Resolutions, bounds, and dimensions for derived categories of varieties

Noah Olander

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2022

Abstract

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In this thesis we solve three problems about derived categories of algebraic varieties: We prove the conjecture [EL21, Conjecture 4.13] of Elagin and Lunts; we positively answer a question raised by the conjecture [Orl09, Conjecture 10] of Orlov, proving new cases of that conjecture in the process; and we extend Orlov's theorem [Orl97, Theorem 2.2] from smooth projective varieties to smooth proper algebraic spaces. These results go toward answering the questions: How rigid is the (triangulated) derived category of coherent sheaves on an algebraic variety, and how much information does it possess about the variety? Our techniques are general and work for algebraic spaces just as well as they do for projective varieties.

Table of Contents

Acknowledgements

To overstate the influence Johan de Jong had on this thesis would be nearly impossible. Everybody in the algebraic geometry community seems to have a story about asking him a question and being blown away by his enthusiasm and love for mathematics. It has been a privilege and an absolute blast to be his student.

Of all the people who overlapped with me at Columbia and influenced this work, I would especially like to thank Alex Perry, who encouraged me to think about some of the problems in this thesis, and Dmitrii Pirozhkov, who read the paper containing Theorem 2.0.1 and immediately realized that it implied Theorem 3.4.1. I cannot thank Raymond Cheng and Shizhang Li enough for all the mathematical discussions we had and for making the community here what it was. I am also very grateful to Roy Magen, Morena Porzio, Anna Abasheva, Carl Lian, and many others.

My life has been enriched by the friends and family who have been in it for some or all of the last five years. I feel a deep gratitude to them for their patience with me during a time which, despite ample warning, I found surprisingly difficult. Let me single out only my parents and Dustin and Tremaine, whose support has been such a constant presence in my life that I don't know what I would do without it.

Dedication

To my family.

Introduction

In this work, we prove three results about derived categories of smooth proper algebraic spaces over a field. One of these results is a generalization of a previously known fact for smooth projective varieties, while the other two answer questions which were also open in the smooth projective case. We use [Sta22, Tag 08EY] as our basic reference for derived categories of spaces and follow all conventions used there. The corresponding material for schemes is [Sta22, Tag 08CU], and the reader is welcome to replace algebraic spaces with schemes throughout – we have chosen to work with spaces as the proofs are no more difficult in this setting.

The results are as follows. Dmitri Orlov proved in [Orl97] that fully faithful functors between derived categories of smooth projective varieties are represented by objects in the derived category of the product. Our first result, Theorem 3.0.1, generalizes this to algebraic spaces which are smooth and proper over a field. Orlov's result is foundational and led him to conjecture in [Orl05] that the derived category of a smooth projective variety determines its rational Chow motive. The conjecture is currently open, and has stimulated research into which invariants of a variety can be read off from its derived category.

One such invariant is the dimension. It has been known for some time that smooth projective varieties with equivalent derived categories have the same dimension, but the following strengthening was unknown even in the smooth projective case: Our second result, Theorem 3.4.1, says that if X, Y are smooth proper algebraic spaces over a field and there is a fully faithful functor from the derived category of X to the derived category of Y, then $\dim(X) \leq \dim(Y)$. This question had attracted some attention as it was known to follow from Orlov's conjecture [Orl09, Conjecture 10], which says that the Rouquier dimension of X is equal to its dimension. Indeed in proving Theorem 3.4.1 we establish new cases of this conjecture, see Theorem 2.0.2.

The Rouquier dimension of X, denoted $\mathrm{Rdim}(X)$, is one of several notions of dimension which can be read off from the derived category of X.

Another one which plays a major role in this thesis is the diagonal dimension $\mathrm{Ddim}(X)$. The two are related by the inequalities

$$\dim(X) \leq \mathrm{Rdim}(X) \leq \mathrm{Ddim}(X) \leq 2 \cdot \dim(X).$$

While the first inequality is conjectured to be an equality, one expects that $\dim(X) < \mathrm{Ddim}(X)$ for most X. Despite this expectation, our third result, Theorem 4.0.1, gives the first proof of this strict inequality for *any* X. The result says that a smooth projective curve of positive genus has diagonal dimension 2, and positively answers the conjecture [EL21, Conjecture 4.13] of Elagin and Lunts.

This paper is organized into four chapters. Chapter 1 is foundational and introduces the notions of strong generation and dimension of triangulated categories we will work with throughout. Chapter 2 is essentially devoted to the proof of Theorem 3.4.1, though for technical reasons, we are unable to give the complete proof until after we know Orlov's Theorem 3.0.1 for smooth proper algebraic spaces, a result to which Chapter 3 is devoted. Finally, Chapter 4 gives the proof of Theorem 4.0.1.

Chapter 1: Resolutions and dimensions

In this chapter we introduce many of the technical notions needed throughout the paper. Section 1.1 gives a precise meaning to the statement that an object K in a triangulated category $\mathcal{T}$ can be built from objects $G_i, i \in I$ using d cones. In Section 1.2 we turn our attention from general triangulated categories to geometric ones. We define the Rouquier dimension and countable Rouquier dimension of a Noetherian algebraic space and the diagonal dimension of a smooth separated algebraic space over a field. We prove basic properties of these notions. Finally, in Section 1.3, we introduce a special system of coherent sheaves on a Noetherian algebraic space which generate its derived category. We are able to use these objects as substitutes for ample line bundles in Chapters 2 and 3.

1.1 Generation time

Let $\mathcal{T}$ be a triangulated category. Let $S \subset \mathrm{Ob}(\mathcal{T})$ be a set of objects. Then Bondal and Van den Bergh inductively define in [BV03] a sequence of full subcategories $\langle S \rangle_n$ of $\mathcal{T}$ as follows.

An object of $\langle S \rangle_1$ is defined to be a direct summand of a finite direct sum of shifts of objects of S.

For $n \geq 1$, an object of $\langle S \rangle_{n+1}$ is defined to be a direct summand of an extension of an object of $\langle S \rangle_n$ by an object of $\langle S \rangle_1$, i.e., an object which is a direct summand of an object E for which there exists a distinguished triangle

$$E' \to E \to E'' \to E'[1]$$

with $E'' \in \langle S \rangle_n$ and $E' \in \langle S \rangle_1$.

One sets $\langle S \rangle = \bigcup_{n \geq 0} \langle S \rangle_n$.

Lemma 1.1.1. *The subcategories $\langle S \rangle_n$ and $\langle S \rangle$ of $\mathcal{T}$ are closed under finite direct sums, shifts, and direct summands. The subcategory $\langle S \rangle$ is closed under taking cones. If $E' \in \langle S \rangle_a$ and $E'' \in \langle S \rangle_b$ and there exists a distinguished triangle*

$$E' \to E \to E'' \to E'[1],$$

then $E \in \langle S \rangle_{a+b}$.

Proof. Omitted. □

Definition 1.1.2. 1. ([BV03]) An object G of $\mathcal{T}$ is called a *strong generator* of $\mathcal{T}$ if there exists $n > 0$ such that $\mathcal{T} = \langle G \rangle_{n+1}$. We sometimes say G is an n-step generator.

2. An object G of $\mathcal{T}$ is called a *classical generator* of $\mathcal{T}$ if $\mathcal{T} = \langle G \rangle$.

3. ([Rou08]) The *Rouquier dimension* of $\mathcal{T}$ is the smallest $n \geq 0$ for which there exists an n-step generator for $\mathcal{T}$, or infinity if there does not exist a strong generator. The Rouquier dimension of $\mathcal{T}$ is denoted $\mathrm{Rdim}(\mathcal{T})$.

4. (Pirozhkov) The *countable Rouquier dimension* of $\mathcal{T}$ is the smallest $n \geq 0$ for which there exists a countable set S of objects of $\mathcal{T}$ such that $\mathcal{T} = \langle S \rangle_{n+1}$, or infinity if there exist no such n and S. It is denoted $\mathrm{CRdim}(\mathcal{T})$.

Example 1.1.3. If $\mathcal{T}$ has only countably many objects up to isomorphism, then $\mathrm{CRdim}(\mathcal{T}) = 0$ as we may take an S which contains every object up to isomorphism. In particular, if X is an algebraic space of finite type over a countable field, then this applies to $D^b_{coh}(\mathcal{O}_X)$.

Example 1.1.4. Let R be a not necessarily commutative left-Noetherian ring. Let $\mathcal{T}$ be the bounded derived category of the category of finitely generated left R-modules. Let M be a finite left R-module of projective dimension d. Then the reader should show that $M \in \langle R \rangle_{d+1}$. More generally, if E is an object of $\mathcal{T}$ whose cohomology modules all have projective dimension at most d, then $E \in \langle R \rangle_{d+1}$. We prove this by induction on d. For $d = 0$ it follows from the fact that a complex with projective cohomology modules is decomposable, and a projective module is a direct summand of $R^{\oplus N}$ for some N. For $d > 0$ we may choose an object P which is a finite direct sum of shifts of R and a morphism $P \to E$ which is surjective on cohomology modules. Then there is a distinguished triangle

$$E' \to P \to E \to E'[1]$$

and the long exact sequence of cohomology for the triangle splits into short exact sequences

$$0 \to H^n(E') \to H^n(P) \to H^n(E) \to 0$$

so that the cohomology modules of E' have projective dimension at most $d-1$. Applying the inductive hypothesis and the definition of $\langle R\rangle_d$ gives the result. In particular we have

$$\mathrm{Rdim}(\mathcal{T}) \leq \text{the left global dimension of } R.$$

We close by proving a pleasant feature of Rouquier dimension. Recall that a full triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ of a triangulated category is called *right admissible* (resp. *left admissible*, resp. *two-sided admissible*) if the inclusion functor admits a right (resp. left, resp. right and left) adjoint.

Lemma 1.1.5. *Let $\mathcal{A} \subset \mathcal{T}$ be a right admissible subcategory of a triangulated category. Then* $\mathrm{Rdim}(\mathcal{A}) \leq \mathrm{Rdim}(\mathcal{T})$ *and* $\mathrm{CRdim}(\mathcal{A}) \leq \mathrm{CRdim}(\mathcal{T})$.

Proof. The proofs of both statements are exactly the same so we prove only the second. Let R be the right adjoint of the inclusion. If $\mathcal{T} = \langle \{E_i\}_{i\in I}\rangle_{n+1}$ then since R is essentially surjective, $\mathcal{A} = \langle \{R(E_i)\}_{i\in I}\rangle_{n+1}$. □

1.2 Dimension in the geometric case

For X a Noetherian algebraic space, we shall write $\mathrm{Rdim}(X) = \mathrm{Rdim}(D^b_{coh}(\mathcal{O}_X))$ and $\mathrm{CRdim}(X) = \mathrm{CRdim}(D^b_{coh}(\mathcal{O}_X))$. We then have the following key finiteness results due to Neeman and Aoki, respectively.

Theorem 1.2.1 ([Nee21])**.** *If X is a Noetherian separated regular scheme of finite dimension, then* $\mathrm{Rdim}(X) < \infty$, *i.e., $D^b_{coh}(\mathcal{O}_X)$ has a strong generator.*

Theorem 1.2.2 ([Aok21])**.** *If X is a Noetherian separated quasi-excellent scheme of finite dimension, then* $\mathrm{Rdim}(X) < \infty$, *i.e., $D^b_{coh}(\mathcal{O}_X)$ has a strong generator.*

For varieties, we also have a lower bound due to Rouquier.

Proposition 1.2.3 ([Rou08, Proposition 7.17])**.** *Assume X is a reduced algebraic space of finite type over a field. Then* $\mathrm{Rdim}(X) \geq \dim(X)$.

The hypotheses given there are slightly stronger but the same exact proof shows this. A big motivation for studying the Rouquier dimension of varieties is provided by the following conjecture of Orlov.

Conjecture 1.2.4 (Orlov)**.** *Let k be a field. Let X be a smooth quasi-projective scheme over k. Then the Rouquier dimension of $D^b_{coh}(\mathcal{O}_X)$ is equal to the dimension of X.*

In [Orl09], Orlov proved this when X has dimension at most one and then made the conjecture in general. So far, the conjecture is only known for certain special classes of varieties. See the introduction of [Pir19] for a list of known cases.

When X is smooth and separated over a field we can also define its diagonal dimension. First we must briefly review the concept of box products. In this thesis, box products arise when X, Y are algebraic spaces over a field k and we have objects E, F of $D_{QCoh}(\mathcal{O}_X), D_{QCoh}(\mathcal{O}_Y)$, respectively. Then we set

$$E \boxtimes F = \mathbf{L}pr_1^* E \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_k Y}} \mathbf{L}pr_2^* F \in D_{QCoh}(\mathcal{O}_{X\times_k Y}),$$

where $pr_1 : X \times_k Y \to X$ and $pr_2 : X \times_k Y \to Y$ are the projections. There is also a notion of box product of quasi-coherent sheaves which is defined in the same way but with underived pullbacks and tensor product. This leads to no ambiguity:

Lemma 1.2.5. *Let X, Y be algebraic spaces over a field k. Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent sheaves on X, Y respectively. Then*

$$\mathcal{T}or_i^{\mathcal{O}_{X\times_k Y}}(pr_1^*\mathcal{F}, pr_2^*\mathcal{G}) = 0$$

for $i > 0$ so the derived and underived box product and the derived box product of $\mathcal{F}$ and $\mathcal{G}$ coincide.

Proof. The second statement follows from the first. Note that we use that the projections are flat so that $\mathbf{L}pr_i^* = pr_i^*$. The first statement reduces to the following algebra problem: A, B are k-algebras and M, N are an A-module and a B-module, respectively. Then show

$$\mathrm{Tor}_i^{A\otimes_k B}(M \otimes_A (A \otimes_k B), N \otimes_B (A \otimes_k B)) = 0$$

for $i > 0$. This follows from

$$\begin{aligned}(M \otimes_A (A \otimes_k B)) \otimes^{\mathbf{L}}_{A\otimes_k B} (N \otimes_B (A \otimes_k B)) &= (M \otimes^{\mathbf{L}}_A (A \otimes_k B)) \otimes^{\mathbf{L}}_{A\otimes_k B} ((A \otimes_k B) \otimes^{\mathbf{L}}_B N) \\ &= M \otimes^{\mathbf{L}}_k N,\end{aligned}$$

by the associativity of derived tensor products. Since k is a field, $M \otimes^{\mathbf{L}}_k N$ lives in degree zero. □

Definition 1.2.6. Let X be a smooth separated algebraic space over a field k. Then the *diagonal dimension* of X, denoted $\mathrm{Ddim}(X)$, is the smallest n such that there exist $E, F \in D^b_{coh}(\mathcal{O}_X)$ such that

$$\mathcal{O}_\Delta \in \langle E \boxtimes F \rangle_{n+1},$$

where $\mathcal{O}_\Delta \in \mathrm{Coh}(\mathcal{O}_{X\times_k X})$ is the structure sheaf of the diagonal.

Lemma 1.2.7. *Let $X = U \coprod V$ be a disjoint union of smooth separated algebraic spaces over a field k. Then*

$$\mathrm{Ddim}(X) = \max\{\mathrm{Ddim}(U), \mathrm{Ddim}(V)\}.$$

Proof. Omitted. □

The definition above appears to depend on the field k, but this dependence is only superficial by the following lemma.

Lemma 1.2.8. *Let X be a smooth separated algebraic space over a field k' which is a finite separable extension of a field k. Then the diagonal dimension of X viewed as a smooth algebraic space over k is the same as the dimension of X viewed as a smooth algebraic space over k'.*

Proof. The pullback diagram

$$\begin{array}{ccc} X \times_{k'} X & \xrightarrow{f} & X \times_k X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \xrightarrow{\Delta_{k'/k}} & \mathrm{Spec}(k') \times_k \mathrm{Spec}(k') \end{array}$$

shows that f is both an open and closed immersion, since the same is true for $\Delta_{k'/k}$.

With this in mind, we have

$$\mathcal{O}_{\Delta_{X/k}} \in \langle G \boxtimes_k H \rangle_{d+1} \iff \mathcal{O}_{\Delta_{X/k'}} = Lf^*(\mathcal{O}_{\Delta_{X/k}}) \in \langle Lf^*(G \boxtimes_k H)\rangle_{d+1} = \langle G \boxtimes_{k'} H \rangle_{d+1},$$

completing the proof. □

The following lemma provides motivation for studying the diagonal dimension.

Lemma 1.2.9. *Let X be a smooth separated algebraic space of finite type over a field k. Then*

$$\mathrm{Rdim}(X) \le \mathrm{Ddim}(X) \le 2 \cdot \dim(X).$$

Proof. For the first inequality, if $\mathcal{O}_\Delta \in \langle E \boxtimes F \rangle_{d+1}$ then for any object $K \in D^b_{coh}(\mathcal{O}_X)$, we have

$$K = \Phi_{\mathcal{O}_\Delta}(K) \in \langle \Phi_{E\boxtimes F}(K)\rangle_{d+1}$$

where Φ_M denotes the Fourier–Mukai transform with kernel M. But

$$\Phi_{E\boxtimes F}(K) = R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K) \otimes^{\mathbf{L}}_k F.$$

This is a direct sum of shifts of the object $F \in D^b_{coh}(\mathcal{O}_X)$. If say X is proper, then the sum is finite and we can conclude immediately that F is a d-step generator of $D^b_{coh}(\mathcal{O}_X)$, completing the proof. In general, F is still a d-step generator, but this follows from the compactness result [BV03, Proposition 2.2.4].

For the second inequality, set $n = \dim(X)$. We will see in Lemma 1.3.3 below that there is an exact sequence

$$\cdots \to \mathcal{E}_i \boxtimes \mathcal{F}_i \to \cdots \to \mathcal{E}_0 \boxtimes \mathcal{F}_0 \to \mathcal{O}_\Delta \to 0$$

with $\mathcal{E}_i, \mathcal{F}_i$ coherent sheaves on X. Since $X \times_k X$ is regular of dimension $2n$, for any coherent sheaf $\mathcal{G}$ on $X \times_k X$ we have $\mathrm{Ext}^{2d+1}_{\mathcal{O}_{X\times_k X}}(\mathcal{O}_\Delta, \mathcal{F})) = 0$. This implies that the projection from the complex

$$\mathcal{E}_{2n} \boxtimes \mathcal{F}_{2n} \to \cdots \to \mathcal{E}_0 \boxtimes \mathcal{F}_0$$

to $\mathcal{O}_\Delta$ splits (the obstruction to splitting lives in an Ext^{2d+1}). Thus

$$\mathcal{O}_\Delta \in \langle \bigoplus_{i=0}^{2n} \mathcal{E}_i \boxtimes \mathcal{F}_i \rangle_{2n+1}.$$

□

In particular, a smooth separated algebraic space of finite type over a field has finite Rouquier dimension. This is also true without the smoothness assumption but we will not prove it here as it won't be necessary for us. Here is a useful consequence:

Lemma 1.2.10. *Let X, Y be smooth proper algebraic spaces over a field k. Then any k-linear exact functor $D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ has exact k-linear left and right adjoints.*

Proof. This follows from [BV03, Theorem 1.3] and the fact we have just seen that $D^b_{coh}(\mathcal{O}_X)$ has a strong generator. □

In the affine case, we can compute the diagonal dimension exactly.

Lemma 1.2.11. *Let X be a smooth affine scheme over a field k. Then*

$$\mathrm{Ddim}(X) = \dim(X).$$

Proof. The inequality $\geq$ follows from Lemma 1.2.9 and Proposition 1.2.3. For the reverse inequality, we will show

$$\mathrm{Ext}^{d+1}_{\mathcal{O}_{X\times_k X}}(\mathcal{O}_\Delta, \mathcal{F}) = H^0(X \times_k X, \mathcal{E}xt^{d+1}_{\mathcal{O}_{X\times_k X}}(\mathcal{O}_\Delta, \mathcal{F})) = 0 \tag{1.2.11.1}$$

for every coherent sheaf $\mathcal{F}$ on $X \times_k X$. The first equality is because $X \times_k X$ is affine. For the second equality we are going to show $\mathcal{E}xt^{d+1}_{\mathcal{O}_{X\times_k X}}(\mathcal{O}_\Delta, \mathcal{F}) = 0$ which can be checked locally, and locally $\mathcal{O}_\Delta$ is defined by a regular sequence of length $\leq d$ so the result follows.

It follows from (1.2.11.1) that the projective dimension of the module $\mathcal{O}_\Delta$ is $\leq d$ so Example 1.1.4 implies that $\mathcal{O}_\Delta \in \langle \mathcal{O}_{X\times_k X}\rangle_{d+1} = \langle \mathcal{O}_X \boxtimes \mathcal{O}_X\rangle_{d+1}$ and we are done. □

The Beilinson resolution of the diagonal can be used to show $\mathrm{Ddim}(\mathbf{P}^n_k) = n$. The same argument works slightly more generally.

Lemma 1.2.12. *Let X be a Severi–Brauer variety over a field k. Then* $\mathrm{Ddim}(X) = \dim(X)$.

Proof. Set $n = \dim(X)$. There is a line bundle $\mathcal{L}$ on $X \times_k X$ which after base change to $\bar{k}$ becomes identified with the line bundle $\mathcal{O}(-1) \boxtimes \mathcal{O}(1)$ on $\mathbf{P}^n_{\bar{k}} \times_{\bar{k}} \mathbf{P}^n_{\bar{k}}$ and an exact sequence

$$pr_1^*\Omega^1_{X/k} \otimes \mathcal{L} \xrightarrow{\varphi} \mathcal{O}_{X\times_k X} \to \mathcal{O}_\Delta \to 0$$

of coherent sheaves on $X \times_k X$. See [Ber05, 1412]. For dimension reasons, the Koszul complex of φ is exact, so that $\mathcal{O}_\Delta \in \langle \bigoplus_{i=0}^n pr_1^*\Omega^i_{X/k} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes i}\rangle_{n+1}$. Note that the summands become box products of vector bundles after base change to $\bar{k}$.

There is a vector bundle $\mathcal{E}$ on X which becomes isomorphic to $\mathcal{O}(1)^{\oplus n+1}$ after base change to $\bar{k}$ (it is the bundle $F(X)$ of [Kol16, 8]). It follows from [Kol16, Lemma 8] that $\mathcal{L}$ is a direct summand of $\mathcal{E}^\vee \boxtimes \mathcal{E}$ on $X \times_k X$, and therefore $\mathcal{L}^{\otimes i}$ is a direct summand of $(\mathcal{E}^\vee)^{\otimes i} \boxtimes \mathcal{E}^{\otimes i}$. Then we are done since

$$\mathcal{O}_\Delta \in \langle \bigoplus_{i=0}^n pr_1^*\Omega^i_{X/k} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes i}\rangle_{n+1} \subset \langle \bigoplus_{i=0}^n \left(\Omega^i_{X/k} \otimes_{\mathcal{O}_X} (\mathcal{E}^\vee)^{\otimes i}\right) \boxtimes \mathcal{E}^{\otimes i}\rangle_{n+1}.$$

□

1.3 Resolution by coherent sheaves

In this section, X denotes a Noetherian algebraic space and $g : U \to X$ is a surjective étale morphism with U an affine scheme. Then by Zariski's Main Theorem [Sta22, Tag 082K] there is a diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & \overline{U} \\ & \searrow^{g} & \downarrow{\scriptstyle \overline{g}} \\ & & X \end{array} \tag{1.3.0.1}$$

where j is an open immersion and $\overline{g}$ is finite. Let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf cutting out the complement of U in $\overline{U}$. We will set $\mathcal{F}_n = \overline{g}_*(\mathcal{I}^n)$ so that we have a system of coherent sheaves

$$\cdots \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_0 \tag{1.3.0.2}$$

on X. In this section we are going to get as much mileage as we can from the following version of Deligne's formula.

Lemma 1.3.1. *There is a system of compatible morphisms $\psi_n : \mathcal{O}_U \to Lg^*(\mathcal{F}_n)$ such that for K an object of $D_{QCoh}(\mathcal{O}_X)$, the map*

$$\mathrm{colim}_n \mathrm{Hom}(\mathcal{F}_n, K) \to R\Gamma(U, Lg^*(K)) = \mathrm{Hom}(\mathcal{O}_U, Lg^*(K)) \tag{1.3.1.1}$$

taking $\varphi : \mathcal{F}_n \to K$ to $Lg^*(\varphi) \circ \psi_n$, is an isomorphism.

Lemma 1.3.2. *Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. Then there is a morphism*

$$\bigoplus_{i \in I} \mathcal{F}_{n_i}[d_i] \to K$$

which is surjective on cohomology sheaves. If K has only one cohomology sheaf in degree zero, then we may take $d_i = 0$ for all i. If K is in $D^b_{coh}(\mathcal{O}_X)$ then we may take the index set I to be finite.

Proof. We produce a morphism $\bigoplus_{i \in I} \mathcal{F}_{n_i} \to K$ which is surjective on H^0 and the general result follows by shifting. Choose a morphism $\bigoplus_{i \in I} \mathcal{O}_U \to Lg^*(K)$ which is surjective on H^0. Then each $\mathcal{O}_U \to Lg^*(K)$ factors as $Lg^*(\varphi_i) \circ \psi_{n_i}$ for sufficiently large n_i. We claim that the morphism

$$\bigoplus_{i \in I} \mathcal{F}_{n_i} \xrightarrow{\varphi_i} K \tag{1.3.2.1}$$

is surjective on H^0. This can be checked after pullback along the flat morphism g, and this follows from the fact that the precomposition with $\bigoplus_{i\in I}\mathcal{O}_U \to \bigoplus_{i\in I}\mathcal{F}_{n_i}$ was surjective by choice. This completes the proof. It will be useful later that a morphism obtained from (1.3.2.1) by increasing each n_i remains surjective. □

In the rest of this section we prove some more technical properties of the objects $\mathcal{F}_n$. We suggest that the reader skip the rest of the section and return later as needed.

Lemma 1.3.3. *Assume X is an algebraic space of finite type over a field k. Let K be an object of $D_{QCoh}(\mathcal{O}_{X\times_k X})$. Then there is a morphism*

$$\bigoplus_{i\in I}\mathcal{F}_{m_i}\boxtimes\mathcal{F}_{n_i}[d_i]\to K$$

which is surjective on cohomology sheaves. If K has only one cohomology sheaf in degree zero, then we may take $d_i=0$ for all i. If K is in $D^b_{coh}(\mathcal{O}_{X\times_k X})$ then we may take the index set I to be finite.

Proof. We apply Lemma 1.3.2 with X replaced by $X\times_k X$ and the diagram (1.3.0.1) replaced by

$$\begin{array}{ccc} U\times_k U & \xrightarrow{j\times j} & \overline{U}\times_k\overline{U} \\ & \searrow^{g\times g} & \downarrow{\scriptstyle \overline{g}\times\overline{g}} \\ & & X\times_k X. \end{array}$$

Then for an ideal sheaf for the complement of $U\times_k U$ in $\overline{U}\times_k\overline{U}$ we may take $\mathcal{J}=pr_1^{-1}\mathcal{I}\cdot\mathcal{O}_{X\times_k X}+pr_2^{-1}\mathcal{I}\cdot\mathcal{O}_{X\times_k X}$. Then Lemma 1.3.2 gives us the Lemma but with the coherent sheaves $(\overline{g}\times\overline{g})_*(\mathcal{J}^N)$ instead of $\mathcal{F}_m\boxtimes\mathcal{F}_n$. It therefore suffices to show every $(\overline{g}\times\overline{g})_*(\mathcal{J}^N)$ is the quotient of a direct sum of sheaves $\mathcal{F}_m\boxtimes\mathcal{F}_n$. Since $\mathcal{F}_m\boxtimes\mathcal{F}_n=(\overline{g}\times\overline{g})_*(\mathcal{I}^m\boxtimes\mathcal{I}^n)$ and the functor $(\overline{g}\times\overline{g})_*$ is exact on the category of quasi-coherent sheaves, it suffices to show each $\mathcal{J}^N$ is a quotient of a direct sum of sheaves of the form $\mathcal{I}^m\boxtimes\mathcal{I}^n$. This is because

$$\mathcal{J}^N=\sum_{m+n=N}pr_1^{-1}(\mathcal{I}^m)\cdot\mathcal{O}_{X\times_k X}\cdot pr_2^{-1}(\mathcal{I}^n)\cdot\mathcal{O}_{X\times_k X}$$

and $pr_1^{-1}(\mathcal{I}^m)\cdot\mathcal{O}_{X\times_k X}\cdot pr_2^{-1}(\mathcal{I}^n)\cdot\mathcal{O}_{X\times_k X}$ is a quotient of $\mathcal{I}^m\boxtimes\mathcal{I}^n$. □

Lemma 1.3.4. *Assume X is an algebraic space which is proper over an Artinian ring and X has depth ≥ 1 at every closed point. Then for any coherent sheaf $\mathcal{F}$ on X we have*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}_n)=0$$

for $n\gg 0$.

Proof. We have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}_n) = \mathrm{Hom}_{\mathcal{O}_{\overline{U}}}(\overline{g}^*\mathcal{F},\mathcal{I}^n)$ and since $\mathcal{I}^n \subset \mathcal{I}^{n-1}$ we see that the Hom groups form a descending chain of finite modules over an Artinian ring. It is therefore enough to show that their intersection is zero. An element of the intersection is represented by a morphism $\overline{g}^*\mathcal{F} \to \mathcal{O}_{\overline{U}}$ with image $\mathcal{G}$ contained in $\bigcap_{n\geq 0}\mathcal{I}^n$. Then $\mathcal{G}$ is a coherent sheaf on $\overline{U}$ which by Krull's intersection theorem has support contained in U. But then $\mathrm{Supp}(\mathcal{G})$ is both proper and affine (over our Artinian base) and therefore it consists of finitely many closed points. If u is one of these closed points, then $\mathcal{O}_{U,u}$ has a submodule supported at the closed point, contradicting the fact that $\mathcal{O}_{U,u}$ has depth ≥ 1 (here we use that $\mathcal{O}^h_{X,g(u)} \to \mathcal{O}^h_{U,u}$ is flat and $g(u)$ is closed so $\mathrm{depth}(\mathcal{O}^h_{X,g(u)}) \geq 1$ by assumption). Thus $\mathcal{G} = 0$ and we are done. □

Chapter 2: Countable Rouquier dimension of regular algebraic spaces

The main goal of this chapter is to prove the following weak version of Orlov's Conjecture 1.2.4.

Theorem 2.0.1. *Let X be a regular (Noetherian) algebraic space. Then* $\operatorname{CRdim}(X) \leq \dim(X)$*. If X is of finite type over an uncountable field, then the two are equal.*

This is proved as Propositions 2.1.3 and 2.1.4 below. We will see in the next chapter that this result implies Theorem 3.4.1, an interesting consequence of Orlov's conjecture which was previously unknown. Note that we have seen in Example 1.1.3 that the restriction to uncountable fields is necessary. Our proof will also show that Orlov's conjecture is true in the quasi-affine case.

Theorem 2.0.2. *Let X be a regular (Noetherian) quasi-affine scheme. Then* $\operatorname{Rdim}(X) \leq \dim(X)$*. If X is of finite type over a field, then the two are equal.*

In the course of proving Theorem 2.0.1, we prove:

Theorem 2.0.3. *Let X be a regular (Noetherian) algebraic space of dimension $d < \infty$. Let $K_0 \to K_1 \to \cdots \to K_{d+1}$ be morphisms in $D^b_{coh}(\mathcal{O}_X)$ whose induced maps on cohomology sheaves vanish. Then the composition $K_0 \to K_{d+1}$ is zero.*

The argument proving this occurs in the proof of [Chr98, Proposition 4.5], so it is possible that Theorem 2.0.3 is known to experts but we have not found a reference.

This chapter is based on the author's paper [Ola21b].

2.1 Proofs

The majority of this section is devoted to proving Proposition 2.1.3, whose proof is modeled on the simpler Example 1.1.4 above. We suggest studying that example first.

Let $\mathcal{A}$ be an abelian category and $\varphi : K \to L$ a map in $D(\mathcal{A})$. We would like to know whether $\varphi = 0$. An obvious necessary condition is that $H^n(\varphi) : H^n(K) \to H^n(L)$ be zero for all n, but this is not sufficient: Consider any non-zero map $A \to A[1]$ with $A \in \mathcal{A}$. However it turns out that if all the maps $H^n(\varphi)$ vanish, we may associate to φ a sequence of Ext classes $\xi_n \in \operatorname{Ext}^1_{\mathcal{A}}(H^n(K), H^{n-1}(L))$ whose vanishing gives another necessary condition for φ to be zero. One may expect that if all the

ξ_n vanish then we could associate to φ a sequence of Ext^2-classes, and so forth. This is not true and the failure is caused by non-zero differentials in a spectral sequence. The following proposition gives a corrected statement. We let FAb denote the category of filtered abelian groups and Ab the category of abelian groups.

Proposition 2.1.1. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for each $K, L \in D^b(\mathcal{A})$ there is a decreasing filtration F on $\mathrm{Hom}_{D(\mathcal{A})}(K, L)$, which is natural in the sense that it arises from a functor $D^b(\mathcal{A})^{opp} \times D^b(\mathcal{A}) \to FAb$ whose composition with the forgetful functor $FAb \to Ab$ is the* Hom*-functor, and which satisfies for $K, L, M \in D^b(\mathcal{A})$:*

1. *$F^0\mathrm{Hom}_{D(\mathcal{A})}(K, L) = \mathrm{Hom}_{D(\mathcal{A})}(K, L)$ and $F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L) = 0$ for $p \gg 0$.*
2. *If $f \in F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L)$ and $g \in F^q\mathrm{Hom}_{D(\mathcal{A})}(L, M)$ then $g \circ f \in F^{p+q}\mathrm{Hom}_{D(\mathcal{A})}(K, M)$.*
3. *$F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L)/F^{p+1}\mathrm{Hom}_{D(\mathcal{A})}(K, L)$ is a subquotient of $\prod_{n \in \mathbf{Z}} \mathrm{Ext}^p_{\mathcal{A}}(H^n(K), H^{n-p}(L))$.*
4. *$F^1\mathrm{Hom}_{D(\mathcal{A})}(K, L) = \{\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(K, L) : H^n(\varphi) = 0$ for all $n \in \mathbf{Z}\}$.*

Remark 2.1.2. The proposition gives an enrichment of $D^b(\mathcal{A})$ in the symmetric monoidal category of filtered abelian groups; recall that the tensor product in this category is defined by $F^k(A \otimes B) = \sum_{i+j=k} \mathrm{Im}(F^iA \otimes F^jB \to A \otimes B)$.

We prove Proposition 2.1.1 in Appendix A. The proof uses a spectral sequence

$$E_1^{p,q} = \prod_{n \in \mathbf{Z}} \mathrm{Ext}^{2p+q}(H^n(K), H^{n-p}(L)) \implies \mathrm{Ext}^{p+q}(K, L),$$

and is not difficult but we need to use the definition of the spectral sequence to prove 2.

Proof of Theorem 2.0.3. Note that $D^b_{coh}(\mathcal{O}_X)$ is a full subcategory of $D(\mathrm{QCoh}(\mathcal{O}_X))$ by [Sta22, Tag 09TN] and $\mathrm{QCoh}(\mathcal{O}_X)$ is an abelian category with enough injectives by [Sta22, Tag 077V]. Thus we may use the filtration

$$\mathrm{Hom}(K_0, K_{d+1}) = F^0 \supset F^1 \supset F^2 \supset \cdots$$

of Proposition 2.1.1. We have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > d$ and $\mathcal{F}, \mathcal{G}$ coherent sheaves on X. This is by [Sta22, Tag 0FZ3] in the case of schemes but the same proof works for spaces. Therefore by 3 of Proposition 2.1.1, $F^{d+1} = F^{d+2} = \cdots$. Since $F^p = 0$ for $p \gg 0$, in fact $F^{d+1} = 0$. Then by 4 each $K_i \to K_{i+1}$ is in $F^1\mathrm{Hom}(K_i, K_{i+1})$, so that by 2 the composition $K_0 \to K_{d+1}$ is in $F^{d+1}\mathrm{Hom}(K_0, K_{d+1}) = 0$ and we are done. ☐

Proposition 2.1.3. *Let X be a regular (Noetherian) algebraic space. Then*

$$\mathrm{CRdim}(X) \leq \dim(X).$$

If furthermore X possesses an ample line bundle $\mathcal{L}$, then

$$D^b_{coh}(\mathcal{O}_X) = \langle \{\mathcal{L}^{\otimes n}\}_{n<0} \rangle_{\dim(X)+1}.$$

Proof. We may assume X has finite dimension d since otherwise there is nothing to prove. Let $\mathcal{F}_n$ be as in (1.3.0.2) in the general case and $\mathcal{L}^{\otimes -n}$ if $\mathcal{L}$ is an ample line bundle on X. Let $K \in D^b_{coh}(\mathcal{O}_X)$. We are going to show $D^b_{coh}(\mathcal{O}_X) = \langle \{\mathcal{F}_n\}_{n>0} \rangle_{d+1}$. Set $K = K_0$. Choose a finite set I and a morphism $\bigoplus_{i \in I} \mathcal{F}_{m_i}[n_i] \to K$ which is surjective on cohomology sheaves. This exists by Lemma 1.3.2 in the first case and by [Ola21b, Lemma 2] in the second. let K_1 be the cone. Note that $K_0 \to K_1$ is zero on cohomology sheaves by construction, and its cone is in $\langle \{\mathcal{F}_n\}_{n>0} \rangle_1$. Now repeat the process with $K = K_1$ and so on to obtain a sequence

$$K_0 \to K_1 \to \cdots \to K_{d+1}$$

such that each $K_i \to K_{i+1}$ is zero on cohomology sheaves and has cone in $\langle \{\mathcal{F}_n\}_{n>0} \rangle_1$. Thus $K_0 \to K_{d+1}$ is zero by Theorem 2.0.3. We will prove by induction that the cone of $K_0 \to K_i$ is in $\langle \{\mathcal{F}_n\}_{n>0} \rangle_i$. For $i = 1$ this is known and for $i = d+1$ this completes the proof: Since $K = K_0 \to K_{d+1}$ is zero the cone is isomorphic to $K_{d+1} \oplus K[1]$ and the category $\langle \{\mathcal{F}_n\}_{n>0} \rangle_{d+1}$ is closed under direct summands and shifts.

So assume known that the cone of $K_0 \to K_i$ is in $\langle \{\mathcal{F}_n\}_{n>0} \rangle_i$. Then by the octahedral axiom there is a distinguished triangle

$$C \to D \to E \to C[1]$$

with C a cone of $K_0 \to K_i$, D a cone of $K_0 \to K_{i+1}$, and E a cone of $K_i \to K_{i+1}$. Since $C \in \langle \{\mathcal{F}_n\}_{n>0} \rangle_i$ and $E \in \langle \{\mathcal{F}_n\}_{n>0} \rangle_1$ it follows that $D \in \langle \{\mathcal{F}_n\}_{n>0} \rangle_{i+1}$, as needed. □

Proof of Theorem 2.0.2. The inequality $\mathrm{Rdim}(X) \leq \dim(X)$ follows from the second statement in Proposition 2.1.3 since $\mathcal{O}_X$ is an ample line bundle on X and all its tensor powers are itself. The reverse inequality in the finite type over a field case is Lemma 1.2.9. □

To complete the proof of Theorem 2.0.1 it remains only to give the following lower bound, whose proof closely follows that of [Rou08, Proposition 7.17].

Proposition 2.1.4. *Let k be an uncountable field. Let X be a reduced algebraic space of finite type over k. Then* $\mathrm{CRdim}(X) \geq \dim(X)$.

Proof. Let $n = \mathrm{CRdim}(X)$ and let $\{E_i\}_{i\in I}$ be a countable family of objects such that $D^b_{coh}(X) = \langle \{E_i\}_{i\in I}\rangle_{n+1}$. Consider the set of closed points $x \in X$ such that for every $i \in I$, the cohomology modules of E_i are locally free at x. Since a variety over k is not a countable union of proper closed subsets ([Liu02, Exercise 2.5.10]), the set contains a closed point x such that $\dim_x(X) = \dim(X)$ – note that our X is an algebraic space but to find x we may restrict our attention to a dense open which is a scheme, so we may even assume X is a scheme in a neighborhood of x. Then we have $(E_i)_x \in \langle \mathcal{O}_{X,x}\rangle_1$ for each i since a complex with projective cohomology modules is decomposable, hence

$$\kappa(x) \in \langle \{(E_i)_x\}_{i\in I}\rangle_{n+1} \subset \langle \mathcal{O}_{X,x}\rangle_{n+1},$$

hence $n \geq \dim(X)$ by [Rou08, Proposition 7.14]. □

Chapter 3: Fully faithful functors are Fourier–Mukai transforms

In this chapter we generalize Orlov's Theorem [Orl97, Theorem 2.2] to smooth proper algebraic spaces over a field. Namely, we prove

Theorem 3.0.1. *Let k be a field. Let X, Y be smooth proper algebraic spaces over k. Let $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ be a fully faithful, exact, k-linear functor. Then there is an object $E \in D^b_{coh}(\mathcal{O}_{X\times_k Y})$ such that F is isomorphic to the Fourier–Mukai transform*

$$\Phi_E(K) = \mathbf{R}pr_{2*}(\mathbf{L}pr_1^* K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_k Y}} E).$$

The object E is unique up to isomorphism.

This chapter is based on [Ola20] which proves the theorem when X, Y are assumed to be schemes. In Section 3.1 we produce a candidate for E such that $\Phi_E(\mathcal{F}) \cong F(\mathcal{F})$ whenever $\mathcal{F}$ is a skyscraper sheaf at a closed point. In Section 3.2 we show that this essentially implies that Φ_E and F agree on the full subcategory $Coh(\mathcal{O}_X) \subset D^b_{coh}(\mathcal{O}_X)$. In Section 3.3, we show that fully faithful functors agreeing on coherent sheaves are isomorphic. Finally, in Section 3.4, we prove Theorem 3.4.1, one of the main results of the thesis as discussed in the introduction.

3.1 Producing a kernel

In this section k is a field, X and Y are smooth proper algebraic spaces over k, and $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ is an exact, k-linear functor which we will soon assume to be fully faithful. The goal is to prove Proposition 3.1.2. The following Lemma will play a similar role in our construction to the role [Orl97, Lemma 2.4] plays in Orlov's.

Lemma 3.1.1. *F is bounded: There is $m > 0$ such that for every coherent sheaf $\mathcal{F}$ on X, $F(\mathcal{F})$ has nonzero cohomology sheaves only in degrees $[-m, m]$.*

Proof. This is a consequence of our proof that $D^b_{coh}(\mathcal{O}_X)$ has a strong generator (Lemma 1.2.9). See [Sta22, Tag 0FZ8] for the details in the scheme case. □

By Lemma 1.3.3 we may construct a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal

$\Delta \subset X \times_k X$

$$\cdots \to \mathcal{E}_n \boxtimes \mathcal{F}_n \to \cdots \to \mathcal{E}_0 \boxtimes \mathcal{F}_0 \to \mathcal{O}_\Delta \to 0 \tag{3.1.1.1}$$

with $\mathcal{E}_i, \mathcal{F}_i$ coherent $\mathcal{O}_X$-modules. Then by applying F to the second box tensor factors, we obtain a complex

$$\cdots \to \mathcal{E}_n \boxtimes F(\mathcal{F}_n) \to \cdots \to \mathcal{E}_0 \boxtimes F(\mathcal{F}_0) \tag{3.1.1.2}$$

(use the Künneth formula to get the differentials). By the Künneth formula,

$$\begin{aligned}
\operatorname{Ext}^k_{\mathcal{O}_{X\times Y}}(\mathcal{E}_i \boxtimes F(\mathcal{F}_i), \mathcal{E}_j \boxtimes F(\mathcal{F}_j)) &= \bigoplus_{m+n=k} \operatorname{Ext}^m_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \operatorname{Ext}^n_{\mathcal{O}_Y}(F(\mathcal{F}_i), F(\mathcal{F}_j)) \\
&= \bigoplus_{m+n=k} \operatorname{Ext}^m_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) \otimes_k \operatorname{Ext}^n_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{F}_j),
\end{aligned}$$

where we have used fully faithfulness for the second equality. This vanishes if $k < 0$, so there is no obstruction to integrating the complex (3.1.1.2) to a right Postnikov system:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{+1} & E_2 & \xleftarrow{+1} & & E_1 & \xleftarrow{+1} & & E_0 \\
& & \searrow & \nearrow & & \searrow & \nearrow & & \searrow^{\cong} \\
\cdots & \longrightarrow & & \mathcal{E}_2 \boxtimes F(\mathcal{F}_2) & \longrightarrow & & \mathcal{E}_1 \boxtimes F(\mathcal{F}_1) & \longrightarrow & \mathcal{E}_0 \boxtimes F(\mathcal{F}_0)
\end{array} \tag{3.1.1.3}$$

The triangles with an arrow of degree +1 are distinguished and the other triangles commute. The objects E_n are unique up to non-unique isomorphism. See [Sta22, Tag 0D7Y]. Note that we use right Postnikov systems instead of left systems as in [Orl97]. This is because our complexes are unbounded to the left.

We also have a trivial Postnikov system

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{+1} & K_2 & \xleftarrow{+1} & & K_1 & \xleftarrow{+1} & & K_0 \\
& & \searrow & \nearrow & & \searrow & \nearrow & & \searrow^{\cong} \\
\cdots & \longrightarrow & & \mathcal{E}_2 \boxtimes \mathcal{F}_2 & \longrightarrow & & \mathcal{E}_1 \boxtimes \mathcal{F}_1 & \longrightarrow & \mathcal{E}_0 \boxtimes \mathcal{F}_0,
\end{array} \tag{3.1.1.4}$$

with K_n the n^{th} naive truncation of the complex (3.1.1.1). Thus K_n has cohomology sheaves only in degrees 0 and $-n$. Since $X \times_k Y$ has finite homological dimension, for $n \gg 0$ we have $K_n \cong \mathcal{O}_\Delta \oplus H^{-n}(K_n)[n]$. Our task in the remainder of the section is to show that E_n has a similar decomposition $E \oplus C_n$ where E is the desired kernel and C_n slides off to $-\infty$.

Let us compute $\Phi_{E_n}(\mathcal{F})$ for $\mathcal{F}$ a coherent $\mathcal{O}_X$-module whose support has dimension 0: Apply the functor

$$K \mapsto \mathbf{R}pr_{2*}(\mathbf{L}pr_1^*(\mathcal{F}) \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} K)$$

to the entire diagram (3.1.1.3) to see that $\Phi_{E_n}(\mathcal{F})$ is an n^{th} convolution of the complex

$$\Gamma(X, \mathcal{F} \otimes \mathcal{E}_\bullet) \otimes_k F(\mathcal{F}_\bullet) \tag{3.1.1.5}$$

(the functor Γ is underived since $\mathcal{F}$ has 0-dimensional support).

On the other hand, we can apply the functor

$$K \mapsto F(\mathbf{R}pr_{2*}(\mathbf{L}pr_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} K))$$

to the entire diagram (3.1.1.4). This transforms the second row of (3.1.1.4) into the complex (3.1.1.5). Thus, for $n \gg 0$, another n^{th} convolution of the complex (3.1.1.5) is

$$\begin{aligned} &F(\mathbf{R}pr_{2*}(\mathbf{L}pr_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} K_n)) \\ &\qquad = F(\mathbf{R}pr_{2*}(\mathbf{L}pr_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta)) \oplus F(\mathbf{R}pr_{2*}(\mathbf{L}pr_1^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} H^{-n}(K_n)))[n] \\ &\qquad = F(\mathcal{F}) \oplus F(\mathcal{K}_n)[n], \end{aligned}$$

where $\mathcal{K}_n$ is a coherent sheaf on X.

Therefore, by uniqueness of convolutions (note that negative Ext's vanish between the terms of the complex (3.1.1.5) since F is fully faithful), we obtain isomorphisms

$$\Phi_{E_n}(\mathcal{F}) \cong F(\mathcal{F}) \oplus F(\mathcal{K}_n)[n].$$

By Lemma 3.1.1, we deduce that there is $m > 0$ so that for every $\mathcal{F}$ coherent with 0-dimensional support and $n \gg 0$, $\Phi_{E_n}(\mathcal{F})$ has cohomology sheaves only in degrees $[-n-m, -n+m] \cup [-m, m]$. But knowing this for every such $\mathcal{F}$ implies that E_n has cohomology sheaves only in degrees $[-n-m, -n+m] \cup [-m, m]$ (see [Sta22, Tag 0FZ9] for details). Thus if we take n to be very large, since $X \times_k Y$ has finite homological dimension, we will have a decomposition $E_n = E \oplus C_n$, where E has cohomology sheaves in degrees $[-m,m]$ and C_n has cohomology sheaves in degrees $[-n-m,-n+m]$. Then it follows that $\Phi_E(\mathcal{F}) \cong F(\mathcal{F})$ and $\Phi_{C_n}(\mathcal{F}) \cong F(\mathcal{K}_n)[n]$. Thus we have proved:

Proposition 3.1.2. *Let X, Y be smooth proper algebraic spaces over a field k. Let $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ be a fully faithful functor which is exact and k-linear. Then there is an object E of $D^b_{coh}(\mathcal{O}_{X\times_k Y})$ such that $F(\mathcal{F}) \cong \Phi_E(\mathcal{F})$ for every coherent sheaf $\mathcal{F}$ on X whose support has dimension 0.*

3.2 Fully faithful functors agreeing on points

The following lemma should be compared with [Orl97, Lemma 2.15].

Lemma 3.2.1. *Let X, Y be smooth proper algebraic spaces over a field k. Let $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ an exact, k-linear functor. Suppose*

$$F : \mathrm{Ext}^*_{\mathcal{O}_X}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \to \mathrm{Ext}^*_{\mathcal{O}_Y}(F(\mathcal{O}_{x_1}), F(\mathcal{O}_{x_2}))$$

is an isomorphism for every pair of closed points $x_1, x_2 \in X$. Then F is fully faithful.

Proof. By Lemma 1.2.10, F has exact k-linear left and right adjoints R and L. By our assumption,

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{O}_X}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) &= \mathrm{Ext}^*_{\mathcal{O}_X}(F(\mathcal{O}_{x_1}), F(\mathcal{O}_{x_2}))\\ &= \mathrm{Ext}^*_{\mathcal{O}_X}(\mathcal{O}_{x_1}, RF(\mathcal{O}_{x_2}))\end{aligned}$$

for every pair x_1, x_2. Since the sheaves $\mathcal{O}_{x_1}$ form a spanning class, we conclude that the unit $\mathbf{1}_{D^b_{coh}(X)} \to RF$ is an isomorphism on the objects $\mathcal{O}_x$. Now let K be an object of $D^b_{coh}(\mathcal{O}_X)$ and $x \in X$ a closed point. Then

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{O}_X}(LF(K), \mathcal{O}_x) &= \mathrm{Ext}^*_{\mathcal{O}_Y}(F(K), F(\mathcal{O}_x))\\ &= \mathrm{Ext}^*_{\mathcal{O}_X}(K, RF(\mathcal{O}_x))\\ &= \mathrm{Ext}^*_{\mathcal{O}_X}(K, \mathcal{O}_x),\end{aligned}$$

induced by the co-unit $LF \to \mathbf{1}_{D^b_{coh}(\mathcal{O}_X)}$. Again, since the objects $\mathcal{O}_x$ form a spanning class, we conclude that $LF(K) \to K$ is an isomorphism for every K, so F is fully faithful. □

Lemma 3.2.2. *Let X, Y be smooth proper algebraic spaces over a field k. Let $F, G : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$ be exact, k-linear functors. Suppose F is fully faithful and $F(\mathcal{O}_x) \cong G(\mathcal{O}_x)$ for every closed point $x \in X$. Then the essential image of G is contained in the essential image of F.*

Proof. By 1.2.10, G has an exact k-linear right adjoint R. Applying the same theorem to F shows that the essential image $\mathcal{A}$ of F is an admissible subcategory of $D^b_{coh}(\mathcal{O}_Y)$. Thus we only have to show that $G(K) \in {}^{\perp}(\mathcal{A}^{\perp})$ for every K in $D^b_{coh}(\mathcal{O}_X)$. So, suppose L is in $\mathcal{A}^{\perp}$. Then

$$\mathrm{Hom}(G(K), L) = \mathrm{Hom}(K, R(L)), \tag{3.2.2.1}$$

so it suffices to show $R(L) = 0$. Applying (3.2.2.1) to the objects $K = \mathcal{O}_x[i]$ gives

$$\mathrm{Ext}^*_{\mathcal{O}_X}(\mathcal{O}_x, R(L)) = \mathrm{Ext}^*_{\mathcal{O}_Y}(G(\mathcal{O}_x), L) = \mathrm{Ext}^*_{\mathcal{O}_Y}(F(\mathcal{O}_x), L) = 0$$

since $L \in \mathcal{A}^{\perp}$. But the objects $\mathcal{O}_x$ form a spanning class so it follows that $R(L) = 0$ as needed. □

Proposition 3.2.3. *Let X be a smooth proper algebraic space over a field k. Let $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_X)$ be an exact, k-linear functor. Suppose $F(\mathcal{F}) \cong \mathcal{F}$ for every coherent sheaf $\mathcal{F}$ on X with 0-dimensional support. Then F is an equivalence of categories. Furthermore, there exists a line bundle $\mathcal{L}$ on X and a k-linear automorphism $f : X \to X$ such that, if we denote by G the functor*

$$G(K) = \mathbf{L}f^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L},$$

then $F|_{Coh(\mathcal{O}_X)} \cong G|_{Coh(\mathcal{O}_X)}$.

Remark. In fact, since $F(\mathcal{O}_x) \cong \mathcal{O}_x$ for every closed point $x \in X$, the automorphism f must be the identity on underlying topological spaces. This often implies that f is the identity, see [Sta22, Tag 0G05]. This is irrelevant to us though: We care only that G is a Fourier–Mukai auto-equivalence.

Proof. Step 1: F is an equivalence of categories.

Applying Lemma 3 to the functors $\mathbf{1}_{D^b_{coh}(X)}$ and F, we see that F is essentially surjective. To prove that F is fully faithful, it suffices by Lemma 2 to prove that

$$F : \mathrm{Ext}^*_{\mathcal{O}_X}(\mathcal{O}_x, \mathcal{O}_y) \to \mathrm{Ext}^*_{\mathrm{O}_X}(F(\mathcal{O}_x), F(\mathcal{O}_y))$$

is an isomorphism for every pair of closed points x, y. It follows from our assumptions that both sides are trivial if $x \neq y$, so there is nothing to prove. If $x = y$, both sides are isomorphic to

$$\bigwedge\nolimits^*_{\mathcal{O}_x} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_x, \mathcal{O}_x),$$

and moreover, F is a k-algebra homomorphism. It will therefore suffice to show F is surjective in degrees 0 and 1 (or equivalently injective or bijective). In degree 0, every non-zero homomorphism $\mathcal{O}_x \to \mathcal{O}_x$ is an isomorphism, and functors preserve isomorphisms, so indeed F is injective in degree 0. Suppose given a non-zero extension class ξ of $\mathcal{O}_x$ by $\mathcal{O}_x$, represented by a non-split short exact sequence

$$0 \to \mathcal{O}_x \to \mathcal{E} \to \mathcal{O}_x \to 0$$

of coherent sheaves. Then $F(\mathcal{E})$ is a coherent sheaf, and $F(\xi)$ is represented by the short exact sequence

$$0 \to F(\mathcal{O}_x) \to F(\mathcal{E}) \to F(\mathcal{O}_x) \to 0$$

of coherent sheaves. Furthermore, we have abstract isomorphisms $F(\mathcal{O}_x) \cong \mathcal{O}_x$ and $F(\mathcal{E}) \cong \mathcal{E}$ by assumption. But now we simply note that an arbitrary extension $\mathcal{E}'$ of $\mathcal{O}_x$ by $\mathcal{O}_x$ is trivial if and only if $\mathcal{E}' \cong \mathcal{O}_x \oplus \mathcal{O}_x$. Thus $F(\mathcal{E}) \cong \mathcal{E} \ncong \mathcal{O}_x \oplus \mathcal{O}_x$, so $F(\xi) \neq 0$.

Step 2: $K \in D^b_{coh}(\mathcal{O}_X)$ is in the subcategory $Coh(\mathcal{O}_X)$ if and only if $F(K)$ is so.

It suffices to show for each coherent sheaf $\mathcal{F}$ on X that $F(\mathcal{F})$ is also a sheaf, since we may then apply the same argument to a quasi-inverse of F. Note that for every $K \in D^b_{coh}(\mathcal{O}_X)$ we have

$$\sup\{n : H^n(K) \neq 0\} = \sup\{n : \mathrm{Hom}(K[n], \mathcal{O}_x) \neq 0 \text{ for some closed point } x \in X\}$$

so by our hypotheses on F, the top cohomology sheaf of K lives in the same degree as the top cohomology sheaf of $F(K)$.

Let $\mathcal{F} \in Coh(\mathcal{O}_X)$. Then $F(\mathcal{F})$ lives in degrees ≤ 0. There is a distinguished triangle

$$\tau_{<0}(F(\mathcal{F})) \xrightarrow{\alpha} F(\mathcal{F}) \to H^0(F(\mathcal{F})) \xrightarrow{+1}$$

and since F is an equivalence, this triangle is isomorphic to F applied to a triangle

$$A \to \mathcal{F} \to B \xrightarrow{+1}.$$

Then since $F(A)$ lives in degrees < 0, the previous paragraph implies that so does A. But then $\mathrm{Hom}(A, \mathcal{F}) = 0$ so $\alpha = 0$. But this implies that $F(\mathcal{F})$ lives in degree zero, as needed.

Step 3: Conclude.

By Step 2, F restricts to a k-linear auto-equivalence of the subcategory $Coh(\mathcal{O}_X)$. But these are classified by Gabriel's Theorem [CG15]: They are all isomorphic to

$$\mathcal{F} \mapsto f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

for some k-linear automorphism f of X and line bundle $\mathcal{L}$ on X, as needed. □

3.3 Fully faithful functors agreeing on coherent sheaves

We recall the notion of almost ample set from [CS14, Section 2.2].

Definition 3.3.1. Let $\mathcal{A}$ be an abelian category. A set of objects $\{P_i\}_{i\in I}$ of $\mathcal{A}$ is *almost ample* if for any $A \in \mathcal{A}$ there is $i \in I$ such that:

1. There is an integer $k > 0$ and a surjection $P_i^{\oplus k} \to A$, and
2. $\mathrm{Hom}(A, P_i) = 0$.

Lemma 3.3.2. *Assume X is an algebraic space proper over an Artinian ring and X has depth ≥ 1 at every closed point. Then the set $\{\mathcal{F}_n\}_{n\geq 0}$ of objects of $Coh(\mathcal{O}_X)$ constructed in Section 1.3 is almost ample.*

Proof. Let $\mathcal{G}$ be a coherent sheaf on X. By Lemma 1.3.2 there is a surjection $\mathcal{F}_n^{\oplus k} \to \mathcal{G}$ for some n, k. By the last sentence of the proof we may increase n and still obtain a surjection. By Lemma 1.3.4, after increasing n sufficiently we may also ensure 2 holds. □

The reason for introducing almost ample sets is that Orlov's proof of [Orl97, Proposition 2.16] goes through almost verbatim with ample sequences replaced by almost ample sets, as was originally pointed out in [CS14, Proposition 3.3]. Here is the result:

Lemma 3.3.3. *Let $\mathcal{A}$ be an abelian category which has an almost ample set. Let $\mathcal{T}$ be a triangulated category. If $F, G : D^b(\mathcal{A}) \to \mathcal{T}$ are exact functors which are isomorphic when restricted to $\mathcal{A}$, and F is fully faithful, then $F \cong G$.*

See also [Sta22, Tag 0FZW]. We now put together Lemmas 3.3.2 and 3.3.3.

Proposition 3.3.4. *Let X be a Noetherian algebraic space. Let $\mathcal{T}$ be a triangulated category. Assume every connected component of X is either*

- *The spectrum of a field, or*
- *Proper over an Artinian ring and having depth ≥ 1 at every closed point.*

Then if $F, G : D^b_{coh}(\mathcal{O}_X) \to \mathcal{T}$ are exact functors which are isomorphic when restricted to $Coh(\mathcal{O}_X)$, and F is fully faithful, then $F \cong G$.

Remark 3.3.5. The conditions hold for example if X is reduced and proper over a field. Also note that if R is a local Artinian ring with residue field κ and X is an algebraic space of finite type and flat over R, then X has depth ≥ 1 at every closed point $\iff$ so does $X_0 = X \otimes_R \kappa$. This follows from [Sta22, Tag 00MF].

Proof. If X has connected components X_i, then $D^b_{coh}(\mathcal{O}_X)$ is the orthogonal direct sum of the subcategories $D^b_{coh}(\mathcal{O}_{X_i})$. Thus if we know the result for the (still fully faithful) compositions $D^b_{coh}(\mathcal{O}_{X_i}) \to D^b_{coh}(\mathcal{O}_X) \to \mathcal{T}$ we can deduce the result for $D^b_{coh}(\mathcal{O}_X) \to \mathcal{T}$. We may therefore assume X is connected. If X is the spectrum of a field, then we deduce the result from the fact that $D^b_{coh}(\mathcal{O}_X)$ is the orthogonal direct sum of the categories $Coh(\mathcal{O}_X)[n]$ just as above. If X is proper over an Artinian ring and has depth ≥ 1 at every closed point the result follows from Lemmas 3.3.2 and 3.3.3. □

Proof of Theorem. Uniqueness: Assume $F \cong \Phi_E$. Consider the functor $\Phi_{\mathcal{O}_\Delta \boxtimes E} : D^b_{coh}(\mathcal{O}_{X \times_k X}) \to D^b_{coh}(\mathcal{O}_{X \times_k Y})$, where Δ is the diagonal of X. A computation shows $E \cong \Phi_{\mathcal{O}_\Delta \boxtimes E}(\mathcal{O}_\Delta)$. Apply $\Phi_{\mathcal{O}_\Delta \boxtimes E}$ to the Postnikov system (3.1.1.4). On the one hand, since $K_n \cong \mathcal{O}_\Delta \oplus H^{-n}(K_n)[n]$ for $n \gg 0$, one has

$$\Phi_{\mathcal{O}_\Delta \boxtimes E}(K_n) \cong E \oplus \Phi_{\mathcal{O}_\Delta \boxtimes E}(H^{-n}(K_n))[n].$$

Since Fourier–Mukai transforms between smooth proper algebraic spaces are always bounded in the sense of Lemma 3.1.1, we obtain the formula

$$E \cong \tau_{\geq a} \Phi_{\mathcal{O}_\Delta \boxtimes E}(K_n)$$

for $n \gg 0$ (here a is any integer small enough that $E \cong \tau_{\geq a} E$).

On the other hand, $\Phi_{\mathcal{O}_\Delta \boxtimes E}(K_n)$ is an n^{th} convolution of the complex

$$\Phi_{\mathcal{O}_\Delta \boxtimes E}(\mathcal{E}_\bullet \boxtimes \mathcal{F}_\bullet) \cong \mathcal{E}_\bullet \boxtimes F(\mathcal{F}_\bullet).$$

This is the complex (3.1.1.3). Recall that it is defined in terms of F alone (not E) and its convolutions E_n are unique as F is fully faithful. Thus for $n \gg 0$ we have $E \cong \tau_{\geq a} E_n$ and this proves uniqueness.

Existence: By Proposition 3.1.2, there is $E \in D^b_{coh}(\mathcal{O}_{X \times_k Y})$ such that $F(\mathcal{F}) \cong \Phi_E(\mathcal{F})$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ with 0-dimensional support. By Lemma 3.2.2, the essential image of Φ_E is contained in the essential image of F, thus the functor $F^{-1} \circ \Phi_E$ makes sense. By combining Propositions 3.2.3 and 3.3.4, we obtain an isomorphism $F^{-1} \circ \Phi_E \cong G$ where G is a Fourier–Mukai auto-equivalence of $D^b_{coh}(\mathcal{O}_X)$. But then $F \cong \Phi_E \circ G^{-1}$, which is Fourier–Mukai. □

3.4 Fully faithful functors and dimension

We can now prove a result promised in the introduction.

Theorem 3.4.1. *Let k be a field. Let X, Y be smooth proper algebraic spaces over k. Assume there exists a fully faithful, exact, k-linear functor $F : D^b_{coh}(\mathcal{O}_X) \to D^b_{coh}(\mathcal{O}_Y)$. Then* $\dim(X) \leq \dim(Y)$.

Proof. Choose any uncountable extension field K/k. By 3.0.1, F is the Fourier–Mukai transform with respect to a kernel $E \in D^b_{coh}(\mathcal{O}_{X \times_k Y})$. Then E_K gives rise to a functor $F_K : D^b_{coh}(\mathcal{O}_{X_K}) \to D^b_{coh}(\mathcal{O}_{Y_K})$ which remains fully faithful: The fact that F is fully faithful may be rephrased as $R \circ F \cong \mathrm{id}$ where R is the right adjoint of F. By the calculus of kernels, this may be rephrased as the existence of an isomorphism

$$\mathbf{R}pr_{13*}(\mathbf{L}pr_{12}^*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_k Y \times_k X}} \mathbf{L}pr_{23}^*(E')) \cong \mathcal{O}_\Delta \tag{3.4.1.1}$$

where $E' = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_k Y}}(E, \mathbf{L}pr_1^*(\omega_X))[\dim(X)]$, viewed as an object of $D^b_{coh}(\mathcal{O}_{Y \times_k X})$, is the kernel of R, and (3.4.1.1) remains valid upon base change to K.

Now F_K is the inclusion of an admissible subcategory by Lemma 1.2.10. We have $\mathrm{CRdim}(D^b_{coh}(\mathcal{O}_{X_K})) = \dim(X_K) = \dim(X)$ by Theorem 2.0.1 and similarly for Y. Thus by Lemma 1.1.5, we have

$$\dim(X) = \mathrm{CRdim}(D^b_{coh}(X_K)) \leq \mathrm{CRdim}(D^b_{coh}(Y_K)) = \dim(Y),$$

as needed. □

Chapter 4: Diagonal dimension of curves

In this chapter, we compute the diagonal dimension of smooth curves, thus proving [EL21, Conjecture 4.13]. The result is

Theorem 4.0.1. *Let X be a finite type smooth separated algebraic space of dimension 1 over a field. Then* $\mathrm{Ddim}(X) = 1$ *if* $H^1(X, \mathcal{O}_X) = 0$ *and* $\mathrm{Ddim}(X) = 2$ *otherwise.*

Note that a connected component of such an X is automatically either an affine or projective variety. Also by Lemma 1.2.9 and Proposition 1.2.3, we know that

$$1 \leq \mathrm{Ddim}(X) \leq 2.$$

The main content is that if X is a smooth projective geometrically connected curve over a field whose genus is at least one, then $\mathrm{Ddim}(X) = 2$. The following lemma explains what this means in more down to earth terms.

Lemma 4.0.2. *Let X be a finite type smooth separated algebraic space of dimension 1 over a field. Then* $\mathrm{Ddim}(X) = 1$ *if and only if there exist objects E, F, G, H of $D^b_{coh}(\mathcal{O}_X)$ and a morphism $E \boxtimes F \to G \boxtimes H$ in $D^b_{coh}(\mathcal{O}_{X \times_k X})$ such that $\mathcal{O}_\Delta$ is a direct summand of its cone.*

Proof. If there exist such E, F, G, H and $E \boxtimes F \to G \boxtimes H$ then $\mathcal{O}_\Delta \in \langle (E \oplus G) \boxtimes (F \oplus H) \rangle_2$, hence $1 \leq \mathrm{Ddim}(X) \leq 1$.

Conversely, if $\mathrm{Ddim}(X) = 1$, choose perfect complexes K, L on X such that $\mathcal{O}_\Delta \in \langle K \boxtimes L \rangle_2$. This means that $\mathcal{O}_\Delta$ is a direct summand of the cone of a morphism

$$\varphi : \bigoplus_{i \in I finite} K \boxtimes L[m_i] \to \bigoplus_{j \in J finite} K \boxtimes L[n_j].$$

Then there is the following trick: $\oplus_i E_i \boxtimes F_i$ is a direct summand of $(\oplus_i E_i) \boxtimes (\oplus_j F_j)$, hence we may find objects E, F, G, H of $D^b_{coh}(\mathcal{O}_X)$ and A, B of $D^b_{coh}(\mathcal{O}_{X\times_k X})$ such that

$$E \boxtimes F = \Big(\bigoplus_{i\in I} K \boxtimes L[m_i]\Big) \oplus A$$

$$G \boxtimes H = \Big(\bigoplus_{j\in J} K \boxtimes L[n_j]\Big) \oplus B.$$

Then there is the morphism

$$\begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} : E \boxtimes F \to G \boxtimes H.$$

The cone of this morphism contains $\mathrm{Cone}(\varphi)$ as a direct summand, which in turn contains $\mathcal{O}_\Delta$ as a direct summand, so we are done. □

This chapter is based on [Ola21a].

4.1 Reduction to the case of vector bundles

In this section, X will be a smooth projective curve over a field k. Our goal is to prove Proposition 4.1.5. We begin by studying the structure of the box product $E \boxtimes F$ with $E, F \in D^b_{coh}(\mathcal{O}_X)$ and setting up some notation regarding this.

Since X is regular and one-dimensional, every $K \in D^b_{coh}(\mathcal{O}_X)$ is decomposable, i.e., $K \cong \bigoplus_{n\in\mathbf{Z}} H^n(K)[-n]$. Thus for $E, F \in D^b_{coh}(\mathcal{O}_X)$ the object $E \boxtimes F$ of $D^b_{coh}(\mathcal{O}_{X\times_k X})$ is also decomposable, and we have the formula

$$H^k(E \boxtimes F) = \bigoplus_{i+j=k} H^i(E) \boxtimes H^j(F)$$

with the box products on the right hand side underived. See 1.2.5. Next, a coherent sheaf on X is a direct sum of a vector bundle and a torsion sheaf, so we see that $H^k(E \boxtimes F)$ is a direct sum of four parts which respectively have the form:

$$\bigoplus bundle \boxtimes bundle, \bigoplus bundle \boxtimes torsion, \bigoplus torsion \boxtimes bundle, \bigoplus torsion \boxtimes torsion.$$

We will sometimes refer to the first part as $H^k(E \boxtimes F)_{free}$, the last part as $H^k(E \boxtimes F)_0$ (for 0-dimensional support), and the sum of the last three parts as $H^k(E \boxtimes F)_{tors}$.

The following two lemmas are simple calculations we will need in the proof of Proposition 4.1.5.

Lemma 4.1.1. *Let E, F be objects of $D^b_{coh}(\mathcal{O}_X)$. Then*

$$R\mathrm{Hom}_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, E \boxtimes F) = R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F \otimes^{\mathbf{L}}_{\mathcal{O}_X} T_X)[-1],$$

and

$$R\mathrm{Hom}_{\mathcal{O}_{X\times X}}(E \boxtimes F, \mathcal{O}_\Delta) = R\mathrm{Hom}_{\mathcal{O}_X}(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F, \mathcal{O}_X).$$

Proof. The second formula is by adjunction between $R\Delta_*$ and $L\Delta^*$. The first is by adjunction between $R\Delta_*$ and $L\Delta^*(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_\Delta(\Delta)[-1]$, that is, by Grothendieck duality. □

Lemma 4.1.2. *Let $\mathcal{E}, \mathcal{F}$ be torsion coherent sheaves on X and $\mathcal{A}$ a vector bundle on $X \times X$. Then*

$$\mathrm{Ext}^i_{\mathcal{O}_{X\times X}}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{A}) = 0$$

for $i = 0, 1$.

Proof. It suffices to show $\mathcal{E}xt^i_{\mathcal{O}_{X\times X}}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{A}) = 0$ for $i = 0, 1$ by the relation between local and global Ext's. This reduces to a local algebra problem: A, B are DVRs with uniformizers π_A, π_B; R is a regular local ring of dimension 2 with local ring homomorphisms $A \to R$ and $B \to R$ such that (π_A, π_B) is a regular system of parameters for R; M, N are finitely generated torsion modules over A, B; and F is a finite free R-module. Then

$$\mathrm{Ext}^i_R((M \otimes_A R) \otimes_R (N \otimes_B R), F) = 0$$

for $i = 0, 1$.

By the structure theorem for finite modules over a DVR, we are allowed to assume $M = A/(\pi_A^m), N = B/(\pi_B^n)$ for $m, n > 0$. Then the Ext groups in question are

$$\mathrm{Ext}^i_R(R/(\pi_A^m, \pi_B^n), F)$$

which vanish for $i = 0, 1$ since (π_A^m, π_B^n) is a regular sequence on the free module F. □

We have two strategies for taking a summand of a cone of a morphism and showing that it is actually a summand of a cone of a simpler morphism (which is exactly what we are trying to do – see Proposition 4.1.5).

Reduction Strategy 1. Here $\mathcal{T}$ is a triangulated category. Suppose we have a morphism $A \to B$ in $\mathcal{T}$ with cone C and that S is a direct summand of C, so that there are morphisms $i : S \to C$ and $p : C \to S$ with $p \circ i = \mathrm{id}_S$. Suppose that there are distinguished triangles

$$A' \to A \to A'' \to A'[1]$$
$$B' \to B \to B'' \to B'[1]$$

in $\mathcal{T}$.

Lemma 4.1.3. *Notation as above. If* $\mathrm{Hom}(B', S) = 0$ *and* $\mathrm{Hom}(S, A''[1]) = 0$*, then actually* S *is a direct summand of the cone of* $A' \to B''$*.*

Proof. Write D for the cone of $A \to B''$. We will first show that S is a direct summand of D. By the octahedral axiom, there is a distinguished triangle

$$B' \to C \to D \to B'[1]$$

where the first arrow is the composition $B' \to B \to C$. What we have to show is that the projection $p : C \to S$ factors through D. This follows via a diagram chase from the assumption that $\mathrm{Hom}(B', S) = 0$.

Thus S is a direct summand of the cone of $A \to B''$ and a dual argument to the one in the first paragraph allows us to replace A with A'. □

Reduction Strategy 2. This is a slightly more specialized situation. Again $\mathcal{T}$ is a triangulated category and this time we have a morphism $A \oplus B \to C \oplus D$ with matrix

$$\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix},$$

and we have a direct summand S of its cone. The cone is isomorphic to

$$\mathrm{Cone}(F) \oplus D \oplus B[1].$$

Lemma 4.1.4. *Notation as above. Assume* $\mathrm{Hom}(S, D) = 0$ *and* $\mathrm{Hom}(B[1], S) = 0$*. Then* S *is a* *direct summand of* Cone(F).

Proof. Say the inclusion $S \to \mathrm{Cone}(F) \oplus D \oplus B[1]$ is given by the maps (a, b, c) and the retraction is given by the maps (d, e, f). Then we have $da + eb + cf = 1$ and $b = 0 = f$ by assumption, hence $da = 1$. □

Proposition 4.1.5. *Suppose there are perfect complexes E, F, G, H on X and a morphism $E \boxtimes F \to G \boxtimes H$ such that $\mathcal{O}_\Delta$ is a direct summand of its cone. Then there are vector bundles $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ on X and a morphism $\mathcal{E} \boxtimes \mathcal{F} \to \mathcal{G} \boxtimes \mathcal{H}$ such that $\mathcal{O}_\Delta$ is a direct summand of its cone.*

Proof. Step 1. $\mathcal{O}_\Delta$ is a direct summand of the cone of $\tau_{\leq 1}(E \boxtimes F) \to \tau_{\geq 0}(G \boxtimes H)$.

This follows from the first reduction strategy since there are no maps $\tau_{<0}(G \boxtimes H) \to \mathcal{O}_\Delta$ and no maps $\mathcal{O}_\Delta \to \tau_{>1}(E \boxtimes F)[1]$ for degree reasons.

Step 2. $\mathcal{O}_\Delta$ is a direct summand of the cone of a map

$$H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)[-1] \to H^0(G \boxtimes H) \oplus H^1(G \boxtimes H)[-1].$$

Here we apply the second reduction strategy using the decompositions

$$\tau_{\leq 1}(E \boxtimes F) = (H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)[-1]) \oplus \tau_{<0}(E \boxtimes F)$$
$$\tau_{\geq 0}(G \boxtimes H) = (H^0(G \boxtimes H) \oplus H^1(G \boxtimes H)[-1]) \oplus \tau_{>1}(G \boxtimes H).$$

For degree reasons, the morphism between them has matrix of the form

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

and also

$$\mathrm{Hom}(\tau_{<0}(E \boxtimes F)[1], \mathcal{O}_\Delta) = 0 = \mathrm{Hom}(\mathrm{O}_\Delta, \tau_{>1}(G \boxtimes H)),$$

so that reduction strategy gives the result.

Step 3. $\mathcal{O}_\Delta$ is a direct summand of the cone of a map

$$H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)_0[-1] \to H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)[-1].$$

First reduction strategy. This applies because

$$\operatorname{Hom}(H^0(G \boxtimes H)_{tors}, \mathcal{O}_\Delta) = 0 = \operatorname{Hom}(\mathcal{O}_\Delta, H^1(E \boxtimes F)/H^1(E \boxtimes F)_0).$$

To prove the first equality we have to show that if $\mathcal{G}, \mathcal{H}$ are coherent sheaves on X with at least one of them a torsion module, then $\operatorname{Hom}(\mathcal{G} \boxtimes \mathcal{H}, \mathcal{O}_\Delta) = 0$. But this Hom is identified with $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G} \otimes \mathcal{H}, \mathcal{O}_X)$ which vanishes since there are no maps from a torsion sheaf to a locally free sheaf. For the second equality we have to show $\operatorname{Hom}(\mathcal{O}_\Delta, \mathcal{E} \boxtimes \mathcal{F}) = 0$ if $\mathcal{E}, \mathcal{F}$ are coherent sheaves on X with at least one of them locally free. By Lemma 4.1.1 this Hom is identified with $H^{-1}(X, \mathcal{E} \otimes^{\mathbf{L}} \mathcal{F} \otimes T_X) = 0$ since the tensor product is underived.

Step 4. $\mathcal{O}_\Delta$ is a direct summand of the cone of a map

$$H^0(E \boxtimes F)_{free} \oplus H^1(E \boxtimes F)_0[-1] \to H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)_0[-1].$$

Second reduction strategy using the decompositions

$$H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)_0[-1] = (H^0(E \boxtimes F)_{free} \oplus H^1(E \boxtimes F)_0[-1]) \oplus H^0(E \boxtimes F)_{tors}$$
$$H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)[-1] = (H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)_0[-1]) \oplus H^1(G \boxtimes H)/H^1(G \boxtimes H)_0[-1].$$

The morphism between them has matrix

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

Most of the vanishing required to prove this is easy and left to the reader, but we will explain why

$$\operatorname{Hom}(H^1(E \boxtimes F)_0, H^1(G \boxtimes H)/H^1(G \boxtimes H)_0) = 0.$$

It suffices to show that if $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ are coherent sheaves on X with $\mathcal{E}, \mathcal{F}$ torsion and at least one of $\mathcal{G}, \mathcal{H}$ a vector bundle, then $\operatorname{Hom}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{G} \boxtimes \mathcal{H}) = 0$. By the Künneth formula, we have

$$\operatorname{Hom}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{G} \boxtimes \mathcal{H}) = \operatorname{Hom}(\mathcal{E}, \mathcal{G}) \otimes_k \operatorname{Hom}(\mathcal{F}, \mathcal{H}) = 0$$

since one of $\mathcal{G}, \mathcal{H}$ is a vector bundle and there are no maps from a torsion sheaf to a vector bundle.

Finally, we have

$$\mathrm{Hom}(H^0(E \boxtimes F)_{tors}[1], \mathcal{O}_\Delta) = 0 = \mathrm{Hom}(\mathcal{O}_\Delta, H^1(G \boxtimes H)/H^1(G \boxtimes H)_0[-1])$$

for degree reasons, so that the second reduction strategy applies.

Step 5. $\mathcal{O}_\Delta$ is a direct summand of the cone of a map

$$\varphi : H^0(E \boxtimes F)_{free} \to H^0(G \boxtimes H)_{free}.$$

This is very similar to the second reduction strategy. We note that the map

$$H^0(E \boxtimes F)_{free} \oplus H^1(E \boxtimes F)_0[-1] \to H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)_0[-1]$$

has diagonal matrix

$$\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}$$

with respect to the direct sum decompositions: $\mathrm{Hom}(H^0(E \boxtimes F)_{free}, H^1(G \boxtimes H)_0[-1]) = 0$ for degree reasons and $\mathrm{Hom}(H^1(E \boxtimes F)_0[-1], H^0(G \boxtimes H)_{free}) = 0$ by Lemma 4.1.2. Therefore, $\mathcal{O}_\Delta$ is a direct summand of $\mathrm{Cone}(\varphi) \oplus \mathrm{Cone}(\psi)$. Let the inclusion be given by maps (a, b) and the retraction by maps (c, d) so that $ca + db = \mathrm{id}_{\mathcal{O}_\Delta}$. We are going to show that db is zero on cohomology sheaves, and then we will be done since ca will be a quasi-isomorphism. But $\mathcal{O}_\Delta$ has only the one nonzero cohomology sheaf so actually we will be done if we can show

$$H^0(d) : H^0(\mathrm{Cone}(\psi)) \to \mathcal{O}_\Delta$$

is zero. This follows since $H^0(\mathrm{Cone}(\psi))$ has zero-dimensional support, being a submodule of $H^1(E \boxtimes F)_0$.

Step 6. Conclusion.

The Proposition now follows from the same trick as Lemma 4.0.2. Since $H^0(E \boxtimes F)_{free}$ and

$H^0(G \boxtimes H)_{free}$ are direct sums of box products of vector bundles, there are vector bundles $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ on X and $\mathcal{A}, \mathcal{B}$ on $X \times X$ such that

$$\mathcal{E} \boxtimes \mathcal{F} = H^0(E \boxtimes F)_{free} \oplus \mathcal{A}$$
$$\mathcal{G} \boxtimes \mathcal{H} = H^0(G \boxtimes H)_{free} \oplus \mathcal{B}.$$

Take the morphism

$$\begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix}$$

between them. Then $\mathcal{O}_\Delta$ is a direct summand of the cone of φ which is a direct summand of the cone of this morphism, and we are done. □

4.2 The vector bundle case

On X a smooth, projective, geometrically connected curve over a field k, a vector bundle $\mathcal{E}$ has a Harder–Narasimhan filtration:

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_N = \mathcal{E}$$

and we will write $\mu_i = \mu_i(\mathcal{E}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. The μ_i are weakly decreasing. It will sometimes be convenient to index instead by slope. For this we will write $\mathcal{E}_i = \mathcal{E}^{\mu_i}$. We will also write $\mathcal{E}_i/\mathcal{E}_{i-1} = \mathrm{gr}^{\mu_i}(\mathcal{E})$. Given a second vector bundle $\mathcal{F}$, we get a filtration of the box-product $\mathcal{E} \boxtimes \mathcal{F}$ with

$$\mathrm{Fil}^\gamma = \sum_{\alpha+\beta \geq \gamma} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta$$

The graded pieces of the filtration are

$$\mathrm{gr}^\gamma = \bigoplus_{\alpha+\beta=\gamma} \mathrm{gr}^\alpha(\mathcal{E}) \boxtimes \mathrm{gr}^\beta(\mathcal{F}).$$

In particular, this is a filtration of $\mathcal{E} \boxtimes \mathcal{F}$ by sub-*bundles*.

Proof of Theorem 4.0.1. By Lemma 1.2.7, we may assume X is connected. If X is affine then $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Ddim}(X) = 1$ by 1.2.11. Otherwise, X is projective, and by Lemma 1.2.8, we may replace k by $H^0(X, \mathcal{O}_X)$ to assume X is geometrically connected over k. If $H^1(X, \mathcal{O}_X) = 0$, then X is a Severi–Brauer variety over k and we conclude by Lemma 1.2.12 that $\mathrm{Ddim}(X) = 1$. Otherwise, X is a smooth projective goemetrically connected curve of genus $g \geq 1$ and we must show $\mathrm{Ddim}(X) = 2$. By Lemma 4.0.2 and Proposition 4.1.5, it suffices to show $\mathcal{O}_\Delta$ is not a direct summand of the cone of a map $\mathcal{E} \boxtimes \mathcal{F} \to \mathcal{G} \boxtimes \mathcal{H}$ for any vector bundles $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ on X. Suppose it is. Apply Reduction Strategy 1 using the short exact sequences

$$0 \to \sum_{\alpha+\beta\geq 2g-2} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \to \mathcal{E} \boxtimes \mathcal{F} \to \mathcal{Q} \to 0$$

$$0 \to \sum_{\alpha+\beta>0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta \to \mathcal{G} \boxtimes \mathcal{H} \to \mathcal{Q}' \to 0.$$

For this to work we need

$$\mathrm{Hom}(\sum_{\alpha+\beta>0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta, \mathcal{O}_\Delta) = 0 = \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{Q}[1]).$$

The first equality is true because $\sum_{\alpha+\beta>0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta$ has a filtration whose graded pieces are direct sums of box products $\mathcal{G}' \boxtimes \mathcal{H}'$ with $\mathcal{G}'$ and $\mathcal{H}'$ semistable bundles with positive sum of slopes, and no such box product has a morphism to $\mathcal{O}_\Delta$:

$$\mathrm{Hom}(\mathcal{G}' \boxtimes \mathcal{H}', \mathcal{O}_\Delta) = \mathrm{Hom}(\mathcal{G}' \otimes \mathcal{H}', \mathcal{O}_X) = \mathrm{Hom}(\mathcal{G}', \mathcal{H}'^\vee) = 0$$

since the source has higher slope than the target. The second equality is true because $\mathcal{Q}$ has a filtration whose graded pieces are direct sums of box products $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}', \mathcal{F}'$ semistable bundles with sum of slopes $> 2g-2$, and no such box product receives a map from $\mathcal{O}_\Delta[-1]$:

$$\mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{E}' \boxtimes \mathcal{F}'[1]) = H^0(X, \mathcal{E}' \otimes \mathcal{F}' \otimes T_X) = 0$$

since $\mu(\mathcal{E}') + \mu(\mathcal{F}') + \deg(T_X) < 0$.

Therefore, $\mathcal{O}_\Delta$ is a direct summand of the cone of a morphism

$$\varphi : \sum_{\alpha+\beta\geq 2g-2} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \to \mathcal{Q}'.$$

We now split into two cases:

Case 1. $g \geq 2$.

In this case $\varphi = 0$. This is because the source has a filtration whose graded pieces are direct sums of objects of the form $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}', \mathcal{F}'$ semistable with sum of slopes $\geq 2g - 2 > 0$ and the target has a filtration whose graded pieces are direct sums of objects of the form $\mathcal{G}' \boxtimes \mathcal{H}'$ with $\mathcal{G}', \mathcal{H}'$ semistable with sum of slopes ≤ 0. Furthermore we have by the Künneth formula

$$\operatorname{Hom}(\mathcal{E}' \boxtimes \mathcal{F}', \mathcal{G}' \boxtimes \mathcal{H}') = \operatorname{Hom}(\mathcal{E}', \mathcal{G}') \otimes_k \operatorname{Hom}(\mathcal{F}', \mathcal{H}')$$

which can only be non-zero if $\mu(\mathcal{E}') \leq \mu(\mathcal{G}')$ and $\mu(\mathcal{F}') \leq \mu(\mathcal{H}')$. But this contradicts the fact that $\mu(\mathcal{E}') + \mu(\mathcal{F}') > \mu(\mathcal{G}') + \mu(\mathcal{H}')$.

Hence $\mathcal{O}_\Delta$ is a direct summand of $H^0(\operatorname{Cone}(\varphi)) = \mathcal{Q}'$ which is a vector bundle, a contradiction.

Case 2. $g = 1$.

In this case we claim that $H^0(\operatorname{Cone}(\varphi))$ is still a vector bundle, leading to the same contradiction. This time the argument of Case 1 does not show that φ is zero but it does show that we obtain a factorization

$$\varphi : \sum_{\alpha+\beta \geq 2g-2=0} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \twoheadrightarrow \operatorname{gr}^0(\mathcal{E} \boxtimes \mathcal{F}) \to \operatorname{gr}^0(\mathcal{G} \boxtimes \mathcal{H}) \subset \mathcal{Q}',$$

and it suffices to show that the cokernel of

$$\psi : \operatorname{gr}^0(\mathcal{E} \boxtimes \mathcal{F}) \to \operatorname{gr}^0(\mathcal{G} \boxtimes \mathcal{H})$$

is a vector bundle. The source is a direct sum of box products $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}', \mathcal{F}'$ semistable bundles with sum of slopes = 0, and similarly for the target. By the argument of Case 1 with the Künneth formula, the only nonzero components of ψ are the maps $\mathcal{E}' \boxtimes \mathcal{F}' \to \mathcal{G}' \boxtimes \mathcal{H}'$ with $\mu(\mathcal{E}') = \mu(\mathcal{G}')$ and $\mu(\mathcal{F}') = \mu(\mathcal{H}')$, i.e., ψ is diagonal with respect to the given direct sum decomposition. Therefore, it suffices to show that the cokernel of each nonzero component $\mathcal{E}' \boxtimes \mathcal{F}' \to \mathcal{G}' \boxtimes \mathcal{H}'$ is a vector bundle. This follows from Lemma 4.2.1 below. □

Lemma 4.2.1. *Let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be semistable bundles on a smooth projective geometrically connected curve X over k with $\mu(\mathcal{E}) = \mu(\mathcal{G})$ and $\mu(\mathcal{F}) = \mu(\mathcal{H})$. Then the kernel and cokernel of any map $\varphi : \mathcal{E} \boxtimes \mathcal{F} \to \mathcal{G} \boxtimes \mathcal{H}$ are vector bundles.*

Proof. It suffices to prove this after pullback along the flat covering $X_{\bar{k}} \to X$. Since the pullback of a semistable bundle on X to $X_{\bar{k}}$ remains semistable (see [HL10, Corollary 1.3.8]), we may assume $k = \bar{k}$.

We must show that the function taking a closed point $p \in X \times_k X$ to $\mathrm{rk}(\varphi \otimes \kappa(p))$ is constant. To prove this, it is enough to show that the rank stays constant on each horizontal and vertical closed fiber $X \times \{y\}, \{x\} \times X$, x, y closed points of X. The restriction of φ to $X \times \{y\} \cong X$ is a map

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus m} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus n},$$

where $m = \mathrm{rk}(\mathcal{F})$ and $n = \mathrm{rk}(\mathcal{H})$. This is a map between semistable bundles of the same slope, hence it has constant rank. The same argument works for the vertical fibers and we are done. □

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Appendix A: Enrichment of $D^b(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category with enough injectives. As in [Ill72, I, Chapter V], let $DF(\mathcal{A})$ denote the filtered derived category of $\mathcal{A}$, whose objects are represented by $\mathcal{A}$-complexes $K^\bullet$ with a *finite* decreasing filtration (i.e., there exists $i > 0$ such that for all n, $F^iK^n = 0$ and $F^{-i}K^n = K^n$). The full subcategory of $DF(\mathcal{A})$ spanned by those K with $\mathrm{gr}^i(K) \in D^+(\mathcal{A})$ for each $i \in \mathbf{Z}$ is denoted $D^+F(\mathcal{A})$, and $D^bF(\mathcal{A}), D^-F(\mathcal{A})$ are defined similarly. We will let Ab denote the category of abelian groups and FAb the category of filtered abelian groups. Given $K \in DF(\mathcal{A})$ we will also write K for its image in $D(\mathcal{A})$ but when taking Hom groups we will use subscripts to make it clear which category we are viewing the objects in.

Lemma A.0.1. *For each $K \in DF(\mathcal{A})$ and $L \in D^+F(\mathcal{A})$, there is a decreasing filtration F on $\mathrm{Hom}_{D(\mathcal{A})}(K, L)$, which is natural in the sense that it arises from a functor (*) fitting into a commutative diagram*

$$\begin{array}{ccc} DF(\mathcal{A})^{opp} \times D^+F(\mathcal{A}) & \xrightarrow{(*)} & FAb \\ \downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle forget} \\ D(\mathcal{A})^{opp} \times D^+(\mathcal{A}) & \longrightarrow & Ab, \end{array}$$

$$(K, L) \longmapsto \mathrm{Hom}_{D(\mathcal{A})}(K, L)$$

and which satisfies for $K \in DF(\mathcal{A}), L, M \in D^+F(\mathcal{A})$:

(1) The filtration on $\mathrm{Hom}_{D(\mathcal{A})}(K, L)$ is finite.

(2) If $f \in F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L)$ and $g \in F^q\mathrm{Hom}_{D(\mathcal{A})}(L, M)$ then $g \circ f \in F^{p+q}\mathrm{Hom}_{D(\mathcal{A})}(K, M)$.

(3) $F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L)/F^{p+1}\mathrm{Hom}_{D(\mathcal{A})}(K, L)$ is a subquotient of $\prod_{n\in\mathbf{Z}} \mathrm{Hom}_{D(\mathcal{A})}(\mathrm{gr}^n(K), \mathrm{gr}^{n+p}(L))$.

(4) Assume $\mathrm{Ext}^n_{D(\mathcal{A})}(\mathrm{gr}^i(K), \mathrm{gr}^j(L)) = 0$ when $n = 0, -1$ and $i > j$, for example if $\mathrm{Ext}^n_{D(\mathcal{A})}(\mathrm{gr}^i(K)[-i], \mathrm{gr}^j(L)[-j]) = 0$ for $n < 0$. Then:

(a) $F^0\mathrm{Hom}_{D(\mathcal{A})}(K, L) = \mathrm{Hom}_{D(\mathcal{A})}(K, L) = \mathrm{Hom}_{DF(\mathcal{A})}(K, L)$.

(b) $F^1\mathrm{Hom}_{D(\mathcal{A})}(K, L) = \{\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(K, L) = \mathrm{Hom}_{DF(\mathcal{A})}(K, L) : \mathrm{gr}^n(\varphi) = 0$ for all $n \in \mathbf{Z}\}$.

Proof. The commutative diagram in the statement is the outer square of a commutative diagram:

$$\begin{array}{ccccc} DF(\mathcal{A})^{opp} \times D^+F(\mathcal{A}) & \xrightarrow{(**)} & DF(\mathbf{Z}) & \xrightarrow{(***)} & FAb \\ \downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle forget} \\ D(\mathcal{A})^{opp} \times D^+(\mathcal{A}) & \longrightarrow & D(\mathbf{Z}) & \xrightarrow{H^0} & Ab. \end{array}$$

$$(K, L) \longmapsto R\mathrm{Hom}(K, L)$$

We need to explain the two top horizontal arrows, but note first that the composition of the bottom two arrows is indeed the Hom-functor.

The functor $(**)$ is the filtered variant of RHom constructed in [Ill72]. Let us briefly recall its construction: For $K \in DF(\mathcal{A})$ and $L \in D^+F(\mathcal{A})$ we may represent K by a complex $K^\bullet$ with a finite filtration and we may represent L by a complex $L^\bullet$ with a finite filtration such that each F^iL^n is an injective object of $\mathcal{A}$ and $L^n = 0$ for $n \ll 0$ (we say $L^\bullet$ is *of injective type*). Then the Hom complex $\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)$ whose i^{th} term is

$$\mathrm{Hom}^i(K^\bullet, L^\bullet) = \prod_{n \in \mathbf{Z}} \mathrm{Hom}(K^n, L^{n+i})$$

has a filtration such that $(f^n) \in \prod_{n\in\mathbf{Z}} \mathrm{Hom}(K^n, L^{n+i})$ lies in F^p iff

$$f^n(F^jK^n) \subset F^{j+p}L^{n+i} \text{ for all } n, j.$$

Since the filtrations on $K^\bullet$ and $L^\bullet$ were assumed finite, the filtration on $\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)$ is also finite so that we have indeed defined an object of $DF(\mathbf{Z})$ which we denote simply as $R\mathrm{Hom}(K, L)$. Since $L^\bullet$ is a bounded below complex of injectives, the underlying object of $D(\mathbf{Z})$ is the usual $R\mathrm{Hom}(K, L)$.

The functor (***) takes an object K to $H^0(K)$ equipped with its induced filtration, namely

$$F^iH^0(K) = \mathrm{Im}(H^0(F^iK) \to H^0(K)).$$

Note that by our convention that filtrations are finite, $(***)$ lands in the full subcategory of FAb consisting of abelian groups with a finite filtration, so that (1) is immediate.

Let us prove (2). Represent K, L, M by finitely filtered complexes $K^\bullet, L^\bullet, M^\bullet$ with $L^\bullet$ and $M^\bullet$ of injective type. Then

$$f \in \mathrm{Im}(H^0(F^p\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)) \to H^0(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet))$$

so we can represent f by a morphism of complexes $\varphi : K^\bullet \to L^\bullet$ such that $\varphi(F^jK^\bullet) \subset F^{j+p}L^\bullet$ for each j, and similarly we can represent g by a morphism of complexes $\psi : L^\bullet \to M^\bullet$ such that $\psi(F^jL^\bullet) \subset F^{j+q}L^\bullet$ for each j. Then the composition $\psi \circ \varphi : K^\bullet \to M^\bullet$ represents $g \circ f : K \to M$ and satisfies $\psi \circ \varphi(F^jK^\bullet) \subset F^{j+p+q}M^\bullet$ for each j, thus $g \circ f \in F^{p+q}\mathrm{Hom}(K, M)$, as needed.

For (3) we consider the spectral sequence associated to the filtered complex $\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)$ constructed in the second paragraph. It converges since the filtration is finite, and it has

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(R\mathrm{Hom}(K, L))) = \prod_{n \in \mathbf{Z}} \mathrm{Ext}^{p+q}_{D(\mathcal{A})}(\mathrm{gr}^n(K), \mathrm{gr}^{n+p}(L))$$

(see [Ill72, V 1.4.9]) and

$$E_\infty^{p,-p} = F^p\mathrm{Hom}_{D(\mathcal{A})}(K, L)/F^{p+1}\mathrm{Hom}_{D(\mathcal{A})}(K, L)$$

directly from the definition of the spectral sequence of a filtered complex. Since $E_\infty^{p,-p}$ is a subquotient of $E_1^{p,-p}$ this proves (3).

The first equality of (4)(a) follows from (3) since under the assumptions the group in (3) vanishes for $p < 0$. The second equality is [BBD82, Proposition 3.1.4(i)]. For (4)(b) consider the distinguished triangle

$$F^1R\mathrm{Hom}(K, L) \to F^0R\mathrm{Hom}(K, L) \to \mathrm{gr}^0R\mathrm{Hom}(K, L) \to .$$

The map $H^0(F^0R\mathrm{Hom}(K, L)) \to H^0(\mathrm{gr}^0R\mathrm{Hom}(K, L))$ identifies with the canonical map $\mathrm{Hom}_{DF(\mathcal{A})}(K, L) \to \prod_{n \in \mathbf{Z}} \mathrm{Hom}_{D(\mathcal{A})}(\mathrm{gr}^n(K), \mathrm{gr}^n(L))$, see [Ill72, V 1.4.6 and 1.4.9] hence using the exact sequence of cohomology,

$$
\begin{aligned}
F^1\mathrm{Hom}_{D(\mathcal{A})}(K,L) &= \mathrm{Im}(H^0(F^1R\mathrm{Hom}(K,L)) \to H^0(R\mathrm{Hom}(K,L))) \\
&= \mathrm{Im}(H^0(F^1R\mathrm{Hom}(K,L)) \to H^0(F^0R\mathrm{Hom}(K,L))) \\
&= \mathrm{Ker}(\mathrm{Hom}_{DF(\mathcal{A})}(K,L) \to \prod_{n\in\mathbf{Z}} \mathrm{Hom}_{D(\mathcal{A})}(\mathrm{gr}^n(K),\mathrm{gr}^n(L)).
\end{aligned}
$$

The second equality is (4)(a). □

Proof of Proposition 2.1.1. There is a fully faithful functor $can : D^b(\mathcal{A}) \to D^bF(\mathcal{A})$ whose composition with the forgetful functor $D^bF(\mathcal{A}) \to D^b(\mathcal{A})$ is isomorphic to the identity and such that $\mathrm{gr}^p(can(K)) = H^{-p}(K)[p]$. It is the quasi-inverse of the equivalence of categories of [BBD82, Proposition 3.1.6]. Informally, can equips K with its canonical filtration $F^iK = \tau_{\leq -i}K$. For $K,L \in D^b(\mathcal{A})$ we view them in $D^bF(\mathcal{A})$ via can and therefore Lemma A.0.1 gives the desired filtration on $\mathrm{Hom}_{D(\mathcal{A})}(K,L)$. All of (1)-(4) are now immediate. □