Higher-Order Logical Pluralism as Metaphysics

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Abstract

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Higher-order metaphysics is in full swing. One of its principle aims is to show that higher-order logic can be our foundational metaphysical theory. A foundational metaphysical theory would be a simple, powerful, systematic theory which would ground all of our metaphysical theories from modality, to grounding, to essence, and so on. A satisfactory account of its epistemology would in turn yield a satisfactory epistemology of these theories. And it would function as the final court of appeals for metaphysical questions. It would play the role for our metaphysical community that ZFC plays for the mathematical community.

I think there is much promise in this project. There is clear value in having a shared foundational theory to which metaphysicians can appeal. And there is reason to think that higher-order logic can play this role. After all, it has long been known that one can do math in higher-order logic. And there is growing reason to think that one can do metaphysics in higher-order logic in much the same way. However, most of the research approaches higher-order logic from a monist perspective, according to which there is 'one true' higher-order logic. And in the midst of the enthusiasm, metaphysicians seem to have overlooked that this approach leaves the program susceptible to epistemological problems that plague monism about other areas, like set theory. The most significant of these is the Benacerraf Problem. This is the problem of explaining the reliability of our higher-order-logical beliefs. The problem is sufficiently serious that, in the set-theoretic

case, it has led to a reconception of the foundations of mathematics, known as pluralism.

In this dissertation I investigate a pluralist approach to higher-order metaphysics. The basic idea is that any higher-order logic which can play the role of our foundational metaphysical theory correctly describes the metaphysical structure of the world, in much the way that the set-theoretic pluralist maintains that any set theory which can play the role of our foundational mathematical theory is true of a mind-independent platonic universe of sets. I outline my view about what it takes for a higher-order logic to play this role, what it means for such a logic to correctly describe the metaphysical structure of the world, and how it is that different higher-order logics which seem to disagree with each other can meet both of these conditions. I conclude that higher-order logical pluralism is the most tenable version of the higher-order logic as metaphysics program.

Higher-order logical pluralism constitutes a radical departure from conventional wisdom, requiring a significant reconception of the nature of validity, modality, and metaphysics in general. It renders moot some of the most central questions in these domains, such as: Is the law of excluded middle valid? Is it the case that necessarily everything is necessarily something? Is the grounding relation transitive? On this picture, these questions no longer have objective answers. They become like the question of whether the Continuum Hypothesis is true, according to the set-theoretic pluralist. The only significant question in the neighborhood of the aforementioned questions is: which metaphysical principles are best suited to the task at hand?

Scope and Overview

In the time of Frege and Russell higher-order logic was the dominant form of logic. But as the twentieth century progressed, in no small part due to Quine's seminal criticisms, first-order logic came to prominence.[1] Higher-order quantification was thought to be unintelligible at worst, and set-theory in disguise at best. And while higher-order logic has nonetheless played a significant role in a variety of significant philosophical projects in the interim (see for instance Gallin [1975]), in the past fifteen years or so higher-order logic has experienced a vibrant renaissance in metaphysical theorizing. One particularly interesting line of thought is that as a result of both the expressive power and simplicity of higher-order logic, it it might be able to occupy a seemingly much needed foundational role in our overall metaphysical view of the world.

Putting the point provocatively we might think that modern metaphysics is in an interestingly similar situation to classical mathematics before the advent and acceptance of set theory. Think of the state of the practice of analysis in the 18th century before the work of Cauchy and Weierstrass. The subject is rife with seemingly unrelated areas of inquiry with no common background theory to which they appeal. There are myriad different metaphysical theories from theories of properties, to propositions, to facts, to events, to functions, to ordered pairs, to classes, to kinds of necessity, and so on, which are studied in isolation, without a common background set of metaphysical axioms. In response to this situation, we might think that adopting higher-order logic as our common background theory may ameliorate this situation, in much the same way that adopting set theory did for the mathematicians.

[1] See for instance, Quine [1970].

Think of the role which ZFC is commonly thought to play. ZFC interprets all of our classical mathematical theories. It interprets geometry, arithmetic, real analysis, and so on. In this way ZFC provides a robust theoretical foundation for all of classical mathematics. It provides a simple and powerful common background set of mathematical assumptions. It functions as the final court of appeals for mathematical questions. And it provides all of the mathematical resources we need to construct our best theories about the world.

Analogously, a foundational metaphysical theory would be a simple, powerful, systematic theory which would ground all of our other metaphysical theories, just as set theory grounds all of our other mathematical theories. We would be able to interpret our various metaphysical theories in it. It would provide a simple and powerful common background set of metaphysical assumptions. It would function as the final court of appeals for metaphysical questions. And it would provide all of the metaphysical resources we need to construct our best theories about the world.

There is a growing train of thought that higher-order logic can occupy this role. There have already been some really interesting strides made towards interpreting our theories of modality[2], grounding[3], essence[4], and validity[5], in higher-order logic. And there is seemingly no reason to think that things will not continue in this vein.

Moreover, as has long been known, higher-order logic provides a firm foundation for theorizing about mathematics.[6] And so it ought to be clear why there is a sense of excitement emanating from this burgeoning program. Proponents take higher-order logic to promise to be the metaphysicians proverbial paradise.

We share in the earnest excitement surrounding this *higher-order logic as metaphysics* program. We think there is clear value in having a common background theory to which we can appeal in the course of our theorizing – just consider the success of the mathematical community! However, most of the research approaches higher-order logic from a monist perspective, according

[2]See Dorr [2016], Bacon [2018], and Williamson [2013] for some good examples.

[3]See Fritz [2020], and [2021].

[4]See Ditter [2022].

[5]See Williamson [2013] - particularly the discussion of metaphysical universality.

[6]For instance, see Russell and Whitehead [1910], Church [1940], and Shapiro [2000].

to which there is 'one true' higher-order logic. And in the midst of the enthusiasm, metaphysicians seem to have overlooked that this approach leaves the program susceptible to epistemological problems that plague monism about other areas, like set theory. The foundational metaphysical project inherits not just the benefits of set-theoretic foundationalism, but also some of the serious problems. The most significant of these is the Benacerraf Problem.[7]

The Benacerraf challenge for a domain D, is the challenge to explain the reliability of our D-beliefs – of showing how it is, freely making use of the assumption that our D-beliefs are both true and justified, that we could not easily have had false D-beliefs (using the method we actually used to form them).[8] There is a strong intuitive pull in favor of needing to meet the challenge. Suppose we cannot even tell an in principle story, freely making use of the assumptions that our beliefs about higher-order logic are both true and defeasibly justified, of how it is that we could not easily have formed false such beliefs by carrying out the same process which actually produced our beliefs about. It would seem then that we were very lucky in coming to believe the true logic. Becoming aware that we are epistemically lucky in this way is a paradigm case of undermining evidence. How would it be different from finding out that expert metaphysician on the basis of whose testimony you formed your logical beliefs, was choosing what to tell you by flipping a coin? If we cannot meet the challenge for a domain, then we have a Benacerraf problem.

The problem is sufficiently serious that, in the set-theoretic case, it has led many to consider a radical reconception of the foundations of mathematics, known as pluralism.[9] The basic thrust of pluralism is that there is not just one universe of sets which is correctly described by one set theory. Rather there are a plurality of universes of sets, exhibiting a great diversity of mathematical features, which are correctly described by different set theories. This is a fascinating and exciting rethinking of the foundations of pure math, which may well afford the pluralist a way to meet the challenge.

Suppose, for instance, you hold the view that every classically consistent set theory is true.

[7] See Benacerraf [1973].

[8] See Benacerraf [1973], Field [1988], and Clarke-Doane [2017].

[9] See Field [1994], Balaguer [1995], Linsky and Zalta [1995], Hamkins [2012], and Priest [In Progress].

Then it may seem straightforward enough to make the case that you could not easily have had a false set-theoretic belief (assuming its truth and justification), as every consistent set theory you might have believed would have been true. In order to believe a false set-theoretic claim, you would have had to believe a claim which is not proved by any consistent set theory. Provided that you think that we could not easily have believed a classically inconsistent set theory, then this would seem to provide an answer to the challenge.[10] As Beall once quipped, if it seems really hard to hit the target, then just make the target a lot bigger![11] It is the heavyduty metaphysical assumptions of pluralism which allow its proponents to explain the reliability of their beliefs.

An epistemically shaky foundation is not one on which we want to build our metaphysical edifices. And insofar as it is beset by a Benacerraf problem, we think that a higher-order logical monism can only give us such a foundation. As such, we think one promising route for the foundationalist about higher-order logic is to give up on the idea that there is one true higher-order logic, and to adopt a kind of pluralism. We think it is reasonable to suppose at the outset that just as in the set-theoretic case, we may be able to use the significant metaphysical assumptions of a higher-order pluralist view to meet the Benacerraf challenge for higher-order logic.

This dissertation comprises four papers on topics concerning higher-order logic and pluralism. They are intended to be self-contained pieces focusing on a philosophically important issue, rather than chapters in a monograph. As such, in places they cover some of the same ground, rather than purely building on each other as chapters would. We have organized them here as best we can to present an overall narrative in which we motivate, locate precedent for, outline, and outline some consequences of, a pluralist approach to higher-order logic.

In the first paper we aim to motivate a pluralist approach to higher-order logic. We begin by outlining the Benacerraf challenge for set theory. As we have said, this is the challenge to give an account of the link between our set-theoretic beliefs and the set-theoretic truths; to give an account of how it is that our set-theoretic beliefs *track* the set-theoretic truths. More precisely, it is the

[10] We are actually somewhat dubious of the way such a view privileges classical logic. But that is currently beside the point.

[11] See Beall [1999] for an elaboration on this point.

challenge to show how it is that our set-theoretic beliefs are reliable, given that we are allowed to assume that our beliefs are both true and defeasibly justified. It is generally accepted that if we cannot meet this challenge, our set-theoretic beliefs are undermined.

After that, we make the case that higher-order logic and set theory make a variety of very similar claims on the world. While the guises in which they appear are different, we claim that they both make claims about the nature of extensional collections and infinite cardinals, for example. We then outline our main claim that in so far as higher-order logic and set theory make these very similar kinds of claims on the world, if there is a Benacerraf problem for set theory, there is also one for higher-order logic. Higher-order logic and set theory are companions in guilt. The gist of our reasoning is that any *solution* to the problem for higher-order logic could be translated into a solution for set theory, and vice versa.

After that, we defend the claim against a variety of potential objections, to the effect that set theory and higher-order logic are different in respects which undermine the companions in guilt thesis. These include that set theory concerns a realm of abstract objects, whereas higher-order logic does not come along with a problematic ontology; that the language of higher-order logic is determinate in a way that the language of first-order set theory is not; and that set theory is a pure theory, while higher-order logic (as it is construed in the higher-order logic as metaphysics program) is an applied theory. And we conclude that, absent an alternative solution to the problem, higher-order logical pluralism, according to which there is no privileged higher-order logic, is the most tenable version of the higher-order logic as metaphysics program.

In the second paper we locate precedent for a pluralist approach to higher-order logic. A pluralist approach to higher-order logic may offer solutions to the epistemological issues facing the foundational project. However, such an approach is very different to most of the current research in this area. As such, we situate the approach in relation to a variety of well-explored views in the philosophy of logic, philosophy of set theory, and metaphysics. These include logical pluralism, multiverse set-theory, the view that the cumulative hierarchy is indefintely extensible, and quantifier variance. Our main point is that the same kinds of reasons which motivate these views also motivate a pluralist approach to higher-order logic. And we also make the case that modulo the intelligibility of higher-order quantification, these views require such an approach.

Another issue we explore is that the literature on these views provides solutions to a variety of prima facie significant problems which face a pluralist approach to higher-order logic. These include issues of normative pluralism stemming from the variety of equally good higher-order consequence relations – it seems that there would be no objective answer to questions concerning the validity of arguments which appear in philosophy, math, and physics; issues regarding the view *collapsing* into monism, which should be familiar from similar worries concerning quantifier variance and set-theoretic pluralism; issues of inexpressibility, which are similar to issues which beset the Tractarian Wittgenstein, and the view that we cannot quantify over absolutely everything; and issues regarding contradiction – an important aspect of pluralism is that there are true higher-order logics which appear to disagree about substantive theorems: how could this be, without lapsing immediately into trivialism?

In the third paper we defend the claim that there is a contentful and coherent pluralist approach to the higher-order logic as metaphysics program. We outline an approach according to which, roughly, any higher-order logic which (i) a community of metaphysicians could use to provide a foundation for their metaphysical practice, (ii) provides all of the metaphysical resources they would ever need in order to construct their best theories about the world, and (iii) is theoretically virtuous, is true. Compare this with a set-theoretic pluralist view according to which any set theory which we could use to play the role of our foundational mathematical theory is true of a mind-independent Platonically existing universe of sets. And we defend the view against the prima facie significant worry that it leads to an untenable degree of pluralism about intuitively non-metaphysical domains, such as physics.

The fourth and final paper is an edited version of a paper co-written with Justin Clarke-Doane which appears in Philosophical Perspectives. We would like to thank them for their kind permission to include this version of the paper here[12]. In the paper, we discuss some of the consequences

[12] See Clarke-Doane and McCarthy [2023].

of higher-order pluralism. In particular, we show how this view defuses an otherwise damning objection to modal pluralism.

We begin by discussing a simple argument that modal metaphysics is misconceived, and responses to it. Unlike Quine's, this argument begins with the simple observation that there are different candidate interpretations of the predicate 'could have been the case'. This is analogous to the observation that there are different candidate interpretations of the predicate 'is a member of'. The argument then infers that the search for metaphysical necessities is misguided in much the way the set-theoretic pluralist claims that the search for the true axioms of set theory is.

We argue that the obvious responses to this argument fail. However, a new response has emerged that purports to prove, from higher-order logical principles, that metaphysical possibility is the broadest kind of possibility applying to propositions, and is to that extent special. We distill two lines of reasoning from the literature, and argue that their import depends on premises that any 'modal pluralist' should deny. Both presuppose a monist approach to higher-order logic. But in the context of higher-order logic, all of the modal pluralist's reasons to eschew modal monism are equally reasons to eschew higher-order logical monism. The modal pluralist already has reasons to adopt a pluralist approach to higher-order logic. We argue that as such, these arguments rest upon premises the modal pluralist can and should reject in a principled manner.

Along the way we address a variety of potential worries, including that, in a higher-order setting, modal pluralism faces an insuperable problem of articulation, collapses into modal monism, is vulnerable to the Russell-Myhill paradox, or even contravenes the truism that there is a unique actual world, and argue that these worries are misplaced. We also sketch two kinds of logical possibility which respectively correspond to consistency in propositional $S4$ and propositional $S5$.

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Chapter 1: From Set-theoretic Pluralism to Higher-Order Logical Pluralism: A Benacerraf Problem for Higher-Order Logic

Abstract

Higher-order metaphysics is in full swing. Higher-order logic is rapidly coming to occupy a central place in metaphysical theorizing. There is good reason to believe it may come to occupy a place in metaphysical theorizing which is very to similar to that which set theory occupies in pure mathematics. It has long been known that one can do math in higher-order logic.[1] This is one of its intriguing and exciting features. Moreover, there is growing reason to think that one can *do* metaphysics in higher-order logic in much the same way.[2] But in the midst of the enthusiasm, metaphysicians seem to have overlooked that these features – particularly the fact that higher-order logic can serve as a foundational mathematical theory – leave the program susceptible to some serious epistemological problems, the most significant of which is the Benacerraf Problem. Roughly speaking, this is the problem of explaining the reliability of our higher-order-logical beliefs. This problem ought to be familiar from the set-theoretic case[3], where it has motivated a reconception of the foundations of mathematics known as pluralism[4]. In this paper, we shall argue that if there is a Benacerraf problem for set theory, there is also one for higher-order logic. They are companions in guilt. We conclude that, absent an alternative solution to the problem, higher-order logical pluralism, according to which there is no privileged higher-order logic, is the most tenable version of the higher-order logic as metaphysics program. The consequences of this conclusion are hard to overstate. Higher-order logical pluralism requires a complete reconception of the nature of metaphysics.

[1]See Russell and Whitehead [1910], and Church [1940].

[2]See Bacon [2018] for an illustration of how this works for modal metaphysics. And Goodman [forthcoming] for a general illustration of the higher-order logic as metaphysics program.

[3]See Benacerraf [1973], Field [1988], Clarke-Doane [2017].

[4]See Field [1994], Balaguer [1995], Linsky and Zalta [1995], and Hamkins [2012].

1.1 Higher-Order Metaphysics

Higher-order logic is experiencing a renaissance in metaphysical theorizing. What is particularly interesting about higher-order logic is the role it seems it might be able to occupy in our overall metaphysical view of the world. Goodsell and Yli-Vakurri have observed that modern metaphysics is in an interestingly similar situation to classical mathematics before the advent and acceptance of ZFC; the subject is rife with seemingly unrelated areas of inquiry with no common background theory to which they appeal. In response to this situation, they contend that higher-order logic has the potential to come to occupy the role of our foundational metaphysical theory. It could come to play the same functional role within metaphysics that ZFC plays in mathematics. In particular, they make this claim for their preferred higher-order logic *Classicism + Choice*.[5] [6]

Think of the role which ZFC is commonly thought to play. ZFC can interpret all of our mathematical theories. ZFC interprets geometry, arithmetic, real analysis, and so on. All of math is either actually just about the cumulative hierarchy, or all of math can be modelled in the cumulative hierarchy. In this way ZFC provides a robust theoretical foundation for all of classical mathematics. This also has an epistemological and a dialectical component. A satisfactory epistemological account of the foundational theory would percolate to all of the theories which depend on it. And it

[5]See Goodsell and Yli-Vakkuri *In Progress*

[6]Classicism is the closure of the common core higher-order logic H under a natural rule of equivalence. H has the following rules and axioms:

- PC: All of the theorems of the propositional calculus
- UI: : $\forall_\sigma x\phi \rightarrow \phi[t/x]$ (where t is a term of type σ and no variable in t gets bound when substituted in ϕ)
- MP: From $A \rightarrow B$ and A, infer B
- Gen: From $A \rightarrow B$, infer $A \rightarrow \forall_\sigma xB$, provided x is not free in A;
- Ref: $A =_\sigma A$;
- LL: $A =_\sigma B \rightarrow (FA \rightarrow FB)$
- $\beta\eta$: $A \leftrightarrow B$, whenever A and B are $\beta\eta$-equivalent (A and B are β-equivalent when A is of the form $\phi[(\lambda v.A)B](x_1, ..., x_n.A)N_1, ..., N_n$, and B is of the form $\phi[A[B/v]][N_1/x_1, ...N_n/x_n]$. And they are η-equivalent when A is of the form $\phi[\lambda v.(Fv)]$, and B is of the form $\phi[F]$.)

Classicism is the logic which results from closing H under the rule E – If $\vdash A \leftrightarrow B$, then $\vdash \lambda v.A = \lambda v.B$. The axiom of Choice is the following: $\exists f\forall X(\exists y(Xy) \rightarrow Xf(X))$. The logic CC – Classicism + Choice – results from adding Choice to Classicism and closing under the rule E.

functions as the final court of appeals for mathematical questions. Whether ZFC actually fulfills this role is a question on which we shall remain neutral. Regardless, the contention is that higher-order logic can play the role for the community of metaphysicians that ZFC is *thought to* play for the mathematicians.

Analogously, a foundational metaphysical theory would be a comprehensive, powerful, systematic theory which would ground all of our other metaphysical theories, just as set theory grounds all of our other mathematical theories. We would be able to use it to interpret our theories of modality, grounding, essence, and so on. A satisfactory account of its epistemology would in turn yield a satisfactory epistemology of these theories. And the it would function as the final court of appeals for metaphysical questions.

There is a growing train of thought, along with Yli-Vakkuri and Goodsell, that higher-order logic can occupy this role. There have already been some really interesting strides made towards interpreting modal theory in higher-order logic.[7] There have also been some made in giving a higher-order account of grounding.[8] Ditter has given a higher-order account of essence.[9] Williamson has done some interesting work in giving a higher-order account of validity.[10] And there is good reason to suspect that this is just the tip of the iceberg.

Higher-order logic also provides a firm foundation for theorizing about mathematics. There are statements in the language of higher-order logic which express the higher-order versions of the axiom of choice, the Continuum Hypothesis, the axiom of replacement, and so on. Various higher-order logics interpret theories of extensional collections, and theories of infinity. Recently Goodsell has shown an elegant way of interpreting arithmetic in the higher-order logic *Classicism + Choice*.[11] And Yli-Vakkuri and Goodsell have shown how to interpret ZFC in the stronger theory, which adds an inaccessible cardinal axiom to *Classicism + Choice*. Given the familiar result that one can interpret all of classical mathematics in ZFC, it follows that this higher-order

[7] See Dorr [2016], Bacon [2018], and Williamson [2013] for some good examples.
[8] See Fritz [2020] and [2021].
[9] See Ditter [2022].
[10] See Williamson [2013] - particularly the discussion of metaphysical universality.
[11] See Goodsell [2022].

logic can interpret all of classical mathematics.

And so it ought to be clear why there is a sense of excitement emanating from this burgeoning program. Proponents take higher-order logic to offer a powerful, elegant, and systematic background theory against which we can formulate all of our metaphysical theories. It would gather all of these disciplines together into one theoretical framework, which would be our metaphysical meta-theory, as it were. We would use it to study and interpret all of our metaphysical theories. It would outline, encompass, and constrain the entire space of metaphysics. They take it to promise to be the metaphysicians proverbial paradise.

For the purposes of this paper we will remain officially neutral about whether this promise will be borne out (though it really does seem promising). Our point will be that if we buy into this promise – if we think that higher order logic can provide a robust foundation for all of our metaphysical theorizing, we face some serious epistemological problems.

1.2 Structural Claims

Higher-order logic is a powerful theory. But this metaphysical strength comes with epistemological cost. As we have gestured towards, higher-order logic can serve as a foundational mathematical theory. We are not claiming that it does this better than set theory. Or that it is the one true foundational mathematical theory. We are just claiming that it can adequately fulfill the role. The relevant aspect of this for us is that we can make seemingly mathematical claims in the language of higher-order logic. We can interpret arithmetic, set theory, real analysis, and so on in higher-order logic. To be clear, we are not claiming that the higher-order logical version of arithmetic and set theory are really about numbers or sets, and more than the proponent of set theory need think that the set-theoretic interpretation of arithmetic is really about numbers. Nor are we more generally claiming that higher-order logic is really set theory in disguise.[12] We are claiming that there is an important similarity between number-theoretic and set-theoretic claims (naively construed) on the one hand, and the higher-order interpretations of these claims on the other. It is the same kind of

[12] See Quine [1970].

similarity which holds between the number-theoretic claims and their set-theoretic interpretation.

Roughly, the idea is that, for most intents and purposes, these kinds of claims make the same demands on the world. Consider arithmetic claims. In particular consider the claim that $1 + 1 = 2$. Naively construed as a number-theoretic claim, this is about abstract entities, relations, and functions. Now, we could really restrict what this claim means, taking it to just be about these particular objects, having no applicability outside of this context. But very few people think of number theory in this way. Much more commonly, it is taken to be applicable to every domain whatsoever. Given this very common attitude that number theory is universally valid, taking this claim to be true involves a commitment to a very general structural feature of the world. And it is this general structural feature which we think can also be expressed in other languages and frameworks. Suppose we endorse the claim that the union of a singleton and another singleton of distinct sets contains exactly two elements. And we think that set theory is universally applicable. Then taking this to be true involves a commitment to a very general structural feature of the world. And we think that this general structural feature is very similar to the feature involved in being committed to the number-theoretic claim. We can make analogous claims in first-order logic, second-order logic, plural logic, and so on. For want of a better term, we shall call these kinds of claims *arithmetic* claims. The point is that if we take arithmetic claims to be universally applicable, then the guise in which they appear is mostly irrelevant. The same point applies to claims about collections, and claims about infinite cardinalities.

We might be tempted to try and formally pin down this similarity. Perhaps as a kind of bi-interpretability relative to ZFC, or some other background theory. Or as a more metaphysically loaded notion such as metaphysically necessary equivalence, or hyperintensional equivalence. We do not want to get bogged down in any such attempt. In our view, none of these kinds of analyses will be satisfactory.

To begin to see this, note that there is a theory of pure Riemannian geometry which is bi-interpretable, relative to ZFC, with the general relativistic theory of physical geometry. But these are not similar in the relevant sense. One is about pure geometric spaces. The other is about physical spacetime. Mere bi-interpretability is not sufficient.

Nor, it seems, is it necessary. It seems that for any very specific formal criterion of bi-interpretability, we can cook up some theories which are not bi-interpretable in this sense, but which nonetheless are relevantly similar. For example, we can articulate very similar geometric claims in a language which only talks about points, and in a language which only talks about lines, even though the theories in question are not strictly speaking, bi-interpretable in a standard sense.[13]

Metaphysically necessary equivalence also misses the mark. Suppose, as several metaphysicians in the higher-order tradition do, that number theory is strictly speaking false, as there are no numbers. But that the natural higher-order interpretations of all of the canonical number-theoretic claims are true. In other words, the structural claims which are expressed by taking number theory to be universally valid are true – there are just no numbers. If this is the case, number theory and higher-order logic are relevantly similar, but they are not metaphysically necessarily equivalent, as one is true, and the other is false. Metaphysically necessary equivalence requires that both theories be true or both false. But being similar in our sense does not require this. The same line of thought applies to hyperintensional equivalence.

Instead we just observe that there clearly is this kind of similarity. In trying to precisely pin it down, we are likely to miss the salient point. We will miss the epistemological forest for the metaphysical trees. We shall call claims which share this similarity with the claims of arithmetic and set theory *structural claims*. So these include number-theoretic and set-theoretic claims, but they also include claims of category theory, cardinal quantificational logic, plural logic, second-order logic, quantified propositional logic, property theories, and higher-order logic in general.

In choosing this name we are not committing ourselves to any kind of structuralist thesis about mathematics. We are not, for instance committing ourselves to the claim that number-theoretic claims are really disguised second-order claims about all omega-sequences.[14] We think that number theory is about numbers, that property theories are about properties, that set theories are about sets, and so on. We are just trying to emphasise the structural similarity between, for instance,

[13] See Barrett and Halvorsen [2017] for details.

[14] For an example of such a view see Hellman [1989].

number-theoretic claims, cardinal quantificational claims, certain set-theoretic claims, and so on, without thinking that this is a formally, or metaphysically very precise point.

1.3 The Benacerraf Problem

The guiding intuition behind this paper is that structural claims, so long as we take them to be objective and mind-independent, stand and fall together with respect to certain crucial epistemological features. Our specific claim is that in so far as we can make a variety of structural claims in the language of higher-order logic which we can also make in set theory, if there is a Benacerraf problem for set theory, there is also one for higher-order logic.[15] We think it is important to note that if we are right, the point would apply more widely than this – we could not solve certain central epistemological challenges for pure math by adopting plural logic, a modal approach, or any other approach according to which there are mind and language independent structural truths.[16] But we shall only focus higher-order logic for the rest of this paper.

The Benacerraf problem ought to familiar from the case of realism about set theory, where it originated. Roughly, it is the challenge to give an account of the link between our set-theoretic beliefs and the set-theoretic truths; to give an account of how it is that our set-theoretic beliefs *track* the set-theoretic truths. It is the challenge to show how it is that we are reliable about the set-theoretic truths.

We might think there are a several, perhaps more obvious epistemological problems which we could raise for higher-order logic. Given their similar role in the foundations of math, we might think we could transpose any of a variety of well-known problems from the philosophy of math to higher-order logic. These include access problems and justification problems. An access problem is roughly the challenge to account for how it is we come into contact with the content of our claims. In the case of pure math, this is the challenge to explain how we come into contact with the objects of the theories – numbers, sets, lines, and so on. And a justification problem is roughly the challenge to account for how it is we can be justified in believing a theory. In the case of pure

[15] See Benacerraf [1973] for the original statement of the problem.

[16] We explicitly disagree then with the general argument in Putnam [1967].

math, this is the challenge to explain how we could be justified in believing ZFC, for instance. It is important to note how this is distinct from, though clearly related to the access problem. We might think that we have a faculty of pure mathematical perception which allows us to interact with mathematical objects, but still not have an account of how it is that beliefs we form using this faculty are justified. Much as we might think that we have a faculty of perception which allows us to access the external world, but still worry about how it is that this generates justified beliefs about the external world.

While both of these are significant challenges which need to be met in order to have an adequate epistemology of higher-order logic, we have chosen to focus our energy on the Benacerraf problem in this paper because we think that reliability, unlike access and justification, is precisely the kind of epistemological feature which is invariant under the various guises in which structural claims appear. This pretty clearly is not the case for access problems. We may have a good account of how we come into contact with the objects of cardinal quantificational logic regular objects – but not for number theory, even though these make the same structural claims. And while justification is a little more tricky, we think it too may vary between different theories which express the same structural claims.

We have chosen to present the problem as a companions in guilt thesis for several reasons. First, it reflects our background view that the epistemology of theories which make structural claims stand and fall together (with respect to reliability). Second, it allows us to draw on the significant literature from the philosophy of math, and brings the metaphysics of higher-order logic into contact with this literature. Third, it is a very easy and natural argument to make. And fourth, given that we think that there is very good reason to think that there is a Benacerraf problem for set theory, the companions in guilt thesis is tantamount to a guilty verdict for higher-order logic.

Let us now outline what we take the Benacerraf problem to be. As we said above, the rough statement of the problem is that it is the challenge to account for how it is that our beliefs about a domain track the truths about the domain. The canonical statement of the problem for pure math can be found in Field [1989]. There are four main steps in Field's account.

1. We grant the truth of our mathematical beliefs. For instance, we assume the truth of ZFC. So this is not a question of truth.

2. We grant that we are defeasibly justified in having these beliefs. For instance, we grant that we are justified in believing the axiom of choice because it is an indispensable theoretical postulate. This is significant. It is explicit in this step that we are not talking about justification. We are granting that our beliefs are justified.

3. We then have to explain the link between our beliefs and the truths, freely making use of the assumptions that our beliefs are true and defeasibly justified. We have to show that the method we used to form those beliefs is reliable. We have to show that having true beliefs was not due merely to luck.

4. If we have reason to think this cannot be done, our beliefs are undermined. If it seems impossible to us to explain how our coming to have the belief that every set has a choice function is a reliable guide to that being the case, we ought to give up that belief.

The challenge for the set-theoretic realist is to show how it is that their set-theoretic beliefs are reliable, given that they are allowed to assume that their beliefs are both true and defeasibly justified. And if they cannot do this, their beliefs are undermined.

Clarke-Doane further elaborates on the nature of the challenge in Clarke-Doane [2020]. There he observes that it is of vital importance for the argument that there is a sense of 'explain the reliability', such that it seems impossible for the realist to explain the reliability of their set-theoretic beliefs, and that this failure undermines those beliefs. He suggests that *safety* can fulfill this role. Let us say that our set-theoretic beliefs are safe, just in case for any one of them, that p, we could not have easily had a false belief as to whether p (using the method that we actually used to determine whether p). With this understanding in hand, we can construe the Benacerraf challenge for set theory as the challenge to show, while freely making use of the assumptions that we are defeasibly justified in our beliefs and that they are true, that we could not easily have had false set-theoretic beliefs (using the method we actually used to form them).[17]

Now there are a variety of other interpretations of reliability in the literature. These include the idea that our beliefs are reliable just in case they are *sensitive*, that they are reliable just in case they are safe and sensitive, and that they are reliable just in case their truth 'figures' in to our best explanation of our coming to have them.[18] However, we shall adopt the safety interpretation here. A full defence of this choice would take us too far afield. But we shall briefly outline the virtues of this interpretation.

First, there is a strong intuitive pull in favor of needing to meet the challenge. Suppose we cannot even tell an in principle story, freely making use of the assumptions that our beliefs about D are both true and defeasibly justified, of how it is that we could not easily have formed false beliefs about D by carrying out the same process which actually produced our beliefs about D. It would seem then that we were lucky in coming to have the correct beliefs. Becoming aware that we are epistemically lucky in this way is a paradigm case of undermining evidence. How would it be different from finding out that expert mathematician on the basis of whose testimony you formed your set-theoretic beliefs, was choosing what to tell you by flipping a coin?

We can further bolster this point by noting that it is very much like a debunking argument. These should be familiar from the case of ethics. There, one might wonder whether the truth of any ethical claims play any role at all in the explanation of our coming to believe them. If not, then it seems that our ethical beliefs are not safe. This has motivated anti-realism about ethics.[19] Debunking arguments have also been raised about consciousness, aesthetics, validity, and many other areas.[20] This is a significant general epistemological problem. And one can think of the Benacerraf problem for set theory and higher-order logic as a debunking argument.

And second, the challenge does not seem to prove too much. It is not a mere skeptical challenge. It is quite different in character. To see this it is crucial to note that the Benacerraf challenge is not the challenge to show that our beliefs are safe. Showing that our beliefs are safe requires

[17] See Clarke-Doane [2020]
[18] See Clarke-Doane [2017].
[19] See Street [2006]
[20] See for instance, Chalmers [2020], and Schechter [2010].

showing that they are true. This would be a mere skeptical challenge. Rather, this is the challenge to show that our beliefs are safe, freely making use of the assumption that they are true and that we are defeasibly justified in believing them. This can clearly be met for a domain D, even if it turns out that our D-beliefs are not safe. Meeting the challenge does not establish safety. It establishes safety conditionally upon the truth and justification of the relevant beliefs. This is why we think the Benacerraf challenge applies across the board to structurally similar theories. It can equally apply to theories even if one is true and one is false, because we are allowed to assume the truth and justification of the theories.

The main idea is that if it seems impossible to meet the challenge for D, then this seems like very good reason to seriously doubt that our D-beliefs are in fact safe. Consider, by way of illustration, our perceptual beliefs. It seems that we can meet the Benacerraf challenge for them. For our perceptual beliefs in general, if we assume that they are true and that we are defeasibly justified in believing them, we can gesture towards a complex neurobiological account of how we track the physical truths about our environment, and an evolutionary account of how we came to have faculties which so track these truths. But this does not build in the actual safety of our perceptual beliefs. All of this is consistent with all of our perceptual beliefs being false, perhaps because we are brains in vats. If our beliefs are systematically false in this way, then we certainly could easily have had false perceptual beliefs.

The important upshot of this consideration is that the challenge is not too strong in general. We have good reason to think it can be met for our perceptual beliefs. Given that it is has strong intuitive motivation this suggests that the challenge needs to be met for pure math. Absent some principled reason why our beliefs about math are allowed to be lucky in a way which we have good reason to think our perceptual beliefs are not, it would seem that the mathematical realist needs to provide an account of how it is that the method which produced their mathematical beliefs could not easily have led them astray. Let us say that there is a Benacerraf problem for set theory just in case we cannot, even in principle, show how it is that we could not easily have had false set-theoretic beliefs, freely making use of the assumptions that our beliefs are both true and defeasibly justified.

1.4 Companions in Guilt

Let us now make the case that if there a Benacerraf problem for set theory, there is also one for higher-order logic. The argument is short and simple. In what follows, the axiom of choice is just an example. We could equally well have chosen any of a variety of claims.

- 1: Suppose that we cannot (even in principle) account for the safety of our belief in the axiom of choice, freely making use of the assumptions that choice is true, and that we are defeasibly justified in believing it – there is a Benacerraf problem for set theory.
- 2: Set theory and higher-order logic make very similar structural claims – both make arithmetic claims, claims about extensional collections, and infinite cardinal claims. In particular, there is a higher-order claim which is very similar to choice.[21]
- 3: If we cannot account for the safety our belief in the axiom of choice (freely making use of the assumption that it is true and that we are justified in believing it), then we cannot account for the safety our belief in the higher-order version of the axiom of choice (freely making use of the assumption that it is true and that we are justified in believing it).
- C: Therefore we cannot account for the safety of our belief in the higher-order choice. And thus higher-order logic has a Benacerraf problem.

The main premises in this argument are 2 and 3. In order to reject the argument one would need to reject one of these. 2 seems extremely plausible. It is not clear to us how one could reject that set theory and higher-order logic make very similar structural claims about the world. We shall not defend this point here, instead we just appeal to the authority of the litany of philosophers and mathematicians who have noted the intimate relation between set theory and higher-order logic. Among whom one can find Godel, Quine, Shapiro, Linnebo, Rayo, and many others.[22]

[21] Here is the claim: $\exists f \forall X(\exists y(Xy) \rightarrow Xf(X))$.

[22] See respectively, Godel [1933], Quine [1970], Shapiro [2012], and Linnebo and Rayo [2012].

It seems then that the main place one may get off the boat is at 3. This premise says that if there is not a solution to the Benacerraf challenge for set theory, there is not a solution to the Benacerraf challenge for higher-order logic. In longer form, it says that if we cannot, even in principle show how it is, freely making use of the assumptions that it is true and that we are defeasibly justified in believing it, that we could not easily have had a false belief about choice, using the method we actually used to formed our belief, then we cannot even in principle show how it is, freely making use of the assumptions that it is true and that we are defeasibly justified in believing it, that we could not easily have had a false belief about higher-order choice, using the method we actually used to formed our belief.

We shall support this claim by considering its contrapositive, which says that if there is a solution to the challenge for higher-order logic, there is a solution to the challenge for set theory. The reason why we think this is true is that given any solution to the challenge for higher-order logic, we think that we can translate this into a solution for set theory. The idea is as follows.

Suppose there is a solution for higher-order logic. Then there is an in principle account of our reliability with respect to our higher-order logical beliefs. Suppose that the correct higher-order logic is *CCI* – the logic which results from adding an axiom of infinity to Yli-Vakkuri and Goodsell's *CC*. Then we would have an account of our reliability with respect to the structural claims expressed by *CCI*, including the axiom of choice.

Now consider a set theory which agrees with the correct higher-order logic when it comes to arithmetic, and extensional collections, infinite cardinal arithmetic, and so on – they prove the same structural claims. In order to meet the challenge for this set theory, we would need to be able to account for how we could not easily have had false beliefs (freely assuming the truth and justification of our set-theoretic beliefs, using the method we actually used). We are able to assume that our set-theoretic beliefs are true, and that we are defeasibly justified in believing them. As such, we are allowed to assume that our account of how our set-theoretic beliefs are justified is correct.

Now, the solution for higher-order logic provides us with an account of how it is we could not easily have had false beliefs about the structural claims which it expresses. Our set theory expresses these same structural claims. We claim then, that if we put our account of the justification of our set-theoretic beliefs together with this account of our reliability with respect to the underlying structural claims, we get an account of our reliability with respect to our set-theoretic beliefs.

By way of illustration, consider the following example. Taking our cue from the shape of the answer in the case of perception – our most successful domain vis a vis answering reliability challenges – we shall need to have (i) a faculty which allows us to glean the relevant facts, and (ii) an account of how it is we came to have a faculty. In the case of perception this involves all of our very complex perceptual faculties which grant us information about the world around us, and an evolutionary account of how and why it is that we came to have perceptual faculties which allow us to do this. So, let us suppose that have a faculty of (higher-order) logical intuition. This is intended to be analogous to the faculty of mathematical intuition or intellectual perception to which philosophers of math often appeal in their epistemologies of math. This faculty allows us to intuit higher-order logical features of the world. And let us further suppose that there is an evolutionary account of how any why it is we came to have this reliable faculty of logical intuition.

Now suppose that we are attempting to meet the challenge for a set theory which makes the same structural claims as our higher-order logic. Suppose that our account of justification appeals to a faculty of mathematical intuition which allows us to intuit complex mathematical features of sets of individuals. We are allowed to assume that this account of justification is correct. Further, we are allowed to assume that our set theory is true, and that the structural features it expresses are indeed the structural features of the world. And from the solution for higher-order logic, we are allowed to assume that there is an evolutionary account of why it is useful to be able to track these structural features. In particular, we are allowed to assume that we could not easily have had false beliefs about any of these structural features. Given the minor premise that we take our set theory to be universally valid, it follows that we could not easily have had false set-theoretic beliefs. If the structural truths, and so the set-theoretic truths, had been different, we would have had correspondingly different beliefs. And any scenario in which we had false beliefs about set theory (using the method we actually used to form them), would also be a scenario in which we had correspondingly false beliefs about the underlying structural claims. Given that the latter could not easily have happened, it follows that neither could the former. In other words, if there is an in principle account of the safety of our higher-order logical beliefs (assuming truth and and justification), there is also an account of the safety of our set-theoretic beliefs (assuming truth and justification).

We think that this case is illustrative of a general pattern. For any higher-order logic L and set theory S which come along with the same structural commitments, given a solution for L, we can pair the account of the reliability of the underlying structural claims with the account of justification for S (which we are allowed to assume is correct) to obtain a solution for S. Any argument for the safety of the higher-order logician's belief, assuming its truth and justification, can easily be translated into an argument for the safety of the set theorist's structurally equivalent belief, assuming its truth and justification. Whether it is an evolutionary argument, an abductive argument, or what have you. It would seem then that the set-theorist can answer the Benacerraf challenge if the higher-order logician can. And vice versa.

If the foregoing considerations are correct, set theory and higher-order logic are companions in guilt when it comes to the Benacerraf challenge. In virtue of higher-order logic making substantive mathematical-like claims on the world, realism about higher-order logic is vulnerable to this epistemological challenge. We need to explain how our beliefs about recherche higher-order claims, such as the axiom of choice, can so well reflect the facts about them. And if we cannot do this, it would seem that realism about higher-order logic is quite precarious. We will conclude this section with a summary of the mains reasons to think that there is a Benacerraf problem for set theory.

Suppose that our set-theoretic realist believes in ZFC. Consider the axiom of choice. The realist can meet the challenge for their belief in choice only if they are able to show that could not easily have had a false belief about whether every set has a choice function. Supposing that the set-theoretic truths are necessary if true, there are three main reasons to think that they cannot do this. First, there has been persistent disagreement about the axiom. Many experts have thought that it was false. Indeed, there has also been serious disagreement among experts about almost every other axiom, including separation[23], foundation[24], and even the axiom of infinity[25]. This suggests that in general, being the content of a robust intuition of a set-theoretic expert is not a reliable guide to being true. And more generally, that the intuitions of set theorists are not a reliable guide to the simple set-theoretic truths.

Second, it seems that the received wisdom that choice is true is quite contingent. It seems that it could easily have been the case that the canonical set theory did not include choice. Perhaps if Banach and Tarski had formulated the axiom, and proved their infamous theorem in 1900, choice would never have gained any momentum.[26] Indeed, there is a general point that which axioms and theorems become canonical depends, in part, on the order of the discovery of their logical relations. Relative to a background theory, say ZF, there are fixed logical relations between set-theoretic claims. For instance, ZF may prove that ϕ entails ψ, that ϕ is equivalent to ψ, or so on. Fix on a claim ϕ and a claim ψ. Suppose that ZF proves that these claims are mutually inconsistent. Consider the set ZF_ϕ of claims which ZF proves ϕ entails, and the ZF_ψ of claims which ZF proves ψ entails. Among these sets, there may be some claims which are very intuitive, and claims which are quite unintuitive.[27] And suppose that a community of set theorists is trying to decide between which of ϕ and ψ to adopt. It seems very reasonable to suppose that which axiom they adopt depends on which members of ZF_ϕ and ZF_ψ they come to learn about first. It seems clearly possible that the community could settle on ϕ instead of ψ in virtue of only being aware of intuitive members of ZF_ϕ while having proved that there are a few unintuitive members of ZF_ψ, while it is the case that if they were in full possession of the ZF_ϕ and ZF_ψ, they would choose ψ. And in this sense, it seems that the content of the community's canonical theory could quite easily have been different than it actually is. These considerations readily transfer to the actual

[23]See Quine [1937]
[24]See Aczel [1988]
[25]See Nelson [1986]
[26]See Banach and Tarski [1924].
[27]For instance, the Banach-Tarski theorem is in ZF_{Choice}.

mathematical community. This suggests that the collective assent of the set-theoretic community is not a reliable guide to a set-theoretic claim being true. In other words, it suggests that being a theorem of the canonical theory is not a reliable guide to the theorem's being true.

And third, it seems that the truth of choice is not determined by any accepted empirical facts, and it will not be decided by any outstanding empirical conjectures. We will never be able to empirically verify whether or not choice is true. Once again, this also applies to many set-theoretic claims. And the point seems stronger and stronger the more recherche the set-theoretic claims become. Just consider the continuum hypothesis, or the large cardinal axioms. It seems very unlikely that these claims could, even in principle, be empirically verified. As such, set theories which differ with respect to these claims may be just as useful for the purposes of constructing empirically adequate physical theories. This suggests that being a theorem of the most physically useful mathematical theory is not a reliable guide to the theorem being true.

These points support the idea that there could easily have been a community of speakers very similar to us, whose mathematical concepts occupy very similar roles to ours except that they reject choice, and who get on in the world just as well as we do. Their mathematical experts have expert intuitions, they have a robust canonical foundational mathematical theory which undergirds the rich workings of a community of mathematicians, and they can formulate empirically adequate physical theories within the boundaries of this canonical theory. Now, if this is indeed the case, then it would seem that our realist cannot show that they could not easily have had a false belief about choice, precisely because it seems that they could easily have been a member of this community. And if this is the case, not only can they not meet the challenge, their belief in choice is actually *unsafe*.

1.5 Objections and Responses

In this section we will consider and respond to some potential objections to the companions in guilt thesis. We will begin with ontology.

1.5.1 Abstract Ontology: Objection

Field's presentation of the problem for set theory involved the postulation of abstract mathematical entities. What Benacerraf, and Field after him, were primarily concerned about was the metaphysically cordoned off realm of pure sets. (Though in later work Field is much less concerned about ontology[28]). How could we ever come to know about such objects? We never causally or otherwise interact with them. And set theory postulates an absolutely staggering number of them, with concomitantly staggeringly precise claims about their features. Higher-order logic on the other hand, does not come along with any distinctive first-order ontology. It is logic afterall. It is supposed to be ontologically neutral. It makes no ontological commitments over and above the claim that there are an infinity of objects.[29] As such, it does not face the same epistemological worry that faces set theory. The source of the worry for set theory is the seeming impossibility of an account of how we could interact with a realm of abstract objects. Higher-order logic does not come along with these commitments. And so does not face the problem.

1.5.2 Abstract Ontology: Response

This objection misses the point. It seems to confuse the reliability problem with an access problem. It doesn't help with respect to reliability that higher-order logic is *logic* – that it does not come along with a distinctive and abstruse ontology – even if it helps ontologically. It is just as much a mystery, supposing that the truth of higher order mathematical claims is mind-independent, how we could reliably come to know these as it is to suppose that we can come to reliably know things about pure set theory. Compare the cases of morality, aesthetics, and consciousness. Surely there is still a reliability issue for these domains, even if one is a Quinean nominalist. The central target of the Benacerraf challenge is objectivity, not objects, to borrow a phrase from Kreisel. And to generalize Kreisel's point even further, the problem is not with the objectivity of set theory specifically, it is with the underlying structural claim. We can think of set-theoretic choice and

[28]See for instance, the foreword in the second edition of Science Without Numbers.

[29]Indeed many higher-order logics do not even make this claim.

higher-order choice as two guises which present the same structural claim. The epistemological problem is with they underlying claim, not with the specifics of any of its guises.

Furthermore, it is not at all clear that higher-order logic really is as ontologically innocent as the potential objection made it out to be. Sure, it does not come along with a host of abstract first-order objects. But if we are taking the higher-order logic project seriously – if we think it is perfectly intelligible to use higher-order quantification – then it is not at all clear why we should not have to adopt a higher-order version of the Quinean dictum that *to be is to be the value of a bound variable*.[30] If second-order and propositional quantification are just as metaphysically perspicuous as first-order quantification, then ought not we accept that there are a variety of kinds of existence? That to be_e is the be the value of a bound first-order variable, that to $be_{e \to t}$ is the be the value of a bound second-order variable, and that to be_t is the be the value of a bound propositional variable? If this is the case, it is simply wrong to think that higher-order logic does not come along with a host of abstruse entities. It does, these are just not first-order entities.

1.5.3 Determinacy of Reference: Objection

The language of first-order set theory is notoriously indeterminate. It is quite straightforward to construct wildly non-standard models of set theory. Famously, Putnam used facts such as this to claim that the language of mathematics is hopelessly indeterminate.[31] However, the same cannot be said of higher-order logic. Or at least, so says the orthodoxy. Suppose that we are considering *Classicism* + *Choice* + *Infinity* together with a syntactic theory such as the classical theory of strings. In this augmented theory we can prove that all of the first-order logical terms of the theory have a unique intended interpretation. Similarly for the second-order terms, and so on. This is in stark contrast to set theory. These are familiar categoricity results. This metasemantic feature is relevant to the epistemological challenge, as it is the indeterminacy of set-theoretic language which makes it susceptible to not being safe. As such, higher-order logic is not equally susceptible, for the reasons we have just outlined.

[30] See Quine [1948].
[31] See Putnam [1980].

1.5.4 Determinacy of Reference: Response

This meta-semantic point is irrelevant (even if true, which we shall question momentarily). It is true that the reasons for the metasemantic worry and the epistemological worry are often the same. Benacerraf, for instance, has a causal account of our mathematical knowledge and of the determinacy of our mathematical language. And so it is the same lack of casual interaction which motivates and undergirds his determinacy worry and his reliability worry.[32] Be this as it may though, these are strictly different issues.

The reliability challenge is an epistemological challenge, not a metasemantic one. Indeed, it is not just that the reliability challenge is distinct from the determinacy of language problem, it is that setting up the reliability challenge presupposes the determinacy of all of the language involved. So if there really were this difference between set theory and higher-order logic, this would leave higher-order logic worse off than set theory! The metasemantic worry is that there is nothing in our practice or in the world that could make it the case that we mean *is a member* of by 'is a member of'. This was Putnam's worry in his influential [1980]. That problem can be addressed by a theory of reference, as in Lewis [1984]. The epistemic worry though, is that even if we suppose the truth and defeasible justification of our set-theoretic beliefs, we can't seem to show that we could not easily have had false such beliefs. In order to set up this problem we need to assume that we can determinately refer to the objects of our theory.

Suppose that that a set-theorist believes in ZFC. They then need to show that they could not easily have had false set-theoretic beliefs, making use of the assumption that ZFC is true, and that they are defeasibly justified in believing it. This would involve showing that they could not easily have believed a different set theory about the same sets of which their set theory is true. For if they could easily have believed such a set theory, then they could easily have had false set-theoretic beliefs. But all of this clearly involves an assumption that their set-theoretic terms determinately refer to the sets out there in the world. And that their set-theoretic beliefs are *determinately* true.

With this in mind we can see that the (putative) determinacy of higher-order claims does not

[32] See Benacerraf [1973].

help in side-stepping the challenge. We can grant that the claims of the higher-order logician are perfectly determinate. That all of the logical particles determinately refer to their semantic values. Still though the epistemological challenge arises. The only ingredient we need for the problem to get going is to be able to coherently suppose that we could easily have believed a different higher-order logic. Or that it is possible for there to have been a community of speakers like us who believed a different higher-order logic, who nonetheless get on just fine in the world.

Finally, even if you are unmoved by our line of thought that the indeterminacy is not the source of the reliability issue, this likely would not refute the companions in guilt claim. As is probably clear from the preceeding paragraphs, it is not at all clear to us that higher-order claims are determinate in a way in which set-theoretic claims are not. The putative proofs of the determinacy of higher-order terms (in a higher-order meta-theory) all explicitly use higher-order terms in the meta-theory. But if these terms are do not determinately refer, then we cannot determinately use them in the proof in the meta-theory. In other words, if we assume that the terms are determinate in the meta-theory, then we can prove that they are determinate in the object language. Admittedly, we cannot even do this much for first-order set theory. Nonetheless, if you are already suspicious of the determinacy of higher-order logic, then you are unlikely to be swayed by this line of reasoning. From this perspective, these 'proofs' just beg the question.

1.5.5 Applied Versus Pure Theories: Objection (i)

So much for the objections that higher-order logic is ontologically innocence, and determinate. There is another clear difference between higher-order logic and set theory. Or at least between higher-order logic and pure set theory. Pure set theory does not concern actual (non-set-theoretic) objects, properties and so on. It is about a pure realm of abstract objects, not with sets of concrete objects. These latter kind of sets are firmly within the purview of impure set theory, which is not primarily what concerns set theorists. Higher-order logic, as it is understood in the metaphysics community, is an applied theory. It concerns actual objects, properties, propositions, and operators. In this way it is analogous with impure set theory. We are on board that there is a reliability challenge for pure set theory. And were we considering higher-order logic as a pure theory, as Russell perhaps did, we would grant that it would also have the problem. But this is decidedly not how we are thinking of higher-order logic – it is an applied theory. Higher-order logic makes substantive mathematical claims on the world, but these are applied mathematical claims. But the reliability challenge can be answered for impure set theory, and for applied mathematics in general.

Our epistemological account of impure set theory is an empirical one, more akin to our epistemology of physics than our epistemology of pure set theory. Our impure set theory is justified in just the same way that our physical theories are. We formulate our best overall theories of the physical world, which happen to be set-theoretic-cum-physical theories. These theories face the tribunal of justification holistically. The success of our best physical theory justifies our impure set theory just as much as it does our theory of particle physics.[33]

Now, as we have seen, a sufficiently strong higher-order logic interprets all of the impure set theory that we need to formulate these theories. As such, we can formulate our best physical theories in the language of higher-order logic plus the basic physical predicates of the physical theories.[34]. In this way higher-order logic can be empirically justified. This is the significant asymmetry with pure set theory. And this is why the reliability challenge does not pose a problem for higher-order logic.

1.5.6 Applied Versus Pure Theories: Response (i)

As it stands, it seems to us that this kind of objection just confuses justification with reliability. We can grant, with Quine, that our overall best physical theory is empirically justified in a holistic fashion, and that impure set theory, which is part of our overall best physical theories, is defeasibly justified in this way. As would a higher-order logic be, were we to formulate our physical theories in this language. However, justification is distinct from reliability. Even if we are defeasibly justified in believing a particular higher-order logic on holistic abductive empirical grounds, this does not thereby mean that our belief in this theory is reliable.

[33]See Quine [1951] for an outline of this kind of view.

[34]See Field [1980], Bacon [2019], and Dorr and Arntzenius [2012] for examples of this

Recall the formulation of the problem from above. We first assume the truth and defeasible justification of an empirically useful higher-order logic. The challenge is to explain how our beliefs about the higher-order logic can so well reflect the higher-order logical facts. It is easily conceivable that a higher-order logic could be empirically justified, and yet not be the true higher-order logic. There could be different higher-order logics which are equally useful for doing physics – as is almost certainly the case for impure set theories. Or it could be that the most empirically useful higher-order logic is not the true higher-order logic. Perhaps because it makes some practically useful, though strictly false, simplifications. Or even perhaps because the higher-order logic is true when restricted to the domain of physics, but does not preserve truth when making inferences about a different domain. The point is that the reliability question is not settled by an account of justification. We could grant that we have a better account of how our higher-order logic is justified than our account of pure set theory. But this is independent of the reliability point.

1.5.7 Applied Versus Pure Theories: Objection (ii)

Be this as it may, there is still a problem here. Just as our beliefs about physics and higher-order logic are justified in the same way, so too can we answer the Benacerraf challenge for both in the same way. The claims of physics in the language of higher-order logic are all about physical objects. As such, the reason our beliefs about higher-order logic are reliable is because they are beliefs about concrete individuals, and we have experience of, and are in causal contact with concrete individuals. If there is a reliability problem for the higher-order logician, it's just the general skeptical challenge which we also face for physics. Whatever story you tell for why it is that we have reliable beliefs regarding physical properties, that's the story we're going to tell about how we have reliable beliefs about higher-order logic. And this is not a story you will be able to tell about pure mathematics.

1.5.8 Applied Versus Pure Theories: Response (ii)

Certain aspects of this point are well taken. There is a sense in which higher-order logic is indeed about concrete objects, in which pure set theory is not. So there isn't a worry about how we interact with these objects. But there is a relevant difference between kinds of claims we can make about concrete objects. On the one hand there are standard perceptual claims, attributing physical properties to concrete objects. These are the kinds of claims we had in mind above when we said that we can sketch an account of how it is that our perceptual beliefs are safe. If the distinctive claims of the higher-order logician were on an epistemic par with these kinds of claims, as the objector is claiming, then indeed the only epistemological challenges which would face them would be of the general skeptical kind. But the only argument the objector has given for this claim of parity is that the higher-order logical claims, just as more basic claims of physics, are about the same objects. This is clearly not sufficient. Just because some claims all attribute features to concrete objects, does not mean they are on an epistemic par. The nature of the features so attributed surely also plays a significant role.

Moreover, as we have already outlined, there is good reason to think that higher-order logic, realistically construed, is not just about electrons, quarks and muons. It is also about properties, properties of properties, propositions, operators, and the like. And that this alone accounts for the difference, vis a vis reliability, between the recherche claims of higher-order logic, and the claim that electrons are negatively charged. But even if we set this point aside, the objection does not ultimately hold water.

Consider again the Quinean nominalist, for whom every claim is about particulars. Nominalism alone does not make them immune to reliability problems. The point is that there still may be reliability problems which arise for claims which involve certain predicates, even if other claims about the very same particulars, which involve different predicates, do not have such problems. Suppose that they want to accept a variety of moral, aesthetic, and mathematical claims, over and above standard perceptual claims – the kinds of claims a moral, aesthetic, and mathematical nihilist will want to reject. If we suppose that we can make sense of a thoroughgoing nominalist interpretation of higher-order logic, so that it really is just about concrete particulars, then we can also suppose that they may want to accept a variety of claims which attribute complex higher-order logical properties to particulars. Consider the following claims: (i) There is a computer on the table in front of me, (ii) Jon's action was good; and (iii) every property of particles has a choice function. For our nominalist, these are true claims solely about concrete objects. There is a key difference between (i) on the one hand, and (ii) and (iii) on the other. Insofar as all three are about concrete objects, we think all can be justified. But, it seems as though we have the beginnings of an account of our reliability about standard perceptual claims – one that involves lots of complicated stuff about our perceptual faculties, and our brains, and evolution. The basic thought is that we track the perceptual truths – we could not easily have had a false belief about whether we are now staring at our computer (provided that this is true and that we are justified in believing it), given the method we actually used. But this starkly contrasts with the belief that Jon's action was wrong, and that every property of particles has a choice function. Even if we are Quinean nominalists, and interpret these claims as such, it still seems like we could easily have had a false belief about these, (supposing again that they are both true, and that we are defeasibly justified in believing both) using the method we actually used to form those beliefs.

1.5.9 Who Needs Infinity: Objection

Let us now consider one last potential objection to the companions in guilt point. There is no denying the point that the force of the reliability challenge, when considering standard set theory, seems to increase as one moves up the cumulative hierarchy. The challenge seems to intrinsically involve some kind of notion of coherent supposition or entertaining of alternative set theories. This is much much easier the further up one goes. As Joel Hamkins' has convincingly argued in his [2012], set theorists are very comfortable working in a great variety of models which extend ZFC in mutually incompatible ways. For instance, they are very comfortable in models of CH and in models of $\neg CH$. This comfort plays a role in how genuine the different interpretations of the $\in$ offered by these different models seem. In fairly stark contrast to this though, there does not seem to be very much plausibility at all to models which disagree about arithmetic. This of course is not to say that there is not some disagreement about these. There is. Just consider ultrafinitism, intuitionistic arithmetic, and paraconsistent arithmetic. But it seems that most working set theorists and philosophers of mathematics alike find models of $PA + \phi$, where ϕ is a statement which is independent of PA but which is false in the standard model of arithmetic, unintuitive, strange, and just false in a way which they do not regard models analogous of $ZFC + \phi$, where ϕ is independent of ZFC. (Bracketing instances of ϕ such as $\neg Con(ZFC)$.

What is the upshot of this for our purposes? Well, the reliability issue seems to have much less traction about finite arithmetic facts, than infinite cardinal facts. And most of the currently popular higher order logics do not incorporate an axiom of infinity, and as such, are expressively weaker than ZFC. Just how much weaker depends on the higher-order logic. But generally they do not go much beyond second-order Peano Arithmetic. As such, one might object that they, unlike set theory, do not make the kinds of substantive mathematical claims about which there really does seem to be a significant reliability challenge.

1.5.10 Who Needs Infinity: Response

There are three main reasons why this claim does not threaten the companions in guilt point. First, the reliability challenge does not just apply to infinite cardinal claims. It also applies to substantive mathematical claims such as choice. And it is clear that higher order logicians are already putting forward theories which have higher order versions of claims such as these as axioms.[35] As such, we cannot avoid the issue by restricting our theories to the realm of the finite, or the countably infinite.

Second, most popular higher-order logics interpret second-order arithmetic. This point is not in dispute. But Steven Simpson has convincingly shown that many substantive disagreements already arise at the level of second-order arithmetic.[36] This ought not to be surprising. Second-order arithmetic is already quite *set-theory-ish*. So just in virtue of interpreting this, higher-order

[35] See for example, Yli-Vakkuri and Goodsell [In Progress].
[36] See Simpson [2006].

logic is already mired in reliability worries.

And third, even if one disagrees with all of this, there are fragments of higher-order logic which interpret other kinds of theories which are susceptible to reliability challenges. In particular, higher-order logic can interpret modal theories. As well as theories of validity. This is completely independent of issues of infinite cardinality. [37] Take any field which higher-order logic interprets. If there is a reliability challenge for that field, then higher-order logic will inherit the problem. And it seems that there is good reason to think that there are such a challenge for modality.[38]

With all of these objections considered, it seems that the companions in guilt thesis is in good standing.

1.6 Pluralism and Safety

What is the upshot if our companions in guilt argument is successful? Well, in that case, either there would be a significant Benacerraf problem for the higher-order logic as metaphysics program, or one would have to answer the challenge for set theory. Broadly speaking, this would seem to leave four options. First, we could deny that there is a problem for set theory. Second, we could give up on the higher-order logic as metaphysics program. Third, we could maintain our realist stance and embrace an epistemic mystery, accepting that our beliefs have been undermined yet maintaining them anyway. And fourth, we could could revise some aspect of the package-view, which would allow us to meet the reliability challenge.

We think that the first and third options are non-starters. There is a compelling case that there is a Benacerraf problem for set theory. And embracing such an epistemic mystery doesn't seem far removed from mysticism. That leaves two options: give up on the project of higher-order logic as metaphysics, or revise some aspect of the approach. We don't really think there is much to be gained by abandoning the project. We all need structural claims in order to theorize about the world. They are indispensable. Whether they come in the guise of set theory, higher-order logic,

[37] Though, there are modal questions which do turn on theses which involve substantive claims about infinite cardinalities. See Fritz [2016].

[38] See Nolan [2017], and Clarke-Doane [2019 (a)] and [2019 (b)] for arguments that there is a reliability challenge for modal theories.

or of some other language. But we shall not push this line here. Let us just make the conditional claim that if we want to embrace the higher-order logic as metaphysics program, we need to revise some part of our approach.

So, what kind of revision might we adopt? Here we think we should look to the philosophy of math for guidance. The Benacerraf challenge has generally been accepted as a very significant problem for set-theoretic realism. In response to it, some realists have decided to fundamentally change their view of set-theoretic reality. They have come to reject an assumption which has gone unmentioned thus far in the paper. They have rejected monism about set theory, and instead have come to accept pluralism of various stripes. The basic thrust of pluralism is that there is not just one universe of sets which is correctly described by one set theory. Rather there are a plurality of universes of sets, which are correctly described by different set theories.[39] This is a fascinating and exciting rethinking of the foundations of pure math.

If the argument of this paper is successful, absent another kind of response, we think the thing to do is to give up on an assumption that has gone unspoken in the literature on higher-order logic – that there is one true higher-order theory.[40] This is what was done in the math case, and even Field grants that it resolves the problem. It would seem that the most promising route for the foundationalist about higher-order logic, would be to adopt some kind of pluralism. To get a sense of why going pluralist would help, let us briefly look at why it helps in the math case.

Consider Clarke-Doane's pluralist view of set theory. According to this view, every Σ_1-sound set theory is true (of different subjects).[41] There are two features of Clarke-Doane's view which are important to note. The first is that the view is not outright contradictory. Or at least not for obvious reasons. Both ZFC and $ZF + \neg C$ are arithmetically sound. And so they are both true. And so both C and $\neg C$ are true. But this is not a contradiction, as it is part of the view that these propositions, though expressed by one and the same syntactic item, are not contradictories. They are about different kinds of sets. We might more perspicuously depict what is going on here

[39]Though not a realist, Field first considered such a view in Field [1994]. For realist views, see Balaguer [1995], Linsky and Zalta [1995], Hamkins [2012], and Priest [In Progress].

[40]See Clarke-Doane and McCarthy [2023] for an exception.

[41]See Clarke-Doane [2020] for more details.

as follows: every set has a choice function, but not every schmet has a schoice function. This point generalizes. Each arithmetically sound set theory truly describes a distinctive set-theoretic universe. And so the propositions expressed by each different such set theory do not contradict one another. They are about different things.

The second thing to note is that he adopts a very cooperate metasemantics for set-theoretic language. Let S be the set theory one believes. Then set-theoretic sentences, out of one's mouth, are automatically about the set-theoretic universe of which S is true. If we believe the formal set theory $ZF+\neg C$, then when we assert set-theoretic sentences, the propositions we express are about a set-theoretic universe which is correctly described by the theory $ZF + \neg C$.

With these points in mind, suppose that we believe the set theory $ZFC+V = L$. In this context, let p be the proposition which is expressed by the sentence $'2^{\aleph_0} = \aleph_1'$. Further, let us suppose that $ZFC + V = L$, and so p, is true. We concede that we easily could have accepted the sentence $'2^{\aleph_0} < \aleph_1'$. Suppose that in that scenario we accepted $ZFC + \neg CH$. The crucial point, and this is where the very cooperative metasemantics comes in, is that in such a scenario, we would not have had a false belief as to whether p. We would not have believed $\neg p$. Indeed we would not have had a false belief at all, but rather a true one. For we would not have had beliefs about the $ZFC+V = L$ universe in which CH is true. We would have had beliefs about the $ZFC+\neg CH$ universe. And our belief that the sets of this universe obey the negation of the continuum hypothesis would be true.

This account generalizes to any arithmetically sound set-theoretic belief. For any arithmetically sound set-theoretic sentence s, had it been part of a theory that we believed, then our beliefs would have been about a universe which that theory is true of. And so we could not easily have had a false, yet arithmetically sound, set-theoretic belief. This takes care of any of the arithmetically sound set-theories we might easily have believed. All that remains are set-theories which are not arithmetically sound. At this point, the argument relies on a conjecture. It is a premise of the argument that we could not easily have believed an arithmetically unsound set theory. If this is correct, and it certainly seems quite plausible, our set-theoretic beliefs are safe. It is the heavyduty metaphysical assumptions of pluralism which allow one to show safety and thus explain reliability.

Now, it is reasonable to suppose that we could similarly use the significant metaphysical assumptions of a higher-order pluralist view in order to show that our higher-order logical beliefs are safe, and thus reliable. In an analogous fashion to the defence just outlined, if we can demarcate a class of higher-order logics which are all true (of different subjects), and also argue that we could not easily have accepted any higher-order logic outside of the class, we can show that our higher-order beliefs are safe. As Beall once quipped, if it seems really hard to hit the target, then just make the target a lot bigger![42] Developing the idea that there is no one true higher-order logic raises new and fascinating challenges, potentially offers some fruitful new answers to old problems, and and intriguing consequences. It is not just a notational variant on any extant set-theoretic pluralist theory.[43]

1.7 Conclusion

Higher-order metaphysicians are currently engaged in a fascinating foundational project. We share in the earnest excitement about the prospect of a foundational metaphysical theory. And we also share the optimism that higher-order logic can occupy this foundational role. As far as the metaphysics goes, higher-order logic seems promising. But we are deeply concerned about the epistemology. As we have outlined in the paper, it seems to us that higher-order logic is just as susceptible to the Benacerraf problem as set theory is.[44] A foundation which is epistemologically shaky, even if it is metaphysically secure, is not one upon which we want to base all of our metaphysical theorizing.

Higher-order pluralism though, if we can get it off the ground, may provide an epistemologically safe foundation for metaphysical theorizing. This is obviously a different kind of foundation than the one the monist is after. Rather than there being just one higher-order logic within which we can theorize, there are a variety. And we are free to choose between them. We could just pick the most useful higher-order logic for whatever theoretical purpose we had, and go about our

[42] See Beall [1999] for an elaboration on this point.

[43] See my *On the Plurality of Higher-Order Logics* for more details about such an approach.

[44] Indeed, one might even argue that because of its generality it is more susceptible. But this is neither here nor there.

constructions, secure in the understanding that our higher-order logical beliefs are safe.

Over and above this epistemic safety, this higher-order pluralist framework has the potential to provide a unified framework within which to study otherwise seemingly unrelated pluralist positions, about math, validity and modality. It might allow us to get a grip on metaphysical pluralism, very generally construed. And it could potentially provide solutions to persistent disagreement about these domains.

The shift to higher-order logical pluralism would constitute a fairly radical rethinking of the nature of validity, modality, and metaphysics in general, much as set-theoretic pluralism has engendered a radical rethinking about the nature of mathematics. It would render moot some of the most central questions in these domains. Is the law of excluded middle valid? Is it the case that necessarily everything is necessarily something? Is the grounding relation transitive? On this picture of reality, these questions no longer have objective answers. They become like the question of whether the continuum hypothesis true from the perspective of the set-theoretic pluralist. It is true in some universes, and false in others. That is all there is to it. It is valid from some higher-order logical perspectives to infer according to the rule of excluded middle, but not from others. And so on. Metaphysics is perspectival, all the way down.

We tentatively suggest that picture of metaphysics which could well emerge from this shift has two main characteristics. First, there is no longer a serious question about whether we can legitimately apply some contested piece of higher-order logic in our theories about the world, particularly in science. Applied metaphysics becomes a purely practical domain. If we want to use a quantum higher-order logic to theorize about superpositions, a necessitist higher-order logic to model some linguistic phenomenon, or a higher-order logic with the negation of *CH* to theorize about the continuum and objective chance, it is perfectly legitimate to do so. The only significant questions here are whether some logical principles are particularly well-suited to the task at hand. And second, the study of *pure* metaphysical questions takes on shape quite similar to Hamkins' study of the set-theoretic multiverse, or Priest's study of mathematical pluralism.[45]

[45] See Hamkins' [2012], and Priest [In Progress].

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Chapter 2: Precedent for Higher-Order Logical Pluralism

Abstract

The higher-order logic as metaphysics program is in full swing. This is a foundational project which takes higher-order logic to stand in the same kind of relation to metaphysical theorizing that ZFC is commonly taken to stand in to our classical mathematical theorizing. Much as with set-theoretic foundationalism, we can approach this project form a variety of different meta-theoretic perspectives. One such approach is a pluralist approach, which may offer solutions to some epistemological issues. We may worry though, given how far this strays from typical research in this area, that this kind of approach is untenable. We do not think that this is case. By way of suggesting this, we are going to situate this kind of approach in relation to a variety of well explored views in the philosophy of logic, the philosophy of math, and metaphysics. Our main point is that the same kinds of reasons which motivate these views also motivate a pluralist approach to higher-order logic. Indeed, modulo accepting the intelligibility of higher-order quantification, we think these views require such an approach. As such, if this kind of approach is untenable, there is good reason to think that each of these other views are too. And if you are tempted by any of these views, you should be similarly tempted by higher-order logical pluralism.

2.1 Introduction

There are several centrally important domains to which pure higher-order logic has been applied, in the same sense that pure geometry has been applied to the structure of spacetime. The two most historically prominent of these are to the validity of arguments, and to the foundations of math.[1] The ideas respectively being that the space of valid inferences, relative to a faithful mapping of natural language into the formal language, is isomorphic to the structure characterized by a specific higher-order logic, and that all of our classical mathematical theories can be interpreted within the structure characterized by a specific higher-order logic.

Quite recently, there is another centrally important domain to which higher-order logic is being applied – the foundations of metaphysics.[2] The idea is roughly that higher-order logic stands in the same kind of relation to our metaphysical theories that ZFC stands in to our classical mathematical theories.[3] The hope is that higher-order logic might be able to provide a metaphysical, epistemological, and dialectical foundation for our theories of modality, grounding, essence, and so on, in much the same way that ZFC provides a foundation for number theory, real analysis, differential geometry, and the entire practice of the classical mathematical community.[4]

Now, with respect to these applications, we may wonder whether there is one higher-order logic which is uniquely well suited. In this paper we will primarily focus on the application of higher-order logic to the foundations of metaphysics. But to the extent that this already includes our theory of validity and the foundations of math, these considerations will readily apply to the other applications also.[5]

[1] See, for instance, Frege [1879], and Russell and Whitehead [1910] respectively.

[2] See Williamson [2013], Dorr [2016], Yli-Vakkuri and Goodsell [In Progress], and Goodman [Forthcoming], for some good examples of this.

[3] See Maddy [2016] for an account of the role of ZFC, see Yli-Vakkuri and Goodsell [In Progress] for a more detailed account of higher-order logic as metaphysics, and see my *On the Plurality of Higher-Order Logics* for more details about the analogy.

[4] See Bacon [2018], Fritz [2021], and Ditter [2022] respectively for interesting higher-order accounts of modality, grounding, and essence.

[5] We have in mind here the views that the best interpretation of logic is the metaphysical interpretation, and the view that higher-order logic provides the correct foundations for pure mathematics. See McSweeney [2019] for an outline of the metaphysical interpretation of validity. As well as Williamson [2013], particularly the discussion of metaphysical universality. And see Church [1940] for an account of higher-order logic as a foundational mathematical

Monism about higher-order logic answers this wondering in the affirmative. According to the monist, there is just one higher-order logic which provides the correct foundations for metaphysical theorizing. In much the same way that the set-theoretic monist thinks that there is just one set theory – usually ZFC (plus some large cardinal axioms) – which provides the correct foundations for mathematical theorizing. Consequently, there is just one higher-order logic which ought to be adopted as our metaphysical meta-theory in our reasoning about the world.

Pluralism about higher-order logic, on the other hand, answers this wondering in the negative. According to the pluralist, there are a variety of different higher-order logics which are equally well suited to be our foundational metaphysical theory. Much like the conventionalist, the fictionalist, and the pluralist about (first-order) logic and math, they think that there are a variety of higher-order logics which can provide all of the resources we need to do math, physics, and metaphysics.[6] There is no denying that this is a radical approach. It departs significantly from the majority of current research on higher-order logic. As such, it may seem that the view is untethered.

However, a pluralist approach to higher-order logic has much in common with a variety of well established and interesting views. In sections (2.2), (2.3), (2.4) and (2.5) we will investigate the relation between higher-order logical pluralism and (first-order) logical pluralism, multiverse set theory, the indefinite extensibility of the cumulative hierarchy, and (first-order) quantifier variance, respectively.[7] We will argue that not only do the same reasons which motivate these views also motivate a pluralist approach to higher-order logic, but modulo the intelligibility of higher-order quantification, these views actually require it. They also provide solutions to a number of significant issues which face higher-order logical pluralism.

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theory.

[6]See Carnap [1937] and Warren [2020], for conventionalist views in this spirit. See Field [1980] and Akiba [2000] for fictionalist views in this spirit. And see Field [1994], Balaguer [1995], Linsky and Zalta [1995], Hamkins [2012], and Priest [In Progress] for pluralist views in this spirit.

[7]Another view which stands in this exact same relation to higher-order logic is modal pluralism, according to which there are a variety of equally metaphysically privileged kinds of necessity and possibility. We outline the relation between these in Clarke-Doane and McCarthy [2023].

2.2 Logical Pluralism

A natural place to look for precedent for a pluralist approach to higher-order logic is in the literature on pluralism about first-order logic. Logical pluralism is the view that there are a variety of logics which are equally good vis a vis the canonical application of logic – evaluating the validity of arguments. As such, there are a variety of logics which true of their intended subjects. Generally speaking, a logical pluralist will think that one can infer according to any of the variety of consequence relations and *get things right*.

There are many different versions of logical pluralism in the literature. See, for a few representative examples, Field [2009], Shapiro [2014], and Kouri-Kissel [2018]. Perhaps the most well known version of logical pluralism is that of Beall and Restall.[8] In a nutshell, Beall and Restall claim that our concept of logical consequence is imprecise, and that different precisifications of the concept give us different genuine logical consequence relations. They provide what they call the Generalized Tarski Thesis (GTT), which they claim captures our imprecise notion of logical consequence:

- (GTT): An argument is $valid_x$ just in case in every $case_x$ in which the premises are true, the conclusion is also true.

GTT is a schema that, when instantiated with an applicable notion of a case, provides a precisification of the concept of logical consequence. Now, for Beall and Restall, there are three main, equally good, logics: classical logic (where cases are classical models), intuitionistic logic (where cases are stages, leaving room for incomplete scenarios), and relevance logic (where cases are situations, which may be contradictory). But prima facie, there is no reason to suppose that these are the only acceptable precisifications. In particular, there is no reason to think that cases are uniquely first-order.

There are many arguments in the literature to the effect that logical consequence does not just relate first-order claims. Think of the inference from that *the ball is red* to the conclusion that *there*

[8] See Beall and Restall [2005].

is some property which the ball has. Or the inference from that *Cian believes that no objects have proper parts* to the conclusion that *Cian believes something*. Just as one expands the notion of logical consequence from propositional logic to first-order logic in order to be able to characterize logical inferences involving the first-order quantifiers, so too can one move from first-order logic to higher-order logic in order to be able to characterize logical inferences involving higher-order quantifiers. It seems our concept of logical consequence is imprecise enough to allow a variety of higher-order precisifications. We can and should examine the GTT with respect to higher-order logic. We could, for instance, interpret 'cases' as classical Henkin models of second-order logic, or as full models of simple-type theory, and investigate the notions of consequence which result.

It is important to note that this point is not hostage to the peculiarities of Beall and Restall's approach to logical pluralism. The point generalizes. Consider, for instance, Shapiro's domain-specific logical pluralist view.[9] According to this view, different logical consequence relations operate in different contexts. So, whether an argument is valid depends upon contextual parameters which may vary. For instance, if we are in a seminar room discussing a proof of some theorem in classical ZFC, 'valid' will mean classically valid. But if we are in a seminar on smooth infinitesimal calculus, and we are discussing whether some proof is valid, 'valid' will mean intuitionistically valid. Whether a proof is valid or not depends upon the context in which we are in.

Just as in the case of Beall and Restall's view, prima facie, there is no reason to suppose that these contexts, particularly mathematical contexts, are uniquely first-order. Certainly not if we are following Shapiro's guiding intuition that interesting mathematical theories should be taken at face value. There are many interesting mathematical theories which are founded in higher-order logic, rather than a first-order logic with some some set theory. Just think of Russell or Church's type-theoretic approach to math. In these mathematical contexts, in which higher-order logic is taken to be the foundational mathematical theory, whether or not an argument is a proof depends on whether the conclusions follow from the premises according to the rules of the background higher-order logic. In other words, it depends on whether or not these higher-order arguments are

[9] See Shapiro [2014].

valid in a distinctively higher-order sense.

Pair this with the observation that there are different higher-order logics which could be adopted as the foundational theory, and we get the conclusion that whether or not some higher-order proof is valid depends upon which higher-order logical context we are in. There are higher-order logics which are built on classical logic, intuitionistic logic, and so on. And so there are higher-order contexts in which 'valid' means higher-order intuitionistic validity, or higher-order classical validity, and so on. There are plenty of such contexts which we can think of. For instance, these different contexts arise quite frequently in the case of automated theorem provers, some of which are programmed using intuitionistic type theory, and others of which are programmed using classical type theory. They arise in metaphysics seminars on higher-order logic when different higher-order logics are being investigated and considered. And perhaps most relevantly to our pluralist approach to higher-order logic, they arise in contexts where metaphysicians have adopted different higher-order logics as their foundational meta-theory for investigating the structure of the world.

Pluralism of various stripes about first-order consequence percolates up into our approach to higher-order logic. This is a general pattern. And a moment's reflection shows this to be an almost unavoidable point. Regardless of the specifics of our view of logical pluralism, if we think in some sense that classical and intuitionistic consequence are equally correct, then it is very hard to see how we could come to privilege a higher-order logic based on classical logic over a higher-order logic based on intuitionistic logic. Conversely, if we think that a monist approach to higher-order logic is the only viable approach, it is hard to see how this would leave any room for pluralism about first-order consequence.

In this manner, logical pluralism strongly suggests pluralism about higher-order logic. However, there is perhaps an even stronger case than this to be made. It may be the case that pluralism about first-order consequence actually requires a pluralist approach to higher-order logic. There is an old tradition of defining first-order consequence using the second-order quantifiers.[10] The basic idea is that a first-order claim is valid just in case it holds for every possible value of every term

[10] For instance, see Tarski [1983], Kreisel [1962], and Williamson [2013].

which occurs in the claim. In order to express this, we can replace every constant in the claim with a variable of the appropriate type, and universally bind each variable. If the only terms in the claim were first-order, this results in a first-order claim. For example, $a = b$ becomes $\forall x \forall y(x = y)$. But the majority of first-order claims contain second-order constants. As such, this process requires the use of second-order variables, and second-order quantifiers. For example, $Fa \vee \neg Fa$ becomes $\forall X \forall x(Xx \vee \neg Xx)$.

If we accept this account of first-order logical consequence, it seems pretty clear that pluralism about first-order consequence requires the truth of seemingly incompatible second-order claims. And that as such, it requires a pluralist approach at the level of the second-order quantifiers. This would in turn require pluralism at the level of the third-order quantifiers to interpret the second-order quantifiers. And so on. The pluralism percolates upwards. Now, a full, unconditional argument to the effect that pluralism about first-order consequence requires a pluralist approach to higher-order quantification would require us to argue that this is the correct account of validity. And while we are tempted by such a view, we shall not argue for it here. However, if we accept the intelligibility of second-order quantification, it is very difficult not to accept that the validity of a first-order claim at least requires the truth of its second-order universal generalization. And this is all it would take to require pluralism at the level of the second-order quantifiers.

One important upshot of situating a pluralist approach to higher-order logic within the context of pluralism about first-order consequence is that this provides us with precedent for the idea that there are a variety of equally legitimate consequence relations with which we might regulate our reasoning. We might worry that a pluralist approach to higher-order logic involves the idea that there is no objective answer to questions concerning the validity of arguments which appear in philosophy, math, and physics. And that given the centrality of valid inferences to these practices, we may well worry that the kind of normative pluralism which ensues is both unstable, and may lead us into confusion and falsity. But these issues of normative pluralism already arise at the level of pluralism about first-order logic. And there are several interesting and potentially fruitful avenues of response which have been explored.[11]

We will not go into much detail here, but one line which we find quite promising is that while there are a variety of equally legitimate available consequence relations, these only come to have normative force on our reasoning once we commit to using them to regulate our reasoning. The normative force comes solely from our commitment, not from any intrinsic feature of the logic. This dovetails nicely with the pluralist idea that there are a variety of equally viable higher-order logics which the metaphysician could adopt as their foundational metaphysical theory. The idea is that in so adopting a higher-order logic, we commit to reasoning within that framework. And it is only in virtue of this that said higher-order logic comes to have normative force on our reasoning. There is no intrinsic feature of any higher-order logic which generates a normative relation to our reasoning. As such, there is no confusion about whether a higher-order logic is normatively binding in a situation. There is no instability. We ought to infer in accordance with the logic we have adopted as our foundational theory.[12]

Even if one does not buy this account, the main point we want to end this section with is that higher-order logic is no worse with respect to issues of normative pluralism than pluralism about first-order logic, which though very contentious, is now a well established and respectable position to hold in the philosophy of logic.

2.3 The Set-Theoretic Multiverse

Since the work of Godel we have known that recursively axiomatizable theories which are strong enough to interpret Robinson arithmetic are either (classically) inconsistent, or incomplete. A corollary of this, supposing that ZFC is consistent, is that the axioms of ZFC do not settle every set-theoretic question. This is not just because of some esoteric, purpose built claims –

[11] See Caret [2016], and Blake-Turner and Russell [2018] for two interesting discussions of some of these avenues.

[12] We should note that this does not mean that we are always bound by this logic. We are always free to adopt a different logic. And after doing so, we would be bound to reason in accordance with the new logic. But it does mean that we may end up having our reasoning normatively bound by an inadequate logic. However, if we choose our logic carefully, our reasoning will be regulated by a logic which can well play the role of a foundational metaphysical theory. And as such, our normative commitments will be in harmony with a perspective which facilitates fruitful theorizing about the world.

such as the claim that the system itself is consistent. There are substantive claims which are left unsettled. Perhaps the most famous of these is the Continuum Hypothesis – the thesis that the powerset of the natural numbers has the same cardinality as that of the real numbers. This means that the exact meaning of the powerset operation is not pinned down by the axioms. There is an inherent fuzziness in taking the powerset of an infinite set. In light of this, mathematicians and philosophers of mathematics have investigated the view that we can sharpen our set-theoretic concepts in a variety of equally good ways.[13]. Let us elaborate a little on one of these views.

According to Hamkins [2012], there are many different concepts of set, corresponding to different sharpenings of the notion of the powerset. Each of these concepts is instantiated in its own set-theoretic universe. And each of these set-theoretic universes exist in exactly the same Platonic way that proponents of the universe view take their preferred universe to exist. Many of these universes have already been named and intensely studied, such as the constructible universe L, and very many forcing extensions, including Cohen's $L[G]$. These universes exhibit a great diversity of set-theoretic truths. For instance, some satisfy the continuum hypothesis, and some satisfy its negation.

Higher-order logic is intimately bound up with set theory. This has been appreciated by a variety of philosophers, including Quine [1970], Godel [1933], Linnebo and Rayo [2012], Shapiro [2012], and others. Just think of Quine's famous quip that second-order logic is set theory in sheep's clothing. Or the following passages of Godel's:

> only one solution [to the paradoxes] has been found, although more then 30 years have elapsed since the discovery of the paradoxes. This solution consists in the theory of types. (I mean the simple theory of types [...].)
>
> It may seem as if another solution were afforded by the system of axioms for the theory of aggregates, as presented by Zermelo, Fraenkel and von Neumann; but it turns out that this system is nothing else but a natural generalization of the theory of types, or

[13]For instance, see Field [1994], Balaguer [1995], Linsky and Zalta [1995], Vaananen [2014], Hamkins [2012], and Priest [In Progress]

> rather, it is what becomes of the theory of types if certain superfluous restrictions are removed.[14]

In particular, there is an intimate relationship between the concept of *all subsets* and the second-order universal quantifier. The gist of this relationship is as follows. If one has a domain D of non-set-theoretic objects, say numbers or groups, and a first-order theory T_D which is satisfied by D, the theory that one gets by augmenting T_D with a fragment of set theory which allows one to apply the powerset operation to D is essentially the same theory which results from augmenting T_D with classical second-order logic. Both the powerset and the second-order universal quantifier allow one to speak about and characterize all of the collections of objects in D. This is the take home point. When applied to any (first-order) domain other than set theory, set theory and second-order logic perform the same role. For instance, second-order number theory, and number theory plus ZFC allow one to prove much the same things about the numbers. See Shapiro [2012] for a more detailed account of this point.

Once this deep similarity been appreciated, it seems that the same reasons which motivate the view that there is an abundance of different interpretations of all subsets similarly suggest a view according to which there is an abundance of different interpretations of the second-order universal quantifier. Just consider the fact that one can prove in ZFC that there is a statement in pure second-order logic which is true if the continuum hypothesis is true.[15] If there are equally good available sharpenings of the concept of all subsets which differ regarding the continuum hypothesis, then it is strongly suggested that there will be equally good, different *sharpenings* of the second-order universal quantifier which differ regarding this claim. If one thinks the powerset operation is indeterminate, one ought to think that the second-order universal quantifier is indeterminate. To

[14] Godel, 1933, pp. 45-6. I took this particular quote from Linnebo and Rayo [2012].

[15] Let $\Theta_{POW}(E,U)$ be the formula $\forall X(\forall x\forall y(\forall z(E(z,x) \leftrightarrow E(z,y)) \rightarrow x = y) \wedge \forall x\forall y(E(x,y) \rightarrow (U(x) \wedge \neg U(y))) \wedge \forall x(X(x) \rightarrow U(x)) \rightarrow \exists y\forall z(X(z) \leftrightarrow E(z,y)))$. Let $\Theta_{PA}(U,G,z)$ be the formula $U(z) \wedge \forall x(U(x) \rightarrow (U(G(x)) \wedge \neg G(x) = z)) \wedge \forall x\forall y((U(x) \wedge U(y) \wedge G(x) = G(y)) \rightarrow x = y) \wedge \forall X([X(z) \wedge \forall x((U(x) \wedge X(x)) \rightarrow X(G(x)))] \rightarrow \forall x(U(x) \rightarrow X(x)))$. Let $\Theta_{\leq}(P,R)$ be the formula $\exists F(\forall x\forall y((F(x) = F(y) \rightarrow x = y) \wedge P(x) \rightarrow R(F(x))))$. Let $\Theta_{EQ}(P,R)$ be the formula $\Theta_{\leq}(P,R) \wedge \Theta_{\leq}(R,P)$. Let $\Theta_{EC}(Y)$ be the formula $\exists F(\forall x\forall y((F(x) = F(y) \rightarrow x = y) \wedge R(F(x))) \wedge \forall x\exists y(R(x) \rightarrow x = F(y)))$. Finally, let Θ_{CH} be the sentence $\exists E\exists U\exists G\exists z(\Theta_{POW}(E,U) \wedge \Theta_{PA}(U,G,z) \wedge \forall Y(\Theta_{EC}(Y) \vee \Theta_{\leq}(Y,U)))$. Now, Θ_{CH} has a model if and only if the Continuum Hypothesis holds. Similarly, there is a sentence $\Theta_{\neg CH}$, which has a model if and only if the Continuum Hypothesis does not hold. These definitions come from Väänänen [2021].

be clear, you do not have to think that higher-order logic is set theory in disguise to accept this point. The point is that doubts about the determinacy of 'all subsets' leads to doubts about the determinacy of second-order quantifiers.

Now, one might be quite skeptical of this point. One may object that higher-order logic is importantly different than set theory. One cannot prove in first-order ZFC that ZFC is (quasi-) categorical. However, one can prove, given a fixed domain of individuals, that the interpretations of the second-order quantifier, and third-order quantifiers, and so on, are categorical using higher-order logic. In other words, one can prove, using higher-order logic, that the second-order universal quantifier is determinate. And one cannot do this for set theory. Given this, one might then argue that the indeterminacy of the powerset does not suggest the indeterminacy of the second-order universal quantifier.

There are a few points to make in response to this. First, the 'proofs' of the determinacy of higher-order terms all explicitly use higher-order terms in the meta-theory. We use the second-order universal quantifier to show that the first-order quantifier has a unique broadest interpretation, just as we use the third order universal quantifier to prove the same about the second-order quantifier, and so on. But this process doesn't start at a unique top level. If we flip this on its head, it is as though we are appealing to a non-well-founded grounding relation. The 'proof' about the first-order quantifier is only substantive if the second-order quantifier is determinate. And the 'proof' about the second-order quantifier is only substantive if the third-order quantifier is determinate. And so on. If we are already suspicious of the determinacy of higher-order logic, we are unlikely to be swayed by this line of reasoning. From this perspective, these 'proofs' just beg the question.

Second, the 'proof' that second-order logic is categorical in this way relies on choosing full semantics rather than Henkin semantics, which in this context seems tantamount to simply assuming that the second-order quantifier is categorical. One can hardly be surprised that one gets categoricity if one builds it in. One does not get this result if one uses Henkin semantics.

And third, given the understanding that there is a set-theoretic multiverse – that the powerset operation is indeterminate – the categoricity 'proof' is illusory. The fuzziness of the powerset operation infects the proof. This just does not show up at the level where the proof takes place. Consider Vaananen's view, according to which intended models of ZFC are multiverse models which contain a number of universe models.[16] Vaananen shows that the class of multiverse models satisfy the exact same theorems as the class of universe models. As such, the truth of the multiverse is hidden, in terms of what one can explicitly prove (in the language of first-order ZFC). However, the universe models within a multiverse model can exhibit much more variation than appears at this level. For instance, the universe models in a multiverse model may differ with respect to the interpretation of the powerset operation. Now, imagine the categoricity proof 'simultaneously' taking place in each of the universe models in the intended multiverse model of set theory. In each of these, we get the result. Nonetheless, the construction of models of second-order logic in each universe model would result in different models. And so were we to 'inspect' the proof from a (multiverse) model-theory of ZFC perspective, we could 'observe' how the categoricity is an illusion.

The same kinds of reasons which motivate various multiverse views in set theory, also motivate a pluralist approach to higher-order logic. But there is a stronger case to be made. Suppose that we accept the intelligibility of second-order quantification, and think that there is a unique, determinate, unrestricted universal second-order quantifier (which obeys a full-comprehension principle). There is then a unique fact of the matter about how many extensional properties of individuals there are. In particular, supposing that there are at least countably infinitely many individuals, there is a unique fact of the matter about how many extensional properties any such individuals has. Applying this to the domain of set theory, we have that there is an unique fact of the matter about how many extensional properties members of any infinite set have. In other words, there is an unique fact of the matter about how many sub-collections there are of any infinite set.

This would not entail, in a strict sense, that the powerset operation is determinate. But it would mean that the only way that there could be different notions of the powerset which are equally legitimate, is if none of them *agreed* in full with the second-order quantifier. Now, of course none

[16] See Vaananen [2014]

of them can fully agree with the second-order quantifier when applied to the entire domain of set theory all at once, on pain of contradiction. But they would also need to not agree with the second-order quantifier when applied to any particular infinite set. Otherwise, some precisifications would include all (in the sense of the determinate second-order quantifier) of the existing sub-collections of a countably infinite set, while others would miss some sub-collections. Were this the case, it would be very hard to see how each of these precisifications are equally legitimate. And we cannot think of a reason why, for any given level of the cumulative hierarchy, some of the precisifications of the powerset operation in the multiverse would not agree with the second-order quantifier about what sub-collections there are. As such, we think that if we are in the business of second-order set-theory, multiverse views require a pluralist approach to higher-order logic.

One upshot of locating a pluralist approach to higher-order logic in this context is that it provides the higher-order pluralist with an answer to a potential collapse argument. According to a pluralist view, a variety of different higher-order logics are true of their intended subject matter. This in turn means that there are a variety of different interpretations of the higher-order logical constants out there in the world, which satisfy different acceptable higher-order logics in a perfectly standard manner. In other words, there are different kinds of propositions, properties, and operators out there in the world. But we may wonder what these claims strictly amount to.

We may initially want to fall back to an objective metatheory, and to realize all of the acceptable higher-order logics in this metatheory. The normal choice for such a project would be ZFC. And this could definitely be done. However, there are two problems with making the analogous move for higher-order logics. First, given the general commitments of the higher-order logic as metaphysics project, which involve that higher-order logic, not set theory, ought to be our foundational metatheory for metaphysical theorizing, it seems illicit to formulate higher-order pluralism in the language of set theory. After all, this would seemingly carry the metaphysical commitment that higher-order logics are really about sets. The upshot of this is that the metatheory to which we would have to retreat in order to formulate and state higher-order pluralism must itself be a higher-order logic. And this leads to the second issue.

Which higher-order logic are we to use in order to state pluralism about higher-order logic itself? Absent a proof that the choice of ground logic does not matter (of the kind Steel offers for his set-theoretic multiverse view in Steel [2014]), such a choice would seem woefully arbitrary. But worse than this, if the logic is sufficiently strong, it seems that we will be able to run a kind of collapse argument which interestingly mirrors Martin's argument that there is a broadest notion of set.[17][18] If we can define a notion of *'being quantifier-like'*, we can argue that there is a *'broadest'* among the quantifier-like properties of properties, by arguing that *'having some quantifier-like property of properties'* is itself quantifier-like, and that it is at least as *'broad'* as all the other quantifier-like properties of properties. And once this conclusion is in hand, we will be able to argue like this:

- 1 : There is a broadest quantifier-like entity of type $(\sigma \to t) \to t)$
- 2 : If there is a broadest quantifier-like entity of type $(\sigma \to t) \to t)$, it is the only candidate interpretation of $\exists_\sigma$
- C: So, there is only one candidate interpretation of $\exists_\sigma$.

Here $\exists_\sigma$ is the quantifier for type σ. These kinds of arguments seem to suggest that the kind of pluralist view we have been discussing is untenable. It collapses into monism.

[17] See Martin [2001].

[18] Any logic which interprets Classicism will be strong enough. The argument is very similar to Bacon's argument that there is a broadest kind of necessity. See Bacon [2018]. Classicism is the closure of the common core higher-order logic H under a natural rule of equivalence. H has the following rules and axioms:

- PC: All of the theorems of the propositional calculus
- UI: : $\forall_\sigma x\phi \to \phi[t/x]$ (where t is a term of type σ and no variable in t gets bound when substituted in ϕ)
- MP: From $A \to B$ and A, infer B
- Gen: From $A \to B$, infer $A \to \forall_\sigma xB$, provided x is not free in A;
- Ref: $A =_\sigma A$;
- LL: $A =_\sigma B \to (FA \to FB)$
- $\beta\eta$: $A \leftrightarrow B$, whenever A and B are $\beta\eta$-equivalent (A and B are β-equivalent when A is of the form $\phi[(\lambda v.A)B](x_1, ..., x_n.A)N_1, ..., N_n$, and B is of the form $\phi[A[B/v]][N_1/x_1, ...N_n/x_n]$. And they are η-equivalent when A is of the form $\phi[\lambda v.(Fv)]$, and B is of the form $\phi[F]$.)

Classicism is the logic which results from closing H under the rule E – If $\vdash A \leftrightarrow B$, then $\vdash \lambda v.A = \lambda v.B$.

We actually agree with this in a sense. From the perspective of any sufficiently strong higher-order logic, we can run a successful collapse argument. Much as we can do in the case of set theory.[19] Hamkins' solution in the case of set theory is to completely jettison the idea of a uniquely correct set-theoretic metatheory from which we can state the pluralist position.[20] Correspondingly, there is no uniquely correct meta-theoretic perspective from which we can run a collapse argument. And this is the line that we shall also adopt here.[21]

A collapse argument may be sound from the perspective of a particular higher-order meta-theory, but all that is established is that there is a broadest quantifier of each type according to that logic. But that logic itself has no claim to giving the uniquely correct claims about the available interpretations higher-order logic. As such, this kind of argument does not establish what will be the case from the perspective of different higher-order meta-theories. We think that this kind of answer is quite promising in dealing with the collapse argument against higher-order logical pluralism.

2.4 Indefinite Extensibility

Multiverse set theory is an elaboration on the idea that there is indeterminacy regarding the width of the cumulative hierarchy. There is also a popular view according to which the height of the cumulative hierarchy is, in a sense, indeterminate. This is the view that the iterative concept of a set is indefinitely extensible. The idea was popularized by Dummett.

> If we can form a definite conception of a totality all of whose members fall under the concept set, we can, by reference to that totality, characterize a larger totality all of whose members fall under it.[22]

The gist here is that one can never collect all of the sets. For any putative plurality of all of the

[19] See Martin [2001].

[20] See Hamkins [2012].

[21] It is also Clarke-Doane and my response to Bacon's argument. See Clarke-Doane and McCarthy [2023]. And it is very similar to the line that Warren adopts in dealing with collapse arguments against (first-order) quantifier variance. See Warren [2015] and [2022].

[22] Dummett [1993], p. 441

sets, there is an operation one can perform – a kind of diagonalization – which yields a set which one can prove was not among the original sets. The upshot of this is that there is no highest level of the cumulative hierarchy. Just how high is the sky is indeterminate.[23] With this understanding in mind, it is not hard to see how this too suggests a similar kind of indeterminacy about higher-order quantifiers.

Consider how high the hierarchy of types extends. We think that the same kinds of arguments which motivate the view that the cumulative hierarchy is indefinitely extensible also motivate the idea that the hierarchy of types is indefinitely extensible. This point is immediate if the kind of higher-order logic we are interested in is of the kind discussed in Linnebo and Rayo [2012]. There, it seems that the higher-order quantifiers of each type are almost being identified with successive applications of the powerset operation. Intuitively then, were we interpreting set theory in this higher-order logic, the sets would not all be found in the domain of the first-order quantifier. They would populate the domains of each type. Moreover, they index the hierarchy of types with the ordinals in one-to-one fashion. Arguments of the same kind which show that the iterative sets are indefinitely extensible also show that the ordinals are indefinitely extensible. So, if the ordinals are indefinitely extensible, so too are the higher-order types.

But we do not think this point is peculiar to the setup employed by Linnebo and Rayo. Typically, at least in metaphysical discussions of higher-order logic, it is assumed that there are a countable infinity of types. The types are indexed by the natural numbers. There are the first-order quantifiers, the second-order quantifiers, the third-order quantifiers, and so on for every natural number. There is a clear similarity between the applying powerset operation, and moving to a higher-order quantifier. Very few people think that the powerset operation can only be applied to finite sets. According to ZFC, the application of the powerset continues on for many infinite steps. Indeed, there is no reason to think it ever stops. Similarly, there is seemingly no reason to suppose that the iteration of higher-order quantifiers just stops at the finite level. Why is there not always a quantifier of a higher ordinality? You might think we can avoid the problem by stipulat-

[23] See Shapiro and Wright [2006] for more details about indefinite extensibility.

ing, in some sense, that there are only quantifiers of finite types. But aside from the fact that this seems just as unnatural as cutting off the ordinals at the finite level, it also comes with a significant meta-semantic bite.

As has been recognized by many authors, in order to give an account of the semantics of a quantifier of a certain type, we need to use a quantifier of a higher type.[24] There is an important sense in which each quantifier is dependent upon quantifiers of higher types. In particular we need to use a cumulative quantifier in order to give the semantics of a higher-order logic with a countable infinity of types. In using such a higher-order logic, it seems we are committed to the use of a cumulative quantifier. Just as we are committed to an infinite set if we want to quantify over all the natural numbers. This in turn requires the use of a still higher-order quantifier. And so on. Once we are on this train it is hard to see how it could have a final destination.

As we can, the reasons which motivate the view that the cumulative hierarchy is indefinitely extensible translate into reasons which motivate the view that the hierarchy of quantifiers is indefinitely extensible. Further, as was the case for multiverse views, we think that if we are in the business of higher-order set theory, we need to approach higher-order logic in a pluralist manner if we want to maintain that the cumulative hierarchy is indefinitely extensible. Though things are a little more subtle in this case.

Let us first note that if we are interpreting set theory in higher-order logic as Linnebo and Rayo do – where sets are taken not just to be entities of type e, but rather to live up throughout the type hierarchy – then it is fairly immediate that a pluralist approach to the higher-order quantifiers is required. Specifically, we would need to take the hierarchy of types to be indefinitely extensible. If the there were an upper bound on the types of quantifiers, there would be a corresponding upper bound on the sets.

But we think the point also holds, though in a more delicate manner, if we take sets to just be entities of type e, the predicate *is a set* to be an entity of type $e \to t$, and the predicate *is an element of* to be an entity of type $e \to (e \to t)$. You might think that if we stick to characterizing indefinite

[24]See Rayo and Uzquiano [1999], Williamson [2003], and Rayo and Williamson [2003] for more details.

extensibility in the language of first-order set theory, we can maintain that the cumulative hierarchy is indefinitely extensible, even in the presence of a completely determine second-order quantifier. All that is required for indefinite extensibility, if the collection method is strictly set-theoretic, is that there is no set of all of the relevant individuals. And there is no set of all sets. As such, given any set of sets, there are relative construction techniques which allow us to construct a larger set. The presence of a determinate second-order quantifier in no way changes this. Things might be completely determinate from the second-order perspective, but indefinitely extensible from the first-order set-theoretic perspective.

Strictly speaking, this is correct. But we think this ignores the spirit of the view that sets are indefinitely extensible. If we have explicitly second-order resources at our disposal, we do not think it makes much sense to just characterize indefinite extensibility in terms of first-order set theory.[25] The spirit of the claim is that the things which satisfy a concept cannot all be collected together. In any manner, whatsoever. If we can collect all of the sets together under one second-order property, and we can completely characterize this property in the higher-order language, the claim that sets are indefinitely extensible – in the sense that the property cannot be characterized by first-order set theory – is of seemingly little substance.

Compare the case of the natural numbers. If we restrict ourselves to finite collections, the natural numbers appear to be indefinitely extensible. For any finite set of natural numbers, we have a construction technique which generates a larger natural number. This would be a substantive claim, if we also thought that there were no infinite collections. But suppose we take ourselves to have all of the resources provided by first-order ZFC. With these resources we can gather together all of the natural numbers into one set. And if we interpret Peano Arithmetic in ZFC, we can determinately characterize structure of the natural numbers. If we can collect all of the natural numbers together in one set, and we can completely characterize this set in the language of set

[25] Unless we are explicitly interpreting our second-order resources in first-order set theory. There is a familiar result that second-order ZFC, with the second-order quantifiers so interpreted, is only quasi-categorical. Understood in this way, the second-order resources do not relevantly outstrip those of first-order set theory. But if we do think that our higher-order resources outstrip the first-order set-theoretic resources, then it does not do justice to the spirit of the claim that sets are indefinite extensibility to think that there is a perfectly determinate extensional second-order property *is a set*.

theory, the claim that the natural numbers are indefinitely extensible – in the sense that the set cannot be characterized using only finitistic resources – is of no substance.

If we want to maintain that the cumulative hierarchy is indefintely extensible, while acknowledging the intelligibility of higher-order quantification, we need to maintain that there is an amount of indeterminacy at the level of the higher-order quantifiers. We would want to rule out a case where the higher-order quantifiers are determinate to the point that they settle how many sets there are. We can express this, we think, in two ways, depending on which kind of indeterminacy we are going in for. On the one hand, we could say that a higher-order logic is acceptable only if models of the theory which result from closing first-order ZFC under it are quasi-categorical. On the other we could say that there are very determinate acceptable higher-order logics – models of the closures of first-order ZFC under them are categorical. But that various acceptable logics *disagree* with each other about the the height of the hierarchy of sets. For instance, it might be the case that one proves that there are not inaccessibly many sets, and another proves that there are.

Just as in the previous two cases, one of the significant upshots of these considerations is that they provide the pluralist with a crucial rejoinder to a prima facie very compelling point against the tenability of a pluralist approach to higher-order logic. A significant feature of the view that the cumulative hierarchy is indefinitely extensible is that it is very difficult to refer to set-theoretic reality, or *the* cumulative hierarchy in a fixed and univocal manner, precisely because it is extensible.

Part and parcel of the pluralist approach to higher-order logic is that there is not a uniquely correct higher-order meta-theory in which we can realize all of the acceptable higher-order logics. The view has it that the various acceptable higher-order logics are incommensurable. In a sense, this amounts to an admission that we cannot fully state our view. But if we accept that the cumulative hierarchy is indefinitely extensible (and thus that the hierarchy of types is indefinitely extensible), this kind of expressibility issue is not unique to the higher-order logical pluralist.

On this view, the monist also cannot say that their preferred higher-order logic has an intended interpretation. No quantifier of finite type can range over the interpretations of every quantifier of finite type. So in order to state their view, the monist has to invoke a cumulative quantifier. But this quantifier cannot give its own interpretation. For that, we need a still higher-order quantifier. And up, and up, we go. If the hierarchy of types is indefinitely extensible, there is no highest-order quantifier to which the monist can appeal.[26] The monist might respond that while they cannot talk about the intended interpretation of their preferred higher-order logic, they can nonetheless do something the pluralist cannot. For any given level of quantifier, they can express that there is an intended interpretation of that quantifier, using a quantifier a the next level of the hierarchy. But the pluralist can also express fragments of their view. If the goal is just to be able to express fragments, then both the monist and the pluralist are good. But if the goal is to be able to completely express their view, the monist and the pluralist are in the same boat. Inexpressibility will rear its head for any semantically open foundational theory. It's everyone's problem.

2.5 Quantifier Variance

Quantifier variance is, roughly speaking, the view that there are a variety of different first-order quantifiers, all of which are equally genuine, yet which differ regarding answers to basic quantificational questions. For instance, there is the quantifier of the mereological nihilist, according to which no things ever compose a further thing.[27] And there is also the quantifier of the mereological vitalist, according to which some thing compose a further thing only when they constitute a life.[28] These quantifiers disagree over the existence of a great many objects, including amoeba, porcupines, humans, and every other organism in the universe. Yet according to the variantist, these may be on a metaphysical par. Humans are in the domain of the vitalist quantifier. They are not in the domain of the nihilist quantifier. If these quantifiers are indeed on a metaphysical par, then there is no deeper fact of the matter about whether humans exist. They exist on the vitalist interpretation of existence, and not on the nihilist interpretation.

According to Warren [2017], there are a variety of different possible communities of speakers

[26] The exact same kind of issue applies to the monist about set theory. The universe of sets cannot be a set. And so the authentic set theorist is in a bind. Either they cannot express their view that there is a universe of sets; or they have to invoke some extra theoretical resources in order to do so, which amounts to an admission that their set theory was not the full foundational theory they claimed it to be.

[27] See Sider [2013].

[28] See Van Inwagen [1990].

of languages with the same syntax as ours, yet in which different quantified sentences are true. These are different languages, with subtle differences in the meanings of a variety of terms, including the quantifiers. Thus, members of different communities can utter homophonic sentences, both speaking truly, yet seemingly disagreeing. Moreover, a variety of these different languages are equally good for the purposes of carrying out our metaphysical investigations. They are capable of expressing every proposition which needs to be expressible. And in virtue of this, they are on a metaphysical par. This is quite a pragmatic take. For the more heavy-duty metaphysically inclined – those inclined towards metaphysical structure – this step may involve something along the lines of there being a variety of languages which cut nature at the joints equally well, in these sense of Sider [2011]. If reality comes along with an ontological structure, then it comes along with a variety of structures – there is not just an ontology room, there is an entire ontology hotel. Whatever one's account of metaphysical privilege, the point is that the quantifier variantist claims that the various quantifiers of the different languages are equally privileged in that sense.

We think the same kinds of reasons which motivate variance about the first-order quantifiers also motivate variance about the higher-order quantifiers. We will first consider the part of the view according to which there are a variety of possible communities who employ unrestricted quantifiers which differ in meaning. And then we will consider the part of the view according to which a variety of these languages are equally good for the purpose of metaphysical theorizing.

Warren's line of thought begins with proof-theoretically characterizing unrestricted first-order quantifiers. This is useful because it allows us to identify operators as such quantifiers across languages. We can do this for the higher-order quantifiers in much the same way. We can say that a term is an unrestricted second-order universal quantifier just in case it satisfies the classical introduction and elimination rules. With this in mind, it seems that there could have been communities of speakers, who speak languages with the same basic syntactic structure, in which different (unrestricted) second-order quantified sentences are true.

For instance, it seems as though there could have been a community of speakers, who have a term which satisfies the introduction and elimination rules, according to which only basic or fundamental properties exist. We can call this second-order quantifier the sparse quantifier. So, members of this community would have assented to the claim that *having positive charge* exists – or that *charge is instantiated*, depending on how one wants to phrase second-order existential claims. But they would have rejected the claim that *being beautiful* exists. Similarly, it seems as though there could have been a community of speakers according to whom the fundamental properties exist, as well as any property which supervenes on the basic properties. So, members of this community would have assented to the claim that *having positive charge* exists, as well as the claim that *being beautiful* exists (supposing that aesthetic properties supervene on the basic physical properties). More generally, we can say that there could have been a community who use the physical second-order quantifier – according to which every physically possible property exists – the metaphysical second-order quantifier, the classical logical second-order quantifier, and so on. Each of these possible languages comes along with its own unrestricted second-order quantifier, and these differ in meaning.

This does not just apply to the second-order quantifier. Consider the propositional quantifier. It seems as though there could have been a community of speakers who have an operator, which satisfies the inference pattern we take to be definitive of unrestricted propositional quantifiers, according to whom only the metaphysically possible propositions exist, and another possible community, who also have an operator which satisfies this pattern, according to whom only the classically logically possible propositions exist, and so on. As such, it seems as though these different possible communities make use of unrestricted propositional quantifiers which differ in meaning. This pattern seems to generalize to all types of higher-order quantifiers.

So, it seems that the reasons which support the thought that there are possible communities of speakers which have unrestricted first-order quantifiers which differ in meaning — for Warren, a top-down, inferentialist meta-semantic view — also seem to support the thought that there are possible communities of speakers which have unrestricted higher-order quantifiers which differ in meaning. However, you might object here that it was important in the case of the first-order quantifiers that these different possible communities all speak languages which not only have very similar syntax to each other, but also very similar syntax to the natural language that we speak. And further that the natural languages we speak do not make use of higher-order quantification. As such, the analogy breaks down.

This point is somewhat well taken. It is true that the relation between natural language and higher-order quantification is quite contentious. But this seems a little beside the point. We are not here arguing that natural language already makes use of higher-order quantification. And that as such, the higher-order framework is the best framework for carrying out our metaphysical theorizing. We are making this specific point conditional on already finding the higher-order framework attractive. The point is that if one is tempted by the higher-order approach, then the same kinds of reasons that motivate a variance view about the first-order quantifiers also motivate a variance view about the higher-order quantifiers.

But with this said, even if we do not make use of higher-order quantification in natural language, this in no way impedes the possibility of these communities of speakers. Thus it in no way impedes the availability of different possible interpretations of the higher-order quantifiers. Nor by the way, would it seem to preclude the possibility of adopting one of these languages. We do such things in the metaphysics room quite often. And so the point about the relation of higher-order quantification to natural language (on which we take no stand here) seems to be beside the point.

Let us now turn the privilege of these possible languages. Prima facie, there is reason to think that some of these different languages would be equally well suited for the purpose of metaphysical investigations. This is supported by considering specific cases. For instance, consider two extensions of the logic $CCIC$[29], one which proves the higher-order version of the continuum hypothesis, and one which proves the higher-order version of the negation of the continuum hypothesis. It is hard to see how one of these could carve nature at the joints, while the other is hopelessly ger-

[29]The axiom of Choice is the following: $\exists f \forall X(\exists y(Xy) \rightarrow Xf(X))$. The logic CC – Classicism + Choice – results from adding Choice to Classicism and closing under the rule E. The logic $CCIC$ results from adding an axiom stating that there are inaccessibly many individuals to CC, and closing it under the rule E. This can be stated as follows. A class X is at least as big as Y if there is no surjective function whose domain is Y and codomain is X. X and Y have equal size if they are at least as big as each other, and X is strictly bigger than Y if X is at least as big as Y and they do not have equal size. Say a class is small if it is strictly smaller than the class of all individuals. There is an inaccessible cardinality of individuals if (i) For any small class X, the class of subclasses of X is small; and (ii) If X is a small class of small classes, then the union of X— the class of members of members of X — is small.

rymandered. As well as a general argument. Given the minor premise that what properties of individuals there are in some way depends on what individuals there are, this point follows readily. If there are different languages in which the meanings of the first-order quantifiers are different, the meanings of the second-order quantifiers in those languages have to be correspondingly different. And if these languages are equally equally metaphysically privileged, it follows that there are languages which are equally metaphysically privileged, in which the meanings of the second-order (and higher-order in general) quantifiers differ.

You might object that while this is plausible in the case of the second-order quantifier, it is not so plausible in the case of the propositional quantifier. The pluralist approach to higher-order logic counts higher-order logics which differ with respect to the grain of propositions as legitimate. For instance, there is the metaphysical propositional quantifier, according to which propositions are precisely as fine-grained as metaphysical equivalence. And there is also Bacon's logical propositional quantifier, according to which propositions are precisely as fine-grained as logical equivalence.[30] It is natural to think that the language in which Bacon's theory is true is able to express more propositions than the language in which the metaphysical theory is true. The same point holds if we compare a language in which the physical propositional quantifier is true to the metaphysical language. If we pair this with Warren's account of the privilege of languages, according to which a language is equally metaphysically privileged just in case it is able to express all of the propositions which need to be expressible, we can see how there may be a problem. How can different languages be equally good for the theorizing about the foundations of metaphysics, if some are clearly strictly more expressive than others?

One thing to note, is that this kind of objection does not apply to all putative cases of propositional quantifier variance. Suppose we hold the plausible view that what singular propositions there are depends upon what individuals there are.[31] And consider the languages of the mereological universalist and the mereological vitalist. Suppose that both of these agree that metaphysical necessity is the broadest necessity, and that propositions are precisely as fine-grained as metaphys-

[30] See Bacon [2020]

[31] See Fritz [2016] for an example of such a view.

ical equivalence. Then the meaning of the propositional quantifiers will differ in these languages. The propositional quantifier of the mereological universalist will include the singular proposition that *the Eiffel Tower is in France*. And the propositional quantifier will not include this proposition. The objection then at most applies to a pair of languages which are such that one contains a propositional quantifier which stands in a very precise kind of relation to the other. Its propositions are strictly more fine-grained than the propositions of the other quantifier. However, even with respect to pairs of languages such as this, we think that we think that this rejoinder, while interesting, ultimately rests on a mistake. It relies on the availability of a neutral background standard of the fineness of grain of the propositions. And such a standard is not available in the context of variance about the higher-order quantifiers.

This is the propositional analogue of the relation between *bigger* and *smaller* languages in the first-order case. There the problem is roughly that of accounting for how it could be that a language whose first-order quantifier is intuitively contained within the quantifier of another language is as metaphysically privileged as the latter language. Surely the more expressive language – the bigger language – is better.[32] Much the same thought arises in the case of set-theoretic pluralism when considering a ground model and it's canonical inner model, such as V and L. The answer in the case of the quantifiers is that in order to make the comparison, one needs to be using a language, using a quantifier. Much as the answer in the case of set theory is that one needs to be making use of a background set-theoretic metatheory in order to be making the comparison.[33] From the perspective of the bigger language, the smaller language is not able to quantify over things which exist. It is ontologically impoverished. And as such, lacks metaphysical privilege. But from the perspective of the smaller language, the bigger language is quantifying over things which do not exist. It is ontologically perfidious. And as such, lacks metaphysical privilege. The exact same point applies here. From the perspective of the language of the logical propositional quantifier, the language of the metaphysical propositional quantifier is unable to express some real distinctions, and as such is expressively impoverished. But from the perspective of the the language of the

[32] See Eklund [2008], Eklund [2009], and Hawthorne [2006] for some examples of positions in this vein.

[33] See Hamkins [2012].

metaphysical propositional quantifier, the language of the logical propositional quantifier makes distinctions where there are none, and as such is expressively perfidious. Neither is privileged over the other.

So, it seems to us that the same kinds of reasons which motivate variance about the first-order quantifier also motivate variance about the higher-order quantifiers. But as with the other views we have discussed, we think that there is perhaps a stronger case to made. We think that the viability of this kind of variance about the first-order quantifiers may actually require a similar variance about higher-order quantifiers. There is a simple argument to this effect. It begins with the premise that in order to have an unrestricted first-order quantifier, we need to have a second-order quantifier. And it then observes that if the various possible languages all had the same second-order quantifier, we would be able to define the broadest first-order quantifier in each language. As such, there would a collapse, and we would end up with the same first-order quantifier in every languages.

We find it very plausible that in order to state, in a language, that the first-order quantifier (of that language) is unrestricted, we need to be able to employ second-order resources – we need a second-order quantifier. This is Williamson's point in *Everything*. The quantifier variantists needs to be able to make this claim. This is not unique to the quantifier variantist. We think the monist also needs to be able to make this claim. Everybody does.

You might think that we can get away with first-order set theory, giving an interpretation of the unrestricted first-order quantifier using our familiar model-theoretic resources. But we don't think this is right. For one thing, using model theory seems to commit us to sets. As such, the domain of the quantifier cannot be a set. It would need to be a class model. But this in turn would seem to commit us to classes. And then in order to interpret the quantifier we would need a domain which contains classes. And so on, and so on.

Perhaps though, we could make use of a set theory which does have a universal set. We could do this while maintaining classical logic if we adopted Quine's *New Foundations*[34]. A model theory based on NF requires other significant concessions though, which seem to get in the way

[34] See Quine [1937].

of the idea that we can quantify over absolutely everything. We won't get too deep into the weeds here, but the gist is that there are 'collections' which we can define which do not exist. These 'collections' seem to be well-defined, and well understood, but according to NF these do not exist. We cannot quantify over them.[35] When we adopt NF in place of ZFC, we trade a universal set for the universal validity of separation. According to NF, not every 'sub-collection' of a set is a set. As such, we are do not think that this really provides a solution.

Another route we might consider is to adopt a paraconsistent approach. We might adopt Naive set theory in order to interpret our first-order quantifier. This set theory also proves that there is a universal set. Of course, this theory is classically inconsistent. But a paraconsistent version can be developed.[36] Unlike the NF approach, we think that the paraconsistent approach actually has promise. It has been shown that we can have dialethic languages which are semantically closed in a variety of ways. They can, for instance, include their own theory of truth. One can provide the semantics for the language within the language. These languages can seemingly have a completely unrestricted quantifier. Of course, these languages are classically inconsistent. But they are paraconsistent. We are mostly concerned with classically consistent logics in this paper. And so while we find this idea intriguing, we shall leave it for now. We will take away the conditional point, that if we are working in a classically consistent setting, it seems as though in order to have an unrestricted first-order quantifier, we need to have second-order resources available.

So, we are reasonably satisfied with the idea that (in a classically consistent setting) in order to have an unrestricted first-order quantifier, we need to have a second-order quantifier. This much applies equally to the quantifier variantist and the quantifier monist. But if we pair this observation with the main thesis of quantifier variance, it seems that there needs to be different second-order quantifiers. For if there were a univocal second-order quantifier, there would be the same maximum interpretation of the first-order quantifier in every such language. In other words, we could run a

[35]For instance, if f is a function from a set X to the powerset $p(X)$ of X, then the diagonal set $[x \in X : x \notin f(x)]$ does not exist. The reason for this is that the formula $x \in X \land x \notin f(x) \land f : X \to p(X)$ is not stratified. Because $\exists y(y \in p(X) \land \langle y, x\rangle \in f \land f : X \to p(X))$ is not stratified.

[36]See Weber [2021] for details.

semantic collapse argument.[37] As such, it seems that in order for the variance position to avoid collapse, it actually needs to embrace a corresponding kind of variance about the higher-order quantifiers.

One important upshot of this is that with this precedence in hand, we can sketch an answer to the worry that a pluralist approach to higher-order logic immediately collapses into trivialism. Recall the way we presented the pluralist approach to higher-order logic. There are different higher-order logics which a foundational metaphysical theorist could adopt to investigate the structure of the world, and do an equally good job. One of the obvious worries with this is that it seems to require that classically contradictory claims would be true. For instance, suppose that both $CCIC$ and $CIC + \neg C$[38] can both well play the role of our foundational metaphysical theory. This would seem to require that the axiom of choice is both true and false. But a quantifier variance approach readily deals with this problem. The seemingly contradictory claims are not made in the same language. They are made in different languages. And as such, they are not contraries. Just as the set-theoretic pluralist is not straightforwardly contradicting themselves when saying that $ZF + C$ and $ZF + \neg C$ are equally good set theories, neither is the higher-order pluralist. There are different higher-order languages which are equally good with respect to the task of correctly describing the higher-order structure of the world. And in some of these languages choice is true, while in others is is false. We could say that what the metaphysician is doing when picking a higher-order logic, is picking a higher-order language in which to theorize. If we think this suggestion is immediately untenable, it is hard to see why we would think quantifier variance is tenable in the first place.

2.6 Conclusion

Higher-order metaphysicians are currently engaged in a fascinating foundational project. We share in the earnest excitement about the prospect of settling on higher-order logic as our foundational metaphysical theory. However, it seems that the dominant monist approach to this project rules out a variety of interesting views including logical pluralism, set-theoretic pluralism, and

[37] See Dorr [2014] for such an argument.

[38] Classicism + Inaccessibly Many Individuals + the negation of Choice.

quantifier variance. Not only do these views seem to require a pluralist approach to higher-order logic, but the reasons which directly motivate each of these views seem to similarly motivate such an approach. Logical pluralism clearly percolates up into a pluralist approach to higher-order logic. Pluralism about set theory naturally translates to pluralism about many higher-order claims. And it seems that all of the regular reasons to adopt quantifier variance apply equally to first and higher-order quantifiers.

As such, we think that proponents of these views have two options. Either they can reject the intelligibility of higher-order quantification, or they should investigate and adopt a pluralist approach to higher-order logic. We shall not take a stand here on which option they should take, though we must confess that rejecting higher-order quantification seems difficult. Higher-order metaphysics is in full swing. And it seems incumbent on us all to engage with it.

Similarly, we think that proponents of the higher-order logic as metaphysics program have two options. Either they can engage with these views, and come up with principled reasons to reject them, or they should investigate and adopt a pluralist approach to higher-order logic. Again we shall not take a stand on which option they should take here. Though we must confess that we find these views very tempting.

We have begun to carry out this kind of investigation in other work.[39] There we outlined a view according to which any higher-order logic which we could use to play the role of our foundational metaphysical theory is true. Roughly, this would involve us being able to use the logic to interpret all of our distinctively metaphysical theories, to elucidate informal metaphysical concepts, to function as the benchmark of the acceptability of a metaphysical theory, to act as the final court of appeals for metaphysical questions, and to provide us with sufficient metaphysical resources to construct empirically adequate, simple, and explanatory theories of any domain we might encounter.

We claim that every higher-order logic which meets these criteria accurately describes the metaphysical structure of the world. And that there are a variety of such logics. Roughly, the way

[39] See my *On the Plurality of Higher-Order Logics*.

this works is that the world comes along with a variety of higher-order structures. This again is similar to Hamkins' multiverse view of set theory. As we have mentioned, according to that view, there are a variety of equally real set-theoretic universes. And the manner in which there are a variety of equally good set theories, is that each one is satisfied by a genuine set-theoretic universe. And so our idea is that a higher-order logic correctly describes the metaphysical structure of the world just in case there is a higher-order structure which satisfies it in a standard Tarskian manner.

We think this approach is motivated by many of the same considerations which motivate logical pluralism, multiverse set theory, and quantifier variance. Moreover, we think that if we were to adopt this pluralist approach to the foundations of metaphysics, we could endorse logical pluralism, multiverse set theory, and quantifier variance. Indeed, we would very likely have to.

Whether or not one ends up persuaded to adopt a pluralist approach to higher-order logic, we would like to stress that we have found that our views on all of these matters, from logical pluralism, to multiverse set theory, to quantifier variance, to higher-order logic as metaphysics, have been greatly enriched by thinking about these topics in relation to one another. We highly recommend this kind of cross-pollination to proponents of all of the views we have discussed in this paper.

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Chapter 3: On the Plurality of Higher-Order Logics

Abstract

Metaphysicians are currently engaged in the exciting project of applying higher-order logic to the foundations of metaphysics, in much the same way that mathematicians apply set theory to the foundations of math. With respect to this project, we may ask whether there is one higher-order logic which is uniquely well suited. A monist answer would be: yes, there is one higher-order logic which gives the uniquely correct account of the foundations of metaphysics, and that when theorizing about everything from physics, to math, to modality we ought to use this logic as our all purpose metatheory. This is similar to a monist view in the philosophy of math according to which there is one theory (usually ZFC + large cardinals) which is uniquely well suited to providing the foundations of mathematics. A pluralist answer, on the other hand, would be: no, there are a variety of different higher-order logics which are equally well suited to be our foundational metaphysical theory. There are a variety of compelling reasons in favor of the pluralist answer. However, prima facie, there are significant reasons to think that such an answer is incoherent. In this paper, we will outline a **pragmatist pluralist** view of higher-order logic, argue that it is contentful and coherent, and defend it against the worry that it leads to an untenable degree of pluralism about intuitively non-metaphysical domains, such as physics.

3.1 Introduction

Everyone accepts that there are a variety of higher-order logics which we can characterize mathematically. Considered as abstract objects, most would agree they are equally legitimate. Of course some may be more interesting or useful than others. But this is a separate point. Far fewer would accept that there are a variety of equally legitimate higher-order logics, employed as they are by modern metaphysicians.

Consider the case of geometry. We can mathematically characterize a variety of pure geometric spaces. Considered as abstract structures, most agree that there are on par. But we may also think of the canonical application of geometry – to the structure of spacetime. Considered as theories of the structure of spacetime, very few would think that there are a variety of equally legitimate geometries.[1]

Within the higher-order logic as metaphysics community, higher-order logics are applied to the foundations of metaphysics.[2] Broadly speaking, the idea is that higher-order logic could stand in the same kind of relation to metaphysical theorizing that ZFC stands in to our classical mathematical theorizing. Roughly, we could embed all of our canonical metaphysical theories in it, have it provide a common stock of metaphysical axioms to which we can all appeal, use it to function as the final court of appeals for metaphysical questions, and use it to provide all of the resources we need for constructing empirically adequate and theoretically virtuous accounts of the metaphysics of any domain.[3]

Considered in this way, as a candidate for our foundational metaphysical theory, we may ask whether there is just one higher-order logic which is uniquely well suited. A monist answer would be: yes, there is one higher-order logic which gives the uniquely correct account of the foundations of metaphysics, and that when theorizing about everything from physics, to math, to modality we ought to use this logic as our all purpose metatheory. This is similar to a monist view in the philosophy of math according to which there is one theory (usually ZFC + large cardinals) which

[1] See Reichenbach [1957] for an example of one such maverick. More on this point later.
[2] See Williamson [2013], Dorr [2016], and Goodsell and Yli-Vakkuri [In Progress].
[3] See Maddy [2016] for a discussion of the foundational role that ZFC plays.

is uniquely well suited to providing the foundations of mathematics. A pluralist answer, on the other hand, would be: no, there are a variety of different higher-order logics which are equally well suited to be our foundational metaphysical theory.

There are a variety of reasons recommending the pluralist answer. Aside from pure interest, skirting arbitrariness, providing a foundation for a variety of very interesting views from (first-order) logical pluralism, to multiverse set theory, to (first-order) quantifier variance,[4] and accommodating the wide ranging experiences of theoretical metaphysicians,[5] it has the potential to provide a solution to the Benacerraf problem for higher-order logic.[6] This is roughly the problem of accounting for the reliability of our logical beliefs. This should be familiar from the case of set theory, where it has motivated a variety of philosophers and set theorists to investigate a pluralist reconception of the foundations of math.[7] However, there are good reasons to doubt whether the pluralist answer is viable. In this paper we will defend the claim that there is a contentful and coherent view in the vicinity of:

- **Pragmatist Pluralism:** A higher-order logic L is true if (i) L provides a foundation for some actual metaphysical practice M, (ii) M provides sufficient metaphysical resources to construct empirically adequate and theoretically virtuous theories of any non-metaphysical domain, and (iii) L is theoretically virtuous.

As we shall see, this is no easy task. It will require careful elaboration of the notions appealed to in (i)-(iii), and a number of refinements in order to navigate around crass relativism, triviality, and collapse. We will also defend the final version of the claim against a significant objection that it leads to too much pluralism about intuitively non-metaphysical domains, such as physics.

[4]See my *Precedent for Higher-Order Logical Pluralism* for more details on this point.

[5]We have in mind here an analogue for metaphysics of Joel Hamkins idea that that multiverse set theory vindicates the lived experience of working set theorists. See Hamkins [2012].

[6]See Benacerraf [1973] and Field [1988] for accounts of this problem for pure math. And see my *From Set-Theoretic Pluralism to Higher-Order Logical Pluralism: A Benacerraf Problem for Higher-Order Logic* for more details about the problem for higher-order logic specifically.

[7]See Field [1994], Balaguer [1995], Linsky and Zalta [1995], Hamkins[2012], Clarke-Doane [2020 (b)], and Priest [In Progress].

3.2 Pragmatist Pluralism

We need to elucidate three aspects of the view: theories being empirically adequate and theoretically virtuous, a metaphysical practice providing sufficient metaphysical resources to construct theories, and a logic providing a foundation for a metaphysical practice. Let us address these in turn.

Intuitively, an empirically adequate and theoretically virtuous theory of a domain D is a theory which we could use to accommodate, systematize, and explain any data we could ever encounter regarding the domain. For instance, an empirically adequate and theoretically virtuous fundamental physical theory is one which we could use to correctly and completely predict the outcome of any experiment we could ever run. We could use it to systematize and explain any physical phenomena we have ever, will ever, and could ever encounter. As a first pass, say that:

- **Empirical adequacy**: T is empirically adequate just in case (i) it is consistent with every physically possible event, and (ii) it correctly predicts every (relevant) feature of every (relevant) physically possible event.

These correspond to correctness and completeness conditions. Consider the domain of fundamental physics. If a theory T is consistent with every physically possible event, then it never incorrectly describes a physical event. It never gives us the wrong answer about the outcomes of any experiment we could run. It never makes a prediction which does not come to pass. This is a crucial feature. However, it does not seem sufficient for empirical adequacy. As a toy illustration consider the empty physical theory – the physical theory which makes no predictions whatsoever. This theory is trivially consistent with every physical event. To be empirically adequate a theory need not only not get things wrong, it needs to get things right. And so a theory also needs to (relevantly) fully describe every (relevant) event. It needs to make the (relevantly) complete predictions regarding every experiment we could ever run.

Empirical adequacy is important. But so is theoretical virtue. As another toy illustration, suppose that Newtonian Mechanics is the correct fundamental physical theory, and consider the (highly) infinitary theory which is just the list of all of its theorems. This infinitary theory would then be empirically adequate – it makes all of the same predictions as the standard formulation. Nonetheless, it does not seem like a very good theory. We want to adopt a standard notion of theoretical virtue here. As a first pass, say that:

- **Theoretical Virtue**: T is theoretically virtuous just in case it is simple, powerful, explanatory, parsimonious, and so on.

This of course is inherently vague. There is no way around that. Nonetheless, we think we should have a reasonably good grasp of whether a theory is theoretically virtuous or not, even though there are likely to be many borderline cases. At this stage, we are comfortable enough with this level of vagueness. Say that:

- **Acceptability**: T is acceptable just in case it is empirically adequate and theoretically virtuous.

Readers familiar with pluralist views likely have already spotted a significant issue. We have said that a theory is acceptable just in case it is consistent with every physically possible event, the correct predictions follow from it, and it is virtuous. But pluralism about higher-order logic engenders pluralism about meta-logical, modal and normative notions. As such, if we appeal to notions of these kinds in the formulation of our pluralist view, we may end up unintentionally committed to pluralism about pluralism.

Compare the case set-theoretic pluralism. Suppose that we want claim that every classically consistent set theory is true. If ZFC is consistent, then so are $ZFC + Con(ZFC)$ and $ZFC + \neg Con(ZFC)$. And so whether ZFC is true seems to vary depending on which set-theoretic metatheory we happen to be using. Aside from metaphysical instability, even Clarke-Doane, who is very inclined towards set-theoretic pluralism, grants that this kind of scenario undermines the epistemic motivation (its potential for solving the Benacerraf problem) for adopting a pluralist approach in the first place.[8] In a sense, this is just the old metaphysical problem of relativism rearing

[8] See Clarke-Doane [2020 (a)].

its head. Recall the old worry that by his own lights, Carnap's claim that every factual claim is either an internal claim or a senseless external claim, is itself either a claim internal to a particular framework, or a senseless external claim. And so his view is either senseless, or only true relative to some frameworks.[9]

Let us illustrate the issue. Suppose a community C, with the metaphysical practice M, have constructed a physical theory T. Suppose that C are about to run an experiment E to test T. Using the logical, meta-logical, and modal notions provided by their practice, C determine that the space of mutually exclusive and exhaustive possible outcomes of E, is O_E. They also determine that T predicts the outcome of E will be p – a specific member of O_E. They run the experiment. And the outcome is indeed p. On this basis, they take T to be a live option for their acceptable physical theory. Now suppose that another community $C*$, with the metaphysical practice $M*$, encounter C, and they have a spirited discussion about physics, in which C tell them about T and the successful experiment E. $C*$, being suitably impressed, decide to repeat the experiment. Unfortunately, things do not go to plan. First, using their logical, meta-logical, and modal notions, they determine that the space of mutually exclusive and exhaustive possible outcomes of E is O_E*, which is different than O_E. No big deal, they think. This is unsurprising. After all, the C and $C*$ subscribe to different logics, and different theories of modality. So, $C*$ decide to make use of the logic and modal theory of C. However, even in doing this, they get answers other than O_E and p. It turns out that the reason for this is that they are still relying on the meta-theoretic resources of $M*$ in applying these theories, rather than on those of M. Frustrated at this point, they decide to just run the experiment. And the result is indeed p.

The point is that communities with different metaphysical practices may disagree about the following kinds of claims:

- Consistency and Consequence Claims: Whether a certain theory physical theory T is *consistent* with a certain observation O. And whether p is a prediction of T. They may disagree about these because their practices are based on different notions of logical consequence.

[9]See Carnap [1950].

- L-Consistency and L-Consequence Claims: Whether a certain theory physical theory T is consistent with a certain observation O, according to a specific higher-order logic L. And whether p is a prediction of T, according to L. They may disagree about these because their practices differ with respect to what follows from what according to a specific set of rules.

- Virtue Claims: Whether a certain theory T is virtuous.

What are we to make of this? Well, it seems to us that the views of $C*$ are beside the point in this situation. M and $M*$ disagree about the higher-order logical facts concerning the space of possible outcomes of E according to T, what follows from T, and what follows from T according to M. But all of this seems irrelevant to the point that C can use T to correctly predict the outcome of E. As could the members of $C*$ if they fully embraced M (meta-theory and all). Intuitively, T is empirically adequate for C if they could use it to correctly predict the outcome of every experiment they could ever run. Regardless of whether another community of metaphysicians would think they are constantly making logical, meta-logical, and modal mistakes.[10]

This point holds even if we stipulate that the logical, meta-logical, and modal theories of C and $C*$ are both true. And that T is true. Given this, there is no determinate fact of the matter about whether the space of mutually exclusive and exhaustive possible outcomes of E is O_E or O_E*. Nor whether it is O_E relative to the logical and modal theories of M. Nor whether T predicts that the outcome of E will be p. But the plurality of meta-logical truths seem irrelevant to whether C *can use* T to correctly predict the outcome of any experiment they run.

Intuitively, this notion of usability seems as clear as day to us. With classical logic, ZFC, and the standard theory of physical possibility as our meta-theories, we can use classical mechanics to correctly predict the outcome of every typical macroscopic experiment, such as dropping bowling balls off of towers. With only the empty logic as our meta-theory, we cannot. With the same classical meta-theories, we cannot use classical mechanics to correctly predict the outcome of the double-slit experiment. But with the same meta-theories, we can use quantum mechanics to

[10]This is, we think, somewhat reminiscent of Kripke's views about the importance of practice with respect to rule-following. See Kripke [1982].

correctly predict the outcome. With the logic whose only inference rule is *From p, infer p*, we can use *true physics* (if such a thing makes sense) to also correctly predict the outcome. And so on. We want to say that the theories which are usable in this intuitive sense, to make correct and (relevantly) complete predictions regarding every actual event, are acceptable. But this may be chimeric. It would be good to be able to more formally capture this notion of usability, in the presence of logical pluralism. We think we have a way that works.

First, we need to suppose that there is an objective fact of the matter about which actual events there are.[11] And second, we need to make use of the observation that whether T (in conjunction with the logical, meta-logical, and modal theories of M) proves that the space of possible outcomes of E is O_E, and that the outcome of E will be p, relative to the meta-theories of M, relative to the meta-theories of M, relative to the meta-theories of M, ... , and so on, is independent of the plurality higher-order logical facts.[12] Say that:

- **M-Coherence:** A theory T is M-coherent just in case for every actual (relevant) event p, the space of events which T (in conjunction with M) proves is the space of mutually exclusive and exhaustive possible succeeding events of p[13] (relative to $M \rightarrow$), includes the event q, which actually succeeds p;

- **M-Predictive:** A theory T is M-predictive just in case for any (relevant) actual event p, *the* event which T predicts succeeds p, relative to $M \rightarrow$, is q, where q is the event that actually succeeds p;

[11] This appears to be unavoidable. We can't see how to get the view off the ground absent this assumption. Eventually, every view needs some objective ground on which to stand. Balaguer [1995] assumes there is an objective notion of logical possibility. Hamkins [2012] appears to assume that ZFC is objectively the case. Clarke-Doane [2020 (b)] assumes that there is an objective notion of *being finite*. So for the purposes of this paper, we shall assume that there is an objective fact of the matter about which physical events there are. We mean this in a very thin sense. We don't assume that there is any objectively correct way to characterize any of these events. Nor about which individuals they contain. We just need to assume that they are out there. And that which ones are identical to which is a fact that the world comes along with.

[12] For the rest of the paper, we will abbreviate 'relative to the meta-theories of M, relative to the meta-theories of M, relative to the meta-theories of M, ...' to 'relative to $M \rightarrow$'

[13] These needn't all be actual events. In fact, presumably most of them will not be. They will be various physically possible events (in the sense of $M's$ notions). As such, these events are mutually exclusive and exhaustive according to M, not in any objective sense.

- **M-Empirical-Adequacy:** T is M-empirically-adequate just in case (i) T is M-coherent, and (ii) T is M-predictive;

- **M-Virtuous:** T is M-virtuous just in case T is theoretically virtuous (in M's sense of theoretical virtue) relative to $M \rightarrow$; and

- **M-Acceptability:** T is M-acceptable just in case (i) T is M-empirically-adequate, and (ii) T is M-virtuous.

We think M-acceptability captures this invariant notion of usability. Of course we might be wrong about this. But we can't think of any counterexamples.[14] The upshot is that while there may be meta-logical, modal, and normative disagreement between the practitioners of true logics, our specification of acceptability does not land us in a relativistic regress. This strategy will prove useful in what follows.[15]

Let us now consider metaphysical resources. The idea of a metaphysical practice providing its practitioners with sufficient resources for constructing theories about a variety of domains is intended to be analogous to the idea of a mathematical practice providing sufficient resources for constructing theories. We should be quite familiar with this. Just think of differential geometry,

[14]For instance, one case which we though might be problematic would be if some community C, used the empty logic as their meta-theory M, and a trivial physical theory T. Consider an arbitrary actual physical event p. According to M and T, the space of mutually exclusive and possible succeeding events (according to M) includes every possible (in the sense of M) event. Provided that every actual event is among the M-possible events, T is M-coherent. (As it would hold for every actual physical event.) But it is not $M-$predictive. It might initially seem that it is (it did to us), as in a sense, it predicts that q (the event which actually succeeds p) will happen. However, it also predicts that every actual event (which it has the means to refer to) will succeed p. And so *the* event it predicts will succeed p is not q – there is no unique event it predicts will happen.

[15]One thing we should note is that (provided we don't think we will end up in a situation where, in some sense, a computer both does and doesn't compute a function) there is a cap on the kind and amount of disagreement there ever will be between true logics. They can only disagree at the level of what theories prove. They cannot actually disagree about what the results of physical operations will be. In a sense, a group of human metaphysicians deriving consequences from their theory T is a physical system, just like collection of particles. We can make this even more concrete by supposing that C have created computers (programmed to run in accordance with M), which they use to crank out consequences of T. If $C*$ cannot correctly predict the outcome of these computers running certain programs, then their theories are empirically inadequate (by their own lights). If $M*$ is a *successful* metaphysical practice for $C*$ (more on this in what follows), then they can in principle use it to construct an empirically adequate theory of the outcome of these computers running programs. If we think the world is such that there is a determinate fact of the matter about physical processes such as these, then in the end, their theories will have predict the correct outcomes. Even if they would still describe the whole situation as one in which the members of C have systematically programmed their computers to incorrectly follow the rules of M.

dynamical systems theory, and Hilbert spaces on the one hand, and constructing our theory of physical geometry, our theories of chaotic systems, and quantum mechanics, respectively, on the other.

The locution of our metaphysical practice providing us with resources for constructing theories may sound unfamiliar. And it may seem a little unclear what these metaphysical resources are. But we don't have anything arcane in mind. We just intend the general kinds of resources which are provided by our actual metaphysical theories, including the theory of validity, arithmetic, the theory of classes, modal theory, higher-order logic, and so on. Now, you might think that the only metaphysical resources which will prove to be indispensable are also mathematical resources – class-theoretic. So that there are no distinctively metaphysical resources. We will not argue this point here, though we disagree. If this your view, then discussion of metaphysics just reduces to the discussion of math.

In the same vein as the case of math, we should be quite familiar with the idea of a metaphysical practice providing us with sufficient resources for constructing theories about domains as diverse as physics and semantics. And also of a practice not providing sufficient resources. Still, it will be useful to illustrate with examples. The idea is most easily illustrated by considering any of a variety of projects, the goal of which is to reconstruct existing theories using only the resources of a metaphysical view which rejects some piece of orthodoxy that is appealed to in the standard versions of the theories.

A famous example of both points can be found in Hartry Field's project in *Science Without Numbers*.[16] As the title suggests, Field is a nominalist who rejects the existence of abstract objects, including all of the seemingly abstract quantities which are referred to by the standard versions of our physical theories. Nonetheless, Field accepts that mathematics is extremely useful for formulating our physical theories. At face value, this seems to commit us to the existence of these

[16] Other notable examples include the intuitionistic project of recapturing classical mathematics, Frege's project of reducing math to logic (see Frege [1884]), the neo-logicist program (see for instance, Wright [1983]), the Quinean/Sellarsian project of recapturing claims which refer to universals as nominalists (see Quine [1948] and Sellars [1963]), and Van Inwagen's project of recapturing common everyday truths about tables and chairs as a mereological vitalist (see Van Inwagen [1990]).

abstract objects. And so Field took it to be incumbent on the nominalist to provide nominalistically acceptable versions of successful physical theories. What he showed was that a metaphysical view which endorses full classical second-order logic[17] and substantivalism – the view that spacetime, and regions of spacetime, exist – has the resources to construct a nominalistically acceptable surrogate of Newtonian Mechanics.[18] We can characterize the geometric structure of spacetime (according to Newtonian mechanics) with a theory in the language of classical second-order logic with the physical predicates Bet and $Cong$. And we can also formulate the differential equations which give the dynamics of the system in this language.[19] This version of the theory only quantifies over regions of spacetime – this is how Field interpreted the second-order variables. As such, we can say that a nominalistic metaphysical practice which endorses full second-order logic, as well as substantivalism, provides sufficient resources to construct Newtonian Mechanics.

In contrast, it has been objected that this practice does not provide sufficient resources to construct the theory of quantum mechanics (or state-space theories in general).[20] Malament has argued that the standard formulation of quantum mechanics is naturally understood as quantifying over possible quantum states of the world. And that there is no nominalistically acceptable surrogate for these possible states of the system. Now, whether this is exactly correct involves some very technical, unsettled points. But the take home point is that if Malament is correct, which there seems to be some reason to think, then the metaphysical practice of the Fieldian nominalist does not provide them with sufficient metaphysical resources for constructing quantum mechanics, nor state-space theories in general.

Just as we can discuss the resources that the practice of Fieldian nominalism provides its practitioners, we can discuss the resources which are provided by every metaphysical practice any group of metaphysicians will ever have. With this in mind, we can say that:

- **Successful Metaphysical Practice**: Any actual metaphysical practice M is *successful* just

[17] Strictly speaking he only appealed to full second-order mereology restricted to the domain of regions of spacetime.
[18] Of course not everyone agrees with this. Malament [1982], for instance, complains that Field's theory does not allow us to capture the modal force of the laws.
[19] See Field [1980].
[20] See Malament [1982].

in case M provides sufficient resources to construct M-acceptable theories of every non-metaphysical domain (relative to $M \rightarrow$).

And finally, let us discuss what is involved in being able to use a logic to provide a foundation for a metaphysical practice. Consider the more familiar case of set theory and mathematics. Think of the role that ZFC plays for the community of classical mathematicians. Of course the relation of ZFC to classical mathematics is a contested point. We do not mean to suggest that it is not. But given space constraints, we are going to partially adopt Maddy's account.[21] The main features we are adopting are these:

- **Shared Standard:** Formal derivation in ZFC is the shared standard of what counts as a proof for the community of mathematicians.

- **Meta-Mathematical Corral:** All of classical mathematics can be embedded in ZFC. Every mathematical concept can be defined in terms of set membership. Every mathematical object can be defined to be a certain set. And via these definitions, all of the theorems of classical mathematics can be derived from the ZFC axioms.

Applying this to the case of metaphysics, we can say that if a community of metaphysicians can use a higher-order logic (i) as an all-purpose metametaphysical theory in which they can embed all of their canonical metaphysical theories of validity, classes, necessity, and so on, and (ii) as a shared standard of rigorous metaphysical argument, they can use it to provide a foundation for their metaphysical practice.[22] Say that:

[21] See Maddy [2016].

[22] A lot of interesting work has already been done showing that the higher-order logic $CCIC$ provides a foundation for the canonical metaphysical practice (which involves the classical theory of validity and identity, the S5 theory of metaphysical necessity, the ZFC theory of classes, the classical theory of mereology, and so on. We would like to stress here that there is no value judgement whatsoever intended by demarcating one subset of the metaphysical community as the canonical community. We mean only to making making a comment about the number of adherents of particular views). H is the logic which has the following rules and axioms:

- PC: All of the theorems of the propositional calculus
- UI: : $\forall_\sigma x\phi \rightarrow \phi[t/x]$ (where t is a term of type σ and no variable in t gets bound when substituted in ϕ)
- MP: From $A \rightarrow B$ and A, infer B

- **Foundations:** L provides a foundation for any actual metaphysical practice M (relative to $M \rightarrow$) just in case (i) formal derivability in L captures M's notion of what follows from what (relative to $M \rightarrow$), and (ii) the $M-$theories can be embedded in L (relative to $M \rightarrow$); and

- **Foundational Logics:** A higher-order logic L is *foundational* just in case L provides a foundation for any actual metaphysical practice M (relative to $M \rightarrow$), and L is M-virtuous.

Crucially, whether L is foundational does not need to be relativized to a specific metaphysical practice. The characterization appears to be invariant, even in the presence of higher-order logical pluralism. One major obstacle has been hurdled.

3.3 Higher-Order Quantifier Variance

With this characterization of the foundational logics in hand, we can take another step towards a contentful and coherent version of higher-order logical pluralism:

- **Pragmatist Pluralism * :** A higher-order logic L is true if it is foundational.

We are immediately faced with another significant problem. According to the view, there may be higher-order logics which appear to disagree about specific claims. For instance, suppose that $CCIC$ and $CIC + \neg C$ are both true. One says that the axiom of choice is true. And the other says that it is false. How is this not just a straightforward contradiction?

- Gen: From $A \rightarrow B$, infer $A \rightarrow \forall_\sigma x B$, provided x is not free in A;
- Ref: $A =_\sigma A$;
- LL: $A =_\sigma B \rightarrow (FA \rightarrow FB)$
- $\beta\eta$: $A \leftrightarrow B$, whenever A and B are $\beta\eta$-equivalent (A and B are β-equivalent when A is of the form $\phi[(\lambda v.A)B](x_1, ..., x_n.A)N_1, ..., N_n$, and B is of the form $\phi[A[B/v]][N_1/x_1, ...N_n/x_n]$. And they are η-equivalent when A is of the form $\phi[\lambda v.(Fv)]$, and B is of the form $\phi[F]$.)

Classicism is the logic which results from closing H under the rule E – If $\vdash A \leftrightarrow B$, then $\vdash \lambda v.A = \lambda v.B$. See Bacon and Dorr [Forthcoming] for a discussion of Classicism. The axiom of Choice is the following: $\exists f \forall X(\exists y(Xy) \rightarrow Xf(X))$. The logic CC – Classicism + Choice – results from adding Choice to Classicism and closing under the rule E. See Goodsell and Yli-Vakkuri [In Progress] for a discussion of this axiom and this logic. The logic $CCIC$ results from adding an axiom stating that there are inaccessibly many individuals to CC, and closing it under the rule E. See Goodsell and Yli-Vakkuri [In Progress] for a discussion of this axiom and this logic. A variety of authors have shown that we can embed the classical theory of validity, classical arithmetic, classical analysis, ZFC, the classical theory of identity, the theory of metaphysical necessity, and so on in this logic. See Williamson [2013], Church [1940], Goodsell and Yli-Vakkuri [In Progress], Dorr [2016], Bacon [2018] for a few examples of this work.

The basic idea, following a variety of philosophers including Kit Fine and Joel Hamkins, is to think that reality is constituted relative to a parameter, rather than absolutely. Fine discusses this kind of view with respect to time, and Hamkins discusses it with respect to set theories.[23] Consider Hamkins' set-theoretic multiverse. According to that view, there are a variety of mind-independent, Platonically existing set-theoretic universes, each satisfying a different set theory in a straightforward Tarskian manner. For Hamkins (it seems), that set-theoretic reality is constituted relative to these different set theories. And so Hamkins can say that '$ZFC + Ch$' is true of set-theoretic reality relative to a $ZFC + Ch$ universe, and that '$ZFC + \neg Ch$' is true of set-theoretic reality relative to a $ZFC + \neg Ch$ universe. There is no contradiction in this. Analogizing, we can say that the world comes along with a variety of higher-order structures, and that each foundational higher-order logic is satisfied (in a standard Tarskian manner) by reality, relative to a different structure.

What does it mean to say that the world comes along with a higher-order logical structure? Intuitively, the idea is that the world comes along with a first-order domain, a second-order domain, a propositional domain, an operator domain, and so on, all of which have various structural features.[24] We say 'intuitively' because strictly speaking, we are not claiming that the world comes along with *a* higher-order structure, as this would imply that this structure is an individual – an entity of type e. The more careful claim involves the use of an infinite cumulative quantifier, which allows us to talk all at once about entities of any finite type, expressing the existence of higher-order structure in this sense. Such a quantifier is already required in order to give a higher-order semantics for higher-order logic.[25] This is reminiscient of Hellman's structuralism without structures.[26]

But more than this is needed. What does it mean to say that the world comes along with a plurality of higher-order structures? Can we use a single cumulative quantifier to express the exis-

[23] See Fine [2005], and Hamkins [2012].

[24] For instance, you might think that the domain of propositions forms a Boolean algebra with respect to negation and conjunction, and that there are strictly more propositional operators than there are propositions.

[25] See Rayo and Williamson [2003] for details.

[26] See Hellman [1996].

tence of the variety of structures? It seems not, as this would leave us vulnerable to a higher-order variant of a 'collapse argument'.[27] There would be nothing barring us from using the cumulative quantifier to 'gather all of structures together' and to define the broadest quantifier of each finite type in terms of this. As such, there would be a quantifier of each finite type which stands out among the rest as the genuine unrestricted quantifier of that type.[28] It seems we may have to use a variety of cumulative quantifiers. But in order to successfully state the pluralist claim about structures, we would need to use all of these different cumulative quantifiers at once. If we can do that, it seems we would once again be in a position to run a collapse argument. We seem to either have to embrace collapse, or inexpressibility.

Strictly speaking, we think this is correct. However, there is a way to elaborate on the inexpressibility horn of this dilemma which has some promise. This same issue arises in a variety of domains, from (first-order) logical pluralism, to multiverse set theory, to (first-order) quantifier variance. Taking our cue from answers in each of these domains[29], we can say that the variety of different metaphysical practices across time come along with correspondingly different languages in which the meanings of various important metaphysical terms, including the higher-order quantifiers, differ. This is basically to adopt a kind of higher-order quantifier variance. The idea is that for each foundational logic, the world comes along with a corresponding higher-order logical structure. And while strictly speaking, we cannot assert the existence of these structures, or the truth of the foundational higher-order logics which they satisfy, we can assert that there are a variety of communities of metaphysicians across time who can. Say that:

- **M-Truth:** A higher-order logic L is M-true just in case there is a successful metaphysical practice M, who speak the language Q, such that 'L is true' is true-in-Q.

- **Pragmatist Pluralism **:** Each foundational higher-order logic is M-true for some successful actual metaphysical practice M.

[27] See Priest [2006] (pp. 203) for an example of an argument in this vein against (first-order) logical pluralism, Martin [2001] for an example of such an argument about sets, and Dorr [2014] for an example of such an argument for first-order quantifiers.

[28] See my *Precedent for Higher-Order Logical Pluralism* (Section 2.3) for more details.

[29] See Shapiro and Kouri-Kissel [2020], Hamkins [2012], and Warren [2015] and [2022], respectively.

This bakes in a measure of incommensurability which defuses the worry that the view leads to contradiction, and helps to answer the collapse worry. There is no actual community which comprises all of the metaphysical communities. And so there is no neutral background metaphysical language in which we can define the broadest unrestricted quantifiers of each type. Correspondingly, there is no structure which encompasses all of the structures, in any objective sense.

As such, we think that if both the kind of non-standard realism, according to which reality is constituted relative to different higher-order logical structures, and the kind of quantifier variance we need to adopt in order to 'express' the view, are viable, then **Pragmatist Pluralism** ** is a contentful and coherent pluralist approach to the foundations of metaphysics.

3.4 Too Much Pluralism?

Even if we are satisfied with the account thus far, there is still a worry that the view leads to an untenable level of pluralism about other domains of inquiry. The view naturally suggests that each theory of any domain D, which is acceptable relative to some successful metaphysical practice, gives an equally good account of D. We might worry that this is an untenable level of pluralism about various domains, from physics, to economics, to modality.

We don't think this is particularly worrying with respect to distinctively metaphysical domains such as modality. Indeed, it would be quite strange to adopt this kind of pluralist approach to higher-order logic and not be happy to accept, for instance, that different modal theories which are part of different successful metaphysical practices, give equally good accounts of modality. The same point applies to our theories of grounding, essence, mereology, identity, set theory, and so on.

But this may seem much more worrying with respect to more concrete domains such as physics. We might put the worry as follows:

- (i) Higher-order pluralism commits us to pluralism about some physical claims.

- (ii) Pluralism about physical claims, even granting language variance, involves contradictory events, such as a particle both having and not having a particular spin.

- (iii) We have never observed such events. Moreover, we have reason to think that there never have been, nor will there ever be, such events.

- (C) As such, higher-order pluralism is untenable.

We shall mostly focus on the second premise. But we should note that (iii) is not obviously correct. There are a variety of views according to which we regularly experience such events. They just seem normal and non-contradictory to us. For instance, according to dialethic theories of change we experience contradictory situations all the time.[30] Moreover, according to several interpretations of quantum mechanics, there are a great variety of microscopic situations in which particles are in a superposition of having and not having some property. While we never observe such situations (observation leads to collapse) these interpretations nonetheless say that they are overwhelmingly common. Indeed, on some construals of quantum mechanics, such as the many minds interpretation, similar macroscopic events occur regularly. For instance, events which we would describe as the dial determinately pointing to the left, are actually events in which the dial is in a superposition of pointing to the left and pointing to the right.[31] So it is not clear that we haven't actually experienced contradictory situations. But we ware willing to grant this for now.

We will give three examples of views which seem to endorse the idea that different acceptable physical theories give equally good accounts of the world, with a view towards showing that this is not so scary. First, consider the notion of simultaneity within the context of general relativity. Each different reference frame comes along with its own notion of simultaneity. Claims about space-time distance between events are reference frame independent. Whereas (some) claims about which events are simultaneous with which are dependent on which reference frame we are in. While one way to understand this is that there are no deep facts about simultaneity, another way is that our current best theory of the geometric structure spacetime tells us that there is a kind of pluralism

[30] See Weber [2021].

[31] For instance, see Albert and Loewer [1988].

about simultaneity.[32] We see no problem in being committed to pluralism about simultaneity, and more generally about physical claims which are reference frame dependent in this way. (If it is a problem, it is everybody's problem.) Such pluralism does not commit us to contradictory situations in any problematic sense.

Second, think of Reichenbach's view of the geometry of spacetime,[33] according to which, there are only determinate facts about its geometric structure relative to a *congruence* relation, of which there are an available variety. In a sense, this takes the relativity of general relativity a step further. Relative to a certain (admittedly very natural) choice, the geometric structure is what we typically take it to be — a pseudo-Riemannian manifold. But relative to another choice – roughly one according to which there are uniform forces which act on certain regions of spacetime to stretch or compress them – the geometric structure is Galilean. On this view, there is no determinate fact of the matter about whether spacetime is pseudo-Riemannian, or Galilean (or of any of a variety of other geometric structures). But theories articulated using either notion make the same empirical predictions – once we have adjusted for our choice of congruence relation, we agree about all of the physical claims. As such, we don't see too much of an issue in being committed to pluralism about physical claims of congruence, and consequently of geometric structure, in this manner. Such pluralism does not seem to commit us to contradictory situations in any problematic sense.

And third, think of the role that infinity plays in theories of physical geometry. We could have a view according to which there are a variety of available notions of infinity, and that there are only determinate facts about the geometric structure of spacetime relative to a notion of infinity. One way of having different notions of infinity would be to embrace a multiverse set theory, according to which a variety of classically consistent extensions of ZFC, including $ZFC + Ch$ and $ZFC + \neg Ch$, are true. On such a view, we still embrace monism about all of our classical mathematical theories, and our physical theories constructed in terms of them, up to the point of ZFC, but beyond this we endorse pluralism. So for instance, any features of arithmetic, analysis, and differential geometry which are independent of ZFC may vary. According to such a view, there would not

[32] See Fine [2005] for such a view.
[33] See Reichenbach [1957].

be a determinate fact of the matter about how many sub-regions there are of a countably infinite spacetime region. Supposing that such claims are beyond the realm of empirical detection, once we adjust for our choice of set theory, we would agree about all of the empirically detectable physical claims. Again, we don't see too much of an issue in being committed such pluralism. It does not seem to commit us to contradictory situations.[34]

Pluralism about physical claims which can be untangled relative to a choice of reference frame, congruence relation, or kind of infinity, does not seem too troubling. We see no reason to think that **Pragmatist Pluralism **** commits us to anything beyond this kind of situation. After all, in order to commit us to pluralism about some physical claim p, there has to be an M-acceptable physical theory T, for some successful metaphysical practice M, which proves p (in the language of M), and an $M*$-acceptable physical theory $T*$, for some successful metaphysical practice $M*$, which proves $\neg p$ (in the language of $M*$). And such theories pass a high bar. They would be like the 'different' formulations of general relativity relativized to different reference frames. As such, we do not think the view commits us to an untenable level of pluralism, even with respect to physics.

3.5 Conclusion

In this paper, we have argued that there is a contentful and coherent pluralist approach to the higher-order logic as metaphysics program. And we have defended it against the prima facie worrying objection that it leads to an untenable level of pluralism about intuitively non-metaphysical domains, such as physics. Of course this does not come close to being a knock-down argument in favor of pluralism. But it does establish that it is a viable contender among various approaches to the foundations of metaphysics. And this is progress enough for now.

[34]And if it does, then multiverse set theory can be empirically disconfirmed. In which case, we could seriously restrict the scope of our pluralism about mathematical concepts to accommodate this empirical fact.

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Chapter 4: Modal Pluralism and Higher-Order Logic

[1]

[1] This is an edited version of a paper I co-authored with Justin Clarke-Doane which appears in Philosophical Perspectives. See Clarke-Doane and McCarthy [2023]. thanks again to them for allowing me to include this version of the paper here.

Abstract

In this article, we discuss a simple argument that modal metaphysics is misconceived, and responses to it.[2] Unlike Quine's, this argument begins with the simple observation that there are different candidate interpretations of the predicate 'could have been the case'. This is analogous to the observation that there are different candidate interpretations of the predicate 'is a member of'. The argument then infers that the search for metaphysical necessities is misguided in much the way the 'set-theoretic pluralist' claims that the search for the true axioms of set theory is. We show that the obvious responses to this argument fail. However, a new response has emerged that purports to prove, from higher-order logical principles, that metaphysical possibility is the broadest kind of possibility applying to propositions, and is to that extent special. We distill two lines of reasoning from the literature, and argue that their import depends on premises that any 'modal pluralist' should deny. Both presuppose a monist approach to higher-order logic, which the modal pluralist, in the context of higher-order logic, should disavow.[3]. We consider the worry that, in a higher-order setting, modal pluralism faces an insuperable problem of articulation, collapses into modal monism, is vulnerable to the Russell-Myhill paradox, or even contravenes the truism that there is a unique actual world, and argue that these worries are misplaced. We also sketch the bearing of the resulting 'Higher-Order Pluralism' on the theory of content. One theme of the discussion is that, if Higher-Order Pluralism is correct, then there is no fixed metatheory from which to characterize higher-order reality.

[2]Thanks to Andrew Bacon, Mark Balaguer, Cian Dorr, Hartry Field, Michael Raven, Alex Roberts, Juhani Yli-Vakkuri, Tim Williamson, Katja Vogt, and Jin Zeng for comments.

[3]Much as the logical pluralist, the multiverse set theorist, and the first-order quantifier variantist should. See my *Precedent for Higher-Order Logical Pluralism*

4.1 The Argument from Modal Pluralism

Modal metaphysics is the theory of how the world could have been. Could God have failed to exist, if God exists, in fact? Could the mind have been distinct from the body? Could you have had different parents? Quine dismissed modal metaphysics as misconceived [1952]. His criticisms are widely agreed to have turned on confusions – between necessity and analyticity, and names and definite descriptions, for example. Once these confusions were resolved, there was little to which to object. So say the orthodox (Soames [2005, Part. VII], Williamson [2016]).

However, the orthodox have proceeded under an assumption of their own – namely, that if modal metaphysics is misconceived, then this is because its questions are unintelligible.[4] In recent years, this assumption has come under scrutiny (Cameron [2009], Sider [2011]). The problem that we wish to press (Clarke-Doane [2019, 2021]) is that there are a plurality of candidate interpretations of the expression 'could have been' giving intuitively opposite verdicts on paradigmatic questions of modal metaphysics ('intuitively' because of course the questions mean subtly different things under the different interpretations).[5] All of these interpretations are counterfactual -– concerning how the world could have been, as opposed to epistemic (concerning how it might be, for all we know or believe), or deontic (concerning how it is permissible for it to be, according to some set of norms).[6] For instance, we can ask how, as a matter of physical possibility, the world could have been, or, apparently, how, as a matter of logical possibility (fixing on a logic), it could have been. The worry is: even if we can also ask how the world could have been in a distinctively metaphysical sense of 'could have been', what is there to learn from this exercise except how select academics use the phrase 'could have been'?

Let us illustrate. A paradigmatic metaphysical necessity – i.e., a necessity under the interpre-

[4]There is another assumption that the orthodox have made, namely, that modal claims purport to state mind-independent facts, or at least facts. See Sidelle [1989], and Blackburn [1986] and Thomasson [2020], respectively, for arguments against this.

[5]Cameron's argument requires that modal operators are quantifiers over Lewis's 'ersatz worlds', while Sider's argument turns on an alleged reduction of modality, and on the availability of a non-modal analysis of logical consequence. The problem that we will consider does not make any of these assumptions.

[6]However, there is no single sense of how the world could have been, if the pluralist view described below is correct.

tation of ‘necessary’ on which modal metaphysics has focused – is that you could not have had different parents (Kripke [1980, 113]). Let us grant, then, that you could not, as a matter of metaphysical possibility, have had different parents. The problem is that you could have had different parents in a more inclusive, logical, sense of ‘could have’. The sentence ‘X’s parents are Donald and Melania Trump’ (where ‘X’ is your name) has a first-order model, after all (more on this below).[7] So, you could not have had different parents in one sense of ‘could have’ and you could have in another. The question of whether you could have had different parents period looks worrisomely like that of whether two lines making less than a 180 degree with a third must intersect, period – i.e., that of whether the Parallel Postulate is true, understood as a question of pure mathematics.[8] Yes, they must, as a matter of Euclidean geometry. But, no, they need not, as a matter of, say, hyperbolic geometry. This question is certainly misconceived!

Indeed, there is a deep analogy between worries about modal metaphysics and worries about the foundations of mathematics (Clarke-Doane [2019, Sec. 9], [2020, C.2], [2022]). The latter concerns itself with finding the true axioms of set theory. Is the Axiom of Choice true? Is the Continuum Hypothesis? What about ‘large’ large cardinal axioms like that of a Measurable Cardinal and beyond? The traditional critique of such questions is that, taken at face-value, they are simply unintelligible. ‘Infinitary’ mathematics makes no sense (Hilbert [1983/1926]). The best that we can do is to ask what follows from different sets of axioms that purport to speak of such objects as all uncountable subsets of the continuum. While this position does require that questions of arithmetic make sense (since questions of what does not follow from what are generally Π_1 undecidables, by Gödel’s Second Incompleteness Theorem), it is sometimes argued that the natural number structure is transparent in a way that uncountable structures are not (Feferman [1979,70]).

However, as in the modal case, there is another critique of the foundations of mathematics that does not turn on considerations of intelligibility, an argument from pluralism. This is that there are a rich plurality of candidate interpretations of the predicate ‘is a member of’ giving intuitively

[7] As will become clear, we do not have in mind a metalinguistic idea by ‘sense’, notwithstanding the reference to models.

[8] Compare van Inwagen, who writes that if p is metaphysically possible, then it is possible “tout court. Possible simpliciter. Possible period ... possib[le] without qualification” [1997, 72].

different answers to such questions as 'does every set have a Choice function?', 'is there a bijection between every uncountable subset of the real numbers and all of them?', and so forth. While the answer might be (determinately) 'yes' under the interpretation that the set-theoretic community happens to have adopted, myriad other interpretations are available. According to many of these, the answer will be 'no'.[9] Even if such questions make sense, they may be misconceived in much the way that the Parallel Postulate question would be if we were so misguided as to ask it. All we would learn by resolving such questions is something about ourselves, rather than learning what set-theoretic reality contains (Field [1998], Clarke-Doane [2020, Ch. 6]). This is a kind of deflation by inflation. It is the richness of reality that under cuts the significance of the universe that happens to dominate the interests of set theorists today.[10]

It is important to emphasize that the pluralist critique does not stem from worries about the determinacy of reference.[11] The worry is not that there is nothing in our practice or in the world that could make it the case that we mean is a member of by 'is a member of'. (This was Putnam's worry in his influential [1980].) That problem can be addressed by a theory of reference, as in Lewis [1983]. On the contrary, even if we all determinately mean $sets_1$ by 'sets', it would be enough to undercut the search for the true axioms of set theory that there are $sets_2$ just like $sets_1$, but satisfying different sentences – whether or not anyone happens to refer to them. (This is why arguments that there is a serious question whether the Continuum Hypothesis is true, like Woodin [2010] or Koellner [2014], make no appeal to metasemantics.) The pluralist critique is instead like Einstein's 'critique' of simultaneity. Even if we all refer to the same property with 'simultaneous' — simultaneous-relative-to-reference-frame-R, say — this does nothing to vindicate the search for

[9]We do not just mean that one can find a (set) model of the negations of the original sentences, if one can find such models of the sentences. That follows from the Completeness Theorem. We mean that one can find an intended (class) model of them. For example, Quine's NF is not just supposed by a pluralist to have an obviously unintended model in the ZFC sets (if $ZFC \vdash Con(NF)$), but an intended model, of the sort that Gödel takes ZFC itself to have. More on this in Section (4.5).

[10]Of course, we recognize a use/mention distinction. The question "if you pass two straight lines through another, and on one side of the original line the sum of the angles that the two lines make with the original is less than 180 degrees, must the two lines intersect on the side which sums to less than 180 degrees?" is not about the word 'lines'. But, understood as pure mathematics, an answer to it would only inform us about the word 'lines'. We already know what geometric spaces there are (among the candidates). We just learn which (classes) of them we are talking about.

[11]We will make claims about the metasemantics of higher order quantifiers. But the argument from pluralism does not depend on them.

what is really simultaneous with what. This is so even if there is a mind-and-language independent answer to the question out of our mouths. The problem is metaphysical. There are too many simultaneity-like relations in the neighborhood, giving intuitively opposite verdicts to the question of whether to events are simultaneous (as is the case with the Parallel Postulate, understood as pure mathematics). Modal pluralism says something similar about 'could have been the case'.[12]

4.2 Responses

How do the orthodox respond to this simple 'argument from modal pluralism' (as we will call it)?[13] To the extent that they respond, they respond with the following. If a sense of 'could have been' is less inclusive than metaphysical possibility, then it is really just another way of talking about what could, as a matter of metaphysical possibility, have been the case had certain conditions obtained. For instance, physical possibility is just metaphysical possibility, given the laws of physics (Williamson [2016, 462]).[14] The analogous response in the set-theoretic case says that talk of less inclusive notions of set is really just talk of the canonical notion of set, restricted somehow. The constructible sets (in the sense of Gödel [1940]), for instance, are just (all of) the sets that are also definable in the language of ZF with parameters of a special kind.

What, though, if a sense of 'could have been' is more inclusive than metaphysical possibility? In that case, the orthodox tell us that it is not *alethic* [Hale, 2013], *real* [Rosen, 2002, 16], *ontic*, [Kment, 2014, 31], or *objective* [Williamson, 2016, 459]. A sense of logical possibility according to which Hesperus could have failed to be identical to Phosphorus, for instance, is a merely *epistemic* or *verbal* kind of possibility. What do the italicized terms mean? Unfortunately, responders rarely tell us, and, when they claim to, their definitions do not rule out broader interpretations of 'could have been the case' than metaphysical possibility. For example, logical necessity is certainly

[12]We do not offer a full account of when one property is 'in the neighborhood of' another here. What will matter is just that the candidate interpretations of higher-order vocabulary are more like candidate interpretations of set-theoretic vocabulary than they are like an interpretation according to which, say, 'cat' means dog.

[13]For a fuller treatment of the responses considered in this section, other than the one from higher-order logic on which we will focus, see Clarke-Doane [2021].

[14]That is, for a proposition p to be physically possible is for the proposition $T \wedge p$ to be metaphysically possible, where T is the conjunction of the laws of nature. There are objections even to this claim (see Fine [2002]), but we will not pursue them here.

alethic in that it may satisfy the axiom $(T) : \Box p \rightarrow p$.[15]

Strohminger and Yli Vakkuri say that "the most straightforward way to characterize objective modality is negatively: it is what the modal words express when they are not used in any epistemic or deontic sense ... " [2017, 825]. But, again, according to this criterion, logical possibility is an objective modality. Williamson adds that objective modalities are also "not sensitive to the guises under which the objects, properties, relations and states of affairs at issue are presented" [2016, 454] and concludes that "identity [and distinctness are] simply objectively necessary ... "[2016, 454].[16] However, not even Williamson's additions give the desired verdicts. Just consider a reading of 'could have been' that validates the Necessity of Identity and Distinctness, but according to which there could have been no mathematical entities, you could have had different parents, and so forth (see, e.g., Priest [2008, 16.2 and 16.3]).[17]

Indeed, the reader might wonder what is accomplished by such exercises in conceptual explication. Suppose we manage to define 'objective' ('real', 'alethic', etc.) so that it is true that metaphysical possibility is the most inclusive objective interpretation of 'possible'. The original problem would get transposed. What is special about that interpretation of 'objective'? So it is not 'objectively' possible that there could have failed to be any numbers. Who cares, given that the world could have been that way (in some sense)? If we are interested in how the world could have been -– what we are calling counterfactual possibility– then appeal to terms like 'objective' only seems to help if we can establish that the objective possibilities exhaust the counterfactual possibilities. (We will examine the prospects for establishing exactly this when we discuss purported

[15]There is also the problem of what to say about senses of 'could have been' that are neither less inclusive nor more inclusive than metaphysical possibility. We will return to problems of incomparability below.

[16]For a sketch of kinds of logical possibility according to which identities can fail see Priest [2008, Ch.17]. (Did not Kripke prove that identities are necessary, appealing only to the idea of rigid designation [1971, 181]? If saying that names are rigid designators is to say that, e.g., 'Hesperus' and 'Phosphorus' refer to what they actually refer to in every world, then showing that 'Hesperus' and 'Phosphorus' are rigid designators does not show that the terms co-refer in every world. It shows that 'Hesperus' refers to Hesperus, and that'Phosphorus' refers to Phosphorus, in every world. If it means that 'Hesperus' and 'Phosphorus' co-refer in every world, if they do in the actual world, then Kripke assumes what he seeks to prove. See Cameron [2006].)

[17]Williamson is explicit that the mathematical truths are necessary (as is Kripke (1980, 37]). For instance, he says that "the structure of the hierarchy of pure sets ... seems to be a metaphysically non-contingent matter" (2017, 199). See also his [2016, 454]. (Again, we do not have in mind a metalinguistic idea by 'reading'. We clarify our meaning below.)

proofs of the existence of a broadest kind of objective possibility below.)

A slightly better response appeals to Lewis's idea of a natural kind – a property that 'carves nature at the joints'. As Sider [2011] points out, there is nothing to bar us from applying this idea to quantifiers, connectives, and operators. Perhaps, then, the orthodox could claim that modal metaphysics is not misconceived because it discovers how the world could have been under the most natural interpretation of 'could have been' (or 'it is possible that') [Nolan 2011, 322].[18] But it is hard to think of a reason to regard metaphysical possibility as more natural than, say, logical possibility. Metaphysical possibility is notoriously gerrymandered by comparison [Sider 2011, Ch. 12].

Some try to argue that what ties the apparently disparate metaphysical necessities – including that there are prime numbers, that you could not have had different parents, and that nothing can be two places at once – together is that they are all 'grounded in the nature of things' (Hale [2013], Lowe [2012], Fine [1994], Kment [2014]), where grounding and nature are hyperintensional phenomena on which we are supposed to have an independent grip. But, unless, a la Anselm, things can exist of their very natures, it could at most be grounded in the nature of things that if there are numbers, then, e.g., there are infinitely-many prime ones (Kment [2006, 267]). It could not be grounded in the nature of numbers that there are infinitely-many prime numbers. Moreover, someone worried that there is nothing special about our interpretation of 'could have been different' should just worry that there is nothing special about our interpretation of 'nature' (Clarke-Doane, [2019, Sec. 6]). Along with nature, let us introduce *shnature*. Even if it is, say, part of your nature that you have the parents that you have, it is no part of your shnature. Our question rearises. Why are not disagreements about nature like disagreements about the Parallel Postulate (understood as a pure mathematical conjecture)?[19]

[18]Nolan himself seems only to think that the relevant interpretation may be one of many natural interpretations.

[19]It is tempting to claim that the nature of things is special as compared to their shnature because we care about them! The problem is that we can all agree on what worlds there are (for the purposes of the critique). We just disagree about what to count as 'the same thing' across them – e.g., about whether to call a world with a lectern just like this one but made of ice this lectern. It is hard to see what factual (rather than practical) question could be at stake, if not just one of natural language semantics. (See Unger [2014, Ch. 4] for a worry along these lines). Also recall Quine's complaint: "When modal logic has been paraphrased in terms of such notions as possible world or rigid designator, where the displaced fog settles is on the question when to identify objects between worlds, or when to treat a designator

The most promising response to the argument from modal pluralism of which we are aware is that any interpretation of 'could have been' that is more inclusive than metaphysical possibility must apply to sentences, not propositions. (This is one way of giving content to the response that more inclusive interpretations are not real, ontic, objective, etc.) Did we not betray as much when we said that 'X's parents are Donald and Melania Trump' has a model? How the world could have been, understood as the question of which sentences have models (or which sentences have true substitution instances relative to some class of substitutions) is different than how the world could have been, interpreted at face-value. It is like the difference between the hypothesis that the Parallel Postulate is true in a structure, and the claim that it is true of physical spacetime.

This rejoinder does assume that there is a determinate question as to what propositions, or states of affairs are 'out there', as it were, and which ones are identical with which. Although one could certainly have doubts about this, we would like to examine what happens if one grants the assumption.[20] Then, if one could argue, from premises that even a 'modal pluralist' should accept, that metaphysical possibility is the most inclusive kind of counterfactual possibility applying to propositions, the pluralist critique would arguably be answered. The answer would be much the same as the answer the Gödelian gives to the objection: why search for the size of the continuum when $ZF(C) + V = L$ already proves that the Continuum Hypothesis is true? The answer is that, assuming a Measurable Cardinal, $V = L$ is a proper inner model of the universe of all sets.

The caveat 'arguably' is needed because even if metaphysical possibility is the broadest kind of counterfactual possibility, it could be indefinitely extensible -– just as the cumulative hierarchical conception of set might be the broadest conception of set while being indefinitely extensible

as rigid, or where to attribute meta-physical necessity" [Quine 1972, 492-493]. (Note that the 'shnature' difficulty is not generic, applicable to disagreements in physics, for example. Bracketing grue-like predicates, being a shgraviton is not instanced. By contrast, shnatures are instanced if natures are, where nature and shnature may be much alike, and comparably natural, except that, for example, it is no part of your shnature that you have the parents that you have. One might respond that, even if the shnature difficulty does not arise in physics, it arises in high-level sciences. Indeed, we believe that this is roughly correct. The question of how to carve things up in biology or psychiatry, for example, is practical, depending on our purposes, in a way that it does not in physics.)

[20]At least provisionally. It will emerge that part and parcel to higher-order logical pluralism is a higher-order quantifier variantism. (Again, this is not to grant that there is a unique such question, just as granting that there is a determinate question of what is simultaneous with what is not to grant that there is a unique such question.)

[Shapiro and Wright 2006].[21] In that case, there would not be a definite collection of all propositions over which metaphysical possibility operates [Rayo Manuscript], and a version of the pluralist critique would survive. There would be no unique stage, as it were, at which metaphysical possibility has been extended a determinate amount, and is broadest as compared to it at any other stage. This might vindicate the import of questions that are 'absolute' as the hierarchy expands. But it would undercut hope of pinning down that hierarchy. We discuss indeterminacy about the class of propositions, and a much more radical kind of pluralism about that class, below.

4.3 Higher-Order Monism: Two Variations

Various authors have argued, or contributed to arguments, that metaphysical possibility is the most inclusive kind of counterfactual possibility applying to propositions (we will not consistently add the qualification 'counterfactual' in what follows).[22] First, various authors have argued that there is a broadest, or most inclusive, kind of possibility that operates on propositions.[23] Second, some have gone on to identify the broadest kind of possibility with metaphysical possibility.[24] We focus on the first thesis since the second is often (though by no means always) assumed as a matter of definition (Kripke [1980, 99], Stalnaker [2003, 203], Lewis [1986], Williamson [2016, 459]).[25] It is worth noting, however, that if 'metaphysical possibility' is defined as the broadest kind of possibility, then it could turn out that traditional modal questions are misconceived in another way. It could be that virtually nothing of metaphysical interest is metaphysically necessary (Clarke-

[21] Although Shapiro and Wright focus on extensions of the 'height' of the cumulative hierarchy, it can also be argued that it is indefinitely extensible by 'width' using forcing. In the simplest case, a generic extension, $M[G]$, of a model, M, adds κ-many new subsets of ω to M, without adding (or subtracting) any ordinals (or collapsing cardinals), resulting in a 'wider' model of the same height.

[22] Indeed, some of the relevant authors do not seem to recognize a difference between counterfactual kinds of possibility and, say, deontic kinds. This is another obscurity afflicting the arguments that we will not pursue.

[23] See Bacon [2018], Bacon and Zeng [2022], Williamson [2016], Dorr [2016], and Yli-Vakkuri and Goodsell [Manuscript].

[24] See Williamson [2016], Dorr, Hawthorne and Yli-Vakkuri [Forthcoming], and Yli-Vakkuri and Goodsell [Manuscript]

[25] This is another juncture at which verbal disagreement threatens. Bacon [2020], Nolan [2011], and Mallozzi [2019] are adamant that metaphysical possibility is not the broadest kind of possibility. (Another reason we focus on the first claim is that its failure would seem to represent a more serious threat to metaphysics as practiced than the mere failure of the second.)

Doane [2019], Mortensen [1989], Nolan [2011]).

The arguments that there is a broadest kind of possibility in question take place in the framework of higher-order logic. Higher-order languages offer a natural tool for investigating counterfactual possibility (for those non-Quineans who have no qualms about higher-order entities). In modal logic, a kind of necessity (or possibility) is typically regimented with a sentential operator representing a phrase like *it is necessary that* (or *it is possible that*). A language with a particular sentential operator is suitable for articulating the theory of a particular kind of necessity. But in order to reason about kinds of necessity in general, it is useful to be able to quantify into the position of sentential operators, and to employ expressions with more complicated types, such as expressions which combine with operators to form sentences. These include operator identity statements, and statements in which operators are predicated with features like being a necessity.

The new arguments infer from allegedly natural logical or modal axioms that there is a broadest kind of possibility. Let us sketch two variations. The first argument, given in Bacon [2018], begins with the assumption that there is one true higher-order logic, HFE. This logic is an extension of the commonly accepted classical core logic H.[26]

HFE[27] proves that propositions form a Boolean Algebra. Propositions p and q are identical in this framework if and only if the proposition that $p \leftrightarrow q$ is identical to the top proposition. Moreover, HFE proves that entities of every type, other than e, form a Boolean algebra. If one is a

[26] H has the following rules and axioms:

- PC: All of the theorems of the propositional calculus
- UI: : $\forall_\sigma x\phi \rightarrow \phi[t/x]$ (where t is a term of type σ and no variable in t gets bound when substituted in ϕ)
- MP: From $A \rightarrow B$ and A, infer B
- Gen: From $A \rightarrow B$, infer $A \rightarrow \forall_\sigma xB$, provided x is not free in A;
- Ref: $A =_\sigma A$;
- LL: $A =_\sigma B \rightarrow (FA \rightarrow FB)$
- $\beta\eta$: $A \leftrightarrow B$, whenever A and B are $\beta\eta$-equivalent (A and B are β-equivalent when A is of the form $\phi[(\lambda v.A)B](x_1, ..., x_n.A)N_1, ..., N_n$, and B is of the form $\phi[A[B/v]][N_1/x_1, ...N_n/x_n]$. And they are η-equivalent when A is of the form $\phi[\lambda v.(Fv)]$, and B is of the form $\phi[F]$.)

[27] HFE is the logic which results from adding the axiom of *Functionality* – $\forall_\sigma x(Xx =_\tau Yy) \rightarrow X =_{\sigma\rightarrow\tau} Y$ to H, and closing under the rule E – From $A \leftrightarrow B$, infer $A =_t B$.

monist about higher-order entities (more on this below), one can hope that HFE gives the correct theory of the algebraic structure of these domains.

Suppose now that one accepts the following definitions:

- An operator X is a weak necessity just in case: (i) X applies to tautologies – XT, where T is the top proposition; and (ii) X is closed under modus ponens – $\forall p \forall q((Xp \wedge X(p \to q)) \to Xq)$.

- An operator X is a kind of necessity just in case for every weak necessity Y, it is Y-necessary that X is a weak necessity.

- A necessity X is at least as strict as another necessity O just in case $Y(Xp \to Op)$, whenever Y is a kind of necessity.

- X is the strictest kind of necessity just in case it is at least as strict as every other kind, and no other kind is at least as strict as it.

- An operator is a kind of possibility just in case it is the dual of a kind of necessity.

- Finally, the broadest possibility is a kind of possibility Y such that (i) every proposition that has any kind of possibility has Y, and (ii) it is necessary, with the force of every necessity, that (i) holds.[28]

Given these definitions, one can prove in HFE that there is broadest kind of possibility (formally $\lambda p(p \neq F)$, where F is the proposition that $p \wedge \neg p$).[29]

[28] These definitions were suggested by Andrew Bacon in personal correspondence.

[29] Here is the idea. First, HFE proves that every contradiction of HFE expresses the same proposition that $p \wedge \neg p$ does. It is part of the definition of a kind of possibility that F does not have any kinds of possibility. So, consider the operator being distinct from F. This only applies to every proposition other than F. If one could prove in HFE that this is a kind of possibility, and that it is necessarily a possibility, for every kind of necessity, then one could prove that being distinct from F is at least as broad as any other broadest kind of possibility. Moreover, given that HFE proves that kinds of possibility that are exactly as broad as each other are thereby identical, one could prove that it is the broadest kind of possibility. (Bacon establishes this in [2018].) So, relative to these definitions, one could prove in higher-order logic that being distinct from F is the broadest kind of possibility.

What of the second argument? Recall that in propositional and first-order settings one can extend extensional logic to modal logic by adding a modal sentential operator. One can do something analogous in a higher order setting. One can extend the language of simple-type theory, for instance, by adding a non-logical constant of type $t \to t$, which is interpreted as a modal operator. Then one can theorize about a specific kind of modality. But one can also extend the language by adding a predicate of operators Nec, of type $(t \to t) \to t$, which is interpreted as being a necessity operator. This lets one theorize about the kinds of necessity.

We will call such theories about kinds of necessity *modality theories*. An illustrative example is TN, due to Bacon and Zeng [2022]. This is a theory in the language of simply-typed functional type theory, augmented with a predicate of operators Nec. The theory results from adding two axioms to H_0 (a slight weakening of H), and closing the theory under a rule of necessitation. The first axiom is the axiom of $Necessity$. The idea is that an operator X is a necessity operator when X is $Closed$ – meaning that it satisfies the K axiom of modal logic with the force of every kind of necessity, and when X is $Logical$ – meaning that X obeys a principle analogous to the rule of necessitation of modal logic. The second is the axiom of $L - Necessity$. This says that $L := \lambda p \forall X(NecX \to Xp)$ -– the operator of being necessary in every sense of necessary – is itself a kind of necessity.[30]

[30] Here are the details:

- (i) K := $\lambda X.\forall pq(X(p \to q) \to Xp \to Xq)$ – An operator X has K just in case it satisfies the K axiom of modal logic, which is just to say that it is closed under Modus Ponens
- (ii) L := $\lambda p.\forall X(NecX \to Xp)$ – A proposition has L just in case it has every kind of necessity; if it is necessary in every sense of necessity.
- (iii) Closed := $\lambda X.(KX \wedge LKX)$ – An operator is Closed just in case it is closed under modus ponens, and that it is closed under modus ponens is necessarily the case for every kind of necessity.
- (iv) N := $\lambda X.\forall p(Lp \to LXp)$ – An operator X has N just in case if some proposition p is necessary in every sense, then Xp is also necessary in every sense.
- (v) Logical:= $\lambda X.(NX \wedge LNX)$ – An operator X is logical just in case it has N, and that it has N is necessary for every kind of necessity.
- (vi) LW := $\lambda p.\forall X(WX \to Xp)$ – One can think of LW as the the collection of all propositions which have all of the operators which have W. So a proposition has LW just in case it has all of the operators which have W. One can think of LW as the operator of possessing all of the W-operators.

Now the axioms of the theory are as follows:

These axioms are said to express features of the class of all necessity operators. They give a theory of the class of modal operators, in something like the sense that the ZFC axioms give a theory of the cumulative hierarchy of sets. We can now argue that, according to this theory, there is a broadest kind of possibility.[31] According to the theory, given any collection of possibility operators, there is a possibility operator which is the disjunction of these possibility operators. So, necessarily, in every sense of necessary, this applies to a proposition just in case some operator in the collection does. This ensures that there is a kind of possibility which is as broad as every actual kind of possibility. All that's left are merely possible kinds of possibility. As to these, the theory proves that it is impossible, in every sense of possible, that there could be a kind of possibility that is broader than every actual kind of possibility. This ensures that no merely possible kind of possibility is broader than any actual kind of possibility. The upshot is that there is a kind of possibility which is as broad as every kind of possibility, and every merely possible kind of possibility, and every merely possible merely possible kind of possibility, and so on. This argument does not build in assumptions about the grain of propositions. The conclusion that there is a broadest possibility follows just from the axioms about the class of modal operators.

If either of these arguments (or some variation on one) succeeds, then the argument from modal pluralism would seem to fail. There would be a broadest kind of propositional possibility. As the broadest, it would be special in the same way that a broadest kind of set (Gödel [1947]) would be. If this is metaphysical possibility, as it is commonly argued to be[32], then modal metaphysics would be in good order – at least as far the simple argument from modal pluralism is concerned.

- (1) Necessity: $NecX \leftrightarrow Logical(X) \wedge Closed(X)$ – The necessity operators are exactly those operators which are (i) closed under modus ponens with the force of every necessity, and (ii) are closed under necessitation for every kind of necessity with the force of every necessity.
- (2) L-Necessity: $NecL$ – Being necessary in every sense of necessity is itself a kind of necessity.
- (3) Necessitation: If TN proves p, then TN proves $NecX \rightarrow Xp$ – This is just a natural generalization of the familiar rule of necessitation from modal logic. TN proves that L is a kind of necessity. Indeed it proves that L is as broad as every kind of necessity. And so if TN proves p, then it proves Lp.

Now TN is $H_0 + Necessity + L - Necessity + Necessitation$.

[31] More precisely, according to the theory there are modal operators that are as broad as every kind of necessity, with the force of every necessity, with the force of every necessity, with the force of every necessity, and so on. See Bacon and Zeng [2022] for details.

[32] Ironically, Bacon and Zeng are not among those who accept this argument!

4.4 Assessment of the Arguments

Do these new arguments respond to the argument from modal pluralism? Prima facie, they do. What of the first argument? Well, it has two components, both of which appear responsive. First, there is the thesis that HFE is the correct higher-order logic. Although there are objections to HFE (Goodman [2019], Dorr [2016]), it turns out that, for a range of higher-order logics, one can prove that there is a broadest kind of possibility, in a very similar fashion.[33] So, not much seems to turn on this component. The second component concerns the definitions of possibility and broadest possibility. Although one can quibble about these, many metaphysicians suppose that there are features that all kinds of possibility share, and that the theorems of the background logic are necessary. So, they grant that there are necessary conditions for being a kind of possibility. Of course, one could still doubt that the class of kinds of possibility could be pinned down. Maybe the specifiable classes also contain propositional operators that are not kinds of possibility. However, not much seems to turn on this either. It might still be provable that the broadest element of the class is a kind of possibility.

The second argument, from modality theory, also appears responsive. There are two features of that theory which undergird the broadest possibility. The first is that, given some kinds of possibility, there is a kind of possibility which is having one of those kinds of possibility. Kinds of possibility would have to be sparse for this to fail. The second feature is that it is impossible, in every sense of possible, for there to be a kind of possibility that is broader than every actual kind of possibility. Although this might seem doubtful, one can make a good case for it. The gist of the case is that one can collect all of the kinds of possibility, the merely possible kinds

[33] In many (and perhaps even all) of the currently popular higher-order logics one can prove this. Many of the currently popular logics are extensions of Classicism (which is a slight weakening of HFE). It turns out that Bacon's proof does not rely on this extra strength. So these logics prove that there is a broadest possibility. Many of the other logics that have been considered, which are weaker than Classicism, such as Goodman's [2019] 'Agglomerative' logic, and the extended version of Dorr's [2016] 'Only Logical Circles' logic, also prove that there is a broadest kind of possibility. This suggests that a logic would need to be quite weak in order not to prove that there is a broadest kind of possibility. So, it may be difficult to avoid a logic in which one can prove that there is a broadest kind of possibility when trying to systematically theorize in a higher-order framework (Dorr, Hawthorne and Yli-Vakkuri [Forthcoming]). (That said, one can construct a weak extension of H_0 that captures some distinctive features of modal pluralism as described below, such as the idea that the notion of absolute possibility is indefinitely extensible.)

of possibility, the merely possible merely possible kinds of possibility, and so on for a countable number of stages, and define a broadest kind of possibility in terms of all of these kinds.[34]

On closer inspection, however, both arguments ring hollow. Their force turns on a monist approach to higher-order logic, which a modal pluralist has good reason to deny. Higher-order logical monism is the view that there is 'one true higher-order logic'. On the monist approach, there is a unique domain of propositions, a unique domain of propositional operators, a unique domain of properties of propositional operators, and so on. Each with a determinate algebraic structure, in which all of the modal propositions, modal operators, properties of modal operators, and so on, respectively reside. According to higher-order logical monism, there is a unique typed-hierarchy of higher-order entities[35] in much the same way that according to set-theoretic monism there is a unique hierarchy of sets.[36]

Let us see how the assumption of higher-order logical monism works in the two arguments. Monism says that there is a unique hierarchy of higher-order entities which is correctly described by some particular higher-order logic. The first argument shows that if HFE (or some other sufficiently strong logic) correctly describes this hierarchy, then there is a broadest kind of possibility. But what if there are (so to speak) many hierarchies, which are correctly described by different

[34]Let us say that Extensibilism is the view that even though there is a kind of possibility that is as broad as every actual kind of possibility, it is possible, in some sense of possible, that there could be a kind of possibility that is broader than every actual kind. (Roughly, there is a merely possible kind of possibility that is broader than any actual kind.) The problem with Extensibilism is that one can define an operator, in the language of TN, that applies to a proposition just in case any of the kinds of possibility do, or any of the merely possible kinds of possibility do, or any of the merely possible merely possible kinds of possibility do, and so on. This operator is at least as broad as any kind of possibility that appears anywhere in the Extensibilist modal hierarchy. And there would seem to be a case for its being a kind of possibility. If it were, then this operator would be the broadest kind of possibility. (See Bacon and Zeng [2022] for more on this argument. They call the operator $L*$.) However, one could respond to this argument that Extensibilism does not go far enough to capture the view that kinds of possibility are indefinitely extensible. To really capture the view one would need to extend the iteration process indefinitely, not just for a countable number of stages. If one were to do this, however, then one could no longer define an operator in TN in terms of all of the merely possible kinds of possibility'.

[35]Strictly speaking, this statement abuses quantifiers, talking as though hierarchies were things of type e. We will continue to speak in this way throughout, just as set-theoretic pluralists speak of universes of sets as though they were objects living inside of V. But we do not really believe that a hierarchy is an entity of type e, any more than the set-theoretic pluralist really believes that an intended (class) model of a given set theory is a set that satisfies some such theory, bracketing the case of set theories with a universal set.

[36]This analogy is alluded to in [Godel 1947]. And is especially clear given that type theory is itself occasionally advocated as a foundation for mathematics, like set theory. See Field [1998], Koellner [2014], Martin [1998] and [2001] and Woodin [2010] for contemporary discussion of monism about set theory.

higher-order logics? In other words, what if there are many different candidate interpretations of higher-order claims, just like the set-theoretic pluralist says that there are many different candidate interpretations of set-theoretic claims? If such a higher-order logical pluralism is true, then there are many hierarchies with various kinds of propositions, propositional operators, properties of propositional operators, and so on, each correctly described by a different higher-order logic – as in the set theory case. In this scenario, the argument at most establishes that, within a fixed class of typed-hierarchies, there is a broadest kind of possibility. It does not establish that in all typed-hierarchies there is. The argument shows nothing about hierarchies that do not obey HFE. The hypothesis that there is a broadest necessity could be true in some hierarchies, but false in others.

The problem with the second argument is similar. It seeks to establish that TN is true in the typed-hierarchy (or under the unique candidate interpretation of the higher-order language). Higher-order pluralism says that there is not a unique hierarchy, just as set-theoretic pluralism says that there is not a unique universe of sets. So, TN may be true in some of these hierarchies, but false in others. Since it does not make assumptions about the grain of propositions, it may be true in a larger portion of the 'pluriverse' of hierarchies than HFE. But there may be hierarchies in which TN is false, either because the underlying logic H_O is, or because some of the modal axioms of TN are. For instance, it is false in a hierarchy in which the axiom Necessity fails because there are operators that have Nec, but do not satisfy the K axiom of modal logic, and in hierarchies in which the principle of Extensibility, considered by Roberts, holds (Roberts [Manuscript]).[37] In the latter kind of hierarchy, there is not a broadest kind of possibility. So, like the first argument, the second at most shows that in TN hierarchies there is a broadest kind of possibility.

Of course, none of this shows that the arguments in question are invalid. The set-theoretic pluralist critique does not invalidate arguments for a broadest concept of set (to be discussed) either. It invites a reconception of their significance. Martin emphasizes this in his influential [2001]. His argument does not have the significance that one might hope that it did, if pluralism about the concept

[37]Roberts himself does not ultimately endorse this principle.

of well-ordering is true. Nor are we saying that modal pluralism is incompatible with higher-order monism. There is a 'boring' version of the view, on analogy with a boring version of set-theoretic pluralism (both discussed in the next section), that a higher-order monist could adopt. But, as we will see, higher-order logical pluralism is the natural complement to a more authentic kind of set-theoretic pluralism. Indeed, monism about second-order logic already engenders monism about the Continuum Hypothesis![38] (There is a second-order sentence with a model just in case CH holds, and a sentence with no model just in case it does not.) This brings to mind Woodin's quip: "[One could always rejoin that] instead of sets we should be studying widgets...[T]he axioms for widgets are obvious and, more-over ... resolve the Continuum Hypothesis" [2001, 690].

The import of the two arguments for the existence of a broadest kind of possibility is thus bound up with a monistic conception of the 'type hierarchy' (or, more carefully, the interpretation of the higher-order language). But why would this be objectionable to someone moved by the simple argument from modal pluralism with which we began? Well, because modal pluralism was specifically motivated by analogy to set-theoretic pluralism. It was motivated by analogy to a view about set theory that is the antithesis of higher-order logical monism. There are a rich plurality of candidate interpretations of the predicate 'is a member of', reflecting different universes of set. Why privilege some one interpretation of 'is a member of', but no one interpretation of, say, geometrical terms like 'point' and 'line'? Similar reasoning applies in the modal case. There are a rich plurality of candidate interpretations of the predicate 'is possible' reflecting, in this higher-order setting, different typed-hierarchies (or different candidate interpretations of the higher-order language). Why privilege one interpretation of this language over others?

To repeat: we are not saying that the arguments for a broadest kind of possibility are bound up with the vapid thesis that there are alternative interpretations of higher-order language (any more than the set-theoretic pluralist is saying that there are alternative interpretations of 'is a member of'). The view is not that if a sentence and its negation both have models, then a debate over what it expresses is misconceived! (That would make debate over dark matter misconceived.) Again,

[38] See McCarthy *Precedent for Higher-Order Logical Pluralism* (Section 2.3) for more details about this point.

the problem in the set-theoretic case is that, if pluralism is true, and CH is true in 'the' universe of sets, V, then there is a (class) model 'just like' V except $\neg CH$ is true there – as in the geometric case, with Euclidean and hyperbolic space, say. Likewise, even if we all refer to simultaneity-relative-to-reference-frame-R with 'simultaneous', the metaphysical fact of Special Relativity that there are myriad simultaneity-like relations (corresponding to all space-like hyper-surfaces) that we could have picked out instead with 'simultaneous' is enough to show that questions of what is really simultaneous with what is misconceived. (Obviously, there may be situations where it matters what is simultaneous-relative-to-reference-frame-R, just as there are situations where it matters what is true in a particular model of set theory. The point is that those are not in general the situations where it was thought to matter what is simultaneous, or what is true in 'the' universe of sets, V, period.) The pluralist critique of modal metaphysics is like this.

It is also important to emphasize that a pluralist need not deny that we take interest in some interpretations of the higher-order language over others. We also take interest in some interpretations of 'point' and 'line' over others (e.g., those that are useful for modeling physical spacetime). But no mathematical realist would suggest that some interpretations of 'point' and 'line', construed as pure mathematical entities, are thereby more real than others. There exist hyperbolic points and lines if there exist Euclidean ones. And so on for arbitrarily curved Riemannian (and pseudo-Riemannian) varieties. The set-theoretic pluralist says the same thing about sets. And the higher-order pluralist says the same thing about higher-order entities.

We have been illustrating how analogies between a monistic conception of the type-hierarchy and a monistic conception of sets tend to undercut the import of arguments for a broadest kind of possibility. But higher-order monism is in a way even worse off than monism about set theory. There is a still more intimate analogy between modal pluralism and a mundane kind of logical pluralism. According to this, there is a relation of, say, intuitionistic validity in what ever sense there is a relation of classical validity. This is independent of whether, as a matter of natural language semantics, intuitionistic consequence satisfies the criteria we associate with the word 'consequence' (and so is independent of whether pluralism in the sense of Beall and Restall [2005] is true). It is also independent of whether we ought to reason using intuitionistic logic. It is a claim of pure metaphysics whose negation is hard to even understand. But logical notions are naturally understood modally, indeed counterfactually. To say that it is logically possible that p is just to say that "things might (logically) ... have been" such that p (Rumfitt [2010, fn. 21]).[39] Moreover, the predicate 'might logically have been' admits of myriad interpretations, just like the predicate 'is a member of'! So, we really have a schema, giving different kinds of logical possibility, depending on the logic. For appropriate substitutions (and refinements),[40] we get classical possibility, intuitionistic possibility, LP-possibility, quantum possibility, FDE-possibility, and so on. Where does it end? There appears to be no principled "place to stop the process of generalisation and broadening" the generic notion of logical possibility – at least among kinds that we can actually use in a sustained way [Beall and Restall, 92]. If so, and if logical notions are counterfactual modal notions, then, trivially, there is no broadest kind of possibility. However, even if logical notions are not themselves modal, the intimate relation between logical and modal notions should give a higher-order monist pause (and the fact that higher-order logic, unlike set theory, is itself a kind of logic should give her even longer pause). Why would there be a broadest kind of possibility, but not a most inclusive kind of consistency?

The take-home message is that the two arguments for the existence of a broadest kind of possibility have bite only if one assumes higher-order logical monism. But this is exactly what, in this higher-order setting, a modal pluralist should deny. The arguments at most establish conclusions about particular kinds of higher-order entity. They are not even set up to make claims about a 'pluriverse' of them, if there is one. So, contrary to our initial assessment, the new response from higher-order logic to the argument from modal pluralism actually appears toothless.

[39] Validity is often taken to be a property of sentences. But this is by no means universal

[40] We need to decide what counts as a designated value when propositions are allowed to be other than true or false, for instance

4.5 What is Higher-Order Pluralism?

We have argued that the new responses to the argument from modal pluralism from higher-order logic do not stand up to scrutiny. Their significance depends on a monist approach to higher-order logic, which a modal pluralist should reject. What, though, is the alternative? Higher-order logical pluralism is the view that there are a variety of typed-hierarchies of higher-order entities, out there in the world in just the same sense in which the monist thinks that their hierarchy is out there in the world, which satisfy a corresponding variety of higher-order logics in a standard Tarskian manner. This is analogous to set-theoretic pluralism, which is the view that there are a variety of set-theoretic universes, which exist in the same Platonic sense the monist takes their universe to exist, which satisfy a corresponding variety of set theories in a standard Tarskian manner. But what precisely does this claim mean? In what language, and against what background theory, do we say this?[41]

It is tempting to respond by simply constructing a higher-order theory. We want to formulate pluralism about propositions, propositional operators, properties of propositional operators, and so on. So, we might formulate pluralism about propositions by saying that there are many different (unrestricted) proposition quantifiers, pluralism about about propositional operators by saying that there are many different propositional operator quantifiers, and, in general, pluralism about entities of type σ by saying that there are many different (unrestricted) quantifiers of type $(\sigma \to t) \to t$. For example, we might formulate the existence of Fregean propositions by writing: $\exists^1_t p, q((p \leftrightarrow q) \to p =^1 q)$, and we might formulate the existence of some non-Fregean kind of proposition by writing: $\exists^2_t p, q((p \leftrightarrow q) \land p \neq^2 q)$. We might then assert our general propositional pluralism by introducing the predicate *is a propositional quantifier*, and maintaining that there are different entities that satisfy this predicate. We could then hope to prove that the conjunction of these claims in $H_1 + H_2 + \ldots + H_n$ (the axioms and rules of H stated for all of the sets of quantifiers) is consistent with whatever non-modal truths that we wish to be monists about (surely there are some!). Such a proof would proceed as in the set-theoretic case.

[41]Thanks especially to Andrew Bacon and Alex Roberts for pressing us on this, and for the criticisms to follow.

An initial worry with this is that the procedure would need to be generalized in a way that it need not be in set theory. One can express set-theoretic pluralism 'all at once' by quantifying into binary predicate-of-individuals position. Nothing similar is possible in the higher-order case -– no quantifier can quantify over all types. But we see no bar to introducing a cumulative quantifier and using this express the existence of different hierarchies. Nor actually, in just using a countable infinity of finite quantifiers at once (as it were), to express the claim. After all, such devices are used at several places in expressing the axioms of higher-order logic in the first place.

But the deeper point is that the disanalogy is premised on monism about predicate-of-individual (second-order) quantifiers. This is something that no set-theoretic pluralist should accept. Monism about second-order quantification is tantamount to monism about 'all subsets'! So, not even a set-theoretic pluralist can adequately express their view in the envisioned way.[42] A pluralist, whether higher-order or set-theoretic, can only ever write down a provisional theory of their pluriverse. As we will see, this is integral to pluralism.[43]

The real problem with the envisioned formulation of pluralism, whether higher-order or set-theoretic, is that one cannot state pluralism about a kind of potential metatheory, like set theory or higher-order logic, using a theory of that kind. Any such metatheory will take itself to be broadest. What makes pluralism about foundational theories interesting is exactly that it makes metatheoretic concepts framework-relative. Nobody denies that geometric concepts are framework-relative. This is because they can all be realized in a single metatheory, like set theory. Set-theoretic pluralism, by contrast, precludes any such stable background arena. From the standpoint of any given set theory, other 'universes' are mere (set) models inside its own V. In other words, such a formulation is vulnerable to a 'collapse argument'.[44]

[42] See McCarthy *Precedent for Higher-Order Logical Pluralism* section (2.3) for more details.

[43] Much less radical kinds of pluralism about propositions do not encounter the same problem of articulation. Fritz [2019], for instance, admits two kinds of proposition. But the resulting view is monist about higher-order logic (with a fixed logic of propositions). It simply admits of fine grained proxies for propositions over which fine grained proxies for kinds of possibility could operate. (This is related to a version of set-theoretic pluralism according to which there is a background concept of set with respect to which all others are restrictions. See below.)

[44] Thanks to Alex Roberts for formulating the following simple pluralist theory, as well as the collapse argument in personal communication. Consider a very weak pluralist theory, according to which, putting it informally, there are just two type hierarchies: one in which entities at all types are individuated by co-extensiveness and one in which entities at all types are individuated by the principles of Classicism (where the type hierarchies are 'pure' in the sense

The upshot is that higher-order logical pluralism, like set-theoretic pluralism, involves a kind of quantifier variantism.[45] So, there is no formal theory in which the higher-order pluralist can assert strictness across hierarchies, just as the set-theoretic pluralist cannot compare broadness of sets across the pluriverse (more on this shortly). The authentic pluralist claims that whether a

that the only basic type is t, the type of formulas). The signature for the theory contains the material conditional $\rightarrow$: $\langle\langle\rangle\langle\rangle\rangle$, negation $\neg : \langle\langle\rangle\rangle$, and two kinds of universal quantifier for each type σ, $\forall^1_\sigma : (\langle\sigma\rangle)$ and $\forall^2_\sigma : (\langle\sigma\rangle)$. There is an infinite set of variables V which is disjoint from the set of constants and a type assignment function on V which guarantees there are countably many variables of each type and that no variable of one type is a variable of another type. Given the standard definition of type σ identity in terms of Leibniz equivalence, we have two different relations of identity in this setting:

- $x =^1_\sigma y := \forall^1 X, Y(Xx \leftrightarrow Yy)$, where $x, y : \sigma and X, Y : \sigma \rightarrow t$
- $x =^2_\sigma y := \forall^1 X, Y(Xx \leftrightarrow Yy)$, where $x, y : \sigma and X, Y : \sigma \rightarrow t$

The theory then looks like this:

- PL: All instances of propositional tautologies.
- MP: From A and $A \rightarrow B$, infer B
- Gen 1: From $A \rightarrow Fx$, infer $A \rightarrow \forall^1_\sigma F$ when x does not occur free in A
- Gen 2: Gen1: From $A \rightarrow Fx$, infer $A \rightarrow \forall^2_\sigma F$ when x does not occur free in A
- UI 1: $\forall^1_\sigma F \rightarrow Fa$
- UI 2: $\forall^2_\sigma F \rightarrow Fa$
- Extensional β :$(\lambda v_1 ... v_n.\phi)a_1 ... a_n \leftrightarrow \phi[a_i/v_i]$, where $v_1, ..., v_n$ are any distinct variables, $a_1, ..., a_n$ are any terms of types such that $v_i, a_i : \tau_i$ for all $1 \leq i \leq n$ and each v_i is substitutable for a_i in ϕ
- Ind 1: $\forall_1 X \forall_1 Y \forall_1 x_1 ... \forall_1 x_n((Xx_1 ... x_n \leftrightarrow Yx_1 ... x_n) \rightarrow X =^1 Y$
- Ind 2a: If $A \leftrightarrow B$ is a theorem not derived using Ind 1, then so is $A =^2_t B$
- Ind 2b: If $Mx =^2_\tau Nx$ (with x not free in M or N) is a theorem not derived using Ind 1, then so is $M =^2_{\sigma\rightarrow\tau} N$

The problem with this theory is that there will be a type of collapse argument which leads to the conclusion that $\forall^1$ and $\forall^2$ are identical1 and identical2 (in what follows let x be a variable which does not occur free in F):

- (1) $\forall^1_\sigma F \rightarrow Fx$ (UI 1)
- (2) $\forall^1_\sigma F \rightarrow \forall^2_\sigma F$ (2, Gen 2)
- (3) $\forall^2_\sigma F \rightarrow \forall^1_\sigma F$ (UI 2, Gen 1)
- (4) $\forall^1_\sigma F =^2 \forall^2_\sigma F$ (2, 3, Ind 2a)
- (5) $\forall^1_\sigma =^2 \forall^2_\sigma$ (4, Ind 2a)
- (6) $\forall^1(\lambda X.(\forall^2_\sigma X \leftrightarrow \forall^1_\sigma X))$ (2, 3, Extensional β , Gen 1)
- (7) $\forall^1_\sigma =^1 \forall^2_\sigma$ (6, Ind 1)

We expect that this type of problem will afflict any statement of higher-order pluralism in a single-sorted language.

[45] Although quantifier variantism in the first-order setting has been discussed at length, quantifiers variantism about higher-order quantifiers seems not to have received the attention that it deserves.

quantifier is unrestricted is perspective (metatheory) dependent. From the perspective of a logic L, L's quantifiers are unrestricted. From the perspective of logic $L*$, $L*$'s quantifiers are. It is a practical question whether to reason in one hierarchy or the other. One can also discuss both hierarchies, using a multi-sorted theory comprising the theories of L and $L*$. But this cannot be from the perspective of L or $L*$.[46]

Here is a picturesque way to characterize the pluralist's position (again, on analogy with set-theoretic pluralism). Consider a higher-order logic, L. We adopt the language of L. We reason in it. We use it to construct theories of various domains which meet our highest theoretical standards. And we successfully describe, interact with, and manipulate the world around us from this perspective. We get things right in this way. Then we consider a different logic $L*$. We adopt its language. We reason in it. We use it to construct theories of various domains which meet our highest theoretical standards. And we successfully describe, interact with, and manipulate the world around us from this perspective. We get things right in this way again! The idea is that for a plurality of logics we may do this, and be right each time. So, one can construe our slogan that there are many candidate interpretations of higher-order claims as shorthand for the claim that one can adopt and reason using different logics and be right in this way each time.[47] Of course, this is nothing like a formal theory. As Balaguer [1998, 6] and Hamkins [2012] emphasize in the set-theoretic case, part and parcel to pluralism is that it cannot be formalized. Any formalization would violate the intended meaning, either because it would privilege a universe or because it would invite pluralism about pluralism.[48] (In this respect, pluralism resembles ultrafinitism (Gaifman [2012, Sec. 2.1]). But whereas the pluralist takes formal resources to be insufficiently powerful, ultrafinitists take them to be too powerful to be meaningful.) Such a position will be unsatisfying to those with monist

[46] This kind of meta-theoretic perspectivalism is resonant with the ethical relativism of Rovane [2013], and the fragmentalism about time developed in Fine [2006].

[47] Like Balaguer's and Hamkins's formulations of set-theoretic pluralism, our formulation in this paper of higher-order pluralism leaves open exactly how generous the pluriverse is, or even whether it has an exact extent. Balaguer claims that any mathematical structure that could exist does exist, but says incongruent things about what makes for a possible mathematical structure. See Clarke-Doane [2020, n. 4]. Hamkins relies on examples of universes to give a feel for the scope of his pluriverse. Where to draw the line requires principles. We do not discuss such principles in this article. However, we argue for some elsewhere. See Clarke-Doane [2019 Sec. 8], [2020, 3.5 and 6.2], and [2022, 4.4] and my *On the Plurality of Higher-Order Logics*.

[48] See my *On the Plurality of Higher-Order Logics* (Section 3.4), for more details on this point.

sensibilities. But we are not trying to convince modal monists to be pluralists. We are arguing that modal pluralism is 'a consistent position' [Koellner Manuscript, 22, italics in original].

In sum, we may distinguish four positions:

1. Higher-order monism plus modal monism (the claim that there is a broadest kind of necessity).

2. Higher-order monism plus modal pluralism (the claim that there is not a broadest kind of necessity)

3. Higher-order pluralism plus modal monism.

4. Higher-order pluralism plus 'there is a broadest necessity' does not hold in every hierarchy.

We are defending (4), and arguing against (1). We claim that (2) does not do justice to modal pluralism (just as set-theoretic pluralism against the background of a weak, even indefinitely extensible, concept of set does not do justice to set-theoretic pluralism). And we claim that (3), while apparently coherent (it could be that 'there is a broadest necessity' holds in every hierarchy), is simply false. It fails, for example, in hierarchies in which the aforementioned Extensibility principle holds.

4.6 Objections and Replies

Having elaborated on the content of higher-order pluralism, let us consider objections to the view. The obvious objection is that there may be a broadest hierarchy in the pluriverse of hierarchies (assuming that these can be ordered by broadness[49]). In other words, there may be a broadest interpretation of the higher-order vocabulary. This would seem to serve the purpose of

[49] To give a feeling for what this might mean consider the following simple case. Suppose that a pluralist held that there was a definite domain of individuals, and a definite domain of propositions which were shared by all of the typed-hierarchies they accepted. Then their pluralism would consist in what higher-order entities - typed functions - they took there to be based on these two domains. They would take there to be no 'objective' answer to this (in the sense that there is no objective answer to the Parallel Postulate question). This is analogous to a pluralism about impure set theory. In this scenario we might say that a hierarchy is broader than another just in case each of its higher-order domains contain the corresponding domains of the other. (We do not endorse this kind of pluralism!)

the higher-order monist's unique interpretation. Indeed, an analog of this worry is familiar from the set-theoretic case. As Martin puts it, "[t]he models postulated by [pluralists] determine a canonical maximal set-theoretic structure, the amalgamation. If one takes those models seriously, then one should regard this canonical structure as the true universe of sets" [2001, 14]. The problem with this argument is that it assumes that we can meaningfully compare competing concepts of set. The difficulty, alluded to above, is that, in order to make the comparison, we must appeal to such a concept [Clarke-Doane 2022, 4.4]! So, the real conclusion is only that "within any fixed set-theoretic background concept" [Hamkins, 427] there is a broadest concept of set. It is not that there is a broadest concept independent of choice of set-theoretic background. The higher-order situation is the same. It may fail to be true that there is a broadest interpretation of the higher-order language (in the sense of 4.4 above), even if relative to some chosen one, there is. Any argument that there is will take place against a background interpretation.[50],[51]

But even if one could meaningfully compare typed hierarchies, it would not follow that there was a broadest such hierarchy for the reason that we pointed out at the end of Section 4.2. Consider the case of logical pluralism. Again, the monist might argue that the union of all logical possibilities, for any kind of logical possibility accepted by the pluralist, gives a most inclusive kind of logical consistency. Even bracketing the problem that one must assume a logic to make the comparison (which is like the one above), the argument assumes that the plurality of kinds of logical possibility is fixed. As the quotation from Beall and Restall illustrates, a pluralist may hold that, for any alleged most inclusive kind of logical consistency, there is a more inclusive one. In that case, while it might be 'objectively' true that a given kind of logical consistency is broader

[50]The difficulty is glaring if logical notions are themselves modal. What follows from what even relative to a logic is itself relative to a logic (Shapiro [2014, ch. 7]. So, if commonplace logical pluralism is true, and logical notions are modal (as discussed in Section IV), then there is certainly no stable fact as to the breadth of a kind of possibility.

[51]Again, the objection that Higher-Order Pluralism requires a metatheory, and any such theory will itself engender a maximal higher order hierarchy, is question-begging in a similar way. As in the set-theoretic case, any metatheory takes itself to be maximal. Every set theory amounts to a metatheory, a model theory in which to discuss theories, with its own interpretation of, e.g., higher-order quantification [Hamkins 2012, Sec. 5]. But not even the staunchest critics of set-theoretic pluralism take that to show that set-theoretic pluralism is false. As Koellner puts it, "the...argument is circular....[It] just presupposes in the meta-language what one set out to establish" [Manuscript, 11]. Indeed, Koellner has in mind here the most radical form of the view that he considers, according to which, roughly, every first-order consistent set-theory is true of its intended subject. This even includes set theories that 'disagree' with us about finiteness – and, hence, consistency, theories themselves, well-formed formulas, and syntax more generally.

than another, there would be no final court of appeals for logical questions. The situation would be like the one in which there is a unique, but indefinitely extensible, hierarchy of sets. A limited version of the pluralist critique would survive.

It might be thought that there must be a broadest hierarchy, on pain of the Russell-Myhill paradox (Russell [1902/1996, Appendix B], Myhill [1958]). This is usually taken to establish that there is an upper bound on the grain of propositions. The thrust of the argument is that the structured view of propositions is the broadest conceivable picture, and that it is classically inconsistent.[52],[53] Hence, propositions cannot be structured. In the present context, one might take this to undergird an argument that there is a broadest hierarchy, if one thinks that the grain of propositions in a hierarchy amounts to its broadness. However, this is not a tenable view. There are many axes other than the grain of proposition along which hierarchies can differ, and these differences would also seem to bear on the relative broadness question (again supposing that such a comparison makes sense). One of these axes is the domain of individuals (things of type e) – different hierarchies can have distinct domains of individuals. In particular, different hierarchies can differ on what merely possible individuals there are. Suppose that H is a predicative hierarchy in which propositions are structured.[54] Now consider a proper inner hierarchy $H*$ of H, in which the propositions are also structured, and whose domain of individuals is a proper subset of the domain of individuals of H. It would be very odd to take $H*$ to be the broadest hierarchy, simply in virtue of its having structured propositions, given that it is a proper inner hierarchy of H! The pluralist does not believe that there is a unique collection of all possible individuals[55] – there

[52]The structured view of propositions says that $\forall XY \forall xy(Xx = Yy \rightarrow X = Y \wedge x = y)$.

[53]Here is a loose version of the argument, which Dorr presents in [2016]. Choose some arbitrary proposition p, say that *snow is white*. Let a heteropredicative proposition be one that predicates of p some property that it itself lacks. Now consider the proposition that *p is heteropredicative*, call it q. Is q heteropredicative? If not, then q must have every property that it predicates of p, and in particular the the property of being heteropredicative; contradiction. So q is heteropredicative: it predicates of p some property F that it, q, lacks. This F cannot be the property of being heteropredicative, which, as we have just seen, q does not lack. So, there must be two distinct—and indeed noncoextensive—properties which this single proposition q predicates of p. Dorr offers the following spin on this. "The argument is essentially Cantorian: one can think of the conclusion as saying that the domain of properties of propositions is larger than the domain of propositions, so that there can be no one-one correspondence between the two domains, and in particular the relation of being a property F and a proposition q such that q is (p) cannot be one-one as required by Propositional Structure" [2016, 28-30].

[54]Structured propositions are consistent in a predicative hierarchy. See [Walsh 2016].

[55]See Rayo [manuscript] for details on this point.

simply is no hierarchy which contains all of the individuals in every other hierarchy. So, there will be another hierarchy $H**$ which stands to H, as H stands to $H*$.[56] And so on forever. Or suppose that we amend the above example so that H is a full impredicative hierarchy in which propositions are close to being structured. Suppose that $H*$ is a predicative inner hierarchy of H, in which all of the entities which 'made' the propositions in H not fully structured are removed. It would, again, be very strange to conclude that $H*$ is broader than H just in virtue of its having structured propositions!

To compound matters, note that hierarchies in which propositions are structured may differ on the grain of other higher-order entities as well. For instance, consider two predicative hierarchies H and $H*$, in which the propositions are fully structured. In $H*$ all of the higher-order domains are individuated extensionally. In H they are individuated by metaphysical equivalence. Again, it would be bizarre to claim that $H*$ is broader than H just by virtue of having structured propositions. However, if pluralism is true, then there will be a hierarchy $H**$ in which higher-order entities are individuated by a finer kind of necessary equivalence than metaphysical equivalence, which stands to H, as H stands to $H*$. So, structured propositions are not the end all be all of broadness. Even if we could meaningfully compare hierarchies – something that, to repeat, we cannot see how to usefully do – having structured propositions would not seem to be definitive.

Perhaps there is a variation on the above criticism, however.[57] It might be thought that propositions would have to be incredibly fine grained in order to support all of the kinds of possibility that we believe in. The background logic would be absurdly weak, too weak to prove the claims that we make in this paper. This is a higher-order variant of what Koellner [Manuscript] calls the 'Problem of Articulation'. However, this criticism misunderstands higher-order pluralism. It is like interpreting set theoretic pluralism as the view that there is a single background kind of set, and all of the kinds of set that the pluralist talks about are restrictions of it (e.g., Gödel's constructible

[56] We would not even accept that all structured hierarchies are broader than non-structured hierarchies. A hierarchy H which does not have fully structured propositions may have a predicative inner hierarchy $H*$ in which propositions are fully structured.

[57] Thanks to Juhani Yli-Vakkuri for pressing us on this.

sets).[58] From a higher-order monist perspective, adopting higher-order logic requires thinking that it governs everything. From a higher-order pluralist perspective though, adopting a logic is to reason inside it – as the set-theoretic pluralist holds that adopting a set theory is to jump into a set theory and reason inside it. One does not thereby restrict the one true hierarchy. There is none! A pluralist 'living' in the hierarchy agrees with the monist that its logic governs everything. But there is a difference. A monist can mimic this phenomenon when jumping into hierarchies that are proper parts of theirs, governed by stronger logics. But they cannot adopt weaker logics, and jump into 'outer hierarchies' of theirs.[59] This is like the fact that the set-theoretic monist can jump into Gödel's constructible universe, but cannot jump into a forcing extension of Gödel's 'one true V'.

From what meta-theoretic perspective, though, do we make the claims we make in this paper? Well, as we have emphasized, the whole point of pluralism, in both the set-theoretic and higher-order contexts, is that there is no formalizable theory which gives the one true story of the subject. There is no privileged perspective on the sets or higher-order entities, respectively. Nevertheless, ZFC set theory (with classical logic) suffices to make most claims that a higher-order pluralist might want to make (as it does in the set-theoretic case). ZFC is an expressively powerful and versatile theory which allows us to talk about set-theoretic representations of a wide swath of higher-order typed hierarchies. For the pluralist, discovering that ZFC proves that different higher-order theories have models tells us more than just that they are consistent (insofar as they think that ZFC is). The higher-order pluralist, like the set-theoretic pluralist, takes this to show (with other caveats in mind, such as being able to play the role of our foundational metaphysical theory) that the theories have intended models – i.e., that they are true of their subject. This is like the conclusion that geometers drew from the realization that hyperbolic geometry could be interpreted in Euclidean geometry. They gave up on the idea that Euclidean space was the 'one true geometry', with hyperbolic space just a simulation within it (Hamkins [2012, 425-6]). Both geometries stand on their own, true of their subjects.

[58]This is, in fact, how Field appears to advocate understanding the view in his [1998]. Nolan [2011] could be interpreted as advocating a similar picture in the modal case.

[59]Hamkins uses the 'jump' language in the set-theoretic case. See, e.g., [2012, 417].

Of course, we do not maintain that the higher-order entities in typed hierarchies are ZFC sets, any more than the set-theoretic pluralist claims that, say, Quine's non-well-founded sets are ZFC sets. We claim that they have models in ZFC, and, therefore, intended ones – i.e., that they are true of their intended subjects. This last claim cannot be made in ZFC.[60] One cannot define truth for a theory in a language other than that of set theory in a set theory. Again, the set-theoretic case is analogous. When the set theoretic pluralist adopts ZFC and proves that NF has a model (if she can![61]), concluding that NF has an intended model – that NF is true of its subject – this claim cannot be made in the language of ZFC. One cannot define truth for NF in ZFC, if one thinks that NF and ZFC are in different languages, as the pluralist does. What the set-theoretic pluralist can do is shift their metatheory. They can observe in ZFC that $Con(NF)$. They can then jump to NF+, where NF+ is NF plus resources sufficient to give a theory of truth for NF. In NF+ they can claim that NF is true. (In switching metatheories, they use 'true' ambiguously. The truth predicate is different in the different metatheories.) As indicated in Section 4.5, we are doing something analogous. We conclude in ZFC that a higher-order logic L has a model (and satisfies some further constraints to be mentioned). We then jump into L+, where L+ is a theory sufficient to define truth in L, and assert in L+ that L is true. This obviously assumes that there are different higher-order languages – different higher-order quantifiers and so on. But, again, that is what the higher-order pluralist maintained all along.

Perhaps the impulse behind some of the objections above is that higher-order pluralism leads to reality pluralism. In particular, propositional pluralism seems to engender pluralism about facts. And is that not tantamount to pluralism about whether it is the case that p, for arbitrary p? It is not. Yes, the higher-order pluralist accepts a plurality of different interpretations of the higher-order language, on which there are a diverse array of higher-order logical facts. But this does not mean that there are no facts that hold under every interpretation. The higher-order pluralist should hold

[60] Thanks to Juhani Yli-Vakkuri for pushing us on this point.

[61] The relative consistency of NF is still officially an open problem. But experts seem to be converging on a view. See: https://mathoverflow.net/questions/132103/the-status-of-the-consistency-of-nf-relative-to-zf. (Coincidentally, consistency-in-a-logic claims are among those about which we are monists, bracketing the fact that one must assume a logic to check those claims. So, we are monists about the natural number structure, but logical pluralists. See Clarke-Doane [2020, Sec. 1.6, 3.5, and 6.2] for the rationale.)

that for any domain of inquiry D, the true D-theories (from the various higher-order perspectives) agree on a kernel of facts. Those facts are invariant and objective. The kernel should contain many everyday claims such as that *Biden is President*, physical claims such as that *electrons are negatively charged*, and even some claims in the language of pure higher-order logic such as that *if there are four things, there are finitely many things.* Much as the set-theoretic pluralist should maintain that there are some set-theoretic claims which hold under every interpretation of the set-theoretic terms.

What is true is that kernel claims must correspond to propositions of different grains across the pluriverse. For instance, it may be true in from every higher-order perspective that *Biden is president*, and that *2+2 = 4*. But these will correspond to one proposition in an Extensional hierarchy, and different propositions in a more fine grained Boolean hierarchy. (This is, again, like the set-theoretic case, where different universes at most agree on sentences.) What matters is that there is agreement among the interpretations.

To repeat, such facts cannot be identified with propositions because these are not shared between interpretations. Indeed, we deny that such facts – e.g., *that Biden is president* – are entities of some type over and above the different kinds of proposition. There is the Fregean proposition *that Biden is president*, the metaphysical proposition *that Biden is president*, and so on, not some further entity *that Biden is president*, to which they all bear a relation. Rather, given the distinctive language of some domain D, like the language of fundamental physics, each of the interpretations satisfies many of the same sentences.[62] The resulting theory is analogous to that of set-theoretic pluralism in the context of impure set theory.

Are there any other objections to modal pluralism, the context of higher-order logic? There are certainly surprising implications of the view. Higher-order pluralism implies a kind of pluralism about content. There are various contents of assertion, belief, doubt, and so forth corresponding to the various typed hierarchies. We could make this explicit by indexing propositional attitude predicates to different hierarchies. The idea can already be illustrated in the first-order context.

[62] See my On the Plurality of Higher-Order Logics (Section 3.4) for a discussion of higher-order logical pluralism and physical pluralism.

Suppose that we think of the content of a sentence as the set of worlds in which it is true. Then, of course, we get different notions of content depending on whether we consider, say, the (first-order) logically possible as opposed to metaphysically possible worlds (holding fixed the modal logic). Orthodoxy has it that the real content is the second set, not the first (Stalnaker [1984]). But what turns on this except how select academics use 'content'? If there are sets of metaphysically possible worlds (understood as abstract objects), then there are sets of logically possible worlds, and there is nothing to preclude us from introducing terms to single the latter out. We can say $\exists^{metaphysical}$ the proposition *that snow is white*, and $\exists^{logical}$ the proposition *that snow is white*. Each of these may be expressed by the sentence 'snow is white'. In general, for every sentence, we bear relations to countless propositions of as many kinds. This too is part and parcel to higher-order pluralism.[63]

4.7 Conclusion

We have defended a simple argument that modal metaphysics is misconceived. Unlike Quine's, this does not require that modal questions are unintelligible. It requires that there are different candidate interpretations of the predicate 'could have been' giving intuitively opposite verdicts on modal questions, none of which is broadest, i.e., most inclusive. Modal pluralism is the analog to an increasingly prevalent view about set theory, according to which there are different candidate interpretations of 'is a member of', giving intuitively different answers to set-theoretic questions, none of which is broadest. We showed that the obvious responses to the argument from modal pluralism fail. It is no use arguing that metaphysical possibility is the broadest 'real' or 'objective' kind of possibility, or that it is the most natural kind of possibility, for instance. However, a new response has emerged that purports to prove, in a higher-order logic, that metaphysical possibility is the broadest kind of possibility applying to propositions. We distilled two lines of reasoning from

[63]Other surprising implications of modal pluralism, whether married to higher-order logic or not, include that there are no objective facts as to what supervenes on what, what counts as an intrinsic property, what the state space of a physical system is, and, arguably, whether one event caused another (or whether an arbitrary counterfactual is true). See Clarke-Doane [2019, Sec, 7], and [2022, 4.5]. (Note that there are independent reasons to accept many of these conclusions. For instance, a causal model under an interpretation only represents a situation relative to a space of possibilities. So, if we take this appearance at face-value, actual causation must be relative to a background space of possibilities, even given modal monism. (See McDonald [2021]. Of course, the view that objective causal relations are not needed in basic physics is familiar.)

the literature, and argued that their import depends on an assumption that pluralists should deny. It depends on a monist approach to higher-order logic, which a modal pluralist has good reason to reject. It might be worried that modal pluralism, so conceived, faces an insuperable problem of articulation, is vulnerable to the Russell-Myhill paradox, or even contravenes the truism that there is a unique actual world. However, we argued that these worries are misplaced. Most of them are simple applications of arguments familiar from the set-theoretic literature – arguments that are widely agreed to fail.

One may certainly have doubts about modal pluralism, whether of the higher-order variety discussed here or not. Modal pluralism is a radical departure from the established views of metaphysicians. It has ramifications beyond modal metaphysics, e.g., for the theory of content. And, in a higher-order context, modal pluralism, like set-theoretic pluralism, introduces a kind of perspectivalism about metatheoretic concepts that have long been assumed to be objective (in the sense that, say, geometric concepts are not). Our purpose has not been to argue for modal pluralism. We do that elsewhere (Clarke-Doane [2019], [2020] and [2022])). Our purpose has been to show that what might have appeared, in a higher-order setting, to be proof that modal pluralism is false is ineffective on inspection, like obvious objections to the view.

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