Ricci flow and positivity of curvature on manifolds with boundary

Tsz-Kiu Aaron Chow

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Abstract

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In this thesis, we explore short time existence and uniqueness of solutions to the Ricci flow on manifolds with boundary, as well as the preservation of natural curvature positivity conditions along the flow.

In chapter 2, we establish the existence and uniqueness for linear parabolic systems on vector bundles for Hölder continuous initial data. We introduce appropriate weighted parabolic Hölder spaces to study the existence and uniqueness problem. Having developed the linear theory, we apply it to establish the existence and uniqueness for the Ricci-DeTurck flow, the harmonic map heat flow, and the Ricci flow with Hölder continuous initial data in Chapter 3.

In chapter 4, we discuss a general preservation result concerning the preservation of various curvature conditions during boundary deformation. Using a perturbation argument, we construct a family of metrics which interpolate between two metrics that agree on the boundary, and such family of metrics preserves various natural curvature conditions under suitable assumptions on the boundary data.

The results from chapters 2 through 4 will be utilized in proving the Main Theorems in chapter 5. In particular, we construct canonical solutions to the Ricci flow on manifolds with boundary from canonical solutions to the Ricci flow on closed manifolds with Hölder continuous initial data via doubling.

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Chapter 1: Introduction

This thesis investigates the deformation of Riemannian metrics on compact smooth manifolds with boundary by Ricci flow. The study of Ricci flow dates back to the work of Hamilton [15], who applied it to prove that every three dimensional closed compact Riemannian manifold with positive Ricci curvature admits a metric of constant positive sectional curvature. Since then, many significant results on the interaction between geometry and topology were established using Ricci flow. One key geometric property of Ricci flow is its ability to preserve various curvature conditions. Hamilton's seminal paper [15] highlighted the preservation of positive Ricci curvature in Ricci flow on closed compact Riemannian manifolds in three dimensions. In the four-dimensional case, Hamilton demonstrated that Ricci flow on closed compact Riemannian manifolds preserves positive curvature operators [13] and positive isotropic curvature [14], which have been utilized in later convergence results for Ricci flow. In higher dimensions, Ricci flow on closed compact Riemannian manifolds preserves variants of isotropic curvature conditions, such as PIC, PIC1, and PIC2, as observed by Brendle [1], Brendle and Schoen [5], and Nguyen [20]. These curvature conditions play vital roles in general convergence of Ricci flow in higher dimensions and in the proof of the Differentiable Sphere Theorem, as demonstrated in the results of Brendle [1, 3] and Brendle-Schoen [5].

Motivated by the aforementioned results, it is reasonable to inquire whether these results can be extended to smooth manifolds with boundary. This thesis is primarily driven by two pertinent questions:

Question 1. *Do existence and uniqueness statements for Ricci flow on manifolds with boundary exist, and what would constitute suitable boundary data?*

Question 2. *Does Ricci flow on manifolds with boundary preserve the natural curvature condi-*

tions, similar to its closed manifold counterparts?

The challenge to **Question 1** is the diffoemorphism invariance of the Ricci curvature. One would need to study carefully the geometry of the boundary in order to set up a well-posed boundary value problem. The challenge to **Question 2** is the failure of usual parabolic maximum principle on the boundary, if one asks for an arbitrary initial metric.

The initial attempt to address **Question 1** was made by Shen [23], who proved short-time existence to the Ricci flow on smooth compact manifold with umbilic boundary:

Theorem 1.1 ([23]). *For any given compact Riemannian manifold with boundary* (M, g_0) *such that* $A_{g_0} = \lambda g_0$*, there is a short time solution to the Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) & on \quad M \\ A_{g(t)} = \lambda g(t) & on \quad \partial M \end{cases}$$

such that $g(0) = g_0$*. Here* $A_{g(t)}$ *stands for the second fundamental form of* ∂M *with respect to the metric* $g(t)$ *and* λ *is a constant.*

However, one might wonder about the deformation of more arbitrary metrics. Subsequently, Pulemotov [21] and Gianniotis [10] established short-time existence results for Ricci flow on smooth compact manifolds with boundary with arbitrary initial metrics. Notably, Gianniotis [10] proved both short-time existence and uniqueness results by prescribing the mean curvature and conformal class of the boundary when formulating the evolution equations. More precisely, Gianniotis proved:

Theorem 1.2 ([10]). *Let* g_0 *be a Riemannian metric on* M*,* $\gamma(x,t)$ *be arbitrary smooth Riemannian metrics on* ∂M *and* η *be an arbitrary function on* $\partial M \times [0, \infty)$*. Assume the compatibility conditions* $[\gamma(\cdot, 0)] = [g_0(\cdot)^T]$ *and* $H_{g_0} = \eta|_{t=0}$*. Then there exists a* $T > 0$ *and a smooth solution to the Ricci* *flow*

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) & on \quad M \times (0,T] \\ H_{g(t)} = \eta(t) & on \quad \partial M \times (0,T] \\ [g(t)^T] = [\gamma(t)] & on \quad \partial M \times (0,T] \end{cases} \cdot$$

As $t \longrightarrow 0$, $g(t)$ converges in the Cheeger-Gromov $C^{1,\alpha}$ sense to g_0 and C^∞ away from the boundary.

In dimension 2, more results were contributed by Brendle [2], Cortissoz and Murcia [8]. However, there are no results so far that address **Question 2**. In order to explore this question, we initially examine the interaction between the boundary geometry and the internal geometry of Riemannian manifolds.

The study of such an interaction can be tracked back to the work by Gromov and Lawson [12], who observed that a metric of positive scalar curvature can be constructed on the double of a compact manifold with boundary which equips with metrics of positive scalar curvature and mean convex boundary. In the work of Shi and Tam [24], the boundary behavior of compact manifolds with nonnegative scalar curvature was studied. They proved that the integral of mean curvature of the boundary of a spin manifold cannot be greater than the integral of mean curvature of its isometrically embedded convex image in Euclidean space. Their work implies positivity of the Brown-York quasilocal mass. On the other hand, the study of boundary effects to internal curvatures also plays an important role in studying manifolds with rough data. Positive mass theorem on manifolds with corners was proved by Miao [18]. In his work, a smoothing procedure for the metric was done by constructing an "intrinsic bending" of Riemannian manifolds along the boundary that keeps the scalar curvature non-decreasing. More work were done for positive mass theorem on manifolds with corner in the literature such as the result by McFeron and Szekelyhidi [17]. Another important result that involves the study of boundary effects to internal curvatures is the counter example constructed by Brendle, Marques and Neves [4] to the Min-Oo's conjecture. More specifically, a perturbation argument was performed in [4] to make the boundary totally geodesic while keeping the scalar curvature non-decreasing. More related results can be found in a recent paper by Gromov [11].

Motivated by these remarkable results, it is intriguing to delve deeper into the impact of boundary data deformation on internal curvature conditions. In addition to addressing **Question 2**, this thesis also investigates the effect of boundary deformation on various curvature conditions on the manifold. In higher dimensions, positivity of curvature operator and various isotropic curvatures serve as natural curvature conditions that can be applied to manifolds. The relevant definitions will be revisited in Chapter 1.2.

Positive isotropic curvature was first introduced in the work of Micallef and Moore [19], and variants of this condition were used to prove the Differentiable Sphere Theorem [5]. For manifolds without boundary, the significance of the aforementioned curvature conditions lies in their preservation by Ricci Flow [1]. Manifolds with metrics that meet these curvature conditions exhibit rigid topological constraints. Brendle proved in [3] that for an initial metric that is strictly PIC, Ricci flow only forms neck-pinched singularities for $n \geq 12$. Furthermore, for an initial metric that is strictly PIC1, Brendle showed in [1] that after rescaling, Ricci flow converges to a metric of constant curvature. Given the importance of these curvature conditions, it is natural to examine their behavior in manifolds with boundaries.

1.1 Main results

Our work substantially improves the result of [23], where we prove short-time existence and uniqueness of solutions to Ricci flow on manifold with boundary in which the boundary become instantaneously umbilic for positive time. We remark that our result does not require the initial metric to have a umbilic boundary. We approach the problem via doubling of the manifold. Extending the initial metric to the doubled manifold by reflection, we obtain an extended metric which is merely Hölder continuous. We seek a solution to Ricci flow on the doubled manifold with a Hölder continuous initial metric that is smooth for positive time. To that end, we proved:

Main Theorem 1. *Let M be a closed compact smooth manifold and $g_0 \in C^{\alpha}(M)$ be a Rimannian metric for some $\alpha \in (0,1)$. Let $k \geq 2$, $\gamma \in (0,\alpha)$ and $\beta \in (\gamma,\alpha)$ be given. Then there exists a $C^{1,\beta}$ diffeomorphism ψ and $T = T(M, \|g_0\|_{\alpha})$, $K = K(M, k, \|g_0\|_{\alpha})$ such that the following holds:*

There is a solution $g(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(M \times [0,T])$ to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) \quad on \quad \tilde{M} \times (0,T]$$

such that $g(0) = \psi^* g_0$ and

$$\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}(M\times[0,T])} \leq K.$$

Here the Banach spaces $\mathcal{X}^{(\beta)}_{k,\gamma}(M \times (0,T])$ is defined in the beginning of Chapter 3 in (3.5). We also show that the above theorem gives a canonical solution to the Ricci flow, in the sense of the following uniqueness theorem:

Main Theorem 2. *Let $\alpha \in (0,1)$ be given. Let M be a closed compact smooth manifold and $g_0 \in C^{\alpha}(M)$ be a Riemannian metric on M. Suppose that the pairs $(g^1(t), \psi^1)$ and $(g^2(t), \psi^2)$ satisfy the conclusion of Main Theorem 1. Then there exists a C^{k+1} diffeomorphism $\varphi : M \to M$ such that*

$$g^2(t) = \varphi^*(g^1(t)),$$

where $\psi^2 = \psi^1 \circ \varphi$. In particular, $((\psi^1)^{-1})^* g^1(t) = ((\psi^2)^{-1})^* g^2(t)$.

Via doubling the above results imply the existence and uniqueness results for Ricci flow on manifolds with boundary:

Main Theorem 3. *Let (M, g_0) be a compact smooth Riemannian manifold with boundary. Let $k \geq 2$, $\beta \in (0,1)$ and $\varepsilon \in (0, 1-\beta)$ be given. Then there exists a $C^{1+\beta}$ diffeomorphism ψ and*

$T = T(M, \hat{g}, \|g_0\|_{\beta+\varepsilon})$, $K = K(M, k, \hat{g}, \|g_0\|_{\beta+\varepsilon})$ such that the following holds:

There is a solution $g(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(M \times [0,T])$ to the Ricci flow on manifold with boundary

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) & on \quad M \times (0,T] \\ A_{g(t)} = 0 & on \quad \partial M \times (0,T] \end{cases} \tag{1.1}$$

such that $g(0) = \psi^* g_0$ and

$$\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}(M \times [0,T])} \leq K.$$

Here $A_{g(t)}$ stands for the second fundamental form of the boundary with respect to the metric $g(t)$. For each $t > 0$, the metric $g(t)$ extends smoothly to the doubled manifold $\tilde{M}$ of M, and the doubled metric lies in $\mathcal{X}^{(\beta)}_{k,\gamma}(\tilde{M} \times [0,T])$. The diffeomorphism ψ also extends to a $C^{1+\beta}$ diffeomorphism on the doubled manifold.

Main Theorem 4. *Let (M, g_0) be a compact smooth Riemannian manifold with boundary. Suppose that the pairs $(g^1(t), \psi^1)$ and $(g^2(t), \psi^2)$ satisfy the conclusion of Main Theorem 3. Then there exists a C^{k+1} diffeomorphism $\varphi : M \to M$ such that φ extends to a C^{k+1} diffeomorphism on the doubled manifold and*

$$g^2(t) = \varphi^*(g^1(t)),$$

where $\psi^2 = \psi^1 \circ \varphi$. In particular, $((\psi^1)^{-1})^* g^1(t) = ((\psi^2)^{-1})^* g^2(t)$.

Main Theorem 1 to Main Theorem 4 were proved in [7]. Next, we show that the canonical solution constructed above preserves natural curvature conditions, provided that the geometry of the boundary is controlled. More precisely, we prove that the flow preserves positive curvature operator, PIC1 and PIC2 conditions, provided that the boundary is convex with respect to the initial metric. Moreover, if the initial metric has a two-convex boundary, we prove that the flow preserves the PIC conditions. To that end, we initially established a broader result related to the preservation of these curvature conditions during the deformation of boundary data.

Main Theorem 5. *Suppose that M is a compact manifold with smooth boundary, and g is a smooth Riemannian metric on M. Suppose that $\tilde{g}$ is another smooth Riemannian metric on M such that $g = \tilde{g}$ on ∂M. Then there exists a family of smooth Riemannian metrics $\{\hat{g}_\lambda\}_{\lambda > \lambda^*}$ on M and a neighborhood U of ∂M such that*

- *$\hat{g}_\lambda$ agrees with g outside U.*
- *$\hat{g}_\lambda$ agrees with $\tilde{g}$ in a neighborhood of ∂M.*

Moreover, $\hat{g}_\lambda$ satisfies:

(i) $\lim_{\lambda\to\infty} \hat{g}_\lambda = g$ *in C^α for any $\alpha \in (0, 1)$.*

(ii) Suppose that (M, g) has a convex boundary and that $(M, \tilde{g})$ has a weakly convex boundary such that $A_g > A_{\tilde{g}} \geq 0$, then

- *(M, g) and $(M, \tilde{g})$ have positive curvature operator $\implies$ $(M, \hat{g}_\lambda)$ has positive curvature operator;*
- *(M, g) and $(M, \tilde{g})$ are PIC1 $\implies$ $(M, \hat{g}_\lambda)$ is PIC1;*
- *(M, g) and $(M, \tilde{g})$ are PIC2 $\implies$ $(M, \hat{g}_\lambda)$ is PIC2.*

(iii) Suppose that (M, g) has a two-convex boundary and that $(M, \tilde{g})$ has a weakly two-convex boundary such that

$$A_g(X, X) + A_g(Y, Y) > A_{\tilde{g}}(X, X) + A_{\tilde{g}}(Y, Y) \geq 0$$

for all orthonormal $X, Y \in T(\partial M)$, then

- *(M, g) and $(M, \tilde{g})$ are PIC $\implies$ $(M, \hat{g}_\lambda)$ is PIC.*

(iv) Suppose that (M, g) has a mean-convex boundary and that $(M, \tilde{g})$ has a weakly mean-convex boundary such that $H_g \geq H_{\tilde{g}} > 0$, then

– *(M, g) and $(M, \tilde{g})$ have positive scalar curvature $\Longrightarrow$ $(M, \hat{g}_\lambda)$ has positive scalar curvature.*

The author was informed that a similar result to the Main Theorem 5 was proved by Schlichting in his Ph.D. thesis [22] which concerns with preserving curvature positivity on the gluing of two Riemannian manifolds along isometric boundaries. Nevertheless, our result allows the boundary to be two-convex for preserving positive isotropic curvature. The proof in Schlichting's thesis is also different from ours.

For the purpose of preserving curvature conditions along Ricci flow on manifolds with boundary, it would be helpful to preserve the curvature conditions of a metric while deforming it to one with a totally geodesic boundary. This procedure would be useful in smoothing manifolds with rough data. Under natural assumptions concerning the boundary data, we verify the existence of such deformations:

Main Theorem 6. *Suppose that M is a compact manifold with smooth boundary, and g is a smooth Riemannian metric on M. Then there is a family of smooth Riemannian metrics $\{\hat{g}_\lambda\}_{\lambda > \lambda^*}$ on M so that $\hat{g}_\lambda = g$ outside a neighborhood of ∂M and satisfies:*

(i) $\lim_{\lambda\to\infty} \hat{g}_\lambda = g$ *in C^α for any $\alpha \in [0, 1)$.*

(ii) $(M, \hat{g}_\lambda)$ has a totally geodesic boundary.

(iii) If (M, g) has a convex boundary, then

- *(M, g) has positive curvature operator $\Longrightarrow$ $(M, \hat{g}_\lambda)$ has positive curvature operator;*
- *(M, g) is PIC1 $\Longrightarrow$ $(M, \hat{g}_\lambda)$ is PIC1;*
- *(M, g) is PIC2 $\Longrightarrow$ $(M, \hat{g}_\lambda)$ is PIC2.*

(iv) If (M, g) has a two-convex boundary, then

- *(M, g) is PIC $\Longrightarrow$ $(M, \hat{g}_\lambda)$ is PIC.*

(v) If (M, g) has a mean-convex boundary, then

– (M, g) *has positive scalar curvature* $\implies$ $(M, \hat{g}_\lambda)$ *has positve scalar curvature.*

Main Theorem 5 and Main Theorem 6 were proved in [6]. We now state the result of preserving curvature conditions along Ricci flow on manifolds with boundary, which was proved in [7].

Main Theorem 7. *Suppose that* $g(t)$ *is a canonical solution to the Ricci flow on manifold with boundary on* $M \times [0, T]$ *given by the Main Theorem 3. Then the following holds:*
If (M, g_0) *has a convex boundary, then*

(i) (M, g_0) *has positive curvature operator* $\implies$ $(M, g(t))$ *has positive curvature operator;*

(ii) (M, g_0) *is PIC1* $\implies$ $(M, g(t))$ *is PIC1;*

(iii) (M, g_0) *is PIC2* $\implies$ $(M, g(t))$ *is PIC2.*

If (M, g_0) *has a two-convex boundary, then*

(iv) (M, g_0) *is PIC* $\implies$ $(M, g(t))$ *is PIC.*

Moreover, if (M, g_0) *has a mean-convex boundary, then*

(v) (M, g_0) *has positive scalar curvature* $\implies$ $(M, g(t))$ *has positive scalar curvature.*

Since the canonical solution we obtained preserves natural curvature conditions under suitable assumptions of the boundary data and can be extended smoothly to the doubled manifold, many results in Ricci flow on closed compact manifolds can be applied to our case. For examples, if the initial metric is PIC1 and has a convex boundary, our results and [1] then imply that the Ricci flow converges to a metric of constant curvature with a totally geodesic boundary after rescaling. From [3], we can also obtain a topological classification of all compact manifolds with boundary of dimension $n \geq 12$ which admit metrics that are PIC and have two-convex boundary and do not contain non-trivial incompressible $(n-1)$-dimensional space forms.

Chapter 5 will be dedicated to proving the Main Theorems. In Chapter 2, we study a linear parabolic equation on vector bundles over M. This result will be employed to establish the existence and uniqueness of solutions for the Ricci-DeTurck flow, the harmonic map heat flow, and the

Ricci flow with Hölder continuous initial data in Chapter 3, using the Banach fixed point theorem. Chapter 4 will feature the proof of a general result related to the preservation of various curvature conditions during boundary deformation. The work from Chapters 2 through 4 will be utilized in proving the Main Theorems in Chapter 5.

1.2 Preliminary and notation

In this section we will record notation, identities, and results that will be utilized throughout this thesis. Let M be a smooth manifold of dimension n with boundary ∂M, let g be a Riemannian metric on M, and let D denote the Levi-Civita connection on M compatible with g. The Riemann curvature tensor of (M, g) is defined by

$$-R(X, Y, Z, W) = g(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, W)$$

for all vector fields X, Y, Z, W. The Ricci tensor of (M, g) is defined by

$$Ric(X, Y) = \sum_{k=1}^{n} R(X, e_k, Y, e_k)$$

for all vector fields X, Y. Here $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M.

Let ν be the inward pointing unit normal field on the boundary ∂M. The second fundamental form of ∂M with respect to the metric g is defined by

$$A_g(X, Y) = g(\nu, D_X Y)$$

for all $X, Y \in T(\partial M)$. With this convention the second fundamental form is positive on the standard sphere $S^n \subset \mathbb{R}^{n+1}$ with respect to the inward pointing unit normal vector $\nu = -x$.

Definition 1.3. *We say that the boundary ∂M is (weakly) m-convex ($1 \leq m \leq n-1$) if the second* *fundamental form A_g is m-positive for all point $p \in \partial M$. That is,*

$$A_g(X_1, X_1) + \cdots + A_g(X_m, X_m) > (\geq)\, 0$$

for all orthonormal $X_1, \ldots, X_m \in T(\partial M)$. For $m = 1$ we recover the notion of (weak) convexity, and for $m = 2$ we recover the notion of (weak) two-convexity.

We may view the Riemann curvature tensor R as a symmetric bilinear form on the space of two-forms. For each point $p \in M$, the curvature operator $R : \bigwedge^2 T_pM \times \bigwedge^2 T_pM :\longrightarrow \mathbb{R}$ is defined by

$$R(X \wedge Y, Z \wedge W) = R(X, Y, Z, W)$$

for all vectors $X, Y, Z, W \in T_pM$. With this point of view, we recall the definitions for positive curvature operators and various positive isotropic curvatures.

Definition 1.4 (curvature conditions)**.**

(i) We say that (M, g) has positive curvature operator if $R(\varphi, \varphi) > 0$ for all nonzero two-vectors $\varphi \in \bigwedge^2 T_pM$ and all $p \in M$.

(ii) We say that (M, g) has positive isotropic curvature (PIC) if $R(z \wedge w, \bar{z} \wedge \bar{w}) > 0$ for all linearly independent vectors $z, w \in T_pM \otimes \mathbb{C}$ such that $g(z, z) = g(w, w) = g(z, w) = 0$ and all $p \in M$.

(iii) We say that (M, g) is PIC1 if $R(z \wedge w, \bar{z} \wedge \bar{w}) > 0$ for all linearly independent vectors $z, w \in T_pM \otimes \mathbb{C}$ such that $g(z, z)g(w, w) - g(z, w)^2 = 0$ and all $p \in M$.

(iv) We say that (M, g) is PIC2 if $R(z \wedge w, \bar{z} \wedge \bar{w}) > 0$ for all linearly independent vectors $z, w \in T_pM \otimes \mathbb{C}$ and all $p \in M$. That means (M, g) has positive complex sectional curvature.

1.3 Function spaces

Throughout Chapter 2 and 3, we assume, unless said otherwise, that M is a closed compact manifold, $\pi : E \longrightarrow M$ is a smooth vector bundle, and $\hat{g}$ a smooth metric on M. For sections η, ζ of E, the expression $\eta * \zeta$ means a bilinear combination with respect to $\hat{g}$. We fix a time $T \leq 1$, which we will make small according to our needs. We fix a finite set of coordinate charts which simultaneously serve as trivializations of the vector bundle $\{U_s, \varphi_s, \tilde{\varphi}_s\}_{s=1,\ldots,m}$ so that $\varphi_s : U_s \longrightarrow \mathbb{R}^n$ is a diffeomorphism and $\tilde{\varphi}_s : \pi^{-1}(U_s) \longrightarrow U_s \times \mathbb{R}^N$ is a trivialization. We may also assume the charts trivialize the vector bundle $\mathrm{Sym}^2(T^*M)$. Our trivializations give local frames $\mathbf{e}_r^s$ for E and local frames $dx^i dx^j$ for $\mathrm{Sym}^2(T^*M)$ on U_s (note the coordinates $x_1, \ldots, x_n$ depend on s).

We want to work on the parabolic Hölder spaces of tensors on M. Consider any tensor bundle of the form $E = T^{(p,q)}M$. If S is a section of E, then in the open set U_s it can be written as $S = \sum_{r=1}^N S_s^r \mathbf{e}_r^s$, where $\{\mathbf{e}_r^s\}$ is a local frame on $E|_{\pi^{-1}(U_s)}$. For any $\alpha \in (0,1)$, we define by $C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$ the space of maps $\eta : M \times [t_1, t_2] \longrightarrow E$ such that $\eta(t) \in C^{\alpha,\alpha/2}(M; E)$ is a α-Hölder continuous section of E for each $t \in [t_1, t_2]$. Given any map $\eta \in C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$, we define the associated C^0 norm to be

$$|\eta|_{0;M\times[t_1,t_2]} := \sum_s \sum_{r=1}^N |\eta_s^r|_{0;\varphi_s(U_s)\times[t_1,t_2]},$$

and we also define the associated parabolic Hölder semi-norm to be

$$[\eta]_{\alpha,\alpha/2;M\times[t_1,t_2]} := \sum_s \sum_{r=1}^N [\eta_s^r]_{\alpha,\alpha/2;\varphi_s(U_s)\times[t_1,t_2]}.$$

Subsequently, we define the parabolic Hölder norm on the space $C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$ to be

$$\|\eta\|_{\alpha,\alpha/2;M\times[t_1,t_2]} := |\eta|_{0;M\times[t_1,t_2]} + [\eta]_{\alpha,\alpha/2;M\times[t_1,t_2]}.$$

Next, for any nonnegative integer $k \geq 0$, the space of maps $C^{k+\alpha,(k+\alpha)/2}(M \times [t_1, t_2]; E)$ can be defined similarly using the same charts $\{U_s\}$, where the derivatives are taken with respect to the fixed connection $\hat{\nabla}$. For instance, the space $C^{k+\alpha,(k+\alpha)/2}(M \times [t_1, t_2]; E)$ consists of maps η : $M \times [t_1, t_2] \to E$ such that $\eta(t) \in C^k(M; E)$ is a C^k-differentiable section of E for each t with respect to $\hat{\nabla}$, and $\hat{\nabla}^k \eta(t) \in C^{\alpha,\alpha/2}(M; (T^*M)^{\otimes k} \otimes E)$ is a α-Hölder continuous section for each t. Then the associated norm can be defined similarly by

$$\|\eta\|_{k+\alpha,\frac{k+\alpha}{2};M\times[t_1,t_2]} := |\eta|_{0;M\times[t_1,t_2]} + \cdots + |\hat{\nabla}^k\eta|_{0;M\times[t_1,t_2]} + [\hat{\nabla}^k\eta]_{\alpha,\alpha/2;M\times[t_1,t_2]}.$$

The elliptic Hölder space $C^{k+\alpha}(M \times [t_1, t_2]; E)$ can also be defined similarly. Note that the Hölder norms defined this way are equivalent in different atlas.

We wish to work on weighted parabolic Hölder spaces in order to prove existence results. In the sequel, we define weighted norms on $M \times (0, T]$. Given a nonnegative integer $k \geq 0$, a Hölder exponent $\alpha \in (0, 1)$ and a positive real number γ, we define:

$$\|\eta\|_{C^{k,\alpha}_\gamma(M\times(0,T])} := \sum_{i=0}^{k} \sup_{\sigma\in(0,T]} \sigma^{\gamma+\frac{i}{2}} |\hat{\nabla}^i\eta|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sum_{i=0}^{k} \sup_{\sigma\in(0,T]} \sigma^{\gamma+\frac{\alpha}{2}+\frac{i}{2}} [\hat{\nabla}^i\eta]_{\alpha,\alpha/2;M\times[\frac{\sigma}{2},\sigma]}, \quad (1.2)$$

Throughout the note, we will work on the Banach spaces that we define below.

Definition 1.5. *Let M be a smooth closed compact manifold, E a vector bundle on M, and $\hat{g}$ a smooth background metric on M. Given a nonnegative integer $k \geq 0$, a Hölder exponent $\alpha \in (0, 1)$ and a positive real number γ, we define a weighted parabolic Hölder space on the space of sections by*

$$\begin{aligned} &C^{k,\alpha}_\gamma(M \times (0, T]; E) \qquad (1.3)\\ &:= \{\eta : M \times (0, T] \to E \mid \eta(t) \in C^k(M; E) \text{ for each } t \in (0, T],\ \|\eta\|_{C^{k,\alpha}_\gamma(M\times(0,T])} < \infty\}. \end{aligned}$$

We now state some easy consequences from the definition of these spaces:

Lemma 1.6.

(1) $\eta \in C_\gamma^{k,\alpha}(M \times (0,T])$ *implies* $\hat{\nabla}^j \eta \in C^{k-j,\alpha}_{\gamma+\frac{j}{2}}(M \times (0,T])$ *for* $j \leq k$*;*

(2) For any $\delta > 0$ *,* $\|\eta\|_{C^{k,\alpha}_{\gamma+\delta}(M\times(0,T])} \leq T^\delta \|\eta\|_{C^{k,\alpha}_\gamma(M\times(0,T])}$*. In particular, this implies that*

$$C^{k,\alpha}_\gamma(M \times (0,T]) \subset C^{k,\alpha}_{\gamma+\delta}(M \times (0,T]).$$

(3) Suppose that $\eta \in C^{k,\alpha}_\gamma(M \times (0,T])$*, then* $\eta \in C^{k,\beta}_\gamma(M \times (0,T])$ *for any* $\beta < \alpha$*;*

(4) Suppose that $\eta \in C^{k,\alpha}_\gamma(M \times (0,T])$ *and* $\zeta \in C^{k,\alpha}_\delta(M \times (0,T])$*. Then*

$$\eta * \zeta \in C^{k,\alpha}_{\gamma+\delta}(M \times (0,T]),$$

where $\eta * \zeta$ *means a bilinear combination with respect to* $\hat{g}$*. Moreover,*

$$\|\eta * \zeta\|_{C^{k,\alpha}_{\gamma+\delta}(M\times(0,T])} \leq K(\hat{g})\ \|\eta\|_{C^{k,\alpha}_\gamma(M\times(0,T])} \|\zeta\|_{C^{k,\alpha}_\delta(M\times(0,T])}.$$

Proof. Statements (1) and (2) follow from the definition. Statement (3) follows from the properties of parabolic Hölder spaces. Now we prove statement (4). Fix $\sigma \in (0,T]$. For any $j \leq k$, we have

$$\hat{\nabla}^j(\eta * \zeta) = \sum_{j_1+j_2=j} \hat{\nabla}^{j_1}\eta * \hat{\nabla}^{j_2}\zeta.$$

Let us denote $M_\sigma = M \times [\frac{\sigma}{2}, \sigma]$ for simplicity, we have

$$\begin{aligned} \sigma^{\gamma+\delta+\frac{j}{2}} |\hat{\nabla}^j(\eta * \zeta)|_{0;M_\sigma} &\leq K(\hat{g}) \sum_{j_1+j_2=j} \sigma^{\gamma+\frac{j_1}{2}} |\hat{\nabla}^{j_1}\eta|_{0;M_\sigma} \sigma^{\delta+\frac{j_2}{2}} |\hat{\nabla}^{j_2}\zeta|_{0;M_\sigma} \\ &\leq K(\hat{g})\ \|\eta\|_{C^{k,\alpha}_\gamma(M\times(0,T])} \|\zeta\|_{C^{k,\alpha}_\delta(M\times(0,T])} \end{aligned}$$

and

$$
\begin{aligned}
&\sigma^{\gamma+\delta+\frac{\alpha}{2}+\frac{j}{2}}[\hat{\nabla}^{j}(\eta * \zeta)]_{\alpha,\frac{\alpha}{2};M_\sigma} \\
&\leq K(\hat{g}) \sum_{j_1+j_2=j} \left(\sigma^{\gamma+\frac{j_1}{2}}|\hat{\nabla}^{j_1}\eta|_{0;M_\sigma}\sigma^{\delta+\frac{\alpha}{2}+\frac{j_2}{2}}[\hat{\nabla}^{j_2}\zeta]_{\alpha,\frac{\alpha}{2};M_\sigma} + \sigma^{\gamma+\frac{\alpha}{2}+\frac{j_1}{2}}[\hat{\nabla}^{j_1}\eta]_{\alpha,\frac{\alpha}{2};M_\sigma}\sigma^{\delta+\frac{j_2}{2}}|\hat{\nabla}^{j_2}\zeta|_{0;M_\sigma}\right) \\
&\leq K(\hat{g})\, \|\eta\|_{C^{k,\alpha}_{\gamma}(M\times(0,T])}\|\zeta\|_{C^{k,\alpha}_{\delta}(M\times(0,T])}.
\end{aligned}
$$

From these, statement (4) follows. □

Chapter 2: Linear Parabolic Equation with C^{α} Initial Data

Although our goal is to solve a boundary-value problem for PDE on a manifold with boundary, it is equivalent to work with PDE with rough initial data on a closed manifold. Fix a real number $I \in (0,1)$. Let $w(x,t), t \in [0,I]$ be a continuous family of Riemannian metrics on M. Given a section $\eta_0 \in C^{\alpha}(M;E)$, we consider the following parabolic system on vector bundle:

$$\begin{cases} \frac{\partial}{\partial t}\eta(x,t) - tr_w \hat{\nabla}^2 \eta(x,t) = F(x,t) & \text{on} \quad M \times (0,T] \\ \eta(x,0) = \eta_0(x) & \text{on} \quad M, \end{cases} \tag{2.1}$$

where $T \leq I$ and $F \in \Gamma(M \times (0,T];E)$, $w \in \Gamma(M \times [0,T]; \mathrm{Sym}^2(T^*M))$. Here $\hat{\nabla}$ is the Levi-Civita connection with respect to the background metric $\hat{g}$. Our goal is to prove solvability of (2.1). To do that, we need the uniform parabolicity assumption on w: there is a $\lambda > 0$ such that

$$\lambda |\xi|^2_{\hat{g}}(x) \geq w^{kl}(x,t)\xi_k(x)\xi_l(x) \geq \frac{1}{\lambda}|\xi|^2_{\hat{g}}(x) \tag{2.2}$$

for any $(x,t) \in M \times [0,I]$ and any $\xi \in \Gamma(TM)$.

We now state the main result of this Chapter. It will be utilized to study the existence of Ricci flow on manifolds with boundary in the next Chapter. The work in this Chapter is taken from [7].

Theorem 2.1. *Let $\alpha, \gamma \in (0,1)$ be given such that $\alpha > \gamma$. Let $k \geq 0$ be an non-negative integer. Suppose that*

(1) $\eta_0 \in C^{\alpha}(M;E)$;

(2) w satisfies the uniform parabolicity condition (2.2);

(3) $\|w\|_{\gamma,\frac{\gamma}{2};M\times[0,I]} + \|\hat{\nabla} w\|_{C^{k-1,\gamma}_{\frac{1}{2}}(M\times(0,I])} \leq A$ *when* $k \geq 1$*;*
or $\|w\|_{\gamma,\frac{\gamma}{2};M\times[0,I]} \leq A$ *when* $k = 0$.

Then there exits a positive constant $K = K(M, k, \hat{g}, A)$ *such that the following holds:*

For each $T \leq I$*, if* $F \in C^{k,\gamma}_{1-\frac{\alpha}{2}}(M \times (0,T]; E)$*, then there is an unique solution* η *to the system (2.1) such that*

$$\eta \in C^{\alpha,\frac{\alpha}{2}}(M \times [0,T]; E),\ \hat{\nabla}\eta \in C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M \times (0,T]; E).$$

Moreover, η *satisfies the estimate*

$$\|\eta\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla}\eta\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \leq K(\|F\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\eta_0\|_{\alpha;M}).$$

We will prove this theorem in the remainder of this Chapter.

2.1 Formulation of the proof

In the remainder of this Chapter, we fix $\alpha, \gamma \in (0,1)$ such that $\alpha > \gamma$. For every $T \in (0, I]$, we define the Banach space

$$\mathcal{W}_k(M \times (0,T]; E) := C^{k,\gamma}_{1-\frac{\alpha}{2}}(M \times (0,T]; E) \times C^{\alpha}(M; E)$$

whose elements are the pairs of sections $h = (F, \eta_0)$, where $F \in C^{k,\gamma}_{1-\frac{\alpha}{2}}(M \times (0,T]; E)$ and $\eta_0 \in C^{\alpha}(M; E)$. We equip $\mathcal{W}_k$ with the norm

$$||h||_{\mathcal{W}_k} := ||F||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + ||\eta_0||_{\alpha;M}.$$

Moreover, we define the norm $\|\cdot\|_{\mathcal{X}_k(M\times[0,T];E)}$ by

$$||\eta||_{\mathcal{X}_k} := ||\eta||_{\alpha,\frac{\alpha}{2};M\times[0,T]} + ||\hat{\nabla}\eta||_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])}.$$

We define the associated Banach space $\mathcal{X}_k(M \times [0,T]; E)$ by

$$\mathcal{X}_k(M \times [0,T]; E) := \{\eta : M \times [0,T] \to E |\ \|\eta\|_{\mathcal{X}_k} < \infty\}.$$

Subsequently $\mathcal{X}_{k+2}$ will serve as the solution space. We basically adapt the method in Chapter IV of [16] to the case of vector bundles. The idea of the proof is as follows:

Let $H : \mathcal{X}_{k+2} \to \mathcal{W}_k$ to be the linear operator that associates any $\eta \in \mathcal{X}_{k+2}$ to

$$H\eta = (L\eta, \eta(\cdot, 0)),$$

where $L\eta = \frac{\partial}{\partial t}\eta - tr_w \hat{\nabla}^2 \eta$. Then Theorem 2.1 can be interpreted to the solvability of

$$H\eta = h$$

for any $h \in \mathcal{W}_k(M \times (0,T]; E)$. It is equivalent to prove the existence of a bounded inverse operator H^{-1}. The key is to construct an operator $R : \mathcal{W}_k \to \mathcal{X}_{k+2}$ which satisfies

$$\begin{cases} HRh = h + Sh \\ RH\eta = \eta + G\eta \end{cases}$$

for some bounded operators $S : \mathcal{W}_k \to \mathcal{W}_k$ and $G : \mathcal{X}_{k+2} \to \mathcal{X}_{k+2}$. If their norms can be controlled such that $||S||, ||G|| < 1$, then it follows from an elementary argument that H^{-1} exists.

2.2 Construction of an approximated solution

Let $h = (F, \eta_0) \in \mathcal{W}_k$ be given. To construct the operator R, we consider a system of PDE on each chart U_s. As in the setup given in Chapter 1.3, we let $\tilde{\varphi}_s : \pi^{-1}(U_s) \to U_s \times \mathbb{R}^N$ be the local trivialization of E on the open set U_s, and let $\{\mathbf{e}_r^s\}_{r=1,..,N}$ be the canonical local frame of $\pi^{-1}(U_s)$

with respect to the trivialization. Then F can be written as $F = \sum_{r=1}^{N} F_s^r \mathbf{e}_r^s$, similarly for η, where we abbreviate $F_s^r(x,t) = F_s^r \circ (\varphi_s^{-1}(x), t)$ for $x \in \varphi_s(U_s) \cong \mathbb{R}^n$. On each chart U_s, we consider the following parabolic system:

$$\begin{cases} \frac{\partial}{\partial t}\eta_s^r(x,t) - w^{kl}(x,t)\frac{\partial^2}{\partial x_k \partial x_l}\eta_s^r(x,t) = F_s^r(x,t) & \text{on} \quad \mathbb{R}^n \times (0,T], \quad r = 1,..,N \\ \eta_s^r(x,0) = (\eta_0)_s^r(x) & \text{on} \quad \mathbb{R}^n, \quad r = 1,..,N. \end{cases} \tag{2.3}$$

We see that the system is equivalent to N uncoupled scalar equations, one for each η_s^r. We will prove the existences and the uniqueness for the uncoupled problems in this subsection. We begin with an auxiliary lemma for linear parabolic PDEs.

Lemma 2.2. *Let $\alpha, \gamma \in (0,1)$ be given such that $\alpha > \gamma$. Suppose that*

(1) $a_{ij}(x,t) \in C^{\gamma,\frac{\gamma}{2}}(\mathbb{R}^n \times [0,T])$ and satisfies the uniform parabolicity condition. i.e. there is $\lambda > 0$ such that $\frac{1}{\lambda}\delta_{ij} < a_{ij}(x,t) < \lambda\delta_{ij}$;

(2) $\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} < \infty$ and $\|u_0\|_{\alpha;\mathbb{R}^n} < \infty$.

Then the initial-value problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - a_{kl}(x,t)\frac{\partial^2}{\partial x_k \partial x_l}u(x,t) = f(x,t) & \textit{on} \quad \mathbb{R}^n \times (0,T] \\ u(x,0) = u_0(x) & \textit{on} \quad \mathbb{R}^n \end{cases} \tag{2.4}$$

has a unique solution u, where $u \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])$ and $Du \in C^{1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n \times (0,T])$. Moreover, u satisfies the estimate

$$\|u\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} + \|D_x u\|_{C^{1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} \leq K(\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n}). \tag{2.5}$$

Here K is a constant depending only on λ, $\|a_{ij}\|_{\gamma,\frac{\gamma}{2}}$.

Proof. In the sequel of the proof, K will denote a constant depending only on $\lambda, \|a_{ij}\|_{\gamma,\frac{\gamma}{2}}, \alpha, \gamma$ unless otherwise specified.

Step1: We use the single layer potential method to construct a unique solution u to (2.4). Given $(\xi, \tau) \in \mathbb{R}^n \times [0, T]$,let $\Gamma(x, t; \xi, \tau)$ to be the fundamental solution to

$$\frac{\partial}{\partial t}u - a_{kl}(x,t)\frac{\partial^2}{\partial x_k \partial x_l}u = 0$$

on $\mathbb{R}^n \times (\tau, T]$ such that $\Gamma(x, t; \xi, \tau) \to \delta(x - \xi)$ as $t \to \tau$ in the sense of distribution. We claim that the formula

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(x,t;\xi,\tau) f(\xi,\tau) d\xi d\tau + \int_{\mathbb{R}^n} \Gamma(x,t;\xi,0) u_0(\xi) d\xi \tag{2.6}$$

gives a solution to the system (2.4), and uniqueness would follow from the maximum principle. It is well known from [9] that the formula (2.6) gives a unique solution to (2.4) provided that f in a Hölder-continuous function on $\mathbb{R}^n \times [0, T]$. In our case, the assumption $\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n \times (0,T])} < \infty$ implies that f is Hölder-continuous on $\mathbb{R}^n \times [\sigma/2, \sigma]$ for any $\sigma \in (0, T]$, and that

$$\tau^{1-\frac{\alpha}{2}}|f|_{0;\mathbb{R}^n\times[\tau/2,\tau]} + \tau^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[f]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\tau/2,\tau]} \leq K$$

for each $\tau \in (0, T]$. Hence

$$\begin{aligned}
\|f(\cdot,\tau)\|_{\gamma;\mathbb{R}^n} &= |f(\cdot,\tau)|_{0;\mathbb{R}^n} + [f(\cdot,\tau)]_{\gamma;\mathbb{R}^n} \\
&\leq |f|_{0;\mathbb{R}^n\times[\tau/2,\tau]} + [f]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\tau/2,\tau]} \\
&\leq K\tau^{\frac{\alpha}{2}-1} + K\tau^{\frac{\alpha}{2}-1-\frac{\gamma}{2}} \\
&\leq K\tau^{\frac{\alpha}{2}-1-\frac{\gamma}{2}}
\end{aligned}$$

for each $\tau \in (0, T]$. In particular, the elliptic Hölder bound for $f(\cdot, \tau)$ is integrable over

$\tau \in (0, t)$ for any $t \in (0, T]$. This implies that the proof of Theorem 9 in Chapter 1 of [9] still works so that (2.6) satisfies the evolution equation in (2.4) for every $t > 0$. To show that (2.6) gives the correct initial condition, it suffices to show that the first integral in RHS of (2.6) tends to zero as $t \to 0$.

Let us write (2.6) as $u := u_1 + u_2$, where u_1 stands for the first term in RHS of (2.6), and u_2 stands for the second term in RHS of (2.6). We recall the estimates for the fundamental solution

$$\left|D_t^r D_x^s \Gamma(x, t; \xi, \tau)\right| \leq K(t-\tau)^{-\frac{n+2r+s}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) \quad , 2r + s \leq 2 \tag{2.7}$$

given in page 376 of [16]. For any $(x, t) \in \mathbb{R}^n \times [0, T]$, we estimate

$$\begin{aligned}
&|u_1(x,t)| \\
&\leq \int_0^t \int_{\mathbb{R}^n} |\Gamma(x,t;\xi,\tau) f(\xi,\tau)| d\xi d\tau \\
&\leq K \int_0^t \int_{\mathbb{R}^n} (t-\tau)^{-\frac{n}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) |f(\xi,\tau)| \; d\xi d\tau \\
&\leq K \int_0^t \int_{\mathbb{R}^n} (t-\tau)^{-\frac{n}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) \tau^{-1+\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \; d\xi d\tau \\
&\leq K \int_0^t \int_0^\infty \rho^{n-1} \exp(-\frac{1}{K}\rho^2) \tau^{-1+\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \; d\rho d\tau \\
&\leq K t^{\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}.
\end{aligned} \tag{2.8}$$

Since $u_2(x, t)$ converges to $u_0(x)$ as $t \to 0$, this shows that $u(x, t)$ converges to $u_0(x)$ as $t \to 0$, and that u is continuous on $\mathbb{R}^n \times [0, T]$. For the integral u_2, it is easy to see that

$$|u_2(x, t)| \leq K \sup_{\mathbb{R}^n} |u_0| \tag{2.9}$$

for any $(x,t) \in \mathbb{R}^n \times [0,T]$. Thus we have obtained a C^0 estimate for $u(x,t)$:

$$\|u\|_{0;\mathbb{R}^n\times[0,T]} \leq K(T^{\frac{\alpha}{2}}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|u_0\|_{0;\mathbb{R}^n}) \tag{2.10}$$

<u>Step2</u>: In this step, we are going to derive

$$\|u\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} \leq K(\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n}). \tag{2.11}$$

We first bound the Hölder semi-norm for u_1. That is, we seek the following inequalities for any $x, y \in \mathbb{R}^n$ and $s, t \in [0,T]$:

$$\begin{cases} |u_1(x,t) - u_1(y,t)| \leq K|x-y|^{\alpha}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \\ |u_1(x,t) - u_1(x,s)| \leq K|t-s|^{\frac{\alpha}{2}}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \end{cases} \tag{2.12}$$

To derive the first inequality in (2.12), we divide $\mathbb{R}^n$ into $A_1 = \{\xi \in \mathbb{R}^n : |x-\xi| > 2|x-y|\}$ and $A_2 = \mathbb{R}^n - A_1$. Using the estimates for fundamental solutions (2.7), we obtain

$$\begin{aligned} &|u_1(x,t) - u_1(y,t)| \qquad (2.13)\\ &\leq \int_0^t \int_{A_1} \sup_{z\in\overline{xy}} |D_x\Gamma(z,t;\xi,\tau)||x-y||f(\xi,\tau)|d\xi d\tau \\ &\quad + \int_0^t \int_{A_2} (|\Gamma(x,t;\xi,\tau)| + |\Gamma(y,t;\xi,\tau)|)|f(\xi,\tau)|d\xi d\tau \\ &\leq K|x-y| \int_0^t \int_{\{|x-\xi|>2|x-y|\}} (t-\tau)^{-\frac{n+1}{2}} \sup_{z\in\overline{xy}} \exp\left(-\frac{|z-\xi|^2}{K(t-\tau)}\right)|f(\xi,\tau)|d\xi d\tau \\ &\quad + K\int_0^t \int_{\{|x-\xi|<2|x-y|\}} (t-\tau)^{-\frac{n}{2}} \left(\exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) + \exp\left(-\frac{|y-\xi|^2}{K(t-\tau)}\right)\right)|f(\xi,\tau)|d\xi d\tau. \end{aligned}$$

Note that if $\xi \in A_1$ and z is a point on the segment $\overline{xy}$, then $|z-\xi| > \frac{1}{2}|x-\xi|$. Thus

$$\sup_{z\in\overline{xy}} \exp\left(-\frac{|z-\xi|^2}{K(t-\tau)}\right) \leq \exp\left(-\frac{|x-\xi|^2}{4K(t-\tau)}\right).$$

We observe that for any $m \geq \alpha$, we have

$$(t-\tau)^{-\frac{m}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) = \frac{1}{|t-\tau|^{\frac{\alpha}{2}}|x-\xi|^{m-\alpha}} \frac{|x-\xi|^{m-\alpha}}{|t-\tau|^{\frac{m-\alpha}{2}}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) \tag{2.14}$$
$$\leq \frac{K}{|t-\tau|^{\frac{\alpha}{2}}|x-\xi|^{m-\alpha}}.$$

Using the above estimate with $m = n+1$ for the first term and $m = n$ for the second term respectively in the last inequality in (2.13), we proceed as in (2.8) to obtain that

$$\begin{aligned} &|u_1(x,t) - u_1(y,t)| \\ &\leq K|x-y| \int_0^t \int_{\{|x-\xi|>2|x-y|\}} \frac{1}{|t-\tau|^{\frac{\alpha}{2}}|x-\xi|^{n+1-\alpha}} |f(\xi,\tau)| d\xi d\tau \\ &\quad + K \int_0^t \Big(\int_{\{|x-\xi|<2|x-y|\}} \frac{1}{|t-\tau|^{\frac{\alpha}{2}}|x-\xi|^{n-\alpha}} |f(\xi,\tau)| d\xi \\ &\qquad + \int_{\{|y-\xi|<3|x-y|\}} \frac{1}{|t-\tau|^{\frac{\alpha}{2}}|y-\xi|^{n-\alpha}} |f(\xi,\tau)| d\xi \Big) d\tau \\ &\leq K|x-y|^{\alpha} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}, \end{aligned} \tag{2.15}$$

where we have used the inequality

$$\int_0^t \frac{\tau^{\frac{\alpha}{2}-1}}{(t-\tau)^{\frac{\alpha}{2}}} d\tau \leq K(\alpha)$$

in deriving the last line. Next, to derive the second inequality in (2.12), we assume without loss of generality that $s < t$. and we divide $\mathbb{R}^n$ into $A_3 = \{\xi \in \mathbb{R}^n : |x-\xi| > \sqrt{t-s}\}$ and $A_4 = \mathbb{R}^n - A_3$. Then

$$\begin{aligned} &|u_1(x,t) - u_1(x,s)| \\ &\leq \int_0^s \int_{A_3} \sup_{\bar{t}\in[s,t]} |D_t\Gamma(x,\bar{t};\xi,\tau)||t-s||f(\xi,\tau)| d\xi d\tau + \int_s^t \int_{A_3} |\Gamma(x,t;\xi,\tau)||f(\xi,\tau)| d\xi d\tau \\ &\quad + \int_0^t \int_{A_4} |\Gamma(x,t;\xi,\tau)||f(\xi,\tau)| d\xi d\tau + \int_0^s \int_{A_4} |\Gamma(x,s;\xi,\tau)||f(\xi,\tau)| d\xi d\tau \end{aligned} \tag{2.16}$$

For instance, using (2.7) and (2.14) with $m = n+1$, we can estimate the second term in the RHS of the above inequality

$$\begin{aligned}
&\int_s^t \int_{A_3} |\Gamma(x,t;\xi,\tau)||f(\xi,\tau)|d\xi d\tau \\
&\leq K \int_s^t (t-\tau)^{\frac{1}{2}} \int_{A_3} (t-\tau)^{-\frac{n+1}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) |f(\xi,\tau)|d\xi d\tau \\
&\leq K \int_s^t (t-s)^{\frac{1}{2}} \int_{|x-\xi|>\sqrt{t-s}} \frac{1}{|t-\tau|^{\frac{\alpha}{2}}|x-\xi|^{n+1-\alpha}} |f(\xi,\tau)|d\xi d\tau \\
&\leq K|t-s|^{\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \int_0^t \frac{\tau^{\frac{\alpha}{2}-1}}{|t-\tau|^{\frac{\alpha}{2}}} d\tau.
\end{aligned}$$

We can estimate the other terms in the RHS of (2.16) similarly. Hence we obtain

$$\begin{aligned}
&|u_1(x,t) - u_1(x,s)| \\
&\leq K|t-s|^{\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \left(\int_0^t \frac{\tau^{\frac{\alpha}{2}-1}}{|t-\tau|^{\frac{\alpha}{2}}} d\tau + \int_0^s \frac{\tau^{\frac{\alpha}{2}-1}}{|s-\tau|^{\frac{\alpha}{2}}} d\tau\right) \\
&\leq K|t-s|^{\frac{\alpha}{2}} \|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}.
\end{aligned}$$

From which (2.12) follows.

In the remainder of this step, we derive the following estimate for the Hölder semi-norm of u_2:

$$[u_2]_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} \leq K\ [u_0]_{\alpha;\mathbb{R}^n}.$$

That is, for any $x, y \in \mathbb{R}^n$ and $s, t \in [0,T]$, we claim that:

$$\begin{cases} |u_2(x,t) - u_2(y,t)| \leq K|x-y|^{\alpha}[u_0]_{\alpha;\mathbb{R}^n} \\ |u_2(x,t) - u_2(x,s)| \leq K|t-s|^{\frac{\alpha}{2}}[u_0]_{\alpha;\mathbb{R}^n} \end{cases}. \tag{2.17}$$

To derive the first inequality, we break $u_2(x,t) - u_2(y,t)$ into two integrals as follows:

$$\begin{aligned} u_2(x,t) - u_2(y,t) = &\int_{\mathbb{R}^n} (\Gamma(x,t;\xi,0) - \Gamma(y,t;\xi,0))(u_0(\xi) - u_0(x))d\xi \\ &+ u_0(x) \int_{\mathbb{R}^n} (\Gamma(x,t;\xi,0) - \Gamma(y,t;\xi,0))d\xi \\ := &\ J_1 + J_2. \end{aligned}$$

For the integral J_1, we use (2.7) to estimate

$$\begin{aligned} |J_1| \leq &\int_{A_1} \sup_{z\in\overline{xy}} |D_x\Gamma(z,t;\xi,0)||x-y||u_0(x) - u_0(\xi)|d\xi \\ &+ \int_{A_2} (|\Gamma(x,t;\xi,0)| + |\Gamma(y,t;\xi,0)|)|u_0(x) - u_0(\xi)|d\xi \\ \leq &\ K|x-y| \int_{\{|x-\xi|>2|x-y|\}} (t-\tau)^{-\frac{n+1}{2}} \exp\left(-\frac{|x-\xi|^2}{4K(t-\tau)}\right) |x-\xi|^\alpha [u_0]_{\alpha;\Omega}\, d\xi \\ &+ K \int_{\{|x-\xi|<2|x-y|\}} (t-\tau)^{-\frac{n}{2}} \left(\exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) + \exp\left(-\frac{|y-\xi|^2}{K(t-\tau)}\right)\right) |x-\xi|^\alpha [u_0]_{\alpha;\Omega}\, d\xi. \end{aligned} \tag{2.18}$$

Similar to the derivation in (2.13) to (2.15), we apply (2.14) with $\alpha = 0$ to above, and we subsequently obtain

$$|J_1| \leq K|x-y|^\alpha [u_0]_{\alpha;\mathbb{R}^n}.$$

On the other hand, we see that the second integral J_2 vanishes since the fundamental solution satisfies $\int_\Omega \Gamma(x,t;\xi,0)d\xi = \int_\Omega \Gamma(y,t;\xi,0)d\xi = 1$.

Next, we preform the same procedure on the term $u_2(x,t) - u_2(x,s)$. We break it into two integrals:

$$u_2(x,t) - u_2(x,s) = \int_{\mathbb{R}^n} (\Gamma(x,t;\xi,0) - \Gamma(x,s;\xi,0))(u_0(\xi) - u_0(x))d\xi$$
$$+ u_0(x)\int_{\mathbb{R}^n} (\Gamma(x,t;\xi,0) - \Gamma(x,s;\xi,0))d\xi$$
$$:= J_3 + J_4.$$

Again, the integral J_4 vanishes. Using a similar argument as in (2.16) and (2.18), we have

$$|J_3| \le \int_{A_3} \sup_{\bar{t}\in[s,t]} |D_t\Gamma(x,\bar{t};\xi,0)||t-s||u_0(x) - u_0(\xi)|d\xi$$
$$+ \int_{A_4} (|\Gamma(x,t;\xi,0)| + |\Gamma(x,s;\xi,0)|)|u_0(x) - u_0(\xi)|d\xi$$
$$\le K|t-s|^{\frac{\alpha}{2}}\,[u_0]_{\alpha;\Omega}. \tag{2.19}$$

From this, (2.17) follows.

Therefore we summarize that

$$[u]_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} \le K(\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + [u_0]_{\alpha;\mathbb{R}^n}). \tag{2.20}$$

From (2.10) and (2.20), the estimate (2.11) follows.

Step3: In this step we claim that the solution $u(x,t)$ satisfies the estimates

$$\|D_x u\|_{C^{1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} \le K(\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + [u_0]_{\alpha;\mathbb{R}^n}). \tag{2.21}$$

Fix a point $z \in \mathbb{R}^n$ and $\sigma \in (0,T]$, we consider the parabolic cylinder

$$P_\sigma(z) := B_{\sqrt{\sigma}}(z) \times [0,\sigma]. \tag{2.22}$$

We define a function v on the parabolic cylinder $P_1(0) = B_1(0) \times [0, 1]$ by scaling:

$$v(y, s) := u(z + \sigma^{\frac{1}{2}} y, \sigma s). \tag{2.23}$$

Let $\chi(s)$ be a three times continuously differentiable cutoff function on $[0, 1]$ such that

$$\chi(s) = \begin{cases} 1, & \text{if} \quad s \in [\frac{1}{2}, 1] \\ 0, & \text{if} \quad s \in [0, \frac{1}{4}] \end{cases} \tag{2.24}$$

and

$$|D_s^j \chi(s)| \leq C \quad , j = 0, 1, 2. \tag{2.25}$$

We now define a function $\tilde{v}$ on $P_1(0)$ by

$$\tilde{v}(y, s) := \chi(s) \left(v(y, s) - v(0, 1) \right).$$

Then $\tilde{v}$ satisfies

$$\begin{cases} \frac{\partial \tilde{v}}{\partial s} - a_{ij}(z + \sigma^{\frac{1}{2}} y, \sigma s) \frac{\partial^2 \tilde{v}}{\partial y_i \partial y_j} = \tilde{f}(y, s) & \text{on} \quad B_1(0) \times (0, 1] \\ \tilde{v}(y, 0) = 0 & \text{on} \quad B_1(0), \end{cases} \tag{2.26}$$

where $\tilde{f}(y, s) = \chi'(s)(v(y, s) - v(0, 1)) + \sigma \chi(s) f(z + \sigma^{\frac{1}{2}} y, \sigma s)$.

Note that $\tilde{f} \in C^{\gamma, \frac{\gamma}{2}}(B_1(0) \times [0, 1])$ after change of variables since by the previous step we have $u \in C^{\alpha, \frac{\alpha}{2}}(B_{\sqrt{\sigma}}(z) \times [0, \sigma])$ and $\alpha \geq \gamma$. Hence by standard parabolic Schauder interior estimate $\tilde{v} \in C^{2+\gamma,\frac{2+\gamma}{2}}(B_{\frac{1}{2}}(0) \times [\frac{1}{2}, 1])$ and we have the following estimate:

$$|\tilde{v}|_{2+\gamma,\frac{2+\gamma}{2};B_{\frac{1}{2}}(0)\times[\frac{1}{2},1]} \leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}})(|\chi'(v - v(0,1))|_{\gamma,\frac{\gamma}{2};B_1(0)\times[0,1]} + \sigma|\chi f|_{\gamma,\frac{\gamma}{2};B_1(0)\times[0,1]}) \tag{2.27}$$
$$\leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}}, C)(|v - v(0,1)|_{\gamma,\frac{\gamma}{2};B_1(0)\times[\frac{1}{4},1]} + \sigma|f|_{\gamma,\frac{\gamma}{2};B_1(0)\times[\frac{1}{4},1]}).$$

Then it follows from the rescaling $u(x,t) = v(\sigma^{-\frac{1}{2}}(x-z), \sigma^{-1}t)$ that

$$\begin{aligned}
&\sum_{i=1}^{2} \sigma^{\frac{i}{2}}|D_x^i u|_{0;B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]} + \sum_{i=1}^{2} \sigma^{\frac{i}{2}+\frac{\gamma}{2}}[D_x^i u]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]} \qquad (2.28)\\
&\leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}}, C)\Big(|u - u(z,\sigma)|_{0;B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]} + \sigma^{\frac{\gamma}{2}}[u - u(z,\sigma)]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}\\
&\qquad + \sigma|f|_{0;B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]} + \sigma^{1+\frac{\gamma}{2}}[f]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}\Big)\\
&\leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}}, C)\Big(|u - u(z,\sigma)|_{0;B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]} + \sigma^{\frac{\gamma}{2}}[u - u(z,\sigma)]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}\\
&\qquad + \sigma^{\frac{\alpha}{2}}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}\Big).
\end{aligned}$$

Observe that $[f]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}$ can be controlled by norms of f on $[\frac{\sigma}{4}, \frac{\sigma}{2}]$ and $[\frac{\sigma}{2}, \sigma]$. Indeed, for any $x \in B_{\sqrt{\sigma}}(z)$ and $\tau, t \in [\frac{\sigma}{4}, \sigma]$, we have

$$\frac{|f(x,t) - f(x,\tau)|}{|t-\tau|^{\frac{\gamma}{2}}} \leq \frac{|f(x,t) - f(x,\frac{\sigma}{2})|}{|t-\tau|^{\frac{\gamma}{2}}} + \frac{|f(x,\tau) - f(x,\frac{\sigma}{2})|}{|t-\tau|^{\frac{\gamma}{2}}}.$$

It suffices to consider the case where $\tau \leq \frac{\sigma}{2} \leq t$, then

$$\begin{aligned}
\frac{|f(x,t) - f(x,\tau)|}{|t-\tau|^{\frac{\gamma}{2}}} &\leq \frac{|f(x,t) - f(x,\frac{\sigma}{2})|}{|t-\frac{\sigma}{2}|^{\frac{\gamma}{2}}} + \frac{|f(x,\tau) - f(x,\frac{\sigma}{2})|}{|\tau-\frac{\sigma}{2}|^{\frac{\gamma}{2}}}\\
&\leq [f]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\frac{\sigma}{2}]} + [f]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{2},\sigma]}.
\end{aligned}$$

This implies

$$\sigma^{1+\frac{\gamma}{2}}[f]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]} \leq \sigma^{\frac{\alpha}{2}}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}.$$

Putting this inequality into (2.28), we obtain

$$\sum_{i=1}^{2}\sigma^{\frac{i}{2}}|D_x^i u|_{0;B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]}+\sum_{i=1}^{2}\sigma^{\frac{i}{2}+\frac{\gamma}{2}}[D_x^i u]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]} \tag{2.29}$$
$$\leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}},C)\Big(|u-u(z,\sigma)|_{0;B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}+\sigma^{\frac{\gamma}{2}}[u-u(z,\sigma)]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}$$
$$+\sigma^{\frac{\alpha}{2}}\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}\Big).$$

Now, note that the Hölder estimate for $[u]_{\alpha,\frac{\alpha}{2}}$ in (2.20) implies that

$$|u-u(z,\sigma)|_{0;B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}\leq K\sigma^{\frac{\alpha}{2}}\,[u]_{\alpha,\frac{\alpha}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}$$
$$\leq K\sigma^{\frac{\alpha}{2}}(\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}+[u_0]_{\alpha;\mathbb{R}^n})$$

and

$$[u-u(z,\sigma)]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}\leq K\sigma^{\frac{\alpha}{2}-\frac{\gamma}{2}}\,[u]_{\alpha,\frac{\alpha}{2};B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{4},\sigma]}$$
$$\leq K\sigma^{\frac{\alpha}{2}-\frac{\gamma}{2}}(\|f\|_{C^{0,\alpha}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}+[u_0]_{\alpha;\mathbb{R}^n}).$$

Putting these facts into (2.29), we obtain

$$\sum_{i=1}^{2}\sigma^{\frac{i}{2}-\frac{\alpha}{2}}|D_x^i u|_{0;B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]}+\sum_{i=1}^{2}\sigma^{\frac{i}{2}-\frac{\alpha}{2}+\frac{\gamma}{2}}[D_x^i u]_{\gamma,\frac{\gamma}{2};B_{\sqrt{\sigma}/2}(z)\times[\frac{\sigma}{2},\sigma]} \tag{2.30}$$
$$\leq K(\|a_{ij}\|_{\gamma,\frac{\gamma}{2}},C)\Big(\;\|f\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}+\|u_0\|_{\alpha;\mathbb{R}^n}\Big).$$

Since $z\in\mathbb{R}^n$ and $\sigma\in(0,T]$ are arbitrary, the desired estimate (2.21) follows. Putting (2.11) and (2.21) together, the lemma is thus proved.

□

Using a standard bootstrap argument, we can improve the regularity of u. We have the following auxiliary lemmas for higher order regularity:

Lemma 2.3. *Let $\alpha, \gamma \in (0,1)$ be given such that $\alpha > \gamma$. Suppose that*

(1) $a_{ij}(x,t) \in C^{\gamma,\frac{\gamma}{2}}(\mathbb{R}^n \times [0,T])$ and satisfies the uniform parabolicity condition. i.e. there is $\lambda > 0$ such that $\frac{1}{\lambda}\delta_{ij} < a_{ij}(x,t) < \lambda\delta_{ij}$;

(2) $\|D_x a_{ij}\|_{C^{k-1,\gamma}_{\frac{1}{2}}(\mathbb{R}^n\times(0,T])} < \infty$ if $k \geq 1$;

(3) $\|f\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} < \infty$ and $\|u_0\|_{\alpha;\mathbb{R}^n} < \infty$.

Then the initial-value problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - a_{kl}(x,t)\frac{\partial^2}{\partial x_k \partial x_l}u(x,t) = f(x,t) & on \quad \mathbb{R}^n \times (0,T] \\ u(x,0) = u_0(x) & on \quad \mathbb{R}^n \end{cases} \tag{2.31}$$

has a unique solution u, where $u \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])$ and $Du \in C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n \times (0,T])$, such that

$$\|u\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} + \|D_x u\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} \leq K(\|f\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n}). \tag{2.32}$$

Here K is a constant depending only on $k, \mathbb{R}^n$, $\|a_{ij}\|_{\alpha,\frac{\alpha}{2}}$ and $\|a_{ij}\|_{C^{k,\alpha}_0}$.

Proof. We intend to prove the lemma by induction on k. The case $k = 0$ follows from Lemma 2.2. We suppose that the conditions (2) and (3) in the lemma holds with k replaced by $k+1$. That is,

- $\|D_x a_{ij}\|_{C^{k,\gamma}_{\frac{1}{2}}(\mathbb{R}^n\times(0,T])} < \infty$;
- $\|f\|_{C^{k+1,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} < \infty$ and $\|u_0\|_{\alpha;\mathbb{R}^n} < \infty$.

Moreover, the induction hypothesis implies that the estimate

$$\|u\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]} + \|D_x u\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} \leq K(\|f\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n})$$

holds. Let us fix $\sigma \in (0, T]$. Let $\chi(t)$ be a three times continuously differentiable cutoff function on $[0, \sigma]$ such that

$$\chi(t) = \begin{cases} 1, & \text{if} \quad s \in [\frac{\sigma}{2}, \sigma] \\ 0, & \text{if} \quad s \in [0, \frac{\sigma}{4}] \end{cases} \tag{2.33}$$

and

$$|D_t^h \chi(t)| \leq C\sigma^{-h} \quad , h = 0, 1, 2. \tag{2.34}$$

It follows that the $\frac{\gamma}{2}$-Hölder norm for χ and χ' have estimates

$$[\chi]_{\frac{\gamma}{2};[0,\sigma]} \leq C\sigma^{-\frac{\gamma}{2}} \quad \text{and} \quad [\chi']_{\frac{\gamma}{2};[0,\sigma]} \leq C\sigma^{-1-\frac{\gamma}{2}}.$$

On $\mathbb{R}^n \times (0, \sigma]$, we define $\bar{u}(x,t) := \chi(t) D_x^{k+1} u(x,t)$, then $\bar{u}$ is the solution to the system

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}(x,t) - a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\bar{u}(x,t) = \bar{f}(x,t) & \text{on} \quad \mathbb{R}^n \times (0, \sigma] \\ \bar{u}(x, 0) = 0 & \text{on} \quad \mathbb{R}^n, \end{cases}$$

where $\bar{f}$ is given by

$$\bar{f} = \chi' D_x^{k+1} u + \chi D^{k+1} f + \chi \sum_{r=0}^{k} \binom{k+1}{r} D^{k+1-r} a_{ij} D^r D_{ij}^2 u. \tag{2.35}$$

Note that by the induction hypothesis we have $\bar{f} \in C^{\gamma, \frac{\gamma}{2}}(\mathbb{R}^n \times [0, \sigma])$. Then we apply Theorem 5.1 in [16] to obtain $D_x^{k+1} u \in C^{2+\gamma, \frac{2+\gamma}{2}}(\mathbb{R}^n \times [\frac{\sigma}{2}, \sigma])$ and the estimate

$$\sigma |D^{k+3} u|_{0;\mathbb{R}^n \times [\frac{\sigma}{2}, \sigma]} + \sigma^{1+\frac{\gamma}{2}} [D^{k+3} u]_{\gamma, \frac{\gamma}{2};\mathbb{R}^n \times [\frac{\sigma}{2}, \sigma]} \leq K\Big(\sigma |\bar{f}|_{0;\mathbb{R}^n \times [0,\sigma]} + \sigma^{1+\frac{\gamma}{2}} [\bar{f}]_{\gamma, \frac{\gamma}{2};\mathbb{R}^n \times [0,\sigma]} \Big).$$

Hence,

$$\begin{aligned}
&\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}}|D^{k+3}u|_{0;\mathbb{R}^n\times[\frac{\sigma}{2},\sigma]}+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[D^{k+3}u]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{2},\sigma]} \\
&\leq K\Big(\ \sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}}|\bar{f}|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\bar{f}]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}\Big).
\end{aligned} \tag{2.36}$$

We can check that (2.36) implies the estimate (2.32). For instance, we can check for the Hölder-semi norm term of $\bar{f}$ in the last line of (2.36). We have

$$\begin{aligned}
\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\chi' D_x^{k+1}u]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} &\leq K\Big(\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k}{2}}|D_x^{k+1}u|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k}{2}+\frac{\gamma}{2}}[D_x^{k+1}u]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}\Big) \\
&\leq K(\|u\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^n\times[0,T]}+\|D_xu\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}) \\
&\leq K(\|f\|_{C^{k-1,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}+\|u_0\|_{\alpha;\mathbb{R}^n})
\end{aligned}$$

and

$$\begin{aligned}
\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\chi D^{k+1}f]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} &\leq K\Big(\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}}|D^{k+1}f|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[D^{k+1}f]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}\Big) \\
&\leq K\|f\|_{C^{k+1,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}.
\end{aligned}$$

Also, for each integer $r\in[0,k]$ we have

$$\begin{aligned}
&\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\chi D^{k+1-r}a_{ij}D^ru_{ij}]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} \\
&\leq K\Big(\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}}|D^{k+1-r}a_{ij}D^{r+2}u|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}|D^{k+1-r}a_{ij}|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}[D^{r+2}u]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} \\
&\qquad+\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[D^{k+1-r}a_{ij}]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}|D^{r+2}u|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]}\Big) \\
&\leq K\|D_xa_{ij}\|_{C^{k,\gamma}_{\frac{1}{2}}(\mathbb{R}^n\times(0,T])}\ \|D_xu\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,\sigma])} \\
&\leq K\|D_xa_{ij}\|_{C^{k,\gamma}_{\frac{1}{2}}(\mathbb{R}^n\times(0,T])}\ \big(\|f\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])}+\|u_0\|_{\alpha;\mathbb{R}^n}\big).
\end{aligned}$$

From these, we have

$$\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\bar{f}]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} \leq K(a_{ij})\left(\|f\|_{C^{k+1,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n}\right). \tag{2.37}$$

And similarly we get

$$\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}}|\bar{f}|_{0;\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} + \sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k+2}{2}+\frac{\gamma}{2}}[\bar{f}]_{\gamma,\frac{\gamma}{2};\mathbb{R}^n\times[\frac{\sigma}{4},\sigma]} \leq K(a_{ij})\left(\|f\|_{C^{k+1,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T])} + \|u_0\|_{\alpha;\mathbb{R}^n}\right). \tag{2.38}$$

Therefore we have completed the induction. □

2.3 Completion of the proof

Now we are on the ground of constructing the operator $R : \mathcal{W}_k \to \mathcal{X}_{k+2}$. To begin with, let $\{\rho_s\}$ be the partition of unity subordinate to the charts $\{U_s\}$. On the basis of Lemma 2.2 and Lemma 2.3, there is a unique solution set $\{\eta^r_s\}$ to the system (2.3) such that for each (r, s),

$$\eta^r_s \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^n \times [0,T]), \quad D_x\eta^r_s \in C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^n \times (0,T])$$

and

$$\|\eta^r_s\|_{\alpha,\frac{\alpha}{2};\varphi_s(U_s)\times[0,T]} + \|D_x\eta^r_s\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\varphi_s(U_s)\times(0,T])} \leq K(\|F\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\eta_0\|_{\alpha;M}). \tag{2.39}$$

We then define $R_s : \mathcal{W}_k \to \Gamma(U_s \times [0,T], \pi^{-1}(U_s))$ by

$$(R_sh)(p,t) = \rho_s(p)\sum_r \eta^r_s(\varphi_s(p),t)\mathbf{e}^s_r(p) \tag{2.40}$$

for any $(p,t) \in U_s \times [0,T]$. Moreover, by setting R_sh to be the zero section outside the support of ρ_s, we can extend R_sh to a section of $\Gamma(M \times [0,T], E)$. Now, we define a map $R : \mathcal{W}_k \to$

$\Gamma(M \times [0,T], E)$ by

$$Rh = \sum_s R_s h. \tag{2.41}$$

From the construction of R it is clear that $Rh \in \mathcal{X}_{k+2}$ for any $h \in \mathcal{W}_k$. By (2.39), Rh satisfies the following estimates:

$$\|Rh\|_{\mathcal{X}_{k+2}(M\times[0,T])} \le K\|h\|_{\mathcal{W}_k(M\times(0,T])}. \tag{2.42}$$

Here K is a constant depending only on k, M, A. Recall that A is the positive constant such that $\|w\|_{\gamma,\frac{\gamma}{2};M\times[0,T]} + \|\hat{\nabla} w\|_{C^{k-1,\gamma}_{\frac{1}{2}}(M\times(0,T])} \le A$ provided in the assumption of the theorem.

Next, we define $S : \mathcal{W}_k \to \mathcal{W}_k$ and $G : \mathcal{X}_{k+2} \to \mathcal{X}_{k+2}$ by

$$\begin{cases} Sh = HRh - h \\ G\eta = RH\eta - \eta \end{cases}. \tag{2.43}$$

Lemma 2.4. *The operators defined above in (2.43) satisfies*

$$||Sh||_{\mathcal{W}_k} \le KT^{\frac{1}{2}}||h||_{\mathcal{W}_k}$$
$$\text{and} \quad ||G\eta||_{\mathcal{X}_{k+2}} \le KT^{\frac{1}{2}}||\eta||_{\mathcal{X}_{k+2}}.$$

Here K is a constant depending only on $k, M, \hat{g}, A$.

Proof. Firstly, note that on the intersection $U_\mu \cap U_\nu$, the transition map for the vector bundle E is given by

$$\tilde{\varphi}_{\mu\nu} := \tilde{\varphi}_\mu \circ (\tilde{\varphi}_\nu)^{-1} : (U_\mu \cap U_\nu) \times \mathbb{R}^N \to (U_\mu \cap U_\nu) \times \mathbb{R}^N.$$

We denote by $\Phi_{\mu\nu} : (U_\mu \cap U_\nu) \to \mathrm{GL}_N\mathbb{R}$ the induced isomorphism which is given by

$$\tilde{\varphi}_{\mu\nu}(p, V) := (p, \Phi_{\mu\nu}(x)V), \quad \forall p \in U_\mu \cap U_\nu,\ V \in \mathbb{R}^N.$$

Then with respect to the canonical basis $\{\mathbf{e}_i^\mu\}$ and $\{\mathbf{e}_j^\nu\}$ of $\pi^{-1}(U_\mu)$ and $\pi^{-1}(U_\nu)$, the matrix components for the map $\Phi_{\mu\nu}$ are given by

$$\mathbf{e}_i^\mu = (\Phi_{\mu\nu})_i^j(\mathbf{e}_j^\nu).$$

With this notation, we can write the components of $R_\mu h$ on $U_\mu \cap U_\nu$ with respect to the basis $\{\mathbf{e}_j^\nu\}$ as

$$(R_\mu h)_\nu^j = \sum_i \rho_\mu \eta_\mu^i (\Phi_{\mu\nu})_i^j.$$

Thus we can write the component $(Rh)_\mu^i$ on U_μ as

$$(Rh)_\mu^i = \sum_\nu (R_\nu h)_\mu^i = \sum_{j,\nu} \rho_\nu (\Phi_{\nu\mu})_j^i \eta_\nu^j. \tag{2.44}$$

Now, let us prove the first inequality in the lemma. For any $h = (F, \eta_0) \in \mathcal{W}_k$, we have

$$Sh = (LRh - F, 0).$$

Let us write $v = Rh$. Then on each chart U_μ, we have $v = \sum_i v_\mu^i \mathbf{e}_i^\mu$ satisfying

$$v_\mu^i = \sum_{j,\nu} \rho_\nu (\Phi_{\nu\mu})_j^i \eta_\nu^j,$$

where $\{\eta_\nu^j\}$ denotes the unique solution of the system (2.3). By (2.44), we have

$$\frac{\partial}{\partial t} v_\mu^i = \sum_{j,\nu} \rho_\nu (\Phi_{\nu\mu})_j^i (\frac{\partial}{\partial t} \eta_\nu^j),$$

and

$$w^{kl}D^2_{kl}v^i_\mu = \sum_{j,\nu}\left\{\rho_\nu(\Phi_{\nu\mu})^i_j(w^{kl}D^2_{kl}\eta^j_\nu) + 2w^{kl}D_k(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot D_l(\eta^j_\nu) + w^{kl}D^2_{kl}(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot\eta^j_\nu\right\}.$$

This implies that $\{v^i_\mu\}$ satisfies the system

$$\begin{cases}\frac{\partial}{\partial t}v^i_\mu(x,t) - w^{kl}D^2_{kl}v^i_\mu(x,t) = \tilde{F}^i_\mu(x,t) & \text{on} \quad \varphi_\mu(U_\mu)\times(0,T], \quad i=1,..,N \\ v^i_\mu(x,0) = 0 & \text{on} \quad \varphi_\mu(U_\mu), \quad i=1,..,N\end{cases},$$

where

$$\tilde{F}^i_\mu = \sum_{j,\nu}\left\{\rho_\nu(\Phi_{\nu\mu})^i_j F^j_\nu + 2w^{kl}D_k(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot D_l(\eta^j_\nu) + w^{kl}D^2_{kl}(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot\eta^j_\nu\right\}. \tag{2.45}$$

Note that v satisfies the estimate

$$\|v\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla}v\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \le K\|h\|_{\mathcal{W}_k(M\times(0,T])}$$

by (2.42). And by $F^i_\mu = \sum_\nu \rho_\nu F^i_\mu = \sum_{j,\nu}\rho_\nu(\Phi_{\nu\mu})^i_j F^j_\nu$, we have the estimate

$$\begin{aligned}&\|\tilde{F}^i_\mu - F^i_\mu\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} \\ &= \left\|\sum_{j,\nu}\left\{2w^{kl}D_k(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot D_l(\eta^j_\nu) + w^{kl}D^2_{kl}(\rho_\nu(\Phi_{\nu\mu})^i_j)\cdot\eta^j_\nu\right\}\right\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} \\ &\le K\sum_{j,\nu}\left\{\|D_x\eta^j_\nu\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} + \|\eta^j_\nu\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])}\right\} \\ &\le K(\|v\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\hat{\nabla}v\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}) \\ &\le KT^{\frac{1}{2}}(\|v\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla}v\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])}).\end{aligned} \tag{2.46}$$

Hence, we have

$$LRh - F = \sum_r \left(\frac{\partial}{\partial t} v^i_\mu(x,t) - w^{kl}\hat{\nabla}_k \hat{\nabla}_l v^i_\mu(x,t) - F^\mu_i(x,t)\right)\mathbf{e}^i_\mu$$
$$= \sum_r \left((\partial\hat{\Gamma} + \hat{\Gamma} *_w \hat{\Gamma}) *_w v + \hat{\Gamma} *_w \hat{\nabla} v + \tilde{F}^i_\mu - F^i_\mu\right)\mathbf{e}^i_\mu,$$

where $\hat{\Gamma}$ are the connection terms with respect to $\hat{g}$. Thus, using Lemma 1.6 and (2.46) we obtain the estimate

$$\begin{aligned}\|Sh\|_{\mathcal{W}_k(M\times(0,T])} &= ||LRh - F||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \\ &\le K\left(\|v\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\hat{\nabla} v\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \sum_{i,\mu}\|\tilde{F}^i_\mu - F^i_\mu\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])}\right) \\ &\le KT^{\frac{1}{2}}(\|v\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla} v\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])}) \\ &\le KT^{\frac{1}{2}}\|h\|_{\mathcal{W}_k(M\times(0,T])},\end{aligned} \tag{2.47}$$

where $K = K(k, M, \hat{g}, A)$.

In the next step, we derive the second inequality in the lemma. For any $\eta \in \mathcal{X}_{k+2}(M\times[0,T];E)$, we denote by $(F,\eta_0) := (L\eta, \eta(\cdot,0)) = H\eta$ and $\zeta := RH\eta$. Then on each chart U_μ, we see that $(F,\eta_0) = (\sum_{r=1}^N F^i_\mu \mathbf{e}^\mu_\mathbf{i},\ \eta(\cdot,0))$ satisfies

$$F^i_\mu(x,t) = \frac{\partial}{\partial t}\eta^i_\mu(x,t) - w^{kl}\hat{\nabla}_k\hat{\nabla}_l\eta^i_\mu(x,t) \quad \text{on} \quad \varphi_\mu(U_\mu)\times(0,T], \quad r = 1,..,N.$$

Now, we denote by $\{\tilde{\eta}^j_\nu\}$ the unique solution of the system

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\eta}^j_\nu(x,t) - w^{kl}(x,t)D^2_{kl}\tilde{\eta}^j_\nu(x,t) = F^j_\nu(x,t) & \text{on} \quad \varphi_\nu(U_\nu)\times(0,T], \quad r = 1,..,N \\ \tilde{\eta}^j_\nu(x,0) = (\eta_0)^j_\nu(x) & \text{on} \quad \varphi_\nu(U_\nu), \quad r = 1,..,N. \end{cases}$$

Then by (2.44), $RH\eta = \zeta = \sum_{r=1}^{N} \zeta_\mu^i \mathbf{e}_\mathbf{i}^\mu$ satisfies

$$\zeta_\mu^i = \sum_{j,\nu} \rho_\nu (\Phi_{\nu\mu})_j^i \tilde{\eta}_\nu^j,$$

and in particular

$$\zeta_\mu^i(x,0) = \sum_{j,\nu} \rho_\nu (\Phi_{\nu\mu})_j^i (\eta_0)_\nu^j(x,0) = (\eta_0)_\mu^i(x,0).$$

Moreover, the components $\{\zeta_\mu^i\}$ satisfy the system

$$\begin{cases} \frac{\partial}{\partial t}\zeta_\mu^i(x,t) - w^{kl} D_{kl}^2 \zeta_\mu^i(x,t) = \tilde{F}_\mu^i(x,t) & \text{on} \quad \varphi_\mu(U_\mu)\times(0,T], \quad i=1,..,N \\ \zeta_\mu^i(x,0) = (\eta_0)_\mu^i(x,0) & \text{on} \quad \varphi_\mu(U_\mu), \quad i=1,..,N, \end{cases}$$

where $\tilde{F}_\mu^i$ is again defined by (2.45) with η replaced by $\tilde{\eta}$. Similar to (2.46), we have the estimate

$$\begin{aligned} & \|\tilde{F}_\mu^i - F_\mu^i\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} && (2.48) \\ &= \left\| \sum_{j,\nu} \left\{ 2w^{kl} D_k(\rho_\nu(\Phi_{\nu\mu})_j^i)\cdot D_l(\tilde{\eta}_\nu^j) + w^{kl} D_{kl}^2(\rho_\nu(\Phi_{\nu\mu})_j^i)\cdot\tilde{\eta}_\nu^j \right\} \right\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} \\ &\leq K \sum_{j,\nu} \left\{ \|D_x\tilde{\eta}_\nu^j\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} + \|\tilde{\eta}_\nu^j\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} \right\} \\ &\leq KT^{\frac{1}{2}} \sum_{j,\nu} \left\{ \|D_x\tilde{\eta}_\nu^j\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])} + \|\tilde{\eta}_\nu^j\|_{\alpha,\frac{\alpha}{2};\mathbb{R}^N\times[0,T]} \right\} \\ &\leq KT^{\frac{1}{2}}(||F||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\eta_0\|_{\alpha;M}) \\ &\leq KT^{\frac{1}{2}}\|\eta\|_{X_{k+2}(M\times[0,T])}, \end{aligned}$$

where the second last inequality follows from Lemma 2.3.

This implies that $G\eta = RH\eta - \eta = \sum_{r=1}^{N}(\zeta_\mu^i - \eta_\mu^i)\mathbf{e}_\mathbf{i}^\mu$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}(\zeta_\mu^i - \eta_\mu^i) - w^{kl}D_{kl}^2(\zeta_\mu^i - \eta_\mu^i) = (\partial\hat{\Gamma} + \hat{\Gamma} *_w \hat{\Gamma}) *_w \eta + \hat{\Gamma} *_w \hat{\nabla}\eta + \tilde{F}_\mu^i - F_\mu^i & \text{on} \quad \varphi_s(U_s) \times (0, T] \\ (\zeta_\mu^i - \eta_\mu^i)(x, 0) = 0 & \text{on} \quad \varphi_s(U_s). \end{cases}$$

Then (2.48) and Lemma 2.3 imply that

$$\begin{aligned} &||G\eta||_{\mathcal{X}_{k+2}(M\times[0,T])} \qquad (2.49)\\ &\leq K\left(||(\partial\hat{\Gamma} + \hat{\Gamma} *_w \hat{\Gamma}) *_w \eta||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + ||\hat{\Gamma} *_w \hat{\nabla}\eta||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \sum_{i,\mu}\|\tilde{F}_\mu^i - F_\mu^i\|_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(\mathbb{R}^N\times(0,T])}\right)\\ &\leq KT^{\frac{1}{2}}\left(\|\eta\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla}\eta\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} + \|\eta\|_{\mathcal{X}_{k+2}(M\times[0,T])}\right)\\ &\leq KT^{\frac{1}{2}}||\eta||_{\mathcal{X}_{k+2}(M\times[0,T])}. \end{aligned}$$

We thereby proved the Lemma. □

To complete the proof of Theorem 2.1, we choose T^* sufficiently small so that the constant given in Lemma 2.4 satisfies $KT^* \leq \frac{1}{2}$. Then for each $T \in (0, T^*]$, Lemma 2.4 implies

$$||Sh||_{\mathcal{W}_k} \leq \frac{1}{2}||h||_{\mathcal{W}_k}, \quad ||G\eta||_{\mathcal{X}_{k+2}} \leq \frac{1}{2}||\eta||_{\mathcal{X}_{k+2}}.$$

Then on the basis of contraction mapping principle and the Fredholm alternative the operators $Id_{\mathcal{W}} + S$ and $Id_{\mathcal{X}} + G$ have bounded inverses. From this and (2.43) we conclude that the operator H has bounded inverse such that

$$R(Id_{\mathcal{W}} + S)^{-1} = (Id_{\mathcal{X}} + G)^{-1}R = H^{-1}. \qquad (2.50)$$

Consequently we have

$$
\begin{aligned}
&\|\eta\|_{\alpha,\frac{\alpha}{2};M\times[0,T]} + \|\hat{\nabla}\eta\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \\
&= ||\eta||_{\mathcal{X}_{k+2}(M\times[0,T])} \\
&= ||H^{-1}h||_{\mathcal{X}_{k+2}(M\times[0,T])} \\
&\leq K||h||_{\mathcal{W}_k} \\
&\leq K\big(||F||_{C^{k,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|\eta_0\|_{\alpha;M}\big)
\end{aligned}
\tag{2.51}
$$

for each $T \in (0, T^*]$. Here K is a constant depending only on $k, M, \hat{g}, A$. Having established the theorem on a small time interval $[0, \min\{T^*, I\}]$, we can prove the theorem on an arbitrary interval $[0, T]$ for each $T \leq I$ by standard parabolic theory. Since for $t > 0$, we have $\eta(\cdot, t) \in C^{k+2+\gamma}(M)$ and $F, w \in C^{k+\gamma,\frac{k+\gamma}{2}}(M \times [t, T])$. Therefore we have established Theorem 2.1.

Chapter 3: Ricci Flow with Hölder continuous initial metrics

Let M be a smooth, compact Riemannian manifold with boundary ∂M. Let g_0 be a smooth Riemannian metric on M. Recall that one the goals of this thesis is to prove short time existence to Ricci flow on manifold with boundary in the following sense:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) & \text{on} \quad M \times (0,T] \\ A_{g(t)} = 0 & \text{on} \quad \partial M \times (0,T], \end{cases} \tag{3.1}$$

where $A_{g(t)}$ is the second fundamental form of ∂M in $(M, g(t))$. In [23], Shen proved short time existence to the above system provided that the initial metric g_0 is totally geodesic. We remark here that we do not impose any condition on the boundary second fundamental form A_{g_0} for the initial metric g_0.

Equivalently, we can prove short time existence to the above boundary value problem via doubling the manifold and solving the corresponding Ricci flow on the doubled manifold but with rough initial data. Let $\tilde{M}$ be the double of M. More precisely, we define $\tilde{M} = {}^{M_1 \cup M_2}/_{\sim}$, where M_1 and M_2 are identical copies of M and $p_1 \sim p_2$ if $p \in \partial M$. Fix a smooth background metric $\hat{g}$ on M such that in a small collar neighborhood of ∂M the metric $\hat{g}$ is isometric to a product $\partial M \times [0, \varepsilon)$. Note that $\hat{g}$ extends to a smooth metric on the doubled manifold $\tilde{M}$ via reflection about ∂M, which we would still denote it as $\hat{g}$. Next, we extend g_0 to a metric $\tilde{g}_0$ on the doubled manifold $\tilde{M}$ via reflection about ∂M. Then $\tilde{g}_0$ is a Lipschitz metric on $\tilde{M}$. In particular, we have $\tilde{g}_0 \in C^\alpha(M; \mathrm{Sym}^2(T^*M))$ for all $\alpha \in (0,1)$. We consider the Ricci flow on $\tilde{M}$:

$$\frac{\partial}{\partial t} \tilde{g}(t) = -2Ric(\tilde{g}(t)) \quad \text{on} \quad \tilde{M} \times (0,T]. \tag{3.2}$$

Note that solving (3.1) is equivalent to solving (3.2) with initial metric $\tilde{g}_0$. Namely if $\tilde{g}(t)$ is a smooth solution to (3.2) on $\tilde{M} \times (0, I]$, then $\tilde{g}(t)$ preserves the $\mathbb{Z}_2$ symmetry of $\tilde{g}_0$ and therefore $\tilde{g}(t)|_{M_i}$ is a smooth solution to (3.1) on $M_i \times (0, I]$ such that $(M_i, \tilde{g}(t)|_{M_i})$ has totally geodesic boundary.

We use the DeTurck's trick to relate the system (3.2) to a modified system which is strictly parabolic. In the sequel of this chapter, we will work on the doubled manifold $(\tilde{M}, \tilde{g}_0)$. For the sake of notation simplicity we will still denote the doubled Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ as (M, g_0) if no confusions would be made. $\hat{\nabla}$ will denote the covariant derivative with respect to the background metric $\hat{g}$. We consider the following Ricci-DeTurck system on the doubled manifold (M, g_0):

$$\begin{cases} \frac{\partial}{\partial t} g = -2Ric(g) + \mathcal{L}_W g & \text{on} \quad M \times (0, T] \\ g(x, 0) = g_0 & \text{on} \quad M. \end{cases} \tag{3.3}$$

Here the vector field W is defined as $W^k = g^{ij}((\Gamma_g)_{ij}^k - (\hat{\Gamma}_{\hat{g}})_{ij}^k)$. Moreover, we have $-2Ric(g) + \mathcal{L}_W g = g^{kl}\hat{\nabla}_k\hat{\nabla}_l g + Q(g, \hat{\nabla} g)$, where $Q(g, \hat{\nabla} g)$ is defined by

$$\begin{aligned} Q(g, \hat{\nabla} g)_{ij} =& - g^{kl} g_{ip} \hat{g}^{pq} \hat{R}_{jkql} - g^{kl} g_{jp} \hat{g}^{pq} \hat{R}_{ikql} \\ &+ \frac{1}{2} g^{kl} g^{pq} (\hat{\nabla}_i g_{pk} \hat{\nabla}_j g_{ql} + 2\hat{\nabla}_k g_{ip} \hat{\nabla}_q g_{jl} - 2\hat{\nabla}_k g_{ip} \hat{\nabla}_l g_{jq} \\ &- 4\hat{\nabla}_i g_{pk} \hat{\nabla}_l g_{jq}). \end{aligned}$$

We remark that the initial metric g_0 is merely Hölder-continuous. We will use the Banach fixed point theorem to prove existence of a short time solution to (3.3). The work in this Chapter is taken from [7].

To begin with, we define a suitable Banach space for the solutions. Let $\alpha, \gamma \in (0, 1)$ be given such that $\alpha > \gamma$ and let $k \geq 1$, for $\eta \in \Gamma(M \times [0,T]; E)$ we define a norm $\|\eta\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}(M\times[0,T])}$ by

$$\|\eta\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}(M\times[0,T])} := ||\eta||_{\alpha,\frac{\alpha}{2};M\times[0,T]} + ||\hat{\nabla}\eta||_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])}. \tag{3.4}$$

Moreover we define the Banach space

$$\mathcal{X}^{(\alpha)}_{k,\gamma}(M \times [0,T]; E) := \{\eta : M \times [0,T] \to E |\ \|\eta\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}(M\times[0,T])} < \infty.\}. \tag{3.5}$$

3.1 Formulation of the existence result to the Ricci-DeTurck flow

Let $\alpha, \gamma \in (0,1)$ be given such that $\alpha > \gamma$ and $g_0 \in C^\alpha(M)$. Denote by $E = \mathrm{Sym}^2(T^*M)$ the tensor bundle of symmetric (0,2)-tensors on the closed manifold M. The system (3.3) is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} g(x,t) - tr_g \hat{\nabla}^2 g(x,t) = Q(g, \hat{\nabla} g)(x,t) & \text{on} \quad M \times (0,T] \\ g(x,0) = g_0 & \text{on} \quad M. \end{cases} \tag{3.6}$$

The results in Chapter 2 will help us to apply the Banach fixed point theorem.

Let $w \in \mathcal{X}^{(\alpha)}_{2,\gamma}(M \times [0,T]; E)$ be given such that $w(\cdot,t)$ is a family of Riemannian metrics on $M \times [0,T]$, we consider the following linear system:

$$\begin{cases} \frac{\partial}{\partial t}\eta(x,t) - tr_{g_0}\hat{\nabla}^2\eta(x,t) = tr_w\hat{\nabla}^2 w(x,t) - tr_{g_0}\hat{\nabla}^2 w(x,t) + Q(w,\hat{\nabla}w)(x,t) & \text{on} \quad M \times (0,T] \\ \eta(x,0) = g_0(x) & \text{on} \quad M \end{cases} \tag{3.7}$$

Note that if a solution η to (3.7) satisfies $\eta = w$, then η solves the nonlinear system (3.6).

Proposition 3.1. *Consider the linear system (3.7). Suppose that*

(1) $w(x,0) = g_0(x)$*;*

(2) $w(\cdot,t)$ *is a family of Riemannian metrics on* $M \times [0,T]$ *such that*

$$\Lambda g_0(x) \geq w(x,t) \geq \frac{1}{\Lambda} g_0(x)$$

for any $(x,t) \in M \times [0,T]$.

(3) $\|w\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])} \leq A$.

Then there is a unique solution $\eta \in \mathcal{X}_{2,\gamma}^{(\alpha)}(M \times [0,T])$ *to the system (3.7) and positive constants* $K_1 = K_1(M, \hat{g}, \|g_0\|_\alpha), K_2 = K_2(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ *such that*

$$\|\eta\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])} \leq K_1(K_2\, T^{\frac{\gamma}{2}} + \|g_0\|_{\alpha;M}).$$

Moreover, there exists $T^* = T^*(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ *such that* $\eta(\cdot,t)$ *is a family of Riemannian metrics on* $M \times [0, \min\{T, T^*\}]$ *satisfying*

$$\Lambda g_0(x) \geq \eta(x,t) \geq \frac{1}{\Lambda} g_0(x)$$

for any $(x,t) \in M \times [0,T]$.

Proof. In the sequel, K_1 will denote a constant depending only on $M, \hat{g}, \|g_0\|_{\alpha;M}$ and K_2 will denote a constant depending only on $M, \Lambda, \hat{g}, \|g_0\|_{\alpha;M}, A.$ we first consider the term

$$Q = w^{-1} * w * \hat{R} + w^{-1} * w^{-1} * \hat{\nabla} w * \hat{\nabla} w.$$

By assumption (2) and the matrix identity $A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}$, we have

$$|w^{-1}|_{0;M\times[0,T]} \leq K_2(\Lambda, \hat{g}) \quad \text{and} \quad [w^{-1}]_{\alpha,\frac{\alpha}{2};M\times[0,T]} \leq K(|w^{-1}|_{0;M\times[0,T]}, [w]_{\alpha,\frac{\alpha}{2};M\times[0,T]}) \leq K_2(\Lambda, \hat{g}, A).$$

This implies

$$
\begin{aligned}
&||Q(w,\hat{\nabla}w)||_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \qquad (3.8)\\
&\le K_2\left(\|w^{-1}*w\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}+\|w^{-1}*w^{-1}*\hat{\nabla}w*\hat{\nabla}w\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}\right)\\
&\le K_2\left(T^{1-\frac{\alpha}{2}}\ \|w^{-1}\|_{\alpha,\frac{\alpha}{2};M\times[0,T]}\|w\|_{\alpha,\frac{\alpha}{2};M\times[0,T]}+T^{\frac{\alpha}{2}}\ \|w^{-1}\|^2_{\alpha,\frac{\alpha}{2};M\times[0,T]}\|\hat{\nabla}w\|^2_{C^{0,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])}\right)\\
&\le K_2\, T^{\frac{\alpha}{2}}.
\end{aligned}
$$

Similarly, using the facts $\alpha>\gamma$ and $|(w-g_0)(x,t)|\le t^{\frac{\alpha}{2}}\ [w-g_0]_{\alpha,\frac{\alpha}{2};M\times[0,T]}$, and the matrix identity $A^{-1}-B^{-1}=B^{-1}(B-A)A^{-1}$, we have

$$
\begin{aligned}
&\|tr_w\hat{\nabla}^2w-tr_{g_0}\hat{\nabla}^2w\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \qquad (3.9)\\
&=\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}}|(w^{-1}-g_0^{-1})*\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}+\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[(w^{-1}-g_0^{-1})*\hat{\nabla}^2w]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\\
&\le K_2\Big(\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}}|w-g_0|_{0;M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}+\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}|w-g_0|_{0;M\times[\frac{\sigma}{2},\sigma]}[\hat{\nabla}^2w]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\\
&\qquad+\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[w-g_0]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big)\\
&\le K_2\Big(\sup_{\sigma\in(0,T]}\sigma|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}+\sup_{\sigma\in(0,T]}\sigma^{1+\frac{\gamma}{2}}[\hat{\nabla}^2w]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}+\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big)\\
&\le K_2\Big(T^{\frac{\alpha}{2}}\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}}|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}\quad+T^{\frac{\alpha}{2}}\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[\hat{\nabla}^2w]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\\
&\qquad+T^{\frac{\gamma}{2}}\sup_{\sigma\in(0,T]}\sigma^{1-\frac{\alpha}{2}}|\hat{\nabla}^2w|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big)\\
&\le K_2\, T^{\frac{\gamma}{2}}.
\end{aligned}
$$

From (3.8), (3.9) and Theorem 2.1, we conclude that there is a unique solution $\eta\in\mathcal{X}^{(\alpha)}_{2,\gamma}(M\times$

$[0,T]$) to the linear system (3.7) and a positive constant $K_1 = K_1(M, \hat{g}, \|g_0\|_\alpha)$ such that

$$\begin{aligned}
&\|\eta\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}(M\times[0,T])} \\
&\leq K_1 \left(||tr_w \hat{\nabla}^2 w - tr_{g_0} \hat{\nabla}^2 w + Q(w, \hat{\nabla} w)||_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|g_0\|_{\alpha;M} \right) \\
&\leq K_1 (K_2\, T^{\frac{\gamma}{2}} + \|g_0\|_{\alpha;M}).
\end{aligned}$$

Next, note that the bound for the semi-Hölder norm $[\eta]_{\alpha,\frac{\alpha}{2};M\times[0,T]}$ implies that

$$\|\eta(x,t) - g_0(x)\|_{0;M\times[0,T]} \leq K(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)\, T^{\frac{\alpha}{2}}.$$

Thus if $T^* = T^*(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ is sufficiently small, then the second conclusion of the proposition also holds. □

3.2 Short time existence and uniqueness to Ricci-DeTurck flow

We now prove the short time existence for the Ricci-DeTurck flow (3.3) by employing the Banach fixed point theorem. Let $\Lambda > 2$ and $A > 10\Lambda K_1 \|g_0\|_{\alpha;M}$ be large constants, where K_1 is the constant given in Proposition 3.1. We define a closed subset in $\mathcal{X}^{(\alpha)}_{2,\gamma}(M\times[0,T])$ by

$$\mathcal{B} := \{ w \in \mathcal{X}^{(\alpha)}_{2,\gamma} |\, w|_{t=0} = g_0,\ \Lambda g_0(\cdot) \geq w(\cdot,t) \geq \frac{1}{\Lambda} g_0(\cdot),\ \|w\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \leq A \}.$$

The subset $\mathcal{B}$ is non-empty by our choice of A provided that $T = T(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ is chosen sufficiently small. We next define an operator $\mathcal{R} : \mathcal{B} \to \mathcal{X}^{(\alpha)}_{2,\gamma}$ by

$$\eta := \mathcal{R}(w), \tag{3.10}$$

where η is the unique solution to the system (3.7) in $\mathcal{X}^{(\alpha)}_{2,\gamma}$. The operator $\mathcal{R}$ is well defined by Proposition 3.1. Moreover, we can further set $T = T(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ sufficiently small such that

$\|\eta\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \leq A$ and $\Lambda g_0(\cdot) \geq \eta(\cdot,t) \geq \frac{1}{\Lambda} g_0(\cdot)$ by Proposition 3.1. Consequently we have $\mathcal{R}(\mathcal{B}) \subset \mathcal{B}$ by our choice of T.

Proposition 3.2. *If $T = T(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ is chosen sufficiently small, the operator $\mathcal{R}$ is a contraction mapping.*

Proof. In the sequel, K will denote a constant depending only on $M, \Lambda, A, \hat{g}, \|g_0\|_{\alpha;M}$. Let $w_1, w_2 \in \mathcal{B}$ and write $\eta_i := \mathcal{R}(w_i)$ for $i = 1, 2$. Then $\eta := \eta_1 - \eta_2$ solves the following system:

$$\begin{cases} \frac{\partial}{\partial t}\eta - tr_{g_0}\hat{\nabla}^2\eta = \tilde{Q} & \text{on} \quad M \times (0,T] \\ \eta(x,0) = 0 & \text{on} \quad M, \end{cases} \tag{3.11}$$

where

$$\tilde{Q} := tr_{w_1}\hat{\nabla}^2 w_1 - tr_{w_2}\hat{\nabla}^2 w_2 - tr_{g_0}\hat{\nabla}^2(w_1 - w_2) + Q(w_1, \hat{\nabla}w_1) - Q(w_2, \hat{\nabla}w_2).$$

Then Theorem 2.1 with $\eta_0 = 0$ asserts that

$$\|\eta\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \leq K\|\tilde{Q}\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])}. \tag{3.12}$$

We first derive the estimate for the term $Q(w_1, \hat{\nabla}w_1) - Q(w_2, \hat{\nabla}w_2)$. Recall that

$$Q(w, \hat{\nabla}w) = w^{-1} * w * \hat{R} + w^{-1} * w^{-1} * \hat{\nabla}w * \hat{\nabla}w,$$

thus we can write the difference as

$$\begin{aligned} & Q(w_1, \hat{\nabla}w_1) - Q(w_2, \hat{\nabla}w_2) \\ & = (w_1^{-1} - w_2^{-1}) * w_1 * \hat{R} + w_2^{-1} * (w_1 - w_2) * \hat{R} \\ & \quad + (w_1^{-1} - w_2^{-1}) * w_1^{-1} * \hat{\nabla}w_1 * \hat{\nabla}w_1 + w_2^{-1} * (w_1^{-1} - w_2^{-1}) * \hat{\nabla}w_1 * \hat{\nabla}w_1 \\ & \quad + w_2^{-1} * w_2^{-1} * (\hat{\nabla}w_1 - \hat{\nabla}w_2) * \hat{\nabla}w_1 + w_2^{-1} * w_2^{-1} * \hat{\nabla}w_2 * (\hat{\nabla}w_1 - \hat{\nabla}w_2). \end{aligned}$$

Using the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ and the fact that $\|w_i\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \le A$, we have for each $\sigma \in (0, T]$,

$$
\begin{aligned}
&|Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} \\
&\le K\Big(|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\alpha-1}|w_1^{-1} - w_2^{-1}|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}-\frac{1}{2}}|\hat{\nabla} w_1 - \hat{\nabla} w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big) \\
&\le K\Big(\sigma^{\alpha-1}|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}-\frac{1}{2}}|\hat{\nabla} w_1 - \hat{\nabla} w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big).
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\sigma^{1-\frac{\alpha}{2}}|Q(w_1, w_1^{-1}, \hat{\nabla} w_1) - Q(w_2, w_2^{-1}, \hat{\nabla} w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} \\
&\le K\sigma^{\frac{\alpha}{2}} \sup_{\sigma\in(0,T]} |w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}} \sup_{\sigma\in(0,T]} \sigma^{\frac{1}{2}-\frac{\alpha}{2}}|\hat{\nabla} w_1 - \hat{\nabla} w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} \\
&\le KT^{\frac{\alpha}{2}}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}}.
\end{aligned}
\tag{3.13}
$$

On the other hand,

$$
\begin{aligned}
&[Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \\
&\le K\Big([w_1 - w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} + |w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\alpha-1}[w_1^{-1} - w_2^{-1}]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \\
&\quad + \sigma^{-\frac{\gamma}{2}+\alpha-1}|w_1^{-1} - w_2^{-1}|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}-\frac{1}{2}}[\hat{\nabla} w_1 - \hat{\nabla} w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} + \sigma^{-\frac{\gamma}{2}+\frac{\alpha}{2}-\frac{1}{2}}|\hat{\nabla} w_1 - \hat{\nabla} w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big).
\end{aligned}
$$

Since $\alpha > \gamma$, the Hölder semi-norm $[w_1 - w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}$ is controlled by $[w_1 - w_2]_{\alpha,\frac{\alpha}{2};M\times[\frac{\sigma}{2},\sigma]}$, thus

$$
\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \le KT^{\frac{\alpha}{2}}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \tag{3.14}
$$

for each $\sigma \in (0, T]$. Putting (3.13) and (3.14) together we obtain

$$
\|Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \le KT^{\frac{\alpha}{2}}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}}. \tag{3.15}
$$

Next, we derive the estimate for the term $tr_{w_1}\hat{\nabla}^2 w_1 - tr_{w_2}\hat{\nabla}^2 w_2 - tr_{g_0}\hat{\nabla}^2(w_1 - w_2)$. We write it as

$$\begin{aligned} & tr_{w_1}\hat{\nabla}^2 w_1 - tr_{w_2}\hat{\nabla}^2 w_2 - tr_{g_0}\hat{\nabla}^2(w_1 - w_2) \\ & = (w_1^{-1} - w_2^{-1}) * \hat{\nabla}^2 w_1 + (w_2^{-1} - g_0^{-1}) * \hat{\nabla}^2(w_1 - w_2). \end{aligned} \tag{3.16}$$

For the first term in (3.16), we use the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ to estimate

$$\begin{aligned} |(w_1^{-1} - w_2^{-1}) * \hat{\nabla}^2 w_1|_{0;M\times[\frac{\sigma}{2},\sigma]} & \le K|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2 w_1|_{0;M\times[\frac{\sigma}{2},\sigma]} \\ & \le K\sigma^{\frac{\alpha}{2}-1}|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}. \end{aligned}$$

Since $w_1(\cdot, 0) = w_2(\cdot, 0)$, we have

$$|(w_1 - w_2)(x, t)| \le t^{\frac{\alpha}{2}}\ [w_1 - w_2]_{\alpha,\frac{\alpha}{2};M\times[0,\sigma]}$$

for any $(x, t) \in M \times [0, \sigma]$. Consequently,

$$\begin{aligned} \sigma^{1-\frac{\alpha}{2}}|(w_1^{-1} - w_2^{-1}) * \hat{\nabla}^2 w_1|_{0;M\times[\frac{\sigma}{2},\sigma]} & \le K\sigma^{\frac{\alpha}{2}}\sigma^{-\frac{\alpha}{2}}|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} \\ & \le K\sigma^{\frac{\alpha}{2}} \sup_{\tau\in(0,T]} \tau^{-\frac{\alpha}{2}}|w_1 - w_2|_{0;M\times[\frac{\tau}{2},\tau]} \\ & \le K\sigma^{\frac{\alpha}{2}}\ \|w_1 - w_2\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}} \end{aligned} \tag{3.17}$$

for any $\sigma \in (0, T]$. On the other hand,

$$\begin{aligned} & [(w_1^{-1} - w_2^{-1}) * \hat{\nabla}^2 w_1]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \\ & \le K\Big(|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]}[\hat{\nabla}^2 w_1]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} + [w_1 - w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2 w_1|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big) \\ & \le K\Big(\sigma^{-\frac{\gamma}{2}+\frac{\alpha}{2}-1}|w_1 - w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}-1}[w_1 - w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\Big). \end{aligned}$$

for any $\sigma \in (0,T]$. By an argument similar to (3.17) and the fact $\alpha > \gamma$, we have

$$\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[(w_1^{-1}-w_2^{-1}) * \hat{\nabla}^2 w_1]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \leq K\Big(\sigma^{\frac{\gamma}{2}}\sigma^{-\frac{\gamma}{2}}|w_1-w_2|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{\frac{\alpha}{2}}[w_1-w_2]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\Big) \tag{3.18}$$

$$\leq K\sigma^{\frac{\gamma}{2}}\,\|w_1-w_2\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}}$$

for any $\sigma \in (0,T]$. For the second term in (3.16), we have

$$\begin{aligned}\sigma^{1-\frac{\alpha}{2}}|(w_2^{-1}-g_0^{-1}) * \hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} &\leq K\sigma^{1-\frac{\alpha}{2}}|w_2-g_0|_{0;M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} \\ &\leq K\sigma\,|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} \\ &\leq K\sigma^{\frac{\alpha}{2}}\sup_{\tau\in(0,T]}\tau^{1-\frac{\alpha}{2}}|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\tau}{2},\tau]} \\ &\leq K\sigma^{\frac{\alpha}{2}}\|w_1-w_2\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}},\end{aligned} \tag{3.19}$$

for any $\sigma \in (0,T]$, since $|w_2-g_0|_{0;M\times[\frac{\sigma}{2},\sigma]} \leq \sigma^{\frac{\alpha}{2}}[w_2]_{\alpha,\frac{\alpha}{2};M\times[0,\sigma]} \leq A\sigma^{\frac{\alpha}{2}}$. Moreover,

$$\begin{aligned}&\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[(w_2^{-1}-g_0^{-1}) * \hat{\nabla}^2(w_1-w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \\ &\leq K\Big(\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}|w_1-g_0|_{0;M\times[\frac{\sigma}{2},\sigma]}[\hat{\nabla}^2(w_1-w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \\ &\qquad + \sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}[w_1-g_0]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]}\Big) \\ &\leq K\Big(\sigma^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\sigma}{2},\sigma]} + \sigma^{1+\frac{\gamma}{2}}[\hat{\nabla}^2(w_1-w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}\Big) \\ &\leq K\Big(\sigma^{\frac{\gamma}{2}}\sup_{\tau\in(0,T]}\tau^{1-\frac{\alpha}{2}}|\hat{\nabla}^2(w_1-w_2)|_{0;M\times[\frac{\tau}{2},\tau]} + \sigma^{\frac{\alpha}{2}}\sup_{\tau\in(0,T]}\tau^{1-\frac{\alpha}{2}+\frac{\gamma}{2}}\,[\hat{\nabla}^2(w_1-w_2)]_{\gamma,\frac{\gamma}{2};M\times[\frac{\tau}{2},\tau]}\Big) \\ &\leq K\sigma^{\frac{\gamma}{2}}\|w_1-w_2\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}}.\end{aligned} \tag{3.20}$$

Putting (3.16) to (3.19) together, we obtain that

$$\|tr_{w_1}\hat{\nabla}^2 w_1 - tr_{w_2}\hat{\nabla}^2 w_2 - tr_{g_0}\hat{\nabla}^2(w_1 - w_2)\|_{C^{0,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \le KT^{\frac{\gamma}{2}}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \tag{3.21}$$

Hence, putting (3.12), (3.15) and (3.20) together, we obtain

$$||\eta_1 - \eta_2||_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \le KT^{\frac{\gamma}{2}}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}}. \tag{3.22}$$

If $T = T(M, \Lambda, \hat{g}, \|g_0\|_\alpha, A)$ is chosen sufficiently small, we then have

$$\|\mathcal{R}(w_1) - \mathcal{R}(w_2)\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}} \le \frac{1}{2}\|w_1 - w_2\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}}.$$

This proves the proposition. □

Therefore, by the Banach fixed point theorem we have proved

Theorem 3.3. *There exist $K = K(M, \hat{g}, \|g_0\|_{\alpha;M})$ and $T = T(M, \hat{g}, \|g_0\|_{\alpha;M})$ such that the following holds:*

There is a unique solution $g \in \mathcal{X}^{(\alpha)}_{2,\gamma}(M \times [0, T])$ to the Ricci-DeTurck flow (3.3) such that

- *$g(\cdot, t)$ is a family of Riemannian metrics;*
- *$\|g\|_{\mathcal{X}^{(\alpha)}_{2,\gamma}(M\times[0,T])} \le K$.*

Next, we can improve the regularity of g by bootstrapping.

Corollary 3.4. *Let $k \ge 2$ be given. There exist $K = K(M, k, \hat{g}, \|g_0\|_{\alpha;M})$ and $T = T(M, \hat{g}, \|g_0\|_{\alpha;M})$ such that the following holds:*

There is a unique solution $g \in \mathcal{X}^{(\alpha)}_{k,\gamma}(M \times [0, T])$ to the Ricci-DeTurck flow (3.3) such that

- *$g(\cdot, t)$ is a family of Riemannian metrics;*
- $\|g\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}(M\times[0,T])} \le K$.

Proof. We will prove the corollary by induction on k. It is clear from Theorem 3.3 that the assertion holds when $k = 2$. Let us assume that $g \in \mathcal{X}^{(\alpha)}_{k,\gamma}(M \times [0, T])$ for some $k \ge 2$. In the sequel, K will denote a constant depending only on M, k, $\hat{g}$, $\|g_0\|_{\alpha;M}$, $\|g\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}}$. Firstly, note that by the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$, the term $\|g^{-1}\|_{C_0^{k-1,\gamma}(M\times(0,T])}$ is controlled by $\|g\|_{C_0^{k-1,\gamma}(M\times(0,T])}$. Moreover, we observe that

$$\|g\|_{C_0^{k-1,\gamma}(M\times(0,T])} = \|g\|_{C_0^{0,\gamma}(M\times(0,T])} + \|\hat{\nabla} g\|_{C_{\frac{1}{2}}^{k-2,\gamma}(M\times(0,T])}.$$

Since $\alpha > \gamma$,

$$\|g\|_{C_0^{0,\gamma}(M\times(0,T])} \le K\|g\|_{\alpha,\frac{\alpha}{2};M\times[0,T]},$$

and it follows from Lemma 1.6 that

$$\|\hat{\nabla} g\|_{C_{\frac{1}{2}}^{k-2,\gamma}(M\times(0,T])} \le KT^{\frac{\alpha}{2}}\|\hat{\nabla} g\|_{C_{\frac{1}{2}-\frac{\alpha}{2}}^{k-1,\gamma}(M\times(0,T])},$$

we have

$$\|g\|_{C_0^{k-1,\gamma}(M\times(0,T])} \le K.$$

Since g solves the Ricci-DeTurck system (3.3) on $M \times (0, T]$, by Lemma 1.6, the induction hypothesis, and the above estimates, we have

$$\begin{aligned}\|g^{-1} * g^{-1} * \hat{\nabla} g * \hat{\nabla} g\|_{C_{1-\frac{\alpha}{2}}^{k-1,\gamma}(M\times(0,T])} &\le KT^{\frac{\alpha}{2}}\ \|g^{-1}\|^2_{C_0^{k-1,\gamma}(M\times(0,T])}\|\hat{\nabla} g\|^2_{C_{\frac{1}{2}-\frac{\alpha}{2}}^{k-1,\gamma}(M\times(0,T])} \\ &\le K\end{aligned}$$

and

$$\|g^{-1} * g * \hat{R}\|_{C^{k-1,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} \leq KT^{1-\frac{\alpha}{2}}\ \|g^{-1}\|_{C^{k-1,\gamma}_{0}(M\times(0,T])}\|g\|_{C^{k-1,\gamma}_{0}(M\times(0,T])}$$
$$\leq K.$$

Theorem 2.1 then implies that

$$\|g\|_{X^{(\alpha)}_{k+1,\gamma}(M\times[0,T])} \leq K\big(\|Q(g,\hat{\nabla}g)\|_{C^{k-1,\gamma}_{1-\frac{\alpha}{2}}(M\times(0,T])} + \|g_0\|_{\alpha}\big)$$
$$\leq K.$$

From these, the assertion follows.

□

3.3 Short time existence of solutions to the Ricci flow

We will prove the short time existence of solution to the Ricci flow with Hölder-continuous initial metrics in this section. More specifically, we show that any solution of the Ricci-DeTurck flow with Hölder-continuous initial data gives rise to a solution of the Ricci flow with Hölder-continuous initial data.

For any $\alpha \in (0,1)$ and $k \geq 0$, we define by $C^{k,\alpha}(M;M)$ the space of $C^{k,\alpha}$ maps $f : M \to M$. Recall in Chapter 1.3 that $\{U_\mu, \varphi_\mu\}_{\mu=1,..,m}$ is a fixed set of coordinate charts on M. On the coordinate chart U_μ, a map $f : M \to M$ has components $(f^1_\mu, \dots, f^n_\mu)$. The Hölder norm for maps $f \in C^{k,\alpha}(M;M)$ is measured with respect to the fixed charts $\{U_\mu, \varphi_\mu\}$. More precisely, given $f \in C^{k,\alpha}(M;M)$, we define the associated $C^{k,\alpha}$ norm to be

$$\|f\|_{C^{k,\alpha}(M;M)} = \sum_{\mu}\sum_{r=1}^{n} |f^r_\mu|_{C^{k,\alpha}(\varphi_\mu(U_\mu))}.$$

In the remainder of this subsection, we will consider the ODE

$$\begin{cases} \frac{\partial}{\partial t}\psi_t = -W(\psi_t,\ t) \\ \psi_T = \mathrm{id} \end{cases}, \tag{3.23}$$

where W is the DeTurck vector field defined by $W^k = g^{ij}((\Gamma_g)^k_{ij} - (\hat{\Gamma}_{\hat{g}})^k_{ij})$, and $g \in \mathcal{X}^{(\alpha)}_{k,\gamma}$ is the solution to the Ricci-DeTurck flow on $M \times (0, T]$. Note that the vector field W is undefined at $t = 0$. Nevertheless we can show that the one-parameter family of maps $\{\psi_t\}$ generated by (3.23) can be extended to a map ψ_0 at $t = 0$. In the sequel of the section, K will denote a constant depending only on $M, \alpha, \hat{g}, \|g_0\|_{\alpha;M}, \|g\|_{\mathcal{X}^{(\alpha)}_{k,\gamma}}$.

Lemma 3.5. *Let $g \in \mathcal{X}^{(\alpha)}_{k,\gamma}$ be the solution to the Ricci-DeTurck flow on $M \times (0, T]$, and W be the DeTurck vector field defined by $W^k = g^{ij}((\Gamma_g)^k_{ij} - (\hat{\Gamma}_{\hat{g}})^k_{ij})$. Then the one-parameter family of maps $\{\psi_t\}$ generated by (3.23) can be extended to a map ψ_0 at $t = 0$ in $C^{1,\beta}(M; M)$ for any $\beta < \alpha$ and satisfies*

$$\|\psi_t\|_{C^{1,\beta}(M;M)} \le K$$

for each $t \in [0, T]$. Moreover, ψ_t remains a diffeomorphism for $t \in [0, T]$ if T is sufficiently small

Proof. Since the $W = g^{-1} * g^{-1} * \hat{\nabla} g$, we have the estimate $\|W\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \le K$. Note that the ODE (3.23) is uniquely solvable on $M \times (0, T]$, we would like to show that the solution ψ_t can be extended to $M \times [0, T]$ in $C^{1,\beta}(M; M)$ for any $\beta < \alpha$.

We now work in local coordinates. Suppose that ψ_t has components ψ^k_t in a chart U, we see that $\partial_i \psi^k_t$ satisfies the ODE

$$\frac{\partial}{\partial t} \partial_i \psi^k_t = -\partial_l W^k(\psi_t, t)\, \partial_i \psi^l_t, \quad i = 1, .., n \tag{3.24}$$

for each $t \in (0, T)$. Let $\beta \in (0, \alpha)$ be given and define $\beta' := \frac{\beta+\alpha}{2}$. We claim that $\|\psi_t\|_{C^{1,\beta'}(M;M)}$ has a uniform bound for every $t \in (0,T)$. Define

$$f(t) := \|\psi_t\|_{C^{1,0}(M;M)},$$

then from (3.24) we have

$$\frac{\partial}{\partial t}\partial_i\psi_t^k \le K\, t^{\frac{\alpha}{2}-1} f(t). \tag{3.25}$$

To integrate the above inequality, it is possible that the integral curves pass into different charts. Fix $x \in M$ and $t \in (0,T]$. Suppose that $\psi_T(x) \in U$ and $\psi_t(x) \in V$. Since there are only finitely many coordinate charts, we may assume without loss of generality that $U \cap V \neq \emptyset$ and there is $s \in (t,T)$ such that $\psi_\tau(x) \in U$ for all $\tau \in [s,T]$ and $\psi_\tau(x) \in V$ for all $\tau \in [t,s]$. By integrating the inequality (3.25) from s to T, and by noting that $\partial_i\psi_T^k = \delta_i^k$, we have

$$|\partial_i\psi_s^k(x) - \delta_i^k| \le \int_s^T K\, \tau^{\frac{\alpha}{2}-1} f(\tau)\, d\tau. \tag{3.26}$$

Moreover, on the intersection $U \cap V$ we may write $\partial_i\psi_\tau^k = \partial_i\tilde{\psi}_\tau^l \frac{\partial y_k}{\partial z_l}$, where (y_k) and (z_l) stand for the local coordinates on the charts U and V respectively, and $\tilde{\psi}_\tau^l$ stand for the components of ψ_τ on V. Since $\partial_i\tilde{\psi}_\tau^l$ satisfies (3.25), it also satisfies a similar inequality as above by integrating (3.25) from t to s. This implies that

$$|\partial_i\tilde{\psi}_t^l(x) - \partial_i\tilde{\psi}_s^l(x)| \le \int_t^s K\, \tau^{\frac{\alpha}{2}-1} f(\tau)\, d\tau. \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$|\partial_i\tilde{\psi}_t^l(x)| \le K + \int_t^T K\, \tau^{\frac{\alpha}{2}-1} f(\tau)\, d\tau. \tag{3.28}$$

We can similarly obtain bounds for $|\tilde{\psi}_t^l(x)|$ by integrating the ODE (3.23). Since $x \in M$ is arbitrary, we obtain

$$f(t) \le K + \int_t^T K\, \tau^{\frac{\alpha}{2}-1} f(\tau)\, d\tau. \tag{3.29}$$

Grönwall's lemma then implies that

$$\|\psi_t\|_{C^{1,0}(M;M)} \le K \exp(K T^{\frac{\alpha}{2}}) \tag{3.30}$$

for any $t \in (0, T]$.

Next, let $x, y \in M$ and fix $t \in (0, T)$. To bound the $C^{1,\beta'}$ norm for ψ_t, we may assume without loss of generality that $\psi_\tau(x)$ stays in the same chart with $\psi_\tau(y)$ for $\tau \in [t, T)$ using the uniform C^1 bound (3.30). Note that it is still possible that the integral curves $\psi_\tau(x)$ and $\psi_\tau(y)$ may pass into different charts. By (3.24) we have

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\Big(\frac{\partial_i \psi_\tau^k(x) - \partial_i \psi_\tau^k(y)}{|x-y|^{\beta'}}\Big) \\
&= -\frac{\partial_l W^k(\psi_\tau(x),\tau) - \partial_l W^k(\psi_\tau(y),\tau)}{|x-y|^{\beta'}}\partial_i \psi_\tau^l(x) - \partial_l W^k(\psi_\tau(y),\tau)\frac{\partial_i \psi_\tau^l(x) - \partial_i \psi_\tau^l(y)}{|x-y|^{\beta'}}
\end{aligned} \tag{3.31}$$

for each $\tau \in (t, T)$. By Lemma 1.6, we have $\|W\|_{C^{k-1,\beta'}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \le K$. Thus the bound for $\hat{\nabla} W$ and (3.30) imply

$$\begin{aligned}
|\partial_l W^k(\psi_\tau(x),\tau) - \partial_l W^k(\psi_\tau(y),\tau)| &\le K\, \tau^{-1+\frac{\alpha-\beta'}{2}}\, |\psi_\tau(x) - \psi_\tau(y)|^{\beta'} \\
&\le K\, \tau^{-1+\frac{\alpha-\beta'}{2}}\, |x-y|^{\beta'}
\end{aligned}$$

and

$$|\partial_l W^k(\psi_\tau(y),\tau)| \le K \tau^{-1+\frac{\alpha}{2}}.$$

Hence

$$\frac{\partial}{\partial \tau}\Big(\frac{\partial_i \psi_\tau^k(x) - \partial_i \psi_\tau^k(y)}{|x-y|^{\beta'}}\Big) \leq K\tau^{-1+\frac{\alpha-\beta'}{2}} |\partial_i \psi_\tau^l(x)| + K\tau^{-1+\frac{\alpha}{2}} \frac{|\partial_i \psi_\tau^l(x) - \partial_i \psi_\tau^l(y)|}{|x-y|^{\beta'}}$$
$$\leq K\tau^{-1+\frac{\alpha-\beta'}{2}} + K\tau^{-1+\frac{\alpha}{2}} \|\psi_\tau\|_{C^{1,\beta'}(M;M)}$$

for each $\tau \in (t, T)$. Subsequently similar to the argument in deriving the C^1 bound (3.30), we can integrate the above inequality to obtain

$$\frac{|\partial_i \psi_t^k(x) - \partial_i \psi_t^k(y)|}{|x-y|^{\beta'}} \leq \int_t^T (K\tau^{-1+\frac{\alpha-\beta'}{2}} + K\tau^{-1+\frac{\alpha}{2}} \|\psi_\tau\|_{C^{1,\beta'}(M;M)}) d\tau$$
$$\leq KT^{\frac{\alpha-\beta'}{2}} + \int_t^T K\tau^{-1+\frac{\alpha}{2}} \|\psi_\tau\|_{C^{1,\beta'}(M;M)} d\tau.$$

Since $x, y \in M$ are arbitrary, we can combine the above inequality and the inequality (3.29) to obtain

$$\|\psi_t\|_{C^{1,\beta'}(M;M)} \leq K + KT^{\frac{\alpha-\beta'}{2}} + \int_t^T K\tau^{-1+\frac{\alpha}{2}} \|\psi_\tau\|_{C^{1,\beta'}(M;M)} d\tau.$$

Then Grönwall's lemma implies

$$\|\psi_t\|_{C^{1,\beta'}(M;M)} \leq K'(T) \tag{3.32}$$

for any $t \in (0, T]$, which is uniform for each $t \in (0, T]$. Since bounded subsets in $C^{1,\beta'}(M; M)$ are precompact in $C^{1,\beta}(M; M)$ as $\beta' > \beta$, upon passing to a subsequence we have $\psi_t \to \psi_0$ in $C^{1,\beta}(M; M)$ as $t \to 0$.

Lastly, ψ_t remains a diffeomorphism on $(0, T]$ provided T is sufficiently small by the inverse function theorem. It remains to show that ψ_0 is also a diffeomorphism. We need a uniform lower bound for the differential $d\psi_t$. Fix $x, y \in M$ and $t \in (0, T]$. Let $c : [0, L] \to M$ be a curve such that

$c(0) = x$ and $c(L) = y$. Define

$$h_c(\tau) = \int_0^L (\psi_\tau^* \hat{g})(c'(s), c'(s))ds.$$

Then $\|\psi_\tau(x) - \psi_\tau(y)\|_{\hat{g}} = \inf_c h_c(\tau)$. Moreover, from the fact that $\sigma^{1-\frac{\alpha}{2}} \|W\|_{0;M\times[\frac{\sigma}{2},\sigma]} \leq K$, we have

$$\begin{aligned} h_c'(\tau) &= \int_0^L (\psi_\tau^*(\mathcal{L}_{-W}\hat{g}))(c'(s), c'(s))ds \\ &\leq K\|\hat{\nabla} W(\tau)\|_{0;M} h_c(\tau) \\ &\leq K\tau^{\frac{\alpha}{2}-1} h_c(\tau) \end{aligned}$$

for each $\tau \in (t, T]$. This implies

$$h_c(T) \leq h_c(t) \exp\left(\int_t^T K\tau^{\frac{\alpha}{2}-1} d\tau\right) \leq K h_c(t).$$

Similarly,

$$h_c(T) \geq \frac{1}{K} h_c(t).$$

In particular, this implies the uniform estimate

$$\frac{1}{K}\|x - y\|_{\hat{g}} \leq \|\psi_t(x) - \psi_t(y)\|_{\hat{g}} \leq K\|x - y\|_{\hat{g}} \tag{3.33}$$

for any $t \in (0, T]$ and any $x, y \in M$. Consequently both ψ_0 and $D\psi_0$ are injective. Thus ψ_0 is a diffeomorphism to its image. Now we show that ψ_0 is surjective. Take any $y \in M$ and choose a sequence $s_j \to 0$. Define a sequence of points by $x_j := \psi_{s_j}^{-1}(y)$. Since M is compact, after passing to a subsequence we have $x_j \to x_\infty \in M$. The uniform estimate (3.33) then implies that $|\psi_{s_j}(x_j) - \psi_{s_j}(x_\infty)| \to 0$. On the other hand, we have $|\psi_{s_j}(x_\infty) - \psi_0(x_\infty)| \to 0$. Hence $|\psi_{s_j}(x_j) - \psi_0(x_\infty)| \to 0$ and we get $\psi_0(x_\infty) = y$.

□

For the family of diffeomorphisms $\{\psi_t\}_{t\in[0,T]}$, it is possible that the curve $t \mapsto \psi_t(x)$ does not stay in the same chart for all $t \in [0,T]$. Hence if $\tilde{\psi} : M \times [0,T] \to M$ is a map defined by $\tilde{\psi}(x,t) = \psi_t(x)$, we cannot measure its parabolic Hölder norm the way we did for the elliptic Hölder norm without any modification. However, our final goal is to show that the pullback metric $\psi_t^* g_t$ satisfies the Ricci flow equation and lies in some weighted space $\mathcal{X}^{(*)}_{*,\gamma}$ provided that $g \in \mathcal{X}^{(\alpha)}_{k,\gamma}$ is a solution to the Ricci-DeTurck flow.

Now, consider the section $d\psi_t : M \to T^*M \otimes (\psi_t)^*(TM)$. We introduce the multi-index notations

$$\mu = (\mu_1, \mu_2).$$

If $(x_i^{\mu_1})_{i=1,..,n}$ and $(y_j^{\mu_2})_{j=1,..,n}$ are local coordinates on the charts U_{μ_1} and U_{μ_2} respectively, then

$$\mathbf{e}^{\mu}_{ij}(t) := dx_i^{\mu_1} \otimes \frac{\partial}{\partial(y_j^{\mu_2} \circ \psi_t)}, \quad i_1, i_2 = 1,..,n$$

form a local frame on $T^*M \otimes (\psi_t)^*(TM)|_{U_{\mu_1}\cap\psi_t^{-1}(U_{\mu_2})}$. Moreover, if $x \in U^t_\mu := U_{\mu_1} \cap \psi_t^{-1}(U_{\mu_2})$, then locally around x the section $d\psi_t$ can be expressed as

$$d\psi_t = (d\psi_t)^{ij}_\mu \mathbf{e}^\mu_{ij}(t), \quad \text{where} \quad (d\psi_t)^{ij}_\mu = \partial_i(\psi_t)^j_\mu$$

Definition 3.6. *With the above notations, let $\{\rho_{\mu_2}\}$ be the partition of unity subordinate to the chart U_{μ_2}, we define functions $\Psi^{ij}_\mu : M \times [0,T] \to \mathbb{R}$ by*

$$\Psi^{ij}_\mu(x,t) = \begin{cases} \rho_{\mu_2}(\psi_t(x))(d\psi_t)^{ij}_\mu(x) & \text{if} \quad \psi_t(x) \in supp(\rho_{\mu_2}) \\ 0 & \text{if} \quad \psi_t(x) \notin supp(\rho_{\mu_2}) \end{cases}.$$

Note that in this way, although the component maps $t \mapsto (d\psi_t)^{ij}_\mu(x)$ may not be defined for all $t \in [0,T]$, the maps $\Psi^{ij}_\mu(x,\cdot)$ are well defined for all $t \in [0,T]$.

Lemma 3.7. *Let $g \in \mathcal{X}^{(\alpha)}_{k+2,\gamma}$ be the solution to the Ricci-DeTurck flow on $M \times (0,T]$, and $\{\psi_t\}_{t\in[0,T]}$*

be the one-parameter family of diffeomorphisms generated by (3.23). Then given any $\beta < \alpha$, the functions Ψ_μ^{ij} which are defined by Definition 3.6 satisfy

(1) $\Psi_\mu^{ij} \in C^{\beta,\frac{\beta}{2}}(M \times [0,T]; \mathbb{R})$ *and* $\hat{\nabla}\Psi_\mu^{ij} \in C^{k-2,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M \times (0,T]; \mathbb{R})$*;*

(2) $\|\Psi_\mu^{ij}\|_{\beta,\frac{\beta}{2};M\times[0,T]} + \|\hat{\nabla}\Psi_\mu^{ij}\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \leq K(M, k, \hat{g}, \|g_0\|_\alpha, \|g\|_{\mathcal{X}^{(\alpha)}_{k+2,\gamma}})$.

Proof. Firstly, Lemma 3.5 has already shown that $\Psi_\mu^{ij}(\cdot, t) \in C^{0,\beta}(M; \mathbb{R})$ for each $t \in [0,T]$ with an uniform upper bound for the elliptic Hölder norm. Thus to show that $\Psi_\mu^{ij} \in C^{\beta,\frac{\beta}{2}}(M \times [0,T]; \mathbb{R})$, it remains to show that $\Psi_\mu^{ij}(x, \cdot)$ is also Hölder-continuous for $t \in [0,T]$ for each $x \in M$. We rewrite the equation (3.24) in the new notations as

$$\frac{\partial}{\partial t}(d\psi_t)_\mu^{ij} = -\partial_l W^j(\psi_t, t)\,(d\psi_t)_\mu^{il}.$$

This gives

$$\frac{\partial}{\partial t}\Psi_\mu^{ij} = -\langle D\rho_{\mu_2}(\psi_t), W\rangle (d\psi_t)_\mu^{ij} - \rho_{\mu_2}(\psi_t)\partial_l W^j(\psi_t, t)\,(d\psi_t)_\mu^{il}.$$

Using the C^1 bound (3.30) and the fact that $\|W\|_{C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \leq K$, we have

$$\frac{\partial}{\partial t}\Psi_\mu^{ij} \leq K t^{-1+\frac{\alpha}{2}}$$

for each $t \in (0,T)$. Hence

$$\begin{aligned}
|\Psi_\mu^{ij}(x,t) - \Psi_\mu^{ij}(x,s)| &\leq \int_s^t K\tau^{-1+\frac{\alpha}{2}}\,d\tau \\
&\leq K(t^{\frac{\alpha}{2}} - s^{\frac{\alpha}{2}}) \\
&\leq K\,|t-s|^{\frac{\alpha}{2}}
\end{aligned}$$

for $t, s \in [0,T]$. This gives $\sup_{t\neq s}\frac{|\Psi_\mu^{ij}(x,t)-\Psi_\mu^{ij}(x,s)|}{|t-s|^{\frac{\alpha}{2}}} \leq K$ for any $x \in M$. Hence, we conclude that

$$\|\Psi_\mu^{ij}\|_{\beta,\frac{\beta}{2};M\times[0,T]} \leq K.$$

In the next step, we prove the remaining assertions. Heuristically as $g \in \mathcal{X}^{(\alpha)}_{k+2,\gamma}$, we have $W \in C^{k+1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}$, and so the highest regularity we can achieve for $\hat{\nabla}\Psi^{ij}_{\mu}$ would be $C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}$. On the other hand, we can easily control the Hölder semi-norm $[\hat{\nabla}^{m-1}\Psi]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}$ by the higher order C^0 norm $|\hat{\nabla}^m\Psi|_{0;M\times[\frac{\sigma}{2},\sigma]}$. And since we have the freedom of choosing k to be any large integer, it doesn't hurt if we omit controlling the highest order Hölder semi-norm $[\hat{\nabla}^k\Psi]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]}$. Nevertheless, we will first control the C^0 norm of $|\hat{\nabla}^m\Psi|_{0;M\times[\frac{\sigma}{2},\sigma]}$ up to the highest order. More precisely, we first prove the estimate

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{m-2}{2}}|\hat{\nabla}^{m-1}\Psi^{ij}_{\mu}|_{0;M\times[\frac{\sigma}{2},T]} \leq K \tag{3.34}$$

by induction for $2 \leq m \leq k+1$. For notation simplicity, we abbreviate the components $(d\psi_t)^{ij}_{\mu}$ of the differential $d\psi_t$ simply by $\partial\psi_t$, the partial derivatives $\partial^{m-1}_x(d\psi_t)^{ij}_{\mu}$ and $\partial^m_x W^i$ by $\partial^m\psi_t$ and $\partial^m W$ respectively, and the functions Ψ^{ij}_{μ} by Ψ. Moreover, with this abbreviation, for each $t \in (0,T]$ the C^0 norm for $\partial^m\psi_t$ is given by

$$\|\partial^m\psi_t\|_{0;M} = \sum_{\mu_2}\sum_{i,j}|\partial^{m-1}_x(d\psi_t)^{ij}_{\mu}|_{C^0(\psi_t^{-1}(U_{\mu_2});\mathbb{R})}.$$

Now suppose that the estimate (3.34) holds for all $2 \leq j \leq m$. Thus the induction hypothesis implies that

$$\|\partial^j\psi_t\|_{0;M} \leq Kt^{\frac{\beta}{2}-\frac{1}{2}-\frac{j-2}{2}}$$

for all $2 \leq j \leq m$ and $t \in [\frac{\sigma}{2},T]$. Using the Francesco Faà di Bruno formula for higher-order chain rule, the following ODE is satisfied by $\partial^{m+1}\psi_t$:

$$\begin{aligned}&\frac{\partial}{\partial t}(\partial^{m+1}\psi_t) \qquad (3.35)\\ &= -\sum_{j_1+2j_2+\cdots+(m+1)j_{m+1}=m+1}(\partial^{j_1+\cdots+j_{m+1}}W)(\psi_t,t)*(\partial\psi_t)^{*j_1}*\cdots*(\partial^{m+1}\psi_t)^{*j_{m+1}}\\ &= -\sum_{j_1+2j_2+\cdots+mj_m=m+1}(\partial^{j_1+\cdots+j_m}W)*(\partial\psi_t)^{*j_1}*\cdots*(\partial^m\psi_t)^{*j_m} - \partial W*\partial^{m+1}\psi_t.\end{aligned}$$

Here $(\partial^i\psi)^{*j}$ stands for $\underbrace{\partial^i\psi * \cdots * \partial^i\psi}_{j \text{ times}}$ and the summation sums over all non-negative integer solutions of $j_1 + 2j_2 + \cdots + mj_m = m + 1$.

Now observe that for any $t \in [\frac{\sigma}{2}, T]$, the C^0 bound for $\partial^j W$, $\partial\psi_t$, and the induction hypothesis imply

$$\begin{aligned}
&\left|(\partial^{j_1+\cdots+j_m}W)(\psi_t, t) * (\partial\psi_t)^{*j_1} * \cdots * (\partial^m\psi_t)^{*j_m}\right| \\
&\leq K\, t^{\frac{\alpha}{2}-\frac{1}{2}-\frac{j_1+\cdots+j_m}{2}} \prod_{r=2}^{m} \|\partial^r\psi_t\|_{0;M}^{j_r} \\
&\leq K\, t^{\frac{\alpha}{2}-\frac{1}{2}-\frac{j_1+\cdots+j_m}{2}} \prod_{r=2}^{m} t^{(\frac{\beta}{2}+\frac{1}{2}-\frac{r}{2})j_r} \\
&= K\, t^{\frac{\alpha}{2}-\frac{m+2}{2}+\frac{\beta}{2}\sum_{r=2}^{m} j_r} \\
&\leq K\, t^{\frac{\alpha}{2}-\frac{m+2}{2}}
\end{aligned} \tag{3.36}$$

for each $\{j_1, .., j_m\}$ satisfying $j_1 + 2j_2 + \cdots + mj_m = m + 1$.

Let us define $f(t) := \|\partial^{m+1}\psi_t\|_{0;M}$. Since $(d\psi_T)^{ij}_{\mu}(x) = \delta^i_j$ for all $x \in \psi_t^{-1}(U_{\mu_2})$, we have $f(T) = 0$. By (3.34), (3.35), (3.36) and the fundamental theorem of calculus, we have

$$\begin{aligned}
f(t) &\leq K\int_t^T \tau^{\frac{\alpha}{2}-\frac{m+2}{2}}d\tau \; + K\int_t^T \tau^{\frac{\alpha}{2}-1}f(\tau)d\tau \\
&\leq Kt^{\frac{\alpha}{2}-\frac{m}{2}} + K\int_t^T \tau^{\frac{\alpha}{2}-1}f(\tau)d\tau
\end{aligned}$$

for each $t \in [\frac{\sigma}{2}, T]$. Hence Grönwall's lemma implies

$$\|\partial^{m+1}\psi_t\|_{0;M} \leq Kt^{\frac{\alpha}{2}-\frac{m}{2}}\exp(KT^{\frac{\alpha}{2}}) \tag{3.37}$$

for each $t \in [\frac{\sigma}{2}, T]$. Now, since $\Psi = (\rho \circ \psi_t) * (\partial\psi_t)$, where we have abbreviated the function ρ_{μ_2}

as ρ. Using the Francesco Faà di Bruno formula again, we obtain

$$\partial^l(\rho \circ \psi_t) = \sum_{i_1+2i_2+\cdots+li_l=l} \rho^{(i_1+\cdots+i_l)}(\psi_t) * (\partial\psi_t)^{*i_1} * \cdots * (\partial^l\psi_t)^{*i_l}$$

for $1 \leq l \leq m+1$. By the induction hypothesis and (3.37), we have the estimate

$$\begin{aligned} |\partial^l(\rho \circ \psi_t)| &\leq K \sum_{i_1+2i_2+\cdots+li_l=l} \left(\prod_{r=2}^{l} \|\partial^r \psi_t\|_{0;M}^{i_r} \right) \\ &\leq K \sum_{i_1+2i_2+\cdots+li_l=l} \left(\prod_{r=2}^{l} t^{(\frac{\beta}{2}+\frac{1}{2}-\frac{r}{2})i_r} \right) \\ &\leq K \sum_{i_1+2i_2+\cdots+li_l=l} t^{-\frac{1}{2}\sum_{r=2}^{l} ri_r} \\ &\leq K \sum_{i_1+2i_2+\cdots+li_l=l} t^{-\frac{l-i_1}{2}} \\ &\leq Kt^{-\frac{l}{2}}. \end{aligned} \tag{3.38}$$

for any $1 \leq l \leq m+1$ and $t \in [\frac{\sigma}{2}, T]$. Then we observe that $\partial^{m+1}\Psi$ satisfies

$$\partial^m\Psi = \sum_{j_1+j_2=m} \partial^{j_1}(\rho \circ \psi_t) * \partial^{j_2+1}\psi_t.$$

Using the estimate (3.38), the induction hypothesis and (3.37), we subsequently obtain

$$|\partial^m\Psi(x,t)| \leq K \sum_{j_1+j_2=m} t^{-\frac{j_1}{2}} t^{\frac{\beta}{2}-\frac{j_2}{2}} \leq Kt^{\frac{\beta}{2}-\frac{m}{2}}.$$

for each $t \in [\frac{\sigma}{2}, T]$. Therefore we conclude that

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{m-1}{2}} |\hat{\nabla}^m \Psi_\mu^{ij}|_{0;M\times[\frac{\sigma}{2},T]} \leq K.$$

Note that in the RHS of (3.37) the exponent of σ is negative for any $m \geq 1$. This is why we cannot hope for higher powers of $\hat{\nabla}^m\Psi_\mu^{ij}$ to converge for any $m \geq 1$ as $t \to 0$. Lastly, to verify that (3.34) holds for $m = 2$, it suffices to consider the following ODE satisfied by $\partial^2\psi_t$:

$$\frac{\partial}{\partial t}(\partial^2\psi_t) = -\,\partial^2 W(\psi_t, t) * \partial\psi_t * \partial\psi_t - \,\partial W(\psi_t, t) * \partial^2\psi_t.$$

For each $t \in [\frac{\sigma}{2}, T]$, the C^0 bound for $\partial^2 W$ and $\partial\psi_t$ imply

$$|\partial^2 W(\psi_t, t) * \partial\psi_t * \partial\psi_t| \leq K\, t^{\frac{\alpha}{2}-\frac{3}{2}}.$$

Now we can proceed as before to obtain (3.34) for the case $m = 2$. This proves (3.34) by induction. In particular, this implies that

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{m-2}{2}}|\hat{\nabla}^{m-1}\Psi_\mu^{ij}|_{0;M\times[\frac{\sigma}{2},\sigma]} \leq K \tag{3.39}$$

for $2 \leq m \leq k+1$.

Next, we show that

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{m-2}{2}+\frac{\gamma}{2}}[\hat{\nabla}^{m-1}\Psi_\mu^{ij}]_{\gamma,\frac{\gamma}{2};M\times[\frac{\sigma}{2},\sigma]} \leq K \tag{3.40}$$

for any $2 \leq m \leq k$. We first observe that the following equation is satisfied by $\partial^{m-1}\Psi$:

$$\begin{aligned}&\frac{\partial}{\partial t}(\partial^{m-1}\Psi)\\ &= \sum_{j_1+j_2=m-1}\left\{\partial^{j_1+1}\psi_t * \sum_{l_1+l_2=j_2}(\partial^{l_1}(\rho\circ\psi_t) * \frac{\partial}{\partial t}(\partial^{l_2}\psi_t)) + \frac{\partial}{\partial t}(\partial^{j_1+1}\psi_t) * \partial^{j_2}(\rho\circ\psi_t)\right\}.\end{aligned}$$

Using (3.35) to (3.37) and the C^0 norm for $\hat{\nabla}W$, we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial t}(\partial^l\psi_t)\right| &\leq Kt^{\frac{\alpha}{2}-\frac{l+1}{2}} + Kt^{\frac{\alpha}{2}-1}\cdot t^{\frac{\alpha}{2}-\frac{l-1}{2}}\\ &\leq Kt^{\frac{\alpha}{2}-\frac{l+1}{2}} + Kt^{\alpha-\frac{l+1}{2}}\\ &\leq Kt^{\frac{\alpha}{2}-\frac{l+1}{2}}\end{aligned}$$

for all $0 \leq l \leq k+1$. Combining the above estimate, (3.37) and (3.38), we obtain

$$
\begin{aligned}
&\left|\frac{\partial}{\partial t}(\partial^{m-1}\Psi)(x,t)\right| \\
&\leq K \sum_{j_1+j_2=m-1} \left\{ t^{\frac{\alpha}{2}-\frac{j_1}{2}} \sum_{l_1+l_2=j_2} t^{-\frac{l_1}{2}} \cdot t^{\frac{\alpha}{2}-\frac{l_2}{2}} + t^{\frac{\alpha}{2}-\frac{j_1+2}{2}} \cdot t^{-\frac{j_2}{2}} \right\} \\
&\leq K \sum_{j_1+j_2=m-1} \left\{ t^{\alpha-\frac{j_1+j_2}{2}} + t^{\frac{\alpha}{2}-\frac{j_1+j_2+2}{2}} \right\} \\
&\leq K(t^{\alpha-\frac{m-1}{2}} + t^{\frac{\alpha}{2}-\frac{m+1}{2}}) \\
&\leq K t^{\frac{\alpha}{2}-\frac{m+1}{2}}.
\end{aligned}
$$

Now, we fix a point $z \in M$, and let $(x,t),(y,s) \in B_{\sqrt{\sigma}}(z) \times [\frac{\sigma}{2}, \sigma]$. We may assume without loss of generality that $\psi_t(x), \psi_s(y) \in \operatorname{supp}(\rho_{\mu_2})$, and $t > s$ are close enough such that $\psi_\tau(y) \in \operatorname{supp}(\rho_{\mu_2})$ for $\tau \in [s,t]$. Using (3.35), (3.36) and (3.39) we have

$$
\begin{aligned}
&\frac{|\partial_x^{m-1}\Psi_\mu^{ij}(x,t) - \partial_x^{m-1}\Psi_\mu^{ij}(y,s)|}{|x-y|^\gamma + |t-s|^{\frac{\gamma}{2}}} \\
&\leq \frac{|\partial_x^{m-1}\Psi_\mu^{ij}(x,t) - \partial_x^{m-1}\Psi_\mu^{ij}(y,t)|}{|x-y|^\gamma} + \frac{|\partial_x^{m-1}\Psi_\mu^{ij}(y,t) - \partial_x^{m-1}\Psi_\mu^{ij}(y,s)|}{|t-s|^{\frac{\gamma}{2}}} \\
&\leq K\, |\partial_x^m \Psi_\mu^{ij}|_{0;M\times[\frac{\sigma}{2},\sigma]}\, |x-y|^{1-\gamma} + K|t-s|^{-\frac{\gamma}{2}} \int_s^t \left|\frac{\partial}{\partial\tau}(\partial_x^{m-1}\Psi_\mu^{ij})(y,\tau)\right| d\tau \\
&\leq K\, \sigma^{\frac{\beta}{2}-\frac{m}{2}} \sigma^{\frac{1}{2}-\frac{\gamma}{2}} + K|t-s|^{-\frac{\gamma}{2}} \int_s^t \tau^{\frac{\alpha}{2}-\frac{m+1}{2}} d\tau \\
&\leq K\, \sigma^{\frac{\beta}{2}-\frac{\gamma}{2}-\frac{m-1}{2}} + K|t-s|^{-\frac{\gamma}{2}} \int_s^t \tau^{\frac{\alpha}{2}-\frac{m+1}{2}} d\tau \\
&\leq K\, \sigma^{\frac{\beta}{2}-\frac{\gamma}{2}-\frac{m-1}{2}} + K|t-s|^{-\frac{\gamma}{2}} (t^{\frac{\alpha}{2}-\frac{m-1}{2}} - s^{\frac{\alpha}{2}-\frac{m-1}{2}}).
\end{aligned}
$$

Now, we can write $\frac{\alpha}{2} - \frac{m-1}{2} = (\frac{\beta}{2} - \frac{\gamma}{2} - \frac{m-1}{2}) + (\frac{\alpha-\beta}{2} + \frac{\gamma}{2})$. Using the fact that $\alpha > \beta$ and $t, s \in [\frac{\sigma}{2}, \sigma]$, we obtain

$$\begin{aligned}
&|t-s|^{-\frac{\gamma}{2}}(t^{\frac{\alpha}{2}-\frac{m-1}{2}}-s^{\frac{\alpha}{2}-\frac{m-1}{2}})\\
&\le K\sigma^{\frac{\beta}{2}-\frac{\gamma}{2}-\frac{m-1}{2}}|t-s|^{-\frac{\gamma}{2}}(t^{\frac{\gamma}{2}}-s^{\frac{\gamma}{2}})\\
&\le K'\sigma^{-\frac{1}{2}+\frac{\beta}{2}-\frac{\gamma}{2}-\frac{m-2}{2}}.
\end{aligned}$$

Hence

$$\frac{|\partial_x^{m-1}\Psi_\mu^{ij}(x,t)-\partial_x^{m-1}\Psi_\mu^{ij}(y,s)|}{|x-y|^\gamma+|t-s|^{\frac{\gamma}{2}}}\le K\sigma^{-\frac{1}{2}+\frac{\beta}{2}-\frac{\gamma}{2}-\frac{m-2}{2}}$$

for any $(x,t),(y,s)\in B_{\sqrt{\sigma}}(z)\times[\frac{\sigma}{2},\sigma]$. Since $z\in M$ is arbitrary, we have proved (3.40).

□

Proposition 3.8. *Let $k\ge 2$ and $\beta,\gamma\in(0,\alpha)$ be given. There exist a $C^{1,\beta}$ diffeomorphism $\psi:M\to M$ and $K=K(M,k,\hat{g},\|g_0\|_{\alpha;M})$, $T=T(M,\hat{g},\|g_0\|_{\alpha;M})$ such that the following holds: There is a solution $\tilde{g}\in\mathcal{X}_{k,\gamma}^{(\beta)}(M\times[0,T])$ to the Ricci flow such that*

$$\tilde{g}(\cdot,0)=\psi^*g_0\quad\textit{and}\quad\|\tilde{g}\|_{\mathcal{X}_{k,\gamma}^{(\beta)}(M\times[0,T])}\le K.$$

Proof.

By Corollary 3.4, we can find $T=T(M,\hat{g},\|g_0\|_\alpha)>0$ sufficiently small such that $g(t)\in\mathcal{X}_{k+3,\gamma}^{(\alpha)}(M\times[0,T])$ is the unique solution to the Ricci-DeTurck system such that

$$\|g\|_{\mathcal{X}_{k+3,\gamma}^{(\alpha)}}\le K.$$

By Lemma 3.5 and Lemma 3.7, we can find a one-parameter family of diffeomorphisms $\{\psi_t\}_{t\in[0,T]}$ which is generated by

$$\begin{cases} \frac{\partial}{\partial t}\psi_t = -W(\psi_t, t) \\ \psi_T = \mathrm{id} \end{cases},$$

such that if the functions Ψ^{ij}_μ are defined by Definition 3.6, then they satisfy

(1) $\Psi^{ij}_\mu \in C^{\beta,\frac{\beta}{2}}(M \times [0,T];\mathbb{R})$ and $\hat{\nabla}\Psi^{ij}_\mu \in C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M \times (0,T];\mathbb{R})$;

(2) $\|\Psi^{ij}_\mu\|_{\beta,\frac{\beta}{2};M\times[0,T]} + \|\hat{\nabla}\Psi^{ij}_\mu\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \leq K(M,k,\hat{g},\|g_0\|_\alpha,\|g\|_{\mathcal{X}^{(\alpha)}_{k+3,\gamma}})$.

Equivalently, we have $\|\hat{\nabla}\Psi^{ij}_\mu\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \leq K$.

We then define $\tilde{g}(t) := (\psi_t)^* g(t)$. Thus $\tilde{g}(0) = \psi_0^* g_0$ where ψ_0 is a $C^{1,\beta}$ diffeomorphism. We have

$$\frac{\partial}{\partial t}\tilde{g} = (\psi_t)^*(-2Ric(g(t)) + L_W g(t)) - L_W \tilde{g}(t) = -2Ric(\tilde{g}(t)).$$

Hence $\tilde{g}$ is a solution to the Ricci flow on a $M \times (0,T]$. In the sequel, let us denote $g(t)$ by g_t. Next, we fix a chart U. Let $x \in U$ and $\psi_t(x) \in V$ for some chart V. Let $\{x_i\}$ and $\{\tilde{x}_i\}$ be the coordinates on the charts U and V respectively. Moreover, we denote by $h_\mu : U_\mu \cap V \to \mathrm{GL}_n(\mathbb{R})$ the induced transition maps between trivializations of the tangent bundle TM; and by $H_s : U_s \cap V \to \mathrm{GL}_N(\mathbb{R})$ the induced transition maps between trivializations of the bundle $\mathrm{Sym}^2(T^*M)$. So on the intersection $U_\mu \cap V$, we have

$$(d\psi_t)^{ij} = (h_\mu)^j_k (d\psi_t)^{ik}_\mu \quad \text{and} \quad g_{kl} = (H_\mu)^{ab}_{kl} \cdot g^\mu_{ab}$$

where $(d\psi_t)^{ij} = \partial_i \psi^j_t$ are the components of $d\psi_t$ on $U \cap \psi_t^{-1}(V)$, and g^μ_{ab} are the components of g

on U_μ. Then

$$\begin{aligned}
\tilde{g}(x,t)_{ij} &= \psi_t^* g_t(x)_{ij} \qquad (3.41)\\
&= g_t(\psi_t(x))((d\psi_t)(\frac{\partial}{\partial x_i}),\ (d\psi_t)(\frac{\partial}{\partial x_j}))\\
&= g_t(\psi_t(x))((d\psi_t)^{ik}\frac{\partial}{\partial \tilde{x}_k}, (d\psi_t)^{jl}\frac{\partial}{\partial \tilde{x}_l})\\
&= (g_t\circ\psi_t)_{kl}(d\psi_t)^{ik}(d\psi_t)^{jl}\\
&= \sum_s \rho_s(\psi_t)(g_t\circ\psi_t)_{kl}\cdot\sum_\mu \rho_\mu(\psi_t)(d\psi_t)^{ik}\cdot\sum_\nu \rho_\nu(\psi_t)(d\psi_t)^{jl}\\
&= \sum_s \rho_s(\psi_t)(g_t\circ\psi_t)^s_{ab}(H_s)^{ab}_{kl}\cdot\sum_\mu \rho_\mu(\psi_t)(h_\mu)^k_p(d\psi_t)^{ip}_\mu\cdot\sum_\nu \rho_\nu(\psi_t)(h_\nu)^l_q(d\psi_t)^{jq}_\nu\\
&= \sum_{\mu,\nu,s} \rho_s(\psi_t)(g_t\circ\psi_t)^s_{ab}\cdot\Psi^{ip}_\mu\Psi^{jq}_\nu\cdot(h_\mu)^k_p(h_\nu)^l_q(H_s)^{ab}_{kl}.
\end{aligned}$$

Now, since $g \in C^{\alpha,\frac{\alpha}{2}}(M\times[0,T])$, $\Psi \in C^{\beta,\frac{\beta}{2}}(M\times[0,T])$ and $\alpha > \beta$, we have $\tilde{g} \in C^{\beta,\frac{\beta}{2}}(M\times[0,T])$ from (3.41). Next, for $1 \le m \le k$, we see from (3.41) that

$$\hat{\nabla}\tilde{g} = \sum_{j_1+j_2+j_3+j_4=1} \hat{\nabla}^{j_1}(\rho\circ\psi_t) * \hat{\nabla}^{j_2}(g_t\circ\psi_t) * \hat{\nabla}^{j_3}\Psi * \hat{\nabla}^{j_4}\Psi.$$

Note that we have $\|\hat{\nabla}\Psi\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \le K$. Since ψ_t is a $C^{1,\beta}$ diffeomorphism, $g_t\circ\psi_t$ has the same regularity as g_t, and so $\|\hat{\nabla}(g_t\circ\psi_t)\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \le \|\hat{\nabla}(g_t\circ\psi_t)\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \le K$. By the definition of Ψ, $\rho\circ\psi_t$ has at least the regularity of Ψ, thus $\|\hat{\nabla}(\rho\circ\psi_t)\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \le K$. Moreover, by Lemma 1.6, there is

$$\begin{aligned}
\|\Psi\|_{C^{k-1,\gamma}_0(M\times(0,T])} &\le K(\|\Psi\|_{\beta,\frac{\beta}{2};M\times(0,T])} + \|\hat{\nabla}\Psi\|_{C^{k-2,\gamma}_{\frac{1}{2}}(M\times(0,T])})\\
&\le K(\|\Psi\|_{\beta,\frac{\beta}{2};M\times(0,T])} + \|\hat{\nabla}\Psi\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])})\\
&\le K.
\end{aligned}$$

Similarly,

$$\|g_t \circ \psi_t\|_{C_0^{k-1,\gamma}(M\times(0,T])} \le K \quad \text{and} \quad \|\rho \circ \psi_t\|_{C_0^{k-1,\gamma}(M\times(0,T])} \le K.$$

Putting everything together, we obtain

$$\begin{aligned}
&\|(\rho\circ\psi_t) * (g_t\circ\psi_t) * \Psi * \hat{\nabla}\Psi\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])}\\
&\le K\|(\rho\circ\psi_t)\|_{C_0^{k-1,\gamma}(M\times(0,T])}\|(g_t\circ\psi_t)\|_{C_0^{k-1,\gamma}(M\times(0,T])}\|\Psi\|_{C_0^{k-1,\gamma}(M\times(0,T])}\|\hat{\nabla}\Psi\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])}\\
&\le K.
\end{aligned}$$

We can similarly derive the bounds for the other terms in the summation of (3.41). Therefore we conclude that

$$\|\hat{\nabla}\tilde{g}\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \le K.$$

This proves that $\tilde{g} \in \mathcal{X}^{(\beta)}_{k,\gamma}(M\times[0,T])$ and the assertion follows. □

3.4 Short time existence and uniqueness to the harmonic map heat flow

In this section, we assume that a solution to the Ricci flow is given. To construct a solution to the Ricci-Deturck flow, we prove the short time existence and the uniqueness to the associated harmonic map heat flow.

Throughout this subsection, let $\alpha \in (0,1)$ be given such that $g_0 \in C^\alpha(M)$. Let $\gamma \in (0,\alpha)$ be given and we fix $\beta \in (\gamma,\alpha)$. Moreover, let $g(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(M\times[0,T])$ be a solution to the Ricci flow on $M\times(0,T]$ and ψ be a $C^{1,\beta}$ diffeomorphism such that

- $g(0) = \psi^* g_0$;
- $\|\psi\|_{C^{1,\beta}(M;M)} \le C$;
- $\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}(M\times[0,T])} = \|g\|_{\beta,\frac{\beta}{2};M\times[0,T]} + \|\hat{\nabla}g\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}} \le C.$

for some constant $C > 0$.

Associated with $g(t)$, we consider the harmonic map heat flow

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \Delta_{g(t),\hat{g}} \phi_t, & \text{on} \quad M \times (0, T] \\ \varphi_0 = \psi, & \text{on} \quad M. \end{cases} \tag{3.42}$$

We seek short time existence and uniqueness to the initial value problem (3.42). To do that, we reformulate the problem into an equivalent equation on TM via the exponential map. Since $\psi \in C^{1,\beta}(M)$, we can find a C^∞ map $\hat{\psi} : (M, g) \to (M, \hat{g})$ such that $d_{\hat{g}}(\psi(x), \hat{\psi}(x)) < \frac{1}{2}\text{inj}(M, \hat{g})$. Here $d_{\hat{g}}$ is the Riemannian distance with respect to the metric $\hat{g}$. Thus we can write

$$\psi(x) = \exp_{\hat{\psi}(x)}(U(x)) \tag{3.43}$$

for some $C^{1+\beta}$ vector field $U(x)$. The exponential map is taken with respect to the metric $\hat{g}$. If we assume that T is sufficiently small so that $d_{\hat{g}}(\phi_t(x), \hat{\psi}(x)) < \frac{3}{4}\text{inj}(M, \hat{g})$, then we can write the harmonic map heat flow $\phi_t(x)$ in the form

$$\phi_t(x) = \exp_{\hat{\psi}(x)}(V(x, t)) \tag{3.44}$$

for some vector field $V(x, t)$. Note that the assumptions on the injectivity radius ensures that $V(x, t)$ is well-defined. Now the idea is to transform the initial value problem (3.42) into an equivalent PDE for the vector field $V(x, t)$ with initial condition $U(x)$. The following lemma gives such an equivalence.

Lemma 3.9. *If $\phi_t(x) \in \Gamma(M \times [0, T]; M)$ is a solution to the harmonic map heat flow (3.42), then the vector field $V(x, t)$ defined by $\phi_t(x) = \exp_{\hat{\psi}(x)}(V(x, t))$ is a solution in $\Gamma(M \times [0, T]; \hat{\psi}^*(TM))$ to* *the initial value problem*

$$\begin{cases} \left(\frac{\partial}{\partial t} - tr_g \hat{\nabla}^2\right) V = \mathcal{P}(V, \hat{\nabla} V) & on \quad M \times (0, T] \\ V(x, 0) = U(x) & on \quad M, \end{cases} \tag{3.45}$$

provided that T is sufficiently small, where

$$\mathcal{P}(V, \hat{\nabla} V)^a := g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij} \, (\hat{\nabla}_k V^a + Z^a_k(V)) + g^{ij} S^a_{ij}(V, \hat{\nabla} V), \tag{3.46}$$

$Z(V)$ and $S(V, \hat{\nabla} V)$ are sections of $T^*M \otimes \hat{\psi}^* TM$ and $T^*M \otimes T^*M \otimes \hat{\psi}^* TM$ respectively. Here $Z^a_k(V)$ is a smooth function in V, whereas $S^a_{ij}(V, \hat{\nabla} V)$ is a smooth function in V and a polynomial of degree 2 in $\hat{\nabla} V$. Moreover, the converse is also true.

Proof. Let $\{x^i\}$, $\{y^a\}$ and $\{z^c\}$ be local coordinates around x, $\psi(x)$ and $\phi_t(x)$ respectively. We first show that

$$d\phi_t(x)(\frac{\partial}{\partial x^i}) = (d \exp_{\hat{\psi}(x)})_{V(x,t)} \left(\hat{\nabla}_i V(x, t) + Z_i(V(x, t))\right), \tag{3.47}$$

where $Z_i = Z_i(x, V(x, t)) \in T_{\hat{\psi}(x)} M$ is a vector field depending smoothly on x and V.

Fix x and t, let $\lambda(\tau) := \exp_{\hat{\psi}(x)}(\tau V(x, t))$ be a geodesic. Let $\gamma(s)$ be a curve such that $\gamma(0) = x$ and $\gamma'(0) = \frac{\partial}{\partial x^i}$. Let $F(s, \tau) := \exp_{\hat{\psi}(\gamma(s))}(\tau V(\gamma(s), t))$ be a variation of λ through geodesics. Then

$$J(\tau) := \frac{\partial F}{\partial s}(s, \tau)\Big|_{s=0}$$

is a Jacobi field with initial conditions $J(0) = d\hat{\psi}(x)(\frac{\partial}{\partial x^i})$, $\hat{\nabla}_\tau J(0) = \hat{\nabla}_X V(x, t)$ such that $J(1) = d\phi_t(x)(\frac{\partial}{\partial x^i})$. Let us decompose the Jacobi field $J := J_1 + J_2$ in a way that $J_1(0) = 0$, $\hat{\nabla}_\tau J_1(0) = \hat{\nabla}_i V(x, t)$ and $J_2(0) = d\hat{\psi}(x)(\frac{\partial}{\partial x^i})$, $\hat{\nabla}_\tau J_2(0) = 0$. Then we have

$$J_1(1) \; = (d \exp_{\hat{\psi}(x)})_{V(x,t)} \left(\hat{\nabla}_i V(x, t)\right).$$

Moreover, the vector $J_2(1)$ depends smoothly on x and $V(x,t)$. To see that, let $\sigma(u) := \exp_{\hat{\psi}(x)}(u\ J_2(0))$ be a geodesic, and let $H(u,\tau) := \exp_{\sigma(u)}\left(\tau\ P_{\sigma(u)}(V(x,t))\right)$ be another geodesic variation of $\lambda(\tau)$ where $P_{\sigma(u)}(V(x,t))$ is the parallel transport of $V(x,t)$ through $\sigma(u)$. Note that the Jacobi field $J_2(\tau)$ arises from the variation H since

$$\frac{\partial H}{\partial u}(u,\tau)\Big|_{u=0,\tau=0} = J_2(0) \quad \text{and} \quad D_\tau \frac{\partial H}{\partial u}(u,\tau)\Big|_{u=0,\tau=0} = 0.$$

Then

$$\tilde{Z}_i(x, V(x,t)) := J_2(1) = \frac{\partial H}{\partial u}(u,1)\Big|_{u=0} \in T_{\exp_{\hat{\psi}(x)}(V)}M$$

is a vector field depending smoothly on x and $V(x,t)$. We then define the vector field $Z_i(x, V(x,t)) \in T_{\hat{\psi}(x)}M$ by

$$(d\exp_{\hat{\psi}(x)})_{V(x,t)}(Z_i(x, V(x,t))) = \tilde{Z}_i(x, V(x,t)).$$

Combining the above results, we obtain the identity (3.47).

Next, let $\bar{\nabla}$ be the connection on $T^*M \otimes \phi_t^*(TM)$ induced by $\nabla_{g(t)}$ and $\phi_t^*\hat{\nabla}$. It is worth noting that although we are using the connection $\bar{\nabla} = \nabla_{g(t)} \otimes 1 + 1 \otimes \phi_t^*\hat{\nabla}$ which applies to sections of the form $dx^i \otimes \phi_t^*\partial_{z^c}$, we will be using the basis $\{(d\exp_{\psi(\hat{x})})_{V(x,t)}(\partial_{y^a})\}$ on the fibre $T_{\phi_t(x)}M$. Moreover, we denote by ω_a^c the connection 1-forms for the connection $\phi_t^*\hat{\nabla}$ with respect to the basis $\{(d\exp_{\psi(\hat{x})})_{V(x,t)}(\partial_{y^a})\}$, thus

$$\hat{\nabla}_{(d\exp_{\hat{\psi}})_V(\partial_{y^a})}(d\exp_{\hat{\psi}})_V(\partial_{y^b}) = \omega_{ab}^c \cdot (d\exp_{\hat{\psi}})_V(\partial_{y^c}).$$

Note that ω_a^c depend smoothly on $V(x,t)$. Then (3.47) implies

$$
\begin{aligned}
&\bar{\nabla} d\phi_t \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= (\phi_t^* \hat{\nabla})_{\frac{\partial}{\partial x^i}} \left(d\phi_t \left(\frac{\partial}{\partial x^j}\right)\right) - d\phi_t \left((\nabla_{g(t)})_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) \\
&= (\phi_t^* \hat{\nabla})_{\frac{\partial}{\partial x^i}} \left((\hat{\nabla}_j V^b + Z_j^b)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^b}\right)\right) - d\phi_t \left((\Gamma_g)_{ij}^k \frac{\partial}{\partial x^k}\right) \\
&= \left(\frac{\partial}{\partial x^i}\hat{\nabla}_j V^b + \frac{\partial}{\partial x^i} Z_j^b\right)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^b}\right) \\
&\qquad + (\hat{\nabla}_j V^b + Z_j^b) \cdot \hat{\nabla}_{(\hat{\nabla}_i V^a + Z_i^a)(d\exp_{\hat{\psi}})_V\left(\frac{\partial}{\partial y^a}\right)} \left((d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^b}\right)\right) \\
&\qquad - (\Gamma_g)_{ij}^k (\hat{\nabla}_k V^c + Z_k^c)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^c}\right) \\
&= \left(\hat{\nabla}_i \hat{\nabla}_j V^c + \hat{\nabla}_i Z_j^c\right)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^c}\right) + (\hat{\nabla}_j V^b + Z_j^b)(\hat{\nabla}_i V^a + Z_i^a) \cdot \omega_{ab}^c \left(\frac{\partial}{\partial y^c}\right) \\
&\qquad + (\Gamma_{\hat{g}} - \Gamma_g)_{ij}^k (\hat{\nabla}_k V^c + Z_k^c)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^c}\right) \\
&= \left(\hat{\nabla}_i \hat{\nabla}_j V^c + (\Gamma_{\hat{g}} - \Gamma_g)_{ij}^k \, (\hat{\nabla}_k V^c + Z_k^c) + \hat{\nabla}_i V^a \, \hat{\nabla}_j V^b \cdot \omega_{ab}^c\right) \, (d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^c}\right) \\
&\qquad + \left(\hat{\nabla}_i Z_j^c + (Z_i^a \hat{\nabla}_j V^b + Z_j^b \hat{\nabla}_i V^a + Z_i^a Z_j^b) \cdot \omega_{ab}^c\right) \, (d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^c}\right).
\end{aligned}
$$

This means that

$$
\begin{aligned}
&\bar{\nabla} d\phi_t \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \left(\hat{\nabla}_j \hat{\nabla}_i V^a + (\Gamma_{\hat{g}} - \Gamma_g)_{ij}^k \, (\hat{\nabla}_k V^a + Z_k^a(V)) + S_{ij}^a(V, \hat{\nabla} V)\right)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^a}\right).
\end{aligned}
\tag{3.48}
$$

Here $S_{ij}^a(V, \hat{\nabla} V) := \hat{\nabla}_i Z_j^a + (Z_i^b \hat{\nabla}_j V^c + Z_j^c \hat{\nabla}_i V^b + Z_i^b Z_j^c) \cdot \omega_{bc}^a + \hat{\nabla}_i V^b \hat{\nabla}_j V^c \cdot \omega_{bc}^a$ is a smooth function in $V(x,t)$ and a polynomial of degree 2 in $\hat{\nabla}V(x,t)$. The Laplacian $\Delta_{g(t),\hat{g}}\phi_t$ is thus given by

$$\begin{aligned}
&\Delta_{g(t),\hat{g}}\phi_t \\
&= \mathrm{tr}_g \nabla d\phi_t \\
&= \left(\mathrm{tr}_g \hat{\nabla}^2 V^a \ + g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij} \ (\hat{\nabla}_k V^a + Z^a_k) + g^{ij} S^a_{ij}\right)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^a}\right).
\end{aligned}$$

On the other hand, it is easy to see that

$$\frac{\partial}{\partial t}\phi_t = (d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial t}V\right) = \left(\frac{\partial}{\partial t}V^a\right)\ (d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^a}\right).$$

Therefore, we conclude that

$$\begin{aligned}
&\frac{\partial \phi_t}{\partial t} - \Delta_{g(t),\hat{g}}\phi_t \\
&= \left(\frac{\partial}{\partial t}V^a - \mathrm{tr}_g \hat{\nabla}^2 V^a \ - g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij} \ (\hat{\nabla}_k V^a + Z^a_k) - g^{ij} S^a_{ij}\right)(d\exp_{\hat{\psi}})_V \left(\frac{\partial}{\partial y^a}\right).
\end{aligned} \tag{3.49}$$

The assertion then follows since the exponential map exp is a smooth diffeomorphism with respect to the metric $\hat{g}$. □

Similar to the proof of the short time existence and uniqueness to the Ricci De-Turck flow, the short time existence and uniqueness to (3.45) can be obtained by applying the Banach fixed point theorem to the following linear system:

$$\begin{cases} \left(\frac{\partial}{\partial t} - tr_g \hat{\nabla}^2\right) V = \mathcal{P}(W, \hat{\nabla}W) & \text{on} \quad M \times (0, T] \\ V(x,0) = U(x) & \text{on} \quad M, \end{cases} \tag{3.50}$$

where

$$\mathcal{P}(W, \hat{\nabla}W)^a := g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij} \ (\hat{\nabla}_k W^a + Z^a_k(W)) + g^{ij} S^a_{ij}(W, \hat{\nabla}W). \tag{3.51}$$

Moreover, we note that the fact $U \in C^{1,\beta}(M)$ implies that $U \in C^{\delta}(M)$ for any exponent $\delta \in (0,1)$ which can be arbitrarily close to 1.

Proposition 3.10. *Let $k \geq 2$ and $\delta \in (0,1)$ be given such that $\delta \geq \beta$. Suppose that*

- $W(x,0) = U(x)$*;*
- $\|W\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \leq B.$

Then there exists a unique solution $V \in \mathcal{X}^{(\delta)}_{k+1,\gamma}(M \times [0,T]; \hat{\psi}^*(TM))$ to the linear system (3.50) and there are positive constants $K_1 = K_1(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}})$, $K_2 = K_2(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B)$ such that

$$\|V\|_{\mathcal{X}^{(\delta)}_{k+1,\gamma}(M\times[0,T])} \leq K_1(K_2\, T^{\frac{1}{2}} + \|U\|_{C^{1,\beta}(M)}).$$

Proof.

We first claim that the assumption $\|W\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \leq B$ implies

$$\|\mathcal{P}(W, \hat{\nabla} W)\|_{C^{k-1,\gamma}_{1-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \leq K(\hat{g}, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B). \tag{3.52}$$

To see that, we estimate

$$\begin{aligned}
&\left\|g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij}\, (\hat{\nabla}_k W + Z_k)\right\|_{C^{k-1,\gamma}_{1-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\leq K(\hat{g})\, \|g^{-1}\|_{C^{k-1,\gamma}_0(M\times(0,T])} \|\Gamma_{\hat{g}} - \Gamma_g\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \|\hat{\nabla} W + Z\|_{C^{k-1,\beta}_{\frac{1}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\leq K(\hat{g}, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B)
\end{aligned}$$

and

$$
\begin{aligned}
&\left\|g^{ij} S^a_{ij}(W, \hat\nabla W)\right\|_{C^{k-1,\gamma}_{1-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\le K(\hat g)\, T^{\frac{\delta}{2}-\frac{\beta}{2}}\ \|g^{-1}\|_{C^{k-1,\gamma}_0(M\times(0,T])} \|\hat\nabla W\|^2_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\le K(\hat g, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B) T^{\frac{\delta}{2}-\frac{\beta}{2}}.
\end{aligned}
$$

Hence the estimate (3.52) is established. By Lemma 1.6, this in particular implies that

$$
\|\mathcal{P}(W, \hat\nabla W)\|_{C^{k-1,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \le K(\hat g, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B)\, T^{\frac{\beta}{2}}.
$$

Since the initial condition in (3.50) satisfies $U \in C^\delta(M; \hat\psi^*(TM))$, then by Theorem 2.1 there exists a unique solution V to the linear system (3.50) such that $V \in C^{\delta,\frac{\delta}{2}}(M \times [0,T])$ and $\hat\nabla V \in C^{k,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M \times (0,T])$. Moreover, V satisfies the estimate

$$
\begin{aligned}
&\|V\|_{\delta,\frac{\delta}{2};M\times[0,T]} + \|\hat\nabla V\|_{C^{k,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\le K_1 \left(\|\mathcal{P}(W, \hat\nabla W)\|_{C^{k-1,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} + \|U\|_{\delta;M} \right) \\
&\le K_1(K_2\, T^{\frac{\beta}{2}} + \|U\|_{C^{1,\beta}(M)}),
\end{aligned}
\tag{3.53}
$$

where $K_1 = K_1(\hat g, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}})$ and $K_2 = K_2(\hat g, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B)$. □

Now, we choose $B > 2K_1\|U\|_{C^{1,\beta}(M)}$ to be a large positive constant. For $k \ge 2$, we define a closed subset $\mathcal{W}$ in $\mathcal{X}^{(\delta)}_{k,\gamma}(M \times [0,T])$ by

$$
\mathcal{W} := \{W \in \mathcal{X}^{(\delta)}_{k,\gamma}(M \times [0,T]) \mid \|W\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \le B\}.
$$

Next, we define an operator $\mathcal{S} : \mathcal{W} \to \mathcal{X}^{(\delta)}_{k,\gamma}(M \times [0,T])$ by

$$V := \mathcal{S}(W),$$

where V is the unique solution to the system (3.50) in $\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])$. By Proposition 3.10 and our choice of B, we can make $\|V\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \leq B$ provided that T is sufficiently small. Consequently $\mathcal{S}(\mathcal{W}) \subset \mathcal{W}$.

Proposition 3.11. *If $T = T(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B)$ is chosen sufficiently small, the operator $\mathcal{S}$ is a contraction mapping.*

Proof. In the sequel, K will denote a constant depending only on $\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, B$.

Let $W_1, W_2 \in \mathcal{W}$ and write $V_i := \mathcal{S}(W_i)$ for $i = 1,2$. Then $V = V_1 - V_2$ solves the system

$$\begin{cases} \left(\frac{\partial}{\partial t} - tr_g \hat{\nabla}^2\right) V = \mathcal{P}(W_1, \hat{\nabla} W_1) - \mathcal{P}(W_2, \hat{\nabla} W_2) & \text{on} \quad M \times (0,T] \\ V|_{t=0} = 0 & \text{on} \quad M. \end{cases} \tag{3.54}$$

We recall that

$$\mathcal{P}(W, \hat{\nabla} W)^a = g^{ij}(\Gamma_{\hat{g}} - \Gamma_g)^k_{ij} \, (\hat{\nabla}_k W^a + Z^a_k(W)) + g^{ij} S^a_{ij}(W, \hat{\nabla} W).$$

Now, we define the tensors $Z(s) \in \Gamma(M; T^*M \otimes \hat{\psi}^* TM)$ and $S(s) \in \Gamma(M; T^*M \otimes T^*M \otimes \hat{\psi}^* TM)$ by

$$Z(s) := Z(sW_1 + (1-s)W_2))$$

and

$$S(s) := S(sW_1 + (1-s)W_2, \; s\hat{\nabla} W_1 + (1-s)\hat{\nabla} W_2).$$

Note that $(1-s)W_1 + sW_2 \in \mathcal{W}$ for all $s \in [0,1]$. From Lemma 3.9, we observe that

- $\nabla_{\mathbf{q}} Z_k^a(x, \mathbf{q})$ is a smooth function in x and $\mathbf{q}$;

- $\nabla_{\mathbf{q}} S_{ij}^a(x, \mathbf{q}, \mathbf{A})$ is a smooth function in x and $\mathbf{q}$, and a polynomial of degree two in $\mathbf{A}$;

- $\nabla_{\mathbf{A}} S_{ij}^a(x, \mathbf{q}, \mathbf{A})$ is a smooth function in x and $\mathbf{q}$, and a polynomial of degree one in $\mathbf{A}$.

Let $\{dx^i \otimes \frac{\partial}{\partial y^a}\}$ be the local frame of $T^*M \otimes \hat{\psi}^*TM$ on the chart U_{ia}, then from the above observations and Lemma 1.6, we have

$$\begin{aligned}
\|\frac{\partial}{\partial s} Z(s)\|_{C_0^{k-2,\gamma}(M\times(0,T])} &= \sum_{i,a} \|\frac{\partial}{\partial s} Z(s)_i^a\|_{C_0^{k-2,\gamma}(U_{ia}\times(0,T])} \\
&= \sum_{i,a} \|\langle D_{\mathbf{q}} Z(s)_i^a,\ W_1 - W_2\rangle\|_{C_0^{k-2,\gamma}(U_{ia}\times(0,T])} \\
&\leq \sum_{i,a} K \sup_{s\in[0,1]} \|D_{\mathbf{q}} Z(s)_k^a\|_{C_0^{k-2,\gamma}(U_{ia}\times(0,T])} \|W_1 - W_2\|_{C_0^{k-2,\gamma}(U_{ia}\times(0,T])} \\
&\leq K\|W_1 - W_2\|_{\mathcal{X}_{k,\gamma}^{(\delta)}(M\times[0,T])}.
\end{aligned}$$

Hence

$$\|Z(W_1) - Z(W_2)\|_{C_0^{k-2,\gamma}(M\times(0,T])} \leq \|\frac{\partial}{\partial s} Z(s)\|_{C_0^{k-2,\gamma}(M\times(0,T])} \leq K\|W_1 - W_2\|_{\mathcal{X}_{k,\gamma}^{(\delta)}(M\times[0,T])}. \quad (3.55)$$

Let $\{dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^a}\}$ be the local frame of $T^*M \otimes T^*M \otimes \hat{\psi}^*TM$ on the chart U_{ija}, then we similarly have

$$
\begin{aligned}
&\|\frac{\partial}{\partial s}S(s)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&= \sum_{i,j,a}\|\frac{\partial}{\partial s}S(s)^a_{ij}\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(U_{ija}\times(0,T])} \\
&\leq \sum_{i,j,a}\|\langle D_{\mathbf{q}}S(s)^a_{ij},\ W_1-W_2\rangle\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(U_{ija}\times(0,T])} + \|\langle D_{\mathbf{A}}S(s)^a_{ij},\ \hat{\nabla}W_1-\hat{\nabla}W_2\rangle\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(U_{ija}\times(0,T])} \\
&\leq \sum_{i,j,a}KT^{\frac{\delta}{2}}\sup_{s\in[0,1]}\|D_{\mathbf{q}}S(s)^a_{ij}\|_{C^{k-2,\gamma}_{1-\delta}(U_{ija}\times(0,T])}\|W_1-W_2\|_{C^{k-2,\gamma}_{0}(U_{ija}\times(0,T])} \\
&\qquad + \sum_{i,j,a}KT^{\frac{\delta}{2}}\sup_{s\in[0,1]}\|D_{\mathbf{A}}S(s)^a_{ij}\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(U_{ija}\times(0,T])}\|\hat{\nabla}W_1-\hat{\nabla}W_2\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(U_{ija}\times(0,T])} \\
&\leq KT^{\frac{\delta}{2}}\|W_1-W_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\|S(W_1,\hat{\nabla}W_1)-S(W_2,\hat{\nabla}W_2)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} &\leq \|\frac{\partial}{\partial s}S(s)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&\leq K\|W_1-W_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])}.
\end{aligned} \tag{3.56}
$$

From (3.55), (3.56) and Lemma 1.6 we obtain

$$\begin{aligned}
&\|\mathcal{P}(W_1, \hat{\nabla} W_1) - \mathcal{P}(W_2, \hat{\nabla} W_2)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&\leq \|g^{-1} * (\Gamma_{\hat{g}} - \Gamma_g) * \hat{\nabla}(W_1 - W_2)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&\quad + \|g^{-1} * (\Gamma_{\hat{g}} - \Gamma_g) * (Z(W_1) - Z(W_2))\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&\quad + \|g^{-1} * (S(W_1, \hat{\nabla} W_1) - S(W_2, \hat{\nabla} W_2))\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \\
&\leq K \Big(T^{\frac{\beta}{2}} \|\Gamma_{\hat{g}} - \Gamma_g\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \|\hat{\nabla}(W_1 - W_2)\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M\times(0,T])} \\
&\qquad + T^{\frac{1}{2}+\frac{\beta}{2}-\frac{\delta}{2}} \|\Gamma_{\hat{g}} - \Gamma_g\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \|Z(W_1) - Z(W_2)\|_{C^{k-2,\gamma}_{0}(M\times(0,T])} \\
&\qquad + \|S(W_1, \hat{\nabla} W_1) - S(W_2, \hat{\nabla} W_2)\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \Big) \\
&\leq K\, T^{\frac{\beta}{2}} \|W_1 - W_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])}.
\end{aligned}$$

Theorem 2.1 then implies

$$\|V\|_{\delta,\frac{\delta}{2};M\times[0,T]} + \|\hat{\nabla} V\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M\times(0,T])} \leq K\, T^{\frac{\beta}{2}} \|W_1 - W_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])}.$$

Therefore,

$$\|V_1 - V_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \leq K\, T^{\frac{\beta}{2}} \|W_1 - W_2\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])}. \tag{3.57}$$

This proves the proposition. □

From the previous proposition and Lemma 3.9, the harmonic map heat flow (3.42) has a unique solution ϕ_t such that $\phi_t \in C^{\delta,\frac{\delta}{2}}(M \times [0,T])$ and $d\phi_t \in C^{k-1,\gamma}_{\frac{1}{2}-\frac{\delta}{2}}(M \times (0,T])$ provided that $g \in \mathcal{X}^{(\beta)}_{k,\gamma}(M \times [0,T])$ and that $T = T(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, \|\psi\|_{C^{1,\beta}(M)})$ is chosen sufficiently small. Furthermore, as the initial condition satisfies $\psi \in C^{1,\beta}(M)$, we can choose δ to be arbitrarily close to

1. It turns out that the regularity of ϕ_t can be improved.

Theorem 3.12. *Let $\lambda \in (\gamma, \beta)$ be given. If $T = T(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, \|\psi\|_{C^{1,\beta}(M)})$ is chosen sufficiently small, then there exists a unique solution $V(x,t)$ to the initial value problem (3.45) such that*

(1) $V \in C^{1+\lambda, \frac{1+\lambda}{2}}(M \times [0,T])$ and $\hat{\nabla}^2 V \in C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(M \times (0,T])$;

(2) $\|V\|_{1+\lambda,\frac{1+\lambda}{2};M\times[0,T]} + \|\hat{\nabla}^2 V\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(M\times(0,T])} \leq K(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, \|U\|_{C^{1,\beta}(M)})$.

Proof. In the sequel of the proof, K will always denote a constant depending only on $\hat{g}$, M, $\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}$, and $\|U\|_{C^{1,\beta}(M)}$.

Let us take $\delta = 1 + \lambda - \beta$ in the definition of $\mathcal{X}^{(\delta)}_{k,\gamma}(M \times [0,T]; \hat{\psi}^*(TM))$. Note that $\lambda \in (\gamma, \beta)$ implies $\delta < 1$. By Proposition 3.11, we can find a unique solution $V \in \mathcal{X}^{(\delta)}_{k,\gamma}(M \times [0,T])$ to the initial value problem (3.45) if T is chosen sufficiently small. Since $\|\hat{\nabla} U\|_{\lambda;M} \leq K$, we claim that

$$\lim_{t\to 0^+} \hat{\nabla}_i V(x,t) = \hat{\nabla}_i U(x). \tag{3.58}$$

For instance, let us consider an arbitrary chart $\varphi : U \to \mathbb{R}^n$, so that on this chart $V = V^a \frac{\partial}{\partial y^a}$ and we abbreviate $V^a(x,t) = V^a(\varphi^{-1}(x), t)$ for $x \in \varphi(U)$. Then V^a solves the equation

$$\begin{cases} \frac{\partial}{\partial t} V^a - g^{kl} D^2_{kl} V^a = \tilde{\mathcal{P}}^a(V, \hat{\nabla} V) & \text{on} \quad \varphi(U) \times (0,T] \\ V^a(x,0) = U^a(x) & \text{on} \quad \varphi(U) \end{cases},$$

where $\tilde{\mathcal{P}}^a(V, \hat{\nabla} V) = \mathcal{P}^a(V, \hat{\nabla} V) + (\partial\hat{\Gamma} + \hat{\Gamma} *_g \hat{\Gamma}) *_g V + \hat{\Gamma} *_g \hat{\nabla} V$. Since $\|V\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \leq K$, the estimate (3.52) implies that

$$\|\tilde{\mathcal{P}}^a(V, \hat{\nabla} V)\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(U\times(0,T])} \leq K. \tag{3.59}$$

Moreover, the uniqueness part of Lemma 2.2 and (2.6) imply that V^a has the form

$$D_iV^a(x,t) = -\int_0^t \int_{\mathbb{R}^n} D_i\Gamma(x,t;\xi,\tau)\tilde{\mathcal{P}}^a(V,\hat{\nabla}V)(\xi,\tau)d\xi d\tau + \int_{\mathbb{R}^n} D_i\Gamma(x,t;\xi,0)U^a(\xi)d\xi. \quad (3.60)$$

Using the estimates of fundamental solution (2.7) and (3.59), we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} |D_i\Gamma(x,t;\xi,\tau)\tilde{\mathcal{P}}^a(\xi,\tau)|d\xi d\tau \qquad (3.61)\\
&\leq K\int_0^t \int_{\mathbb{R}^n} (t-\tau)^{-\frac{n+1}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) |\tilde{\mathcal{P}}^a(\xi,\tau)|\ d\xi d\tau\\
&\leq K\int_0^t \int_{\mathbb{R}^n} (t-\tau)^{-\frac{n+1}{2}} \exp\left(-\frac{|x-\xi|^2}{K(t-\tau)}\right) \tau^{-\frac{1}{2}+\frac{\lambda}{2}} \|\tilde{\mathcal{P}}^a\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(U\times(0,T])}\ d\xi d\tau\\
&\leq K\int_0^t \int_0^\infty (t-\tau)^{-\frac{1}{2}}\rho^{n-1}\exp(-\frac{1}{K}\rho^2)\tau^{-\frac{1}{2}+\frac{\lambda}{2}}\ d\rho d\tau\\
&\leq Kt^{\frac{\lambda}{2}}.
\end{aligned}$$

For the second integral in (3.60), we note that by [16, (11.13)] the fundamental solution $\Gamma(x,t;\xi,0)$ can be written in the form

$$\Gamma(x,t;\xi,0) = \Gamma_0(x-\xi,t;\xi,0) + \Gamma_1(x,t;\xi,0),$$

where the function $\Gamma_0(x-\xi,t;\xi,0)$ is defined in [16, (11.2)] which is the fundamental solution obtained by freezing the operator $\frac{\partial}{\partial t} - g^{kl}D^2_{kl}$ at the point $(\xi,0)$. Moreover, Γ_0 also satisfies the estimates (2.7) by [16, (11.3)]. On the other hand, since $g^{kl}\in C^{\beta,\frac{\beta}{2}}(\mathbb{R}^n\times[0,T])$, the minor term $\Gamma_1(x,t;\xi,0)$ satisfies the estimate

$$\left|D_x\Gamma_1(x,t;\xi,0)\right| \leq Kt^{-\frac{n+1-\beta}{2}}\exp\left(-\frac{|x-\xi|^2}{Kt}\right) \quad (3.62)$$

by [16, P.377]. Since $\int_{\mathbb{R}^n} D_z\Gamma_0(z,t;\xi,0)dz = 0$ for any fixed ξ by [16, (11.5)], we also have

$\int_{\mathbb{R}^n} D_x\Gamma_1(x,t;\xi,0)d\xi = 0$. Hence (3.62) implies

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} D_i\Gamma_1(x,t;\xi,0)U^a(\xi)d\xi\right| \\
&= \left|\int_{\mathbb{R}^n} D_i\Gamma_1(x,t;\xi,0)(U^a(\xi)-U^a(x))d\xi\right| \\
&\le K\int_{\mathbb{R}^n} t^{-\frac{n+1-\beta}{2}}\exp\left(-\frac{|x-\xi|^2}{Kt}\right)\|\hat{\nabla}U\|_{0;M}|x-\xi|\;d\xi \\
&\le K\int_0^\infty t^{\frac{\beta}{2}}\rho^n\exp(-\frac{1}{K}\rho^2)\;d\rho \\
&\le Kt^{\frac{\beta}{2}}.
\end{aligned}
\tag{3.63}
$$

We next write

$$
\begin{aligned}
&\int_{\mathbb{R}^n} D_i\Gamma_0(x-\xi,t;\xi,0)U^a(\xi)d\xi \\
&= \int_{\mathbb{R}^n} D_i\Gamma_0(x-\xi,t;x,0)U^a(\xi)d\xi \\
&\qquad + \int_{\mathbb{R}^n}(D_i\Gamma_0(x-\xi,t;\xi,0)-D_i\Gamma_0(x-\xi,t;x,0))(U^a(\xi)-U^a(x))d\xi \\
&:= I_1 + I_2.
\end{aligned}
\tag{3.64}
$$

For the second term I_2, we apply the estimate [16, (11.4)] to obtain

$$
\begin{aligned}
|I_2| &\le K\int_{\mathbb{R}^n}|x-\xi|^\beta t^{-\frac{n+1}{2}}\exp\left(-\frac{|x-\xi|^2}{Kt}\right)|U^a(x)-U^a(\xi)|\;d\xi \\
&\le K\int_{\mathbb{R}^n}|x-\xi|^{1+\beta}t^{-\frac{n+1}{2}}\exp\left(-\frac{|x-\xi|^2}{Kt}\right)\|\hat{\nabla}U\|_{0;M}\;d\xi \\
&\le K\int_0^\infty t^{\frac{\beta}{2}}\rho^n\exp(-\frac{1}{K}\rho^2)\;d\rho \\
&\le Kt^{\frac{\beta}{2}}.
\end{aligned}
\tag{3.65}
$$

For the first term I_1, by noting that $D_i\Gamma_0(x-\xi,t;x,0) = -D_{\xi^i}\Gamma_0(x-\xi,t;x,0)$, we have

$$I_1 = \int_{\mathbb{R}^n} \Gamma_0(x-\xi,t;x,0) D_i U^a(\xi) d\xi. \tag{3.66}$$

Now, by putting (3.61) to (3.65) into (3.60), we obtain

$$\left| D_i V^a(x,t) - \int_{\mathbb{R}^n} \Gamma_0(x-\xi,t;x,0) D_i U^a(\xi) d\xi \right| \le K t^{\frac{\lambda}{2}}.$$

Note that $\Gamma_0(x-\xi,t;x,0) \to \delta(x-\xi)$ as $t \to 0^+$ in the sense of distribution. Hence by taking $t \to 0^+$, we obtain (3.58).

Next, we observe that $\hat{\nabla}_i V(x,t)$ solves the system

$$\begin{cases} \left(\frac{\partial}{\partial t} - tr_g \hat{\nabla}^2\right)(\hat{\nabla}_i V) = \tilde{\mathcal{P}}(V, \hat{\nabla} V, \hat{\nabla}^2 V) & \text{on} \quad M \times (0,T] \\ \hat{\nabla}_i V(x,0) = \hat{\nabla}_i U(x) & \text{on} \quad M, \end{cases} \tag{3.67}$$

where

$$\tilde{\mathcal{P}}(V, \hat{\nabla} V, \hat{\nabla}^2 V) = \hat{\nabla} g^{-1} * \hat{\nabla}^2 V + g^{-1} * \hat{R} * \hat{\nabla} V + g^{-1} * \hat{\nabla}\hat{R} * V + \hat{\nabla}_i(\mathcal{P}(V, \hat{\nabla} V))$$

and the initial condition makes sense in view of (3.58).

Since $\|V\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \le K$, the estimate (3.52) implies that

$$\|\hat{\nabla}_i(\mathcal{P}(V, \hat{\nabla} V))\|_{C^{k-2,\gamma}_{\frac{3}{2}-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \le K. \tag{3.68}$$

It is also easy to see that

$$\|g^{-1} * \hat{R} * \hat{\nabla} V + g^{-1} * \hat{\nabla}\hat{R} * V\|_{C^{k-2,\gamma}_{\frac{3}{2}-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \le K. \tag{3.69}$$

Moreover, $\|V\|_{\mathcal{X}^{(\delta)}_{k,\gamma}(M\times[0,T])} \le K$ implies that $\|\hat{\nabla}^2 V\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \le K$. This gives

$$\|\hat{\nabla} g^{-1} * \hat{\nabla}^2 V\|_{C^{k-2,\gamma}_{\frac{3}{2}-\frac{\beta}{2}-\frac{\delta}{2}}(M\times(0,T])} \le K\, \|\hat{\nabla} g^{-1}\|_{C^{k-2,\gamma}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} \|\hat{\nabla}^2 V\|_{C^{k-2,\gamma}_{1-\frac{\delta}{2}}(M\times(0,T])} \le K'. \tag{3.70}$$

As $1-\frac{\lambda}{2} = \frac{3}{2}-\frac{\beta}{2}-\frac{\delta}{2}$ and $\lambda < \beta$, we obtain by putting (3.68) to (3.70) that

$$\left\|\tilde{\mathcal{P}}(V, \hat{\nabla} V, \hat{\nabla}^2 V)\right\|_{C^{k-2,\gamma}_{1-\frac{\lambda}{2}}(M\times(0,T])} \le K.$$

Since $\|\hat{\nabla}_i U\|_{\lambda;M} \le K$, we apply Theorem 2.1 with $\alpha = \lambda$ to the system (3.67) to obtain

$$\|\hat{\nabla} V\|_{\lambda,\frac{\lambda}{2};M\times[0,T]} + \|\hat{\nabla}^2 V\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(M\times(0,T])} \le K.$$

From this, the assertion follows.

□

Now, associated with the differential $d\phi_t$ of the harmonic map flow ϕ_t, we define the functions

$$\Phi^{ij}_{\mu} : M \times [0,T] \to \mathbb{R}$$

as in Definition 3.6. Then Theorem 3.12 and Lemma 3.9 imply

$$\|\Phi^{ij}_{\mu}\|_{\lambda,\frac{\lambda}{2};M\times[0,T]} + \|\hat{\nabla}\Phi^{ij}_{\mu}\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(M\times(0,T])} \le K(\hat{g}, M, \|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}}, \|\psi\|_{C^{1,\beta}(M)}) \tag{3.71}$$

for $\lambda \in (\gamma, \beta)$.

Next, we show that a solution to the Ricci flow gives rise to a solution to the Ricci-DeTurck flow via the harmonic map heat flow.

Proposition 3.13. *Let $\lambda \in (\gamma, \beta)$ be given. Let $\{\phi_t\}_{t\in[0,T]}$ be a one-parameter family of diffeomorphisms which evolves under the harmonic map heat flow (3.42). For each $t \in [0,T]$, we define a metric $\tilde{g}(t)$ by $\tilde{g}(t) := (\phi_t^{-1})^*(g(t))$. Then, $\tilde{g}(t) \in \mathcal{X}_{k,\gamma}^{(\lambda)}(M \times [0,T])$ solves the Ricci-DeTurck system (3.3). Moreover, $\tilde{g}$ satisfies the estimate*

$$\|\tilde{g}\|_{\mathcal{X}_{k,\gamma}^{(\lambda)}(M\times[0,T])} \leq K(\hat{g}, M, \|g\|_{\mathcal{X}_{k,\gamma}^{(\beta)}}, \|\psi\|_{C^{1,\beta}(M)}). \tag{3.72}$$

Proof. Since

$$\Delta_{g(t),\hat{g}}\phi_t(p) = \Delta_{(\phi_t)^*\tilde{g}(t),\ \hat{g}}\phi_t(p) = \Delta_{\tilde{g}(t),\ \hat{g}}\mathrm{id}(\phi_t(p)) = -W(\phi_t(p))$$

for all $(p,t) \in M \times (0,T]$, where $W^k = \tilde{g}^{ij}(\Gamma_{ij}^k(\tilde{g}) - \Gamma_{ij}^k(\hat{g}))$, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\phi}_t)^*\tilde{g}(t)\Big|_p &= (\phi_t)^*\Big(\frac{\partial}{\partial t}\tilde{g}(t)\Big|_p + \frac{d}{ds}\Big|_{s=0}(\varphi_s)^*\tilde{g}(t)\Big|_p\Big) \\ &= (\phi_t)^*\Big(\frac{\partial}{\partial t}\tilde{g}(t) - \mathcal{L}_W\tilde{g}(t)\Big)\Big|_p.\end{aligned}$$

But since $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$, by the diffeomorphism invariance of Ricci curvature we obtain

$$\frac{\partial}{\partial t}\tilde{g}(t) - \mathcal{L}_W\tilde{g}(t) = -2Ric(\tilde{g}(t)).$$

Hence $\tilde{g}$ satisfies the Ricci-DeTurck system with initial condition $\tilde{g}(0) = g_0$. Lastly, since Theorem 3.12 and Lemma 3.9 imply

$$\|\Phi_\mu^{ij}\|_{\lambda,\frac{\lambda}{2};M\times[0,T]} + \|\hat{\nabla}\Phi_\mu^{ij}\|_{C^{k-1,\gamma}_{\frac{1}{2}-\frac{\lambda}{2}}(M\times(0,T])} \leq K(\hat{g}, M, \|g\|_{\mathcal{X}_{k,\gamma}^{(\beta)}}, \|\psi\|_{C^{1,\beta}(M)}),$$

we have $\tilde{g} \in C^{\lambda,\frac{\lambda}{2}}$, and we can then proceed as in Proposition 3.8 to obtain that $\tilde{g} \in \mathcal{X}_{k,\gamma}^{(\lambda)}(M\times[0,T])$ and the desired estimates. □

Chapter 4: Positivity of Curvature on Manifolds with Boundary

This Chapter is devoted to study the preservation of multiple positive curvature conditions in Definition 1.4 while deforming boundary data. To that end, the main result in this Chapter is Corollary 4.8, which plays a crucial role in the proof of Main Theorem 5 to Main Theorem 7. The work in this Chapter is taken from [6].

Throughout this Chapter, let M be a smooth manifold of dimension n with boundary ∂M, let g be a Riemannian metric on M, and let D denote the Levi-Civita connection on M compatible with g. The Riemann curvature tensor of (M, g) is defined by

$$-R(X, Y, Z, W) = g(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W)$$

for all vector fields X, Y, Z, W. Let ν be the inward pointing unit normal field on the boundary ∂M. The second fundamental form of ∂M with respect to the metric g is defined by

$$A_g(X, Y) = g(\nu, D_X Y)$$

for all $X, Y \in T(\partial M)$. The argument in this Chapter is based on choosing a suitable perturbation to the metric g.

4.1 Auxiliary results

Lemma 4.1. *Consider the Riemannian metrics of the form $\hat{g} = g + h$, where h satisfies the pointwise estimate $|h|_g \leq \frac{1}{2}$. Let $p \in M$ be any point and $\{E_i\}$ be a (local) geodesic orthonormal frame with* *respect to g around p, then the Riemann curvature tensor of $\hat{g}$ satisfies*

$$\hat{R}_{ijkl} = R_{ijkl} + \frac{1}{2}\, g^{pq}\, R_{ijkp}\, h_{ql} - \frac{1}{2}\, g^{pq}\, R_{ijlp}\, h_{kq} + E_{ijkl} + F_{ijkl}$$

where

$$E_{ijkl} = \frac{1}{2}\Big[- (D^2_{i,k} h)_{jl} + (D^2_{i,l} h)_{jl} + (D^2_{j,k} h)_{il} - (D^2_{j,l} h)_{ik}\Big]$$

$$\begin{aligned} F_{ijkl} =& \frac{1}{4}\, \hat{g}^{pq}\, [(D_j h)_{kp} + (D_k h)_{jp} - (D_p h)_{jk}]\, [(D_i h)_{lq} + (D_l h)_{iq} - (D_q h)_{il}] \\ & - \frac{1}{4}\, \hat{g}^{pq}\, [(D_i h)_{kp} + (D_k h)_{ip} - (D_p h)_{ik}]\, [(D_j h)_{lq} + (D_l h)_{jq} - (D_q h)_{jl}] \end{aligned}$$

Here D is the Levi-Civita connection with respect to g.

Proof. Let g and $\hat{g}$ be two Riemannian metrics. Let D denote the Levi-Civita connection associated with g, and let $\hat{D}$ denote the Levi-Civita connection associated with $\hat{g}$. Then

$$\hat{D}_X Y = D_X Y + \Lambda(X, Y),$$

where the tensor Λ is given by

$$2\, \hat{g}(\Lambda(X, Y), Z) = (D_X \hat{g})(Y, Z) + (D_Y \hat{g})(X, Z) - (D_Z \hat{g})(X, Y).$$

This gives

$$\hat{D}_X \hat{D}_Y Z = D_X D_Y Z + \Lambda(X, D_Y Z) + D_X(\Lambda(Y, Z)) + \Lambda(X, \Lambda(Y, Z)).$$

Consequently,

$$
\begin{aligned}
\hat{g}(\hat{D}_X\hat{D}_Y Z, W) &= \hat{g}(D_X D_Y Z, W) + \hat{g}(\Lambda(X, D_Y Z), W) \\
&\quad + \hat{g}(D_X(\Lambda(Y, Z)), W) + \hat{g}(\Lambda(X, \Lambda(Y, Z)), W) \\
&= \hat{g}(D_X D_Y Z, W) + \hat{g}(\Lambda(X, D_Y Z), W) \\
&\quad + X(\hat{g}(\Lambda(Y, Z), W)) - (D_X\hat{g})(\Lambda(Y, Z), W) \\
&\quad - \hat{g}(\Lambda(Y, Z), D_X W) + \hat{g}(\Lambda(X, \Lambda(Y, Z)), W).
\end{aligned}
$$

Using the identity $(D_X\hat{g})(U, V) = \hat{g}(\Lambda(X, U), V) + \hat{g}(U, \Lambda(X, V))$, we obtain

$$
(D_X\hat{g})(\Lambda(Y, Z), W) = \hat{g}(\Lambda(X, \Lambda(Y, Z)), W) + \hat{g}(\Lambda(Y, Z), \Lambda(X, W)),
$$

hence

$$
\begin{aligned}
\hat{g}(\hat{D}_X\hat{D}_Y Z, W) &= \hat{g}(D_X D_Y Z, W) + \hat{g}(\Lambda(X, D_Y Z), W) \\
&\quad + X(\hat{g}(\Lambda(Y, Z), W)) - \hat{g}(\Lambda(Y, Z), \Lambda(X, W)) \\
&\quad - \hat{g}(\Lambda(Y, Z), D_X W).
\end{aligned}
$$

In the following, we fix a point p, and work in geodesic normal coordinates around the point p. Putting $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, $W = \partial_l$ gives

$$
\hat{g}(\hat{D}_{\partial_i}\hat{D}_{\partial_j}\partial_k, \partial_l) = \hat{g}(D_{\partial_i}D_{\partial_j}\partial_k, \partial_l) + \partial_i(\hat{g}(\Lambda(\partial_j, \partial_k), \partial_l)) - \hat{g}(\Lambda(\partial_j, \partial_k), \Lambda(\partial_i, \partial_l))
$$

at p. Next, we switch i and j and take the difference. Note that

$$
-\hat{g}(\hat{D}_{\partial_i}\hat{D}_{\partial_j}\partial_k, \partial_l) + \hat{g}(\hat{D}_{\partial_j}\hat{D}_{\partial_i}\partial_k, \partial_l) = \hat{R}_{ijkl}
$$

and

$$-\hat{g}(D_{\partial_i} D_{\partial_j} \partial_k, \partial_l) + \hat{g}(D_{\partial_j} D_{\partial_i} \partial_k, \partial_l) = g^{pq}\, R_{ijkp}\, \hat{g}_{ql}$$

at p. Moreover, using the definition of Λ, we obtain

$$\begin{aligned}
&- \partial_i(\hat{g}(\Lambda(\partial_j, \partial_k), \partial_l)) + \partial_j(\hat{g}(\Lambda(\partial_i, \partial_k), \partial_l)) \\
&= -\frac{1}{2}\,\big[(D^2_{i,j}\hat{g})_{kl} + (D^2_{i,k}\hat{g})_{jl} - (D^2_{i,l}\hat{g})_{jk}\big] + \frac{1}{2}\,\big[(D^2_{j,i}\hat{g})_{kl} + (D^2_{j,k}\hat{g})_{il} - (D^2_{j,l}\hat{g})_{ik}\big] \\
&= -\frac{1}{2}\, g^{pq}\, R_{ijkp}\, \hat{g}_{ql} - \frac{1}{2}\, g^{pq}\, R_{ijlp}\, \hat{g}_{kq} \\
&\quad + \frac{1}{2}\,\big[- (D^2_{i,k}\hat{g})_{jl} + (D^2_{i,l}\hat{g})_{jl} + (D^2_{j,k}\hat{g})_{il} - (D^2_{j,l}\hat{g})_{ik}\big]
\end{aligned}$$

at p. Finally,

$$\begin{aligned}
&\hat{g}(\Lambda(\partial_j, \partial_k), \Lambda(\partial_i, \partial_l)) - \hat{g}(\Lambda(\partial_i, \partial_k), \Lambda(\partial_j, \partial_l)) \\
&= \frac{1}{4}\, \hat{g}^{pq}\, [(D_j\hat{g})_{kp} + (D_k\hat{g})_{jp} - (D_p\hat{g})_{jk}]\, [(D_i\hat{g})_{lq} + (D_l\hat{g})_{iq} - (D_q\hat{g})_{il}] \\
&\quad - \frac{1}{4}\, \hat{g}^{pq}\, [(D_i\hat{g})_{kp} + (D_k\hat{g})_{ip} - (D_p\hat{g})_{ik}]\, [(D_j\hat{g})_{lq} + (D_l\hat{g})_{jq} - (D_q\hat{g})_{jl}]
\end{aligned}$$

at p. Putting these facts together, we obtain

$$\begin{aligned}
\hat{R}_{ijkl} &= \frac{1}{2}\, g^{pq}\, R_{ijkp}\, \hat{g}_{ql} - \frac{1}{2}\, g^{pq}\, R_{ijlp}\, \hat{g}_{kq} \\
&\quad + \frac{1}{2}\,\big[- (D^2_{i,k}\hat{g})_{jl} + (D^2_{i,l}\hat{g})_{jl} + (D^2_{j,k}\hat{g})_{il} - (D^2_{j,l}\hat{g})_{ik}\big] \\
&\quad + \frac{1}{4}\, \hat{g}^{pq}\, [(D_j\hat{g})_{kp} + (D_k\hat{g})_{jp} - (D_p\hat{g})_{jk}]\, [(D_i\hat{g})_{lq} + (D_l\hat{g})_{iq} - (D_q\hat{g})_{il}] \\
&\quad - \frac{1}{4}\, \hat{g}^{pq}\, [(D_i\hat{g})_{kp} + (D_k\hat{g})_{ip} - (D_p\hat{g})_{ik}]\, [(D_j\hat{g})_{lq} + (D_l\hat{g})_{jq} - (D_q\hat{g})_{jl}]
\end{aligned}$$

at p. Hence, if we put $h = \hat{g} - g$, then

$$\begin{aligned}
\hat{R}_{ijkl} = {} & R_{ijkl} + \frac{1}{2}\, g^{pq}\, R_{ijkp}\, h_{ql} - \frac{1}{2}\, g^{pq}\, R_{ijlp}\, h_{kq} \\
& + \frac{1}{2}\Big[-(D^2_{i,k}h)_{jl} + (D^2_{i,l}h)_{jl} + (D^2_{j,k}h)_{il} - (D^2_{j,l}h)_{ik} \Big] \\
& + \frac{1}{4}\, \hat{g}^{pq}\, [(D_j h)_{kp} + (D_k h)_{jp} - (D_p h)_{jk}]\, [(D_i h)_{lq} + (D_l h)_{iq} - (D_q h)_{il}] \\
& - \frac{1}{4}\, \hat{g}^{pq}\, [(D_i h)_{kp} + (D_k h)_{ip} - (D_p h)_{ik}]\, [(D_j h)_{lq} + (D_l h)_{jq} - (D_q h)_{jl}]
\end{aligned}$$

at p.

□

Lemma 4.2. *Let S and T be real symmetric bilinear forms on T_xM . If S and T are (weakly) positive, then $S \owedge T$ is (weakly) positive on $\bigwedge^2 T_xM^{\mathbb{C}}$.*

Proof. By assumption, we can write S, T as a sum of rank-one operators:

$$S = \sum_{a=1}^{n} v^a \otimes v^a, \quad T = \sum_{b=1}^{n} w^b \otimes w^b.$$

Then for any two-vectors $\varphi = \varphi^{ij} e_i \wedge e_j \in \bigwedge^2 T_xM^{\mathbb{C}}$, we have

$$\begin{aligned}
(S \owedge T)(\varphi, \bar{\varphi}) &= 4\varphi^{ij}\bar{\varphi}^{kl} S_{ik} T_{jl} \\
&= 4 \sum_{a,b} \varphi^{ij}\bar{\varphi}^{kl} v^a_i v^a_k w^b_j w^b_l \\
&= 4 \sum_{a,b} |\varphi^{ij} v^a_i w^b_j|^2 \\
&\geq 0.
\end{aligned}$$

□

Lemma 4.3 (c.f. [3], Lemma A.2)**.** *Let S and T be real symmetric bilinear forms on T_xM. If S and T are (weakly) two-positive with respect to the metric g, then $S \owedge T$ is (weakly)* PIC *with respect to the metric g.*

Proof. Let $\zeta, \eta \in T_xM^{\mathbb{C}}$ be linear independent vectors such that

$$g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0.$$

We shall show that $(S \owedge T)(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$. We can find vectors $z, w \in \text{span}\{\zeta, \eta\}$ such that $g(z, \bar{z}) = g(w, \bar{w}) = 1$, $g(z, \bar{w}) = 0$ and $S(z, \bar{w}) = 0$. The identities $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$ give $g(z, z) = g(z, w) = g(w, w) = 0$. Consequently, we can find an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_xM$ such that

$$z = e_1 + ie_2, \quad w = e_3 + ie_4.$$

Using the identities $S(\bar{z}, w) = S(z, \bar{w}) = 0$, we obtain

$$\begin{aligned}
(S \owedge T)(z, w, \bar{z}, \bar{w}) &= S(z, \bar{z})T(w, \bar{w}) + S(w, \bar{w})T(z, \bar{z}) \\
&= (S_{11} + S_{22})(T_{33} + T_{44}) + (S_{33} + S_{44})(T_{11} + T_{22}) \\
&\geq 0.
\end{aligned}$$

Since $\text{span}\{z, w\} = \text{span}\{\zeta, \eta\}$, we conclude that $(S \owedge T)(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$. □

4.2 Preserving curvature conditions

In this section, g and $\tilde{g}$ are Riemannian metrics on M such that $g - \tilde{g} = 0$ along ∂M. Now, we describe our choice of perturbation as in [4]. We fix a neighborhood U of ∂M and a smooth boundary defining function $\rho : M \to [0, \infty)$ by taking it to be the distance function from ∂M

with respect to the metric g. Then we have $|D\rho|_g = 1$. Since $g - \tilde{g} = 0$ along ∂M, we can find a symmetric (0,2)-tensor S such that $\tilde{g} = g + \rho S$ in a neighborhood of ∂M and $S = 0$ outside U. Then for all $X, Y \in T(\partial M)$, the second fundamental forms satisfy

$$\frac{1}{2}S(X,Y) = A_g(X,Y) - A_{\tilde{g}}(X,Y),$$
$$D^2\rho(X,Y) = -A_g(X,Y).$$

This implies that the identity

$$A_g(X,Y) - A_{\tilde{g}}(X,Y) = \frac{1}{2}S(X,Y) = -D^2\rho(X,Y) - A_{\tilde{g}}(X,Y) \tag{4.1}$$

holds on the boundary ∂M for all $X, Y \in T(\partial M)$.

Let $\chi : [0,\infty) \to [0,1]$ a smooth cut-off function with the following properties (c.f. [4], Lemma 17):

- $\chi(s) = s - \frac{1}{2}s^2$ for $s \in [0, \frac{1}{2}]$;
- $\chi(s)$ is constant for $s \geq 1$;
- $\chi''(s) < 0$ for $s \in [0,1)$.

Moreover, let $\beta : (-\infty, 0] \to [0,1]$ be a smooth cutoff function such that

- $\beta(s) = \frac{1}{2}$ for $s \in [-1, 0]$;
- $\beta(s) = 0$ for $s \in (-\infty, -2]$.

Now, if $\lambda > 0$ is sufficiently large, we can define a smooth metric $\hat{g}_\lambda$ on M by

$$\hat{g}_\lambda = \begin{cases} g + \lambda^{-1}\chi(\lambda\rho)S & \text{for} \quad \rho \geq e^{-\lambda^2} \\ \tilde{g} - \lambda\rho^2\beta(\lambda^{-2}\log\rho)S & \text{for} \quad \rho < e^{-\lambda^2}. \end{cases} \tag{4.2}$$

In the sequel, we will show that $\hat{g}_\lambda$ preserves various curvature conditions of g and $\tilde{g}$ for sufficiently large λ. Note that we have $\hat{g}_\lambda = \tilde{g}$ in the region $\{\rho \leq e^{-2\lambda^2}\}$ and $\hat{g}_\lambda = g$ outside U.

Now we give a lower bound to the curvature operator of $\hat{g}_\lambda$. We first consider the region $\{\rho \geq e^{-\lambda^2}\}$.

Proposition 4.4. *Suppose that (M, g) has a convex boundary and that $(M, \tilde{g})$ has a weakly convex boundary such that $A_g > A_{\tilde{g}} \geq 0$. Let ϵ be an arbitrary positive real number. If $\lambda > 0$ is sufficiently large, then*

$$Rm_{\hat{g}_\lambda}(x)(\varphi, \bar{\varphi}) - Rm_g(x)(\varphi, \bar{\varphi}) \geq -\epsilon|\varphi|_g^2$$

for any (2,0)-tensor $\varphi \in \bigwedge^2 T_xM^{\mathbb{C}}$ and any $x \in M$ in the region $\{\rho(x) \geq e^{-\lambda^2}\}$.

Proof. First we fixed a point $x \in M$ so that $\rho(x) \geq e^{-\lambda^2}$. Let $\{e_i\}$ be a geodesic normal frame around x. Without loss of generality let us assume that the two-vector $\varphi \in \bigwedge^2 T_xM^{\mathbb{C}}$ is normalized so that $|\varphi|_g = 1$. We can write $\varphi = \sum_{i,j} \varphi^{ij} e_i \wedge e_j$ so that φ^{ij} is anti-symmetric. Einstein summation will be adopted freely so that $\varphi^{ij} A_{jk}$ means that we are summing over $j = 1, .., n.$. In the region $\{\rho(x) \geq e^{-\lambda^2}\}$, we have $\hat{g}_\lambda = g + h_\lambda$, where

$$h_\lambda = \lambda^{-1}\chi(\lambda\rho)S.$$

The tensor h_λ satisfies

$$(D_{e_j} h_\lambda)(e_k, e_l) = \chi'(\lambda\rho) D_j\rho \cdot S_{kl} + \lambda^{-1}\chi(\lambda\rho) D_j S_{kl} \tag{4.3}$$

and

$$(D^2_{e_i,e_j} h_\lambda)(e_k, e_l) = \lambda\chi''(\lambda\rho)D_i\rho D_j\rho \cdot S_{kl} + \chi'(\lambda\rho)D_iD_j\rho \cdot S_{kl} + \chi'(\lambda\rho)D_j\rho \cdot D_iS_{kl} + \chi'(\lambda\rho)D_i\rho \cdot D_jS_{kl} + \lambda^{-1}\chi(\lambda\rho)D_iD_jS_{kl}. \tag{4.4}$$

Since φ is a (2,0)-tensor, φ induces a linear map $[\varphi] : T_xM^* \to T_xM$ via the action $[\varphi]w := \varphi^{ij}w(e_i)e_j$. Using the notation of Lemma 4.1, we compute

$$\begin{aligned}\varphi^{ij}\bar{\varphi}^{kl}E_{ijkl} &= -2\varphi^{ij}\bar{\varphi}^{kl}(D^2_{i,k}h)_{jl} \\ &= -2\lambda\chi''(\lambda\rho)S([\varphi]d\rho, \overline{[\varphi]d\rho}) - 2\varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)D_iD_k\rho S_{jl} \\ &\quad - 2\varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(D_k\rho D_iS_{jl} + D_i\rho D_kS_{jl}) + O(\lambda^{-1})\end{aligned}$$

and

$$\begin{aligned}&\varphi^{ij}\bar{\varphi}^{kl}F_{ijkl} \\ &= \frac{1}{2}\varphi^{ij}\bar{\varphi}^{kl}\hat{g}^{pq}(D_jh_{kp} + D_kh_{jp} - D_ph_{jk}) \\ &\qquad \cdot (D_ih_{lq} + D_lh_{iq} - D_qh_{il}) \\ &= \frac{1}{2}\varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)^2\hat{g}^{pq}(D_j\rho S_{kp} + D_k\rho S_{jp} - D_p\rho S_{jk}) \\ &\qquad \cdot (D_i\rho S_{lq} + D_l\rho S_{iq} - D_q\rho S_{il}) + O(\lambda^{-1}) \\ &= \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)^2\hat{g}^{pq}\Big[D_j\rho D_l\rho S_{kp}S_{iq} - \frac{1}{2}D_p\rho D_q\rho S_{ik}S_{jl} \\ &\qquad + (D_j\rho S_{lp} + D_l\rho S_{jp})D_q\rho S_{ik}\Big] + O(\lambda^{-1}).\end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&= \varphi^{ij}\bar{\varphi}^{kl}E_{ijkl} + \varphi^{ij}\bar{\varphi}^{kl}F_{ijkl} + O(\lambda^{-1}) \\
&= 2\lambda(-\chi''(\lambda\rho))S([\varphi]d\rho, \overline{[\varphi]d\rho}) \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(-2D_iD_k\rho\, S_{jl} - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) \\
&\quad - 2\varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(D_i\rho\, D_kS_{jl} + D_k\rho\, D_iS_{jl}) \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)^2\hat{g}^{pq}(D_j\rho S_{lp} + D_l\rho S_{jp})D_q\rho S_{ik} \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)^2\hat{g}^{pq}D_j\rho D_l\rho S_{kp}S_{iq} + O(\lambda^{-1}).
\end{aligned}$$

This gives

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2\lambda(-\chi''(\lambda\rho))S([\varphi]d\rho, \overline{[\varphi]d\rho}) \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(-2D_iD_k\rho\, S_{jl} - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) \\
&\quad - C\chi'(\lambda\rho)|[\varphi]d\rho| - C\lambda^{-1},
\end{aligned} \tag{4.5}$$

where C is independent of λ. By assumption, $A_g - A_{\tilde{g}} > 0$ along the ∂M. Therefore, the restriction of S to the tangent space of ∂M is positive definite. Hence, we can find a tensor $\tilde{S}$ such that $\tilde{S} - S$ is a large (but fixed) multiple of $d\rho \otimes d\rho$, and $\tilde{S}$ is positive definite at each point on ∂M. Let us fix a small number $a > 0$ such that $\tilde{S} - ag$ is positive definite in a small neighborhood of ∂M. This implies

$$\tilde{S}([\varphi]d\rho, \overline{[\varphi]d\rho}) \geq a\,|[\varphi]d\rho|^2$$

in a neighborhood of ∂M. On the other hand, since $[\varphi]$ is anti-symmetric, we have $([\varphi]d\rho)^i D_i\rho = \varphi^{ij}D_j\rho D_i\rho = 0$. In other words, the vector $[\varphi]d\rho$ is annihilated by the one-form $d\rho$. Since $\tilde{S} - S$ is a multiple of $d\rho \otimes d\rho$, it follows that

$$S([\varphi]d\rho, \overline{[\varphi]d\rho}) = \tilde{S}([\varphi]d\rho, \overline{[\varphi]d\rho}) \geq a\,|[\varphi]d\rho|^2 \tag{4.6}$$

in a neighborhood of ∂M. The above estimate with (4.5) give

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(-2D_iD_k\rho\, S_{jl} - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) \\
&\quad - C\chi'(\lambda\rho)|[\varphi]d\rho| - C\lambda^{-1}.
\end{aligned} \tag{4.7}$$

Since $\chi(0) = 0$, we can find a real number $s_0 \in [0, 1)$ such that

$$\chi'(s_0)\sup_U\left|\varphi^{ij}\bar{\varphi}^{kl}(2D_iD_k\rho\, S_{jl} + \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) + C|[\varphi]d\rho|\right| < \epsilon. \tag{4.8}$$

In the region $\{\rho \geq s_0\lambda^{-1}\}$, we have

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 \\
&\quad - \sup_{s\geq s_0}\chi'(s)\sup_U\left|\varphi^{ij}\bar{\varphi}^{kl}(2D_iD_k\rho\, S_{jl} + \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) + C|[\varphi]d\rho|\right| - C\lambda^{-1} \\
&= 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 \\
&\quad - \chi'(s_0)\sup_U\left|\varphi^{ij}\bar{\varphi}^{kl}(2D_iD_k\rho\, S_{jl} + \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}S_{ik}S_{jl}) + C|[\varphi]d\rho|\right| - C\lambda^{-1}.
\end{aligned}$$

Thus it follows from (4.8) that

$$\inf_{\rho\geq s_0\lambda^{-1}}(Rm_{\hat{g}_\lambda}(\varphi, \bar{\varphi}) - Rm_g(\varphi, \bar{\varphi})) \geq -\epsilon$$

if $\lambda > 0$ is sufficiently large.

We next consider the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$. We can find a neighborhood V of ∂M and a diffeomorphism Φ such that $\Phi : V \cong \partial M \times [0, \delta)$. Let T be a smooth (0,2)-tensor in a neighborhood of ∂M defined by $T = \Phi^*(A_{\tilde{g}} + d\rho \otimes d\rho)$. Thus T is weakly positive definite in a neighborhood of ∂M. We observe by (4.1) that the restriction of $S_{ik} + 2D_iD_k\rho + T_{ik}$ to the tangent space to ∂M vanishes. Therefore, we may write $S_{ik} + 2D_iD_k\rho + T_{ik} = \omega_i D_k\rho + \omega_k D_i\rho$ at each point on ∂M, where ω is a suitable 1-form. Hence, in a small neighborhood of the boundary, we have $|S_{ik} + 2D_iD_k\rho + T_{ik} - \omega_i D_k\rho - \omega_k D_i\rho| \leq C\rho$. This implies

$$\begin{aligned}
|\varphi^{ij}\bar{\varphi}^{kl}(S_{ik} + 2D_iD_k\rho + T_{ik})S_{jl}| &\leq C|[\varphi]d\rho| + C\rho \\
&\leq C|[\varphi]d\rho| + C\lambda^{-1}
\end{aligned}$$

in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$. Putting these facts with (4.7) together, we obtain

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(1 - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda})S_{ik}S_{jl} \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)T_{ik}S_{jl} - C\chi'(\lambda\rho)|[\varphi]d\rho| - C\lambda^{-1}
\end{aligned}$$

in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$.

Next, we may assume without loss of generality that the frame $\{e_i\}$ diagonalizes the (0,2)-tensor $\tilde{S}$. Since $\tilde{S} - ag$ is positive definite and T is weakly positive definite, we have

$$\varphi^{ij}\bar{\varphi}^{kl}T_{ik}\tilde{S}_{jl} \geq 0$$

in a neighborhood of ∂M by Lemma 4.2. Since $\tilde{S} - S$ is a multiple of $d\rho \otimes d\rho$, it follows that

$$\varphi^{ij}\bar{\varphi}^{kl}T_{ik}S_{jl} \geq \varphi^{ij}\bar{\varphi}^{kl}T_{ik}\tilde{S}_{jl} - C\,|[\varphi]d\rho| \geq -C\,|[\varphi]d\rho|$$

in a neighborhood of ∂M.

On the other hand, since $\tilde{S} - ag$ is positive definite, we know that

$$\varphi^{ij}\bar{\varphi}^{kl}\tilde{S}_{ik}\tilde{S}_{jl} \geq a^2|\varphi|^2 = a^2$$

in a neighborhood of ∂M. Since $\tilde{S} - S$ is a multiple of $d\rho \otimes d\rho$, it follows that

$$\varphi^{ij}\bar{\varphi}^{kl}S_{ik}S_{jl} \geq \varphi^{ij}\bar{\varphi}^{kl}\tilde{S}_{ik}\tilde{S}_{jl} - C\,|[\varphi]d\rho| \geq a^2 - C\,|[\varphi]d\rho|$$

in a neighborhood of ∂M. To summarize, we obtain

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \qquad (4.9)\\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 + a^2\chi'(\lambda\rho)(1 - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda})\\
&\quad - N\chi'(\lambda\rho)|[\varphi]d\rho| - N\lambda^{-1}\\
&= |[\varphi]d\rho|\Big[2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho| - N\chi'(\lambda\rho)\Big]\\
&\quad + a^2\chi'(\lambda\rho)(1 - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}) - N\lambda^{-1}
\end{aligned}$$

in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$. Here, N is a positive constant depending only on (M, g) but independent of λ. This implies that

$$Rm_{\hat{g}_\lambda}(\varphi, \bar{\varphi}) - Rm_g(\varphi, \bar{\varphi}) \geq -\frac{N^2\chi'(\lambda\rho)^2}{8a\lambda \cdot \inf_{0\leq s\leq s_0}(-\chi''(s))} + a^2\chi'(\lambda\rho)(1 - \frac{1}{2}|D\rho|^2_{\hat{g}_\lambda}) - N\lambda^{-1}.$$

Here we have used $x(Ax - B) \geq -\frac{B^2}{4A}$ for positive A and the fact that χ is between 0 and 1. Since $|D\rho|_{\hat{g}_\lambda} = |D\rho|_g = 1$ on the boundary ∂M, by continuity we have $(1 - \frac{1}{2}|D\rho|^2_{\hat{g}_\lambda}) > \frac{1}{4}$ in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$ if λ is sufficiently large. By the definition of χ, we also have $\inf_{0\leq s\leq s_0}(-\chi''(s)) > 0$. Thus, we conclude that

$$\inf_{e^{-\lambda^2}\leq\rho\leq s_0\lambda^{-1}}(Rm_{\hat{g}_\lambda}(\varphi,\bar{\varphi}) - Rm_g(\varphi,\bar{\varphi})) \geq \frac{1}{4}a^2\chi'(\lambda\rho)$$

if λ is sufficiently large. Combing these facts, we complete the proof. □

Corollary 4.5. *Suppose that (M, g) has a two-convex boundary and that $(M, \tilde{g})$ has a weakly two-convex boundary such that*

$$A_g(X, X) + A_g(Y, Y) > A_{\tilde{g}}(X, X) + A_{\tilde{g}}(Y, Y) \geq 0$$

for all orthonormal $X, Y \in T(\partial M)$. Let ϵ be an arbitrary positive real number. If $\lambda > 0$ is sufficiently large, then

$$Rm_{\hat{g}_\lambda}(x)(\varphi,\bar{\varphi}) - Rm_g(x)(\varphi,\bar{\varphi}) \geq -\epsilon|\varphi|_g^2$$

for any isotropic 2-vector $\varphi = z \wedge w \in \bigwedge^2 T_xM^{\mathbb{C}}$ satisfying $\hat{g}_\lambda(z, z) = \hat{g}_\lambda(w, w) = \hat{g}_\lambda(z, w) = 0$ and any $x \in M$ in the region $\{\rho(x) \geq e^{-\lambda^2}\}$.

Proof. As in Proposition 4.4, we recall (4.5):

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl}\\
&\geq 2\lambda(-\chi''(\lambda\rho))S([\varphi]d\rho, \overline{[\varphi]d\rho})\\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(-2D_iD_k\rho\, S_{jl} - \frac{1}{2}\chi'(\lambda\rho)|D\rho|_{\hat{g}_\lambda}^2 S_{ik}S_{jl})\\
&\quad - C\chi'(\lambda\rho)|[\varphi]d\rho| - C\lambda^{-1}.
\end{aligned}$$

By assumption, the difference $A_g - A_{\tilde{g}}$ is strictly 2-positive on ∂M. Therefore, the restriction of S to the tangent space of ∂M is strictly 2-positive (with respect to the metric g). Hence, we can find a tensor $\tilde{S}$ such that $\tilde{S} - S$ is a large (but fixed) multiple of $d\rho \otimes d\rho$, and $\tilde{S}$ is strictly 2-positive (with respect to g) at each point on ∂M. Let us fix a small number $a > 0$ such that $\tilde{S} - 2ag$ is 2-positive (with respect to g) in a small neighborhood of ∂M. Then $\tilde{S} - ag$ is 2-positive with respect to $\hat{g}_\lambda$, if λ is sufficiently large. Since φ is an isotropic 2-vector with respect to $\hat{g}_\lambda$, we know that $[\varphi]d\rho$ is an isotropic vector with respect to $\hat{g}_\lambda$. Since $\tilde{S} - ag$ is 2-positive with respect to $\hat{g}_\lambda$, we obtain

$$\tilde{S}([\varphi]d\rho, \overline{[\varphi]d\rho}) \geq a\,|[\varphi]d\rho|^2$$

in a neighborhood of ∂M. On the other hand, since $[\varphi]$ is anti-symmetric, we have $([\varphi]d\rho)^i D_i\rho = \varphi^{ij} D_i\rho D_j\rho = 0$. In other words, the vector $[\varphi]d\rho$ is annihilated by the 1-form $d\rho$. Since $\tilde{S} - S$ is a multiple of $d\rho \otimes d\rho$, it follows that

$$S([\varphi]d\rho, \overline{[\varphi]d\rho}) = \tilde{S}([\varphi]d\rho, \overline{[\varphi]d\rho}) \geq a\,|[\varphi]d\rho|^2$$

in a neighborhood of ∂M. Then we proceed as in Proposition 4.4, we can find a real number $s_0 \in (0, 1)$ so that the estimate

$$\inf_{\rho \geq s_0\lambda^{-1}} (Rm_{\hat{g}_\lambda}(\varphi, \bar{\varphi}) - Rm_g(\varphi, \bar{\varphi})) \geq -\epsilon$$

holds if $\lambda > 0$ is sufficiently large.

Next, proceed as in Proposition 4.4, we have

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)(1 - \frac{1}{2}\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda})S_{ik}S_{jl} \\
&\quad + \varphi^{ij}\bar{\varphi}^{kl}\chi'(\lambda\rho)T_{ik}S_{jl} - C\chi'(\lambda\rho)|[\varphi]d\rho| - C\lambda^{-1}
\end{aligned}$$

in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$, where the (0,2)-tensor T is defined in the same way as in Proposition 4.4. In this case, T is weakly 2-positive as $A_{\tilde{g}}$ does. Since $\tilde{S}$ is 2-positive with respect to the metric g, by Lemma 4.3 we know that the Kulkarni-Nomizu products $\tilde{S} \owedge \tilde{S}$ and $T \owedge \tilde{S}$ are strictly PIC and weakly PIC respectively with respect to the metric g. Let us fix a small number $b > 0$ so that $\tilde{S} \owedge \tilde{S} - 5b\, g \owedge g$ is PIC with respect to the metric g. Hence, if λ is sufficiently large, then $\tilde{S} \owedge \tilde{S} - 4b\, g \owedge g$ and $T \owedge \tilde{S} + b\, g \owedge g$ are PIC with respect to the metric $\hat{g}_\lambda$. Since φ is an isotropic 2-vector with respect to $\hat{g}_\lambda$, it follows that

$$\varphi^{ij}\bar{\varphi}^{kl}\tilde{S}_{ik}\tilde{S}_{jl} \geq 4b|\varphi|^2 = 4b$$

and

$$\varphi^{ij}\bar{\varphi}^{kl}T_{ik}\tilde{S}_{jl} \geq -b|\varphi|^2 = -b.$$

Since $\tilde{S} - S$ is a multiple of $d\rho \otimes d\rho$, we obtain

$$\varphi^{ij}\bar{\varphi}^{kl}S_{ik}S_{jl} \geq \varphi^{ij}\bar{\varphi}^{kl}\tilde{S}_{ik}\tilde{S}_{jl} - C\,|[\varphi]d\rho| \geq 4b - C\,|[\varphi]d\rho|$$

and

$$\varphi^{ij}\bar{\varphi}^{kl}T_{ik}S_{jl} \geq \varphi^{ij}\bar{\varphi}^{kl}T_{ik}\tilde{S}_{jl} - C\,|[\varphi]d\rho| \geq -b - C\,|[\varphi]d\rho|$$

in a neighborhood of ∂M. To summarize, we find

$$\begin{aligned}
&\varphi^{ij}\bar{\varphi}^{kl}(R_{\hat{g}_\lambda})_{ijkl} - \varphi^{ij}\bar{\varphi}^{kl}R_{ijkl} \\
&\geq 2a\lambda(-\chi''(\lambda\rho))\,|[\varphi]d\rho|^2 + b\chi'(\lambda\rho)(3 - 2\chi'(\lambda\rho)|D\rho|^2_{\hat{g}_\lambda}) \\
&\quad - N\chi'(\lambda\rho)|[\varphi]d\rho| - N\lambda^{-1}
\end{aligned}$$

in the region $\{e^{-\lambda^2} \leq \rho \leq s_0\lambda^{-1}\}$. The assertion then follows as in Proposition 4.4.

□

Next, we estimate the curvature operator of $\hat{g}_\lambda$ in the region $\{\rho < e^{-\lambda^2}\}$.

Proposition 4.6. *Suppose that* (M, g) *has a convex boundary and that* $(M, \tilde{g})$ *has a weakly convex boundary such that* $A_g > A_{\tilde{g}} \geq 0$*. Let* ϵ *be an arbitrary positive real number. If* $\lambda > 0$ *is sufficientlt large, then*

$$Rm_{\hat{g}_\lambda}(x)(\varphi, \bar{\varphi}) - Rm_{\tilde{g}}(x)(\varphi, \bar{\varphi}) \geq -\epsilon|\varphi|^2_{\tilde{g}}$$

for any (2,0)-tensor $\varphi \in \bigwedge^2 T_xM^{\mathbb{C}}$ *and any* $x \in M$ *in the region* $\{\rho(x) < e^{-\lambda^2}\}$.

Proof. In the region $\{\rho < e^{-\lambda^2}\}$, we have $\hat{g}_\lambda = \tilde{g} + \tilde{h}_\lambda$, where $\tilde{h}_\lambda$ is defined by

$$\tilde{h}_\lambda = -\lambda\rho^2\beta(\lambda^{-2}\log\rho)S.$$

Let $\{e_i\}$ be a geodesic normal frame around x. And without loss of generality let $\varphi \in \bigwedge^2 T_xM^{\mathbb{C}}$ be a normalized (2,0)-tensor, we can write $\varphi = \sum_{i<j} \varphi^{ij} e_i \wedge e_j$. Then we have

$$\begin{aligned}(\tilde{D}_{e_j}\tilde{h}_\lambda)(e_k, e_l) &= -\left[2\lambda\rho\beta(\lambda^{-2}\log\rho) + \lambda^{-1}\rho\beta'(\lambda^{-2}\log\rho)\right]D_j\rho\cdot S_{kl} \\ &\quad - \lambda\rho^2\beta(\lambda^{-2}\log\rho)D_jS_{kl} \\ &= O(\lambda^{-1})\end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
(\tilde{D}^2_{e_i,e_j}\tilde{h}_\lambda)(e_k,e_l) = &-\big[2\lambda\beta(\lambda^{-2}\log\rho) + 3\lambda^{-1}\beta'(\lambda^{-2}\log\rho) + \lambda^{-3}\beta''(\lambda^{-2}\log\rho)\big]D_i\rho D_j\rho\cdot S_{kl} \qquad (4.11)\\
&-\big[2\lambda\rho\beta(\lambda^{-2}\log\rho) + \lambda^{-1}\rho\beta'(\lambda^{-2}\log\rho)\big]D_iD_j\rho\cdot S_{kl}\\
&-\big[2\lambda\rho\beta(\lambda^{-2}\log\rho) + 2\lambda^{-1}\rho\beta'(\lambda^{-2}\log\rho)\big]D_j\rho\cdot D_iS_{kl}\\
&-\big[2\lambda\rho\beta(\lambda^{-2}\log\rho) + 2\lambda^{-1}\rho\beta'(\lambda^{-2}\log\rho)\big]D_i\rho\cdot D_jS_{kl}\\
&-\lambda\rho^2\beta(\lambda^{-2}\log\rho)D_iD_jS_{kl}\\
= &-2\lambda\beta(\lambda^{-2}\log\rho)D_i\rho D_j\rho\cdot S_{kl} + O(\lambda^{-1}).
\end{aligned}
$$

Using Lemma 4.1, we obtain

$$
Rm_{\hat{g}_\lambda}(\varphi,\bar{\varphi}) - Rm_g(\varphi,\bar{\varphi}) \geq 2\lambda\beta(\lambda^{-2}\log\rho)S([\varphi]d\rho,\overline{[\varphi]d\rho}) - L\lambda^{-1}. \qquad (4.12)
$$

Here, L is a positive constant independent of λ.

Recall that $S(X,X) > 0$ for all $X \in T(\partial M)^{\mathbb{C}}$. By continuity, we have

$$
S([\varphi]d\rho,\overline{[\varphi]d\rho}) \geq 0
$$

in a neighborhood of ∂M. Hence, if $\lambda > 0$ is sufficiently large, then we have

$$
\inf_{\rho<e^{-\lambda^2}}(Rm_{\hat{g}_\lambda}(\varphi,\bar{\varphi}) - Rm_{\tilde{g}}(\varphi,\bar{\varphi})) \geq -\epsilon.
$$

From this, the assertion follows.

□

Following the same line of the above Proposition and Corollary 4.5, we also have

Corollary 4.7. *Suppose that (M,g) has a two-convex boundary and that $(M,\tilde{g})$ has a weakly two-convex boundary such that*

$$A_g(X,X) + A_g(Y,Y) > A_{\tilde{g}}(X,X) + A_{\tilde{g}}(Y,Y) \geq 0$$

for all orthonormal $X, Y \in T(\partial M)$. Let ϵ be an arbitrary positive real number. If $\lambda > 0$ is sufficientlt large, then

$$Rm_{\hat{g}_\lambda}(x)(\varphi,\bar{\varphi}) - Rm_{\tilde{g}}(x)(\varphi,\bar{\varphi}) \geq -\epsilon|\varphi|^2_{\tilde{g}}$$

for any isotropic 2-vector $\varphi = z \wedge w \in \bigwedge^2 T_x M^{\mathbb{C}}$ satisfying $\hat{g}_\lambda(z,z) = \hat{g}_\lambda(w,w) = \hat{g}_\lambda(z,w) = 0$ and any $x \in M$ in the region $\{\rho(x) < e^{-\lambda^2}\}$.

We now sum up the results in this section:

Corollary 4.8. *Let $\epsilon > 0$ be an arbitrary positive real number.*

(i) Suppose that (M,g) has a convex boundary and that $(M,\tilde{g})$ has a weakly convex boundary such that $A_g > A_{\tilde{g}} \geq 0$. If $\lambda > 0$ is sufficiently large, then we have the point-wise inequality

$$Rm_{\hat{g}_\lambda}(x)(\varphi,\bar{\varphi}) \geq \min\{Rm_g(x)(\varphi,\bar{\varphi}), Rm_{\tilde{g}}(x)(\varphi,\bar{\varphi})\} - \epsilon \max\{|\varphi|^2_g, |\varphi|^2_{\tilde{g}}\}$$

for all $\varphi \in \bigwedge^2 T_x M^{\mathbb{C}}$ and each point $x \in M$.

(ii) Suppose that (M,g) has a two-convex boundary and that $(M,\tilde{g})$ has a weakly two-convex boundary such that

$$A_g(X,X) + A_g(Y,Y) > A_{\tilde{g}}(X,X) + A_{\tilde{g}}(Y,Y) \geq 0$$

for all orthonormal $X, Y \in T(\partial M)$. If $\lambda > 0$ is sufficiently large, then we have the point-wise *inequality*

$$Rm_{\hat{g}_\lambda}(x)(\varphi, \bar{\varphi}) \geq \min\{Rm_g(x)(\varphi, \bar{\varphi}), Rm_{\tilde{g}}(x)(\varphi, \bar{\varphi})\} - \epsilon \max\{|\varphi|_g^2, |\varphi|_{\tilde{g}}^2\}$$

for all isotropic 2-vector $\varphi = z \wedge w \in \bigwedge^2 T_x M^{\mathbb{C}}$ *satisfying* $\hat{g}_\lambda(z, z) = \hat{g}_\lambda(w, w) = \hat{g}_\lambda(z, w) = 0$ *and each point* $x \in M$.

(iii) Suppose that (M, g) *has a mean-convex boundary and that* $(M, \tilde{g})$ *has a weakly mean-convex boundary such that* $H_g \geq H_{\tilde{g}} > 0$*. If* $\lambda > 0$ *is sufficiently large, then we have the point-wise inequality*

$$R_{\hat{g}_\lambda}(x) \geq \min\{R_g(x), R_{\tilde{g}}(x)\} - \epsilon$$

for each point $x \in M$.

Assertion (iii) in above corollary was given by Theorem 5 in [4].

Chapter 5: Proof of Main Results

5.1 Proof of Main Theorem 1 to Main Theorem 4

We first prove the existence of a canonical solution to Ricci flow on the doubled manifold with C^α initial metric. The following result implies Main Theorem 1 and Main Theorem 2.

Theorem 5.1. *Let $\tilde{M}$ be a closed compact smooth manifold and $\tilde{g}_0 \in C^\alpha(\tilde{M})$ be a Riemannian metric for some $\alpha \in (0,1)$. Let $k \geq 2$, $\gamma \in (0,\alpha)$ and $\beta \in (\gamma,\alpha)$ be given. Then there exists a $C^{1,\beta}$ diffeomorphism ψ and $T = T(\tilde{M}, \hat{g}, \|\tilde{g}_0\|_\alpha)$, $K = K(\tilde{M}, k, \hat{g}, \|\tilde{g}_0\|_\alpha)$ such that the following holds:*

There is a solution $g(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(\tilde{M} \times [0,T])$ to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) \quad on \quad \tilde{M} \times (0,T]$$

such that $g(0) = \psi^* \tilde{g}_0$ and

$$\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}(\tilde{M}\times[0,T])} \leq K.$$

Moreover, the solution is canonical in the sense that if $\bar{g}(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(\tilde{M} \times [0,T])$ is another solution to the Ricci flow and $\bar{\psi} \in C^{1,\beta}(\tilde{M})$ is a diffeomorphism such that $\bar{g}(0) = \bar{\psi}^* \tilde{g}_0$, then there is a C^{k+1} diffeomorphism ϕ such that $\bar{g}(t) = \phi^* g(t)$ which satisfies $\bar{\psi} = \psi \circ \phi$.

Proof. By Proposition 3.8, we can find $T = T(\tilde{M}, \hat{g}, \|\tilde{g}_0\|_\alpha) > 0$ sufficiently small and $K = K(\tilde{M}, k, \hat{g}, \|\tilde{g}_0\|_\alpha)$ and a $C^{1,\beta}$ diffeomorphism ψ such that there is a solution $g(t)$ to the Ricci flow satisfying the estimate

$$\|g\|_{\mathcal{X}^{(\beta)}_{k,\gamma}} \leq K.$$

and $g(0) = \psi^* \tilde{g}_0$.

Now, suppose that $\bar{g}(t) \in \mathcal{X}^{(\beta)}_{k,\gamma}(\tilde{M} \times [0,T])$ is another solution to the Ricci flow and $\bar{\psi} \in C^{1+\beta}(\tilde{M})$ is a diffeomorphism such that $\bar{g}(0) = \bar{\psi}^* \tilde{g}_0$. Let ϕ_t denote the solution to the harmonic map heat flow starting from $\phi_0 = \psi$ with respect to $g(t)$ and $\hat{g}$. Let $\bar{\phi}_t$ denote the solution to the harmonic map heat flow starting from $\bar{\phi}_0 = \psi$ with respect to $\bar{g}(t)$ and $\hat{g}$. By Proposition 3.11 these solutions exist and are unique. Subsequently Theorem 3.12 and Proposition 3.13 imply that

$$h(t) := (\phi_t^{-1})^* g(t) \quad \text{and} \quad \bar{h}(t) := (\bar{\phi}_t^{-1})^* \bar{g}(t)$$

are both solutions to the Ricci-DeTurck flow (3.3) such that $h, \bar{h} \in \mathcal{X}^{(\lambda)}_{k,\gamma}(\tilde{M} \times [0,T])$ for some $\lambda \in (\gamma, \beta)$ and $h(0) = \bar{h}(0) = \tilde{g}_0$. Since the solution to the Ricci-DeTurck flow in $\mathcal{X}^{(\lambda)}_{k,\gamma}$ are unique by Theorem 3.3, we have $h(t) = \bar{h}(t)$ on $M \times [0,T]$. Now observe that

$$\Delta_{g(t),\hat{g}} \phi_t \Big|_{\psi^{-1}(p)} = \Delta_{h(t),\hat{g}} \mathrm{id} \Big|_{\phi_t \circ \psi^{-1}(p)},$$

hence $\phi_t \circ \psi^{-1}$ is a solution to the ODE

$$\begin{cases} \frac{\partial}{\partial t} \phi_t \circ \psi^{-1}(p) = \Delta_{h(t),\hat{g}} \mathrm{id}\big|_{\phi_t \circ \psi^{-1}(p)} \\ \varphi_0 \circ \psi^{-1} = \mathrm{id}. \end{cases} \tag{5.1}$$

Analogously, $\bar{\varphi}_t \circ \bar{\psi}^{-1}$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} \bar{\varphi}_t \circ \bar{\psi}^{-1}(p) = \Delta_{\bar{h}(t),\hat{g}} \mathrm{id}\big|_{\bar{\varphi}_t \circ \bar{\psi}^{-1}(p)} \\ \bar{\varphi}_0 \circ \bar{\psi}^{-1} = \mathrm{id}. \end{cases} \tag{5.2}$$

Since we know that $h(t) = \bar{h}(t)$, it follows that $\phi_t \circ \psi^{-1}$ and $\bar{\phi}_t \circ \bar{\psi}^{-1}$ satisfy the same ODE with the same initial condition. Consequently $\phi_t \circ \psi^{-1} = \bar{\phi}_t \circ \bar{\psi}^{-1}$ on $M \times [0,T]$. In other words,

$\phi_t^{-1} \circ \bar{\phi}_t = \psi^{-1} \circ \bar{\psi}$ is constant in t. Let us take the desired diffeomorphism ϕ to be $\phi := \phi_t^{-1} \circ \bar{\phi}_T$. Then from Theorem 3.12 we know that ϕ is a C^{k+1} diffeomorphism satisfying $\bar{\psi} = \psi \circ \phi$ and

$$\bar{g}(t) = (\phi_t^{-1} \circ \bar{\phi}_t)^* g(t) = \phi^* g(t).$$

□

The previous theorem implies Main Theorem 3 and Main Theorem 4 via doubling:

Theorem 5.2 (Main Theorem 3)**.** *Let (M, g_0) be a compact smooth Riemannian manifold with boundary. Let $k \geq 2$, $\beta \in (0,1)$, $\gamma \in (0, \beta)$ and $\epsilon \in (0, 1-\beta)$ be given. Then there exists a $C^{1,\beta}$ diffeomorphism ψ and $T = T(M, \hat{g}, \|g_0\|_{\beta+\varepsilon})$, $K = K(M, k, \hat{g}, \|g_0\|_{\beta+\varepsilon})$ such that the following holds:*

There is a solution $g(t) \in \mathcal{X}_{k,\gamma}^{(\beta)}(M \times [0,T])$ to the Ricci flow on manifold with boundary

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) & on \quad M \times (0, T] \\ A_{g(t)} = 0 & on \quad \partial M \times (0, T] \end{cases} \tag{5.3}$$

such that $g(0) = \psi^* g_0$ and

$$\|g\|_{\mathcal{X}_{k,\gamma}^{(\beta)}(M \times [0,T])} \leq K.$$

For each $t > 0$, the metric $g(t)$ extends smoothly to the doubled manifold $\tilde{M}$ of M, and the doubled metric lies in $\mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0,T])$. The diffeomorphism ψ also extends to a $C^{1,\beta}$ diffeomorphism on the doubled manifold.

Proof. Let $\tilde{M}$ be the doubling of M, and $\tilde{g}_0$ be the extension of g_0 in $\tilde{M}$ via reflection about the boundary ∂M. Since $\tilde{g}_0$ is Lipschitz, we can find $\beta + \varepsilon \in (\beta, 1)$ such that $\tilde{g}_0 \in C^{\beta+\varepsilon}(M)$. Fix a smooth background metric $\hat{g}$ on M such that in a small collar neighborhood of ∂M the metric $\hat{g}$ is isometric to a product $\partial M \times [0, \varepsilon]$.

By Proposition 3.8, we can find $T = T(\tilde{M}, \hat{g}, \|\tilde{g}_0\|_{\beta+\varepsilon}) > 0$ sufficiently small and $K = K(\tilde{M}, k, \hat{g}, \|\tilde{g}_0\|_{\beta+\varepsilon})$ and a $C^{1,\beta}$ diffeomorphism $\tilde{\psi}$ such that there is a solution $\tilde{g}(t)$ to the Ricci flow on $\tilde{M} \times (0, T]$ satisfying the estimate

$$\|\tilde{g}\|_{\mathcal{X}^{(\beta)}_{k,\gamma}} \leq K.$$

and $\tilde{g}(0) = \tilde{\psi}^* \tilde{g}_0$.

Note that by uniqueness and diffeomorphism invariance the solution to the Ricci-DeTurck flow on the doubled manifold $\tilde{M}$ with initial metric $\tilde{g}_0$ is invariant under the natural natural $\mathbb{Z}_2$ action given by the reflection about ∂M which switches the two halves of $\tilde{M}$. Thus the diffeomorphism $\tilde{\psi}$ we obtained by solving the ODE (3.19) is equivariant under this $\mathbb{Z}_2$ action. This implies that the solution $\tilde{g}(t)$ to the Ricci flow is also invariant under the $\mathbb{Z}_2$ action. In particular, it follows from the $\mathbb{Z}_2$ symmetry that $g = \tilde{g}|_M \in \mathcal{X}^{(\beta)}_{k,\gamma}(M \times [0, T])$ has totally geodesic boundary on M, and thus a solution to (5.3). Moreover, $\psi = \tilde{\psi}|_M : M \to M$ is a $C^{1,\beta}$ diffeomorphism such that $g(0) = \psi^* g_0$. □

Theorem 5.3 (Main Theorem 4)**.** *Let (M, g_0) be a compact smooth Riemannian manifold with boundary. Suppose that the pairs $(g^1(t), \psi^1)$ and $(g^2(t), \psi^2)$ satisfy the conclusion of Theorem 5.2. Then there exists a C^{k+1} diffeomorphism $\varphi : M \to M$ such that φ extends to a C^{k+1} diffeomorphism on the doubled manifold and*

$$g^2(t) = \varphi^*(g^1(t)),$$

where $\psi^2 = \psi^1 \circ \varphi$. In particular, $((\psi^1)^{-1})^* g^1(t) = ((\psi^2)^{-1})^* g^2(t)$.

Proof. By assumptions, the pairs $(g^1(t), \psi^1)$ and $(g^2(t), \psi^2)$ can be extended to the doubled manifold so that Theorem 5.1 can be applied. Since the whole construction of the harmonic map heat flow is invariant under the natural $\mathbb{Z}_2$ action given by reflection, the assertion thus follows from restricting the conclusion of Theorem 5.1 to M. □

5.2 Proof of Main Theorem 5

Assertion (i) in Main Theorem 5 follows from the definition of $\hat{g}_\lambda$ in (4.2). Assertion (iv) follows from Corollary 4.8 for sufficiently large $\lambda > 0$.

Proof of (ii):

Suppose that (M, g) has a convex boundary and that $(M, \tilde{g})$ has a weakly convex boundary such that $A_g > A_{\tilde{g}} \geq 0$. Suppose also that (M, g) and $(M, \tilde{g})$ are PIC1. Let $\varphi = z \wedge w \in \bigwedge^2 T_x M^{\mathbb{C}}$ satisfies the condition $\hat{g}_\lambda(z,z)\hat{g}_\lambda(w,w) - \hat{g}_\lambda(z,w)^2 = 0$. This condition is equivalent to the condition $\sum_{i,k} \varphi^{ik}\varphi^{ki} = 0$ with respect to the metric $\hat{g}_\lambda$. We wish to prove that $Rm_{\hat{g}_\lambda}(x)(\varphi, \bar{\varphi}) > 0$. We divide the proof into two cases: (1) x is in the region $\{\rho \geq e^{-\lambda^2}\}$; (2) x is in the region $\{\rho < e^{-\lambda^2}\}$.

For the first case, we have $\hat{g}_\lambda = g + h_\lambda$ where $h_\lambda = \lambda^{-1}\chi(\lambda\rho)S$. We fix an orthonormal frame $\{e_i\}$ with respect to g such that this frame diagonalizes h_λ. i.e. $h_{ij} = \delta_{ij}\mu_j$, where μ_j are eigenvalues of h. We then evolve the frame $\{e_i\}$ in $T_x M$ by

$$\begin{cases} \frac{d}{ds}E_i(s) &= -\frac{1}{2}h_\lambda \circ E_i(s) \\ E_i(0) &= e_i. \end{cases} \tag{5.4}$$

Then we see that $\{E_i(s)\}$ remains orthonormal with respect to $g_s = g + sh_\lambda$. In particular, we have $\{E_i(1)\}$ being orthonormal with respect to $\hat{g}_\lambda = g + h_\lambda$.

Now, we set $\varphi_s = \sum_{ij} \varphi^{ij} E_i(s) \wedge E_j(s)$ so that $\varphi_1 = \varphi$ and we define $\psi = \varphi_0 = \sum_{ij} \varphi^{ij} E_i(0) \wedge E_j(0)$. Hence we see that the condition $\sum_{i,k} \varphi^{ik}\varphi^{ki} = 0$ implies the condition $\sum_{i,k} \psi^{ik}\psi^{ki} = 0$ with respect to the metric g. Since (M, g) is PIC1, it follows that $Rm_g(x)(\psi, \bar{\psi}) > 0$. In view of Corollary 4.8, it suffices to show that $Rm_g(x)(\varphi, \bar{\varphi}) > 0$.

By writing $E_i(s) = A_i^k(s)e_k$, we observe that

$$\frac{d}{ds}E_i(s) = \frac{d}{ds}A_i^k(s)e_k = -\frac{1}{2}h_l^k A_i^l(s)e_k = -\frac{1}{2}\sum_k \mu_k A_i^k(s)e_k$$

for any $s \in [0, 1]$. Then

$$\frac{d}{ds}\varphi_s = 2\varphi^{ij}E_i'(s) \wedge E_j(s) = -\mu_k \varphi^{ij}A_i^k(s)e_k \wedge E_j(s).$$

So we have the estimate

$$\begin{aligned}
&\left|\frac{d}{ds}Rm_g(\varphi_s, \bar{\varphi}_s)\right| \\
&= \sum_k |\mu_k| \left| Rm_g(\varphi^{ij}A_i^k(s)e_k \wedge E_j(s), \bar{\varphi}_s) + Rm_g(\varphi_s, \overline{\varphi^{ij}A_i^k(s)e_k \wedge E_j(s)}) \right| \\
&\leq \frac{C}{\lambda}.
\end{aligned}$$

Here C is a positive constant depending only on (M, g). This implies that

$$\left|R_g(\varphi_1, \bar{\varphi}_1) - R_g(\varphi_0, \bar{\varphi}_0)\right| \leq \int_0^1 \frac{C}{\lambda}d\tau \leq \frac{C}{\lambda}.$$

Hence,

$$Rm_g(x)(\varphi, \bar{\varphi}) \geq \frac{1}{2}Rm_g(x)(\psi, \bar{\psi}) > 0 \tag{5.5}$$

if $\lambda > 0$ is sufficiently large.

For case (2), we have $\hat{g}_\lambda = \tilde{g} - \lambda\rho^2\beta(\lambda^{-2}\log\rho)S$. Thus $h_\lambda = -\lambda\rho^2\beta(\lambda^{-2}\log\rho)S$. Following the same argument in case (1) and the fact that

$$|h_\lambda| \leq C(g)e^{-\lambda^2} \leq \frac{C(g)}{\lambda}$$

in the region $\rho < e^{-\lambda^2}$ for sufficiently large $\lambda > 0$, we also have

$$Rm_{\tilde{g}}(x)(\varphi, \bar{\varphi}) \geq \frac{1}{2} Rm_{\tilde{g}}(x)(\psi, \bar{\psi})$$

for sufficiently large λ. Since $(M, \tilde{g})$ is also PIC1, this gives $Rm_{\tilde{g}}(x)(\varphi, \bar{\varphi}) > 0$.

Combining the two cases together, Corollary 4.8 implies that

$$Rm_{\hat{g}_\lambda}(x)(\varphi, \bar{\varphi}) > 0$$

for sufficiently large $\lambda > 0$. From this, we conclude that $(M, \hat{g}_\lambda)$ is also PIC1 for sufficiently large λ. The other assertions in statement (ii) of the Main Theorem 5 can be proved similarly.

□

Proof of (iii):

Suppose that (M, g) has a two-convex boundary and that $(M, \tilde{g})$ has a weakly two-convex boundary such that

$$A_g(X, X) + A_g(Y, Y) > A_{\tilde{g}}(X, X) + A_{\tilde{g}}(Y, Y) \geq 0$$

for all orthonormal $X, Y \in T(\partial M)$. Suppose also that (M, g) and $(M, \tilde{g})$ are PIC. Let $\varphi = z \wedge w \in \bigwedge^2 T_x M^{\mathbb{C}}$ be an isotropic 2-vector satisfying $\hat{g}_\lambda(z, z) = \hat{g}_\lambda(w, w) = \hat{g}_\lambda(z, w) = 0$. This condition is equivalent to the condition $\sum_k \varphi^{ik} \varphi^{kj} = 0$ with respect to the metric $\hat{g}_\lambda$. Using the same argument in the previous proof, it follows from Corollary 4.8 that we have $R_{\hat{g}_\lambda}(\varphi, \bar{\varphi}) > 0$ for sufficiently large $\lambda > 0$. Thus $(M, \hat{g}_\lambda)$ is also PIC. The assertion then follows. □

5.3 Proof of Main Theorem 6

Since the boundary ∂M is an embedded hypersurface in M, we can use the Tubular Neighborhood Theorem to find an open neighborhood U of ∂M and a Riemannian metric $\tilde{g}$ on U, such that $g - \tilde{g} = 0$ at each point of ∂M, and that the boundary ∂M is totally geodesic with respect to $\tilde{g}$.

It follows from the Lemma below that we can choose $\tilde{g}$ such that it satisfies the same curvature conditions as g does. Having fixed the neighborhood U, we define $\hat{g}_\lambda$ as in (4.2). The metric $\hat{g}_\lambda$ is well-defined for sufficiently large λ. Finally, Main Theorem 6 follows from applying Main Theorem 5 with choosing the metric $\tilde{g}$ to be the one constructed in the following Lemma.

Lemma 5.4. *Suppose that (M, g) is a smooth, compact Riemannian manifold with boundary ∂M. Then there is a neighborhood U of ∂M and a Riemannian metric $\tilde{g}$ on U such that*

(i) $\tilde{g} - g = 0$ *on* ∂M.

(ii) ∂M *is totally geodesic with respect to* $\tilde{g}$.

(iii) If (M, g) *has a convex boundary, then*

- (M, g) *has positive curvature operator* $\implies$ $(U, \tilde{g})$ *has positive curvature operator;*
- (M, g) *is PIC1* $\implies$ $(U, \tilde{g})$ *is PIC1;*
- (M, g) *is PIC2* $\implies$ $(U, \tilde{g})$ *is PIC2.*

(iv) If (M, g) *has a 2-convex boundary, then*

- (M, g) *is PIC* $\implies$ $(U, \tilde{g})$ *is PIC.*

(v) $(U, \tilde{g})$ *has positive scalar curvature.*

Proof. By the tubular neighbourhood theorem, there is a neighbourhood U of ∂M that is diffeomorphic to an open set of the normal bundle of ∂M. Specifically, there is a diffeomorphism $\Phi : U \to V$ where $V = \{(x, s) \in \partial M \times \mathbb{R}_+ : |s| < \delta\}$ and $\Phi(\partial M) = \partial M \times \{0\}$. Let $\theta > 0$ to be a large constant specified later, we define $V_\theta \subset V$ by $V_\theta = \{(x, s) \in V : |s| < \delta\theta^{-3}\}$. Define on V_θ a product metric $ds^2 + (\cos^2(\theta s))g_{\partial M}$. Now we define a metric $\tilde{g}$ on $U_\theta = \Phi^{-1}(V_\theta)$ by

$$\tilde{g} = \Phi^*(ds^2 + (\cos^2(\theta s))g_{\partial M}). \tag{5.6}$$

It is easy to see that condition (i) is satisfied. Next, let $\{E_1, .., E_n\}$ be a local frame of TU_θ such that $\{d\Phi(E_a)\}_{a=1,..,n-1}$ is an O.N. frame with respect to $g_{\partial M}$ and $d\Phi(E_n) = \frac{\partial}{\partial s}$. From now on, the subscript $a, b, c, ..$ varies over 1 to $n-1$. Using the Koszul formula, we compute the connection terms of $\tilde{g}$:

$$\begin{aligned}
\tilde{g}(\tilde{D}_{E_a}E_b, E_c) &= \cos^2(\theta s) g_{\partial M}(D_{d\Phi(E_a)} d\Phi(E_b), d\Phi(E_c)), \\
\tilde{g}(\tilde{D}_{E_a}E_b, E_n) &= \frac{1}{2}\theta \sin(2\theta s)\delta_{ab}, \\
\tilde{g}(\tilde{D}_{E_a}E_n, E_b) &= -\frac{1}{2}\theta \sin(2\theta s)\delta_{ab}, \\
\tilde{g}(\tilde{D}_{E_a}E_n, E_n) &= 0, \\
\tilde{g}(\tilde{D}_{E_n}E_a, E_b) &= -\frac{1}{2}\theta \sin(2\theta s)\delta_{ab}, \\
\tilde{g}(\tilde{D}_{E_n}E_a, E_n) &= 0, \\
\tilde{D}_{E_n}E_n &= 0.
\end{aligned} \tag{5.7}$$

Identifying Φ as the identity map, we then have

$$\begin{aligned}
\tilde{D}_{E_a}E_b &= D_{E_a}E_b + \frac{1}{2}\theta \sin(2\theta s)\delta_{ab}E_n, \\
\tilde{D}_{E_a}E_n &= -\theta \tan(\theta s)E_a, \\
\tilde{D}_{E_n}E_a &= -\theta \tan(\theta s)E_a, \\
\tilde{D}_{E_n}E_n &= 0.
\end{aligned} \tag{5.8}$$

From this, we calculate the components of the Riemann curvature tensor of $\tilde{g}$:

$$\begin{aligned}
\tilde{R}_{abcd} &= \cos^2(\theta s)(R_{g_{\partial M}})_{abcd} - \theta^2 \sin^2(\theta s)\cos^2(\theta s)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \\
\tilde{R}_{nbnd} &= \theta^2 \cos^2(\theta s)\delta_{bd}, \\
\tilde{R}_{abnd} &= 0.
\end{aligned} \tag{5.9}$$

Now, it is easy to see from (5.8) that

$$\tilde{D}_{E_a}E_b = D_{E_a}E_b + \frac{1}{2}\theta\sin(0)\delta_{ab}E_n = D_{E_a}E_b$$

on ∂M. Hence condition (ii) is satisfied by $\tilde{g}$.

Next, we fix a point $p \in U_\theta$. Note that the diffeomorphism Φ maps p to a point $(\bar{p}, s) \in \partial M \times \mathbb{R}_+$. Let $\varphi \in \bigwedge^2 T_p U_\theta$ be a unit two-vector with respect to $\tilde{g}$. Since the vector fields $\tilde{E}_a = \frac{1}{\cos(\theta s)}E_a$ and $\tilde{E}_n = E_n$ form an orthonormal frame in U_θ with respect to $\tilde{g}$, we can write

$$\begin{aligned}\varphi &= \sum_{a<b}\varphi^{ab}\tilde{E}_a \wedge \tilde{E}_b + \sum_{a<n}\varphi^{an}\tilde{E}_a \wedge \tilde{E}_n \\ &= \sum_{a<b}\frac{1}{\cos^2(\theta s)}\varphi^{ab}E_a \wedge E_b + \sum_{a<n}\frac{1}{\cos(\theta s)}\varphi^{an}E_a \wedge E_n.\end{aligned}$$

And we also denote $\varphi^T = \sum_{a<b}\varphi^{ab}\tilde{E}_a \wedge \tilde{E}_b$ and $\varphi^N = \sum_{a<n}\varphi^{an}\tilde{E}_a \wedge \tilde{E}_n$. Then by (5.9) we have

$$\begin{aligned}R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) &= \sum_{a<b}\sum_{c<d}\frac{1}{\cos^4(\theta s)}\varphi^{ab}\bar{\varphi}^{cd}\tilde{R}_{abcd}(p) + \sum_{a<n}\sum_{c<n}\frac{1}{\cos^2(\theta s)}\varphi^{an}\bar{\varphi}^{cn}\tilde{R}_{ancn}(p) \qquad (5.10)\\ &= \sum_{a<b}\sum_{c<d}\frac{1}{\cos^2(\theta s)}\varphi^{ab}\bar{\varphi}^{cd}(R_{g_{\partial M}})_{abcd}(\bar{p}) \\ &\quad - \theta^2\tan^2(\theta s)\sum_{a<b}\sum_{c<d}\varphi^{ab}\bar{\varphi}^{cd}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \theta^2\sum_{a<n}\sum_{c<n}\varphi^{an}\bar{\varphi}^{cn}\delta_{ac} \\ &= cos^2(\theta s)R_{g_{\partial M}}(\bar{p})(d\Phi(\varphi^T), d\Phi(\bar{\varphi}^T)) - \theta^2\tan^2(\theta s)|\varphi^T|^2_{\tilde{g}} + \theta^2|\varphi^N|^2_{\tilde{g}}.\end{aligned}$$

Here $d\Phi(\varphi^T) = \sum_{a<b}\frac{1}{\cos^2(\theta s)}\varphi^{ab}d\Phi(E_a) \wedge d\Phi(E_b)$. By the Gauss equation, we have

$$\begin{aligned}&R_{g_{\partial M}}(\bar{p})(d\Phi(\varphi^T), d\Phi(\bar{\varphi}^T)) \qquad (5.11)\\ &= R_g(\bar{p})(d\Phi(\varphi^T), d\Phi(\bar{\varphi}^T)) + \sum_{a<b}\sum_{c<d}\frac{1}{\cos^4(\theta s)}\varphi^{ab}\bar{\varphi}^{cd}(A_{bd}A_{ac} - A_{ad}A_{bc}).\end{aligned}$$

Here $A = A_g$ is the second fundamental form of the boundary with respect to g. Now, associating to φ we define a two-vector $\psi \in \bigwedge^2 T_{\bar{p}}M$ by

$$\psi := \sum_{a<b} \frac{1}{\cos^2(\theta s)} \varphi^{ab} d\Phi(E_a) \wedge d\Phi(E_b) + \sum_{a<n} \frac{1}{\cos^2(\theta s)} \varphi^{an} d\Phi(E_a) \wedge \nu, \tag{5.12}$$

where ν is the unit normal vector field on ∂M with respect to g. Hence $\{d\Phi(E_a), \nu\}$ forms an orthonormal frame around $\bar{p} \in \partial M$. Thus,

$$\begin{aligned} & R_g(\bar{p})(d\Phi(\varphi^T), d\Phi(\bar{\varphi}^T)) \qquad (5.13)\\ &= R_g(\bar{p}) \left(\psi - \sum_{a<n} \frac{1}{\cos^2(\theta s)} \varphi^{an} d\Phi(E_a) \wedge \nu, \quad \overline{\psi - \sum_{a<n} \frac{1}{\cos^2(\theta s)} \varphi^{an} d\Phi(E_a) \wedge \nu} \right) \\ &> R_g(\bar{p})(\psi, \bar{\psi}) - C_1(g) |\varphi^N|_{\tilde{g}}. \end{aligned}$$

Here $C(g)$ is a uniform constant depending only on (M, g). Upon combining (5.10), (5.11) and (5.13) we obtain the following estimate

$$\begin{aligned} R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) > \frac{1}{2} R_g(\bar{p})(\psi, \bar{\psi}) + \sum_{a<b} \sum_{c<d} \varphi^{ab} \bar{\varphi}^{cd} (A_{bd} A_{ac} - A_{ad} A_{bc}) \qquad (5.14)\\ - \theta^2 \tan^2(\theta s) |\varphi^T|^2_{\tilde{g}} + \theta^2 |\varphi^N|^2_{\tilde{g}} - C_1(g) |\varphi^N|_{\tilde{g}}. \end{aligned}$$

Next, by noting that $\theta^3 s < \delta \implies \theta \tan(\theta s) < B\theta^2 s < B\delta\theta^{-1}$ for some positive constant B, we have

$$\begin{aligned} R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) > \frac{1}{2} R_g(\bar{p})(\psi, \bar{\psi}) + \sum_{a<b} \sum_{c<d} \varphi^{ab} \bar{\varphi}^{cd} (A_{bd} A_{ac} - A_{ad} A_{bc}) - \frac{B^2 \delta^2}{\theta^2} |\varphi^T|^2_{\tilde{g}} \qquad (5.15)\\ + \theta^2 |\varphi^N|^2_{\tilde{g}} - C_1(g) |\varphi^N|_{\tilde{g}}. \end{aligned}$$

Now, we are ready to prove statements (iii) and (iv) of the Lemma.

Proof of (iii):

Suppose that (M, g) has convex boundary, then the second term in the RHS of (5.15) is positive, thus

$$R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) > \frac{1}{2} R_g(\bar{p})(\psi, \bar{\psi}) + (\inf_{\partial M} \mu_1)^2 |\varphi^T|_{\tilde{g}}^2 - \frac{B^2 \delta^2}{\theta^2} |\varphi^T|_{\tilde{g}}^2 + \theta^2 |\varphi^N|_{\tilde{g}}^2 - C_1(g) |\varphi^N|_{\tilde{g}}, \tag{5.16}$$

where μ_1 is the smallest eigenvalue of A_g. If (M, g) is PIC1, then we let $\varphi = z \wedge w \in \bigwedge^2 T_p U^{\mathbb{C}}$ be a unit two-vector such that $\tilde{g}(z, z)\tilde{g}(w, w) - \tilde{g}(z, w)^2 = 0$. This condition is equivalent to the condition $\sum_{i<j} \varphi^{ij} \varphi^{ij} = 0$ with respect to the metric $\tilde{g}$. By the definition of ψ, this implies that $\sum_{i<j} \psi^{ij} \psi^{ij} = 0$ with respect to the metric g. Since (M, g) is PIC1, $R_g(\bar{p})(\psi, \bar{\psi}) > 0$, we obtain

$$\begin{aligned}
R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) &> (\inf_{\partial M} \mu_1)^2 |\varphi|_{\tilde{g}}^2 - \frac{B^2 \delta^2}{\theta^2} |\varphi^T|_{\tilde{g}}^2 + (\theta^2 - (\inf_{\partial M} \mu_1)^2) |\varphi^N|_{\tilde{g}}^2 - C_1(g) |\varphi^N|_{\tilde{g}} \\
&> (\inf_{\partial M} \mu_1)^2 |\varphi|_{\tilde{g}}^2 - \frac{B^2 \delta^2}{\theta^2} |\varphi^T|_{\tilde{g}}^2 - \frac{C_1(g)^2}{\theta^2} \\
&> \frac{1}{2} (\inf_{\partial M} \mu_1)^2 |\varphi|_{\tilde{g}}^2,
\end{aligned}$$

provided that θ is sufficiently large. Hence $(U, \tilde{g})$ is also PIC1. The other two assertions under statement (iii) can be proved similarly.

Proof of (iv):

Suppose now that (M, g) has a two-convex boundary and is PIC. Let $\varphi = z \wedge w \in \bigwedge^2 T_p U^{\mathbb{C}}$ be a unit two-vector such that $\tilde{g}(z, z) = \tilde{g}(w, w) = \tilde{g}(z, w) = 0$. By a similar argument as in the proof of (iii), the definition of ψ and the fact that (M, g) is PIC imply the positivity of $R_g(\bar{p})(\psi, \bar{\psi})$. We will show that the second term in RHS of (5.15) is positive modulus an error term involving the normal components. At the point $\bar{p} \in \partial M$, the restriction of A_g to the tangent space $T_{\bar{p}}(\partial M)$ is 2-positive. We define a (0,2)-tensor $\tilde{A}_g$ by $\tilde{A}_g = A_g + \nu^* \otimes \nu^*$, thus $\tilde{A}_g$ is strictly 2-positive on $T_{\bar{p}} M$. Consequently, the Kulkarni-Nomizu product $\tilde{A}_g \owedge \tilde{A}_g$ is strictly PIC by Lemma 4.3. In fact, the above procedure can be done uniformly at each point on the boundary. On the other hand, if we extend the definition of φ so that $\varphi^{ji} = -\varphi^{ij}, i < j$, the condition $\tilde{g}(z, z) = \tilde{g}(w, w) = \tilde{g}(z, w) = 0$ is equivalent to the condition $\sum_k \varphi^{ik} \varphi^{kj} = 0$ with respect to the metric $\tilde{g}$. By the definition of ψ, this implies that $\sum_k \psi^{ik} \psi^{kj} = 0$ with respect to the metric g. Therefore, we can find a uniform constant

$a > 0$ such that

$$\tilde{A}_g \owedge \tilde{A}_g(\psi, \bar{\psi}) > a|\psi|_g^2$$

on $T_{\bar{p}}M$. It follows from the definition of $\tilde{A}_g$ that

$$\sum_{a<b}\sum_{c<d} \psi^{ab}\bar{\psi}^{cd}(A_{bd}A_{ac} - A_{ad}A_{bc}) > \frac{a}{2}|\psi|_g^2 - C_2(g)|\psi^N|_g^2, \tag{5.17}$$

where $C_2(g)$ is a uniform constant depending only on (M, g). From the definition of ψ, we then have

$$\sum_{a<b}\sum_{c<d} \varphi^{ab}\bar{\varphi}^{cd}(A_{bd}A_{ac} - A_{ad}A_{bc}) > \frac{a}{2}|\varphi|_{\tilde{g}}^2 - C_2(g)|\varphi^N|_{\tilde{g}}^2. \tag{5.18}$$

Combining (5.15), (5.18) and the positivity of $R_g(\bar{p})(\psi, \bar{\psi})$, we obtain

$$\begin{aligned} R_{\tilde{g}}(p)(\varphi, \bar{\varphi}) &> \frac{a}{2}|\varphi|_{\tilde{g}}^2 - \frac{B^2\delta^2}{\theta^2}|\varphi^T|_{\tilde{g}}^2 + (\theta^2 - C_2(g))|\varphi^N|_{\tilde{g}}^2 - C_1(g)|\varphi^N|_{\tilde{g}} \\ &> \frac{a}{2}|\varphi|_{\tilde{g}}^2 - \frac{B^2\delta^2}{\theta^2}|\varphi^T|_{\tilde{g}}^2 - \frac{C_1(g)^2}{\theta^2} \\ &> \frac{a}{4}|\varphi|_{\tilde{g}}^2, \end{aligned}$$

provided that θ is sufficiently large. Hence $(U, \tilde{g})$ is also PIC.

Proof of (v):

We have

$$\begin{aligned}
R_{\tilde g} &= \sum_{i,j=1}^{n} \tilde R(\tilde E_i, \tilde E_j, \tilde E_i, \tilde E_j) \\
&= \sum_{a,b=1}^{n-1} \frac{1}{\cos^4(\theta s)} \tilde R_{abab} + 2\sum_{a=1}^{n-1} \frac{1}{\cos^2(\theta s)} \tilde R_{anan} \\
&= \sum_{a,b=1}^{n-1} \frac{1}{\cos^2(\theta s)} (R_{g_{\partial M}})_{abab} - (n-1)^2\theta^2 \tan^2(\theta s) + 2(n-1)\theta^2 \\
&> \sum_{a,b=1}^{n-1} \frac{1}{\cos^2(\theta s)} (R_{g_{\partial M}})_{abab} - (n-1)^2 \frac{B^2\delta^2}{\theta^2} + 2(n-1)\theta^2 \\
&> 0,
\end{aligned}$$

if θ is sufficiently large.

□

5.4 Proof of Main Theorem 7

We can show that various important curvature conditions are preserved along the Ricci flow on manifolds with boundary in our formulation. We first prove a compactness result.

Lemma 5.5. *Let $\alpha, \beta \in (0,1)$ be given such that $\beta < \alpha$. Moreover, let $\gamma \in (0,\alpha)$ and $\delta \in (0,\beta)$ be given such that $\delta < \gamma$. Then the bounded subsets in the space $C^{k,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M \times (0,T])$ are precompact in the space $C^{k,\delta}_{\frac{1}{2}-\frac{\beta}{2}}(M \times (0,T])$.*

Proof. Note that by Lemma 1.6 we have the inclusion $C^{k,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}} \subset C^{k,\delta}_{\frac{1}{2}-\frac{\beta}{2}}$. Let $\{\eta_j\}$ be a bounded sequence in $C^{k,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M \times (0,T])$ so that $\|\eta_j\|_{C^{k,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}} \le L$. In particular,

$$\|\hat\nabla^k \eta_j\|_{0;M\times[\sigma/2,\sigma]} \le \frac{L}{\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k}{2}}} \quad \text{and} \quad [\hat\nabla^k \eta_j]_{\gamma,\frac{\gamma}{2};M\times[\sigma/2,\sigma]} \le \frac{L}{\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{k}{2}+\frac{\gamma}{2}}}$$

for any $\sigma \in (0,T]$. Thus $\{\hat\nabla^k \eta_j\}$ is equicontinuous on $M \times [\sigma/2,\sigma]$ and uniformly bounded. By Arzela-Ascoli it contains a subsequence $\{\tilde\eta_j\}$ such that $\{\hat\nabla^k \tilde\eta_j\}$ is uniformly convergent on compact subsets of $M \times [\sigma/2, \sigma]$. By the same argument we can also proceed to find a subsequence of $\{\tilde{\eta}_j\}$ which is uniformly convergent on bounded subsets of $M \times [\sigma/2, \sigma]$ together with its lower order x-derivatives. Moreover, by a standard diagonal argument we can further find a subsequence which is uniformly convergent on bounded subsets of $M \times (0, T]$. We still denote this subsequence by $\{\tilde{\eta}_j\}$ and its limit by η. Thus η satisfies the same bounds as η_j given above. Define $\zeta_j = \tilde{\eta}_j - \eta$. To finish the proof we will show that $\zeta_j \to 0$ in $C^{k,\delta}_{\frac{1}{2}-\frac{\beta}{2}}(M \times (0, T])$. For any $0 \leq r \leq k$, we have

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{r}{2}} \|\hat{\nabla}^r \zeta_j\|_{0;M\times[\sigma/2,\sigma]} \leq T^{\frac{\alpha-\beta}{2}} \sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{r}{2}} \|\hat{\nabla}^r \zeta_j\|_{0;M\times[\sigma/2,\sigma]}. \tag{5.19}$$

Moreover,

$$\sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{r}{2}+\frac{\gamma}{2}} [\hat{\nabla}^r \zeta_j]_{\gamma,\frac{\gamma}{2};M\times[\sigma/2,\sigma]} \leq \|\zeta_j\|_{C^{k,\gamma}_{\frac{1}{2}-\frac{\alpha}{2}}(M\times(0,T])} \leq 2L \tag{5.20}$$

for any $\sigma \in (0, T]$. Now, we choose a local chart U in M and let $x, y \in U$ so that the (0,2)-tensors ζ_j are represented in local coordinates as $\zeta_j = \sum_{m=1}^{N} \zeta_{jm} \mathbf{e}_m$. Let $\epsilon > 0$ be an arbitrary small number, we may assume without loss of generality that $\epsilon < \sigma$.

If $d((x, t), (y, s)) < \epsilon$, where the distance $d((x, t), (y, s)) = |x - y| + |t - s|^{1/2}$ is taken within the chart, then (5.20) implies

$$\begin{aligned}
&\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{r}{2}+\frac{\delta}{2}} \frac{|\hat{\nabla}^r \zeta_{jm}(x,t) - \hat{\nabla}^r \zeta_{jm}(y,s)|}{d((x,t),(y,s))^{\delta}} \\
&= \sigma^{\frac{1}{2}-\frac{\alpha}{2}+\frac{r}{2}+\frac{\gamma}{2}} \frac{|\hat{\nabla}^r \zeta_{jm}(x,t) - \hat{\nabla}^r \zeta_{jm}(y,s)|}{d((x,t),(y,s))^{\gamma}} \cdot d((x,t),(y,s))^{\gamma-\delta} \sigma^{\frac{\alpha}{2}-\frac{\beta}{2}+\frac{\delta}{2}-\frac{\gamma}{2}} \\
&\leq 2LT^{\frac{\alpha-\beta}{2}} \epsilon^{\frac{\gamma-\delta}{2}}.
\end{aligned} \tag{5.21}$$

If $d((x, t), (y, s)) \geq \epsilon$, then

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{r}{2}+\frac{\delta}{2}} \frac{|\hat{\nabla}^r \zeta_{jm}(x,t) - \hat{\nabla}^r \zeta_{jm}(y,s)|}{d((x,t),(y,s))^{\delta}} \leq K(T) \|\hat{\nabla}^r \zeta_j\|_{0;M\times[\sigma/2,\sigma]} \epsilon^{-\delta}. \tag{5.22}$$

Combining (5.21) and (5.22), we obtain

$$\sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{r}{2}+\frac{\delta}{2}}[\hat{\nabla}^r \zeta_j]_{\delta,\frac{\delta}{2};M\times[\sigma/2,\sigma]} \leq K(T)(2L\epsilon^{\frac{\gamma-\delta}{2}} + \|\hat{\nabla}^r \zeta_j\|_{0;M\times[\sigma/2,\sigma]}\epsilon^{-\delta}). \tag{5.23}$$

By taking $j \to \infty$ and then $\epsilon \to 0$ in (5.23), we obtain

$$\lim_{j\to\infty} \sigma^{\frac{1}{2}-\frac{\beta}{2}+\frac{r}{2}+\frac{\delta}{2}}[\hat{\nabla}^r \zeta_j]_{\delta,\frac{\delta}{2};M\times[\sigma/2,\sigma]} = 0. \tag{5.24}$$

for each $\sigma \in (0, T]$. Therefore, (5.19) and (5.24) conclude that

$$\lim_{j\to\infty} \|\zeta_j\|_{C^{k,\delta}_{\frac{1}{2}-\frac{\beta}{2}}(M\times(0,T])} = 0 \tag{5.25}$$

□

Theorem 5.6 (Main Theorem 7)**.** *Suppose that $g(t)$ is a canonical solution to (5.3) on $M \times [0, T]$ given by Main Theorem 3. Then the following holds:*

If (M, g_0) has a convex boundary, then

(i) (M, g_0) has positive curvature operator $\implies$ $(M, g(t))$ has positive curvature operator;

(ii) (M, g_0) is PIC1 $\implies$ $(M, g(t))$ is PIC1;

(iii) (M, g_0) is PIC2 $\implies$ $(M, g(t))$ is PIC2.

If (M, g_0) has a two-convex boundary, then

(iv) (M, g_0) is PIC $\implies$ $(M, g(t))$ is PIC.

Moreover, if (M, g_0) has a mean-convex boundary, then

(v) (M, g_0) has positive scalar curvature $\implies$ $(M, g(t))$ has positive scalar curvature.

Proof. By Main Theorem 6, there is a family of smooth Riemannian metrics $\{\hat{g}_\lambda\}_{\lambda>\lambda^*}$ on M which converges to g_0 in C^α for any $\alpha \in [0, 1)$ and satisfies:

(i) $(M, \hat{g}_\lambda)$ has a totally geodesic boundary.

(ii) If (M, g_0) has a convex boundary, then

- (M, g_0) has positive curvature operator $\implies$ $(M, \hat{g}_\lambda)$ has positive curvature operator;
- (M, g_0) is PIC1 $\implies$ $(M, \hat{g}_\lambda)$ is PIC1;
- (M, g_0) is PIC2 $\implies$ $(M, \hat{g}_\lambda)$ is PIC2.

(iii) If (M, g_0) has a two-convex boundary, then

- (M, g_0) is PIC $\implies$ $(M, \hat{g}_\lambda)$ is PIC.

(iv) If (M, g_0) has a mean-convex boundary, then

- (M, g_0) has positive scalar curvature $\implies$ $(M, \hat{g}_\lambda)$ has positive scalar curvature.

Note that by Corollary 4.8 the positivity conditions in the above statement are uniform in all sufficiently large λ.

Let $\tilde{M}$ be the doubled manifold of M, and fix a background metric $\hat{g}$ such that in a small collar neighborhood of ∂M the metric is isometric to $\partial M \times [0, \varepsilon]$. We extend the metrics g_0 to $\tilde{M}$ via reflection about the boundary ∂M. Moreover, from the construction in Main Theorem 6, in a neighborhood of the boundary $\hat{g}_\lambda$ has the form of a product metic, thus the metric $\hat{g}_\lambda$ can be extended smoothly to the doubled manifold $\tilde{M}$. Then g_0 is a Lipschitz metric on $\tilde{M}$ and $\hat{g}_\lambda$ is a smooth metric on $\tilde{M}$.

Now, we assume that (M, g_0) has convex boundary. If (M, g_0) has positive curvature operator/ PIC1/ PIC2, then $(\tilde{M}, \hat{g}_\lambda)$ has positive curvature operator/ PIC1/ PIC2 for all sufficiently large $\lambda > 0$ and these positivity conditions are uniformly bounded below for λ large. Let $\tilde{g}_\lambda(t)$ be the solution to Ricci-DeTurck flow on $\tilde{M} \times (0, T]$ starting from $\tilde{g}_\lambda(0) = \hat{g}_\lambda$. Let ϕ_t solves the harmonic map heat flow so that $\tilde{g}(t) = (\phi_t^{-1})^* g(t)$ is a solution to Ricci-DeTurck flow on $\tilde{M} \times (0, T]$ starting from $\tilde{g}(0) = g_0$. Proposition 3.13 then implies that $\tilde{g}(t) \in \mathcal{X}_{2,\gamma}^{(\alpha)}$ for some exponent $\alpha \in (0,1)$ and $\gamma \in (0,\alpha)$. Next we apply Theorem 3.3 to obtain the estimates

$$\|\tilde{g}_\lambda(t)\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])} \leq K(M, \hat{g}, \|\hat{g}_\lambda\|_{\alpha;M}).$$

Since $\hat{g}_\lambda \to g_0$ in $C^\alpha(M)$, we can find a uniform constant $K(M, \hat{g}, \|g_0\|_{\alpha;M})$ such that

$$\|\tilde{g}_\lambda(t)\|_{\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])} \leq K(M, \hat{g}, \|g_0\|_{\alpha;M}). \tag{5.26}$$

Next, we pick some $\beta \in (0,\alpha)$ and $\delta \in (0,\gamma)$ such that $\delta < \beta$. Since bounded subsets in $C^{\alpha,\frac{\alpha}{2}}(M\times[0,T])$ are precompact in $C^{\beta,\frac{\beta}{2}}(M\times[0,T])$, Lemma 5.5 then implies that bounded subsets in the Banach space $\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])$ are precompact in $\mathcal{X}_{2,\delta}^{(\beta)}(M\times[0,T])$. Now, by the estimate (5.26) the metrics $\tilde{g}_\lambda$ are uniformly bounded in $\mathcal{X}_{2,\gamma}^{(\alpha)}(M\times[0,T])$ for all large λ. Upon passing to a subsequence we have $\tilde{g}_\lambda \to \tilde{g}_\infty$ in $\mathcal{X}_{2,\delta}^{(\beta)}(M\times[0,T])$ as $\lambda \to \infty$. By the continuity of coefficients in the Ricci-DeTurck flow, $\tilde{g}_\infty(t)$ is also a solution to the Ricci-DeTurck flow with $\tilde{g}_\infty(0) = g_0$ such that

$$\|\tilde{g}_\infty(t)\|_{\mathcal{X}_{2,\delta}^{(\beta)}(M\times[0,T])} \leq K(M, \hat{g}, \|g_0\|_{\alpha;M}).$$

Since $\delta < \beta$, Theorem 3.3 implies that such a solution is unique, therefore we have $\tilde{g}_\infty(t) = \tilde{g}(t)$. On the other hand, since $\hat{g}_\lambda$ are smooth metrics, the curvature conditions are preserved under the Ricci flow, thus $\tilde{g}_\lambda(t)$ also has positive curvature operator/ PIC1/ PIC2 for all sufficiently large λ by diffeomorphism invariance of curvatures. This in particular implies that $\tilde{g}(t) = \tilde{g}_\infty(t)$ also has positive curvature operator/ PIC1/ PIC2 for each $t > 0$. Therefore, since $g(t) = (\phi_t)^*\tilde{g}(t)$, by diffeomorphism invariance of curvatures these curvature conditions also hold for $g(t)$ for each $t > 0$. This proves statements (i) to (iii) in the theorem. Statements (iv) and (v) can be proved similarly. The theorem then follows.

□

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