic group, and N for the operator $CH_*(\mathcal{A})$ obtained by considering the action of N as above on each direct summand. Let $b' = \widehat{d}$ denote the bar differential. The maps defined so far satisfy the relations

$$b(1-t) = (1-t)b'$$
$$b'N = Nb$$

meaning that we can form the following bicomplex.

$$\begin{array}{ccccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longleftarrow & CH_1(\mathcal{A}) & \xleftarrow[1-t]{} & CH_1(\mathcal{A}) & \xleftarrow[N]{} & CH_1(\mathcal{A}) & \xleftarrow[1-t]{} & CH_1(\mathcal{A}) & \longleftarrow \\
 & {\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & & {\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & \\
\longleftarrow & CH_0(\mathcal{A}) & \xleftarrow[1-t]{} & CH_0(\mathcal{A}) & \xleftarrow[N]{} & CH_0(\mathcal{A}) & \xleftarrow[1-t]{} & CH_0(\mathcal{A}) & \longleftarrow \\
 & {\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & & {\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & \\
\longleftarrow & CH_{-1}(\mathcal{A}) & \xleftarrow[1-t]{} & CH_{-1}(\mathcal{A}) & \xleftarrow[N]{} & CH_{-1}(\mathcal{A}) & \xleftarrow[1-t]{} & CH_{-1}(\mathcal{A}) & \longleftarrow \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

All of these maps descend to $CH_*^{\text{red}}(\mathcal{A})$, so we can define a reduced bicomplex analogously. To define an analogue of Connes' operator when $\mathfrak{m}_0$ is nonzero, we need to specify a slightly different contracting homotopy for the bar complex than is typically used in the uncurved case.

Lemma 3.1.6. *[75, Lemma 5.4] There is a contracting homotopy $\tilde{s}$ for the complex*

$$(CH_*(\mathcal{A}), \widehat{d})$$

which is defined by decomposing

$$CH_*(\mathcal{A}) = \ker(\widehat{d}) \oplus V$$

for some Λ_{nov}-module V, and setting

$$\widetilde{s}(\alpha) = \begin{cases} 1 \otimes \alpha\,, & \alpha \in V \\ 1 \otimes \alpha - \mathfrak{m}_0 \otimes 1 \otimes \alpha\,, & \alpha \in \ker \widehat{d}. \end{cases}$$

□

The Connes B operator is then

$$B = (1 - t)\widetilde{s}N$$

and we can form the (b, B)-bicomplex.

$$
\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\longleftarrow & CH_1(\mathcal{A}) & \xleftarrow[B]{} & CH_2(\mathcal{A}) & \xleftarrow[B]{} & CH_3(\mathcal{A}) & \longleftarrow \\
 & b\uparrow & & b\uparrow & & b\uparrow & \\
\longleftarrow & CH_0(\mathcal{A}) & \xleftarrow[B]{} & CH_1(\mathcal{A}) & \xleftarrow[B]{} & CH_2(\mathcal{A}) & \longleftarrow \\
 & b\uparrow & & b\uparrow & & b\uparrow & \\
\longleftarrow & CH_{-1}(\mathcal{A}) & \xleftarrow[B]{} & CH_0(\mathcal{A}) & \xleftarrow[B]{} & CH_1(\mathcal{A}) & \longleftarrow \\
 & \uparrow & & \uparrow & & \uparrow &
\end{array}
$$

On CH_*^{red}, the terms of $\widetilde{s}$ of the form $\mathfrak{m}_0 \otimes 1 \otimes \alpha$ vanish, and thus B descends to the usual Connes operator, i.e.

$$B(a_1 \otimes \cdots \otimes a_k) = \sum 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_1 \cdots \otimes a_{i-1}$$

on the reduced Hochschild chain space.

Remark 3.1.7. To account for possibly non-unital A_∞-algebras, one could instead work with the non-unital Hochschild complex, which also carries a homological S^1-action [52, §3.2]. Since the Floer cochain complexes we construct are strictly unital, this will not be necessary in this paper.

If u is a formal variable of degree 2, we define $b_{eq} = b + uB$, and define the positive, negative, and periodic cyclic chain complexes

$$
\begin{aligned}
CC_*^+(\mathcal{A}) &\coloneqq (CH_*(\mathcal{A}) \otimes_R R((u))/uR[[u]], b_{eq}) \\
CC_*^-(\mathcal{A}) &\coloneqq (CH_*(\mathcal{A}) \widehat{\otimes}_R R[[u]], b_{eq}) \\
CC_*^\infty(\mathcal{A}) &\coloneqq (CH_*(\mathcal{A}) \widehat{\otimes}_R R((u)), b_{eq})
\end{aligned}
$$

respectively. The positive and negative cyclic chain complexes are obtained by restricting to the positive and negative columns of the (b, B)-complex, respectively, and the periodic cyclic complex is obtained from the full bicomplex. Denote by $HC_*^{+/-/\infty}(\mathcal{A})$ the homology of these complexes, called the positive, negative, or periodic cyclic homology. Similarly, one defines the reduced cyclic complexes $CC_*^{\circ,\mathrm{red}}(\mathcal{A})$ and the reduced cyclic cohomologies $HC_*^{\circ,\mathrm{red}}(\mathcal{A})$, where $\circ \in \{+, -, \infty\}$.

Dualizing the reduced (b, B)-complex, gives us the reduced (b^*, B^*)-complex

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
\longrightarrow & & CH^2_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^1_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^0_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \longrightarrow & \\
 & & {\scriptstyle b^*}\uparrow & & {\scriptstyle b^*}\uparrow & & {\scriptstyle b^*}\uparrow & & \\
\longrightarrow & & CH^1_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^0_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^{-1}_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \longrightarrow & \\
 & & {\scriptstyle b^*}\uparrow & & {\scriptstyle b^*}\uparrow & & {\scriptstyle b^*}\uparrow & & \\
\longrightarrow & & CH^0_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^{-1}_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \xrightarrow{B^*} & CH^{-2}_{\mathrm{red}}(\mathcal{A},\mathcal{A}^\vee) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array} \tag{3.1.11}$$

where $(CH^*_{\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee, b^*)$ is the dual complex to the Hochschild complex. We obtain the reduced negative cyclic cochain complex $CC^*_{-,\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee)$ of $\mathcal{A}^\vee$ by taking the bicomplex consisting of the nonpositive columns of the above bicomplex. Similarly, the reduced positive cyclic cochain complex $CC^*_{+,\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee)$ is obtained by taking the bicomplex consisting of the positive columns of the bicomplex. The double complex consisting of all columns is called the periodic cyclic cochain complex, and it is denoted $CC^*_{\infty,\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee)$.

The proof of [77, Lemma 3.6] carries over to the filtered case to give a condition under which the connecting homomorphism is an isomorphism.

Lemma 3.1.8. *If there is an integer N such that $HH^*_{\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee) = 0$ whenever $* > N$, then for any integer n, there is an isomorphism*

$$HC^{n+1}_{+,\mathrm{red}}(\mathcal{A}) \xrightarrow{\sim} HC^n_{-,\mathrm{red}}(\mathcal{A}^\vee)$$

induced by B^*. □

The reduced Hochschild cohomology $HH^*_{\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee)$ is defined by taking the cohomology of the R-dual of the reduced Hochschild complex $CC_*^{\mathrm{red}}(\mathcal{A})$.

3.1.3 ∞-inner products and ∞-cyclic potentials

The following terminology is due to Tradler [78].

Definition 3.1.9. An A_∞-bimodule homomorphism $\phi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ is called an ∞-*inner product*.

For an uncurved A_∞-algebra, one possible definition of a (weak proper) Calabi–Yau structure is a bimodule quasi-isomorphism $\mathcal{A}_\Delta \to \mathcal{A}^\vee$ [52, Remark 49]. This data can, equivalently, be packaged as a negative cyclic cohomology class by Lemma 3.1.8. Even for a curved A_∞-algebra, one can obtain an ∞-inner product from a cocycle in $CC^*_{\mathrm{red},-}(\mathcal{A}, \mathcal{A}^\vee)$.

Lemma 3.1.10 (Cho-Lee [76])**.** *Consider a negative cyclic cocycle $\phi \in CC^*_{-,\mathrm{red}}(\mathcal{A}, \mathcal{A}^\vee)$, whose restriction to the $-i$th column of* (3.1.11) *is denoted ϕ_i. Then the sequence of maps*

$$\phi_{p,q}\colon \mathcal{A}^{\otimes p} \otimes \underline{\mathcal{A}} \otimes \mathcal{A}^{\otimes q} \to \mathcal{A}^\vee$$

defined by

$$\phi_{p,q}(\alpha \otimes \underline{v} \otimes \beta)(w) = \psi_0(\alpha \otimes v \otimes \beta)(w) - \psi_0(\beta \otimes w \otimes \alpha)(v) \tag{3.1.12}$$

where

$$\alpha = a_1 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes p}$$
$$\beta = b_1 \otimes \cdots \otimes b_q \in \mathcal{A}^{\otimes q}$$

and $v, w \in \mathcal{A}$, is an A_∞-bimodule homomorphism also denoted $\phi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$. □

The proof that ϕ is a bimodule homomorphism uses the negative cyclic cocycle condition $b^*\phi_i = B^*\phi_{i+1}$, and this is the main connection between the trivialization of the S^1-action on the Fukaya category and our construction of the open Gromov–Witten potential. This particular chain-level correspondence between (negative) cyclic cocycles and bimodule homomorphisms is convenient, because the homomorphisms ϕ constructed this way admit useful symmetries.

Corollary 3.1.11. *The bimodule homomomorphisms* $\phi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ *of Lemma 3.1.10 are skew-symmetric and closed, meaning, respectively, that*

- *for* α, β, v, *and* w *as in the statement of Lemma 3.1.10, we have that*

$$\phi_{p,q}(\alpha \otimes \underline{v} \otimes \beta)(w) = (-1)^{\kappa} \phi_{q,p}(\beta, w, \alpha)(v) \tag{3.1.13}$$

where

$$\kappa = \left(\sum_{i=1}^{p} |a_i|' + |v|'\right) \cdot \left(\sum_{j=1}^{q} |b_j|' + |w|'\right)$$

and;

- *for* $a_1 \otimes \cdots \otimes a_{\ell+1} \in \mathcal{A}^{\otimes \ell+1}$ *and any triple* $1 \leq i < j < k \leq \ell + 1$*, we have that*

$$\begin{aligned}(-1)^{\kappa_i}\phi(\cdots \otimes \underline{a_i} \otimes \cdots)(a_j) + (-1)^{\kappa_j}\phi(\cdots \otimes \underline{a_j} \otimes \cdots)(a_k)\\ + (-1)^{\kappa_k}\phi(\cdots \otimes \underline{a_k} \otimes \cdots \otimes)(a_i) = 0\end{aligned} \tag{3.1.14}$$

where the sign is determined by

$$\kappa_* = (|a_1|' + \cdots + |a_*|') \cdot (|a_{*+1}|' + \cdots + |a_k|')$$

and where the inputs are cyclically ordered. □

Remark 3.1.12. A cyclic pairing on $\mathcal{A}$ can be thought of as a closed skew-symmetric ∞-inner product $\psi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ for which $\psi_{p,q} = 0$ whenever $p > 0$ or $q > 0$.

We can relate the ∞-inner product obtained from a negative cyclic cocycle ϕ to the trace associated to $B^*\phi$.

Lemma 3.1.13. *For an* ∞*-inner product obtained via Lemma 3.1.13, we have the identity*

$$\phi_0(1 \otimes \mathfrak{m}_2(a_1, a_2)) = \phi_{0,0}(\underline{a_1})(a_2). \tag{3.1.15}$$

Proof. We have that $b^*\phi(1, a_1, a_2) = B^*\psi(1, a_1, a_2) = 0$ for a Hochschild cochain ψ because ϕ is a negative cyclic cocycle. A direct calculation shows that

$$\begin{aligned} 0 &= b^*\phi_0(1 \otimes a_1 \otimes a_2) \\ &= \phi_0(\mathfrak{m}_2(1 \otimes a_1) \otimes a_2) - \phi_0(1 \otimes \mathfrak{m}_2(a_1 \otimes a_2)) + (-1)^{|a_2|' \cdot (|a_1|'+1)}\phi_0(\mathfrak{m}_2(a_2 \otimes 1) \otimes a_1) \\ &= \phi_0(a_1 \otimes a_2) - \phi_0(1 \otimes \mathfrak{m}_2(a_1 \otimes a_2)) + (-1)^{|a_1|' \cdot |a_2|' + |a_2|' + |a_2|}\phi_0(a_2 \otimes a_1). \end{aligned}$$

□

We can think of the expression defined in Lemma 3.1.13 as the formula for a cyclic pairing on a canonical model of $\mathcal{A}$, in analogy in Kontsevich–Soibelman's theorem in the unfiltered setting. An ∞-inner product ϕ is said to be *homologically nondegenerate* if for any nonzero $[a_1] \in H^*(\mathcal{A}, \mathfrak{m}_{1,0})$ there is an element $[a_2] \in H^*(\mathcal{A}, \mathfrak{m}_{1,0})$ such that $\phi_{0,0}(\underline{a_1})(a_2)$ on the chain level, for some representatives of these classes.

Lemma 3.1.14 (Cho–Lee [76])**.** *Let ϕ be a closed skew-symmetric homologically nondegenerate ∞-inner product on $\mathcal{A}$. Then there is a canonical model for $\mathcal{A}$ carrying a strictly cyclic pairing such that we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{A} & \longleftarrow & H^*(\mathcal{A}, \mathfrak{m}_{1,0}) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow \\ \mathcal{A}^* & \longrightarrow & H^*(\mathcal{A}, \mathfrak{m}_{1,0})^* \end{array}$$

where the right arrow comes from the strictly cyclic pairing, and the top arrow is a quasi-isomomorphism.

□

Any ϕ which is both skew-symmetric and closed satisfies a weak analogue of cyclic symmetry.

Lemma 3.1.15 (Cho–Lee [54])**.** *Let $b, y \in \mathcal{A}$ and suppose that $|b| = 1$. Then for any $N \geq 0$, we* *have that*

$$N \sum_{p+q+k=N} \phi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k(b^{\otimes k})} \otimes b^{\otimes q})(y) \tag{3.1.16}$$

$$= \sum_{\substack{p+q+k=N \\ r+s=k-1}} \phi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k(b^{\otimes r} \otimes y \otimes b^{\otimes s})} \otimes b^{\otimes q})(b) \tag{3.1.17}$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=p-1}} \phi_{p,q}(b^{\otimes r} \otimes y \otimes b^{\otimes s} \otimes \underline{\mathfrak{m}_k(b^{\otimes k})} \otimes b^{\otimes q})(b) \tag{3.1.18}$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=q-1}} \phi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k(b^{\otimes k})} \otimes b^{\otimes r} \otimes y \otimes b^{\otimes s})(b). \tag{3.1.19}$$

□

The potential of a cyclic A_∞-algebra is defined as follows.

Definition 3.1.16. If $\phi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ is an ∞-inner product, then the ∞-cyclic potential $\Phi'\colon F_{>0}\mathcal{A} \to R$ is a function on the set of elements of $\mathcal{A}$ of positive valuation defined by

$$\Phi'(x) := \sum_{N=0}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \phi_{p,q}(x^{\otimes p} \otimes \underline{\mathfrak{m}_k(x^{\otimes k})} \otimes x^{\otimes q})(x). \tag{3.1.20}$$

The sum in (3.1.20) converges since $\mathcal{A}$ is gapped and b has positive valuation.

Although it is not strictly necessary (cf. Theorem 3.4.9), we can use Lemma 3.1.15 to show that Φ' respects gauge-equivalence classes of bounding cochains. Recall that [47, Chapter 4.2] constructs, for any A_∞-algebra $\mathcal{A}$ over a field of characteristic 0, a model for the cylinder over $\mathcal{A}$ denoted $Poly([0,1], \mathcal{A})$ whose elements are pairs of formal polynomials in the Novikov variable T with coefficients that are functions on the interval $[0,1]$. A consequence (cf. [47, Proposition 4.3.5] and [47, Lemma 4.3.7]) of this construction is that a pair of bounding cochains $b_0, b_1 \in \widehat{\mathcal{M}}(\mathcal{A})$ are gauge-equivalent if and only if there is a path of elements

$$b_t = \sum_i b_i(t) T^{\lambda_i} \in \mathcal{A}$$

with $\lim \lambda_i = \infty$ such that

- $b_i(t)$ is a polynomial in the variable t; and
- for each fixed t, the element $b_t \in \mathcal{A}$ is a bounding cochain.

Theorem 3.1.17. *For any gauge-equivalent bounding cochains $b_0, b_1 \in \widehat{\mathcal{M}}(\mathcal{A})$, one has that $\Phi'(b_0) = \Phi'(b_1)$.*

Proof. Choosing a path b_t as above, we compute

$$\begin{aligned}
\frac{d}{dt}\Phi'(b_t) &= \sum_{N=0}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1}\phi_{p,q}\left(b_t^{\otimes p} \otimes \underline{\mathfrak{m}_k(b_t^{\otimes k})} \otimes b_t^{\otimes q}\right)\left(\frac{db_t}{dt}\right) \\
&+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ r+s=k-1}} \frac{1}{N+1}\phi_{p,q}\left(b_t^{\otimes p} \otimes \underline{\mathfrak{m}_k\left(b_t^{\otimes r} \otimes \frac{db_t}{dt} \otimes b_t \otimes s\right)} \otimes b_t^{\otimes q}\right)(b_t) \\
&+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ r+s=p-1}} \frac{1}{N+1}\phi_{p,q}\left(b_t^{\otimes r} \otimes \frac{db_t}{dt} \otimes b_t^{\otimes s} \otimes \underline{\mathfrak{m}_k(b_t^{\otimes k})} \otimes b_t^{\otimes q}\right)(b_t) \\
&+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ r+s=q-1}} \frac{1}{N+1}\phi_{p,q}\left(b_t^{\otimes p} \otimes \underline{\mathfrak{m}_k(b_t^{\otimes k})} \otimes b_t^{\otimes r} \otimes \frac{db_t}{dt} \otimes b_t^{\otimes s}\right)(b_t) \\
&= \sum_{N=0}^{\infty} \sum_{p+q+k=N} \phi_{p,q}\left(b_t^{\otimes p} \otimes \underline{\mathfrak{m}_k(b_t^{\otimes k})} \otimes b_t^{\otimes q}\right)\left(\frac{db_t}{dt}\right) = 0
\end{aligned}$$

where the second equality follows from Lemma 3.1.15, and the last equality follows from the Maurer–Cartan equation. □

3.2 Lagrangian Floer theory

The purpose of this section is to review the Morse–theoretic model for the Lagrangian Floer cochcain complex. In this discussion, we fix notation for pseudoholomorphic pearly trees that will be helpful when we construct the cyclic open-closed map. We will also explain how to construct A_∞-structures on the Morse complex of a cylinder, which we need to study the invariance of the open Gromov–Witten potential.

3.2.1 Pseudoholomorphic pearly trees

Suppose that $M = (M^{2n}, \omega)$ is a closed connected symplectic manifold. Let $\mathcal{J}(M)$ denote the space of ω-tame almost compatible structures on M, and let $J \in \mathcal{J}(M)$. For the rest of this section, fix a closed connected Lagrangian embedding $L \subset M$, where L is equipped with a spin structure $\mathfrak{s}$ and a $GL(1, \Bbbk)$ local system. Additionally, choose Morse–Smale pairs (f_L, g_L) and (f_M, g_M) on L and M, respectively. The sets of critical points of f_L and f_M are denoted $\operatorname{crit}(f_L)$ and $\operatorname{crit}(f_M)$.

Assumption 3.2.1. The Morse functions f_L and f_M both have a unique local minimum and a unique local maximum.

We can define the Morse cochain complexes $(CM^*(L; R), d)$ and $(CM^*(M; Q), d)$ in terms of these Morse–Smale pairs, with coefficients in R and Q, respectively, whose differentials count isolated gradient flow lines joining two critical points.

Given a class $\beta \in H_2(M, L; \mathbb{Z})$ and nonnegative integers $k, \ell \geq 0$, consider the (uncompactified) moduli space $\mathcal{M}_{k+1,\ell}(L; \beta)$ of all J-holomorphic disks $u\colon (D^2, \partial D^2) \to (M, L)$ with $k + 1$ cyclically ordered boundary marked points $z_0, \dots, z_k$ and ℓ ordered interior marked points $w_1, \dots, w_\ell$. Similarly, for $\beta \in H_2(M; \mathbb{Z})$, let $\mathcal{M}_\ell(\beta)$ denote the (uncompactified) moduli space of J-holomorphic spheres in M with ℓ marked points denoted $w_1, \dots, w_\ell$. Define the boundary evaluation maps

$$\begin{aligned} \mathrm{evb}_j^\beta\colon \mathcal{M}_{k+1,\ell}(L; \beta) &\to L \\ \mathrm{evb}_j^\beta(u) &= u(z_j) \end{aligned}$$

for $j = 0, \dots, k$. Similarly, define the interior evaluation maps by

$$\begin{aligned} \mathrm{evi}_j^\beta\colon \mathcal{M}_{k+1,\ell}(L; \beta) &\to M \\ \mathrm{evi}_j^\beta(u) &= u(w_j) \end{aligned}$$

There are also interior evaluation maps $\mathrm{evi}_j^\beta\colon \mathcal{M}_\ell(\beta) \to M$ defined similarly.

The A_∞-operations on $CM^*(L;R)$ will count configurations consisting of trees of elements in the moduli spaces $\mathcal{M}_{k+1,\ell}(\beta)$ joined by gradient flow lines of (perturbations of) f_L. The combinatorial structures underlying such configurations are described by oriented metric ribbon trees whose vertices are partitioned depending on whether they parametrize sphere or disk components of a pearly tree.

Definition 3.2.2. A *bicolored* tree is a tree T with vertex set $V(T)$ and edge set $E(T)$, together with a partition of its vertices

$$V(T) = V_\circ(T) \sqcup V_\bullet(T)$$

called the disk vertices and sphere vertices of T, respectively. We require that T come equipped with a choice of a subtree $T_\circ$ whose vertex set $V(T_\circ)$ coincides with $V_\circ(T)$. Let $E_\circ(T) \coloneqq E(T_\circ)$ and $E_\bullet(T) \coloneqq E(T) \setminus E_\circ(T)$. Finally, let $e_0^\circ, \dots, e_k^\circ$ denote the (combinatorially) semi-infinite edges contained in $T_\circ$, and let $e_1^\bullet, \dots, e_\ell^\bullet$ denote the remaining semi-infinite edges.

In the above we also allow the exceptional case of a tree T with $V(T) = \emptyset$ consisting of a single infinite edge.

Definition 3.2.3. An *oriented metric ribbon tree* consists of a bicolored tree T, equipped with

- a ribbon structure on $T_\circ$; i.e. a cyclic ordering of all edges in $E_\circ(T)$ adjacent to any vertex in $T_\circ$ (which induces a cyclic ordering of $e_0^\circ, \dots, e_k^\circ$) and an ordering of the all edges in $E_\bullet(T)$ adjacent to any vertex of T;
- a metric on T, which is described by a length function $\lambda\colon E(T) \to \mathbb{R}_{\geq 0}$;
- an orientation on T determined by orienting e_0° so that it is an outgoing edge, and orienting all remaining edges of T so that they point toward e_0°;
- a class $\beta_v \in H_2(M, L)$ for each $v \in V_\circ(T)$ and a class $\beta_v \in H_2(M)$ for each $v \in V_\bullet(T)$.

For any $v \in V(T)$, let $\mathrm{val}_\circ(v)$ denote the number of edges in $E_\circ(T)$ adjacent to v and $\mathrm{val}_\bullet(v)$ denote the number of edges in $E_\bullet(T)$ adjacent to v.

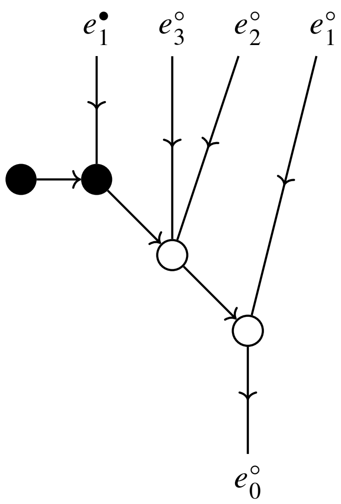

Figure 3.1: An oriented metric ribbon tree.

Definition 3.2.4. We say that T is stable if for each $v \in V(T)$ for which $\omega(\beta_v) = 0$, either

- $v \in V_\circ(T)$ and $\mathrm{val}_\circ(v) + 2 \cdot \mathrm{val}_\bullet(v) \geq 3$; or
- $v \in V_\bullet(T)$ and $\mathrm{val}_\bullet(v) \geq 3$.

From now on, we will only ever consider moduli spaces of stable trees, both in this construction and in all others like it. There is a moduli space of stable oriented metric ribbon trees, which can be compactified by allowing the length an edge to go to infinity and break. Given two oriented metric ribbon trees T_1 and T_2, we can attach endpoints to the edge e_0° of T_1 and to the edge e_i°, for some $i > 0$, of T_2, and glue the two endpoints together to form a new tree. Since we have glued the output edge of T_1 to an input edge of T_2, the glued tree carries a ribbon structure and orientation.

We associate to each vertex of T the moduli space $\mathcal{M}_{\mathrm{val}_\circ(v)-1,\mathrm{val}_\bullet(v)}(\beta_v)$ if $v \in E_\circ(T)$, or $\mathcal{M}_{\mathrm{val}_\bullet(v)}(\beta_v)$ if $v \in E_\bullet(T)$. For brevity, we write $\mathcal{M}(\beta_v)$ for either of these moduli spaces. Let $E^f_{\circ/\bullet}(T)$ denote the sets of combinatorially finite edges of $E_{\circ/\bullet}(T)$. We will construct an evaluation map

$$\mathrm{ev}_{T^f} \colon \prod_{v \in V(T)} \mathcal{M}(\beta_v) \to \prod_{e \in E^f_\circ(T)} (L \times L) \times \prod_{e \in E^f_\bullet(T)} (M \times M)$$

using the bicoloring of T.

If $e \in E(T)$ is a combinatorially finite edge, let $s(e), t(e) \in V(T)$ denote the source and target of e, respectively. When $e \in E_\circ(T)$, there is an integer k_t such that e comes k_tth in the cyclic ordering of edges adjacent to $t(e)$. Our orientation conventions imply that e is always the zeroth edge of $s(e)$. Similarly, when $e \in E_\bullet(T)$, there are integers k_s and k_t which are defined analogously using the orderings of the edges in $E_\bullet(T)$.

Let $\vec{u} = (u_v)_{v\in V(T)}$ denote an element of $\prod_{v\in V(T)} \mathcal{M}(\beta_v)$. For $e \in E_\circ(T)$, define

$$\begin{aligned}\mathrm{ev}_e \colon \prod_{v\in V(T)} \mathcal{M}(\beta_v) &\to L \times L\\ \mathrm{ev}_e(\vec{u}) &= (\mathrm{evb}_0(u_{s(e)}), \mathrm{evb}_{k_t}(u_{t(e)})).\end{aligned}$$

For $e \in E_\bullet(T)$, define

$$\begin{aligned}\mathrm{ev}_e \colon \prod_{v\in V(T)} \mathcal{M}(\beta_v) &\to M \times M\\ \mathrm{ev}_e(\vec{u}) &= (\mathrm{evi}_{k_s}(u_{s(e)}), \mathrm{evi}_{k_t}(u_{t(e)})).\end{aligned}$$

Finally, set

$$\mathrm{ev}_{T^f}(\vec{u}) = \prod_{e\in E^f_\circ(T)} \mathrm{ev}_e(\vec{u}) \times \prod_{e\in E^f_\bullet(T)} \mathrm{ev}_e(\vec{u}).$$

We extend this to a full evaluation map for T by taking into account the semi-infinite edges. According to our orientation conventions, the edge e_0° has one endpoint denoted $s(e_0^\circ)$, and all other semi-infinite edges $e_j^{\circ/\bullet}$, where $j = 1, \ldots, k$ or $j = 1, \ldots, \ell$, have one endpoint denoted $t(e_j^{\circ/\bullet})$. With this understood, define the evaluation maps $\mathrm{ev}_j^{\circ/\bullet}$ to be the evaluation map determined by the position of $e_j^{\circ/\bullet}$ in the ordering of edges adjacent to its endpoint. We can now associate to T an evaluation map

$$\mathrm{ev}_T(\vec{u}) = \prod_{j=1}^{\ell} \mathrm{ev}_j^{\bullet}(u_{t(e_j^{\bullet})}) \times \prod_{j=1}^{k} \mathrm{ev}_j^{\circ}(u_{t(e_j^{\circ})}) \times \mathrm{ev}_{T^f}(\vec{u}) \times \mathrm{ev}_0^{\circ}(u_{s(e_0^{\circ})}). \tag{3.2.1}$$

Having defined ev_T, we will define the moduli spaces of pearly trees by pulling back a submanifold in the codomain of this map. We must also assign to each edge of T a Morse function of the following type.

Definition 3.2.5. Fix a Morse function $f_0 \colon Y \to \mathbb{R}$ on a compact manifold Y. We say that a Morse function $f \colon Y \to \mathbb{R}$ is *f_0-admissible* if it is a C^2-small perturbation of f_0 for which $\mathrm{crit}(f_0) = \mathrm{crit}(f)$ and which agrees with f_0 in a neighborhood of its critical points.

For each $e \in E_\circ(T)$, choose an f_L-admissible Morse function $f_{L,e}$ and for each $e \in E_\bullet(T)$, choose an f_M-admissible Morse function $f_{M,e}$. Given a combinatorially finite edge $e \in E_\circ^f(T)$, let $\phi_t^{f_{L,e}}$ denote the time-t gradient flow of $f_{L,e}$, and for $e \in E_\bullet^f(T)$, let $\phi_t^{f_{M,e}}$ denote the time-t gradient flow of $f_{M,e}$. These yield embeddings

$$\begin{aligned} (L \setminus \mathrm{crit}(f_L)) \times \mathbb{R}_{\geq 0} &\hookrightarrow L \times L \\ (x,t) &\mapsto (x, \phi_t^{f_{L,e}}(x)) \end{aligned} \tag{3.2.2}$$

and

$$\begin{aligned} (M \setminus \mathrm{crit}(f_M)) \times \mathbb{R}_{\geq 0} &\hookrightarrow M \times M \\ (x,t) &\mapsto (x, \phi_t^{f_{M,e}}(x)) \end{aligned} \tag{3.2.3}$$

whose images are denoted G_e. Given $x \in \mathrm{crit}(f_L)$, let $W^u(x)$ and $W^s(x)$ denote its unstable and stable manifolds, and define $W^u(y)$ and $W^s(y)$ for $y \in \mathrm{crit}(f_M)$ analogously.

Definition 3.2.6. Let $x = (x_1, \ldots, x_k)$ be a sequence of inputs in $\mathrm{crit}(f_L)$ and $y = (y_1, \ldots, y_\ell)$ be a sequence of inputs in $\mathrm{crit}(f_M)$, and let $x_0 \in \mathrm{crit}(f_L)$ denote an output critical point. Define the

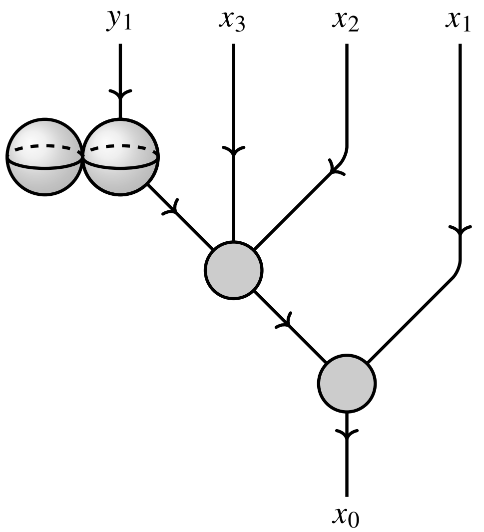

Figure 3.2: A pseudoholomorphic pearly tree with underlying oriented metric ribbon tree depicted in Figure 3.1.

moduli space of pearly trees in the class $\beta \in H_2(M, L)$ with these inputs and output to be

$$\mathcal{M}(x_0, x; y; \beta) := \coprod_T \mathrm{ev}_T^{-1}\left(\prod_{j=1}^{\ell} W^u(y_j) \times \prod_{j=1}^{k} W^u(x_j) \times \prod_{e \in E_\circ^f(T) \sqcup E_\bullet^f(T)} G_e \times W^s(x_0)\right)$$

where the disjoint union is taken over the set of all oriented metric ribbon trees T with $\sum_{v \in V(T)} \beta_v = \beta$. This admits a natural Gromov compactification

$$\overline{\mathcal{M}}(x_0, x; y; \beta). \tag{3.2.4}$$

We assume that J can be chosen so that these moduli spaces are regular.

Assumption 3.2.7. There exists $J \in \mathcal{J}(M)$ such that all of the moduli spaces (3.2.4) of virtual dimension at most 1 are compact orbifolds with boundary of the expected dimension.

Since we have assumed that L is spin, the moduli spaces (3.2.4) are oriented.

Definition 3.2.8. Given sequences x and y as above, define

$$\mathfrak{q}_{k,\ell}^{\beta}(x_1, \ldots, x_k; y_1, \ldots, y_\ell) = \sum_{x_0 \in \mathrm{crit}(f_L)} \mathrm{hol}_\nabla(\beta) \# |\mathcal{M}(x_0, x; y; \beta)| \cdot x_0$$

where $\#|\mathcal{M}(x_0, x; y; \beta)|$ is the signed count of elements in the zero-dimensional moduli space $\mathcal{M}(x_0, x; y; \beta)$.

These extend linearly to arbitrary inputs in $CM^*(L; R)$ and $CM^*(M; Q)$. Define operations

$$\begin{aligned} \mathfrak{q}_{k,\ell} &: CM^*(L; R)^{\otimes k} \otimes CM^*(M : Q)^{\otimes \ell} \to CM^*(L; R) \\ \mathfrak{q}_{k,\ell} &= \sum_{\beta \in H_2(M,L;\mathbb{Z})} e^{\mu(\beta)/2} T^{\omega(\beta)} \mathfrak{q}^{\beta}_{k,\ell}. \end{aligned}$$

For $\gamma \in CM^*(L; R)$ with $d\gamma = 0$ and $|\gamma| = 2$, where d denotes the differential on the Morse cochain complex, define the bulk-deformed operators

$$\mathfrak{m}^{\gamma}_k(x) = \sum_{\ell \geq 0} \frac{1}{\ell!} \mathfrak{q}_{k,\ell}(x; \gamma^{\otimes \ell}).$$

We call such a class γ a *bulk parameter*. The next lemma follows from Assumption 3.2.7, by standard arguments.

Lemma 3.2.9. *For any $\gamma \in CM^*(M)$ with $|\gamma| = 2$ and $d\gamma = 0$, the pair*

$$(CM^*(L), \{\mathfrak{m}^{\gamma}_k\}_{k=0}^{\infty})$$

is a strictly unital gapped filtered A_∞-algebra. The unit element is given by the unique minimum of f_L. □

For a discussion of the existence of a unit, see the proof of [53, Lemma 5.2.4].

3.2.2 Models for cylinder objects

In [18] and [49], the behavior of the open Gromov–Witten potential as the almost complex structure on M varies is understood in terms of pseudo-isotopies, which can be viewed as A_∞-structures on the space of differentials forms on $L \times [-1, 1]$. In this subsection, we develop the analogous notion in the pearly model.

Suppose that we are given two almost complex structures $J_{\pm 1} \in \mathcal{J}(M)$, two Morse–Smale pairs $(f_L^{\pm 1}, g_L^{\pm 1})$ on L, and two Morse–Smale pairs on $(f_M^{\pm 1}, g_M^{\pm 1})$ on M, both of which satisfy Assumption 3.2.7. Consider two smooth functions

$$F_L\colon L \times (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$$
$$F_M\colon M \times (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$$

where, if $F_L^t(x) \coloneqq F_L(x,t)$ and $F_M^t(x) \coloneqq F_M(x,t)$, then for some $\epsilon > 0$, we have that

$$F_L^t = \begin{cases} f_L^{-1}, & t \in (-1-\epsilon, -1+\epsilon] \\ f_L^1, & t \in [1-\epsilon, 1+\epsilon) \end{cases}$$

and similarly for F_M. We can also assume without loss of generality that F_L^t and F_M^t are independent of t and are Morse functions $f_L^0\colon L \to \mathbb{R}$ and $f_M^0\colon M \to \mathbb{R}$, respectively, provided that $t \in (-\epsilon, \epsilon)$. We modify F_L and F_M to obtain Morse functions on $L \times [-1,1]$ and $M \times [-1,1]$ by choosing a Morse function $\rho\colon (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$ such that

- ρ has index 0 critical points at ± 1 and an index 1 critical point at 0;
- ρ is sufficiently increasing on $(-1,0)$ and sufficiently decreasing on $(0,1)$ that

$$\begin{cases} \frac{\partial F_M}{\partial t}(y,t) + \rho'(t) > 0 \text{ and } \frac{\partial F_L}{\partial t}(x,t) + \rho'(t) > 0\,, & t \in (-1,0)\,,\ y \in M\,,\ x \in L \\ \frac{\partial F_M}{\partial t}(y,t) + \rho'(t) < 0 \text{ and } \frac{\partial F_L}{\partial t}(x,t) + \rho'(t) < 0\,, & t \in (0,1)\,,\ y \in M\,,\ x \in L \end{cases}$$

- $F_L^t(x) + \rho(t)$ and $F_M^t(y) + \rho(t)$ are Morse functions on L and M for all $t \in (-\epsilon, \epsilon)$.

It follows from these conditions on ρ that the functions

$$\widetilde{f}_L(x,t) \coloneqq F_L(x,t) + \rho(t)\colon L \times (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$$
$$\widetilde{f}_M(y,t) \coloneqq F_M(y,t) + \rho(t)\colon M \times (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$$

are Morse functions whose critical point sets are

$$\operatorname{crit}(\widetilde{f}_L) = \operatorname{crit}(f_L^{-1}) \times \{-1\} \cup \operatorname{crit}(f_L^0) \times \{0\} \cup \operatorname{crit}(f_L^1) \times \{1\}$$
$$\operatorname{crit}(\widetilde{f}_M) = \operatorname{crit}(f_M^{-1}) \times \{-1\} \cup \operatorname{crit}(f_M^0) \times \{0\} \cup \operatorname{crit}(f_M^1) \times \{1\}$$

with Morse indices given by

$$\begin{aligned}
\operatorname{ind}_{F_L}(x, \pm 1) &= \operatorname{ind}_{f_L^{\pm 1}}(x) \\
\operatorname{ind}_{F_L}(x, 0) &= \operatorname{ind}_{f_L^0}(x) + 1 \\
\operatorname{ind}_{F_M}(y, \pm 1) &= \operatorname{ind}_{f_M^{\pm 1}}(y) \\
\operatorname{ind}_{F_M}(y, 0) &= \operatorname{ind}_{f_M^0}(y) + 1
\end{aligned}$$

By a partition of unity argument, one can construct a Riemannian metric $\widetilde{g}_L$ on $L \times [-1, 1]$ such that

- the restrictions of $\widetilde{g}_L$ to $L \times [-1, -1 + \epsilon)$ and $L \times (1 - \epsilon, 1]$ agree with the products of the metrics $g_L^{\mp 1}$ with the standard metric on the interval;
- for all $t \in (-\epsilon, \epsilon)$, the restriction of $\widetilde{g}_L$ to $L \times \{t\} \cong L$ is a Riemannian metric g_L^0 on L which is independent of t, and (f_L^0, g_L^0) is a Morse–Smale pair;
- $(\widetilde{f}_L, \widetilde{g}_L)$ is a Morse–Smale pair on $L \times [-1, 1]$.

We can repeat this construction with M in place of L to obtain a Morse–Smale pair $(\widetilde{f}_M, \widetilde{g}_M)$. Define the Morse cochain complexes $(CF^*(L \times [-1, 1]), d)$ and $(CF^*(M \times [-1, 1]), d)$ with respect to these Morse–Smale pairs.

To define A_∞-operations on $CM^*(L \times [-1, 1])$, we consider moduli spaces of disks defined using time-dependent almost complex structures. Let $\underline{J} = \{J_t\}_{t \in [-1,1]}$ be a path in $\mathcal{J}(M)$, with the property that the moduli spaces of pearly trees of virtual dimension at most 1 defined using the Morse–Smale pairs (f_L^i, g_L^i) and (f_M^i, g_M^i) and the almost complex structures J_i, for $i \in \{-1, 0, 1\}$, satisfy Assumption 3.2.7.

For any $\beta \in H_2(M, L; \mathbb{Z})$ or $\beta \in H_2(M; \mathbb{Z})$, define *time-dependent moduli spaces*

$$\widetilde{\mathcal{M}}_{k+1,\ell}(\beta) := \{(u,t) : u \in \mathcal{M}_{k+1,\ell}(\beta; J_t)\}$$
$$\widetilde{\mathcal{M}}_{\ell}(\beta) := \{(u,t) : u \in \mathcal{M}_{\ell}(\beta; J_t)\}$$

Let $\mathrm{evb}^{\beta}_{j,t} : \mathcal{M}_{k+1,\ell}(\beta; J_t) \to L$ and $\mathrm{evi}^{\beta}_{j,t} : \mathcal{M}_{k+1,\ell}(\beta; J_t) \to M$ denote the boundary and interior evaluation maps at time t, respectively. The time-dependent moduli spaces carry boundary evaluation maps

$$\widetilde{\mathrm{evb}}^{\beta}_j : \widetilde{\mathcal{M}}_{k+1,\ell}(L;\beta) \to L \times [-1,1]$$
$$\widetilde{\mathrm{evb}}^{\beta}_j(u,t) = (\mathrm{evb}^{\beta}_{j,t}(u), t)$$

for $j = 0, \ldots, k$. There are similarly defined interior evaluation maps

$$\widetilde{\mathrm{evi}}^{\beta}_j : \widetilde{\mathcal{M}}_{k+1,\ell}(L;\beta) \to M \times [-1,1]$$
$$\widetilde{\mathrm{evi}}^{\beta}_j(u,t) = (\mathrm{evi}^{\beta}_{j,t}(u), t)$$
$$\widetilde{\mathrm{evi}}^{\beta}_j : \widetilde{\mathcal{M}}_{\ell}(L;\beta) \to M \times [-1,1]$$
$$\widetilde{\mathrm{evi}}^{\beta}_j(u,t) = (\mathrm{evi}^{\beta}_{j,t}(u), t)$$

for all $j = 0, \ldots, \ell$. These let us define analogues of the evaluation maps (3.2.1), enabling the following definition.

Definition 3.2.10. Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_k)$ be a sequence of inputs in $\mathrm{crit}(\widetilde{f_L})$ and $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_\ell)$ be a sequence of inputs in $\mathrm{crit}(\widetilde{f_M})$, and let $\tilde{x}_0 \in \mathrm{crit}(\widetilde{f_L})$ be an output critical point. The compactified moduli spaces

$$\overline{\mathcal{M}}(\tilde{x}_0, \tilde{x}; \tilde{y}; \beta)$$

are defined the same way as the moduli spaces (3.2.4), except that, given an oriented metric ribbon tree T,

- the vertices $v \in V_\circ(T)$ are assigned elements of $\widetilde{\mathcal{M}}_{k+1,\ell}(\beta)$ and the vertices $v \in V_\bullet(T)$ are assigned elements of $\widetilde{\mathcal{M}}_\ell(\beta)$; and
- the edges of T are assigned small generic perturbations of $\widetilde{f}_L$ or $\widetilde{f}_M$ which agree with $\widetilde{f}_L$ and $\widetilde{f}_M$, respectively, near the critical points.

The time-dependent analogue of Assumption 3.2.7 is the following.

Assumption 3.2.11. There is a path $\underline{J}$ joining J_{-1} and J_1 such that all moduli spaces (3.2.10) of virtual dimension ≤ 1 are compact orbifolds with boundary of the expected dimension.

Remark 3.2.12. In the de Rham or singular chains model for Lagrangian Floer theory, one would require submersivity the boundary evaluation maps on $\widetilde{\mathcal{M}}_{k+1,\ell}(\beta)$ to define A_∞-structures on $L\times I$. If this is the case, then regularity of the moduli spaces would implies that all of the moduli spaces $\widetilde{\mathcal{M}}_{k+1,\ell}(\beta; J_t)$ satisfy Assumption 3.2.7. The existence of such paths $\underline{J}$ cannot be established using standard transversality arguments.

Under this assumption, we define bulk-deformed A_∞-structures on $CM^*(L\times[-1,1];R)$ just as we did on $CM^*(L;R)$. Given a cocycle $\widetilde{\gamma} \in CM^*(M \times [-1,1])$, our choice of Morse function on $M\times[-1,1]$ implies that its restrictions to $M\times\{\mp 1\}$ are Morse cocycles $\gamma^{\mp 1} \in CM^*(M;(f_M^{\mp 1}, g_M^{\mp 1}))$ for which $|\gamma^{\mp 1}| = |\gamma|$.

Lemma 3.2.13. *Let $\widetilde{\gamma} \in CM^*(M \times [-1,1])$ be a class with $|\widetilde{\gamma}| = 2$ and $d\widetilde{\gamma} = 0$. Then there exist bulk-deformed operators*

$$\widetilde{\mathfrak{m}}_k^{\widetilde{\gamma}} \colon CM^*(L \times [-1,1];R)^{\otimes k} \to CM^*(L \times [-1,1];R)$$

of degree $2-k$ such that $(CM^*(L;R), \{\widetilde{\mathfrak{m}}_k^{\widetilde{\gamma}}\})$ is a gapped filtered A_∞-algebra.

Furthermore, this A_∞-algebra has

$$(CM^*(L;(f_L^{\mp 1}, g_L^{\mp 1})), \mathfrak{m}_k^{\gamma^{\mp 1}})$$

as A_∞-subalgebras. □

3.3 The cyclic open-closed map

For the next two sections, let $\mathcal{A} = (CF^*(L), \{\mathfrak{m}_k^\gamma\})$ denote the possibly bulk-deformed curved A_∞-algebra constructed in Section 3.2.1. Denote by $(CM_*(M), \partial)$ the Morse chain complex constructed using the Morse–Smale pair already chosen in the construction of $\mathcal{A}$. We will construct a sequence of maps

$$\mathcal{OC}_m \colon CH_*(\mathcal{A}) \to CM_{n-*+2m}(M)$$

where the map $\mathcal{OC}_0$ is the usual open-closed map. The main result of this section says that these can be assembled to give a chain map $\mathcal{OC} \colon CC_*^+(\mathcal{A}) \to CM_*(M)$. It follows (cf. Assumption 3.2.1) that this induces a class in $HC^*_{+,\mathrm{red}}(\mathcal{A})$, and an ∞-inner product on $\mathcal{A}$ in turn.

Lemma 3.3.1. *For each $m \geq 0$, we have that*

$$\mathcal{OC}_{m-1} \circ B + \mathcal{OC}_m \circ b = 0. \tag{3.3.1}$$

To construct $\mathcal{OC}_m$, we need to consider, for $m, k, \ell \in \mathbb{Z}_{\geq 0}$, two types of (uncompactified) moduli spaces of domains

$$\mathcal{P}_{m,k,\ell} \tag{3.3.2}$$

$$\mathcal{P}^{S^1}_{m,k,\ell} \tag{3.3.3}$$

consisting of disks with marked points satisfying some additional constraints. The maps $\mathcal{OC}_m$ are defined by counting pearly trees with a single vertex corresponding to a disk with domain in (3.3.2), while the moduli spaces (3.3.3) are used in a similar way to define auxiliary operations arising in the proof of Lemma 3.3.1.

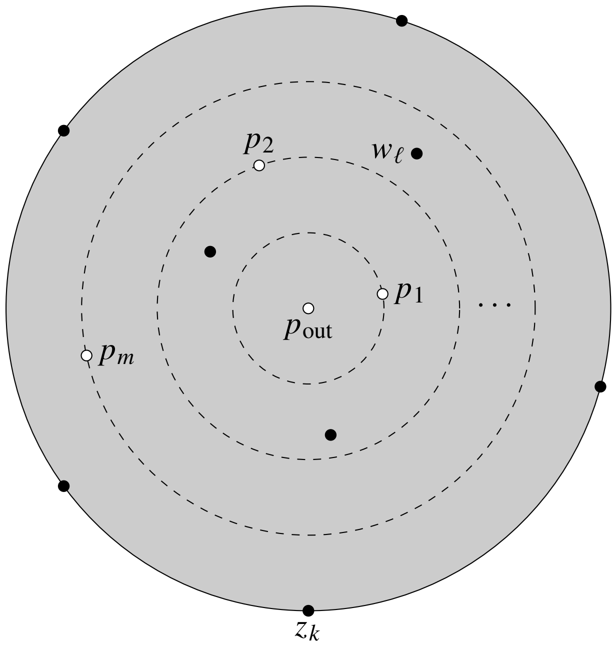

Figure 3.3: An element of (3.3.2). The black interior marked points are the ones which are neither auxiliary nor the output.

The elements of (3.3.2) are disks with k cyclically ordered boundary marked points denoted $z_1, \ldots, z_k$ and $\ell + m + 1$ interior marked points denoted

$$w_1, \ldots, w_\ell, p_{\text{out}}, p_1, \ldots, p_m.$$

The last m of these marked points are called auxiliary, and p_{out} is called the output marked point. Additionally, on the unit disk representative of such a disk which takes z_k to 1 and p_{out} to 0, the norms of the points p_i are required to satisfy

$$0 < |p_1| < \cdots < |p_m| < \frac{1}{2}. \tag{3.3.4}$$

Define $\theta_i := \arg(p_i)$ to be the argument of p_i taken with respect to the unit disk representative.

Elements of (3.3.3) are disks with k cyclically ordered boundary marked points $z_1, \ldots, z_k$ and

$\ell + m + 2$ interior marked points denoted

$$w_1, \dots, w_\ell, p_{\text{out}}, p_1, \dots, p_{m+1}$$

where the last $m + 1$ of these marked points are auxiliary. On the unit disk representative taking z_k to 1 and p_{out} to 0, the norms of the auxiliary marked points are required to satisfy

$$0 < |p_1| < \cdots < |p_m| < |p_{m+1}| = \frac{1}{2}.$$

We have an abstract identification $\mathcal{P}^{S^1}_{m,k,\ell} \cong S^1 \times \mathcal{P}_{m,k,\ell}$ where the S^1-coordinate is given by θ_{m+1}. Under this identification, we orient (3.3.3) by giving S^1 the opposite of its boundary orientation, and giving the product the opposite of the product orientation.

We form uncompactified moduli spaces of pseudoholomorphic disks in M with boundary on L whose domains belong to the moduli spaces (3.3.2) or (3.3.3). These are denoted

$$\mathcal{P}_{m,k,\ell}(\beta) \tag{3.3.5}$$

$$\mathcal{P}^{S^1}_{m,k,\ell}(\beta) \tag{3.3.6}$$

Each of these moduli spaces has naturally defined evaluation maps at each of the boundary and interior marked points.

To define the pearly trees relevant to the open-closed map, we need to modify our orientation convention for trees.

Definition 3.3.2. Let T be a bicolored tree in the sense of Definition (3.2.2) equipped with a ribbon structure, a metric, and a labeling of its vertices by classes $\beta_v \in H_2(M, L)$ for $v \in E_\circ(T)$ and $\beta_v \in H_2(M)$ for $v \in E_\bullet(T)$. Suppose that the edge set of T can be written as

$$E_\circ(T) = \{e^\circ_1, \dots, e^\circ_k\}$$

$$E_\bullet(T) = \{e^\bullet_1, \dots, e^\bullet_\ell, e^\bullet_{\text{out}}\}$$

We say that T is of *open-closed type* if its orientation is obtained by declaring that

- $e^\bullet_{\text{out}}$ is an outgoing edge adjacent to the vertex $v_{\text{out}} := s(e^\bullet_{\text{out}}) \in E_\circ(T)$, and all other edges of T point toward v_{out}.

The choices of homology classes and the valences of each vertex determine associated moduli spaces $\mathcal{M}(\beta_v) := \mathcal{M}_{\text{val}_\circ(v)-1,\text{val}_\bullet(v)}(\beta_v)$ if $v \in E_\circ(T)$ or $\mathcal{M}(\beta_v) := \mathcal{M}_{\text{val}_\bullet(v)}(\beta_v)$ if $v \in E_\bullet(T)$. To the vertex v_{out}, we associate one of the moduli spaces

$$\mathcal{P}_{m,\text{val}_\circ(v_{\text{out}}),\text{val}_\bullet(v_{\text{out}})}(\beta_{v_{\text{out}}}) \tag{3.3.7}$$

$$\mathcal{P}^{S^1}_{m,\text{val}_\circ(v_{\text{out}}),\text{val}_\bullet(v_{\text{out}})+1}(\beta_{v_{\text{out}}}). \tag{3.3.8}$$

In particular, auxiliary marked points do not have corresponding edges of T, but all other interior and boundary marked points do. The bicoloring on T induces an evaluation map ev_T defined similarly to (3.2.1). Finally, for each edge $e \in E_\circ(T)$, choose an f_L-admissible Morse function on L, and for each edge $e \in E_\bullet(T)$, choose an f_M-admissible Morse function on M. For each combinatorially finite edge $e \in E^f_{\circ/\bullet}(T)$, we can define embeddings as in (3.2.2) and (3.2.3), the images of which are still denoted G_e. Given the data as above, we can now define the moduli spaces of pearly trees contributing to the cyclic open-closed map.

Definition 3.3.3. Let $x = (x_1, \ldots, x_k)$ be a sequence of input critical points in $\text{crit}(f_L)$, let $y = (y_1, \ldots, y_\ell)$ be a sequence of input critical points in $\text{crit}(f_M)$, and let $y_{\text{out}} \in \text{crit}(f_M)$ be an output critical point. The *moduli spaces of open-closed pearly trees* of class $\beta \in H_2(M, L)$ are denoted

$$\mathcal{P}_m(x; y, y_{\text{out}}; \beta)$$

$$\mathcal{P}^{S^1}_m(x; y, y_{\text{out}}; \beta)$$

and are defined to be

$$\coprod_T \text{ev}_T^{-1}\left(\prod_{j=1}^{\ell} W^u(y_j) \times \prod_{j=1}^{k} W^u(x_j) \times \prod_{e \in E^f_\circ(T) \sqcup E^f_\bullet(T)} G_e \times W^s(y_{\text{out}})\right)$$

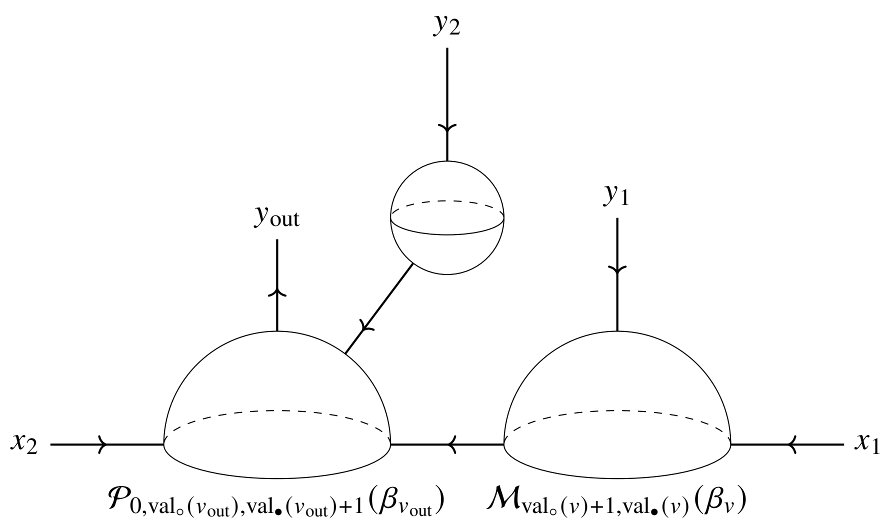

Figure 3.4: A pearly tree in $\mathcal{P}(x_1, x_2; y_{\text{out}}, y_1, y_2; \beta)$.

where the output vertex v_{out} is associated an element of (3.3.5) or (3.3.6), respectively. Both of these spaces admit natural Gromov compactifications

$$\overline{\mathcal{P}}_m(x; y, y_{\text{out}}; \beta) \tag{3.3.9}$$

$$\overline{\mathcal{P}}^{S^1}_{m,k,\ell}(x; y, y_{\text{out}}; \beta). \tag{3.3.10}$$

The regularity of (3.3.9) and (3.3.10) do not immediately follow from Assumption 3.2.7, so we must also assume that:

Assumption 3.3.4. The moduli spaces (3.3.9) and (3.3.10) (defined with respect to the almost complex structure J subject to Assumption 3.2.7) of open-closed pearly trees of virtual dimension ≤ 1 are compact oriented orbifolds of the expected dimension.

This assumption is sufficient for us to define the components of the cyclic open-closed map.

Definition 3.3.5. Given sequences of input critical points x and y as in Definition 3.3.3, define

$$\mathfrak{oc}^{\beta}_{m,k,\ell}(x; y) := \sum_{y_0 \in \text{crit}(f_M)} (-1)^{\star_k} \operatorname{hol}_\nabla(\beta) \# |\mathcal{P}_{m,k,\ell}(x; y, y_{\text{out}}; \beta)| y_{\text{out}}$$

$$\mathfrak{oc}^{S^1,\beta}_{m,k,\ell}(x;y) \coloneqq \sum_{y_{\text{out}}\in\text{crit}(f_M)} (-1)^{\star^{S^1}_k} \text{hol}_\nabla(\beta)\#|\mathcal{P}^{S^1}_{m,k,\ell}(x;y,y_{\text{out}};\beta)|y_{\text{out}}$$

and extend these R-linearly. Here $\#|\mathcal{P}_{m,k,\ell}(x;y,y_{\text{out}};\beta)|$ and $\#|\mathcal{P}^{S^1}_{m,k,\ell}(x;y,y_{\text{out}};\beta)|$ are signed counts of elements in the respective moduli spaces, and the signs are determined by

$$\star_k = \sum_{j=1}^{k}(n+j)|\alpha_j|'$$
$$\star^{S^1}_k = \star_k + \maltese_k - 1.$$

Define operations

$$\mathfrak{oc}_{m,k,\ell}(x;y) \coloneqq \sum_{\beta\in H_2(M,L;\mathbb{Z})} \mathfrak{oc}^{\beta}_{m,k,\ell}(x;y)$$
$$\mathfrak{oc}^{S^1}_{m,k,\ell}(x;y) \coloneqq \sum_{\beta\in H_2(M,L;\mathbb{Z})} \mathfrak{oc}^{S^1,\beta}_{m,k,\ell}(x;y).$$

For any bulk deformation parameter γ, define the bulk-deformed operations

$$\mathcal{OC}_{m,k}(x) \coloneqq \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\mathfrak{oc}_{m,k,\ell}(x;\gamma^{\otimes\ell}) \tag{3.3.11}$$
$$\mathcal{OC}^{S^1}_{m,k}(x) \coloneqq \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\mathfrak{oc}^{S^1}_{m,k,\ell}(x;\gamma^{\otimes\ell}). \tag{3.3.12}$$

We prove Lemma 3.3.1 by examining the boundary strata of the moduli spaces of open-closed pearls. First, we describe the boundary strata of 3.3.9.

Lemma 3.3.6. *Consider a sequence of inputs $y = y_1\otimes\cdots\otimes y_\ell \in CM^*(M;Q)^{\otimes\ell}$, an input sequence $x = x_1\otimes\cdots\otimes x_k \in CM^*(L;R)^{\otimes k}$, an output $y_{\text{out}} \in CM_*(M;Q)$, and a class $\beta \in H_2(M,L;\mathbb{Z})$ such that* (3.3.9) *is* 1*-dimensional. Let $I_1 \sqcup I_2 = \{1,\ldots,\ell\}$ denote a partition of the set of positive integers $\leq \ell$ into disjoint ordered subsets, and let $\beta = \beta_1+\beta_2 \in H_2(M,L;\mathbb{Z})$. Then the boundary* *of* (3.3.9) *is covered by the images under the natural inclusions of the following products of zero-dimensional moduli spaces:*

$$\mathcal{M}(x_{\text{out}}; x_{i+1}, \ldots, x_{i+j}; y_{I_1}, y_{\text{out}}; \beta_1) \times \mathcal{P}_m(x_1, \ldots, x_i, x_{\text{out}}, x_{i+j+1}, \ldots, x_k; y_{I_2}, y_{\text{out}}; \beta_2) \quad (3.3.13)$$

$$\mathcal{P}^{S^1}_{m-1}(x; y, y_{\text{out}}; \beta) \quad (3.3.14)$$

$$\mathcal{P}^{i,i+1}_{m}(x; y, y_{\text{out}}; \beta) \quad (3.3.15)$$

where (3.3.15) *consists of the subset of pearls whose output vertex is decorated by a pseudoholomorphic disk whose domain is of the sort contained in* (3.3.2)*, except that for some* $1 \leq i \leq m-1$*, the norms of the auxiliary marked points in the unit disk representative satisfy* $|p_i| = |p_{i+1}|$.

Proof. The first type of boundary breaking is of the same sort that occurs in one-dimensional moduli spaces of ordinary pearly trajectories. These behave as expected because we have used a fixed J to define the Cauchy–Riemann equation. Because of the constraints on their norms, the auxiliary marked points must always remain on the same component of any Gromov limit of a sequence such of curves. The boundary components (3.3.14) and (3.3.15) arise from sequences of pearly trajectories coming from a sequence of holomoprhic disks in which the norms of the auxiliary marked points change. □

Lemma 3.3.7. *For each* $m \geq 0$*, we have that*

$$\mathcal{OC}^{S^1}_{m-1} + \mathcal{OC}_m \circ b = 0\,.$$

Proof. Consider a sequence of input critical points $x = (x_1, \ldots, x_k)$ in $\text{crit}(f_L)$ and an output critical point $y_{\text{out}} \in \text{crit}(f_M)$ for which the moduli spaces $\mathcal{P}_m(x; \gamma^{\otimes \ell}, y_{\text{out}}; \beta)$ are one-dimensional, where γ is a bulk parameter. Lemma 3.3.6 implies that we can write

$$0 = \mathcal{OC}^{S^1}_{m-1} + \mathcal{OC}_m \circ b + \sum_{i=1}^{m-1} {}^{i,i+1}\mathcal{OC}_m$$

where the operations ${}^{i,i+1}OC_m$ are defined by counting pearly trees in moduli spaces of the form (3.3.15). The first summand corresponds to the boundary components (3.3.14), and the second summand corresponds to the boundary components (3.3.13).

All of the operations in the last summand vanish, because there is a forgetful map

$$\mathcal{P}_m^{i,i+1}(x, y, y_{\text{out}}; \beta) \to \mathcal{P}_{m-1}(x, y, y_{\text{out}}; \beta) \tag{3.3.16}$$

which forgets the auxiliary marked point p_{i+1}, and shifts the labels of all remaining auxiliary marked points down by 1. This implies that the elements of $\mathcal{P}_m^{i,i+1}(x, y, y_{\text{out}}; \beta)$ are never isolated, as this forgetful map always has one-dimensional fibers. The lemma now follows from a sign analysis of the sort carried out in [52]. □

Remark 3.3.8. For us, the existence of the forgetful map (3.3.16) follows because we have defined all moduli spaces of pseudoholomorphic disks using a fixed almost complex structure. In [52], the conditions imposed on domain-dependent perturbations for open-closed moduli spaces imply that the relevant analogue (3.3.16) exists. By [48, Remark 3.1], we cannot expect to construct Kuranishi structures which are compatible with forgetful maps of interior marked points. Nevertheless, one might hope for an independent proof in that setting that there are no isolated pearly trees in $\mathcal{P}_m^{i,i+1}(x, y, y_{\text{out}}; \beta)$.

In [52, Proposition 12], Deligne–Mumford type compactifications of (3.3.3) are decomposed into sectors corresponding to the angle of the auxiliary marked point p_1. The following lemma is proved by showing, roughly, that this decomposition into sectors yields a decomposition of the moduli spaces (3.3.10) of pearly trees.

Lemma 3.3.9. *For each $m \geq 0$, we have that*

$$OC_m^{S^1} = OC_m \circ B\,.$$

Proof. Let

$$\mathcal{P}_{m,k+1,\ell,\tau_i}$$

to be the uncompactified moduli space of disks with $k+1$ cyclically ordered boundary marked points

$$z_1, \ldots, z_i, z_0, z_{i+1}, \ldots, z_k$$

along with ℓ interior marked points, m auxiliary interior marked points, and an output interior marked point p_{out}. The norms (on the unit disk representative) of the auxiliary marked points are required to satisfy

$$0 < |p_1| < \cdots < |p_m| < \frac{1}{2}.$$

There is a bijection

$$\tau_i \colon \mathcal{P}_{m,k+1,\ell,\tau_i} \to \mathcal{P}_{m,k,\ell}$$

given by cyclically permuting boundary labels.

Now consider, for all $1 \leq i \leq k$, the moduli spaces

$$\mathcal{P}_{m,k,\ell}^{S^1_{i,i+1}}$$

which are the open subsets of $\mathcal{P}_{m,k,\ell}^{S^1}$ with the property that $\arg(p_1)$ lies between $\arg(z_i)$ and $\arg(z_{i+1})$, where the indices are taken mod k.

We also have an auxiliary-rescaling map

$$\pi_{k+1}^i \colon \mathcal{P}_{m,k+1,\ell,\tau_i} \to \mathcal{P}_{m,k,\ell}^{S^1_{i,i+1}}$$

which places a marked point p_{m+1} of norm $\frac{1}{2}$ with the line between p_{out} and z_{k+1}, and deletes z_{k+1}. Taking the union of these maps gives an orientation-preserving embedding

$$\coprod_i \mathcal{P}_{m,k+1,\ell,\tau_i} \xrightarrow{\coprod_i \pi_{k+1}^i} \coprod_i \mathcal{P}_{m,k,\ell}^{S^1_{i,i+1}} \hookrightarrow \mathcal{P}_{m,k,\ell}^{S^1}.$$

It is clear that image of this embedding covers all but a codimension 1 subset of the target. Specifically, the complement of the image is the locus of disks for which $\arg(p_1) = \arg(z_i)$ for some $1 \leq i \leq k$.

This implies that all elements of the zero-dimensional moduli spaces (3.3.9) can be taken to have disks at the output vertices whose domains lie in the image of this embedding, possibly after perturbing the Morse functions f_L and f_M. By counting isolated pseudoholomorphic pearly trees with underlying bicolored metric ribbon trees of open-closed type and output vertex decorated by elements of moduli spaces of the form

$$\mathcal{P}_{m,k+1,\ell,\tau_i}(\beta)$$

we can define operations $\mathcal{OC}_{m,k,\tau_i}$. More precisely, elements of this moduli space consist of pseudoholomorphic disks $(D^2, \partial D^2) \to (M, L)$ representing the class $\beta \in H_2(M, L; \mathbb{Z})$ whose domains lie in $\mathcal{P}_{m,k+1,\ell,\tau_i}$.

Hence, if we assume without loss of generality that f_L has a unique minimum which represents the unit $1 \in CM^*(L; R)$, there is an equality of chain-level operations

$$\begin{aligned} \mathcal{OC}^{S^1}_{m,k}(x_1 \otimes \cdots \otimes x_k) &= \sum_{i=0}^{m-1} \mathcal{OC}_{m,k,\tau_i}(x_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_k) \\ &= \sum_{i=0}^{m-1} \mathcal{OC}_{m,k+1}(1 \otimes x_{i+1} \otimes \cdots \otimes x_k \otimes x_1 \otimes \cdots \otimes x_i). \end{aligned}$$

In this identity, the inputs for $\mathcal{OC}_{m,k,\tau_i}$ at the marked point z_0 must be 1 for degree reasons. The last equality holds because the bijections τ_i induce bijections between the relevant spaces of pearly trees. The result now follows from a sign analysis of the operations on the right hand side, of the sort carried out in [52]. □

Proof of Lemma 3.3.1. This is an immediate consequence of Lemma 3.3.7 and Lemma 3.3.9. □

There is also a version of the cyclic open-closed map on the possibly bulk-deformed A_∞-

algebras

$$\widetilde{\mathcal{A}} = (CM^*(L \times [-1,1]; R), \{\widetilde{\mathfrak{m}}_k^{\widetilde{\gamma}}\}_{k=0}^{\infty})$$

as defined in Section 3.2.2 using a path of almost complex structures $\underline{J} = \{J_t\}_{t\in[-1,1]}$. The definitions of these maps require time-dependent analogues of (3.3.5) and (3.3.6). Namely consider the moduli spaces

$$\widetilde{\mathcal{P}}_{m,k,\ell}(\beta) \coloneqq \{(u,t)\colon u \in \mathcal{P}_{m,k,\ell}(\beta; J_t)\} \tag{3.3.17}$$

$$\widetilde{\mathcal{P}}^{S^1}_{m,k,\ell}(\beta) \coloneqq \{(u,t)\colon u \in \mathcal{P}^{S^1}_{m,k,\ell}(\beta; J_t)\}. \tag{3.3.18}$$

Given a tree T of open-closed type, we can define open-closed pearly trees on cylinder objects.

Definition 3.3.10. Let $\widetilde{x} = (\widetilde{x}_1, \ldots, x_k)$ be a sequence of input critical points in $\mathrm{crit}(\widetilde{f_L})$, let $y = (\widetilde{y}_1, \ldots, \widetilde{y}_\ell)$ be a sequence of input critical points in $\mathrm{crit}(\widetilde{f_M})$, and let $\widetilde{y}_{\mathrm{out}} \in \mathrm{crit}(\widetilde{f_M})$ be an output critical point. The *time-dependent moduli spaces of open-closed pearly trees* of class $\beta \in H_2(M, L)$ are denoted

$$\mathcal{P}_m(\widetilde{x}; \widetilde{y}, \widetilde{y}_{\mathrm{out}}; \beta)$$

$$\mathcal{P}_m^{S^1}(\widetilde{x}; \widetilde{y}, \widetilde{y}_{\mathrm{out}}; \beta)$$

and are defined to be

$$\coprod_T \mathrm{ev}_T^{-1}\left(\prod_{j=1}^{\ell} W^u(y_j) \times \prod_{j=1}^{k} W^u(x_j) \times \prod_{e \in E_\circ^f(T) \sqcup E_\bullet^f(T)} G_e \times W^s(y_{\mathrm{out}})\right)$$

where the output vertex v_{out} is associated an element of (3.3.17) or (3.3.18), respectively. Both of these spaces admit natural Gromov compactifications

$$\overline{\mathcal{P}}_m(x; y, y_{\mathrm{out}}; \beta) \tag{3.3.19}$$

$$\overline{\mathcal{P}}_m^{S^1}(x; y, y_{\mathrm{out}}; \beta). \tag{3.3.20}$$

As usual, we work under a regularity assumption on these time-dependent pearly moduli spaces.

Assumption 3.3.11. The moduli spaces (3.3.19) and (3.3.20) of virtual dimension ≤ 1 are compact orbifolds of the expected dimension.

Given this assumption, a discussion completely parallel to the one carried out above shows that we can construct the desired cyclic open-closed map on the cylinder.

Lemma 3.3.12. *There exists a sequence of linear maps*

$$\widetilde{\mathcal{OC}}_m \colon CH_*(\widetilde{\mathcal{A}}) \to CM_{n+1-*+2m}(M \times [-1, 1])$$

which satisfy

$$\widetilde{\mathcal{OC}}_{m-1} \circ B + \widetilde{\mathcal{OC}}_m \circ b = 0$$

for all $m \geq 0$. □

3.4 The open Gromov–Witten potential

In this section, we explain how to construct the open Gromov–Witten potential using the ∞-inner product induced by the cyclic open-closed map. We have so far not discussed disks without marked points. Since we have defined the Lagrangian Floer theory using pearly configurations, the appropriate analogue of the inhomogeneous $\mathfrak{m}_{-1}$ as it appears in [18] and [49] should of course be a count of pearly trees, not just of disks.

3.4.1 Inhomogeneous terms

Denote by $\mathcal{M}_{-1,\ell}(\beta; J)$ the moduli spaces of pseudoholomorphic disks $u\colon (D^2, \partial D^2) \to (M, L)$ in the class $\beta \in H_2(M, L; \mathbb{Z})$ with no boundary marked points and interior marked points labeled

$$w_1, \ldots, w_\ell$$

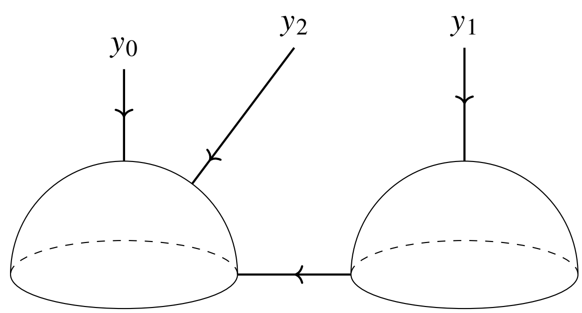

Figure 3.5: A pearly tree contributing to $\mathfrak{m}_{-1}$.

in order for any integer $\ell \geq 0$. There are evaluation maps

$$\mathrm{evi}_j\colon \mathcal{M}_{-1,\ell}(\beta; J) \to M$$

at the interior marked points. The pearly trees which contribute to the inhomogeneous term of our open Gromov–Witten potential should be trees with no outputs which take no inputs from the Morse cochain of L.

Definition 3.4.1. Let T be a bicolored tree equipped with a metric, a ribbon structure, and a labeling of its vertices by by classes $\beta_v \in H_2(M, L)$ for $v \in E_\circ(T)$ and $\beta_v \in H_2(M)$ for $v \in E_\bullet(T)$. We say that T is of *inhomogeneous type* if $E_\circ(T)$ contains no semi-infinite edges, and if its oriented such that

- all semi-infinite edges in $E_\bullet(T)$ are are incoming edges; and
- every disk vertex $v \in E_\circ(T)$ has at most one outgoing adjacent edge.

For each T of inhomogeneous type, choose f_M-admissible and f_L-admissible Morse functions on each edge $E_\bullet(T)$ and $E_\circ(T)$, respectively.

Definition 3.4.2. Given input critical points $y_1, \ldots, y_\ell \in \mathrm{crit}(f_M)$, define

$$\mathcal{M}_{-1}(y_1 \otimes \cdots \otimes y_\ell; \beta) \tag{3.4.1}$$

to be the moduli space of all pearly trees with the given inputs whose domain is parametrized by a tree T of inhomogeneous type.

The precise definition of these moduli spaces also uses an evaluation map ev_T associated to T, but the submanifold that we pull back under this map does not contain any stable manifold factor (which would correspond to an output). We need a regularity assumption on these moduli spaces which is the analogue of Assumption 3.2.7.

Assumption 3.4.3. The moduli spaces (3.4.1) of virtual dimension 0 are compact oriented zero-dimensional orbifolds.

This assumption lets us define operations by counting the elements of the zero-dimensional moduli spaces of this type.

Definition 3.4.4. Let $y_1, \dots, y_\ell \in \mathrm{crit}(f_M)$ be a sequence of input critical points and y_{out} be a critical point of index 0, with the property that (3.4.1) is zero-dimensional, and set

$$\mathfrak{q}^{\beta}_{-1,\ell}(y_1 \otimes \cdots \otimes y_\ell) = \sum_{y_{\mathrm{out}} \in \mathrm{crit}(f_M)} \mathrm{hol}_\nabla(\beta)\#|\mathcal{M}_{-1}(y_1 \otimes \cdots \otimes y_\ell; \beta)| \in R.$$

Extend these to operations

$$\mathfrak{q}_{-1,\ell} \colon CM^*(M; Q)^{\otimes \ell} \to R$$

by setting

$$\mathfrak{q}_{-1,\ell} \coloneqq \sum_{\beta \in H_2(M,L;\mathbb{Z})} \mathfrak{q}^{\beta}_{-1,\ell}.$$

Finally, given a bulk parameter $\gamma \in CM^*(M)$, set

$$\mathfrak{m}^{\gamma}_{-1} \coloneqq \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \mathfrak{q}_{-1,\ell}(\gamma^{\otimes \ell}). \tag{3.4.2}$$

Given a path $\underline{J} = \{J_t\}_{t\in[-1,1]}$ of almost complex structures, there are also moduli spaces

$$\widetilde{\mathcal{M}}_{-1,\ell}(\beta;\underline{J}) \coloneqq \{(u,t)\colon u \in \mathcal{M}_{-1,\ell}(\beta;J_t)\}$$

which have naturally defined evaluation maps at the interior marked points. Using these evaluation maps, one defines moduli spaces of pearly trees

$$\mathcal{M}_{-1}(\widetilde{y}_1 \otimes \cdots \otimes \widetilde{y}_\ell;\beta;\underline{J}) \tag{3.4.3}$$

with underlying tree T of inhomogeneous type.

Assumption 3.4.5. The moduli spaces (3.4.3) of virtual dimension at most 1 are compact orbifolds of the expected dimension.

We need regularity for 1-dimensional time-dependent moduli spaces, since these arise in the proof of Theorem 3.4.9, whereas the 1-dimensional moduli spaces of ordinary pearly trees with no inputs do not. With this we define the time-dependent inhomogeneous terms $\widetilde{\mathfrak{m}}^{\widetilde{\gamma}}_{-1}$ in the obvious way.

If T is a tree for which the planar part $T_\circ$ consists of a single vertex, the boundary of an associated pearly configuration can collapse to a point in L. We will now introduce notation for the count of such configurations.

For $\beta \in H_2(M;\mathbb{Z})$, define the moduli spaces of $\underline{J}$-holomorphic spheres

$$\widetilde{\mathcal{M}}_{\ell+1}(\beta;\underline{J}) \coloneqq \{(u,t)\colon u \in \mathcal{M}_{\ell+1}(\beta;J_t)\}$$

and label the marked points on the domain $0,\ldots,\ell$. Consider bicolored metric ribbon trees T for which $T_\circ = \emptyset$ with $\ell+1$ semi-infinite edges, labeled $e^\bullet_{\mathrm{out}}, e^\bullet_1,\ldots,e^\bullet_\ell$, where $e^\bullet_{\mathrm{out}}$ is outgoing all other semi-infinite edges are incoming. The combinatorially finite edges of T are oriented so that they point to $e^\bullet_0$. Associate a class $\beta_v \in H_2(M;\mathbb{Z})$ to each vertex $v \in V(T) = V_\bullet(T)$. Given a sequence of input critical points $\widetilde{y}_1, \ldots, \widetilde{y}_\ell \in \mathrm{crit}(\widetilde{f}_M)$, we can define moduli spaces of pearly trees

$$\mathcal{M}(y_{\mathrm{out}}, y_1, \ldots, y_\ell; \beta; \underline{J}) \tag{3.4.4}$$

by assigning $\widetilde{f}_M$-admissible Morse functions to each edge of T and using the evaluation maps associated to T. As is routine by now, we impose a regularity assumption on these moduli spaces.

Assumption 3.4.6. The moduli spaces (3.4.4) of virtual dimension 0 are compact orbifolds of dimension 0.

We set

$$\widetilde{\mathfrak{q}}^{\beta}_{\emptyset,\ell}(y_1, \ldots, y_\ell) := \sum_{y_{\mathrm{out}}} \#|\mathcal{M}(y_{\mathrm{out}}, y_1, \ldots, y_\ell; \beta; \underline{J})| \cdot y_{\mathrm{out}}$$

where the sum is over all y_{out} such that (3.4.4) is 0-dimensional. Extend these to operations

$$\widetilde{\mathfrak{q}}_{\emptyset,\ell} \colon CM^*(M \times [-1,1]; Q)^{\otimes \ell} \to CM^*(M \times [-1,1]; Q)$$

in the usual way. The Lagrangian embedding $\iota \colon L \hookrightarrow M$ lets us pull back the values of $\widetilde{\mathfrak{q}}_{\emptyset,\ell}$ to $CM^*(L \times [-1,1])$, the result of which is denoted

$$\iota^* \widetilde{\mathfrak{q}}_{\emptyset,\ell}(\widetilde{y}_1 \otimes \cdots \otimes \widetilde{y}_\ell). \tag{3.4.5}$$

By the construction of the Morse function $\widetilde{f}_L$ used to construct $CM^*(L \times [-1,1])$, we can assume that $CM^*(L \times [-1,1]; R)$ has a single generator in degree $n+1$. This is because we can take the Morse function f_0 on L to have a single maximum without loss of generality. The element $\widetilde{GW} \in R$ is defined to be the coefficient of this degree $n+1$ generator in the Morse cochain (3.4.5).

3.4.2 Wall-crossing

Having defined the inhomogeneous terms $\mathfrak{m}^{\gamma}_{-1}$ in our setting, we can now define the open Gromov–Witten potential as it is defined in the cyclic case. The higher order terms of the open

Gromov–Witten potential will be as in Definition 3.1.16. The ∞-inner product we use is induced from the trace map

$$CC_*^{+,\mathrm{red}} \xrightarrow{\mathcal{OC}} CM_{*-n}(M;R) \otimes_R R((u))/uR[[u]] \to R$$

where the last map projects to the u^0-factor and then projects to $R = H_*(\mathrm{pt};R)$. This can be thought of as a positive cyclic cocycle, which induces a negative cyclic cohomology class by Lemma 3.1.8, and in turn an ∞-inner product by Lemma 3.1.10.

Definition 3.4.7. For a Lagrangian submanifold $L \subset M$ with a flat $GL(1,\Bbbk)$-connection ∇ and a bulk parameter γ, let $\mathcal{A} = (CF^*(L;R), \{\mathfrak{m}_k^\gamma\}_{k=0}^\infty)$ be the bulk-deformed pearly A_∞-algebra, and let $\phi\colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ denote the ∞-inner product induced by the cyclic open-closed map. The ∞-*open Gromov–Witten potential* of (L,∇) is the function $\Phi\colon \mathcal{MC}(\mathcal{A}) \to H_0(M;R)$ defined by the convergent power series

$$\Phi(b) := \mathfrak{m}_{-1}^\gamma + \sum_{N=0}^\infty \sum_{p+q+k=N} \frac{1}{N+1} \phi_{p,q}(b^{\otimes p} \otimes \underline{\mathfrak{m}_k^\gamma(b^{\otimes k})} \otimes b^{\otimes q})(b). \tag{3.4.6}$$

To show that this choice of inhomogeneous term is appropriate, we need to verify that $\Phi(b)$ has the expected behavior under variations of the almost complex structure J used to construct the open-closed map and the A_∞-operations. For this purpose, we need a notion of gauge-equivalence of weak bounding cochains which is compatible with the A_∞-structures we have constructed on cylinder objects, adapted from [49, Definition 3.12].

Definition 3.4.8. Suppose that we have a bulk parameter $\widetilde{\gamma} \in \widetilde{\mathcal{A}} = CM^*(M \times [-1,1]; Q)$, where curved the A_∞-structure is defined using $\underline{J}$, and a class $\widetilde{b} \in \widetilde{\mathcal{A}}$ of degree $|b| = 1$. Further assume that there is a constant $c \in R$ such that

$$\widetilde{\mathfrak{m}}_0^b = c \cdot 1.$$

Then the pairs (b_{-1}, γ_{-1}) and (b_1, γ_1) obtained by restricting to $L \times \{\mp 1\}$ and $M \times \{\mp 1\}$ are said to be gauge-equivalent.

The rest of this section is occupied by the proof of our main result.

Theorem 3.4.9 (Wall crossing formula)**.** *Let $\underline{J} = \{J_t\}_{t\in[-1,1]}$ be a path of almost complex structures satisfying Assumption 3.2.11, and suppose we are given gauge-equivalent pairs $(b_{\mp 1}, \gamma_{\mp 1})$ which are gauge-equivalent in the sense of Definition 3.4.8. Then the open Gromov–Witten potentials defined with respect to the almost complex structures J_{-1} and J_1 and Morse functions f_L^{-1} and f_L^1 satisfy*

$$\Phi_{-1}(b_{-1}) = \Phi_1(b_1) + \widetilde{GW}.$$

Proof. Let $\phi^{\pm 1}$ denote the ∞-inner products on $CM^*(L; R; f_L^{\pm 1})$ constructed from the cyclic open-closed maps $\mathcal{OC}^{\pm 1}$ defined using the almost complex structures $J_{\pm 1}$, and let $\widetilde{\phi}$ denote the ∞-inner product on $CM^*(L\times[-1,1]; R)$ defined using $\underline{J}$. Additionally, let $\widetilde{b}$ and $\widetilde{\gamma}$ be a bounding cochain and bulk parameter on $CM^*(L\times[-1,1]; R)$ realizing the gauge-equivalence between b_{-1} and b_1.

Let $\pi\colon M\times[-1,1]\to[-1,1]$ denote the projection map, and let

$$\pi_*\colon (CM_*(M\times[-1,1]), \partial) \to (CM_*([-1,1]), \partial)$$

denote the induced map on Morse cochains, where the Morse compex on $[-1,1]$ is defined using the Morse function ρ on $(-1-\epsilon, 1+\epsilon)$. Since this is a chain map, it follows that

$$\partial\pi_*(\widetilde{\mathcal{OC}}_0(\widetilde{b}^{\otimes p}\otimes\widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k})\otimes\widetilde{b}^{\otimes q}\otimes\widetilde{b}) = \pi_*(\partial\widetilde{\mathcal{OC}}(\widetilde{b}^{\otimes p}\otimes\widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k})\otimes\widetilde{b}^{\otimes q}\otimes\widetilde{b})) \tag{3.4.7}$$

where the expression on the left hand side can be written as

$$\begin{aligned}&\partial\pi_*(\widetilde{\mathcal{OC}}_0(\widetilde{b}^{\otimes p}\otimes\widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k})\otimes\widetilde{b}^{\otimes q}\otimes\widetilde{b})\\ &= \mathcal{OC}_0^1(b_1^{\otimes p}\otimes\mathfrak{m}_k^1(b_1^{\otimes k})\otimes b_1^{\otimes q}\otimes b_1) - \mathcal{OC}_0^{-1}(b_{-1}^{\otimes p}\otimes\mathfrak{m}_k^{-1}(b_{-1}^{\otimes k})\otimes b_{-1}^{\otimes q}\otimes b_{-1})\end{aligned} \tag{3.4.8}$$

by construction of the chain map π_*.

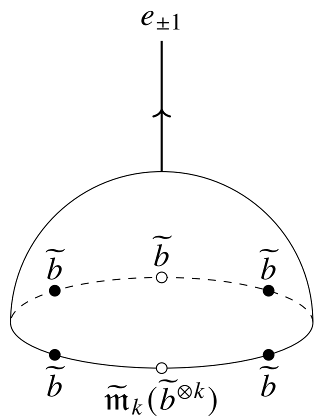

Figure 3.6: Elements of (3.4.9). The marked points corresponding to the input and output of $\widetilde{\phi}$ are the white dots.

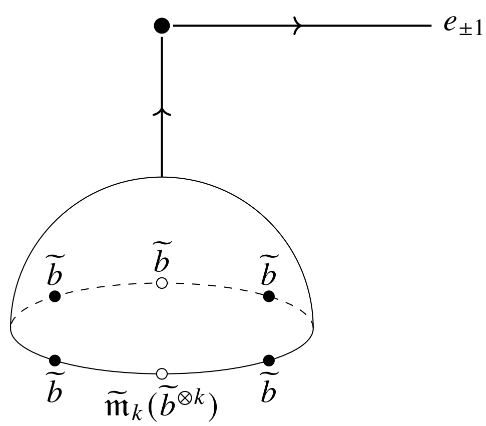

Figure 3.7: Boundary stratum of (3.4.9) involving breaking on the output edge.

On the other hand, consider the one-dimensional moduli spaces of trees with underlying domain of open-closed type

$$\overline{\mathcal{P}}_{0,p+q+1,\ell}(\widetilde{b}^{\otimes p} \otimes \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}) \otimes \widetilde{b}^{\otimes q} \otimes \widetilde{b}; e_{\pm 1}). \tag{3.4.9}$$

Examining the boundary strata of these spaces, we see that the boundary components where the output edge breaks into a broken negative gradient flow line in $M \times [-1, 1]$ (cf. (3.7)) contributes

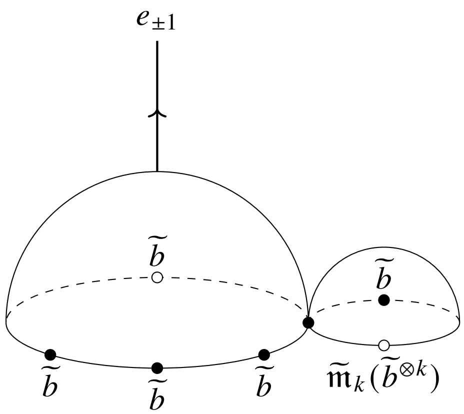

Figure 3.8: Elements of the boundary stratum contributing to term (3.4.12).

to the Morse differential of the open-closed map, i.e.

$$\partial \widetilde{\mathcal{OC}}(\widetilde{b}^{\otimes p} \otimes \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}) \otimes \widetilde{b}^{\otimes q} \otimes \widetilde{b}).$$

The other boundary strata involve breakings of gradient flow lines on $L \times [-1, 1]$ or of pseudoholomorphic disks into nodal disks with boundary on $L \times [-1, 1]$. These are represented schematically in Figures (3.8), (3.9), (3.10), and (3.11). There is a parallel description of the moduli spaces

$$\overline{\mathcal{P}}_{0,p+q+1,\ell}(\widetilde{b}^{\otimes p} \otimes \widetilde{b} \otimes \widetilde{b}^{\otimes q} \otimes \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}); e_{\pm 1}). \tag{3.4.10}$$

By (3.4.8) and the definition of the ∞-inner product, it follows that

$$\Phi'_1(b_1) - \Phi'_{-1}(b_-) \tag{3.4.11}$$

$$= \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{1}{N+1} \sum_{r+s=k_1-1} \widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes r} \otimes \widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}) \otimes \widetilde{b}^{\otimes s})} \otimes \widetilde{b}^{\otimes q})(\widetilde{b}) \tag{3.4.12}$$

$$+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{1}{N+1} \sum_{r+s=p-1} \widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes r} \otimes \widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}) \otimes \widetilde{b}^{\otimes s} \otimes \underline{\mathfrak{m}_{k_1}(\widetilde{b}^{\otimes k_1})} \otimes \widetilde{b}^{\otimes q})(\widetilde{b}) \tag{3.4.13}$$

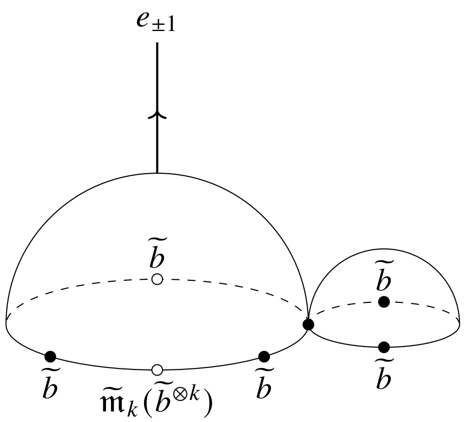

Figure 3.9: Boundary strata contributing to terms (3.4.13) and (3.4.14).

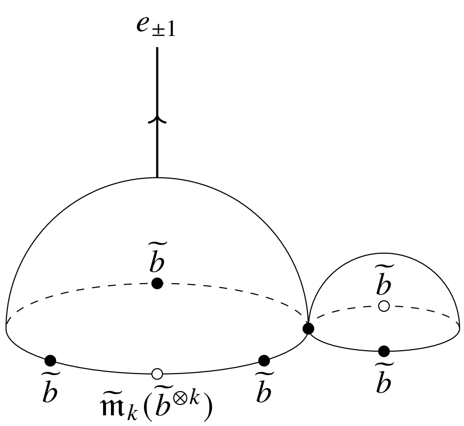

Figure 3.10: Boundary stratum contributing to the term (3.4.15).

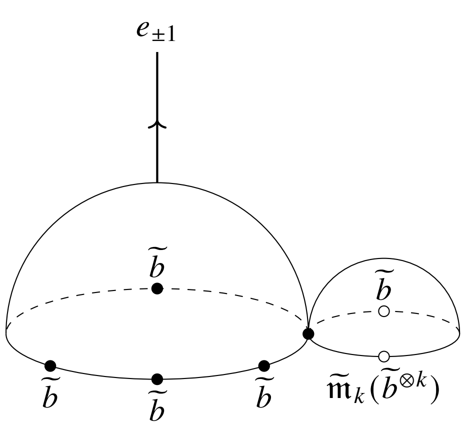

Figure 3.11: Canceling boundary strata.

$$+\sum_{N=0}^{\infty}\sum_{\substack{p+q+k=N\\k_1+k_2=k+1}}\frac{1}{N+1}\sum_{r+s=q-1}\widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes p}\otimes\underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})}\otimes\widetilde{b}^{\otimes r}\otimes\widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2})\otimes\widetilde{b}^{\otimes s})(\widetilde{b}) \tag{3.4.14}$$

$$+\sum_{N=0}^{\infty}\sum_{\substack{p+q+k=N\\k_1+k_2=k+1}}\frac{k_2}{N+1}\widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes p}\otimes\underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})}\otimes\widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2})) \tag{3.4.15}$$

Because the value of $\widetilde{\phi}$ on inputs of the form under consideration are expressed as the difference

$$\widetilde{OC}_0(\widetilde{b}^{\otimes p}\otimes\widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k})\otimes\widetilde{b}^{\otimes q}\otimes\widetilde{b})-\widetilde{OC}(\widetilde{b}^{\otimes p}\otimes\widetilde{b}\otimes\widetilde{b}^{\otimes q}\otimes\widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}))$$

it follows that boundary strata of the sort depicted in Figure 3.11 appear in both (3.4.9) and (3.4.10). Since $\widetilde{\phi}_{p,q}$ is defined by taking a difference corresponding to these two moduli spaces, the contributions of Figure (3.11) cancel, and thus they do not appear in the sum above.

We can rewrite the sum of (3.4.12), (3.4.13), and (3.4.14) as

$$\sum_{N=0}^{\infty}\sum_{\substack{p+q+k=N\\k_1+k_2=k+1}}\frac{N+1-k_2}{N+1}\widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes p}\otimes\underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})}\otimes\widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2})) \tag{3.4.16}$$

by Lemma 3.1.15. The sum of (3.4.15) and (3.4.16) can be rewritten, using the Maurer–Cartan equation, as

$$\sum_{p,q\geq 0}\widetilde{\phi}_{p,q}(\widetilde{b}^{\otimes p}\otimes\underline{(\widetilde{\mathfrak{m}}_0-c\cdot 1)}\otimes\widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_0-c\cdot 1)\,. \tag{3.4.17}$$

Since $\Bbbk$ is of characteristic 0, it follows from Lemma 3.1.14 that the terms of (3.4.17) for which $p>0$ or $q>0$ all vanish. Thus we are left with

$$\widetilde{\phi}_{0,0}(\underline{\widetilde{\mathfrak{m}}_0})(\widetilde{\mathfrak{m}}_0)=\widetilde{\phi}_0(1,\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0,\widetilde{\mathfrak{m}}_0))\,. \tag{3.4.18}$$

by Lemma 3.1.13 and the linearity of $\widetilde{\phi}_{0,0}$. The right hand side of (3.4.18) can itself be rewritten as

$$OC_0(\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0)) \tag{3.4.19}$$

by the construction of the negative cyclic cocycle.

We can also analyze the inhomogeneous terms similarly. One type of boundary component that can occur in the one-dimensional moduli spaces (3.4.3) consists of a broken configuration consisting of two pearly trees in (3.2.10), both of which have one output and no inputs, meaning that they would both contribute to $\mathfrak{m}_0$. Notice, however, that such broken configurations correspond exactly to those which contribute to (3.4.19), because $\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0)$ is of top degree, so that only constant disks can contribute to the product.

Examining the remaining boundary strata of (3.4.3) shows that

$$\mathfrak{m}^1_{-1} - \mathfrak{m}^0_{-1} + OC_0(\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0)) + \widetilde{GW} = 0.$$

The third term in this sum coincides with (3.4.19), and the remaining terms give the leading terms and wall-crossing term. □

Remark 3.4.10. Our appeal to Lemma 3.1.14 is the only place, other than in the definition of the infinity cyclic potential itself, where we use the fact that our ground field has characteristic 0.

3.5 Comparison with Solomon and Tukachinsky's invariants

Showing that the open Gromov–Witten potential of Definition 3.4.7 agrees with the open Gromov–Witten potential of [49] would most likely require the construction of a very well-behaved quasi-isomorphism between the pearly A_∞-algebra for $L \subset M$ and the de Rham version of the A_∞-algebra for L. Alternatively, one might hope to compare our invariants with those of [79]. Instead of pursuing this, we will sketch the analogue of our construction under the technical assumptions of [49], illustrating in the process how our constructions simplify in the presence of a strictly cyclic pairing.

Recall that [49] assumes that

- the moduli spaces $\mathcal{M}_{k+1,\ell}(\beta)$ are compact orbifolds with corners for all $k \geq -1$ and $\ell \geq 0$ and the boundary evaluation maps evb_0 on these spaces at the zeroth boundary marked points are submersions.

Under this assumption, the closed-open operations $\mathfrak{q}^{\beta}_{k,\ell}$ are defined by pulling back differential forms $\alpha_1, \ldots, \alpha_k \in \Omega^*(L; R)$ and $\gamma_1, \ldots, \gamma_\ell \in \Omega^*(M; Q)$ to $\mathcal{M}_{k+1,\ell}(\beta)$ under the corresponding evaluation maps, taking the wedge product of these forms, and pushing forward by evb_0. Here the pushforward of differential forms is given by integration along the fiber, which is where submersivity is required. These yield bulk-deformed A_∞-operations on $\Omega^*(L; R)$.

Let $\gamma \in \Omega^*(L; R)$ be a bulk parameter, and let $\mathcal{A}$ denote $\Omega^*(L; R)$ equipped with the resulting A_∞-operations. In [58], the cyclic open-closed map on the de Rham complex was constructed under these regularity assumptions. There, the open-closed map $\mathcal{OC}_0$ is characterized by the property that

$$\langle \eta, \mathcal{OC}_0(\alpha)\rangle_M = (-1)^{|\alpha_0|(\sum_{i\geq 1}|\alpha_i|'+1)}\langle \mathfrak{q}^{\gamma}_{k,1}(\alpha_1 \otimes \cdots \otimes \alpha_k; \eta), \alpha_0\rangle_L \tag{3.5.1}$$

for any reduced Hochschild cohain $\alpha = \underline{\alpha_0} \otimes \alpha_1 \otimes \cdots \otimes \alpha_k \in \mathcal{A} \otimes (\mathcal{A}[1])^{\otimes k}$ and any differential form $\eta \in \Omega^*(M; Q)$. Since the integration pairing on L is strictly cyclic, we can extend the open-closed map u-linearly to obtain a cyclic open-closed map $\mathcal{OC}$. This induces a strictly cyclic ∞-inner product ψ on $\mathcal{A}$, whose values are determined only by the open-closed map.

The potential of Definition 3.4.7 in this case reduces to

$$\Psi(b) = \mathfrak{m}^{\gamma}_{-1} + \sum_{k=0}^{\infty} \frac{1}{k+1} \psi_{0,0}(\underline{\mathfrak{m}^{\gamma}_k(b^{\otimes k})})(b).$$

Since we have obtained ψ from a negative cyclic cocycle, it follows that

$$\psi_{0,0}((\underline{\mathfrak{m}^{\gamma}_k(b^{\otimes k})})(b) = \psi_0(1, \mathfrak{m}^{\gamma}_2(\mathfrak{m}^{\gamma}_k(b^{\otimes k}), b))$$

where ψ_0 refers to the part of the negative cyclic cocycle residing in the zeroth column of the (b^*, B^*)-bicomplex (3.1.11). Since the negative cyclic cocyle is obtained from the cyclic open-closed map under the isomorphism of Lemma 3.1.8, it follows that the open Gromov-Witten potential can be rewritten as

$$\Psi(b) = \mathfrak{m}_{-1}^{\gamma} + \sum_{k=0}^{\infty} \frac{1}{k+1} \mathcal{OC}_0(\mathfrak{m}_2^{\gamma}(\mathfrak{m}_k^{\gamma}(b^{\otimes k}) \otimes b)).$$

By the top-degree property [80, Proposition 3.12] of Solomon and Tukachinsky's A_∞-algebra, we have that

$$\mathfrak{m}_2^{\gamma}(\mathfrak{m}_k^{\gamma}(b^{\otimes k}) \otimes b) = \mathfrak{m}_k^{\gamma}(b^{\otimes k}) \wedge b.$$

Using (3.5.1), we compute

$$\langle 1, \mathcal{OC}_0(\mathfrak{m}_k^{\gamma}(b^{\otimes k}) \wedge b) \rangle_M = \langle \mathfrak{q}_{0,1}^{\gamma}(1), \mathfrak{m}_k^{\gamma}(b^{\otimes k}) \wedge b \rangle_L = \langle \mathfrak{m}_k^{\gamma}(b^{\otimes k}), b \rangle_L.$$

To summarize, we have proven the following.

Theorem 3.5.1. *For any $L \subset M$ subject to the assumptions of [49], the ∞-OGW potential defined over the de Rham complex recovers the OGW potential of [49] up to an overall sign.*

It is also possible to give an independent proof of the wall-crossing formula over the de Rham complex using pseudo-isotopies of A_∞-algebras defined in [48] or [18]. By [81, Lemma 21.31], these arise from A_∞-structures on the de Rham complex of $L \times I$ as constructed in [80].

3.6 Regularity hypotheses

We have made several regularity assumptions throughout the main body of this paper, all of which are summarized below.

- Assumption 3.2.1: all Morse functions used to define Morse (co)chain complexes of L and M have a unique local minimum and a unique local maximum.

- Assumption 3.2.7: there is a $J \in \mathcal{J}(M)$ such that the moduli spaces of pseudoholomorphic pearly trees (3.2.4) of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension.

- Assumption 3.2.11: for any two $J_{\pm 1}$ satisfying assumption 3.2.7, there is a path $\underline{J} = \{J_t\}_{t\in[-1,1]}$ such that the moduli spaces (3.2.10) of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension. Note that the definition of these moduli spaces requires that we have constructed Morse–Smale pairs on $L \times [-1, 1]$ and $M \times [-1, 1]$, as detailed in Section 3.2.2.

- Assumption 3.3.4: the moduli spaces of open-closed pearly trees (3.3.9) and (3.3.10) of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension. Here the moduli spaces are defined using the same almost complex structure of Assumption 3.2.7 appearing in the definition of the A_∞-operations.

- Assumption 3.3.11: the moduli spaces of open-closed pearly trees on the cylinder (3.3.19) and (3.3.20) of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension. Here the moduli spaces are defined using the same path of almost complex structures of Assumption 3.2.11.

- Assumption 3.4.3: the moduli spaces (3.4.1) of J-holomorphic pearly trees with no inputs in L of virtual dimension 0 are transversely cut out 0-dimensional manifolds.

- Assumption 3.4.5: the moduli spaces (3.4.3) of $\underline{J}$-holomorphic pearly trees with no inputs in $L \times [-1, 1]$ of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension.

- Assumption 3.4.6: the moduli spaces (3.4.4) of $\underline{J}$-holomorphic pearly trees in $M \times [-1, 1]$ with only sphere components of virtual dimension at most 1 are transversely cut out orbifolds of the expected dimension.

We remark that there exist $J \in \mathcal{J}(M)$ and paths $\underline{J} = \{J_t \in \mathcal{J}(M)\}_{t\in[-1,1]}$ simultaneously satisfying all of these assumptions if one of the following conditions on $L \subset M$ is satisfied.

(i) $L \subset M$ satisfies the assumptions of [49] for some fixed J. For the time-dependent moduli spaces, there should exist a path $\underline{J}$ in $\mathcal{J}(M)$ which satisfies the assumptions of [49] at all times.

(ii) L is a monotone Lagrangian in a monotone symplectic manifold and $J \in \mathcal{J}(M)$ and $\underline{J} = \{J_t \in \mathcal{J}(M) \colon t \in [-1, 1]\}$ are generic.

In the case of (i), the assumptions of [49] imply that all of the moduli spaces $\mathcal{M}_{k+1,\ell}(\beta; J)$ is pseudo-holomorphic disks in M with boundary on L are already smooth orbifolds with corners of the expected dimension. Thus it is clear that one can choose Morse functions satisfying Assumption 3.2.1 for which Assumptions 3.2.7 and 3.4.3 are satisfied. The spaces of disks 3.3.5 and 3.3.6 can be identified with subdomains of moduli spaces already covered by the assumptions of [49], giving us Assumption 3.3.4 immediately. The assumptions involving pearly trees in $M \times [-1, 1]$ can also be checked similarly.

In the monotone case (ii), Assumption 3.2.7 can be checked using the techniques of [53], with no modifications. Roughly, this works by decomposing any J-holomorphic disk on L as a sum of simple disks in homology, and then using the constraint on virtual dimensions to argue that all pearly trees contributing to the A_∞-operations are equipped with simple disks at all vertices. The verification of all other assumptions on pearly trees of disks can be carried out in the same way. In particular, the extra decorations on the domains of the open-closed moduli spaces introduce no additional complications. Assumption 3.4.6 in the monotone setting follows from the discussion of the quantum product in [53].

Chapter 4

Mirror symmetry for lines in the mirror quinitic

4.1 Mirrors to points in the mirror quintic

Since all quintic threefolds are symplectomorphic, we will usually work in a Dwork quintic threefold X^5_τ, where τ is a small real number. In particular, complex conjugation on $\mathbb{CP}^4$ restricts to an anti-symplectic involution on X^5_τ, which will allow us to apply the results of [36] to prove that certain Lagrangian submanifolds are unobstructed.

Studying tropical Lagrangian submanifolds in the quintic using homological mirror symmetry requires some knowledge of how the mirror functor interacts with the putative SYZ fibration. This section draws a partial connection between these two pictures of mirror symmetry, as we show that certain smooth Lagrangian tori in the quintic (which are traditionally thought of as nonsingular SYZ fibers), correspond to skyscraper sheaves under the mirror functor of [15].

For any Dwork quintic X^5_τ, recall from [36, Example 1.8] and [35, §3.1] that there is a natural family of Lagrangian tori in X^5_τ, which are constructed as follows. The central fiber X^5_0 in the Dwork pencil is the union of coordinate hyperplanes in $\mathbb{CP}^4$. Each of these hyperplanes can be identified with $\mathbb{CP}^3$, so the smooth locus of X^5_0 consists of the union of 5 disjoint copies of $(\mathbb{C}^*)^3$. There is a moment map $\mathrm{Log} \coloneqq \mathrm{Log}_e \colon (\mathbb{C}^*)^3 \to \mathbb{R}^3$ which is just the SYZ fibration discussed in the introduction. We obtain Lagrangian tori in X^5_τ, for sufficiently small τ, by deforming these moment fibers under symplectic parallel transport.

Definition 4.1.1. Consider the meromorphic function

$$s := \frac{\prod_{j=1}^{5} x_j}{\sum_{j=1}^{5} x_j^5}$$

on $\mathbb{CP}^4$. Let f denote the real part of this function, and consider its gradient ∇f with respect to the metric on $\mathbb{CP}^4$ induced by the Fubini–Study form and the integrable complex structure. If $L_{0,q} \in (\mathbb{C}^*)^3$ denotes a smooth moment fiber in one of the components of X_0 over a point $q \in \mathbb{R}^3$, let $L_{\tau,q}$ denote its image in X_τ^5 under the gradient flow of f.

The Lagrangian torus $L_{\tau,q}$ will live near one of the coordinate hyperplanes of $\mathbb{CP}^4$, meaning that there are actually five Lagrangian tori obtained this way. We do not include this in the notation since the choice of coordinate hyperplane will usually be irrelevant or otherwise clear from context.

Remark 4.1.2. We emphasize that our main results, and in particular Theorem 1.2.6, do not assume the existence of an SYZ fibration on the quintic. Instead, we only need to consider certain smooth Lagrangian tori, and in particular we do not need to consider the singular fibers or discriminant locus of an SYZ fibration.

Remark 4.1.3. As shown in [35, §3.2], the normalized gradient of f is given by

$$\frac{\nabla f}{|\nabla f|^2} = \mathrm{Re}\left(\frac{x_1^5 + x_2^5 + x_3^5 + 1}{x_1 x_2 x_3} \frac{\partial}{\partial x_5}\right)$$

when restricted to the chart $\{x_4 = 1\,,\ x_5 = 0\}$. In particular, we can think of the Lagrangian tori $L_{\tau,q}$, for small τ, as the graph of a holomorphic function restricted to a moment fiber on $(\mathbb{C}^*)^3$.

Since these Lagrangian tori in X_τ^5 come in a family (as q varies), they will not all be exact in the complement of a divisor, and thus they cannot all be objects of the relative Fukaya category as considered by [82]. For this reason, it is more convenient for us to construct mirrors to points by identifying an appropriate family of A-branes supported on a single Lagrangian torus, which we will arrange to be exact.

We define the torus $L_{\tau,0}$ is to be the result of parallel transporting the SYZ fiber $L_{0,0}$ over the origin in a copy of $(\mathbb{C}^*)^3$. The fibration is given in (1.2.1). In one of the hyperplanes $\{x_5 = 0\} \subset X^5_0$, the torus $L_{0,0}$ is given by

$$\{|x_1| = |x_2| = |x_3| = |x_4| = 1 \colon x_1x_2x_3x_4 = 1\} \subset \{x_1x_2x_3x_4 = 1\} \cong (\mathbb{C}^*)^3 \subset \mathbb{CP}^3 \qquad (4.1.1)$$

meaning that it is the unique Lagrangian torus in X^5_0 preserved by all permutations of the first four coordinates on $\mathbb{CP}^4$. It follows from Remark 4.1.3, that $L_{\tau,0}$ possesses the same symmetries.

Definition 4.1.4. Let $D_\tau \subset X^5_\tau$ denote the divisor given by the intersection

$$D_\tau \coloneqq X^5_\tau \cap X^5_0 \qquad (4.1.2)$$

of X^5_τ with the coordinate hyperplanes in $\mathbb{CP}^4$.

By construction, $L_{\tau,0}$ lies in the complement of D_τ, so we will be able to think of it as an object of the relative Fukaya category [15]. Since $X^5_\tau \setminus D_\tau$ is an affine variety, it carries a Liouville structure induced by pulling back the standard symplectic form on $(\mathbb{C}^*)^4$. We fix a primitive for this symplectic form which makes $L_{\tau,0} \subset X^5_\tau \setminus D_\tau$ an exact Lagrangian submanifold. The split-generators used in the proof of homological mirror symmetry of [15] are represented by immersed Lagrangian spheres, and thus they are exact with respect to any choice of primitive.

Proposition 4.1.5. *For $\tau \in \mathbb{R}$, the Lagrangian branes obtained by equipping $L_{\tau,0}$ with rank one complex local systems correspond to skyscraper sheaves in the derived category $D^b\,\mathrm{Coh}(\mathcal{X}^{5,\vee})$ of the mirror quintic with respect to the mirror functor of [15].*

A previous version of this article claimed an analogue of this result using the mirror functor of [17], but the proof of homological mirror symmetry as written in op. cit. does not apply to the quintic threefold and its mirror.

Proof. First note that complex conjugation on X^5_τ acts on $L_{\tau,0}$ and acts on $H^1(L_{\tau,0};\mathbb{C})$ as $-\mathrm{id}$

(cf. [36, Example 1.8]). Thus, by [36], equipping $L_{\tau,0}$ with any rank one $\mathbb{C}$-local system yields an unobstructed Lagrangian brane (with bounding cochain 0).

The split-generators for the Fukaya category described in [15] are naturally thought of as Lagrangian branes in the Fermat quintic threefold X^5_∞. Letting M^5 denote the pair of pants

$$M^5 := \{z_1 + z_2 + z_3 + z_4 + z_5 = 0\} \subset \mathbb{CP}^4 \setminus \bigcup_{j=1}^{5} \{z_j = 0\}$$

and noting that $X^5_\infty \setminus D_\infty$ is the affine Fermat quintic

$$X^5_\infty \setminus D_\infty = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{CP}^4 \setminus \bigcup_{j=1}^{5} \{x_j = 0\}$$

we recall that there is a covering map

$$\begin{aligned} X^5_\infty \setminus D_\infty &\to M^5 \\ x_j &\mapsto x_j^5 \end{aligned}$$

Sheridan's split-generators [15] for $\mathcal{F}(X^5_\infty)$ are obtained by taking lifts of an immersed Lagrangian sphere $S^3 \looparrowright M^5$ constructed in [32]. The projection of M^5 to the moment fiber of $(\mathbb{C}^*)^4$ defines a *coamoeba* whose boundary is a polyhedral 3-sphere with self-intersections. Sheridan's immersed sphere is, intuitively, characterized as a lift of this sphere to the pair of pants. The result of the proposition will follow from comparing the Lagrangian torus $L_{\tau,0}$ to a boundary facet of the coamoeba.

Since all smooth quintic threefolds are symplectomorphic, we can think of the Lagrangian tori $L_{\tau,0} \subset X^5_\tau$ as Lagrangian submanifolds of X^5_∞. It will be convenient to specify a symplectomorphism between the very affine quintic threefolds $X^5_\tau \setminus D_\tau$ and $X^5_\infty \setminus D_\infty$. Representing the big torus in $(\mathbb{C}^*)^4 \subset \mathbb{CP}^4$ by $\{x_1x_2x_3x_4x_5 = \xi\}$, for some generic constant $\xi \in \mathbb{C}^*$ of small norm, we can represent the very affine Dwork quintic as the vanishing locus of a polynomial as follows

$$X_\tau^5 \setminus D_\tau = X_\tau^5 \cap \left\{ \prod_{j=1}^{5} x_j = \xi \right\} = \left\{ \xi - \frac{\tau^{1/5}}{5} \sum_{j=1}^{5} x_j^5 = 0 \right\} \subset \mathbb{CP}^4 \setminus \bigcup_{j=1}^{5} \{x_j = 0\} . \tag{4.1.3}$$

Scaling the constant term of (4.1.3) by small positive real constants determines a symplectomorphism between $X_\tau^5 \setminus D_\tau$ and $X_\infty^5 \setminus D_\infty$ which carries $L_{\tau,0}$ to a Lagrangian torus in the affine Fermat quintic which we can take to be given by the set of points $\{|x_1| = |x_2| = |x_3| = \epsilon\}$, for a constant $\epsilon \in \mathbb{C}^*$ of small norm, as can be seen from Remark 4.1.3.

The image of $L_{\tau,0}$ under this covering map is a Lagrangian torus in the pair of pants whose argument projection is a boundary facet of the coamoeba, using the description in [32, Proposition 2.1], and this corresponds to the structure sheaf of a smooth point on the mirror variety $\{z_1 z_2 z_3 z_4 z_5 = 0\} \subset \mathbb{C}^5$. Consequently $L_{\tau,0}$, thought of as an object of $\mathcal{F}(X_\tau^5)$, is mirror to the structure sheaf of a point by the description of the mirror equivalence in [15] as a deformation of a mirror functor discussed in [32, Theorem 7.4]. □

An easy computation in a Weinstein neighborhood shows that if one equips $L_{\tau,0}$ with two different rank 1 local systems, then the Floer cohomology of the resulting pair of Lagrangian branes (in the quintic) vanishes. This Floer cohomology group corresponds to the Ext group between two skyscraper sheaves under mirror symmetry, implying that no two distinct local systems on $L_{\tau,0}$ yield mirrors to the same point of the mirror quintic.

4.2 Tropical Lagrangians in the quintic threefold

In this section, we will construct the immersed Lagrangian of Theorem 1.2.6. A brief outline of its construction is as follows.

A Weinstein neighborhood of the Lagrangian torus of Proposition 4.1.5 yields an open neighborhood in the Dwork quintic X_τ^5 in which we can include copies of tropical Lagrangians. Using this chart, we can identify a copy of (a cover of) the tropical Lagrangian of [29, Theorem 1.1] in the quintic. By attaching certain Lagrangian cones to this noncompact Lagrangian in X^5_τ, we obtain a Lagrangian denoted $\widetilde{L}^5_{\text{sing}}$ which has isolated conical singularities, all of which are modeled on the Harvey–Lawson cone. The immersed Lagrangian $\widetilde{L}^5_{\text{im}} \to X^5_\tau$ lies in a small Weinstein neighborhood of $\widetilde{L}^5_{\text{sing}}$, and is obtained by 'doubling' the singular Lagrangian as in [29, §5.3]. This immersed Lagrangian can also be thought of as the result of attaching immersed Lagrangian handles to (a cover of) the Lagrangian immersion L_{im} constructed in [29, Theorem 1.2]. This section draws heavily from [29], but we have attempted to keep our exposition in the current paper mostly self-contained.

Remark 4.2.1. The notation for Lagrangian immersions in this paper differs slightly from the notation of [29, Theorem 1.2]. In loc. cit., we denoted by $\widetilde{L}_{\text{im}}$ the domain of the immersed Lagrangian lift of a 4-valent tropical vertex. The image of this immersion was denoted L_{im}. In this paper, we will use the symbol $\widetilde{L}_{\text{im}}$ to refer to a particular *cover* of L_{im} instead. We also work with a 4-valent tropical curve whose legs point in the opposite direction, but this makes no essential difference to the results proved above.

4.2.1 Singular tropical Lagrangians near an SYZ fiber

We begin by discussing the cover of L_{sing} that will serve as a local model for the lifts of the van Geemen lines.

Remark 4.2.2. In $(\mathbb{C}^*)^3$, the periodized conormals to the legs of V can be written as

$$N^*V_i/N^*_{\mathbb{Z}}V_i = \left\{|u_j| = |u_k| \text{ and } u_j \in [1, \infty)\right\} \tag{4.2.1}$$

for $\{i, j, k\} = \{1, 2, 3\}$, and

$$N^*V_i/N^*_{\mathbb{Z}}V_i = \{|u_1| = |u_2| = |u_3| \text{ and } u_1u_2u_3 \in (0, 1]\} \ . \tag{4.2.2}$$

Instead of considering L_{sing} in our discussion of the quintic threefold, we will need to consider a certain cover of it, denoted $\widetilde{L}_{\text{sing}}$. More precisely, this is a Lagrangian lift of the tropical curve $\widetilde{V}$ obtained from V by giving all of its edges weight 5. One can construct $\widetilde{L}_{\text{sing}}$ from L_{sing} using the procedure described in [28, §5.2]. Recall that L_{sing} was constructed in terms of a subset of T^3 called the *coamoeba*, which is identified as the image of L_{sing} under the bundle projection $T^*T^3 \to T^3$. In this language, the coamoeba of $\widetilde{L}_{\text{sing}}$ is obtained from the coamoeba of L_{sing} by taking its preimage under the 125-fold covering map $T^3 \to T^3$ corresponding to the subgroup $(5\mathbb{Z})^3 \subset \mathbb{Z}^3$. It is easy to see that $\widetilde{L}_{\text{sing}}$ has 125 cone points. Let $\widetilde{L}'$ denote the (cusped hyperbolic) 3-manifold obtained by taking the complement of singular points on $\widetilde{L}_{\text{sing}}$.

Remark 4.2.3. Using Lemma 2.3.18, we can describe the cover $\widetilde{L}' \to L'$ explicitly. By choosing a basepoint in the center of Figure 2.7, we can obtain an element $\widehat{m}_i \in \pi_1(L')$ from each meridian of the minimally-twisted five-component chain link. Consider the normal closure in $\pi_1(L')$ of the subgroup generated by the elements $\{\widehat{m}_0, \widehat{m}_1^5, \widehat{m}_2^5, \widehat{m}_3^5, \widehat{m}_4^5\}$. A straightforward calculation shows that the quotient of $\pi_1(L')$ by this subgroup is isomorphic to $(\mathbb{Z}/5)^3$. It is also clear from the construction of $\widetilde{L}'$ that the image of $\pi_1(\widetilde{L}') \to \pi_1(\widetilde{L})$ contains this subgroup. In [29], we wrote L' as the union of two ideal cubes in the hyperbolic upper half space $\mathbb{H}^3$, and it is interesting to note that $\widetilde{L}'$ can be written as the union of 250 such cubes. These are glued together in a manner determined by the combinatorics of the lift of the *coamoeba* used to construct L_{sing} [29, §4.2], but we will not need this fact.

If we view $\widetilde{L}_{\text{sing}}$ as a subset of $(\mathbb{C}^*)^3$, then $\widetilde{L}_{\text{sing}} \setminus \pi_{\text{SYZ}}^{-1}(B)$ can be written as the union of the following subsets

$$\left\{ |u_j| = |u_k| \text{ and } u_\ell = re^{i\theta} \middle| r \in [1,\infty) \text{ and } \theta = \frac{2m\pi}{5} \text{ for } m \in \mathbb{Z} \right\} \subset (\mathbb{C}^*)^3 \tag{4.2.3}$$

whenever $\{j, k, \ell\} = \{1, 2, 3\}$, and

$$\left\{ |u_1| = |u_2| = |u_3| \text{ and } u_1u_2u_3 = re^{i\theta} \middle| r \in (0,1] \text{ and } \theta = \frac{2m\pi}{5} \text{ for } m \in \mathbb{Z} \right\} \subset (\mathbb{C}^*)^3 . \tag{4.2.4}$$

These are of course just lifts of (4.2.1) and (4.2.2), respectively, under the symplectic covering map $(\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3$.

The Darboux ball B_0 centered at the cone point of L_{sing} has 125 lifts, denoted

$$\widetilde{B}_{\ell,0} \subset T^*T^3 \tag{4.2.5}$$

for all $\ell = 1, \dots, 125$. These are all Darboux charts centered at the cone points of $\widetilde{L}_{\text{sing}}$.

4.2.2 A singular tropical Lagrangian in the quintic

Consider the Lagrangian torus $L_{\tau,0} \subset X^5_\tau$ of Proposition 4.1.5, and fix a Weinstein neighborhood $\text{Wein}(L_{\tau,0}) \subset X^5_\tau$, together with a symplectomorphism $N^*_\epsilon T^3 \to \text{Wein}(L_{\tau,0})$ whose domain is a tubular neighborhood of the zero section in T^*T^3 of radius $\epsilon > 0$. By Theorem 1.2.1 (cf. Remark 2.3.12), we can isotope L_{sing}, and hence $\widetilde{L}_{\text{sing}}$, so that it coincides with the periodized conormal Lagrangians outside of $N^*_\epsilon T^3 \subset T^*T^3$. Abusing notation, we will identify $\widetilde{L}_{\text{sing}}$ with its image in $\text{Wein}(L_{\tau,0})$. To compactify this noncompact Lagrangian in X^5_τ, we will construct Lagrangian cones away from $\text{Wein}(L_{\tau,0})$, and show that they patch smoothly with $\widetilde{L}_{\text{sing}}$. This gluing is controlled by the combinatorics of the tropical curve $\widetilde{V}$, as in the construction of tropical Lagrangians of [20].

Let $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. Observe that there are five points in X^5_τ with homogeneous coordinates $x_i = x_j = x_k = 0$ on $\mathbb{CP}^4$. Let $\widetilde{v}_{\ell,m}$, where $m = 1, \dots, 5$, denote these points. We define charts near each $\widetilde{v}_{\ell,m}$ as follows. If we set $x_\ell = 1$, then there is a ball in $B_0(\epsilon) \subset \mathbb{C}^3$ centered at 0, where $\epsilon > 0$ denotes the radius of the ball, such that if $(x_i, x_j, x_k) \in B_0(\epsilon)$, then any x_5 for which $[x_1 : x_2 : x_3 : x_4 : x_5]$ lies in X^5_τ is nonzero. The value of x_5 then determines a section of the (trivial) 5-fold cover of $B_0(\epsilon)$. Let

$$\widetilde{B}_{\ell,m} \subset X^5_\tau \tag{4.2.6}$$

denote the sheet of this cover containing the point $\widetilde{v}_{\ell,m}$. The ball $\widetilde{B}_{\ell,m}$ is a Darboux ball centered at $\widetilde{v}_{\ell,m}$, since the symplectic form on X^5_τ is pulled back from the Fubini–Study form on $\mathbb{CP}^4$.

In each of the balls (4.2.6), there is a Lagrangian cone defined by

$$L_{\ell,m} \coloneqq \left\{[x_1 : x_2 : x_3 : x_4 : x_5] \mid |x_i| = |x_j| = |x_k| \text{ and } x_i x_j x_k \in \mathbb{R}_{\geq 0}\right\} \cap \widetilde{B}_{\ell,m}\,. \tag{4.2.7}$$

These are just the images of the Harvey–Lawson cone in $\widetilde{B}_{\ell,m}$. To complete the construction of $\widetilde{L}^5_{\text{sing}}$, we will show that these cones can be patched smoothly with the copy of $\widetilde{L}_{\text{im}}$ in $\text{Wein}(T^3_{t,0})$.

Theorem 4.2.4. *There is a singular Lagrangian $\widetilde{L}^5_{\text{sing}}$ in X^5_τ, where τ is any sufficiently small real number, which coincides with a copy of $\widetilde{L}_{\text{sing}}$ contained in* $\text{Wein}(L_{\tau,0}) \subset X^5_\tau$. *The singular locus of $\widetilde{L}^5_{\text{sing}}$ consists of* 145 *singular points modeled on the Harvey–Lawson cone, and the complement of these points in $\widetilde{L}^5_{\text{sing}}$ is diffeomorphic to the hyperbolic* 3*-manifold $\widetilde{L}'$ as discussed in Remark 4.2.3.*

There are 125 cone points that come from lifting the cone point of L_{sing}, and another 20 cone points attached to the ends of $\widetilde{L}_{\text{sing}}$, specifically 5 cone points corresponding to each leg of the tropical curve V.

Proof. Consider the tropical curve $\widetilde{V} \subset Q \cong \mathbb{R}^3$. Choose a point $q \in \widetilde{V}_\ell$ which is not the vertex. If we view $(\mathbb{C}^*)^3$ as the big torus in $\mathbb{CP}^3 \cong \{x_5 = 0\}$, then $\mathbb{R}^3$ can be identified with the interior of the moment polytope for (this copy of) $\mathbb{CP}^3$. Consider the moment fiber $L_{0,q}$ over q inside $\mathbb{CP}^3$. As before, we can parallel transport this to a Lagrangian torus $L_{\tau,q} \subset X^5_\tau$. Choose a Weinstein neighborhood $\text{Wein}(L_{\tau,q})$ contained in X^5_τ, whose domain is a small tubular neighborhood of the 0-section in T^*T^3.

If $W \subset Q$ is a 1-dimensional integral affine subspace containing $\widetilde{V}_\ell$, then we can consider the intersection periodized conormal $N^*W/N^*_{\mathbb{Z}}W$ with the tubular neighborhood specified above. This will give us the portion of the periodized conormal lying over a finite segment of $\widetilde{V}_\ell$ containing q. Taking a sequence of such points on $\widetilde{V}_\ell$, we can assume that the union of Weinstein neighborhoods $\text{Wein}(L_{\tau,q})$ is connected and intersects each of the balls $\widetilde{B}_{\ell,m}$, as well as $\text{Wein}(L_{\tau,q})$.

Examining (4.2.7) shows that the intersection of the union of cones $\bigcup_m \widetilde{L}_{\ell,m}$ with the Lagrangian segments in $\text{Wein}(L_{\tau,q})$ constructed above pulls back to a segment of one of the subsets (4.2.3) or (4.2.4) inside $(\mathbb{C}^*)^3 \subset \{x_5 = 0\}$ under symplectic parallel transport. In particular the union of the cones with the Lagrangian segment in $\mathrm{Wein}(L_{\tau,q})$, where $q \in Q$ is sufficiently far away from 0, is smooth except at the cone points. For the same reason, one sees that the union of these pieces with the copy of $\widetilde{L}_{\text{sing}} \subset \mathrm{Wein}(L_{\tau,0})$ is smooth (away from the cone points) as well. By taking the union of all local Lagrangian pieces considered so far, we obtain $\widetilde{L}^5_{\text{sing}}$. □

Remark 4.2.5. There are several obvious similarities between the proof of Theorem 4.2.4 and Mak–Ruddat's construction of tropical Lagrangians in mirror quintic threefolds of [20]. The Weinstein neighborhoods of moment fibers in our setting can be thought of as analogues of the charts used by Mak–Ruddat to construct the parts of their tropical Lagrangians lying away from the boundary of the moment polytope. Because we have attached Lagrangian cones to the noncompact ends of $\widetilde{L}_{\text{sing}}$, we do not require an analogue of their construction of a Lagrangian solid torus. Consequently, we have not undertaken a detailed study of the discriminant locus or singular fibers as in op. cit. In the language of the Gross–Siebert program, the relevant dual intersection complex would not be simple, and so constructing tropical Lagrangians in these terms would be different than in [20].

Remark 4.2.6. Since five subsets in (4.2.3), for some fixed indices j, k, and ℓ are carried to each other under multiplication of the ℓth coordinate of $(\mathbb{C}^*)^3$ by fifth roots of unity, one might expect that they should all approach the same cone point. Since the points $\widetilde{v}_{\ell,m}$ are also carried to each other by multiplication by fifth roots of unity in the ℓth coordinate, however, each end of $\widetilde{L}_{\text{sing}}$ approaches a different cone point when compactified in X^5_τ.

The proof of Theorem 1.2.6 involves understanding disks of small symplectic area with boundary on $\widetilde{L}^5_{\text{im}}$. It is illuminating to recast these arguments as calculations of *local* Lagrangian Floer cohomology in a Weinstein neighborhood of $\widetilde{L}^5_{\text{sing}}$ inside X^5_τ. We will construct a Weinstein neighorhood $\widetilde{W}_5 := \mathrm{Wein}(\widetilde{L}^5_{\text{sing}})$ which is invariant under the action of $(\mathbb{Z}/5)^3$ on X^5_τ. It was shown by Joyce [37] that any dilation-invariant Lagrangian cone C in $\mathbb{C}^n$ with link Σ has a dilation-invariant Weinstein neighborhood, meaning that there is an open neighborhood $U_C \subset T^*(\Sigma \times \mathbb{R}_{>0})$ of the 0-

section and a symplectic embedding $\Phi_C \colon U_C \to \mathbb{C}^n$. We also have that Φ_C restricts to the inclusion map along the 0-section, and that it intertwines the $\mathbb{R}_{>0}$-actions on $T^*(\Sigma \times \mathbb{R}_{>0})$ and $\mathbb{C}^n$.

Let $\widetilde{v}_{\ell,m}$ denote one of the singular points of $\widetilde{L}^5_{\text{sing}}$, and let $\Psi_{\ell,m} \colon B_0(\epsilon) \to X^5_\tau$ denote one of the Darboux balls (4.2.5) or (4.2.6), where we take the domain to be a ball of radius $\epsilon > 0$ centered at $0 \in \mathbb{C}^3$. Denote by $\psi_{\ell,m} \colon \mathbb{C}^3 \to T_{x_s} X^5_\tau$ the linear isomorphism induced from the differential of $\Psi_{\ell,m}$ at the origin.

We can write $\Psi^{-1}_{\ell,m}(\widetilde{L}^5_{\text{sing}})$ as the image of the 0-section in $T^*(\Lambda_{HL} \times \mathbb{R}_{>0})$, where we recall that Λ_{HL} denotes the link of the Harvey–Lawson cone. This induces a map $\phi_{\ell,m} \colon \Lambda_{HL} \times \mathbb{R}_{>0} \to B_0(\epsilon)$ which parametrizes the 0-section, implying that $\Psi_{\ell,m} \colon \phi_s$ has image contained in $\widetilde{L}' \subset X^5_\tau$. Define the compact subset

$$K := \widetilde{L}^5_{\text{sing}} \setminus \bigcup_{\ell,m} \widetilde{B}_{\ell,m}$$

of X^5_τ.

Lemma 4.2.7. *There is an open neighborhood $U_{\widetilde{L}'}$ of the 0-section in $T^*\widetilde{L}'$ and a symplectic embedding $\Phi_{\widetilde{L}'} \colon U_{\widetilde{L}'} \to X^5_\tau$ which restricts to the inclusion over the 0-section. This embedding also satisfies*

$$\Phi_{\widetilde{L}'} \circ (d\Psi_{\ell,m} \circ \phi_{\ell,m}) = \Psi_{\ell,m} \circ \Phi_{C_{HL}} \tag{4.2.8}$$

meaning that the Darboux balls centered at the cone points of $\widetilde{L}^5_{\text{sing}}$ patch smoothly with the image of $U_{\widetilde{L}'}$ to give a symplectic subdomain $\widetilde{W}^5$ of X^5_τ. Moreover, we can choose all symplectic embeddings as above so that $\widetilde{W}^5$ is invariant under the action of $(\mathbb{Z}/5)^3$.

Proof. The construction of $\Phi_{\widetilde{L}'}$ and $U_{\widetilde{L}'}$ satisfying (4.2.8) is straightforward using the techniques of [37] and [29, Section 5.1]. Thus we can take $\widetilde{W}^5$ to be the union of $\Phi_{\widetilde{L}'}(U_{\widetilde{L}'})$ with the balls $\widetilde{B}_{\ell,m}$. To prove that $\widetilde{W}^5$ is $(\mathbb{Z}/5)^3$-invariant, first note that the cone points $\widetilde{v}_{\ell,m}$, for $\ell = 1, 2, 3, 4$, are permuted by the group action. Because there are only finitely many such points, we can choose the Darboux balls centered at these points so that the disjoint union of all of the balls is preserved under the group action.

The action of $(\mathbb{Z}/5)^3$ on X^5_τ is the restriction of a corresponding $(\mathbb{Z}/5)^3$-action on $\mathbb{CP}^4$. This group action also restricts to an action of $(\mathbb{Z}/5)^3$ on $(\mathbb{C}^*)^3$, and this is in fact the action by deck transformations of the covering map $(\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3$ arising in the construction of $\widetilde{V}$. In particular $\widetilde{L}_{\text{sing}}$ is preserved under this action and, by Remark 4.1.3, its symplectic parallel transport is also $(\mathbb{Z}/5)^3$-invariant. Hence wee can assume without loss of generality that the image of $\Phi_{\widetilde{L}'}$ is preserved by the $(\mathbb{Z}/5)^3$-action as well. □

The quotient of $\widetilde{W}^5$ by the action of $(\mathbb{Z}/5)^3$ admits a similar description.

Lemma 4.2.8. *The quotient space $W^5 := \widetilde{W}^5/(\mathbb{Z}/5)^3$ is a smooth symplectic manifold. It contains a singular Lagrangian L^5_{sing} given by the image of $\widetilde{L}^5_{\text{sing}}$ under the quotient map. The Lagrangian L^5_{sing} has five cone points, denoted $v_0, \dots, v_4$, all of which are modeled on the cone point of a Harvey–Lawson cone.*

Proof. Consider the disjoint union

$$\coprod_{m=1}^{5} \widetilde{B}_{\ell,m}$$

for a fixed $\ell = 1, 2, 3, 4$. We can choose a generating set for $(\mathbb{Z}/5)^3$ with the property that one of the generators cyclically permutes the balls $\widetilde{B}_{\ell,m}$, and for which the action of the other two generators on $\widetilde{B}_{\ell,m} \cong \mathbb{C}^2$ is given by the action of

$$(\mathbb{Z}/5)^2 \cong \{(a_1, a_2, a_3) \in (\mathbb{Z}/5)^3 \mid a_1 + a_2 + a_3 = 0\}$$

on $\mathbb{C}^3$. This group action is associated to a branched cover of $\mathbb{C}^3$ by itself. The action of $(\mathbb{Z}/5)^3$ on $\Phi_{\widetilde{L}'}(U_{\widetilde{L}'})$ is free, and thus W^5 is a smooth manifold. Abstractly, it is obtained by gluing five symplectic balls to a copy of T^*L' such that the Legendrians $\Lambda_{HL} \subset S^5$ in the boundaries of these balls are identified with the cusps of L', thought of as the 0-section. □

We denote the Darboux ball centered at the cone point v_ℓ of L^5_{sing} by

$$B_\ell \subset W^5 . \tag{4.2.9}$$

This is consistent with the notation (2.3.1) for the Darboux chart near the cone point of L_{sing} established above.

4.2.3 An immersed tropical Lagrangian in the quintic

The Lagrangian immersion $\widetilde{L}^5_{\mathrm{im}}$ is closely related to the immersion L_{im} studied in [29]. More precisely, one can think of $\widetilde{L}^5_{\mathrm{im}}$ as being obtained from a cover $\widetilde{L}_{\mathrm{im}}$ of L_{im} by attaching immersed 'doubles' of the Harvey–Lawson cone, as constructed in [29, §5.3]. A consequence of this is that we will be able to view $\widetilde{L}^5_{\mathrm{im}}$ as an object of the relative Fukaya category of the quintic, which will later allow us to quote the results of [15].

Recall that the construction of the double takes place in a small neighborhood of the origin in $\mathbb{C}^3$. We can think of this as a neighborhood of $0 \in \mathbb{C}^3 \setminus \{y_1 y_2 y_3 = 1\}$, which we can identify with the variety

$$Y \coloneqq \left\{(y_1, y_2, y_3, u) \in \mathbb{C}^3 \times \mathbb{C}^* \mid y_1 y_2 y_3 = v\right\} . \tag{4.2.10}$$

This is, not coincidentally, one of the local Gross–Siebert models studied in [33]. Consider the projection

$$\begin{aligned} w &: Y \to \mathbb{C} \\ w(y_1, y_2, y_3, u) &= y_1 y_2 y_3 . \end{aligned} \tag{4.2.11}$$

The fiber $D \coloneqq w^{-1}(0)$ is the union of coordinate hyperplanes in $\mathbb{C}^3$. As shown in [33, Lemma

2.2], the coordinates (y_1, y_2, w) on $Y \setminus D$ induce a commutative diagram

$$\begin{array}{ccc} Y \setminus D & \xrightarrow{\cong} & \mathbb{C}^2 \times (\mathbb{C}^* \setminus \{1\}) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbb{C}^2 \times \mathbb{C} \end{array} \tag{4.2.12}$$

where the top arrow is a symplectomorphism.

Let L_{arc} denote the immersed Lagrangian arc in $\mathbb{C} \setminus \{0, 1\}$ depicted in Figure 2.10. Notice $L_{\text{arc}} \subset \mathbb{C}$ bounds a holomorphic teardrop, i.e. a disk with one corner, through the origin. By choosing L_{arc} appropriately, we can assume that this teardrop is arbitrarily small. If we restrict to a neighborhood of $0 \in \mathbb{C}^3$, then the portion of the Harvey–Lawson cone contained in this neighborhood projects, under w, to a portion of the nonnegative real axis in $\mathbb{C}$. Using (4.2.12), we can define an immersed Lagrangian in $\mathbb{C}^3$.

Definition 4.2.9. Let $T^2_{Cl} \subset \mathbb{C}^2$ denote the standard Clifford torus, and consider the product $T^2 \times L_{\text{arc}} \subset \mathbb{C}^2 \times (\mathbb{C}^* \setminus \{1\})$. Let $L_Y \subset Y \setminus D$ denote the image of this Lagrangian under the symplectomorphism at the top of (4.2.12).

Near the cone points $\widetilde{v}_{\ell,m}$, the Lagrangian immersion $\widetilde{L}^5_{\text{im}}$ will be given by the images in $\widetilde{B}_{\ell,m}$ of the Lagrangian $L_Y \cap B_0(\epsilon) \subset B_0(\epsilon)$ contained in a small ball centered at the origin.

Remark 4.2.10. Although the construction of L_Y depends on an ordering of coordinates on $\mathbb{C}^3$, the fibers of L_Y over L_{arc}, as subsets of $\mathbb{C}^3$, are independent of this choice. Therefore we can construct $\widetilde{L}^5_{\text{im}}$ using any ordering of coordinates on $\mathbb{C}^3$. It will only be important to specify a choice of coordinates later, when we study the Floer theory of this Lagrangian.

The part of $\widetilde{L}^5_{\text{im}}$ lying outside the balls $\widetilde{B}_{\ell,m}$ will consist of two copies of $\widetilde{L}'$.

Lemma 4.2.11. *There is a Morse function $h\colon L' \to \mathbb{R}$ with 2 index 0 critical points, 6 index 1 critical points, and 4 index 2 critical points. Furthermore, there exist collar neighborhoods $T^2 \times \mathbb{R} \to L'$ of each cusp of L' in which the gradient vector field of h points outward in the*

$\mathbb{R}$-direction. Here the gradient vector field is taken with respect to the metric on L' determined by the symplectic form on T^*L' and a compatible almost complex structure.

Proof. The Morse function h is associated to the decomposition of L' into two ideal cubes shown in Figure 2.5. The numbers of index-0, -1, and -2 critical points of h correspond to the numbers of 3-cells, 2-cells, and 1-cells in the ideal cubulation. □

Definition 4.2.12. Let $\widetilde{h}\colon \widetilde{L}' \to \mathbb{R}$ be the Morse function obtained by precomposing h with the covering map $\widetilde{L}' \to L'$.

Since $\widetilde{h}$ is obtained from h by lifting it to a covering space, it admits a similar description near the cusps of $\widetilde{L}'$.

Corollary 4.2.13. *There exist collar neighborhoods $T^2 \times \mathbb{R} \to \widetilde{L}'$ of each cusp of $\widetilde{L}'$ in which the gradient vector field of $\widetilde{h}$ points outward in the $\mathbb{R}$-direction.* □

Let $\Gamma(d\widetilde{h}) \subset T^*\widetilde{L}'$ denote the graph of the 1-form $d\widetilde{h}$. The graphs $\Gamma(d\widetilde{h})$ and $\Gamma(-d\widetilde{h})$ intersect each other transversely, and the intersection points correspond to the critical points of $\widetilde{h}$. Choosing $\widetilde{h}$ with small C^1-norm means that we can assume that both of these graphs are arbitrarily C^0-close to the 0-section, and in particular that they are contained in the neighborhood $U_{\widetilde{L}'}$.

Lemma 4.2.14. *The union*

$$\widetilde{L}^5_{\mathrm{im}} \coloneqq \Phi_{\widetilde{L}'}(\Gamma(d\widetilde{h}) \cup \Gamma(-d\widetilde{h})) \cup \bigcup_{\ell,m} \Psi_{\ell,m}(L_Y) \subset \widetilde{W}^5 \tag{4.2.13}$$

is the image of a smooth Lagrangian immersion, possibly after applying Hamiltonian isotopies to L_Y away from the origin in $\mathbb{C}^3$.

Proof. This is similar to the discussion in [29, §5.3]. Modifying L_Y by Hamiltonian isotopies gives us enough freedom to guarantee that $\widetilde{L}^5_{\mathrm{im}}$ is smooth away from the self-intersection points. □

Notice that $\widetilde{L}^5_{\mathrm{im}}$ is contained in the very affine quintic hypersurface $X^5_\tau \setminus D_\tau$, where we recall that D_τ is the intersection of the quintic with the toric boundary of $\mathbb{CP}^4$. By construction, L_Y avoids the coordinate hyperplanes (which coincide with the coordinate hyperplanes of $\mathbb{CP}^4$ in our choices of coordinates). A straightforward adaptation of the proof of [29, Lemma 5.5] yields the following.

Lemma 4.2.15. *The Lagrangian immersion $\widetilde{L}^5_{\mathrm{im}} \subset X^5_\tau \setminus D_\tau$ is exact with respect to the Liouville form on X^5_τ for which $L_{\tau,q}$ is exact.* □

Remark 4.2.16. The domain of $\widetilde{L}^5_{\mathrm{im}}$ is a closed 3-manifold that has a JSJ decomposition with two irreducible pieces, both of which are diffeomorphic to $\widetilde{L}'$. More precisely, we can describe the domain of $\widetilde{L}^5_{\mathrm{im}}$ as the 3-manifold obtained by gluing together two copies of $\widetilde{L}'$ by deleting small collar neighborhoods of their cusps. Abusing notation, we will also call this manifold with boundary $\widetilde{L}'$. The curves $\{m_i, \ell_i\}$ drawn in Figure 2.7 lift to curves in $\widetilde{L}'$ as in Remark 4.2.3. We glue these two manifolds together in a way that respects these homology classes.

Since we defined the Morse function $\widetilde{h}$ by pulling back a Morse function on L', we have the following.

Lemma 4.2.17. *The Lagrangian $\widetilde{L}^5_{\mathrm{im}}$ is invariant under the action of $(\mathbb{Z}/5)^3$, and thus it descends to a Lagrangian immersion whose image is $L^5_{\mathrm{im}} \subset W^5$.* □

The intersection of L^5_{im} with one of the balls B_ℓ can be identified with a copy of L_Y. The intersection of L^5_{im} with $W^5 \setminus \bigcup_\ell B_\ell$ can be identified with a copy of

$$\Gamma(dh) \cup \Gamma(-dh) \subset T^*L'. \tag{4.2.14}$$

There is a Lagrangian immersion whose image is L^5_{im} and whose domain is a quotient of the domain of $\widetilde{L}^5_{\mathrm{im}}$ under the action of $(\mathbb{Z}/5)^3$. Specifically, the domain of this immersion is the 3-manifold obtained by gluing two copies of the minimally-twisted five component chain link complement together using an identification of their boundaries which respects the distinguished generators of $H_1(L'; \mathbb{Z})$, as in Remark 4.2.16.

Notice that there is an immersed Lagrangian $L_{\mathrm{im}} \subset T^*T^3$ gotten by gluing a copy of L_Y in the Darboux ball B_0 of (2.3.1) to the images of the graphs (4.2.14) of $\pm dh$ inside a Weinstein neighborhood of L_{sing}. This is essentially the Lagrangian of [29, Theorem 1.2].

Remark 4.2.18. In [29] we constructed L_{im} using a different Morse function, which we denote by h' in this remark. The choice of Morse function in the present paper simplifies the computations of Floer theory carried out in §4.3.4. We will now explain why the conclusion of Theorem 1.2. in op. cit. still holds if one uses the Morse function h constructed in Lemma 4.2.11 instead. Recall that h' is obtained from the ideal triangulation on L' subordinate to the ideal cubulation used to construct h (cf. Figure 2.5).

There are three natural projections $T^*T^3 \to T^*T^2$, which we can use to naturally identify the Lagrangian pair of pants in T^*T^2 constructed by Matessi [28] with smooth submanifolds of L'. In Figure 2.5, they can be thought of as the union of the two triangles with black edges and vertices at the jth, kth and 4th vertices, where $j, k \in \{1, 2, 3\}$. By generically perturbing h if necessary, we can assume that it restricts to a Morse function on each of these three copies of the 2-dimensional pair of pants. Inspecting the proof of [29, Lemma 6.4], which describes the image of L_{im} under certain Lagrangian correspondences in $(T^*T^3)^- \times (T^*T^2)$, shows that the conclusion of this Lemma holds with no changes when we replace h' with h. Thus the main results of [29], in particular the formulas for the support of the mirror sheaf to L_{im}, hold for the version of L_{im} constructed using h with no additional changes to the proofs.

To make sense of the Floer theory of $\widetilde{L}^5_{\text{im}}$, we need to equip it with a grading, in the sense of [44], and a spin structure. A suitable grading is obtained from a grading on L_{im} in a straightforward way.

Lemma 4.2.19. *There is a grading $\alpha^\#\colon \widetilde{L}^5_{\text{im}} \to \mathbb{R}$ which is approximately equal to 0 near the critical points of $-\widetilde{h}$ and approximately equal to 1 near the critical points of $\widetilde{h}$. Moreover, it is lifted from a grading on L^5_{im}.*

Proof. In [29, Lemma 5.6], we constructed a grading on L_{im} with the desired values near the critical points of $h\colon L' \to \mathbb{R}$, which we could take to be constant when restricted to the part of L_{im} lying over the 1-cones of the underlying tropical curve V. This lifts to a grading on $\widetilde{L}_{\text{im}} \subset T^*T^3$. Under the identification of a Weinstein neighborhood of the zero section T^*T^3 with $\text{Wein}(L_{\tau,0}) \subset$

X^5_τ, it follows that we can define a grading on the part of $\widetilde{L}^5_{\text{im}}$ lying in this neighborhood. This extends to a grading on the rest of $\widetilde{L}_{\text{im}}$ in a unique way. □

We can equip $\widetilde{L}^5_{\text{im}}$ with a spin structure by lifting the spin structure on L' specified in [29, §4.1] to a spin structure on $\widetilde{L}'$, and gluing the spin structures on the two copies of $\widetilde{L}'$ in the JSJ decomposition of $\widetilde{L}^5_{\text{im}}$. We can further assume that this spin structure coincides with the one obtained by lifting the spin structure on L_{im} used in [29] and extending it over the copies of L_Y in the Darboux charts $\widetilde{B}_{\ell,m}$, for $\ell \in \{1,2,3,4\}$. This does not describe a unique spin structure on $\widetilde{L}^5_{\text{im}}$, but we will specify a particular choice for one during the proof of Proposition 4.3.3.

Having fixed a grading on $\widetilde{L}^5_{\text{im}}$, we can determine its Floer cochain space. See Appendix 4.8 for a review of the relevant definitions.

Lemma 4.2.20. *As a graded Λ_0-module, the Floer cochain complex $CF^*(\widetilde{L}^5_{\text{im}})$ has underlying $\mathbb{C}$-vector space given $\overline{CF}^*(\widetilde{L}^5_{\text{im}})$ given by*

$$\Omega^*(\widetilde{L}^5_{\text{im}}) \oplus \bigoplus_{\ell,m} \Omega^*(T^2)[1] \oplus \bigoplus_{\ell,m} \Omega^*(T^2)[-2] \oplus CM^*(\widetilde{L}')[-1] \oplus CM^*(\widetilde{L}', \partial\widetilde{L}')[1] \qquad (4.2.15)$$

where Ω^* denotes the de Rham cochain space, and CM^* denotes the Morse cochain space. The direct sums range over the set

$$\{(\ell, m) \in \mathbb{Z}\times\mathbb{Z} \mid \ell \in \{0,\dots,4\},\ m \in \{1,\dots,125\} \text{ if } \ell = 0 \text{ and } m \in \{1,\dots,5\} \text{ if } \ell \neq 0\}$$

which has 145 *elements.*

We think of $CM^*(\widetilde{L}')$ and $CM^*(\widetilde{L}', \partial\widetilde{L}')$ as copies of the spaces of differential forms on zero-dimensional manifolds given by the Morse critical points of $\widetilde{h}$ or $-\widetilde{h}$, respectively.

Proof of Lemma 4.2.20. Consider the fiber product $\widetilde{L}^5_{\text{im}} \times_{X^5_\tau} \widetilde{L}^5_{\text{im}}$, and note that its switching components are either copies of T^2 or transverse double points.

The 1-dimensional Lagrangian $L_{\text{arc}} \subset \mathbb{C}$ has a single transverse double point, which should contribute two generators to its Floer cochain space. Since L_{arc} bounds an isolated teardrop, i.e. a disk with one corner on a switching component, it follows that one of these generators must have degree 2, and the Poincaré dual generator must have degree −1. Taking the product to form L_Y shows that each of these critical points are replaced by copies of $\Omega^*(T^2)$ with the corresponding grading shifts. Here we can assume that the orientation local system, which appears in the general definition (4.8.4) of the $\mathbb{C}$-vector space underlying the Floer cochain complex, is trivial, since the immersion is locally described as the product of an embedded Lagrangian torus in $\mathbb{C}^2$ with an immersed Lagrangian submanifold with at worst transverse double points.

Note that the 0-dimensional switching components all correspond to pairs (p_-, p_+), where p_- is a Morse critical point of $-\widetilde{h}$ and p_+ is the corresponding critical point of $\widetilde{h}$. It follows from [44] that the grading shifts associated to these points are determined by the Morse indices. More precisely

$$\begin{aligned}\deg(p_-) &= \deg_{\text{Morse}}(p_-) - \alpha^{\#}(p_-) + \alpha^{\#}(p_+)\\ &= \deg_{\text{Morse}}(p_-) - \alpha^{\#}(p_-) + \alpha^{\#}(p_+) + 1\\ &= \deg_{\text{Morse}}(p_-) + 1\end{aligned}$$

Symmetrically, one has that

$$\deg(p_+) = \deg_{\text{Morse}}(p_+) + 1$$

for generators on the positive sheet. These yield the expected grading shifts on the copies of $CM^*(\widetilde{L}')$ and $CM^*(\widetilde{L}', \partial\widetilde{L}')$. □

4.2.4 An embedded tropical Lagrangian in the quintic

We can use $\widetilde{L}^5_{\text{sing}}$ to construct three families of *embedded* Lagrangian submanifolds $\widetilde{L}^5_{\text{sm}}(i;\epsilon)$, for $i = 1, 2, 3$, of X^5_τ which can be thought of as the tropical Lagrangian lifts of smooth tropical curves in the quintic, similar to the tropical Lagrangians in the *mirror quintic* constructed by Mak–

Ruddat [20]. Instead of relying on Mak–Ruddat's construction of Lagrangian solid tori near the singular fibers of an SYZ fibration, however, we will instead make use of Lagrangian solid tori which are thought of as asymptotically conical fillings of the links of the cone points of $\widetilde{L}^5_{\text{sing}}$.

The link of the Harvey–Lawson cone C_{HL} has three asymptotically conical fillings given in coordinates by

$$C_{HL}(i;\epsilon) = \{|y_i|^2 - \epsilon = |y_j|^2 = |y_k|^2\,,\ y_1y_2y_3 \in \mathbb{R}_{\geq 0}\} \tag{4.2.16}$$

for all $\{i, j, k\} = \{1, 2, 3\}$. We can associate to each of these Lagrangian solid tori an embedded Lagrangian submanifold of T^*T^3 which is C^0-close to L_{sing} and which can be thought of as the lift of a tropical smoothing of $V \subset Q$. By a tropical smoothing of V, we mean one of the tropical curves $V(i;\epsilon)$, for $i = 1, 2, 3$, where $V(1;\epsilon)$ is given explicitly by

$$\begin{aligned} V(1;\epsilon) = \left\{\left(0, t + \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) : t \in [0,\infty)\right\} &\cup \left\{\left(0, \frac{\epsilon}{2}, t + \frac{\epsilon}{2}\right) : t \in [0,\infty)\right\} \\ &\cup \left\{\left(0, -t + \frac{\epsilon}{2}, -t + \frac{\epsilon}{2}\right) : t \in [0,\epsilon]\right\} \\ &\cup \left\{\left(t, -\frac{\epsilon}{2}, -\frac{\epsilon}{2}\right) : t \in [0,\infty)\right\} \\ &\cup \left\{\left(-t, -t - \frac{\epsilon}{2}, -t - \frac{\epsilon}{2}\right) : t \in [0,\infty)\right\} \end{aligned} \tag{4.2.17}$$

and where $V(2;\epsilon)$ and $V(3;\epsilon)$ are obtained by cyclically permuting coordinates on Q.

To construct the smoothings of L_{sing}, recall from the proof of [29, Lemma 4.7] that the link of the singular point in L_{sing} maps to a 2-sphere in Q centered at the origin. The restriction of this projection map to such a Legendrian link is generically 2-to-1, except over a tetrahedral graph embedded on the sphere, over which it is 1-to-1. The vertices of this graph lie on the 1-dimensional cones of V (cf. Figure 4.2). In the Darboux coordinates we have chosen near B_0 in (2.3.1), the asymptotically conical fillings (4.2.16) will project to subsets of Q containing the origin bounded by (smoothings of) tetrahedra with vertices on the semi-infinite 1-dimensional cones of the smooth

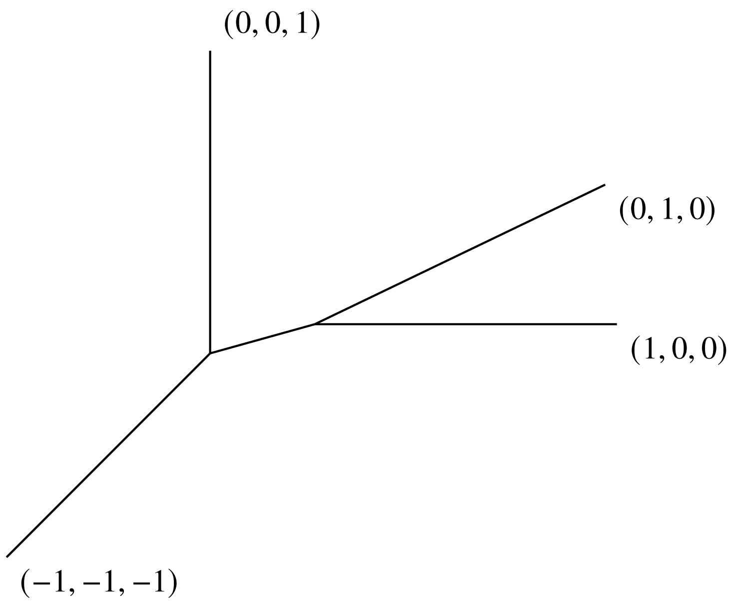

Figure 4.1: A tropical curve $V(i;\epsilon)$

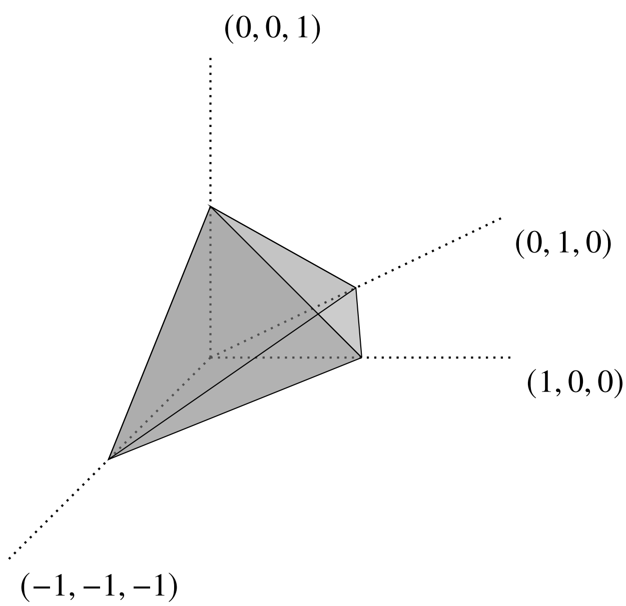

Figure 4.2: The image in Q of a link of the cone point of L_{sing} is obtained by smoothing the edges of this tetrahedron. The dotted lines are the edges tropical curve V.

tropical curves $V(i;\epsilon)$. We can represent the subsets

$$\{|y_i|^2 - \epsilon = |y_j|^2 = |y_k|^2\,,\ y_1 y_2 y_3 \in \mathbb{R}_{>0}\}$$

as graphs of closed 1-forms on the complement $T^2 \times (0,1) \cong C_{HL} \setminus \{0\}$ of the singular point of the Harvey–Lawson cone. Let $\partial_0 L'$ denote the link of the singular point in L_{sing}. Since the map $H^1(L';\mathbb{R}) \to H^1(\partial_0 L';\mathbb{R})$ is a surjection, we can extend such a 1-form to a closed 1-form δ_i on the complement $L' = L_{\text{sing}} \setminus \{(0,0)\}$ of the cone point in L_{sing}.

Without loss of generality, we can assume that δ_i takes a certain standard form outside of the compact subset $\pi_{\text{SYZ}}^{-1}(B)$ of Theorem 1.2.1 of $T^*Q/T^*_{\mathbb{Z}}Q$, in which L_{sing} coincides with the periodized conormals to the legs V_ℓ of V. More precisely, δ_i should coincide with a closed 1-form on each of the periodized conormals $N^*V_\ell/N^*_{\mathbb{Z}}V_\ell \cong T^2 \times \mathbb{R}_{>0}$ which is constant in the $\mathbb{R}_{>0}$-direction. Deforming the periodized conormals by such a 1-form corresponds to translating them in Q. Thus, perturbing $L_{\text{sing}} \setminus \{(0,0)\}$ by δ_i and taking the union of the resulting (non-properly) embedded Lagrangian with $C_{HL}(i;\epsilon)$ yields an embedded Lagrangian submanifold in $T^*Q/T^*_{\mathbb{Z}}Q$ which projects to a neighborhood of $V(i;\epsilon)$ in Q. These give embedded Lagrangian submanifolds in X^5_τ as follows.

Theorem 4.2.21. *For each $i = 1,2,3$, there is a family of embedded Lagrangian submanifolds $\widetilde{L}^5_{\text{sm}}(i;\epsilon)$ contained in $\widetilde{W}^5$, where $\epsilon > 0$ is sufficiently small, which are topologically obtained by replacing neighborhoods of the cone points $\widetilde{v}_{\ell,m}$ in $\widetilde{L}^5_{\text{sing}}$ with copies of one of the fillings* (4.2.16).

Proof. Consider the singular Lagrangian submanifold $L^5_{\text{sing}} \subset W^5$. Recall that the smooth locus of L^5_{sing} is diffeomorphic to the minimally-twisted five-component chain link complement L', and by abuse of notation let ∂L_ℓ denote the link of the ℓth cusp of L', thought of here as a link of one of the conical singular points of L^5_{sing}, where $\ell = 0,1,2,3,4$. The 1-form δ_i on L' can then be pulled back to a closed 1-form $\widetilde{\delta}_i$ on $\widetilde{L}'$.

In the charts $\widetilde{B}_{\ell,m}$, for $\ell = 1,2,3,4$, the 1-form $\widetilde{\delta}_i$ restricts to a 1-form on the smooth part of the Harvey–Lawson cone $C_{HL} \setminus \{0\}$ from which one obtains the asymptotically conical smooth fillings. From this we obtain the embedded Lagrangian submanifolds $\widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)$ in the statement of the theorem. □

We will often omit i and ϵ from the notation for these embedded Lagrangian submanifolds and the 1-forms used to construct them. As we will see in §4.3.5, any of the Lagrangian submanifolds of Theorem 4.2.21 will support an isomorphic family of objects in the Fukaya category of the quintic, so the specific choices of i and ϵ will be irrelevant for our purposes.

4.3 Mirrors to lines in the mirror quintic

In this section, we will prove Theorem 1.2.6. We established a local version of this result in [29, Theorem 1.2]. Equipping the Lagrangian torus $L_{\tau,0}$ with a $GL(1,\mathbb{C})$-flat connection yields an object of the Fukaya category mirror to a point. Viewing this torus as an object of the *affine* Fukaya category, it follows from homological symmetry at large radius that these objects correspond to an analytic chart on the mirror. It is then almost immediate from the results of [29] that Lagrangian branes supported on $\widetilde{L}^5_{\mathrm{im}}$ of a suitable form (cf. Proposition 4.3.3) are mirror to sheaves supported on a line. By [29, Corollary 6.1], this sheaf has rank 2 away from a Zariski open subset (of $\mathbb{CP}^1$). To check that the sheaf has rank 2 at all points on the line, we will calculate the Floer differential of $\widetilde{L}^5_{\mathrm{im}}$ with an unobstruced local system to low order. Because $\widetilde{L}^5_{\mathrm{im}}$ bounds many disks of low energy, this turns out to be enough for us to obtain an upper bound on the ranks the stalks of the sheaf.

The rest of this section is organized as follows. In §4.3.1, we will identify a subspace of the deformation space of $\widetilde{L}^5_{\mathrm{im}}$ in the Fukaya category of the quintic. We will then explain the calculations at large radius which determine the supports of the mirrors to these objects in §4.3.3. Finally, §4.3.4 completes the proof of Theorem 1.2.6. Combined with our results about the support of the mirror sheaf, the local computation of §4.3.4 actually determines the full Floer homology of $\widetilde{L}^5_{\mathrm{im}}$.

4.3.1 Local systems

It is clear from the construction of $\widetilde{L}^5_{\text{im}}$ that it bounds holomorphic teardrops, i.e. disks with a corner on a switching component. By equipping $\widetilde{L}^5_{\text{im}}$ with suitable local systems, we can ensure that the algebraic counts of these teardrops vanish, and in fact that $\mathfrak{m}_0$ vanishes for all of these local systems. We will identify a 1-dimensional space of unobstructed rank 1 complex local systems. We do not attempt to compute the full deformation space of $\widetilde{L}^5_{\text{im}}$ in this paper, though we conjecture that all other brane structures on $\widetilde{L}^5_{\text{im}}$ give trivial objects of the Fukaya category. For simplicity, we will only consider certain local systems on $\widetilde{L}^5_{\text{im}}$ which come from local systems on the minimally-twisted five-component chain link complement. Working within this restricted class of brane structures already suffices to recover partial A-model analogues of nontrivial results [24] on the moduli spaces of lines in the mirror quintic.

An easy calculation with the Mayer–Vietoris sequence shows that $H_1(L^5_{\text{im}};\mathbb{Z})$, the first homology of the quotient $\widetilde{L}^5_{\text{im}}/(\mathbb{Z}/5)^3$, is a free $\mathbb{Z}$-module of rank 9. Fix a basis for $H_1(L^5_{\text{im}};\mathbb{Z})$ of the form

$$\{m_0, m_1, m_2, m_3, m_4\} \cup \{s_1, s_2, s_3, s_4\} \tag{4.3.1}$$

where $m_0, \ldots, m_4$ denote the homology classes of the meridians in L', as in Figure 2.7, and $s_1, \ldots, s_4$ are classes which do not lie in the span of the meridians. Explicitly, we can assume that s_i is represented by a simple closed curve which intersects the zeroth boundary torus and the ith boundary torus of L' transversely and once each.

Consider a local system, thought of as a representation $H_1(L^5_{\text{im}};\mathbb{Z}) \to \mathbb{C}^*$, of the form

$$\begin{cases} m_i \mapsto \mu_i & i = 0, \ldots, 4 \\ s_j \mapsto 1 & j = 1, \ldots, 4\,. \end{cases} \tag{4.3.2}$$

In this situation, we let λ_i denote the image of the longitude ℓ_i, for all $i = 0, \ldots, 4$. Let ∇ be a rank

1 local system on $\widetilde{L}^5_{\mathrm{im}}$ induced by a local system on L^5_{im} of the form (4.3.2).

The following lemma classifies all holomorphic teardrops on $\widetilde{L}^5_{\mathrm{im}}$, defined with respect to the *integrable* almost complex structure. There are obvious families of teardrops bounded by $\widetilde{L}^5_{\mathrm{im}}$ which are the images of teardrops on the local immersed Lagrangian $L_Y \subset Y$. The teardrops in Y arise as sections of $w\colon Y \to \mathbb{C}$ as in (4.2.11). Observe that any holomorphic teardrop on L_Y which contributes to $\mathfrak{m}_0$ must come in a family of virtual dimension 2.

Lemma 4.3.1. *Any holomorphic teardrop on L_Y which contributes to $\mathfrak{m}_0$ can be written as the product, with respect to* (4.2.12)*, of the teardrop bounded by L_{arc} with (cf. Figure 2.10) with a holomorhpic disk in $\mathbb{C}^2$ bounded by the Clifford torus. The latter disk is either constant, or contained in one of the coordinate axes of $\mathbb{C}^2$.*

Proof. Since L_Y is a product Lagrangian in $\mathbb{C} \times \mathbb{C}^2$, all holomorphic disks with boundary on L_Y split. For any such disk to have index 2, its second factor must be one of the disks listed in the statement of the Lemma. □

All disks described in Lemma 4.3.1 are regular, by the argument of [33, Lemma 2.10], and each moduli space of teardrops can naturally be identified with a 2-torus.

Lemma 4.3.2. *Any holomorphic teardrop in $\widetilde{W}^5$ bounded by $\widetilde{L}^5_{\mathrm{im}}$ is one of the teardrops of Lemma 4.3.1.*

Proof. Let $u\colon \overline{\Delta} \setminus \{1\}$ be such a teardrop. The boundary of u would have to lift to a path on (the domain of) $\widetilde{L}^5_{\mathrm{im}}$ which passes between the two branches. Thus, the image of u would have to intersect one of the neighborhoods $\widetilde{B}_{\ell,m}$. There are holomorphic maps $w_{\ell,m}\colon \widetilde{B}_{\ell,m} \to \mathbb{C}$ induced from (4.2.11). By the open mapping theorem, the image of $w_{\ell,m} \circ u$ must be an open subset of $\mathbb{C}$. Since the image of $\widetilde{L}_{\mathrm{sing}}$ under this map is an arc with a self-intersection point, it follows that the image of u must be contained in $\widetilde{B}_{\ell,m}$. □

By reordering the coordinates on the domains of the Darboux charts $\widetilde{B}_{\ell,m}$ if necessary, cf. Remark 4.2.10, we can calculate $\mathfrak{m}_0$ for $\widetilde{L}^5_{\mathrm{im}}$ to low order.

Proposition 4.3.3. *Let ∇ be a rank one $\mathbb{C}$-local system on $\widetilde{L}^5_{\mathrm{im}}$ which is induced from a local system on L^5_{im} of the form* (4.3.2)*. Then $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$, equipped with an appropriate choice of spin structure, is unobstructed with bounding cochain $0 \in CF^*(\widetilde{L}^5_{\mathrm{im}})$ in $\widetilde{W}^5$ if the holonomies $\{\mu_i, \lambda_i\}_{i=0}^4$ satisfy*

$$1 + \mu_0^{-1} + \mu_0^{-1}\lambda_0^{-1} = 0 \tag{4.3.3}$$

$$-1 - \mu_1^5 + \lambda_1^{-5} = 0 \tag{4.3.4}$$

$$-1 - \mu_2^{-5} - \lambda_2^5 = 0 \tag{4.3.5}$$

$$-1 - \mu_3^{-5} + \lambda_3^5 = 0 \tag{4.3.6}$$

$$-1 + \mu_4^5 + \mu_4^5\lambda_4^5 = 0\,. \tag{4.3.7}$$

There is a subspace of the space of local systems satisfying (4.3.3)–(4.3.7) *and containing the local system $\nabla^{\mathrm{vG}}_\omega$, to be described in Example 4.3.5 below which can be identified with a punctured genus* 6 *curve.*

Definition 4.3.4. A local system on $\widetilde{L}^5_{\mathrm{im}}$ satisfying the conclusion of Proposition 4.3.3 is said to be *locally unobstructed*.

Before proving Proposition 4.3.3, we will check that the set of locally unobstructed local systems is nonempty by exhibiting local systems mirror to van Geemen lines.

Example 4.3.5. [Very affine van Geemen lines] Let $\omega, a \in \mathbb{C}^*$ be constants satisfying

$$1 + \omega + \omega^2 = 0$$

$$a^5 = 27\,.$$

Consider the local system on L^5_{im} with holonomy

$$\begin{aligned}
\mu_0 &= \omega\,; & \lambda_0 &= \omega\\
\mu_1 &= -\omega\,; & \lambda_1^{-1} &= -\frac{a}{3}(1-\omega)\\
\mu_2^{-1} &= \frac{a}{3}(1-\omega^2)\,; & \lambda_2 &= -\omega^2\\
\mu_3^{-1} &= -\omega\,; & \lambda_3 &= -\frac{a}{3}(1-\omega)\\
\mu_4^{-1} &= -\omega^2\,; & \lambda_4 &= -\omega\,.
\end{aligned} \tag{4.3.8}$$

Since there are two primitive third roots of unity (4.3.8) determines two local systems, which we refer to interchangeably. It is easy to check that these satisfy (4.3.3)–(4.3.7). Thus this local system lifts to one on $\widetilde{L}_{\mathrm{im}}$ via the covering map $\widetilde{L}_{\mathrm{im}} \to L_{\mathrm{im}}$. By (4.3.2) this yields a local system on $\widetilde{L}^5_{\mathrm{im}}$ as well, which is the local system $\nabla^{\mathrm{vG}}_{\omega}$ mentioned in the statement of Proposition 4.3.3. The proof of [29, Theorem 1.2] shows that equipping $\widetilde{L}_{\mathrm{im}} \subset T^*T^3$ with a local system with holonomy as in (4.3.8) gives an object of the wrapped Fukaya category of T^*T^3 which is mirror to a coherent sheaf supposed on the line $(\mathbb{C}^*)^3$ cut out by

$$\begin{aligned}
\lambda_3 u_2 - \mu_2^{-1} u_3 + 1 &= 0\\
-\mu_2^{-1}\lambda_2^{-1} u_1 + \mu_1^{-1}\lambda_1^{-1} u_2 + 1 &= 0\\
\mu_3\lambda_3 u_1 - \lambda_1^{-1} u_3 + 1 &= 0\,.
\end{aligned}$$

In this example, these linear forms specialize to

$$\begin{aligned}
-\frac{a}{3}(1-\omega)u_2 - \frac{a}{3}(1-\omega^2)u_3 + 1 &= 0\\
-\frac{a}{3}(1-\omega)u_1 - \frac{a}{3}(1-\omega^2)u_2 + 1 &= 0\\
-\frac{a}{3}(1-\omega^2)u_1 - \frac{a}{3}(1-\omega)u_3 + 1 &= 0\,.
\end{aligned}$$

The projective closure of this line in $\mathbb{CP}^3$ is easily seen to be the limiting cycle, in X^5_0, of a van

Geemen line C_0^{ω}. By interchanging ω with ω^2, we obtain another local system $\nabla^{\mathrm{vG}}_{\omega^2}$, to which we can associate a curve in $(\mathbb{C}^*)^3$ whose projective closure in X_0^5 is $C_0^{\omega^2}$.

Proof of Proposition 4.3.3. All three families of teardrops contained in $\widetilde{B}_{\ell,m}$, as in Lemma 4.3.1, have the same output in the de Rham complex $\Omega^*(T^2)$ of the switching component. The definitions of the balls $\widetilde{B}_{\ell,m}$ and Lemma 2.3.18 imply that the contributions of these teardrops to $\mathfrak{m}_0$ weighted by holonomies given in (4.3.3)–(4.3.7), up to signs.

We claim that there is a spin structure on $\widetilde{L}^5_{\mathrm{im}}$ with respect to which these teardrops contribute with the signs given in the statement of the Proposition. We will describe this as the lift of a spin structure on the quotient L^5_{im}. As previously discussed, the spin structure we choose for L^5_{im} should restrict to the spin structure on L_{im} specified in [29, Lemma 4.2]. This determines the signs appearing in (4.3.3) (cf. [29, Lemma 5.7]).

In (4.3.7), the terms with holonomy $\pm\mu_4^5$ and $\pm\mu_4\lambda_4^5$ must carry the same sign, since the spin structure on the minimally-twisted five-component chain link complement of [29, Lemma 4.2] is preserved by a hyperbolic isometry which interchanges the homology classes m_4 and $m_4 + \lambda_4$. We can then choose the spin structure on L^5_{im} such that the remaining disk, which we can assume has boundary lying on a curve representing the class $s_4 \in H_1(L^5_{\mathrm{im}})$, contributes with the opposite sign. Again by [29, Lemma 4.2], the terms of (4.3.4), (4.3.5), and (4.3.6) weighted by nontrivial holonomy contribute with opposite signs. Because $H_1(L^5_{\mathrm{im}})$ has rank 9, the spin structure can be chosen so that it has the desired behavior at all four necks of L^5_{im}.

In particular the first term in each expression corresponds to the product of the teardrop bounded by L_Y with a constant disk. The signs are determined using the symmetries of the spin structure on L (cf. Section 6.3 and Remark 6.3 of [29]).

It is clear that the subvariety of $(\mathbb{C}^*)^5$ cut out by (4.3.3)-(4.3.7) is at most 2-dimensional, since the value of μ_0 (which determines the value of λ_0) and the value of μ_i^5 or λ_i^5, for any $i = 1, 2, 3, 4$, will uniquely determine a local system satisfying these relations using Lemma 2.3.17. Moreover, we calculate

$$
\begin{aligned}
1+\mu_1^5=\lambda_1^{-5} &= \mu_0^5\mu_2^{-5}\\
&= \mu_0^5\lambda_3^5\mu_4^{-5}\\
&= \mu_0^5(1+\mu_3^{-5})\mu_4^{-5}\\
&= \mu_0^5(1+\mu_0^{-5}\lambda_4^{-5})\mu_4^{-5}.
\end{aligned}
$$

Multiplying by μ_4^5 gives us

$$
\begin{aligned}
\mu_4^5+\mu_1^5\mu_4^5 &= \mu_0^5(1+\mu_0^{-5}\lambda_4^{-5})\\
\mu_4^5+\lambda_0^{-5} &= \mu_0^5+\lambda_4^{-5}
\end{aligned}
$$

which we rewrite using (4.3.3) and (4.3.7) as

$$
\frac{1}{1+\lambda_4^5}-\lambda_5^4=\mu_0^5-\lambda_0^{-5}\,.
$$

This implies that the values of μ_0^5 and λ_0^5 determine a value of λ_4^5 satisfying (4.3.3)–(4.3.7), and thus that the space of unobstructed local systems is 1-dimensional (as we have already seen that it is nonempty). For values of μ_0 and λ_0, contained outside of a finite subset of the pair of pants in $(\mathbb{C}^*)^2$ cut out by (4.3.3), the value of λ_4^5 we obtain will be nonzero. It follows that the space of local systems satisfying (4.3.3)–(4.3.7) is naturally a Zariski open subset of a 25-fold cover the standard pair of pants in $(\mathbb{C}^*)^2$. An application of the Riemann–Hurwitz formula shows that this cover has genus 6. □

Remark 4.3.6. As pointed out in Remark 4.2.10, the intersection of $\widetilde{L}^5_{\mathrm{im}}$ with the balls $\widetilde{B}_{\ell,m}$ depends on an ordering of coordinates on the domains of these charts. One observes, however, that reordering coordinates in the domains corresponds to scaling (4.3.3)–(4.3.7) by nonzero scalars, so any choices of coordinates used to construct L_Y give us Lagrangians with *canonically* isomorphic deformation spaces.

Remark 4.3.7. The appearance of a genus six curve is consistent with the results of [24], though its description here is very different from the genus six curves of op. cit. One expects the mirror map, which by [6] induces the change of variables underlying the mirror functor, to embed the punctured curve of Proposition 4.3.3 into the moduli space of lines on the mirror quintic in a nontrivial way. Some of the punctures in the curve of Proposition 4.3.3, in particular the punctures corresponding to lifts of punctures in the pair of pants (4.3.3), should, informally, correspond to taking limits of local systems on $\widetilde{L}^5_{\mathrm{im}}$ such that the holonomy around certain loops approaches 0 or ∞. This suggests that A-branes corresponding to these punctures in the moduli space of lines should be supported on different Lagrangians.

4.3.2 Unobstructedness

Recall from [43], or Appendix 4.8, that the obstruction term $\mathfrak{m}_0$ can be thought of as a cochain in $CF^2(L)$, for any clean Lagrangian immersion $\iota\colon L \to M$. Moreover, we can represent it as a sum of elements

$$\mathfrak{m}_0^0 + \mathfrak{m}_0^s \in \left((\Omega^2(L))\widehat{\otimes}_{\mathbb{C}}\Lambda_0\right) \oplus \bigoplus_{\substack{a\in A\setminus\{0\}\\ k-\deg(L_a)=2}} \left((\Omega^k(L_a))\widehat{\otimes}_{\mathbb{C}}\Lambda_0\right)$$

where Λ_0 is the Novikov ring (4.8.1) over $\mathbb{C}$ and $\widehat{\otimes}_{\mathbb{C}}$ denotes the completed tensor product. More succinctly, we split $\mathfrak{m}_0$ as a sum of elements in summands of the Floer cohcain space corresponding to the diagonal and switching components of the fiber product $L \times_\iota L$.

In the case of $\widetilde{L}^5_{\mathrm{im}} \to X^5_\tau$, we will show that these two summands of the curvature both vanish. The vanishing of $\mathfrak{m}_0^s$ in this setting follows from the definition of a locally unobstructed local system (Definition 4.3.4) combined with an SFT compactness theorem for disks with Lagrangian boundary appearing in [83].

Proposition 4.3.8. *The switching part $\mathfrak{m}_0^s$ of the obstruction class of $\widetilde{L}^5_{\mathrm{im}}$ vanishes.*

Proof. Fix a class $\beta \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}}; \mathbb{Z})$. Consider the moduli space $\mathcal{M}_1(\widetilde{L}^5_{\mathrm{im}}; \beta)$ of holomorphic teardrops, i.e. holomorphic disks with one boundary marked point asymptotic to a switching component, and let $[u]$ be an element of this moduli space. For topological reasons, the boundary of u is an arc which must pass through one of the necks in $\widetilde{L}^5_{\mathrm{im}}$, and thus the image of the interior of u intersects one of the balls $\widetilde{B}_{\ell,m}$, for $\ell = 0, 1, 2, 3, 4$.

The boundary $\widetilde{S}_{\ell,m} = \partial \widetilde{B}_{\ell,m}$ is a hypersurface of contact type which divides X^5_τ into two connected components. In particular, there is a tubular neighborhood $N \cong (-\epsilon, \epsilon) \times \widetilde{S}_{\ell,m}$ and a contact form α on $\widetilde{S}_{\ell,m}$ such that the symplectic form on X^5_τ restricts to $d(e^{kt}\alpha)$ on N, where $t \in (-\epsilon, \epsilon)$ and k is a positive integer. We can take α to be the standard contact form on S^5. By construction, we can write $\widetilde{L}^5_{\mathrm{im}} \cap N = \Lambda \times (-\epsilon, \epsilon)$ for a Legendrian $\Lambda \subset S^5$.

We will apply an SFT compactness theorem for disks with Lagrangian boundary, specifically [83, Theorem 3.13], in this situation. The discussion around [83, Example 3.1] shows that the SFT compactness theorem applies in the setting described in the last paragraph. Neck stretching along $\widetilde{S}_{\ell,m}$ shows that given any teardrop u as in the first paragraph, there is a holomorphic teardrop in $\widetilde{B}_{\ell,m}$ with interior punctures asymptotically converging to Reeb orbits in the standard contact S^5. Moreover the boundary of u has the same homology class in $H_1(\widetilde{L}^5_{\mathrm{im}})$ as the boundary of this new teardrop.

We will think of this punctured teardrop as a curve contained in the Gross–Siebert space Y of (4.2.10). This can be thought of as a meromorphic function $\overline{u}\colon \Delta \to Y$. We can replace this with a holomorphic teardrop as follows. Using the product decomposition of (4.2.12), we can write $\overline{u} = (\overline{u}_i, \overline{u}_j, \overline{w})$. Each of these component functions is meromorphic. The singular points of these functions are all (at worst) poles at points in $\operatorname{Int} \Delta$ corresponding to the punctures. The punctures cannot correspond to essential singularities, since the condition that they are asymptotic to Reeb orbits would contradict Picard's great theorem, for instance. If P_i denotes the (finite) set of poles of $\overline{u}_i$, we can form a holomorphic function by setting

$$\widetilde{u}_i(z) \coloneqq \overline{u}_i(z) \cdot \prod_{p \in P_i} (z - p)^{\mathrm{ord}(p)} .$$

This is a holomorphic disk with boundary on simple closed curve in $\mathbb{C}$ enclosing the origin. We also define $\widetilde{u}_j$ and $\widetilde{w}$ analogously. Note that $\widetilde{u}_j$ and $\widetilde{w}$ will have boundary on 1-dimensional submanifolds of $\mathbb{C}$ that are isotopic to the unit circle or to L_{arc}, respectively. Thus we obtain a holomorphic disk $\widetilde{u} := (\widetilde{u}_i, \widetilde{u}_j, \widetilde{w})$ with boundary on a different Lagrangian submanifold, denoted L'_Y, contained in Y. Although L'_Y is only Lagrangian isotopic to L_Y, it is clear that teardrops bounded by L'_Y can be classified in the same way as teardrops on L_Y. In particular, $\widetilde{u}$ is a section over the teardrop $\widetilde{w}$ per the proof of Lemma 4.3.1. From this we can determine the homology class of $\partial\widetilde{u}$ in $H_1(L'_Y)$, and in turn the homology class of $\partial\overline{u} \in H_1(L_Y)$. The latter must be the homology class of the boundary of a disk described in Lemma 4.3.1, as is the case for the boundary ∂u of the original teardrop as well.

From the teardrop u, we can form two other teardrops with boundary on $\widetilde{L}^5_{\text{im}}$ by cyclically permuting the coordinates (x_i, x_j, x_k) on $\mathbb{CP}^4$, where we recall our convention that $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. By the result of the previous paragraph, we can determine the first homology classes of the boundaries of each of these teardrops. Since we have equipped $\widetilde{L}^5_{\text{im}}$ with a locally unobstructed local system, it follows that the contributions of these three teardrops to $\mathfrak{m}^s_0$ cancel. □

Remark 4.3.9. The technique of treating punctured curves arising from neck stretching as meromorphic functions with poles is inspired by the proof of [83, Theorem 6.27], and our arguments can be thought of as a weaker version of those in [83, §6.3].

To show that diagonal obstruction term $\mathfrak{m}^0_0$ vanishes, we will adapt the results of [36]. The results of op. cit. apply to an embedded Lagrangian $L \subset M$ fixed, as a set, by an anti-symplectic involution $\mathfrak{d}$. Under additional assumptions about the action of $\mathfrak{d}$ on $H^*(L)$, [36, Theorem 1.2] shows that $CF^*(L)$ is unobstructed (and in fact that it is formal), by calculating the sign of $\mathfrak{d}$ as it acts on the moduli spaces $\mathcal{M}_{k+1}(L)$. These arguments to apply in our setting show that $\mathfrak{m}^0_0$ for $\widetilde{L}^5_{\text{im}} \to X^5_\tau$ vanishes with essentially no changes, since the diagonal obstruction term only counts disks with smooth boundary. We only need to explain how a canonical model for the Fukaya A_∞-algebra of an immersed Lagrangian is constructed, and to check that complex conjugation (which acts on X^5_τ for real τ) acts suitably on the de Rham cohomology of $H^*(\widetilde{L}^5_{\text{im}})$.

Lemma 4.3.10. *The involution $\mathfrak{d}$ preserves $\widetilde{L}^5_{\mathrm{im}}$ as a set.*

Proof. First we will show that the singular Lagrangian $\widetilde{L}^5_{\mathrm{sing}}$ is $\mathfrak{d}$-invariant. Recall from [29] that the tropical Lagrangian L_{sing} is fixed by complex conjugation on $(\mathbb{C}^*)^3$, which just acts by the inverse map on the T^3-fibers. Note that complex conjugation also restricts to the inverse map on the 2-torus fibers of the periodized conormal bundles. There is a corresponding involution on T^*T^3. Thus by Remark 4.2.3, the cover $\widetilde{L}_{\mathrm{sing}}$ is also $\mathfrak{d}$-invariant.

All of the Lagrangian tori $L_{\tau,q} \subset X^5_\tau$ is preserved by $\mathfrak{d}$, and its action on $\mathrm{Wein}(L_{\tau,0})$ can be identified with the action by complex conjugation on T^*T^3. Similarly, the Lagrangian cones near X^5_τ used to construct $\widetilde{L}^5_{\mathrm{sing}}$ are also preserved by $\mathfrak{d}$, and its action on the link of each cone is easily seen to coincide with the action of conjugation on the ends of the tropical Lagrangians. Therefore $\mathfrak{d}$ respects all of the gluings carried out in the proof of Theorem 4.2.4, meaning that it acts on $\widetilde{L}^5_{\mathrm{sing}}$. In fact, the action of $\mathfrak{d}$ restricts to an action of the smooth locus $\widetilde{L}'$ as well.

The Morse function $\widetilde{h} \colon \widetilde{L}' \to \mathbb{R}$ can be made $\mathfrak{d}$-invariant, since, by construction, the action by complex conjugation on $L_{\mathrm{sing}} \subset T^*T^3$ swaps the two ideal cubes drawn in Figure 2.5. This means that the graphs $\Gamma(d\widetilde{h})$ and $\Gamma(-d\widetilde{h})$ contained in the Weinstein neighborhood $\Phi_{\widetilde{L}'}(U_{\widetilde{L}'})$ are swapped by the action of $\mathfrak{d}$. Correspondingly, the action of complex conjugation the neighborhood Y defined in (4.2.10) descends to complex conjugation on $\mathbb{C}$ under the map $Y \to \mathbb{C}$ appearing in (4.2.12). Since the ends of L_Y are glued to the two branches $\Gamma(d\widetilde{h})$ and $\Gamma(-d\widetilde{h})$, it follows that the action of $\mathfrak{d}$ respects this gluing. □

Consider a holomorphic disk $u \colon (D^2, \partial D^2) \to (X^5_\tau, \widetilde{L}^5_{\mathrm{im}})$ in the homology class $\beta \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}}; \mathbb{Z})$. There is a map on the corresponding moduli space

$$\widetilde{\mathfrak{d}} \colon \mathcal{M}_1(\widetilde{L}^5_{\mathrm{im}}; \beta) \to \mathcal{M}_1(\widetilde{L}^5_{\mathrm{im}}; -\delta_*\beta) \tag{4.3.9}$$

induced by $\mathfrak{d}$ and defined as follows. Recall that an element of $\mathcal{M}_1(\widetilde{L}^5_{\mathrm{im}}; \beta)$ is represented by a holomorphic map $u \colon (\Sigma, j) \to (X^5_\tau, J)$ Let (Σ, j) from a (nodal) bordered Riemann surface (Σ, j) for which $[u] = \beta \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}})$ and a boundary marked point $z_0 \in \partial\Sigma$. Let $\psi_\Sigma \colon \overline{\Sigma} \to \Sigma$ denote the antiholomorphic involution whose underlying map of sets is the identity. Then the map of moduli spaces (4.3.9) is defined by

$$\widetilde{\mathfrak{d}}(\Sigma, u, z_0) := (\overline{\Sigma}, \delta \circ u \circ \psi_\Sigma, z_0)\,. \tag{4.3.10}$$

The proof of [36, Proposition 5.6] extends to the case of Lagrangian immersions with clean self-intersection to show the following.

Lemma 4.3.11. *The sign of* (4.3.9) *is given by*

$$sgn(\widetilde{\mathfrak{d}}) = sgn(\mathfrak{d}|_{\widetilde{L}^5_{\mathrm{im}}}) + 1 \tag{4.3.11}$$

provided that $\beta \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}}; \mathbb{Z})$ is a relative homology class as described above. □

Since the irreducible locus of any such moduli space consits only of maps whose domain is a disk without corners, the orientation calculation from the proof of [36, Proposition 5.6] is unaffected. Because the boundary strata of the moduli spaces of disks with boundary on $\widetilde{L}^5_{\mathrm{im}}$ are oriented coherently, the sign of (4.3.9) is the same on the boundary strata.

Note that because $\widetilde{L}^5_{\mathrm{im}}$ is graded, the Maslov class evaluated at any $\beta \in H_2(X^5_\tau; \widetilde{L}^5_{\mathrm{im}})$ as above vanishes, explaining the relative simplicity of the formula in (4.3.11).

A canonical model for the Lagrangian Floer cochain complex can be constructed as described in [47] or [36]. Consider the $\mathbb{C}$-vector space $\overline{C}^* := \overline{CF}^*(\widetilde{L}^5_{\mathrm{im}})$ of (4.8.4). Let $\overline{D}^*$ denote the finite-dimensional $\mathbb{C}$-vector space obtained by taking the cohomology of $\overline{C}^*$ with respect to the de Rham differential, denote $\mathfrak{m}_{1,0}$. By the Hodge decomposition theorem, there exist linear maps

$$i\colon \overline{D} \to \overline{C}$$
$$p\colon \overline{C} \to \overline{D}$$
$$h\colon \overline{C} \to \overline{C}$$

satisfying

$$p \circ \mathfrak{m}_{1,0} = 0\,, \qquad \mathfrak{m}_{1,0} \circ i = 0\,,$$
$$p \circ i = \mathrm{id}\,, \qquad \mathfrak{m}_{1,0} \circ h + h \circ \mathfrak{m}_{1,0} = i \circ p - \mathrm{id}\ .$$

The existence of such maps implies that there is a gapped filtered A_∞-structure on the $\mathfrak{m}_{1,0}$-cohomology $H^*(CF^*(\widetilde{L}^5_{\mathrm{im}}), \mathfrak{m}_{1,0})$ of $CF^*(\widetilde{L}^5_{\mathrm{im}})$ which is quasi-isomorphic to $CF^*(\widetilde{L}^5_{\mathrm{im}})$. Note that the decomposition of $\mathfrak{m}_0$ into $\mathfrak{m}_0^0 + \mathfrak{m}_0^s$ still makes sense in $H^*(CF^*(\widetilde{L}^5_{\mathrm{im}}), \mathfrak{m}_{1,0})$, since $\mathfrak{m}_{1,0}$ respects the splitting of $CF^*(\widetilde{L}^5_{\mathrm{im}})$.

Proposition 4.3.12. *We have that $\mathfrak{m}_0^0 \in H^2(CF^*(\widetilde{L}^5_{\mathrm{im}}), \mathfrak{m}_{1,0})$ vanishes, implying that $\mathfrak{m}_0^0 \in CF^*(\widetilde{L}^5_{\mathrm{im}})$ vanishes as well.*

Proof. Consider the classes $m_i, s_j \in H_1(L^5_{\mathrm{im}})$ described in (4.3.1), and notice that they lift to a set of generators for $H_1(\widetilde{L}^5_{\mathrm{im}})$. By examining the action of complex conjugation on these classes in $H_1(L^5_{\mathrm{im}})$, it is easy to see that $\mathfrak{d}$ induces $-\mathrm{id}$ on $H_1(\widetilde{L}^5_{\mathrm{im}})$. From this one can also see that $\mathfrak{d}$ induces the identity on $H_2(\widetilde{L}^5_{\mathrm{im}})$, since $H_2(\widetilde{L}^5_{\mathrm{im}})$ is generated by products of the lifts of the classes in (4.3.1) and the lifts of the longitudes. Dualizing, we see that $\mathfrak{d}$ induces the identity on the de Rham cohomology $H^2(\widetilde{L}^5_{\mathrm{im}})$ as well. Thus by Lemma 4.3.11 we conclude that $\mathfrak{m}_0^0 = \mathfrak{d}^*\mathfrak{m}_0^0 = -\mathfrak{m}_0^0$, meaning that $\mathfrak{m}_0^0 = 0$. □

Corollary 4.3.13. *For any locally unobstructed local system ∇, the Lagrangian brane $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is unobstructed with bounding cochain zero.* □

This corollary says that locally unobstructed local systems are unobstructed. We will refer to them as such hereafter. Observe that the gappedness of the A_∞-algebra implies that an unobstructed local system is locally unobstructed.

4.3.3 Supports of mirror sheaves

Since we have checked that $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is unobstructed for suitable choices of ∇, we can take these to be objects of the relative Fukaya category $\mathcal{F}(X^5_\tau, D_\tau)$ by Lemma 4.2.15, so by Assumption 4.9.1, we can consider the image of $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ under the mirror functor of [15], which we denote by $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}}, \nabla)} \in D^b_{dg}\,\mathrm{Coh}(X^{5,\vee})$. The support of this complex of coherent sheaves can be determined by calculating the Floer cohomology of $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ with objects of $\mathcal{F}(X^5_\tau, D_\tau)$ supported on the Lagrangian torus $L_{\tau,0}$. By Proposition 4.1.5, equipping $L_{\tau,0}$ with any $\mathbb{C}$-local system gives a Lagrangian brane mirror to a point. The stalk of $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}}, \nabla)}$ at such a point can be identified with Floer cohomology group

$$HF^0((\widetilde{L}^5_{\mathrm{im}}, \nabla), (L_{\tau,0}, \nabla_p)) \tag{4.3.12}$$

under homological mirror symmetry.

We can reduce to computation of the Floer cohomology groups (4.3.12) in the quintic X^5_τ, to a computation in the very affine quintic $X^5_\tau \setminus D_\tau$ using the open mapping theorem.

Lemma 4.3.14. *Suppose that $u\colon \mathbb{R} \times [0,1] \to X^5_\tau$ is a holomorphic strip which contributes to the Floer differential on $CF^*((L_{\tau,0}, \nabla_p), (\widetilde{L}^5_{\mathrm{im}}, \nabla))$. Then the image of u does not intersect the divisor D_τ.*

Proof. Suppose that $u\colon \mathbb{R} \times [0,1] \to X^5_\tau$ is a holomorphic strip whose image intersects D_τ. This implies that the boundary component of u mapped to $\widetilde{L}^5_{\mathrm{im}}$ must pass through one of the necks of $\widetilde{L}^5_{\mathrm{im}}$, i.e. through a copy of L_Y in one of the charts $\widetilde{B}_{\ell,m}$. As in the proof of Proposition 4.3.8, we can neck stretch along $\partial\widetilde{B}_{\ell,m}$ to produce a holomorphic strip u' with interior punctures. Since all points in $\widetilde{L}^5_{\mathrm{im}} \cap L_{\tau,0}$ lie away from $\widetilde{B}_{\ell,m}$ it follows that the boundary arc of u' must exit the ball. We can rescale u', using the same argument in the proof of Proposition 4.3.8, to produce a holomorphic strip on $\widetilde{u}$ with a boundary component on $\widetilde{L}^5_{\mathrm{im}}$ and a boundary component on $L_{\tau,0}$. This strip has the property that its image intersects $\partial\widetilde{B}_{\ell,m}$ in an arc.

If we consider the restriction $\widetilde{u}'$ of $\widetilde{u}$ to $\widetilde{u}^{-1}(\widetilde{B}_{\ell,m})$, then the composition of this map with $w\colon \widetilde{B}_{\ell,m} \to \mathbb{C}$, which is the projection defined in (4.2.11), is holomorphic. Since $\widetilde{u}$ is not contained in $\widetilde{B}_{\ell,m}$, its holomorphicity, and in turn the holomorphicity of u, contradicts the open mapping principle. $\square$

Exactness of $L_{\tau,0}$ and $\widetilde{L}^5_{\mathrm{im}}$ allows us to control the areas of disks contributing to the Floer differentials on $CF^*((L_{\tau,0}), (\widetilde{L}^5_{\mathrm{im}}, \nabla))$, thereby reducing the calculation of Floer cohomology to [29, Theorem 1.2].

Lemma 4.3.15. *Suppose that $u\colon \mathbb{R}\times[0,1] \to X^5_\tau \setminus D_\tau$ is a holomorphic strip which contributes to the Floer differential on $CF^*((L_{\tau,0}, \nabla_p), (\widetilde{L}^5_{\mathrm{im}}, \nabla))$. Then the image of u is contained in a Weinstein neighborhood* $\mathrm{Wein}(L_{\tau,0})$.

Proof. Since $\widetilde{L}^5_{\mathrm{im}}$ and $L_{\tau,0}$ are both exact in $X^5_\tau \setminus D_\tau$ the area of any strip is determined by its two corners. Thus, rescaling the 1-form used to construct $\widetilde{L}_{\mathrm{sing}}$ if necessary, we can bound the area of any strip contributing to the Floer differential above by an arbitrarily small constant. Consequently, the image of u cannot exit $\mathrm{Wein}(L_{\tau,0})$. $\square$

Hereafter, we will only consider local systems on $L_{\tau,0}$ which are pulled back from local systems on T^3 under the 125-fold cover $L_{\tau,0} \to L_{\tau,0}$ with deck group $(\mathbb{Z}/5)^3$.

Corollary 4.3.16. *There is an isomorphism*

$$HF^*((L_{\tau,0}, \nabla_p), (\widetilde{L}^5_{\mathrm{im}}, \nabla)) \cong HF^*((T^3, \nabla_p), (\widetilde{L}_{\mathrm{im}}, \nabla)) \tag{4.3.13}$$

of $\mathbb{C}$-vector spaces. The Floer group on the left hand side is computed in $X^5_\tau \setminus D_\tau$, whereas the Floer group on the right hand side is computed in T^*T^3, where T^3 is thought of as the 0*-section.*

Proof. By Lemma 4.3.15, any holomorphic strip which contributes to the Floer differential on $CF^*((L_{\tau,0}, \nabla_p), (\widetilde{L}^5_{\mathrm{im}}, \nabla))$ can also naturally be thought of as a disk in T^*T^3 contributing to the Floer differential $CF^*((T^3, \nabla_p), (\widetilde{L}_{\mathrm{im}}, \nabla))$. The local system ∇ on $\widetilde{L}_{\mathrm{im}}$ is induced from the local system ∇ on $\widetilde{L}^5_{\mathrm{im}}$ in the obvious way. $\square$

The main results of [29] as written concern the support of the Lagrangian immersion L_{im}, rather than that of $\widetilde{L}_{\text{im}}$. We will relate the groups $HF^*((T^3,\nabla_p),(L_{\text{im}},\nabla))$ to the groups on the right hand side of (4.3.13) using a covering argument.

By the construction of $\widetilde{L}_{\text{im}}$ and our choices of local systems ∇ on $\widetilde{L}^5_{\text{im}}$ and ∇_p on $L_{\tau,0}$, it follows that $(\mathbb{Z}/5)^3$ acts on the Floer cochain space $CF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))$, and thus on homology $HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))$.

Lemma 4.3.17. *The action of* $(\mathbb{Z}/5)^3$ *on* $HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))$ *is trivial, i.e. the space of invariants can be written as*

$$HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))^{(\mathbb{Z}/5)^3} = HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))\,. \tag{4.3.14}$$

Proof. The action of $(\mathbb{Z}/5)^3$ on Floer cohomology is induced by an action of $(\mathbb{Z}/5)^3$ on T^*T^3 by symplectomorphisms. Specifically, this group acts by rotations in the T^3-direction, which are Hamiltonian isotopies. Hence the action on homology is trivial. □

On the other hand, we have that the group of $(\mathbb{Z}/5)^3$-invariants can be written as

$$HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))^{(\mathbb{Z}/5)^3} = H^*(CF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))^{(\mathbb{Z}/5)^3},\mathfrak{m}_1)\,,$$

the cohomology of the subcomplex of $(\mathbb{Z}/5)^3$-invariant chains. The following is immediate from the $(\mathbb{Z}/5)^3$-equivariance of the chain complex.

Lemma 4.3.18. *The cohomology group* $H^*(CF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla))^{(\mathbb{Z}/5)^3},\mathfrak{m}_1)$ *can be identified with* $HF^*((T^3,\overline{\nabla}_p),(L_{\text{im}},\overline{\nabla}))$*, where* $\overline{\nabla}_p$ *and* $\overline{\nabla}$ *denote the local systems on* T^3 *and* L_{im} *from which the local systems* ∇_p *and* ∇ *are induced. Consequently*

$$HF^*((T^3,\nabla_p),(\widetilde{L}_{\text{im}},\nabla)) \cong HF^*((T^3,\overline{\nabla}_p),(L_{\text{im}},\overline{\nabla}))\,.$$

□

Remark 4.3.19. One could also compute the Floer-theoretic support of $\widetilde{L}_{\mathrm{im}}$ in T^*T^3 directly by considering its geometric compositions with the Lagrangian correspondences discussed in Remark 4.2.18. Instead of reducing the computation of $HF^*((T^3, \nabla_p), (\widetilde{L}_{\mathrm{im}}, \nabla))$ to a computation of the Floer-theoretic support (cf. [29]) of a tropical pair of pants, one would instead need to consider the Floer-theoretic support of the Lagrangian lift of a non-smooth, trivalent tropical curve quintic plane curve. In this case, one can see directly that the nonvanishing Floer cohomology groups have rank 2, in accordance with the results of Lemma 4.3.18 and [29, Corollary 6.1]. We chose to describe a more abstract strategy of proof because we will also make use of it when we discuss the self Floer cohomology of $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$.

It follows from Lemma 4.3.18 and [29, Theorem 1.2] that the mirror sheaf to $(\widetilde{L}_{\mathrm{im}}, \nabla)$ in $(\mathbb{C}^*)^3/(\mathbb{Z}/5)^3$ is supported on a rational curve with four punctures. The analogous result for the mirror sheaf to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ follows almost immediately.

Proposition 4.3.20. *The mirror sheaf to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is supported on a line, i.e. it is the pushforward of a sheaf under an embedding of $\mathbb{P}^1$ in the mirror quintic. The stalks of this sheaf have rank 2 in a Zariski open subset of $\mathbb{P}^1$. The line supporting the mirror to $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega})$, as defined in Example 4.3.8, is a van Geemen line.*

Proof. It is immediate from our discussion above that the sheaf on $X^{5,\vee}$ mirror to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is supported on a Zariski open subset of a line. That the stalks are generically of rank 2 follows from [29, Corollary 6.1]. Since the mirror is a complex of *coherent* sheaves, the ranks of its stalks cannot decrease on a Zariski closed subset, and thus its support must be an entire embedded $\mathbb{P}^1$. We have already checked that $\nabla^{\mathrm{vG}}_{\omega}$ is unobstructed. Recall that the mirror functor of [15] is obtained as a versal deformation of a fully faithful A_∞ embedding

$$\mathrm{Perf}(X^{5,\vee}_0) \to D^\pi(\mathcal{F}(X^5_\tau)) .$$

We can thus determine the support of the mirror sheaf to $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega})$ by determining the restriction of the support to the central fiber. The lemma now follows from the explicit description of the support of the mirror from [29, §6.3] and Example 4.3.8. □

4.3.4 Local Floer cohomology and the second Chern class

To prove Theorem 1.0.1, we need a stronger version of Proposition 4.3.20. More specifically, if we can show that the mirror sheaf to $(\widetilde{L}^5_{\text{im}}, \nabla^{\text{vG}}_{\omega})$ is the *pushforward of a vector bundle* on $\mathbb{P}^1$, we will be able to conclude that its algebraic second Chern class is an integer multiple of the support, which is critical for understanding the extensions of Hodge structure associated to this object. Let $\mathcal{L}_{(\widetilde{L}^5_{\text{im}},\nabla)}$ denote the mirror object to the brane $(\widetilde{L}^5_{\text{im}}, \nabla)$, where ∇ is unobstructed. We have shown in Proposition 4.3.20 that this object is supported on a line in $X^{5,\vee}$, and we let $i\colon \mathbb{P}^1 \to X^{5,\vee}$ denote the inclusion of the support. Then we have an isomorphism

$$\mathcal{L}_{(\widetilde{L}^5_{\text{im}},\nabla)} \cong i_*(i^{-1}\mathcal{L}_{(\widetilde{L}^5_{\text{im}},\nabla)})\,. \tag{4.3.15}$$

Since $i^{-1}\mathcal{L}_{(\widetilde{L}^5_{\text{im}},\nabla)}$ is a complex of *coherent* sheaves on $\mathbb{P}^1$, it can be written as a direct sum of line bundles and skyscraper sheaves. The results of [29, §6.3] give us a lower bound on the rank of the stalks this sheaf.

We can rule out the presence of skyscraper summands in this sheaf by establishing an upper bound on the rank of $HF^*(\widetilde{L}^5_{\text{im}}, \nabla^{\text{vG}}_{\omega})$. Luckily, it is possible to achieve this while only computing the differentials on the first page of the energy spectral sequence of Proposition 4.8.12. This computation, combined with the mirror symmetry considerations above, then gives us enough information to completely determine the Floer cohomology of $(\widetilde{L}^5_{\text{im}}, \nabla)$, and the precise object to which it is mirror.

Theorem 4.3.21. *The Floer cohomology of* $(\widetilde{L}^5_{\text{im}}, \nabla)$*, where* ∇ *is an unobstructed local system, is* *the graded Λ_0-module given by*

$$HF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla) \cong \begin{cases} \Lambda_0 & * = -1, 4 \\ \Lambda_0^3 & * = 0, 3 \\ \Lambda_0^4 & * = 1, 2 \\ 0 & \textit{otherwise.} \end{cases} \tag{4.3.16}$$

Moreover $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ *is mirror to a direct sum of two copies of the same line bundle on a line in the mirror quintic, whose gradings differ by* 1.

By construction, the differentials on the E_2-page of the energy spectral sequence of Proposition 4.8.12 are determined by the terms of the Floer differential counting (nonconstant) disks of the lowest energy bounded by $\widetilde{L}^5_{\mathrm{im}}$. The Weinstein neighborhood $\widetilde{W}^5$ contains all of these low-energy disks, allowing us to rephrase the computation of the E_2-differential as a computation of local Floer homology. More precisely, the local Floer homology $HF^*_{\widetilde{W}^5}(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ calculated inside $\widetilde{W}^5$ completely determines the E_3-page of the energy spectral sequence.

Lemma 4.3.22. *There is a constant* $\epsilon_0 > 0$ *such that every holomorphic disk* u *with corners and boundary on* $\widetilde{L}^5_{\mathrm{im}}$ *for which* $\omega([u]) \leq \epsilon_0$ *is contained in* $\widetilde{W}^5$*. In particular the only such holomorphic disks which contribute to the Floer differential are either*

(i) holomorphic teardrops as in Lemma 4.3.1 with an additional smooth boundary marked point; or

(ii) holomorphic strips with two corners on different switching components (at least one of which corresponds to a Morse critical point of $\widetilde{h}$*).*

Proof. Using the same arguments as in the proof of Lemma 4.2.15, one can show that $\widetilde{L}^5_{\mathrm{im}}$ is exact for some choice of primitive on $\widetilde{W}^5$. An easy consequence of this is that the areas of these disks are controlled by the choice of Morse function $\widetilde{h}\colon \widetilde{L}' \to \mathbb{R}$ and the area of the disk bounded by L_{arc}. These can both be made arbitrarily small by scaling $\widetilde{h}$ appropriately.

The classification of strips on $\widetilde{L}^5_{\text{im}}$ follows immediately from this, where in particular the strips in (ii) correspond to gradient flowlines of $\pm\widetilde{h}'$. Notice that the 2-fold covers of the teardrops on $\widetilde{L}^5_{\text{im}}$ do not contribute to the Floer differential for degree reasons. □

Since $\widetilde{L}^5_{\text{im}}$ is built using 125-fold covers $\widetilde{L}'$ of the minimally-twisted five-component chain link complement, its de Rham cohomology $H^*(\widetilde{L}^5_{\text{im}};\Lambda_0)$ will have high rank as a free Λ_0-module. This makes directly computing even the E_2-differentials in the energy spectral sequence difficult. To remedy this, we will using a covering argument along the lines of Lemma 4.3.17, which allows us to compute the low energy terms of the Floer differential for the quotient Lagrangian L^5_{im} instead.

Lemma 4.3.23. *The group $(\mathbb{Z}/5)^3$ acts trivially on $HF^*(\widetilde{L}^5_{\text{im}},\nabla)$. Thus we have isomorphisms*

$$HF^*_{X^5_\tau}(\widetilde{L}^5_{\text{im}},\nabla) = HF^*_{X^5_\tau}(\widetilde{L}^5_{\text{im}},\nabla)^{(\mathbb{Z}/5)^3} = H^*(CF^*_{X^5_\tau}(\widetilde{L}^5_{\text{im}},\nabla)^{(\mathbb{Z}/5)^3},\mathfrak{m}_{\mathfrak{G}}) \tag{4.3.17}$$

where the subscripts indicate that all Floer chain complexes and cohomology groups are taken in the Weinstein neighborhood $\widetilde{W}^5$, and

$$H^*(CF^*_{\widetilde{W}^5}(\widetilde{L}^5_{\text{im}},\nabla)^{(\mathbb{Z}/5)^3},\mathfrak{m}_{\mathfrak{G}}) \cong HF^*_{W^5}(L^5_{\text{im}},\overline{\nabla}) \tag{4.3.18}$$

where the group on the right is the Floer cohomology of $(L^5_{\text{im}},\overline{\nabla})$ in the quotient W^5.

Proof. The proof mostly uses the arguments of the previous subsection. Recall that the action of $(\mathbb{Z}/5)^3$ is inherited from an action of $(\mathbb{Z}/5)^3$ on $\mathbb{CP}^4$. The very affine quintic $X^5_\tau\setminus D_\tau$ is contained in the big torus $(\mathbb{C}^*)^4\subset\mathbb{CP}^4$. As before, we have that $(\mathbb{Z}/5)^3$ acts on $(\mathbb{C}^*)^4$ by Hamiltonian isotopies, and it follows that it acts on $X^5_\tau\setminus D_\tau$, and hence on X^5_τ by Hamiltonian isotopies as well. This proves the nontrivial equality (4.3.17).

The isomorphism of (4.3.18) follows from Lemma 4.3.22 and a completely analogous classification of holomorphic strips in W^5 with boundary on L^5_{im}. In particular, $(\mathbb{Z}/5)^3$ still acts on the Floer cochain space of $CF^*(\widetilde{L}^5_{\text{im}},\nabla)$, and the subcomplex of invariant Floer cochains is naturally identified with the Floer complex $CF^*(L^5_{\text{im}},\overline{\nabla})$. □

The lemma above reduces the problem of computing $HF^*_{\widetilde{W}^5}(\widetilde{L}^5_{\mathrm{im}}, \nabla)$, to the problem of computing the (local) Floer cohomology group $HF^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})$ in W^5.

Proposition 4.3.24. *The Floer cohomology groups $HF^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})$ coincide with those given in* (4.3.16).

We will break the proof of this proposition into several smaller lemmas. As a $\mathbb{Z}$-graded Λ_0-module, the Floer cochain complex of $(L^5_{\mathrm{im}}, \overline{\nabla})$ in W^5 is the completed tensor product

$$CF^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla}) = \overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})\widehat{\otimes}_{\mathbb{C}}\Lambda_0$$

where $\overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})$ is the $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$\Omega^*(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^{4}(\Omega^*(T^2)[1] \oplus \Omega^*(T^2)[-2]) \oplus CM^*(L')[-1] \oplus CM^*(L', \partial L')[1] \,.$$

This implies that the E_2-page of the energy spectral sequence is determined by the $\mathbb{C}$-vector space

$$\begin{aligned} &H^\bullet(\overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})) \\ &:= H^*(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^{4}(H^*(T^2)[1] \oplus H^*(T^2)[-2]) \oplus CM^*(L')[-1] \oplus CM^*(L', \partial L')[1] \end{aligned}$$

obtained by taking de Rham cohomology. The terms of the E_2-page are given by

$$E_2^{p,q} := H^p(\overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})) \otimes (Q^{q\epsilon_0}\Lambda_0/Q^{(q+1)\epsilon_0}\Lambda_0) \tag{4.3.19}$$

where ϵ_0 denotes the constant of Lemma 4.3.22. The differential $E_2^{p,q} \to E_2^{p+1,q+1}$ is determined from the Floer differential on $CF^*(L^5_{\mathrm{im}})$ by setting

$$\delta_2^{p,q}[x] = [\mathfrak{m}_1(x)] \in E_2^{p+1,q+1}$$

for any $[x] \in E_2^{p,q}$. Notice that in W^5, the energy spectral sequence collapses at the E_2-page by exactness, so computing the differentials on this page amounts to computing the full Floer cohomology groups.

The differential $\delta_2^{p,q}$ is determined by $\mathfrak{m}_1^p$, the degree p part of the differential

$$\mathfrak{m}_1^p \colon CF^p(L^5_{\mathrm{im}}) \to CF^{p+1}(L^5_{\mathrm{im}})\,.$$

Again by the exactness of L^5_{im} in W^5, we can think of this as a $\mathbb{C}$-linear map. Consequently, the E_2-differentials in the spectral sequence are determined by $\mathbb{C}$-linear maps

$$\delta^p \colon H^p(\overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})) \to H^{p+1}(\overline{CF}^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla})) \tag{4.3.20}$$

induced from the Floer differential $\mathfrak{m}_1^p$ as in (4.3.19). More specifically, δ^p can be written as a sum of linear maps

$$\delta^p = \sum_{\beta \in H_2(W^5, L^5_{\mathrm{im}})} \delta^p_\beta$$

where δ^p_β is defined as in (4.3.19) using $\mathfrak{m}^p_{1;\beta}$. We then have that $\delta_2^{p,q}$ is represented by

$$\delta_2^{p,q} = \sum_{\beta \in H_2(W^5, L^5_{\mathrm{im}})} \delta^p_\beta \otimes Q^{\omega(\beta)}\, \mathrm{id}_{\Lambda_0}$$

on $E_2^{p,q}$ for all q. In the following three lemmas, we will compute the maps δ^p for $p = 3, 2, 1$.

Lemma 4.3.25. *The map*

$$\delta^3 \colon H^3(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^{4} H^1(T^2)[-2] \oplus CM^2(L')[-1] \to \bigoplus_{\ell=0}^{4} H^2(T^2)[-2] \tag{4.3.21}$$

induced by $\mathfrak{m}_1^3$ *has rank* 4.

Proof. This component of the E_2-differential is induced, at the chain level, by the Floer differential

$$\mathfrak{m}_1^3 \colon \Omega^3(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^{4} \Omega^1(T^2)[-2] \oplus CM^2(L')[1] \to \bigoplus_{\ell=0}^{4} \Omega^2(T^2)[-2] \, .$$

The only holomorphic strips that contribute to this part of the Floer correspond to gradient trajectories of h starting at index 2 critical points and approaching the cusps of L'. This implies that the only nontrivial component of the Floer differential is

$$CM^2(L')[-1] \to \bigoplus_{\ell=0}^{4} \Omega^2(T^2)[-2] \, .$$

The critical points of h in degree 2 correspond to edges of the cube in Figure 2.9. There are two gradient trajectories which emanate from each of these critical points: one of these approaches the 0th cusp, and the other approaches the ith cusp, where $i \in \{1, 2, 3, 4\}$. From this, the image of $\mathfrak{m}_1^3$ is easily seen to be a free module of rank 4, from which the statement of the lemma follows. □

Calculating other components of the E_2-differential requires the determination of some nontrivial values of the Floer differential. Let $\beta \in H_2(W^5, L^5_{\mathrm{im}}; \mathbb{Z})$ denote a relative second homology class represented by one of the disks of Lemma 4.3.22, and consider the (Gromov compactified) moduli spaces

$$\mathcal{M}_2(\beta) \tag{4.3.22}$$

of such disks. By the argument in the proof of Lemma 2.10 of [33], all moduli spaces (4.3.22) are transversely cut out. These moduli spaces are either 3-dimensional, in the case of Lemma 4.3.22(i), or 0-dimensional, in the case of Lemma 4.3.22(ii). When these moduli spaces are 3-dimensional they can be identified with the product of a 2-torus with a closed interval.

Recall from Appendix 4.8 that the Floer differential is defined by pulling back differential forms on $CF^*(L^5_{\mathrm{im}})$ under the evaluation map at one of the boundary marked points, and pushing forward the resulting differential form under the evaluation map at the other marked point. The boundary evaluation maps are submersions, which we see because all switching components are either 0-dimensional, or they are contained in neighborhoods in which L^5_{im} can be written as the product of a 1-dimensional Lagrangian with T^2, which is a Lie group, so that submersivity follows by the argument of [80, Example 1.5].

Also recall that L^5_{im} is obtained by gluing two smooth manifolds with boundary diffeomorphic to the disjoint union of five copies of T^2. Below, we will call the images of these boundary tori in L^5_{im} the *splitting tori*. With the above understood, we can now compute the remaining terms of the E_2-differential.

Lemma 4.3.26. *The $\mathbb{C}$-linear map δ^2 induced by $\mathfrak{m}^2_1$ has domain and codomain*

$$\begin{array}{c} H^2(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^4 H^0(T^2)[-2] \oplus CM^3(L', \partial L')[1] \oplus CM^1(L')[-1] \\ \downarrow \\ H^3(L^5_{\mathrm{im}}) \oplus \bigoplus_{\ell=0}^4 H^1(T^2)[-2] \oplus CM^2(L')[-1] \end{array} \tag{4.3.23}$$

and has rank 8.

Proof. At the chain-level, the E_2-differentials are induced by the Floer differential $\mathfrak{m}^2_1$ in degree 2. The only strips whose output is a class in $\Omega^3(\widetilde{L}^5_{\mathrm{im}})$ come from teardrops as in Lemma 4.3.22(i). Since these teardrops are weighted by holonomies and signs whose sum vanishes, it follows that their total contribution to the differential vanishes. Therefore $\Omega^3(\widetilde{L}^5_{\mathrm{im}})$ does not lie in the image of the Floer differential. Similarly, the component of the Floer differential mapping out of $CM^3(L', \partial L')[1]$ is trivial.

Fix a basis for $H_1(L^5_{\mathrm{im}})$ consisting of the classes $s_1, \dots, s_4 \in H_1(L^5_{\mathrm{im}})$ as in (4.3.1), together with classes $\ell_0, \dots, \ell_4$ representing the longitudes in L'. Using the de Rham isomorphism and Poincaré duality, we can identify these with generators for the de Rham cohomology $H^2(\widetilde{L}^5_{\mathrm{im}})$. Fix differential forms in $\Omega^2(\widetilde{L}^5_{\mathrm{im}})$ representing these classes. The images of these classes under $\mathfrak{m}^2_1$ are all closed 1-forms in $\bigoplus_{\ell=0}^4 \Omega^1(T^2)[-2]$ by general properties of integration along the fiber. We their images in $\bigoplus_{\ell=0}^4 H^1(T^2)[-2]$ span a subspace of dimension 5, since all of the forms dual to the classes $s_1, \dots, s_4$ which are not generated by the longitudes in L' map to zero.

The Floer cocycles in $CM^1(L')[-1]$ correspond to faces of the cube in Figure 2.9. The only holomorphic strips contributing to the part of $\mathfrak{m}_1^2$ that maps out of this summand are as in Lemma 4.3.22(ii), and correspond to gradient trajectories of h starting at index 1 critical points and approaching the cusps of L'. There are two such trajectories starting at any such critical point and emanating towards the zeroth cusp, as can be seen from Figure 2.9. The corresponding holomorphic strips contribute with the opposite signs, so their total contribution to the Floer differential is trivial. There are two more gradient trajectories starting at any such critical point, but they will approach different cusps of L'.

In total, the images of the classes considered above span an 8-dimensional subspace of the codomain of (4.3.23). To see this, notice the the images of generators in $CM^1(L')[-1]$, together with the images of the forms corresponding to $\ell_1, \dots, \ell_4$, span all of $\bigoplus_{\ell=1}^4 H^1(T^2)[1]$. It then follows from Lemma 2.3.17 that the image of the form corresponding to ℓ_0 lies in the span of these classes. This implies that $\mathfrak{m}_1^2$ induces a map of rank 8 on de Rham cohomology. □

The last E_2-differentials that we will need to compute come from the Floer differential in degree 1.

Lemma 4.3.27. *The $\mathbb{C}$-linear map δ^1 induced by $\mathfrak{m}_1^1$ has domain and codomain*

$$\begin{array}{c} H^1(L^5_{\text{im}}) \oplus \bigoplus_{\ell=0}^4 H^2(T^2)[1] \oplus CM^0(L')[-1] \oplus CM^2(L',\partial L')[1] \\ \big\downarrow \\ H^2(L^5_{\text{im}}) \oplus \bigoplus_{\ell=0}^4 H^0(T^2)[2] \oplus CM^1(L')[-1] \to CM^3(L',\partial L')[1] \end{array}$$

and has rank 10.

Proof. First observe that the Floer differential is nontrivial on the components

$$CM^2(L',\partial L')[1] \to CM^3(L',\partial L')[1]$$

$$CM^0(L')[-1] \to CM^1(L')[-1]$$

since there are strips as in Lemma 4.3.22(ii) which correspond to gradient flow trajectories connecting the two index 0 points to the index 1 critical points. Both of the above components of the Floer differential have rank 1 (cf. Figure 2.9).

By Lemma 4.3.22, the only other possibly nontrivial components of $\mathfrak{m}_1^1$ are

$$\bigoplus_{\ell=0}^{4} \Omega^2(T^2)[1] \to CM^3(L', \partial L')[1] \tag{4.3.24}$$

$$\bigoplus_{\ell=0}^{4} CM^0(L')[-1] \to \bigoplus_{\ell=0}^{4} \Omega^0(T^2)[-2] \tag{4.3.25}$$

which count holomorphic strips described in Lemma 4.3.22(ii), or

$$\Omega^1(L^5_{\text{im}}) \to \bigoplus_{\ell=0}^{4} \Omega^0(T^2)[-2] \tag{4.3.26}$$

$$\bigoplus_{\ell=0}^{4} \Omega^2(T^2)[1] \to \Omega^2(L^5_{\text{im}}) \tag{4.3.27}$$

which count holomorphic disks as in Lemma 4.3.22(i).

Observe that (4.3.25) vanishes, because for any holomorphic strip with a corner at one of the two generators of $CM^0 * (L')[-1]$ and another corner on a T^2-switching component, there is a corresponding holomorphic strip with a corner at the other generator of $CM^0 * (L')[-1]$, and these two strips are counted with opposite signs. Since (4.3.24) counts precisely the same strips, it also vanishes. Consequently, these components of the Floer differential induce trivial maps on the E_2-page of the spectral sequence.

We will compute the terms of the E_2-differential corresponding to (4.3.26). To that end, choose closed 1-forms in $\Omega^1(L^5_{\text{im}})$ whose cohomology classes correspond to the generators in (4.3.1) under the de Rham isomorphism. The only disks which contribute to (4.3.26) come from teardrops as in Lemma 4.3.22(i). Let $\beta_\ell \in H_2(W^5, L^5_{\text{im}})$ denote the relative homology class represented by such a disk. The moduli space of strips in this class, where one marked point is the corner of the teardrop, and the other is at a smooth point on the boundary, is denoted $\mathcal{M}_2(\beta_\ell)$, as in (4.3.22). Let $\text{evb}_0^{\beta_\ell}$

denote the evaluation map at the corner, and $\mathrm{evb}_1^{\beta_\ell}$ denote the evaluation map at the smooth marked point. If δ_{s_i} denotes the differential form dual to class s_i, then the form

$$(\mathrm{evb}_0^{\beta_\ell})_*(\mathrm{evb}_1^{\beta_\ell})^*\sigma_i \in \Omega^0(T^2)[-2] \tag{4.3.28}$$

is a volume form on the ℓth splitting torus. The images of the classes δ_{s_i} under $\mathfrak{m}_1^1$ form a 4-dimensional subspace, spanned by sums of volume forms. Note that for appropriately chosen classes s_i, the moduli space $\mathcal{M}_2(\beta_\ell)$ will contribute nontrivially to the Floer differential of δ_{s_i} if and only if β_ℓ is represented by the product of a teardrop on L_{arc} with a constant disk. In particular, this value of the Floer differential does not vanish. On the other hand, if δ_{m_i} is a differential form dual m_i, for $i = 0, \ldots, 4$, then its image under the Floer differential vanishes. Hence the component (4.3.26) descends to a map of rank 4 on de Rham cohomology. The map on homology induced by (4.3.27) is the dual of this map, and so it also has rank 4. In total, this shows that the map in the statement of the lemma has rank 10. □

Proof of Proposition 4.3.24. The results of the previous three lemmas, combined with the collapse of the energy spectral sequence computed in W^5 at the E_2-page, show that

$$HF^*_{W^5}(L^5_{\mathrm{im}}, \overline{\nabla}) \cong \begin{cases} \Lambda_0 & * = 4 \\ \Lambda_0^3 & * = 3 \\ \Lambda_0^4 & * = 2\,. \end{cases} \tag{4.3.29}$$

The values of the Floer cohomology groups for $* = -1, 0, 2$ follow from this by Poincaré duality. □

Proof of Theorem 4.3.21. Observe that by Lemma 4.3.23, the result of Proposition 4.3.24 gives an upper bound on the ranks of $HF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla)$. On the other hand, we have already seen in Proposition 4.3.20 that the mirror to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is a sheaf whose stalks all have rank at least 2. Since each of these supports is a line contained in a 1-dimensional family in a Calabi–Yau threefold, it follows that the rank cannot be greater than 2 at any point in the support, or else the total rank of $HF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ would be higher than the rank of $HF^*_{\widetilde{W}^5}(\widetilde{L}^5_{\mathrm{im}}, \nabla)$. As before, let $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}$ denote the mirror object to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$, and let $i\colon \mathbb{P}^1 \to X^{5,\vee}$ denote the inclusion map of its support. We know, by (4.3.15), that $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}$ is the pushforward of the sheaf $i^{-1}\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}$ on $\mathbb{P}^1$. In particular, it is the pushforward of a vector bundle of rank 2.

The grading of the Λ_0-module $HF^*_{\widetilde{W}^5}(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ computed in Proposition 4.3.24, together with the fact that any vector bundle on $\mathbb{P}^1$ splits, implies that the mirror object to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ is a direct sum of two rank 1 vector bundles on a line, both of which have the same degree, but with gradings that differ by 1. □

From the result of Theorem 4.3.21 and the Grothendieck–Riemann–Roch theorem, we can compute the algebraic second Chern classes of the sheaves mirror to $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$.

Corollary 4.3.28. *The mirror sheaves $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}$ to the objects $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ in $\mathcal{F}(X^5_\tau)$ have algebraic second Chern classes given by*

$$c_2(\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}) = -2[C_\nabla]$$

where C_∇ is the support of $\mathcal{L}_{(\widetilde{L}^5_{\mathrm{im}},\nabla)}$.

Proof. See [84, p. 29]. □

4.3.5 Lagrangian surgery and direct summands

Theorem 4.3.21 and the classification of coherent sheaves on $\mathbb{P}^1$ shows that any object of the form $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$, where ∇ is a local system satisfying (4.3.3)–(4.3.7), splits as a direct sum in the split-closed derived Fukaya category. In this subsection, we will show that the summands of such objects can be identified with Lagrangian branes supported on $\widetilde{L}^5_{\mathrm{sm}}$, which will be enough to apply Theorem 1.3.7 to calculate their open Gromov–Witten potentials. This is achieved by showing that two copies of $\widetilde{L}^5_{\mathrm{sm}}$ can be obtained from a copy of $\widetilde{L}^5_{\mathrm{im}}$ under Lagrangian isotopy and a clean (anti-)surgery.

Let $\widetilde{L}^5_{\mathrm{sm}} := \widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)$ denote one of the Lagrangian submanifolds constructed in Theorem 4.2.21, and let $\widetilde{\delta}$ the closed 1-form on $\widetilde{L}'$ used to construct it. By taking ϵ sufficiently small, we can assume that $\widetilde{\delta}$ is arbitrarily C^1-small. Observe that $\widetilde{\delta}$ extends to a 1-form on the domain of $\widetilde{L}^5_{\mathrm{im}}$, since we already assumed that it can be expressed as the product of a 1-form on T^2 with a constant function on the real line near the cusps of $\widetilde{L}'$. Let $\widetilde{\delta}_{\mathrm{im}} \in \Omega^1(\widetilde{L}^5_{\mathrm{im}})$ denote the 1-form obtained by patching two copies of $\widetilde{\delta}$ defined on the sheets of $\widetilde{L}^5_{\mathrm{im}}$.

Lemma 4.3.29. *Let $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ denote an object of $\mathcal{F}(X^5_\tau)$, where ∇ is an unobstructed local system. Deforming $\widetilde{L}^5_{\mathrm{im}}$ in a Weinstein neighborhood (contained in $\widetilde{W}^5$) through the graphs $\widetilde{L}^5_{\mathrm{im}}(t) := \Gamma(t\widetilde{\delta}_{\mathrm{im}})$ of $t\widetilde{\delta}$, for $t \in [0,1]$ and equipping $\widetilde{L}^5_{\mathrm{im}}(1)$ with a suitable rank one Λ-local system $\nabla^{\mathrm{im}}_\Lambda$ yields a Lagrangian brane $(\widetilde{L}^5_{\mathrm{im}}(1), \nabla^{\mathrm{im}}_\Lambda)$ for which we have a quasi-isomorphism*

$$CF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla) \simeq CF^*(\widetilde{L}^5_{\mathrm{im}}(1), \nabla^{\mathrm{im}}_\Lambda)\,. \tag{4.3.30}$$

Remark 4.3.30. Before proceeding with the proof, we will describe the effect of this Lagrangian isotopy in the Darboux charts $\widetilde{B}_{\ell,m}$. Recall that these charts are identified with open neighborhoods of the origin in Y of (4.2.10), and that in these charts $\widetilde{L}^5_{\mathrm{im}}$ coincides with the Lagrangian submanifold L_Y of Definition 4.2.9. In these charts, isotoping the intersection of $\widetilde{L}^5_{\mathrm{im}}$ through the graphs $\Gamma(t\widetilde{\delta}_{\mathrm{im}})$ determines a Lagrangian isotopy $L_Y(t)$ of Lagrangian submanifolds of Y. Each $L_Y(t)$ still projects to L_{arc}, as drawn in Figure 2.10, under the projection $Y \to \mathbb{C}$ of (4.2.11).

Proof of Lemma 4.3.29. Observe that $\widetilde{L}^5_{\mathrm{im}}$ is isotoped to $\widetilde{L}^5_{\mathrm{im}}(1)$ in a way that determines a bijective correspondence between holomorphic disks bounded by the two immersed Lagrangian submanifolds. More precisely, for any class $\beta \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}})$, there is a corresponding class $\beta_1 \in H_2(X^5_\tau, \widetilde{L}^5_{\mathrm{im}}(1))$ and bijections between the moduli spaces of holomorphic disks representing these homology classes. The contribution of any such disk to the Fukaya A_∞-algebra of $\widetilde{L}^5_{\mathrm{im}}(1)$ differs from its contribution to the A_∞-algebra of $\widetilde{L}^5_{\mathrm{im}}$ by a factor of $Q^{-\int_{\partial\beta}\widetilde{\delta}_{\mathrm{im}}}$. This factor and ∇ then determine the holonomy representation of the local system $\nabla^{\mathrm{im}}_\Lambda$ on $\widetilde{L}^5_{\mathrm{im}}(1)$. □

We remark that the local system $\nabla^{\mathrm{im}}_\Lambda$ is non-unitary over the Novikov field, but it is clear from

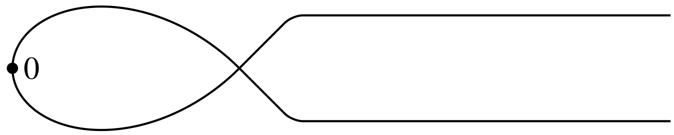

Figure 4.3: The images of $L_Y(i;\epsilon)_\pm$ in $\mathbb{C}$. These Lagrangian solid tori intersect each other cleanly in a circle, lying in the fiber over 0, and a 2-torus, lying in the fiber over the other intersection point.

its construction, and in particular (4.3.30), that all sums involved in the definition of the A_∞-algebra converge.

A key observation is that $\widetilde{L}^5_{\mathrm{im}}(1) \cap \widetilde{B}_{\ell,m} = L_Y(1)$ admits a description as a clean Lagrangian surgery on two smoothings of the Harvey–Lawson cone. More precisely, recall that δ restricts to a 1-form on the complement of the cone point $C_{HL} \setminus \{0\}$ in the Harvey–Lawson cone. This consequently determines a smooth Lagrangian solid torus $L_Y(i;\epsilon)$ in Y given by one of (4.2.16). The exact 1-forms $\pm d\widetilde{h}$ used to construct $\widetilde{L}^5_{\mathrm{im}}$ restrict to exact 1-forms on $C_{HL} \setminus \{0\}$, and we can use these to construct a pair of Lagrangian solid tori $L_Y(i;\epsilon)_\pm$ intersecting cleanly in a circle and a 2-torus. These two solid tori project to arcs in $\mathbb{C}$ under (4.2.11) as shown in Figure 4.3. The S^1 self-intersection component lies above the origin, and the T^2-component lies above the other intersection point in $\mathbb{C}$. The latter component corresponds to the switching component of $L_Y(1)$. The following is immediate from this local description.

Lemma 4.3.31. *By performing a clean Lagrangian surgery along the S^1-component of*

$$L_Y(i;\epsilon)_- \cap L_Y(i;\epsilon)_+$$

we obtain an immersed Lagrangian submanifold of Y Hamiltonian isotopic to $L_Y(1)$. □

Similarly, by using the exact 1-form $\pm d\widetilde{h}$ to perturb the graph $\Gamma(t\delta)$ used to construct $\widetilde{L}^5_{\mathrm{sm}}$, we obtain two embedded Lagrangian submanifolds $\widetilde{L}^5_{\mathrm{sm},\pm}$ of X^5_τ which intersect each other cleanly. Both of these are Hamiltonian isotopic to $\widetilde{L}^5_{\mathrm{sm}}$, and the switching components of the self-intersection correspond either to

- switching components of $\widetilde{L}^5_{\mathrm{im}}$; or

- switching components of $L_Y(i;\epsilon)_- \cap L_Y(i;\epsilon)_+$ diffeomorphic to S^1.

This follows from the fact that, by construction, the union $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ agrees with (a copy of) $L_Y(1)_- \cup L_Y(1)_+$ inside the charts $\widetilde{B}_{\ell,m}$ and with $\widetilde{L}^5_{\mathrm{im}}(1)$ in the complement of these charts.

Definition 4.3.32. We give $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ the structure of a Lagrangian brane as follows.

- Both of $\widetilde{L}^5_{\mathrm{sm},\pm}$ are equipped with the local system ∇_Λ and spin structure inherited from $\widetilde{L}'$ by Dehn filling;
- the gradings on $\widetilde{L}^5_{\mathrm{sm},\pm}$ by a shift of 1; and
- the immersion $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ is equipped with bounding cochain 0.

In particular, the bounding cochain vanishes on the S^1-switching components along which surgery is performed.

The main result of this subsection is a verification that the A_∞-algebra of this Lagrangian brane structure on $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ is quasi-isomorphic to the A_∞-algebra of the corresponding object supported on $\widetilde{L}^5_{\mathrm{im}}(1)$. Since the former A_∞-algebra splits as a direct sum, this realizes the direct sum decomposition in the derived Fukaya category implicit in Theorem 4.3.21. This is inspired by the arguments of [85], in that we use SFT neck stretching to compare holomorphic disks on $\widetilde{L}^5_{\mathrm{im}}(1)$ with holomorphic disks on $\widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)_- \cup \widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)_+$. The results of [85] do not apply directly in this setting, but a very simplified version of the techniques use in op. cit. will suffice for our purposes. To be more precise, we can appeal the form of the switching loci of $\widetilde{L}^5_{\mathrm{im}}(1)$ and $\widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)_- \cup \widetilde{L}^5_{\mathrm{sm}}(i;\epsilon)_+$ to entirely rule out the existence of disks with corners on the S^1 self-intersection component of the latter Lagrangian immersion. This implies that we do not need to equip $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ with a bounding cochain with nontrivial support on the S^1-switching components.

Lemma 4.3.33. *The Lagrangian brane on $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$ specified in Definition 4.3.32 is quasi-isomorphic to the corresponding object $(\widetilde{L}^5_{\mathrm{im}}(1), \nabla^{\mathrm{im}}_\Lambda)$ in the split-closed derived Fukaya category.*

Proof. Let $\mathcal{A}^5$ denote the union of split-generators for $\mathcal{F}(X^5_\tau)$ constructed in [15], where we have identified X^5_τ with X^5_∞. It suffices to show that there is a quasi-isomorphism

$$CF^*(\mathcal{A}^5, (\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}, \nabla_\Lambda)) \simeq CF^*(\mathcal{A}^5, (\widetilde{L}^5_{\mathrm{im}}(1), \nabla^{\mathrm{im}}_\Lambda)$$

of A_∞-modules over $CF^*(\mathcal{A}^5)$.

To that end, let $u \colon \Sigma \longrightarrow X^5_\tau$ be a holomorphic map from a boundary punctured Riemann surface which contributes to $CF^*(\mathcal{L}^5, (\widetilde{L}^5_{\mathrm{im}}(1), \nabla^{\mathrm{im}}_\Lambda)$. If u has a boundary component which passes through a T^2-neck of $\widetilde{L}^5_{\mathrm{im}}(1)$, then by the SFT neck-stretching argument used in the proofs of Proposition 4.3.8 and Lemma 4.3.14, we would be able to produce a holomorphic teardrop bounded by $\widetilde{L}^5_{\mathrm{im}}(1)$ contained in one of the charts $\widetilde{B}_{\ell,m}$. Remark 4.3.30 and Proposition 4.3.1 implies that no such teardrop can exist. The same argument, using SFT neck-stretching and the open mapping theorem, shows that if u is a boundary punctured Riemann surface which constributes to the A_∞-module structure on $CF^*(\mathcal{L}^5, (\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}, \nabla_\Lambda))$, then it cannot have a corner at the S^1-switching component of $\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}$. In particular, we have a natural bijection between the sets of intersection points of $\mathcal{A}^5$ with the two Lagrangian branes in question, since $\mathcal{A}^5$ is contained away from the balls $\widetilde{B}_{\ell,m}$, and between the sets of holomoprhic disks contributing to the $CF^*(\mathcal{A}^5)$-module structures. ☐

One could also make this argument using Nohara–Ueda's split-generators [86] for the Fukaya category of the quintic. Combining this with Lemma 4.3.29 implies the following.

Corollary 4.3.34. *There is a quasi-isomorphism of A_∞-algebras*

$$CF^*(\widetilde{L}^5_{\mathrm{sm},-} \cup \widetilde{L}^5_{\mathrm{sm},+}, \nabla_\Lambda) \simeq CF^*(\widetilde{L}^5_{\mathrm{im}}, \nabla)$$

where the brane structures are as given in Definition 4.3.32. ☐

Remark 4.3.35. The quasi-isomorphism we have constructed tell us that there is an A_∞-functor

$$(\widetilde{L}^5_{\mathrm{sm}}, \nabla_\Lambda) \to D^\pi \mathcal{F}(X^5) \tag{4.3.31}$$

from the A_∞-category with one object and hom set $CF^*(\widetilde{L}^5_{\mathrm{sm}}, \nabla_\Lambda)$ to the split-closed derived Fukaya category. This functor takes the object in the former category to a direct summand of $(\widetilde{L}^5_{\mathrm{im}}, \nabla)$ mirror to the pushforward of a line bundle on a curve.

Corollary 4.3.36. *The functor* (4.3.31) *carries* $(\widetilde{L}^5_{\mathrm{sm}}, \nabla_\Lambda)$ *to an object of the Fukaya category mirror to the pushforward of a line bundle on C, where C is a generic line in the mirror quintic. The second Chern class of this object is represented by* $-[C]$*. Then there is an isomorphism of vector spaces*

$$HF^*(\widetilde{L}^5_{\mathrm{sm}}, \nabla_\Lambda) \cong H^*(S^1 \times S^2; \Lambda)$$

over the Novikov field, where the right hand side refers to the ordinary cohomology of $S^1 \times S^2$*.*

4.4 Open Gromov–Witten theory for Lagrangian immersions

The approach to defining the open Gromov–Witten potential taken in [18, 49, 14, 41] is to correct that naïve count $\mathfrak{m}^J_{-1}$ of pseudoholomorphic disks in a closed symplectic manifold M with boundary on a Lagrangian L, which depends nontrivially on the almost complex structure J, by equipping L with a bounding cochain. For ease of notation, we will only define the open Gromov–Witten potential when that L is a graded spin Lagrangian immersion with clean self-intersections in a Calabi–Yau 3-fold. This implies that the moduli spaces $\mathcal{M}_0(\beta; J)$ have virtual dimension 0, so for any nonzero $\beta \in H_2(M, L; \mathbb{Z})$, we define

$$\mathfrak{m}^{\beta;J}_{-1} := \int_{\mathcal{M}_0(\beta;J)} 1 \in \mathbb{C} \tag{4.4.1}$$

and we set $\mathfrak{m}_{-1}^{0;J} = 0$. These define an element of the Novikov ring

$$\mathfrak{m}_{-1}^{J} \coloneqq \sum_{\beta \in H_2(M,L;\mathbb{Z})} \mathfrak{m}_{-1}^{\beta;J} Q^{\omega(\beta)} \in \Lambda_+ \tag{4.4.2}$$

of positive valuation. For embedded Lagrangian submanifolds, a construction of the open Gromov–Witten potential was proposed by Fukaya [18]. When L is embedded, the integration pairing $\langle \cdot, \cdot \rangle$ on $CF^*(L) = \Omega^*(L) \widehat{\otimes}_{\mathbb{C}} \Lambda_+$, given by

$$\langle \alpha, \beta \rangle = (-1)^{\deg \beta} \int_M \alpha \wedge \beta \tag{4.4.3}$$

is strictly cyclically symmetric, meaning that

$$\langle \mathfrak{m}_k(\alpha_1 \otimes \cdots \otimes \alpha_k), \alpha_0 \rangle = (-1)^{\clubsuit_k} \langle \mathfrak{m}_k(\alpha_0 \otimes \cdots \otimes \alpha_{k-1}), \alpha_k \rangle \tag{4.4.4}$$

where $\clubsuit_k = (\deg' \alpha_0) \sum_{j=1}^{k} (\deg' \alpha_j)$ and $\deg' \alpha_j = \deg \alpha_j + 1$. When (4.4.4) holds, one can define a function $\Psi \colon \mathcal{M}(L) \to \Lambda_+$ on the moduli space of bounding cochains for $CF^*(L)$ by

$$\Psi(b) \coloneqq \mathfrak{m}_{-1} + \sum_{k=0}^{\infty} \frac{1}{k+1} \langle \mathfrak{m}_k(b^{\otimes k}), b \rangle \,. \tag{4.4.5}$$

Strict cyclicity is used to show that $\Psi(b)$ only depends on the gauge-equivalence class of $b \in \mathcal{M}(L)$ [18, Proposition 2.2].

The proof of (4.4.4) in [48] uses the existence of a system of Kuranishi structures on the moduli spaces $\mathcal{M}_{k+1}(\beta)$ of disks with $k+1$ boundary marked points representing the class $\beta \in H_2(M, L)$ that is compatible with the forgetful maps of boundary marked points [48, Corollary 3.1]. Compatibility with forgetful maps is used there to establish that the Kuranishi structures are invariant under cyclic permutations of boundary marked points, since they are pulled back from the Kuranishi structures on $\mathcal{M}_0(\beta)$. We cannot speak of such forgetful maps for disks with boundary on an *immersed* Lagrangian, since some of the boundary marked points of such a disk are mapped to self-intersection points of the immersion. As such, it is unclear how to construct cyclically symmetric perturbations of the moduli spaces of disks in the immersed setting, and so one cannot immediately extend the construction of the open Gromov–Witten potential in [18] to immersed Lagrangians.

The main results of [41], however, show that only the existence of a *homotopy cyclic* pairing is required to define the open Gromov–Witten potential, and such a pairing can be extracted from the trace associated to the cyclic open-closed map (cf. Assumption 4.9.2). Let $\mathcal{A} := CF^*(L, \nabla)$ denote the curved A_∞-algebra of a graded clean Lagrangian immersion $\iota_L \colon L \looparrowright M$ equipped with a rank one U_Λ-local system ∇ as constructed in Appendix 4.8. A trace as given in Assumption 4.9.2 can be thought of as a positive cyclic cocycle in $CC^*_+(\mathcal{A})$, from which one can construct a *homotopy cyclic* ∞*-inner product*, or a strong homotopy inner product as it is called in [51], on $\mathcal{A}$ following [76].

An ∞-inner product on $\mathcal{A}$, in the sense of Tradler [78], is an A_∞-bimodule hommormophism $\psi \colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ from the diagonal bimodule over $\mathcal{A}_\Delta$ to the (linear) dual. For a more thorough review of these notions and of Hochschild and cyclic (co)homology for gapped filtered A_∞-algebras, see [41, §2]. Recall that such an A_∞-bimodule homomorphism consists of linear maps $\{\psi_{p,q} \colon \mathcal{A}^{\otimes p} \otimes \underline{\mathcal{A}} \otimes \mathcal{A} \to \mathcal{A}^\vee\}_{p,q \in \mathbb{Z}_{\geq 0}}$, where the bimodule factor has been underlined for readability, for which the associated map of tensors algebras commute with the A_∞-bimodule structure maps. The precise meaning of this condition is reviewed in [41, (2.10)]. Such a ψ is said to be homotopy cyclic if it is *skew-symmetric*, *closed*, and *homologically nondegenerate*, which mean, respectively, that

- for $\alpha \in \mathcal{A}^{\otimes p}$, $\beta \in \mathcal{A}^{\otimes q}$, and $v, w \in \mathcal{A}$, we have that

$$\psi_{p,q}(\alpha \otimes \underline{v} \otimes \beta)(w) = (-1)^\kappa \psi_{q,p}(\beta, w, \alpha)(v) \tag{4.4.6}$$

  where

$$\kappa = \left(\sum_{i=1}^{p} |a_i|' + |v|' \right) \cdot \left(\sum_{j=1}^{q} |b_j|' + |w|' \right)$$

- for $a_1 \otimes \cdots \otimes a_{\ell+1} \in \mathcal{A}^{\otimes \ell+1}$ and any triple $1 \leq i < j < k \leq \ell + 1$, we have that

$$\begin{aligned} &(-1)^{\kappa_i} \psi(\cdots \otimes \underline{a_i} \otimes \cdots)(a_j) + (-1)^{\kappa_j} \psi(\cdots \otimes \underline{a_j} \otimes \cdots)(a_k) \\ &+ (-1)^{\kappa_k} \psi(\cdots \otimes \underline{a_k} \otimes \cdots \otimes)(a_i) = 0 \end{aligned} \tag{4.4.7}$$

  where the sign is determined by

$$\kappa_* = (|a_1|' + \cdots + |a_*|') \cdot (|a_{*+1}|' + \cdots + |a_k|')$$

  and where the inputs are cyclically ordered; and

- the pairing on the de Rham cohomology $H^*(\mathcal{A}, \mathfrak{m}_{1,0})$ is nondegenerate.

Having obtained a positive cyclic cocycle from Assumption 4.9.2, one obtain a negative cyclic cocycle using the connecting homomorphism in the long exact sequence for cyclic cohomology (cf. [41, Lemma 2.8]). From ψ_0, the part of this cocycle that is dual to the inclusion of Hochschild cycles, one defines $\psi \colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ by setting

$$\psi_{p,q} := \psi_0(\alpha \otimes v \otimes \beta)(w) - \psi_0(\beta \otimes w \otimes \alpha)(v)\,. \tag{4.4.8}$$

That the ∞-inner product ψ given by (4.4.8) is skew-symmetric and closed is immediate from the definition. For the cocycle associated to the cyclic open-closed map, one can check that the ∞-inner product ψ obtained this way induces the Poincaré duality pairing on de Rham cohomology, which shows that it is homologically nondegenerate. The Poincaré duality pairing for an immersed Lagrangian comes from a chain-level integration pairing on the Fukaya A_∞-algebra of a clean Lagrangian immersion defined as follows.

Definition 4.4.1. Let $\iota \colon L \to M$ be a clean Lagrangian immersion, where L is closed. Let A denote the index set for the components of $L \times_\iota L$. The integration pairing on $\overline{CF}^*(L)$ is defined as follows. If $a_\pm \in A$ are a pair of labels that are swapped under the natural involution, then for a pair of forms

$$\alpha_\pm \in \Omega^*(L_{a_\pm}; \Theta^-_{a_\pm}) .$$

we define

$$\langle \alpha_-, \alpha_+ \rangle := (-1)^{\deg \alpha_+} \int_{L_{a_-}} \alpha_- \wedge \alpha_+ . \tag{4.4.9}$$

Set $\langle \alpha_-, \alpha_+ \rangle = 0$ if $\alpha_\pm$ are forms on switching components that are not related this way. This naturally extends to a pairing on $CF^*(L)$.

Strict cyclic symmetry of such a pairing would imply that it induces a homotopy cyclic inner product $\psi \colon \mathcal{A}_\Delta \to \mathcal{A}^\vee$ for which all terms $\psi_{p,q}$ with $p > 0$ or $q > 0$ vanish [51]. This morally explains why the open Gromov–Witten potential defined below generalizes the one defined in [18], as discussed in [41, §6].

Definition 4.4.2 ([41])**.** The ($\infty-$)open Gromov–Witten potential on $CF^*(L, \nabla)$ is the function $\Psi \colon \mathcal{M}(L, \nabla) \to \Lambda_+$ given by

$$\Psi_J(b) := \mathfrak{m}^J_{-1} + \Psi'_J(b) \tag{4.4.10}$$

$$:= \mathfrak{m}^J_{-1} + \sum_{N=0}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \psi_{p,q}(b^{\otimes p} \otimes \mathfrak{m}_k(b^{\otimes k}) \otimes b^{\otimes q})(b) \tag{4.4.11}$$

where ψ is the homotopy cyclic infinity inner product associated to the cyclic open-closed trace (Assumption 4.9.2).

The gauge-invariance of Ψ in characteristic 0 follows from [41, Theorem 2.17]. For general Lagrangians L, it will still not be the case that $\Psi(b)$ is independent of the almost complex structure J used to define it. Instead, Ψ will only satisfy a wall-crossing formula involving closed $\underline{J}$-holomorphic curves, where $\underline{J} = \{J_t\}_{t\in[0,1]}$ is a 1-parameter family of almost complex structures. This wall-crossing formula is most cleanly stated using pseudo-isotopies of A_∞-algebras, as reviewed in §4.8.5. It is most convenient for us to think of a pseudo-isotopy as a (gapped filtered) A_∞-structure on the space of differential forms on $[0,1] \times (L \times_{\iota_L} L)$. The structure maps are denoted $\{\widetilde{\mathfrak{m}}_k\}_{k\geq 0}$.

Remark 4.4.3 (Bounding cochains on a pseudo-isotopy)**.** A pseudo-isotopy determines A_∞-quasi-isomorphisms

$$\mathfrak{c}^t \colon (CF^*(L,\nabla), \{\mathfrak{m}_k^{J_0}\}_{k\geq 0}) \to (CF^*(L,\nabla), \{\mathfrak{m}_k^{J_t}\}_{k\geq 0})$$

for all $t \in [0,1]$. An A_∞-quasi-isomorphism is comprised of a sequence of linear maps

$$\mathfrak{c}_k^t \colon (CF^*(L,\nabla)[1])^{\otimes k} \to CF^*(L,\nabla)[1]$$

of degree 0 for all $k \geq 0$. We set $b_t = \mathfrak{c}_*^t(b_0)$ for all $t \geq 0$, where the pushforward of bounding cochains is defined by $\mathfrak{c}_*^t(b_0) = \sum_{k=0}^{\infty} \mathfrak{c}_k^t(b_0)$. This path of bounding cochains $\widetilde{b} = \{b_t\}_{t\in[0,1]}$ is a bounding cochain for the pseudo-isotopy, by the so-called pointwise condition [81, Definition 21.27].

Theorem 4.4.4 (Wall-crossing)**.** *For a path $\underline{J}$ as above and any path of bounding cochains $\widetilde{b} = \{b_t\}_{t\in[0,1]}$ as above, we have that*

$$\Psi_{J_0}(b_0) = \Psi_{J_1}(b_1) + \widetilde{GW}(L)\,. \tag{4.4.12}$$

The number $\widetilde{GW}(L) \in \Lambda_0$ is defined in (4.8.18) *as the count of $\underline{J}$-holomorphic closed rational curves intersecting L in a point.*

Proof. The proof closely follows the proof of [41, Theorem 5.2], with some modifications owing to the different chain-level model used in the present context. Recall that the terms of $\Psi'_{J_i}(b_i)$ are defined in terms of the open-closed map $\mathcal{OC}_0$. Let $\widetilde{\mathcal{A}}$ denote the pseudo-isotopy defined using $\underline{J}$. By Assumption 4.9.2, there is an ∞-inner product $\widetilde{\psi}$ on $\widetilde{\mathcal{A}}$ defined using the trace associated to

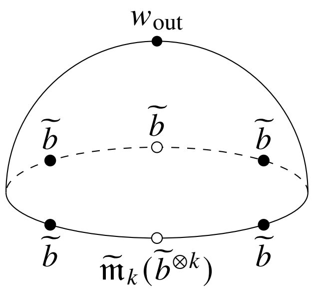

Figure 4.4: Elements of (4.8.16), with boundary marked points labeled by the relevant inputs. The marked points corresponding to the input and output of $\widetilde{\psi}$ are the white dots.

the cyclic open-closed map on a pseudo-isotopy. Applying Stokes' theorem [48, Proposition 4.2] to the moduli spaces $\widetilde{\mathcal{M}}_{k,1}(L;\vec{a};\beta;\underline{J})$ of (4.8.16) and the definition of the ∞-inner products shows that

$$\Psi'_1(b_1) - \Psi'_{-1}(b_-) \tag{4.4.13}$$

$$= \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{1}{N+1} \sum_{r+s=k_1-1} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes r} \otimes \widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}) \otimes \widetilde{b}^{\otimes s})} \otimes \widetilde{b}^{\otimes q})(\widetilde{b}) \tag{4.4.14}$$

$$+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{1}{N+1} \sum_{r+s=p-1} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes r} \otimes \widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}) \otimes \widetilde{b}^{\otimes s} \otimes \underline{\mathfrak{m}_{k_1}(\widetilde{b}^{\otimes k_1})} \otimes \widetilde{b}^{\otimes q})(\widetilde{b}) \tag{4.4.15}$$

$$+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{1}{N+1} \sum_{r+s=q-1} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})} \otimes \widetilde{b}^{\otimes r} \otimes \widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}) \otimes \widetilde{b}^{\otimes s})(\widetilde{b}) \tag{4.4.16}$$

$$+ \sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{k_2}{N+1} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})} \otimes \widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2}))\,. \tag{4.4.17}$$

Because the value of $\widetilde{\psi}$ on inputs of the form under consideration are expressed as the difference

$$\widetilde{OC}_0(\widetilde{b}^{\otimes p} \otimes \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}) \otimes \widetilde{b}^{\otimes q} \otimes \widetilde{b}) - \widetilde{OC}_0(\widetilde{b}^{\otimes p} \otimes \widetilde{b} \otimes \widetilde{b}^{\otimes q} \otimes \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}))$$

we prove the identity above by applying Stokes' theorem to two copies of the moduli space (4.8.16). Since $\widetilde{\psi}_{p,q}$ is defined by taking a difference corresponding to these two moduli spaces, the contributions of Figure (3.11) cancel, and thus they do not contribute to the sum above.

We can rewrite the sum of (4.4.14), (4.4.15), and (4.4.16) as

$$\sum_{N=0}^{\infty} \sum_{\substack{p+q+k=N \\ k_1+k_2=k+1}} \frac{N+1-k_2}{N+1} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_{k_1}(\widetilde{b}^{\otimes k_1})} \otimes \widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_{k_2}(\widetilde{b}^{\otimes k_2})) \tag{4.4.18}$$

using [41, Lemma 2.15]. By the Maurer–Cartan eqation, the sum of (3.4.15) and (4.4.18) gives

$$\sum_{p,q\geq 0} \widetilde{\psi}_{p,q}(\widetilde{b}^{\otimes p} \otimes \underline{\widetilde{\mathfrak{m}}_0} \otimes \widetilde{b}^{\otimes q})(\widetilde{\mathfrak{m}}_0)\,. \tag{4.4.19}$$

By passing to a canonical model for $\widetilde{\mathscr{A}}$ in which the ∞-inner product coincides with the Poincaré pairing, which exists by [76], we can rewrite this as

$$\mathcal{OC}_0(\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0)) = \langle \widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0 \rangle \tag{4.4.20}$$

where the pairing on the right hand side is an integration pairing on $CF^*([0,1]\times L)$. Here we have used the fact that for degree reasons, only constant disks can contribute to $\widetilde{\mathfrak{m}}_2(\widetilde{\mathfrak{m}}_0, \widetilde{\mathfrak{m}}_0)$.

We can also analyze the leading terms $\mathfrak{m}_{-1}^{J_i}$ of $\Psi_{J_i}(b_i)$ simultaneously. To that end, consider the moduli spaces $\widetilde{\mathcal{M}}_{-1}(\beta; \underline{J})$. One type of boundary stratum in this moduli space is given by the fiber product of two moduli spaces of the form $\widetilde{\mathcal{M}}_0(\vec{a}; \beta; \underline{J})$ along the evaluation map at the boundary marked point. This evaluation map can have a switching component of the immersion as its codomain, which is the only new geometric subtlety as compared to the case of embedded Lagrangians. Notice that the contributions of these terms are precisely given by (4.4.20), with the opposite sign. The contributions of the remaining boundary strata are given by the wall-crossing term $\widetilde{GW}(L)$ that counts closed rational curves, just as for embedded Lagrangians. □

When $\iota\colon L \to M$ is nullhomologous, we can obtain an invariant from $\Psi^J(b)$ by considering a smooth singular 4-chain $\Gamma \in H_4(M, L; \mathbb{Z})$ with boundary $\partial\Gamma = L$.

Corollary 4.4.5. *If $[L] = 0$ in $H_3(M, L; \mathbb{Z})$ and $\Gamma \in H_4(M, L; \mathbb{Z})$ is a bounding 4-chain for L,*

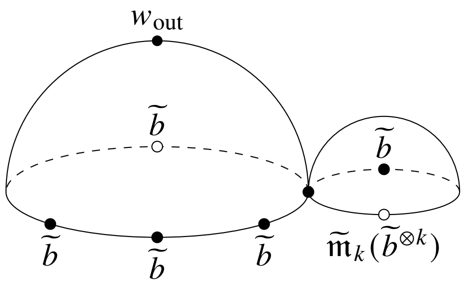

Figure 4.5: Elements of the boundary stratum contributing to term (4.4.14).

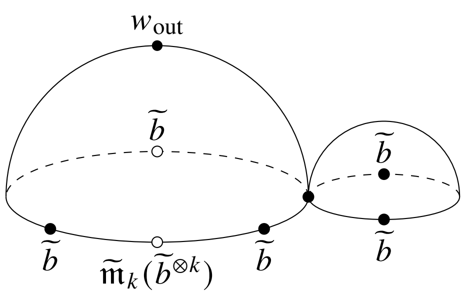

Figure 4.6: Boundary strata contributing to terms (4.4.15) and (4.4.16).

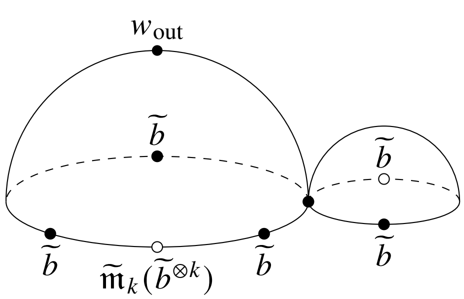

Figure 4.7: Boundary stratum contributing to the term (4.4.17).

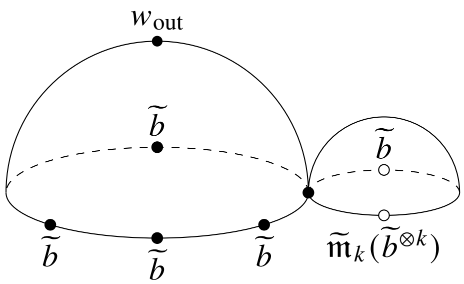

Figure 4.8: Canceling boundary strata.

then

$$\Psi_{L,\Gamma}(b) \coloneqq \Psi^J(b) - \int_\Gamma \mathfrak{m}_\emptyset^J \tag{4.4.21}$$

is independent of the almost complex structure J, where $\mathfrak{m}_\emptyset^J$ is defined in (4.8.15).

Proof. Consider the pseudo-isotopy associated to a path $\underline{J}$ of ω-compatible almost complex structures, and let $p_M\colon [0,1] \times M \to M$ denote the projection. By applying Stokes' theorem to the moduli spaces $\widetilde{\mathcal{M}}_1^{cl}(\beta)$ defined in §4.8.6, we find that

$$\mathfrak{m}_\emptyset^{J_1} - \mathfrak{m}_\emptyset^{J_0} = -d((p_M)_*\widetilde{\mathfrak{m}}_\emptyset)\,. \tag{4.4.22}$$

Integrating both sides of (4.4.22) over Γ and applying Stokes' theorem shows that

$$\int_\Gamma \mathfrak{m}_\emptyset^{J_1} - \int_\Gamma \mathfrak{m}_\emptyset^{J_0} = -\widetilde{GW}\,. \tag{4.4.23}$$

□

4.5 Background on Hodge structures and cyclic homology

In this section, we will review some background material on variations of semi-infinite Hodge structure (VSHS), including those obtained from the negative cyclic homology of an A_∞-category, and the VSHSs appearing in closed string mirror symmetry, following [6].

Consider a complete discrete valuation ring R with valuation ν and maximal ideal $\mathfrak{m}$ and residue field $\mathbb{C}$. Denote by $\mathbb{K}$ its field of fractions. We call $\mathcal{M} := \operatorname{Spec}\mathbb{K}$ a formal punctured disk. We also denote by q an element $q \in R$ with $\nu(q) = 1$. This determines an isomorhpism $R \cong \mathbb{C}[[q]]$. This can also be thought of as a coordinate on $\mathcal{M}$. Also define $\mathcal{O}_{\mathcal{M}} := \mathbb{K}$ and $T\mathcal{M} := \operatorname{Der}_{\mathbb{C}} \mathcal{M}$.

Variations of semi-infinite Hodge structure, introduced by Barranikov [87], can roughly be thought of as variations of Hodge structure, in the classical sense, without an integral lattice, provided that $\mathcal{M}$ trivial grading. It is nonetheless more natural to describe the VSHSs associated to negative cyclic homology and quantum cohomology slightly differently.

Definition 4.5.1. Let u be a formal variable of degree 2, and consider the ring $\mathbb{K}[[u]]$. For any $f \in \mathbb{K}[[u]]$, we define the element

$$f^{\wedge}(u) := f(-u)\,. \tag{4.5.1}$$

A ($\mathbb{Z}$-graded unpolarized) *VSHS* over the formal punctured disk $\mathcal{M}$ is a pair $\mathcal{H} = (\mathcal{E}, \nabla)$ where:

- $\mathcal{E}$ is a graded, finitely-generated free $\mathbb{K}[[u]]$-module;
- ∇ is a flat connection

$$\nabla\colon T\mathcal{M} \otimes \mathcal{E} \to u^{-1}\mathcal{E}$$

  of degree 0.

Furthermore, a *polarization* of dimension $n \in \mathbb{Z}/2$ for $\mathcal{H}$ is a pairing

$$(\cdot,\cdot)\colon \mathcal{E} \otimes \mathcal{E} \to \mathbb{K}[[u]]$$

of degree 0 which is:

- sesquilinear:

$$(s_1 + s_2, t) = (s_1, t) + (s_2, t)$$
$$(t, s_1 + s_2) = (t, s_1) + (t, s_2)$$

and

$$(f s_1, s_2) = f(s_1, s_2) = (s_1, f^\wedge s_2)$$

for all $f \in \mathbb{K}[[u]]$ and $s_1, s_2, t \in \mathcal{E}$;

- covariantly constant:

$$X(s_1, s_2) = (\nabla_X s_1, s_2) + (s_1, \nabla_X s_2)$$

for all $X \in T\mathcal{M}$; and

- graded symmetric:

$$(s_1, s_2) = (-1)^{n+\deg s_1}(s_2, s_1)^\wedge .$$

Moreover, we require that the induced pairing of $\mathbb{K}$-modules

$$\mathcal{E}/u\mathcal{E} \otimes_{\mathbb{K}} \mathcal{E}/u\mathcal{E} \to \mathbb{K}$$

is nondegenerate.

The relation between the notion of a VSHS and classical variations of Hodge structure is established in [6].

Lemma 4.5.2 ([6, Lemma 2.7])**.** *A $\mathbb{Z}$-graded unpolarized VSHS $\mathcal{H} = (\mathcal{E}, \nabla)$ over a punctured formal disk $\mathcal{M}$ is equivalent to the following data:*

- *a free, finite rank, $\mathbb{Z}/2$-graded $\mathbb{K}$-module $\mathcal{V} := \mathcal{V}_{\mathrm{even}} \oplus \mathcal{V}_{\mathrm{odd}}$;*

- *a flat connection ∇ on each $\mathcal{V}_\sigma$; and*

- *a pair of decreasing Hodge filtrations $\{F^{\geq p}\mathcal{V}_{\mathrm{even}}\}$ and $\{F^{\geq p-\frac{1}{2}}\mathcal{V}_{\mathrm{odd}}\}$ which satisfy Griffiths transversality:*

$$\nabla F^{\geq p}\mathcal{V}_\sigma \subset F^{\geq p-1}\mathcal{V}_\sigma .$$

An n-dimensional polarization on $\mathcal{H}$ is equivalent to a pair of covariantly constant bilinear pairings

$$(\cdot,\cdot)\colon \mathcal{V}_\sigma \otimes \mathcal{V}_\sigma \to \mathbb{K}$$

such that $(\alpha, \beta) = (-1)^n(\beta, \alpha)$, and for which

$$(F^{\geq p}\mathcal{V}_\sigma, F^{\geq q}\mathcal{V}_\sigma) = 0$$

if $p + q > 0$. Furthermore, the induced pairing on the associated graded modules

$$\mathrm{Gr}_F^p \mathcal{V}_\sigma \otimes \mathrm{Gr}_F^{-p} \mathcal{V}_\sigma \to \mathbb{K}$$

is nondegenerate for all p. □

It will be illuminating to sketch the proof of one direction of this correspondence following the proof of [6, Lemma 2.7].

Proof sketch. If $\mathcal{H} = (\mathcal{E}, \nabla)$ is as in Definition 4.5.1, then the construction of the $\mathbb{K}$-module and Hodge filtration of Lemma 4.5.2 can be summarized as follows. We define a $\mathbb{K}[u, u^{-1}]$-module by setting $\widetilde{\mathcal{E}} := \mathcal{E} \otimes_{\mathbb{K}[[u]]} \mathbb{K}((u))$. Multiplication by u induces isomorphisms

$$\widetilde{\mathcal{E}}_k \xrightarrow{u\cdot} \widetilde{\mathcal{E}}_{k+2} \tag{4.5.2}$$

so we set

$$\mathcal{V}_{[k]} := \widetilde{\mathcal{E}}_k \,.$$

The Hodge filtration on $\mathcal{V}_{[k]}$ is given by powers of u:

$$F^{\geq p-\frac{k}{2}}\mathcal{V}_{[k]} := \left(u^{\geq p} \cdot \mathcal{E}\right)_k \subset \widetilde{\mathcal{E}}_k \,.$$

The connection on $\mathcal{V}_{[k]}$ is obtained from the connection on $\mathcal{E}$, and Griffiths transversality follows because ∇ carries $\mathcal{E}$ to $u^{-1}\mathcal{E}$. The pairing is inherited from the polarization on $\mathcal{E}$, up to rescaling by a constant prefactor so that it respects (4.5.2). □

Let $\mathcal{A}$ be a strictly unital uncurved $\mathbb{K}$-linear A_∞-category. Further assume that $\mathcal{A}$ is proper and that it carries a weak proper Calabi–Yau structure.

Remark 4.5.3. When $\mathcal{A}$ is the Fukaya category of unobstructed Lagrangian branes, the weak proper Calabi–Yau structure is given by the negative cyclic open-closed map under Assumption 4.9.2. On homology, this amounts to the Poincaré pairing, as noted in the previous section. If $\mathcal{A}$ is a dg-enhancement of the derived category of a smooth Calabi–Yau variety over $\mathbb{C}$, the weak proper Calabi–Yau structure is determined by Serre duality and a choice of volume form. For this purpose, we will always use the Hodge-theoretically normalized volume form, in the terminology of [56] (see also [6, §2.4])

In this setting, a VSHS on the negative cyclic homology $HC_*^-(\mathcal{A})$ is constructed by Sheridan [88]. The description of the VSHS in following Theorem uses the characterization of Definition 4.5.1.

Theorem 4.5.4 ([88]). *Recall that $HC_*^-(\mathcal{A})$ is a $\mathbb{K}[[u]]$-module. This can be equipped with a flat connection ∇^{GGM} called the Getzler–Gauss–Manin connection. There is a $\mathbb{K}[[u]]$ sesquilinear pairing*

$$\langle \cdot, \cdot \rangle_{res} \colon HC_*^-(\mathcal{A}) \otimes HC_*-(\mathcal{A}) \to \mathbb{K}[[u]]$$

called the Mukai pairing, which is graded symmetric of dimension n when $\mathcal{A}$ admits an n-dimensional weak proper Calabi–Yau structure.

The negative cyclic homology is only a *pre-VSHS a priori*, but for the Fukaya category or derived category, in the geometric setting to be considered below, the results of [6] imply that it is in fact a VSHS. In the case of the Fukaya category, this requires Assumption 4.9.2, which is used to construct a weak proper Calabi–Yau structure.

Extracting (closed string) Hodge-theoretic mirror symmetry [2] from homological mirror symmetry amounts to comparing this VSHS to geometrically defined VSHSs associated to quantum cohomology and to a suitable family of Calabi–Yau varieties near a large complex structure limit point.

Let (X, ω) be a connected integral symplectic Calabi–Yau 3-fold, meaning that $[\omega] = H^2(X; \mathbb{Z})$ and $c_1(TX) = 0$. Also let $R_A := \mathbb{C}[[Q]]$ and $\mathbb{K}_A := \mathbb{C}((Q))$, so that R_A is a subring of the Novikov ring Λ_0.

Definition 4.5.5. The small A-model VSHS, denoted $\mathcal{H}^A(X, \omega) := (\mathcal{E}, \nabla, (\cdot, \cdot))$ is given by the data

$$\begin{aligned} \mathcal{E} &:= H^*(X; \mathbb{C}) \widehat{\otimes}_{\mathbb{C}} \mathbb{K}_A[[u]][n] \\ \nabla_{Q\partial_Q} \alpha &:= Q\partial_Q \alpha - u^{-1}[\omega] \star \alpha \\ (\alpha, \beta) &:= \int_X \alpha \cup \beta^\wedge \end{aligned}$$

where $[\omega] \star \alpha$ denotes the small quantum product and β^* is defined in (4.5.1).

Remark 4.5.6. Suppose that $h^{1,1}(X) = h^{2,2}(X) = 1$. Then there is a basis $\{e_3, e_2, e_1, e_0\}$ for the even degree part of $H^*(X;\mathbb{C})$ given by

$$e_3 = [X]\,; \quad e_2 = [D]\,; \quad e_1 = -[\ell]\,; \quad e_0[\mathrm{pt}]$$

where $[D]$ is the hyperplane class dual to the complexified Kähler form on X, and e_1 is chosen to have intersection number $e_1 \cdot e_2 = 1$. These extend to sections of $\mathcal{E}$.

In this basis, a connection matrix for the quantum connection can be expressed as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\Phi'' & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.5.3}$$

where Φ'' is the closed Gromov–Witten potential with two interior constraints.

The B-model VSHS is described in more classical terms, per Lemma 4.5.2. Let $R_B := \mathbb{C}((z))$ and $\mathbb{K}_B := \mathbb{C}((z))$. Let $X^\vee \to \mathcal{M}_B$ be a smooth projective connected scheme of relative dimension 3, with trivial relative canonical sheaf. We further assume that $X^\vee$ is *maximally unipotent* (cf. [6, §1.1]).

Definition 4.5.7. The small B-model ($\mathbb{Z}$-graded polarized) VSHS $\mathcal{H}^B(X^\vee)$ consists of

- the relative de Rham cohomology $\mathcal{V}$ of $X^\vee$ over the formal punctured disk $\mathcal{M}_B$, with a $\mathbb{Z}/2$-grading induced from the cohomological $\mathbb{Z}$-grading;
- the filtration

$$F^{\geq s}\mathcal{V} := \bigoplus_p H^p\left(\Omega^{\geq p+2s}_{X^\vee/\mathcal{M}_B}\right) \tag{4.5.4}$$

which comes from the classical Hodge filtration;

- the Gauss–Manin connection; and
- the integration pairing.

Remark 4.5.8. The filtration can be written

$$F^{\geq\frac{3}{2}}\mathcal{V} \subset F^{\geq\frac{1}{2}}\mathcal{V} \subset F^{\geq-\frac{1}{2}}\mathcal{V} \subset F^{\geq-\frac{3}{2}}\mathcal{V}\,.$$

With this convention for the filtration, the lowest level $F^{\geq\frac{3}{2}}\mathcal{V} = H^0\left(\Omega^3_{X^\vee/\mathcal{M}_B}\right)$ is the space of (relative) holomorphic volume forms on $X^\vee$. Under classical conventions, this level of the filtration would be denoted $F^3\mathcal{V}$ (cf. [56, 12]).

Remark 4.5.9. Suppose $F^{\geq\frac{3}{2}}\mathcal{V}$ is one-dimensional and that rank of $F^{\geq p+\frac{1}{2}}\mathcal{V}$ increases by 1 at each level, as is the case for the mirror quintic. In this setting [59] constructs a basis $\{e_3, e_2, e_1, e_0\}$ for the odd-degree part of the module $\mathcal{V}$. These sections are such that $e_i \in F^{\geq i-\frac{3}{2}}\mathcal{V}_{\mathrm{odd}}$, and in particular we can take $e_3 = \Omega \in F^{\geq\frac{3}{2}}\mathcal{V}_{\mathrm{odd}}$ to the Hodge-theoretically normalized volume form. This basis is constructed using the fact that $0 \in \Delta$ is a point of maximally unipotent monodromy, where the monodromy is associated to the Gauss–Manin connection on the B-side.

In this basis, the connection matrix for the Gauss–Manin connection can be expressed as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\mathfrak{C} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.5.5}$$

where $\mathfrak{C}$ is the Yukawa coupling [56].

These VSHSs are related in the following way.

Theorem 4.5.10 (cf. [6, Theorem A])**.** *Suppose that X and $X^\vee$ as above are homologically mirror.*

Under Assumption 4.9.3 and Assumption 4.9.2, there are isomorphisms of VSHS given by

$$\begin{array}{ccc} HC_*^-(\mathcal{F}(X)) & \longrightarrow & HC_*^-(D^b_{dg}\,\mathrm{Coh}(X^\vee)) \\ \Big\downarrow OC^- & & \Big\downarrow \widetilde{\mathfrak{I}}_{\mathrm{HKR}} \\ \mathcal{H}^A(X,\omega) & & \mathcal{H}^B(X^\vee) \end{array} \tag{4.5.6}$$

where the horizontal arrow is given by homological mirror symmetry and Morita invariance.

The isomorphism $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ is obtained from the HKR (Hochschild–Kostant–Rosenberg) isomorphism [89]

$$\widetilde{I}_{\mathrm{HKR}} \colon HC_*^-(D^b_{dg}\,\mathrm{Coh}(X^\vee)) \cong HC_*^-(X^\vee) \to \mathcal{H}^B(X^\vee)$$

which is in turn a lit of the HKR isomorphism on Hochschild homology. We define $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ to be the composition of $\widetilde{I}_{\mathrm{HKR}}$ with the square root of the Todd class

$$H^*(\Omega^{-*}X^\vee) \xrightarrow{\mathrm{td}^{1/2}(X^\vee)} H^*(\Omega^{-*}X^\vee)\,.$$

Tu [57, Remark 0.3] shows that $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ is an isomorphism of polarized VSHS.

Remark 4.5.11. As explained in [6, §1.10], the mirror map appearing in Hodge-theoretic mirror symmetry (cf. [56]) arises as a change of coordinates relating $\mathbb{C}((Q))$ to $\mathbb{C}((z))$ in homological mirror symmetry, and the natural Calabi–Yau structure on $\mathcal{F}(X)$ associated to the negative cyclic open-closed map is mirror to the Calabi–Yau structure on the derived category determined by the Hodge-theoretically normalized volume form. These facts make it possible to recover closed string enumerative mirror symmetry from homological mirror symmetry, and they will also allow us to recover open string enumerative mirror symmetry.

4.6 Background on Extensions of VSHS

Morrison showed that enumerative mirror symmetry is encapsulated by an isomorphism of VSHS [2]. Similarly, we will see that open enumerative mirror symmetry, as in e.g. [11], can be reformulated as the existence of an isomorphism of *extensions of VSHS*. In this section, we will explain how to classify extensions of VSHS following [38] in a language consistent with the discussion of VSHS in the last section. We will also discuss the classical construction of an extension of VHS associated to a homologically trivial algebraic cycle, as discussed in [12].

4.6.1 Extensions as normal functions

In this subsection, we will work with the characterization of VSHSs from Lemma 4.5.2. Let $\mathcal{V}$ be a polarized VSHS over the formal disk $\mathcal{M}$, and consider an extension

$$0 \to \mathcal{V} \xrightarrow{a} \mathcal{V}' \xrightarrow{b} \mathbb{K} \to 0 \tag{4.6.1}$$

where $\mathbb{K}$ carries the trivial connection and $F^{\geq p}\mathbb{K} = \mathbb{K}$ and $F^{\geq p+1} = 0$ for a fixed $p \in \frac{1}{2}\mathbb{Z}$. Sufficiently well-behaved extensions of this form can be classified by *normal functions* as we will now explain following [38, §6.1].

Definition 4.6.1. We say that a variation of semi-infinite Hodge structures $\mathcal{V}$ is *regular singular* if there is an R-submodule $\mathcal{V}_R \subset \mathcal{V}$ with $\nabla_{q\partial_q}\mathcal{V}_R \subset \mathcal{V}_R$. An extension of VSHS as in (4.6.1) is said to be regular singular if both $\mathcal{V}$ and $\mathcal{V}'$ are.

The *Deligne lattice* of $\mathcal{V}$ is an R-submodule $\widetilde{\mathcal{V}} \subset \mathcal{V}$ which is characterized by the requirement that the residue of the connection

$$N \colon \widetilde{\mathcal{V}}_0 \to \widetilde{\mathcal{V}}_0$$

has eigenvalues in $[0, 1)$, where $\widetilde{\mathcal{V}}_0 := \widetilde{\mathcal{V}}/q\widetilde{\mathcal{V}}$ denote the fiber of $\widetilde{\mathcal{V}}$ over 0. There is a filtration

$\widetilde{F}^{\geq p}\widetilde{\mathcal{V}}$ on the Deligne lattice induced from the filtration on $\mathcal{V}$.

Taking Deligne lattices of the modules in (4.6.1) gives a short exact sequence of R-modules

$$0 \to \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}' \to R \to 0 \tag{4.6.2}$$

and restricting to the central fibers gives us a short exact sequence of $\mathbb{C}$-vector spaces

$$0 \to \widetilde{\mathcal{V}}_0 \to \widetilde{\mathcal{V}}'_0 \to \mathbb{C} \to 0\,. \tag{4.6.3}$$

If we let M denote the residue of the connection

$$M : \widetilde{\mathcal{V}}'_0 \to \widetilde{\mathcal{V}}'_0$$

then (4.6.3) is a short exact sequence of vector spaces with endomorphisms, where the endomorphism acting on $\mathbb{C}$ is 0.

A regular singular extension of VSHS is called *holomorphically flat* if (4.6.3) splits. In other words, this condition means that we can write

$$M = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to this splitting. By [38, Lemma 6.6], this is equivalent to the existence of an element $h \in \ker(\nabla^{\mathcal{V}'}_{q\partial_q}) \subset \widetilde{\mathcal{V}'} \subset \mathcal{V}'$ such that $b(h) = 1 \in R$. Hugtenburg classifies holomorphically flat regular singular extensions of VSHS in analogy with the classification of extensions of VHS due to Carlson [55].

Definition 4.6.2. In the setting of (4.6.1), the *kth intermediate Jacobian* is defined to be

$$\widetilde{\mathcal{J}}^k := \frac{\widetilde{\mathcal{V}}}{\widetilde{F}^{\geq k}\widetilde{\mathcal{V}} \oplus \ker(\nabla_{\partial_q})}\,.$$

A *normal function* is an element $\nu \in \widetilde{\mathcal{J}}^k$ of the intermediate Jacobian such that for a lift $\widetilde{\nu} \in \widetilde{\mathcal{V}}$ we have $\nabla_{q\partial_q}\widetilde{\nu} \in \widetilde{F}^{\geq k-1}\widetilde{\mathcal{V}}$.

Proposition 4.6.3 ([38, Proposition 6.12])**.** *There is a bijection between the set of holomorphically flat regular singular extension of VSHS with filtrations as in* (4.6.1) *and the set of normal functions in* $\widetilde{\mathcal{J}}^k$. □

We sketch a proof of this Lemma, as some knowledge of the construction of a normal function will be helpful in the proof of Theorem 1.0.1 when we compare extensions of VSHS in the B-model.

Proof sketch. Assume given an extension of VSHSs as in (4.6.1) which is regular singular and for which the short exact sequence (4.6.3) of complex vector spaces with endomorphims splits. Since the extension is regular singular, we an choose an element $f \in \widetilde{F}^{\geq k}\widetilde{\mathcal{V}}'$ with $b(f) = 1$. The existence of a splitting implies the existence of an element $h \in \ker\left(\nabla^{\mathcal{V}'}_{q\partial_q}\right)$ as discussed above. A lift of the normal function is obtained by setting

$$\widetilde{\nu} := f - h \in \widetilde{\mathcal{V}}$$

and one can check that the image $\nu \in \widetilde{\mathcal{J}}^k$ of $\widetilde{\nu}$ in the quotient is independent of the choices for h and f.

Conversely, given a normal function $\nu \in \widetilde{\mathcal{J}}^k$, we will construct an extension of VSHS with underlying $\mathbb{K}$-module

$$0 \to \mathcal{V} \to \mathbb{K} \oplus \mathcal{V} \to \mathbb{K} \to 0\,.$$

Fix a lift $\widetilde{\nu}$ of the normal function. The connection on $\mathbb{K}\oplus\mathcal{V}$ is given by $q\partial_q \oplus \nabla^{\mathcal{V}}$ and the filtration is determined by setting

$$\mathcal{F}^{\geq i}(\mathbb{K} \oplus \mathcal{V}) := (\{0\} \oplus F^{\geq i}\mathcal{V}) + (F^{\geq i}\mathbb{K} \oplus \mathbb{K}\widetilde{\nu})$$

where the filtration on $\mathbb{K}$ is such that $\mathcal{F}^{\geq k}\mathbb{K} = \mathbb{K}$ and $F^{\geq k+1}\mathbb{K} = 0$, consistent with (4.6.1). Griffiths transverality follows from the definition of a normal function, and one can also check the VSHS obtained this way does not depend on the choice of lift. □

4.6.2 Normal functions from algebraic cycles

Let $\pi\colon \mathcal{X}^\vee \to \Delta^*$ be a family of smooth Calabi–Yau threefolds over $\mathbb{C}$, where Δ^* is the unit disk in $\mathbb{C}$ with the origin removed. Further assume that this family admits a semistable continuation over the unit disk Δ, and that $0 \in \Delta$ is a point of maximally unipotent monodromy (i.e. a large complex structure limit point in the complex moduli space). Note that up to a change of base, one can obtain a relative scheme $X^\vee \to \mathcal{M}_B$ as considered in [6].

Let $X_z^\vee$ denote the fiber of $\mathcal{X}^\vee$ over $z \in \Delta^*$, and assume that we are given algebraic curves $i\colon C_z^i \to X_z$, for $i = 0, 1$, which are the fibers of smooth families $C^i \to \Delta^*$. Suppose that these families admit semistable continuations $\overline{C}^i$ over Δ. These can be can be thought of as algebraic curves $C^i \subset X^\vee$ in the scheme over $\mathcal{M}_B$. If $[C_z^0]$ and $[C_z^1]$ lie in the same homology class, then for any integer $m > 0$ we can consider a family of algebraic cycles mC whose fibers are the homologically trivial algebraic cycles $mC_z^0 - mC_z^1$.

From such a family of homologically trivial algebraic cycles, one can construct an extension of VHS using classical techniques [55]. We will summarize this construction following [13] and [12], in particular explaining how the normal function thus obtained gives a normal function in the sense of Definition 4.6.2.

Since each $m[C_z] \in H_2(X_z^\vee;\mathbb{Z})$ vanishes, we have a short exact sequence

$$0 \to H_3(X_z^\vee;\mathbb{Z}) \to H_3(X_z^\vee, mC_z;\mathbb{Z}) \to \mathbb{Z} \to 0 \tag{4.6.4}$$

for all z, where the rightmost entry is thought of as a generator for $H_2(mC_z;\mathbb{Z})$. This allows us to associate a normal function to mC as follows. In each fiber, choose any 3-chain $\Gamma_z \in H_3(X_z^\vee, mC_z;\mathbb{Z})$ with $\partial\Gamma_z = mC_z$.

Lemma 4.6.4. *The integral*

$$\nu_z(\eta) \coloneqq (2\pi i)^2 \int_{\Gamma_z} \eta\,; \qquad \eta \in F^{\geq \frac{1}{2}} H^3(X_z^\vee;\mathbb{C})$$

is independent of Γ_z *up to periods of the form* $\int_{A_z} \eta$ *for* $A_z \in H_3(X_z^\vee;\mathbb{Z})$*. In particular,* ν *can be thought of as an element of the (classical) intermediate Jacobian*

$$(F^{\geq\frac{1}{2}} H^3(X_z^\vee;\mathbb{C}))^\vee / H^3(X_z^\vee;\mathbb{Z})^\vee \cong \frac{H^3(X_z^\vee;\mathbb{C})}{F^{\geq\frac{1}{2}} H^3(X^\vee;\mathbb{C}) \oplus H^3(X_z^\vee;\mathbb{Z})}\,. \tag{4.6.5}$$

□

The group (4.6.5) is a fiber of an intermediate Jacobian. Let $h = (2\pi i)^2 \delta_{\Gamma_z}$, where δ_{Γ_z} is a differential form representing the class in $H^3(X_z^\vee \setminus C_z;\mathbb{C})$ Poincaré dual to Γ_z, and choose a form $f \in H^3(X_z^\vee \setminus C_z; (2\pi i)^2\mathbb{Z})$ such that $df = (2\pi i)^2 \delta_{C_z}$, which exists because C_z is nullhomologous. Then we can write

$$\nu_z(\eta) = (2\pi i)^2 \int_{X_z^\vee} (h - f) \wedge \eta = (2\pi i)^2 \int_{X_z^\vee} h \wedge \eta = (2\pi i)^2 \int_{\Gamma_z} \eta \tag{4.6.6}$$

for any $\eta \in F^{\geq\frac{1}{2}} H^3(X_z^\vee;\mathbb{C}) = H^{3,0}(X_z^\vee) \oplus H^{2,1}(X_z^\vee)$. The integral

$$\int_{X_z^\vee} f \wedge \eta$$

vanishes by type considerations. Notice that the difference $h - f \in H^3(X_z^\vee)$ is well-defined modulo $F^{\geq\frac{1}{2}} H^3(X_z^\vee;\mathbb{C}) \oplus H^3(X_z^\vee; (2\pi i)^2\mathbb{Z})$. By carrying out this construction in each fiber $X_z^\vee$ over $z \in \Delta^*$, we obtain a function on Δ^*. By a theorem of Griffiths, this function is holomorphic on Δ^* and horizontal [13, Lemma 7.9], where the latter condition is the analogue of the defining condition in Definition 4.6.2.

When $\mathcal{X}^\vee$ is the mirror quintic family, $\nu_z(\eta)$ takes a particularly simple form. Slightly more generally, we will assume that the Hodge numbers of $\mathcal{X}^\vee$ are as follows.

Assumption 4.6.5. The Hodge numbers of $X_z^\vee$ satisfy

$$h^{3,0}(X_z^\vee) = h^{2,1}(X_z^\vee) = h^{1,2}(X_z^\vee) = h^{0,3}(X_z^\vee) = 1 \tag{4.6.7}$$

and $X_z^\vee$ is simply connected.

Assumption 4.6.5 enables us to express ν in terms of $[C_z]$ and the Hodge-theoretically normalized volume form. As noted in Remark 4.5.9, this Assumption allows us to write the connection matrix for the Gauss–Manin connection in the form (4.5.5).

As explained in [12, §3.1], one can choose a canonical lift of ν by considering the monodromy logarithm, which we write as

$$\widetilde{\nu} = \mathcal{W}_1 e_1 + \mathcal{W}_0 e_0 \, .$$

This is a section of the local system $R^3\pi_*\mathbb{Z} \otimes \mathcal{O}_{\Delta^*}$, where $\pi\colon \mathcal{X}^\vee \to \Delta^*$ is the projection map for the family. Here we can write

$$\mathcal{W}_0(z) = \int_{\Gamma_z} \Omega_z \tag{4.6.8}$$

where Ω_z denotes the restriction of the Hodge-theoretically normalized relative volume form Ω to $X_z^\vee$. Using (4.5.5) and horizontality, it follows that $W_1 = z\partial_z \mathcal{W}_0$, meaning that the normal function, and hence the extension of Hodge structures, is completely determined by (4.6.8). It is shown in [12] that $\mathcal{W}_0$ can be written in the form

$$\mathcal{W}_0(z) = \frac{1}{(2\pi i)^2}\frac{\lambda}{r^2}\log^2(z) + \frac{1}{2\pi i}\frac{s}{r}\log(z) + c + w(z)$$

where $w(z)$ extends to a holomorphic function which vanishes at $0 \in \Delta$, up to passing to an r-fold cover

$$\Delta^* \to \Delta^*$$
$$z \mapsto z^r$$

to account for branching of C near the origin.

Remark 4.6.6. The integer r is the order of the large complex structure limit monodromy when restricted to the trivial VHS associated to C. Here λ and s are both integers. It turns out that for (multiples of) the van Geemen lines, we have that $r = 1$, because they are preserved by the monodromy map on $X_z^\vee$. A direct calculation carried out by Jockers–Morrison–Walcher shows that $\lambda = s = 0$. These terms of the normal function are not included in the data of a VSHS, so we will not explain their significance any further.

We obtain an element of the $\frac{1}{2}$th intermediate Jacobian of Definition 4.6.2 by considering the power series in $\mathbb{C}[[z]]$ corresponding to the *holomorphic part* $w(z)$ of the normal function. Denote by

$$\mathcal{H}^B(\mathcal{X}^\vee, C) \tag{4.6.9}$$

the extension of the trivial VSHS $\mathbb{K}_B[[u]]$ by $\mathcal{H}^B(\mathcal{X}^\vee)$. Removing the $\log(z)$ and $\log^2(z)$ terms is necessary because normal functions, as we have defined them in Definition 4.6.2, are required to lie in a quotient of the Deligne lattice, which is a module over $\mathbb{C}[[z]]$, not $\mathbb{C}((z))$.

Remark 4.6.7. While the constant term c is part of the normal function considered in Definition 4.6.2, we note that any constant section is flat, so the extension of VSHS does not depend on the constant term of the normal function. Classically, the constant term determines the integral lattice on an extension of VHS, so this reflects the fact that a VSHS does not include the data of an integral lattice. This is the reason that we are unable to directly give an interpretation of the integral structure on the A-model using homological mirror symmetry. Analogously, in the closed string case one cannot obtain the Gamma conjecture directly from Theorem 4.5.10.

4.7 Open Gromov–Witten invariants and relative period integrals

We will prove Theorem 1.0.1 in this section by establishing an analogue of Theorem 4.5.10 for extensions of VHS. Stating a fully general open string analogue of this Theorem would require an appropriate notion of a mirror pair of objects. It would be difficult to formulate this notion in a canonical way compatible with homological mirror symmetry, since the support of a sheaf is not a categorical invariant, i.e. autoequivalences of the derived category need not preserve the support of an object. Thus we will content ourselves with stating our comparisons of VSHSs in terms of a specific choice of mirror functor. This is part of the reason that the statement of Theorem 1.2.6 referred to the quasi-equivalence constructed in [15].

Assumption 4.7.1. (X, ω) is a symplectic Calabi–Yau threefold and $X^\vee \to \mathcal{M}_B$ is a relative scheme of dimension 3 coming from a family of smooth Calabi–Yau threefolds with maximally unipotent monodromy. There is a fixed quasi-equivalence

$$\mathcal{F}(X, \omega) \simeq D^b_{dg}\operatorname{Coh}(X^\vee) \tag{4.7.1}$$

where the coefficient fields are related by the mirror map $\mathbb{K}_A \to \mathbb{K}_B$.

Moreover let $(L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1) \in \mathcal{F}(X, \omega)$ be a pair of cleanly immersed Lagrangian branes in X such L_0 and L_1 intersect cleanly and such that the Lagrangian immersion $L := L_0 \sqcup L_1 \looparrowright X$ is nullhomologous. The disjoint union carries a natural local system and bounding cochain, and the resulting Lagrangian brane is denoted (L, ∇, b).

We assume that (L_i, ∇_i, b_i) corresponds under (4.7.1) to the pushforward of a vector bundles of rank m, denoted $\mathcal{L}_i$, supported on a curve $C^i \subset X^\vee$, and that the restrictions C^i_z of these curves to the smooth fibers $X^\vee_z$ lie in the same homology class. In other words, we have a family $C_z :=$

$C_z^0 - C_z^1$ of homologically trivial algebraic cycles in $X_z^\vee$.

Considering a pair of branes, rather than a single brane as in [38], does not lead to a loss of generality given the other parts of Assumption 4.7.1, since there are no nullhomologous algebraic *curves* on the B-side. Keeping track of the homology classes of curves will be crucial to our comparison of B-model extensions of VSHS.

We will also impose a rather restrictive assumption on the Hodge numbers of the A-model symplectic manifold X, which are satisfied by the quintic.

Assumption 4.7.2. The Hodge numbers of X satisfy

$$h^{0,0}(X) = h^{1,1}(X) = h^{2,2}(X) = h^{3,3}(X) = 1$$

and X is simply connected. In particular $X^\vee$ satisfies Assumption 4.6.5 by mirror symmetry.

This assumption enables us to make use of a basis for the VSHS $\mathcal{H}^A(X, \omega)$ as described in Remark 4.5.6, which in turn lets us easily construct an A-model extension of VSHS. Despite their restrictive nature, Assumptions 4.7.1 and 4.7.2 encompass the examples of the real quintic [11] and the Lagrangian $\widetilde{L}^5_{\mathrm{sm}}$, which are the only examples of open mirror pairs, as defined in [12], in the literature.

One would expect the A-model analogue of the extension of VHS associated to a nullhomologous algebraic cycle to come from Solomon–Tukachinsky's relative quantum cohomology [14] of a nullhomologous Lagrangian. We cannot appeal to their construction directly, since our definition of the open Gromov–Witten potential differs from theirs at the chain level. Although we could define a relative quantum connection using the results of §1.3.4, we would require a different proof of the flatness of this connection.

We will avoid this issue by constructing an extension of VSHS the trivial VSHS $\mathbb{K}_A$ by $\mathcal{H}^A(X, \omega)$ at the homological level by specifying a normal function in terms of the (∞-cyclic) open Gromov–Witten potential. The isomorphisms of Theorem 4.5.10, and in particular Assumption 4.7.1, then allow us to construct a B-model extension of the trivial VSHS $\mathbb{K}_B$ by $\mathcal{H}^B(X^\vee)$. Our description of

B-model normal functions is then rigid enough that we can prove this extension of VSHSs to be equivalent to the one described in the previous subsection.

Fix a bounding 4-chain Γ for the nullhomllogous Lagrangian immersion L. In what follows, we will write Ψ for the open Gromov–Witten potential $\Psi_{L,\Gamma}(b)$ of Corollary 4.4.5, which we will now think of as a function of the Novikov variable Q.

Proposition 4.7.3. *Let* $\{e_3, e_2, e_1, e_0\}$ *denote the basis for the even degree part of* $\mathcal{H}^A(X, \omega)$ *specified in Remark 4.5.6. Then*

$$\widetilde{\nu}_A \coloneqq Q\partial_Q \Psi(Q) e_1 + \Psi(Q) e_0$$

descends to a well-defined normal function ν_A *for* $\mathcal{H}^A(X, \omega)$.

Proof. Horizontality follows from the form of the connection matrix in Remark 4.5.6, from which we can directly compute that

$$\nabla \widetilde{\nu}_A = -(Q\partial_Q)^2 \Psi(Q) e_1 \, .$$

That this is well-defined, i.e. independent of the almost complex structure on X, follows because L is nullhomologous by Corollary 4.4.5. Moreover, replacing Γ with another bounding 4-chain does not affect ν_A as an element in the intermediate Jacobian. To see this, note that if Γ_1 and Γ_2 are two choices of bounding 4-chains, then their difference can be represented as a closed smooth singular 4-cycle, which implies that OGW potentials obtained from these two chains only differ by classes in $\widetilde{F}^{\geq k}\widetilde{\mathcal{V}}$. □

Definition 4.7.4. Denote by $\mathcal{H}^A(X)_L$ the extension of VSHSs associated to $\nu_{\mathcal{F}}$.

Remark 4.7.5. For a nullhomologous embedded Lagrangian, the extension of VSHSs constructed this way is just the relative quantum cohomology of [14].

Using our characterization of the open Gromov–Witten potential, we can lift this to a normal function for the negative cyclic homology of $\mathcal{F}(X, \omega)$.

Proposition 4.7.6. *Under assumptions 4.8.5, 4.9.3, and 4.9.2, there is a normal function $\nu_{\mathcal{F}}$ in $HC_*^-(\mathcal{F}(X,\omega))$ such that*

$$OC^-(\nu_{\mathcal{F}}) = \nu_A\,.$$

It follows that the negative cyclic open-closed map induces an isomorphisms between the extensions of VSHS obtained from these normal functions.

Proof. Notice that the terms of $-(Q\partial_Q)^2\Psi_b(Q)$ involving b lie in the image of the OC^- by construction, since the ∞-inner product used to define Ψ is defined using values of the cyclic open-closed map. Assumption 4.8.5, the divisor axiom, and Assumption 4.7.2 imply that we can write

$$-(Q\partial_Q)^2\mathfrak{m}_{-1} = -\mathfrak{q}_{-1,2}(\omega\otimes\omega)\,.$$

On the other hand, the right hand side can be expressed in terms of operators with horocyclic constraints via

$$\mathfrak{q}_{-1,2}(\omega\otimes\omega) = \langle 1, \mathfrak{q}_{0,2;\perp_0}(\omega\otimes\omega)\rangle$$

and the expression on the right hand side is in the image of the open-closed map. Hence we have shown that $\nabla\widetilde{\nu}_A$ lies in the image of the open-closed map. Using Assumption 4.9.3, which says that OC^- is an isomorphism and that it intertwines the Getzler–Gauss–Manin and quantum connections establishes the lift of an element $\widetilde{\nu}_F \in HC_*^-(\mathcal{F}(X,\omega))$ whose open-closed image is $\widetilde{\nu}_A$, from which we obtain the normal function $\nu_{\mathcal{F}}$. The assumption that OC^- is an isomorphism of VSHSs implies that $\nu_{\mathcal{F}}$ is a normal function as well. □

Definition 4.7.7. Let $HC_*^-(\mathcal{F}(X))_L$ denote the extension of VSHSs determined by $\nu_{\mathcal{F}}$.

Remark 4.7.8. We have not checked whether $\nu_{\mathcal{F}}$ is the normal function associated to the extension of VSHSs constructed from relative negative cyclic homology defined in [38, §2.3]. This statement is essentially the content of [38, §5], but those arguments are phrased at the chain-level, and thus do not immediately adapt to our setting.

Corollary 4.7.9. *Recall that $\mathcal{L}_i \in D^b_{dg}\operatorname{Coh}(X^\vee)$ denotes the mirror sheaf to (L_i, ∇_i, b_i), for $i =$* $0, 1$*, which is a rank m vector bundle over C_i. with respect to the particular mirror functor* (4.7.1). *Then we have isomorphisms of (extensions of) VSHS that fit into the following diagram*

$$\begin{array}{ccc} HC^-_*(\mathcal{F}(X))_L & \longrightarrow & HC^-_*(D^b_{dg}\operatorname{Coh}(X^\vee))_{\mathcal{L}_0-\mathcal{L}_1} \\ \big\downarrow{\scriptstyle OC^-} & & \big\downarrow{\scriptstyle \widetilde{\mathfrak{I}}_{\mathrm{HKR}}} \\ \mathcal{H}^A(X,\omega)_L & & \mathcal{H}^B(X^\vee)_{\mathcal{L}_0-\mathcal{L}_1}\,. \end{array} \tag{4.7.2}$$

Here the horizontal arrow is induced by our choice of mirror functor, and the vertical arrows come from OC^- and $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ (cf. Proposition 4.6.3).

Here the B-model VSHSs are defined such that all arrows in the diagram are isomorphisms. The corollary is immediate from Theorem 4.5.10. We can think of the vector spaces underlying the categorical extensions of VSHS in (4.7.2) as the *relative negative cyclic homology* of [38].

To obtain predictions for open Gromov–Witten invariants from this, we must prove that:

Proposition 4.7.10. *The extension of VSHS $\mathcal{H}^B(X^\vee)_{\mathcal{L}_0-\mathcal{L}_1}$ is isomorphic to the extension of VSHS associated to the family of cycles mC.*

Proof. We begin by determining the image of the sheaf $\mathcal{L}_i$ under the HKR isomorphism. Recall that $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ is induced from the HKR isomorphism

$$HH_*(X^\vee) \to H^*(X^\vee)\,.$$

On the other hand [60, Theorem 4.5] implies that the ordinary Chern character on K-theory $\mathrm{ch}\colon K_0(X^\vee) \to H^*(X^\vee)$ factors as the composition of the Chern character $K_0(X^\vee) \to HH_*(X^\vee)$ with the HKR isomorphism. Thus we see that image of $\mathcal{L}_i$ under $\widetilde{\mathfrak{I}}_{\mathrm{HKR}}$ is its algebraic second Chern class $c_2(\mathcal{L}_i)$. Our assumption that $\mathcal{L}_i$ is the pushforward of a vector bundle implies that the algebraic Chern class is represented by mC^i, where m is the rank of the vector bundle and C^i is the support of the sheaf.

There is a $\mathbb{C}$-local system on Δ^* associated to $\mathcal{H}^B(X^\vee)_{\mathcal{L}_0-\mathcal{L}_1}$, and the previous paragraph implies that it is given fiberwise by (4.6.4). As discussed in the proof of Proposition 4.6.3 and around (4.6.6), a normal can be determined by specifying a suitable pair of elements in $\mathcal{H}^B(X^\vee)_{\mathcal{L}_0-\mathcal{L}_1}$ which are both mapped to $1 \in \mathbb{K}_B$. By (4.6.4), such elements are given by relative chains with boundary mC. Hence by Poincaré duality and type considerations, the normal function for $\mathcal{H}^B(X^\vee)_{\mathcal{L}_0-\mathcal{L}_1}$ can be written as (4.6.6). As an element of the intermediate Jacobian, the normal function thus described is independent of the choice of bounding chain. □

We can now complete the proofs of Theorems 1.3.7 and 1.0.1 from the introduction. bo

Proof of Theorem 1.3.7. This is an immediate consequence of Corollary 4.7.9 and Proposition 4.7.10. □

Proof of Theorem 1.0.1. Using the notation of this section, let L_0 and L_1 both denote the Lagrangian branes specified in the statement of Theorem 1.0.1. More precisely, these are copies of $\widetilde{L}^5_{\mathrm{sm}}$ equipped with local systems obtained from those in Example 4.3.8 via Definition 4.3.32. The difference of open Gromov–Witten potentials of these two branes can be thought of as the open Gromov–Witten potential associated to the immersed Lagrangian consisting of the union of these two branes, where the orientation on L_1 is reversed. To define the open Gromov–Witten potential, we use the degenerate 4-chain interpolating between these two copies of the same Lagrangian submanifold.

Using the A_∞-functor (4.3.31) and the result of Theorem 4.3.21, we can think of these branes as mirrors to pushforwards of line bundles on the van Geemen lines, so that the second Chern classes of these objects are represented by $-[C^{\omega^i}]$. The result of Theorem 4.3.21 says that Assumption 4.7.1 is satisfied is satisfied in this context, so we can appeal to Theorem 1.3.7.

A solution to an inhomogeneous Picard–Fuchs equation associated to the van Geemen line C^ω_z was computed by Walcher [19, (6.12)], and is written there after a change of coordinates given by the mirror map. The value of the relative period integral (4.6.8) over a chain with boundary $C_z^{\omega} - C_z^{\omega^2}$ is twice the value of the solution to the Picard–Fuchs equation (cf. [19, (2.9)]). Since the branes $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega})$ and $(\widetilde{L}^5_{\mathrm{im}}, \nabla^{\mathrm{vG}}_{\omega^2})$ are mirror to vector bundles on the van Geemen lines of rank $m = 1$, it follows that the values of the open Gromov–Witten invariants differ from the values calculated by Walcher by an overall factor of 2. □

4.8 Background on immersed Floer theory

This appendix is meant to fix notation and to collect some mostly standard facts regarding the Floer complex of an immersed Lagrangian with clean self-intersections. In doing so, we will also explain how the constructions of [43] should be modified in the presence of a grading and rank 1 local system, and we will construct a spectral sequence using the energy filtration that converges to the Floer cohomology of a Lagrangian immersion.

The Lagrangian Floer cochain spaces we consider have coefficients in various versions of the Novikov ring.

Definition 4.8.1. The *Novikov ring* over $\mathbb{C}$ is the formal power series ring

$$\Lambda_0 = \left\{ \sum_{j=0}^{\infty} a_j Q^{\lambda_j} : a_j \in \mathbb{C}\,,\ \lambda_i \in \mathbb{R}_{\geq 0}\,,\ \lim_{j\to\infty} = \infty \right\} . \tag{4.8.1}$$

Here Q is called the Novikov variable, and the grading on Λ_0 is trivial. There is a natural valuation $v\colon \Lambda_0 \to \mathbb{R}_{\geq 0}$ given by

$$v\left(\sum_{j=0}^{\infty} a_j Q^{\lambda_j}\right) = \min\left\{\lambda_j : a_j \neq 0\right\} . \tag{4.8.2}$$

Let $\Lambda_+ \subset \Lambda_0$ denote the subset of elements with strictly positive valuation. Correspondingly, the set of *unitary Novikov elements* $U_\Lambda \subset \Lambda_0$ is the set of valuation 0 elements. More explicitly, we have that

$$U_\Lambda := \left\{ a_0 + \sum_{j=1}^{\infty} a_j Q^{\lambda_j} \in \Lambda \colon a_0 \in \mathbb{C}^* \right\}.$$

The *Novikov field* Λ is obtained from Λ_0 by localizing at the ideal (Q), meaning that

$$\Lambda = \left\{ \sum_{j=0}^{\infty} a_j Q^{\lambda_j} \colon a_j \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lim_{j\to\infty} = \infty \right\}. \tag{4.8.3}$$

Note that any element of $\mathbb{C}^* \subset \Lambda_0$ is unitary. In general, to obtain $\mathbb{Z}$-graded A_∞-algebras, one would require the Novikov ring to carry a nontrivial grading to account for the Maslov indices of disks, but since we only consider *graded* Lagrangian immersions in this paper, we do not need this.

4.8.1 Closed-open operators for immersed Lagrangians

Let L be a closed Lagrangian immersion with clean self-intersections in a symplectic manifold (M, ω). Now suppose that we have equipped L with a spin structure. As explained in [90, Ch. 8], the choice of spin structure on L induces *orientation local systems* on each of the switching components L_a. Let Θ_a^- denote the complex line bundle on L_a associated to the orientation local system. For any $a \in A$, let $\Omega^*(L_a; \Theta_a^-)$ denote the space of smooth differential forms valued in Θ_a^-.

Definition 4.8.2. As a graded module over Λ_0, we define the Floer cochain space of a graded spin Lagrangian immersion with clean self-intersections $\iota \colon L \to M$ to be the space obtained from the graded $\mathbb{C}$-vector space

$$\overline{CF}^*(L) := \Omega^*(L_0) \oplus \bigoplus_{a\in A} \Omega^*(L_a; \Theta_a^-)[\deg(L_{a_-})] \tag{4.8.4}$$

by taking the completed tensor product with Λ, i.e.

$$CF^*(L) := \overline{CF}^*(L) \widehat{\otimes}_{\mathbb{C}} \Lambda_0 \,. \tag{4.8.5}$$

Fix an ω-compatible almost complex structure J on M. The A_∞-structure maps on $CF^*(L)$ count J-holomorphic disks with boundary and corners on the image $\iota(L)$ of the Lagrangian immersion.

Definition 4.8.3. Let $k \geq -1$ and $\ell \geq 0$ be integers. A J-holomorphic disk with corners of degree $\beta \in H_2(M, \iota(L); \mathbb{Z})$ consists of the data $(u, \vec{z}, \vec{w}, \vec{a}, \widetilde{\partial u})$, where

(i) $u\colon D^2 \to M$ is J-holomorphic, and $u(\partial D^2) \subset \iota(L)$;

(ii) $[u] = \beta$;

(iii) $\vec{z} = (z_1, \ldots, z_k)$ is a cyclically ordered collection of mutually distinct boundary marked points, ordered counterclockwise (with respect to the boundary orientation inherited form the complex structure);

(iv) $\vec{w} = (w_1, \ldots, w_\ell)$ is an ordered collection of mutually distinct interior marked points;

(v) $\vec{a} = (a_0, \ldots, a_k) \in A^{k+1}$ assigns a component of $L \times_\iota L$ to each boundary marked point z_i;

(vi) $\widetilde{\partial u}\colon \partial D^2 \setminus \{z_0, \ldots, z_k\} \to L$ is a smooth map satisfying

$$\iota \circ \widetilde{\partial u} \equiv u|_{\partial D^2}$$

which asymptotically approaches L_{a_i} as one approaches a boundary marked points, i.e.

$$\left(\lim_{z \to z_i^-} \widetilde{\partial u}(z), \lim_{z \to z_i^+} \widetilde{\partial u}(z) \right) \in L_{a_i}$$

for all $i = 0, \ldots, k$.

(vii) The set of biholomorphic maps $\phi\colon D^2 \to D^2$ for which

(a) $u \circ \phi = u$;

(b) $\phi(z_i) = z_i$ and $\phi(w_j) = w_j$;

(c) $\widetilde{\partial u} \circ \phi = \widetilde{\partial u}$

is finite.

Two J-holomorphic disks with corners are said to be equivalent if they are related by an automorphism as in (vii). The *(uncompactified) moduli space* $\overset{\circ}{\mathcal{M}}_{k+1,\ell}(L; \vec{a}; \beta; J)$ of J-holomorphic disks is the set of all such equivalence classes. In the case $k = -1$, this should be understood as the moduli space of disks with no boundary marked points, and hence no corners. We will drop L and J from the notation for these moduli spaces when it will not cause confusion.

The moduli spaces defined in Definition 4.8.3 have Gromov compactifications denoted $\mathcal{M}(L; \vec{a}; \beta)$, which are discussed in detail in [43, §3.2]. The only essential difference between the embedded and immersed Lagrangians is that disks with boundary on the latter can form nodes which lie on the switching components, but the data of (vi) above determines how such nodal disks can be glued. The elements of these moduli spaces are represented by bordered stable maps with nodes. Note that by stability, the moduli spaces $\mathcal{M}_{1,0}(L; \vec{a}; 0)$ and $\mathcal{M}_{2,0}(L; \vec{a}; 0)$ are empty.

There are natural evaluation maps

$$\begin{aligned} \mathrm{evb}_i^{\vec{a};\beta} &\colon \mathcal{M}_{k+1,\ell}(L; \vec{a}; \beta) \to L_{a_i} \\ \mathrm{evi}_j^{\vec{a};\beta} &\colon \mathcal{M}(L; \vec{a}; \beta) \to M \end{aligned}$$

for $i = 0, \ldots, k$ and $j = 1, \ldots, \ell$ at the boundary and interior marked points on these moduli spaces. On the loci of irreducible curves, these are given explicitly by the formulas

$$\begin{aligned} \mathrm{evb}_i^{\vec{a};\beta} \colon \overset{\circ}{\mathcal{M}}_{k+1,\ell}(L; \vec{a}; \beta) &\to L_{a_i} \subset L \times_\iota L \\ [(u, \vec{z}, \vec{w}, \vec{a}, \widetilde{\partial u})] &\mapsto \left(\lim_{z \to z_i^-} \widetilde{\partial u}(z), \lim_{z \to z_i^+} \widetilde{\partial u}(z) \right) \end{aligned}$$

and

$$\mathrm{evi}_j^{\vec{a},\beta} \colon \overset{\circ}{\mathcal{M}}_{k+1,\ell}(L; \vec{a}; \beta) \to M$$

$$[(u, \vec{z}, \vec{w}, \vec{a}, \widetilde{\partial u})] \mapsto u(w_j)\,.$$

By [43, Theorem 3.24, Proposition 3.30], one can construct a system of Kuranishi structures on the moduli spaces $\mathcal{M}_{k+1,0}(L; \vec{a}; \beta)$, along with continuous families of perturbations on these moduli spaces which allow one to treat the evaluation maps evb_0 as smooth submersions. This means that there is a well-defined pushforward of differential forms.

Remark 4.8.4. Combining this with the results of [48], one can also obtain such Kuranishi structures for the moduli spaces $\mathcal{M}_{k+1,\ell}(L; \vec{a}; \beta)$ with little effort. More precisely, we equip these moduli spaces with Kuranishi structures obtained from those on $\mathcal{M}_{k+1}(L; \vec{a}; \beta)$ via pullback with respect to the forgetful maps of interior marked points, as in [48]. Defining the Kuranishi structures this way means that the evaluation maps $\mathrm{evi}_j^{\vec{a};\beta}$ along interior marked points will *not* be weakly submersive, as explained in [48, Remark 3.3], but we expect that this is not needed, since it is possible to define open-closed maps valued in differential currents, as we explain below. Since we have allowed ourselves to consider Lagrangian immersions, such a definition should be sufficient for defining cyclic open-closed maps on the full Fukaya category of Definition 4.8.9.

Fix $\beta \in H_2(M, L)$ and integers $k, \ell \geq 0$ for which $(k, \ell, \beta) \notin \{(0,0,0), (1,0,0)\}$. Also fix a sequence $\vec{a} = (a_0, \ldots, a_k)$. We define degree 1 operators

$$\mathfrak{q}_{k,\ell;\vec{a};\beta} \colon (\overline{CF}^*(L)[1])^{\otimes k} \otimes (\Omega^*(M)[2])^{\otimes \ell} \to \overline{CF}^*(L)[1]$$

$$\mathfrak{q}_{k,\ell;\vec{a};\beta}(\otimes_{i=1}^k \alpha_i; \otimes_{j=1}^\ell \gamma_j) := (-1)^* (\mathrm{evb}_0^{\vec{a};\beta})_* \left(\bigwedge_{i=1}^k (\mathrm{evb}_i^{\vec{a};\beta})^* \alpha_i \wedge \bigwedge_{j=1}^\ell (\mathrm{evi}_j^{\vec{a};\beta})^* \gamma_j \right)$$

where $\alpha_i \in \Omega^*(L_{a_i})$ for all $i = 1, \ldots, k$. The sign is determined by the formula

$$* = \sum_{i=1}^k i(\deg \alpha_i + 1) + 1 \tag{4.8.6}$$

where $\deg \alpha_i$ denotes its degree in $\overline{CF}^*(L)$. These determine maps

$$\mathfrak{q}_{k,\ell;\beta}\colon (\overline{CF}^*(L)[1])^{\otimes k} \to \overline{CF}^*(L)[1]$$

in an obvious way by summing over all sequences $\vec{a} \in A^{k+1}$. Also set

$$\mathfrak{q}_{0,0;0} = 0$$
$$\mathfrak{q}_{1,0;0} = d$$

where d denotes sum of de Rham differentials on the components of $\overline{CF}^*(L)$.

Suppose that L has been equipped with a U_Λ-local system denoted ∇. We can extend these to operators

$$\mathfrak{q}_{k,\ell}\colon (CF^*(L)[1])^{\otimes k} \otimes (\Omega^*(M;\Lambda_+)[2])^{\otimes \ell} \to CF^*(L)[1] \tag{4.8.7}$$

by setting

$$\mathfrak{q}_{k,\ell} := \sum_{\beta \in H_2(M;\iota(L))} \mathrm{hol}_\nabla(\partial\beta)\mathfrak{q}_{k,,\ell;\beta} Q^{\omega(\beta)} \otimes \mathrm{id}_{\Lambda_0}\ . \tag{4.8.8}$$

Let G_L denote the smallest submonoid of $\mathbb{R}_{\geq 0}$ which contains the symplectic area of every J-holomoprhic disk with corners and boundary on $\iota(L)$. As a consequence of Remark 4.8.4, these operations satisfy a *divisor axiom* with respect to forgetting interior marked points.

Lemma 4.8.5 (Divisor axiom)**.** *For any $\gamma_1 \in \Omega^2(X)$ with $d\gamma = 0$, the operators $\mathfrak{q}_{k,\ell;\beta}$ should satisfy the divisor axiom*

$$\mathfrak{q}_{k,\ell;\beta}(\otimes_{i=1}^k \alpha_i; \otimes_{j=1}^\ell \gamma_j) = \left(\int_\beta \gamma_1\right) \cdot \mathfrak{q}_{k,\ell-1;\beta}(\otimes_{i=1}^k \alpha_i; \otimes_{j=2}^\ell \gamma_j)\ . \tag{4.8.9}$$

□

The divisor axiom is used to relate derivatives of the open Gromov–Witten potential to the open Gromov–Witten invariants with interior constraints.

Imitating the proof of [43, Proposition 3.35], while keeping track of gradings and holonomy, we obtain the following.

Proposition 4.8.6. *For all $k \geq 0$, define $\mathfrak{m}_k := \mathfrak{q}_{k,0}$. Then $CF^*(L,\nabla) := (CF^*(L);\{\mathfrak{m}_k\}_{k=0}^{\infty})$ has the structure of a strictly unital G_L-gapped filtered A_∞-algebra, as defined in [47, Ch. 3.2]. In particular, the operators $\mathfrak{m})_k$ satisfy the curved A_∞-relations* (3.1.8).

The strict unit above is given by the constant 0-form on the diagonal component with value 1. We have denoted the A_∞-algebra by $CF^*(L,\nabla)$ to emphasize the role of the local system. Gappedness is a consequence of Gromov compactness. By [43, Remark 3.44], the zeroth order part of $\mathfrak{m}_2$ is given by

$$\mathfrak{m}_{2,0}(\alpha_1,\alpha_2) = (-1)^{\deg\alpha_1}\alpha_1 \wedge \alpha_2 \tag{4.8.10}$$

where the right hand side denotes the wedge product of two differential forms on the same connected component of $L \times_\iota L$.

4.8.2 Horocyclic operators

To understand how the open Gromov–Witten invariants are determined by the Fukaya category, we will need to introduce operators

$$\mathfrak{q}_{k,\ell;\perp_i} \colon (CF^*(L;\nabla)[1])^{\otimes k} \otimes (\Omega^*(M;\Lambda)[2])^{\otimes \ell} \to CF^*(L;\nabla)[1]$$

using moduli spaces of disks with horocyclic constraints, as in [38]. Similar moduli spaces are used by [14] to prove flatness of the relative quantum connection. The irreducible loci of the moduli spaces we consider are the subsets $\mathring{\mathcal{M}}_{k+1,\ell,\perp_i}(\vec{a};\beta) \subset \mathring{\mathcal{M}}_{k+1,\ell}(\vec{a};\beta)$ defined by requiring the boundary marked point z_i and the interior marked points w_1 and w_2 to lie on a horocycle in the domain (i.e. a circle in the disk tangent to a point on the boundary, in this case z_i). We require that (z_i, w_1, w_2) are ordered counterclockwise on the horocycle. The moduli space $\mathring{\mathcal{M}}_{k+1,\ell,\perp_i}(\vec{a};\beta)$ can be written as a fiber product

$$\mathring{\mathcal{M}}_{k+1,\ell,\perp_i}(\vec{a};\beta) = I \times_{D^2} \mathring{\mathcal{M}}_{k+1,\ell}(\vec{a};\beta)\,.$$

Thus, given a Kuranishi structure on $\mathcal{M}_{k+1,\ell}(\vec{a};\beta)$, it should also be possible to construct one on $\mathcal{M}_{k+1,\ell,\perp_i}(\vec{a};\beta)$. For any $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_k$ with $\alpha_i \in \Omega^*(L_{a_i})$ and $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_\ell \in \Omega^*(M)^{\otimes \ell}$ set

$$\mathfrak{q}_{k,\ell;\vec{a};\beta;\perp_i}(\alpha;\gamma) \coloneqq (-1)^{*_\perp}(\mathrm{evb}_0^{\vec{a};\beta})_* \left(\bigwedge_{j=1}^{\ell} (\mathrm{evi}_j^{\vec{a};\beta})^* \gamma_j \wedge \bigwedge_{i=1}^{k} (\mathrm{evb}_i^{\vec{a};\beta})^* \alpha_i \right)$$

where

$$*_\perp \coloneqq * + \sum_{i=1}^{k} (\deg \alpha_i + 1) + \sum_{j=1}^{\ell} \deg \gamma_j$$

for $*$ as in (4.8.6). Then set

$$\mathfrak{q}_{k,\ell;\perp_i} \coloneqq \sum_{\vec{a}\in A^k} \sum_{\beta \in H_2(M,L;\mathbb{Z})} \mathrm{hol}_\nabla(\partial\beta)\, \mathfrak{q}_{k,\ell;\vec{a};\beta;\perp_i} Q^{\omega(\beta)} \otimes \mathrm{id}_{\Lambda_0}\ .$$

4.8.3 The Fukaya category and the open-closed map

Definition 4.8.7. Let $b \in \overline{CF}^1(L)\widehat{\otimes}_{\mathbb{C}}\Lambda_+$ be a degree 1 Floer cocycle of strictly positive valuation (cf. Definition 4.8.1). We say that it is a *bounding cochain* if it is a solution to the *Maurer–Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = 0\,. \tag{4.8.11}$$

Let $\mathcal{M}(L, \nabla)$ denote the space of equivalence classes of bounding cochains on $CF^*(L, \nabla)$ up to *gauge-equivalence*, in the sense of [47, Definition 4.3.1].

In the presence of a bounding cochain, one can define uncurved A_∞-operations

$$\mathfrak{m}_k^b(\alpha_1 \otimes \cdots \otimes \alpha_k) := \sum_{\substack{\ell \geq 0 \\ \ell_0 + \cdots + \ell_k = \ell}} \mathfrak{m}_{k+\ell}(b^{\otimes \ell_0} \otimes \alpha_1 \otimes b^{\ell_1} \otimes \cdots \otimes b^{\ell_{k-1}} \otimes \alpha_k \otimes b^{\ell_k})$$

where the sum is over all k-part partitions of arbitrary length. In this situation, we define the *Lagrangian Floer homology* $HF^*(L, \nabla, b)$ to be the cohomology of $CF^*(L)$ with respect to the deformed differential $\mathfrak{m}_1^b$. The objects of the Fukaya category should roughly be *unobstructed* Lagrangians. We formalize this notion in the following definition.

Definition 4.8.8. An *(unobstructed) Lagrangian brane* consists of a Lagrangian immersion $\iota \colon L \to M$ with clean self-intersections, together with a rank one U_Λ-local system ∇, a spin structure on L, a grading $\alpha^\# \colon L \to \mathbb{R}$, and a bounding cochain $b \in CF^1(L)$.

The construction of (any finite subcategory of) the Fukaya category $\mathcal{F}(M)$, with coefficients in the Novikov field Λ, can be reduced to the construction of the A_∞-algebra of a single *immersed* Lagrangian. Specifically, fix a finite collection of unobstructed Lagrangian branes $\mathbb{L} := \{L_c \colon c \in \mathfrak{D}\}$, where $\mathfrak{D}$ is an index set, which intersect each other cleanly. In particular, there is a clean Lagrangian immersion $L_{\mathfrak{D}} := \coprod_{c \in \mathfrak{D}} L_c$ obtained by taking the union of Lagrangian immersions over $\mathfrak{D}$. We give this a brane structure by requiring the bounding cochain on $L_{\mathfrak{D}}$ to vanish on components of $L_{\mathfrak{D}} \times_M L_{\mathfrak{D}}$ corresponding to components of $L_c \cap L_{c'}$ for $c \neq c' \in \mathfrak{D}$ (and to restrict to the given bounding cochains on each L_c). The local system, spin structure, and grading are defined on the domain of an immersion, so the correct choices of such data for $L_{\mathfrak{D}}$ are clear.

Definition 4.8.9. The Fukaya category $\mathcal{F}(M)$ has objects $\operatorname{Ob} \mathcal{F}(M) := \mathbb{L}$. The hom sets and A_∞-composition maps on $\mathcal{F}(M)$ are inherited from $CF^*(L_{\mathfrak{D}}) \otimes \Lambda$ (cf. [43, §3.4]).

Remark 4.8.10. This construction depends on a choice of branes $\mathbb{L}$. In practice, we can usually take a collection of objects whose disjoint union satisfies Abouzaid's generation criterion. For

Calabi–Yau hypersurfaces in toric Fano varieties, a set of generators is identified in [15], and we can adjoin any finite collection of Lagrangian branes to this collection. This means that the split-derived category $D^{\pi}\mathcal{F}(M)$ obtained from $\mathcal{F}(M)$ will be independent of the choice of $\mathbb{L}$ provided that it contains a suitable generating set. In particular, this construction should suffice for many applications in homological mirror symmetry. We remark that the proofs of our main results essentially only use *homological* data, so we do not lose information by passing to the derived category.

Remark 4.8.11. Recall that the proofs of homological mirror symmetry in [15, 16, 17] all involve computations in the *relative Fukaya category* $\mathcal{F}(X, D)$, as constructed in [82]. Let (X, ω) be a closed symplectic manifold and $D \subset X$ a simple normal crossings divisor Poincaé dual to D. The objects of the relative Fukaya category are the same as the objects of the Fukaya category of compact (exact) Lagrangians $\mathcal{F}(X \setminus D)$, but the A_∞-operations on $\mathcal{F}(X, D)$ count pseudoholomorphic disks in X weighted by their intersection number with D. Moreover, it is natural to require that the objects of $\mathcal{F}(X \setminus D)$ only carry $\mathbb{C}^*$-local systems, since the Fukaya category of exact Lagrangians is naturally defined over $\mathbb{C}$, not just the Novikov field. A suitable analogue of the A_∞ category of [82, Definition 1.6], which has coefficients in a universal Novikov field, can be obtained by defining $\mathcal{F}(X)$ as in Definition 4.8.9 using a collection of objects that avoids D. Thus, in practice, we will not distinguish between the full and relative Fukaya categories.

One benefit of this construction of the Fukaya category is that it allows us to define the open-closed map

$$\mathcal{OC}_0\colon HH_*(\mathcal{F}(M)) \to H^{*+n}(M; \Lambda_0) \tag{4.8.12}$$

in terms of the open-closed map for the A_∞-algebra of an immersed Lagrangian. One would expect that this should be defined by pulling back differential forms on a Lagrangian immersion L to an open-closed moduli space, and pushing the wedge product of these forms forward along the evaluation map at an interior output marked point w_{out} (using integration over the fiber). The techniques of [18] do not, however, imply that the evaluation maps evi_j at the interior marked points are submersions, but we can use this pairing to define the open-closed map whose chain-level outputs are differential currents. Given any sequence $\vec{a} = (a_1, \dots, a_k) \in A^k$, relabel the boundary marked points on $\mathcal{M}_{k,1}(L;\vec{a};\beta)$ by $\vec{z} = (z_1, \dots, z_k)$ and the interior marked point by w_{out} to get evaluation maps

$$\mathrm{evb}_i^{\vec{a};\beta} \colon \mathcal{M}_{k,1}(L;\vec{a};\beta) \to L_{a_i}$$

for all $i = 1, \dots, k$, and

$$\mathrm{evi}^{\vec{a};\beta} \colon \mathcal{M}_{k,1}(L;\vec{a};\beta) \to M\,.$$

Let $\mathcal{D}^*(M;\mathbb{C})$ denote the space of differential currents on M with coefficients in $\mathbb{C}$. For a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$, where $k \geq 1$ and $\alpha_i \in \Omega^*(L_{a_i})$ for all $i = 1, \dots, k$, we define the differential current

$$\mathfrak{p}_k^{\vec{a};\beta}(\alpha) \coloneqq (\mathrm{evi}^{\vec{a};\beta})_* \left(\bigwedge_{i=1}^{k} (\mathrm{evb}_i^{\vec{a};\beta})^* \alpha_i \right) \in \mathcal{D}^*(M;\mathbb{C})\,.$$

Here the pushforward is only a current because we cannot assume that $\mathrm{evi}^{\vec{a};\beta}$ is a submersion in any sense per Remark 4.8.4. As before, we can extend this to a map

$$\begin{aligned} \mathfrak{p}_k \colon (CF^*(L;\nabla)[1])^{\otimes k} &\to \mathcal{D}^*(M;\Lambda) \\ \mathfrak{p}_k &\coloneqq \sum_{\vec{a} \in A^k} \sum_{\beta \in H_2(M,L)} \mathrm{hol}_\nabla(\partial\beta)\, \mathfrak{p}_k^{\vec{a};\beta} Q^{\omega(\beta)}\,. \end{aligned}$$

Examining the boundary strata of $\mathcal{M}_{k,1}(\beta)$ (cf. [91, Lemma 2.14]) shows that this induces a map

$$\mathcal{OC}_0 \colon HH_*(CF^*(L,\nabla)) \to H^{*+n}(M;\Lambda_0) \tag{4.8.13}$$

on the Hochschild homology of $CF^*(L,\nabla)$ called the open-closed map. Letting $L_{\mathfrak{D}}$ denote the

Lagrangian immersion of Definition 4.8.9, we obtain (4.8.12).

4.8.4 Energy spectral sequence

Applying the algebraic results of [47, Ch. 6] to the current setting, we can construct a spectral sequence from the energy filtration on $CF^*(L)$. Note that the description of the E_2-page of the spectral sequence simplifies as compared to [47, Theorem D] because we have equipped L with a grading (and the grading on our Novikov ring is trivial).

Proposition 4.8.12. *Let ∇ be a rank one U_Λ-local system on L, and let b be a bounding cochain for (L, ∇). Then for any sufficiently small positive real number $\epsilon_0 > 0$, there is a spectral sequence for which*

(i)

$$E_2^{p,q} = H^p(\overline{CF}^*(L)) \otimes (Q^{q\epsilon_0}\Lambda_0/Q^{(q+1)\epsilon_0}\Lambda_0)$$

where $H^p(\overline{CF}^*(L))$ is a complex vector space given by taking the degree p cohomology of $\overline{CF}^*(L)$ equipped with the de Rham differential; and

(ii) there is a filtration $F^\bullet HF^*(L, \nabla, b)$ on the Floer cohomology of L such that

$$E_\infty^{p,q} \cong F^q HF^p(L, \nabla, b)/F^{q+1}HF^p(L, \nabla, b)\,.$$

Proof. The existence of an appropriate constant $\epsilon_0 > 0$ follows because $CF^*(L, \nabla)$ is G_L-gapped. By [47, Theorem 5.4.2], one can construct a canonical model for $CF^*(L, \nabla)$, meaning that it is weakly finite in the sense of [47, Definition 6.3.27]. The result then follows from [47, Theorem 6.3.28]. □

4.8.5 Pseudo-isotopies

Studying the behavior of the open Gromov–Witten potential under changes of the almost complex structure requires an explicit geometric model of the cylinder object $CF^*(L,\nabla)\times[0,1]$ in the category of gapped filtered A_∞-algebras. Consider the graded $\mathbb{C}$-vector space

$$\overline{CF}^*(L\times[0,1]) := \Omega^*([0,1]\times L_0)\oplus\bigoplus_{a\in A\setminus\{0\}}\Omega^*([0,1]\times L_a;\Theta_a^-)[\deg(L_a)]$$

from which we obtain the Λ_+-module

$$CF^*([0,1]\times L) := \overline{CF}^*([0,1]\times L)\widehat{\otimes}_{\mathbb{C}}\Lambda_+\,.$$

The results of [43, §14] imply that for any path $\underline{J}=\{J_t\}_{t\in[0,1]}$ of ω-compatible almost complex structures on M, one can construct a gapped filtered A_∞-structure $CF^*(L\times[0,1])$. Consider the moduli spaces

$$\widetilde{\mathcal{M}}_{k+1}(L;\vec{a};\beta;\underline{J}) := \{(t,[(u,\vec{a},\vec{z},\widetilde{\partial u})])\colon t\in[0,1]\,,\ [(u,\vec{a},\vec{z},\widetilde{\partial u})]\in\mathcal{M}_{k+1}(L;\vec{a};\beta;J_t)\}$$

which are defined for any $k\geq -1$. There are natural evaluation maps

$$\widetilde{\mathrm{evb}}_i^{\vec{a};\beta}\colon \widetilde{\mathcal{M}}_{k+1}(L,\vec{a};\beta;\underline{J})\to[0,1]\times(L\times_\iota L)\,.$$

By [43, Proposition 14.17], the evaluation maps $\widetilde{\mathrm{evb}}_0^{\vec{a};\beta}$ at the zeroth boundary marked points can be treated as smooth submersions, meaning that one can make sense of integration over the fiber with respect to these maps. For all $k\geq 0$ with $(k,\beta)\neq(1,0)$ and any $\vec{a}=(a_0,\dots,a_k)\in A^{k+1}$, we define degree 1 linear maps

$$\widetilde{\mathfrak{m}}_{k;\vec{a};\beta}\colon(\overline{CF}^*([0,1]\times L)[1])^{\otimes k}\to\overline{CF}^*([0,1]\times L)[1]$$

$$\widetilde{\mathfrak{m}}_{k;\vec{a};\beta}(\widetilde{\alpha}_1 \otimes \cdots \otimes \widetilde{\alpha}_k) := (-1)^*(\widetilde{\mathrm{evb}}_0^{\vec{a};\beta})_* \left(\bigwedge_{i=1}^{k} (\widetilde{\mathrm{evb}}_i^{\vec{a};\beta})^* \widetilde{\alpha}_i \right) .$$

The sign $(-1)^*$ is defined the same way as in (4.8.6). By summing over all sequences $\vec{a} \in A^{k+1}$, we obtain operations $\widetilde{\mathfrak{m}}_{k,\beta}$. As before, set $\widetilde{\mathfrak{m}}_{0,0} = 0$ and $\widetilde{\mathfrak{m}}_{1,0}$. Now suppose that L is equipped with a rank one U_Λ-local system. We extend these operations to $CF^*([0,1] \times L)[1]$ to obtain the A_∞-structure maps

$$\widetilde{\mathfrak{m}}_k \colon (CF^*([0,1] \times L, \nabla)[1])^{\otimes k} \to CF^*([0,1] \times L, \nabla)[1]$$
$$\widetilde{\mathfrak{m}}_k := \sum_\beta \mathrm{hol}_\nabla(\partial\beta) \widetilde{\mathfrak{m}}_{k;\beta} Q^{\omega(\beta)} .$$

These operations define a pseudo-isotopy in the sense of [43, Definition 3.36] by [81, Lemma 21.31].

There is also an analogue of the open-closed map on a pseudo-isotopy. These are defined using moduli spaces of $\underline{J}$-holomorphic curves with an interior marked point, where $k \geq 1$ and $\vec{a} \in A^k$, defined by

$$\widetilde{\mathcal{M}}_{k,1}(L; \vec{a}; \beta; \underline{J}) := \{(t, [u]) \colon t \in [0,1] ,\ [u] \in \mathcal{M}_{k,1}(L; \vec{a}; \beta; J_t)\} .$$

where we abuse notation and write $[u]$ for

$$[(u, \vec{a}, \vec{z}, w_{\mathrm{out}}, \widetilde{\partial u})] \in \mathcal{M}_{k,1}(L; \vec{a}; \beta; J_t) .$$

We have relabeled the boundary marked points $\vec{z} = (z_1, \ldots, z_k)$ and the interior marked point w_{out}. These moduli spaces carry evaluation maps

$$\widetilde{\mathrm{evb}}_i^{\vec{a};\beta} \colon \widetilde{\mathcal{M}}_{k,1}(L; \vec{a}; \beta) \to L_{a_i}$$
$$\widetilde{\mathrm{evi}}^{\vec{a};\beta} \colon \widetilde{\mathcal{M}}(L; \vec{a}; \beta) \to M .$$

Since $\widetilde{\mathrm{evi}}$ cannot be treated as a submersion, we have linear operators $\widetilde{\mathfrak{p}}_k^{\vec{a};\beta}$, for all $\beta \in H_2(M, L)$, valued in differential currents $\mathcal{D}^*([0,1] \times M; \mathbb{C})$ defined by

$$\widetilde{\mathfrak{p}}_k^{\vec{a};\beta}(\widetilde{\alpha}) := (\widetilde{\mathrm{evi}}^{\vec{a};\beta})_* \left(\bigwedge_{i=1}^{k} (\widetilde{\mathrm{evb}}_i^{\vec{a};\beta})^* \widetilde{\alpha}_i \right) \in \mathcal{D}^*([0,1] \times M; \mathbb{C})\,.$$

for any sequence $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$ with $\widetilde{\alpha}_i \in \Omega^*([0,1] \times L_{a_i})$ for all $i = 1, \ldots, k$. These extend to linear maps

$$\widetilde{\mathfrak{p}}_k \colon (CF^*([0,1] \times L; \nabla)[1])^{\otimes k} \to \mathcal{D}^*([0,1] \times M; \Lambda)$$
$$\widetilde{\mathfrak{p}}_k := \sum_{\vec{a} \in A^k} \sum_{\beta \in H_2(M,L)} \mathrm{hol}_\nabla(\partial\beta) \widetilde{\mathfrak{p}}_k^{\vec{a};\beta} Q^{\omega(\beta)}\,.$$

Similarly to the time-independent case (cf. [91, Lemma 2.18]), these operators can be assembled to obtain a map

$$\widetilde{\mathcal{OC}}_0 \colon HH_*(CF^*([0,1] \times L, \nabla)) \to H^{*+n}([0,1] \times M; \Lambda_0)\,. \tag{4.8.14}$$

4.8.6 Wall-crossing terms

We will also need to consider counts of closed curves when studying the dependence of the open Gromov–Witten invariants on the almost complex structure J used to define them. Let Σ be a closed connected nodal Riemann surface of genus zero and let J be an ω-compatible almost complex structure on M. For any nonzero $\beta \in H_2(M; \mathbb{Z})$, consider the moduli space $\mathcal{M}_1^{cl}(\beta; J)$ of stable J-holomorphic maps $u \colon \Sigma \to M$ with one marked point, modulo reparametrizations. The elements of this moduli space are denoted $[(u, w)]$, where $w \in \Sigma$ denotes the marked point. Let $\mathrm{evi}^\beta \colon \mathcal{M}_1^{cl}(\beta; J) \to M$ denote the evaluation map defined by $\mathrm{evi}^\beta([(u, w)]) = u(w)$. The results of [48] imply that this moduli space can be equipped with a Kuranishi structure with respect to which evi^β can be treated as a smooth submersion. With this we define operators

$$\begin{aligned} \mathfrak{m}_\emptyset^\beta &\coloneqq (\mathrm{evi}^\beta)_*(1) \in \Omega^*([0,1]\times M) \\ \mathfrak{m}_\emptyset &\coloneqq \sum_{\beta\in H_2(M;\mathbb{Z})} \mathfrak{m}_\emptyset^\beta Q^{\omega(\beta)} \in \Lambda_+ \,. \end{aligned} \tag{4.8.15}$$

For a time-dependent ω-compatible almost complex structure $\underline{J} = \{J_t\}_{t\in[0,1]}$, we define the moduli space

$$\widetilde{\mathcal{M}}_1^{cl}(\beta;\underline{J}) \coloneqq \{(t,[(u,w)])\colon\ [(u,w)] \in \mathcal{M}_1^{cl}(\beta;J_t)\,,\ t\in[0,1]\}\,. \tag{4.8.16}$$

There is a natural evaluation map

$$\begin{aligned} \widetilde{\mathrm{evi}}^\beta\colon \widetilde{\mathcal{M}}_1^{cl}(\beta;\underline{J}) &\to [0,1]\times M \\ \widetilde{\mathrm{evi}}^\beta(t,[u,w]) &\coloneqq (t,u(w))\,. \end{aligned}$$

The results of [48] imply that this moduli space can be equipped with a Kuranishi structure with respect to which $\widetilde{\mathrm{evi}}^\beta$ can be treated as a smooth submersion. With this we define operators

$$\begin{aligned} \widetilde{\mathfrak{m}}_\emptyset^\beta &\coloneqq (\widetilde{\mathrm{evi}}^\beta)_*(1) \in \Omega^*([0,1]\times M) \\ \widetilde{\mathfrak{m}}_\emptyset &\coloneqq \sum_{\beta\in H_2(M;\mathbb{Z})} \widetilde{\mathfrak{m}}_\emptyset^\beta Q^{\omega(\beta)} \in \Lambda_+ \,. \end{aligned} \tag{4.8.17}$$

For a Lagrangian immersion $\iota_L\colon L \to M$, we define a *wall-crossing term*

$$\widetilde{GW}(L) \coloneqq \int_L \iota_L^* \widetilde{\mathfrak{m}}_\emptyset \tag{4.8.18}$$

which can be interpreted as a count of closed curves in M which intersect L in a point.

4.9 Assumptions on the cyclic open-closed map

To construct the open Gromov–Witten invariants of a graded Lagrangian in a Calabi–Yau 3-fold and relate them to the Fukaya category, we need to assume that the Fukaya category of [43] satisfies some additional algebraic properties. Analogues of these assumptions, in a different model for the Fukaya category, have been announced in [6]. All of these assumptions require one to consider moduli spaces of disks with immersed Lagrangian boundary conditions and *interior* marked points, which are absent from [43].

Assumption 4.9.1. The full Fukaya category $\mathcal{F}(X)$ of Definition 4.8.9, defined with a suitable choice of objects, satisfies all of the assumptions listed in [16, §2.5].

These roughly say that $\mathcal{F}(X)$ should have enough algebraic structure to mimic the proof of Abouzaid's generation criterion [45]. Such structures include closed-open and open-closed maps, about which we must make some additional assumptions for our treatment of (open) Gromov–Witten theory.

The closed-open operations are essentially given by the $\mathfrak{q}$-operators on $CF^*(L, \nabla)$. To relate the counts of disks $\mathfrak{m}_{-1}$ to the open-closed map, we will need to introduce operators defined using moduli spaces of disks with horocyclic constraints. These operations are usually used to show that the closed-open and open-closed maps give $HH_*(\mathcal{F}(M))$ the structure of a $QH^*(X)$-module, and thus verifying the following assumption would most likely be a byproduct of verifying Assumption 4.9.2 below.

Genus zero Gromov–Witten invariants are related to the Fukaya category in [6] and [38] via a *(negative) cyclic open-closed map*. A construction of a cyclic open-closed map (on the relative Fukaya category) satisfying the assumptions below was announced in [6]. We only require that such a map has been constructed with values in the cohomology of X, as opposed to the de Rham complex.

For a review of Hocschild and cyclic homology, see e.g. [88, 52, 41]. Let $\mathcal{A}$ be a (possibly curved) A_∞-algebra, and let $(CH_\bullet(\mathcal{A}), b)$ denote its Hochschild chain complex, where b is the cyclic bar differential. Also let B denote the Connes operator on $CH_\bullet(\mathcal{A})$. These operators give $CH_\bullet(\mathcal{A})$ the structure of a strict S^1-complex (cf. [52, §2]). When $\mathcal{A}$ is the curved A_∞-algebra associated to a Lagrangian immersion equipped with a local systems, we require the existence of a cyclic open-closed map which intertwines this action with the trivial S^1-action on quantum cohomology. Versions of the cyclic open-closed map are used both to define the open Gromov–Witten invariants following [41], and to extract them from the Fukaya category.

Assumption 4.9.2. Let $\mathcal{A} := CF^*(L, \nabla)$ denote the (curved) A_∞-algebra associated to a clean graded Lagrangian immersion in a Calabi–Yau n-fold. Then there is a sequence of maps $\{\mathcal{OC}_m\}_{m=0}^\infty$ of the form

$$\mathcal{OC}_m \colon CH_*(\mathcal{A}) \to QH^{*+n-2m}(X)$$

where $\mathcal{OC}_0$ induces the open-closed map of (4.8.12). Additionally, these maps should satisfy

$$\mathcal{OC}_{m-1} \circ B + \mathcal{OC}_m \circ b = 0\,.$$

In principle, we could mimic the construction of [52] or [41] to verify Assumption 4.9.2, though we prefer not to make any assumptions about how the cyclic open-closed map is defined at the chain level, given that one would also need to verify Assumption 4.9.3. Let u be a formal variable of degree 2, and consider the positive cyclic chain complex

$$CC^+_\bullet(\mathcal{A}) = (CH_\bullet(\mathcal{A}) \otimes_\Lambda \Lambda((u))/u\Lambda[[u]], b_{eq})$$

where $b_{eq} := b + uB$. The maps of Assumption 4.9.2 determine a chain map

$$\mathcal{OC}^+ \colon CC^+_\bullet(\mathcal{A}) \to QH^{\bullet+n}(X) \otimes_\Lambda \Lambda((u))/u\Lambda[[u]]$$
$$\mathcal{OC}^+ := \sum_{m=0}^\infty \mathcal{OC}_m u^m\,.$$

By projecting with the u^0-factor and integrating over X, we obtain a trace map

$$\widetilde{tr}\colon CC_*^+(\mathcal{A}) \to \Lambda[-n]\,. \tag{4.9.1}$$

The construction of the open Gromov–Witten invariants in [41] only requires (4.9.1).

One also has a negative cyclic chain complex

$$CC_\bullet^-(\mathcal{A}) = (CH_\bullet(\mathcal{A})\widehat{\otimes}_\Lambda\Lambda[[u]], b_{eq})\,.$$

Let $HC_\bullet^-(\mathcal{F}(X))$ denote the negative cyclic homology of Fukaya category equipped with the *Getzler–Gauss–Manin* connection ∇^{GGM}. Similarly, let

$$QH^*(X;\Lambda)[[u]] \coloneqq QH^*(X;\Lambda)\widehat{\otimes}_\Lambda\Lambda[[u]]$$

denote the quantum cohomology of X. The quantum connection is given in terms of the (small) quantum product $\omega \star (\cdot)$ by

$$\nabla^{\mathrm{QDE}} = Q\partial_Q(\cdot) + u^{-1}\omega \star (\cdot)\,.$$

Assumption 4.9.3. The negative cyclic open-closed map

$$\mathcal{OC}^-\colon HC_\bullet^-(\mathcal{F}(X)) \to QH^{\bullet+n}(X)[[u]]$$

induced by the maps in Assumption 4.9.2 respects connections, in the sense that

$$\mathcal{OC}^- \circ \nabla^{\mathrm{GGM}}_{Q\partial_Q} = \nabla^{\mathrm{QDE}}_{Q\partial_Q} \circ \mathcal{OC}^-\,.$$

Remark 4.9.4. Let $\mathcal{A}$ be a curved A_∞-algebra $\mathcal{A}$ for which $H^*(\mathcal{A}, \mathfrak{m}_{1,0})$ is finite-dimensional. We say that a chain map $tr\colon CH_*(\mathcal{A}) \to \Lambda[-n]$ is a weak proper Calabi–Yau structure if the composition

$$H^*(\mathcal{A}, \mathfrak{m}_{1,0}) \otimes H^{n-*}(\mathcal{A}, \mathfrak{m}_{1,0}) \xrightarrow{\mathfrak{m}_{2,0}(\cdot,\cdot)} H^n(\mathcal{A}, \mathfrak{m}_{1,0}) \to HH_n(\mathcal{A}) \xrightarrow{tr} \Lambda \tag{4.9.2}$$

is a perfect pairing. Restricting the trace $\widetilde{tr}$ of (4.9.1) to Hochschild chains yields a weak proper Calabi–Yau structure in the sense above, and is said to be a *stronger* proper Calabi–Yau structure. We do not necessarily claim that this is the optimal definition of a proper Calabi–Yau structure on a curved A_∞-algebra, but in the presence of a trace as in Assumption 4.9.2, such a structure will exist.

Remark 4.9.5 (Forgetting interior marked points)**.** The verifications that the cyclic open-closed maps are chain maps in [52] and [41] both invoke forgetful maps of interior marked points, which is reasonable in our setting in view of Remark 4.8.4. Assumptions 4.9.2 and 4.9.3 are both phrased in such a way that they would follow from the construction of a chain-level cyclic open-closed map valued in *differential currents*, as in. Hugtenburg's construction [58] of the (cyclic) open-closed map for a single embedded Lagrangian is phrased in terms of Poincaré duality on X essentially for this reason. By Definition 4.8.9, we can use this strategy to define cyclic open-closed maps on the entire Fukaya category provided that we can do so on the Floer cochain space of a single *immersed* Lagrangian.

References

[1] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, “An exactly soluble superconformal theory from a mirror pair of calabi–yau manifolds,” *Nucl. Phys. B*, vol. 359, pp. 21–74, 1991.

[2] D. Morrison, “Compactifications of moduli spaces inspired by mirror symmetry, Journées de Géométrie Algébrique d’Orsay (Orsay, 1992),” *Astérisque*, vol. 218, pp. 243–271, 1993.

[3] A. Givental, “A Mirror Theorem for Toric Complete Intersections,” in *Topological Field Theory, Primitive Forms and Related Topics*, ser. Prog. Math. Vol. 160, Springer, 1995, pp. 141–175.

[4] B. H. Lian, K. Liu, and S.-T. Yau, “Mirror principle I,” *Asian J. Math*, vol. 1, no. 4, pp. 729–763, 1997.

[5] M. Kontsevich, “Homological algebra of mirror symmetry,” in *Proceedings of the International Congress of Matematicians*, Zürich, 1994, pp. 120–139.

[6] S. Ganatra, T. Perutz, and N. Sheridan, *Mirror symmetry: From categories to curve counts*, arXiv:1510.03839, 2015.

[7] E. Witten, “Chern-Simons gauge theory as a string theory,” in *The Floer Memorial Volume*, ser. Prog. Math. Vol. 133, Springer, 1995, pp. 637–678.

[8] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, arXiv:hep-th/0012041, 2000.

[9] J. Walcher, “Opening Mirror Symmetry on the Quintic,” *Commun. Math. Phys.*, vol. 276, pp. 671–689, 2007.

[10] D. R. Morrison and J. Walcher, “D-branes and normal functions,” *Adv. Theor. Math. Phys.*, vol. 13, no. 2, pp. 553–598, 2009.

[11] R. Pandharipande, J. P. Solomon, and J. Walcher, “Disk enumeration on the quintic 3-fold,” *J. Amer. Math. Soc.*, vol. 21, pp. 1169–1209, 2008.

[12] L. Hahn and J. Walcher, *A-model Implications of Extended Mirror Symmetry*, arXiv:2201.08745, 2022.

[13] C. Voisin, *Hodge Theory and Complex Algebraic Geometry II*. Cambridge University Press, 2003.

[14] J. P. Solomon and S. B. Tukachinsky, “Relative quantum cohomology,” *J. Eur. Math. Soc.*, vol. 26, pp. 3497–3575, 2024.

[15] N. Sheridan, “Homological mirror symmetry for Calabi–Yau hypersurfaces in projective space,” *Invent. Math.*, vol. 199, pp. 1–186, 2015.

[16] N. Sheridan and I. Smith, “Homological mirror symmetry for generalized Greene–Plesser mirrors,” *Invent. Math.*, vol. 224, pp. 627–682, 2021.

[17] S. Gantra, A. Hanlon, J. Hicks, D. Pomerleano, and N. Sheridan, *Homological mirror symmetry for Batyrev mirror pairs*, arXiv:2406.05272, 2024.

[18] K. Fukaya, “Counting pseudo-holomorphic discs in Calabi–Yau 3-folds,” *Tohoku Math. J.*, vol. 63, no. 4, pp. 697–727, 2011.

[19] J. Walcher, “On the arithmetic of D-brane superpotentials. Lines and conics on the mirror quintic,” *Commun. Number Theory Phys.*, vol. 6, no. 2, pp. 279–377, 2012.

[20] C.-Y. Mak and H. Ruddat, “Tropically constructed Lagrangians in mirror quintic threefolds,” *Forum of Mathematics, Sigma*, vol. 8, 2020.

[21] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” *Nucl. Phys. B*, vol. 577, no. 3, pp. 419–438, 2000.

[22] C.-C. M. Liu, *Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov–Witten invariants for an* S^1*-equivariant pair*, arXiv:math.SG/0210257, 2002.

[23] A. Albano and S. Katz, “Van Geemen’s families of lines on special quintic threefolds,” *Manuscripta Mathematica*, vol. 70, pp. 183–188, 1991.

[24] P. Candelas, X. de la Ossa, B. van Geemen, and D. van Straten, “Lines on the Dwork pencil of quintic threefolds,” *Adv. Theor. Math. Phys.*, vol. 16, no. 6, pp. 1779–1836, 2012.

[25] G. Mikhalkin, “Examples of tropical-to-Lagrangian correspondence,” *European Journal of Mathematics*, vol. 5, no. 3, pp. 1033–1066, 2019.

[26] J. Hicks, “Tropical Lagrangian hypersurfaces are unobstructed,” *Journal of Topology*, vol. 13, no. 4, pp. 1409–1454, 2020.

[27] J. Hicks, “Tropical Lagrangians in toric del-Pezzo surfaces,” *Sel. Math.*, vol. 27, no. 1, pp. 1–50, 2021.

[28] D. Matessi, “Lagrangian pairs of pants,” *International Math Research Notices*, vol. 2021, no. 15, pp. 11 306–11 356, 2021.

[29] S. Haney, *Cusped hyperbolic Lagrangians as mirrors to lines in three-space*, arXiv:2402.03296, 2024.

[30] R. Harvey and J. H. B. Lawson, "Calibrated geometries," *Acta Math.*, vol. 148, pp. 47–157, 1982.

[31] P. Seidel, "Homological mirror symmetry for the genus two curve," *J. Algebraic Geom.*, 2011.

[32] N. Sheridan, "On the homological mirror symmetry conjecture for pairs of pants," *J. Differential Geometry*, vol. 89, pp. 271–367, 2011.

[33] M. Abouzaid and Z. Sylvan, *Homological Mirror Symmetry for local SYZ singularities*, arXiv:2107.05068, 2021.

[34] J. Hicks, "Realizability in tropical geometry and unobstructedness of Lagrangian submanifols," *Geometry and Topology*, 2024.

[35] W.-D. Ruan, "Lagrangian torus fibrations of quintic Calab-Yau hypersurfaces I: Fermat type quintic case," in *Winter School on Mirror Symmetry, Vector Budnles and Lagrangian Submanifolds*, ser. AMS/IP Studies in Advanced Mathematics, C. Vafa and S.-T. Yau, Eds., vol. 23, AMS and International Press, 2001.

[36] J. P. Solomon, "Involutions, obstructions and mirror symmetry," *Adv. Math.*, vol. 367, 2020.

[37] D. Joyce, "Special Lagrangian submanifolds with isolated conical singularities I. Regularity," *Annals of Global Analysis and Geometry*, 2004.

[38] K. Hugtenburg, "Open Gromov–Witten invariants from the Fukaya category," *Adv. Math.*, vol. 441, 2024.

[39] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds* (Graduate Texts in Mathematics). Springer, 2003, vol. 219.

[40] N. M. Dunfield and W. P. Thurston, "The virtual Haken conjecture: Experiments and examples," *Geom. Topol.*, vol. 7, pp. 399–441, 2003.

[41] S. Haney, *Infinity inner products and open Gromov–Witten invariants*, arXiv:2406.08693, 2024.

[42] G. Laporte and J. Walcher, "Monodromy of an Inhomogeneous Picard–Fuchs Equation," *SIGMA*, vol. 056, 2012.

[43] K. Fukaya, *Unobstructed immersed Lagrangian correspondence and filtered A infinity functor*, 2021. arXiv: `1706.02131 [math.SG]`.

[44] P. Seidel, "Graded Lagrangian submanifolds," *Bulletin de la Société Mathématique de France*, vol. 128, no. 1, pp. 103–149, 2000.

[45] M. Abouzaid, "A geometric criterion for generating the Fukaya category," *Publications mathématiques de l'IHÉS*, vol. 112, pp. 191–240, 2010.

[46] D. Joyce, *Kuranishi homology and Kuranishi cohomology*, arXiv:0707.3572, 2008.

[47] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*. Providence and Somerville: American Mathematical Society and International Press, 2009.

[48] K. Fukaya, "Cyclic symmetry and adic convergence in Lagrangian Floer theory," *Kyoto J. Math.*, vol. 50, no. 3, pp. 521–590, 2010.

[49] J. Solomon and S. Tukachinsky, "Point-like bounding cochain in open Gromov–Witten theory," *Geom. Func. Anal.*, vol. 31, no. 5, pp. 1245–1320, 2021.

[50] M. Kontsevich and Y. Soibelman, *Notes on A_∞-Algebras, A_∞-Categoris and Non-Commutative Geometry*. Berlin and Heidelberg: Springer, 2008.

[51] C.-H. Cho, "Strong homotopy inner products of an A_∞-algebra," *IMRN*, vol. 2008, no. 41, 2008.

[52] S. Ganatra, "Cyclic homology, S^1-equivariant Floer cohomology, and Calabi–Yau structures," *Geometry & Topology*, vol. 27, pp. 3461–3548, 2023.

[53] P. Biran and O. Cornea, *Quantum structures for Lagrangian submanifolds*, arXiv:0708.4221, 2007.

[54] C.-H. Cho and S. Lee, "Potentials of homotopy cyclic A_∞-algebras," *Homology, Homotopy, and Applications*, vol. 14, no. 1, pp. 203–220, 2012.

[55] J. A. Carlson, "The geometry of the extension class of a mixed Hodge structure," in *Algebraic Geometry–Bowdoin 1985, Part 2*, ser. Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, 1987, pp. 199–222.

[56] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*. American Mathematical Society, 1999.

[57] J. Tu, *A remark on the B-model Comparison of Hodge structures*, arXiv:2404.13805, 2024.

[58] K. Hugtenburg, "The cyclic open-closed map, u-connections and R-matrices," *Sel. Math (N.S.)*, vol. 30, no. 29, 2024.

[59] A. Schwarz, V. Vologodsky, and J. Walcher, *Integrality of Framing and Geometric Origin of 2-functions*, arXiv:1702.07153, 2017.

[60] A. Căldăraru, "The Mukai pairing-II: The Hochschild–Kostant–Rosenberg isomorphism," *Adv. Math.*, vol. 20, pp. 34–66, 2004.

[61] G. Alston and E. Bao, "Immersed Lagrangian Floer cohomology via pearly trajectories," *J. Geom. Phys.*, vol. 169, pp. 1–21, 2021.

[62] L. Lazzarini, "Relative frames on J-holmorphic curves," *J. Fixed Point Theory Appl.*, vol. 9, no. 2, pp. 213–256, 2011.

[63] A. Perrier, *Structure of J-holomorphic disks with immersed lagrangian boundary conditions*, 2018. arXiv: `1808.01849 [math.SG]`.

[64] M. Abouzaid, "A geometric criterion for generating the Fukaya category," *Publ. Math. Inst. Hautes Études Sci.*, vol. 112, pp. 191–240, 2010.

[65] Y. Gao, *Wrapped Floer cohomology and Lagrangian correspondences*, 2017. arXiv: `1703.04032 [math.SG]`.

[66] M. Akaho and D. Joyce, "Immersed Lagrangian Floer theory," *J. Diff. Geom.*, vol. 86, no. 3, pp. 381–500, 2010.

[67] R. Benedetti and C. Petronio, "Spin structures on 3-manifolds via arbitrary triangulations," *Algebraic & Geometric Topology*, 2014.

[68] R. Casals and E. Zaslow, "Legendrian weaves: N-graph calculus, flag moduli and applications," *Geometry & Topology*, 2022.

[69] D. Treumann and E. Zaslow, "Cubic planar graphs and Legendrian surface theory," *Advances in Theoretical and Mathematical Physics*, vol. 22, no. 5, pp. 1289–1345, 2018.

[70] A. Weinstein, "Symplectic manifolds and their Lagrangian submanifolds," *Advances in Math*, 1971.

[71] B. Martelli, C. Petrionio, and F. Roukema, "Exceptional Dehn surgery on the minimally twisted five-chain link," *Communications in Analysis and Geometry*, vol. 22, no. 4, pp. 689–735, 2014.

[72] A. Oancea, "The Künneth formula in Floer homology for manifolds with restricted contact type boundary," *Matt. Ann.*, vol. 334, no. 1, pp. 65–89, 2006.

[73] K. Wehrheim and C. T. Woodward, "Floer cohomology and geometric composition of Lagrangian correspondences," *Adv. Math.*, vol. 230, no. 1, pp. 177–228, 2012.

[74] Y. Gao, *Functors of wrapped Fukaya categories from Lagrangian correspondences*, 2017. arXiv: 1712.00225 [math.SG].

[75] C.-H. Cho, "On the obstructed Lagrangian Floer theory," *Advances in Mathematics*, vol. 229, no. 2, pp. 804–853, 2012.

[76] C.-H. Cho and S. Lee, "Notes on Kontsevich–Soibelman's Theorem about Cyclic A_∞-algebras," *IMRN*, vol. 2011, no. 14, pp. 3095–3140, 2011.

[77] A. Hamilton and A. Lazarev, "Symplectic A_∞-algebras and string topology operations," in *Geometry, Topology, and Mathematical Physics: S.P. Novikov's Seminar: 2006-2007*, American Mathematical Society, 2008, pp. 147–157.

[78] T. Tradler, "Infinity inner products," *J. Homotopy Relat. Struct.*, vol. 3, no. 1, pp. 245–271, 2008.

[79] X. Chen, "A Geometric description of Solomon–Tukachinsky's construction of open Gromov–Witten invariants," *Peking Mathematical Journal*, vol. 5, pp. 279–348, 2022.

[80] J. Solomon and S. Tukachinsky, "Differential forms, Fukaya A_∞-algebras, and open Gromov–Witten axioms," *J. Sympl. Geom.*, vol. 20, no. 4, pp. 927–994, 2022.

[81] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Kuranishi structures and virtual fundamental chains*. Singapore: Springer, 2020.

[82] N. Sheridan, "Constructing the relative Fukaya category," *Journal of Symplectic Geometry*, vol. 21, no. 5, pp. 997–1076, 2024.

[83] S. Chanda, *Floer Cohomology and Higher Mutations*, arXiv:2301.08311, 2023.

[84] R. Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles* (Universitext). Springer, 1998.

[85] J. Palmer and C. Woodward, *Invariance of immersed Floer cohomology under Lagrangian surgery*, arXiv:1903.01943v10, 2019.

[86] Y. Nohara and K. Ueda, "Homological mirror symmetry for the quintic 3-fold," *Geom. Topol.*, vol. 16, pp. 1967–2001, 2012.

[87] S. Barannikov, "Semi-infinite Variations of Hodge Structures," *IMRN*, vol. 2001, no. 23, pp. 1243–1264, 2001.

[88] N. Sheridan, "Formulae in noncommutative Hodge theory," *Journal Homotopy and Related Structures*, vol. 15, no. 1, pp. 249–299, 2020.

[89] C. Weibel, “The Hodge Filtration and Cyclic Homology,” *K Theory*, vol. 12, no. 2, pp. 145–164, 1997.

[90] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*. Providence and Somerville: American Mathematical Society and International Press, 2009.

[91] S. Tukachinsky, *Open Gromov–Witten theory for cohomologically incompressible Lagrangians*, arXiv:2311.18511, 2023.