Spectral Moments of Rankin-Selberg L-functions

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Abstract

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Spectral moment formulae of various shapes have proven to be very successful in studying the statistics of central L-values. In this article, we establish, in a completely explicit fashion, such formulae for the family of $GL(3) \times GL(2)$ Rankin-Selberg L-functions using the period integral method. The Kuznetsov and the Voronoi formulae are not needed in our argument. We also prove the essential analytic properties and explicit formulae for the integral transform of our moment formulae. It is hoped that our method will provide insights into moments of L-functions for higher-rank groups.

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[2] also his handwritten notes!

[3] It exists and is real... in this case!

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Dedication

To Sannie & Charles.

Chapter 1: Introduction

1.1 Background & Literature

1.1.1 $GL(1)$.

Historically speaking, the interests in studying *Moments (or Mean Value Theorems)* of L-functions were initiated by the seminal works of Hardy-Littlewood [1, 2] and Ingham [3] in the 1920's. They succeeded to establish the asymptotic formulae

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim T\log T \quad \text{and} \quad \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2}\, T\log^4 T \tag{1.1}$$

for the Riemann ζ-function, which is the prototype of all L-functions. Not only are these moment estimates interesting in their own right, but they have fundamental applications to the distributions of zeros of the ζ-function and to divisor problems. The monographs of Titchmarsh [4], Ivić [5, 6] have provided excellent exposition regarding this direction of research.

Up till this date, the asymptotic formulae in (1.1) remain the only unconditional ones for moments of the form

$$I_{2k}(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \qquad (k > 0).$$

Nevertheless, assuming the Riemann Hypothesis, we do have good understanding of the order of magnitude of $I_{2k}(T)$ $(\asymp_k T(\log T)^{k^2})$ thanks to a number of works including Soundararajan [7], Harper [8], Radziwiłł-Soundararajan [9], Heath-Brown [10]. We also have good conjectures on what the asymptotic formula of $I_{2k}(T)$ should be, see Diaconu-Goldfeld-Hoffstein [11] and

Conrey-Farmer-Keating-Rubinstein-Snaith [12].

1.1.2 $GL(2)$.

Another type of moment problem concerns the values of *automorphic L-functions* at the *central point* $s = 1/2$. These L-values are of great arithmetic significance and thus have been at the center stage of many branches of number theory in the past decades. A variety of interesting perspectives furnish our understanding of the nature of the central L-values. For example, one may wish to take a *statistical* look at them — questions about (non-)vanishing and sizes of these L-values are fundamental. Moments of L-functions have offered a particularly effective way to approach problems of this sort. Techniques from analytic number theory have proven to be very successful in estimating the sizes of moments of all kinds. Moreover, spectacular results can be obtained when moment estimates join forces with arithmetic geometry and automorphic representations.

For instance, an important object in arithmetic geometry is the rank of elliptic curves, or more generally, abelian varieties. Analytic methods in moments of L-functions have contributed major advances in understanding the asymptotic growths of these ranks, see Chapter 26 of Iwaniec-Kowalski [13] for a nice introduction of these results and the relevant techniques.

Another nice illustration of this line of investigation is the landmark result of Conrey-Iwaniec [14]. Let χ be a real primitive Dirichlet character $(\bmod\, q)$ with q odd and square-free. The main object of [14] is the *cubic moment* of $GL(2)$ automorphic L-functions of the congruence subgroup $\Gamma_0(q)$ twisted by χ. An upper bound of Lindelöf strength in the q-aspect was established therein. When combining this upper bound with the celebrated results of Waldspurger [15], Kohnen-Zagier [16], Katok-Sarnak [17], Guo [18], the famous Burgess $3/16$-bound for Dirichlet L-functions was improved for the first time since the 1960's. In fact, Conrey-Iwaniec [14] proved the bound

$$L\left(\frac{1}{2}, \chi\right) \ll_\epsilon q^{\frac{1}{6}+\epsilon}. \tag{1.2}$$

Understanding the effects of a sequence of intricate transformations, which can be of arithmetic or analytic nature, constitutes a significant part of moment calculations. In the context of [14], such a sequence would end up in a beautiful and surprising **exact identity** connecting the two aforementioned, seemingly unrelated types of moments. This was elaborated in Petrow [19] and Frolenkov [20]. Let us also mention the relevant works of Ivić [21, 22]. Indeed, the cubic moment of $GL(2)$ automorphic L-functions 'dualizes' to the fourth moment of $GL(1)$ L-functions and the identity roughly takes the shape [1]

$$\sum_{f:\ GL(2)} w(t_f)\, L\left(\frac{1}{2}, f\right)^3 \ = \int_{-\infty}^{\infty} \widehat{w}(t) \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \, dt \ + \ (***), \tag{1.3}$$

where the average on the left is over an orthonormal basis of Maass (or holomorphic) forms of level one. For the sake of illustrating the main features, we have suppressed certain polar contributions (denoted by $(***)$) and the shape of the integral transform $\widehat{w}(t)$ above. Further curious features of the identity (1.3) include that the total degrees on the two sides are unequal and it transforms a moment about the *spectral aspect* to another one about the *t-aspect* instead.

Besides its structural elegance, the identity (1.3) comes with immediate applications. It leads to sharp moment estimates as a consequence of exact evaluation and further results such as:

- Asymptotic formula for $\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \, dt$ with the best known power-saving error term, see Motohashi [23, 24];

- Strong subconvexity bounds for the Dirichlet L-function $L\left(\frac{1}{2}+it, \chi\right)$ and the $GL(2)$ L-functions $L\left(\frac{1}{2}+it, f\right)$ (or $L\left(\frac{1}{2}+it, f\otimes\chi\right)$) in various aspects, see [14, 21, 22, 25, 19, 20, 26, 27] etc.;

- Equidistribution of Heegner points in shrinking sets, see Young [28];

[1] The works [14] and [19] focused on the twist and the level aspects of the moment problem. As a result, the identity (1.3) in their cases would alter accordingly to incorporate these non-archimedean structures. Similar phenomenon will occur in our case as well, see Chapter 8.

- Bounds for Fourier coefficients of cusp forms of half-integral weights, see Conrey-Iwaniec [14], Petrow [19]. [2]

As an extra benefit, it cleans up the analysis in the traditional 'approximate' [3] approach. In Petrow [19], such an identity was termed a **'Motohashi-type identity'**. Indeed, Motohashi [23, 24] discovered an identity of this sort but with the choice of test function made on the fourth moment side instead. It greatly enhances our understanding of the Riemann ζ-function. For extensions of Motohashi's work to Dirichlet L-functions, see Young [31], as well as the recent works of Blomer-Humphries-Khan-Milinovich [32], Topacogullari [33] and Kaneko [34].

1.1.3 $GL(3)$.

Conrey-Iwaniec [14] studied the cubic moment of (1.3) via harmonic analysis of $GL(2)$. In the introduction of [14], they nevertheless envisioned the possibility (and challenges) of studying the same problem from the perspective of harmonic analysis of $GL(3)$. [4] Indeed, the cubic moment of $GL(2)$ L-functions can be regarded as the first moment of $GL(3)\times GL(2)$ L-functions in the $GL(2)$ spectral aspect, where the $GL(3)$ automorphic form is chosen to be the minimal parabolic Eisenstein series. If one wishes to adapt the method developed in [14] to the $GL(3)\times GL(2)$ moment, a key ingredient from the theory of $GL(3)$ automorphic forms would clearly be the *Voronoi formula* for $GL(3)$, i.e., the functional equation for the L-series

$$\sum_{n=1}^{\infty} \frac{\lambda_{\Phi}(n)}{n^s}\, e^{2\pi i n a/q}, \tag{1.4}$$

where $(a,q)=1$ and $\lambda_{\Phi}(n)$'s are the Hecke eigenvalues [5] of the $GL(3)$ automorphic form Φ. It took some time to understand the structures and the mechanisms behind the $GL(3)$ Voronoi formula, which was undoubtedly more complex than its $GL(2)$ counterpart and was a subject

[2] The results obtained in this way is stronger than the ones obtained in Iwaniec [29] and Duke [30].

[3] i.e., replacing the L-functions in a moment problem by Dirichlet polynomials, or the approximate functional equations, see [13] Chapter 5.2.

[4] an instance of so-called 'higher-rank' groups.

[5] Or equivalently, $\lambda_{\Phi}(n)=\mathcal{B}_{\Phi}(1,n)$, the $GL(3)$ Fourier coefficients.

under intensive investigation in the 2000's, see the works of Sarnak-Watson (unpublished), Miller-Schmid [35, 36], Goldfeld-Li [37, 38], Ichino-Templier [39] etc.

However, the higher-rank set-up does come with benefits and new applications. For example, one may also consider the first moment of $GL(3) \times GL(2)$ Rankin-Selberg L-functions but with a cusp form of $GL(3)$. This led to the breakthrough work of Li [40] (about ten years after Conrey-Iwaniec [14]) in which she obtained the first-ever subconvexity bound (in the t-aspect) for the $GL(3)$ L-functions [6]. This was achieved by combining techniques of [14] together with some new analysis of her own. Furthermore, Li's techniques further inspired many other works, say Buttcane-Khan [41, 42] on the estimation of the L^4-norm of $GL(2)$ Maass forms [7], as well as Blomer [43], Qi [44], Huang [45], Nunes [46], Lin-Nunes-Qi [47] etc.

1.2 Main Results

The main purpose of this thesis is to propose and illustrate a strategy towards moment problems and identities of Motohashi type. We are motivated by the following natural question:

Question 1.2.1. *What are the structural or conceptual causes of moment duality like (1.3)?*

Through the lenses of *period integrals*, we are able to answer this question [8] in the context of Li [40] and uncover the key identity behind her important work. Incidentally, this provides new examples to the recent framework of *'Period Reciprocity'* (other classes of examples include Zacharias [48, 49], Nunes [50]).

More importantly, our approach is more streamlined than the traditional ones described in Section 1.1. In particular, a number of intermediate objects that are known to cause serious technical complications in the higher-rank settings will not appear in our argument, see Section 1.3 for further discussions. As a result, our method is favorable in isolating the essential ingredients behind

[6] which are self-dual and globally unramified.

[7] which served as good evidence towards Berry's Random Wave Conjecture.

[8] There could well be many other ways to address this question!

the Motohashi phenomenon and should offer a better view towards generalizations to higher-rank groups.

Our main result can be stated as follows.

Theorem 1.2.2. *Let*

- Φ *be a fixed, Hecke-normalized Maass cusp form of* $SL_3(\mathbb{Z})$ *with the Langlands parameters* $(\alpha_1, \alpha_2, \alpha_3) \in (i\mathbb{R})^3$, *and* $\widetilde{\Phi}$ *be the dual form of* Φ*;*

- $(\phi_j)_{j=1}^{\infty}$ *be an orthogonal basis of even, Hecke-normalized Maass cusp forms of* $SL_2(\mathbb{Z})$ *which satisfy* $\Delta\phi_j = \left(\frac{1}{4} - \beta_j^2\right)\phi_j$*;*

- $L(s, \phi_j \otimes \Phi)$ *and* $L(s, \Phi)$ *be the Rankin-Selberg* L*-function of the pair* (ϕ_j, Φ) *and the standard* L*-function of* Φ *respectively, where* L^* *denotes the corresponding complete* L*-functions;*

- $\mathcal{C}_\eta$ $(\eta > 40)$ *be the class of holomorphic functions* H *defined on the vertical strip* $|\operatorname{Re}\beta| < 2\eta$ *such that* $H(\beta) = H(-\beta)$ *and* H *has rapid decay:*

$$H(\beta) \ll e^{-10|\beta|} \qquad (|\operatorname{Re}\beta| < 2\eta). \tag{1.5}$$

- *For* $H \in \mathcal{C}_\eta$, *the function* $(\mathcal{F}_\Phi H)(s_0, s)$ *is the integral transform defined in (4.5).* [9]

[9] Actually, it only depends on the Langlands parameters of Φ.

Then on the domain $\frac{1}{4}+\frac{1}{200}<\sigma<\frac{3}{4}$, we have the following moment identity:

$$\frac{1}{2}\sum_{j=1}^{\infty} H(\beta_j)\ \frac{L^*\left(s,\phi_j\otimes\widetilde{\Phi}\right)}{\langle\phi_j,\phi_j\rangle}\ +\ \frac{1}{8\pi}\int_{\mathbb{R}} H(i\mu)\ \frac{L^*\left(s+i\mu,\widetilde{\Phi}\right)L^*(1-s+i\mu,\Phi)}{|\zeta^*(1+2i\mu)|^2}\,d\mu$$
$$=\ \frac{\pi^{-3s}}{4}\ L(2s,\Phi)\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^{3}\Gamma\left(\frac{s+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{s-\beta-\alpha_i}{2}\right)\frac{d\beta}{2\pi i}$$
$$+\ \frac{1}{4}\ L(2s-1,\Phi)\ (\mathcal{F}_\Phi H)(2s-1,s)$$
$$+\ \frac{1}{4}\int_{(1/2)}\zeta(2s-s_0)\,L(s_0,\Phi)\cdot(\mathcal{F}_\Phi H)(s_0,s)\ \frac{ds_0}{2\pi i}. \tag{1.6}$$

Remark 1.2.3. The assumption of Φ being tempered at ∞ (i.e., the Langlands parameters being purely imaginary) merely serves as a simplification of our exposition. One should be able to remove it easily. In fact, all Maass cusp forms of $SL_3(\mathbb{Z})$ are conjectured to be tempered at ∞. The non-tempered forms constitute a density zero set (see Miller [51]).

In Section 7, we provide more expressions for $(\mathcal{F}_\Phi H)(s_0,s)$. In particular,

Theorem 1.2.4. *For $\frac{1}{2}+\frac{1}{100}<\sigma<1$, we have*

$$(\mathcal{F}_\Phi H)(2s-1,s)\ =\ \pi^{\frac{1}{2}-s}\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}$$
$$\cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^{3}\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\beta}{2}\right)\frac{d\beta}{2\pi i}. \tag{1.7}$$

The calculation behind Theorem 1.2.4 is in the style of Bump [52] and Stade [53]. More generally, the transform can be expressed in terms of *hypergeometric functions* of special types. Recently, the articles [54, 55], [56], [57] have brought in the powerful asymptotic analysis of hypergeometric functions into the study of moments and obtain sharp estimates in the spectral aspect. Also, our class of admissible test functions in Theorem 1.2.2 is large enough for such prospects for the family of $GL(3)\times GL(2)$ L-functions, see Remark 1.3.2 for a further discussion.

Remark 1.2.5. As already hinted in Section 1.1.3, one obtains a new proof and a new explanation of the Motohashi-type identity (1.3) [10] from the perspective of $GL(3)$ up to some minor modifications of our argument for Theorem 1.2.2.

Indeed, if the fixed cusp form Φ in Theorem 1.2.2 is replaced by the minimal parabolic Eisenstein series of $SL_3(\mathbb{Z})$, the spectral side is precisely the $GL(2)$ cubic moment, whereas on the dual side the factor $L(s_0, \Phi)$ becomes a product of three ζ-functions. Also, there will be extra polar contributions from the degenerate part of the Fourier expansion of the Eisenstein series and the continuation of the continuous spectrum.

Moreover, if one replaces Φ by a maximal parabolic Eisenstein series twisted by a Maass cusp form ϕ of $SL_2(\mathbb{Z})$, one obtains an identity of the shape

$$\sum_{j=1}^{\infty} H(\beta_j) \, \frac{L^*\left(\frac{1}{2}, \phi \otimes \phi_j\right) L^*\left(\frac{1}{2}, \phi_j\right)}{\langle \phi_j, \phi_j \rangle} \;=\; \int_{\mathbb{R}} \widehat{H}(t) \, L\left(\frac{1}{2} + it, \phi\right) \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \, dt \;+\; (***), \tag{1.8}$$

where the transform $\widehat{H}(t)$ is exactly the one described in Theorem 1.2.2. As a result, the $GL(3) \times GL(2)$ viewpoint allows one to encapsulate several instances of the Motohashi phenomena simultaneously. For the extra structural benefits, see Chapter 1.3 & 1.4.

Remark 1.2.6. In Chapter 8, we shall incorporate twists of $GL(2)$ Hecke eigenvalues into the spectral average of Theorem 1.2.2 so as to open up possibilities for broader applications. A curious feature is that such (multiplicative) twists are transformed to additive twists of arithmetic nature on the dual side, but the archimedean component will play a role in this.

Beyond applications, such kind of 'twisted moments' is interesting on their own rights because they contain unexpected interchanges of structures. As pointed out by Conrey [58], this type of 'reciprocity' phenomena is already implicit in the investigations of Selberg, Heath-Brown [4] and Iwaniec-Sarnak [59] on the Dirichlet L-functions. Moving onto the context of $GL(2)$ moments, the *'Level Reciprocity'* phenomenon has recently gained considerable amount of attention, see

[10] $GL(2)$ cubic moment $\leftrightarrow$ $GL(1)$ fourth moment.

Blomer-Khan [60, 61], Zacharias [49, 48], Nunes [50], Andersen-Kıral [62].

1.3 Features of Our Method

1.3.1 Structural Aspects

An important feature of our method is that we are able to uncover the dual moment, i.e., the right side of (1.6), quickly and naturally thanks to the structural advantages provided by the **period integral**:

$$\left\langle P(*;h)\ ,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))} \tag{1.9}$$

for $s \in \mathbb{C}$, where $P(*;h)$ is a Poincaré series of $SL_2(\mathbb{Z})$ (with a test function h), Φ is a fixed Maass cusp form of $SL_3(\mathbb{Z})$, $(\mathbb{P}_2^3\Phi)(g) := \Phi\begin{pmatrix} g & \\ & 1\end{pmatrix}$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))}$ is the Petersson inner product.

Actually, Theorem 1.2.2 is an equality of two Rankin-Selberg unfoldings of $GL(2)$ in two distinct directions. This is possible because of the 'disparity of ranks' present in our case. More precisely,

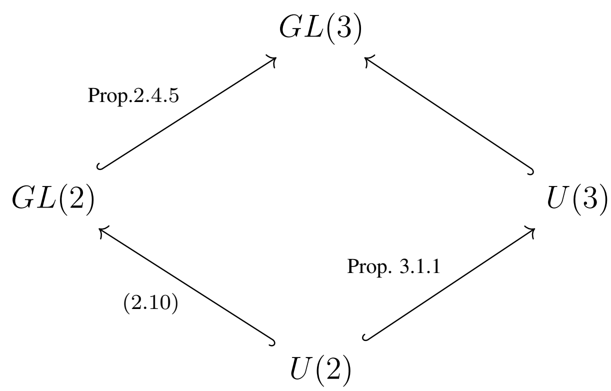
(A) ($GL(2)$ side) An unfolding using the Fourier expansion of the Maass form Φ, which has the effect of *'projecting'* Φ from $GL(3)$ to $GL(2)$, gives the spectral average of $GL(3)\times GL(2)$ L-functions;

(B) ($GL(3)$ side) Another unfolding using the Poincaré series $P(*;h)$ leads to a certain *'incomplete unipotent integration'* of $GL(3)$ for the Maass form Φ, which gives the dual continuous moment.

Observe that from Proposition 2.3.3 and 3.1.1, our Motohashi-type identities are natural consequences of interpreting the trivial embedding

$$g \in GL(2) \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix} \in GL(3)$$

with respect to the diagram:



where U's in the diagram stand for the unipotent subgroups, i.e.,

$$U(2) := \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \quad \text{and} \quad U(3) := \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}.$$

The two arrows on the left side correspond to the spectral calculation, whereas the two on the right side correspond to the dual calculation.

Observant readers will also notice the roles of the integration over the center of $GL(2)$ on the spectral side (see (2.19)), or equivalently, the y_0-intgeral of (3.1) on the dual side:

- it gives the (complete) $GL(3) \times GL(2)$ L-functions; [11]
- it gives the dual line integral for the continuous moment (upon taking a Mellin inversion).

In other words, the curious duality between the discrete and the continuous moments in our formulae of Motohashi-type is actually embedded right from the beginning of our method, i.e., the

[11] The y_0-integral is termed the 'Rank-lowering operator' in some literature, say Goldfeld-Thillainatesan [63].

expression (1.9). For complete details, see Chapter 3 and Chapter 4 of this thesis.

Remark 1.3.1. The diagram above and the accompanied interpretation are motivated by the *'Strong Gelfand Formation'* first proposed by Reznikov [64]. This formalism enhances our understanding of a zoo of *spectral identities* in the literature, especially their origins. Our work adds new instances to Reznikov's formalism. For more interesting examples, see Michel-Venkatesh [65, 66].

1.3.2 Technical Aspects

The construction of the $GL(3) \times GL(2)$ Rankin-Selberg period fits well with the inner product method of two Poincaré series à la Kuznetsov (say) — the function $\mathbb{P}_2^3\Phi$ can be regarded as a kind of Poincaré series for $SL_2(\mathbb{Z})$ by the Fourier expansion of Φ. [12] While Proposition 3.1.1 is the main structural input of this thesis, the inner product (1.9) offers quite a few technical benefits. In particular, it allows us to insert:

- the factor $|\det *|^{\bar{s}-\frac{1}{2}}$ which is crucial for convergence and analytic continuation;
- a sufficiently general class of test functions into Theorem 1.2.2.

It turns out that the regularity of the test functions will play a role in gaining sufficient decay of the integral transform (see Proposition 6.3.1), as well as providing room to offset the exponential decay coming from the Γ-factors [13] of which their occurrence is *intrinsic* to period integral methods.

Remark 1.3.2. For instance, to prove a smooth Weyl law for the L-values $L\big(1/2, \phi_j \otimes \widetilde{\Phi}\big)$ in the $GL(2)$ spectral aspect as in Ivić [22], one may pick the test function

$$H(\beta) \ = \ H_{R,\eta}(\beta;\Phi) \ := \ e^{(\beta/R)^2} \cdot \frac{\Gamma(2\eta+\beta)\Gamma(2\eta-\beta)}{\prod\limits_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2}+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{\frac{1}{2}-\beta-\alpha_i}{2}\right)}, \tag{1.10}$$

where η is a large, absolute constant and $R \gg 1$. In (1.10),

[12] In this thesis, we did not make use of this perspective.

[13] of $GL(3) \times GL(2)$ type in our case.

- the factor $e^{(\beta/R)^2}$ serves as a smooth cut-off for $|\beta_j| < R$ and gives the needed decay in (1.5);
- the factors $\prod_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2}+\beta-\alpha_i}{2}\right) \Gamma\left(\frac{\frac{1}{2}-\beta-\alpha_i}{2}\right)$ in (1.10) cancel out those in $L^*\left(1/2, \phi_j \otimes \widetilde{\Phi}\right)$ in the spectral expansion and diagonal contribution in (1.6);
- the factors $\Gamma\left(2\eta+\beta\right)\Gamma(2\eta-\beta)$ balance off the exponential growth from the spectral measure $d\beta/|\Gamma(\beta)|^2$ and the harmonic weights: $||\phi_j||^{-2}$ and $|\zeta^*(1+2i\mu)|^{-2}$. Also, a large enough region of holomorphy for the function in (1.10) is maintained so that $h(y) := H^\flat(y)$ has sufficient decay at 0 and ∞.

In Theorem 1.2.2, the choice (1.10) gives a main term of size $\asymp_\eta L\left(1, \Phi\right) \cdot R^{4\eta+1}$ as $R \to \infty$. One may also replace the factor $e^{(\beta/R)^2}$ by $e^{((\beta+iM)/R)^2} + e^{((\beta-iM)/R)^2}$ in (1.10) as in Li [40] for averaging over short intervals (and for applications towards subconvexity), where $M = R^\gamma$ and $0 \leqslant \gamma < 1$.

Remark 1.3.3. The regularity assumptions imposed on our test functions follow directly from that of the Kontorovich-Lebedev inversion (see Chapter 2.1), which was found to be rather handy in a number of applications, e.g., [67, 68], [69, 70].

In this thesis, we have made no attempt in optimize the regularity though in principle this should be doable — by a limiting process and the absolute convergence of both sides of (1.6). This will take a bit of extra work.

The method of this thesis makes the dual structures completely visible for linear algebra reasons. Not only is our method more direct, but generalizations to higher-rank situations are much more likely when compared to the traditional 'Kuznetsov-Voronoi' approach.

The Kuznetsov trace formula, or more generally the relative trace formula, has been a cornerstone in the analytic theory of L-functions for $GL(2)$ during the past few decades. It is an equality between the spectral average of Fourier coefficients over a basis of automorphic forms and the geometric expansion consisting of exponential sums (known as the *Kloosterman sums*) and oscillatory integrals. Unfortunately, the geometric expansion becomes substantially more complicated once we reach $GL(3)$ and it surely presents huge obstacles for the traditional approach. Therefore, this prompts us to re-think our strategies towards moment problems.

Upon further reflection, the author believes that the *Bruhat decomposition* is a source of complications. Such a decomposition is the main reason why Kloosterman sums and certain oscillatory integrals appear on the geometric side.

Remark 1.3.4 (Archimedean Oscillatory Integrals). In $GL(2)$, a couple of coincidences allow us to identify the oscillatory integrals with some well-studied special functions, see [24], [71]. However, such a phenomenon does not exist in $GL(3)$ and there turn out to be many unexpected analytic difficulties, see Buttcane [72, 73]. Furthermore, the complicated formulae for the oscillatory integrals make the Kuznetsov trace formula for $GL(3)$ challenging to apply, see Blomer-Buttcane [74].

Remark 1.3.5 (Non-archimedean Kloosterman Sums). The Kloosterman sums encode the arithmetic of moments of L-functions and are subject to delicate transformations. On one hand, the Kloosterman sums of $GL(3)$ are complicated, for example, [14]

$$\begin{aligned} S(m_1, m_2, n_1, n_2; D_1, D_2) \\ := \sum_{B_1 \pmod{D_1}} \sum_{C_1 \pmod{D_1}} \sum_{B_2 \pmod{D_2}} \sum_{C_2 \pmod{D_2}} \\ e\left(\frac{m_1 B_1 + n_1(Y_1 D_2 - Z_1 B_2)}{D_1}\right) \cdot e\left(\frac{m_2 B_2 + n_2(Y_2 D_1 - Z_2 B_1)}{D_2}\right). \end{aligned} \tag{1.11}$$

On the other hand, the changes of structures brought by the summation formulae of $GL(n)$, or the Voronoi formulae (see [37, 39] for instance), can be quite subtle when $n \geqslant 3$. In fact, it was essential to apply the $GL(3)$ Voronoi formula twice in [40], which were remarkably non-involutary,

[14] where the definitions of Y_i, Z_i's along with some congruence and coprimality conditions are suppressed, see [72] for detail.

when studying the spectral moment of the left side of (1.6) using the traditional approach. Attempts beyond $GL(3)$ are very limited, see [75] and [76].

Thus, we are motivated to look for an approach that is **free from the Bruhat decomposition** and thus any involvement of geometric sums and integrals. The period integral method, along the line of Blomer [77] for example, has provided some instances in which such a goal is indeed attainable. This is the approach adopted in this article. Altogether, the Kuznetsov formula, the Voronoi formula, and the approximate functional equation, which belong to the standard toolbox in analytic number theory (see [14, 19, 40] for instance), **are completely avoided** in our proof of Theorem 1.2.2.

In view of Remark 1.3.4, although our calculation on the dual side is about $GL(3)$, the integrals under consideration (see (3.9)) are relatively simple, say when compared to those in the $GL(3)$ Kuznetsov of [78] (In fact, $GL(2)$ Kuznetsov to some extent) or the $GL(3)$ Voronoi formulae (see [37]). Moreover, the crucial archimedean ingredient in our proof generalizes to $GL(n)$. It is known as *Stade's formula* (see [53]), which allows us to rewrite the archimedean part completely in terms of integrals Γ-functions and it possesses remarkable recursive structures. In our case, it turns out to be sufficient to work with such representation and obtain the needed analytic continuation for the identity (1.6).

In view of Remark 1.3.5, it would be favourable if one can detect the structures of moments more directly without having exponential sums as intermediate objects. This is achieved in the context of Theorem 1.2.2, which is a direct consequence of no Bruhat decomposition is ever involved in our argument.

1.4 Further Comparisons

1.4.1 With Dirichlet Series Approaches

In terms of technical ingredients, we have the following three different facets towards the cubic moment problem of $GL(2)$ L-functions: [15]

Articles	Method	Spectral Averaging	Summation Formulae	Transforms	Essential Special Functions
Conrey-Iwaniec [14]/ Petrow [19]	Dirichlet Series	$GL(2)$-Kuznetsov	Poisson thrice	Fourier	J, K-Bessel
Li [40] / Lü [79]	Dirichlet Series	$GL(2)$-Kuznetsov	$GL(3)$-Voronoi twice	Fourier	J, K-Bessel, $GL(3)$-Bessel
This thesis	Period Integral	$GL(2)$-unfolding	Proposition 3.1.1	Mellin	K-Bessel, $GL(3)$-Whittaker

The actions of the $GL(3)$ automorphy behind the scene distinguish the three approaches and are reflected in the step 'Summation Formulae' in the table above. The period integral set-up of this thesis suggests the following maneuver:

- performing *two* Poisson summations, leading to a product of *two* L-functions on the dual side of (1.6);
- 'meshing' these two Poisson summations together via appropriate use of the $GL(3)$ automorphy.

This is quite distinct from the paradigm of the method of Dirichlet series which begins with the Kuznetsov formula. The object of interest is then a certain additively-twisted Dirichlet series (say (1.4)) for the off-diagonal part. To facilitate our discussions, let's consider the case of Li [40] and restrict ourselves to the special instance when the conductor q in (1.4) is *prime*. Then the elementary formula [16]

$$e^{\frac{2\pi i a n}{q}} = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi}(an)\tau(\chi) \qquad (n,q) = 1 \tag{1.12}$$

[15] Lü [79] adapted the techniques of [40] to this particular setting.
[16] A straight-forward consequence of the orthogonality relation.

(with $\tau(\chi)$ being the Gauss sum) implies the fact that the Dirichlet series (1.4) is nothing but a linear combination of the automorphic L-series

$$L(s, \Phi \otimes \chi) := \sum_{n=1}^{\infty} \frac{\lambda_\Phi(n)\chi(n)}{n^s} \qquad \operatorname{Re} s \gg 1 \tag{1.13}$$

over the set of multiplicative characters $\chi \neq \chi_0 \pmod q$. The natural way to proceed further so as to bring in changes of structures is to appeal to the *functional equation* for $L(s, \Phi \otimes \chi)$ à la Jacquet-Piatetski-Shapiro-Shalika [80]. This gives essentially the $GL(3)$ Voronoi formula in the simplest case. (Here, we sidestep the analysis of the root numbers and the relevant Hecke relations in Goldfeld-Li [37].) Actually, by tracing through the argument of [80], the functional equation for $L(s, \Phi \otimes \chi)$ is a (non-trivial [17]) consequence of the automorphy of Φ with respect to a single Weyl element $\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$. In other words, the work of Li [40] [18] exploits this particular kind of automorphy, whereas our period integral set-up takes advantage of the flexibility of applying the automorphy with respect to other elements in $SL_3(\mathbb{Z})$ (actually infinitely many of them!) which turns out to be more adapted to the dual structure of Theorem 1.2.2.

Remark 1.4.1 (A few more words on Voronoi)**.** The circle of ideas of deducing arithmetic consequences from functional equations of multiplicatively-twisted L-functions were explored in a series of works due to Duke-Iwaniec [81, 82, 83, 84] and Luo-Rudnick-Sarnak [85].

In terms of proving the Voronoi formula, while the sketch above is intuitive, it does not work for general conductors and this turns out to be a serious obstruction as pointed out by Miller-Schmid [35]. A proof of the general Voronoi formula based on the method of (double) Dirichlet series was found by Kıral-Zhou [86], which was quite different from the one sketched above.

It is possible to view the Kuznetsov approach of Li [40] and the approach of this thesis under

[17] See also [39] Section 4, [38] Section 5, or [36] Appendix A.
[18] or more generally the Kuznetsov-Voronoi method.

the same set-up (1.9):

$$\left\langle P(*;h)\ ,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\overline{s}-\frac{1}{2}}\right\rangle_{L^2(SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))}.$$

The former case uses a *single* unfolding, i.e., that for $\mathbb{P}_2^3\Phi$, then applies the spectral and the geometric expansions for the Fourier coefficients of the Poincaré series $P(*;h)$ on the $GL(2)$ side. In particular, the diagonal term of the moment identity (1.6) is extracted from the identity contribution in the $GL(2)$ Bruhat decomposition. In our case, we take advantage of yet another unfolding (that of $P(*;h)$). This time, however, the diagonal term is of harmonic origin, namely it originates from the *zeroth frequency* of a Poisson summation applied to the $GL(3)$ *cusp form* Φ. The diagonal terms in both cases coincide as they should.

1.4.2 With Other Period Integral Approaches

Very recently, the Motohashi phenomenon for $GL(2)$, i.e., (1.3), is also nicely accounted under various insightful frameworks in automorphic representations. See the works of Nelson [87], Wu [88] and Balkanova-Frolenkov-Wu [57]. In [88, 57], the authors developed a relative trace formula of Godement-Jacquet type and performed a refined analysis on the integral transforms. In [87], the author was able to overcome many significant analytic challenges intrinsic to the regularized period method proposed by Michel-Venkatesh [66].

In the following, let us include a sketch of the ideas of [66] (taken from Section 4.3 of [65]) to facilitate comparison of features and to enhance our understandings of the Motohashi phenomenon. We shall not make any attempt to make the argument rigorous as this is rather technical (see [87]).

Michel-Venkatesh 'deduced' (1.3) by evaluating the period integral

$$\int_0^\infty |E^*(iy)|^2\ d^\times y \tag{1.14}$$

in two ways, where $E^*(z) := E^*(z;1/2)$ is the (complete) Eisenstein series of $SL_2(\mathbb{Z})$ at $s = 1/2$.

Certainly, the integral (1.14) (and some more below) diverges. However, we shall pretend the usual rules of analysis remain valid.

Firstly, the spectral expansion of $GL(2)$ gives

$$\int_0^\infty |E^*(iy)|^2 \, d^\times y \;=\; \sum_{\phi\colon\, GL(2)} \left\langle\, |E^*|^2, \phi \right\rangle \int_0^\infty \phi(iy)\, d^\times y \;+\; (\cdots). \tag{1.15}$$

The $GL(2) \times GL(2)$ Rankin-Selberg integral and the $GL(2)$ Hecke integral readily give

$$\left\langle\, |E^*|^2, \phi \right\rangle \;=\; L^*\left(\frac{1}{2}, \phi\right)^2 \qquad \text{and} \qquad \int_0^\infty \phi(iy)\, d^\times y \;=\; L^*\left(\frac{1}{2}, \phi\right) \tag{1.16}$$

respectively when ϕ is a cusp form. Then upon suppressing any discussion of the residual and the continuous parts which involve periods of (purely) Eisenstein series, equation (1.15) should give us the spectral cubic moment of $GL(2)$ L-functions.

Secondly, the Mellin transform of $E^*(iy)$, denoted by $\widetilde{E^*}(s)$, should be interpreted as $\zeta^*(s)^2$ at the heuristic level. At least when $\operatorname{Re} s > 1$ and when the constant term is subtracted from $E^*(iy)$, an elementary calculation shows that this is indeed the case. Of course, this naive subtraction is too brutal for multitude of reasons, say it 'destroys' the computation done in (1.16). In any case, we apply the Mellin-Plancherel formula [19] naively to the integral (1.14), we have

$$\int_0^\infty |E^*(iy)|^2 \, d^\times y \;=\; \int_{(1/2)} \left|\widetilde{E^*}(s)\right|^2 \frac{ds}{2\pi i} \;=\; \int_{\mathbb{R}} \left|\zeta^*\left(\frac{1}{2} + it\right)^2\right|^2 \frac{dt}{2\pi}, \tag{1.17}$$

which is the dual fourth moment of the Riemann ζ-function.

Remark 1.4.2. As discussed in [65], one may also wish to replace the Eisenstein series E^* in (1.14) by a Maass cusp form f of $GL(2)$. Then one expects a spectral identity for the integral

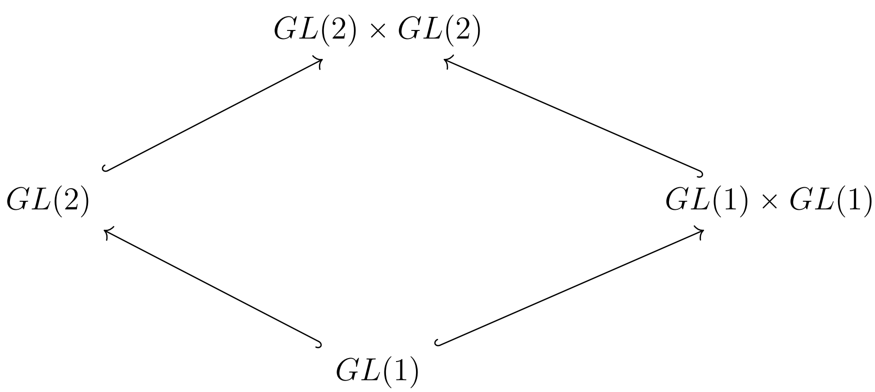
[19] regarded as the 'spectral expansion' with respect to the group $GL(1) \times GL(1)$ in the diagram below.

moment

$$\int_{\mathbb{R}} \left| L^* \left(\frac{1}{2} + it, f \right) \right|^2 \, dt.$$

The spectral projection $\langle |f|^2, \phi \rangle$ can be understood using the celebrated Ichino-Watson formula when ϕ is a basis element in the cuspidal spectrum. Even though the integral (1.14) becomes convergent in the present case, the terms '$(\cdots)$' in the spectral expansion (1.15) still require suitable regularizations.

Similar to Section 1.3, the sketch above can also be summarized in a diagram:



In fact, our method is quite different from the Michel-Venkatesh method in a number of ways:

	Michel-Venkatesh [66] / Nelson [87]	This Thesis (Minimal Parabolic case)
Reciprocity	$GL(1) \leftrightarrow GL(2)$	$GL(2) \leftrightarrow GL(3)$
Periods	$GL(2) \times GL(2)$ Rankin-Selberg period, $GL(2)$ Hecke period	$GL(3) \times GL(2)$ Rankin-Selberg period, Incomplete Whittaker period of $GL(3)$
Numerology (Spectral)	$6 = (1 \boxplus 1) \times (1 \boxplus 1) \boxplus 2$	$6 = (1 \boxplus 1 \boxplus 1) \times 2$
Numerology (Dual)	$4 = (1 \boxplus 1) \times (1 \boxplus 1)$	$4 = (1 \boxplus 1 \boxplus 1) \boxplus 1$
Subgroups of interest	$GL(2)$ Levi	$GL(2)$ Unipotent
Harmonic Analysis of $GL(1)$	$L^2(\mathbb{R}^\times)$	$L^2(\mathbb{Z}\backslash\mathbb{R})$

Remark 1.4.3. In this thesis, we use the term 'Incomplete Whittaker period' for (3.2) so as to distinguish from the 'Pseudo-Whittaker period' in Ichino-Templier [39]. The former period is

taken with respect to the unipotent subgroup $\left\{\begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}$, whereas the one for the latter involves the complementary $\left\{\begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}$ instead.

Remark 1.4.4. Another interesting feature of the (generalized) Motohashi phenomenon (i.e., Theorem 1.2.2) is that three distinct types of L-functions are linked together in a single identity. By 'distinct', we mean when viewing in terms of their respective period integrals:

1. The Riemann ζ-function — the Hecke integral of Jacobi's θ-function:

$$\int_0^\infty \frac{1}{2}\left(\theta(iy) - 1\right) y^{s/2}\, d^\times y; \tag{1.18}$$

2. The $GL(3) \times GL(2)$ L-function — the Rankin-Selberg integral (2.19) *without* Eisenstein series;

3. The $GL(3)$ L-function — a Hecke integral but with extra unipotent padding: [20] [21]

$$\int_0^\infty \int_0^1 \int_0^1 \Phi\begin{pmatrix} y & & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} y^{s-1} e^{-2\pi i u_{2,3}}\, du_{1,3}\, du_{2,3}\, d^\times y. \tag{1.19}$$

It is not clear to the author how to connect the three period integrals above in the context of Theorem 1.2.2. As discussed in Chapter 1.4.1, our solution is to package the ζ-function and the L-function of Φ together into a certain Whittaker period instead and we believe that this is a natural way to proceed.

Our solution comes with the extra benefit that the Eisenstein cases (see Remark 1.2.5) can be covered, almost mutatis mutandis, as in the case of cusp forms. This is true because unipotent

[20] One may also consider a variant of (1.19) with a double Mellin transform instead, as in Chapter 6 of Goldfeld [89]. Then one can perform a more explicit archimedean computation for $GL(3)$ thanks to the Bump-Takhtadzhyan-Vinogradov formula (i.e., Proposition 2.1.1).

[21] Here, we consider Φ to be a **general** spherical Maass cusp form. When Φ is self-dual, one can show that Φ comes from the sym^2-lift of a $GL(2)$ Maass form. In this case, there is a more familiar integral representation of $GL(2) \times GL(2)$ type involving the θ-function. This was due to Shimura and Gelbart-Jacquet. For more details, see Chapter 7.3 of Goldfeld [89].

integration and the method of analytic continuation serve as the basis of our approach. The absolute convergence of the period integral and the two Rankin-Selberg unfoldings remain in place when $\operatorname{Re} s \gg 1$ — with the degenerate terms subtracted off initially and to be taken care of at a later stage.

1.5 Work In-progress & Further Research Directions

In this thesis, we have restricted ourselves to the simplest possible setting to illustrate the main ideas with focus on the archimedean aspect. In our upcoming work, we plan to

1. Develop the moment formula behind the more general version of the period integral (1.9), i.e.,

$$\left\langle P^a,\ \mathbb{P}_n^N \Phi \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(GL_n(F)\backslash GL_n(\mathbb{A}_F))}, \tag{1.20}$$

   where $\mathbb{P}_n^N$ is defined in Cogdell [90], [22] P^a is a Poincaré series of $GL(n)$ and Φ is an automorphic form of $GL(N)$. Our method uses integral representation, which will allow local treatment and handling of ramifications.

2. Refine our investigation of the archimedean aspect of the problem, say along the line of the series of papers [72, 73, 74, 69, 70]. The main goal is to provide good estimations for the integral transforms.

1.6 Outline & Notations

In Chapter 2, we collect some essential notions and results for later parts of the article. In particular, we strive to maintain all notations to be with convenient normalizations for the analytic theory of $GL(3)$ automorphic forms. Also, we want to ensure the compatibility of the conventions between the $GL(2)$ and the $GL(3)$ theories.

[22] which explains the use of the notation $\mathbb{P}_2^3$ before.

The proof of Theorem 1.2.2 is divided into five sections. In Chapter 3, we prove the key identity of this article (see Corollary 3.1.2). In Chapter 4, we develop such an identity into moments of L-functions on the region of absolute convergence. In particular, the intrinsic structure of the problem allows one to easily see the shape of the dual moment (see Proposition 4.0.2).

In Chapter 5, we obtain the region of holomorphy and growth of the archimedean transform. In Chapter 6, a step-by-step analytic continuation argument is performed based on the analytic information obtained in Chapter 5. In Chapter 7, we provide several explicit formulae of the transforms. In Chapter 8, we provide a 'twisted' version of Theorem 1.2.2.

Throughout this thesis, we make use of the following notations frequently:

- $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2) \ \ (s \in \mathbb{C})$;
- $e(x) := e^{2\pi i x} \ \ (x \in \mathbb{R})$;
- $\Gamma_n := SL_n(\mathbb{Z}) \ \ (n \geqslant 2)$.

Moreover, we follow

Convention 1.6.1.

1. *All Maass cusp forms will be simultaneous eigenfunctions of the Hecke operators and will be either even or odd. Also, their first Fourier coefficients are equal to* 1. *In this case, the forms are said to be* ***Hecke-normalized****. Note that there are no odd forms for* $SL_3(\mathbb{Z})$*, see Proposition 9.2.5 of [89].*

2. *Our fixed Maass cusp form* Φ *of* $SL_3(\mathbb{Z})$ *is assumed to be* ***tempered at*** ∞*, i.e., its Langlands parameters are purely imaginary.*

3. *Denote by* θ *the best progress towards the Ramanujan conjecture for the Maass cusp forms of* $SL_3(\mathbb{Z})$*. We have* $\theta \leqslant \frac{1}{2} - \frac{1}{10}$*, see Theorem 12.5.1 of [89].*

Chapter 2: Preliminaries

2.1 (Spherical) Whittaker Functions & Transforms

The Whittaker function of $GL_2(\mathbb{R})$ is more familiar and is given by $W_\beta(y) := 2\sqrt{y}K_\beta(2\pi y)$ for $\beta \in \mathbb{C}$ and $y > 0$. For the group $GL_3(\mathbb{R})$, we first introduce the function [1]

$$I_\alpha(y_0, y_1) \;=\; I_\alpha\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} := \; y_0^{1-\alpha_3}y_1^{1+\alpha_1}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)} := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$. Then the Whittaker function for $GL_3(\mathbb{R})$, denoted by $W_\alpha(y_0, y_1) = W_\alpha\left(\begin{smallmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{smallmatrix}\right)$, can be defined using *Jacquet's integral*:

$$\prod_{1\leqslant j<k\leqslant 3} \Gamma_{\mathbb{R}}(1 + \alpha_j - \alpha_k) \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} I_\alpha\left[\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right]$$
$$\cdot\, e(-u_{1,2} - u_{2,3})\, du_{1,2}\, du_{1,3}\, du_{2,3}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$. See Chapter 5.5 of [89] for details. Moreover, it admits the following useful Mellin-Barnes representation:

[1] The normalization here for the I_α-function is more convenient than that of eq. 5.1.1 in [89].

Proposition 2.1.1. *Assume* $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ *is tempered, i.e.,* $\operatorname{Re} \alpha_i = 0$ *(*$i = 1, 2, 3$*). Then for any* $\sigma_0, \sigma_1 > 0$*,*

$$W_{-\alpha}(y_0, y_1) = \frac{1}{4} \int_{(\sigma_0)} \int_{(\sigma_1)} G_\alpha(s_0, s_1) y_0^{1-s_0} y_1^{1-s_1} \frac{ds_0}{2\pi i} \frac{ds_1}{2\pi i}, \qquad y_0, y_1 > 0, \tag{2.1}$$

where

$$G_\alpha(s_0, s_1) := \frac{\prod_{i=1}^{3} \Gamma_{\mathbb{R}}(s_0 + \alpha_i) \Gamma_{\mathbb{R}}(s_1 - \alpha_i)}{\Gamma_{\mathbb{R}}(s_0 + s_1)}. \tag{2.2}$$

Proof. See (6.1.4)–(6.1.5) of [89]. For a proof, see Chapter X of [91]. □

Corollary 2.1.2. *For any* $-\infty < A_0, A_1 < 1$*, we have*

$$|W_{-\alpha}(y_0, y_1)| \ll y_0^{A_0} y_1^{A_1}, \qquad y_0, y_1 > 0, \tag{2.3}$$

where the implicit constant depends only on α, A_0, A_1*.*

Proof. Follows directly from Proposition 2.1.1. □

We will need the evaluation of the $GL_3(\mathbb{R}) \times GL_2(\mathbb{R})$ Rankin-Selberg integral:

Proposition 2.1.3. *Let* W_β *and* W_α *be the Whittaker functions of* $GL_2(\mathbb{R})$ *and* $GL_3(\mathbb{R})$ *respectively. For* $\operatorname{Re} s \gg 0$*, we have*

$$\int_0^\infty \int_0^\infty W_\beta(y_1) \cdot \overline{W_\alpha}(y_0, y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \frac{dy_0 dy_1}{y_0 y_1^2} = \frac{1}{4} \cdot \prod_{j=1}^{2} \prod_{K=1}^{3} \Gamma_{\mathbb{R}}\left(s + \beta_j + \overline{\alpha_K}\right). \tag{2.4}$$

Proof. See [52]. □

The following pair of integral transforms plays an important role in the archimedean aspect of this article.

Definition 2.1.4. Let $h : (0, \infty) \to \mathbb{C}$ and $H : i\mathbb{R} \to \mathbb{C}$ be measurable functions with $H(\beta) = H(-\beta)$. Let $W_\beta(y) := 2\sqrt{y} K_\beta(2\pi y)$. Then the Kontorovich-Lebedev transform of h is defined by

$$h^{\#}(\beta) \ := \ \int_0^\infty h(y) \cdot W_\beta(y) \, \frac{dy}{y^2}, \tag{2.5}$$

whereas its inverse transform is defined by [2]

$$H^\flat(y) \ = \ \frac{1}{4\pi i} \int_{(0)} H(\beta) \cdot W_\beta(y) \, \frac{d\beta}{|\Gamma(\beta)|^2}, \tag{2.6}$$

provided the integrals converge absolutely.

Definition 2.1.5. Let $\mathcal{C}_\eta$ be the class of holomorphic functions H on the vertical strip $|\operatorname{Re}\beta| < 2\eta$ such that

1. $H(\beta) = H(-\beta)$,

2. H has rapid decay in the sense that

$$H(\beta) \ \ll \ e^{-10|\beta|} \qquad\qquad (|\operatorname{Re}\beta| \ < \ 2\eta). \tag{2.7}$$

In this article, we take $\eta > 40$ without otherwise specified. By contour-shifting and Stirling's formula, we have

Proposition 2.1.6. *For any $H \in \mathcal{C}_\eta$, the integral (2.6) defining $H^\flat$ converges absolutely. Moreover, we have*

$$H^\flat(y) \ \ll \ \min\{y, y^{-1}\}^\eta \qquad (y \ > \ 0). \tag{2.8}$$

Proof. See Lemma 3.2 of [92]. □

[2]The normalization constant $1/4\pi i$ in (2.6) is consistent with that in [24], [71].

Proposition 2.1.7. *Under the same assumptions of Proposition 2.1.6, we have*

$$(h^{\#})^{\flat}(g) \;=\; h(g) \qquad \textit{and} \qquad (H^{\flat})^{\#}(\beta) \;=\; H(\beta). \tag{2.9}$$

Proof. See [92]. □

2.2 Automorphic Forms of $GL(2)$ and $GL(3)$

Let

$$\mathfrak{h}^2 \;:=\; \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} : u \in \mathbb{R},\ y > 0 \right\}$$

with its invariant measure given by $dudy/y^2$. An automorphic form $\phi : \mathfrak{h}^2 \to \mathbb{C}$ of Γ_2 satisfies $\Delta\phi = \left(\frac{1}{4} - \beta^2\right)\phi$ for some $\beta = \beta(\phi) \in \mathbb{C}$, where $\Delta := -y^2\left(\partial_x^2 + \partial_y^2\right)$. It is convenient to identify β with the pair $(\beta, -\beta) \in \mathfrak{a}_{\mathbb{C}}^{(2)}$.

For $a \in \mathbb{Z} - \{0\}$, the a-th Fourier coefficient of ϕ, denoted by $\mathcal{B}_\phi(a)$, is defined by

$$(\widehat{\phi})_a(y) \;:=\; \int_0^1 \phi\left[\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] e(-au)\, du \;=\; \frac{\mathcal{B}_\phi(a)}{\sqrt{|a|}} \cdot W_{\beta(\phi)}(|a|y). \tag{2.10}$$

In the case of the Eisenstein series of Γ_2, i.e.,

$$\phi \;=\; E(z;\mu) \;:=\; \frac{1}{2} \sum_{\gamma \in U_2(\mathbb{Z})\backslash\Gamma_2} I_\mu(\operatorname{Im}\gamma z) \qquad (z \;\in\; \mathfrak{h}^2), \tag{2.11}$$

where $I_\mu(y) := y^{\mu+\frac{1}{2}}$, it is well-known that

$$\mathcal{B}(a;\mu) \;=\; \frac{|a|^\mu \sigma_{-2\mu}(|a|)}{\zeta^*(1+2\mu)} \qquad \text{and} \qquad \Delta E(*;\mu) \;=\; \left(\frac{1}{4} - \mu^2\right) E(*;\mu), \tag{2.12}$$

where $\zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\sigma_{-2\mu}(|a|) := \sum_{d|a} d^{-2\mu}$. Moreover, the series (2.11) converges absolutely when $\operatorname{Re}\mu > 1/2$ and admits a meromorphic continuation to $\mathbb{C}$.

Next, let

$$\mathfrak{h}^3 := \left\{ \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} : u_{i,j} \in \mathbb{R},\ y_k > 0 \right\}.$$

Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3 as defined in Definition 5.1.3 of [89]. In particular, there exists $\alpha = \alpha(\Phi) \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ such that for any $D \in Z(U\mathfrak{gl}_3(\mathbb{C}))$, [3] we have

$$D\Phi = \lambda_D \Phi \qquad \text{and} \qquad DI_\alpha = \lambda_D I_\alpha$$

for some $\lambda_D \in \mathbb{C}$. The triple $\alpha(\Phi)$ is said to be the Langlands parameters of Φ. There is also the notion of Fourier coefficients for Φ thanks to the multiplicity-one theorem of Shalika. For a proof, see Theorem 6.1.6 of [89]. Once again, we follow the convention of [89] (pp. 261).

Definition 2.2.1. Let $m = (m_1, m_2) \in (\mathbb{Z} - \{0\})^2$ and $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form. For any $y_0, y_1 > 0$, define

$$(\widehat{\Phi})_{(m_1,m_2)} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} := \int_0^1 \int_0^1 \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] \cdot e\left(-m_1 u_{2,3} - m_2 u_{1,2}\right)\, du_{1,2}\, du_{1,3}\, du_{2,3}. \quad (2.13)$$

[3] The center of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$.

Then the (m_1, m_2)-th **Fourier coefficient** of Φ is the complex number $\mathcal{B}_\Phi(m_1, m_2)$ for which

$$(\widehat{\Phi})_{(m_1,m_2)} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} = \frac{\mathcal{B}_\Phi(m_1, m_2)}{|m_1 m_2|} W^{\mathrm{sgn}(m_2)}_{\alpha(\Phi)} \begin{pmatrix} (|m_1|y_0)(|m_2|y_1) & & \\ & |m_1|y_0 & \\ & & 1 \end{pmatrix} \tag{2.14}$$

holds for any $y_0, y_1 > 0$.

2.3 Automorphic L-functions

The Maass cusp forms Φ and ϕ below are Hecke-normalized and their Langlands parameters are $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ and $\beta \in \mathfrak{a}_{\mathbb{C}}^{(2)}$ respectively.

Definition 2.3.1. Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3. We define the standard L-function of Φ by the Dirichlet series

$$L(s, \Phi) := \sum_{n=1}^{\infty} \frac{\mathcal{B}_\Phi(1, n)}{n^s} \qquad (\mathrm{Re}\, s \gg 1). \tag{2.15}$$

Let $\widetilde{\Phi}(g) := \Phi\left({}^t g^{-1}\right)$ be the dual form of Φ. The L-function $L(s, \Phi)$ admits an entire continuation and satisfies the functional equation:

$$L^*(s, \Phi) := \prod_{K=1}^{3} \Gamma_{\mathbb{R}}(s + \alpha_K) \cdot L(s, \Phi) \\ = \prod_{K=1}^{3} \Gamma_{\mathbb{R}}(1 - s - \alpha_K) \cdot L\left(1 - s, \widetilde{\Phi}\right) := L^*\left(1 - s, \widetilde{\Phi}\right). \tag{2.16}$$

For a proof, see Section 6.5 of [89] or [80].

Definition 2.3.2. Let ϕ (resp. Φ) be a Maass cusp form of Γ_2 (resp. Γ_3). We define the Rankin-

Selberg L-function of ϕ and Φ by the Dirichlet series

$$L\left(s,\phi\otimes\Phi\right) := \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\mathcal{B}_{\phi}(m_2)\mathcal{B}_{\Phi}(m_1,m_2)}{\left(m_1^2 m_2\right)^s} \qquad (\operatorname{Re}s \gg 1). \tag{2.17}$$

The corresponding complete L-function is given by

$$L^{*}\left(s,\phi\otimes\Phi\right) := L_{\infty}\left(s,\phi\otimes\Phi\right)\cdot L\left(s,\phi\otimes\Phi\right) := \prod_{k=1}^{2}\prod_{K=1}^{3}\Gamma_{\mathbb{R}}\left(s+\beta_k+\alpha_K\right)\cdot L\left(s,\phi\otimes\Phi\right). \tag{2.18}$$

It admits an integral representation and is given by the following Proposition using an unfolding argument. From this, the analytic properties of (2.17) can be deduced.

Proposition 2.3.3. *Suppose Φ (resp. ϕ) is a Maass cusp form of Γ_3 (resp. Γ_2). For $\operatorname{Re}s \gg 1$, if ϕ is even, then*

$$\left\langle \phi,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} = \frac{1}{2}\,L^{*}\left(s,\phi\otimes\widetilde{\Phi}\right) \tag{2.19}$$

where $(\mathbb{P}_2^3\Phi)(g) := \Phi\begin{pmatrix} g & \\ & 1\end{pmatrix}$ $(g\in GL_2(\mathbb{R}))$; whereas (2.19) is 0 if ϕ is odd.

Proof. We take this opportunity to correct some minor inaccuracies in Section 12.2 of [89]. Indeed, the assumptions on the parities are missing in [89] and the inner product should be taken over the quotient $\Gamma_2\backslash GL_2(\mathbb{R})$ instead of $\Gamma_2\backslash\mathfrak{h}^2$.

As a brief sketch, we replace $\mathbb{P}_2^3\Phi$ by its Fourier-Whittaker expansion (Theorem 5.3.2 of [89]) on the left side of (2.19) and unfold. Then one may extract the Dirichlet series (2.17) by using (2.10) and (2.13). The integral of Whittaker functions can be computed by Proposition 2.1.3. All of the series and the integrals converge absolutely whenever $\operatorname{Re}s \gg 1$. $\square$

Proposition 2.3.4. *For* $\mathrm{Re}(s \pm \mu) \gg 1$*, we have*

$$\left\langle E(*;\mu),\ \left(\mathbb{P}_2^3\Phi\right)\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} \;=\; \frac{1}{2}\,\frac{L^*\left(s+\mu,\ \widetilde{\Phi}\right)L^*\left(s-\mu,\ \widetilde{\Phi}\right)}{\zeta^*(1+2\mu)}. \tag{2.20}$$

Proof. Parallel to Proposition 2.3.3, but replacing $\mathcal{B}_\phi(m_2)$ in (2.17) by (2.12) instead. The right side of (2.20) follows from comparing the relevant Euler products of the L-series, see [89] for details. □

Remark 2.3.5. In fact, the rapid decay of Φ at ∞ guarantees the inner products of Proposition 2.3.3-2.3.4 converge absolutely whenever $s \in \mathbb{C}$ and away from the poles of $E(*;\mu)$. In particular, the equalities (2.19) and (2.20) hold for such s and μ by analytic continuation.

With the same set-up of Proposition 2.3.3, the involution $g \mapsto {}^t g^{-1}$ and (2.19) give the functional equation

$$L^*\left(s, \phi\otimes\widetilde{\Phi}\right) \;=\; L^*\left(1-s, \phi\otimes\Phi\right)$$

whenever ϕ is even.

2.4 Calculation on the Spectral Side

As indicated in the introduction, our approach differs from the 'Kuznetsov-Voronoi' approach right from the start — we will *not* make use of the Dirichlet series (2.17). Instead, the moment of $GL(3)\times GL(2)$ L-functions is first interpreted in terms of the period integral of Proposition 2.3.3 using a Poincaré series.

Definition 2.4.1. Let $a \geqslant 1$ be an integer and $h \in C^\infty(0,\infty)$. The Poincaré series of Γ_2 is defined by

$$P^a(z;h) \;:=\; \sum_{\gamma\in U_2(\mathbb{Z})\backslash\Gamma_2} h(a\,\mathrm{Im}\,\gamma z)\cdot e\left(a\,\mathrm{Re}\,\gamma z\right) \qquad (z \in \mathfrak{h}^2) \tag{2.21}$$

provided it converges absolutely.

It is not hard to see that if the bounds

$$h(y) \ll y^{1+\epsilon} \quad (\text{as } y \to 0) \quad \text{and} \quad h(y) \ll y^{\frac{1}{2}-\epsilon} \quad (\text{as } y \to \infty) \tag{2.22}$$

are satisfied, then the Poincaré series $P^a(z;h)$ converges absolutely, is an L^2-function of at most polynomial growth at ∞. In this article, we take $h := H^\flat$ with $H \in \mathcal{C}_\eta$ and $\eta > 40$. By Proposition 2.1.6, conditions (2.22) clearly holds. We will often use the shorthand $P^a := P^a(*;h)$.

Lemma 2.4.2. *Let ϕ be a Maass cusp form of Γ_2, $\Delta\phi = \left(\frac{1}{4} - \beta^2\right)\phi$, and $\mathcal{B}_\phi(a)$ be the a-th Fourier coefficient of ϕ. Then*

$$\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ |a|^{1/2}\cdot\overline{\mathcal{B}_\phi(a)}\cdot h^\#\left(\overline{\beta}\right).$$

Proof. Replace P^a in $\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)}$ by its definition and unfold, we find that

$$\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ \int_0^\infty h(ay)\cdot\overline{\widehat{(\phi)}_a(y)}\,\frac{dy}{y^2}.$$

The result follows at once upon plugging-in (2.10) and making the change of variable $y \to |a|^{-1}y$.

□

Similarly, the following holds away from the poles of $E(*;\mu)$:

Lemma 2.4.3.

$$\left\langle P^a,\ E(*;\mu)\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ |a|^{1/2}\cdot\frac{|a|^{\overline{\mu}}\sigma_{-2\overline{\mu}}(|a|)}{\zeta^*(1+2\overline{\mu})}\cdot h^\#\left(\overline{\mu}\right). \tag{2.23}$$

Proposition 2.4.4 (Selberg's Spectral Expansion)**.** *Suppose $f \in L^2(\Gamma_2\backslash\mathfrak{h}^2)$ and $\langle f, 1\rangle = 0$. Then*

$$f(z) \ = \ \sum_{j=1}^\infty \frac{\langle f,\phi_j\rangle}{\langle \phi_j,\phi_j\rangle}\cdot\phi_j(z) + \frac{1}{4\pi}\int_{-\infty}^\infty \left\langle f,\ E(*;i\mu)\right\rangle\cdot E(z;i\mu)\,d\mu \qquad (z \in \mathfrak{h}^2) \tag{2.24}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle := \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)}$ and $(\phi_j)_{j\geqslant 1}$ is any orthogonal basis of Maass cusp forms for Γ_2.

Proof. See Theorem 3.16.1 of [89]. □

Proposition 2.4.5. *Let Φ be a Maass cusp form of Γ_3 and P^a be a Poincaré series of Γ_2. Then for any $s \in \mathbb{C}$,* [4]

$$2 \cdot |a|^{-1/2} \cdot \left\langle P^a,\ (\mathbb{P}_2^3 \Phi) \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2 \backslash GL_2(\mathbb{R}))}$$
$$= \sum_{j=1}^{\infty}{}' \ h^{\#}\left(\overline{\beta_j}\right) \cdot \frac{\overline{\mathcal{B}_j(a)} \cdot L^*\left(s, \phi_j \otimes \widetilde{\Phi}\right)}{\langle \phi_j, \phi_j \rangle}$$
$$+ \frac{1}{4\pi} \int_{\mathbb{R}} h^{\#}(i\mu) \frac{\sigma_{-2i\mu}(|a|)|a|^{-i\mu} L^*\left(s + i\mu, \widetilde{\Phi}\right) L^*\left(1 - s + i\mu, \Phi\right)}{|\zeta^*(1 + 2i\mu)|^2} \, d\mu, \tag{2.25}$$

where the sum is restricted to an orthogonal basis (ϕ_j) of even Hecke-normalized Maass cusp forms for Γ_2 with $\Delta \phi_j = \left(\frac{1}{4} - \beta_j^2\right) \phi_j$ and $\mathcal{B}_j(a) := \mathcal{B}_{\phi_j}(a)$. [5]

Proof. Substitute the spectral expansion of P^a as in (2.24) into $\left\langle P^a,\ (\mathbb{P}_2^3 \Phi) \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2 \backslash GL_2(\mathbb{R}))}$. The inner products involved have been computed in Lemma 2.4.2-2.4.3 and Proposition 2.3.3-2.3.4. □

Remark 2.4.6. In many applications, it is important to have refined control in the spectral component and study the analytic properties of the relevant integral transforms. As a result, the ability of making flexible choices of test functions on the spectral side is crucial. This is one of the strengths of the Kuznetsov formula over the period integral methods (say [77, 50, 48, 49]) and might partly explain why the former seems to be more ubiquitous in the current literature.

Although our method is period-integral based (see (1.9)), we are able to put a large class of test functions on the spectral side as in the Kuznetsov approach, by using the pair of transforms introduced in Definition 2.1.4. Such transforms have been generalized to $GL(n)$ in a simple and explicit

[4] See Remark 2.3.5.
[5] also the a-th Hecke eigenvalue of ϕ_j.

fashion in [92]. They have played important roles in the recent development of the Kuznetsov formulae of higher-rank (see [67], [68], [69, 70]).

Our method preserves the advantages of both the Kuznetsov and the period integral approaches — the former being the precision in the archimedean aspect whereas the latter being the structural insights in the nonarchimedean aspect.

Remark 2.4.7. Readers may wonder about the possiblity of using an automorphic kernel in place of a Poincaré series in studying the moment of L-functions in Theorem 1.2.2. Although this offers extra flexibility in incorporating new structures, the analysis behind the integral transforms (the spherical transforms) becomes quite complicated, see [72] for the case of $GL(3)$. The approach using Poincaré series seems to be more adapted to the analytic number theory of higher-rank groups.

Chapter 3: Basic Identity for Dual Moment

3.1 Unipotent Integration

We are ready to work on the dual side of our moment formula. As a simplification of our argument, we shall only consider $P = P^a(*; h)$ with $a = 1$ in the following. For discussions of the general case, see Appendix 8. Suppose $\operatorname{Re} s > 1 + \frac{\theta}{2}$. We begin by replacing P by its definition in the inner product $\left\langle P,\ \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$. We find upon unfolding:

$$\left\langle P, \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$$
$$= \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \cdot \int_0^1 \overline{\Phi} \left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(u_{1,2})\, du_{1,2}\, \frac{dy_0 dy_1}{y_0 y_1^2}. \tag{3.1}$$

The main task of this section is to compute the inner, 'incomplete' unipotent integral in (3.1). We wish to evaluate it in terms of the Fourier coefficients of Φ (see Definition 2.2.1) as they are relevant in the constructions of various L-functions associated to Φ, say those discussed in Section 2.3.

Certainly, this can be obtained by plugging in the *full* Fourier expansion of [80] (see [89] Theorem 5.3.2) and look for possible simplifications. This is in fact not necessary. We prefer a self-contained and conceptual treatement. It simply follows from two one-dimensional Fourier expansions and the automorphy of Φ. In essence, this is where 'summation formulae' take place in our approach, and they are nicely packaged in an elementary, clean, and global fashion.

Proposition 3.1.1. *For any automorphic function Φ of Γ_3, we have, for any $y_0, y_1 > 0$,*

$$\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$
$$= \sum_{a_0, a_1 = -\infty}^{\infty} (\widehat{\Phi})_{(a_1,1)} \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right]. \tag{3.2}$$

Proof. Firstly, we Fourier-expand along the abelian subgroup $\left\{\begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix}\right\}$:

$$\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$

$$= \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} \Phi\left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e(-u_{1,2} - a_0 \cdot u_{1,3})\, du_{1,2}\, du_{1,3}. \tag{3.3}$$

Secondly, for each $a_0 \in \mathbb{Z}$, consider a unimodular change of variables of the form

$$(u_{1,2}, u_{1,3}) = (u'_{1,2}, u'_{1,3}) \cdot \begin{pmatrix} 1 & \\ -a_0 & 1 \end{pmatrix}. \tag{3.4}$$

One can readily observe that

$$\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix}.$$

Together with the automorphy of Φ with respect to Γ_3, we have

$$\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e(-a_2 \cdot u_{1,2})\, du_{1,2}$$

$$= \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2 \backslash \mathbb{R}^2} \Phi\left[\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix}\begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e(-u'_{1,2})\, du'_{1,2}\, du'_{1,3}. \tag{3.5}$$

The result follows from the third and the final Fourier expansion along the abelian subgroup $\left\{\left(\begin{smallmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{smallmatrix}\right)\right\}$:

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$

$$= \sum_{a_0,a_1=-\infty}^{\infty} \int_0^1\int_0^1\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]$$

$$\cdot e(-u_{1,2} - a_1 \cdot u_{2,3})\, du_{1,2}\, du_{1,3}\, du_{2,3}.$$

□

We then explicate Proposition 3.1.1 when Φ is a Maass cusp form of Γ_3. This constitutes the *basic identity* of the present article. Theorem 1.2.2 is a natural consequence of this identity and the diagonal/ off-diagonal structures on the dual side become apparent (see Proposition 4.0.2).

Corollary 3.1.2. *Suppose Φ is a Maass cusp form of Γ_3. Then*

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$

$$= \sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|} \cdot W_{\alpha(\Phi)}\left(|a_1| y_0, y_1\right)$$
$$+ \sum_{a_0\neq 0}\sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|} \cdot W_{\alpha(\Phi)}\left(\frac{|a_1|y_0}{1+(a_0y_0)^2},\ y_1\sqrt{1+(a_0y_0)^2}\right)$$
$$\cdot e\left(-\frac{a_0a_1y_0^2}{1+(a_0y_0)^2}\right). \tag{3.6}$$

Proof. By cuspidality, $(\widehat{\Phi})_{(0,1)} \equiv 0$. Indeed,

$$\begin{aligned}
\left(\widehat{\Phi}\right)_{(0,1)}(g) &:= \int_0^1\int_0^1\int_0^1 \Phi\left[\left(\begin{smallmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{smallmatrix}\right) g\right] e\left(-u_{1,2}\right)\, du_{1,2}\, du_{1,3}\, du_{2,3} \\
&= \int_0^1 \left\{ \int_0^1\int_0^1 \Phi\left[\left(\begin{smallmatrix} 1 & & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{smallmatrix}\right) g\right]\, du_{1,3}\, du_{2,3} \right\} e\left(-u_{1,2}\right)\, du_{1,2} \\
&= 0.
\end{aligned}$$

The result follows from a straight-forward linear algebra calculation:

$$\begin{aligned}
&\begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \\
&\qquad \equiv \begin{pmatrix} 1 & & \\ & 1 & -\frac{a_0 y_0^2}{1+(a_0 y_0)^2} \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{y_0}{1+(a_0 y_0)^2} \cdot y_1 \sqrt{1+(a_0 y_0)^2} & & \\ & \frac{y_0}{1+(a_0 y_0)^2} & \\ & & 1 \end{pmatrix}
\end{aligned}$$

under the right quotient by $O_3(\mathbb{R}) \cdot \mathbb{R}^\times$. This can be verified by the mathematica command *IwasawaForm[]* in the *GL(n)pack* (*gln.m*): [1]

```
SetDirectory[NotebookDirectory[]];<< gln.m
A={{1,0,0}, {0,1,0}, {0, -a[0],1}};
Y={{y[0]*y[1], 0,0},{0, y[0], 0},{0,0,1}};
R = IwasawaForm[A.Y];
IwasawaXMatrix[R]   IwasawaXVariables[R]
IwasawaYMatrix[R]   IwasawaYVariables[R]
```

□

[1]The user manual and the package can be downloaded from Kevin A. Broughan's website: `https://www.math.waikato.ac.nz/~kab/glnpack.html`.

3.2 Initial Simplification and Absolute Convergence

We temporarily restrict ourselves to the vertical strip

$$1+\frac{\theta}{2} \;<\; \sigma \;:=\; \operatorname{Re} s \;<\; 4. \tag{3.7}$$

As we shall see, this guarantees the absolute convergence of all sums and integrals.

Suppose $H \in \mathcal{C}_\eta$ with $\eta > 40$ (see Proposition 2.1.6). Then the bound (2.8) for $h := H^\flat$ implies its Mellin transform $\widetilde{h}(w) := \int_0^\infty h(y) y^w \, d^\times y$ is holomorphic on the vertical strip $|\operatorname{Re} w| < \eta$. Substituting (3.6) into (3.1), and apply the changes of variables $y_0 \to |a_1|^{-1} y_0$, $y_0 \to |a_0|^{-1} y_0$ to the first, second piece of the resultant,

$$\begin{aligned}\Big\langle P,\ \mathbb{P}_2^3\Phi\cdot|\det *|^{\bar{s}-\frac12}\Big\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} \\ \;=\; 2\cdot L(2s,\Phi)\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac12}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\,\frac{dy_0dy_1}{y_0y_1^2}+OD_\Phi(s),\end{aligned} \tag{3.8}$$

where we define

Definition 3.2.1.

$$\begin{aligned}OD_\Phi(s) \;:=\; \sum_{a_0\neq 0}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|}\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac12}\cdot e\left(\frac{a_1}{a_0}\cdot\frac{y_0^2}{1+y_0^2}\right) \\ \cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\cdot\frac{y_0}{1+y_0^2},\ y_1\sqrt{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}.\end{aligned} \tag{3.9}$$

Proposition 3.2.2. *When $H \in \mathcal{C}_\eta$ and $4 > \sigma > \frac{1+\theta}{2}$, we have*

$$\int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \, \frac{dy_0 dy_1}{y_0 y_1^2} = \frac{\pi^{-3s}}{8} \cdot \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \prod_{i=1}^{3} \Gamma\left(\frac{s+\beta-\alpha_i}{2}\right) \Gamma\left(\frac{s-\beta-\alpha_i}{2}\right) \frac{d\beta}{2\pi i}. \tag{3.10}$$

Proof. From Proposition 2.1.7, we have

$$\int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \, \frac{dy_0 dy_1}{y_0 y_1^2} = \frac{1}{2} \cdot \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \int_0^\infty \int_0^\infty W_\beta(y_1) W_{-\alpha(\Phi)}(y_0, y_1) (y_0^2 y_1)^{s-\frac{1}{2}} \, \frac{dy_0 dy_1}{y_0 y_1^2} \, \frac{d\beta}{2\pi i}.$$

The y_0, y_1-integrals can be evaluated by Proposition 2.1.3 and (3.10) follows. Moreover, the right side of (3.10) is holomorphic on $\sigma > 0$. □

Proposition 3.2.3. *The off-diagonal $OD_\Phi(s)$ converges absolutely when $4 > \sigma > 1 + \frac{\theta}{2}$ and $H \in \mathcal{C}_\eta$ $(\eta > 40)$.*

Proof. Upon inserting absolute values, breaking up the y_0-integral into $\int_0^1 + \int_1^\infty$, and applying the bounds (2.3) [2] and $|\mathcal{B}_\Phi(1, a_1)| \ll |a_1|^\theta$, observe that

$$|OD_\Phi(s)| \ll \sum_{a_0=1}^\infty \sum_{a_1=1}^\infty \frac{1}{a_0^{2\sigma-1} a_1^{1-\theta}} \left(\int_{y_0=1}^\infty + \int_{y_0=0}^1 \right) \int_{y_1=0}^\infty |h(y_1)| \cdot (y_0^2 y_1)^{\sigma-\frac{1}{2}} \left(\frac{a_1 a_0^{-1} y_0}{1+y_0^2} \right)^{A_0} \cdot \left(y_1 \sqrt{1+y_0^2} \right)^{A_1} \frac{dy_0 dy_1}{y_0 y_1^2},$$

where the implicit constant depends only on Φ, A_0, A_1 with $-\infty < A_0, A_1 < 1$.

[2] We are allowed to choose different A_0, A_1 in different ranges of the y_0, y_1-integrals.

The convergence of both of the series is guaranteed if

$$A_0 \; < \; -\theta \quad \text{and} \quad \sigma \; > \; 1 - \frac{A_0}{2}. \tag{3.11}$$

Claim 3.2.4. *If we have (3.11) and*

$$A_1 \; < \; A_0 - 2\sigma + 1, \tag{3.12}$$

then the y_0-integrals converge.

Indeed, observe that $2\sigma + A_0 - 2 > -1$ by (3.11), and

$$\int_{y_0=0}^{1} y_0^{2\sigma+A_0-2} \left(1+y_0^2\right)^{\frac{A_1}{2}-A_0} \, dy_0 \; \asymp_{A_0,A_1} \; \int_{y_0=0}^{1} y_0^{2\sigma+A_0-2} \, dy_0.$$

The last integral converges. Also, (3.11)-(3.12) imply $A_1 < \min\{1, 2A_0\}$ and thus,

$$\int_{y_0=1}^{\infty} y_0^{2\sigma+A_0-2} \left(1+y_0^2\right)^{\frac{A_1}{2}-A_0} \, dy_0 \; \leqslant \; \int_{y_0=1}^{\infty} y_0^{2\sigma+A_1-A_0-2} \, dy_0.$$

Now, the last integral converges because of (3.12).

For the y_1-intgeral, the integrals

$$\int_{y_1=1}^{\infty} |h(y_1)| y_1^{\sigma+A_1-\frac{5}{2}} \, dy_1 \quad \text{and} \quad \int_{y_1=0}^{1} |h(y_1)| y_1^{\sigma+A_1-\frac{5}{2}} \, dy_1$$

converge whenever $H \in \mathcal{C}_\eta$ (we then have (2.8)) and

$$\eta \; > \; \left|\sigma + A_1 - \frac{3}{2}\right|. \tag{3.13}$$

Let $\delta := \sigma - 1 - (\theta/2) \; (> 0)$. In view of (3.11) and (3.12), we may take $A_0 := -\theta - \delta$ and $A_1 := -2\theta - 1 - 4\delta$. Also, (3.13) trivially holds as $\eta > 40$ and $\sigma < 4$. The result follows. $\square$

Remark 3.2.5. Readers will have no trouble in realizing the resemblance of (1.9) to the well-known inner product construction for the Kuznetsov formula. However, there are some differences. One of them has been mentioned: our moment identity is an equality between two unfoldings instead of that between spectral and geometric expansions.

The other is on the technical aspect. In the Kuznetsov formula, it is possible to annihilate the oscillatory factors therein to obtain a primitive form of the trace formula with some applications, see [67], [93], [68]. However, such a treatment is far from sufficient in our case — we have not analytically continued into the critical strip in Proposition 3.2.3! In other words, the oscillatory factor in $OD_{\Phi}(s)$ is of intrinsic importance to our problem. It arises naturally from the abstract characterization of Whittaker functions.

Chapter 4: Structures of the Off-diagonal

Fix $\epsilon := 1/100$ (say) [1], $0 < \phi < \pi/2$, [2] and consider the domain $1 + \frac{\theta}{2} + \epsilon < \sigma < 4$ in this section. We define a perturbed version of $OD_{\Phi}(s)$ as follows:

$$OD_{\Phi}(s;\phi) \;:=\; \sum_{a_0\neq 0}\sum_{a_1\neq 0} \frac{\mathcal{B}_{\Phi}(1,a_1)}{|a_0|^{2s-1}|a_1|} \int_0^{\infty}\int_0^{\infty} h(y_1)\cdot (y_0^2 y_1)^{s-\frac{1}{2}} W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right| \frac{y_0}{1+y_0^2},\; y_1\sqrt{1+y_0^2}\right) \cdot e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2};\phi\right) \frac{dy_0 dy_1}{y_0 y_1^2}, \tag{4.1}$$

where

$$e\left(x;\phi\right) \;:=\; \int_{(\epsilon)} |2\pi x|^{-u} e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\,\frac{du}{2\pi i} \qquad (x\in\mathbb{R}-\{0\}). \tag{4.2}$$

Remark 4.0.1. The goals of this section is to obtain an expression of $OD_{\Phi}(s;\phi)$ that

- reveals the structure of the dual moment;
- that it can be analytically continued into the critical strip;
- and will allow us to pass to the limit $\phi \to \pi/2$.

In view of these, it is natural to work on the dual side of Mellin transforms. Also, we will be able to separate variables as an added benefit. The main result of this section is as follows:

Proposition 4.0.2 (Dual Moment)**.** *Let $H \in \mathcal{C}_{\eta}$ ($\eta > 40$) and $\phi \in (0,\pi/2)$. On the vertical strip*

$$1 + \frac{\theta}{2} + \epsilon \;<\; \sigma \;<\; 4, \tag{4.3}$$

[1] We will stick with this choice of ϵ for the rest of this article.

[2] This should not be pose any confusion with the basis of cusp forms (ϕ_j) of Γ_2.

we have

$$OD_\Phi(s;\phi) \ = \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta\left(2s-s_0'\right) L\left(s_0',\Phi\right) \cdot \sum_{\delta=\pm} \left(\mathcal{F}_\Phi^{(\delta)} H\right)\left(s_0',\ s;\ \phi\right) \ \frac{ds_0'}{2\pi i}, \tag{4.4}$$

where the transform of H *is given by*

$$\left(\mathcal{F}_\Phi^{(\delta)} H\right)\left(s_0',\ s;\ \phi\right) \ := \int_{(15)} \int_{(\epsilon)} \widetilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \mathcal{G}_\Phi^{(\delta)}\left(s_1, u;\ s_0', s;\ \phi\right) \ \frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{4.5}$$

with $h := H^\flat$ *and the kernel function being*

$$\mathcal{G}_\Phi^{(\delta)}\left(s_1, u;\ s_0', s;\ \phi\right) \ := G_\Phi\left(s_0' - u, s_1\right) \cdot (2\pi)^{-u} e^{i\delta\phi u}\Gamma(u) \cdot \frac{\Gamma\left(\frac{u+1-2s+s_1-s_0'}{2}\right)\Gamma\left(\frac{2s-s_0'-u}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0'\right)}. \tag{4.6}$$

Here, $G_\Phi := G_{\alpha(\Phi)}$ *was defined in (2.2).*

Proof. Plug-in the expression of $W_{-\alpha(\Phi)}$ described in Proposition 2.1.1 into $OD_\Phi(s;\phi)$ with

$$\sigma_1 \ := \ 15 \quad \text{and} \quad 1+\theta \ < \ \sigma_0 \ < \ 2\sigma - 1 - \epsilon. \tag{4.7}$$

Inserting absolute values to the resulting expression, the sums and integrals are bounded by

$$\begin{aligned} \sum_{\delta:=\operatorname{sgn}(a_0a_1)=\pm} & \left(\sum_{a_0\neq 0} \frac{1}{|a_0|^{2\sigma-\sigma_0-\epsilon}}\right)\left(\sum_{a_1\neq 0} \frac{|\mathcal{B}_\Phi(1,a_1)|}{|a_1|^{\sigma_0+\epsilon}}\right) \\ & \cdot \left(\int_{(\sigma_0)}\int_{(\sigma_1)} |G_\Phi(s_0,s_1)| \ |ds_0||ds_1|\right)\left(\int_{(\epsilon)} \left|e^{i\delta\phi u}\Gamma(u)\right| \ |du|\right) \\ & \cdot \left(\int_0^\infty y_0^{-\sigma_0-2\epsilon+2\sigma}(1+y_0^2)^{\sigma_0+\epsilon-\frac{1+\sigma_1}{2}} \ d^\times y_0\right) \\ & \cdot \left(\int_0^\infty |h(y_1)| \cdot y_1^{\sigma-\sigma_1-\frac{1}{2}} \ d^\times y_1\right). \end{aligned} \tag{4.8}$$

Observe that:

- by Stirling's formula, the s_0, s_1, u-integrals converge as long as

$$\sigma_0,\ \sigma_1,\ \epsilon\ >\ 0, \qquad \phi\ \in\ (0, \pi/2); \tag{4.9}$$

- the y_0-integral converges as long as

$$\sigma_0 + 2\epsilon\ <\ 2\sigma\ <\ \sigma_1 - \sigma_0 + 1; \tag{4.10}$$

- by the bound $|\mathcal{B}_\Phi(1, a_1)|\ \ll\ |a_1|^\theta$, the a_0-sum and the a_1-sum converge as long as

$$2\sigma - 1\ >\ \sigma_0 + \epsilon\ >\ 1 + \theta. \tag{4.11}$$

Under (4.7), items (4.9), (4.10), (4.11) hold. Moreover, the y_1-integral converges by (2.8) and $H \in \mathcal{C}_\eta$ ($\eta > 40$). Now, upon rearranging sums and integrals, and notice that $\mathcal{B}_\Phi(1, a_1) = \mathcal{B}_\Phi(1, -a_1)$, we have

$$\begin{aligned} OD_\Phi(s;\phi)\ =\ 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)}\ & \frac{G_\Phi(s_0, s_1)}{4}\cdot(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\ \left(\int_0^\infty h(y_1)y_1^{s-s_1-\frac{1}{2}}\,d^\times y_1\right) \\ & \cdot\left(\int_0^\infty y_0^{-s_0-2u+2s}(1+y_0^2)^{s_0+u-\frac{1+s_1}{2}}\ d^\times y_0\right) \\ & \cdot\left(\sum_{a_0=1}^\infty\sum_{a_1=1}^\infty\frac{\mathcal{B}_\Phi(1, a_1)}{a_0^{2s-1}a_1}\left(\frac{a_1}{a_0}\right)^{1-s_0-u}\right)\ \frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}. \end{aligned} \tag{4.12}$$

Recall the integral identity

$$\int_0^\infty y_0^v(1+y_0^2)^A\ d^\times y_0\ =\ \frac{1}{2}\ \frac{\Gamma\left(-A-\frac{v}{2}\right)\Gamma\left(\frac{v}{2}\right)}{\Gamma(-A)} \tag{4.13}$$

for $0 < \operatorname{Re} v < -2 \operatorname{Re} A$. It follows that

$$OD_\Phi(s;\phi) \;=\; 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)} \zeta\left(2s-s_0-u\right) L\left(s_0+u;\Phi\right)\cdot \widetilde{h}\left(s-s_1-\frac{1}{2}\right) \cdot \frac{G_\Phi(s_0,s_1)}{4}\cdot (2\pi)^{-u} e^{i\delta\phi u}\Gamma(u) \cdot \frac{1}{2}\,\frac{\Gamma\left(s-\frac{s_0}{2}-u\right)\Gamma\left(\frac{1+s_1-s_0}{2}-s\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0-u\right)}\,\frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}. \tag{4.14}$$

We pick the contour $(\sigma_0) := (1+\theta+\epsilon)$ (we thus impose (4.3)). To isolate the nonarchimedean part of $OD_\Phi(s;\phi)$, we make the change of variable $s_0' \;=\; s_0+u$. Upon plugging-in the expression for $G_\Phi\left(s_0'-u, s_1\right)$ (see (2.2)), we obtain (4.4)-(4.6). By the absolute convergence proven above, we also conclude that the integral transform $\left(\mathcal{F}_\Phi^{(\delta)} h\right)(s_0',\ s;\ \phi)$ is holomorphic on the domain

$$\sigma \;<\; 4 \qquad \text{and} \qquad 1+\theta+\epsilon \;<\; \sigma_0' \;<\; 2\sigma-1. \tag{4.15}$$

This completes the proof. □

Remark 4.0.3. In the following, we replace s_0' by s_0 and have (4.15) superseding (4.7) correspondingly.

Proposition 4.0.4. *For $4 > \sigma > (3+\theta)/2$ and $H \in \mathcal{C}_\eta$, we have*

$$\lim_{\phi\to\pi/2} OD_\Phi(s;\phi) \;=\; OD_\Phi(s). \tag{4.16}$$

Proof. Let $\epsilon := 1/100$, $\sigma_1 := 15$, and pick any σ_0 satisfying

$$\frac{3}{2}+\theta+\epsilon \;<\; \sigma_0 \;<\; 2\sigma-1-\epsilon. \tag{4.17}$$

Denote by $\mathcal{C}_\epsilon$ the indented path consisting of the line segments:

$$-\frac{1}{2}-\epsilon-i\infty \ \to\ -\frac{1}{2}-\epsilon-i \ \to\ \epsilon-i \ \to\ \epsilon+i \ \to\ -\frac{1}{2}-\epsilon+i \ \to\ -\frac{1}{2}-\epsilon+i\infty.$$

Replace $e(x;\phi)$ in (4.12) by the expression:

$$e(x;\phi) \ = \ \int_{\mathcal{C}_\epsilon} |2\pi x|^{-u} e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\,\frac{du}{2\pi i}. \tag{4.18}$$

Note that $\left|e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\right| \ \ll_\epsilon\ (1+|\operatorname{Im} u|)^{-1-\epsilon}$ for $u\in\mathcal{C}_\epsilon$ and $\phi\in(0,\pi/2]$. Insert absolute values in (4.12). The resulting sums and integrals converge absolutely when $\phi\in(0,\pi/2]$ and (4.17) holds, which can be seen by the same argument following (4.8). Apply Dominated Convergence and shift the contour of the u-integral to $-\infty$, the residual series obtained is exactly $e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2}\right)$. This completes the proof. □

Now, $OD_\Phi(s;\phi)$ is in terms of integrals of Mellin-Barnes type. Note that the Γ-factors from Proposition 2.1.1 and (4.2) alone are not sufficient for our goals (see Remark 4.0.1 and (4.9), (4.10), (4.11)). The three extra Γ-factors brought by the y_0-integral, which 'mix' all variables of integrations, will play an important role in Section 5-6.

Chapter 5: Analytic Properties of the Archimedean Transform

In (4.4), the factors $\zeta(2s - s_0)$ and $L(s_0, \Phi)$ are known to admit holomorphic continuation and have polynomial growth in vertical strips (except on the line $2s - s_0 = 1$). It remains to study the archimedean part of (4.4), i.e., the integral transform

$$\left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(s_0,\ s;\ \phi) \ := \ \int_{(15)} \int_{(\epsilon)} \widetilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \mathcal{G}_{\Phi}^{(\delta)}(s_1, u;\ s_0, s;\ \phi)\ \frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{5.1}$$

where $h := H^\flat$ and $\mathcal{G}_{\Phi}^{(\delta)}(\cdots)$ as defined in (4.6). In Section 4, we have shown that when $\phi \in (0, \pi/2)$, the function $(s_0, s) \mapsto \left(\mathcal{F}_{\Phi}^{(\delta)} h\right)(s_0,\ s;\ \phi)$ is holomorphic on the domain (4.15), i.e.,

$$\sigma \ < \ 4 \qquad \text{and} \qquad 1 + \theta + \epsilon \ < \ \sigma_0 \ < \ 2\sigma - 1.$$

In this section, we establish a larger region of holomorphy for $(s_0, s) \mapsto \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(s_0, s; \phi)$ that holds for $\phi \in (0, \pi/2]$. We write

$$s \ = \ \sigma + it, \qquad s_0 = \sigma_0 + it_0, \qquad s_1 = \sigma_1 + it_1, \qquad \text{and} \qquad u = \ \epsilon + iv,$$

with $\epsilon := 1/100$. It is sufficient to consider s inside the rectangular box $\epsilon < \sigma < 4$ and $|t| \leqslant T$, for any given $T \geqslant 1000$. Moreover, $\alpha_k := i\gamma_k \in i\mathbb{R}$ $(k = 1, 2, 3)$ by our assumptions on Φ. The main result of this section can be stated as follows:

Proposition 5.0.1. *Suppose $H \in \mathcal{C}_\eta$.*

1. *For any $\phi \in (0, \pi/2]$, the transform $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0, s; \phi)$ is holomorphic on the domain*

$$\sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \quad \textit{and} \quad 2\sigma - \sigma_0 - \epsilon \ > \ 0. \tag{5.2}$$

2. *Whenever $(\sigma_0, \sigma) \in (5.2)$, $|t| < T$, and $\phi \in (0, \pi/2)$, the transform $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \phi)$ has exponential decay as $|t_0| \to \infty$.* [1]

Remark 5.0.2. The domain (5.2) is chosen in a way that the function $(s_0, s) \mapsto \mathcal{G}_\Phi^{(\delta)}(s_1, u;\ s_0, s;\ \phi)$ is holomorphic on (5.2) when $\operatorname{Re} s_1 = \sigma_1 \geqslant 15$ and $\operatorname{Re} u = \epsilon$. Moreover, if we have both $15 \leqslant \sigma_1 \leqslant \eta - \frac{1}{2}$ and (5.2), then $s - s_1 - 1/2$ lies inside the region of holomorphy of $\widetilde{h}$.

Indeed, by inspecting the Γ's in the numerator of (4.6), all of the polar divisors are avoided:

1. $\Gamma(u) \prod_{i=1}^{3} \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)$ — because $\operatorname{Re} u = \epsilon > 0$ and $\operatorname{Re}(s_1 - \alpha_i) = \sigma_1 > 0$;
2. $\prod_{i=1}^{3} \Gamma\left(\frac{s_0 + \alpha_i - u}{2}\right)$ — because $\operatorname{Re}(s_0 + \alpha_i - u) = \sigma_0 - \epsilon > 0$;
3. $\Gamma\left(\frac{2s - s_0 - u}{2}\right)$ — because $\operatorname{Re}(2s - s_0 - u) = 2\sigma - \sigma_0 - \epsilon > 0$;
4. $\Gamma\left(\frac{u + 1 - 2s + s_1 - s_0}{2}\right)$ — because $\operatorname{Re}(u + 1 - 2s + s_1 - s_0) \ > \ \epsilon + 1 - 2\sigma + 15 - (2\sigma - \epsilon) \ > \ 2\epsilon$.

Proof. The proof is based on a careful application of the Stirling estimate

$$|\Gamma(a + ib)| \ \asymp_a \ (1 + |b|)^{a - \frac{1}{2}} e^{-\frac{\pi}{2}|b|} \qquad (a \ \neq \ 0,\ -1,\ -2,\ \ldots,\ \ b \ \in \ \mathbb{R}) \tag{5.3}$$

to the kernel function $\mathcal{G}_\Phi^{(\delta)}(s_1, u;\ s_0, s;\ \phi)$. The following set of conditions will be repeated

[1] The explicit estimate is stated in the proof below and the implicit constant depends only on T and Φ.

throughout the proof:

$$\begin{cases} \quad 0 < \phi \leqslant \pi/2, \\ \sigma_0 > \epsilon, \quad \sigma < 4, \quad 2\sigma - \sigma_0 - \epsilon > 0, \\ \operatorname{Re} s_1 = \sigma_1 \geqslant 15, \quad \operatorname{Re} u = \epsilon. \end{cases} \tag{5.4}$$

Assuming (5.4), apply (5.3) to the kernel function (4.6). It follows that

$$\begin{aligned} \left| \mathcal{G}_\Phi^{(\delta)}(s_1, u;\ s_0, s;\ \phi) \right| \asymp\ & (1+|v|)^{\epsilon - \frac{1}{2}} e^{-\left(\frac{\pi}{2} - \phi\right)|v|} \cdot \prod_{k=1}^{3} (1 + |t_1 - \gamma_k|)^{\frac{\sigma_1 - 1}{2}} e^{-\frac{\pi}{4}|t_1 - \gamma_k|} \\ & \cdot \prod_{k=1}^{3} (1 + |t_0 - v + \gamma_k|)^{\frac{\sigma_0 - \epsilon - 1}{2}} e^{-\frac{\pi}{4}|t_0 - v + \gamma_k|} \\ & \cdot (1 + |2t - t_0 - v|)^{\frac{2\sigma - 1 - \sigma_0 - \epsilon}{2}} e^{-\frac{\pi}{4}|2t - t_0 - v|} \\ & \cdot (1 + |v - 2t + t_1 - t_0|)^{\frac{\epsilon - 2\sigma + \sigma_1 - \sigma_0}{2}} e^{-\frac{\pi}{4}|v - 2t + t_1 - t_0|} \\ & \cdot (1 + |t_1 - 2t_0|)^{-\left(\frac{\sigma_1}{2} - \sigma_0\right)} e^{\frac{\pi}{4}|t_1 - 2t_0|} \\ & \cdot (1 + |t_0 + t_1 - v|)^{-\frac{\sigma_0 + \sigma_1 - \epsilon - 1}{2}} e^{\frac{\pi}{4}|t_0 + t_1 - v|}, \end{aligned} \tag{5.5}$$

where the implicit constant depends at most on σ_1. [2] Let $\mathcal{P}_s^\Phi(t_0, t_1, v)$ be the 'polynomial part' of (5.5) and

$$\begin{aligned} \mathcal{E}_s^\Phi(t_0, t_1, v) := \sum_{k=1}^{3} \{|t_1 - \gamma_k| + |t_0 - v + \gamma_k|\} + |2t - t_0 - v| + |v - 2t + t_1 - t_0| \\ - |t_1 - 2t_0| - |t_0 + t_1 - v|. \end{aligned}$$

We first examine the exponential phase $\mathcal{E}_s^\Phi(t_0, t_1, v)$ of (5.5) as it determines the effective

[2] Note that the domain (5.2) for (σ, σ_0) is bounded and thus the estimate is uniform in $\sigma, \sigma_0, \epsilon$. This will be assumed for all estimates in the rest of this section.

support of $\left(\mathcal{F}_{\Phi}^{(\delta)}H\right)(s_0, s; \phi)$. By the triangle inequality and the fact $\gamma_1 + \gamma_2 + \gamma_3 = 0$, we have

$$\left|\mathcal{G}_{\Phi}^{(\delta)}\left(s_1, u;\ s_0, s;\ \phi\right)\right| \ll_{\sigma_1} e^{\pi T} \cdot \mathcal{P}_s^{\Phi}\left(t_0, t_1, v\right) \cdot \exp\left(-\frac{\pi}{4}\mathcal{E}(t_0, t_1, v)\right) \cdot e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \tag{5.6}$$

with

$$\mathcal{E}(t_0, t_1, v) \ := \ 3|t_1| + 3|t_0 - v| - |t_1 - 2t_0| + |v + t_1 - t_0| + |t_0 + v| - |t_0 + t_1 - v|, \tag{5.7}$$

whenever we have (5.4) and $|t| \leqslant T$,

Claim 5.0.3. *For any $t_0, t_1, v \in \mathbb{R}$, we have $\mathcal{E}(t_0, t_1, v) \ \geqslant \ 0$. Equality holds if and only if*

$$t_1 \ = \ 0 \qquad \textit{and} \qquad t_0 - v \ = \ 0. \tag{5.8}$$

Proof. Adding up the inequalities $|t_1| + |t_0 - v| \geqslant |t_0 + t_1 - v|$ and $|v + t_1 - t_0| + |t_0 + v| \geqslant |t_1 - 2t_0|$, we have

$$\mathcal{E}(t_0, t_1, v) \ \geqslant \ 2\left(|t_1| + |t_0 - v|\right) \ \geqslant \ 0. \tag{5.9}$$

The equality case is apparent. □

Claim 5.0.4. *When (5.4) and $|t| \leqslant T$ hold, the integral*

$$\iint\limits_{\substack{(\operatorname{Re} s_1, \operatorname{Re} u) = (\sigma_1, \epsilon),\\ (t_1, v):\text{(5.11) holds}}} \widetilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \mathcal{G}_{\Phi}^{(\delta)}\left(s_1, u;\ s_0, s;\ \phi\right) \frac{du}{2\pi i} \frac{ds_1}{2\pi i} \tag{5.10}$$

has exponential decay as $|t_0| \to \infty$, where

$$|t_1| \ > \ \log^2(3 + |t_0|) \qquad \textit{or} \qquad |v - t_0| \ > \ \log^2(3 + |t_0|). \tag{5.11}$$

Proof. In case of (5.11), we have

$$\mathcal{E}(t_0, t_1, v) \ > \ \log^2(3 + |t_0|) + |t_1| + |t_0 - v| \tag{5.12}$$

from (5.9). The polynomial part $\mathcal{P}_s^{\Phi}(t_0, t_1, v)$ can be crudely bounded by

$$\mathcal{P}_s^{\Phi}(t_0, t_1, v) \ \ll_{\Phi, \sigma_1, T} \ [(1 + |t_1|)\,(1 + |v - t_0|)\,(1 + |t_0|)]^{A(\sigma_1)}, \tag{5.13}$$

where $A(\sigma_1) > 0$ is some constant.

Putting (5.12), (5.13), and the bound $e^{-\left(\frac{\pi}{2} - \phi\right)|v|} \leqslant 1$ $(\phi \in (0, \pi/2])$ into (5.6), we obtain

$$\left|\mathcal{G}_{\Phi}^{(\delta)}\left(s_1, u;\ s_0, s;\ \phi\right)\right| \ \ll_{\Phi, \sigma_1, T} \ (1 + |t_0|)^{A(\sigma_1)} e^{-\frac{\pi}{4}\log^2(3 + |t_0|)}$$
$$\cdot\, [(1 + |t_1|)\,(1 + |v - t_0|)]^{A(\sigma_1)} e^{-\frac{\pi}{4}[|t_1| + |t_0 - v|]} \tag{5.14}$$

whenever (5.11), (5.4), and $|t| \leqslant T$ hold. The boundedness of $\widetilde{h}$ on vertical strips implies that (5.10) is

$$\ll_{\sigma_1, \Phi, T} \ (1 + |t_0|)^{A(\sigma_1)} e^{-\frac{\pi}{4}\log^2(3 + |t_0|)}. \tag{5.15}$$

This proves Claim 5.0.4. $\square$

Now, let $\phi \in (0, \pi/2]$ and consider $\left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(s_0,\ s;\ \phi)$ as a function on the bounded domain

$$(\sigma_0, \sigma) \ \in \ (5.2), \qquad |t|,\ |t_0| \ \leqslant \ T. \tag{5.16}$$

When $|t_1| > \log^2(3 + T)$ or $|v| > T + \log^2(3 + T)$, observe that (5.11) is satisfied and from (5.14),

$$\left|\mathcal{G}_{\Phi}^{(\delta)}\left(s_1, u;\ s_0, s;\ \phi\right)\right| \ \ll_{\Phi, T} \ [(1 + |t_1|)\,(1 + |v|)]^{A(15)} \cdot e^{-\frac{\pi}{4}[|t_1| + |v|]}. \tag{5.17}$$

The last function is clearly jointly integrable with respect to t_1, v, and by Remark 5.0.2, the function $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \phi)$ is holomorphic on the domain (5.16). Since the choice of T is arbitrary, we arrive at the first conclusion of Proposition 5.0.1.

In the remaining part, we prove the second assertion of Proposition 5.0.1. We estimate the contribution from

$$|t_1| \ \leqslant\ \log^2(3+|t_0|) \qquad \text{and} \qquad |v-t_0| \ \leqslant\ \log^2(3+|t_0|). \tag{5.18}$$

(The complementary part has been treated in Claim 5.0.4.)

It suffices to restrict ourselves to the effective support (5.8). The polynomial part can be essentially computed by substituting $t_1 := 0$ and $v := t_0$. More precisely, when (5.18) and $|t_0| \gg_T 1$ hold, there are only two possible scenarios for the factors $1+|(\cdots)|$ in (5.5): either $1+|(\cdots)| \asymp |t_0|$, or $\log^{-C}(3+|t_0|) \ll 1+|(\cdots)| \ll \log^C(3+|t_0|)$ for some absolute constant $C>0$.

To see this, we use $x > \log^2 x$ for $x \geqslant 3$ and the following sets of elementary inequalities:

1. for $|t_0| \gg 1$, we have $1+|v| \ =\ 1+|t_0|+O(|t_0-v|) \ \asymp\ |t_0|$; and

$$\log^{-2}(3+|t_0|) \ \leqslant\ 1+|t_1-\alpha_i| \ \leqslant (1+|t_1|)\,(1+|\alpha_i|) \ \ll_\Phi\ \log^2(3+|t_0|);$$

2. Similar to above, $\log^{-2}(3+|t_0|) \ \leqslant\ 1+|t_0-v+\alpha_i| \ \ll_\Phi\ \log^2(3+|t_0|)$;

3. $1+|2t-t_0-v| \ \ll_T\ 1+|t_0+v| \ \ll\ |t_0|$; and for $|t|<T<\frac{1}{8}|t_0|$ we have

$$\begin{aligned} 1+|2t-t_0-v| \ &\geqslant\ 1+|t_0+v|-2T \ \geqslant\ 1+2|t_0|-|t_0-v|-2T \\ &\geqslant\ 1+2|t_0|-\log^2(3+|t_0|)-2T \ \geqslant\ \frac{1}{2}|t_0|; \end{aligned}$$

4. $\log^{-4}(3+|t_0|) \ \leqslant\ 1+|v-2t+t_1-t_0| \ \ll_T\ (1+|v-t_0|)\,(1+|t_1|) \ \ll\ \log^4(3+|t_0|)$;

5. for $|t_0| \gg 1$, we have $1+|t_1-2t_0| \;=\; 1+2|t_0| + O\left(\log^2(3+|t_0|)\right) \;\asymp\; |t_0|$;

6. $\log^{-2}(3+|t_0|) \;\leqslant\; 1+|t_0+t_1-v| \;\ll\; \log^2(3+|t_0|)$.

In case of (5.18), apply the bounds

$$e^{-\frac{\pi}{4}\mathcal{E}(t_0,t_1,v)} \;\leqslant\; 1 \quad \text{and} \quad e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \;\leqslant\; e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \quad \text{for} \quad |t_0| \;\gg\; 1$$

to (5.6). As a result, if we also have (5.4), $|t|<T$, and $|t_0|>8T$, then

$$\left|\mathcal{G}_{\Phi}^{(\delta)}\left(s_1,u;\; s_0,s;\; \phi\right)\right| \;\ll_{\sigma_1,\Phi,T}\; |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{B(\sigma_1)}|t_0| \tag{5.19}$$

and

$$\iint\limits_{\substack{(\operatorname{Re}s_1,\operatorname{Re}u)=(\sigma_1,\epsilon),\\ (t_1,v):\ (5.18)\text{ holds}}} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot \mathcal{G}_{\Phi}^{(\delta)}\left(s_1,u;\; s_0,s;\; \phi\right)\frac{du}{2\pi i}\frac{ds_1}{2\pi i}$$

$$\ll_{\sigma_1,\Phi,T}\; |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{4+B(\sigma_1)}|t_0|, \tag{5.20}$$

where $B(\sigma_1)>0$ is some constant. If $\phi<\pi/2$, then there is exponential decay in (5.20) as $|t_0|\to\infty$. Therefore, the second conclusion of the proposition follows from (5.20) and (5.15) (putting $\sigma_1=15$). $\square$

Chapter 6: Analytic Continuation of the Off-diagonal

6.1 Step 1:

We first obtain a holomorphic continuation of $OD_\Phi(s;\phi)$ up to $\operatorname{Re} s > \frac{1}{2} + \epsilon$ by shifting the contour of the s_0-integral to the left.

Fix any $\phi \in (0, \pi/2)$ and $T \geqslant 1000$. We first restrict ourselves to

$$1 + \frac{\theta}{2} + 2\epsilon \;<\; \sigma \;<\; 4, \qquad |t| \;<\; T. \tag{6.1}$$

Clearly, the pole $s_0 = 2s - 1$ of $\zeta(2s - s_0)$ is on the right of the contour $\operatorname{Re} s_0 = 1 + \theta + 2\epsilon$ of the integral (4.4).

Let $T_0 \gg 1$. The rectangle with vertices $2\epsilon \pm iT_0$ and $(1 + \theta + 2\epsilon) \pm iT_0$ in the s_0-plane lies inside the region of holomorphy (5.2) of $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$. The contribution from the horizontal segments $[2\epsilon \pm iT_0, (1 + \theta + 2\epsilon) \pm iT_0]$ tends to 0 as $T_0 \to \infty$ by the exponential decay of $(\mathcal{F}_\Phi^{(\delta)} H)(s_0,\ s;\ \phi)$ (see Proposition 5.0.1.[1]) As a result, we may shift the line of integration to $\operatorname{Re} s_0 = 2\epsilon$ and no pole is crossed. Hence,

$$OD_\Phi(s;\phi) \;=\; \frac{1}{4} \int_{(2\epsilon)} \zeta\left(2s - s_0\right) L\left(s_0, \Phi\right) \cdot \sum_{\delta = \pm} \left(\mathcal{F}_\Phi^{(\delta)} H\right)\left(s_0,\ s;\ \phi\right) \frac{ds_0}{2\pi i} \tag{6.2}$$

on (6.1). The right side of (6.2) is holomorphic on

$$\frac{1}{2} + \epsilon \;<\; \sigma \;<\; 4, \qquad |t| \;<\; T \tag{6.3}$$

[1] which surely counteracts the polynomial growth from $L(s_0, \Phi)$ and $\zeta(2s - s_0)$.

and serves as an analytic continuation of $OD_{\Phi}(s;\phi)$ to (6.3) by using (5.2). Notice the fact that $\sigma > \frac{1}{2} + \epsilon$ implies the holomorphy of $\zeta(2s - s_0)$.

6.2 Step 2: Crossing the Polar Line (Shifting the s_0-integral again)

Consider a subdomain of (6.3):

$$\frac{1}{2} + \epsilon \ < \ \sigma \ < \ \frac{3}{4}, \qquad |t| \ < \ T. \tag{6.4}$$

Different from Step 1, the pole $s_0 = 2s - 1$ is now inside the rectangle with vertices

$$2\epsilon \pm iT_0, \quad \frac{1}{2} \pm iT_0$$

provided $T_0 > 4T$. Such a rectangle lies in the region of holomorphy (5.2) of $(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$. When $\phi < \pi/2$, the exponential decay of $(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$ once again allows us to shift the line of integration from $\operatorname{Re} s_0 = 2\epsilon$ to $\operatorname{Re} s_0 = 1/2$, crossing the pole of $s_0 \mapsto \zeta(2s - s_0)$ which has residue -1. In other words,

$$\begin{aligned} OD_{\Phi}(s;\phi) \ &= \frac{1}{4}\, L(2s-1, \Phi) \sum_{\delta = \pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H \right) (2s-1, \ s; \ \phi) \\ &\qquad + \frac{1}{4} \int_{(1/2)} \zeta\left(2s - s_0\right) L\left(s_0, \Phi\right) \cdot \sum_{\delta = \pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H \right) (s_0, \ s; \ \phi) \ \frac{ds_0}{2\pi i}. \end{aligned} \tag{6.5}$$

On the line $\operatorname{Re} s_0 = 1/2$, observe that $s \mapsto \left(\mathcal{F}_{\Phi}^{(\delta)} H \right) (s_0, \ s; \ \phi)$ is holomorphic on $\sigma > \frac{1}{4} + \frac{\epsilon}{2}$ because of (5.2) ; whereas $s \mapsto \zeta(2s - s_0)$ is holomorphic on $\sigma < 3/4$ because of $2\sigma - \sigma_0 < 1$. As a result, the function $s \mapsto \int_{(1/2)} (\cdots) \frac{ds_0}{2\pi i}$ in (6.5) is holomorphic on the vertical strip

$$\frac{1}{4} + \frac{\epsilon}{2} \ < \ \sigma \ < \ \frac{3}{4}, \tag{6.6}$$

which is sufficient for our purpose.

However, Proposition 5.0.1 only asserts that the function $s \mapsto \left(\mathcal{F}_\Phi^{(\delta)} H\right)(2s-1, s; \phi)$ is holomorphic on $\frac{1}{2} + \epsilon < \sigma < 4$. This issue will be addressed in Section 6.4.

6.3 Step 3: Putting Back $\phi \to \pi/2$ — Shifting the s_1-integral and Refining Step 1-2

By estimate (5.14) and Dominated Convergence,

$$\lim_{\phi \to \pi/2} \left(\mathcal{F}_\Phi^{(\delta)} H\right)(2s-1,\ s;\ \phi) \ = \ \left(\mathcal{F}_\Phi^{(\delta)} H\right)(2s-1,\ s;\ \pi/2) \tag{6.7}$$

for $\frac{1}{2} + \epsilon < \sigma < 4$ and $|t| < T$. However, for the continuous part of (6.5), we need a follow-up of Proposition 5.0.1 in order to pass to the limit $\phi \to \pi/2$. Essentially, thanks to the structure of the Γ's in Proposition 2.1.1 and the analytic properties of the Mellin transform $\widetilde{h}$ of h, it is possible to shift the line of integration of the s_1-integral before estimating so as to gain sufficient polynomial decay for the transform $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \pi/2)$.

Proposition 6.3.1. *Let $H \in \mathcal{C}_\eta$. There exists a constant $B_\eta > 0$ such that whenever*

$$(\sigma_0, \sigma) \ \in \ (5.2), \quad |t| \ < \ T, \quad \textit{and} \quad |t_0| \ \gg_T \ 1,$$

we have the estimate

$$\left|\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \pi/2)\right| \ \ll \ |t_0|^{8-\frac{\eta}{2}} \log^{B_\eta} |t_0|, \tag{6.8}$$

where the implicit constant depends only on η, T, Φ.

Proof. On domain (5.2), i.e.,

$$\sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \quad \text{and} \quad 2\sigma - \sigma_0 - \epsilon \ > \ 0,$$

observe that the vertical strip $\operatorname{Re} s_1 \in [15, \eta - \frac{1}{2}]$ contains no pole of the function

$$s_1 \;\mapsto\; \mathcal{G}_{\Phi}^{(\delta)}\left(s_1, u;\; s_0, s;\; \phi\right),$$

and it lies within the region of holomorphy of $\widetilde{h}$ (see Remark 5.0.2). The estimate (5.14) allows us to shift the line of integration from $\operatorname{Re} s_1 = 15$ to $\operatorname{Re} s_1 = \eta - \frac{1}{2}$ in the expression (4.5) for the transform $\left(\mathcal{F}_{\Phi}^{(\delta)} H\right)$. Notice that the estimates done in Proposition 5.0.1 works for $\phi = \pi/2$ too. In particular, from (5.20) and (5.15), the bound (6.8) follows by taking $\sigma_1 := \eta - \frac{1}{2}$ therein. This completes the proof. □

We now summarize the previous discussions. Suppose $(3+\theta)/2 \;<\; \sigma \;<\; 4$. By Proposition 4.0.4, equation (4.4) and equation (6.2), we have

$$\begin{aligned} OD_{\Phi}(s) \;&=\; \lim_{\phi \to \pi/2} OD_{\Phi}(s; \phi) \\ &=\; \lim_{\phi \to \pi/2} \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta\left(2s - s_0\right) L\left(s_0, \Phi\right) \cdot \sum_{\delta = \pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)\left(s_0,\; s;\; \phi\right) \frac{ds_0}{2\pi i} \\ &=\; \lim_{\phi \to \pi/2} \frac{1}{4} \int_{(2\epsilon)} \zeta\left(2s - s_0\right) L\left(s_0, \Phi\right) \cdot \sum_{\delta = \pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)\left(s_0,\; s;\; \phi\right) \frac{ds_0}{2\pi i}. \end{aligned} \tag{6.9}$$

Next, Proposition 6.3.1 ensures enough polynomial decay and hence the absolute convergence of (6.10) at $\phi = \pi/2$:

$$OD_{\Phi}(s) \;=\; \frac{1}{4} \int_{(2\epsilon)} \zeta\left(2s - s_0\right) L\left(s_0, \Phi\right) \cdot \sum_{\delta = \pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)\left(s_0,\; s;\; \pi/2\right) \frac{ds_0}{2\pi i}. \tag{6.10}$$

Now, (6.10) serves as an analytic continuation of $OD_{\Phi}(s)$ to the domain $1/2 + \epsilon < \sigma < 4$.

On the smaller domain $1/2+\epsilon<\sigma<3/4$, the expressions (6.9) and (6.5) are equal. Then

$$OD_{\Phi}(s) \ = \ (6.9) \ = \frac{1}{4}\, L(2s-1,\Phi) \sum_{\delta=\pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(2s-1,\ s;\ \pi/2) + \frac{1}{4} \int_{(1/2)} \zeta\left(2s-s_0\right) L\left(s_0,\Phi\right) \cdot \sum_{\delta=\pm} \left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(s_0,\ s;\ \pi/2)\ \frac{ds_0}{2\pi i} \tag{6.11}$$

by Dominated Convergence and Proposition 5.0.1. The last integral is holomorphic on

$$\frac{1}{4}+\frac{\epsilon}{2} \ < \ \sigma \ < \ \frac{3}{4}.$$

To ease the notations, let's write

$$\left(\mathcal{F}_{\Phi} H\right)(s_0,s) \ := \ \left(\mathcal{F}_{\Phi}^{+} H\right)(s_0,s;\ \pi/2) + \left(\mathcal{F}_{\Phi}^{-} H\right)(s_0,s;\ \pi/2)\,.$$

Then by the duplication and the reflection formula of Γ-functions in the form

$$2^{-u}\Gamma(u) \ = \ \frac{1}{2\sqrt{\pi}} \cdot \Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+1}{2}\right) \qquad \text{and} \qquad \Gamma\left(\frac{1+u}{2}\right)\Gamma\left(\frac{1-u}{2}\right) \ = \ \pi\sec\frac{\pi u}{2},$$

we readily have

$$\left(\mathcal{F}_{\Phi} H\right)(s_0,s) \ = \sqrt{\pi} \int_{(\eta-1/2)} \widetilde{h}\left(s-s_1-\frac{1}{2}\right) \pi^{-s_1} \frac{\prod_{i=1}^{3} \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \cdot \int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1-(s_0-u)}{2}+\frac{1}{2}-s\right)\cdot \prod_{i=1}^{3} \Gamma\left(\frac{(s_0-u)+\alpha_i}{2}\right)\Gamma\left(s-\frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{(s_0-u)+s_1}{2}\right)} \frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}. \tag{6.12}$$

6.4 Step 4: Continuation of the Residual Term — Shifting the u-integral

Proposition 6.4.1. *Let $H \in \mathcal{C}_\eta$. The function $s \mapsto (\mathcal{F}_\Phi H)(2s-1,s)$ can be holomorphically continued to the vertical strip $\epsilon < \sigma < 4$ except at three simple poles:*

$$s = \frac{1-\alpha_i}{2} \qquad (i = 1,2,3),$$

where $(\alpha_1,\alpha_2,\alpha_3)$ are the Langlands parameters of the Maass cusp form Φ.

Proof. We will prove a stronger result in Proposition 7.0.3. However, a simpler argument suffices for the time being. Suppose $\frac{1}{2}+\epsilon < \sigma < 4$ and $s_0 = 2s-1$. In (6.12), we shift the line of integration from $\operatorname{Re} u = \epsilon$ to $\operatorname{Re} u = -1.9$ and we pick up the pole at $u = 0$:

$$\begin{aligned}(\mathcal{F}_\Phi H)(2s-1,s) &= 2\sqrt{\pi}\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\\ &\cdot\int_{(\eta-\frac{1}{2})}\widetilde{h}\left(s-s_1-\frac{1}{2}\right)\frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)\Gamma\left(\frac{s_1}{2}+1-2s\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)\Gamma\left(s-\frac{1}{2}+\frac{s_1}{2}\right)}\frac{ds_1}{2\pi i}\\ &+\sqrt{\pi}\int_{(\eta-\frac{1}{2})}\int_{(-1.9)}\widetilde{h}\left(s-s_1-\frac{1}{2}\right)\pi^{-s_1}\frac{\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)}\\ &\cdot\frac{\Gamma\left(\frac{u+s_1}{2}+1-2s\right)\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}-\frac{u}{2}\right)\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(s-\frac{1}{2}+\frac{s_1-u}{2}\right)}\\ &\frac{du}{2\pi i}\frac{ds_1}{2\pi i}.\end{aligned}\tag{6.13}$$

By Stirling's formula and the same argument following (5.17), the integrals above represent holomorphic functions on $\epsilon < \sigma < 4$. □

6.5 Conclusion of Theorem 1.2.2

Apply Proposition 6.4.1 to (6.11) and observe that the poles of $s \mapsto (\mathcal{F}_\Phi H)(2s-1, s)$ are exactly the trivial zeros of the the arithmetic factor $L(2s-1, \Phi)$ in (6.5). We conclude that the product of functions $s \mapsto L(2s-1, \Phi) \cdot (\mathcal{F}_\Phi H)\,(2s-1, s)$ is holomorphic on $\epsilon < \sigma < 4$ and thus (6.11) provides a holomorphic continuation of $OD_\Phi(s)$ to the vertical strip $\frac{1}{4} + \frac{\epsilon}{2} < \sigma < \frac{3}{4}$. By the rapid decay of Φ at ∞, the inner product $s \mapsto \left\langle P,\ \mathbb{P}_2^3 \Phi \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2 \backslash GL_2(\mathbb{R}))}$ represents an entire function. Putting (2.4.5), (3.10) and (6.11) together, we arrive at Theorem 1.2.2.

Chapter 7: Explicit Evaluations of the Transform

Spectral summation formulae have been prominent in the development of analytic number theory since the 1980's. Classical examples include Petersson's, Selberg's, and Kuznetsov's trace formulae (see [13, 71]). The strengths of these formulae are encoded in the analytic properties of the integral transforms therein which are archimedean in nature. It would be desirable to obtain very explicit expressions for these transforms [1] in terms of known *special functions* and indeed this step plays quite an essential role. Also, experiences for trace formulae have demonstrated that such kind of expressions are much better to work with than the 'primitive' ones [2], for the oscillatory phases of the latter are often very complicated.

The period identities of this thesis (e.g., Theorem 1.2.2), albeit of somewhat different nature than the classical ones in the previous paragraph, are clearly instances of spectral summation formulae as well. Extra work is needed though in our case because the objects of interest are now L-functions for which their behaviors are the most interesting inside the critical strip. A delicate analytic continuation argument is thus necessary but has been done in the previous chapters. [3] Although it takes up a good portion of this thesis, it seems to have decent returns in terms of both structural and technical aspects as in the previous works [60, 61, 75, 23, 24]. [4] In fact, our analysis is more basic [5] than the aforementioned works as we never appealed to the Kuznetsov trace formula and its associated transforms. For instance, it should simplify the corresponding treatment of the archimedean component in [19].

[1] Their kernel functions in particular.

[2] by which we mean expressions like (16.9) of [13]

[3] In the cuspidal case (for Φ), the main work was done for the dual side. For the Eisenstein cases, a bit of extra work is also needed for the spectral side (see [60, 61]), but still it will be much simpler than the one for the dual side.

[4] In particular, we captured the oscillatory factor during the process of analytic continuation (Remark 3.2.5).

[5] in the sense that the only non-trivial estimate and transform involved in Chapter 5 were the Stirling formula and the Mellin transform.

For the purpose of continuation, a certain explicit form of the integral transform, i.e., (4.5), is already necessary. In this chapter, we shall make further progress by obtaining other formulae for the integral transform $(\mathcal{F}_\Phi H)(s_0, s)$. Notice that we have not obtained any formula for the kernel function of $\mathcal{F}_\Phi H$ at this point, but this will be done shortly in Proposition 7.0.2. Fortunately, this particular task turns out to be not too technical in our case and the kernel function admits an essential simplification, at least when the arguments are aligned in a nice configuration (see Proposition 7.0.3). However, readers should be cautious that these may not be always easily achievable — an example would be the Kuznetsov trace formula for $GL(3)$ (see [72, 73]).

The spectral moments of L-functions investigated in this thesis have attracted great interests in the past. Judging from the experiences of prior works [14, 40, 43, 19, 28, 94] (say) which employed more traditional techniques in analytic number theory, the tasks of obtaining good approximations or (semi-)exact formulae to the integral transforms were quite challenging for a number of reasons. On the other hand, the mentioned works took advantage of the abundance of transformation identities for the $GL(2)$ special functions. Not only does this feature ceases to exist when we get to $GL(3)$ or above, difficulties would arise in the identification of the key underlying structures at the archimedean place if a multitude of special functions identities are in use. It seems to the author that these two issues were noticed and addressed somewhat in [23, 24, 72, 73] (in different contexts).

That said, there have been some recent successes in the analytic number theory of higher-rank groups. For example, Goldfeld et. al. [67, 68] obtained (harmonic-weighted) spherical Weyl laws for $GL(3)$ and $GL(4)$ with good power-saving error terms; there are the works of Buttcane [72, 73] and Blomer-Buttcane [74] on the Kuznetsov formulae for $GL(3)$. What lies at the core of these results are the *Mellin-Barnes* integrals representing various special functions of higher-rank. It is not surprising that the one for the *Whittaker functions* (e.g., Proposition 2.1.1) is the most fundamental of all. Indeed, Whittaker functions serve as the 'harmonics' of the spectral (or the Kontorovich-Lebedev) transforms, as well as the 'bases' of automorphic forms of $GL_n(\mathbb{R})$ (via the Fourier expansions). Fortunately, generalizations of Proposition 2.1.1 to the general $GL_n(\mathbb{R})$

have been obtained in the very important works of Stade [53, 95]. He then computed explicitly certain Rankin-Selberg integrals of Whittaker functions which serve as crucial technical inputs towards the Kontorovich-Lebedev inversion for $GL_n(\mathbb{R})$. In his works, Stade was able to simplify very complicated integrals skillfully via transformation formulae of integrals of Barnes type.

In this thesis, we shall follow their footsteps by directly working with the Mellin-Barnes integrals. Judging from their experiences, this way of handling the archimedean aspects of problems is more likely to generalize and there remain prospects for more in-depth investigations. Moreover, it is not hard to realize [6] that the type of the integral transform of Theorem 1.2.2 is actually rather different from those of the Kuznetsov formulae or the Iwaniec-Sarnak moment formulae [96]. However, this fact seems to be not obvious if one adopts the classical 'approximate' approaches starting from the Kuznetsov formulae. [7]

Lemma 7.0.1. *Suppose $H \in \mathcal{C}_\eta$ and $h := H^\flat$. On the vertical strip $-\frac{1}{2} < \operatorname{Re} w < \eta$, we have*

$$\widetilde{h}(w) \ := \ \int_0^\infty h(y) y^w \, d^\times y \ = \ \frac{\pi^{-w-\frac{1}{2}}}{4} \int_{(0)} H(\beta) \cdot \frac{\Gamma\left(\frac{w+\frac{1}{2}+\beta}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\beta}{2}\right)}{|\Gamma(\beta)|^2} \frac{d\beta}{2\pi i}, \tag{7.1}$$

Proof. Since $H \in \mathcal{C}_\eta$, both sides of (7.1) converge absolutely on the strip $-1/2 < \operatorname{Re} w < \eta$ by Stirling's formula and Proposition 2.1.7. Substituting the definition of h as in (2.6) into $\widetilde{h}(w)$, the result follows from

$$\int_0^\infty W_\beta(y) y^w \, d^\times y \ = \ \frac{\pi^{-w-\frac{1}{2}}}{2} \Gamma\left(\frac{w+\frac{1}{2}+\beta}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\beta}{2}\right), \tag{7.2}$$

where $\operatorname{Re} \beta \ = \ 0$ and $\operatorname{Re} w \ > \ -1/2$.

□

[6] already in (4.5), but to be spelled out further below.

[7] further complicated by the oscillatory behavior of the Bessel function $J_{2it}(x)$ in general.

Proposition 7.0.2. *Suppose $H \in \mathcal{C}_\eta$. On the domain* [8] [9]

$$\begin{cases} \sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \\ 2\sigma - \sigma_0 - \epsilon \ > \ 0, \\ \sigma_0 + 2\sigma - 1 - \epsilon \ > \ 0, \\ 1 + \epsilon - \sigma_0 - \sigma \ > \ 0, \end{cases}$$

we have

$$\left(\mathcal{F}_\Phi H\right)(s_0, s) \ = \frac{\pi^{\frac{1}{2}-s}}{4} \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \mathcal{K}_\Phi(s_0, s; \beta) \, \frac{d\beta}{2\pi i}, \tag{7.3}$$

where the kernel function $\mathcal{K}_\Phi(\cdots)$ is given explicitly by the double Barnes integrals

$$\mathcal{K}_\Phi(s_0, s; \beta) \ := \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1-(s_0-u)}{2} + \frac{1}{2} - s\right) \cdot \prod_{i=1}^{3}\Gamma\left(\frac{(s_0-u)+\alpha_i}{2}\right)\Gamma\left(s - \frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{(s_0-u)+s_1}{2}\right)} \frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}, \tag{7.4}$$

and the contours follow Barnes' convention. Explicitly, they can be taken as $\operatorname{Re} u = \epsilon$ *and* $\operatorname{Re} s_1 = \sigma_1$ *with*

$$\sigma_0 + 2\sigma - 1 - \epsilon \ < \ \sigma_1 \ < \ \sigma. \tag{7.5}$$

Proof. Suppose

$$\sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \quad \text{and} \quad 2\sigma - \sigma_0 - \epsilon \ > \ 0 \tag{7.6}$$

[8] Certainly non-empty — it includes our point of interest $(\sigma_0, \sigma) = (1/2, 1/2)$.

[9] The restriction $\sigma < 4$ below is actually redundant, but it is included here for book-keeping.

as in Proposition 5.0.1. Recall the expression (6.12) for $(\mathcal{F}_\Phi H)(s_0, s)$. This time, we shift the line of integration of the s_1-integral to $\mathrm{Re}\, s_1 = \sigma_1$ satisfying

$$\sigma_1 < \sigma \tag{7.7}$$

and no pole is crossed during this shift as long as

$$\sigma_1 > 0 \quad \text{and} \quad \sigma_1 > \sigma_0 + 2\sigma - 1 - \epsilon. \tag{7.8}$$

Now, assume (7.3). The restrictions (7.6), (7.7), (7.8) hold and such a line of integration for the s_1-integral exists. Upon shifting the line of integration to such a position, substituting (7.1) into (6.12) and the result follows. □

In Section 4, the u-integral was introduced for various technical reasons. However, the u-integral turns out containing nice symmetries upon bringing in new Γ-factors and is an integral part of the archimedean computation. Indeed, recall the *Second Barnes Lemma*: for

$$a, b, c, d, e, f \in \mathbb{C} \quad \text{with} \quad f = a + b + c + d + e,$$

we have

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(w+a)\Gamma(w+b)\Gamma(w+c)\Gamma(d-w)\Gamma(e-w)}{\Gamma(w+f)} \frac{dw}{2\pi i} = \frac{\Gamma(d+a)\Gamma(d+b)\Gamma(d+c)\Gamma(e+a)\Gamma(e+b)\Gamma(e+c)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)} \tag{7.9}$$

(see Theorem 2.4.3 of [97]). We then have

Proposition 7.0.3. *Suppose* $\frac{1}{2}+\epsilon<\sigma<1$. *Then*

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;=\; \pi^{\frac{1}{2}-s}\cdot\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)} \cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^{3}\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\beta}{2}\right)\frac{d\beta}{2\pi i}. \tag{7.10}$$

Proof. Suppose $\frac{1}{2}+\epsilon<\sigma<\frac{2+\epsilon}{3}$. Then (7.3) is satisfied with $\sigma_0=2\sigma-1$ and by Proposition 7.0.2,

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;=\; \frac{\pi^{\frac{1}{2}-s}}{4}\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot\int_{(\sigma_1)}\frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)} \cdot\int_{(\epsilon)}\frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+s_1}{2}+1-2s\right)\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i-u}{2}\right)}{\Gamma\left(s-\frac{1}{2}+\frac{s_1-u}{2}\right)} \frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}\,\frac{d\beta}{2\pi i}. \tag{7.11}$$

For the u-integral, apply the change of variable $u\to-2u$ and (7.9) with

$$(a,b,c,d,e)\;\to\;\left(s-\frac{1}{2}+\frac{\alpha_1}{2},\;s-\frac{1}{2}+\frac{\alpha_2}{2},\;s-\frac{1}{2}+\frac{\alpha_3}{2},\;0,\;\frac{s_1}{2}+1-2s\right),$$

we obtain

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;=\; \frac{\pi^{\frac{1}{2}-s}}{2}\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\int_{(\sigma_1)}\frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)}$$
$$\cdot\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\Gamma\left(\frac{1}{2}-s+\frac{s_1+\alpha_i}{2}\right)}{\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}$$
$$\frac{ds_1}{2\pi i}\,\frac{d\beta}{2\pi i}$$
$$= \frac{\pi^{\frac{1}{2}-s}}{2}\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)$$
$$\cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\int_{(\sigma_1)}\frac{\prod_{i=1}^{3}\Gamma\left(\frac{1}{2}-s+\frac{s_1+\alpha_i}{2}\right)\cdot\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)}$$
$$\frac{ds_1}{2\pi i}\,\frac{d\beta}{2\pi i}. \tag{7.12}$$

For the s_1-integral, apply the change of variable $s_1 \to 2s_1$ and (7.9) the second time but with

$$(a,b,c,d,e) \;\to\; \left(\frac{1}{2}-s+\frac{\alpha_1}{2},\;\frac{1}{2}-s+\frac{\alpha_2}{2},\;\frac{1}{2}-s+\frac{\alpha_3}{2},\;\frac{s+\beta}{2},\;\frac{s-\beta}{2}\right), \tag{7.13}$$

we obtain

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;=\; \pi^{\frac{1}{2}-s}\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^{3}\frac{\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\beta}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}\,\frac{d\beta}{2\pi i}. \tag{7.14}$$

By analytic continuation, (7.10) holds for $\frac{1}{2}+\epsilon<\sigma<1$ and this completes the proof. □

More generally, it is possible to express the transform in terms of *hypergeometric functions* of a special type. We define

$$
{}_4\widehat{F}_3\left(\begin{matrix} A_1 & A_2 & A_3 & A_4 \\ & B_1 & B_2 & B_3 \end{matrix}\;\middle|\; z\right) := \frac{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)\Gamma(A_4)}{\Gamma(B_1)\Gamma(B_2)\Gamma(B_3)}\,{}_4F_3\left(\begin{matrix} A_1 & A_2 & A_3 & A_4 \\ & B_1 & B_2 & B_3 \end{matrix}\;\middle|\; z\right)
$$

$$
:= \sum_{n=0}^{\infty} \frac{\Gamma(A_1+n)\Gamma(A_2+n)\Gamma(A_3+n)\Gamma(A_4+n)}{\Gamma(B_1+n)\Gamma(B_2+n)\Gamma(B_3+n)} \frac{z^n}{n!}. \tag{7.15}
$$

By Theorem 2.1.2 of [97], the series converges absolutely when $|z| < 1$ and $A_1, A_2, A_3, A_4 \notin \mathbb{Z}_{\leqslant 0}$; and on $|z| = 1$ if $\mathrm{Re}\,(B_1 + B_2 + B_3 - A_1 - A_2 - A_3 - A_4) > 0$. In fact, our hypergeometric functions are of *Saalschütz* type, i.e.,

$$
B_1 + B_2 + B_3 - A_1 - A_2 - A_3 - A_4 = 1.
$$

They possess many functional relations and integral representations.

Proposition 7.0.4. *Suppose $H \in \mathcal{C}_\eta$ and $h := H^\flat$. On the region*

$$
\sigma_0 > \epsilon, \quad \sigma < 4, \quad \textit{and} \quad 2\sigma - \sigma_0 - \epsilon > 0,
$$

we have

$$\frac{1}{2\pi^{3/2}}\,\left(\mathcal{F}_\Phi H\right)(s_0,s) \;=\; \int_{-i\infty}^{i\infty} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot \frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)\cos\frac{\pi}{2}\left(2s+s_0-s_1\right)}$$

$$\cdot\ {}_4\widehat{F}_3\left(\begin{matrix} s-\frac{s_0}{2} & \frac{s_0+\alpha_1}{2} & \frac{s_0+\alpha_2}{2} & \frac{s_0+\alpha_3}{2} \\ 1/2 & \frac{s_0+s_1}{2} & s+\frac{1}{2}+\frac{s_0-s_1}{2} & \end{matrix}\,\middle|\, 1\right)\frac{ds_1}{2\pi i}$$

$$-\int_{-i\infty}^{i\infty} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot \frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)\cos\frac{\pi}{2}\left(2s+s_0-s_1\right)}$$

$$\cdot\ {}_4\widehat{F}_3\left(\begin{matrix} \frac{1}{2}-s_0+\frac{s_1}{2} & \frac{1}{2}-s+\frac{s_1+\alpha_1}{2} & \frac{1}{2}-s+\frac{s_1+\alpha_2}{2} & \frac{1}{2}-s+\frac{s_1+\alpha_3}{2} \\ \frac{1}{2}-s+s_1 & 1-s-\frac{s_0-s_1}{2} & \frac{3}{2}-s-\frac{s_0-s_1}{2} & \end{matrix}\,\middle|\, 1\right)\frac{ds_1}{2\pi i}, \tag{7.16}$$

where the contour can be taken explicitly as $\mathrm{Re}\, s_1 = \eta - \frac{1}{2}$ *(say).*

Proof. By Stirling's formula, we can shift the line of integration of the u-integral in (6.12) to $-\infty$. The residual series obtained can then be identified in terms of hypergeometric series as asserted in the present proposition. This can also be verified by *InverseMellinTransform[]* command in mathematica. More systematically, one rewrites the u-integral in the form of a Meijer's G-function. The conversion between Meijer's G-functions and generalized hypergeometric functions is known as *Slater's theorem*, see Chapter 8 of [98]. □

Chapter 8: Some Further Extensions

The focus of this article has been the archimedean aspect of the spectral moment (1.6). However, it is desirable to study the more general *'twisted moments'*, which are obtained by incorporating extra non-archimedean features. In this Appendix, the twists under consideration are the Hecke eigenvalues of $GL(2)$, see (2.25) with $a \geqslant 1$ being any integer. Not only does this display new arithmetic structures, but this is also essential for many further applications, such as mollification, amplification, effective determination, or building moments of L-functions of higher degrees, see [56, 20, 55].

There will be some book-keeping needed for the dual, non-archimedean calculation of the inner product

$$\left\langle P^{a},\ \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}.$$

We will indicate the necessary modifications for interested readers. We begin by revisiting Section 3 & 4. Following the proof of Proposition 3.1.1, we have

$$\begin{aligned}
&\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] \cdot\ e(-a\cdot u_{1,2})\, du_{1,2} \\
&\qquad = \sum_{d\mid a_2}\ \sum_{\substack{a_0\in\mathbb{Z}\\ \gcd(a_0,d)=1}}\ \sum_{a_1=-\infty}^{\infty} (\widehat{\Phi})_{(a_1,a/d)} \left[\begin{pmatrix} 1 & & \\ & * & * \\ & -a_0 & d\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] \qquad (8.1)
\end{aligned}$$

with $\begin{pmatrix} * & * \\ -a_0 & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The diagonal contribution comes from $a_0 = 0$. It is given by

$$a^{\frac{1}{2}-s}\left(\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a,a_1)}{|a_1|^{2s}}\right)\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}\left(y_0,y_1\right)\frac{dy_0dy_1}{y_0y_1^2}. \tag{8.2}$$

The double integral has been computed in Proposition 3.2.2. On the other hand, we have

$$\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a,a_1)}{|a_1|^{2s}} \;=\; 2\cdot L(2s,\Phi)\sum_{r|a}\frac{\mu(r)\mathcal{B}_\Phi(a/r,1)}{r^{2s}}. \tag{8.3}$$

by the Hecke relation of Φ (see Theorem 6.4.11 of [89]):

$$\mathcal{B}_\Phi(a,a_1) \;=\; \sum_{r|(a,a_1)}\mu(r)\mathcal{B}_\Phi(a/r,1)\mathcal{B}_\Phi(1,a_1/r).$$

The off-diagonal contribution comes from $a_0 \neq 0$. In this case,

$$\begin{pmatrix} 1 & & \\ & \alpha & \beta \\ & -a_0 & d \end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & & \\ & 1 & \frac{\beta}{d}-\frac{1}{da_0}\frac{(a_0y_0/d)^2}{1+(a_0y_0/d)^2} \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{y_0y_1}{\sqrt{(a_0y_0)^2+d^2}} & & \\ & \frac{y_0}{(a_0y_0)^2+d^2} & \\ & & 1 \end{pmatrix} \tag{8.4}$$

under the right quotient $O_3(\mathbb{R})\cdot\mathbb{R}^\times$. As a result, Definition 3.2.1 now generalizes to

$$\begin{aligned} OD_\Phi^{(a)}(s) \;:=\; & a^{\frac{1}{2}-s}\sum_{d|a} d^{2s}\sum_{\substack{a_0\neq 0\\ \gcd(a_0,d)=1}}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi\left(a/d,a_1\right)}{|a_0|^{2s-1}|a_1|}\cdot e\left(-\frac{a_1\overline{a_0}}{d}\right) \\ & \cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{da_0}\right|\cdot\frac{y_0}{1+y_0^2},\; y_1\sqrt{1+y_0^2}\right) \\ & \cdot e\left(\frac{a_1}{da_0}\cdot\frac{y_0^2}{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}, \end{aligned} \tag{8.5}$$

where $a_0\overline{a_0} \equiv 1 \pmod d$.

Remark 8.0.1. The essential difference between $OD_\Phi^{(a)}(s)$ and $OD_\Phi(s)$ lies in the arithmetic factor $e\left(-a_1\overline{a_0}/d\right)$.

The version of Proposition 4.0.2 here is

$$OD_\Phi^{(a)}(s;\phi) \;=\frac{1}{4}\int_{(1+\theta+2\epsilon)} \mathcal{L}_\Phi^{(a)}\left(s_0,s\right)\cdot\sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\;s;\;\phi\right)\,\frac{ds_0}{2\pi i}, \tag{8.6}$$

where for $\mathrm{Re}(2s-s_0)>1$ and $\mathrm{Re}\, s_0>1+\theta$, we define

$$\mathcal{L}_\Phi^{(a)}\left(s_0,s\right) \;:=\; a^{\frac{1}{2}-s}\sum_{d|a} d^{2s+s_0-1}\sum_{\ell\;(\bmod\; d)}^{*}\;\sum_{\substack{a_0\neq 0\\ a_0\,\equiv\,\ell\;(\bmod\; d)}}\;\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi\left(a/d,a_1\right)}{|a_0|^{2s-s_0}|a_1|^{s_0}}\cdot e\left(-\frac{a_1\overline{\ell}}{d}\right). \tag{8.7}$$

Remark 8.0.2. The transform $\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\;s;\;\phi\right)$ is the same as the one defined in (4.5). As a result, the work carried out in Section 5 & 7 can be applied directly to the present context.

In other words, it remains to consider the analytic properties of $\mathcal{L}_\Phi^{(a)}\left(s_0,s\right)$. In view of this, we introduce two special Dirichlet series:

- For $\mathrm{Re}\, s>1$ and $a\neq 0,-1,-2,\ldots$, the Hurwitz ζ-function is defined as

$$\zeta(s,a)\;:=\;\sum_{n=0}^{\infty}(n+a)^{-s}. \tag{8.8}$$

  It admits a holomorphic continuation to $\mathbb{C}$ except at $s=1$. It has a simple pole at $s=1$ and the residue is 1. Moreover, it satisfies a functional equation

$$\zeta\left(1-s,a\right)\;=\;\frac{\Gamma(s)}{(2\pi)^s}\sum_{\pm}e^{\mp\frac{i\pi s}{2}}\sum_{n=1}^{\infty}\frac{e(\pm na)}{n^s} \tag{8.9}$$

  when $\mathrm{Re}\, s>1$ and $0<a\leqslant 1$. See Chapter 12 of [99].

- For $\mathrm{Re}\, s > 1+\theta$ and $a/c \in \mathbb{Q}$, the additively-twisted L-series of Φ by a/c is defined as

$$L\left(s, \frac{a}{c}; \Phi\right) := \sum_{n=1}^{\infty} \frac{\mathcal{B}_\Phi(1,n)}{n^s} \cdot e\left(\frac{na}{c}\right). \tag{8.10}$$

It admits an entire continuation and its functional equation is precisely the *Voronoi formula of $GL(3)$*, see [37].

The L-series $\mathcal{L}_\Phi^{(a)}(s_0, s)$ can be re-written as

$$2a^{\frac{1}{2}-s} \sum_{\pm} \sum_{d|a} \sum_{r|(a/d)} \frac{\mu(r)}{r^{s_0}} d^{2s_0-1} \mathcal{B}_\Phi\left(\frac{a}{dr}, 1\right) \sideset{}{^*}\sum_{\ell \pmod{d}} \zeta\left(2s - s_0, \frac{\ell}{d}\right) L\left(s_0; \mp\frac{r\bar{\ell}}{d}; \Phi\right). \tag{8.11}$$

As a result, it admits a holomorphic continuation to $\mathbb{C}^2$ except on $2s - s_0 = 1$. When $\mathrm{Re}\, s > 1+\theta$, the residue of $\mathcal{L}_\Phi^{(a)}(s_0, s)$ at $s_0 = 2s-1$ is equal to

$$-2a^{\frac{1}{2}-s} \sum_{d|a} d^{4s-3} \sum_{\pm} \sum_{a_1=1}^{\infty} \frac{\mathcal{B}_\Phi(a/d, a_1)}{a_1^{2s-1}} \cdot S(0, \mp a_1; d), \tag{8.12}$$

where

$$S(0, \mp a_1; d) := \sideset{}{^*}\sum_{\ell \pmod{d}} e\left(\mp a_1 \bar{\ell}/d\right)$$

is the Ramanujan sum. Now, the argument of Section 6 carries over to the present case.

Remark 8.0.3. In conjunction with the Dirichlet series method, the moment identity developed in this Appendix can serve as a different starting point for studying the simultaneous moment of $GL(3) \times GL(2)$ and $GL(2)$ L-functions than that of Li [100] (a companion paper of [40]). This approach should provide a better understanding of the sequence of summation formulae and transforms done in [100]. This is of separate interest and we will leave it to a future paper.

References

[1] G. H. Hardy and J. E. Littlewood, "Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes," *Acta Math.*, vol. 41, no. 1, pp. 119–196, 1916.

[2] ——, "The Approximate Functional Equation in the Theory of the Zeta-Function, with Applications to the Divisor-Problems of Dirichlet and Piltz," *Proc. London Math. Soc. (2)*, vol. 21, pp. 39–74, 1923.

[3] A. E. Ingham, "Mean-Value Theorems in the Theory of the Riemann Zeta-Function," *Proc. London Math. Soc. (2)*, vol. 27, no. 4, pp. 273–300, 1927.

[4] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second. The Clarendon Press, Oxford University Press, New York, 1986, pp. x+412, Edited and with a preface by D. R. Heath-Brown, ISBN: 0-19-853369-1.

[5] A. Ivić, *The Riemann zeta-function*, ser. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985, pp. xvi+517, The theory of the Riemann zeta-function with applications, ISBN: 0-471-80634-X.

[6] A. Ivić, *Lectures on mean values of the Riemann zeta function*, ser. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1991, vol. 82, pp. viii+363, ISBN: 3-540-54748-7.

[7] K. Soundararajan, "Moments of the Riemann zeta function," *Ann. of Math. (2)*, vol. 170, no. 2, pp. 981–993, 2009.

[8] A. J. Harper, "Sharp conditional bounds for moments of the Riemann zeta function," *Arxiv Preprint*, 2013.

[9] M. Radziwił ł and K. Soundararajan, "Continuous lower bounds for moments of zeta and L-functions," *Mathematika*, vol. 59, no. 1, pp. 119–128, 2013.

[10] D. R. Heath-Brown, "Fractional moments of the Riemann zeta function," *J. London Math. Soc. (2)*, vol. 24, no. 1, pp. 65–78, 1981.

[11] A. Diaconu, D. Goldfeld, and J. Hoffstein, "Multiple Dirichlet series and moments of zeta and L-functions," *Compositio Math.*, vol. 139, no. 3, pp. 297–360, 2003.

[12] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, "Integral moments of L-functions," *Proc. London Math. Soc. (3)*, vol. 91, no. 1, pp. 33–104, 2005.

[13] H. Iwaniec and E. Kowalski, *Analytic number theory*, ser. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004, vol. 53, pp. xii+615, ISBN: 0-8218-3633-1.

[14] J. B. Conrey and H. Iwaniec, "The cubic moment of central values of automorphic L-functions," *Ann. of Math. (2)*, vol. 151, no. 3, pp. 1175–1216, 2000.

[15] J.-L. Waldspurger, "Sur les coefficients de Fourier des formes modulaires de poids demi-entier," *J. Math. Pures Appl. (9)*, vol. 60, no. 4, pp. 375–484, 1981.

[16] W. Kohnen and D. Zagier, "Values of L-series of modular forms at the center of the critical strip," *Invent. Math.*, vol. 64, no. 2, pp. 175–198, 1981.

[17] S. Katok and P. Sarnak, "Heegner points, cycles and Maass forms," *Israel J. Math.*, vol. 84, no. 1-2, pp. 193–227, 1993.

[18] J. Guo, "On the positivity of the central critical values of automorphic L-functions for $\mathrm{GL}(2)$," *Duke Math. J.*, vol. 83, no. 1, pp. 157–190, 1996.

[19] I. N. Petrow, "A twisted Motohashi formula and Weyl-subconvexity for L-functions of weight two cusp forms," *Math. Ann.*, vol. 363, no. 1-2, pp. 175–216, 2015.

[20] D. Frolenkov, "The cubic moment of automorphic L-functions in the weight aspect," *J. Number Theory*, vol. 207, pp. 247–281, 2020.

[21] A. Ivić, "On sums of Hecke series in short intervals," *J. Théor. Nombres Bordeaux*, vol. 13, no. 2, pp. 453–468, 2001.

[22] ——, "On the moments of Hecke series at central points," *Funct. Approx. Comment. Math.*, vol. 30, pp. 49–82, 2002.

[23] Y. Motohashi, "An explicit formula for the fourth power mean of the Riemann zeta-function," *Acta Math.*, vol. 170, no. 2, pp. 181–220, 1993.

[24] Y. Motohashi, *Spectral theory of the Riemann zeta-function*, ser. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1997, vol. 127, pp. x+228, ISBN: 0-521-44520-5.

[25] Z. Peng, *Zeros and central values of automorphic L-functions*. ProQuest LLC, Ann Arbor, MI, 2001, p. 73, Thesis (Ph.D.)–Princeton University, ISBN: 978-0493-16728-2.

[26] I. Petrow and M. P. Young, "A generalized cubic moment and the Petersson formula for newforms," *Math. Ann.*, vol. 373, no. 1-2, pp. 287–353, 2019.

[27] ——, "The Weyl bound for Dirichlet L-functions of cube-free conductor," *Ann. of Math. (2)*, vol. 192, no. 2, pp. 437–486, 2020.

[28] M. P. Young, "Weyl-type hybrid subconvexity bounds for twisted L-functions and Heegner points on shrinking sets," *J. Eur. Math. Soc. (JEMS)*, vol. 19, no. 5, pp. 1545–1576, 2017.

[29] H. Iwaniec, "Fourier coefficients of modular forms of half-integral weight," *Invent. Math.*, vol. 87, no. 2, pp. 385–401, 1987.

[30] W. Duke, "Hyperbolic distribution problems and half-integral weight Maass forms," *Invent. Math.*, vol. 92, no. 1, pp. 73–90, 1988.

[31] M. P. Young, "The fourth moment of Dirichlet L-functions," *Ann. of Math. (2)*, vol. 173, no. 1, pp. 1–50, 2011.

[32] V. Blomer, P. Humphries, R. Khan, and M. B. Milinovich, "Motohashi's fourth moment identity for non-archimedean test functions and applications," *Compos. Math.*, vol. 156, no. 5, pp. 1004–1038, 2020.

[33] B. Topacogullari, "The fourth moment of individual Dirichlet L-functions on the critical line," *Math. Z.*, vol. 298, no. 1-2, pp. 577–624, 2021.

[34] I. Kaneko, "Motohashi's formula for the fourth moment of individual Dirichlet L-functions and applications," *Arxiv Preprint*, 2021.

[35] S. D. Miller and W. Schmid, "Automorphic distributions, L-functions, and Voronoi summation for $\mathrm{GL}(3)$," *Ann. of Math. (2)*, vol. 164, no. 2, pp. 423–488, 2006.

[36] ——, "A general Voronoi summation formula for $GL(n, \mathbb{Z})$," in *Geometry and analysis. No. 2*, ser. Adv. Lect. Math. (ALM), vol. 18, Int. Press, Somerville, MA, 2011, pp. 173–224.

[37] D. Goldfeld and X. Li, "Voronoi formulas on $\mathrm{GL}(n)$," *Int. Math. Res. Not.*, Art. ID 86295, 25, 2006.

[38] ——, "The Voronoi formula for $\mathrm{GL}(n, \mathbb{R})$," *Int. Math. Res. Not. IMRN*, no. 2, Art. ID rnm144, 39, 2008.

[39] A. Ichino and N. Templier, "On the Voronoĭ formula for $\mathrm{GL}(n)$," *Amer. J. Math.*, vol. 135, no. 1, pp. 65–101, 2013.

[40] X. Li, "Bounds for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions and $\mathrm{GL}(3)$ L-functions," *Ann. of Math. (2)*, vol. 173, no. 1, pp. 301–336, 2011.

[41] J. Buttcane and R. Khan, "L^4-norms of Hecke newforms of large level," *Math. Ann.*, vol. 362, no. 3-4, pp. 699–715, 2015.

[42] ——, "On the fourth moment of Hecke-Maass forms and the random wave conjecture," *Compos. Math.*, vol. 153, no. 7, pp. 1479–1511, 2017.

[43] V. Blomer, "Subconvexity for twisted L-functions on $\mathrm{GL}(3)$," *Amer. J. Math.*, vol. 134, no. 5, pp. 1385–1421, 2012.

[44] Z. Qi, "Subconvexity for twisted L-functions on $_3$ over the Gaussian number field," *Trans. Amer. Math. Soc.*, vol. 372, no. 12, pp. 8897–8932, 2019.

[45] B. Huang, "Hybrid subconvexity bounds for twisted L-functions on GL(3)," *Sci. China Math.*, vol. 64, no. 3, pp. 443–478, 2021.

[46] R. M. Nunes, "Subconvexity for $GL(3)$ L-functions," *Arxiv Preprint*, 2017.

[47] Y. Lin, R. Nunes, and Z. Qi, "Strong subconvexity for self-dual $GL(3)$ L-functions," *Arxiv Preprint*, 2021.

[48] R. Zacharias, "Periods and reciprocity I," *Int. Math. Res. Not. IMRN*, no. 3, pp. 2191–2209, 2021.

[49] ——, "Periods and reciprocity II," *Arxiv Preprint*, 2020.

[50] R. M. Nunes, "Spectral reciprocity via Integral Representations," *Arxiv Preprint*, 2020.

[51] S. D. Miller, "On the existence and temperedness of cusp forms for $\mathrm{SL}_3(\mathbb{Z})$," *J. Reine Angew. Math.*, vol. 533, pp. 127–169, 2001.

[52] D. Bump, "Barnes' second lemma and its application to Rankin-Selberg convolutions," *Amer. J. Math.*, vol. 110, no. 1, pp. 179–185, 1988.

[53] E. Stade, "Mellin transforms of $\mathrm{GL}(n, \mathbb{R})$ Whittaker functions," *Amer. J. Math.*, vol. 123, no. 1, pp. 121–161, 2001.

[54] O. Balkanova and D. Frolenkov, "The mean value of symmetric square L-functions," *Algebra Number Theory*, vol. 12, no. 1, pp. 35–59, 2018.

[55] ——, "Moments of L-functions and the Liouville-Green method," *J. Eur. Math. Soc. (JEMS)*, vol. 23, no. 4, pp. 1333–1380, 2021.

[56] O. Balkanova, G. Bhowmik, D. Frolenkov, and N. Raulf, "Mixed moment of $GL(2)$ and $GL(3)$ L-functions," *Proc. Lond. Math. Soc. (3)*, vol. 121, no. 2, pp. 177–219, 2020.

[57] O. Balkanova, D. Frolenkov, and H. Wu, "On Weyl's subconvex bound for cube-free Hecke characters: Totally real case," *Arxiv Preprint*, 2021.

[58] J. B. Conrey, "The mean-square of Dirichlet L-functions," *Arxiv Preprint*, 2007.

[59] H. Iwaniec and P. Sarnak, "Dirichlet L-functions at the central point," in *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, de Gruyter, Berlin, 1999, pp. 941–952.

[60] V. Blomer and R. Khan, "Twisted moments of L-functions and spectral reciprocity," *Duke Math. J.*, vol. 168, no. 6, pp. 1109–1177, 2019.

[61] ——, "Uniform subconvexity and symmetry breaking reciprocity," *J. Funct. Anal.*, vol. 276, no. 7, pp. 2315–2358, 2019.

[62] N. Andersen and E. M. Kı ral, "Level reciprocity in the twisted second moment of Rankin-Selberg L-functions," *Mathematika*, vol. 64, no. 3, pp. 770–784, 2018.

[63] D. Goldfeld and M. Thillainatesan, "Rank lowering linear maps and multiple Dirichlet series associated to $\mathrm{GL}(n, \mathbb{R})$," *Pure Appl. Math. Q.*, vol. 2, no. 2, Special Issue: In honor of John H. Coates. Part 2, pp. 601–615, 2006.

[64] A. Reznikov, "Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms," *J. Amer. Math. Soc.*, vol. 21, no. 2, pp. 439–477, 2008.

[65] P. Michel and A. Venkatesh, "Equidistribution, L-functions and ergodic theory: On some problems of Yu. Linnik," in *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, pp. 421–457.

[66] ——, "The subconvexity problem for GL_2," *Publ. Math. Inst. Hautes Études Sci.*, no. 111, pp. 171–271, 2010.

[67] D. Goldfeld and A. Kontorovich, "On the $\mathrm{GL}(3)$ Kuznetsov formula with applications to symmetry types of families of L-functions," in *Automorphic representations and L-functions*, ser. Tata Inst. Fundam. Res. Stud. Math. Vol. 22, Tata Inst. Fund. Res., Mumbai, 2013, pp. 263–310.

[68] D. Goldfeld, E. Stade, and M. Woodbury, "An orthogonality relation for $\mathrm{GL}(4, \mathbb{R})$ (with an appendix by Bingrong Huang)," *Forum Math. Sigma*, vol. 9, Paper No. e47, 83, 2021.

[69] J. Buttcane, "Kuznetsov, Petersson and Weyl on $\mathrm{GL}(3)$, I: The principal series forms," *Amer. J. Math.*, vol. 142, no. 2, pp. 595–626, 2020.

[70] ——, "Kuznetsov, Petersson and Weyl on $GL(3)$, II: The generalized principal series forms," *Math. Ann.*, vol. 380, no. 1-2, pp. 231–266, 2021.

[71] H. Iwaniec, *Spectral methods of automorphic forms*, Second, ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002, vol. 53, pp. xii+220, ISBN: 0-8218-3160-7.

[72] J. Buttcane, "On sums of $SL(3,\mathbb{Z})$ Kloosterman sums," *Ramanujan J.*, vol. 32, no. 3, pp. 371–419, 2013.

[73] ——, "The spectral Kuznetsov formula on $SL(3)$," *Trans. Amer. Math. Soc.*, vol. 368, no. 9, pp. 6683–6714, 2016.

[74] V. Blomer and J. Buttcane, "On the subconvexity problem for L-functions on $\mathrm{GL}(3)$," *Ann. Sci. Éc. Norm. Supér. (4)*, vol. 53, no. 6, pp. 1441–1500, 2020.

[75] V. Blomer, X. Li, and S. D. Miller, "A spectral reciprocity formula and non-vanishing for L-functions on $\mathrm{GL}(4) \times \mathrm{GL}(2)$," *J. Number Theory*, vol. 205, pp. 1–43, 2019.

[76] V. Chandee and X. Li, "The second moment of $GL(4) \times GL(2)$ L-functions at special points," *Adv. Math.*, vol. 365, pp. 107060, 39, 2020.

[77] V. Blomer, "Period integrals and Rankin-Selberg L-functions on $GL(n)$," *Geom. Funct. Anal.*, vol. 22, no. 3, pp. 608–620, 2012.

[78] D. Bump, S. Friedberg, and D. Goldfeld, "Poincaré series and Kloosterman sums for $\mathrm{SL}(3,\mathbf{Z})$," *Acta Arith.*, vol. 50, no. 1, pp. 31–89, 1988.

[79] Q. Lu, "Bounds for the spectral mean value of central values of L-functions," *J. Number Theory*, vol. 132, no. 5, pp. 1016–1037, 2012.

[80] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, "Automorphic forms on $\mathrm{GL}(3)$. I, II," *Ann. of Math. (2)*, vol. 109, no. 2, pp. 213–258, 1979.

[81] W. Duke and H. Iwaniec, "Estimates for coefficients of L-functions. I," in *Automorphic forms and analytic number theory (Montreal, PQ, 1989)*, Univ. Montréal, Montreal, QC, 1990, pp. 43–47.

[82] ——, "Estimates for coefficients of L-functions. II," in *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, Univ. Salerno, Salerno, 1992, pp. 71–82.

[83] ——, "Estimates for coefficients of L-functions. III," in *Séminaire de Théorie des Nombres, Paris, 1989–90*, ser. Progr. Math. Vol. 102, Birkhäuser Boston, Boston, MA, 1992, pp. 113–120.

[84] ——, “Estimates for coefficients of L-functions. IV,” *Amer. J. Math.*, vol. 116, no. 1, pp. 207–217, 1994.

[85] W. Luo, Z. Rudnick, and P. Sarnak, “On Selberg’s eigenvalue conjecture,” *Geom. Funct. Anal.*, vol. 5, no. 2, pp. 387–401, 1995.

[86] E. M. Kı ral and F. Zhou, “The Voronoi formula and double Dirichlet series,” *Algebra Number Theory*, vol. 10, no. 10, pp. 2267–2286, 2016.

[87] P. D. Nelson, “Eisenstein series and the cubic moment for $PGL(2)$,” *Arxiv Preprint*, 2020.

[88] H. Wu, “On Motohashi’s formula,” *Arxiv Preprint*, 2021.

[89] D. Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2006, vol. 99, pp. xiv+493, With an appendix by Kevin A. Broughan, ISBN: 978-0-521-83771-2; 0-521-83771-5.

[90] J. W. Cogdell, H. H. Kim, and M. R. Murty, *Lectures on automorphic L-functions*, ser. Fields Institute Monographs. American Mathematical Society, Providence, RI, 2004, vol. 20, pp. xii+283, ISBN: 0-8218-3516-5.

[91] D. Bump, *Automorphic forms on* $\mathrm{GL}(3, \mathbf{R})$, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1984, vol. 1083, pp. xi+184, ISBN: 3-540-13864-1.

[92] D. Goldfeld and A. Kontorovich, “On the determination of the Plancherel measure for Lebedev-Whittaker transforms on $\mathrm{GL}(n)$,” *Acta Arith.*, vol. 155, no. 1, pp. 15–26, 2012.

[93] F. Zhou, “Weighted Sato-Tate vertical distribution of the Satake parameter of Maass forms on $\mathrm{PGL}(N)$,” *Ramanujan J.*, vol. 35, no. 3, pp. 405–425, 2014.

[94] E. M. Kiral, I. Petrow, and M. P. Young, “Oscillatory integrals with uniformity in parameters,” *J. Théor. Nombres Bordeaux*, vol. 31, no. 1, pp. 145–159, 2019.

[95] E. Stade, “Archimedean L-factors on $\mathrm{GL}(n) \times \mathrm{GL}(n)$ and generalized Barnes integrals,” *Israel J. Math.*, vol. 127, pp. 201–219, 2002.

[96] H. Iwaniec and P. Sarnak, “The non-vanishing of central values of automorphic L-functions and Landau-Siegel zeros,” *Israel J. Math.*, vol. 120, no. part A, pp. 155–177, 2000.

[97] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999, vol. 71, pp. xvi+664, ISBN: 0-521-62321-9; 0-521-78988-5.

[98] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and series. Vol. 3*. Gordon and Breach Science Publishers, New York, 1990, p. 800, More special functions, Translated from the Russian by G. G. Gould, ISBN: 2-88124-682-6.

[99] T. M. Apostol, *Introduction to analytic number theory*, ser. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976, pp. xii+338.

[100] X. Li, "The central value of the Rankin-Selberg L-functions," *Geom. Funct. Anal.*, vol. 18, no. 5, pp. 1660–1695, 2009.