Quantum de Sitter Entropy and Sphere Partition Functions: $\mathcal{A}$-Hypergeometric Approach to All-Loop Order

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Abstract

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In order to find quantum corrections to the de Sitter entropy, a new approach to higher loop Feynman integral computations on the sphere is presented. Arbitrary scalar Feynman integrals on a spherical background are brought into the generalized Euler integral ($\mathcal{A}$-hypergeometric series/GKZ systems) form by expressing the massive scalar propagator as a bivariate radial Mellin transform of the massless scalar propagator in one higher dimensional Euclidean flat space. This formulation is expanded to include massive and massless vector fields by construction of similar embedding space propagators. Vector Feynman integrals are shown to be sums over generalized Euler integral formed of underlying scalar Feynman integrals. Granting existence of general spin embedding space propagators, general spin Feynman integrals are shown, by the construction of a "master" integral, to also be sums over generalized Euler integral representations of scalar Feynman integrals. Finding exact embedding space propagator expressions for fields of integer spin $\geqslant 2$ and half integer spin is left to future work.

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Chapter 1: Introduction

Observations of distant cosmological objects (light [1] and matter [2–4]) have all but categorically confirmed that this universe is undergoing an accelerated expansion. A positive cosmological constant, $\Lambda > 0$, in the Einstein-Hilbert action implies such an accelerated expansion:

$$\mathsf{S} = \frac{1}{8\pi\,\mathsf{G}} \int \mathrm{d}^{d+1}x\,\sqrt{-\mathsf{g}}\,\left(\tfrac{1}{2}\,\mathsf{R} - \Lambda\right). \tag{1.1}$$

The first order variation of S with respect to a small metric perturbation results in Einstein's field equations in vacuum:

$$\delta\mathsf{S} = 0 \implies \mathsf{R}_{\mu\nu} - \mathsf{g}_{\mu\nu}\left(\tfrac{1}{2}\,\mathsf{R} - \Lambda\right) = 0. \tag{1.2}$$

With a constant positive curvature of $R = \frac{2\,(d+1)}{(d-1)}\,\Lambda$ and Ricci scalar $R_{\mu\nu} = \frac{2}{(d-1)}\,\Lambda\,g_{\mu\nu}$ that is proportional to the metric tensor $g_{\mu\nu}$, $(d+1)$-dimensional de Sitter space, $\mathrm{dS}_{(d+1)}$, is the maximally symmetric vacuum solution to Einstein's equations.

$\mathrm{dS}_{(d+1)}$ can be embedded in Lorentzian flat space of one higher dimension, $\mathbb{R}^{1,(d+1)}$, with a mostly positive metric $\eta = (-1,\,+1,\cdots+1)$, as a hyperboloid sheet that gets parameterised in flat coordinates, X, as

$$-X_0^2 + X_1^2 + X_2^2 + \cdots + X_{d+1}^2 = \ell^2, \tag{1.3}$$

where ℓ is a length scale that is related to the cosmological constant by

$$\ell^2 = \frac{1}{\Lambda}\,\frac{d\,(d-1)}{2}. \tag{1.4}$$

A natural parameterisation of the flat coordinates covering the entire dS hyperboloid is

$$(X_0,\, X) = \ell\,(\sinh t,\, \cosh t\,\hat{\Omega}_d), \tag{1.5}$$

where $\hat{\Omega}_d$ refers to coordinates of the d-sphere. As depicted in fig. (1.1), the spacial dimensions $X \equiv (X_1, \cdots, X_{d+1})$ simply form a d-sphere with a time dependent radius, $|X| = \ell\,\cosh t$. The metric in these coordinates is time dependent:

$$\mathrm{d}s^2 = \ell^2\,(-\,\mathrm{d}t^2 + \cosh^2 t\;\mathrm{d}\hat{\Omega}_d^2), \tag{1.6}$$

where $\mathrm{d}\hat{\Omega}_d^2$ is the distance measure on the d-sphere. Null, real and imaginary distances between two points imply light-like, space-like and time-like separations respectively.

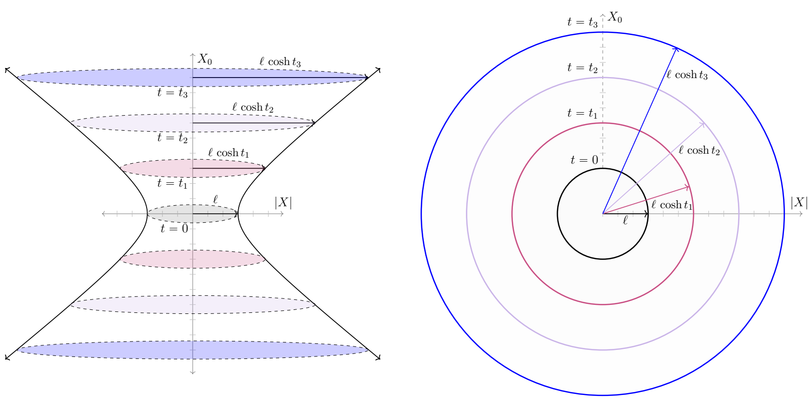


Figure 1.1: Time slices of the $(d+1)$-dimensional de Sitter hyperboloid : $\mathbb{R}\times S^d$

Representing dS in so-called static coordinates, which is characterised by a time independent metric (i.e. time translations leave the metric invariant):

$$\begin{aligned}(X_0,\, X,\, X_{d+1}) &= (\sqrt{\ell^2 - r^2}\,\sinh t,\, r\,\hat{\Omega}_{d-1},\, \pm\sqrt{\ell^2 - r^2}\,\cosh t), \qquad r \leqslant \ell\\ \mathrm{d}s^2 &= -\left(1 - \frac{r^2}{\ell^2}\right)\,\ell^2\;\mathrm{d}t^2 + \frac{1}{(1-\frac{r^2}{\ell^2})}\;\mathrm{d}r^2 + r^2\;\mathrm{d}\Omega_{d-1}^2,\end{aligned} \tag{1.7}$$

splits the space-time into causally disconnected North and South patches, as depicted in fig. (1.2), enclosed by an event horizon, a $(d-1)$-sphere of radius $r = \ell$ and area $A_{\text{hor}} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\,\ell^{d-1}$.

Static coordinates, however, do not cover the entirety of dS, requiring the patches labelled Past and Future to instead be parameterised as

$$(X_0,\, X,\, X_{d+1}) = (\pm\sqrt{r^2-\ell^2}\,\cosh t,\, r\,\hat{\Omega}_{d-1},\, \sqrt{r^2-\ell^2}\,\sinh t), \quad r > \ell. \tag{1.8}$$

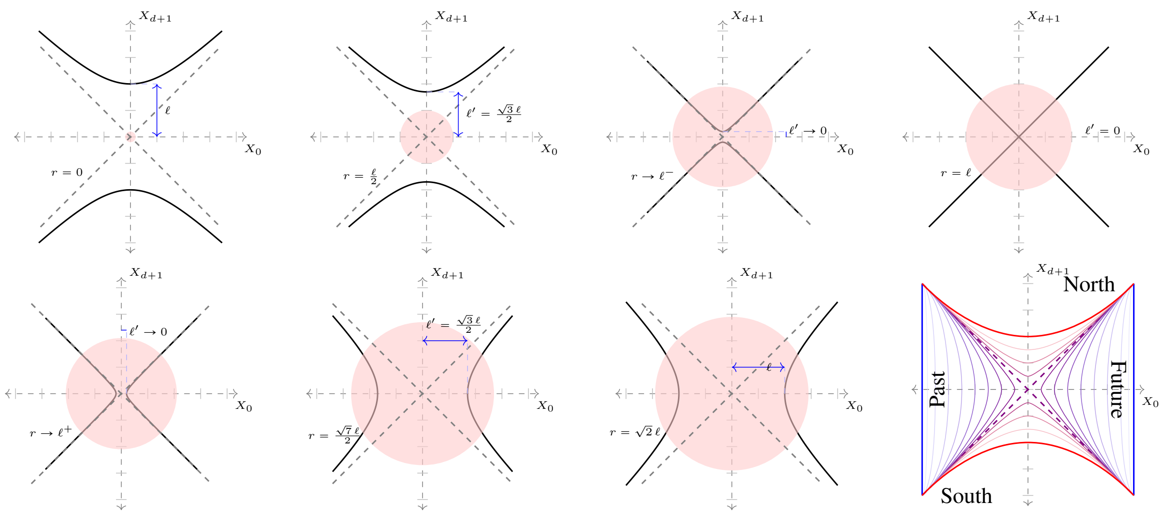


Figure 1.2: North and South Static Patches of De Sitter

The red circles are representative of $r\,\hat{\Omega}_{d-1}$ in eq. (1.7). The dashed lines are asymptotes depicting the horizon at $r = \ell$.

There is perhaps more to be said about what de Sitter space doesn't have or allow than it does. Since it isn't asymptotically flat, there is no scattering matrix. It doesn't have a boundary and so cannot be described in terms of boundary conditions/correlators, unlike Anti-de Sitter space. There is also no conserved positive energy: Picking a global notion of time on dS makes it so that it flows in the 'right' direction, i.e. from the past to future, only in the southern static patch. So although the time-like Killing vector of the southern static patch's coordinates can be used to define a Hamiltonian, it would not maintain its positivity if extended to the entire space [5, 6]. Finally, even inserting an observer to simply look at the space isn't as trivial as it may sound because it will influence the metric itself and physical details of the observer, which are totally extrinsic to the properties of dS, become relevant. Fortunately, there is at least one unambiguously defined non-trivial calculable quantity in empty de Sitter space, its entropy.

The macroscopic dS entropy, $\mathcal{S}$, is given by the logarithm of the Euclidean path integral [7]

$$\mathcal{S} = \log \mathcal{Z}, \tag{1.9}$$

defined on the $(d+1)$-sphere, resulting from Wick rotation of the time-like coordinate ($t \to -\mathrm{i}\,\tau$) of the southern static patch of $\mathrm{dS}_{(d+1)}$,

$$\begin{aligned} \mathsf{S}_{\mathrm{Lor}} \to \mathsf{S}_{\mathrm{Eucl}} &= \frac{1}{8\pi\,\mathsf{G}} \int \mathrm{d}^{d+1}x\,\sqrt{\mathsf{g}}\,\big(\Lambda - \tfrac{1}{2}\,\mathsf{R}\big), \\ \mathcal{Z} &= \int \mathrm{D}\mathsf{g}\, e^{i\mathsf{S}_{\mathrm{Lor}}[\mathsf{g}]} \to \int \mathrm{D}\mathsf{g}\, e^{-\mathsf{S}_{\mathrm{Eucl}}[\mathsf{g}]}, \end{aligned} \tag{1.10}$$

which, when expanded around the saddle point corresponding to the round sphere, S^{d+1} of radius ℓ, leads to the tree-level result [8, 9]:

$$\mathcal{S}_{\mathrm{tree}} = \frac{A_{\mathrm{hor}}}{4\,G}. \tag{1.11}$$

The tree-level entropy, $\mathcal{S}_{\mathrm{tree}}$, however, doesn't provide any additional characterisation of dS, beyond its dependence on an input parameter or dimensionless coupling constant, far from showcasing the expected non-triviality of $\mathcal{S}$.

1.1 Higher loop corrections to de Sitter Entropy: *why?*

Before searching for *the* microscopic model of quantum gravity in dS, it is important to define what *a* microscopic model of quantum gravity in dS is, namely what features is it expected to show, how to recognise it if one does come across it, what is the duck test?

The dS entropy on its own is a just number and doesn't convey much beyond setting a scale. If it is to be compared with something, it would have to look within itself. Less philosophically, non-local quantum corrections to dS entropy, i.e. those that cannot be absorbed into local field redefinitions, contain important information regarding the space itself, with each higher order correction translating information regarding the geometry of the space into numbers.

1.1.1 Non-local Quantum Corrections

As discussed in detail in [10], these non-local quantum corrections to $\mathcal{S}$, represented as a series expansion in terms of $\mathcal{S}_{\text{tree}}$, are the features being sought in a microscopic model. A simple depiction of this idea can be found at 1-loop order of 3D ($d = 2$) gravity with the horizon fluctuating around S^1 of radius ℓ:

$$\begin{aligned} \mathcal{S}^{(0)} &= \frac{\pi}{4\,\mathsf{G}}(3\,\ell - \Lambda\,\ell^3) \xrightarrow{\partial_\ell \mathcal{S}^{(0)}=0} \frac{2\pi}{4\,\mathsf{G}}\,\ell, \\ \mathcal{S}^{(1)} &= \underbrace{-\frac{9\pi}{2\,\epsilon}\,\ell}_{\text{divergent}} + \underbrace{5\,\log(-2\pi\mathrm{i}\,) - 3\,\log\left(\frac{2\pi}{4\,\mathsf{G}}\,\ell\right)}_{\text{finite}}, \end{aligned} \tag{1.12}$$

where ϵ is some UV cutoff. In terms of the Ricci scalar R, traceless Ricci tensor $\mathsf{Q}_{\mu\nu} = \mathsf{R}_{\mu\nu} - \frac{1}{d+1}\,\mathsf{R}\,\mathsf{g}_{\mu\nu}$ and Weyl tensor $\mathsf{W}_{\mu\nu\rho\sigma}$, $\{\mathsf{R},\,\mathsf{Q},\,\mathsf{W}\} \propto \ell^{-2}$, the most general diffeomorphism invariant form of a shift to the action S_{Eucl} in eq. (1.10) is

$$\mathsf{S}_{\text{c}} = \int \sqrt{\mathsf{g}}\,\left(c_\Lambda\,\Lambda - c_{\mathsf{R}}\,\mathsf{R} - \ell_c^2\,\left(c_{\mathsf{R}^2}\,\mathsf{R}^2 + c_{\mathsf{Q}^2}\,\mathsf{Q}^2 + c_{\mathsf{W}^2}\,\mathsf{W}^2\right) + \mathcal{O}(\ell^{-6})\right), \tag{1.13}$$

where ℓ_c is a length scale and c are dimensionless constants. On the round sphere, both Q and W vanish, and $\mathsf{R}\Big|_{S^{d+1}} = \frac{d\,(d+1)}{\ell^2}$. S_{c} serves to parameterise all possible curvature corrections, counter-terms, and local metric field redefinitions of the form $\mathsf{g}_{\mu\nu} \to \mathsf{g}_{\mu\nu} + \ell_c^2\,(c_0\,\Lambda\,\mathsf{g}_{\mu\nu} + c_1\,\mathsf{R}\,\mathsf{g}_{\mu\nu} + c_2\,\mathsf{Q}_{\mu\nu})$. The contribution of S_c at 1-loop order to the entropy is

$$\mathcal{S}_c = -2\pi^2\,c_\Lambda\,\Lambda\,\ell^3 + 12\pi^2\,c_{\mathsf{R}}\,\ell + 2\pi^2\,\sum_{n=2}^{\infty} c_{\mathsf{R}^n}\,6^n\,\ell_c^{2n-2}\,\ell^{3-2n}, \tag{1.14}$$

with terms $\propto \{\ell^3,\,\ell,\,\ell^{-1},\,\ell^{-3}\cdots\}$ i.e. only odd powers of ℓ. Setting $c_\Lambda = 0,\ c_{\mathsf{R}} = \frac{3}{8\pi\,\epsilon}$ ensures that the renormalized values of Λ, G remain the same as at tree-level. The finite terms of $\mathcal{S}^{(1)}$ in eq. (1.12) remain unchanged by local operations and constitute invariant data of the quantum theory. An accurate microscopic model of 3D gravity would be expected to replicate these terms.

This idea applies to theories with matter content too. For example, inclusion of a massive scalar field with some curvature coupling to the above setup:

$$\mathsf{S}_\phi = \frac{1}{2}\int \sqrt{\mathsf{g}}\,\phi\left(-\nabla^2 + m^2 + \frac{1-\eta}{6}\,\mathsf{R}\right)\phi, \quad \nu^2 \equiv m^2\,\ell^2 - \eta \tag{1.15}$$

supplies an additional contribution to the entropy at 1-loop order equalling:

$$\mathcal{S}^{(1)}_\phi = \underbrace{\frac{\pi}{2\,\epsilon^3}\,\ell^3 - \frac{\pi}{4\,\epsilon}\,\nu^2\,\ell}_{\text{divergent}} + \underbrace{\frac{\pi}{6}\,\nu^3 + \frac{\nu^2\,\log(1-e^{-2\pi\,\nu})}{2} - \frac{\nu\,\mathrm{Li}_2(e^{-2\pi\,\nu})}{2\pi} - \frac{\mathrm{Li}_3(e^{-2\pi\,\nu})}{(2\pi)^2}}_{\text{finite}}, \tag{1.16}$$

where Li_n are polylogarithms. The renormalisation condition $\lim\limits_{\ell\to\infty}\partial_\ell\mathcal{S}^{(1)} = 0$ sets the counter-terms in eq. (1.13) to

$$\mathsf{S}_c \to \int\sqrt{\mathsf{g}}\left(\frac{1}{4\pi\,\epsilon^3} - \frac{m^2}{8\pi\,\epsilon} + \frac{m^3}{12\pi} - \left(\frac{3}{8\pi\,\epsilon} - \frac{\eta}{48\pi\,\epsilon} + \frac{m\,\eta}{48\pi}\right)\mathsf{R}\right) + \cdots . \tag{1.17}$$

Thus, entropy at 1-loop order $\mathcal{S}^{(1)}$ represented as an expansion in orders of ℓ:

$$\begin{aligned}\mathcal{S}^{(1)} = {}& \underbrace{5\,\log(-2\pi\mathrm{i}) - 3\,\log\left(\frac{2\pi}{4\,\mathsf{G}}\,\ell\right)}_{\text{gravity : eq. (1.12)}} + \underbrace{\frac{\pi\,\eta}{4}\,m\,\ell - \frac{\pi}{6}\,m^3\,\ell^3}_{\ell^{\text{odd}}} \\ & {}^{\ell^{\text{even}}}\left[\begin{aligned} & + \frac{\pi}{4}\sqrt{\eta}\,\cot(\pi\sqrt{\eta})\,m^2\,\ell^2 + \frac{\pi}{16}\left(\pi\,\csc^2(\pi\sqrt{\eta}) - \frac{\cot(\pi\sqrt{\eta})}{\sqrt{\eta}}\right)m^4\,\ell^4 \\ & - \frac{\mathrm{Li}_3(e^{-2\pi\mathrm{i}\,\sqrt{\eta}})}{4\pi^2} - \frac{\mathrm{i}\sqrt{\eta}\,\mathrm{Li}_2(e^{-2\pi\mathrm{i}\,\sqrt{\eta}})}{2\pi} - \frac{\eta\,\log(1-e^{-2\pi\mathrm{i}\,\sqrt{\eta}})}{2} + \frac{\mathrm{i}\,\eta^{\frac{3}{2}}}{6} + \cdots\end{aligned}\right. \\ & + \mathcal{O}(\ell^5),\end{aligned} \tag{1.18}$$

has some counter-term dependence appearing as terms in odd powers of ℓ but the terms in even powers of ℓ are invariant non-local quantum corrections.

As a major step towards generating such model constraining data, character integral and closed form expressions of quantum corrections at 1-loop order for arbitrary field content on spherical and (A)dS backgrounds are presented in [10]. Their group theoretic approach is, however, not generalisable to account for non-trivial interactions.

It is at this stage that higher loop Feynman integral computations become necessary. Some works advancing the search for quantum corrections to dS entropy with this aim are [11–15].

1.2 Higher loop corrections to de Sitter Entropy: *how?*

Having established the motivation to compute higher loop Feynman integrals on the Euclideanisation of $(d+1)$-dimensional de Sitter, i.e. the $(d+1)$-sphere, this thesis presents a new remarkably succinct formulation of these integrals that is well suited for analytic and numerical integration.

The propagator of a massive scalar field on S^{d+1} (of radius $\ell \equiv 1$) is [16–18]:

$$G(\hat{X}, \hat{Y}) = \frac{\Gamma(\Delta)\,\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}}\,{}_2F_1(\Delta,\, d-\Delta;\, \tfrac{d+1}{2};\, \tfrac{1+\cos\theta}{2}), \quad \cos\theta = \hat{X}\cdot\hat{Y}, \tag{1.19}$$

where the mass m and mass parameter Δ are related by

$$m^2 = \Delta\,(d-\Delta). \tag{1.20}$$

Using this propagator, the integral describing the so-called 3-melon Feynman diagram:

$\mathcal{I}_{2,3}^{[\Delta]} =$

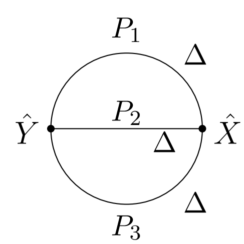


Figure 1.3: 3-Melon Feynman Diagram with mass parameter Δ

consisting of 3 massive scalar propagators of the same mass parameter Δ connecting 2 internal vertices $(\hat{X},\, \hat{Y})$ is

$$\begin{aligned}\mathcal{I}_{2,3}^{[\Delta]} &= \int \mathrm{d}\Omega_{\hat{X}}\; \mathrm{d}\Omega_{\hat{Y}}\, \left(G(\hat{X}\cdot\hat{Y})\right)^3 \\ &= \Omega_{d+1}\,\Omega_d \left(\frac{\Gamma(\Delta)\,\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}}\right)^3 \int_0^\pi \mathrm{d}\theta\, \sin^d\theta \left({}_2F_1(\Delta,\, d-\Delta;\, \tfrac{d+1}{2};\, \tfrac{1+\cos\theta}{2})\right)^3 .\end{aligned} \tag{1.21}$$

Since hypergeometric functions aren't particularly amenable to analytical integration, apart from some special cases of Δ or d in which ${}_2F_1$ reduces to simpler functions, there is no obvious way to simplify this expression further.

In flat space, Feynman integrals featuring massive scalar propagators (when written in terms of Bessel K functions $\propto (\frac{m}{|X-Y|})^{\frac{d-1}{2}}\, K_{\frac{d-1}{2}}(m\,|X-Y|)$) may look similarly daunting in position space but they are quickly remedied by using the momentum space representation of the massive scalar propagator: $\int \frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}}\, \frac{e^{-\mathrm{i}\; P\,(X-Y)}}{(P^2+m^2)}$, which allows better analytic treatment of Feynman integrals, especially when considering them in general dimensions.

Unlike in flat space, there is no global momentum space representation of massive scalar propagators on the sphere. Shifting to angular momentum space, there is an eigenfunction expansion of this propagator:

$$G(\hat{X},\, \hat{Y}) = \sum_{Y\in\Upsilon} \frac{1}{L\,(L+d) + \Delta\,(d-\Delta)}\, Y^*_{L,m}(\hat{X})\, Y_{L,m}(\hat{Y}), \tag{1.22}$$

where $Y_{L,m}$ are $(d+1)$-dimensional scalar spherical harmonics labelled by (L, m), $L\,(L+d)$ is their eigenvalue, and the sum is over their entire orthonormal basis, Υ, [19]. However, using such an expansion to rewrite the 3-melon integral results in sums over coefficients stemming from integrals over multiple spherical harmonics, i.e. Wigner 3-j and general SO$(d+2)$ 3-j symbols, and, if considering Feynman diagrams with more propagators, higher numbered-j symbols. Not only are such sums 'harder' than momentum space integrals, they aren't suited for dimensional regularisation.

Further, the integral measure itself grows more complicated with an increasing number of internal points in the Feynman diagram. Though, this can be somewhat sidestepped by clever reparameterisations like shifting to stereographic coordinates, the simplicity of the Gaussian integrals over internal points of their flat space counterparts remains unmatched.

The main result of this thesis is a set of rules (chapter 4) that allow Feynman integrals on the sphere to be read off from the Feynman diagram itself, directly resulting in an integral form which has an algorithmic solution, hence completely avoiding the aforementioned problems.

Applying these rules to the 3-melon diagram in fig. (1.3), its "incidence matrix" is

$$L_{2,3} = \left(\begin{array}{c|cc} & \hat{X} & \hat{Y} \\ \hline P_1 & \lambda_1 & -\mu_1 \\ P_2 & \lambda_2 & -\mu_2 \\ P_3 & \lambda_3 & -\mu_3 \end{array}\right), \tag{1.23}$$

where each pair of parameters $\{\lambda,\, \mu\}$ are associated with a propagator P connecting $\hat{X}$ and $\hat{Y}$. Then the Feynman integral is given by

$$\mathcal{I}^{[\Delta]}_{2,3} \propto \int_0^\infty \frac{\mathrm{d}\lambda_1}{\lambda_1}\,\frac{\mathrm{d}\lambda_2}{\lambda_2}\,\frac{\mathrm{d}\lambda_3}{\lambda_3}\,\frac{\mathrm{d}\mu_1}{\mu_1}\,\frac{\mathrm{d}\mu_2}{\mu_2}\,\frac{\mathrm{d}\mu_3}{\mu_3}\,\frac{(\lambda_1\,\lambda_2\,\lambda_3)^{d-\Delta}\,(\mu_1\,\mu_2\,\mu_3)^{\Delta}}{\big(f_{2,3}\big)^{\frac{d+2}{2}}}, \tag{1.24}$$

where $f_{2,3}$ is a polynomial in $(\lambda,\, \mu)$ defined as

$$\begin{aligned} f_{2,3} :&= \det(\mathbb{1}_2 + L^{\mathrm{T}}_{2,3}\, L_{2,3}) = \det\begin{pmatrix} 1+\lambda_1^2+\lambda_2^2+\lambda_3^2 & -(\lambda_1\,\mu_1+\lambda_2\,\mu_2+\lambda_3\,\mu_3) \\ -(\lambda_1\,\mu_1+\lambda_2\,\mu_2+\lambda_3\,\mu_3) & 1+\mu_1^2+\mu_2^2+\mu_3^2 \end{pmatrix} \\ &= 1 + \sum_{i=1}^{3}(\lambda_i^2+\mu_i^2) + (\lambda_1\,\mu_2 - \lambda_2\,\mu_1)^2 + (\lambda_2\,\mu_3 - \lambda_3\,\mu_2)^2 + (\lambda_3\,\mu_1 - \lambda_1\,\mu_3)^2. \end{aligned} \tag{1.25}$$

Following the same pattern, the n-melon Feynman integral is

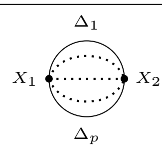
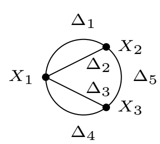
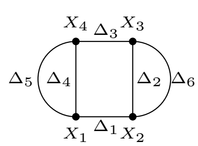
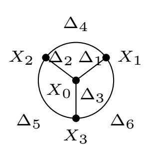
$$\begin{aligned} \mathcal{I}^{[\Delta]}_{2,n} &\propto \int_0^\infty \frac{1}{\big(f_{2,n}\big)^{\frac{d+2}{2}}} \prod_{i=1}^{n} \frac{\mathrm{d}\lambda_i}{\lambda_i}\,\frac{\mathrm{d}\mu_i}{\mu_i}\,\lambda_i^{d-\Delta}\,\mu_i^{\Delta}\,, \\ f_{2,n} &= 1 + \sum_{i=1}^{n}(\lambda_i^2+\mu_i^2) + \tfrac{1}{2}\sum_{i,j=1}^{n}(\lambda_i\,\mu_j - \lambda_j\,\mu_i)^2. \end{aligned} \tag{1.26}$$

In a similar vein, arbitrary scalar Feynman diagrams can be directly converted into parametric integrals taking the form:

$$\mathcal{I}_F \propto \int_0^\infty \left(\det(\mathbb{1} + L^{\mathrm{T}} L)\right)^{-\frac{d+2}{2}} \prod_{i=1}^{n_P} \frac{\mathrm{d}\lambda_i}{\lambda_i} \frac{\mathrm{d}\mu_i}{\mu_i} \lambda_i^{d-\Delta_i} \mu_i^{\Delta_i}, \tag{1.27}$$

where the incidence matrix, L, can be read off from the Feynman diagram. These integrands are symmetric under the exchange $(\lambda \leftrightarrow \mu)$. Some examples of incidence matrices are given in table 1.1, where the mass parameters Δ label the momenta P.

Table 1.1: Incidence matrices of some Feynman Diagrams

Feynman Diagram	Incidence matrix L
p-melon : 	$\begin{pmatrix} \lambda_1 & -\mu_1 \\ \lambda_2 & -\mu_2 \\ \cdots & \cdots \\ \lambda_p & -\mu_p \end{pmatrix}$
pacman : 	$\begin{pmatrix} \lambda_1 & -\mu_1 & 0 \\ \lambda_2 & -\mu_2 & 0 \\ \lambda_3 & 0 & -\mu_3 \\ \lambda_4 & 0 & -\mu_4 \\ 0 & \lambda_5 & -\mu_5 \end{pmatrix}$
pillbox : 	$\begin{pmatrix} \lambda_1 & -\mu_1 & 0 & 0 \\ 0 & \lambda_2 & -\mu_2 & 0 \\ 0 & 0 & \lambda_3 & -\mu_3 \\ -\mu_4 & 0 & 0 & \lambda_4 \\ \lambda_5 & 0 & 0 & -\mu_5 \\ 0 & -\mu_6 & \lambda_6 & 0 \end{pmatrix}$
peace : 	$\begin{pmatrix} \lambda_1 & -\mu_1 & 0 & 0 \\ \lambda_2 & 0 & -\mu_2 & 0 \\ \lambda_3 & 0 & 0 & -\mu_3 \\ 0 & \lambda_4 & -\mu_4 & 0 \\ 0 & 0 & \lambda_5 & -\mu_5 \\ 0 & -\mu_6 & 0 & \lambda_6 \end{pmatrix}$

It is also possible to include external legs in Feynman diagrams with minimal modifications to this formulation (section 4.1.3). The basic stucture of the Feynman integral remains the same with additional polynomials (usually 1 but possibly more for reducible diagrams) in $(\lambda,\, \mu)$ appearing in the denominator of the integrand.

This simplification to sphere scalar Feynman integral representations hinges upon the use of a "momentum-space"-like representation of the massive scalar propagator on the sphere (section 3.1):

$$G(\hat{X},\,\hat{Y}) \propto \int_0^\infty \frac{\mathrm{d}\mu}{\mu}\,\mu^\Delta \int \frac{\mathrm{d}^{d+2}P}{(2\pi)^{d+2}} \frac{e^{-\mathrm{i}\,P\,(\hat{X}-\mu\,\hat{Y})}}{P^2}, \tag{1.28}$$

where the momentum[1] P is integrated over Euclidean *flat* space $\mathbb{R}^{d+2}$ that serves as the embedding space for S^{d+1} and $\hat{X},\,\hat{Y}$ are embedding space coordinates of points on S^{d+1}.

This formulation can be extended to higher spin Feynman integrals too (section 4.1.2), by using higher spin propagators expressions similar to the "momentum-space"-like scalar propagator expression. To this end, massive and massless vector propagator expressions compatible with this formulation have been found (section 3.2).

As may have been noticed, there is a consistent pattern to these Feynman integral representations. Namely, they all appear to be multivariate generalisations of Mellin transformations, the univariate version of which, for some function $f(s)$, is

$$(\mathcal{M}\circ f)(\Delta) = \int_0^\infty \frac{\mathrm{d}s}{s}\,s^\Delta\,f(s). \tag{1.29}$$

In the case at hand, the analogue of the function f always takes the form of a polynomial raised to some arbitrary exponent. These types of integrals are a subset of so-called generalized Euler integrals. More rigorously, the structure of a generalized Euler integral always follows the pattern:

$$\mathcal{F}_{[\alpha,\,\beta]}(z) = \int_{x\in\sigma} \frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_n}{x_n} \frac{x_1^{\beta_1}\cdots x_n^{\beta_n}}{(z_1\,m_1+z_2\,m_2+\cdots+z_N\,m_N)^\alpha}, \quad m_i = x_1^{s_{1,i}}\cdots x_n^{s_{n,i}}, \tag{1.30}$$

where m_i are monomials in the variables $\{x_1,\,\cdots x_n\}$ with non-negative integer exponents, s_i, the integration contour of x avoids the singularities of the integrand, $\{\alpha,\,\beta\}$ are generic parameters, and the integral $\mathcal{F}_{[\alpha,\,\beta]}$ is a function of the coefficients z in the polynomial. Interpreting the sphere Feynman integrals to be in this form, only certain special values of $\alpha,\,\beta,\,z$ are actually physically relevant, making this generalisation appear excessive.

However, there is a huge benefit to it. Apart from a natural consequence of this description being

[1] There is no direction to this momentum, however, some direction may be appropriated to it for notational convenience. There is also no momentum conservation at the vertices.

manifest dimensional regularisation, these integrals show many scaling symmetries, by virtue of which they satisfy certain PDEs known as Gel'fand-Kapranov-Zelevinsky (GKZ) systems, [20]. If and only if these integrals are taken to be in their completely generalized form, the solution space to the aforementioned PDEs exactly equals the integrals (i.e. if the full generalisation isn't considered, the solutions to the PDEs will be a superset of the functions describing the integral).

The solutions to these systems of equations can be found algorithmically and take the form of multivariate hypergeometric series, called **$\mathcal{A}$-hypergeometric functions**, [20]. Restricting these series solutions to the relevant physical parameters solves the Feynman integral.

1.3 Past to Future Work

Starting from as early as [21], all the way up to [22–27], practically culminating in [10], which gives closed form expressions for 1-loop integrals of arbitrary field content, higher than 1-loop integrals on a de Sitter background have been recognised to be particularly challenging, with some recent results at 2-loops being [28, 29].

On the other hand, approaches to higher loop Feynman integral computations in flat space have flowed towards the idea of solving differential equations to do integrals, [30–46], eventually, as predicted by [47], converging into the Euler integral-GKZ ideal-$\mathcal{A}$-hypergeometric language, [48–52], with increasingly higher loop computations (both analytic and numerical) being explored using this technology, [53–59]. This is greatly supported by the Lee-Pomeransky representation of Feynman integrals, [60], of which the formulation suggested in this work is reminiscent.

The next step to further the Feynman integral construction presented in this thesis is building propagator expressions that are compatible with the formulation. Though very different from what is being sought, inspiration was (and more still can be) found from the many works exploring propagators in de Sitter space through a variety of other approaches. Symmetric traceless transverse (STT) eigenvectors of arbitrary spin (integer and half-integer) on a sphere, including explicit constructions thereof, with their eigenvalues and degeneracies, have been presented in [61–63]. An expansion in eigenfunctions of the Laplacian, the spherical representation of $\mathrm{SO}(d+2)$ irreps, can be used to exactly build propagators on a sphere [19, 64, 65]. Such a summation over vector eigenmodes has been used to find massive vector 2-point functions in [66].

Massive and massless vector Wightman functions have been found in [67] by requiring them to satisfy the equations of motion and then, based on requirements set by the vacuum state [18], regularity/"boundary" conditions, picking out the appropriate solution, [68, 69]. This procedure was extended to graviton 2-point functions in dS by the same authors in [70, 71]. Some more general discussions of (d)S propagators and position space vector propagator results can be found in [72–77]. Some (d)S graviton and massive spin-2 results on propagators are discussed in [78–86], many separating the propagator into transverse, vector and scalar sectors.

1.4 Overview

Chapter 2 provides a cursory review of the theory of generalized Euler integrals, GKZ systems of partial differential equations that annihilate them, and their solutions, $\mathcal{A}$-hypergeometric functions. The GKZ ideal is motivated and introduced in section 2.1. A basic explanation of ideals in Weyl algebras, specifically holonomic ideals (which GKZ ideals are), their general properties and approaches to building their solutions is given in section 2.2. The focus then shifts to the main content of this chapter, the properties and constructions of $\mathcal{A}$-hypergeometric solutions to GKZ systems in section 2.3.

The main results of chapter 3 are the propagators on S^{d+1}, eqs. (3.24), (3.83) and (3.87), expressed as quotients of momentum space representations of the Euclidean flat space massless propagators in one higher dimensional embedding space, $\mathbb{R}^{d+2}$. The construction of massive scalar propagators, starting from simplistic observation, $\mathbb{R}^{d+2}/\mathbb{R}_+ = S^{d+1}$, is discussed in section 3.1. In a similar vein but with more care devoted to the details of the embedding space representation of the Mellin transformed basis of fields, described in sections A.1 and A.2, the vector propagators are built in section 3.2 by gauge fixing the higher dimensional massless vector field in different ways (section 3.2.1) to get to the massive (section 3.2.2) and massless (section 3.2.3) cases.

Chapter 4 is devoted to the construction of sphere Feynman integrals in embedding space. Section 4.1.1 contains the set of rules to convert any scalar Feynman diagram into its generalized Euler integral form. In section 4.1.2, it is shown that higher spin integrals can also be turned into generalized Euler integrals, as long as the higher spin propagators featured in the Feynman diagrams (both internal and external) can be represented as parameter derivatives of the scalar propagator in embedding space i.e. eq. (3.29). The construction of a "master" generalized Euler integral form of any Feynman diagram is described. Based on observations made in sections C.1.1 and C.1.2 and some experimental explorations into higher spin propagator constructions[2], it is hypothesized that propagators of arbitrary spin can be represented in this embedding space form. Granting this hypothesis, the master integral form encodes information of all possible Feynman diagrams with the same graphical structure/incidence matrix but different spin propagators. As described in section 4.1.3, all (scalar) correlation functions can be laconically written as eq. (4.30), with the same caveats regarding extentions to higher spin applying. Constructions of scalar Feynman integrals upto 3-loops are given in section 4.2, and the relation of vector Feynman integrals to the underlying scalar Feynman integrals are discussed in section 4.3. A short review of the current status and the next few concrete steps is presented in chapter 5.

Readers are also referred to section A.1 for coordinate conventions consistently maintained throughout this text, section A.2 for details on the Mellin transformation used to convert (scale invariant) fields in $\mathbb{R}^{d+2}$ to fields on S^{d+1}, table A.1 for an inexhaustive notation index, and table A.2 for some formulae that may be considered to be implicitly known.

[2]It is trivial to write a multitude of embedding space forms of higher spin propagators but the gauge conditions being portrayed would be obscure. In the language of [70], only the transverse part of the propagator can be established to be correct. For actual computational purposes, such propagators would be incomplete.

Chapter 2: Generalized Euler Integrals and $\mathcal{A}$-hypergeometric Functions

Generalized Euler integrals form a class of integrals that appears conspicuously often in the study of quantum field theories perturbatively viewed as sums over Feynman integrals/diagrams/graphs, in the path integral formulation

$$\mathcal{Z} = \int \mathrm{d}\Phi \, e^{-S_{\mathrm{Eucl}}[\Phi]}, \tag{2.1}$$

and in most applications of the principle of least action based on expansions around relevant physical saddle points.

The term generalized Euler integrals was introduced in [20], along with the algorithmically solvable system of equations (and their solution space) governing them, the $\mathcal{A}$-hypergeometric system (and functions).

Famously appearing in the computation of periods determining the complex structure of Calabi - Yau manifolds as a means of deriving and solving Picard - Fuchs equations, the prototypical example being the construction of the mirror of the quintic Calabi - Yau manifold on $\mathbb{CP}^4$, the application of this theory is far from new to physicists [87–94].

Generalized Feynman integrals in flat space are Euler integrals considered in specific limits and over contours that are physically relevant. They satisfy systems of linear holonomic partial differential equations [47], their Laurent series expansion coefficients correspond to their periods [49], and can be expressed as Mellin-Barnes integrals, eq. (2.132), which have representations as hypergeometric series [50], with their correspondence to $\mathcal{A}$-hypergeometric systems discussed in more detail, and used as a means of studying their singular loci (Landau varieties) and explicit computation in [48, 51–55, 57–59].

Given the prevalence of generalized Euler integrals, eq. (2.2), in physics, and specifically within the context of the current text, in the computation of Feynman integrals on a spherical background and by analytic continuation, on de Sitter space, application of the theory of $\mathcal{A}$ - hypergeometric systems becomes indispensible, especially when seeking consistency in the ability to solve and evaluate, analytically and/or numerically, the aforementioned integrals.

This chapter reviews the fundamental ideas and basic building blocks of the theory of Euler integrals and $\mathcal{A}$ - hypergeometric systems, mainly based on [20, 95, 96], with the inclusion of some further generalisions proven in [97]. Far from exhaustive, it can at best be considered an introduction or refresher to the topic and a means of establishing prerequisite understanding and notational conventions for the following chapters 3 and 4, which heavily reference and utilize these ideas.

Section 2.1 begins with intiutive examples featuring some prevalent symmetry structures in physically relevant GKZ systems, then proceeds to introduce, motivate and explain the construction of $\mathcal{A}$-Hypergeometric ideal and its related terminology, and finally presents a simplified generic solution algorithm, all in manner to allow practical application, satisfying the aforementioned prerequisite. Section 2.2 discusses ideals in Weyl algebras and their solution spaces in some greater generality to highlight the special properties of $\mathcal{A}$-Hypergeometric ideals. Section 2.3 shows some approaches to finding and constructing solutions to GKZ systems and is exposited through a running example, the physical significance of which will become apparent in chapter 4.

For further reading and detailed discussions, also see [97–100]. A recent review is [101]. GKZ ideals can be restructured and studied as Pfaffian systems, as shown in [102, 103].

2.1 $\mathcal{A}$ - Hypergeometric System

Euler integrals can be symbolically written as

$$I(z) = \int_\sigma \frac{\mathrm{d}x}{x} \frac{x^\beta}{P(x;\, A;\, z)^\alpha}. \tag{2.2}$$

This compact notation, hiding within it many perturbative possibilities of eq. (2.1), is decompressed as follows:

1. The set of integration variables, $x_1,\, x_2, \cdots x_n \in \mathbb{C}$, is denoted by x. σ is a positively oriented closed contour within the domain of x minus the zeroes of the polynomials, here $\mathbb{C}^n \backslash \{P = 0\}$. $\alpha,\, \beta$ are vectors of complex parameters, which may be generic or fixed to some arbitrary value. The non-zero variables $z \in \mathbb{C}\backslash\{0\}$ parameterise the coefficients of the polynomials in x, P.

2. A multivariate monomial in n variables x_i with exponents β_i in the multi-index notation is

$$x^\beta \equiv \prod_{i=1}^{n} x_i^{\beta_i} = \prod_{i=1}^{n} e^{\beta_i \,\log x_i}, \quad \beta \in \mathbb{C}^n. \tag{2.3}$$

3. A polynomial, P, in n variables x_i with N monomial terms can be represented by an $n \times N$ matrix A with non-negative integer entries, where each column consists of the exponents of the multivariate monomials, and an N-dimensional coefficient list/vector z.

$$P(x\,;\, A\,;\, z) = \sum_{k=1}^{N} z_k \, x^{A_k}, \quad A = \left(A_1 \; A_2 \; \cdots \; A_N \right), \quad A_i \in \mathbb{N}_0^n \tag{2.4}$$

   P is a generic (usually sparse) polynomial and its associated matrix representation, A, is the support. P can be interpreted as a function of the coefficients z. Generic polynomials and their supports can be used interchangeably, with the former usually being preferred when the coefficients are fixed and the latter when they are variable. For example,

$$A = \left(\begin{smallmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \end{smallmatrix}\right), \quad z = \{1,\, 2,\, 4\}, \quad P(x\,;\, A\,;\, z) = 1 + 2\, x_1^2 + 4\, x_1\, x_2^2 \tag{2.5}$$

   Within the context of eq. (2.2), $P(x;\, A;\, z)^\alpha$ is understood to be in multi-index notation, i.e. a product of arbitrarily many polynomials raised to powers α, each defined by some matrices A

and monomial coefficients z.

$$P(x;\, A;\, z)^{\alpha} = \prod_{l=1}^{L} P(x;\, A^{(l)};\, z^{(l)})^{\alpha_l} \tag{2.6}$$

4. Usage of such notation isn't limited to just variables but can be extended to operators and functions also, e.g.

$$\partial_x^w = \prod_{i=1}^{n} \partial_{x_i}^{w_i}, \quad \Gamma(w) = \prod_{i=1}^{n} \Gamma(w_i), \quad P(\partial_x\,;\, (\begin{smallmatrix} 2 & 3 \\ 0 & 1 \end{smallmatrix})\,;\, z) = z_1\, \partial_{x_1}^2 + z_2\, \partial_{x_1}^3\, \partial_{x_2} \tag{2.7}$$

or as used in the case at hand $\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n}$.

5. The Euler operator, θ_x, is defined as

$$\theta_x := x\,\partial_x = \partial_{\log x}, \quad \theta_x\, x^{\alpha} = \alpha\, x^{\alpha}, \quad \theta_x\, (\log x)^{\alpha} = \alpha\, (\log x)^{\alpha - 1} \tag{2.8}$$

The rising and falling Pochhammer symbols or factorials are

$$\begin{aligned} a^{(n)} &:= a\,(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \\ a_{(n)} &:= a\,(a-1)\cdots(a-n+1) = \frac{\Gamma(a)}{\Gamma(a-n)} \quad \text{resp.} \end{aligned} \tag{2.9}$$

When it is obvious which variable is being referred to, operators like ∂_{x_i}, θ_{z_j} will be written as ∂_i, θ_j instead. Altogether, a falling form of the Euler operator that often graces discussions involving Euler integrals can be written in an extremely condensed and convenient form

$$\theta_{(m)} = \prod_{i=1}^{n} \prod_{j=0}^{m-1} (\theta_{x_i} - j). \tag{2.10}$$

Interpreting these integrals as functions of the coefficients, z, of the polynomials in their integrand, which are allowed to be generically valued in their domain, is what makes the Euler integrals generalized. They satisfy a class of holonomic linear partial differential equations in z, often dubbed GKZ systems, that admit holomorphic solutions. The solution space, when represented in terms of generalized hypergeometric functions, is referred to as and by $\mathcal{A}$-hypergeometric functions, where $\mathcal{A}$ is a matrix describing the set of polynomials featured in the integral.

2.1.1 Introductory Examples

Euler integrals are far from unfamiliar in their form, and the partial differential equations they satisfy can be found by intuitive arguments based on homogeneity and scaling symmetries. So it is instructive to first consider some examples which at least partially showcase the benefits of representing Feynman integrals in this more generalized and superficially superfluous format.

Example 2.1.1. A needlessly complicated representation of a bivariate monomial $I(z_0,\, z_1)$ is

$$I(z_0,\, z_1) = \int_\sigma \frac{\mathrm{d}x}{x} \, \frac{x^\beta}{(z_0 + z_1\, x^2)^\alpha}, \quad z_0,\, z_1 \in \mathbb{C}\backslash\{0\} \tag{2.11}$$

where σ is a closed contour in $\mathbb{C}$, and α, β are complex valued parameters. By rescaling[1] the variable x by the factor $\sqrt{\frac{z_0}{z_1}}$, $I(z_0,\, z_1)$ simplifies to

$$I(z_0,\, z_1) = z_0^{-\alpha+\frac{\beta}{2}}\, z_1^{-\frac{\beta}{2}} \int_\sigma \frac{\mathrm{d}x}{x}\, \frac{x^\beta}{(1+x^2)^\alpha} = C_\sigma\, z_0^{-\alpha+\frac{\beta}{2}}\, z_1^{-\frac{\beta}{2}} \tag{2.12}$$

where C_σ is a proportionality constant dependent on the integration contour. Identification of the scaling symmetries of this integral leads to the same conclusion.

$$I(\lambda\, z_0,\, \lambda\, z_1) = \lambda^{-\alpha}\, I(z_0,\, z_1), \quad I(z_0,\, \lambda\, z_1) = \lambda^{-\frac{\beta}{2}}\, I(z_0,\, z_1) \tag{2.13}$$

form a basis of the scaling symmetries of $I(z_0,\, z_1)$ and can equivalently be represented as

$$D_0 \circ I(z_0,\, z_1) = D_1 \circ I(z_0,\, z_1) = 0, \quad D_0 = \theta_{z_0} + \theta_{z_1} + \alpha, \quad D_1 = \theta_{z_1} + \tfrac{\beta}{2}. \tag{2.14}$$

[1] At $z_0\, z_1 = 0$, the integral becomes singular and reducible, and doesn't merit discussion in the present context.

All scaling properties of $I(z_0,\, z_1)$ are spanned by linear combinations of $D_0,\, D_1,$ better represented in the basis:

$$D_0' = \theta_{z_0} + \alpha - \tfrac{\beta}{2}, \quad D_1' = \theta_{z_1} + \tfrac{\beta}{2}. \tag{2.15}$$

The simultaneous solution to these PDEs gives the exponents of $z_0,\, z_1$. Thus the solution space of the integral $I(z_0,\, z_1)$ is once again found to be eq. (2.12).

When the integral is convergent, some open integration intervals can also be considered. For example, when $\mathrm{Re}(\beta) > 1$, $\mathrm{Re}(\alpha - \frac{\beta}{2}) > 0$, and $z_0,\, z_1 \in \mathbb{R}^+$, some integration intervals that are more commonly seen in physics and their corresponding proportionality constants are

$$\begin{aligned} &\sigma_1 = x \in (0,\, \infty), && C_{\sigma_1} = \frac{\Gamma(\frac{\beta}{2})\,\Gamma(\alpha - \frac{\beta}{2})}{2\,\Gamma(\alpha)} \\ &\sigma_2 = x \in (-\infty,\, \infty), && C_{\sigma_2} = (1 - (-1)^{\beta})\, C_{\sigma_1} \qquad \text{etc.} \end{aligned} \tag{2.16}$$

Example 2.1.2. $I(z)$, $z \in \mathbb{C}$, is an integral over a closed contour in $\mathbb{C}$, σ:

$$I(z) = \int_\sigma \frac{\mathrm{d}x}{x}\, \frac{x^\beta}{(z_1 + z_2\, x^2)^{\alpha_1}\, (z_3 + z_4\, x^2)^{\alpha_2}}, \quad z_2 \neq z_4, \quad z_1,\, z_2,\, z_3,\, z_4 \in \mathbb{C}\backslash\{0\}. \tag{2.17}$$

The scaling behaviour of $I(z)$, also known as homogeneity conditions, are represented by

$$(\theta_1 + \theta_2 + \alpha_1) \circ I = (\theta_3 + \theta_4 + \alpha_2) \circ I = (\theta_2 + \theta_4 + \tfrac{\beta}{2}) \circ I = 0 \tag{2.18}$$

which expectedly give rise to a set of linear equations between the exponents of z, allowing the relative dependence of $I(z)$ on any 3 of $z_1,\, z_2,\, z_3,\, z_4$ to be fixed. A solution basis to eq. (2.18) is

$$I^{(1)}(z) = z_2^{-\alpha_1}\, z_3^{\frac{\beta}{2} - \alpha_1 - \alpha_2}\, z_4^{\alpha_1 - \frac{\beta}{2}}\, \tilde{I}^{(1)}(\tfrac{z_1\, z_4}{z_2\, z_3}) \tag{2.19}$$

$$I^{(2)}(z) = z_1^{-\alpha_1}\, z_3^{\frac{\beta}{2} - \alpha_2}\, z_4^{-\frac{\beta}{2}}\, \tilde{I}^{(2)}(\tfrac{z_1\, z_4}{z_2\, z_3}) \tag{2.20}$$

$$I^{(3)}(z) = z_1^{-\alpha_1 - \alpha_2 + \frac{\beta}{2}}\, z_2^{\alpha_2 - \frac{\beta}{2}}\, z_4^{-\alpha_2}\, \tilde{I}^{(3)}(\tfrac{z_1\, z_4}{z_2\, z_3}) \tag{2.21}$$

$$I^{(4)}(z) = z_1^{\frac{\beta}{2} - \alpha_1}\, z_2^{-\frac{\beta}{2}}\, z_3^{-\alpha_2}\, \tilde{I}^{(4)}(\tfrac{z_1\, z_4}{z_2\, z_3}) \tag{2.22}$$

for some as yet unknown set of functions $\tilde{I}$. Vectors consisting of the exponents of z in the overall factors of these representations are called roots. Roots are vectors in N-dimensional affine space. In this case, there are four 4 dimensional roots:

$$\begin{aligned} r_1 &= \{0,\, -\alpha_1,\, \tfrac{\beta}{2} - \alpha_1 - \alpha_2,\, \alpha_1 - \tfrac{\beta}{2}\}, \quad & r_2 &= \{-\alpha_1,\, 0,\, \tfrac{\beta}{2} - \alpha_2,\, -\tfrac{\beta}{2}\}, \\ r_3 &= \{-\alpha_1 - \alpha_2 + \tfrac{\beta}{2},\, \alpha_2 - \tfrac{\beta}{2},\, 0,\, -\alpha_2\}, & r_4 &= \{\tfrac{\beta}{2} - \alpha_1,\, -\tfrac{\beta}{2},\, -\alpha_2,\, 0\} \end{aligned} \tag{2.23}$$

with associated family of the solutions:

$$I^{(i)}(z) = z^{r_i}\, \tilde{I}^{(i)}(z_t), \quad z_t \equiv \frac{z_1\, z_4}{z_2\, z_3}. \tag{2.24}$$

The dependence of $\tilde{I}^{(i)}$ on just z_t and not the individual variables $z_1\, z_2,\, z_3,\, z_4$ can be confirmed by appropriately rescaling the variable x. For example, $x \to x\sqrt{\frac{z_1}{z_2}}$ implies

$$I^{(4)}(z) = z_1^{\frac{\beta}{2}-\alpha_1}\, z_2^{-\frac{\beta}{2}}\, z_3^{-\alpha_2}\, \tilde{I}^{(4)}(z_t), \quad \tilde{I}^{(4)}(z_t) = \int_\sigma \frac{\mathrm{d}x}{x} \frac{x^\beta}{(1+x^2)^{\alpha_1}\,(1+z_t\,x^2)^{\alpha_2}} \tag{2.25}$$

where $\tilde{I}^{(4)}(z_t)$ can be seen to remain unchanged under T,

$$T:\, (z_1,\, z_2,\, z_3,\, z_4) \to (\lambda_1\, z_1,\, \lambda_2\, z_2,\, \lambda_3\, z_3,\, \tfrac{\lambda_2\,\lambda_3}{\lambda_1}\, z_4). \tag{2.26}$$

Scale transformations like T are called torus actions. z_t represents the relative scaling symmetries of the variables z. It remains invariant under $T:\, z_t \to z_t$, and is expectedly called a toric invariant. Similarly, all $\tilde{I}$ can be verified to be functions of only the toric invariant z_t. I is annihilated by the class of operators

$$\partial_T^n \equiv \partial_{z_1}^n\, \partial_{z_4}^n - \partial_{z_2}^n\, \partial_{z_3}^n \;\forall\, n \in \mathbb{N}. \tag{2.27}$$

This can be verified by writing the integral I in its Schwinger parametric form.

$$\begin{aligned}
I &= \int_0^\infty \frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1\,y_2}\,\frac{y_1^{\alpha_1}}{\Gamma(\alpha_1)}\,\frac{y_2^{\alpha_2}}{\Gamma(\alpha_2)}\int_\sigma \frac{\mathrm{d}x}{x}\,x^\beta\,e^{-y_1\,(z_1+z_2\,x^2)}\,e^{-y_2\,(z_3+z_4\,x^2)} \\
\partial_T^n \circ I &= \int_0^\infty \frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1\,y_2}\,\frac{y_1^{\alpha_1}}{\Gamma(\alpha_1)}\,\frac{y_2^{\alpha_2}}{\Gamma(\alpha_2)}\int_\sigma \Big(\partial_{z_1}^n\,\partial_{z_4}^n - \partial_{z_2}^n\,\partial_{z_3}^n\Big)\,\frac{\mathrm{d}x}{x}\,x^\beta\,e^{-y_1\,(z_1+z_2\,x^2)}\,e^{-y_2\,(z_3+z_4\,x^2)} \\
&= \int_0^\infty \frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1\,y_2}\,\frac{y_1^{\alpha_1}}{\Gamma(\alpha_1)}\,\frac{y_2^{\alpha_2}}{\Gamma(\alpha_2)}\int_\sigma \frac{\mathrm{d}x}{x}\,x^\beta\,e^{-y_1\,(z_1+z_2\,x^2)}\,e^{-y_2\,(z_3+z_4\,x^2)} \\
&\quad \times \Big((-y_1)^n\,(-y_2\,x^2)^n - (-y_1\,x^2)^n(-y_2)^n\Big) = 0.
\end{aligned} \tag{2.28}$$

Assuming that $I(z)$ has a Laurent series expansion

$$I(z) \equiv z^s \sum_{n\in\mathbb{N}} c_n\,z_t^n = \sum c_n\,z_1^{s_1+n}\,z_2^{s_2-n}\,z_3^{s_3-n}\,z_4^{s_4+n} \tag{2.29}$$

such that it satisfies the class of PDEs $\partial_T^n \circ I = 0$,

$$\partial_T \circ I = \frac{z^s}{z_1\,z_4}\sum_{n\in\mathbb{N}} c_n\,(s_1+n)\,(s_4+n)\,z_t^n - \frac{z^s}{z_2\,z_3}\sum_{n\in\mathbb{N}} c_n\,(s_2-n)\,(s_3-n)\,z_t^n = 0 \tag{2.30}$$

the following relations need to be satisfied:

$$\begin{aligned}
&z^s \sum_{n\in\mathbb{N}} \Big(c_n\,(s_1+n)\,(s_4+n) - c_{n-1}\,(s_2-n+1)\,(s_3-n+1)\Big)\,z_t^n = 0 \\
&z^s\,c_0\,s_1\,s_4 = 0.
\end{aligned} \tag{2.31}$$

Since z_t is generic and $\not\equiv 0$, this implies either s_1 or $s_4 = 0$ and

$$\frac{c_{n+1}}{c_n} = \frac{(s_2-n)\,(s_3-n)}{(s_1+n+1)\,(s_4+n+1)} \implies c_n = c_0\,\frac{\Gamma(n-s_2)\,\Gamma(n-s_3)}{\Gamma(n+s_1+1)\,\Gamma(n+s_4+1)}. \tag{2.32}$$

The representation of $I(z)$ corresponding to $s_1 = 0$ is eq. (2.19) with the root r_1 given in eq. (2.23).

Thus, $I^{(1)}(z)$ is proportional to

$$
\begin{aligned}
I^{(1)}(z) = \; & z_2^{-\alpha_1}\, z_3^{\frac{\beta}{2}-\alpha_1-\alpha_2}\, z_4^{\alpha_1-\frac{\beta}{2}} \sum_{n\in\mathbb{N}} c_0\, \frac{\Gamma(n+\alpha_1)\,\Gamma(n-\frac{\beta}{2}+\alpha_1+\alpha_2)}{\Gamma(n+\alpha_1-\frac{\beta}{2}+1)}\, \frac{z_t^n}{n!} \\
& \propto z_2^{-\alpha_1}\, z_3^{\frac{\beta}{2}-\alpha_1-\alpha_2}\, z_4^{\alpha_1-\frac{\beta}{2}}\; {}_2F_1(\alpha_1,\, \alpha_1+\alpha_2-\tfrac{\beta}{2};\, \alpha_1-\tfrac{\beta}{2}+1;\, z_t).
\end{aligned}
\tag{2.33}
$$

Another representation of $I(z)$, eq. (2.22), with the root r_4 is similarly produced by the alternate solution $s_4 = 0$:

$$
I^{(4)}(z) \propto z_1^{\frac{\beta}{2}-\alpha_1}\, z_2^{-\frac{\beta}{2}}\, z_3^{-\alpha_2}\; {}_2F_1(\tfrac{\beta}{2},\, \alpha_2;\, \tfrac{\beta}{2}-\alpha_1+1;\, z_t). \tag{2.34}
$$

The series expansion of $I(z)$ presented in eq. (2.29) is assumed to be in non-negative powers of z_t. An expansion in non-positive powers,

$$
I(z) \equiv z^s \sum_{n\in\mathbb{N}} c_n\, z_t^{-n} = \sum_{n\in\mathbb{N}} c_n\, z_1^{s_1-n}\, z_2^{s_2+n}\, z_3^{s_3+n}\, z_4^{s_4-n} \tag{2.35}
$$

that satisfies $\partial_T^n \circ I = 0$, analogously results in the conditions $s_2\, s_3 = 0$ and

$$
c_n = c_0\, \frac{\Gamma(n-s_1)\,\Gamma(n-s_4)}{\Gamma(n+s_2+1)\,\Gamma(n+s_3+1)}. \tag{2.36}
$$

Comparing eqs. (2.20) and (2.21) and their corresponding roots r_2, r_3 to the cases $s_2 = 0$, $s_3 = 0$ respectively implies

$$
\begin{aligned}
& I^{(2)}(z) \propto z_1^{-\alpha_1}\, z_3^{\frac{\beta}{2}-\alpha_2}\, z_4^{-\frac{\beta}{2}}\; {}_2F_1(\alpha_1,\, \tfrac{\beta}{2};\, \tfrac{\beta}{2}-\alpha_2+1;\, z_t^{-1}) \\
& I^{(3)}(z) \propto z_1^{-\alpha_1-\alpha_2+\frac{\beta}{2}}\, z_2^{\alpha_2-\frac{\beta}{2}}\, z_4^{-\alpha_2}\; {}_2F_1(\alpha_1+\alpha_2-\tfrac{\beta}{2},\, \alpha_2;\, \alpha_2-\tfrac{\beta}{2}+1;\, z_t^{-1}).
\end{aligned}
\tag{2.37}
$$

Thus, the integral $I(z)$ takes the form of 4 independent solutions of the Gauss hypergeometric

ODE, around $z_t = 0,\, \infty$:

$$\begin{aligned} I(z) &= C_1\, z_2^{-\alpha_1}\, z_3^{\frac{\beta}{2}-\alpha_1-\alpha_2}\, z_4^{\alpha_1-\frac{\beta}{2}}\; {}_2F_1(\alpha_1,\, \alpha_1+\alpha_2-\tfrac{\beta}{2};\, \alpha_1-\tfrac{\beta}{2}+1;\, z_t) \\ &+ C_2\, z_1^{-\alpha_1}\, z_3^{\frac{\beta}{2}-\alpha_2}\, z_4^{-\frac{\beta}{2}}\; {}_2F_1(\alpha_1,\, \tfrac{\beta}{2};\, \tfrac{\beta}{2}-\alpha_2+1;\, \tfrac{1}{z_t}) \\ &+ C_3\, z_1^{-\alpha_1-\alpha_2+\frac{\beta}{2}}\, z_2^{\alpha_2-\frac{\beta}{2}}\, z_4^{-\alpha_2}\; {}_2F_1(\alpha_2,\, \alpha_1+\alpha_2-\tfrac{\beta}{2};\, \alpha_2-\tfrac{\beta}{2}+1;\, \tfrac{1}{z_t}) \\ &+ C_4\, z_1^{\frac{\beta}{2}-\alpha_1}\, z_2^{-\frac{\beta}{2}}\, z_3^{-\alpha_2}\; {}_2F_1(\alpha_2,\, \tfrac{\beta}{2};\, \tfrac{\beta}{2}-\alpha_1+1;\, z_t) \end{aligned} \tag{2.38}$$

where $C_{1,\,2,\,3,\,4}$ are proportionality constants that are dependent on the integration contour.

Example 2.1.3. The simplest class of Euler integrals in 2 variables with 2 toric invariants is represented by

$$I(z) = \int_\sigma \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{x_1\, x_2}\, \frac{x_1^{\beta_1}\, x_2^{\beta_2}}{(z_1 + z_2\, x_1\, x_2 + z_3\, x_1^2 + z_4\, x_2^2 + z_5\, x_1^2\, x_2^2)^\alpha}. \tag{2.39}$$

The homogeneity conditions are

$$\begin{aligned} &D_1 = \sum_{i=1}^{5} \theta_i + \alpha, \quad D_2 = \theta_2 + 2\,\theta_3 + 2\,\theta_5 + \beta_1, \quad D_3 = \theta_2 + 2\,\theta_4 + 2\,\theta_5 + \beta_2 \\ &D_1 \circ I = D_2 \circ I = D_3 \circ I = 0. \end{aligned} \tag{2.40}$$

The toric symmetries can be found by representing this integral in its Schwinger parametric form,

$$I(z) = \int_\sigma \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{x_1\, x_2}\, x_1^{\beta_1}\, x_2^{\beta_2} \int_0^\infty \frac{\mathrm{d}y}{y}\, \frac{y^\alpha}{\Gamma(\alpha)}\, e^{-y\,(z_1+z_2\,x_1\,x_2+z_3\,x_1^2+z_4\,x_2^2+z_5\,x_1^2\,x_2^2)} \tag{2.41}$$

and finding relations between its partial derivatives wrt the variables z:

$$\begin{aligned} &\partial_{z_1} I(z) = (-y)\, I, \quad \partial_{z_2} I(z) = (-y\, x_1\, x_2)\, I, \quad \partial_{z_3} I(z) = (-y\, x_1^2)\, I, \\ &\partial_{z_4} I(z) = (-y\, x_2^2)\, I, \quad \partial_{z_5} I(z) = (-y\, x_1^2\, x_2^2)\, I \end{aligned} \tag{2.42}$$

where for the sake of notational brevity I has been used instead of the entire integrand. All toric symmetries of this integral:

$$
\begin{aligned}
\partial_{z_1}\,\partial_{z_5}\circ I &= (-y)\,(-y\,x_1^2\,x_2^2)\,I = (-y\,x_1^2)\,(-y\,x_2^2)\,I = \partial_{z_3}\,\partial_{z_4}\circ I(z) \\
\partial_{z_1}\,\partial_{z_5}\circ I &= (-y)\,(-y\,x_1^2\,x_2^2)\,I = (-y\,x_1x_2)^2\,I = \partial_{z_2}^2\circ I(z)
\end{aligned}
\tag{2.43}
$$

can be represented by a basis of 2 differential operators:

$$
(\partial_3\,\partial_4 - \partial_1\,\partial_5)\circ I = (\partial_1\,\partial_5 - \partial_2^2)\circ I = 0 \tag{2.44}
$$

corresponding to a working basis of toric invariants

$$
t_1 = \frac{z_3\,z_4}{z_1\,z_5},\quad t_2 = \frac{z_1\,z_5}{z_2^2}. \tag{2.45}
$$

Alternate bases of PDEs and toric invariants are entirely equivalent choices as long as they commute with each other and span all toric symmetries of the integral, e.g.

$$
(\partial_1\,\partial_5 - \partial_3\,\partial_4)\circ I = (\partial_3\,\partial_4 - \partial_2^2)\circ I = 0,\quad t_1' = \frac{z_1\,z_5}{z_3\,z_4} = t_1^{-1},\quad t_2' = \frac{z_3\,z_4}{z_2^2} = t_1\,t_2 \tag{2.46}
$$

$$
(\partial_2^2 - \partial_3\,\partial_4)\circ I = (\partial_1\,\partial_5 - \partial_2^2)\circ I = 0,\quad t_1'' = \frac{z_2^2}{z_3\,z_4} = (t_1\,t_2)^{-1},\quad t_2'' = \frac{z_1\,z_5}{z_2^2} = t_2. \tag{2.47}
$$

The linear equations relating the exponents of the variables implied by eq. (2.40) can be used to fix 3 of 5 exponents in 4 (not $\binom{5}{3} = 10$) different ways. The choice of the dependent and independent variables isn't completely free in this scenario (unlike example 2.1.2) and instead fixed to certain combinations. There are as many roots as the number of these allowed combinations and just as many linearly independent series solutions. Using eq. (2.45) as the basis of toric invariants,

$$
\begin{aligned}
I_1(z) &= z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\, z_2^{-\beta_2}\, z_3^{\frac{\beta_2-\beta_1}{2}}\, \tilde{I}_1(z), \quad \tilde{I}_1(z) = \int_\sigma \frac{\mathrm{d}x}{x}\frac{x^\beta}{(1+x_1\,x_2+x_1^2+t_1\,t_2\,x_2^2+t_2\,x_1^2\,x_2^2)^\alpha} \\
I_2(z) &= z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\, z_2^{-\beta_1}\, z_4^{\frac{\beta_1-\beta_2}{2}}\, \tilde{I}_2(z), \quad \tilde{I}_2(z) = \int_\sigma \frac{\mathrm{d}x}{x}\frac{x^\beta}{(1+x_1\,x_2+t_1\,t_2\,x_1^2+x_2^2+t_2\,x_1^2\,x_2^2)^\alpha} \\
I_3(z) &= z_2^{\beta_1-2\alpha}\, z_3^{\frac{\beta_2-\beta_1}{2}}\, z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\, \tilde{I}_3(z), \quad \tilde{I}_3(z) = \int_\sigma \frac{\mathrm{d}x}{x}\frac{x^\beta}{(t_2+x_1\,x_2+x_1^2+t_1\,t_2\,x_2^2+x_1^2\,x_2^2)^\alpha} \\
I_4(z) &= z_2^{\beta_2-2\alpha}\, z_4^{\frac{\beta_1-\beta_2}{2}}\, z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\, \tilde{I}_4(z), \quad \tilde{I}_4(z) = \int_\sigma \frac{\mathrm{d}x}{x}\frac{x^\beta}{(t_2+x_1\,x_2+t_1\,t_2\,x_1^2+x_2^2+x_1^2\,x_2^2)^\alpha}
\end{aligned}
\tag{2.48}
$$

corresponding to the roots

$$
\begin{aligned}
r_1 &= \{\tfrac{\beta_1+\beta_2}{2}-\alpha,\, -\beta_2,\, \tfrac{\beta_2-\beta_1}{2},\, 0,\, 0\}, \quad r_2 = \{\tfrac{\beta_1+\beta_2}{2}-\alpha,\, -\beta_1,\, 0,\, \tfrac{\beta_1-\beta_2}{2},\, 0\} \\
r_3 &= \{0,\, \beta_1-2\alpha,\, \tfrac{\beta_2-\beta_1}{2},\, 0,\, \alpha-\tfrac{\beta_1+\beta_2}{2}\}, \quad r_4 = \{0,\, \beta_2-2\alpha,\, 0,\, \tfrac{\beta_1-\beta_2}{2},\, \alpha-\tfrac{\beta_1+\beta_2}{2}\}.
\end{aligned}
\tag{2.49}
$$

All non-trivial behaviour is confined to $\tilde{I}(z)$ which is purely a function of 2 toric invariants regardless of the representation. So it isn't unreasonable to expect I to have a Laurent series expansion in powers of the toric invariants of the form

$$
I_i(z) \equiv z^{r_i} \sum c_n\, t^n = z^{r_i} \sum c_{n_1,\,n_2}\, t_1^{n_1}\, t_2^{n_2}. \tag{2.50}
$$

The above series representation of I exhibits the toric symmetries in eq. (2.44) when

$$
\begin{aligned}
\tilde{I}_1(z) &\propto F_4(\tfrac{\beta_2}{2}\,;\, \tfrac{\beta_2+1}{2}\,;\, \tfrac{\beta_2-\beta_1}{2}+1\,;\, \tfrac{\beta_1+\beta_2}{2}-\alpha+1\,;\, 4\,t_1\,;\, 4\,t_2) \\
\tilde{I}_2(z) &\propto F_4(\tfrac{\beta_1}{2}\,;\, \tfrac{\beta_1+1}{2}\,;\, \tfrac{\beta_1+\beta_2}{2}+1\,;\, \tfrac{\beta_1-\beta_2}{2}-\alpha+1\,;\, 4\,t_1\,;\, 4\,t_2) \\
\tilde{I}_3(z) &\propto F_4(\tfrac{2\alpha-\beta_1}{2}\,;\, \tfrac{2\alpha-\beta_1+1}{2}\,;\, \tfrac{\beta_2-\beta_1}{2}+1\,;\, \alpha-\tfrac{\beta_2-\beta_1}{2}+1\,;\, 4\,t_1\,;\, 4\,t_2) \\
\tilde{I}_4(z) &\propto F_4(\tfrac{2\alpha-\beta_2}{2}\,;\, \tfrac{2\alpha-\beta_1+1}{2}\,;\, \tfrac{\beta_1-\beta_2}{2}+1\,;\, \alpha-\tfrac{\beta_1-\beta_2}{2}+1\,;\, 4\,t_1\,;\, 4\,t_2)
\end{aligned}
\tag{2.51}
$$

where $F_4(\alpha_1\,;\,\alpha_2\,;\,\beta_1\,;\,\beta_2\,;\,t_1\,;\,t_2)$ is a Horn hypergeometric function defined as

$$\begin{aligned} F_4(\alpha\,;\,\beta\,;\,t) &= \sum_{n_1,\,n_2=0}^{\infty} \frac{\alpha_1^{(n_1+n_2)}\,\alpha_2^{(n_1+n_2)}}{\beta_1^{(n_1)}\,\beta_2^{(n_2)}}\,\frac{t_1^{n_1}}{n_1!}\,\frac{t_2^{n_2}}{n_2!} \\ &= \frac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \sum_{n_1,\,n_2=0}^{\infty} \frac{\Gamma(\alpha_1+n_1+n_2)\,\Gamma(\alpha_2+n_1+n_2)}{\Gamma(\beta_1+n_1)\,\Gamma(\beta_2+n_2)}\,\frac{t_1^{n_1}}{n_1!}\,\frac{t_2^{n_2}}{n_2!} \end{aligned} \tag{2.52}$$

with the domain of convergence $\sqrt{t_1}+\sqrt{t_2}<1$. The linear independence of these 4 representations is easily verified by checking their behaviours in particular limits.

1. Assuming $\mathrm{Re}(\beta_1-\beta_2),\ \mathrm{Re}(2\,\alpha-\beta_1-\beta_2)>0$, in the limit $z_4,\ z_5\to 0$, $I_2=I_3=I_4=0$,

$$I_1(z) = \int_\sigma \frac{\mathrm{d}x}{x}\,\frac{z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\,z_2^{-\beta_2}\,z_3^{\frac{\beta_2-\beta_1}{2}}\,x^{\beta}}{(1+x_1\,x_2+x_1^2)^{\alpha}} = C_1\,z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\,z_2^{-\beta_2}\,z_3^{\frac{\beta_2-\beta_1}{2}} \tag{2.53}$$

   where $z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\,z_2^{-\beta_2}\,z_3^{\frac{\beta_2-\beta_1}{2}}$ is the starting monomial of the series.

2. Assuming $\mathrm{Re}(\beta_2-\beta_1)>0,\ \mathrm{Re}(2\,\alpha-\beta_1-\beta_2)>0$, in the limit $z_3,\ z_5\to 0$, $I_1=I_3=I_4=0$,

$$I_2(z) = \int_\sigma \frac{\mathrm{d}x}{x}\,\frac{z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\,z_2^{-\beta_2}\,z_4^{\frac{\beta_1-\beta_2}{2}}\,x^{\beta}}{(1+x_1\,x_2+x_2^2)^{\alpha}} = C_2\,z_1^{\frac{\beta_1+\beta_2}{2}-\alpha}\,z_2^{-\beta_2}\,z_4^{\frac{\beta_1-\beta_2}{2}}\,. \tag{2.54}$$

3. Assuming $\mathrm{Re}(\beta_1-\beta_2),\ \mathrm{Re}(\beta_1+\beta_2-2\,\alpha)>0$, in the limit $z_1,\ z_4\to 0$, $I_1=I_2=I_4=0$,

$$I_3(z) = \int_\sigma \frac{\mathrm{d}x}{x}\,\frac{z_2^{\beta_1-2\alpha}\,z_3^{\frac{\beta_2-\beta_1}{2}}\,z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\,x^{\beta}}{(x_1\,x_2+x_1^2+x_1^2\,x_2^2)^{\alpha}} = C_3\,z_2^{\beta_1-2\alpha}\,z_3^{\frac{\beta_2-\beta_1}{2}}\,z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\,. \tag{2.55}$$

4. Assuming $\mathrm{Re}(\beta_2-\beta_1),\ \mathrm{Re}(\beta_1+\beta_2-2\,\alpha)>0$, in the limit $z_1,\ z_3\to 0$, $I_1=I_2=I_3=0$,

$$I_4(z) = \int_\sigma \frac{\mathrm{d}x}{x}\,\frac{z_2^{\beta_2-2\alpha}\,z_4^{\frac{\beta_1-\beta_2}{2}}\,z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\,x^{\beta}}{(x_1\,x_2+x_2^2+x_1^2\,x_2^2)^{\alpha}} = C_4\,z_2^{\beta_2-2\alpha}\,z_4^{\frac{\beta_1-\beta_2}{2}}\,z_5^{\alpha-\frac{\beta_1+\beta_2}{2}}\,. \tag{2.56}$$

For generically valued $\{\alpha,\ \beta_1,\ \beta_2\}$, the 4 starting monomials are independent. The proportionality constants $C_{1,\,2,\,3,\,4}$ can be found comparatively easily in these specific limits and then carried over to the complete series solution:

$$I(z) = \sum_{i=1}^{4} C_i \, z^{r_i} \, \tilde{I}_i(z) \tag{2.57}$$

where r_i are the roots, eq. (2.49), and $\tilde{I}_i(z)$, eq. (2.51), are Horn hypergeometric functions of the toric invariants $t_1 = \frac{z_3 \, z_4}{z_1 \, z_5}$, $t_2 = \frac{z_1 \, z_5}{z_2^2}$, parameterised by the roots.

Example 2.1.4. Euler integrals, eq. (2.2), defined at arbitrary choices of α, β and/or z may have properties which deviate from the generalized versions. For example, the following integral I, over some contour $\sigma \in \mathbb{C}^3$, is defined at a specific choice of β_3 and is to be evaluated at fixed z.

$$I = \int_\sigma \mathrm{d}x_1 \; \mathrm{d}x_2 \; \mathrm{d}x_3 \, \mathscr{I}, \quad \mathscr{I} = \frac{x_1^{\beta_1} \, x_2^{\beta_2}}{g(x)^\alpha} \tag{2.58}$$

where the polynomial $g(x)$ is

$$g(x) = 1 - x_1^2 - x_2^2 + x_1^2 \, x_2^2 + \frac{1}{x_1 x_2} - \frac{x_1}{x_2} - \frac{x_2}{x_1} - x_3 - x_1 \, x_2 \, x_3. \tag{2.59}$$

One way of evaluating this integral is to revert to the generalized format,

$$I = \int_\sigma \mathrm{d}x \, \frac{x_1^{\beta_1+\alpha} \, x_2^{\beta_2+\alpha} \, x_3^{\beta_3}}{g(x; \, z)^\alpha}\Big|_{z=\bar{z}, \, \beta_3=0}, \quad \bar{z} = \{1, \, -1, \, -1, \, 1, \, -1, \, -1, \, 1, \, -1, \, -1\} \tag{2.60}$$

$$g(x; z) = z_1 + z_2 x_1^2 + z_3 x_2^2 + z_4 x_1 x_2 + z_5 x_1^3 x_2 + z_6 x_1 x_2^3 + z_7 x_1^3 x_2^3 + z_8 x_1 x_2 x_3 + z_9 x_1^2 x_2^2 x_3$$

and evaluating $I(z)$ in the limit $z \to \bar{z}$ and $\beta_3 \to 0$. However, this arbitrary choice of parameters allows another approach. The differential equations satisfied by the integrand $\mathscr{I}$ are

$$\partial_{1,2} \, \mathscr{I} = \left(\frac{\beta_{1,2}}{x_{1,2}} - \alpha \, \frac{\partial_{1,2} \, g(x)}{g(x)} \right) \mathscr{I}, \quad \partial_3 \, \mathscr{I} = -\alpha \, \frac{\partial_3 g(x)}{g(x)} \, \mathscr{I}. \tag{2.61}$$

Upon eliminating the variable x_3 from them, $\mathscr{I}$ is found to satisfy

$$\partial_1 \left(\frac{x_1}{\beta_1 - \beta_2} \, \mathscr{I} \right) + \partial_2 \left(\frac{x_2}{\beta_2 - \beta_1} \, \mathscr{I} \right) + \partial_3 \left(\frac{2}{\beta_1 - \beta_2} \, \frac{x_2^2 - x_1^2}{x_1 x_2} \, \mathscr{I} \right) = \mathscr{I}. \tag{2.62}$$

Thus, I is a surface integral of a vector field:

$$\left(\frac{x_1}{\beta_1 - \beta_2} \mathscr{I},\ \frac{x_2}{\beta_2 - \beta_1} \mathscr{I},\ \frac{2}{\beta_1 - \beta_2} \frac{x_2^2 - x_1^2}{x_1 x_2} \mathscr{I} \right), \tag{2.63}$$

and if σ is a closed contour not enclosing any poles, it equals 0. Setting σ to an open contour, e.g. $x \in (\mathbb{R}_+)^3$, and assuming $\mathrm{Re}(\alpha) > 0$ and $|\,\mathrm{Re}(\alpha)| > |\,\mathrm{Re}(\beta_1 + 1)|,\ |\,\mathrm{Re}(\beta_2 + 1)|$, I reduces to an integral over 2 variables:

$$I = \tfrac{2}{\beta_1 - \beta_2} \int_{x_1, x_2} \tfrac{x_1^2 - x_2^2}{x_1 x_2} \mathscr{I} \Big|_{x_3 = 0} = \tfrac{2}{\beta_1 - \beta_2} \int_{x_1, x_2} \left(\tfrac{x_1}{x_2} - \tfrac{x_2}{x_1} \right) \tfrac{x_1^{\beta_1}\, x_2^{\beta_2}}{(1 - x_1^2 - x_2^2 + x_1^2 x_2^2 + \frac{1}{x_1 x_2} - \frac{x_1}{x_2} - \frac{x_2}{x_1})^\alpha}. \tag{2.64}$$

Example 2.1.5. ${}_2F_1(a_1,\, a_2;\, b;\, x)$ is defined as a solution to the Gauss hypergeometric ODE

$${}_2D_1 = \Big(x\,(1 - x)\,\partial_x^2 + (b - x\,(a_1 + a_2 + 1))\,\partial_x - a_1\,a_2 \Big), \quad {}_2D_1 \circ {}_2F_1(x) = 0. \tag{2.65}$$

$I(a_1,\, a_2,\, b\,;\, p,\, q\,;\, x)$, a holomorphic function of x, is the analytic continuation of ${}_2F_1(x)$ and one of its integral representations, specifically the Euler integral representation, is

$$I(a_1,\, a_2,\, b\,;\, p,\, q\,;\, x) = \int_q^p \mathrm{d}t\, f(t,\, x), \quad f(t,\, x) = \frac{1}{t\,(1 - t)} \left(\frac{t}{1 - t} \right)^{a_2} \frac{(1 - t)^b}{(1 - x\,t)^{a_1}} \tag{2.66}$$

where $p,\, q \in \{0,\, 1,\, \frac{1}{x},\, \infty\}$. The integrand $f(t, x)$ satisfies

$$\partial_t f = \left(\frac{(a_2 - 1)}{t} - \frac{(b - a_2 - 1)}{1 - t} + \frac{a_1\, x}{1 - x\,t} \right) f, \quad \partial_x f = \frac{a_1\, t}{(1 - x\,t)}\, f. \tag{2.67}$$

Upon clearing the denominators to bring them into a commonly preferred form, operators D_1, D_2 are found to annihilate the integrand $f(t,\, x)$:

$$\begin{aligned} D_1 &= t\,(1 - t)\,(1 - x\,t)\,\partial_t - (a_2 - 1)\,(1 - t)\,(1 - x\,t) \\ &\quad + (b - a_2 - 1)\,t\,(1 - x\,t) - a_1\,x\,t\,(1 - t) \\ D_2 &= (1 - x\,t)\,\partial_x - a_1\,t. \end{aligned} \tag{2.68}$$

All linear combinations of D_1 and D_2 also annihilate the integrand,

$$D_1 \oplus D_2 = d_1\, D_1 + d_2\, D_2, \quad D_1 \oplus D_2 \circ f(t,\, x) = 0 \tag{2.69}$$

where $d_1,\, d_2$ are any operators constructed from polynomial combinations of $x,\, \partial_x,\, t,\, \partial_t$. A well intentioned choice of $d_1,\, d_2$ results in

$$D_{(1,2)} = (1 - a_1) \Big(x\, {}_2D_1 + \partial_t\, (a_1\, t + x\, \partial_x - \partial_x) \Big) \tag{2.70}$$

where all t (and ∂_t) dependence is confined to a total derivative term wrt t and the operator ${}_2D_1$ is the same as what is given in eq. (2.65), defined only in terms of $x,\, \partial_x$. Integrating the identically null valued $D_{(1,2)} \circ f(t,\, x) = 0$ wrt t over some contour in $\mathbb{C}$, σ, gives

$$(1 - a_1) \left(\int_\sigma \mathrm{d}t\, x\, {}_2D_1 \circ f(t,\, x) + \int_\sigma \mathrm{d}t\, \partial_t\, (a_1\, t + x\, \partial_x - \partial_x) \circ f(t,\, x) \right) = 0. \tag{2.71}$$

Assuming $a_1 \neq 0,\, 1$ and $x \neq 0$, this simplifies to

$$\begin{aligned} {}_2D_1 \circ \int_\sigma \mathrm{d}t\, f(t,\, x) &= \int_\sigma \mathrm{d}t\, \partial_t \left(\frac{1}{x}\, \partial_x - \partial_x - \frac{a_1\, t}{x} \right) \circ f(t,\, x) \\ &= -a_1 \int_\sigma \mathrm{d}t\, \partial_t\, F(t,\, x), \qquad F(t,\, x) = \frac{t^b\, (1-t)^{b-a_2}}{(1 - x\, t)^{a_1+1}}. \end{aligned} \tag{2.72}$$

When $\mathrm{Re}(b)$, $\mathrm{Re}(b - a_2)$, $\mathrm{Re}(a_1 + 1)$ and $\mathrm{Re}(2b - a_1 - a_2 - 1) > 0$, $F(t,\, x)$ tends to 0 at the singular points of $f(t,\, x)$, $\Sigma_0 = \{0,\, 1,\, \frac{1}{x},\, \infty\}$. If σ is a closed contour not enclosing Σ_0, or an open interval $(p,\, q)$, where $\{p,\, q\} \in \Sigma_0$, the RHS equals 0, thus proving the original claim that eq. (2.66) is an integral representation of ${}_2F_1(x)$.

The dimension of the solution space of a degree n ODE

$$\sum_{i=0}^{n} f_i(x)\, \partial_x^i \circ I(x) = 0 \tag{2.73}$$

on $U \subset \mathbb{C}$ equals n for $f_n(U) \neq 0$ (and/or including any regular singularities) and $f_0(x)$ is the indicial polynomial. Since the ODE in the present context is second order, ${}_2F_1(x)$, or rather $I(x)$, has 2 linearly independent non-degenerate Laurent series solutions for $x\,(1-x) \neq 0$. The series expansion of $I(x)$ around $x = 0$ such that ${}_2D_1 \circ I(x) = 0$ is

$$I(x) = x^{\alpha} \sum_{n=0}^{\infty} c_n\, x^n, \quad b_n = \frac{c_{n+1}}{c_n} = \frac{(\alpha+n+a_1)\,(\alpha+n+a_2)}{(\alpha+n+1)\,(\alpha+n+b)}, \quad \alpha\,(\alpha+b-1) = 0. \tag{2.74}$$

Such a rational relation between the coefficients c_n, b_n, is called the Bernstein - Sato polynomial. The equation satisfied by the roots α is called the indicial equation and it expectedly has 2 solutions, each giving rise to a linearly independent series, together spanning the entire solution space:

$$\begin{aligned} I(x) \propto &\sum_{n=0}^{\infty} \tfrac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{\Gamma(b+n)}\, \tfrac{x^n}{n!} \oplus x^{1-b} \sum_{n=0}^{\infty} \tfrac{\Gamma(a_1+1-b+n)\,\Gamma(a_2+1-b+n)}{\Gamma(2-b+n)}\, \tfrac{x^n}{n!} \\ = &\ {}_2F_1(a_1,\, a_2\,;\, b\,;\, x) \oplus x^{1-b}\ {}_2F_1(a_1+1-b,\, a_2+1-b\,;\, 2-b;\, x). \end{aligned} \tag{2.75}$$

It is no coincidence that the superficially different looking integrals eqs. (2.17) and (2.66) describe the same class of functions. As was shown in [20] over 3 decades ago, Euler integrals are classified by their reduced form and eq. (2.17) does reduce to eq. (2.66) under a change of variables that is obvious in hindsight.

Example 2.1.6. The most general 2$^{\text{nd}}$ order polynomial in n variables x is

$$P(x\,;z) = \sum_{i=1}^{n} z_i\, x_i^2 + \sum_{i<j}^{n} z_{i,j}\, x_i\, x_j. \tag{2.76}$$

The class of Euler integrals featuring this polynomial,

$$I(z) = \int_{\sigma} \frac{\mathrm{d}x}{x}\, \frac{x^{\beta}}{(z_0+P)^{\alpha}} \tag{2.77}$$

satisfies $n+1$ homogeneity conditions that can be found by considering the scale transformations

$z \to \rho\, z$, and $x_i \to \rho\, x_i$, $i \in [n]$:

$$D_0 = \theta_0 + \alpha - \tfrac{1}{2} \sum \beta_i, \quad D_i = 2\,\theta_i + \sum_{j=1}^{i-1} \theta_{j,\,i} + \sum_{j=i+1}^{n} \theta_{i,\,j} + \beta_i. \tag{2.78}$$

Thus, the integral can be represented as the product of a monomial in z, with trivial dependence on z_0, and a hypergeometric function, $\tilde{I}$, of $\frac{n\,(n-1)}{2}$ toric invariants, t,

$$I(z) = z_0^{\frac{1}{2}\sum\beta - \alpha}\, z^r\, \tilde{I}(t). \tag{2.79}$$

The toric symmetries are spanned by 3 types of PDEs:

$$D_{i[2]} = \partial_{i_1}\, \partial_{i_2} - \partial^2_{i_1,\,i_2}, \quad D_{i[3]} = \partial_{i_1,\,i_2}\, \partial_{i_2,\,i_3} - \partial_{i_1,\,i_3}\, \partial_{i_2}, \quad D_{i[4]} = \partial_{i_1,\,i_2}\, \partial_{i_3,\,i_4} - \partial_{i_1,\,i_4}\, \partial_{i_2,\,i_3}. \tag{2.80}$$

Akin to example 2.1.3, each PDE has a representative toric invariant. It is immediately apparent that there are more toric PDEs than independent toric invariants. However, unlike example 2.1.3, which has an intuitive basis of toric invariants, eq. (2.45), it is not immediately apparent which of the over-complete set of toric invariants suggested by eq. (2.80) form an appropriate linearly independent basis. For the sake of concrete illustration, let $n = 3$, $z = \{z_1,\, z_2,\, z_3,\, z_{1,\,2},\, z_{1,\,3},\, z_{2,\,3}\}$. The homogeneity conditions are simple:

$$\begin{aligned} &D_0 = \theta_0 + \alpha - \tfrac{1}{2} \sum \beta, \quad D_1 = 2\,\theta_1 + \theta_{1,\,2} + \theta_{1,\,3} + \beta_1 \\ &D_2 = 2\,\theta_2 + \theta_{1,\,2} + \theta_{2,\,3} + \beta_2, \quad D_3 = 2\,\theta_3 + \theta_{1,\,3} + \theta_{2,\,3} + \beta_3. \end{aligned} \tag{2.81}$$

The toric PDEs are spanned by 6 differential operators

$$\begin{aligned} &D_{1,\,2} = \partial_1\, \partial_2 - \partial^2_{1,\,2}, \quad D_{1,\,3} = \partial_1\, \partial_3 - \partial^2_{1,\,3}, \quad D_{2,\,3} = \partial_2\, \partial_3 - \partial^2_{2,\,3} \\ &D_{1,\,2,\,3} = \partial_{1,\,2}\, \partial_{2,\,3} - \partial_{1,\,3}\, \partial_2, \quad D_{2,\,3,\,1} = \partial_{1,\,3}\, \partial_{2,\,3} - \partial_{1,\,2}\, \partial_3, \quad D_{3,\,1,\,2} = \partial_{1,\,3}\, \partial_{1,\,2} - \partial_{2,\,3}\, \partial_1 \end{aligned} \tag{2.82}$$

even though the basis of toric invariants consists of only 3 elements. One such basis is

$$t_1 = \frac{z_1\, z_{2,3}}{z_{1,2}\, z_{1,3}}, \quad t_2 = \frac{z_2\, z_{1,3}}{z_{1,2}\, z_{2,3}}, \quad t_3 = \frac{z_3\, z_{1,2}}{z_{1,3}\, z_{2,3}}. \tag{2.83}$$

Operating in this basis, the solution space of $D \circ I(z) = 0$ will consist of Laurent series solutions in both positive and negative powers of each toric invariant, of which obviously only one can be convergent. Further, since all possible combinations get explored, there will always be at least one convergent series solution at any value of t for generically valued roots, r:

$$I(z) = z_0^{\frac{1}{2}\sum\beta-\alpha}\, z^r \sum_{n\in\mathbb{Z}} c_{n_1,\,n_2,\,n_3}\, t_1^{n_1}\, t_2^{n_2}\, t_3^{n_3}. \tag{2.84}$$

There are 4 independent roots, r, to this system (exponent of z_0 is dropped in the following)

$$\begin{aligned} r_{(1)} &= \{0,0,0,\tfrac{\beta_3-\beta_1-\beta_2}{2},\tfrac{\beta_2-\beta_1-\beta_3}{2},\tfrac{\beta_1-\beta_2-\beta_3}{2}\}, & r_{(2)} &= \{0,0,\tfrac{\beta_1+\beta_2-\beta_3}{2},0,-\beta_1,-\beta_2\}, \\ r_{(3)} &= \{0,\tfrac{\beta_1+\beta_3-\beta_2}{2},0,-\beta_1,0,-\beta_3\}, & r_{(4)} &= \{\tfrac{\beta_2+\beta_3-\beta_1}{2},0,0,-\beta_2,-\beta_3,0\}. \end{aligned} \tag{2.85}$$

The entire solution space isn't represented in terms of the same basis of toric invariants. The series originating from the first root is indeed in terms of the originally suggested basis, eq. (2.83),

$$\begin{aligned} I_1(z) = {}& z_{1,2}^{\frac{\beta_3-\beta_1-\beta_2}{2}}\, z_{1,3}^{\frac{\beta_2-\beta_1-\beta_3}{2}}\, z_{2,3}^{\frac{\beta_1-\beta_2-\beta_3}{2}} \sum_{n\in\mathbb{N}_0} \frac{(t_1)^{n_1}}{n_1!}\,\frac{(t_2)^{n_2}}{n_2!}\,\frac{(t_3)^{n_3}}{n_3!} \\ &\times \frac{\Gamma(\frac{\beta_3-\beta_1-\beta_2}{2}+1)}{\Gamma(\frac{\beta_3-\beta_1-\beta_2}{2}+n_3-n_1-n_2+1)}\,\frac{\Gamma(\frac{\beta_2-\beta_1-\beta_3}{2}+1)}{\Gamma(\frac{\beta_2-\beta_1-\beta_3}{2}+n_2-n_1-n_3+1)}\,\frac{\Gamma(\frac{\beta_1-\beta_2-\beta_3}{2}+1)}{\Gamma(\frac{\beta_1-\beta_2-\beta_3}{2}+n_1-n_2-n_3+1)} \end{aligned} \tag{2.86}$$

but that is not the case for the other 3, which after shifting the summation range to $\mathbb{N}_0$ are

$$\begin{aligned} I_2(z) &= \frac{z_3^{\frac{\beta_1+\beta_2-\beta_3}{2}}}{z_{1,3}^{\beta_1}\, z_{2,3}^{\beta_2}} \sum_{n\in\mathbb{N}_0} \frac{\Gamma(\beta_1+2n_1+n_3)\,\Gamma(\beta_2+2n_2+n_3)}{\Gamma(\frac{\beta_1+\beta_2-\beta_3}{2}+n_1+n_2+n_3+1)}\,\frac{(t_1\,t_3)^{n_1}}{n_1!}\,\frac{(t_2\,t_3)^{n_2}}{n_2!}\,\frac{(t_3)^{n_3}}{n_3!} \\ I_3(z) &= \frac{z_2^{\frac{\beta_1+\beta_3-\beta_2}{2}}}{z_{1,2}^{\beta_1}\, z_{2,3}^{\beta_3}} \sum_{n\in\mathbb{N}_0} \frac{\Gamma(\beta_1+2n_1+n_2)\,\Gamma(\beta_3+2n_3+n_2)}{\Gamma(\frac{\beta_1+\beta_3-\beta_2}{2}+n_1+n_2+n_3+1)}\,\frac{(t_1\,t_2)^{n_1}}{n_1!}\,\frac{(t_2)^{n_2}}{n_2!}\,\frac{(t_2\,t_3)^{n_3}}{n_3!} \\ I_4(z) &= \frac{z_1^{\frac{\beta_2+\beta_3-\beta_1}{2}}}{z_{1,2}^{\beta_2}\, z_{1,3}^{\beta_3}} \sum_{n\in\mathbb{N}_0} \frac{\Gamma(\beta_2+2n_2+n_1)\,\Gamma(\beta_3+2n_3+n_1)}{\Gamma(\frac{\beta_1+\beta_3-\beta_2}{2}+n_1+n_2+n_3+1)}\,\frac{(t_1)^{n_1}}{n_1!}\,\frac{(t_1\,t_2)^{n_2}}{n_2!}\,\frac{(t_1\,t_3)^{n_3}}{n_3!}. \end{aligned} \tag{2.87}$$

The list of roots, eq. (2.85), isn't exhaustive. Shifting it by any element of the kernel $\mathcal{K}$ leaves the hypergeometric system unchanged

$$\mathcal{K} = \mathbb{Z}\{1,0,0,-1,-1,1\} \oplus \mathbb{Z}\{0,1,0,-1,1,-1\} \oplus \mathbb{Z}\{0,0,1,1,-1,-1\}. \tag{2.88}$$

Using these linear combinations, all possible toric invariants and roots can be found. Each root corresponds to a singular point of the integral (zero of the polynomial), encoding the zero locus of the polynomial within the root system of the Euler integral. Thus, depending on the contour σ, different roots and their associated solutions dominate.

Example 2.1.7. Similar in construction but far more simplistic than the famous mirror quintic is the family of elliptic curves $P_\psi = x_1^3 + x_2^3 + x_3^3 - 3\,\psi\, x_1\, x_2\, x_3$, parameterised by ψ, with period Ω invariant under linear scale transformations $(x_1,\, x_2,\, x_3) \to \lambda\,(x_1,\, x_2,\, x_3)$, and hence well-defined on $\mathbb{P}^2$ as

$$\Omega_\psi = \int_{\gamma_P} \frac{-x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge x_3}{P_\psi} \tag{2.89}$$

where γ_P is a small loop around the surface $P_\psi = 0$ [104]. Considering the polynomial in its generalized form $P_{a,\psi} = a_1\, x_1^3 + a_2\, x_2^3 + a_3\, x_3^3 - 3\,\psi\, x_1\, x_2\, x_3$, $\Omega_{a,\psi}$ satisfies

$$\begin{aligned} &(\theta_{a_1} + \theta_{a_2} + \theta_{a_3} + \theta_\psi + 1)\,\Omega_{a,\psi} = 0, \quad (3\,\theta_a + \theta_\psi + 1)\,\Omega_{a,\psi} = 0 \\ &\left(\partial_{a_1}\,\partial_{a_2}\,\partial_{a_3} + (\tfrac{1}{3}\,\partial_\psi)^3\right)\,\Omega_{a,\psi} = 0. \end{aligned} \tag{2.90}$$

The scaling equations require $\Omega_{a,\psi}$ to take the form $\frac{1}{3\,\psi}\,\bar{\Omega}(\varphi)$, where $\varphi = \frac{a_1\,a_2\,a_3}{(3\psi)^3}$ and the toric PDE simplifies to an ODE in φ

$$(\tfrac{1}{3}\,\theta_\varphi)^3\,\bar{\Omega}(\varphi) = \left(\theta_\varphi^2 - \theta_\varphi + \tfrac{2}{9}\right)\,\theta_\varphi\,\varphi\,\bar{\Omega}(\varphi). \tag{2.91}$$

Series solutions of this ODE around $\varphi = 0$ can be iteratively built upon a linearly independent basis of solutions of $(\frac{1}{3}\,\theta_\varphi)^3\,\bar{\Omega}(\varphi) = 0$ that serve as the starting monomials.

A basis of n solutions of $\theta_z^n\, f(z) = 0$ are $f_k(z) = (\frac{1}{2\pi\mathrm{i}}\,\log z)^k$, $k \in [0, \cdots, n-1]$. Rotating the argument by an angle of 2π relates the solutions as $f_n(e^{2\pi \mathrm{i}}\, z) = (\frac{1}{2\pi\mathrm{i}}\, \log z + 1)^n = \sum_{k=0}^{n} \binom{n}{k} f_k(z)$. Thus, the columns of the monodromy matrices, M_n, are binomial coefficients:

$$M_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad M_3 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad M_4 = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \quad M_5 = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \text{ and so on.} \tag{2.92}$$

In this case, the starting monomials are hence $\bar{\Omega}_0(\varphi) = \{1,\, \frac{1}{2\pi\mathrm{i}}\, \log\varphi,\, (\frac{1}{2\pi\mathrm{i}}\, \log\varphi)^2\}$. The most general zeroth and first order terms of the series are

$$\begin{aligned} \bar{\Omega}_0(\varphi) &= c_{0,0} + c_{0,1}\, \tfrac{1}{2\pi\mathrm{i}}\, \log\varphi + c_{0,2}\, (\tfrac{1}{2\pi\mathrm{i}}\, \log\varphi)^2 \\ \bar{\Omega}_1(\varphi) &= \varphi\, (c_{1,0} + c_{1,1}\, \tfrac{1}{2\pi\mathrm{i}}\, \log\varphi + c_{1,2}\, (\tfrac{1}{2\pi\mathrm{i}}\, \log\varphi)^2). \end{aligned} \tag{2.93}$$

Requiring $\left(\theta_\varphi^2 - \theta_\varphi + \frac{2}{9}\right)\, \theta_\varphi\, \varphi\, \bar{\Omega}_0(\varphi) = (\frac{1}{3}\, \theta_\varphi)^3\, \bar{\Omega}_1(\varphi)$ fixes the constants c_1 to

$$\{c_{1,0},\, c_{1,1},\, c_{1,2}\} = \{3\, (2\, c_{0,0} + 5\, c_{0,1} - 6\, c_{0,2}),\, 6\, (c_{0,1} + 5\, c_{0,2}),\, 6\, c_{0,2}\}. \tag{2.94}$$

Subsequent terms in the series can be iteratively found in the same way.

2.1.2 $\mathcal{A}$ - Hypergeometric Ideal

Any Euler integral can be resolved into a set of linear PDEs with series solutions that are in one-to-one correspondence to a set of vectors, known as roots. Finding them is equivalent to solving the integral, and as previously advertised and presented in the following, the procedure to find them is not only algorithmic, but also better visualised in the appropriate mathematical language.

A generalized Euler integral, $I_\sigma[\alpha,\, \beta; P](z)$, is an integral over n integration variables $x \in \mathbb{C}^n$ over a closed contour σ, with the integrand $\mathscr{I}(\alpha,\, \beta,\, P)$ defined by generic complex valued vectors $\alpha \in \mathbb{C}^m$, $\beta \in \mathbb{C}^n$ and m polynomials P, interpreted as function of N variables $z \in \mathbb{C}\backslash\{0\}$.

$$I[\alpha,\, \beta; P](z) = \int_\sigma \mathscr{I}(\alpha,\, \beta;\, P), \quad \mathscr{I}(\alpha,\, \beta;\, P) = \prod_{i=1}^{n} \frac{\mathrm{d}x_i}{x_i}\, x_i^{\beta_i} \prod_{j=1}^{m} P_j(x;\, A^{(j)};\, z^{(j)})^{-\alpha_j} \tag{2.95}$$

By definition, this induces the shift relations:

$$I[\alpha,\,\beta;\,x^{\omega}\,P](z) = I[\alpha,\,\beta+\alpha\,\omega;\,P](z), \quad x^{\omega}\,P = \prod_{i=1}^{m} x^{\omega_i}\,P_i \tag{2.96}$$

allowing infinitely many trivially equivalent representations of I, at least one of which has the polynomials in the form $P_i = 1 + P'_i(x,\,z)$. Equivalent shift relations given in eqs. (2.109), (2.111), (2.113) and (2.115) are also induced by derivatives wrt z. The support of a polynomial $P(x\,;\,z)$ is $\{x \in \mathbb{C}^n \mid P(x\,;\,z) \neq 0\}$. Extrapolating to the integrand, its support, $\Sigma(P)$ is

$$\Sigma(P) = (\mathbb{C}^*)^n \setminus \bigcup_{i\in[m]} \{P_i(x\,;\,z) \neq 0\}. \tag{2.97}$$

The contour σ is an n-cycle defined as a formal sum of maps from the standard n-simplex to $\Sigma(P)$:

$$\sigma^{(n)} = \sum \sigma_i^{(n)} : \mathbf{\Delta}^n \to \Sigma(P), \quad \mathrm{d}\sigma = 0, \quad \mathbf{\Delta}^n = \{v \Big| \sum_{i=0}^{n} v_i = 1, \quad v_i \in \mathbb{R}_+\}, \tag{2.98}$$

schematically pictured in fig. (2.3) (when $q = n$). The space of integrals $I[\alpha,\,\beta;\,P]$ is isomorphic to the solution space of the left ideal, $\mathcal{I}_{\mathcal{A}} \subseteq D_N$, where $D_N = \mathbb{C}[z,\,\partial_z]$ is the N-dimensional Weyl algebra. $\mathcal{I}_{\mathcal{A}}$, often dubbed the $\mathcal{A}$ hypergeometric ideal, GKZ hypergeometric ideal, or GKZ differential equations, can be represented as a D_N-linear combination of the left ideals $H_{\mathcal{A}}$ and $J_{\mathcal{A}}$,

$$\mathcal{I}_{\mathcal{A}} = D_N \circ J_{\mathcal{A}} + D_N \circ H_{\mathcal{A}} = \langle \mathcal{G}_{\mathcal{A}} \rangle, \quad \mathcal{I}_{\mathcal{A}} \circ I(z) = 0. \tag{2.99}$$

There exists no non-trivial element in $D_N \backslash \mathcal{I}_{\mathcal{A}}$ which annihilates $I[\alpha,\,\beta;\,P]$, making $\mathcal{I}_{\mathcal{A}}$ a maximal ideal in D_N. $J_{\mathcal{A}} \subset \mathbb{C}\langle\partial\rangle$ is the toric ideal and $H_{\mathcal{A}} \subset \mathbb{C}\langle\theta\rangle$ is torus fixed, respectively generated by

$$J_{\mathcal{A}} = \left\langle \partial^{u^+} - \partial^{u^-} \,\middle|\, \mathcal{A}\,u^+ = \mathcal{A}\,u^-,\ \{u^+,\,u^-\} \in \mathbb{N}_0^N \right\rangle \tag{2.100}$$

$$H_{\mathcal{A}} = \langle \mathcal{A}\,\theta + \gamma \rangle, \quad \gamma = \{\alpha_1,\,\cdots\,\alpha_m,\,\beta_1,\cdots\,\beta_n\} \tag{2.101}$$

where the vector γ is non-resonant, and $\mathcal{A}$ is an $(n+m) \times N$ matrix formed of the support matrices

A of the polynomials P

$$\mathcal{A} = \begin{pmatrix} \mathbb{1}_1 & 0 & \cdots & 0 \\ 0 & \mathbb{1}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbb{1}_m \\ A^{(1)} & A^{(2)} & \cdots & A^{(m)} \end{pmatrix}, \quad \mathbb{1}_j \equiv \{1, \cdots, 1\}. \tag{2.102}$$

There are an infinitely many ways to represent the same Euler integral.

$$\begin{aligned} I(z) &= \int \frac{\mathrm{d}x_1\ \mathrm{d}x_2}{x_1\, x_2} \frac{x_1^{\beta_1}\, x_2^{\beta_2}}{(z_1+z_2x_1+z_3x_2^2)^{\alpha_1}(z_4+z_5x_2+z_6x_1^2)^{\alpha_2}} = \int \frac{\mathrm{d}x_1\ \mathrm{d}x_2}{x_1\, x_2} \frac{x_1^{\beta_1+\alpha_1}\, x_2^{\beta_2}}{(z_1x_1+z_2x_1^2+z_3x_1\, x_2^2)^{\alpha_1}(z_4+z_5x_2+z_6x_1^2)^{\alpha_2}} \\ &= \int \frac{\mathrm{d}x_1\ \mathrm{d}x_2}{x_1\, x_2} \frac{2\, x_1^{2\,\beta_1}\, x_2^{\beta_2}}{(z_1+z_2x_1^2+z_3x_2^2)^{\alpha_1}(z_4+z_5x_2+z_6x_1^4)^{\alpha_2}} = \cdots \end{aligned}$$

Following along with the above procedure, each of these technically equal representations produce different possible $\mathcal{A}$ matrices,

$$\mathcal{A}^{(1)} = \begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&1&0&0&2&0 \\ 0&0&2&0&0&1 \end{pmatrix}, \quad \mathcal{A}^{(2)} = \begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 1&2&1&0&2&0 \\ 0&0&2&0&0&1 \end{pmatrix}, \quad \mathcal{A}^{(3)} = \begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&2&0&0&4&0 \\ 0&0&2&0&0&1 \end{pmatrix}. \tag{2.103}$$

Irreducibility Conditions on $\mathcal{A}$

For any given Euler integral, there always exists a class of reduced form representations, such that its associated $\mathcal{A}$ matrix satisfies the following conditions.

1. The union of the columns of each A_i generates $\mathbb{Z}^n$ and there is at least one column which is 0 (i.e. there exists a non-zero constant term in the polynomial). These conditions together imply that union of the columns of $\mathcal{A}_i$ generate $\mathbb{Z}^{n+m}$.

$$\begin{aligned} \exists\ c_i \in \mathbb{Z} \mid \xi = \sum c_i\, A_i\ \forall\ \xi \in \mathbb{Z}^n, &\quad \exists\ j \mid A_{ij} = 0 \\ \Longrightarrow \exists\ c_i \in \mathbb{Z} \mid \xi = \sum c_i\, \mathcal{A}_i\ \forall\ \xi \in \mathbb{Z}^{n+m} & \end{aligned} \tag{2.104}$$

2. There exists a group homomorphism $h : \mathbb{Z}^{n+m} \to \mathbb{Z}$ such that all $\mathcal{A}_i \mapsto 1$. One row of $\mathcal{A}$ can always be brought to a constant (which is by default chosen to be 1 in eq. (2.102)) under appropriate unimodular row transformations.

The converse is also true, i.e. given a finite subset $\mathcal{A}$ of $\mathbb{Z}^n$ satisfying the above conditions, the solution space of the system of holonomic differential equations eq. (2.99) is spanned by Euler integrals of the form eq. (2.95) defined over generically valued vectors α, β. Further, the solution space associated with $\mathcal{A}$ is isomorphic to that of all $G\,\mathcal{A}$, where G is an invertible $n \times n$ matrix, with the parameter vector γ similarly transforming to $G\,\gamma$.

Thus, it is worth noting that even if $\mathcal{A}$ doesn't satisfy eq. (2.104), it is enough for it to be of rank $n+m$, i.e. full rank, as shown in [97], thereby relaxing the constraints on $\mathcal{A}$ matrices in given [20]. Returning to the example in eq. (2.103), even though only $\mathcal{A}^{(1)}$ satisfies these conditions, the GKZ ideal, eq. (2.99), of all three $\mathcal{A}^{(1,\,2,\,3)}$ is the same, generated by:

$$\begin{aligned}&\theta_1+\theta_2+\theta_3+\alpha_1, \quad \theta_4+\theta_5+\theta_6+\alpha_2, \quad \theta_2+2\,\theta_5+\beta_1, \quad 2\,\theta_3+\theta_6+\beta_2\\ &\partial_1^2\,\partial_5-\partial_2^2\,\partial_4, \qquad\qquad \partial_3\,\partial_4^2-\partial_1\,\partial_6^2.\end{aligned} \tag{2.105}$$

If $\mathcal{A}$ doesn't satisfy the homogeneity condition, i.e. cannot be brought into a form with a row of $\mathbb{1}$, then the ideal $\mathcal{I}_{\mathcal{A}}$ is instead found to annihilate a confluent hypergeometric integral, which includes exponentials of polynomials:

$$I[\beta;\,P](z)=\int_\sigma \frac{\mathrm{d}x}{x}\,x^\beta\prod_{j=1}^m e^{P_j(x;\,A^{(j)};\,z^{(j)})}. \tag{2.106}$$

However, the confluent integrals $I[\beta;\,P]$ may not span the entire solution space of their associated GKZ system of equations.

Example 2.1.8. The confluent integral $\int_0^\infty \frac{\mathrm{d}t}{t}\,t^\Delta\,e^{z_1\,t+z_2\,t^2}$ is a solution of $\langle\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}+\Delta,\ \partial_1^2-\partial_2\rangle$. This can be confirmed by eliminating t (like in example 2.1.5) from

$$\begin{aligned}&\partial_1-t, \quad \partial_2-t^2, \quad \partial_t-\frac{\Delta-1}{t}-z_1-2\,z_2\,t\cong\partial_t\,t-(\Delta+z_1\,\partial_1+2\,z_2\,\partial_1^2)\\ &(\Delta+z_1\,\partial_1+2\,z_2\,\partial_1^2)\int_0^\infty \tfrac{\mathrm{d}t}{t}\,t^\Delta\,e^{z_1\,t+z_2\,t^2}=\int_0^\infty \partial_t\,t\,\tfrac{\mathrm{d}t}{t}\,t^\Delta\,e^{z_1\,t+z_2\,t^2}=0, \quad \mathrm{Re}(z_2)<0\\ &(\Delta+z_1\,\partial_1+2\,z_2\,\partial_1^2)=(\Delta+z_1\,\partial_1+2\,z_2\,\partial_2)=(\Delta+\theta_1+2\,\theta_2).\end{aligned} \tag{2.107}$$

The solution space expectedly consists of ${}_1F_1$ functions

$$z_2^{-\frac{\Delta}{2}} \sum_n \frac{\Gamma(n+\frac{\Delta}{2}+1)}{\Gamma(n+1)\,\Gamma(n+\frac{1}{2})} \left(-\tfrac{z_1^2}{4\,z_2}\right)^n \oplus \tfrac{1}{4}\, z_1\, z_2^{-\frac{1+\Delta}{2}} \sum_n \frac{\Gamma(n+\frac{1+\Delta}{2}+1)}{\Gamma(n+1)\,\Gamma(n+\frac{3}{2})} \left(-\frac{z_1^2}{4z_2}\right)^n. \quad (2.108)$$

Non-resonance Condition

The vector γ is non-resonant when $\gamma \notin \mathbb{Z}^{n+m} + \mathbb{C}$-linear span of the codimension-1 faces (i.e. $n+m-1$ dimensional faces/facets) of the convex cone of $\mathcal{A}$. An example representation of the projection of resonant values of γ onto the real plane is shown by the dashed arrows in fig. (2.1).

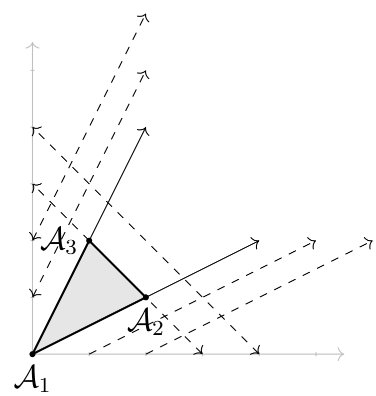


Figure 2.1: Non-resonance Condition

The shaded region is the convex hull of $\mathcal{A}$ with vertices $\{\mathcal{A}_1, \mathcal{A}_2, \cdots \mathcal{A}_N\}$. The interior of the arrows originating from $\mathcal{A}_1$ is the $\mathbb{R}$ linear span thereof, i.e. the convex cone of $\mathcal{A}$, with the arrows also forming the codimension-1 faces (facets) of the convex cone. Dashed arrows are representative of the projection onto the real plane of the facets shifted by all $\mathbb{Z}^{n+m}$.

2.1.3 Equivalence of Euler Integrals and Solution Space of the GKZ Ideal

It can be verified explicitly that all the elements of the left ideal $\mathcal{I}_{\mathcal{A}}$ annihilate the integral $I(z)$. The action of a general element in $\mathbb{C}\langle\partial_z\rangle$, ∂^{u_+}, on the integral $I(z)$ is

$$\partial^{u_+} \circ I(z) = \int_\sigma \tfrac{\mathrm{d}x}{x}\, x^\beta \prod_{j=1}^{m} (-\alpha_j)_{(\sum u_{j+})}\, x^{A^{(j)} u_{j+}}\, P_j(x;\, A^{(j)};\, z^{(j)})^{-\alpha - \sum u_{j+}}. \quad (2.109)$$

This implies $\partial^{u_+} \circ I(z) = \partial^{u_-} \circ I(z)$ iff

$$\sum u_{j+} = \sum u_{j-} = \text{constant}, \quad A^{(j)}\, u_{j_+} = A^{(j)}\, u_{j_-} \quad \forall\, j \in [m], \tag{2.110}$$

a set of conditions which are better represented as $J_{\mathcal{A}}$ as given in eq. (2.100), with the option of replacing any $\mathbb{1}$ with any positive integer multiple thereof in eq. (2.102) to produce equivalent representations of $\mathcal{A}$. The action of the generators of $\mathbb{C}\langle\theta\rangle$, $\theta_{z_i^{(j)}}$, on $I(z)$ is

$$\theta_{z_i^{(j)}} \circ I(z) = \int_\sigma \tfrac{\mathrm{d}x}{x}\, x^\beta\, \tfrac{-\alpha_j\, z_i^{(j)}\, x^{A_i^{(j)}}}{P_j(x;\, A^{(j)};\, z^{(j)})} \prod_{j=1}^{m} P_j(x;\; A^{(j)};\; z^{(j)})^{-\alpha_j} \tag{2.111}$$

$$\begin{aligned} &= \int_\sigma \prod_{j_1 \neq k} \tfrac{\mathrm{d}x_{j_1}}{x_{j_1}}\, x_{j_1}^{\beta_{j_1}} \prod_{j_2 \neq j}^{m} P_{j_2}(x;\; A^{(j_2)};\; z^{(j_2)})^{-\alpha_{j_2}} \\ &\times\; \mathrm{d}x_k\, x_k^{\beta_k}\, \tfrac{(-\alpha_j)}{P_j(x;\, A^{(j)};\, z^{(j)})^{\alpha_j}}\, \tfrac{z_i^{(j)}\, \partial_{x_k}\, x^{A_i^{(j)}}}{P_j(x;\, A^{(j)};\, z^{(j)})}\, \tfrac{1}{A^{(j)}{}_{ik}} \quad \forall\, k \in [n]. \end{aligned} \tag{2.112}$$

One straightforward way of eliminating all x from these N PDEs when presented in the form of eq. (2.111) is to simply sum them up per polynomial,

$$\sum_i \theta_{z_i^{(j)}} \circ I(z) = -\alpha_j\, I(z) \quad \forall\, j \in [m] \tag{2.113}$$

producing m homogeneity relations. Summing rescaled versions of all N PDEs presented in the form of eq. (2.112) per each variable x gives

$$\sum_j \sum_i A^{(j)}{}_{ik}\, \theta_{z_i^{(j)}} \circ I(z) = \int_\sigma \prod_{j_1 \neq k} \tfrac{\mathrm{d}x_{j_1}}{x_{j_1}}\, x_{j_1}^{\beta_{j_1}}\; \mathrm{d}x_k\, x_k^{\beta_k}\, \partial_{x_k} \prod_{j_2 \neq j}^{m} P_{j_2}(x;\; A^{(j_2)};\; z^{(j_2)})^{-\alpha_{j_2}}. \tag{2.114}$$

Since σ is a n-cycle within the support Σ, $\mathscr{I}$ is smooth, which together imply

$$\begin{aligned} \sum_j \sum_i A^{(j)}{}_{ik}\, \theta_{z_i^{(j)}} \circ I(z) &= -\int_\sigma \prod_{j_1 \neq k} \tfrac{\mathrm{d}x_{j_1}}{x_{j_1}}\, x_{j_1}^{\beta_{j_1}}\; \mathrm{d}x_k\, (\partial_{x_k} x_k^{\beta_k}) \prod_{j_2 \neq j}^{m} P_{j_2}(x; A^{(j_2)}; z^{(j_2)})^{-\alpha_{j_2}} \\ &= -\beta_k\, I(z) \quad \forall\, k \in [n] \end{aligned} \tag{2.115}$$

thus producing n more homogeneity relations. Altogether these relations are concisely represented by $H_{\mathcal{A}}$ as given in eq. (2.101). This verifies that eq. (2.99) does annihilate eq. (2.95), or equivalently, proves that the solution space of $\mathcal{I}_{\mathcal{A}}$ contains $I[\alpha,\, \beta;\, P]$.

The solution space of the ideal, $\mathcal{I}_{\mathcal{A}}$, is spanned by Euler integrals, $I[\alpha,\, \beta;\, P]$, eq. (2.95), whilst assuming the vector $\gamma \equiv \{\alpha,\, \beta\}$ satisfies the Non-resonance Condition. This statement and a stronger generalisation thereof are proven in [20] by the construction of three sheaves that are then shown to be irreducible and isomorphic. A loose sketch of these constructions is as follows:

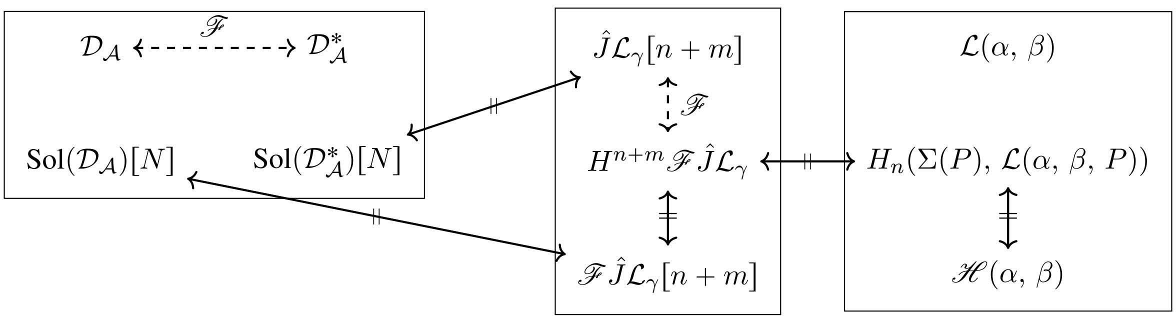


Figure 2.2: Equivalence classes of GKZ systems and their interrelations

Systems connected by solid arrows are isomorphic and those connected by dashed arrows are Fourier transforms of each other.

$\mathcal{D}_{\mathcal{A}}$

$\mathcal{D}_V$ is the sheaf of rings of differential operators with holomorphic coefficients on $V \equiv \mathbb{C}^N \ni z$. The geometric Fourier transform, $\mathscr{F}$, of V, V^*, induces the coordinate transformations:

$$z \mapsto -\partial^*, \quad \partial \mapsto z^*, \quad \theta \mapsto -(\theta^* + 1). \tag{2.116}$$

Thus, $\mathcal{I}_{\mathcal{A}}$ defined on V (see eq. (2.99) for notation) transforms to $\mathcal{I}^*_{\mathcal{A}}$ defined on V^*,

$$\mathcal{I}_{\mathcal{A}} = \langle \partial^{u+} - \partial^{u-} \rangle + \langle \mathcal{A}\theta + \gamma \rangle \mapsto \mathcal{I}^*_{\mathcal{A}} = \langle z^{*u+} - z^{*u-} \rangle + \langle -\mathcal{A}\,(\theta^* + 1) + \gamma \rangle. \tag{2.117}$$

The quotient module $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_V/\mathcal{D}_V \circ \mathcal{I}_{\mathcal{A}}$ similarly transforms to $\mathcal{D}^*_{\mathcal{A}} = \mathcal{D}_{V*}/\mathcal{D}_{V*} \circ \mathcal{I}^*_{\mathcal{A}}$. The first set of generators of $\mathcal{I}^*_{\mathcal{A}}$ are algebraic, to wit: $J^* = \langle z^{*u+} - z^{*u-}\rangle$ are homogeneous binomials by design, and so $\mathcal{I}^*_{\mathcal{A}} \circ f(z^*) = 0$ cannot have any analytic functions as solutions.

This means that $J^*_{\mathcal{A}}$ must be identically 0, which is the case on the loci of z^* defined by the zero set of the aforementioned binomials, and so $\mathcal{D}^*_{\mathcal{A}}$ is not supported at these values. This subspace of V^* is the characteristic variety, $\text{ch}(V^*) \equiv \bar{V}^*$. Its dual is $\bar{V}$ and the complement of $\bar{V}$ in V, $V_0 = V\backslash\bar{V}$, is the proper domain of definition of z with non-singular solutions.

The orbit of the point $(\mathbb{1}_{n+m})$ by the torus $(x_1, \cdots x_{n+m}) \in (\mathbb{C}^*)^{n+m}$ is represented in V^* as $x^{\mathcal{A}} \equiv (x^{\mathcal{A}_1}, \cdots x^{\mathcal{A}_N}) \in \bar{V}^*_0 \subset \bar{V}^*$, with its embedding in V^* being $\hat{J} : \bar{V}^*_0 \to V^*$. The torus action of $x \in (\mathbb{C}^*)^{n+m}$ on $V^* \ni z^*$ is $(x^{\mathcal{A}_1}\, z^*_1, \cdots x^{\mathcal{A}_N}\, z^*_N)$, and it obviously leaves $\text{ch}(V^*)$ unchanged. Note that the orbit $\bar{V}^*_0$ is open. It doesn't include $z^{*u+} = z^{*u-} = 0$ including but not limited to the trivially singular $z^* = 0$.

Example 2.1.9. For the sake of some demystification, example 2.1.3 is recast in this language (see example 2.1.10 for preliminary setup). $z \in (\mathbb{C})^5 = V$, $z^* \in V^*$, and $\mathcal{D}_V$, $\mathcal{D}_{V*}$ consist of all possible differential operators that are polynomial in ∂_z, ∂_{z^*} and rational in z, z^* respectively. The characteristic variety is the zero set of J^*, $\bar{V}^* = \{z^*_1\, z^*_5 = z^{*2}_2 = z^*_3\, z^*_4\}$. It can be explicitly checked that $\bar{V}^*$ remains invariant under the torus action T parameterised by $\lambda \in (\mathbb{C}^*)^3$

$$z \to Tz, \quad z^* \to T^{-1} z^*, \quad T = (\lambda_1,\, \lambda_1\, \lambda_2\, \lambda_3,\, \lambda_1\, \lambda_2^2,\, \lambda_1\, \lambda_3^2,\, \lambda_1\, \lambda_2^2\, \lambda_3^2) \tag{2.118}$$

and the integral itself exhibits the expected homogeneity relation $I(Tz) = \lambda_1^{-\alpha}\, \lambda_2^{-\beta_1}\, \lambda_3^{-\beta_3} I(z)$. Thus, the orbit of $\mathbb{1}_3$ in V^*, $\bar{V}^*_0$, as parameterised by $\lambda \in (\mathbb{C}^*)^3$, doesn't include the hyperplane $z^*_1\, z^*_5 = z^*_2 = z^*_3\, z^*_4 = 0$, and its Zariski closure is indeed $\bar{V}^*$.

Another quotient module is defined on $\bar{V}^*_0$, $\mathcal{D}^*_{H*} = \mathcal{D}_{\bar{V}^*_0}/\mathcal{D}_{\bar{V}^*_0} \circ H^*$, $H^* = \langle -\mathcal{A}\,(\theta^* + 1) + \gamma\rangle$, which is locally isomorphic to $\mathcal{D}^*_{\mathcal{A}}$ by definition. Its extension to V^* by $\hat{J}$, denoted $\hat{J}\mathcal{D}^*_{H*}$, is irreducible and isomorphic to $\mathcal{D}^*_{\mathcal{A}}$. Since $\mathcal{D}^*_{\mathcal{A}}$ is irreducible, so is its Fourier transform $\mathcal{D}_{\mathcal{A}}$, the original D-module of interest.

$\mathcal{L}_\gamma$

The solution space of $\mathcal{D}^*_{H*}$ is locally isomorphic to $\mathcal{L}_\gamma$, a local system represented in $\bar{V}^*_0$ by branches of functions of the form x^γ, i.e. with monodromy exponents γ around $x = 0$.

Example 2.1.9 (continued)**.** Requiring $H^* \circ \bar{I}(\bar{z}^*) = 0$ on the open orbit $\bar{V}^*_0 \ni \bar{z}^*$ as defined in eq. (2.118) implies that $\bar{I}(\bar{z}^*) = \frac{\lambda_1^\alpha \lambda_2^{\beta_1} \lambda_3^{\beta_2}}{(\lambda_1 \lambda_2 \lambda_3)^5}$, i.e. with the monodromy exponents $\gamma = \{\alpha,\, \beta_1,\, \beta_2\}$.

This local system is extended to V^* by $\hat{J}$ to the sheaf $\hat{J}\mathcal{L}_\gamma$ and it is isomorphic to the solution space of $\hat{J}\mathcal{D}^*_{H*}$ and hence that of $\mathcal{D}^*_{\mathcal{A}}$ (specifically the N^{th} element of the solution complex),

$$\mathrm{Sol}(\mathcal{D}^*_{\mathcal{A}})[N] \cong \hat{J}\mathcal{L}_\gamma[n+m] \implies \mathrm{Sol}(\mathcal{D}_{\mathcal{A}})[N] \cong \mathscr{F}\hat{J}\mathcal{L}_\gamma[n+m] \tag{2.119}$$

also confirming the irreducibility of $\hat{J}\mathcal{L}_\gamma$ and its Fourier transform.

$\mathscr{H}(\alpha,\, \beta)$

$\Sigma = \{(x,\, P) \in (\mathbb{C}^*)^n \times \mathbb{C}^N \mid x \in \Sigma(P)\}$ (see eq. (2.98) for notation) is a disjoint union representing all polynomials and their supports (at every supported point x, P is given by N complex valued terms). π is the projection $\pi : \Sigma \to \mathbb{C}^N$. $\mathcal{L}(\alpha,\, \beta,\, P) = (\sigma,\, \mathscr{I}(\alpha,\, \beta;\, P))$ is a sheaf defined on $\Sigma(P)$, with its sections being complex valued functions $\mathscr{I}$. Elements of $\mathscr{H}(\alpha,\, \beta)$ at some point P are given by the n^{th} homology, $H_n(\Sigma(P),\, \mathcal{L}(\alpha,\, \beta,\, P))$, of the chain $C_q\big(\Sigma(P),\; \mathcal{L}(\alpha,\, \beta,\, P)\big)$, where $\mathcal{L}(\alpha,\, \beta, P) = \big(\sigma^{(q)},\, \mathscr{I}(\alpha,\, \beta,\, P)\big)$, with a core result of [20] being that $\mathscr{H}(\alpha,\, \beta)$ is indeed isomorphic to $H^{n+m}(\mathscr{F}\,\hat{J}\,\mathcal{L}_\gamma)$.

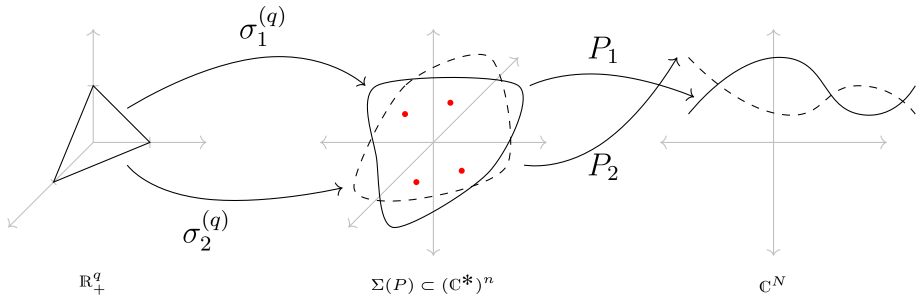


Figure 2.3: $\mathcal{L}(\mathbf{\Delta}^q \times \mathbb{C}^N)$: contour $\sigma^{(q)} = \sigma_1^{(q)} + \sigma_2^{(q)}$ from simplex $\mathbf{\Delta}^q$ to $\Sigma(P)$ to $\mathbb{C}^N$

Red points in $(\mathbb{C}^*)^n$ are indicative of the zeroes of the polynomials P and are not included in $\Sigma(P)$. $\sigma_1 \circ P_1$, $\sigma_2 \circ P_2$ are complex valued functions on the simplex $\mathbf{\Delta}^q$.

2.1.4 Summary of Generic Solution Algorithm

The following algorithm finds the $\mathcal{A}$-hypergeometric series describing the solution space of any GKZ ideal granting complete genericity of the parameters. If followed blindly, it may not yield the most efficient or elucidating results, but yield it will.

Example 2.1.10. The setup of example 2.1.3 is reused to illustrate.

$$I(z) = \int_\sigma \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2} \frac{x_1^{\beta_1}\,x_2^{\beta_2}}{(z_1 + z_2\,x_1\,x_2 + z_3\,x_1^2 + z_4\,x_2^2 + z_5\,x_1^2\,x_2^2)^\alpha} \tag{2.39}$$

1. Given a generalized Euler integral, that may be converted into a form satisfying the Irreducibility Conditions on $\mathcal{A}$,

$$I(z) = \int_\sigma \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2} \frac{x_1^{\frac{\beta_1+\beta_2}{2}}\,x_2^{\beta_2}}{(z_1 + z_2x_1\,x_2 + z_3\,x_1 + z_4x_1\,x_2^2 + z_5\,x_1^2\,x_2^2)^\alpha} \tag{2.120}$$

its $\mathcal{A}$ matrix and γ vector are found to be

$$\mathcal{A} = \begin{pmatrix} 1&1&1&1&1 \\ 0&1&1&1&2 \\ 0&1&0&2&2 \end{pmatrix}, \quad \gamma = \{\alpha,\, \tfrac{\beta_1+\beta_2}{2},\, \beta_2\}. \tag{2.121}$$

If confirmed to be convergent, an Euler integral over an open contour or a Feynman integral (usually with arbitrary fixed values of z and γ, and a contour over $\mathbb{R}/\mathbb{R}_+$) can be treated in the same manner. They are first assumed to be generalized, and then their appropriate limiting behaviours are considered.

2. $\mathcal{K}$ is a reduced basis representation in $\mathbb{Z}^N$ of the kernel of $\mathcal{A}$. The generators of the toric ideal J are $\partial^{|\mathcal{K}_+|} - \partial^{|\mathcal{K}_-|}$, where $\mathcal{K}_\pm$ refer to the sets of positive/negative integer valued elements in $\mathcal{K}$.

$$\mathcal{K} = \left(\begin{smallmatrix} -1 & 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 0 & 1 \end{smallmatrix}\right), \quad J = \langle \partial_3\, \partial_4 - \partial_1\, \partial_5,\ \partial_1\, \partial_5 - \partial_2^2 \rangle \tag{2.122}$$

Putting it together with eq. (2.101) gives the $\mathcal{A}$ hypergeometric ideal.

$$\mathcal{I} = J + H, \quad H = \langle \sum_{i=1}^{5} \theta_i + \alpha,\ \theta_2 + \theta_3 + \theta_4 + 2\,\theta_5 + \tfrac{\beta_1+\beta_2}{2},\ \theta_2 + 2\,\theta_4 + 2\,\theta_5 + \beta_2 \rangle \tag{2.123}$$

3. A Gröbner basis of J can be found by using Macaulay 2 [105] (with many useful packages and their usage described in [106]) or any other relevant math software. M is a monomial ideal of J, formed by only retaining a single monomial from all the binomials that generate J. The choice of monomial to be retained can be made by either picking the term with the higher weight wrt *any* weight vector $w \in \mathbb{R}^N$ or greater precedence wrt any order $\prec$. For example, the monomial ideal wrt $w = \{0,\ 1,\ 1,\ 0,\ 0\}$ is

$$J\Big|_{(w)} = \langle \partial_3\, \partial_4\Big|_{(1)} - \partial_1\, \partial_5\Big|_{(0)},\ \partial_1\, \partial_5\Big|_{(0)} - \partial_2^2\Big|_{(2)} \rangle \implies M = \langle \partial_3\, \partial_4,\ \partial_2^2 \rangle. \tag{2.124}$$

4. Given a monomial ideal $M = \langle \partial^a \rangle$, its distraction is $\tilde{M} = \langle \theta_{(a)} \rangle$ (see eq. (2.10) for notation).

$$\tilde{M} = \langle \theta_3\, \theta_4,\ \theta_2\, (\theta_2 - 1) \rangle \tag{2.125}$$

The roots, $\{s\}$, are the simultaneous solution to $\tilde{M} + H$.

$$s = \{\tfrac{\beta_2}{2} - \alpha,\, 0,\, 0,\, \tfrac{\beta_1 - \beta_2}{2},\, -\tfrac{\beta_1}{2}\}, \qquad \{\tfrac{\beta_1}{2} - \alpha,\, 0,\, \tfrac{\beta_2 - \beta_1}{2},\, 0,\, -\tfrac{\beta_2}{2}\},$$
$$\{\tfrac{\beta_2 - 1}{2} - \alpha,\, 1,\, 0,\, \tfrac{\beta_1 - \beta_2}{2},\, -\tfrac{\beta_1 + 1}{2}\}, \quad \{\tfrac{\beta_1 - 1}{2} - \alpha,\, 1,\, \tfrac{\beta_2 - \beta_1}{2},\, 0,\, -\tfrac{\beta_2 + 1}{2}\} \tag{2.126}$$

These roots are not the same as the ones presented in example 2.1.3, which correspond to a different monomial ideal $\langle \partial_3\, \partial_4,\ \partial_1\, \partial_5 \rangle$. Another possible monomial ideal is $\langle \partial_1\, \partial_5,\ \partial_2^2 \rangle$ with another set of associated roots. Though the choice of monomial ideal and hence set of roots is non-unique, the solution spaces spanned by each set are identical.

5. The integral I is spanned by a linear combination of the series:

$$I = \sum_s N_s\, z^s \sum_{t \in \mathbb{Z}} \frac{\Gamma(s+1)}{\Gamma(s + t_1\, \mathcal{K}_1 + t_2\, \mathcal{K}_2 + 1)}\, z^{t_1\, \mathcal{K}_1 + t_2\, \mathcal{K}_2}, \quad N_s \in \mathbb{C}. \tag{2.127}$$

Considering limits of the original integral with $z \to 0$ matching the zeroes in the roots can be used to find the normalisation constants, N_s.

2.2 Ideals to Solutions

Given the isomorphism between Euler integrals and $\mathcal{A}$-hypergeometric systems' solution space, the next step is to find the latter and to strip away the magic from section 2.1.4.

An N-dimensional Weyl algebra, D_N, is a free associative non-commutative algebra defined over a field of characteristic zero, here $\mathbb{C}$,

$$D_N = \mathbb{C}\langle x_1,\, x_2,\, \cdots x_N,\, \partial_1,\, \partial_2,\, \cdots \partial_N \rangle \tag{2.128}$$

modulo the commutation relations $[x_i,\, x_j] = [\partial_i,\, \partial_j] = 0$ and $[\partial_i,\, x_j] = \delta_{ij}$. The ambiguity of multiple equivalent representations of the same elements in D_N is fixed by demanding that every element be normal ordered by commuting all ∂ to the left,

$$d = \sum_{(\alpha,\,\beta)\in E} c_{\alpha,\,\beta}\, x^{\alpha}\, \partial^{\beta}, \quad c_{\alpha,\,\beta} \neq 0 \;\forall\; (\alpha,\, \beta) \in E, \quad d \in D_N \tag{2.129}$$

making D_N the space of differential operators $\mathbb{C}[\partial_x]$ with coefficients in the ring of polynomials $\mathbb{C}[x]\,:\mathbb{C}^N \to \mathbb{C}$ with a vector space isomorphism $\Psi : \mathbb{C}[x,\, \partial_x] \to \mathbb{C}[x,\, \xi]$, where $\mathbb{C}[x,\, \xi]$ is a $2N$ dimensional commutative ring. Products and Poisson brackets on D_N are pushed forward by Ψ to

$$\Psi(f(x,\,\xi))\,\Psi(g(x,\,\xi)) = \sum_{k\in\mathbb{N}_0^n} \tfrac{1}{k_1!\cdots k_n!}\,\Psi\big(\tfrac{\partial^k f}{\partial \xi^k}\,\tfrac{\partial^k g}{\partial x^k}\big), \quad [f,\, g]_{\mathrm{PB}} = \sum_{i=1}^{n} \tfrac{\partial f}{\partial \xi_i}\,\tfrac{\partial g}{\partial x_i} - \tfrac{\partial g}{\partial \xi_i}\,\tfrac{\partial f}{\partial x_i}. \tag{2.130}$$

D_N is a subset of the ring of differential operators with coefficients in rational functions, $\mathcal{D}_N$, by definition. $\mathbb{C}[x]$, $\mathbb{C}[\partial]$, $\mathbb{C}[\theta]$ are commutative sub-rings in D_N, the relations between which can be concretised by using identities like

$$\begin{aligned} &x^{\alpha}\,\partial^{\alpha} = \theta_{(\alpha)}, && \partial^{\alpha}\,x^{\alpha} = \theta^{(\alpha)}, && \theta_i := x_i\,\partial_i \\ &f(\theta)\,x^{\alpha} = x^{\alpha}\,f(\theta+\alpha), && x^{\alpha}\,f(\theta)\,\partial^{\beta} \to \theta^{(\alpha)}\,f(\theta - b)\,\theta_{(\beta)}, && f \in \mathbb{C}[\theta]. \end{aligned} \tag{2.131}$$

The Mellin and inverse Mellin transforms are defined as

$$\mathcal{M} \circ f\,(s) = \int_0^{\infty} \frac{\mathrm{d}x}{x}\, x^s\, f(x), \quad \mathcal{M}^{-1} \circ f\,(x) = \oint_s x^{-s}\, f(s), \quad \oint_s \equiv \int_{\delta - i\infty}^{\delta + i\infty} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \tag{2.132}$$

where $\delta \in \mathbb{R}$ is such that the integral is absolutely convergent with $f(s)$ being analytic over the line integral and uniformly tending to 0 at the end points. The Mellin transform induces an isomorphism from $\mathbb{C}[x,\, \theta]$ to the ring $\langle s,\, \delta_s \rangle$, where δ_s is the difference operator satisfying the commutation relation $\delta_s\, s = (s+1)\,\delta_s$, via the transformations $s + 1 \mapsto -\theta,\ \delta_s \mapsto x$, i.e. matching the geometric Fourier transform defined in eq. (2.116). As is to be expected, Euler integrals and GKZ systems can also be represented in the form of Mellin-Barnes integrals [107].

2.2.1 Varieties of Ideals

A left ideal or D-ideal, $\mathcal{I}$, is a subset of the Weyl algebra, D_N, satisfying the properties:

$$(1) \quad 0 \in \mathcal{I}, \quad (2) \quad f_1,\, f_2 \in \mathcal{I} \Rightarrow f_1 + f_2 \in \mathcal{I}, \quad (3) \quad f \in I,\, d \in D_N \Rightarrow d \circ f \in \mathcal{I}. \tag{2.133}$$

A right ideal, $\mathcal{I}_{\mathrm{R}}$, is equivalently defined with the third condition changed to $f \circ d \in \mathcal{I}_{\mathrm{R}}$. The $\mathcal{A}$-hypergeometric system, $\mathcal{I}_{\mathcal{A}}$, forms a left ideal in D_N and its solution space corresponds to the D-module $\mathcal{D}_N/\mathcal{I}_{\mathcal{A}}$.

A variety, V, defined by a set of polynomials, $\{f(x)\}$, is their zero locus/solution space in affine $\mathbb{C}^N$. This implies intersections and unions of varieties (which can be defined by the union and products of the sets of defining polynomials respectively) are also affine varieties. This can be extended to include rational functions $g_i = \frac{1}{f_i}$ by introducing dummy variables y_i, replacing the rational functions with $g_i' = y_i\, f_i - 1$ and subsequently eliminating y_i [108]. The ideal of a variety, $\mathcal{I}(V)$, is the set of functions which have the variety as the solution space, i.e.

$$\mathcal{I}(V) = \{f \mid f(v) = 0 \;\forall\; v \in V\}. \tag{2.134}$$

The Zariski closure of some subset $U \subseteq \mathbb{C}^N$ is the smallest variety containing U, i.e. if there exists a variety containing U, it contains the Zariski closure. The variety of an ideal, $V(\mathcal{I})$, is the Zariski closure of the zero locus of the polynomials generating the ideal. The radical of an ideal, $\sqrt{\mathcal{I}}$, is a superset of $\mathcal{I}$ that contains of all elements such that

$$f^n \in \mathcal{I},\; n \in \mathbb{N} \implies f \in \sqrt{\mathcal{I}}. \tag{2.135}$$

The radical, $\sqrt{\mathcal{I}}$, equals $\mathcal{I}(V(\mathcal{I}))$ when the field is algebraically closed. A trivial example: the variety of the ideal $\mathcal{I}_0 = \langle x^2 \rangle \in \mathbb{C}[x]$ is the origin $V(\mathcal{I}_0) = \{0\}$. However, the ideal of $\{0\}$ is $\mathcal{I}(V(\mathcal{I}_0)) = \langle x \rangle$, which isn't equal to the original ideal but includes it and is its radical $\sqrt{\mathcal{I}_0}$.

An ideal is prime if for every $(f_1\, f_2) \in \mathcal{I}$, either $f_1 \in \mathcal{I}$ or $f_2 \in \mathcal{I}$, but not both. An ideal $\mathcal{I}$ is primary if there exists an $m \in \mathbb{N}$ such that $g^m \in Q$ for all $f \notin Q$, $f\, g \in Q$. Any proper ideal has an irreducible primary ideal decomposition, i.e. it can be represented as an intersection of finitely many primary ideals Q_i such that

$$\mathcal{I} = \bigcap_i Q_i, \quad Q_k \nsubseteq Q_i \bigcap Q_j, \quad \sqrt{Q_i} \neq \sqrt{Q_j}, \quad i \neq j \neq k. \tag{2.136}$$

The set of $\sqrt{Q}$ are the associated prime ideals. $V(\mathcal{I})$ is the union of the irreducible pieces $V(\sqrt{Q})$, and $V(\mathcal{I}(V)) = V$. An elimination ideal, $\mathcal{I}_{x'}$, is defined as $\mathcal{I} \bigcap \mathbb{C}[x']$, i.e. it consists of all elements of $\mathcal{I}$ from which the variables $x \backslash x'$ are eliminated. The integrand of an Euler integral belongs to the Weyl algebra $\mathbb{C}\langle x,\, \partial_x,\, z,\, \partial_z\rangle$. Given $\mathcal{I}$ that annihilates the integrand, the ideal annihilating the Euler integral is the elimination ideal $\mathcal{I}_{z,\, \partial_z}$ (see example 2.1.5).

An ideal in a commutative ringis freely generated by a finite basis $\mathcal{I} = \langle f_1,\, f_2,\, \cdots f_\omega\rangle, \omega < \infty$. D-ideals in D_N have finite bases too, as can be intuited given the isomorphism Ψ. The associated graded ring of D_N, $\mathrm{gr}_{(u,\, v)}(D_N)$, wrt the weight vector $(u,\, v) \in \mathbb{R}^{2N}$, $u_i + v_i \geqslant 0$ is generated by

$$\{x\} \bigcup \{\partial_i \,|\, u_i + v_i > 0\} \bigcup \{\xi_i \,|\, u_i + v_i = 0\} \tag{2.137}$$

with the initial form of a general element d, being its restriction to $\mathrm{gr}_{(u,\, v)}(D_N)$,

$$\mathrm{in}_{(u,\, v)}(d) = \sum_{u\,\alpha + v\,\beta = \max E_{(u,\, v)}} c_{\alpha,\, \beta}\; x^\alpha \prod_{u_i + v_i = 0} \partial_i^{\beta_i} \prod_{u_i + v_i > 0} \xi_i^{\beta_i} \in \mathrm{gr}_{(u,\, v)}(D) \tag{2.138}$$

and the initial ideal of a left ideal $\mathcal{I}$ forming a left ideal in $\mathrm{gr}_{(u,\, v)}(D_N)$

$$\mathrm{in}_{(u,\, v)}(\mathcal{I}) = \{\mathrm{in}_{(u,\, v)}(f) \mid \forall\, f \in \mathcal{I}\}. \tag{2.139}$$

Weight vectors that produce the same initial ideal are equivalent. An ordering prescription, $\prec$, attaches a symbolic weight to the generators of D_N, e.g. $\cdots x_2 \prec x_1 \prec \cdots \partial_2 \prec \partial_1$ (lexicographic),

$\partial \prec x$ (reverse lexicographic), etc. $\prec$ is called a term order when it satisfies the property:

$$1 \prec x^{\alpha}\,\partial^{\beta} \prec x^{\alpha+s}\,\partial^{\beta+t}, \quad \alpha+\beta \geqslant 1, \quad \forall\,(s,\,t) \in \mathbb{N}^{2n}, \quad 1 \prec \log x. \tag{2.140}$$

An induced order, $\prec_{(w)}$, uses $\prec$ as a tie breaker between terms of equal weight, for example the graded lexicographic order $\prec_{(\mathbb{1})}$ induces the order $x_2 \prec x_1 \prec x_1\,x_2^2 \prec x_1^2\,x_2$ in D_2, where the last 2 terms have the same total degree so the degree of the lexicographically 'highest' ordered variable, x_1, is used for tie-breaking. An initial monomial of a general element d, $\text{in}_{\prec}(d)$, according to the order $\prec$ is the "largest" term in d such that

$$\text{in}_{\prec}(d) = c_{\bar{\alpha},\,\bar{\beta}}\,x^{\bar{\alpha}}\,\partial^{\bar{\beta}}, \quad c_{\bar{\alpha},\,\bar{\beta}}\,x^{\bar{\alpha}}\,\partial^{\bar{\beta}} \succeq c_{\alpha,\,\beta}\,x^{\alpha}\,\partial^{\beta}\;\forall\,\{\alpha,\,\beta\} \in E. \tag{2.141}$$

When the coefficient of $\text{in}_{\prec}(d)$ is 1, d is said to be in monic form. The monomial ideal of $\mathcal{I}$ wrt $\prec$, $\text{in}_{\prec}(\mathcal{I})$, consists of the initial monomials of all the elements in $\mathcal{I}$. Ordering prescriptions allow the equivalent of subtraction (S-pair) and division (Normal form) operations to be defined in D_N. The S-pair of $f,\,g$ wrt $\prec$, with $\text{in}_{\prec}(f) = f_{\alpha,\,\beta}\,x^{\alpha}\,\xi^{\beta}$, $\text{in}_{\prec}(g) \equiv g_{a,\,b}\,x^{a}\,\xi^{b}$, is

$$\text{Sp}_{\prec}(f,\,g) = x^{\alpha'}\,\partial^{\beta'}\,f - \frac{f_{\alpha,\,\beta}}{g_{a,\,b}}\,x^{a'}\,\partial^{b'}g, \quad \alpha' = \max(a,\,\alpha) - \alpha, \cdots. \tag{2.142}$$

For example the S-pair of $f = x_1 + 2\,x_2\,\partial_1$, $g = x_2 + x_1^2\,\partial_2$ wrt $x \prec \partial$ is

$$\begin{aligned} \text{in}_{\prec}(f) &= 2\,x_2\,\partial_1, \quad \text{in}_{\prec}(g) = x_1^2\,\partial_2, \\ \text{Sp}_{\prec}(f,\,g) &= x_1^2\,\partial_2\,f - 2\,x_2\,\partial_1\,g = 2\,x_1^2\,\partial_1 - 4\,x_1\,x_2\,\partial_2. \end{aligned} \tag{2.143}$$

The Normal form, $\text{NF}_{\prec}(d)$ by $G \subset D_N$, equals the result of the recursive algorithm: Start with $\text{NF}_{\prec}(d) = 0$, (i) While $\exists\,g \in G$ such that $\text{in}_{\prec}(d) = x^p\,\text{in}_{\prec}(g)\,\partial^q$, $p, q \in \mathbb{N}_0$, $d = \text{Sp}_{\prec}(d, g)$. (ii) $\text{NF}_{\prec}(d) = \text{in}_{\prec}(d) + \text{NF}_{\prec}(d - \text{in}_{\prec}(d))$. When $\text{NF}_{\prec}(d)$ by $G \subset D_N$ equals 0, d has a representation in G. For example $\text{NF}_{\prec}(x_1^4\,\partial_2^3 + x_2^2\,\partial_1^2)$ by $G = \{x_1\,\partial_2,\;\; x_2\,\partial_1\}$ wrt $x \prec \partial$ is expectedly 0.

$$
\begin{aligned}
&\text{in}_{\prec}(d) = x_1^4\,\partial_2^3 \in D \circ g_1 \\
&\quad \Longrightarrow \; d \to d' = \text{Sp}_{\prec}(d,\, g_1) = x_1^4\,\partial_2^3 + x_2^2\,\partial_1^2 - x_1^3\,\partial_2^2\, g_1 = x_2^2\,\partial_1^2 \\
&\text{in}_{\prec}(d') = x_2^2\,\partial_1^2 \in D \circ f_2 \\
&\quad \Longrightarrow \; d' \to d'' = \text{Sp}_{\prec}(d,\, g_2) = x_2^2\,\partial_1^2 - x_2\,\partial_1\, g_2 = 0
\end{aligned}
\tag{2.144}
$$

A Gröbner basis, $\mathcal{G}$, of $\mathcal{I}$ wrt the weight $(u,\, v)$ is a finite basis of $\mathcal{I}$ that respects the grading $\text{gr}_{(u,\,v)}(D_N)$, i.e. the initial forms of $\mathcal{G}_{(u,\,v)}$, $\text{in}_{(u,\,v)}(\mathcal{G}_{(u,\,v)})$, form a basis of the initial ideal $\text{in}_{(u,\,v)}(\mathcal{I})$ in $\text{gr}_{(u,\,v)}(D_N)$. Similarly, the initial monomials of a Gröbner basis, $\mathcal{G}_{\prec}$, of $\mathcal{I}$ wrt an order $\prec$ form a basis of the monomial ideal $\text{in}_{\prec}(\mathcal{I})$.

$$
\mathcal{I} = \langle \mathcal{G}_* \rangle, \quad \text{in}_*(\mathcal{I}) = \big\langle \text{in}_*(\mathcal{G}_*) \big\rangle, \quad * = (u, v), \;\; \prec
\tag{2.145}
$$

Given any finite subset of D_N and an order $\prec$ on it, there exists a non-negative integer weight $(u,\, v)$ such that the initial forms wrt $(u,\, v)$ equal the initial monomials wrt $\prec$. A Gröbner basis $\mathcal{G}$ wrt $(u,\, v)$ or $\prec$ admits a standard representation of all elements of $\mathcal{I}$ such that

$$
f = \sum d_i\, g_i, \quad f \in \mathcal{I}, \quad d_i \in D_N, \quad g_i \in \mathcal{G}_*, \quad \text{in}_*(f) \geq \text{in}_*(d_i\, g_i) \quad \forall\, i.
\tag{2.146}
$$

The homogenised Weyl algebra, $D_N^{(h)}$, is generated by the elements of the Weyl algebra D_N and an additional homogenisation variable h. h commutes with all the generators of D_N and acts as a generalisation of $1 = x^0\,\partial^0 \in D_N$, such that the commutation relation $[\partial_i,\, x_j] = \delta_{ij}$ in D_N changes to $[\partial_i,\, x_j] = h^2\,\delta_{ij}$ in $D_N^{(h)}$. A weight vector, $w = (t,\, u,\, v)$, in $\mathbb{R}^{2N+1}$, with $u + v \geqslant 2\,t$ grades $D_N^{(h)}$, where t is the weight of h. The homogenisation of an element d, $H(d)$, makes the total degree of every term in d equal,

$$
H(d) = \sum_{(\alpha,\,\beta)\in E} c_{\alpha\beta}\, h^{\deg(d)-|\alpha|-|\beta|}\, x^\alpha\, \partial^\beta, \quad \deg(d) = \max_{(\alpha,\,\beta)\in E}(|\alpha| + |\beta|).
\tag{2.147}
$$

So $H(d)$ equals its initial form wrt the weight $w = (\mathbb{1})$. The maximum number of monomials in any $H(d)$ is set by $\deg(d)$, formally precluding any infinitely long elements from a finite dimensional $D^{(h)}$. The homogenised ideal, $H(\mathcal{I})$, is expectedly defined as $\{H(f)\ \forall\ f \in \mathcal{I}\}$. Setting $h = 1$ dehomogenises $D_N^{(h)}$ to D_N and can be viewed as a restriction of $D_N^{(h)}$ to a slice in affine $\mathbb{R}^{2N+1}$.

There always exists a universal Gröbner basis for any ideal in $D_N^{(h)}$ that satisfies eq. (2.145) for generic weights and orders. An irredundant and unique form thereof is the reduced Gröbner basis, $\mathcal{G}^{(h)}$, such that for any pair of distinct elements $g,\ g' \in \mathcal{G}^{(h)}$, no term of $\Psi(g')$ is divisible by $\text{in}_*(g)$.[2] Every element of the ideal has a unique standard representation in $\mathcal{G}^{(h)}$. These properties are also carried over to D_N by the dehomogenisation of $\mathcal{G}^{(h)}$, i.e. by the sequence of operations: $\mathcal{I} \in D_N \to H(\mathcal{I}) \in D_N^{(h)} \to \mathcal{G}^{(h)} \in D_N^{(h)} \to \mathcal{G}^{(h)}\big|_{h=1} = \mathcal{G} \in D_N$.

2.2.2 Newton Polytopes to Gröbner Cones

An n-dimensional polytope, Q, has vertices on the lattice $\mathbb{Z}^n$ and has normalised volume such that a regular simplex, Σ_n, with vertices given by $(0,\ e_1,\ \cdots\ e_n)$ has volume 1, where e_i are the basis vectors of the lattice.

$$\Sigma_n = \{v \in \mathbb{R}_+^n \mid \sum_{i=1}^{n} v_i = 1\}, \quad \text{Vol}(\Sigma_n) = 1, \quad \text{Vol}(Q) = n!\ \text{Vol}_{\text{Euclidean}}(Q) \tag{2.148}$$

The Newton polytope of an element d as defined in eq. (2.129) is the convex hull of (i.e. smallest convex space enclosing) the points $(\alpha,\ \beta) \in E$ in affine $\mathbb{R}^{2N}$ space. It is represented as an intersection of half planes defined by vectors w_i and constants χ_i,

$$\text{NP}(d) = \bigcap_i \{v \in \mathbb{R}^{2N} \mid v \cdot w_i \geqslant \chi_i\}. \tag{2.149}$$

[2] Divisibility is defined exactly as expected: g_1 is divisible by g_2 in D if $\exists\, d \in D$ such that $d \circ g_2 = g_1$.

For example, the Newton polytope of $d = 1 + x + \partial + x\,\partial \in D_1$ is a unit square in $\mathbb{R}^2$ is

$$\mathrm{NP}_{1^2} = \{v \in \mathbb{R}^2 \mid v \cdot (1,\,0) \geqslant 0,\ v \cdot (0,\,1) \geqslant 0,\ v \cdot (-1,\,0) \geqslant -1,\ v \cdot (0,\,-1) \geqslant -1\} \quad (2.150)$$

which is a rather long form representation of $\{(x,\,y) \in \mathbb{R}^2 \mid 1 \geqslant x \geqslant 0,\ 1 \geqslant y \geqslant 0\}$ with normalised volume 2. NP_{1^2} is also the Newton polytope of the polynomial $1 + x_1 + x_2 + x_1\,x_2$.

The NP associated with the $\mathcal{A}$-hypergeometric ideal $\mathcal{I}_{\mathcal{A}}$ is the convex hull of the points with coordinates given by the columns vectors of $\mathcal{A}$. Any translations in affine space correspond to multiplying the polynomials in an Euler integral by some monomial, both being expectedly redundant operations.

The face, $\mathrm{F}_w(Q)$, of a polytope, Q, wrt a weight w is the section of the polytope with the maximum w-weight:

$$\mathrm{F}_w(Q) = \{v \in Q \mid v \cdot w \geqslant v' \cdot w \;\forall\; v' \in Q\}. \quad (2.151)$$

A facet is an $n - 1$ dimensional face of n dimensional Q. An edge is a 1 dimensional face. The edges (also facets in this case) and vertices of the 2D unit square, NP_{1^2}, are its faces wrt weight vectors $(0,\,\pm 1)$, $(\pm 1,\,0)$ and $(\pm 1,\,\pm 1)$, $(\pm 1,\,\mp 1)$ respectively. If Q is the Newton polytope of d, then every face of Q corresponds to an initial form of d.

The equivalence class of weights that generate the same face of a polytope is called a normal cone. The sum of the dimensions of a face and its normal cone equals the dimension of the polytope. A cone, $\mathcal{C}$, in affine space can be described by some basis vectors v_i as $\mathcal{C} = \{\sum w^i \cdot v_i\}$, $w^i \in \mathbb{R}_+$. It is strongly convex when $\mathcal{C} \bigcap -\mathcal{C} = \emptyset$. The polar/dual cone $\mathcal{C}^*$ is $\{v \mid v \cdot \mathcal{C} \geqslant 0\}$.

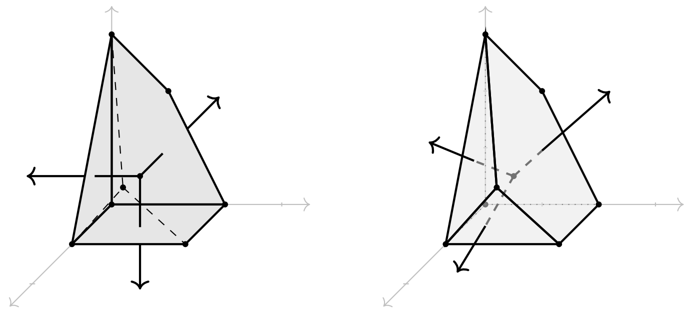


Figure 2.4: Newton polytope, Facets, and Normal Cones

The Newton polytope of the polynomial $1 + x + y^2 + z^3 + y\,z^2 + x\,y^2 + x\,y\,z$ is pictured. Its facets/codimension-1 faces are shaded in and the arrows are representative vectors of the facets' normal cones.

Weight vectors within the same equivalency class sweep out a cone in $\mathbb{R}^{2N}$ and form a Gröbner cone, with each cone corresponding to an initial ideal. A fan can be roughly described as a collection of cones. The small Gröbner fan consists of the all Gröbner cones of weight vectors of the form $(-w,\, w)$, with the fan covering all $\text{in}_{(-w,\, w)}(\mathcal{I})$. The Gröbner fan consists of all possible Gröbner cones and so effectively enumerates all possible initial ideals. It is necessarily finite implying the existence of a finite universal Gröbner basis. The Gröbner fan of $\mathcal{I} \subset D$ is the $t = 0$ slice of the Gröbner fan of $H(\mathcal{I}) \subset D^{(h)}$.

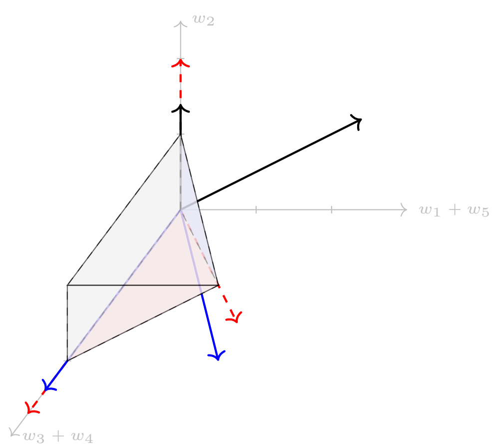


Figure 2.5: Gröbner cone of Toric ideal : eq. (2.124) in example 2.1.10

The choice of weight vector in eq. (2.124) is obviously non-unique and any weight vector of the form $w = \{w_i\}$ such that $w_3 + w_4 > w_1 + w_5$ (interior of blue arrows) and $2\,w_2 > w_1 + w_5$ (interior of black arrows) belongs to the equivalence class of $\{0,\ 1,\ 1,\ 0,\ 0\}$. The dashed red arrows serve as a basis for the Gröbner cone, enclosing it.

2.2.3 Holonomic Characteristics

The characteristic ideal is the initial ideal with respect to the weight $(0, \mathbb{1})$, $\text{in}_{(0, \mathbb{1})}(\mathcal{I})$, defined in the commutative ring $\mathbb{C}[x, \xi]$. The zero set of the characteristic ideal in affine $\mathbb{C}^{2N}$ space is the characteristic variety, $\text{ch}(\mathcal{I})$, equalling the characteristic variety of the D-module $\mathcal{D}/\mathcal{I}$.

For a commutative ideal, $\mathcal{J}$, the number of monomials of total degree $\leqslant k$ which aren't divisible by $\text{in}_{<}(\mathcal{J})$ is polynomial for sufficiently large k. When $\mathcal{J} = \text{in}_{(0, \mathbb{1})}(\mathcal{I})$, it is called the Hilbert polynomial, $p(k)$, of the ideal $\mathcal{I}$. It is known to have the form $p(k) = \frac{m}{N!} k^N + \cdots$.

The degree of the Hilbert polynomial, N, is the (Krull) dimension of $\mathcal{I}$. A 0-dimensional ideal is called Artinian and has a finite zero set i.e. points, corresponding to the roots. When $\mathcal{I}$ is homogeneous, m is called the degree of $\mathcal{I}$. A D-ideal, $\mathcal{I} \subset D_N$, is holonomic if its characteristic ideal has dimension N.

An ideal is integrable if it is closed under Poisson brackets, a characteristic ideal being so trivially. If an ideal is integrable, so is its characteristic ideal and the radical of its characteristic ideal. Thus, the characteristic variety, which is the zero set of the characteristic ideal, is also integrable (see eq. (2.136)). A linear space in $\mathbb{C}^{2N}$ ceases to be integrable if both $x_i = \xi_i = 0$ on it. Thus, it must have a dimension of at least N. The characteristic ideal of a proper D-ideal also has dimension $\geqslant N$. A proper D-ideal $\mathcal{I}$ is holonomic iff the dimension of $\text{in}_{(u, v)}(I)$ is N for generic $(u, v) \in \mathbb{R}^{2N}_{+}$ such that $u + v > 0$.

The dimension of the complex vector space of holomorphic solutions to $\mathcal{I} \circ f = 0$ on a simply connected domain in $\mathbb{C}^n \setminus \text{S}(I)$ is $\text{rank}(\mathcal{I})$, i.e the number of $\mathbb{C}$-linearly independent holomorphic solutions outside the singular locus. The singular locus, $\text{S}(\mathcal{I})$, is the Zariski closure of the image of $\text{ch}(\mathcal{I}) \setminus \{\xi = 0\}$ under the coordinate projection $\mathbb{C}^{2N} \rightarrow \mathbb{C}^N : (x, \xi) \mapsto x$, i.e. it is the smallest variety enclosing the zero set of the projected ideal, which is found by eliminating ξ from the characteristic ideal. The rank of holonomic ideals is finite and defined as the vector space dimension over the field $\mathbb{C}$:

$$\text{rank}(\mathcal{I}) := \dim(\mathbb{C}(x)\,[\xi]/\mathbb{C}(x)\,[\xi] \circ \text{in}_{(0,\,\mathbb{1})}(\mathcal{I})). \tag{2.152}$$

If $M \subset \mathbb{C}[\xi]$ is a monomial ideal wrt the order $\prec$ of the restriction of the characteristic ideal $\text{in}_{(0,\,\mathbb{1})}(\mathcal{I})$ by $x \mapsto 1$ to $\mathbb{C}[\xi]$, the rank is also given by

$$\text{rank}(\mathcal{I}) = \dim(\mathbb{C}[\xi]/M) = \#\{\xi^\alpha \notin M,\ \alpha \in \mathbb{N}_0^N\}. \tag{2.153}$$

Example 2.2.1. Given a left ideal $\mathcal{I} = \langle x_1\,\partial_2,\ x_2\,\partial_1\rangle$ in D_2, the characteristic ideal, $\text{in}_{\prec_{(0,\,\mathbb{1})}}(\mathcal{I})$ is $\langle x_1\,\xi_2,\ x_2\,\xi_1\rangle$. The monomial ideal, M, upon projection $x \mapsto 1$ is $\langle \xi_1,\, \xi_2\rangle$. Obviously M can generate every element all of $\mathbb{C}[\xi_1,\, \xi_2]$ apart from 1. Thus, $\text{rank}(\mathcal{I}) = 1$.

2.2.4 Torus Fixed Ideals

The action of the N-dimensional algebraic torus $T = (\mathbb{C}^*)^N$ on D_N is

$$T \times D_N \to D_N \ :\ (t_i,\, \partial_i) \mapsto t_i\,\partial_i \ :\ (t_i,\, x_i) \mapsto t_i^{-1}\,x_i. \tag{2.154}$$

A D-ideal which remains invariant under the torus action, $T \circ \mathcal{I} = \mathcal{I}$, is torus fixed/invariant. $\mathbb{C}[\theta]$ represents all torus fixed elements of D_N, with a general D-ideal $\mathcal{I}$ being torus fixed iff it consists of elements of the form $x^\alpha\, f(\theta)\,\partial^\beta$, $\alpha,\,\beta \in \mathbb{N}$, $f(\theta) \in \mathbb{C}[\theta]$ or equivalently iff $\text{in}_{(-w,\,w)}(\mathcal{I}) = \mathcal{I}$ for all non-negative weights $w \in \mathbb{R}_+^N$. For sufficiently generic w, $\text{in}_{(-w,\,w)}(\mathcal{I})$ is torus fixed by default. The distraction, $\tilde{\mathcal{I}}$, of a torus fixed $\mathcal{I}$ is

$$\tilde{\mathcal{I}} = \mathbb{C}[x][\partial] \circ \mathcal{I} \bigcap \mathbb{C}[\theta] = \langle \theta_{(b)}\, f(\theta - b)\rangle, \quad \mathcal{I} = \langle x^\alpha\, f(\theta)\,\partial^\beta\rangle. \tag{2.155}$$

$\mathcal{I}$ is a Frobenius ideal if $\mathcal{I} \subseteq \mathbb{C}[\theta]$. It is Artinian if $\mathbb{C}[\theta]/\mathcal{I}$ is finite dimensional, with the dimension equalling the rank of $\mathcal{I}$. The indicial ideal is a generalisation of a Frobenius ideal, defined as

$$\text{ind}_{(w)}(\mathcal{I}) = \mathbb{C}[x][\partial] \circ \text{in}_{(-w,\,w)}(\mathcal{I}) \bigcap \mathbb{C}[\theta], \quad w \in \mathbb{R}^N. \tag{2.156}$$

It is a holonomic Frobenius ideal with rank equal to the rank of the initial ideal $\text{in}_{(-w,\,w)}(\mathcal{I})$. The zeros of the indicial ideal in affine $\mathbb{C}^N$ are called exponents. When counted with multiplicity, there are rank$(\mathcal{I})$ exponents. In short, for a holonomic ideal $\mathcal{I}$:

$$\text{rank}(\mathcal{I}) \underset{= \text{ for regular}}{\geqslant} \text{rank}(\text{in}_{(-w,\,w)}(\mathcal{I})) = \text{rank}(\text{ind}_{(w)}(\mathcal{I})) = \#\text{Exponents}. \tag{2.157}$$

The b-function, $b(\omega)$, or indicial polynomial of an ideal $\mathcal{I}$ can be algorithmically computed by eliminating $\{\theta\}$ from the intersection of $\mathbb{C}\{\omega - w\,\theta\}$ and $\text{in}_{(-w,\,w)}\mathcal{I} \bigcap \mathbb{C}[\theta]$, found by applying eq. (2.131) on the Gröbner basis of $\mathcal{I}$ wrt weight $(-w,\, w)$.

2.2.5 Solutions of Holonomic D-ideals

Holonomic D-ideals have convergent series solutions of the form f around the point c,

$$\mathcal{I} \circ f = 0, \quad f = \mathbb{C}[(x-c),\, \log(x-c)] = \sum c_{\alpha\beta}\, x^{\alpha}\, (\log x)^{\beta}, \quad \alpha \in \mathbb{C}, \quad \beta \in \mathbb{N}_0 \tag{2.158}$$

if the singular locus, $\text{S}(\mathcal{I})$, is a normal crossing divisor at c, i.e. it can be locally represented by a polynomial of the form

$$\text{S}(\mathcal{I})\Big|_{|x-c| \ll 1} \cong (x-c)^s \cong x^s\,(1 + \sum_{u>0} c_u\, x^u). \tag{2.159}$$

The Newton polytope of $\text{S}(\mathcal{I})$, $\text{NP}(\mathcal{I})$, is the convex hull of the points $A = \{A_i\}$ in $\mathbb{R}^n$ such that

$$\text{S}(\mathcal{I}) = \sum c_i\, x^{A_i}, \quad c_i \in \mathbb{C}^*. \tag{2.160}$$

A cone, $\mathcal{C}_{A_i}$, originating from a vertex A_i enclosing NP($\mathcal{I}$) is unimodular if it can be represented by n integer vectors u_i such that

$$\mathcal{C}_{A_i} = \sum_{i=1}^{n} c_i\, u_i, \quad c_i \in \mathbb{R}_+, \quad u_i \in \mathbb{Z}^n \quad U = \{u_1,\, u_2, \cdots u_n\}, \quad |\det U|^2 = 1. \tag{2.161}$$

Its polar/dual cone, $\mathcal{C}^*$, is generated by the vectors $(U^{-1})^{\mathrm{Tr}} \equiv U^*$.

Example 2.2.2. Appell's $F_2(x,\, y)$ is defined as the solution to the differential equations generated by $\langle \theta_x^2 - x\,(\theta_x + \theta_y + a)\,(\theta_x + b_1),\, \theta_y^2 - y\,(\theta_x + \theta_y + a)\,(\theta_y + b_2)\rangle$. This representation is a Gröbner basis wrt any weight $(-w,\, w)$, $w > 0$. Its singular locus is $x\,y\,(1-x)\,(1-y)\,(1-x-y)$ [96].

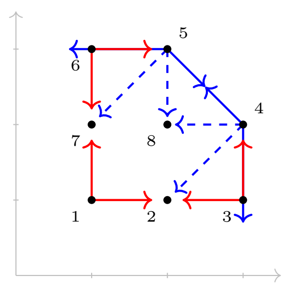


Figure 2.6: Newton Polytope of S($\mathcal{I}$), Unimodular Cone, Polar Cones of Appell's F_2

The interior of the black dots in fig. (2.6) is the newton polytope. The red arrows indicate the generators of the unimodular cones that coincide with their dual cones originating from the vertices $\{1,\, 3,\, 6\}$. The blue and dashed blue arrows encapsulate the unimodular and polar cones respectively at vertices $\{4,\, 5\}$.

$$\begin{aligned} &\mathcal{C}_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \mathcal{C}_3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \mathcal{C}_4 = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right), \quad \mathcal{C}_5 = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right), \quad \mathcal{C}_6 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \\ &\mathcal{C}_1^* = \mathcal{C}_1, \qquad \mathcal{C}_3^* = \mathcal{C}_3, \qquad \mathcal{C}_4^* = \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right), \quad \mathcal{C}_5^* = \left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right), \quad \mathcal{C}_6^* = \mathcal{C}_6 \end{aligned} \tag{2.162}$$

Vertices $\{2,\, 7,\, 8\}$ cannot serve as origins of convex unimodular cones enclosing the NP.

The degree/w-weight of a monomial including logarithmic terms, $x^\alpha \log x^\beta$, wrt some non-negative weight vector $w \in \mathbb{R}^N$ is defined as Re$(w \cdot \alpha)$. Min$_w(f)$ is the minimum of the w-weights of all the terms in f. The sum of the terms which have this minimum w-weight form the initial series, $\text{in}_w(f)$. The initial series can be further ordered wrt some prescription $\prec$ and the initial monomial of the initial series is the starting monomial of the series.

Given a series solution f of $\mathcal{I}$, the initial series, $\text{in}_{(w)}(f)$, is a solution to the initial ideal $\text{in}_{(-w,\,w)}(\mathcal{I})$, which is also referred to as a Gröbner deformation of $\mathcal{I}$. Obviously,

$$\text{rank}\big(\text{in}_{(-w,\,w)}(\mathcal{I})\big) \leqslant \text{rank}\big(\mathcal{I}\big). \tag{2.163}$$

If the D-ideal $\mathcal{I}$ is regular holonomic, the equality holds. A basis of $\mathbb{C}$-linearly independent series solutions, $N(\mathcal{I}) = \{f_i\}$, of regular holonomic ideal $\mathcal{I}$

$$f = \sum_{N(\mathcal{I})} c_i\, f_i, \quad \forall\, \mathcal{I} \circ f = 0, \quad \sum_{N(\mathcal{I})} c_i\, f_i = 0 \implies c_i = 0, \quad c_i \in \mathbb{C} \tag{2.164}$$

are called canonical solutions wrt $\prec_w$, if every basis solution $f_i \in N(\mathcal{I})$ has a *unique* starting monomial, which doesn't appear in any $f_j \in N(\mathcal{I})$, $j \neq i$. There are rank($\mathcal{I}$) elements in $N(\mathcal{I})$, and so just as many $\mathbb{C}$-linearly independent *finite* initial series and unique starting monomials. The initial monomial wrt $\prec_w$ of any series solution of $\mathcal{I}$ falls within the set of starting monomials. The canonical series are convergent when $0 < |x^{u_i}| \ll 1$ which is the same condition as $U \cdot (-\log|x|) \gg 0$, where U is as described in eq. (2.161). This means, the series converges when

$$(-\log|x|) \in w + \mathcal{C}^* \tag{2.165}$$

for some point w, which influences the starting monomials. The degree of logarithmic terms in these solutions is at most rank$(\mathcal{I}) - 1$. Given a set of starting monomials $\{\tilde{f}_i\}$, the canonical series are of the form

$$f = \tilde{f}_i \sum_{t_i \in \mathbb{Z}} \sum_{|s| \in Q} c_{t,\,s}\, x^{t_i\, u_i^*}\, (\log x)^s, \quad Q = \{0,\, 1,\, 2, \cdots \text{rank}(\mathcal{I}) - 1\} \tag{2.166}$$

where $|s| = \sum s_i$, $\{u_1^*, \cdots u_n^*\}$ is the basis of $\mathcal{C}^*$, and the constants $c_{t,\,s}$ can be solved for inductively by requiring the series to satisfy $\mathcal{I} \circ f = 0$ order by order.

Example 2.2.2 (continued)**.** Since this is a system of rank 4, the maximum degree of the logarithmic terms will be $\leqslant 3$. For any given weight $(-w, w)$, $w \in \mathbb{R}^2_+$, the initial ideal is $\langle \theta_x^2, \theta_y^2 \rangle$, yielding the starting monomials $\tilde{f} = \{1, \log x, \log y, \log x \log y\}$. Using the basis of $\mathcal{C}_1^*$, the series are represented by

$$f = \tilde{f}_i \sum_{t_i \in \mathbb{Z}} \sum_{s_i \in \{0,1\}} c_{t,s}\, x^{t_1} y^{t_2} (\log x)^{s_1} (\log y)^{s_2} \tag{2.167}$$

The action of the rest of the ideal $\langle -x(\theta_x + \theta_y + a)(\theta_x + b_1), -y(\theta_x + \theta_y + a)(\theta_y + b_2) \rangle$ on the starting monomials is

$$\begin{aligned}
\mathcal{I}_1 \tilde{f}_1 &= -a\, b_1\, x, & \mathcal{I}_2 \tilde{f}_1 &= -a\, b_2\, y \\
\mathcal{I}_1 \tilde{f}_2 &= -(a + b_1 + a\, b_1 \log x)\, x, & \mathcal{I}_2 \tilde{f}_2 &= -(a\, b_2 \log x + b_2)\, y \\
\mathcal{I}_1 \tilde{f}_3 &= -(b_1 + a\, b_1 \log y)\, x, & \mathcal{I}_2 \tilde{f}_3 &= -(a + b_2 + a\, b_2 \log y)\, y \\
\mathcal{I}_1 \tilde{f}_4 &= -\big(1 + b_1 \log x + (a + b_1) \log y & \mathcal{I}_2 \tilde{f}_4 &= -\big(1 + b_2 \log y + (a + b_2) \log x \\
&\quad + a\, b_1 \log x \log y\big)\, x, & &\quad + a\, b_2 \log x \log y\big)\, y.
\end{aligned} \tag{2.168}$$

The series solution can be found inductively, like in example 2.1.7, to get

$$\begin{aligned}
c_{1,0,0,0} &= \tfrac{a\, b_1 (a(2 b_1 - 1) - (b_1 + 1))}{(1 + a + b_1 - 2 a b_1)^2} c_{0,0,0,0} - \tfrac{b_1 (b_1 + 1)}{(1 + a + b_1 - 2 a b_1)^2} c_{0,0,0,1} \\
c_{1,0,1,0} &= -\tfrac{a^2 b_1^2 (a(2 b_1 - 1) - (b_1 + 1))}{(1 + a + b_1 - 2 a b_1)^2} c_{0,0,0,0} - \tfrac{a b_1^2 (a(2 b_1 - 1) - 2(b_1 + 1))}{(1 + a + b_1 - 2 a b_1)^2} c_{0,0,0,1} \\
c_{1,0,0,1} &= -\tfrac{a\, b_1}{1 + a + b_1 - 2 a b_1} c_{0,0,0,1} \\
c_{1,0,1,1} &= \tfrac{a^2 b_1^2}{1 + a + b_1 - 2 a b_1} c_{0,0,0,1} \\
c_{0,1,0,0} &= \tfrac{a\, b_2 (a(2 b_2 - 1) - (b_2 + 1))}{(1 + a + b_2 - 2 a b_2)^2} c_{0,0,0,0} - \tfrac{b_2 (b_2 + 1)}{(1 + a + b_2 - 2 a b_2)^2} c_{0,0,1,0} \\
c_{0,1,0,1} &= -\tfrac{a^2 b_2^2 (a(2 b_2 - 1) - (b_2 + 1))}{(1 + a + b_2 - 2 a b_2)^2} c_{0,0,0,0} - \tfrac{a b_2^2 (a(2 b_2 - 1) - 2(b_2 + 1))}{(1 + a + b_2 - 2 a b_2)^2} c_{0,0,1,0} \\
c_{0,1,1,0} &= -\tfrac{a\, b_2}{1 + a + b_2 - 2 a b_2} c_{0,0,1,0} \\
c_{0,1,1,1} &= \tfrac{a^2 b_2^2}{1 + a + b_2 - 2 a b_2} c_{0,0,1,0}
\end{aligned} \tag{2.169}$$

and so on. However, induction quickly starts to become rather cumbersome even at 2nd order, and hence isn't particularly suitable for analytic evaluation of solutions in more than one variable.

When the roots of the indicial ideal (i.e. exponents), β, form the starting monomials, x^β, the canonical series solutions are called the Nilsson ring, $N_w(\mathcal{I})$. In this case, the unimodular cone of relevance becomes the Gröbner cone wrt the weight w, $\mathcal{C}_w(\mathcal{I})$, satisfying $\text{in}_{(-w,\,w)}(\mathcal{I}) = \text{in}_{(-w',\,w')}(\mathcal{I})$ for all $w' \in \mathcal{C}_w(\mathcal{I})$, which can in turn be used to find the polar cone $\mathcal{C}_w^*(\mathcal{I})$ and its basis vectors u^*. However, there is apparently a slightly easier procedure to find u^*. Given a Gröbner basis, $\mathcal{G}_w$, a cone $\mathcal{C}_w(\mathcal{G})$, defined as

$$\text{in}_{(-w,\,w)}(\mathcal{G}_w) = \text{in}_{(-w',\,w')}(\mathcal{G}_w), \qquad \forall\; w' \in \mathcal{C}_w(\mathcal{G}) \tag{2.170}$$

includes the proper Gröbner cone $\mathcal{C}_w(\mathcal{I})$ and can instead be used to compute the basis u^*.

2.3 Solutions of $\mathcal{A}$-Hypergeometric Systems

The $\mathcal{A}$-hypergeometric system of equations, $\mathcal{I}_\mathcal{A}$, with $\mathcal{A}$ as defined in eq. (2.102) i.e. with $\mathbb{1}$ as one of the rows, is a regular holonomic ideal. This automatically makes the generating set of the toric ideal $J_\mathcal{A}$ in eq. (2.100) consist of homogenous binomials, also represented as a toric variety of dimension $N-1$ in $\mathbb{P}^{N-1}$.

Example 2.3.1. A running example throughout this section is based on the integral

$$\begin{aligned} I_1 &= \int \frac{\mathrm{d}x}{x} \frac{x^\beta}{(z_1+z_2\,x_1^2+z_3\,x_2^2+z_4\,x_3^2+z_5\,x_4^2+z_6\,x_1^2\,x_4^2+z_7\,x_2^2\,x_3^2+z_8\,x_1\,x_2\,x_3\,x_4)^{\beta_0}} \\ &= \int \frac{\mathrm{d}x}{8\,x} \frac{x_1^{\frac{\beta_1+\beta_4}{2}}\,x_2^{\frac{\beta_2+\beta_3}{2}}\,x_3^{\beta_3}\,x_4^{\frac{\beta_3+\beta_4}{2}}}{(z_1+z_2\,x_1+z_3\,x_2+z_4\,x_2\,x_3^2\,x_4+z_5\,x_1\,x_4+z_6\,x_1^2\,x_4+z_7\,x_2^2\,x_3^2\,x_4+z_8\,x_1\,x_2\,x_3\,x_4)^{\beta_0}} \equiv I_1', \end{aligned} \tag{2.171}$$

as such chosen because it will come to have physical significance (see section 4.2.3). Its associated $\mathcal{A}$ system with generic parameters:

$$\gamma_1 = \{\beta_0,\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_4\}, \quad \gamma_1' = \{\beta_0,\ \tfrac{\beta_1+\beta_4}{2},\ \tfrac{\beta_2+\beta_3}{2},\ \beta_3,\ \tfrac{\beta_3+\beta_4}{2}\}, \tag{2.172}$$

is given by

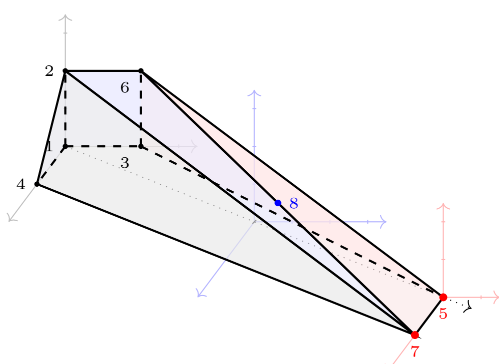
$$\mathcal{A}_1 = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 0&2&0&0&0&2&0&1 \\ 0&0&2&0&0&0&2&1 \\ 0&0&0&2&0&0&2&1 \\ 0&0&0&0&2&2&0&1 \end{pmatrix}, \quad \mathcal{A}'_1 = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&0&1&2&0&1 \\ 0&0&1&1&0&0&2&1 \\ 0&0&0&2&0&0&2&1 \\ 0&0&0&1&1&1&1&1 \end{pmatrix}$$
$$\mathcal{J}_1 = \left\langle \partial_2\,\partial_5 - \partial_1\,\partial_6,\ \partial_3\,\partial_4 - \partial_1\,\partial_7,\ \partial_6\,\partial_7 - \partial_8^2 \right\rangle \tag{2.173}$$

where both $\mathcal{A}$ and $\mathcal{A}'$ have the same kernel and irreducible toric ideal $\mathcal{J}_1$. The $\mathcal{A}$ hypergeometric ideal is $\langle \mathcal{G}_1 \rangle + \mathcal{H}_1,\ \mathcal{H}_1 = \langle \mathcal{A}_1\,\theta + \gamma_1 \rangle$, where $\mathcal{G}_1$ is the universal Gröbner basis of $\mathcal{J}_1$.

$$\mathcal{G}_1 = \mathcal{J}_1 + \langle \partial_1\,\partial_8^2 - \partial_2\,\partial_5\,\partial_7,\ \partial_1\,\partial_8^2 - \partial_3\,\partial_4\,\partial_6,\ \partial_2\,\partial_5\,\partial_7 - \partial_3\,\partial_4\,\partial_6,$$
$$\partial_3\,\partial_4\,\partial_8^2 - \partial_2\,\partial_5\,\partial_7^2,\ \partial_2\,\partial_5\,\partial_8^2 - \partial_3\,\partial_4\,\partial_6^2 \rangle \tag{2.174}$$

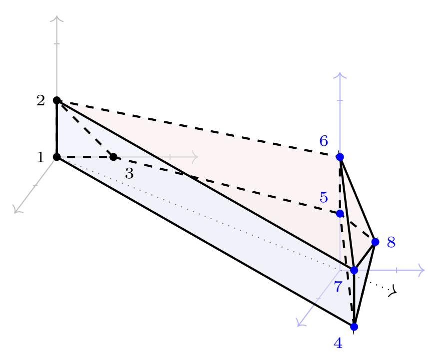
with zero set of $\mathcal{J}_1$ embedded in $\mathbb{P}^7$ as $\{1 :\ p_2^2 :\ p_3^2 :\ p_4^2 :\ p_5^2 :\ p_2^2\,p_5^2 :\ p_3^2\,p_4^2 :\ p_2\,p_3\,p_4\,p_5\}$.

When $\mathcal{A}$ is an $N \times N$ matrix, the volume of the Newton polytope, $\mathrm{Vol}(\mathcal{A})$, is simply $\det \mathcal{A}$. If the dimension of the newton polytope of $\mathcal{A}$ is $< n$, it is degenerated, else it is full dimensional. However, since $\mathcal{A}$ is required to satisfy the Irreducibility Conditions on $\mathcal{A}$ in the current context, it will always produce a full dimensional polytope.



(a) NP_1: Newton Polytope of $\mathcal{A}_1$

Vertex 8 falls on the surface of the convex hull evidencing its degeneracy.



(b) $\mathrm{NP'}_1$: Newton Polytope of $\mathcal{A}'_1$

The irreducibility conditions imposed on $\mathcal{A}'_1$ ensure that there are no degenerate points.

Figure 2.7: Newton Polytopes of $\mathcal{A}_1$, $\mathcal{A}'_1$ matrices in eq. (2.173)

The black, blue and red 3D axes are slices at $0,\ 1$ and 2 resp. in the 4^{th} dimension, represented by the dotted line. Solid dots are colour coded accordingly and label the vertices of the polytopes corresponding to the columns of $\mathcal{A}$.

The newton polytopes in fig. (2.7) are 4-dimensional and can be represented as an intersection of half spaces. For example, NP_1 is defined by $\{2 \geqslant \eta_i \geqslant 0,\ \eta_1 + \eta_4 \leqslant 4,\ \eta_2 + \eta_3 \leqslant 4,\ \sum \eta \leqslant 4\}$.

The degree of a toric ideal, $J_{\mathcal{A}}$, is equal to $\mathrm{Vol}(\mathcal{A})$. In general, the degree of $\mathcal{I}_{\mathcal{A}} \geqslant \mathrm{Vol}(\mathcal{A})$ but equality exists if γ is sufficiently generic and/or $J_{\mathcal{A}}$ is Cohen-Macaulay, i.e. it has a square-free monomial ideal.

$$\text{rank}(\mathcal{I}_{\mathcal{A}}) \geqslant \text{degree}(J_{\mathcal{A}}) = \mathrm{Vol}(\mathcal{A}) \tag{2.175}$$

The singular locus of $\mathcal{I}_{\mathcal{A}}$ is the zero set of the principal $\mathcal{A}$-determinant, $\mathrm{S}_{\mathcal{A}}$, of the polynomials in n-variables, $\{P_1,\ P_2 \cdots P_m\}(x;\ \mathcal{A};\ z)$, defined by $\mathcal{A}$. It is defined as the zero locus of the product of the polynomials from which the variables x have been eliminated. Using the Cayley trick, which at the level of the polynomials boils down to the Feynman parameterisation, this can also be written in terms of one polynomial, $P_{\mathcal{A}} = \sum_{i=1}^{m} y_i\, P_i\,(x,\ A_i,\ z)$, in variables x and y,

$$\mathrm{S}_{\mathcal{A}} = \bigcup_{i=1}^{m} \{P_i\,(x,\ A_i,\ z) = 0\} \bigcap \mathbb{C}\langle z\rangle = \{P_{\mathcal{A}}\,(x,\ y,\ \mathcal{A},\ z) = 0\} \bigcap \mathbb{C}\langle z\rangle. \tag{2.176}$$

This makes $S_{\mathcal{A}}$ the resultant of $n + m$ functions $\{\theta_x\, P_{\mathcal{A}},\ \theta_y\, P_{\mathcal{A}}\}$ (i.e. the intersection of the zero loci of the polynomials and their derivatives). [95] is the prototypical textbook on this topic.

2.3.1 Nilsson Series

In order to construct series solutions of the Nilsson ring wrt w, the initial and indicial ideals, $\text{in}_{(-w,\,w)}(\mathcal{I}_{\mathcal{A}})$ and $\text{ind}_{(w)}(\mathcal{I}_{\mathcal{A}})$, need to be computed. This is computation has a huge time complexity in general, however subsets of these ideals, dubbed fake initial and fake indicial ideals respectively are easier to compute. M is a monomial ideal wrt $\prec_w$ of the toric ideal $J_{\mathcal{A}}$ and $\tilde{M}$ is its distraction.

$$\begin{aligned} M &= \text{in}_{(-w,\,w)}(J_{\mathcal{A}}), \quad \text{in}^{\text{fake}}_{(-w,\,w)}(\mathcal{I}_{\mathcal{A}}) = M + H_{\mathcal{A}} \subseteq \text{in}_{(-w,\,w)}(\mathcal{I}_{\mathcal{A}}) \\ \tilde{M} &= \text{ind}_{(w)}(J_{\mathcal{A}}), \qquad \text{ind}^{\text{fake}}_{(w)}(\mathcal{I}_{\mathcal{A}}) = \tilde{M} + H_{\mathcal{A}} \subseteq \text{ind}_{(w)}(\mathcal{I}_{\mathcal{A}}) \end{aligned} \tag{2.177}$$

When the vector γ is held to be sufficiently generic, equality holds, making the roots of the fake indicial ideal, called fake exponents, equal to the exponents of the original indicial ideal. The monomial ideal, M_1, and its distraction, $\tilde{M}_1$, of J_1 in eq. (2.174) wrt $w_1 = (0,1,1,0,0,0,0,1)$ are

$$M_1 = \left\langle \partial_2\,\partial_5,\ \partial_3\,\partial_4,\ \partial_8^2 \right\rangle, \quad \tilde{M}_1 = \left\langle \theta_2\,\theta_5,\ \theta_3\,\theta_4,\ (\theta_8)_{(2)} \right\rangle. \tag{2.178}$$

A different choice of weight, for instance, $(0,1,1,1,1,1,1,0)$, would produce a different monomial ideal, $\langle \partial_2\,\partial_5,\ \partial_3\,\partial_4,\ \partial_6\,\partial_7 \rangle$ and distraction $\langle \theta_2\,\theta_5,\ \theta_3\,\theta_4,\ \theta_6\,\theta_7 \rangle$. However, the nature of the generic solution space will not change.

When considering specific ranges of variables, it is useful to choose a weight with eq. (2.165) in mind to ensure convergence. For example, granting some foreknowledge of the physically relevant values of z eq. (2.209) in example 2.3.1, a weight belonging to the class $w_2 = \{0,0,0,0,0,1,1,1\}$ can be considered, in line with eq. (2.165), resulting in

$$\begin{aligned} M_2 &= \langle \partial_1\,\partial_7,\ \partial_1\,\partial_6,\ \partial_3\,\partial_4\,\partial_6,\ \partial_6\,\partial_7,\ \partial_1\,\partial_8^2,\ \partial_2\,\partial_5\,\partial_7^2 \rangle \\ \tilde{M}_2 &= \langle \partial_1\,\partial_6,\ \partial_1\,\partial_7,\ \partial_3\,\partial_4\,\partial_6,\ \partial_6\,\partial_7,\ \partial_2\,\partial_5\,\partial_7\,(\partial_7 - 1),\ \partial_1\,\partial_8\,(\partial_8 - 1) \rangle. \end{aligned} \tag{2.179}$$

2.3.2 Horn Hypergeometric Functions

The elements of the kernel of $\mathcal{A}$, $u \in \mathcal{K} \subset \mathbb{Z}^N$, satisfy $\mathcal{A}\,u = 0$. Given a complete basis $\{u_i\}$, the span of $\mathcal{K}$ is represented as $\{t^i\,u_i\}$, $t_i \in \mathbb{Z}$. For notational ease, every element of the kernel, u, can be separated into two non-negative integer vectors $u_+,\ u_-$ such that $u = u_+ - u_-$ paralleling eq. (2.100). A formal series solution, which isn't holomorphic in general, associated with a starting monomial x^s, with s satisfying $\mathcal{A}\,s + \gamma = 0$ is

$$I^{(s)} = \sum_{u \in \mathcal{K}} \frac{(s)_{(u_-)}}{(s+u)_{(u_+)}}\,x^{s+u}. \tag{2.180}$$

Formal solutions of $\mathcal{I}_{\mathcal{A}}$ take the form of Horn hypergeometric functions, hence the name. A multidimensional Horn hypergeometric series is a Laurent series $\sum c(\omega)\,x^{\omega},\ \ \omega \in \mathbb{Z}^N,\ c(\omega) \in \mathbb{C}^*$, such that there exist some non-zero rational functions $b_i : \mathbb{Z}^N \to \mathbb{C}^*$, called b-function(s):

$$b_i(\omega) = \frac{c(\omega + e_i)}{c(\omega)} \; \forall \; i \in [N] \tag{2.181}$$

with e_i being the standard basis vectors of $\mathbb{Z}^N$ and b_i satisfying the consistency condition:

$$\tfrac{c(\omega+e_i+e_j)}{c(\omega)} = \tfrac{c(\omega+e_i)}{c(\omega)} \tfrac{c(\omega+e_i+e_j)}{c(\omega+e_i)} = \tfrac{c(\omega+e_j)}{c(\omega)} \tfrac{c(\omega+e_i+e_j)}{c(\omega+e_j)} \implies \tfrac{b_j(\omega+e_i)}{b_j(\omega)} = \tfrac{b_i(\omega+e_j)}{b_i(\omega)}. \tag{2.182}$$

b_i form a 1-cocycle from $\mathbb{Z}^N$ to $\mathbb{C}^*$ where the group action is additive in abelian $\mathbb{Z}^N$ and multiplicative in $\mathbb{C}^*$ (specifically a subset of $\mathbb{C}^*$ spanning ratios of $c(\omega)$).

2.3.3 Exponents and Log-free Series Solutions

A Gale transform, $\mathcal{K} \subset \mathbb{Z}^N$, of $\mathcal{A}$ is a reduced basis of the kernel of $\mathcal{A}$ such that

$$\begin{aligned} &\mathcal{K} = \left(u_1 \cdots u_i \cdots u_K \right), \quad \mathcal{A}\, u_i = 0, \quad u_i \in \mathbb{Z}^N, \\ &\text{if } \mathcal{A}\, v = 0, \quad \exists\, t \in \mathbb{Z}^N \text{ s.t. } v = \sum_{i=1}^{K} t_i\, u_i. \end{aligned} \tag{2.183}$$

It is obviously non-unique. A Gale transform of $\mathcal{A}_1$, $\mathcal{A}_1'$ in eq. (2.173) is

$$\mathcal{K}_1 = (u_1\; u_2\; u_3) = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}^{\mathrm{Tr}}. \tag{2.184}$$

Thus a general term of $\mathcal{K}_1$ can represented as

$$v \equiv \{-(t_1 + t_2),\; t_1,\; t_2,\; t_2,\; t_1,\; -(t_1 + t_3),\; -(t_2 + t_3),\; 2\,t_3\}, \quad t \in \mathbb{Z}^3. \tag{2.185}$$

A standard pair (∂^a, σ), $a \in \mathbb{N}_0^N$, $\sigma \subseteq [N]$, of a monomial ideal $M = \langle \partial^a \rangle$, is defined such that (i) $a_i = 0 \;\forall\; i \in \sigma$ (ii) $\forall\; b_j \in \mathbb{N}_0$, $j \in \sigma$, $\partial^a\, \partial^b \notin M$ (iii) $\forall\; l \notin \sigma$, there exists $b_j \in \mathbb{N}_0$ such that $\partial^a \prod_{l \in \sigma'} \partial_l^{b_l} \prod_{j \in \sigma} \partial^{b_j} \in M$, where the sets of σ are triangulations of the Newton polytope. $\mathcal{T}(M)$

is the set of all standard pairs of a monomial ideal M. M_1 in eq. (2.178) has the standard pairs

$$\begin{aligned}\mathcal{T}(M_1) = \{1, \{1,2,3,6,7\}\},\ \{1, \{1,2,4,6,7\}\},\ \{1, \{1,3,5,6,7\}\},\ \{1, \{1,4,5,6,7\}\},\\ \{\partial_8, \{1,2,3,6,7\}\},\ \{\partial_8, \{1,2,4,6,7\}\},\ \{\partial_8, \{1,3,5,6,7\}\},\ \{\partial_8, \{1,4,5,6,7\}\},\end{aligned} \tag{2.186}$$

and the associated triangulation of its Newton polytope is $\{1,2,3,6,7\}$, $\{1,2,4,6,7\}$, $\{1,3,5,6,7\}$, $\{1,4,5,6,7\}$, each with a volume of 16. There are 16 possible triangulations, each corresponding to a different monomial ideal. However, only 5 of these monomial ideals are generated by just the representative terms in J_1. Although there do exist weights for which the initial forms of the Gröbner basis $\mathcal{G}_1$ are restricted to the initial forms of J_1, it is not the case in general.

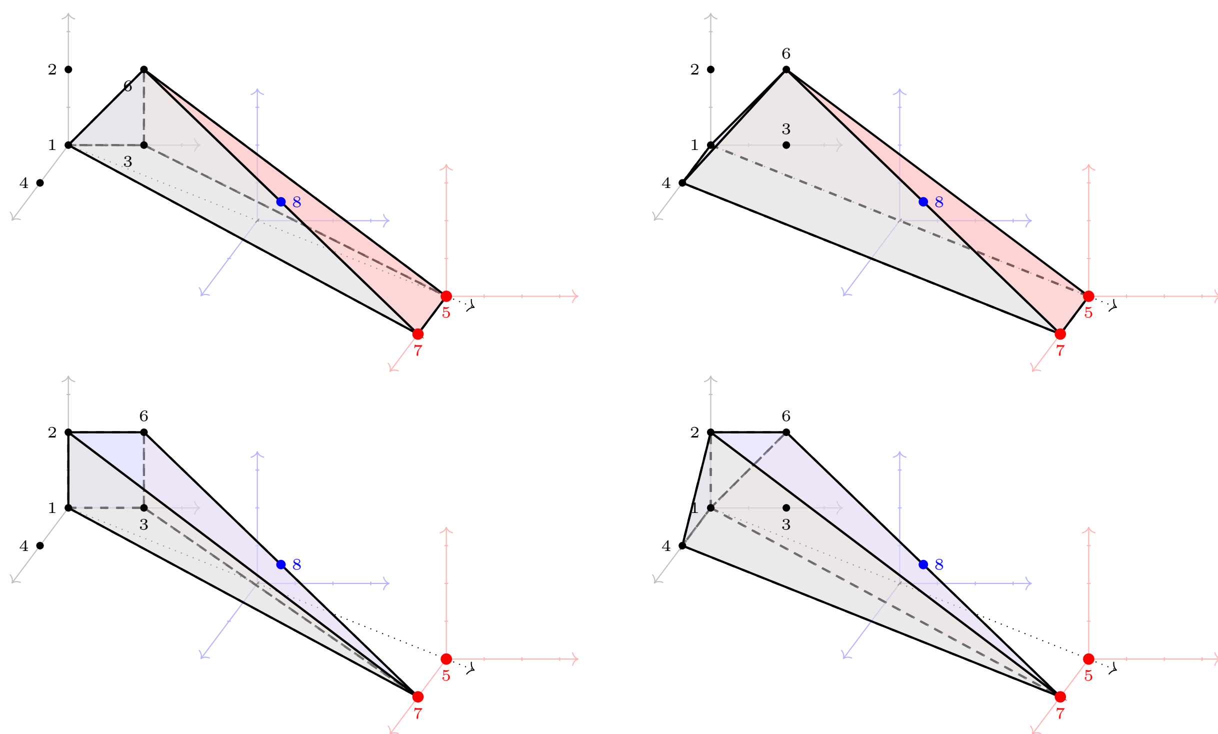


Figure 2.8: Triangulations of the Newton Polytope in fig. (2.7a)
The triangulations correspond to eq. (2.186). Each triangulation has vertex 8 on one of its edges.

If the $n + m \times n + m$ sized matrices formed by columns of $\mathcal{A}$, $\mathcal{A}_\sigma$, that constitute a triangulation $(\partial^a, \sigma) \in \mathcal{T}(M)$ are non-singular (i.e. have non-zero determinants), the triangulation is regular. This can be confirmed for the case at hand. When this determinant, $|\mathcal{A}_\sigma| = 1$, the triangulation is called unimodular. When $\text{in}_{(w)}(J)$ is radical for generic weight w, i.e. square-free for generic weight vector w, the regular triangulation, $\sigma \in \Sigma_w$, of the Newton polytope is unimodular.

$$\text{in}_{(w)}(J) = \langle \partial_i \mid i \notin \Sigma_w \rangle = \bigcap_{\sigma \in \Sigma_w} \langle \partial_j \mid j \notin \sigma \rangle \tag{2.187}$$

For non-resonant γ, the number of standard pairs equals the number of exponents and hence the rank of the system. So eq. (2.173) will have a solution basis of 8 series. The primary ideal decomposition of M in terms of $\mathcal{T}(M)$ is

$$M = \bigcap_{(\partial^a,\, \sigma) \in \mathcal{T}(M)} \langle \partial_i^{a_i+1} \mid i \notin \sigma \rangle. \tag{2.188}$$

The indicial ideal is based upon the decomposition of the monomial ideal and is hence equal to

$$\text{ind}_w(\mathcal{I}) = \bigcap_{(\partial^a,\, \sigma) \in \mathcal{T}(M)} \langle (\theta_i - a_i) \mid i \notin \sigma \rangle, \quad M = \text{in}_{(-w,\, w)}(\mathcal{I}). \tag{2.189}$$

So the decomposition of M_1 is

$$\begin{aligned} M_1 = \bigcap \langle \partial_4,\, \partial_5,\, \partial_8 \rangle,\, \langle \partial_3,\, \partial_5,\, \partial_8 \rangle,\, \langle \partial_2,\, \partial_4,\, \partial_8 \rangle,\, \langle \partial_2,\, \partial_3,\, \partial_8 \rangle, \\ \langle \partial_4,\, \partial_5,\, \partial_8^2 \rangle,\, \langle \partial_3,\, \partial_5,\, \partial_8^2 \rangle,\, \langle \partial_2,\, \partial_4,\, \partial_8^2 \rangle,\, \langle \partial_2,\, \partial_3,\, \partial_8^2 \rangle \end{aligned} \tag{2.190}$$

and indicial ideal $\tilde{M}_1$ can be verified to be $\langle \theta_2\, \theta_5,\, \theta_3\, \theta_4,\, (\theta_8)_2 \rangle$. The root/exponent, $s_{(\partial^a,\, \sigma)}$, associated with a standard pair, $(\partial^a,\, \sigma)$ is defined as

$$s_{(\partial^a,\, \sigma)} \equiv \tilde{s}, \quad \mathcal{A} \cdot \tilde{s} + \gamma = 0, \quad \tilde{s}_i = a_i, \quad i \notin \sigma. \tag{2.191}$$

Alternately, the (fake) indicial equation $\text{ind}_{(w)}(\mathcal{I}) = 0$ can be solved to find the (fake) exponents.

For example, the indicial ideal in eqs. (2.177) and (2.178) results in the exponents

$$
\begin{bmatrix}
s_1 = -\beta_0+\frac{1}{2}(\beta_1+\beta_2) & \frac{\beta_4-\beta_1}{2} & \frac{\beta_3-\beta_2}{2} & 0 & 0 & -\frac{\beta_4}{2} & -\frac{\beta_3}{2} & 0 \\
s_2 = -\beta_0+\frac{1}{2}(\beta_1+\beta_3) & \frac{\beta_4-\beta_1}{2} & 0 & \frac{\beta_2-\beta_3}{2} & 0 & -\frac{\beta_4}{2} & -\frac{\beta_2}{2} & 0 \\
s_3 = -\beta_0+\frac{1}{2}(\beta_2+\beta_4) & 0 & \frac{\beta_3-\beta_2}{2} & 0 & \frac{\beta_1-\beta_4}{2} & -\frac{\beta_1}{2} & -\frac{\beta_3}{2} & 0 \\
s_4 = -\beta_0+\frac{1}{2}(\beta_3+\beta_4) & 0 & 0 & \frac{\beta_2-\beta_3}{2} & \frac{\beta_1-\beta_4}{2} & -\frac{\beta_1}{2} & -\frac{\beta_2}{2} & 0 \\
s_5 = -\beta_0+\frac{1}{2}(\beta_1+\beta_2) & \frac{\beta_4-\beta_1}{2} & \frac{\beta_3-\beta_2}{2} & 0 & 0 & -\frac{\beta_4+1}{2} & -\frac{\beta_3+1}{2} & 1 \\
s_6 = -\beta_0+\frac{1}{2}(\beta_1+\beta_3) & \frac{\beta_4-\beta_1}{2} & 0 & \frac{\beta_2-\beta_3}{2} & 0 & -\frac{\beta_4+1}{2} & -\frac{\beta_2+1}{2} & 1 \\
s_7 = -\beta_0+\frac{1}{2}(\beta_2+\beta_4) & 0 & \frac{\beta_3-\beta_2}{2} & 0 & \frac{\beta_1-\beta_4}{2} & -\frac{\beta_1+1}{2} & -\frac{\beta_3+1}{2} & 1 \\
s_8 = -\beta_0+\frac{1}{2}(\beta_3+\beta_4) & 0 & 0 & \frac{\beta_2-\beta_3}{2} & \frac{\beta_1-\beta_4}{2} & -\frac{\beta_1+1}{2} & -\frac{\beta_2+1}{2} & 1
\end{bmatrix}. \tag{2.192}
$$

Each triangulation of the Newton polytope, or alternately, each possible monomial ideal will lead to a different *set* of exponents, however, some standard pairs and, hence, exponents may overlap. Thus, the set of all possible standard pairs $\bigcup \mathcal{T}(M)$ produces all possible pertubative expansions of the solution around the origin. Here, there are 54 standard pairs in $\bigcup \mathcal{T}(M)$ and hence, just as many exponents. When the considered triangulation, $\sigma \in \Sigma_{(w)}$, wrt some weight w is unimodular, the fake exponents are simply

$$
s_\sigma = -\mathcal{A}_\sigma^{-1}\,\gamma. \tag{2.193}
$$

The b-function for generic γ and weight w is

$$
b_{(w)}(\omega) = \prod_{s\in\alpha}(\omega - w\,s), \quad \alpha = \{s_{(\partial^a,\,\sigma)} \mid \quad (\partial^a,\,\sigma) \in \mathcal{T}(M\} \tag{2.194}
$$

where α is the set of (fake) exponents associated with the standard pairs in $\mathcal{T}(M)$. The parametric b-functions supply information regarding the singular hyperplanes of γ:

$$
b_i(\omega) = \prod_{s\in\alpha}(\omega - s_i), \quad \alpha = \{s_{(\partial^a,\,\sigma)} \mid \quad (\partial^a,\,\sigma) \in \bigcup \mathcal{T}(M)\}. \tag{2.195}
$$

A log-free series solution with the starting monomial z^s is

$$
I^{(s)}(z) = \sum_{u\in\mathcal{K}} \frac{z^{s+u}}{(s+u)_{(u)}} = z^s \sum_{u\in\mathcal{K}} z^u \left(\frac{\Gamma(s+1)}{\Gamma(u+s+1)} \quad \text{or} \quad \frac{(-1)^u\,\Gamma(-u-s)}{\Gamma(-s)}\right) \tag{2.196}
$$

where the summation range of u can be set by requiring the Γ functions in the denominator to have arguments in $\mathbb{C}^*\backslash\mathbb{Z}^-$.

Since the number of standard pairs equals the rank, the set of log-free series solutions $I^{(\alpha)} = \{I^{(s)} \mid s \in \alpha\}$ form the Nilsson ring for *generic* γ, with each starting monomial being different by definition making the generated series linearly independent. e.g.

$$I_1^{(k)} = \sum_{t_1, t_2, t_3 \in \mathbb{Z}} \prod_{i=1}^{8} \frac{\Gamma(s_{ki}+1)}{\Gamma(u_{1i}\, t_1 + u_{2i}\, t_2 + u_{3i}\, t_3 + s_{ki} + 1)}\, z_i^{s_{ki}+u_{1i}\, t_1+u_{2i}\, t_2+u_{3i}\, t_3} \tag{2.197}$$

which can be concisely denoted as

$$I_1^{(k)} = z^s \sum_{t \in \mathbb{Z}^3} \frac{\Gamma(s_k+1)}{\Gamma(u \cdot t + s_k + 1)}\, \omega^t, \quad \omega \equiv \{z^{u_1},\, z^{u_2},\, z^{u_3}\} = \left(\tfrac{z_2\, z_5}{z_1\, z_6},\, \tfrac{z_3\, z_4}{z_1\, z_7},\, \tfrac{z_8^2}{z_6\, z_7}\right) \tag{2.198}$$

where ω are the toric invariants.

Specific γ vectors may have special features, which may prevent the direct translation of the generic solution. If an exponent $s \in \mathbb{N}^N$ for some arbitrary γ, the associated canonical series is finite, i.e. polynomial.

A fake exponent that isn't an exponent has at least one negative integer component for some given γ. The negative support of a fake exponent is defined as

$$\text{nsupp}(s) = \{i \in \{1, \cdots N\} \mid s_i \in \mathbb{Z}^-\}. \tag{2.199}$$

s has *minimal* negative support iff there is no $u \in \mathcal{K}$ such that $\text{nsupp}(s+u) \subset \text{nsupp}(u)$. The series solution for roots with non-empty minimal negative supports are not directly defined by eq. (2.196), since the coefficients would hit poles of the Γ functions and become ill-defined, but instead by eq. (2.180) with the summation range of u restricted to $\mathcal{K}^{(s)} \subset \mathcal{K}$:

$$\mathcal{K}^{(s)} = \{u \in \mathcal{K} \mid \text{nsupp}(s) = \text{nsupp}(s+u)\}. \tag{2.200}$$

The log free series solution consists of the series $\{I^{(s)}\}$, where s belongs to the set of roots with minimal negative support. For example, the arbitrary (and physically relevant) choice of

$$\gamma_1 = \{\tfrac{d}{2}+1,\, \bar{\Delta}_1 = \tfrac{d}{2} - \mathrm{i}\,\nu_1,\, \bar{\Delta}_2 = \tfrac{d}{2} - \mathrm{i}\,\nu_2,\, \Delta_1 = \tfrac{d}{2} + \mathrm{i}\,\nu_1,\, \Delta_2 = \tfrac{d}{2} + \mathrm{i}\,\nu_2\}, \quad \nu_+ \equiv \tfrac{\nu_1+\nu_2}{2} \tag{2.201}$$

in eq. (2.172) produces fake exponents with negative integer terms:

$$\begin{bmatrix}
\alpha_1 = & -1-\mathrm{i}\,\nu_+ & \mathrm{i}\,\nu_+ & \mathrm{i}\,\nu_+ & 0 & 0 & -\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & 0 \\
\alpha_2 = & -1 & \mathrm{i}\,\nu_+ & 0 & -\mathrm{i}\,\nu_+ & 0 & -\frac{\Delta_2}{2} & -\frac{\bar{\Delta}_2}{2} & 0 \\
\alpha_3 = & -1 & 0 & \mathrm{i}\,\nu_+ & 0 & -\mathrm{i}\,\nu_+ & -\frac{\bar{\Delta}_1}{2} & -\frac{\Delta_1}{2} & 0 \\
\alpha_4 = & -1+\mathrm{i}\,\nu_+ & 0 & 0 & -\mathrm{i}\,\nu_+ & -\mathrm{i}\,\nu_+ & -\frac{\bar{\Delta}_1}{2} & -\frac{\bar{\Delta}_2}{2} & 0 \\
\alpha_5 = & -1-\mathrm{i}\,\nu_+ & \mathrm{i}\,\nu_+ & \mathrm{i}\,\nu_+ & 0 & 0 & -\frac{\Delta_2+1}{2} & -\frac{\Delta_1+1}{2} & 1 \\
\alpha_6 = & -1 & \mathrm{i}\,\nu_+ & 0 & -\mathrm{i}\,\nu_+ & 0 & -\frac{\Delta_2+1}{2} & -\frac{\bar{\Delta}_2+1}{2} & 1 \\
\alpha_7 = & -1 & 0 & \mathrm{i}\,\nu_+ & 0 & -\mathrm{i}\,\nu_+ & -\frac{\bar{\Delta}_1+1}{2} & -\frac{\Delta_1+1}{2} & 1 \\
\alpha_8 = & -1+\mathrm{i}\,\nu_+ & 0 & 0 & -\mathrm{i}\,\nu_+ & -\mathrm{i}\,\nu_+ & -\frac{\bar{\Delta}_1+1}{2} & -\frac{\bar{\Delta}_2+1}{2} & 1
\end{bmatrix}. \tag{2.202}$$

Assuming $d,\ \nu_{1,2}$ are generically valued, four of the log-free solutions, $\{I^{(1,4,5,8)}\}$, are exponents. Fake exponents $\alpha_{(2,3,6,7)}$ have the same negative support $\{1\}$. It is minimal because no element of $\mathcal{K}_1$, eq. (2.185), can shift the roots such that the size of their negative support set will reduce, i.e. there exists no $u' \in \mathcal{K}_1$ such that $\mathrm{nsupp}(\alpha_{(2,3,6,7)} + u') \neq \varnothing$, the only possible subset of $\{1\}$.

Requiring the negative support to remain the same restricts the kernel to eq. (2.185) with $t \in \mathbb{N}_0^3$. Thus, the series solution corresponding to $\alpha_{(2)}$ is

$$I_1^{(2)} = z^{\alpha_2} \sum_{t \in \mathbb{N}_0^3} \frac{(t_1+t_2)!}{t_1!\,t_2!\,2\,t_3!} \frac{\Gamma(1+\mathrm{i}\,\nu_+)}{\Gamma(t_1+1+\mathrm{i}\,\nu_+)} \frac{\Gamma(1-\mathrm{i}\,\nu_+)}{\Gamma(t_2+1-\mathrm{i}\,\nu_+)} \frac{\Gamma(t_1+t_3+\frac{\Delta_2}{2})}{\Gamma(\frac{\Delta_2}{2})} \frac{\Gamma(t_2+t_3+\frac{\bar{\Delta}_2}{2})}{\Gamma(\frac{\bar{\Delta}_2}{2})}\, \omega^t. \tag{2.203}$$

In order to construct the complete Nilsson ring if there are fake exponents without negative support or overlapping exponents, a generic deformation to γ in the form of $\gamma + \epsilon\,\gamma'$ can be considered and the initial terms/monomials expanded in orders of small ϵ. Appropriate linear combinations of these terms give distinct initial terms, with the terms of lowest order in ϵ serving as starting monomials to linearly independent solutions in the Nilsson ring. This is implemented in the example by giving a small generic deformation to γ_1 in eqs. (2.172) and (2.201), say $\beta_0 \to \beta_0 + \epsilon$, to forego any considerations regarding fake exponents.

2.3.4 Normalisation Constants

Non-degenerate limiting values of the Euler integral can be used to find the normalisation constants of the series solutions $I^{(s)}$. These normalisation constants take the place of boundary conditions that typically appear in solutions to differential equations.

A direct approach to finding them is to consider $z \to 0$ limits matching the zeroes of the exponents, assuming the integrals in the limiting cases are convergent on the relevant contour.

$$\lim_{z_j \to 0} I = \lim_{z_j \to 0} N_i \, I^{(s_i)}, \quad (s_i)_j = 0 \; \forall \; j \tag{2.204}$$

For example, root s_1 suggests the limit $\{z_4, z_5, z_8\} \to 0$. In this limit, the toric invariants $\omega = 0$, so all the series collapse to just their initial terms z^s. The *generalized* Euler integral (i.e. in terms of the remaining variables z) in this limit can be uniquely identified with the initial series of $I_1^{(1)}$ on the basis of their monodromy exponents around $\{z = 0\}$. Consequently, they are indicative of the singular hyperplanes of γ_1 at which the particular series will diverge.

With the integration range chosen to be $x \in (\mathbb{R}_+)^4$, the normalisation constant N_1 of the series $I_1^{(1)}$ turns out to be

$$N_1 = \frac{\Gamma\left(\frac{\beta_3}{2}\right) \Gamma\left(\frac{\beta_4}{2}\right) \Gamma\left(\frac{\beta_1 - \beta_4}{2}\right) \Gamma\left(\frac{\beta_2 - \beta_3}{2}\right) \Gamma\left(\gamma_0 - \frac{\beta_1 + \beta_2}{2}\right)}{16 \, \Gamma(\beta_0)}. \tag{2.205}$$

Similarly, as suggested by the exponent s_5, the $\{z_4, z_5\} \to 0$ limit of the integral can be evaluated and upon comparison of the monodromy exponents with those of the initial series, the integral is found to be dependent only on $I_1^{(1)} \oplus I_1^{(5)}$. Given the previous piece of information, N_5 is then

$$N_5 = -\frac{\Gamma\left(\frac{\beta_3 + 1}{2}\right) \Gamma\left(\frac{\beta_4 + 1}{2}\right) \Gamma\left(\frac{\beta_1 - \beta_4}{2}\right) \Gamma\left(\frac{\beta_2 - \beta_3}{2}\right) \Gamma\left(\gamma_0 - \frac{\beta_1 + \beta_2}{2}\right)}{16 \, \Gamma(\beta_0)}. \tag{2.206}$$

This example is relatively simple since the only non-radical term in the generators of $\mathrm{in}_w(J_1)$ is

∂_8^2, which constrains the inductive process of finding these normalisation constants to a tolerable depth of 1. This can be bypassed by requiring $\partial_{z_8} I_1 = \sum N_i\, \partial_{z_8}\, I_1^{(i)}$ in the limit $\{z_4,\, z_5,\, z_8\} \to 0$, which will once again allow unique identification of the LHS with the starting monomial of $I_1^{(5)}$. In general, the minimal system of linear equations supplying the normalisation constants is:

$$\lim_{z_j \to 0} \partial_{z_j}^{(s_i)_j}\, I = \lim_{z_j \to 0}\, N_i\, \partial_{z_j}^{(s_i)_j}\, I^{(s_i)} = \lim_{z_j \to 0}\, N_i\, (s_i)_j!\, I^{(s_i)}, \quad (s_i)_j \in \mathbb{N}_0\ \forall\, j. \tag{2.207}$$

The symmetry structure of the exponents in eq. (2.192) can be used to deduce the remaining:

$$N_{2/6} = N_{1/5}\Big|_{\beta_2 \leftrightarrow \beta_3}, \quad N_{3/7} = N_{1/5}\Big|_{\beta_1 \leftrightarrow \beta_4}, \quad N_{4/8} = N_{1/5}\Big|_{\beta_2 \leftrightarrow \beta_3,\, \beta_1 \leftrightarrow \beta_4}. \tag{2.208}$$

2.3.5 Restriction of Solution Spaces

Although solution spaces of $\mathcal{I} \subset D_N \equiv \mathbb{C}\langle z,\, \partial_z \rangle$ are defined in terms of generic variables z, more often than not, interest is limited to only specific values thereof. This implies that the modules, $\mathcal{D} = \mathcal{D}_N/\mathcal{I}$, need to be restricted to particular slices of z, say $\{z_k = \bar{z}_k\}$. The corresponding restriction ideal $\mathcal{I}'$ is the intersection of $(\mathcal{I} + \mathrm{R}_z)$, $\mathrm{R}_z \equiv \mathbb{C}\langle z_k - \bar{z}_k \rangle$, with D_{N-k}, and has a finite Gröbner basis. If $\mathcal{I}$ is holonomic, so is $\mathcal{I}'$, and this property of holonomy presents itself in $\mathcal{D}$ and its restriction by R_z, $\mathcal{D}'$, too. [109, 110] are referred to for an algorithm to restrict by $\{z_k = 0\}$. Restriction of Pfaffian systems of Feynman integrals, specifically by $\{z_k = 1\}$, is discussed in [102, 103] as means of reducing the rank of the solution space.

The naive approach of restricting the generic solution $I^{(s)}(z)$ of $\mathcal{I}$ to the subspace $\{z_k = \bar{z}_k\}$ remains valid. Granting that the Euler integral at $I(\bar{z}_k)$ is convergent, even if individual series $I^{(s)}(\bar{z}_k)$ diverge, their sum will be necessarily converging (see section 2.3.6).

For example, eq. (2.171) in a physically relevant (see section 4.2.3) limit is to be restricted to

$$z = \{1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, -2\}, \quad \omega = \{1,\, 1,\, 4\}. \tag{2.209}$$

In order to find a $\log$-free solution, the integral is evaluated by considering the entire generic solution space as follows: A general term of the kernel $\mathcal{K}_1$, eq. (2.185), is symmetric under the exchanges $\upsilon_3 \leftrightarrow \upsilon_4$ and $\upsilon_2 \leftrightarrow \upsilon_5$, translating to $\beta_2 \leftrightarrow \beta_3$ and $\beta_1 \leftrightarrow \beta_4$ respectively in terms of the roots in eq. (2.192). Relabeling the parameters for notational convenience as $\rho \equiv \{\beta_1,\, \beta_4\}$, $\sigma \equiv \{\beta_2,\, \beta_3\}$, $\bar{\sigma} \equiv \sigma_1 - \sigma_2$, $\bar{\rho} \equiv \rho_1 - \rho_2$, a general root can be represented as

$$s_{j,\,k,\,\Theta} = \{\tfrac{\rho_j+\sigma_k}{2} - \beta_0,\; (-1)^j\tfrac{\bar{\rho}}{2},\; (-1)^k\tfrac{\bar{\sigma}}{2},\; 0,\; 0,\; -\tfrac{\rho_j+(-1)^j\bar{\rho}+\Theta}{2},\; -\tfrac{\sigma_k+(-1)^k\bar{\sigma}+\Theta}{2},\; \Theta\}, \qquad (2.210)$$

where $j,\, k \in \{1,\, 2\}$ and $\Theta = \{0,\, 1\}$. The initial term corresponding to a root labelled $s_{j,\,k,\,\Theta}$ is $(-2)^{\Theta}$. To ensure genericity, γ_0 is given a small deformation $\beta_0 \to \frac{d+2}{2} + \epsilon$, shifting the negative integers appearing in eq. (2.202).

Summing over t_1 and setting $\{z_1,\, z_2,\, z_5,\, z_6\} = 1$ in the generic solution reduces the holonomic rank of the characteristic variety of $\mathcal{I}_1$, ch$(\mathcal{I}_1)$, by 4 to 4 (picking a partial tie-breaking order $z_1,\, z_5,\, z_6 \succ z_4,\, z_7$ and $z_2 \succ z_3,\, z_8$ and applying eq. (2.153) makes this clear). In particular, the series corresponding to roots $s_{1,\,k,\,\Theta}$ and $s_{2,\,k,\,\Theta}$ no longer remain independent for each $\{k,\, \Theta\}$, with the divergent parts cancelling against each other order-by-order. It is useful to note at this stage that the series become well-defined and non-degenerate in the $\epsilon \to 0$ limit, thus allowing straightforward subsequent evalution (section C.2.1), resulting in

$$I_1 \propto (\sin(\pi\,\Delta_1) + \sin(\pi\,\bar{\Delta}_1))\,\Gamma(\Delta_1)\,\Gamma(\bar{\Delta}_1) - (\sin(\pi\,\Delta_2) + \sin(\pi\,\bar{\Delta}_2))\,\Gamma(\Delta_2)\,\Gamma(\bar{\Delta}_2). \quad (2.211)$$

2.3.6 Limiting Values of Hypergeometric Functions

Convergence of an Euler integral, $I(z)$, must obviously also be accompanied by the convergence of the series solution representing it, $\sum N_s\, I^{(s)}(z)$. Thus, even if individual series $I^{(s)}(\bar{z})$ may appear to diverge when analytically evaluated at specific limiting values of the variables $z = \bar{z}$, these divergences precisely cancel out against each other. This can be observed by studying the divergent parts of each series $I^{(s)}(\bar{z})$ order by order.

For example, at the simplest order, this can be observed by using the Euler transformation of the classical Gauss hypergeometric function ${}_2F_1$:

$$\begin{aligned}{}_2F_1(a_1,\, a_2\,; b\,; x) = \tfrac{\Gamma(b)\,\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\,\Gamma(b-a_2)}\,{}_2F_1(a_1,\, a_2\,; a_1+a_2+1-b\,; 1-x) \\ + \tfrac{\Gamma(b)\,\Gamma(a_1+a_2-b)}{\Gamma(a_1)\,\Gamma(a_2)\,(1-x)^{a_1+a_2-b}}\,{}_2F_1(b-a_1,\, b-a_2\,; 1+b-a_1-a_2\,; 1-x).\end{aligned} \tag{2.212}$$

When $\mathrm{Re}(b-a_1-a_2) > 0$, the series has a well known limiting value at $x = 1$. However, even when such is not the case i.e. $\mathrm{Re}(b-a_1-a_2) \leqslant 0$, the series representation can be separated into convergent and divergent parts. A general univariate hypergeometric function ${}_pF_q(a;\, b;\, x)$ is

$${}_pF_q(a_1,\, a_2, \cdots a_p; b_1,\, b_2, \cdots b_q; x) := \sum_{n=0}^{\infty} \tfrac{a^{(n)}}{b^{(n)}}\,\tfrac{x^n}{n!} = \sum_{n=0}^{\infty} \tfrac{a_1^{(n)}\,a_2^{(n)}\cdots a_p^{(n)}}{b_1^{(n)}\,b_2^{(n)}\cdots b_q^{(n)}}\,\tfrac{x^n}{n!}, \tag{2.213}$$

normalised to equal 1 at $x = 0$. If $\mathrm{Re}(\sum b - \sum a) \leqslant 0$, ${}_pF_q(a;\, b;\, x)$ diverges as $x \to 1$. As suggested in [111], using the limiting value of ${}_2F_1$ at $x = 1$, ${}_{p+1}F_p(a;\, b;\, x)$ can be rewritten as

$$\begin{aligned}{}_{p+1}F_p(a; b; x) &= \sum_{n=0}^{\infty} \tfrac{a_1^{(n)}\cdots a_p^{(n)}}{b_1^{(n)}\cdots b_{p-2}^{(n)}}\,\tfrac{\Gamma(b_p)\,\Gamma(b_{p-1})}{\Gamma(a_{p+1})}\,\tfrac{{}_2F_1(b_p-a_{p+1},\, b_{p-1}-a_{p+1},\, b_p+b_{p-1}-a_{p+1}+n,\, 1)}{\Gamma(b_p+b_{p-1}-a_{p+1}+n)}\,\tfrac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \tfrac{a_1^{(n)}\cdots a_p^{(n)}}{b_1^{(n)}\cdots b_{p-2}^{(n)}}\,\tfrac{\Gamma(b_p)\,\Gamma(b_{p-1})}{\Gamma(a_{p+1})}\,\tfrac{{}_2F_1(a_{p+1}-b_p,\, a_{p+1}-b_{p-1},\, a_{p+1}+n,\, 1)}{\Gamma(b_p+b_{p-1}-a_{p+1}+n)}\,\tfrac{x^n}{n!}.\end{aligned} \tag{2.214}$$

and the ${}_2F_1$ function reexpanded to find a recursion relation

$$\begin{aligned}{}_{p+1}F_p(a; b; x) = \sum_{m=0}^{\infty} \tfrac{\Gamma(b_p)\,\Gamma(b_{p-1})\,\Gamma(b_p-a_{p+1}+m)\,\Gamma(b_{p-1}-a_{p+1}+m)}{\Gamma(a_{p+1})\,\Gamma(b_{p-1}-a_{p+1})\,\Gamma(b_p-a_{p+1})\,\Gamma(b_p+b_{p-1}-a_{p+1}+m)\,m!} \\ \times\, {}_pF_{p-1}\left({a_1, \cdots a_p \atop b_1, \cdots b_{p-2},\, (b_p+b_{p-1}-a_{p+1}+m)} \middle| x \right)\end{aligned} \tag{2.215}$$

that can be inductively carried on to ${}_2F_1$, allowing the divergences of ${}_{p+1}F_q$ functions at unit argument to be separated as series in $(1-x)$ with initial terms $\{1,\, (1-x)^{\sum b - \sum a}\}$, and when $(\sum b - \sum a)$ is an integer, also $\log(1-x)$. It is expected that this generalisation can be used find the aforementioned cancellations of divergences. For example, see section C.2.1.

2.3.7 Creation and Annihilation Operators

Appropriate combinations of the shift relations induced by derivatives ∂_z form the toric ideal and annihilate the integral, with each individually acting as raising operators on the vector γ, specifically a non-negative integer vector u in eq. (2.109) induces shifts of the form:

$$\begin{aligned} \partial_{z_k^{(j)}} \circ I(z): \quad & \{\alpha,\, \beta\} \to \{\alpha + 1,\, \beta + A_k^{(j)}\} \\ \partial^u \circ I(z): \quad & \{\alpha,\, \beta\} \to \{\alpha + \sum u_j,\, \beta + \sum_{j=1}^{m} A^{(j)} \cdot u_j\}. \end{aligned} \tag{2.216}$$

The homogeneity relations eq. (2.101) can be viewed as products of these raising operators and their inverses, i.e. operators that induce downward shifts in the γ vector, called ‘creation’ operators [112, 113]. Since the GKZ ideal is the maximal ideal in D_N, all relations satisfied by the Euler integral can and will be encoded within it, and so these creation operators can be found from the GKZ system. Recently reviewed in [114], the fundamental idea behind the construction of creation operators is to find a relation of the form

$$\mathcal{C}_i\, \partial_i\, I = b_i(\gamma)\, I = b_i(\theta)\, I \tag{2.217}$$

where the $b_i(\gamma)$, eq. (2.195), describes singular hyperplanes of the parameter vector γ away from the vertex $\mathcal{A}_i$ of the Newton polytope. $b_i(\theta)$ is found by using the homogeneity relations, eq. (2.101), and then creative use of the toric ideal, eq. (2.100), results in a form divisible by ∂_i, thus allowing identification of $\mathcal{C}_i$.

For example, consider the integral with integrand, $\mathscr{I}$, and it associated GKZ equations:

$$\begin{aligned} \mathscr{I} &= \frac{\lambda^{\Delta}}{(z_0 + z_1\,\lambda_1 + z_2\,\lambda_2 + z_3\,\lambda_1\,\lambda_2)^{\alpha}} \\ \text{GKZ ideal} &= \sum \theta + \alpha, \quad \theta_1 + \theta_3 + \Delta_1, \quad \theta_2 + \theta_3 + \Delta_2, \quad \partial_1\partial_2 - \partial_0\partial_3. \end{aligned} \tag{2.218}$$

Taking any one of the scaling symmetry equations to start, say $(z_1\,\partial_1 + z_3\,\partial_3)\,\mathscr{I}(\Delta_1,\, \Delta_2,\, \alpha) =$

$-\Delta_1\,\mathscr{I}(\Delta_1,\,\Delta_2,\,\alpha)$, the intention is to convert it into the form: $\mathcal{C}_i\,\partial_i\mathscr{I}(\gamma) = f(\gamma,\,\gamma')\,\mathscr{I}(\gamma')$, enabling the identification of an operator $\mathcal{C}$ that shifts the γ vectors instead of holding it constant like the GKZ equations are designed to. Applying ∂_0 on both sides of the chosen equation:

$$\begin{aligned} &\partial_0\,(z_1\,\partial_1 + z_3\,\partial_3)\,\mathscr{I}(\Delta_1,\,\Delta_2,\,\alpha) = -\Delta_1\,(-\alpha)\,\mathscr{I}(\Delta_1,\,\Delta_2,\,\alpha+1) \\ \Longrightarrow\; &(z_1\,\partial_0 + z_3\,\partial_2)\,\partial_1\mathscr{I}(\Delta_1,\,\Delta_2,\,\alpha) = (z_1\,\partial_0 + z_3\,\partial_2)\,(-\alpha)\,\mathscr{I}(\Delta_1+1,\,\Delta_2,\,\alpha+1) \\ &= -\Delta_1\,(-\alpha)\,\mathscr{I}(\Delta_1,\,\Delta_2,\,\alpha+1). \end{aligned} \tag{2.219}$$

Following this process, the creation operators can be read off from:

$$\begin{aligned} &(z_1\,\partial_0 + z_3\,\partial_2)\,\mathscr{I}(\Delta_1+1) = -\Delta_1\,\mathscr{I}(\Delta_1) \\ &(z_2\,\partial_0 + z_3\,\partial_1)\,\mathscr{I}(\Delta_2+1) = -\Delta_2\,\mathscr{I}(\Delta_2) \\ &(z_1\,\partial_0 + z_3\,\partial_2)\,(z_0\,\partial_1 + z_2\,\partial_3)\,\mathscr{I}(\alpha+1) = -\Delta_1\,(\Delta_1-\alpha)\,\mathscr{I}(\alpha). \end{aligned} \tag{2.220}$$

A simple way to envision the singular hyperplaces of γ is via the poles of the normalisation constants, though the information found in each one if far come complete. For example, some variable rescalings change eq. (2.171) to

$$I_1 = \int \frac{\mathrm{d}x}{x}\,\frac{z_1^{-\beta_0+\frac{\beta_1+\beta_2+\beta_3+\beta_4}{2}}\,z_2^{-\frac{\beta_1}{2}}\,z_3^{-\frac{\beta_2}{2}}\,z_4^{-\frac{\beta_3}{2}}\,z_5^{-\frac{\beta_4}{2}}\,x^{\beta}}{(1+x_1^2+x_2^2+x_3^2+x_4^2+q_1\,x_1^2\,x_4^2+q_2\,x_2^2\,x_3^2+q_3\,x_1\,x_2\,x_3\,x_4)^{\beta_0}}, \tag{2.221}$$

where $\{q_1,\,q_2,\,q_3\} \equiv \{\frac{z_1\,z_6}{z_2\,z_5},\,\frac{z_1\,z_7}{z_3\,z_4},\,\frac{z_1\,z_8}{\sqrt{z_2\,z_3\,z_4\,z_5}}\}$. This makes some relevant singular surfaces of the γ vector evident, namely $(\beta_0 - \frac{\sum_1^4\beta}{2}),\,\frac{\beta_1}{2},\,\frac{\beta_2}{2},\,\frac{\beta_3}{2},\,\frac{\beta_4}{2} = 0$. Considering the exponents in the rescaled integral, the homogeneous ideal, $\mathcal{H}_1$, simply becomes $\langle\theta_6 - \theta_7,\,2\,\theta_6 + \theta_8\rangle$, i.e. the integral is purely a function of $\frac{z_8^2}{z_6\,z_7} = \frac{q_3^2}{q_1\,q_2}$, with the overall normalisation $\frac{\Gamma(\beta_0-\frac{\sum_1^4\beta}{2})\,\Gamma(\frac{\beta_1}{2})\,\Gamma(\frac{\beta_2}{2})\,\Gamma(\frac{\beta_3}{2})\,\Gamma(\frac{\beta_4}{2})}{16\,\Gamma(\beta_0)}$, found in the $\{q_1,\,q_2,\,q_3\} \to 0$ limit. It can be deduced from the arguments of the Γ functions in the normalisation that the integral has singularities not only at the aforementioned surfaces but at all negative integer shifts thereof.

Though appearing first in this text, Generalized Euler integrals and GKZ systems, were not the starting point in the development of this work. The actual integrals, the physically relevant ones in this context that is, were constructed first. After much searching, it appears that these methods and/or those closely related to it (like Mellin-Barnes integral representations, Pfaffian Systems, Creation operators), may be the only way to tackle the propagator and Feynman integral representations that are presented next.

Chapter 3: Propagators on the Sphere and de Sitter Space

Feynman diagrams associated with scalar fields on Minkowski space $\mathbb{R}^{(\mathbb{D}-1,1)}$ or Euclidean space $\mathbb{R}^{\mathbb{D}}$ have integral representations, related by Wick rotation, that can be brought into the form of Euler integrals, hence having $\mathcal{A}$-hypergeometric series representations. A Feynman integral consisting of n_P scalar propagators and n_l loops takes the form

$$\mathcal{F}_{n_P}^{n_l} = \int_{P_{\text{int}}}^{\mathbb{R}^{(\mathbb{D}-1,1)}} \prod_{i=1}^{n_P} \frac{1}{(P_i^2 - m_i^2)^{\eta_i}} \xrightarrow{P^{\mathbb{D}} \to -\mathrm{i}\, P^{\mathbb{D}}} \int_{P_{\text{int}}}^{\mathbb{R}^{\mathbb{D}}} \prod_{i=1}^{n_P} \frac{1}{(P_i^2 + m_i^2)^{\eta_i}} \tag{3.1}$$

where the product is over all momentum space propagators, the integral is over internal undetermined momenta P_{int}, and the parameter η allows further generalisation. Using Schwinger parameterisation, with limits over the sum and product suppressed for notational brevity,

$$\mathcal{F}_{n_P}^{n_l} = \int_{P_{\text{int}}}^{\mathbb{R}^{\mathbb{D}}} \int_z^* \prod \frac{z_i^{\eta_i}}{\Gamma(\eta_i)}\, e^{-\sum z_i\,(P_i^2+m_i^2)} \tag{3.2}$$

the integral over P_{int} can be represented as a tractable Gaussian integral,

$$\begin{aligned} \mathcal{F}_{n_P}^{n_l} &=: \int_{P_{\text{int}}}^{\mathbb{R}^{\mathbb{D}}} \int_z^* \prod \frac{z_i^{\eta_i}}{\Gamma(\eta_i)}\, e^{-(P_{\text{int}}{}^{\mathrm{T}}\, U\, P_{\text{int}} + P_{\text{int}}{}^{\mathrm{T}}\, W + W^{\mathrm{T}}\, P_{\text{int}} + \mathcal{P})} \\ &= \int_z^* \prod \frac{z_i^{\eta_i}}{\Gamma(\eta_i)} \frac{e^{-(\mathcal{P} - W^{\mathrm{T}}\, U^{-1}\, W)}}{(\det U)^{\frac{\mathbb{D}}{2}}} \end{aligned} \tag{3.3}$$

where $\mathcal{P}$ and matrices U, W are functions of the external momenta P_{ext}, masses m and Schwinger parameters z. $\det U \equiv \mathcal{U}$ and $\det U\, (\mathcal{P} - W^{\mathrm{T}}\, U^{-1}\, W) \equiv \mathcal{F}$ are called the first and second Symanzik polynomials and are homogenous in z of orders n_l and $n_l + 1$ respectively. When convergent, such integrals can be represented as (Lee-Pomeransky representation [60]):

$$
\begin{aligned}
\mathcal{F}^{n_l}_{n_P} &= \frac{\Gamma(\sum \eta_i - n_l \frac{\mathbb{D}}{2})}{\prod \Gamma(\eta_i)} \int_z^* \prod z_i^{\eta_i}\, \delta(1 - \Sigma\, z_i)\, \frac{\mathcal{F}^{n_l \frac{\mathbb{D}}{2} - \sum \eta_i}}{\mathcal{U}^{(n_l+1)\frac{\mathbb{D}}{2} - \sum \eta_i}} \\
&= \frac{\Gamma(\frac{\mathbb{D}}{2})}{\Gamma((n_l+1)\frac{\mathbb{D}}{2} - \sum \eta_i)\prod \Gamma(\eta_i)} \int_z^* \frac{\prod z_i^{\eta_i}}{(\mathcal{U}+\mathcal{F})^{\frac{\mathbb{D}}{2}}},
\end{aligned}
\tag{3.4}
$$

taking the form of an Euler integral, as described in eq. (2.2). General tensorial flat space Feynman diagrams can be reduced to a sum over scalar integrals [115], which can be explicitly confirmed by taking the form of general higher spin flat space propagators into account [116]. Thus, parametric representations of flat space Feynman integrals are indeed a subset of generalized Euler integrals, and can be represented as solutions of $\mathcal{A}$-hypergeometric systems [47, 117]. Some recent works discussing advances in flat space feynman integrals are [57, 118].

The propagator of a scalar field on S^{d+1} of mass $m^2 = \Delta\,(d-\Delta)$ is known to be

$$
G(\hat{X},\, \hat{Y}) = \frac{\Gamma(\Delta)\,\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}\,\Gamma(\frac{d+1}{2})}\, {}_2F_1(\Delta,\, d-\Delta;\, \tfrac{d+1}{2};\, \tfrac{1+\hat{X}\cdot\hat{Y}}{2}),
\tag{3.5}
$$

some of its earliest sources being [16–18], that find it as a solution to the differential equation satisfied by it or construct it explicitly as a sum over eigenmodes satisfying the scalar wave equation. The choice of solution spaces in either procedure fixes the vacuum state.

In constrast to flat space, computations of higher loop Feynman integrals using eq. (3.5) have two major stumbling blocks. First, integrals of the Gauss hypergeometric function ${}_2F_1$ are in general not analytically conducive (though eq. (3.5) does reduce to simpler functions on odd dimensional spheres, table B.1, and at specific mass parameters in the complementary series on even dimensional spheres, table B.2), and second, when considering more than two points on the sphere, the distance measures between these points become non-trivial.

However, taking inspiration from the simplicity of the Gaussian integral form in eq. (3.2), it becomes worthwhile to explore alternate avenues to eventually make higher loop computations on the sphere algorithmically possible, just like their flat space counterparts [52–55]. Recognising the $(d+1)$-sphere as a quotient of $\mathbb{R}^{\mathbb{D}}$ ($\mathbb{D} \equiv d+2$) by $\mathbb{R}_+$ and building upon this observation is a possible detour around the aforementioned issues, as will be illustrated in the following by the construction and use of the embedding space representations of the massive scalar, massive vector, and massless vector (photon) propagators on S^{d+1} that will, by design, produce sphere Feynman integrals in the generalized Euler integral form.

Though possibly superfluous to some readers, details of the coordinate systems and conventions used to describe the sphere S^{d+1} and its embedding space $\mathbb{R}^{\mathbb{D}}$ have been noted in section A.1. Further, section A.2 is crucial cross-reference as it describes the procedure used to transform fields on $\mathbb{R}^{\mathbb{D}}$ to S^{d+1} and back, in keeping with these coordinate conventions.

In section 3.1, a quick refresher of the massive scalar propagator on the sphere is presented, before laying groundwork for embedding space representations of sphere propagators, essentially formed by recovering the scalar Laplacian on S^{d+1} from the Laplacian on $\mathbb{R}^{\mathbb{D}}$ as a quotient. Along with verifying the presented propagator, its various possible position space forms are noted. Their limiting behaviors imply they satisfy different conditions on the vacuum state [25, 68, 119].

Similarly, section 3.2 begins with the properties of propagators of the massive and massless (photon) vector (gauge fixed) actions on S^{d+1}, including the standard approaches to constructing them, i.e. by explicitly summing over eigenmodes of the Laplacian and/or solving the ODEs defining it away from the coincident point limit, first presented in [67]. Unfortunately, the two-point (Wightman) functions presented in [66, 67], weren't able to serve as 'propagators' on the sphere. It can be verified by explictly integrating them against test functions in simple cases (say finding the eigenvalue of the first Killing vector on S^3), resulting in the expected realisation that they don't account for some δ-function at $\theta = 0$. Further, as is expected, the graviton two-point (Wightman) functions in [70, 71] also can't be used as graviton propagators.

It is followed up with the action of the massless vector in embedding space and gauges that allow its proper reduction to the massive and massless vector Laplacians on S^{d+1}, subsequently deriving the corresponding propagators. The special case of a massless vector S^3 is considered separately. When compared to the aforementioned literature on this topic, it is found that the vector propagators expectedly do differ by what can be described as a δ-function/longitudinal piece but match away from the coincident point limit.

3.1 Scalar Fields

The action representing a massive scalar field Φ on a $(d+1)$-sphere is

$$S^{[0]} = \int^{S^{d+1}} \bar{\Phi} \left(-\nabla^2 + m^2 \right) \Phi \tag{3.6}$$

where $\bar{\Phi} = \Phi, \ \Phi^*$ for real and complex fields respectively. The propagator of Φ, $G(\hat{X}, \hat{Y}) \equiv \left(-\nabla^2 + m^2 \right)^{-1}$, is defined such that it exhibits the property

$$\left(-\nabla^2 + m^2 \right) G(\hat{X}, \hat{Y}) = \delta(\hat{X} - \hat{Y}) \implies \Phi(\hat{Y}) = \int_{\hat{X}} G(\hat{X}, \hat{Y}) \left(-\nabla^2 + m^2 \right) \Phi(\hat{X}). \tag{3.7}$$

The construction of the position space δ function and the relation between the Euclidean and dS propagators is reviewed in [17]. Given a complete orthonormal basis of eigenvectors Φ_ω, labelled by ω, with eigenvalues λ_ω of the scalar Laplacian, $-\nabla^2$,

$$\exists c_\omega \mid \Phi(\hat{X}) = \sum_\omega c_\omega \, \Phi_\omega, \quad -\nabla^2 \, \Phi_\omega(\hat{X}) = \lambda_\omega \, \Phi_\omega(\hat{X}), \quad \int_{\hat{X}} \bar{\Phi}_\omega(\hat{X}) \, \Phi_{\omega'}(\hat{X}) = \delta_{\omega, \omega'}, \tag{3.8}$$

the propagator is

$$G(\hat{X}, \hat{Y}) = \sum_\omega \frac{\bar{\Phi}_\omega(\hat{X}) \, \Phi_\omega(\hat{Y})}{\lambda_\omega + m^2} \tag{3.9}$$

implying

$$\int_{\hat{X}, \hat{Y}} \Phi_\omega(\hat{X}) \, G(\hat{X}, \hat{Y}) \, \bar{\Phi}_{\omega'}(\hat{Y}) = \frac{1}{\lambda_\omega + m^2} \delta_{\omega, \omega'}. \tag{3.10}$$

Spin-s symmetric transverse traceless eigenvectors of the Laplacian on S^{d+1} have eigenvalues labelled by $n \geqslant 0$, $\lambda_n + s = (n+s)\,(n+d+s)$ [62]. As such, it is useful to define scalar mass parameters $\Delta,\ \bar{\Delta}$, such that $\lambda_n + m^2 = (n+\Delta)\,(n+\bar{\Delta})$,

$$m^2 =: \Delta\,\bar{\Delta}, \quad \Delta := \tfrac{d}{2} + \mathrm{i}\,\nu, \quad \bar{\Delta} := (d-\Delta) = \tfrac{d}{2} - \mathrm{i}\,\nu, \quad \nu = \sqrt{m^2 - \tfrac{d^2}{4}} \tag{3.11}$$

where $\bar{\Delta} = \Delta^*$ for $\nu \in \mathbb{R}$.

The scalar propagator $G(\hat{X}, \hat{Y})$ satisfies the equations of motion when $\hat{X} \neq \hat{Y}$. Since there are no unique points on the sphere and it is rotationally invariant, $G(\hat{X}, \hat{Y})$ is purely a function of the geodesic distance, θ:

$$\sigma \equiv \hat{X}\cdot\hat{Y}, \quad \theta \equiv \cos^{-1}\sigma, \quad \mathtt{w} \equiv w^2 \equiv \tfrac{1+\sigma}{2} = \cos^2\tfrac{\theta}{2}. \tag{3.12}$$

The scalar Laplacian on S^{d+1} of unit radius in terms of the geodesic distance is

$$\begin{aligned} -\nabla^2 &= -\frac{1}{\sin^d\theta}\,\partial_\theta\,\sin^d\theta\,\partial_\theta = -\partial_\theta^2 - d\,\cot\theta\,\partial_\theta \\ &= -(1-\mathtt{w})\,\mathtt{w}\,\partial_{\mathtt{w}}^2 - (d+1)\,(\tfrac{1}{2}-\mathtt{w})\,\partial_{\mathtt{w}}. \end{aligned} \tag{3.13}$$

A standard procedure to construct two-point functions is to solve the equations of motion, eq. (3.7), away from $\theta = 0$ [67, 68],

$$\Big((1-\mathtt{w})\,\mathtt{w}\,\partial_{\mathtt{w}}^2 + (d+1)\,\big(\tfrac{1}{2}-\mathtt{w}\big)\,\partial_{\mathtt{w}} - \Delta\,(d-\Delta)\Big)\,G(\mathtt{w}) = 0, \tag{3.14}$$

which is noted to remain unchanged when all $\mathtt{w}$ are changed to $\mathtt{w}' = 1 - \mathtt{w}$ (corresponding to $\theta \to \pi - \theta$). Thus, the solution space to this ODE is

$$\begin{aligned} G(\mathtt{w}) \propto\ & {}_2F_1(\Delta,\, d-\Delta\,;\, \tfrac{d+1}{2}\,;\, \mathtt{w}) \oplus \mathtt{w}^{\frac{1-d}{2}}\, {}_2F_1(\tfrac{1}{2}+\mathrm{i}\,\nu\,,\, \tfrac{1}{2}-\mathrm{i}\,\nu\,;\, 1-\tfrac{d-1}{2}\,;\, \mathtt{w}) \\ & \oplus {}_2F_1(\Delta,\, d-\Delta\,;\, \tfrac{d+1}{2}\,;\, \mathtt{w}') \oplus \mathtt{w}'^{\frac{1-d}{2}}\, {}_2F_1(\tfrac{1}{2}+\mathrm{i}\,\nu\,,\, \tfrac{1}{2}-\mathrm{i}\,\nu\,;\, 1-\tfrac{d-1}{2}\,;\, \mathtt{w}'). \end{aligned} \tag{3.15}$$

Only the first solution is continuous at $\mathtt{w} = 0,\ \theta = \pi$ corresponding to the Euclidean, or upon analytic continuation, the Bunch-Davies vacuum state, [18].

In the $\ell \to \infty$ limit of S^{d+1}, the space appears flat, just like it does in the $\theta \to 0$ limit. The normalisation of the chosen solution can hence be set by comparing the flat space limit to the flat space scalar propagator. The embedding space formalism, presented next, will naturally produce the proper normalisation, inheriting the correct factors from the higher dimensional flat space.

3.1.1 Embedding Space Formalism of the Massive Scalar Propagator

The action of a massless scalar field $\Phi(X)$ in $\mathbb{R}^{\mathbb{D}}$ written in flat and spherical coordinates $(t,\, \hat{X})$ is

$$\mathbb{S}^{[0]} = \int_X \partial^I \bar{\Phi}\, \partial_I \Phi = \int_X \bar{\Phi}\, (-\partial^2)\, \Phi = \int_X \bar{\Phi}\, e^{-2t} \Big(-(d+\partial_t)\,\partial_t - g^{\mu\nu}\,\partial_\mu\,\partial_\nu \Big)\, \Phi \tag{3.16}$$

with the Green's function

$$\begin{aligned} \mathbb{G}(X,\, Y) \equiv \mathbb{G}(e^t\,\hat{X},\, e^s\,\hat{Y}) &= \frac{1}{4\pi^{\mathbb{D}}} \int_P \frac{e^{-2\,\mathrm{i}\,P\,(X-Y)}}{P^2} = \frac{1}{4\pi^{\mathbb{D}}} \int_P \frac{e^{-2\,\mathrm{i}\,P\,(e^t\,\hat{X} - e^s\,\hat{Y})}}{P^2} \\ &= \frac{\Gamma(\frac{d}{2})}{4\pi^{\mathbb{D}}}\, \frac{1}{|X-Y|^d} \end{aligned} \tag{3.17}$$

by definition satisfying

$$\begin{aligned} \Phi(Y) &= \int_X \mathbb{G}(X,\, Y)\, (-\partial^2)\, \Phi(X) \\ \Phi(e^s\,\hat{Y}) &= \int_t e^{d\,t} \int_{\hat{X}} \mathbb{G}(X,\, Y)\, \Big(-(d+\partial_t)\,\partial_t - g^{\mu\nu}\,\partial_\mu\,\partial_\nu \Big)\, \Phi(e^t\,\hat{X}). \end{aligned} \tag{3.18}$$

By representing the scalar fields, Φ, in terms of their radial Mellin transforms, $\Phi_{[\Delta]}$, in eq. (3.18)

$$\oint_\Delta e^{-\Delta\, s}\, \Phi_{[\Delta]}(\hat{Y}) = \oint_\Delta \int_{\hat{X}} \int_t e^{(d-\Delta)\,t}\, \mathbb{G}(X,\, Y)\, \Big(-\nabla^2 + (d-\Delta)\,\Delta \Big)\, \Phi_{[\Delta]}(\hat{X}) \tag{3.19}$$

the massive scalar Laplacian on S^{d+1}, given in eq. (3.6) with mass as defined in eq. (3.11), can be recognised. A couple of simple variable substitutions, namely $t \to t+s$, $P \to P\, e^{-s}$, brings this into a suggestive form that distributes over Δ,

$$\oint_{\Delta} e^{-\Delta s}\, \underbrace{\Phi_{[\Delta]}(\hat{Y})} = \oint_{\Delta} e^{-\Delta s}\, \overbrace{\int_{\hat{X}} \int_t e^{(d-\Delta)\, t}\, \mathbb{G}(e^t\, \hat{X},\, \hat{Y}) \Big(- \nabla^2 + (d-\Delta)\,\Delta \Big) \Phi_{[\Delta]}(\hat{X})} \tag{3.20}$$

with the bracketed terms completely independent of s or alternately $|Y|$, resulting in

$$\Phi_{[\Delta]}(\hat{Y}) = \int_{\hat{X}} \int_t e^{(d-\Delta)\, t}\, \mathbb{G}(e^t\, \hat{X},\, \hat{Y}) \Big(- \nabla^2 + (d-\Delta)\,\Delta \Big)\, \Phi_{[\Delta]}(\hat{X}). \tag{3.21}$$

Comparing eq. (3.21) to eq. (3.7), the massive scalar propagator on the $d+1$-sphere is found to be

$$G(\hat{X},\, \hat{Y}) = \int_t e^{(d-\Delta)\, t}\, \mathbb{G}(e^t\, \hat{X},\, \hat{Y}) = \int_{\lambda}^{*} \lambda^{\bar{\Delta}}\, \mathbb{G}(\lambda\, \hat{X},\, \hat{Y}) = \int_{\mu}^{*} \mu^{\Delta}\, \mathbb{G}(\hat{X},\, \mu\, \hat{Y}) \tag{3.22}$$

or equivalently,

$$\begin{aligned} G(\hat{X},\, \hat{Y}) &= \int_{\lambda}^{*} |\lambda\, X|^{\bar{\Delta}}\, |Y|^{\Delta}\, \mathbb{G}(\lambda\, X,\, Y) = \int_{\mu}^{*} |X|^{\bar{\Delta}}\, |\mu\, Y|^{\Delta}\, \mathbb{G}(X,\, \mu\, Y) \\ &= \int_{q}^{*} q^{i\nu}\, |X|^{\bar{\Delta}}\, |Y|^{\Delta}\, \mathbb{G}(\tfrac{1}{\sqrt{q}}\, X,\, \sqrt{q}\, Y). \end{aligned} \tag{3.23}$$

It can be represented in a more symmetric form,

$$\begin{aligned} G^{[\Delta]}(\hat{X},\, \hat{Y}) &:= \frac{1}{\mathrm{Vol}\, \mathbb{R}^*}\, \frac{1}{4\pi^{\mathbb{D}}} \int_{\lambda,\, \mu}^{*} |\lambda\, X|^{\bar{\Delta}}\, |\mu\, Y|^{\Delta} \int_P \frac{e^{-2\mathrm{i}\; P\, (\lambda\, X - \mu\, Y)}}{P^2} \\ &= \frac{1}{\mathrm{Vol}\, \mathbb{R}^*}\, \frac{1}{4\pi^{\mathbb{D}}} \int_{\lambda,\, \mu,\, \tau}^{*} |\lambda\, X|^{\bar{\Delta}}\, |\mu\, Y|^{\Delta}\, \tau \int_P e^{-\tau\, P^2 - 2\mathrm{i}\; P\, (\lambda\, X - \mu\, Y)} \end{aligned} \tag{3.24}$$

where the division by $\mathrm{Vol}\, \mathbb{R}^*$ indicates the prescription to fix the scaling redundancy $\hat{\mathrm{T}}$ in the integration variables when the mass parameters satisfy $\Delta + \bar{\Delta} = d$,

$$\hat{\mathrm{T}} \,:\, G^{[\Delta]}(\hat{X},\, \hat{Y}) \to G^{[\Delta]}(\hat{X},\, \hat{Y}) \,:\, \{P,\, \tau,\, \lambda,\, \mu\} \mapsto \{\rho^{-1}\, P,\, \rho^2 \tau,\, \rho\, \lambda,\, \rho\, \mu\}, \quad \rho \in \mathbb{R}_+. \tag{3.25}$$

For the sake of notational brevity, this prescription will be denoted by $⨍$. It is emphasised that although the domain of definition of the propagator in eq. (3.24) has been expanded to the entirety of the embedding space $\mathbb{R}^{\mathbb{D}}$, it is still *independent* of $|X|, \;\; |Y|$. Generalisation to a wider class of propagators parameterised by η ($\Delta + \bar{\Delta} + 2\,\eta = \mathbb{D}$), $G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y})$,

$$G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y}) \;\; = \frac{1}{\mathrm{Vol}\,\mathbb{R}^{*}} \frac{1}{4^{\eta}\pi^{\mathbb{D}}} \int^{*}_{\lambda,\mu} \int_{P} |\lambda\, X|^{\bar{\Delta}}\, |\mu\, Y|^{\Delta}\, \frac{e^{-2\mathrm{i}\;P\,(\lambda X - \mu Y)}}{P^{2\,\eta}}, \tag{3.26}$$

now exhibiting the expanded scale invariance $\hat{\mathrm{T}}_{\eta}\,:\, G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y}) \to G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y})$,

$$\hat{\mathrm{T}}_{\eta}\, : \{P,\, X,\, Y,\, \tau,\, \lambda,\, \mu\} \mapsto \{\rho_1^{-1}\, P,\, \rho_2^{-1}\, X,\, \rho_3^{-1}\, Y,\, \rho_1^{2}\tau,\, \rho_1\, \rho_2\, \lambda,\, \rho_1\, \rho_3\, \mu\} \tag{3.27}$$

where $\{\rho_1,\, \rho_2,\, \rho_3\} \in \mathbb{R}_{+}$, allows the integral to be interpreted as a generalized Euler integral with *generic* parameters by assuming the dimension $\mathbb{D}$ can take any value in $\mathbb{C}$, i.e. by evaluating the Feynman integrals containing this propagator in dimensional regularisation. Further, $G_{\eta}(\hat{X},\, \hat{Y})$ can be used to define the extension of $\delta(\hat{X} - \hat{Y})$ on S^{d+1} to the embedding space, $\mathbb{R}^{\mathbb{D}}$, as

$$\begin{aligned} \delta(\hat{X} - \hat{Y}) &= \lim_{\eta \to 0} G_{\eta}(\hat{X},\, \hat{Y}) \\ &= \lim_{\eta \to 0} \frac{1}{4^{\eta}\pi^{\mathbb{D}}} \fint^{*}_{\lambda,\mu,\tau} \int_{P} |\lambda\, X|^{\mathbb{D}-\Delta}\, |\mu\, Y|^{\Delta}\, \frac{\tau^{\eta}}{\Gamma(\eta)}\, e^{-\tau\, P^{2} - 2\mathrm{i}\;P\,(\lambda\, X - \mu\, Y)}, \end{aligned} \tag{3.28}$$

for any value of Δ. For the sake of future notational convenience, $\mathcal{F} \circ G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y})$ is defined as

$$\mathcal{F} \circ G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y}) := \frac{1}{4^{\eta}\pi^{\mathbb{D}}} \fint^{*}_{\lambda,\mu} \int_{P} |\lambda\, X|^{\bar{\Delta}}\, |\mu\, Y|^{\Delta}\, \mathcal{F}\, \frac{e^{-2\mathrm{i}\;P\,(\lambda\, X - \mu\, Y)}}{P^{2\,\eta}} \tag{3.29}$$

When $|\mathcal{F}|$ is polynomial in $\{\lambda\, |P|\, |X|,\, \mu\, |P|\, |Y|,\, \tau\, |P|^{2}\}$, $\mathcal{F} \circ G^{[\Delta]}_{\eta}(\hat{X},\, \hat{Y})$ is $\hat{\mathrm{T}}_{\eta}$ invariant.

For contour integration involving extension of $\lambda,\, \mu$ to $\mathbb{C}$: note that it is possible to extend the integration range of the parameters from $\mathbb{R}_{+}$ to $\mathbb{R}$ by a simple change of variables, related by $G\big|_{\mathbb{R}} = 4\,(-1)^{\Delta + \bar{\Delta} + 1}\, \sin(\pi\bar{\Delta})\, \sin(\pi\Delta)\, G\big|_{\mathbb{R}_{+}}$. In addition, considering a gauge fixed form of this integral for illustration, eq. (3.32), G falls off at large $|\lambda|,\, |\mu|$.

It is should also be noted that although the above formulae were seemingly derived for general Δ, that is not actually the case. There was the ongoing assumption of convergence of the inverse

Mellin transformations in section A.2.2 supplying the relation of the scalar field Φ to the Mellin transformed basis $\Phi_{[\Delta]}$, which requires $\mathrm{Re}(\Delta) > 0$. This is in line with the fact that in de Sitter space massless scalars are forbidden [120, 121]. This lower bound on the validity of the Mellin transformed basis of fields obviously exists for all spin, not just scalars.

3.1.2 Equivalent Forms of the Scalar Propagator

The family of de Sitter invariant vacuum states, usually presented as a superposition of solutions, eq. (3.15), can instead be captured by the embedding space propagator formulation, in different "gauges" used to fix the scale invariance. The scale invariance $\hat{\mathrm{T}}$ in eq. (3.25) can be fixed by any of the following (inexhaustive) list of conditions that are presented with the inclusion of the relevant Faddeev-Popov determinant:

$$\text{(i) } |P|\,\delta(|P|-1), \quad \text{(ii) } 2\,\tau\,\delta(\tau-1), \quad \text{(iii) } \lambda\,\delta(\lambda-1) \cong |n+1|\,\lambda\,\mu^n\,\delta(\lambda\,\mu^n-1), \tag{3.30}$$

where $n \neq -1$. As it turns out, (ii) and (iii) are equivalent conditions related by trivial variable rescalings. The different conditions can be used to study the behaviour of the propagator as an expansion around different values of $\sigma = \cos\theta$ or equivalently $\mathtt{w} = \cos^2\frac{\theta}{2}$.

(i) Restricting the momentum integral to S^{d+1}, yields a form of $G(\hat{X},\,\hat{Y})$

$$\begin{aligned} G(\hat{X},\,\hat{Y}) &= \frac{1}{4\pi^{\mathbb{D}}}\int_{\hat{P}}\int_{\lambda}^{*}\lambda^{\bar{\Delta}}\,e^{-2\mathrm{i}\,\lambda\,\hat{P}\cdot\hat{X}}\int_{\mu}^{*}\mu^{\Delta}\,e^{2i\,\mu\,\hat{P}\cdot\hat{Y}} \\ &= \frac{1}{4\pi^{\mathbb{D}}}\int_{\hat{P}}\frac{\Gamma(\bar{\Delta})\,\Gamma(\Delta)}{(2\mathrm{i}\,\hat{X}\cdot\hat{P})^{\bar{\Delta}}\,(-2\mathrm{i}\,\hat{P}\cdot\hat{Y})^{\Delta}} \end{aligned} \tag{3.31}$$

resembling products of bulk to boundary (and back to bulk) propagators in AdS, [122].

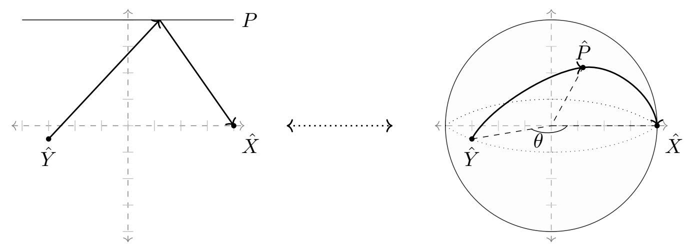


Figure 3.1: dS Scalar Propagator as an analogue of AdS Bulk to Boundary Propagators

(ii) Applying the condition $\tau = 1$

$$G(\hat{X},\,\hat{Y}) = \frac{1}{2\pi^{\frac{\mathbb{D}}{2}}} \int_{\lambda,\mu}^{*} \lambda^{\bar{\Delta}}\,\mu^{\Delta}\,e^{-|\lambda\,\hat{X}-\mu\,\hat{Y}|^2} \tag{3.32}$$

allows the integral to be evaluated as an expansion around $\sigma = -1$, i.e. $\mathtt{w} = 0$,

$$G(\hat{X},\,\hat{Y}) = \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2})}{4\pi^{\frac{\mathbb{D}}{2}}} \int_{r}^{*} \frac{r^{\bar{\Delta}}\,(1-r)^{\Delta}}{(1-4\,r\,(1-r)\,\mathtt{w})^{\frac{\bar{\Delta}+\Delta}{2}}}, \qquad \int_{r}^{*} \equiv \int_{0}^{1} \frac{\mathrm{d}r}{r\,(1-r)} \tag{3.33}$$

by expanding the integrand as a series in increasing powers of $\mathtt{w}$, r. The series remains convergent for $|4\,r\,(1-r)\,\mathtt{w}| < 1$, which is the case for $\mathtt{w},\ r \in (0,1)$.

$$G(\hat{X},\,\hat{Y}) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2}+n)}{4\pi^{\frac{\mathbb{D}}{2}}} \frac{4^n\,\mathtt{w}^n}{n!} \int_{r}^{*} r^{\bar{\Delta}+n}\,(1-r)^{\Delta+n} \tag{3.34}$$

The integral over r is in the Feynman parameterisation form and is easily evaluated to

$$\begin{aligned} G(\hat{X},\,\hat{Y}) &= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2}+n)}{4\pi^{\frac{\mathbb{D}}{2}}} \frac{\Gamma(\bar{\Delta}+n)\,\Gamma(\Delta+n)}{\Gamma(\bar{\Delta}+\Delta+2n)} \frac{4^n\,\mathtt{w}^n}{n!} \\ &= \frac{1}{2^{\bar{\Delta}+\Delta+1}\pi^{\frac{\mathbb{D}-1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(\bar{\Delta}+n)\,\Gamma(\Delta+n)}{\Gamma(\frac{\bar{\Delta}+\Delta+1}{2}+n)} \frac{\mathtt{w}^n}{n!} \end{aligned} \tag{3.35}$$

which is the series representation of

$$G(\hat{X},\,\hat{Y}) = \frac{\Gamma(\bar{\Delta})\,\Gamma(\Delta)}{2^{\bar{\Delta}+\Delta+1}\,\pi^{\frac{\mathbb{D}-1}{2}}\,\Gamma(\frac{\bar{\Delta}+\Delta+1}{2})}\,{}_2F_1(\bar{\Delta},\,\Delta,\,\tfrac{\bar{\Delta}+\Delta+1}{2},\,\mathtt{w}) \tag{3.36}$$

matching eq. (3.5) when the $\bar{\Delta} + \Delta = d$. Another approach to proceeding from eq. (3.32) onwards is to rescale the variable $\lambda \to \lambda\,\mu$, the resulting form of which is equivalent to that stemming from the third condition $\delta(\mu - 1)$, eq. (3.37).

(iii) It is useful to illustrate that $G(\hat{X},\, \hat{Y})$ resulting from $\delta(\lambda - 1)$

$$G(\hat{X},\, \hat{Y}) = \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2})}{4\pi^{\frac{\mathbb{D}}{2}}} \int_{\mu}^{*} \frac{\mu^{\Delta}}{(1 + \mu^2 - 2\,\mu\,\sigma)^{\frac{\bar{\Delta}+\Delta}{2}}} \tag{3.37}$$

is indeed equivalent to eq. (3.5) by considering its expansion around $\sigma = -1$ once again. This can be done by representing the integral as

$$\begin{aligned} G(\hat{X},\, \hat{Y}) &= \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2})}{4\pi^{\frac{\mathbb{D}}{2}}} \int_{\mu}^{*} \frac{\mu^{\Delta}}{((1 + \mu)^2 - 4\,\mu\,\mathtt{w})^{\frac{\bar{\Delta}+\Delta}{2}}} \\ &= \frac{\Gamma(\frac{\bar{\Delta}+\Delta}{2})}{2\pi^{\frac{\mathbb{D}}{2}}} \int_{\mu}^{*} \frac{\mu^{2\,\Delta}}{\big((1 - 2\,\mu\,w + \mu^2)\,(1 + 2\,\mu\,w + \mu^2)\big)^{\frac{\bar{\Delta}+\Delta}{2}}} \end{aligned} \tag{3.38}$$

and evaluating it by using the techniques reviewed in chapter 2. The $\mathcal{A}$-hypergeometric system is defined by

$$\begin{aligned} &\mathcal{A} = \left(\begin{array}{ccc|ccc} z_{0,1} & z_{1,1} & z_{2,1} & z_{0,2} & z_{1,2} & z_{2,2} \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{array}\right), \qquad \gamma = \begin{pmatrix} \frac{\bar{\Delta}+\Delta}{2} \\ \frac{\bar{\Delta}+\Delta}{2} \\ 2\Delta \end{pmatrix}, \\ &\mathcal{J} = \langle \partial_{\bullet,2}^2 - \partial_{\bullet,1}\,\partial_{\bullet,1}\ ;\ \partial_{1,\bullet}\,\partial_{2,\star} - \partial_{1,\star}\,\partial_{2,\bullet} \rangle, \quad z = \{1,\, -2w,\, 1,\, 1,\, 2w,\, 1\} \end{aligned} \tag{3.39}$$

where $\bullet$, $\star$ are placeholders referring to the different coefficients in the first or second polynomials. Some possible bases of the kernel, $\mathcal{K}$, are

$$\mathcal{K}^{\mathrm{T}} = \left(\begin{array}{cccccc} t_1 & t_2 & -(t_1{+}t_2) & -(t_1{+}t_3) & 2t_3{-}t_2 & t_1{+}t_2{-}t_3 \\ \hline t_1{+}t_2{-}t_3 & 2t_3{-}t_2 & -(t_1{+}t_3) & -(t_1{+}t_2) & t_2 & t_1 \end{array}\right). \tag{3.40}$$

Using a weight vector belonging to the class $\{1,\, 1,\, 0,\, 0,\, 1,\, 1\}$, results in the root system $\mathcal{R}$:

$$\mathcal{R} = \left(\begin{array}{ccc|ccc} 0 & 0 & -\frac{\bar{\Delta}+\Delta}{2} & -\bar{\Delta} & 0 & \frac{\bar{\Delta}-\Delta}{2} \\ 0 & 0 & -\frac{\bar{\Delta}+\Delta}{2} & -\bar{\Delta}-\frac{1}{2} & 1 & \frac{\bar{\Delta}-\Delta-1}{2} \\ \frac{\Delta-\bar{\Delta}}{2} & 0 & -\Delta & -\frac{\bar{\Delta}+\Delta}{2} & 0 & 0 \\ \frac{\Delta-\bar{\Delta}-1}{2} & 1 & -\Delta-\frac{1}{2} & -\frac{\bar{\Delta}+\Delta}{2} & 0 & 0 \end{array}\right). \tag{3.41}$$

The second and fourth series are odd in w and so expectedly vanish in the relevant physical limit. The normalisation, $\mathcal{N}$, of the series will be

$$\mathcal{N} = \left\{ \frac{\Gamma(\bar{\Delta})\,\Gamma(\frac{\Delta-\bar{\Delta}}{2})}{2\,\Gamma(\frac{\bar{\Delta}+\Delta}{2})},\ -\frac{\Gamma(\bar{\Delta}+\frac{1}{2})\,\Gamma(\frac{\Delta-\bar{\Delta}+1}{2})}{2\,\Gamma(\frac{\bar{\Delta}+\Delta}{2})},\ \frac{\Gamma(\Delta)\,\Gamma(\frac{\bar{\Delta}-\Delta}{2})}{2\,\Gamma(\frac{\bar{\Delta}+\Delta}{2})},\ -\frac{\Gamma(\Delta+\frac{1}{2})\,\Gamma(\frac{\bar{\Delta}-\Delta+1}{2})}{2\,\Gamma(\frac{\bar{\Delta}+\Delta}{2})} \right\}. \tag{3.42}$$

The first and third series differ only by a Δ-dependent factor, and together sum precisely to eq. (3.5). An expansion around $\mathtt{w} = 1$ can be found by considering the Euler transformation of ${}_2F_1$, eq. (B.1). The benefit of using the integral representation of $G(\hat{X},\, \hat{Y})$ is that it naturally encodes all forms of analytic continuation within it.

For example, considering a weight vector $\cong\ \{1,\, 0,\, 1,\, 1,\, 0,\, 1\}$ results in an expansion in negative powers of $\mathtt{w}$, i.e. an expansion around $\mathtt{w} \to \infty$:

$$\begin{aligned} G(\hat{X},\, \hat{Y}) = {} & \frac{\left((-1)^{\frac{d+1}{2}}\ \mathrm{csch}(\pi\nu)\ \sin(\pi\,\Delta) - 1\right)}{4\pi^{\frac{d}{2}}\,\Gamma(1+\mathrm{i}\,\nu)\ \sin(\pi(\frac{d}{2}+2\mathrm{i}\,\nu))}\,\frac{\Gamma(\Delta)}{(2\mathtt{w})^{\Delta}}\,{}_2F_1(\Delta,\, \tfrac{1}{2}+\mathrm{i}\,\nu,\, 1+2\mathrm{i}\,\nu,\, \tfrac{1}{\mathtt{w}}) \\ & + (\Delta \leftrightarrow \bar{\Delta}), \end{aligned} \tag{3.43}$$

satisfying the condition that G fall off with increasing $\mathtt{w}$, as is required for Anti-de Sitter space-time. Alternately, exploring the behaviour of eq. (3.37) around the limits $\sigma = 0$

$$G(\hat{X},\, \hat{Y}) = \frac{\Gamma(\frac{\Delta}{2})\,\Gamma(\frac{\bar{\Delta}}{2})}{8\pi^{\frac{d+2}{2}}}\,{}_2F_1(\tfrac{\bar{\Delta}}{2},\, \tfrac{\Delta}{2},\, \tfrac{1}{2},\, \sigma^2) + \sigma\,\frac{\Gamma(\frac{\Delta+1}{2})\,\Gamma(\frac{\bar{\Delta}+1}{2})}{4\pi^{\frac{d+2}{2}}}\,{}_2F_1(\tfrac{\bar{\Delta}+1}{2},\, \tfrac{\Delta+1}{2},\, \tfrac{3}{2},\, \sigma^2) \tag{3.44}$$

and $\sigma \to \infty$:

$$G(\hat{X},\, \hat{Y}) = \frac{\Gamma(\bar{\Delta})\,\Gamma(\frac{\Delta-\bar{\Delta}}{2})}{4\pi^{\frac{d+2}{2}}}\,{}_2F_1(\tfrac{\Delta}{2},\, \tfrac{\Delta+1}{2},\, 1+\tfrac{\Delta-\bar{\Delta}}{2},\, \tfrac{1}{\sigma^2}) + (\Delta \leftrightarrow \bar{\Delta}). \tag{3.45}$$

3.2 Vector Fields

The action representing a massive vector field A on a $d+1$-sphere is

$$
\begin{aligned}
S^{[1]} &= \tfrac{1}{2}\int^{S^{d+1}} \bar{A}^{\mu}\left(-\,g_{\mu\nu}\,\nabla^2 + \nabla_{\nu}\,\nabla_{\mu} + g_{\mu\nu}\,m^2\right) A^{\nu} \\
&= \tfrac{1}{2}\int^{S^{d+1}} \bar{A}^{\mu}\left(-\,g_{\mu\nu}\,\nabla^2 + \nabla_{\mu}\,\nabla_{\nu} + g_{\mu\nu}\,(m^2+d)\right) A^{\nu}
\end{aligned}
\tag{3.46}
$$

where $\bar{A} = A,\ A^*$ for real and complex fields respectively.

When massless, the gauge symmetry $A^{\mu} \sim A^{\mu} + \nabla^{\mu}\Psi$, for any scalar field Ψ, is fixed in what is commonly known as the R_{ξ} gauge (Landau gauge: $\xi \to 0$, Feynman gauge: $\xi = 1$) by $\frac{1}{2\xi}(\nabla\cdot\bar{A})\,(\nabla\cdot A)$, resulting in the gauge fixed action:

$$
\begin{aligned}
S^{[1]}_{\text{GF}} &= \tfrac{1}{2}\int^{S^{d+1}} \bar{A}^{\mu}\left(-\,g_{\mu\nu}\,\nabla^2 + (1-\tfrac{1}{\xi})\,\nabla_{\mu}\,\nabla_{\nu} + d\right) A^{\nu} \\
&= \tfrac{1}{2}\int^{S^{d+1}} \bar{A}^{\mu}\left(-\,g_{\mu\nu}\,\nabla^2 + (1-\tfrac{1}{\xi})\,\nabla_{\nu}\,\nabla_{\mu} + \tfrac{d}{\xi}\right) A^{\nu}.
\end{aligned}
\tag{3.47}
$$

The propagator $G^{\nu\nu'}(\hat{X},\,\hat{Y})$ is defined such that it satisfies

$$
\begin{aligned}
&K_{\mu\nu}\,G^{\nu\nu'}(\hat{X},\,\hat{Y}) = \delta_{\mu}{}^{\nu'}\,\delta(\hat{X}-\hat{Y}), \quad K_{\mu\nu} = (-\nabla^2 + m^2 + d)\,g_{\mu\nu} + \nabla_{\mu}\,\nabla_{\nu}, \\
&A^{\mu}(\hat{X}) = \int_{\hat{Y}} G^{\mu\mu'}(\hat{X},\,\hat{Y})\,K_{\mu'\nu'}\,A^{\nu'}(\hat{Y}),
\end{aligned}
\tag{3.48}
$$

where $\partial_{\mu},\ \partial_{\mu'}$ form the bases of the tangent spaces at $\hat{X},\ \hat{Y}$ respectively. Thus, $\delta_{\mu}{}^{\nu'}$ should be interpreted to be

$$
\delta_{\mu}{}^{\nu'} = \Lambda_{\mu}{}^{\mu'}\,\delta^{\nu'}_{\mu'} = \Lambda^{\nu'}{}_{\nu}\,\delta^{\nu}_{\mu}
\tag{3.49}
$$

where $\delta^{\nu}_{\mu},\ \delta^{\nu'}_{\mu'}$ are the usual Kronecker delta functions, and $\Lambda_{\mu}{}^{\mu'}$ and $\Lambda^{\nu'}{}_{\nu}$ parallel transport the basis vectors $\partial_{\mu} \in T_{\hat{X}}(S^{d+1})$ at $\hat{X}$ to the tangent space at $\hat{Y}$, $T_{\hat{Y}}(S^{d+1})$, along some path $\gamma_{\hat{X},\hat{Y}}$ and vice versa. $\Lambda^{\mu}{}_{\mu'}$ is obviously path-dependent, and so the choice of $\gamma_{\hat{X},\hat{Y}}$ needs to be canonicalised in order to properly define the parallel propagator, $\Lambda_{\mu}{}^{\mu'}$. One such choice of $\gamma_{\hat{X},\hat{Y}}$ is the *signed* geodesic θ, with the associated unit normalised tangent vectors and parallel propagator along the geodesic (depicted in fig. (3.2)) given by [67]

$$\begin{aligned} \theta_\mu &= \nabla_{\hat{X}^\mu}\theta, \qquad \nabla_{\hat{X}^\mu}\theta_\nu = \cot\theta\,(g_{\mu\nu} - \theta_\mu\,\theta_\nu), \\ \theta_{\nu'} &= -\nabla_{\hat{Y}^{\nu'}}\theta, \quad \Lambda_\mu{}^{\nu'} = -\sin\theta\,\nabla_{\hat{X}^\mu}\theta^{\nu'} - \theta_\mu\,\theta^{\nu'}. \end{aligned} \tag{3.50}$$

These unit vectors may, alternately, be defined as:

$$\begin{aligned} \sigma_\mu &= \nabla_{\hat{X}^\mu}\cos\theta = -\sin\theta\,\theta_\mu, \quad \sigma_{\nu'} = \nabla_{\hat{Y}^{\nu'}}\cos\theta = -\sin\theta\,\theta_{\nu'} \\ \sigma_{\mu\nu'} &= \nabla_{\hat{X}^\mu}\nabla_{\hat{Y}^{\nu'}}\cos\theta = \Lambda_{\mu\nu'} + 2\,\sin^2(\tfrac{\theta}{2})\,\theta_\mu\,\theta_{\nu'}. \end{aligned} \tag{3.51}$$

Thus, the only two tensorial objects the vector propagator may depend upon are: $\{\theta_\mu\,\theta_{\nu'},\ \Lambda_{\mu\nu'}\}$ or equivalently $\{\sigma_\mu\,\sigma_{\nu'},\ \sigma_{\mu\nu'}\}$.

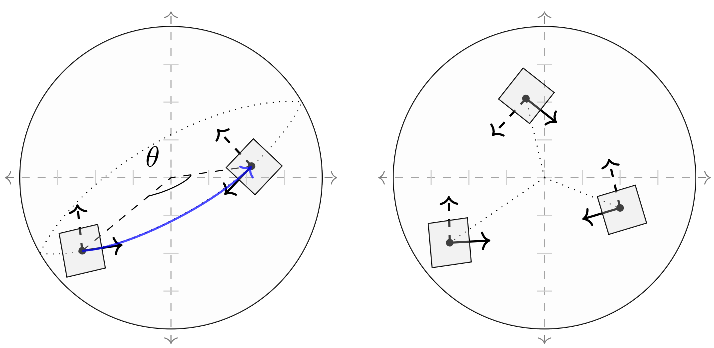


Figure 3.2: "Canonical" choice of Tangent Spaces on the Sphere wrt Geodesics

In the first image, normalised unit vectors are tangent to the geodesic. The parallel propagator carries the tangent space at one point to the other along the geodesic in the direction of these unit vectors. However, if such a geodesic was absent/not considered like in the second image, all arbitrary choices of tangent spaces would be indistinguishable.

An orthonormal basis of eigenmodes of A^μ consists of transverse spherical vector harmonics satisfying $\nabla \cdot A = 0$ and longitudinal modes, which are normalised covariant derivations of scalar spherical harmonics. The longitudinal modes form a basis of the kernel of the massless vector

Laplacian. The mass parameter Δ is

$$
\begin{aligned}
&m^2 =: (\Delta - 1)\,(\bar{\Delta} - 1), \quad \bar{\Delta} = d - \Delta, && \Delta \underset{m=0}{=} 1 \\
&m^2 = (\tfrac{d}{2} - 1)^2 + \nu^2, \qquad m^2 + d = \Delta\,\bar{\Delta} + 1, && \nu \underset{m=0}{=} \mathrm{i}\,\tfrac{d-2}{2},
\end{aligned}
\tag{3.52}
$$

defined as such because the eigenvalue of n^{th} transverse mode of the massive Laplacian operator equals $(n + \Delta + 1)\,(n + \bar{\Delta} + 1)$. Given a complete orthonormal basis of transverse A^ν_ω and longitudinal $\Phi^\nu_{\omega'} = \nabla^\nu \Phi_{\omega'}$ eigenvectors, labelled by ω, ω', the propagator $G^{\nu\nu'}(\hat{X}, \hat{Y})$ for a massive field can be written as a sum over transverse and longitudinal components:

$$
\begin{aligned}
G^{\nu\nu'}(\hat{X}, \hat{Y}) &= \sum_\omega \frac{A^\nu_\omega(\hat{X})\,\bar{A}^{\nu'}_\omega(\hat{Y})}{\lambda_\omega + d + m^2} + \sum_{\omega'} \frac{\nabla^\nu \Phi_{\omega'}(\hat{X})\,\nabla^{\nu'}\bar{\Phi}_{\omega'}(\hat{Y})}{m^2} \\
&= \sum_\omega \frac{A^\nu_\omega(\hat{X})\,\bar{A}^{\nu'}_\omega(\hat{Y})}{\lambda_\omega + d + m^2} + \frac{1}{m^2}\,\nabla^\nu_{\hat{X}}\,\nabla^{\nu'}_{\hat{Y}} \lim_{\epsilon \to 0} G^{[\epsilon]\backslash 0}(\hat{X}, \hat{Y})
\end{aligned}
\tag{3.53}
$$

where $\nabla_{\hat{X}/\hat{Y}}$ is the covariant derivative at $\hat{X}/\hat{Y}$ and the operators commute with each other when $\hat{X} \neq \hat{Y}$, Ω_{d+1} is the volume of S^{d+1} and

$$
G^{[\Delta]\backslash 0}(\hat{X}, \hat{Y}) := G^{[\Delta]}(\hat{X}, \hat{Y}) - \frac{1}{\Delta\,(d - \Delta)\,\Omega_{d+1}},
\tag{3.54}
$$

in order to exclude the zero mode of $\Phi_{\omega'}$ from the sum because the corresponding longitudinal mode is absent. In the R_ξ gauge (eq. (3.47)), the propagator $G^{[1]\nu\nu'}(\hat{X}, \hat{Y})$ of a massless vector field is similarly written as a sum over transverse and longitudinal pieces:

$$
\begin{aligned}
G^{[1]\nu\nu'}(\hat{X}, \hat{Y}) &= \sum_\omega \frac{A^\nu_\omega(\hat{X})\,\bar{A}^{\nu'}_\omega(\hat{Y})}{\lambda_\omega + d} + \xi \sum_{w'} \frac{\nabla^\nu \Phi_{\omega'}(\hat{X})\,\nabla^{\nu'}\bar{\Phi}_{\omega'}(\hat{Y})}{\lambda_{\omega'}} \\
&= \sum_\omega \frac{A^\nu_\omega(\hat{X})\,\bar{A}^{\nu'}_\omega(\hat{Y})}{\lambda_\omega + d} + \xi\,\nabla^\nu_{\hat{X}}\,\nabla^{\nu'}_{\hat{Y}} \lim_{\epsilon,\,\epsilon' \to 0} \int_{\hat{S}} G^{[\epsilon]\backslash 0}(\hat{X}, \hat{S})\,G^{[\epsilon']\backslash 0}(\hat{S}, \hat{Y}).
\end{aligned}
\tag{3.55}
$$

A useful result in this context is given in section 4.2.4 corresponding to the Feynman diagram fig. (4.1b). The transverse and longitudinal components of $G^{\nu\nu'}$ are independent and satisfy the

ODE eq. (3.48) away from $\theta = 0$ independently. Thus, the transverse components $G_T^{\nu\nu'}(\theta \neq 0)$ given by the sums in eqs. (3.53) and (3.55), that are as yet undetermined, can be found as the solution to the aforementioned ODE additionally satisfying the conditions $\nabla_\nu G_T^{\nu\nu'} = \nabla_{\nu'} G_T^{\nu\nu'} = 0$.

These definitions are, however, not enough to properly characterise the coincident point limit of $G^{\nu\nu'}$, in particular the trace of this object, making the formulation lacking for loop computations. This can be verified by explicit computation of some simple test cases, e.g. the inner product of the massive vector propagator on an odd-dimensional sphere with the $n = 1$ transverse modes, i.e. killing vectors, denoted by $\psi^{(1)}$: $\psi_\nu^{(1),(a)} = (\delta_\nu^a)$ and $\psi_\nu^{(1),(ab)} = (x^a\,\delta_\nu^b - x^b\,\delta_\nu^a)$ modulo normalisation, should be

$$\left\langle \psi_\nu^{(1)}(x)\, G^{\nu\nu'}(x,y)\, \psi_{\nu'}^{(1)}(y) \right\rangle \overset{?}{=} \frac{\frac{(d+1)(d+2)}{2}}{(\bar{\Delta}+1)(\Delta+1)}, \tag{3.56}$$

but for $d = 2, 4$ deviates from the expected value by $-\frac{2}{\nu^2}$, $-\frac{3}{1+\nu^2}$ respectively. The embedding space description of $G^{\nu\nu'}$ doesn't have this drawback, as has been explicitly verified for general transverse and longitudinal modes in section C.1.2.

3.2.1 Embedding Space Representation of (Gauge Fixed) Massless Vector Action

The action of a massless vector field $\mathbb{A}_I$ in $\mathbb{R}^{\mathbb{D}}$:

$$\mathbb{S}^{[1]} = \tfrac{1}{4}\int_X \bar{\mathbb{F}}^{IJ}\, \mathbb{F}_{IJ} = \tfrac{1}{2}\int_X \bar{\mathbb{A}}^I \left(-\partial^2\,\delta_{IJ} + \partial_J\,\partial_I \right) \mathbb{A}^J \tag{3.57}$$

exhibits a gauge symmetry $\delta\mathbb{A}_M = \partial_M \Psi$, $\delta\bar{\mathbb{A}}_M = \partial_M \bar{\Psi}$, parameterised by some scalar field Ψ, $\bar{\Psi}$, that leaves the field strength $\mathbb{F}$, $\bar{\mathbb{F}}$ and action $\mathbb{S}^{[1]}$ unchanged. In what is commonly known as the R_ξ gauge, with gauge fixing action $\frac{1}{2\,\xi}(\nabla\cdot\bar{\mathbb{A}})(\nabla\cdot\mathbb{A})$, the propagator $\mathbb{G}_{IJ}(X, Y)$ of a massless vector field is

$$\mathbb{G}_{IJ}(X,\,Y) = \int_P \left(\delta_{IJ} - (1-\xi)\,\tfrac{P_I\,P_J}{P^2}\right) \frac{e^{-\mathrm{i}\,P\,(X-Y)}}{P^2}. \tag{3.58}$$

The 2-point function of the field strength $\mathbb{F}$ remains independent of the gauge parameter ξ:

$$\langle \mathbb{F}_{IJ}(X)\, \mathbb{F}_{I'J'}(Y) \rangle = \int_P 4\, P_{[I}\, \delta_{J][J'}\, P_{I']}\, \frac{e^{-\mathrm{i}\, P\,(X-Y)}}{P^2}. \tag{3.59}$$

In spherical coordinates, with field strength $\mathbb{F}$ defined as

$$\mathbb{F}_{\mu\nu} = \nabla_\mu \mathbb{A}_\nu - \nabla_\nu \mathbb{A}_\mu, \quad \mathbb{F}_\mu := \mathbb{F}_{\mu t} = \nabla_\mu \mathbb{A}_t - \nabla_t \mathbb{A}_\mu, \tag{3.60}$$

the same action is given by

$$\begin{aligned} \mathbb{S}^{[1]} &= \tfrac{1}{2} \int_X \bar{\mathbb{F}}^\mu\, \mathbb{F}_\mu + \tfrac{1}{2} \bar{\mathbb{F}}^{\mu\nu}\, \mathbb{F}_{\mu\nu} = \tfrac{1}{2} \int_X \mathbb{g}^{\mu\nu}\, \mathbb{g}^{tt}\, \bar{\mathbb{F}}_\mu\, \mathbb{F}_\nu + \tfrac{1}{2}\, \mathbb{g}^{\mu\sigma}\, \mathbb{g}^{\nu\rho}\, \bar{\mathbb{F}}_{\mu\nu}\, \mathbb{F}_{\sigma\rho} \\ &= \tfrac{1}{2} \int_X -\mathbb{g}_{tt}\, \bar{\mathbb{A}}^t\, \nabla^\rho\, \nabla_\rho\, \mathbb{A}^t + \bar{\mathbb{A}}^\mu\, \nabla_t\, \nabla_\mu\, \mathbb{A}^t + \bar{\mathbb{A}}^t\, \nabla_\nu\, \nabla_t\, \mathbb{A}^\nu + \bar{\mathbb{A}}^\mu\, \mathbb{K}_{\mu\nu}\, \mathbb{A}^\nu \\ &= \tfrac{1}{2} \int_t e^{\mathbb{D}t} \int_{\hat{X}} -\bar{\mathbb{A}}^t\, \nabla^2\, \mathbb{A}^t + \bar{\mathbb{A}}^\mu\, (\partial_t + d)\, \nabla_\mu\, \mathbb{A}^t + \bar{\mathbb{A}}^t\, (\partial_t + 2)\, \nabla_\nu\, \mathbb{A}^\nu + \bar{\mathbb{A}}^\mu\, K_{\mu\nu}\, \mathbb{A}^\nu \end{aligned} \tag{3.61}$$

where the spherical components of the Laplacian operators $\mathbb{K}_{\mu\nu},\, K_{\mu\nu}$ are

$$\begin{aligned} \mathbb{K}_{\mu\nu} &= -\mathbb{g}_{\mu\nu}\, (\nabla^t\, \nabla_t + \nabla^\rho\, \nabla_\rho) + \nabla_\nu\, \nabla_\mu \\ K_{\mu\nu} &= -g_{\mu\nu}\, \nabla^2 + \nabla_\nu\, \nabla_\mu - g_{\mu\nu}\, (\partial_t^2 + (d+2)\, \partial_t + 2\, d). \end{aligned} \tag{3.62}$$

The gauge symmetry in these coordinates is expectedly generated by the transformations:

$$\delta \mathbb{A}^\mu = \nabla^\mu \Psi, \quad \delta \mathbb{A}^t = e^{-2t}\, \partial_t\, \Psi = \nabla^t \Psi, \quad \delta \bar{\mathbb{A}}^\mu = \nabla^\mu \bar{\Psi}, \quad \delta \bar{\mathbb{A}}^t = e^{-2t}\, \partial_t \bar{\Psi} = \nabla^t \bar{\Psi}. \tag{3.63}$$

Representing $\mathbb{A}^\nu$, $\mathbb{A}^t$ in terms of their radial Mellin transformed fields $A^\nu_{[\Delta]}$, $\chi_{[\Delta]}$, as defined in section A.2.2, the action in eq. (3.61) becomes

$$\begin{aligned}
\mathbb{S}^{[1]} &= \tfrac{1}{2} \oint_\Delta \int_t e^{(\bar{\Delta}+1)t} \int_{\hat{X}} g_{\mu\nu} \left(\nabla^\mu \bar{\mathbb{A}}^t + (\bar{\Delta}-1)\,\bar{\mathbb{A}}^\mu\right)\left(\nabla^\nu \chi_{[\Delta]} + (\Delta-1)\,A^\nu_{[\Delta]}\right) \\
&\qquad + \bar{\mathbb{A}}^\mu \underbrace{\left(-\,g_{\mu\nu}\,\nabla^2 + \nabla_\nu\,\nabla_\mu\right)}\,A^\nu_{[\Delta]} \\
&= \tfrac{1}{2} \oint_\Delta \int_t e^{(\bar{\Delta}+1)t} \int_{\hat{X}} \bar{\mathbb{A}}^t\,(-\nabla^2)\,\chi_{[\Delta]} - (\Delta-1)\,\bar{\mathbb{A}}^t\,\nabla_\nu\,A^\nu_{[\Delta]} + (\bar{\Delta}-1)\,\bar{\mathbb{A}}^\mu\,\nabla_\mu\,\chi_{[\Delta]} \\
&\qquad + \bar{\mathbb{A}}^\mu \underbrace{\left(-\,g_{\mu\nu}\,\nabla^2 + \nabla_\nu\,\nabla_\mu + g_{\mu\nu}\,(\Delta-1)\,(\bar{\Delta}-1)\right)}\,A^\nu_{[\Delta]}
\end{aligned} \tag{3.64}$$

where the bracketed terms are the massless and massive vector Laplacians on S^{d+1} respectively, given in eq. (3.46) with mass as defined in eq. (3.52). By interpreting the remaining integral over t as a Mellin transform of $\bar{\mathbb{A}}$, $\mathbb{S}^{[1]}$ can be represented as

$$\begin{aligned}
\mathbb{S}^{[1]} = \tfrac{1}{2} \oint_\Delta \int_{\hat{X}} g_{\mu\nu} &\left(\nabla^\mu \bar{\chi}_{[\bar{\Delta}]} + (\bar{\Delta}-1)\,\bar{A}^\mu_{[\bar{\Delta}]}\right)\left(\nabla^\nu \chi_{[\Delta]} + (\Delta-1)\,A^\nu_{[\Delta]}\right) \\
&+ \bar{A}^\mu_{[\bar{\Delta}]}\left(-\,g_{\mu\nu}\,\nabla^2 + \nabla_\nu\,\nabla_\mu\right) A^\nu_{[\Delta]}
\end{aligned} \tag{3.65}$$

with the gauge symmetry in these new coordinates $(\Delta, \hat{X})$ being

$$\begin{aligned}
\delta A^\nu_{[\Delta]} &= \nabla^\nu \Psi_{[\Delta]}, \quad \delta\chi_{[\Delta]} = -(\Delta-1)\,\Psi_{[\Delta]} \\
\delta \bar{A}^\mu_{[\bar{\Delta}]} &= \nabla^\mu \bar{\Psi}_{[\bar{\Delta}]}, \quad \delta\bar{\chi}_{[\bar{\Delta}]} = -(\bar{\Delta}-1)\,\bar{\Psi}_{[\bar{\Delta}]}.
\end{aligned} \tag{3.66}$$

BRST gauge fixing

A general form of the gauge fixing action is

$$\begin{aligned}
\mathbb{S}^{[1]}_{\text{GF}} &= \tfrac{1}{2} \int \left(\bar{\beta}_1\,\nabla_\mu \bar{\mathbb{A}}^\mu + \bar{\beta}_2\,\nabla_t \bar{\mathbb{A}}^t\right)\left(\beta_1\,\nabla_\mu \mathbb{A}^\mu + \beta_2\,\nabla_t \mathbb{A}^t\right) + \mathbb{S}^{[1]}_{\text{ghost}} \\
\mathbb{S}^{[1]}_{\text{ghost}} &= -\int \mathbb{c}^\dagger\left(\bar{\beta}_1\,\nabla_\mu \nabla^\mu \bar{\mathbb{c}} + \bar{\beta}_2\,\nabla_t \nabla^t \bar{\mathbb{c}}\right) + \bar{\mathbb{c}}^\dagger\left(\beta_1\,\nabla_\mu \nabla^\mu \mathbb{c} + \beta_2\,\nabla_t \nabla^t \mathbb{c}\right).
\end{aligned} \tag{3.67}$$

where $\mathbb{c}, \bar{\mathbb{c}}$ and $\mathbb{c}^\dagger, \bar{\mathbb{c}}^\dagger$ are ghost and anti-ghost fields. Upon integrating by parts,

$$\mathbb{S}^{[1]}_{\rm GF} = -\tfrac{1}{2}\int \bar{\mathbb{A}}^t {K'}_{tt}\mathbb{A}^t + \bar{\mathbb{A}}^t {K'}_{t\nu}\mathbb{A}^\nu + \bar{\mathbb{A}}^\mu {K'}_{\mu t}\mathbb{A}^t + \bar{\mathbb{A}}^\mu {K'}_{\mu\nu}\mathbb{A}^\nu + \mathbb{S}^{[1]}_{\rm ghost} \tag{3.68}$$

where

$$\begin{aligned}
{K'}_{tt} &= \bar{\beta}_2\,\beta_1\,(d+1)\,\partial_t + \bar{\beta}_2\,\beta_2\,\partial_t\,(\partial_t+1)\\
{K'}_{t\nu} &= \bar{\beta}_2\,\beta_1\,\partial_t\,\nabla_\nu, \qquad {K'}_{\mu t} = \big(\bar{\beta}_1\,\beta_1\,(d+2) + \bar{\beta}_1\,\beta_2\,\partial_t\big)\,\nabla_\mu\\
{K'}_{\mu\nu} &= \bar{\beta}_1\,\beta_1\,\nabla_\mu\nabla_\nu + \bar{\beta}_1\,\big(\beta_1-\beta_2\big)\,g_{\mu\nu}\,\partial_t.
\end{aligned} \tag{3.69}$$

Once again representing the fields $\bar{\mathbb{A}}$, $\mathbb{A}$ in terms of their radially Mellin transformed basis, the gauge fixing action in $(\Delta,\,\hat{X})$ coordinates is

$$\begin{aligned}
\mathbb{S}^{[1]}_{\rm GF} = \tfrac{1}{2}\oint_\Delta\int_{\hat{X}} &\bar{\beta}_2\,(\Delta+1)\,\big(\beta_1\,(d+1)-\beta_2\,\Delta\big)\,\bar{\chi}_{[\bar{\Delta}]}\,\chi_{[\Delta]}\\
&+ \bar{\beta}_2\,\beta_1\,(\Delta+1)\,\bar{\chi}_{[\bar{\Delta}]}\,\nabla_\nu A^\nu_{[\Delta]} + \bar{\beta}_1\,\big(\beta_1\,(d+2)-\beta_2\,(\Delta+1)\big)\,\nabla_\mu\bar{A}^\mu_{[\Delta]}\,\chi_{[\Delta]}\\
&- \bar{\beta}_1\,\beta_1\,\bar{A}^\mu_{[\Delta]}\,\nabla_\mu\,\nabla_\nu A^\nu_{[\Delta]} + \bar{\beta}_1\,\big(\beta_1-\beta_2\big)\,(\Delta+1)\,g_{\mu\nu}\,\bar{A}^\mu_{[\Delta]}\,A^\nu_{[\Delta]}.
\end{aligned} \tag{3.70}$$

Deciding to leave the mass term of $\bar{A},\,A$ unchanged in the total gauge fixed action constrains $\mathbb{S}^{[1]}_{\rm GF}$ to either of the following:

$$\begin{aligned}
\mathbb{S}^{[1]}_{\rm GF}\Big|_{\beta_1=\beta_2} &= \tfrac{\beta_1}{2}\oint_\Delta\int_{\hat{X}} \bar{\beta}_2\,(\Delta+1)\,(\bar{\Delta}+1)\,\bar{\chi}_{[\bar{\Delta}]}\,\chi_{[\Delta]} - \bar{\beta}_1\,\bar{A}^\mu_{[\Delta]}\,\nabla_\mu\,\nabla_\nu A^\nu_{[\Delta]}\\
&\qquad\qquad + \bar{\beta}_2\,(\Delta+1)\,\bar{\chi}_{[\bar{\Delta}]}\,\nabla_\nu A^\nu_{[\Delta]} + \bar{\beta}_1\,(\bar{\Delta}+1)\,\nabla_\mu\bar{A}^\mu_{[\Delta]}\,\chi_{[\Delta]}\\
\mathbb{S}^{[1]}_{\rm GF}\Big|_{\bar{\beta}_1=0} &= \tfrac{\bar{\beta}_2}{2}\oint_\Delta\int_{\hat{X}} (\Delta+1)\,\bar{\chi}_{[\bar{\Delta}]}\,\Big(\big(\beta_1\,(d+1)-\beta_2\,\Delta\big)\,\chi_{[\Delta]} + \beta_1\,\nabla_\nu A^\nu_{[\Delta]}\Big).
\end{aligned} \tag{3.71}$$

Upon requiring the cross terms $\chi - \bar{A}$ and $\bar{\chi} - A$ to vanish in the total gauge fixed action:

$$\begin{aligned}
\mathbb{S}^{[1]}_{\rm GF}\Big|_{R_\xi} &= \tfrac{1}{2}\oint_\Delta\int_{\hat{X}} (\Delta-1)\,(\bar{\Delta}+1)\,\bar{\chi}_{[\bar{\Delta}]}\,\chi_{[\Delta]} - \tfrac{(\bar{\Delta}-1)}{(\bar{\Delta}+1)}\,\bar{A}^\mu_{[\Delta]}\,\nabla_\mu\,\nabla_\nu A^\nu_{[\Delta]}\\
&\qquad + (\Delta-1)\,\bar{\chi}_{[\bar{\Delta}]}\,\nabla_\nu A^\nu_{[\Delta]} + (\bar{\Delta}-1)\,\nabla_\mu\bar{A}^\mu_{[\Delta]}\,\chi_{[\Delta]}.
\end{aligned} \tag{3.72}$$

The total gauge fixed action, $\mathbb{S}^{[1]} + \mathbb{S}^{[1]}_{\mathrm{GF}}\Big|_{R_\xi}$, represented in $(\Delta,\,\hat{X})$ coordinates at $\Delta = 1$ comes to resemble the massless vector action on S^{d+1} in the R_ξ gauge, eq. (3.47), specifically at $\xi = \frac{d}{d-2}$.

Massive

Evidently, a gauge choice of eq. (3.65) in which the scalar fields $\bar{\chi},\,\chi$ (and when $\Delta,\,\bar{\Delta} \neq 1$, equivalently $\bar{\mathbb{A}}_t,\,\mathbb{A}_t$) vanish, the gauge fixed action will take the form of eq. (3.46), representing a massive vector field on S^{d+1}. However, the radial gauge ($\mathbb{A}_t = 0$) does not completely fix the gauge symmetry of eq. (3.61), instead reducing it to $\mathbb{A}_\mu \sim \mathbb{A}_\mu + \nabla_\mu \Psi(\hat{X})$, requiring some additional condition to completely fix this gauge symmetry, the most natural being $\nabla^\mu \mathbb{A}_\mu = 0$. Regardless of the choice of this condition, the gauge invariant quantity $\mathbb{F}_\mu$ will remain the same,

$$\mathbb{F}^{\mathrm{rad}}_\mu = -\partial_t \mathbb{A}_\mu \to -\partial_t \left(\mathbb{A}_\mu + \nabla_\mu \Psi(\hat{X})\right) = -\partial_t \mathbb{A}_\mu = \oint_\Delta (\Delta - 1)\, e^{-(\Delta-1)\,t}\, A_{[\Delta]\mu} \tag{3.73}$$

consequently allowing its evolution to be studied in terms of a propagator in any gauge. Changing back to flat coordinates for convenience,

$$\begin{aligned} &\mathbb{F}^{\mathrm{rad}}_I = \hat{\mathbb{e}}^\mu_I\, \mathbb{F}^{\mathrm{rad}}_\mu = \oint_\Delta e^{-\Delta\, t}\, F_{[\Delta]I}, \quad F_{[\Delta]I} := \hat{e}^\mu_I\, (\Delta - 1)\, A_{[\Delta]\mu} = (\Delta - 1)\, A_{[\Delta]I}, \\ &\mathbb{S}^{[1]}_{\mathrm{rad}} = \tfrac{1}{2} \int_X \bar{\mathbb{F}}^\mu\, \mathbb{F}_\mu + \cdots = \tfrac{1}{2} \int_t \int_{\hat{X}} e^{d\,t}\, \bar{\mathbb{F}}_\mu\, \mathbb{F}_\nu + \cdots = \tfrac{1}{2} \int_X \bar{\mathbb{F}}^{\mathrm{rad}}_I\, e^{-2\,t}\, \delta^{IJ}\, \mathbb{F}^{\mathrm{rad}}_J + \cdots\,, \end{aligned} \tag{3.74}$$

its associated 2-point function, found using eq. (3.59), is

$$\mathbb{G}^{\mathrm{rad}}_{II'}(X,\,Y) = \left\langle \mathbb{F}^{\mathrm{rad}}_I(X)\, \mathbb{F}^{\mathrm{rad}}_{I'}(Y) \right\rangle = \int_P 4\, X^J\, P_{[I}\, \delta_{J][J'}\, P_{I']}\, Y^{J'}\, \frac{e^{-\mathrm{i}\, P\,(X-Y)}}{P^2}. \tag{3.75}$$

Thus, in the same vein as eqs. (3.18) to (3.21) of the scalar case, the propagator equation

$$\mathbb{F}^{\mathrm{rad}}_{I'}(Y) = \int_X \mathbb{G}^{\mathrm{rad}}_{I'I}\, e^{-2\,t}\, \delta^{IJ}\, \mathbb{F}^{\mathrm{rad}}_J(X) \tag{3.76}$$

represented in terms of Mellin transformed fields

$$\oint_{\Delta} e^{-\Delta\, s}\, F_{[\Delta]\, I'}(\hat{Y}) = \oint_{\Delta} e^{-\Delta\, s} \int_{\hat{X}} F^{I}_{[\Delta]}(\hat{X}) \\ \times \int_{t} e^{(d-\Delta+1)\, t} \int_{P} 4\, \hat{X}^{J}\, P_{[I}\, \delta_{J][J'}\, P_{I']}\, \hat{Y}^{J'}\, \frac{e^{-\mathrm{i}\, P\, (e^{t}\hat{X}-\hat{Y})}}{P^2} \tag{3.77}$$

distributes over Δ into $|Y| = e^{s}$ independent parts to give

$$\left\langle F_{[\Delta]\, I}(\hat{X})\, F_{[\Delta]\, I'}(\hat{Y}) \right\rangle = \int_{t} e^{(d-\Delta+1)\, t} \int_{P} 4\, \hat{X}^{J}\, P_{[I}\, \delta_{J][J'}\, P_{I']}\, \hat{Y}^{J'}\, \frac{e^{-\mathrm{i}\, P\, (e^{t}\hat{X}-\hat{Y})}}{P^2}, \tag{3.78}$$

and, upon making the identification eq. (3.74), the propagator of a massive vector field on S^{d+1} as described by eq. (3.65) in the radial gauge (further detailed section 3.2.2).

Massless

A massless vector field A on S^{d+1} is represented by eq. (3.65) in a gauge which results in the decoupling of $\bar{A}$, A and $\bar{\chi}$, χ. However, at $\Delta = 1$, i.e. when the corresponding vector field $A_{[\Delta=1]}$ on S^{d+1} is massless, neither is the gauge condition $\chi = 0$ proper nor is the identification eq. (3.74) valid. Alternately, applying the tangentiality condition on the field $\mathbb{A}$, $X \cdot \mathbb{A} = 0$, decouples the radial $\mathbb{A}_t$ and spherical $\mathbb{A}_\mu$ components of the vector field by enforcing $\mathbb{A}_t = 0$ identically on the projected field,

$$\mathbb{A}^{\tan}_{I} = \hat{\mathbb{t}}^{J}_{I}\, \mathbb{A}_{J}, \tag{3.79}$$

where $\hat{\mathbb{t}}$ is the tangential projection operator, as defined in eq. (A.10), and $\mathbb{F}_\mu$ takes the form:

$$\mathbb{F}_\mu \Big|_{\mathbb{A}=\mathbb{A}^{\tan}} = \nabla_\mu \mathbb{A}_t - \nabla_t \mathbb{A}_\mu \Big|_{\mathbb{A}_t=0} = -\nabla_t \mathbb{A}_\mu = -\partial_t \mathbb{A}_\mu + \mathbb{A}_\mu = \oint_{\Delta} \Delta\, e^{-(\Delta-1)\, t}\, A_{[\Delta]\mu}, \tag{3.80}$$

where the slice of $\mathbb{F}_\mu$ at $\Delta = 1$ is the radial average. Proceeding similarly to the Massive case, the relevant 2-point function of $\mathbb{F}_\mu$ in flat coordinates, constructed in terms of eq. (3.58), is

$$\begin{aligned} \mathbb{G}^{\mathrm{rad,\,tan}}_{I I'}(X,\,Y) &= \left\langle \mathbb{F}^{\mathrm{rad}}_I(X)\,\mathbb{F}^{\mathrm{tan}}_{I'}(Y) \right\rangle = X^J\,(\delta^K_I\,\partial_{X^J} - \delta^K_J\,\partial_{X^I})\,Y^{J'}\,\partial_{Y^{J'}}\,\mathbb{G}_{K K'}\,\hat{\mathbb{t}}^{K'}_{I'} \\ &= \int_P \Big(|PX|\,|PY|\,\delta_{I K'} - |PY|\,P_I\,X_{K'}\Big)\left(\delta^{K'}_{I'} - \tfrac{Y^{K'}\,Y_{I'}}{|Y|^2}\right)\frac{e^{-\mathrm{i}\,P\,(X-Y)}}{P^2} \end{aligned} \tag{3.81}$$

and its reduction to the sphere at $\Delta = 1$ is

$$\begin{aligned} \left\langle F_{[\Delta=1]\,I}(\hat{X})\,F_{[\Delta=1]\,I'}(\hat{Y}) \right\rangle = \int_t e^{d\,t} \int_P \Big(P_R\,\hat{X}^R\,\delta_{I K'} - P_I\,\hat{X}_{K'}\Big)\,P_{R'}\,\hat{Y}^{R'} \\ \times \left(\delta^{K'}_{I'} - \hat{Y}^{K'}\,\hat{Y}_{I'}\right)\frac{e^{-\mathrm{i}\,P\,(e^t\hat{X}-\hat{Y})}}{P^2}. \end{aligned} \tag{3.82}$$

The identifications eqs. (3.74) and (3.80) give the correspondence of eq. (3.82) to the propagator of a massless vector field on S^{d+1} in the $R_{\xi=\frac{d}{d-2}}$ gauge, where the effective gauge parameter ξ is found by comparing the masslike term $\bar{\mathbb{F}}^\mu\,\mathbb{F}_\mu\Big|_{\Delta=1} \sim (d-2)\,\bar{A}^\mu\,A_\mu$ in this radial gauge-tangential projection hybrid setup to that in eq. (3.47), i.e. $\frac{d}{\xi}\,\bar{A}^\mu\,A_\mu$, matching eq. (3.72) (further detailed in section 3.2.3).

3.2.2 Massive Vector Propagator

The propagator of the massive vector field with mass parameter as given in eq. (3.52), as originating from the 2-point function of the field strength in $(\Delta,\,\hat{X})$ coordinates, eq. (3.78), is

$$\begin{aligned} &G_{I I'}(\hat{X},\,\hat{Y}) = \mathcal{G}_{I I'} \circ G^{[\Delta]}(\hat{X},\,\hat{Y}), \qquad G_{\nu\nu'}(\hat{X},\,\hat{Y}) = \hat{e}^I_\nu\,\hat{e}^{I'}_{\nu'}\,G_{I I'}(\hat{X},\,\hat{Y}) \\ &\mathcal{G}_{I I'} = \frac{4\,\lambda\,\mu}{(\bar{\Delta}-1)\,(\Delta-1)}\,\hat{X}^J\,P_{[I}\,\delta_{J][J'}\,P_{I']}\,\hat{Y}^{J'} \\ &\quad = \frac{4\,\lambda\,\mu}{(\bar{\Delta}-1)\,(\Delta-1)}\Big(|PX|\,|PY|\,\delta_{I I'} - |PX|\,Y_I\,P_{I'} - P_I\,X_{I'}\,|PY| + |XY|\,P_I\,P_{I'}\Big), \end{aligned} \tag{3.83}$$

following the notation in eq. (3.29). The operator $\mathcal{G}_{I I'}$ is manifestly bilocally tangential ($X^I\,\mathcal{G}_{I I'} = \mathcal{G}_{I I'}\,Y^{I'} = 0$) and transverse ($\partial^I\mathcal{G}_{I I'} = \partial^{I'}\mathcal{G}_{I I'} = 0$). Its scale invariance, as required by eq. (3.29), can be made more manifest in a variety of ways:

$$\begin{aligned}\mathcal{G}_{II'} &= \frac{1}{(\Delta-1)(\bar{\Delta}-1)}\Big(\theta_\lambda\,\theta_\mu\,\delta_{II'} - \theta_\lambda\,Y_I\,\partial_{Y^{I'}} - \theta_\mu\,X_{I'}\,\partial_{X^I} + |XY|\,\partial_{X^I}\,\partial_{Y^{I'}}\Big)\\ &= \frac{\bar{\Delta}\,\Delta}{(\bar{\Delta}-1)\,(\Delta-1)}\Big(\delta_{II'} - \tfrac{1}{|PY|}\,Y_I\,P_{I'} - \tfrac{1}{|PX|}\,P_I\,X_{I'} + \tfrac{1}{|PX|\,|PY|}\,|XY|\,P_I\,P_{I'}\Big).\end{aligned} \tag{3.84}$$

where eq. (4.24) was used. Although $\mathcal{G}_{II'}$ is defined over the entire embedding space, it is actually dependent on *only* $(\hat{X},\,\hat{Y})$, i.e. $\mathcal{G}_{II'}(X,\,Y) = \mathcal{G}_{II'}(\hat{X},\,\hat{Y})$. Purely in terms of position space operators, it is

$$\mathcal{G}_{II'} = \frac{1}{(\bar{\Delta}-1)\,(\Delta-1)}\,\hat{X}^J\,\partial_{[\hat{X}^I}\,\delta_{J][J'}\,\partial_{\hat{Y}^{I'}]}\,\hat{Y}^{J'}. \tag{3.85}$$

However, since the derivatives in the above do not commute with the rest of the expression, in practice it is more useful to consider an equivalent formulation based on eq. (4.22):

$$\begin{aligned}\mathcal{G}_{II'} = \frac{1}{(\bar{\Delta}-1)\,(\Delta-1)}\Big(&\bar{\Delta}\,\Delta\,\delta_{II'} + 2\mathrm{i}\;\bar{\Delta}\,\mu\,Y_I\,P_{I'} - 2\mathrm{i}\;\Delta\,\lambda\,P_I\,X_{I'}\\ &+ \tfrac{2\mathrm{i}\;\lambda\,\mu}{\sqrt{\alpha_X\,\alpha_Y}}\,P_I\,P_{I'}\,\partial_\beta\,e^{-2\mathrm{i}\;\sqrt{\alpha_X\,\alpha_Y}\,\beta\,|XY|}\Big|_{\beta=0}\Big).\end{aligned} \tag{3.86}$$

3.2.3 Massless Vector Propagator

The propagator of the massless vector field, i.e. with mass parameter $\Delta = 1$, in the R_ξ gauge, $\xi = \frac{d}{d-2}$, given in eq. (3.47), as implied by eq. (3.82), is

$$\begin{aligned}G^{(o)}_{II'}(\hat{X},\,\hat{Y}) = \mathcal{G}^{(o)}_{II'} \circ G^{[\Delta=1]}(\hat{X},\,\hat{Y}), \qquad G^{(o)}_{\nu\nu'}(\hat{X},\,\hat{Y}) = \hat{e}^I_\nu\,\hat{e}^{I'}_{\nu'}\,G^{(o)}_{II'}(\hat{X},\,\hat{Y})\\ \mathcal{G}^{(o)}_{II'} = \frac{4\,\lambda\,\mu}{d-2}\,|PY|\,\Big(|PX|\,\delta_{II'} - P_I\,X_{I'} - \tfrac{|PX|}{|Y|^2}\,Y_I\,Y_{I'} + \tfrac{|XY|}{|Y|^2}\,P_I\,Y_{I'}\Big)\end{aligned} \tag{3.87}$$

The last 2 terms of $\mathcal{G}^{(o)}_{II'}$ will be irrelevant in proper formulations of closed Feynman integrals containing this propagator because they will, by default, be paired with tangential objects. Its tangentiality is obvious: $X^I\,\mathcal{G}^{(o)}_{II'} = \mathcal{G}^{(o)}_{II'}\,Y^{I'} = 0$. Some reformulation akin to the massive case confirms its scale invariance too:

$$\mathcal{G}^{(o)}_{II'} = \frac{d-1}{d-2}\Big(\delta_{II'} - \frac{P_I\, X_{I'}}{|PX|} - \frac{Y_I\, Y_{I'}}{|Y|^2} + \frac{|XY|}{|Y|^2|PX|}\, P_I\, Y_{I'}\Big). \tag{3.88}$$

Though less pleasing to the eye, it is more practical to use in Feynman integrals as:

$$\begin{aligned}\mathcal{G}^{(o)}_{II'} = \frac{1}{d-2}\Big(&(d-1)\,\delta_{II'} - 2\mathrm{i}\;\lambda\, P_I\, X_{I'} - (d-1)\,\frac{Y_I\, Y_{I'}}{|Y|^2}\\ &- \frac{\lambda}{\sqrt{\alpha_X\,\alpha_Y}}\,\frac{P_I\, Y_{I'}}{|Y|^2}\,\partial_\beta\, e^{-2\mathrm{i}\,\sqrt{\alpha_X\,\alpha_Y}\,\beta\,|XY|}\Big|_{\beta=0}\Big).\end{aligned} \tag{3.89}$$

For $d = 2$, i.e. on S^3, the massless propagator is constructed in the $\xi \to \infty$ limit of the R_ξ gauge in embedding space:

$$\mathcal{G}^{(o)}_{II'} = \lim_{\xi\to 0}\Big(\delta^K_I - \frac{X^K\, X_I}{|X|^2}\Big)\Big(\delta_{KK'} + \frac{P_K\, P_{K'}}{\xi\, P^2}\Big)\Big(\delta^{K'}_{I'} - \frac{Y^{K'}\, Y_{I'}}{|Y|^2}\Big). \tag{3.90}$$

3.2.4 Position Space Form of Vector Propagators

The position space form of the massive vector propagator, $G_{\nu\nu'}(\hat{X},\, \hat{Y})$ in eq. (3.83), can be found by applying the operator form of $\mathcal{G}_{II'}$ in eq. (3.85) on the scalar propagator, eq. (3.5). It is useful to recall that derivatives of the hypergeometric function ${}_2F_1$ simply induce shifts to its parameters, written concisely by using Pochhammer symbols, eq. (2.9) as

$$\partial^n_x\, {}_2F_1(a,\, b,\, c,\, x) = \frac{a^{(n)}\, b^{(n)}}{c^{(n)}}\, {}_2F_1(a+n,\, b+n,\, c+n,\, x). \tag{3.91}$$

Working through the details, the position space *propagator*, in terms of the geodesic distance $\theta = \cos^{-1}(\sigma)$, the unit vectors defined along it, eqs. (3.50) and (3.51), and the massive scalar propagator $G_{[\Delta]}$ (where the effective scalar mass $m^2_{\text{sc}} = \Delta\,\bar{\Delta} = m^2_{\text{vec}} + d - 1$), is found to be

$$G_{\mu\nu'}(\sigma) = \underbrace{\tfrac{1}{m^2}\Big((\Lambda_{\mu\nu'} + \theta_\mu\,\theta_{\nu'})\,(\sin^2\theta\,\partial_\sigma - \sigma) + \theta_\mu\,\theta_{\nu'}\Big)\,\partial_\sigma G_{[\Delta]}}_{\text{matching Wightman function of [67]}} + \tfrac{1}{m^2}\,\sigma_{\mu\nu'}\,\underbrace{(-\nabla^2 + \bar{\Delta}\,\Delta)\,G_{[\Delta]}}_{\text{eq. (3.7)}} \tag{3.92}$$

matches [67] away from $\theta = 0$, with the second line showing the expected position space δ-function. The explicit form of the scalar propagator was not used and so the different gauges of section 3.1.2 can be directly translated into this context too.

In the same way, the massless vector propagator, eq. (3.87), in position space is found to be $G^{(o)}_{\mu\nu'} = (\frac{d-1}{d-2}\,\sigma_{\mu\nu'} + \frac{1}{d-2}\sigma_\mu\,\sigma_{\nu'}\,\partial_\sigma)\,G_{[\Delta=1]}$. On S^3, eq. (3.90) takes the rather formal form: $G^{(o)}_{\mu\nu'} = \sigma_{\mu\nu'}\,G_{[\Delta=1]} + \lim_{\Delta,\bar{\Delta}\to 0}\hat{\delta}_{\mu\nu'}$, where $\hat{\delta}_{\mu\nu'} \propto (\sigma_{\mu\nu'}\,\partial_\sigma + \sigma_\mu\,\sigma_{\nu'}\,\partial^2_\sigma)\,G_{[\Delta]}$.

The propagator expressions in this chapter were conjectured and verified to satisfy

$$\Big\langle \psi'(\hat{X})\,G(\hat{X},\,\hat{Y})\,\psi(\hat{Y})\Big\rangle = \frac{1}{\lambda_\psi + m^2}\,\Big\langle \psi'(\hat{X})\,\psi(\hat{X})\Big\rangle, \tag{3.93}$$

for general eigenfunctions of the Laplacian, $\psi,\ \psi'$, where λ_ψ is the eigenvalue of ψ, well before any derivation (see section C.1). Initially, they were verified by integrating them against particular test functions. In order to make these integrals as general as possible for better verifiability, a change to the usual sphere Feynman integral setup became necessary.

Additionally, the propagator expressions were intentionally confined to a form that would be remain compatible with this new setup. And so once again, though presented before, the sphere Feynman integral construction that is discussed next, took form before the derivations of the embedding space propagator expressions.

Chapter 4: Feynman Integrals on the Sphere as Euler Integrals

Loop Feynman integral computations on de Sitter, over the course of their 4 decade plus history, have only recently moved beyond 1-loop to some scalar 2-loop integrals. As was discussed in chapter 1 and at the beginning of chapter 3, the difficulty in proceeding with higher loop computations is easily perceivable.

It is an understatement to call the parameterisation of triangles in curved spacetime cumbersome. For example, on S^3, the massive scalar propagator takes a very neat form: $\frac{\sinh((\pi-\theta)\,\nu)}{4\pi\,\sinh(\pi\nu)\,\sin\theta}$. Given its simplicity, it becomes possible to compute the n-melon (i.e. 2 internal points with n-propagators connecting them) Feynman integral surprisingly easily:

$$\begin{aligned}\mathcal{I}_n &= \Omega_3\,\Omega_2 \int_0^\pi \mathrm{d}\theta\,\sin^2\theta \prod_{i=1}^n \frac{\sinh((\pi-\theta)\,\nu_i)}{4\pi\,\sinh(\pi\nu_i)\,\sin\theta}\\ &= \lim_{d\to 2}\frac{1}{(2\pi)^{n-3}\prod\sinh(\pi\nu)}\int_0^\pi \mathrm{d}\theta\,\sin^{d+1-n}\theta\prod_{i=1}^n(e^{\theta\,\nu_i}-e^{-\theta\,\nu_i}),\end{aligned} \tag{4.1}$$

which can be computed term by term in dimensional regularisation by assuming $d>n-2$ before analytically continuing d to 2. Some low n results, which are so easily extensible to higher n that transcription is the hard part, are:

$$\begin{aligned}&\mathcal{I}_1 = \frac{1}{1+\nu^2},\quad \mathcal{I}_2 = \frac{\nu_1\,\coth(\pi\,\nu_1)-\nu_2\,\coth(\pi\,\nu_2)}{4\pi\,(\nu_1^2-\nu_2^2)}\\ &\mathcal{I}_3 = \lim_{\epsilon\to 0}\frac{1}{48\,\pi^2\,\epsilon}-\frac{\ln 2+\gamma_{\mathrm{E}}}{16\,\pi^2}+\frac{\sinh(\pi\,\delta\nu_{12,3})\left(\psi^{(0)}(\frac{1+\mathrm{i}\,\delta\nu_{12,3}}{2})+\psi^{(0)}(\frac{1-\mathrm{i}\,\delta\nu_{12,3}}{2})\right)}{128\,\pi^2\,\sinh(\pi\nu_1)\,\sinh(\pi\nu_2)\,\sinh(\pi\nu_3)}\\ &\quad+(\delta\nu_{12,3}\to\delta\nu_{23,1})+(\delta\nu_{12,3}\to\delta\nu_{31,2}),\qquad \delta\nu_{12,3}\equiv\nu_1+\nu_2-\nu_3\\ &\mathcal{I}_{\bar{3}} = \lim_{\epsilon\to 0}\frac{1}{48\,\pi^2\,\epsilon}-\frac{\ln 2+\gamma_{\mathrm{E}}}{16\,\pi^2}-\frac{\left(\psi^{(0)}(\frac{1+3\mathrm{i}\,\nu}{2})+\psi^{(0)}(\frac{1-3\mathrm{i}\,\nu}{2})\right)}{32\,\pi^2}\\ &\quad+\frac{3\left(\psi^{(0)}(\frac{1+\mathrm{i}\,\nu}{2})+\psi^{(0)}(\frac{1-\mathrm{i}\,\nu}{2})-\psi^{(0)}(\frac{1+3\mathrm{i}\,\nu}{2})-\psi^{(0)}(\frac{1-3\mathrm{i}\,\nu}{2})\right)}{128\,\pi^2\,\sinh^2(\pi\nu)}.\end{aligned} \tag{4.2}$$

However, inspite of this simplified propagator, even a 1-loop diagram (with 3 internal vertex insertions) starts to become highly impractical:

$$
\begin{aligned}
I_3^1 &= \Omega_3\,\Omega_2\,\Omega_1 \int_0^{\pi} \mathrm{d}\theta_1\, \sin\theta_1 \int_0^{\pi} \mathrm{d}\theta_2\, \sin^2\theta_2 \int_0^{2\pi} \mathrm{d}\theta_3\, \sin\theta_3 \\
&\times \frac{\sinh(\theta_1\,\nu_1)}{4\pi\,\sinh(\pi\nu_1)}\,\frac{\sinh((\pi-\theta')\,\nu_2)}{4\pi\,\sinh(\pi\nu_2)\,\sin\theta'}\,\frac{\sinh((\pi-\theta'')\,\nu_3)}{4\pi\,\sinh(\pi\nu_3)\,\sin\theta''} \\
\theta' &= \cos^{-1}(\cos\theta_2\,\cos\theta_3), \quad \theta'' = \pi - \cos^{-1}(\cos(\theta_2-\theta_1)\,\cos\theta_3).
\end{aligned}
\tag{4.3}
$$

In comparison Feynman integral computations in flat space benefit from their Gaussian structure when it comes to integrating over internal vertices, which seems to be indispensible to higher loop computations, as already noted in and around eq. (3.2). Thus, inspired by the success of [60] in flat space, a new formulation of sphere Feynman integrals is presented in this chapter. It hinges upon the embedding space representation of propagators from chapter 3. The "Feynman rules" converting a Feynman diagram to its integral form are presented in section 4.1.

Section 4.1.1 explains the construction of scalar loop integrals and provides a systematic approach to converting them into generalized Euler integrals. Each integral is described by a single polynomial (the denominator in the integrand of eq. (4.12), which is factorisable if the diagram is reducible), in $2\,n_P$ variables (n_P being the total number of propagators). Specifically, it is the determinant of a matrix, eq. (4.7), encoding the graphical structure of the Feynman diagram, making its origin similar to the first Symanzik polynomial in eq. (3.3).

Though currently only valid for spin-1 fields, section 4.1.2 describes how to modify the formulation with a perturbing matrix, eq. (4.25), to account for terms observed in higher spin computations. This results in a "master" integral form which, granting that higher spin propagators can also be brought into the scalar-based embedding space form, eq. (3.29), can be used to find all spin Feynman integrals with the same incidence matrix.

Section 4.1.3 extends the previous setup to generalized correlation functions by accounting for possible external legs in the Feynman diagrams. The additional component appearing in eq. (4.30) becomes the analogue of the second Symanzik polynomial in eq. (3.3).

The proposed procedure is implemented in section 4.2 for all scalar diagrams upto 3-loops and some vector diagrams that are representative of their relation to scalar integrals. In particular, vector integrals are found to be sums over corresponding scalar integrals.

4.1 Embedding Space Formulation of Feynman Integrals on S^{d+1}

When integrating over internal vertices in embedding space coordinates, it is vital to use the proper measure of S^{d+1}, eq. (A.7). It reproduces the correct volume in the "$\alpha = 1$ gauge":

$$\fint^{\mathbb{R}^{\mathbb{D}}} \frac{\mathrm{d}^{\mathbb{D}} X}{|X|^{\mathbb{D}}} = \frac{1}{\mathrm{Vol}\,\mathbb{R}^*} \int_\alpha^* \int_X \frac{\alpha^{\frac{\mathbb{D}}{2}}\, e^{-\alpha X^2}}{\Gamma(\frac{\mathbb{D}}{2})} = \int_X \frac{2\, e^{-X^2}}{\Gamma(\frac{\mathbb{D}}{2})} = \frac{2\pi^{\frac{\mathbb{D}}{2}}}{\Gamma(\frac{\mathbb{D}}{2})} = \Omega_{d+1}. \tag{4.4}$$

4.1.1 Scalar Feynman Integrals

A general scalar Feynman integral on S^{d+1} with n_P propagators and n_V internal vertices ($n_F = n_P + n_V$), takes the form

$$\mathcal{I}_F = \int\limits_{\hat{X}}^{S^{d+1}} \int\limits_{P}^{\mathbb{R}^{\mathbb{D}}} G_F(\hat{X}) = \int\limits_{X}^{\mathbb{R}^{\mathbb{D}}} \frac{\mathrm{d}^{\mathbb{D}} X}{|X|^{\mathbb{D}}} \int\limits_{P}^{\mathbb{R}^{\mathbb{D}}} G_F(X) \tag{4.5}$$

where G_F is the set/product[1] of propagators forming the Feynman diagram. When each propagator in G_F is represented in terms of eq. (3.24), upon Schwinger parameterisation, $\mathcal{I}_F$ can be consistently simplified to a gaussian integral over $W = \{P,\, X\}$:

$$\mathcal{I}_F = \frac{\mathcal{N}_F}{\mathrm{Vol}\,\mathcal{G}} \int_\varsigma^* \int\limits_{W}^{\mathbb{R}^{\mathbb{D}}} \lambda^{\bar{\Delta}}\, \mu^{\Delta}\, \tau^{\eta}\, \alpha^{\Delta_X}\, e^{-W^{\mathrm{T}} U(\varsigma)\, W}, \tag{4.6}$$

where $\mathcal{G} = (\mathbb{R}^*)^{n_F}$ is the group of scale transformations, the set of integration variables is $\varsigma = \{\lambda_{1,\cdots n_P},\, \mu_{1,\cdots n_P},\, \tau_{1,\cdots n_P},\, \alpha_{1,\cdots n_V}\}$, η is a parameter generalising the scalar propagator, given in eq. (3.26), by default set to 1, Δ are the mass parameters, Δ_X are the weights associated

[1] The multi-index notation is used wherever applicable. So here G_F is also used to denote $\prod G_i$ for all $G_i \in G_F$.

with internal vertices X, $\mathcal{N}_F$ is the normalisation constant, and $U(\varsigma)$ is a symmetric $n_F \times n_F$ matrix, referred to as the weighted incidence matrix, encoding the Feynman diagram F. $U(\varsigma)$ is constructed according to the following rules:

1. $U_{i,i} = \tau_i$ for $i \in \{1, \cdots n_P\}$ and $U_{n_P+j, n_P+j} = \alpha_i$ for $j \in \{1, \cdots n_V\}$.
2. If propagator P_i originates from vertex X_j ($X_j \to P_i$), then $U_{i, n_P+j} = U_{n_P+j, i} = i\,\lambda_i$.
3. If propagator P_i terminates at vertex X_j ($X_j \leftarrow P_i$), then $U_{i, n_P+j} = U_{n_P+j, i} = -i\,\mu_i$.
4. All other elements of U are 0.

Given its symmetric form, it may better visualised as

$$U(\varsigma) = \left(\begin{array}{c|c} \tau_{n_P} & \mathrm{i}\, L(\lambda,\, \mu) \\ \hline \mathrm{i}\, L^{\mathrm{T}}(\lambda,\, \mu) & \alpha_{n_V} \end{array}\right) \tag{4.7}$$

where τ_{n_P} and α_{n_V} are $(n_P \times n_P)$ and $(n_V \times n_V)$ weighted diagonal matrices, and $L(\lambda,\, \mu)$, referred to as the incidence matrix, is a $(n_P \times n_V)$ matrix representing the propagator and vertex connections, constructed as follows:

1. For every $X_j \to P_i$, $L_{ij} = \lambda_i$.
2. For every $X_j \leftarrow P_i$, $L_{ij} = \mu_i$.
3. All other elements of L are 0.

The vertex weights are defined as

$$\Delta_{X_j} = \tfrac{1}{2}\big(\mathbb{D} - \sum_{i \,|\, X_j \to P_i} \bar{\Delta}_i - \sum_{i \,|\, X_j \leftarrow P_i} \Delta_i\big). \tag{4.8}$$

The normalisation constant is

$$\mathcal{N}_F = \frac{1}{4^\eta\, \pi^{n_P\, \mathbb{D}}} \frac{1}{\Gamma(\eta)} \frac{1}{\Gamma(\Delta_X)}, \quad \mathcal{N}_F\Big|_{\eta=1} = \frac{1}{4^\eta\, \pi^{n_P\, \mathbb{D}}} \frac{1}{\Gamma(\Delta_X)}. \tag{4.9}$$

Upon fixing the scaling redundancy $\mathcal{G}$ by setting all $\tau = \alpha = 1$

$$\mathcal{I}_F = 2^{n_F}\, \mathcal{N}_F \int_{\varsigma}^{*} \int\limits_{W}^{\mathbb{R}^{\mathbb{D}}} \lambda^{\bar{\Delta}}\, \mu^{\Delta}\, e^{-W^{\mathrm{T}}\, \bar{U}(\lambda,\, \mu)\, W}, \quad \bar{U}(\lambda,\, \mu) = U(\varsigma)\big|_{\tau=\alpha=1} \tag{4.10}$$

and integrating over the momentum and position spaces W, the integral becomes

$$\mathcal{I}_F = \bar{\mathcal{N}}_F \int_{\lambda,\, \mu}^{*} \frac{\lambda^{\bar{\Delta}}\, \mu^{\Delta}}{(\det \bar{U})^{\frac{\mathbb{D}}{2}}} = \frac{\bar{\mathcal{N}}_F}{\Gamma(\frac{\mathbb{D}}{2})} \int_{\lambda,\, \mu}^{*} \lambda^{\bar{\Delta}}\, \mu^{\Delta}\, z^{\frac{\mathbb{D}}{2}}\, e^{-z\, \det \bar{U}(\lambda,\, \mu)} \tag{4.11}$$

where unit-weighted $\bar{U}(\lambda,\, \mu)$ is now a sparse matrix with unit diagonal and

$$\bar{\mathcal{N}}_F = (2\, \pi^{\frac{\mathbb{D}}{2}})^{n_F}\, \mathcal{N}_F, \quad \bar{\mathcal{N}}_F\Big|_{\eta=1} = (4\pi^{\mathbb{D}})^{\frac{n_V - n_P}{2}}\, \frac{1}{\Gamma(\Delta_X)}. \tag{4.12}$$

Presented in this form, it is clear that all scalar Feynman integrals on the sphere have generalized Euler integral representations, eq. (2.2), in terms of at most $2\, n_P$ integration variables, and hence correspond to $\mathcal{A}$-hypergeometric functions. The complexity of these representations are significantly greater than their flat space counterparts, eq. (3.4), which have at most n_P integration variables.

4.1.2 Higher Spin Feynman Integrals

A general Feynman integral consisting of higher spin propagators, that can be represented as $\mathcal{F} \circ G_{\eta}^{[\Delta]}$ defined in eq. (3.29), similarly takes the form

$$\mathcal{I}_F = \frac{\mathcal{N}_F}{\mathrm{Vol}\, \mathcal{G}} \int_{\varsigma}^{*} \int\limits_{W}^{\mathbb{R}^{\mathbb{D}}} \lambda^{\bar{\Delta}}\, \mu^{\Delta}\, \tau^{\eta}\, \alpha^{\Delta_X}\, f(\varsigma,\, W)\, e^{-W^{\mathrm{T}}\, U(\varsigma)\, W} \tag{4.13}$$

where $f(\varsigma,\, W)$ is polynomial in

$$\lambda_i\, P_i\, X_j\Big|_{X_j \to P_i}, \quad \mu_i\, P_i\, X_j\Big|_{X_j \leftarrow P_i}, \quad \tfrac{\lambda_i\, \mu_i}{\tau_i}\, X_j X_k\Big|_{X_j \to P_i \to X_k}, \quad \alpha_i X_i^2, \quad \tau_i P_i^2 \tag{4.14}$$

and products thereof, and hence invariant under scale transformations generated by $\mathcal{G}$. X^2 and P^2 can simply be absorbed into the Schwinger parameterisation by α, τ, eventually leading to simple shifts in the parameters appearing in the normalisation constant:

$$X^{2\,\Delta'_X} \implies \Delta_X \to \Delta_X - \Delta'_X, \quad P^{2\,\eta'} \implies \eta \to \eta - \eta'. \tag{4.15}$$

Since the inner products of position and momentum vectors belong to the set of elements in $\{W^{\mathrm{T}}\, W\}$, a generating function $W^{\mathrm{T}}\, U'(\varsigma,\, \varsigma')\, W$, where $\varsigma' = \{\lambda',\, \mu',\, \beta\}$, can be used to perturb the Gaussian integral $e^{-W^{\mathrm{T}}\, U(\varsigma)\, W}$, allowing $f(\varsigma,\, W)$ to be replaced by an operator $f[\partial_{\varsigma'}]$ that is polynomial in $\langle \partial_{\lambda'},\, \partial_{\mu'},\, \partial_\beta \rangle$ such that

$$f(\varsigma,\, W) = f[\partial_{\varsigma'}]\, e^{-W^{\mathrm{T}}\, U'(\varsigma,\, \varsigma')\, W}\Big|_{\varsigma'=0}. \tag{4.16}$$

One such construction of $U'(\varsigma,\, \varsigma')$ and $f[\partial_{\varsigma'}]$ is as follows:

F	$f(W)$	$f[\partial_{\varsigma'}]$	$U'(\varsigma,\, \varsigma')$
$X_j \to P_i$	$\lvert P_i\, X_j \rvert^{\bar{s}_i}$	$(-2\mathrm{i}\ \lambda_i)^{-\bar{s}_i}\, \partial^{\bar{s}_i}_{\lambda'_i}$	$U'_{i,\, n_P+j} = U'_{n_P+j,\, i} = \mathrm{i}\ \lambda_i\, \lambda'_i$
$X_j \leftarrow P_i$	$\lvert P_i\, X_j \rvert^{s_i}$	$(2\mathrm{i}\ \mu_i)^{-s_i}\, \partial^{s_i}_{\mu'_i}$	$U'_{i,\, n_P+j} = U'_{n_P+j,\, i} = -\mathrm{i}\ \mu_i\, \mu'_i$
$X_j,\ X_k$	$\lvert X_j X_k \rvert^{\omega}$	$(-2\mathrm{i}\ \sqrt{\alpha_j\, \alpha_k})^{-\omega}\, \partial^{\omega}_{\beta}$	$U'_{n_P+j,\, n_P+k} = U'_{n_P+k,\, n_P+j} = -\mathrm{i}\ \sqrt{\alpha_j\, \alpha_k}\, \beta$
$P_j,\ P_k$	$\lvert P_j P_k \rvert^{\omega'}$	$(-2\mathrm{i}\ \sqrt{\tau_j\, \tau_k})^{-\omega'}\, \partial^{\omega'}_{\beta}$	$U'_{j,\, k} = U'_{k,\, j} = -\mathrm{i}\ \sqrt{\tau_j\, \tau_k}\, \beta$

(4.17)

If there exists a propagator P_i from X_j to X_k i.e. $X_j \to P_i \to X_k$

$$\lvert X_j X_k \rvert^{\omega} \to (\tfrac{\tau_i}{-2\mathrm{i}\ \lambda_i\, \mu_i})^{\omega}\, \partial^{\omega}_{\beta}\, e^{-W^{\mathrm{T}}\, U'\, W}\Big|_{U'=0}, \quad U'_{n_P+j,\, n_P+k} = U'_{n_P+k,\, n_P+j} = -\mathrm{i}\ \tfrac{\lambda_i\, \mu_i}{\tau_i}\, \beta \tag{4.18}$$

can also be used but note that this operator doesn't commute with the remaining prescription in eq. (4.17). Further, a couple of trivial changes of variables allows $f[\partial_{\lambda'},\, \partial_{\mu'}]$ to be simplified to

$$\partial^{\bar{s}_i}_{\lambda'_i} \to (-1)^{\bar{s}_i}\, \frac{\Gamma(\bar{\Delta}_i)}{\Gamma(\bar{\Delta}_i - \bar{s}_i)}, \quad \partial^{s_i}_{\mu'_i} \to (-1)^{s_i}\, \frac{\Gamma(\Delta_i)}{\Gamma(\Delta_i - s_i)}, \tag{4.19}$$

leaving only some $f[\partial_\beta]$ behind. Thus, given a general monomial

$$f(W) = X^{2\,\Delta'_X}\, P^{2\,\eta'}\, |P\,X|^{\bar{s}}\, |P\,X'|^{s}\, |X\,X'|^{\omega}\, |P\,P'|^{\omega'}, \tag{4.20}$$

the Feynman integral

$$\mathcal{I}_F = \frac{\mathcal{N}_F}{\mathrm{Vol}\,\mathcal{G}} \int_{\varsigma}^{*} \int_{W}^{\mathbb{R}^{\mathbb{D}}} \lambda^{\bar{\Delta}}\, \mu^{\Delta}\, \tau^{\eta}\, \alpha^{\Delta_X}\, f(W)\, e^{-W^{\mathrm{T}}\, U(\varsigma)\, W} \tag{4.21}$$

can be rewritten as

$$\begin{aligned} \mathcal{I}_F &= \bar{\mathcal{N}}_F(f) \int_{\lambda,\,\mu,\,z}^{*} \lambda^{\bar{\Delta}-\bar{s}}\, \mu^{\Delta-s}\, z^{\frac{\mathbb{D}}{2}}\, f_{\omega+\omega'}(\lambda,\,\mu)\, e^{-z\,\det \bar{U}(\lambda,\,\mu)} \\ \bar{\mathcal{N}}_F(f) &= \frac{\bar{\mathcal{N}}_F}{\Gamma(\frac{\mathbb{D}}{2})}\, \frac{(-1)^{s+\omega+\omega'}}{(2i)^{\bar{s}+s+\omega+\omega'}}\, \frac{\Gamma(\eta)}{\Gamma(\eta-\eta')}\, \frac{\Gamma(\Delta_X)}{\Gamma(\Delta_X-\Delta'_X)}\, \frac{\Gamma(\bar{\Delta})}{\Gamma(\bar{\Delta}-\bar{s})}\, \frac{\Gamma(\Delta)}{\Gamma(\Delta-s)}, \end{aligned} \tag{4.22}$$

where $f_{\omega+\omega'}(\lambda,\,\mu)$ is a polynomial,

$$f_{\omega+\omega'}(\lambda,\,\mu) = e^{z\,\det(\bar{U}(\lambda,\,\mu)+\bar{U}'(\beta))}\, \partial_\beta^{\omega+\omega'}\, e^{-z\,\det(\bar{U}(\lambda,\,\mu)+\bar{U}'(\beta))}, \quad \bar{U}'(\beta) \;\; = U'(\beta)\Big|_{\tau=\alpha=1}. \tag{4.23}$$

In particular, f_ω is spanned by the zeroth to ω^{th} Hermite polynomials in $(\lambda,\,\mu)$. Thus, eq. (4.13) is a sum over scalar Feynman integrals with generic parameters, and hence also a generalized Euler integral. Interestingly enough this type of formulation also allows negative orders of $|PX|$ ($X \to P$) to be considered:

$$\lambda \circ G = \tfrac{\lambda}{-2\mathrm{i}\,\lambda\,|PX|}\partial_{\lambda'} \circ G\, e^{-W^{\mathrm{T}}\, U'(\lambda')\, W}\Big|_{\lambda'=0} = \tfrac{\bar{\Delta}}{2\mathrm{i}\;|PX|} \circ G \implies \tfrac{1}{|PX|} \circ G = \tfrac{2\mathrm{i}}{\bar{\Delta}}\, \lambda \circ G \tag{4.24}$$

and similarly if $X \leftarrow P$, $\frac{1}{|PX|} \circ G = \frac{-2\mathrm{i}}{\Delta}\, \mu \circ G$, implying that eq. (4.22) is true for $\bar{s},\, s \in \mathbb{Z}$, not just $\mathbb{N}$. Thus, the reduced form of $\bar{U}'$, the perturbation incidence matrix, is sparse and effectively takes the form

$$\bar{U}'(\varsigma,\,\varsigma') = \mathrm{i}\left(\begin{array}{c|c} L'_P(\beta) & L'(\lambda\lambda',\,\mu\mu') \\ \hline L'^{\mathrm{T}}(\lambda\lambda',\,\mu\mu') & L'_V(\beta) \end{array}\right) \to \bar{U}'(\beta) = \mathrm{i}\left(\begin{array}{c|c} L'_P(\beta) & 0 \\ \hline 0 & L'_V(\beta) \end{array}\right) \tag{4.25}$$

where $L'_V,\, L'_P$ are $n_V \times n_V,\, n_P \times n_P$ symmetric matrices with non-zero entries $L'_{ij} = L'_{ji} = \beta_{ij}$ corresponding to the terms $|X_i\, X_j|$ and $P_i\, P_j$ in $f(W)$, as defined in eq. (4.20).

Thus, the perturbation incidence matrix $\bar{U}'(\beta)$ with a maximal $L'_V,\, L'_P$, i.e. with no non-zero entries, forms a gaussian "master" integral, $\mathcal{I}_F(\beta)$. $\mathcal{I}_F(\beta)$, once solved in complete generality of the defining parameters $\{\mathbb{D},\, \Delta,\, \bar{\Delta}\}$, can be used to find all Feynman integrals with the same incidence matrices $L(\lambda,\, \mu)$ as its derivatives with respect to the perturbative parameters β at $\beta = 0$.

4.1.3 Generalized Correlation Functions

A Feynman integral representing a generalized n-point scalar correlation function $\mathcal{J}_F$ on S^{d+1}, with n_P internal propagators (P), n_Y internal vertices (Y), and n external legs of the propagators (Q) and external vertices ($\hat{X}$), can also be brought into the Euler integral format. Using the now familiar embedding space formulation of the scalar propagator, $\mathcal{J}_F(\hat{X})$, can be written as

$$\mathcal{J}_F(\hat{X}) = \frac{1}{\mathrm{Vol}(\mathbb{R}^*)^{n+n_Y}} \int_Q \int_Y \frac{\mathrm{d}^{\mathbb{D}}Y}{|Y|^{\mathbb{D}}} \int^*_{\lambda,\,\mu} \lambda_Q^{\bar{\Delta}_Q}\, |\mu_Q Y|^{\Delta_Q}\, \frac{e^{-2\mathrm{i}\, Q\,(\lambda_Q\, \hat{X} - \mu_Q\, Y)}}{4\pi^{\mathbb{D}}\, Q^2}\, G_P(Y), \tag{4.26}$$

which upon integrating out the external propagator momenta Q (where η_Q has been tacitly assumed to be 1 but can be easily reinstated if needed),

$$\mathcal{J}_F(\hat{X}) = \frac{(2\pi^{\frac{\mathbb{D}}{2}})^{-n}}{\mathrm{Vol}(\mathbb{R}^*)^{n_Y}} \int^*_{\lambda,\,\mu} \lambda_Q^{\bar{\Delta}_Q}\, \mu_Q^{\Delta_Q} e^{-\lambda_Q^2} \int_Y \frac{\mathrm{d}^{\mathbb{D}}Y}{|Y|^{\mathbb{D}}}\, |Y|^{\Delta_Q}\, e^{-(\mu_Q^2\, Y^2 - 2\,\lambda_Q\, \mu_Q\, Y\cdot\hat{X})}\, G_P(Y) \tag{4.27}$$

comes to resemble eq. (4.6) when represented in its Schwinger parameteric form:

$$\begin{aligned} \mathcal{J}_F(\hat{X}) &= \frac{\mathcal{N}_F}{\mathrm{Vol}(\mathbb{R}^*)^{n_Y+n_P}} \int^*_{\lambda,\,\mu,\tau_P,\alpha_P} \lambda^{\bar{\Delta}}\,\mu^{\Delta} e^{-\lambda_Q^2} \int_W \tau_P^{\eta_P}\,\alpha_Y^{\Delta_Y}\, e^{-(W^{\mathrm{T}}\,U\,W - V^{\mathrm{T}}\,W - W^{\mathrm{T}}\,V)} \\ \mathcal{N}_F &= \frac{1}{(2\pi^{\frac{\mathbb{D}}{2}})^{(n+2\,n_P)}\,\Gamma(\eta_P)\,\Gamma(\Delta_Y)}, \end{aligned} \tag{4.28}$$

where the weighted incidence matrix U is appropriately modified to represent the gaussian integral

$$U = \left(\begin{array}{c|c} \tau_P & \mathrm{i}\,L(\lambda_P,\,\mu_P) \\ \hline \mathrm{i}\,L^{\mathrm{T}}(\lambda_P,\,\mu_P) & \alpha_Y + \mu_Q^2 \end{array} \right) \tag{4.29}$$

and V is a vector with the components $\{\underbrace{0\cdots 0}_{n_P},\,\underbrace{\lambda_Q\,\mu_Q\,\hat{X}}_{n_Y}\}$. Fixing all scaling symmetries and integrating over W, leads to

$$\begin{aligned} \mathcal{J}_F(\hat{X}) &= \bar{\mathcal{N}_F} \int^*_{\lambda,\,\mu} \lambda^{\bar{\Delta}}\,\mu^{\Delta}\,e^{-\lambda_Q^2}\,\frac{e^{V^{\mathrm{T}}\,\bar{U}^{-1}\,V}}{(\det \bar{U})^{\frac{\mathbb{D}}{2}}}, \quad \bar{U} = \left(\begin{array}{c|c} 1 & \mathrm{i}\,L(\lambda_P,\,\mu_P) \\ \hline \mathrm{i}\,L^{\mathrm{T}}(\lambda_P,\,\mu_P) & 1+\mu_Q^2 \end{array} \right) \\ \bar{\mathcal{N}_F} &= \frac{1}{(2\pi^{\frac{\mathbb{D}}{2}})^{(n+n_P-n_Y)}\,\Gamma(\eta_P)\,\Gamma(\Delta_Y)}, \end{aligned} \tag{4.30}$$

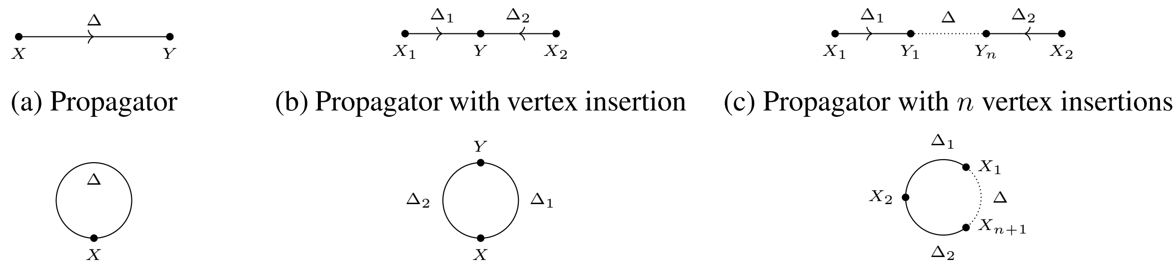
from which one of the parameters in $(\lambda_Q,\,\mu_Q)$ can always be trivially integrated out, finally resulting in an Euler integral representation of the generalized n-point scalar correlation function, $\mathcal{J}_F(\hat{X})$, in terms of $(2\,n_P + 2\,n_Q - 1)$ integration variables.

This setup can be extended to include higher spin fields in the same vein as section 4.1.2, with some obvious additions to the prescription as needed, e.g.

$$Q_I \to \tfrac{1}{-2\mathrm{i}\,\lambda_Q}\partial_{\hat{X}^I}, \quad |V_i^{\mathrm{T}}\cdot W| \to \tfrac{1}{2}\partial_v\, e^{v\,(V^{\mathrm{T}}\,W + W^{\mathrm{T}}\,V)}\Big|_{v=0}. \tag{4.31}$$

This brings all generalized correlation functions into the Euler integral format.

4.2 Explicit Constructions of Scalar Feynman Integrals



(a) Propagator (b) Propagator with vertex insertion (c) Propagator with n vertex insertions

(d) 1-loop (e) 1-loop with 1 vertex insertion (f) 1-loop with n vertex insertions

Figure 4.1: 2-point function and related Feynman diagrams

4.2.1 Coincident Point Limit and 1 Loop Character Integral

The coincident point limit of the scalar propagator, i.e. at $\theta = 0$ in say eq. (3.37), is expectedly divergent:

$$G(\sigma = 1) = \frac{\Gamma(\frac{d}{2})}{4\pi^{\frac{\mathbb{D}}{2}}} \int_{\mu}^{*} \frac{\mu^{\Delta}}{(1 + \mu^2 - 2\,\mu)^{\frac{d}{2}}} = \frac{\Gamma(\frac{d}{2})}{4\pi^{\frac{\mathbb{D}}{2}}} \int_{\mu}^{*} \frac{\mu^{\Delta}}{|1-\mu|^{d}}, \tag{4.32}$$

appearing in the integral as a pole of the integrand at $\mu = 1$. The 1-loop integral, fig. (4.1d), is simply $\mathcal{I}_{1,1} = \Omega_{d+1} \times G(\sigma = 1) = \frac{1}{d}\int_{\mu}^{*} \frac{\mu^{\Delta}}{|1-\mu|^{d}}$. Though in this case needlessly convoluted, strictly following the rules given in section 4.1.1 results in the same:

$$\begin{aligned}
\mathcal{I}_{1,1} &= \frac{1}{(\mathrm{Vol}\,\mathbb{R}^{*})^2}\,\frac{1}{4\pi^{\mathbb{D}}} \int_{\lambda,\,\mu}^{*} \lambda^{\bar{\Delta}}\,\mu^{\Delta} \int_{P,X} \frac{e^{-2\mathrm{i}\;P\,X\,(\lambda-\mu)}}{P^2\,X^2} \\
&= \int_{\lambda,\,\mu}^{*} \frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(1+(\lambda-\mu)^2)^{\frac{\mathbb{D}}{2}}} = \frac{1}{d}\int_{\mu}^{*} \frac{\mu^{\Delta}}{(1+\mu^2-2\,\mu)^{\frac{d}{2}}} = \frac{1}{d}\int_{\mu}^{*} \frac{\mu^{\Delta}}{|1-\mu|^{d}} \\
&= \frac{1}{d}\int_{0}^{1} \frac{\mathrm{d}\mu}{\mu}\,\frac{\mu^{\Delta}+\mu^{\bar{\Delta}}}{|1-\mu|^{d}},
\end{aligned} \tag{4.33}$$

recreating the character integral formulation of scalar 1-loop integrals from [10].

4.2.2 1-Melon : 2-Point Function Integrated

The simplest possible scalar Feynman integral is the integral of the 2-point function over all space, a 1-melon so to say:

$$
\begin{aligned}
\mathcal{I}_{2,1} &= \frac{1}{4\pi^{\mathbb{D}}} \fint_{X,Y}^{\mathbb{R}^{\mathbb{D}}} \fint_{\lambda,\mu}^{*} |\lambda X|^{\bar{\Delta}} |\mu Y|^{\Delta} \int_P^{\mathbb{R}^{\mathbb{D}}} \frac{e^{-2\mathrm{i}\, P\,(\lambda X - \mu Y)}}{P^2} \\
&= \frac{2\pi^{\frac{\mathbb{D}}{2}}}{\Gamma(\frac{\mathbb{D}-\Delta}{2})\,\Gamma(\frac{\mathbb{D}-\bar{\Delta}}{2})} \int_{\lambda,\mu}^{*} \frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(1+\lambda^2+\mu^2)^{\frac{\mathbb{D}}{2}}} = \frac{2\pi^{\frac{\mathbb{D}}{2}}}{\Gamma(\frac{\mathbb{D}}{2})} \frac{\Gamma(\frac{\Delta}{2})\,\Gamma(\frac{\bar{\Delta}}{2})}{4\,\Gamma(\frac{\mathbb{D}-\Delta}{2})\,\Gamma(\frac{\mathbb{D}-\bar{\Delta}}{2})} = \frac{\Omega_{d+1}}{\Delta\,\bar{\Delta}}.
\end{aligned}
\tag{4.34}
$$

4.2.3 2-Melon

The 2-melon Feynman diagram, fig. (4.1e), with 2 propagators and 2 internal vertices forming a 1-loop diagram, is encoded in the 4×4 unit-weighted incidence matrix:

$$
\bar{U}_{2,2} = \begin{pmatrix} 1 & 0 & \mathrm{i}\,\lambda_1 & -i\mu_1 \\ 0 & 1 & \mathrm{i}\,\lambda_2 & -\mathrm{i}\,\mu_2 \\ \mathrm{i}\,\lambda_1 & \mathrm{i}\,\lambda_2 & 1 & 0 \\ -\mathrm{i}\,\mu_1 & -\mathrm{i}\,\mu_2 & 0 & 1 \end{pmatrix}. \tag{4.35}
$$

Using eq. (4.11), the corresponding Feynman integral is

$$
\begin{aligned}
\mathcal{I}_{2,2} &= \mathcal{N}_{2,2} \int_{\lambda,\mu}^{*} \frac{\lambda_1^{\bar{\Delta}_1}\,\mu_1^{\Delta_1}\,\lambda_2^{\bar{\Delta}_2}\,\mu_2^{\Delta_2}}{(1+\lambda_1^2+\lambda_2^2+\mu_1^2+\mu_2^2+\lambda_1^2\,\mu_2^2+\lambda_2^2\,\mu_1^2-2\,\lambda_1\,\lambda_2\,\mu_1\,\mu_2)^{\frac{\mathbb{D}}{2}}}, \\
\mathcal{N}_{2,2} &= \frac{1}{\Gamma(\frac{\mathbb{D}-\Delta_1-\Delta_2}{2})\,\Gamma(\frac{\mathbb{D}-\bar{\Delta}_1-\bar{\Delta}_2}{2})}.
\end{aligned}
\tag{4.36}
$$

Example 2.3.1 deals with the generalized Euler integral version of $\mathcal{I}_{2,2}$ in great detail. At the physically relevant parameters of $\mathcal{I}_{2,2}$, it is found to be eq. (2.211). In particular, it is

$$
\mathcal{I}_{2,2} = \frac{\Gamma(-d)\big((\sin\pi\Delta_1+\sin\pi\bar{\Delta}_1)\Gamma(\Delta_1)\Gamma(\bar{\Delta}_1)-(\sin\pi\Delta_2+\sin\pi\bar{\Delta}_2)\Gamma(\Delta_2)\Gamma(\bar{\Delta}_2)\big)}{\pi(\nu_1^2-\nu_2^2)}. \tag{4.37}
$$

4.2.4 Propagator with 1 Vertex Insertion

The correlation function, $G_{[\Delta_1,\,\Delta_2]}(\hat{X}_1,\,\hat{X}_2)$, depicted in fig. (4.1b), is given by the integral:

$$G_{[\Delta_{(1,2)}]} = \frac{1}{(4\pi^{\mathbb{D}})^2}\fint_{\lambda,\mu}^{*} \lambda_1^{\bar{\Delta}_1}\,\mu_1^{\Delta_1}\lambda_2^{\bar{\Delta}_2}\,\mu_2^{\Delta_2}\int_P^{\mathbb{R}^{\mathbb{D}}}\fint_Y^{\mathbb{R}^{\mathbb{D}}}\frac{e^{-2\mathrm{i}\,P_1\,(\lambda_1\,\hat{X}_1-\mu_1\,Y)-2\mathrm{i}\,P_2\,(\lambda_2\,\hat{X}_2-\mu_2\,Y)}}{P_1^2\,P_2^2\,Y^{\mathbb{D}-\mathbb{\Delta}}}, \tag{4.38}$$

where $\mathbb{\Delta} = \Delta_1 + \Delta_2,\ \bar{\mathbb{\Delta}} = \bar{\Delta}_1 + \bar{\Delta}_2$. Integrating over the internal vertex Y and momenta P, it can be brought into the euler integral form:

$$\begin{aligned} G_{[\Delta_{(1,2)}]} &= \frac{1}{2\pi^{\frac{\mathbb{D}}{2}}\,\Gamma(\frac{\mathbb{D}-\mathbb{\Delta}}{2})}\int_{\lambda,\mu}^{*}\frac{\lambda_1^{\bar{\Delta}_1}\,\mu_1^{\Delta_1}\lambda_2^{\bar{\Delta}_2}\,\mu_2^{\Delta_2}}{(1+\mu_1^2+\mu_2^2)^{\frac{\mathbb{D}-\bar{\mathbb{\Delta}}}{2}}}e^{-(\lambda_1^2+\lambda_2^2+\lambda_1^2\,\mu_2^2+\lambda_2^2\,\mu_1^2-2\,\sigma_{12}\,\lambda_1\,\lambda_2\,\mu_1\,\mu_2)} \\ &= \frac{\Gamma(\frac{\bar{\mathbb{\Delta}}}{2})}{4\pi^{\frac{\mathbb{D}}{2}}\,\Gamma(\frac{\mathbb{D}-\mathbb{\Delta}}{2})}\int_{\lambda,\mu}^{*}\frac{\mu_1^{\Delta_1}\,\mu_2^{\Delta_2}\,\lambda_1^{\bar{\Delta}_1}\,(1+\mu_1^2+\mu_2^2)^{-\frac{\mathbb{D}-\bar{\mathbb{\Delta}}}{2}}}{(1+\mu_1^2+\lambda_1^2+\lambda_1^2\,\mu_2^2-2\,\sigma_{12}\,\lambda_1\,\mu_1\,\mu_2)^{\frac{\bar{\mathbb{\Delta}}}{2}}}. \end{aligned} \tag{4.39}$$

encoded in the $\mathcal{A}_{12}$ matrix and γ_{12} vector:

$$\mathcal{A}_{12} = \begin{pmatrix} 1&1&1&0&0&0&0&0\\ 0&0&0&1&1&1&1&1\\ 0&2&0&0&2&0&0&1\\ 0&0&2&0&0&0&2&1\\ 0&0&0&0&0&2&2&1 \end{pmatrix}, \qquad \gamma_{12} = \{\tfrac{\mathbb{D}-\bar{\mathbb{\Delta}}}{2},\,\tfrac{\bar{\mathbb{\Delta}}}{2},\,\Delta_1,\,\Delta_2,\,\bar{\Delta}_1\}. \tag{4.40}$$

Picking a weight in the class of $\{0,1,1,0,0,1,0,1\}$ to form the indicial ideal and find the roots (see section C.2.2 for details), the log-free series solution as an expansion around $\sigma_{12} = 0$ is found to be:

$$\begin{aligned} G_{[\Delta_{(1,2)}]} = \frac{1}{2\pi^{\frac{\mathbb{D}}{2}}}\Bigg(&\frac{\Gamma(\frac{\bar{\Delta}_2}{2})\,\Gamma(\frac{\Delta_2}{2})\,{}_2F_1(\frac{\bar{\Delta}_2}{2},\,\frac{\Delta_2}{2},\,\frac{1}{2},\,\sigma_{12}^2)-[2\to 1]}{4\,(d-\Delta_1-\Delta_2)\,(\Delta_1-\Delta_2)} \\ &+\sigma\,\frac{\Gamma(\frac{\bar{\Delta}_2+1}{2})\,\Gamma(\frac{\Delta_2+1}{2})\,{}_2F_1(\frac{\bar{\Delta}_2+1}{2},\,\frac{\Delta_2+1}{2},\,\frac{3}{2},\,\sigma_{12}^2)-[2\to 1]}{2\,(d-\Delta_1-\Delta_2)\,(\Delta_1-\Delta_2)}\Bigg), \end{aligned} \tag{4.41}$$

which, upon comparison to eq. (3.44), can be seen to precisely match the expected result:

$$G_{[\Delta_{(1,2)}]}(\sigma_{12}) = \frac{G_{[\Delta_1]}(\sigma_{12})}{(\nu_2^2-\nu_1^2)} + \frac{G_{[\Delta_2]}(\sigma_{12})}{(\nu_1^2-\nu_2^2)}. \tag{4.42}$$

This result is related to the 2-melon result from section 4.2.3, eq. (4.37), by the identification $\hat{X}_2 = \hat{X}_1$ and integrating over this now internal vertex, i.e. $\mathcal{I}_{2,2} = \Omega_{d+1}\times\lim_{\sigma_{12}\to 1}G_{[\Delta_{(1,2)}]}(\sigma_{12})$, where

$$\Omega_{d+1} \times \lim_{\sigma\to 1} G_{[\Delta]}(\sigma) = \frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})} \times \frac{\Gamma(\Delta)\,\Gamma(\bar{\Delta})}{(4\pi)^{\frac{d+1}{2}}\,\Gamma(\frac{d+1}{2})} \times \frac{\Gamma(\frac{d+1}{2})\,\Gamma(\frac{d+1}{2}-d)}{\Gamma(\frac{d+1}{2}-\Delta)\,\Gamma(\frac{d+1}{2}-\bar{\Delta})}$$
$$= \Gamma(\Delta)\,\Gamma(\bar{\Delta}) \times \frac{2^{-d}\sqrt{\pi}}{\Gamma(1+\frac{d}{2})\,\Gamma(\frac{-d}{2})} \times \frac{\Gamma(\frac{1-d}{2})\,\Gamma(\frac{-d}{2})}{\Gamma(\frac{1}{2}-\mathrm{i}\,\nu)\,\Gamma(\frac{1}{2}+\mathrm{i}\,\nu)} \tag{4.43}$$
$$= 2\,\sin(\tfrac{\pi\,d}{2})\,\cosh(\pi\,\nu)\,\Gamma(\Delta)\,\Gamma(\bar{\Delta})\,\frac{\Gamma(1-d)}{d\,\pi}$$
$$= (\sin(\pi\,\Delta)+\sin(\pi\,\bar{\Delta}))\,\Gamma(\Delta)\,\Gamma(\bar{\Delta})\,\frac{\Gamma(1-d)}{d\,\pi}.$$

4.2.5 Propagator and 1 loop with n Vertex Insertions

The integral corresponding to fig. (4.1c) at $n = 2$, i.e. consisting of 2 internal points and 3 propagators, is

$$G_{[\Delta_{(1,2,3)}]} = \frac{1}{(4\pi^{\mathbb{D}})^3} ⨏_{\lambda,\mu} ⨏_{Y} \int_P \frac{\lambda_1^{\bar{\Delta}_1}\,|\mu_1\,Y_1|^{\Delta_1}|\lambda_2\,Y_1|^{\bar{\Delta}_2}\,|\mu_2\,Y_2|^{\Delta_2}\,|\lambda_3\,Y_2|^{\bar{\Delta}_3}\,\mu_3^{\Delta_3}}{P_1^2\,P_2^2\,P_3^2} \times e^{-2\mathrm{i}\,P_1\,(\lambda_1\,\hat{X}_1-\mu_1\,Y_1)-2\mathrm{i}\,P_2\,(\lambda_2\,Y_1-\mu_2\,Y_2)-2\mathrm{i}\,P_3\,(\lambda_3\,Y_2-\mu_3\,\hat{X}_2)}, \tag{4.44}$$

where the argument of $G_{[\Delta_{(\cdots)}]}$ is suppressed when it is $(\hat{X}_1,\,\hat{X}_2) \equiv \sigma_{12}$. Within this integral, eq. (4.38) can be recognised by putting $\hat{Y}_2$ back on the sphere:

$$G_{[\Delta_{(1,2,3)}]} = \frac{1}{4\pi^{\mathbb{D}}} ⨏_{\lambda_3,\mu_3} \int_{\hat{Y}_2} \int_{P_3} \frac{\lambda_3^{\bar{\Delta}_3}\,\mu_3^{\Delta_3}}{P_3^2} e^{-2\mathrm{i}\,P_3\,(\lambda_3\,\hat{Y}_2-\mu_3\,\hat{X}_2)}\,G_{[\Delta_{(1,2)}]}(\hat{X}_1,\,\hat{Y}_2). \tag{4.45}$$

Using eq. (4.42), it can be simplified by to

$$G_{[\Delta_{(1,2,3)}]} = \frac{1}{(\nu_1^2-\nu_2^2)}\,\frac{1}{4\pi^{\mathbb{D}}} ⨏_{\lambda_3,\mu_3} \int_{\hat{Y}_2} \int_{P_3} \frac{\lambda_3^{\bar{\Delta}_3}\,\mu_3^{\Delta_3}}{P_3^2} e^{-2\mathrm{i}\,P_3\,(\lambda_3\,\hat{Y}_2-\mu_3\,\hat{X}_2)} \times (G_{[\Delta_2]}(\hat{X}_1,\,\hat{Y}_2) - G_{[\Delta_1]}(\hat{X}_1,\,\hat{Y}_2)). \tag{4.46}$$

Upon reinstating the integral formulation of these propagators and the scale invariant measure of Y_2, this integral simply becomes

$$G_{[\Delta_{(1,2,3)}]} = \frac{1}{(\nu_1^2 - \nu_2^2)} \left(G_{[\Delta_{2,3}]} - G_{[\Delta_{1,3}]}\right). \tag{4.47}$$

Expanding it out, it becomes:

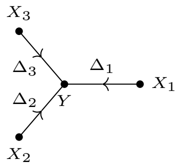
$$G_{[\Delta_{(1,2,3)}]} = \frac{G_{[\Delta_1]}}{(\nu_2^2 - \nu_1^2)(\nu_3^2 - \nu_1^2)} + \frac{G_{[\Delta_2]}}{(\nu_1^2 - \nu_2^2)(\nu_3^2 - \nu_2^2)} + \frac{G_{[\Delta_3]}}{(\nu_1^2 - \nu_3^2)(\nu_2^2 - \nu_3^2)}. \tag{4.48}$$

In the same vein as the relation between eqs. (4.37) and (4.42), the Feynman integral of fig. (4.1f) at $n = 3$ is

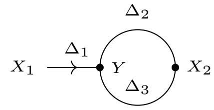
$$\begin{aligned} \mathcal{I}_{3,3} &= \Omega_{d+1} \times \lim_{\sigma_{12}\to 1} G_{[\Delta_{(1,2,3)}]}(\sigma_{12}) \\ &= \tfrac{\Gamma(1-d)}{d\,\pi} \Big(\tfrac{(\sin(\pi\,\Delta_1)+\sin(\pi\,\bar{\Delta}_1))\,\Gamma(\Delta_1)\,\Gamma(\bar{\Delta}_1)}{(\nu_2^2-\nu_1^2)(\nu_3^2-\nu_1^2)} + \tfrac{(\sin(\pi\,\Delta_2)+\sin(\pi\,\bar{\Delta}_2))\,\Gamma(\Delta_2)\,\Gamma(\bar{\Delta}_2)}{(\nu_1^2-\nu_2^2)(\nu_3^2-\nu_2^2)} \\ &\qquad + \tfrac{(\sin(\pi\,\Delta_3)+\sin(\pi\,\bar{\Delta}_3))\,\Gamma(\Delta_3)\,\Gamma(\bar{\Delta}_3)}{(\nu_1^2-\nu_3^2)(\nu_2^2-\nu_3^2)} \Big). \end{aligned} \tag{4.49}$$

This process can be inductively continued to find closed form expressions of the Feynman diagrams in figs. (4.1c) and (4.1f) for arbitrary n:

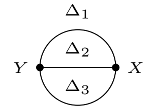
$$\begin{aligned} G_{[\Delta_{(1,\cdots n)}]} &= \sum_{i=1}^{n} \frac{G_{[\Delta_i]}}{\prod\limits_{j\neq i}(\nu_j^2 - \nu_i^2)} \\ \mathcal{I}_{n,n} &= \frac{\Gamma(1-d)}{d\,\pi} \sum_{i=1}^{n} \frac{(\sin(\pi\,\Delta_i) + \sin(\pi\,\bar{\Delta}_i))\,\Gamma(\Delta_i)\,\Gamma(\bar{\Delta}_i)}{\prod\limits_{j\neq i}(\nu_j^2 - \nu_i^2)}. \end{aligned} \tag{4.50}$$



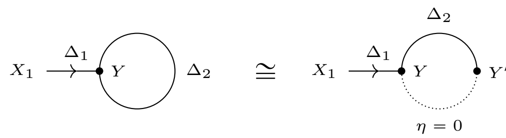
(a) 3-point function



(b) Tadpole with vertex insertion



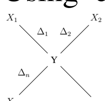
(c) 3-melon



(d) Construction of tadpole with δ-function propagator

Figure 4.2: 3-point function and related Feynman diagrams

4.2.6 n-Point Function and n-Melon

Using eq. (4.30), a general scalar n-point function (corresponding to the feynman diagram ) is given by:

$$\mathcal{J}_n(\hat{X}) = \bar{\mathcal{N}}_n \int_{\lambda,\mu}^{*} \lambda^{\bar{\Delta}} \mu^{\Delta} e^{-\lambda^2} \frac{e^{\frac{(\sum \lambda \mu \hat{X})^2}{(1+\sum \mu^2)}}}{(1+\sum \mu^2)^{\frac{\mathbb{D}}{2}}} = \bar{\mathcal{N}}_n \int_{\lambda,\mu}^{*} \lambda^{\bar{\Delta}} \mu^{\Delta} \frac{e^{-\lambda^2 (1+\sum \mu^2)+(\sum \lambda \mu \hat{X})^2}}{(1+\sum \mu^2)^{\frac{\mathbb{D}-\sum \bar{\Delta}}{2}}}, \tag{4.51}$$

where

$$\bar{\mathcal{N}}_n = \frac{1}{(2\pi^{\frac{\mathbb{D}}{2}})^{(n-1)} \Gamma(\frac{\mathbb{D}-\sum \Delta}{2})}. \tag{4.52}$$

As noted under eq. (4.30), any one parameter λ_i can always be integrating out in such constructions:

$$\begin{aligned} \mathcal{J}_n(\hat{X}) &= \bar{\mathcal{N}}_n \int_{\lambda,\mu}^{*} \frac{\lambda^{\bar{\Delta}} \mu^{\Delta} \delta(\lambda_i - 1)}{(1+\sum \mu^2)^{\frac{\mathbb{D}-\sum \bar{\Delta}}{2}} (\lambda^2 (1+\sum \mu^2) - (\sum \lambda \mu \hat{X})^2)^{\frac{\sum \bar{\Delta}}{2}}}, \\ \bar{\mathcal{N}}_n &= \frac{\Gamma(\frac{\sum \bar{\Delta}}{2})}{2 (2\pi^{\frac{\mathbb{D}}{2}})^{(n-1)} \Gamma(\frac{\mathbb{D}-\sum \Delta}{2})}. \end{aligned} \tag{4.53}$$

The n-melon diagram, $\mathcal{I}_{2,n}$, is related to the n-point function, $\mathcal{J}_n(\hat{X})$, by the identification of all $\hat{X}$ to the same point and integrating over this internal vertex, i.e. $\mathcal{I}_{2,n} = \Omega_{d+1} \times \lim_{\sigma \to 1} \mathcal{J}_n$. Represented by the incidence matrix:

$$L_{1i} = \lambda_i, \quad L_{2i} = -\mu_i, \quad U_{2,n} = \begin{pmatrix} \mathbb{1}_n & \mathrm{i}\begin{pmatrix} \lambda_1 & -\mu_1 \\ \cdots & \cdots \\ \lambda_n & -\mu_n \end{pmatrix} \\ \mathrm{i}\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ -\mu_1 & \cdots & -\mu_n \end{pmatrix} & \mathbb{1}_2 \end{pmatrix}, \tag{4.54}$$

using eq. (4.11), its integral form is

$$\mathcal{I}_{2,n} = \frac{1}{(4\pi^{\mathbb{D}})^{\frac{n-2}{2}}\,\Gamma(\frac{\mathbb{D}-\sum\bar{\Delta}}{2})\,\Gamma(\frac{\mathbb{D}-\sum\Delta}{2})} \int_{\lambda,\mu}^{*} \frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(1+\vec{\lambda}^2+\vec{\mu}^2+\frac{1}{2}\sum_{i,j}(\lambda_i\,\mu_j-\lambda_j\,\mu_i)^2)^{\frac{\mathbb{D}}{2}}}. \tag{4.55}$$

4.2.7 3-Point Function and 3-Melon

Specifically, the 3-point function is

$$\mathcal{J}_3(\hat{X}) = \frac{\Gamma(\frac{\sum\bar{\Delta}}{2})}{8\pi^{\mathbb{D}}\,\Gamma(\frac{\mathbb{D}-\sum\Delta}{2})} \int_{\lambda,\,\mu}^{*} \frac{\lambda_1^{\bar{\Delta}_1}\,\lambda_2^{\bar{\Delta}_2}\,\mu_1^{\Delta_1}\,\mu_2^{\Delta_2}\,\mu_3^{\Delta_3}}{(1+\sum\mu^2)^{\frac{\mathbb{D}-\sum\bar{\Delta}}{2}}\,(1+f_3(\hat{X}))^{\frac{\sum\bar{\Delta}}{2}}}, \tag{4.56}$$

where

$$\begin{aligned} f_3(\hat{X}) &= (1+\lambda^2)\,(1+\sum\mu^2) - (\lambda_1\,\mu_1\,\hat{X}_1+\lambda_2\,\mu_2\,\hat{X}_2+\mu_3\,\hat{X}_3)^2 - 1 \\ &= \mu_1^2+\mu_2^2+\lambda_1^2(1+\mu_2^2+\mu_3^2)+\lambda_2^2(1+\mu_1^2+\mu_3^2) \\ &- 2\,(\lambda_1\,\mu_1\,\lambda_2\,\mu_2\,\sigma_{12}+\lambda_1\,\mu_1\,\mu_3\,\sigma_{13}+\lambda_2\,\mu_2\,\mu_3\,\sigma_{23}), \end{aligned} \tag{4.57}$$

its irreducible form being

$$\mathcal{J}_3(\hat{X}) = \frac{\Gamma(\frac{\sum\bar{\Delta}}{2})}{16\pi^{\mathbb{D}}\,\Gamma(\frac{\mathbb{D}-\sum\Delta}{2})} \int_{\lambda,\,\mu}^{*} \frac{\lambda_1^{\bar{\Delta}_1}\,\lambda_2^{\bar{\Delta}_2}\,\mu_1^{\Delta_1}\,\mu_2^{\Delta_2}\,\mu_3^{1}}{(1+\mu_3\,(1+\mu_1^2+\mu_2^2))^{\frac{\mathbb{D}-\sum\bar{\Delta}}{2}}\,(1+f_3'(\hat{X}))^{\frac{\sum\bar{\Delta}}{2}}}, \tag{4.58}$$

where

$$\begin{aligned} f_3'(\hat{X}) &= \lambda_1^2+\mu_1^2-2\,\lambda_1\,\mu_1\,\sigma_{13}+\lambda_2^2+\mu_2^2-2\,\lambda_2\,\mu_2\,\sigma_{23} \\ &\quad + \mu_3\,\Big(\lambda_1^2+\lambda_2^2+\lambda_1^2\,\mu_2^2+\lambda_2^2\,\mu_1^2-2\,\lambda_1\,\mu_1\,\lambda_2\,\mu_2\,\sigma_{12}\Big). \end{aligned} \tag{4.59}$$

The GKZ system describing the generalisation of this integral has 68 roots and as many $\mathcal{A}$-hypergeometric series. The closely related 3-melon diagram equalling

$$\begin{aligned}\mathcal{I}_{2,3} &= \frac{1}{2\pi^{\frac{\mathbb{D}}{2}}\,\Gamma(\frac{\mathbb{D}-\sum\bar{\Delta}}{2})\,\Gamma(\frac{\mathbb{D}-\sum\Delta}{2})}\int_{\lambda,\mu}^{*}\frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(f_{2,3})^{\frac{\mathbb{D}}{2}}},\\ f_{2,3} &= 1+\sum_{i=1}^{3}(\lambda_i^2+\mu_i^2)+(\lambda_1\,\mu_2-\lambda_2\,\mu_1)^2+(\lambda_2\,\mu_3-\lambda_3\,\mu_2)^2+(\lambda_3\,\mu_1-\lambda_1\,\mu_3)^2,\end{aligned} \tag{4.60}$$

has the same series complexity.

4.2.8 Tadpole

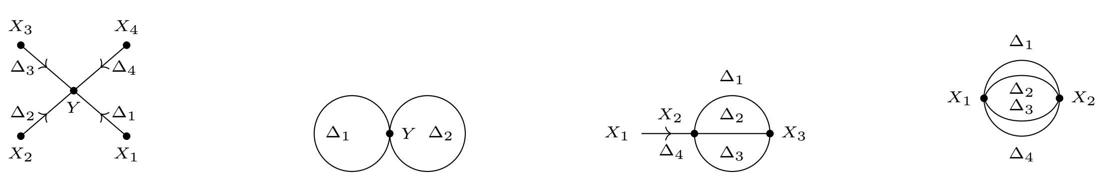
It is easy to evaluate fig. (4.2d) and express it in terms of the 1-loop integral:

$$\begin{aligned}\mathcal{I}_{1,2} &= \frac{\Gamma(\frac{\bar{\Delta}_1}{2})\,\Gamma(\frac{\Delta_1}{2})}{8\pi^{\frac{\mathbb{D}}{2}}\,\Gamma(1-\frac{\bar{\Delta}_1}{2})\,\Gamma(\frac{\mathbb{D}-\bar{\Delta}_1}{2})}\int_{\lambda,\mu}^{*}\frac{\lambda_2^{\bar{\Delta}_2}\,\mu_2^{\Delta_2}}{(1+(\lambda_2-\mu_2)^2)^{1+\frac{\bar{\Delta}_1}{2}}}\\ &= \frac{\Gamma(\frac{\bar{\Delta}_1}{2})\,\Gamma(\frac{\Delta_1}{2})\,\Gamma(1-\frac{\Delta_1}{2})\,\Gamma(\frac{d}{2})}{16\pi^{\frac{\mathbb{D}}{2}}\,\Gamma(1-\frac{\bar{\Delta}_1}{2})\,\Gamma(1+\frac{\Delta_1}{2})\,\Gamma(\frac{\mathbb{D}-\bar{\Delta}_1}{2})}\int_{\mu}^{*}\frac{\mu^{\Delta_2}}{|1-\mu|^{d}}.\end{aligned} \tag{4.61}$$

Using eq. (4.42), fig. (4.2b) is:

$$\mathcal{I}_{1,2,[1]} = \frac{2\pi^{d+2}}{\nu_1^2-\nu_2^2}\Big(\mathcal{I}_{1,2,[0]}\Big|_{\Delta_2}-\mathcal{I}_{1,2,[0]}\Big|_{\Delta_1}\Big), \tag{4.62}$$

and of course, section 4.2.5 can be used to extend it to n-vertex insertions.



(a) 4-point function (b) 2-loops (c) 3-melon tadpole (d) 4-melon

Figure 4.3: 4-point function and related Feynman diagrams

4.2.9 2 Simply Connected Loops

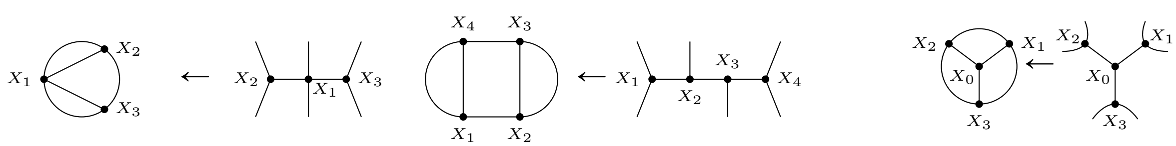
Just like the tadpole diagram, fig. (4.3b) is also easily evaluated in terms of the 1-loop integral:

$$\begin{aligned}\mathcal{I}_{1,2,[0,0]} &= \frac{\Gamma(\frac{2-d}{2})}{(4\pi^{\mathbb{D}})^{\frac{1}{2}}}\int^{*}_{\lambda,\mu}\frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(1+(\lambda_1-\mu_1)^2+(\lambda_2-\mu_2)^2)^{\frac{\mathbb{D}}{2}}}\\ &= \frac{\Gamma(\frac{2-d}{2})}{4\pi^{\frac{\mathbb{D}}{2}}}\int^{*}_{\lambda,\mu}\frac{\lambda_1^{d/2}\lambda_2^{\bar{\Delta}}\,\mu^{\Delta}}{(1+\lambda_1(1-\mu_1)^2+(\lambda_2-\mu_2)^2)^{\frac{\mathbb{D}}{2}}}\\ &= \frac{\Gamma(\frac{2-d}{2})\csc(\frac{\pi\,d}{2})}{4\,d\,\pi^{\frac{d}{2}}}\int^{*}_{\mu}\frac{\mu^{\Delta}}{|1-\mu_1|^d\,|1-\mu_2|^d}.\end{aligned} \tag{4.63}$$

The addition of vertices to each loop is just as simple, since the propagator expressions of fig. (4.1c) can be inductively found for arbitrary n, as shown in section 4.2.5. So say, n_1, n_2 vertices were dropped onto the 2 loops of this diagram, then

$$\mathcal{I}_{1,2,[n_1,n_2]} = \Omega_{d+1}\times\lim_{\sigma\to 1}G_{[\Delta_1\cdots n_1]}(\sigma)\times G_{[\Delta_1\cdots n_2]}(\sigma). \tag{4.64}$$

4.2.10 3-Loops : Pacman, Pillbox, Peace



(a) Pacman (b) Pillbox (c) Peace

Figure 4.4: Irreducible 3-loop Feynman diagrams as limits of tree-level correlation functions

The conversion of these diagrams to their Euler integral form was already described in the introduction itself. Directly using the incidence matrices from table 1.1, they are:

$$\begin{aligned}
\mathcal{I}_{\text{pac}} &= \frac{1}{4\pi^{\mathbb{D}}\,\Gamma\big(\frac{\mathbb{D}-\bar{\Delta}_1-\bar{\Delta}_2-\bar{\Delta}_3-\bar{\Delta}_4}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\bar{\Delta}_5-\Delta_1-\Delta_2}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_3-\Delta_4-\Delta_5}{2}\big)} \\
&\quad\times \int_{\lambda,\,\mu}^{*} \lambda^{\bar{\Delta}}\,\mu^{\Delta}\,\left(\det\begin{pmatrix} 1+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 & -(\lambda_1\,\mu_1+\lambda_2\,\mu_2) & -(\lambda_3\,\mu_3+\lambda_4\,\mu_4) \\ -(\lambda_1\,\mu_1+\lambda_2\,\mu_2) & 1+\lambda_5^2+\mu_1^2+\mu_2^2 & -\lambda_5\,\mu_5 \\ -(\lambda_3\,\mu_3+\lambda_4\,\mu_4) & -\lambda_5\,\mu_5 & 1+\mu_3^2+\mu_4^2+\mu_5^2 \end{pmatrix}\right)^{-\frac{d+2}{2}} \\
\mathcal{I}_{\text{pil}} &= \frac{1}{4\pi^{\mathbb{D}}\,\Gamma\big(\frac{\mathbb{D}-\bar{\Delta}_1-\Delta_4-\bar{\Delta}_5}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_1-\bar{\Delta}_2-\Delta_6}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_2-\bar{\Delta}_3-\bar{\Delta}_6}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_3-\bar{\Delta}_4-\Delta_5}{2}\big)} \\
&\quad\times \int_{\lambda,\,\mu}^{*} \lambda^{\bar{\Delta}}\,\mu^{\Delta}\,\left(\det\begin{pmatrix} 1+\lambda_1^2+\lambda_5^2+\mu_4^2 & -\lambda_1\,\mu_1 & 0 & -(\lambda_4\,\mu_4+\lambda_5\,\mu_5) \\ -\lambda_1\,\mu_1 & 1+\lambda_2^2+\mu_1^2+\mu_6^2 & -(\lambda_2\,\mu_2+\lambda_6\,\mu_6) & 0 \\ 0 & -(\lambda_2\,\mu_2+\lambda_6\,\mu_6) & 1+\lambda_3^2+\lambda_6^2+\mu_2^2 & -\lambda_3\,\mu_3 \\ -(\lambda_4\,\mu_4+\lambda_5\,\mu_5) & 0 & -\lambda_3\,\mu_3 & 1+\lambda_4^2+\mu_3^2+\mu_5^2 \end{pmatrix}\right)^{-\frac{d+2}{2}} \\
\mathcal{I}_{\text{pea}} &= \frac{1}{4\pi^{\mathbb{D}}\,\Gamma\big(\frac{\mathbb{D}-\bar{\Delta}_1-\bar{\Delta}_2-\bar{\Delta}_3}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_1-\bar{\Delta}_4-\Delta_6}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_2-\Delta_4-\bar{\Delta}_5}{2}\big)\,\Gamma\big(\frac{\mathbb{D}-\Delta_3-\Delta_5-\bar{\Delta}_6}{2}\big)} \\
&\quad\times \int_{\lambda,\,\mu}^{*} \lambda^{\bar{\Delta}}\,\mu^{\Delta}\,\left(\det\begin{pmatrix} 1+\lambda_1^2+\lambda_2^2+\lambda_3^2 & -\lambda_1\,\mu_1 & -\lambda_2\,\mu_2 & -\lambda_3\,\mu_3 \\ -\lambda_1\,\mu_1 & 1+\lambda_4^2+\mu_1^2+\mu_6^2 & -\lambda_4\,\mu_4 & -\lambda_6\,\mu_6 \\ -\lambda_2\,\mu_2 & -\lambda_4\,\mu_4 & 1+\lambda_5^2+\mu_2^2+\mu_4^2 & -\lambda_5\,\mu_5 \\ -\lambda_3\,\mu_3 & -\lambda_6\,\mu_6 & -\lambda_5\,\mu_5 & 1+\lambda_6^2+\mu_3^2+\mu_5^2 \end{pmatrix}\right)^{-\frac{d+2}{2}} .
\end{aligned} \tag{4.65}$$

4.3 Representative Constructions of Vector Feynman Integrals

The following examples are representative of the general procedure that relates vector Feynman integrals to purely scalar Feynman integrals.

4.3.1 Vector 1 Loop Character Integral

The 1-loop Feynman integral in fig. (4.1d), involving the trace of the vector propagator in the coincident point limit, equals

$$\mathcal{I}_{1,1}^{(1)} = \int_{\hat{X}} \mathcal{G}_{II'}\,\delta^{II'} \circ G(\hat{X},\,\hat{X}). \tag{4.66}$$

Massive

The trace of $\mathcal{G}$ in eq. (3.83), using the eq. (4.22) (on a part of it) reduces to

$$\mathcal{G}_{II'}\,\delta^{II'} = \frac{d\,\bar{\Delta}\,\Delta}{(\bar{\Delta}-1)\,(\Delta-1)} + \frac{4\,\lambda\,\mu\,X^2\,P^2}{(\bar{\Delta}-1)\,(\Delta-1)} \tag{4.67}$$

The first term simply introduces an additional factor to the underlying 1-loop scalar integral, $\mathcal{I}_{1,1}$, given in eq. (4.33). The second vanishes, as shown in the following:

$$\begin{aligned}\int_{\hat{X}} \frac{4\,\lambda\,\mu\,X^2\,P^2}{(\bar{\Delta}-1)\,(\Delta-1)} \circ G(\hat{X},\,\hat{X}) &= \int_{\hat{X}} \displaystyle{\not}\!\!\int_{\lambda,\mu}^{*} \int_{P} \frac{4\,\lambda^{\bar{\Delta}+1}\,\mu^{\Delta+1}}{(\bar{\Delta}-1)\,(\Delta-1)\,(\lambda-\mu)^{\mathbb{D}}} e^{-2\mathrm{i}\,|P\hat{X}|} \\ &= \int_{\hat{X}} \displaystyle{\not}\!\!\int_{\lambda,\mu}^{*} \int_{P} \frac{4\,\mu^{\Delta+1}}{(\bar{\Delta}-1)\,(\Delta-1)\,(1-\mu)^{\mathbb{D}}} e^{-2\mathrm{i}\,|P\hat{X}|} \\ &\propto \int_{\hat{X}} \int_{\mu}^{*} \frac{4\,\mu^{\Delta+1}}{(\bar{\Delta}-1)\,(\Delta-1)\,(1-\mu)^{\mathbb{D}}} \cancel{\int_{P} e^{-2\mathrm{i}\,|P\hat{X}|}}.\end{aligned} \tag{4.68}$$

Massless

The trace of $\mathcal{G}$ in eq. (3.87), using eq. (4.22), reduces to a factor:

$$\mathcal{G}^{(o)}_{II'}\,\delta^{II'} = \frac{4\,\lambda\,\mu}{(d-2)}\,|PX|^2 \left(\delta_{II'}\,\delta^{II'} - 1\right) \cong \frac{(d+1)\,(d-1)}{(d-2)}. \tag{4.69}$$

4.3.2 1 Loop of 2-Vector Propagators : 2-Melon

The 2-melon Feynman integral for vectors is:

$$\mathcal{I}^{(1)}_{2,2} = \int_{\hat{X},\hat{Y}} \mathcal{G}_{IK'}(\hat{X},\,\hat{Y})\,\mathcal{G}^{K'I}(\hat{Y},\,\hat{X}) \circ G_{\Delta_1}(\hat{X},\,\hat{Y})\,G_{\Delta_2}(\hat{Y},\,\hat{X}). \tag{4.70}$$

Massless

When considering massless propagators, this reduces to a simple change in factor:

$$\mathcal{I}^{(1)}_{2,2} = \frac{d\,(d-1)^2}{(d-2)^2}\,\mathcal{I}_{2,2}, \tag{4.71}$$

where $\mathcal{I}_{2,2}$ is the scalar integral from section 4.2.3.

Massive

Crunching through the indices and making some initial reductions, this integral splits into 10 types of scalar Feynman integrals, all structured similar to $\mathcal{I}_{(2,2)}$ from the scalar case in section 4.2.3. Three of them are straight forward shifts to the values of $\Delta,\ \bar{\Delta}$:

$\mathcal{N}$	$\lambda_1^{\cdots}$	$\mu_1^{\cdots}$	$\lambda_2^{\cdots}$	$\mu_2^{\cdots}$
$(d-2)\,\Delta_1\,\bar{\Delta}_1\,\Delta_2\,\bar{\Delta}_2$	$\bar{\Delta}_1$	Δ_1	$\bar{\Delta}_2$	Δ_2
$(\bar{\Delta}_1-1)\,\bar{\Delta}_1\,(\bar{\Delta}_2-1)\,\bar{\Delta}_2$	$\bar{\Delta}_1-1$	Δ_1+1	$\bar{\Delta}_2-1$	Δ_2+1
$(\Delta_1-1)\,\Delta_1\,(\Delta_2-1)\,\Delta_2$	$\bar{\Delta}_1+1$	Δ_1-1	$\bar{\Delta}_2+1$	Δ_2-1

(4.72)

with each scalar integral contribution taking the form:

$$\frac{\mathcal{N}}{(\bar{\Delta}_1-1)\,(\bar{\Delta}_2-1)\,(\Delta_1-1)\,(\Delta_2-1)}\,\mathcal{I}_{(2,2)}[\Delta_1,\ \bar{\Delta}_1,\ \Delta_2\,\bar{\Delta}_2]. \tag{4.73}$$

Denoting the maximal perturbation incidence matrix as $\bar{U}(\beta) = \{\beta_{ij}\}$ in order to use the "master" integral form for notational convenience, the other integrals, can be written as:

$$\frac{\mathcal{N}}{(\bar{\Delta}_1-1)\,(\bar{\Delta}_2-1)\,(\Delta_1-1)\,(\Delta_2-1)}\,\partial^{\omega}_{\beta}\,\mathcal{I}_{(2,2)}[\Delta_1,\ \bar{\Delta}_1,\ \Delta_2\,\bar{\Delta}_2][\bar{U}(\beta)], \tag{4.74}$$

where

$$\mathcal{I}_{(2,2)}[\bar{U}(\beta)] = \bar{\mathcal{N}}_F \int_{\lambda,\,\mu}^{*} \frac{\lambda^{\bar{\Delta}}\,\mu^{\Delta}}{(\det(\bar{U}_{(2,2)} + \bar{U}(\beta)))^{\frac{\mathbb{D}}{2}}}, \tag{4.75}$$

and are given by

(4.76)

$\mathcal{N}$	$\lambda_1^{\cdots}$	$\mu_1^{\cdots}$	$\lambda_2^{\cdots}$	$\mu_2^{\cdots}$	$\bar{U}(\beta)$
$4\,\bar{\Delta}_2\,\Delta_1\,(d+1-\bar{\Delta}_1-\Delta_2)$	$\bar{\Delta}_1+1$	Δ_1	$\bar{\Delta}_2$	Δ_2+1	$\beta_{1,2}$
$4\,\Delta_2\,\bar{\Delta}_1\,(d+1-\Delta_1-\bar{\Delta}_2)$	$\bar{\Delta}_1$	Δ_1+1	$\bar{\Delta}_2+1$	Δ_2	$\beta_{1,2}$
$8\,\bar{\Delta}_1\,\bar{\Delta}_2$	$\bar{\Delta}_1$	Δ_1+1	$\bar{\Delta}_2$	Δ_2+1	$\beta_{1,2},\ \beta_{3,4}$
$8\,\Delta_1\,\Delta_2$	$\bar{\Delta}_1+1$	Δ_1	$\bar{\Delta}_2+1$	Δ_2	$\beta_{1,2},\ \beta_{3,4}$
16	$\bar{\Delta}_1+1$	Δ_1+1	$\bar{\Delta}_2+1$	Δ_2+1	$\beta_{1,2}^2,\ \beta_{3,4}^2$
$\lim\limits_{\eta\to 0}\frac{4\,\bar{\Delta}_2\,\Delta_2}{\Gamma(\eta)}$	$\bar{\Delta}_1+1$	Δ_1+1	$\bar{\Delta}_2$	Δ_2	$\beta_{3,4}$
$\lim\limits_{\eta\to 0}\frac{4\,\bar{\Delta}_1\,\Delta_1}{\Gamma(\eta)}$	$\bar{\Delta}_1$	Δ_1	$\bar{\Delta}_2+1$	Δ_2+1	$\beta_{3,4}$

where each β_{ij} also implies β_{ji}, the exponent attached to them is notational: β^n implies $f[\partial_\beta^n]$ is used to construct the perturbing polynomial, and if not mentioned, those elements of $\bar{U}(\beta)$ are 0.

Quite simply because transcribing any integral constructions more complex than this will not only be highly unsavory but also result in something entirely illegible, interested readers are invited to explore this formulation further on their own.

Chapter 5: Discussion

The core motivation behind this thesis was to provide a systematic way to represent and compute higher loop Feynman integrals on a spherical background, while avoiding the obstacles that appear when attempting to compute them in position space. This involved the construction of a new class of propagator expressions in chapter 3. Using these expressions, Feynman integrals on the sphere were lifted to embedding space in chapter 4, eventually bringing integrals that were entirely intractable in position space into the domain of Generalized Euler Integrals, that afford systematic solvability, as reviewed in chapter 2.

As an immediate next step towards the extension of this formulation, the graviton propagator has to be found, ideally in the de Donder gauge or Polchinski's generalisation thereof, [123]. It has been observed that when considering symmetric transverse traceless eigenmodes in eq. (C.11), any radially invariant tensor $\mathcal{O}$ will result in either the proper mode integral eigenvalue (i.e. inverse of the Laplacian eigenvalue) or will annihilate that inner product, along the lines of eq. (C.12).

However, the main barrier to building higher spin propagators is ensuring that the corresponding higher dimensional massless field is gauged fixed in the precise way that replicates the massive action on the sphere as its quotient, with massless fields requiring additional gauge conditions or projections. As is to be expected, this difference in choice of gauge can only be detected in the propagators' behaviour with respect to the longitudinal eigenmodes. As is evident in BRST gauge fixing of the flat higher dimensional massless vector action, the current approach is ad hoc. Overall, the derivation process using the Mellin transformed basis of fields is yet to be systematized. A general all spin embedding space representation of propagators will vastly improve the applicability of the presented 'Feynman to generalized Euler integral' formulation.

Apart from the aesthetical benefit of representing physically relevant quantities stemming from

Feynman integrals as solutions to GKZ PDE's that impose symmetry constraints on them, there are many practical benefits to this approach, not all of which have been fully explored. Special cases of mass and dimension in which these systems collapse can be not only be considered in detail (for example, GKZ systems of conformally coupled scalars take simpler forms) but also reverse engineered (specific combinations of mass parameters result in sudden simplifications when the polynomials end up getting positive integer exponents, i.e. $\frac{1}{P^{\alpha}} \to P^{\mathbb{N}}$).

Making efficient use of restriction algorithms will signficantly reduce the number and complexity of series solutions to sphere Feynman integrals, which otherwise radically grow in size with increasing number of propagators. Of course this complexity doesn't magically vanish and instead transfers to the notoriously large complexity of Gröbner bases computations. At any rate, these systems of PDEs cross pen-and-paper solvability very quickly, so larger scale computational efforts are in order.

The physically relevant limits supplied by sphere Feynman integrals always flow to some sort of maximally singular point in the parameter space. The precise physical relevance of this 'confluence of singularities' in the more abstract coordinates of GKZ systems is worth looking into, in order to gain more concrete understanding of the physical principles governing the convergence radii of the series solutions. This would also allow better numerical treatment of these sums. Even more aspirationally, a deeper understanding of this maximally singular physically relevant point may allow direct comparison of the GKZ ideals of Feynman integrals resulting from dS entropy computations with some equivalent symmetry generators of hypothesised microscopic models, foregoing the need to actually compute these integrals.

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Appendix A: Coordinate Systems and Notational Conventions

A.1 Coordinate Systems

A.1.1 $(d+1)$-Sphere

In order to maintain notational simplicity, the dimensions of the sphere are denoted by $d+1 \equiv D \equiv \mathbb{D}-1$, S^{d+1}. General coordinates on the sphere are denoted by w^μ, with the associated metric being $g_{\mu\nu}$. Flat orthonormal coordinates, $\hat{X}(w)$, satisfy $|\hat{X}|^2 = 1$ and have the associated metric:

$$\mathrm{d}\hat{s}^2 = \delta_{IJ}\,\mathrm{d}\hat{X}^I\,\mathrm{d}\hat{X}^J = g_{\mu\nu}\,\mathrm{d}w^\mu\,\mathrm{d}w^\nu. \tag{A.1}$$

Tensors in flat and spherical coordinates are related by

$$T^{\cdots}_{I\cdots} = \hat{e}^\mu_I\,T^{\cdots}_{\mu\cdots}, \quad \hat{e}^\mu_I \equiv \tfrac{\partial w^\mu}{\partial \hat{X}^I}, \quad T^{I\cdots}_{\cdots} = \hat{e}^I_\mu\,T^{\mu\cdots}_{\cdots}, \quad \hat{e}^I_\mu \equiv \tfrac{\partial \hat{X}^I}{\partial w^\mu}. \tag{A.2}$$

An integral over the sphere is denoted by $\displaystyle\int_{S^{d+1}} \equiv \int_{\hat{X}} \equiv \int_w \sqrt{g}$. The volume of S^{d+1} is $\Omega_{d+1} = \frac{2\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})}$. The laplacian of an s-indexed tensor T in $\hat{X}(w)$ is simply $\partial^2_{\hat{X}} T$.

A.1.2 Embedding Space $\mathbb{R}^{\mathbb{D}}$

General coordinates on Euclidean space $\mathbb{R}^{\mathbb{D}}$, serving as embedding space for S^{d+1} in present context, are denoted by x^I with metric $\mathbb{g}_{IJ}$. In flat coordinates X, the metric is simply δ_{IJ}. Spherical coordinates (t, w), $t \in \mathbb{R}$, are related to flat coordinates by $X = e^t \hat{X}(w)$, $\hat{X} = \frac{X}{|X|}$ and have the metric

$$\mathrm{d}s^2 = \delta_{IJ}\,\mathrm{d}X^I\,\mathrm{d}X^J = \mathbb{g}_{IJ}\,\mathrm{d}x^I\,\mathrm{d}x^J = e^{2t}(\,\mathrm{d}t^2 + \,\mathrm{d}\hat{s}^2) = e^{2t}(\,\mathrm{d}t^2 + g_{\mu\nu}\,\mathrm{d}w^\mu\,\mathrm{d}w^\nu). \tag{A.3}$$

Specifically, the metric in spherical coordinates is related to that on the embedded sphere by

$$\mathbb{g}_{tt} = e^{2t}, \quad \mathbb{g}_{t\mu} = 0, \quad \mathbb{g}_{\mu\nu} = e^{2t} g_{\mu\nu}, \tag{A.4}$$

with the Christoffel symbols $\mathbb{\Gamma}$ of the associated Levi-Civita connection $\mathbb{\nabla}$ given by

$$\mathbb{\Gamma}^{t}{}_{tt} = 1, \quad \mathbb{\Gamma}^{t}{}_{\mu\nu} = -g_{\mu\nu}, \quad \mathbb{\Gamma}^{t}{}_{t\mu} = 0, \quad \mathbb{\Gamma}^{\mu}{}_{tt} = 0, \quad \mathbb{\Gamma}^{\mu}{}_{t\nu} = \delta^{\mu}_{\nu}, \quad \mathbb{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} \tag{A.5}$$

where $\Gamma^{\mu}{}_{\nu\rho}$ are Christoffel symbols of the Levi-civita connection ∇ on S^{d+1} and greek indices are explicitly used to indicate that they do not include t. They are related by

$$\begin{aligned}
&\mathbb{\nabla}^t \mathbb{T} = e^{-2t}\, \mathbb{\nabla}_t \mathbb{T}, \quad \mathbb{\nabla}_\mu \mathbb{\nabla}_t \mathbb{T} = \mathbb{\nabla}_t \mathbb{\nabla}_\mu \mathbb{T}, \quad \mathbb{\nabla}_t \mathbb{T}^{b_1\cdots b_m}_{a_1\cdots a_n} = (\partial_t + m - n)\, \mathbb{T}^{b_1\cdots b_m}_{a_1\cdots a_n}, \\
&(\mathbb{\nabla}_\mu - \nabla_\mu)\, \mathbb{T}^{t'} = -g_{\mu\nu}\, \mathbb{T}^{\nu}, \quad (\mathbb{\nabla}_\mu - \nabla_\mu)\, \mathbb{T}^{\nu'} = \delta^{\nu'}_{\mu}\, \mathbb{T}^{t} \\
&(\mathbb{\nabla}_\mu - \nabla_\mu)\, \mathbb{T}_t = -\mathbb{T}_\mu, \quad (\mathbb{\nabla}_\mu - \nabla_\mu)\, \mathbb{T}_\nu = g_{\mu\nu}\, \mathbb{T}_t
\end{aligned} \tag{A.6}$$

for some arbitrarily indexed tensor $\mathbb{T}$. The Riemann curvature tensor $\mathbb{R}$ remains trivial. A integrals over $\mathbb{R}^{\mathbb{D}}$ and S^{d+1} in embedding space coordinates are denoted by

$$\int^{\mathbb{R}^{\mathbb{D}}} \equiv \int_X \equiv \int_x \sqrt{\mathbb{g}} \equiv \int_{\hat{X}} \int_t e^{\mathbb{D}t}, \quad \int_{\hat{X}}^{S^{d+1}} \equiv ⨍_X = \int_X^{\mathbb{R}^{\mathbb{D}}} \frac{\mathrm{d}^{\mathbb{D}} X}{|X|^{\mathbb{D}}}. \tag{A.7}$$

Tensors in flat and spherical coordinates are related by

$$\begin{aligned}
&\mathbb{T}^{\cdots}_{I\cdots} = \frac{X_I}{|X|^2}\, \mathbb{T}^{\cdots}_{t\cdots} + \hat{\mathbb{e}}^{\mu}_{I}\, \mathbb{T}^{\cdots}_{\mu\cdots}, \quad && \mathbb{T}^{I\cdots}_{\cdots} = X^I\, \mathbb{T}^{t\cdots}_{\cdots} + \hat{\mathbb{e}}^{I}_{\mu}\, \mathbb{T}^{\mu\cdots}_{\cdots} \\
&\mathbb{T}^{\cdots}_{t\cdots} = X^J\, \mathbb{T}^{\cdots}_{J\cdots}, && \mathbb{T}^{t\cdots}_{\cdots} = \frac{X_J}{|X|^2}\, \mathbb{T}^{J\cdots}_{\cdots} \\
&\mathbb{T}^{\cdots}_{\mu\cdots} = \hat{\mathbb{e}}^{I}_{\mu}\, \hat{\mathbb{t}}^{J}_{I}\, \mathbb{T}^{\cdots}_{J\cdots}, && \mathbb{T}^{\mu\cdots}_{\cdots} = \hat{\mathbb{e}}^{\mu}_{I}\, \hat{\mathbb{t}}^{I}_{J}\, \mathbb{T}^{J\cdots}_{\cdots}
\end{aligned} \tag{A.8}$$

where

$$\hat{\mathbb{e}}^{I}_{\mu} \equiv \frac{\partial X^I}{\partial w^\mu} = |X|\, \hat{e}^{I}_{\mu}, \quad \hat{\mathbb{e}}^{\mu}_{I} \equiv \frac{\partial w^\mu}{\partial X^I} = \frac{1}{|X|}\, \hat{e}^{\mu}_{I}, \quad \hat{\mathbb{e}}^{I}_{\mu}\, \hat{\mathbb{e}}^{\mu}_{J} = \delta^{I}_{J}, \quad \hat{\mathbb{e}}^{I}_{\mu}\, \hat{\mathbb{e}}^{\nu}_{I} = \delta^{\nu}_{\mu} \tag{A.9}$$

and $\hat{\mathbb{t}}$ is a tangential projection operator

$$\hat{\mathbb{t}}_{IJ} := \delta_{IJ} - \frac{X_I X_J}{|X|^2}, \quad \hat{\mathbb{t}}_{IJ} X^I = \hat{\mathbb{t}}_{IJ} X^J = 0, \quad \hat{\mathbb{t}}_{IJ}\, \hat{\mathbb{t}}^J_K = \hat{\mathbb{t}}_{IJ}\, \delta^J_K = \hat{\mathbb{t}}_{IK}, \quad \hat{\mathbb{t}}^I_I = \mathbb{D} - 1 \tag{A.10}$$

which acts as the embedding space representation of $\delta_{\mu\nu}$ ($\delta^\mu_\mu = \hat{\mathbb{t}}^I_I$) on S^{d+1}.

A.2 Mellin Transformed Basis of Fields

A.2.1 Restriction to the Sphere

There are many ways to define the restriction of a field $\mathbb{T}$ on $\mathbb{R}^{\mathbb{D}}$ to T on S^{d+1}, all of which are technically equivalent. In other words, T can be viewed to be $\mathbb{T}$ in a particular "gauge". One simplistic gauge, for example, is $T^{\cdots}_{\cdots}(w) = \mathbb{T}^{\cdots\neq t}_{\cdots\neq t}\Big|_{t=0} = \int_t \mathbb{T}^{\cdots\neq t}_{\cdots\neq t}\, \delta(t)$. A more sound choice is to define T as a radial average parameterised by Δ:

$$\begin{aligned}
\Phi_{[\Delta]}(\hat{X}) &:= \int_t e^{\Delta t}\, \mathbb{\Phi}(X) \\
A_{[\Delta]\mu} &:= \int_t e^{(\Delta-1)t}\, \mathbb{A}_\mu, & A^\mu_{[\Delta]} &= \int_t e^{(\Delta+1)t}\, \mathbb{A}^\mu \\
\chi_{[\Delta]} &:= \int_t e^{(\Delta-1)t}\, \mathbb{A}_t = \int_t e^{(\Delta+1)t}\, \mathbb{A}^t \\
T_{[\Delta]\mu\nu} &:= \int_t e^{(\Delta-2)t}\, \mathbb{T}_{\mu\nu}, & T^{\mu\nu}_{[\Delta]} &= \int_t e^{(\Delta+2)t}\, \mathbb{T}^{\mu\nu} \\
\xi_{[\Delta]\mu} &:= \int_t e^{(\Delta-2)t}\, \mathbb{T}_{\mu t} = \int_t e^{\Delta t}\, \mathbb{T}_\mu{}^t, & \xi^\mu_{[\Delta]} &= \int_t e^{(\Delta+2)t}\, \mathbb{T}^{\mu t} = \int_t e^{\Delta t}\, \mathbb{T}^\mu{}_t \\
\chi_{[\Delta]} &:= \int_t e^{(\Delta-2)t}\, \mathbb{T}_{tt} = \int_t e^{\Delta t}\, \mathbb{T}^t{}_t = \int_t e^{(\Delta+2)t}\, \mathbb{T}^{tt} & &\cdots \text{ and so on}
\end{aligned} \tag{A.11}$$

where χ and ξ^μ are additional scalars and vector fields defined in order to capture the entire (or if focus is restricted to the sphere, irrelevant) behaviour of the embedding space fields. Thus, when representing spin-s fields on the sphere in embedding space, the additional fields of spin $< s$ can be viewed as parameterising gauge transformations of the spin-s field. These transformations are separate from the *physical* gauge transformations, and instead are an additional mathematical redundancy stemming from the embedding space representation.

A.2.2 Reextension to Embedding Space

The embedding space fields $\mathbb{T}$ can be recovered from the Mellin transformed basis of field T by an inverse Mellin transformation with respect to the radial component. Alternately, fields on the sphere can be extended to embedding space by assuming the tangentiality condition on the resultant embedding space field, $\mathbb{T} \cdot X = 0$. This is straightforward for scalar fields, Φ,

$$\Phi(X) = \oint_\Delta |X|^{-\Delta}\, \Phi_{[\Delta]}(\hat{X}). \tag{A.12}$$

Note that for these integrals to converge, a lower bound on Δ (and $\bar{\Delta}$) is assumed: $\mathrm{Re}(\Delta) > 0$. A vector field $\mathbb{A}$ in spherical coordinates

$$\begin{aligned} \mathbb{A}_\mu &= \oint_\Delta |X|^{-(\Delta-1)}\, A_{[\Delta]\mu}, \quad \mathbb{A}^\mu = \oint_\Delta |X|^{-(\Delta+1)}\, A_{[\Delta]}{}^\mu \\ \mathbb{A}_t &= \oint_\Delta |X|^{-(\Delta-1)}\, \chi_{[\Delta]}, \quad\;\; \mathbb{A}^t = \oint_\Delta |X|^{-(\Delta+1)}\, \chi_{[\Delta]} \end{aligned} \tag{A.13}$$

can be used to recover its flat space representation:

$$\mathbb{A}_I = \oint_\Delta |X|^{-\Delta}\, \frac{X_I}{|X|} \chi_{[\Delta]} + \oint_\Delta |X|^{-\Delta}\, A_{[\Delta]I}, \quad A_{[\Delta]I} = \hat{e}^\mu_I\, A_{[\Delta]\mu}. \tag{A.14}$$

Note the change to Δ-weight of $|X|$. The tangential projection operator, $\hat{\mathbb{t}}$, can be used to project out the radially dependent part of the field, χ. This can be similarly extended to symmetric 2-tensors, in spherical

$$\mathbb{T}_{\mu\nu} = \oint_\Delta |X|^{-(\Delta-2)}\, T_{[\Delta]\mu\nu}, \quad \mathbb{T}_{\mu t} = \oint_\Delta |X|^{-(\Delta-2)}\, \xi_{[\Delta]\mu}, \quad \mathbb{T}_{tt} = \oint_\Delta |X|^{-(\Delta-2)}\, \chi_{[\Delta]} \tag{A.15}$$

and flat coordinates

$$\mathbb{T}_{IJ} = \oint_\Delta |X|^{-\Delta} \left(\frac{X_I X_J}{|X|^2}\, \chi_{[\Delta]} + \frac{X_I}{|X|}\, \xi_{[\Delta]J} + \frac{X_J}{|X|}\, \xi_{[\Delta]I} + T_{[\Delta]IJ} \right). \tag{A.16}$$

A.3 Notation and Commonly Used Relations

Table A.1: Notation

$\mathbb{D}-1=D=d+1$	dimensions
$S^{\mathbb{D}-1}=S^{D}=S^{d+1}$	$d+1$ dimensional sphere of unit radius
$\mathrm{d}S_{d+1}$	$d+1$ dimensional de Sitter space
$\mathbb{R}^n : X=(X^1,\cdots X^n)$ $\mathbb{R}^{\mathbb{D}} \cong \mathbb{R}^* \times S^{d+1}$	Flat euclidean space with metric $\mathrm{d}s^2=\delta_{ij}\,\mathrm{d}X^i\,\mathrm{d}X^j$ $\mathbb{D}$ dimensional euclidean space serving as embedding space for S^{d+1}
$\Lambda,\ \ell$	Cosmological constant, Length scale
$\Delta,\ \bar{\Delta}$	Mass parameters related by $d=\Delta+\bar{\Delta}$
$\int_{\sigma}^{*},\ \fint_{\sigma}^{*},\ \intx,\ \intx_{\sigma}^{*}$ $\int_t$ $\oint_{\Delta}$	$\int_0^\infty \frac{\mathrm{d}\sigma}{\sigma},\ \int_0^1 \frac{\mathrm{d}\sigma}{\sigma\,(1-\sigma)},\ \frac{1}{\mathrm{Vol}\,\mathbb{R}^*}\int,\ \frac{1}{\mathrm{Vol}\,\mathbb{R}^*}\int_0^\infty \frac{\mathrm{d}\sigma}{\sigma}$ $\int_{\lambda}^{*}=\fint_{s}^{*},\quad s=\frac{\lambda}{\lambda+1},\quad \lambda=\frac{s}{1-s}$ $\int_{-\infty}^{\infty}$ $\mathrm{d}t$ when the range of integration is obvious in context $\int_{\delta-i\infty}^{\delta+i\infty}\frac{\mathrm{d}\Delta}{2\pi i}$, where $\delta\in\mathbb{R}$ belongs to a range making the integral convergent.
$\sigma,\ \sigma_{1,2},\ \theta,\ \mathtt{w},\ w$	$\hat{X}\cdot\hat{Y},\ \hat{X}_1\cdot\hat{X}_2,\ \cos^{-1}(\hat{X}\cdot\hat{Y}),\ \frac{1+\hat{X}\cdot\hat{Y}}{2},\ \sqrt{\frac{1+\hat{X}\cdot\hat{Y}}{2}}$

Table A.2: Parameterisations and Commonly Used Formulae

Schwinger parameterisation	$\frac{1}{f^s} = \int_0^\infty \frac{\mathrm{d}z}{z} \frac{z^s}{\Gamma(s)} e^{-z f}$
Feynman parameterisation	$\frac{1}{f_1^{s_1} \cdots f_n^{s_n}} = \int_0^1 \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_n}{y_n} \frac{y_1^{s_1}}{\Gamma(s_1)} \cdots \frac{y_n^{s_n}}{\Gamma(s_n)} \frac{\Gamma(\sum s_i)\, \delta(1-\sum y_i)}{(\sum y_i\, f_i)^{\sum s_i}}$ $= \frac{\Gamma(\sum s)}{\Gamma(s_1)\cdots\Gamma(s_n)} \int_0^\infty \frac{\mathrm{d}y_2}{y_2} \cdots \frac{\mathrm{d}y_n}{y_n} \frac{y_2^{s_2}\cdots y_n^{s_n}}{(f_1+y_2\, f_2\cdots+y_n\, f_n)^{\sum s_i}}$
Mellin transform Inverse Mellin transform	$\mathcal{M} \circ f\,(\Delta) = \int_r^* r^\Delta\, f(r)$ $f(r) = \oint_\Delta r^{-\Delta}\, \mathcal{M} \circ f\,(\Delta), \quad \oint_\Delta = \int_{\delta-i\infty}^{\delta+i\infty} \frac{\mathrm{d}\Delta}{2\pi i}$
Mellin representation of Γ function	$\Gamma(z) = \int^* s^z\, e^{-s} = \mathcal{M} \circ e^{-x}$
Euler's reflection	$\Gamma(1-z)\,\Gamma(z) = \pi\, \csc(\pi\, z)$
Legendre duplication	$\Gamma(z)\,\Gamma(z+\frac{1}{2}) = \frac{2\sqrt{\pi}}{2^{2z}}\,\Gamma(2z)$
Limiting value of ${}_2F_1$	${}_2F_1(a,\, b;\, c;\, 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0$

Appendix B: Sphere Propagators

B.1 Position Space Forms of Scalar Propagators

Table B.1: Scalar propagators on S^{odd} at geodesic distance θ with mass parameter $\Delta = \frac{d}{2} + \mathrm{i}\,\nu$

$S^{2\,n+1}$	Scalar Propagator
S^1	$-\frac{\cos(\Delta(\pi-\theta))}{4\,\Delta\,\sin(\pi\Delta)} = \frac{\cosh((\pi-\theta)\,\nu)}{4\,\nu\,\sinh(\pi\nu)}$
S^3	$\frac{\sin(\Delta(\pi-\theta)+\theta)}{4\pi\,\sin(\pi\Delta)\,\sin\theta} = \frac{\sinh((\pi-\theta)\,\nu)}{4\pi\,\sinh(\pi\nu)\,\sin\theta}$
S^5	$\frac{3\big((\Delta-1)\sin(\Delta(\pi-\theta)+3\theta)-(\Delta-3)\sin(\Delta(\pi-\theta)+\theta)\big)}{8\pi^2\,\sin(\pi\Delta)\,\sin^3\theta}$ $= \frac{3\big(\nu\,\cosh((\pi-\theta)\,\nu)\,\sin\theta+\sinh((\pi-\theta)\,\nu)\,\cos\theta\big)}{4\pi^2\,\sinh(\pi\nu)\,\sin^3\theta}$
S^7	$\frac{15\big((\Delta-1)\,(\Delta-2)\,\cos 4\theta-3\,(\Delta-1)\,(\Delta-4)\,\cos 2\theta+2(\Delta-2)\,(\Delta-4)\big)}{8\pi^3\,\sin(\pi\Delta)\,\sin^5\theta}\,\sin(\Delta\,(\pi-\theta))\,\cos\theta$ $+\frac{15\big((\Delta-1)\,(\Delta-2)\,\cos 4\theta-(\Delta-1)\,(\Delta-8)\,\cos 2\theta+6\big)}{8\pi^3\,\sin(\pi\Delta)\,\sin^5\theta}\,\cos(\Delta\,(\pi-\theta))\,\sin\theta$ $= \frac{15\Big(3\,\nu\,\sin 2\theta\,\cosh((\pi-\theta)\,\nu)+((2-\nu^2)\,\cos 2\theta+\nu^2+4)\,\sinh((\pi-\theta)\,\nu)\Big)}{8\pi^3\,\sinh(\pi\nu)\,\sin^5\theta}$

Table B.2: Scalar propagators on S^{even} at geodesic distance θ at some mass parameters Δ

S^{2n}	Δ	Scalar Propagator
S^2	$\frac{1}{2}$, $\frac{3}{2}$	$\frac{K(\mathtt{w})}{2\pi^{\frac{3}{2}}}, \frac{K(\mathtt{w})-2\,E(\mathtt{w})}{2\pi^{\frac{3}{2}}}$
S^4	$\frac{1}{2}$, $\frac{3}{2}$	$\frac{\tan\frac{\theta}{2}\,K(\mathtt{w})+2\,\cot\theta\,E(\mathtt{w})}{2\pi^{\frac{5}{2}}\,\sin\theta}, \frac{E(\mathtt{w})-\sin^2\frac{\theta}{2}\,K(\mathtt{w})}{\pi^{\frac{5}{2}}\,\sin^2\theta}$
S^6	$\frac{3}{2}$	$\frac{4\left(2\,(1-\mathtt{w}\,(1-\mathtt{w}))\,E(\mathtt{w})-(2-\mathtt{w})\,(1-\mathtt{w})\,K(\mathtt{w})\right)}{\pi^{\frac{7}{2}}\,\sin^4\theta}$
	$\frac{5}{2}$	$\frac{4\left((2-3\,\mathtt{w})\,(1-\mathtt{w})\,K(\mathtt{w})-2\,(1-2\,\mathtt{w})\,E(\mathtt{w})\right)}{\pi^{\frac{7}{2}}\,\sin^4\theta}$

Euler transformations of the scalar propagator $G(\hat{X},\,\hat{Y})$:

$$\begin{aligned}
G &\to \frac{\Gamma(\Delta)\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})}\,\frac{1}{(1-\mathtt{w})^{\frac{d-1}{2}}}\,{}_2F_1(\tfrac{\bar{\Delta}-\Delta+1}{2},\,\tfrac{\Delta-\bar{\Delta}+1}{2};\,\tfrac{d+1}{2};\,\mathtt{w})\\
G &\to \frac{\Gamma(\Delta)\Gamma(d-\Delta)\Gamma(\frac{1-d}{2})}{(4\pi)^{\frac{d+1}{2}}\,\Gamma(\frac{\bar{\Delta}-\Delta+1}{2})\,\Gamma(\frac{\Delta-\bar{\Delta}+1}{2})}\,{}_2F_1(\Delta,\,d-\Delta;\,\tfrac{d+1}{2};\,1-\mathtt{w})\\
&\quad+\frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}}\,\frac{1}{(1-\mathtt{w})^{\frac{d-1}{2}}}\,{}_2F_1(\tfrac{\bar{\Delta}-\Delta+1}{2},\,\tfrac{\Delta-\bar{\Delta}+1}{2};\,\tfrac{3-d}{2};\,1-\mathtt{w}).
\end{aligned}\tag{B.1}$$

Pfaff transformation of the scalar propagator $G(\hat{X},\,\hat{Y})$:

$$\text{Pfaff}:\;G\to\frac{\Gamma(\Delta)\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})}\,\frac{1}{(1-\mathtt{w})^{\Delta}}\,{}_2F_1(\Delta,\,\tfrac{\Delta-\bar{\Delta}+1}{2};\,\tfrac{d+1}{2};\,\tfrac{\sigma+1}{\sigma-1}).\tag{B.2}$$

Appendix C: Feynman Integrals

C.1 Eigenmodes of the Laplacian on the Sphere in Embedding Space

Spin-s symmetric transverse traceless (STT) fields, forming the eigenmodes of the Laplacian on S^{d+1}, can be represented by radially normalised STT harmonics in embedding space $\mathbb{R}^{\mathbb{D}}$, as

$$\Psi^{s,n}_{p,q}(X,\,U) = \frac{|pX|^n}{n!\,|X|^n}\frac{\big(|pX|\,|qU| - |pU|\,|qX|\big)^s}{s!\,|X|^s}, \tag{C.1}$$

symmetrised by defining the eigenmodes to be $\Psi^{s,n}_{p,q\,a_1\cdots a_s}(X) = \partial_{U^{a_1}}\cdots\partial_{U^{a_s}}\,\Psi(X,\,U)$. The required conditions of transversality, tracelessness, tangentiality and harmonicity of the underlying unnormalised eigenmode

$$\begin{aligned} &\text{transverse} \quad \partial_U\cdot\partial_X\,\Psi = 0, \quad \text{traceless:} \quad \partial_U^2\,\Psi = 0 \\ &\text{tangential:} \quad X\cdot\partial_U\,\Psi = 0, \quad \text{harmonic :} \quad \partial_X^2\left(|X|^{n+s}\,\Psi\right) = 0 \end{aligned} \tag{C.2}$$

are satisfied when $\mathbb{D}$-dimensional vectors $(p,\,q)$, $p\sim\mathbb{C}^*\,p,\;q\sim\mathbb{C}^*\,q$, are orthogonal and light-like $p^2 = q^2 = p\cdot q = 0$. Using a mode generating function $\Psi\!\!\!\Psi$

$$\Psi\!\!\!\Psi_{p,q}(X,\,U) = \sum_n\sum_s \Psi_{p,q}(X,\,U) = e^{\frac{|pX|}{|X|}}\,e^{\frac{|pX|\,|qU|-|pU|\,|qX|}{|X|}} \tag{C.3}$$

to compare the identically equal objects:

$$\sum_{n,n',s,s'} \fint_X^{\mathbb{R}^{\mathbb{D}}}\int_U e^{-U^2}\Psi^*_{p,q}(X,\,U)\,\Psi_{p',q'}(X,\,U) = \fint_X^{\mathbb{R}^{\mathbb{D}}}\int_U e^{-U^2}\,\Psi\!\!\!\Psi^*_{p,q}(X,\,U)\,\Psi\!\!\!\Psi_{p',q'}(X,\,U), \tag{C.4}$$

the overlap of these eigenmodes is found to be:

$$\begin{gathered}\left\langle \Psi^{s,n}_{p,q}\, \Psi^{s',n'}_{p',q'} \right\rangle = \delta_{ss'}\, \delta_{nn'}\, \frac{2\pi^{\frac{\mathbb{D}}{2}}}{\Gamma(\frac{\mathbb{D}}{2}+n+s)}\, \frac{(n+s+1)!\, s!}{2^{n+s}\,(n+1)!\, n!}\, \Lambda^s\, (\bar{p}\cdot p')^n \\ \Lambda = (\bar{q}\cdot q')\,(\bar{p}\cdot p') - (\bar{q}\cdot p')\,(q'\cdot\bar{p}).\end{gathered} \tag{C.5}$$

Longitudinal spin-s eigenmodes modes are derivatives of transverse fields and are given by

$$\Upsilon^{s,n}_{p,q}(X,\, U) = |X|^{s-r}\, (U\cdot\partial_X)^{s-r}\, \Psi^{r,n}_{p,q}(X,\, U), \quad \forall\, r < s, \quad n \geqslant (s-r). \tag{C.6}$$

For example, longitudinal vector eigenmodes are

$$\Upsilon^{1,n}_{p}(X,\, U) = \frac{|pX|^{n-1}}{(n-1)!\, |X|^{n-1}} \left(|pU| - \frac{|pX|\, |XU|}{|X|^2} \right). \tag{C.7}$$

C.1.1 Verification of Scalar Propagator in Embedding Space Formalism

Integrating eigenmodes with the scalar propagator can be used to verify eq. (3.9). A sum these eigenmode integrals can be modelled by the mode integral

$$\begin{aligned}\Theta &\equiv \sum_{n,n'} \Gamma(\tfrac{\mathbb{D}+n-\bar{\Delta}}{2})\, \Gamma(\tfrac{\mathbb{D}+n'-\Delta}{2})\, \langle \bar{\Psi}^{(n)}_p(X)\, G(X,\, Y)\, \Psi^{(n)}_{p'}(Y)\rangle \\ &= \frac{1}{4\pi^{\mathbb{D}}} \fint^{*} \int_W \alpha_X^{\frac{\mathbb{D}-\bar{\Delta}}{2}}\, \alpha_Y^{\frac{\mathbb{D}-\Delta}{2}}\, \tau\, \lambda^{\bar{\Delta}}\, \mu^{\Delta} e^{-f(W)} \sum_{n,n'} \frac{(\sqrt{\alpha_X}\, \bar{p}\cdot X)^n}{n!}\, \frac{(\sqrt{\alpha_Y}\, p'\cdot Y)^{n'}}{n'!} \\ &= 2\pi^{\frac{\mathbb{D}}{2}} \sum_n \frac{(\bar{p}\cdot p')^n}{2^{n+2}\, n!\, \Gamma(\frac{\mathbb{D}}{2}+n)}\, \Gamma(\tfrac{\bar{\Delta}}{2}+n)\, \Gamma(\tfrac{\Delta}{2}+n)\end{aligned} \tag{C.8}$$

where

$$f(W \equiv \{P,\, X,\, Y\}) = \alpha_X\, X^2 + \alpha_Y\, Y^2 + \tau\, P^2 + 2\mathrm{i}\; P\, (\lambda\, X - \mu\, Y). \tag{C.9}$$

Using eq. (C.5) to normalise Θ, it can be confirmed that

$$\langle \bar{\Psi}^{(n)}_p(X)\, G(X,\, Y)\, \Psi^{(n)}_{p'}(Y)\rangle = \frac{1}{(n+\Delta)(n'+\bar{\Delta})}\, \langle \bar{\Psi}^{(n)}_p(X)\, \Psi^{(n)}_{p'}(X)\rangle. \tag{C.10}$$

C.1.2 Verification of Vector Propagator in Embedding Space Formalism

Similarly, the massive and massless vector propagator eigenvalue equations, eq. (3.48), can be verified too, but with much greater tedium. Given some operator $\mathcal{O}^{IJ}$, the normalised inner product

$$\langle \mathcal{O} \rangle = \frac{\langle \bar{\Psi}^{(n)}_{p}{}_{I}(X)\, \mathcal{O}^{IJ} \circ G(X,\, Y)\, \Psi^{(n)}_{p'}{}_{J}(Y) \rangle}{\langle \bar{\Psi}^{(n)}_{p}{}_{I}(X)\, \Psi^{(n)I}_{p'}(X) \rangle} \tag{C.11}$$

with transverse $\langle \mathcal{O} \rangle_{\mathrm{T}}$ and longitudinal $\langle \mathcal{O} \rangle_{\mathrm{L}}$ vector eigenmodes is given by

$\mathcal{O}^{IJ}$	$\langle \mathcal{O} \rangle_{\mathrm{T}}$	$\langle \mathcal{O} \rangle_{\mathrm{L}}$
$4\,\lambda\,\mu\,\vert X\vert\,\vert Y\vert\, P^I\, P^J$	0	$\frac{n(n+d)}{(n+\bar{\Delta})(n+\Delta)}$
$\frac{P^I\, P^J}{P^2}$	0	$\frac{n(n+d)}{\kappa_{n,\,\Delta}\,\kappa_{n,\,\bar{\Delta}}}$
$\delta^{IJ} \cong \frac{4\,\lambda\,\mu\,\vert PX\vert\,\vert PY\vert}{\Delta\,\bar{\Delta}}\delta^{IJ}$	$(\lambda^{(1)}_{n,\Delta})^{-1}$	$\frac{(\Delta+1)\,(\bar{\Delta}+1)+n\,(n+d)}{\kappa_{n,\,\Delta}\,\kappa_{n,\,\bar{\Delta}}}$
$4\,\lambda\,\mu\,\vert XY\vert\, P^I\, P^J$	$(\lambda^{(1)}_{n,\Delta})^{-1}$	$\frac{(\Delta+1)\,(\bar{\Delta}+1)+n\,(n+d)(n\,(n+d)+\bar{\Delta}\,\Delta-d-2)}{\kappa_{n,\,\Delta}\,\kappa_{n,\,\bar{\Delta}}}$
$\frac{4\,\lambda\,\mu\,\vert PX\vert}{\bar{\Delta}}\, Y^I\, P^J = \frac{4\,\lambda\,\mu\,\vert PY\vert}{\Delta}\, X^J\, P^I$	$(\lambda^{(1)}_{n,\Delta})^{-1}$	$\frac{(\bar{\Delta}+1)\,(\Delta+1)-n\,(n+d)}{\kappa_{n,\,\Delta}\,\kappa_{n,\,\bar{\Delta}}}$

(C.12)

where

$$\lambda^{(1)}_{n,\Delta} := (n + \Delta + 1)\,(n + d - \Delta + 1), \qquad \kappa_{n,\,\Delta} := (n + \Delta + 1)\,(n + \Delta - 1). \tag{C.13}$$

Using these results, the massive vector propagator is found to have the eigenvalues, $\frac{1}{\lambda^{(1)}_{n,\Delta}}$ for transverse modes and $\frac{1}{(\Delta-1)\,(\bar{\Delta}-1)}$ for longitudinal modes, satisfying expectations. The massless vector propagator has the eigenvalues $\frac{1}{(n+2)\,(n+d)}$ and $\frac{d}{(d-2)}\,\frac{1}{n\,(n+d)}$ for transverse and longitudinal modes respectively, once again matching the requirements.

C.2 Series Solutions to Feynman Integrals

C.2.1 2 Propagator Melon: Series

Summing over t_1 of the series solution:

$$\phi_1 = \frac{\Gamma\left(\frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}\right)\Gamma\left(-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)\Gamma\left(t_3+\frac{\Delta_2}{2}\right)\Gamma\left(t_2+t_3+\frac{\Delta_1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\bar{\Delta}_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+1)\Gamma\left(t_2-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+t_2} z_4^{t_2} z_7^{-\frac{\Delta_1}{2}-t_2-t_3} z_8^{2t_3} \omega_1^{\frac{\Delta_2}{2}+t_3}\, {}_2F_1\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\bar{\Delta}_2}{2}, t_3+\frac{\Delta_2}{2}; -\frac{\bar{\Delta}_1}{2}+\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_2 = \frac{\Gamma\left(\frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)\Gamma\left(\frac{\Delta_1}{2}-\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_3+\frac{\Delta_2}{2}\right)\Gamma\left(t_2+t_3+\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\Delta_1}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+1)\Gamma\left(t_2+\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{t_2} z_4^{\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+t_2} z_7^{-\frac{\bar{\Delta}_2}{2}-t_2-t_3} z_8^{2t_3} \omega_1^{\frac{\Delta_2}{2}+t_3}\, {}_2F_1\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\Delta_1}{2}, t_3+\frac{\Delta_2}{2}; -\frac{\bar{\Delta}_1}{2}+\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_3 = \frac{\Gamma\left(\frac{\Delta_2}{2}-\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}\right)\Gamma\left(-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)\Gamma\left(t_3+\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(t_2+t_3+\frac{\Delta_1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_2}{2}-\frac{\Delta_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+1)\Gamma\left(t_2-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+t_2} z_4^{t_2} z_7^{-\frac{\Delta_1}{2}-t_2-t_3} z_8^{2t_3} \omega_1^{\frac{\bar{\Delta}_1}{2}+t_3}\, {}_2F_1\left(t_3+\frac{\bar{\Delta}_1}{2}, t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_2}{2}-\frac{\Delta_2}{2}; \frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_4 = \frac{\Gamma\left(\frac{\Delta_2}{2}-\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)\Gamma\left(\frac{\Delta_1}{2}-\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_3+\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(t_2+t_3+\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+1)\Gamma\left(t_2+\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{t_2} z_4^{\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+t_2} z_7^{-\frac{\bar{\Delta}_2}{2}-t_2-t_3} z_8^{2t_3} \omega_1^{\frac{\bar{\Delta}_1}{2}+t_3}\, {}_2F_1\left(t_3+\frac{\bar{\Delta}_1}{2}, t_2+\frac{\mathbb{D}}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}; \frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_5 = -\frac{\Gamma\left(\frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}\right)\Gamma\left(-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)\Gamma\left(t_3+\frac{\Delta_2}{2}+\frac{1}{2}\right)\Gamma\left(t_2+t_3+\frac{\Delta_1}{2}+\frac{1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\bar{\Delta}_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+2)\Gamma\left(t_2-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+t_2} z_4^{t_2} z_7^{-\frac{\Delta_1}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \omega_1^{\frac{\Delta_2}{2}+t_3+\frac{1}{2}}\, {}_2F_1\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\bar{\Delta}_2}{2}, t_3+\frac{\Delta_2}{2}+\frac{1}{2}; -\frac{\bar{\Delta}_1}{2}+\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_6 = -\frac{\Gamma\left(\frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)\Gamma\left(\frac{\Delta_1}{2}-\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_3+\frac{\Delta_2}{2}+\frac{1}{2}\right)\Gamma\left(t_2+t_3+\frac{\bar{\Delta}_2}{2}+\frac{1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\Delta_1}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+2)\Gamma\left(t_2+\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{t_2} z_4^{\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+t_2} z_7^{-\frac{\bar{\Delta}_2}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \omega_1^{\frac{\Delta_2}{2}+t_3+\frac{1}{2}}\, {}_2F_1\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_1}{2}-\frac{\Delta_1}{2}, t_3+\frac{\Delta_2}{2}+\frac{1}{2}; -\frac{\bar{\Delta}_1}{2}+\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_7 = -\frac{\Gamma\left(\frac{\Delta_2}{2}-\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}\right)\Gamma\left(-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)\Gamma\left(t_3+\frac{\bar{\Delta}_1}{2}+\frac{1}{2}\right)\Gamma\left(t_2+t_3+\frac{\Delta_1}{2}+\frac{1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_2}{2}-\frac{\Delta_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+2)\Gamma\left(t_2-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{-\frac{\bar{\Delta}_2}{2}+\frac{\Delta_1}{2}+t_2} z_4^{t_2} z_7^{-\frac{\Delta_1}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \omega_1^{\frac{\bar{\Delta}_1}{2}+t_3+\frac{1}{2}}\, {}_2F_1\left(t_3+\frac{\bar{\Delta}_1}{2}+\frac{1}{2}, t_2+\frac{\mathbb{D}}{2}-\frac{\bar{\Delta}_2}{2}-\frac{\Delta_2}{2}; \frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}+1; \omega_1\right)$$

$$\phi_8 = -\frac{\Gamma\left(\frac{\Delta_2}{2}-\frac{\bar{\Delta}_1}{2}\right)\Gamma\left(\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)\Gamma\left(\frac{\Delta_1}{2}-\frac{\bar{\Delta}_2}{2}\right)\Gamma\left(t_3+\frac{\bar{\Delta}_1}{2}+\frac{1}{2}\right)\Gamma\left(t_2+t_3+\frac{\bar{\Delta}_2}{2}+\frac{1}{2}\right)\Gamma\left(t_2+\frac{\mathbb{D}}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}\right)}{16\Gamma\left(\frac{\mathbb{D}}{2}\right)\Gamma(t_2+1)\Gamma(2t_3+2)\Gamma\left(t_2+\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+1\right)}$$

$$\times\, z_3^{t_2} z_4^{\frac{\bar{\Delta}_2}{2}-\frac{\Delta_1}{2}+t_2} z_7^{-\frac{\bar{\Delta}_2}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \omega_1^{\frac{\bar{\Delta}_1}{2}+t_3+\frac{1}{2}}\, {}_2F_1\left(t_3+\frac{\bar{\Delta}_1}{2}+\frac{1}{2}, t_2+\frac{\mathbb{D}}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}; \frac{\bar{\Delta}_1}{2}-\frac{\Delta_2}{2}+1; \omega_1\right).$$

Setting $\omega_1 = 1$ and expanding the ${}_2F_1$ functions into convergent and divergent parts, the divergent parts cancel against each other, reducing to

$$
\begin{aligned}
\phi_{1+3} &= z_3^{\frac{i\nu_1}{2}+\frac{i\nu_2}{2}+t_2} z_4^{t_2} z_7^{-\frac{d}{4}-\frac{i\nu_1}{2}-t_2-t_3} z_8^{2t_3} \frac{\Gamma\left(-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}\right)\Gamma\left(\frac{i\nu_1}{2}+\frac{i\nu_2}{2}+1\right)\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}+t_2+t_3+\frac{i\nu_1}{2}\right)}{16\Gamma\left(\frac{d}{2}+1\right)\Gamma(2t_3+1)\Gamma\left(\frac{d}{4}+t_2+t_3+\frac{i\nu_2}{2}+1\right)} \\
\phi_{2+4} &= z_3^{t_2} z_4^{-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}+t_2} z_7^{-\frac{d}{4}+\frac{i\nu_2}{2}-t_2-t_3} z_8^{2t_3} \frac{\Gamma\left(-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}+1\right)\Gamma\left(\frac{i\nu_1}{2}+\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}+t_2+t_3-\frac{i\nu_2}{2}\right)}{16\Gamma\left(\frac{d}{2}+1\right)\Gamma(2t_3+1)\Gamma\left(\frac{d}{4}+t_2+t_3-\frac{i\nu_1}{2}+1\right)} \\
\phi_{5+7} &= -z_3^{\frac{i\nu_1}{2}+\frac{i\nu_2}{2}+t_2} z_4^{t_2} z_7^{-\frac{d}{4}-\frac{i\nu_1}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \frac{\Gamma\left(-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}\right)\Gamma\left(\frac{i\nu_1}{2}+\frac{i\nu_2}{2}+1\right)\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_2+t_3+\frac{i\nu_1}{2}+\frac{1}{2}\right)}{16\Gamma\left(\frac{d}{2}+1\right)\Gamma(2t_3+2)\Gamma\left(\frac{d}{4}+t_2+t_3+\frac{i\nu_2}{2}+\frac{3}{2}\right)} \\
\phi_{6+8} &= -z_3^{t_2} z_4^{-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}+t_2} z_7^{-\frac{d}{4}+\frac{i\nu_2}{2}-t_2-t_3-\frac{1}{2}} z_8^{2t_3+1} \frac{\Gamma\left(-\frac{i\nu_1}{2}-\frac{i\nu_2}{2}+1\right)\Gamma\left(\frac{i\nu_1}{2}+\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_2+t_3-\frac{i\nu_2}{2}+\frac{1}{2}\right)}{16\Gamma\left(\frac{d}{2}+1\right)\Gamma(2t_3+2)\Gamma\left(\frac{d}{4}+t_2+t_3-\frac{i\nu_1}{2}+\frac{3}{2}\right)}.
\end{aligned}
$$

Then summing over t_2 and once again following the same process, the series reduce to

$$
\begin{aligned}
\phi_{1+2+3+4} &= \frac{\pi\,\mathrm{csch}(\pi\nu_+) z_8^{2t_3}}{8\Gamma\left(\frac{d}{2}+1\right)(\nu_1-\nu_2)} \frac{\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}\right)-\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_1}{2}\right)}{\Gamma(2t_3+1)} \\
\phi_{5+6+7+8} &= \frac{\pi\,\mathrm{csch}(\pi\nu_+) z_8^{2t_3+1}}{8\Gamma\left(\frac{d}{2}+1\right)(\nu_1-\nu_2)} \frac{\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_1}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_1}{2}+\frac{1}{2}\right)-\Gamma\left(\frac{d}{4}+t_3-\frac{i\nu_2}{2}+\frac{1}{2}\right)\Gamma\left(\frac{d}{4}+t_3+\frac{i\nu_2}{2}+\frac{1}{2}\right)}{\Gamma(2t_3+2)}
\end{aligned}
$$

Finally summing over t_3 results in the stated solution:

$$
\mathcal{I}_{2P} = \frac{\Gamma(-d)\left((\sin\pi\Delta_1+\sin\pi\bar{\Delta}_1)\Gamma(\Delta_1)\Gamma(\bar{\Delta}_1)-(\sin\pi\Delta_2+\sin\pi\bar{\Delta}_2)\Gamma(\Delta_2)\Gamma(\bar{\Delta}_2)\right)}{\pi(\nu_1^2-\nu_2^2)}. \tag{C.14}
$$

C.2.2 Propagator with 1 Vertex Insertion: Root System

The roots used to construct this series solution are

$$
\begin{pmatrix}
\frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}-1 & 0 & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & -\frac{\Delta_2}{2} & 0 \\
-1 & 0 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{\Delta_1}{2} & \frac{\Delta_1}{2}-\frac{d}{2} & 0 \\
-1 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & 0 & \frac{\Delta_2}{2}-\frac{d}{2} & -\frac{\Delta_2}{2} & 0 \\
-\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2}-1 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & 0 & \frac{\Delta_2}{2}-\frac{d}{2} & \frac{\Delta_1}{2}-\frac{d}{2} & 0 \\
\frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2}-1 & 0 & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{\Delta_1}{2}-\frac{1}{2} & -\frac{\Delta_2}{2}-\frac{1}{2} & 1 \\
-1 & 0 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & -\frac{\Delta_1}{2}-\frac{1}{2} & -\frac{d}{2}+\frac{\Delta_1}{2}-\frac{1}{2} & 1 \\
-1 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & -\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2} & 0 & -\frac{d}{2}+\frac{\Delta_2}{2}-\frac{1}{2} & -\frac{\Delta_2}{2}-\frac{1}{2} & 1 \\
-\frac{d}{2}+\frac{\Delta_1}{2}+\frac{\Delta_2}{2}-1 & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & \frac{d}{2}-\frac{\Delta_1}{2}-\frac{\Delta_2}{2} & 0 & 0 & -\frac{d}{2}+\frac{\Delta_2}{2}-\frac{1}{2} & -\frac{d}{2}+\frac{\Delta_1}{2}-\frac{1}{2} & 1
\end{pmatrix}. \tag{C.15}
$$

They were initially taken to be generic by taking $\mathbb{D} = d + 2 + \epsilon$ and the series were evaluated in the limit $\epsilon \to 0$, similar to section C.2.1.