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# Some Problems in Topological Graph Theory

Jonathan L. Gross
COLUMBIA UNIVERSITY

Frank Harary
UNIVERSITY OF MICHIGAN

**ABSTRACT**

A list of 31 problems presented here reflects some of the main trends in topological graph theory.

## 0. INTRODUCTION

During the past 10 years or so, about 100 different authors representing a total of about 20 different countries have used recognizably topological methods to obtain graph theoretic results. Although the majority of these authors are best known as graph theorists, many others among them are primarily topologists, algebraists, or computer scientists. The great emphasis on genus in the present collection of problems is about proportional to that in the general literature. Some of the problems are included for their seeming strategic value for obtaining additional results, others because they would strengthen the ties to areas such as group theory or computational complexity, and others mainly due to the longstanding interest of the present authors in their solution.

For further information on the context of some of these problems, it may be useful to consult the recent surveys of topological graph theory by Stahl [37] and by White and Beineke [44]. All unexplained graph theoretic terminology here follows the usage of Harary [22].

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## 1. GENUS OF CAYLEY GRAPHS AND SCHREIER GRAPHS

Much of the progress in obtaining graph embeddings has been for various kinds of Cayley graphs. For instance, the complete graphs, whose genus was determined in the classic work by Ringel and Youngs [36], are Cayley graphs. Schreier graphs are generalizations of Cayley graphs that include, as proved by Gross [15], all regular graphs of even degree or of odd degree and sufficiently high connectivity.

For completeness, we now define Cayley graphs and Schreier graphs. Let $A$ be a group and $X$ be a set of generators for $A$. The vertices of the (right) *Cayley graph* $C(A, X)$ are the elements of the group $A$. For any $a_1, a_2 \in A$ the vertices $a_1$ and $a_2$ are adjacent in $C(A, X)$ if either $a_1^{-1} a_2 \in X$ or $a_2^{-1} a_1 \in X$. One observes that under this definition, $C(A, X)$ is a graph, and not a digraph. Sometimes the set $X$ includes a redundant generator $x$, in the sense that $x$ is a product of some members of $X - \{x\}$. Otherwise, the generating set $X$ is called *minimal*. If $|X| = 1$, then $A$ is cyclic and, obviously, the genus of $C(A, X)$ is zero. To inaugurate a systematic approach to the genus of Cayley graphs, following problem is suggested:

**Problem 1.1.** Let $P$ be a minimal generating set for the cyclic group $Z_n$. What is the genus of the Cayley graph $C(Z_n, P)$?

Let $B$ be a subgroup of a group $A$, and let $X$ be a generating set for $A$. The vertices of the (right) *Schreier graph* $S(A/B, X)$ are the right cosets of the subgroup $B$ in the group $A$. The right cosets $Ba_1$ and $Ba_2$ are adjacent in $S(A/B, X)$ if either $a_1^{-1} a_2 \in X$ or $a_2^{-1} a_2 \in X$. The main interest in Schreier graphs is for the case when the subgroup $B$ is not normal, because if $B$ is normal, then $A/B$ is a group, and $S(A/B, X)$ is simply the Cayley graph $C(A/B, X)$.

A *cubic graph* is regular of degree three, and a *quartic* graph is regular of degree four. From results of Gross [15] it follows that every 2-connected cubic graph is a Schreier graph and that every quartic graph is a Schreier graph.

**Problem 1.2.** Calculate the genus of all 2-connected cubic graphs.

A solution to Problem 1.2 would yield the genus of every cubic graph, because of the theorem on the additivity of genus due to Battle et al. [4]. A possible reduction of Problem 1.2, to calculating the genus of 3-connected cubic graphs, is apparent in recent work of Decker et al. [8].

**Problem 1.3.** Calculate the genus of all quartic graphs.

Since Problems 1.2 and 1.3 are likely to be very difficult, it is appropriate to provide another problem whose solution would be a stepping stone. One recalls Petersen's theorem [32] that a graph (or pseudograph) is 2-factorable iff it is regular of even degree. Thus, quartic graphs are 2-factorable. Petersen [32] also proved that every bridgeless cubic graph is decomposable into a 1-factor and a 2-factor. By doubling the 1-factor, i.e., for each edge in the 1-factor, inserting an additional edge with the same two endpoints, one obtains a 2-factorable multigraph that is regular of degree four. The stepping stone problem is concerned with the properties of cycle decompositions that are consistent with a 2-factorization.

Let $\mathcal{F} = F_1, \ldots, F_n$ be a 2-factorization for a graph $G$. For $i = 1, \ldots, n$ let $k(F_i)$ be the number of components of the 2-factor $F_i$. Define the number

$$\psi(G) = \max_{\mathcal{F}} \sum_{i=1}^{n} k(F_i).$$

If the graph $G$ has $p$ vertices and degree $2d$, then clearly

$$d \leq \psi(G) \leq pd/3.$$

As indicated in Sec. 4 of Gross [15], the permutation voltage graph construction of Gross and Tucker [19] could be used to obtain an embedding of a regular graph $G$ of even degree with at least $\psi(G)+1$ faces, thereby establishing an upper bound on the genus of $G$.

**Problem 1.4.** Find a good lower bound for $\psi(G)$, where $G$ is a regular graph of even degree (especially degree four).

In applying the method of current graphs, either combinatorial (see Ringel [34] or White [41]) or topological (see Gross and Alpert [16]), it is a great convenience to be able to use a current graph of index one. From the dual viewpoint of voltage graphs (see Gross [14] or White [42]), it is a great convenience to be able to assign the voltages to a bouquet of circles. Accordingly the following problem is proposed:

**Problem 1.5.** Characterize algebraically the generating sets $X$ and groups $A$ such that the genus of the Cayley graph $C(A, X)$ can be realized without adjacency modifications from an index one current graph (or dually, from a voltage assignment on a bouquet of circles).

## 2. GENUS OF A GROUP

The *genus* of a *group* is the least genus of any of its Cayley graphs. A *minimum generating set* for a group $G$ is a generating set with the least possible number of generators. Obviously, a minimum generating set is minimal, but minimal sets, e.g., {2, 3} for $Z_6$, need not be minimum. Knowing that the genus of a group could be calculated from consideration of only its minimum generating sets would substantially simplify the calculation in many cases.

**Problem 2.1.** Is the genus of a finite group always realized by a Cayley graph for some minimum generating set?

Babai [3] has proved that the genus of a finite group cannot be exceeded by the genus of any of its subgroups, thereby solving a problem of White [41, p. 80]. A related problem is the following:

**Problem 2.2.** Can the genus of a quotient of a finite group be larger than the genus of the group itself? Conjecture: No.

Proulx [33] has classified the toroidal Cayley graphs, thereby isolating the toroidal groups. Previously, Maschke [31] classified the planar groups. There are infinitely many toroidal groups and infinitely many planar groups. However, Tucker [38] has proved that for $n \geq 2$ there are only finitely many groups of genus $n$. No groups of genus two have yet been discovered.

**Problem 2.3.** For which integers $n \geq 2$ are there no groups of genus $n$?

**Problem 2.4.** Prove or disprove the following conjecture, due to Babai: For every integer $n > 2$ there are only finitely many vertex-transitive graphs of genus $n$.

Jungerman and White [29] have calculated the genus of most finite abelian groups. Far less is known about the genus of nonabelian groups, but curiously, the smallest group whose genus is unknown is abelian. If the next problem were solved, then the smallest groups of unknown genus would have order 32.

**Problem 2.5.** What is the genus of $Z_3 + Z_3 + Z_3$? Conjecture: This genus in 10.

Proulx [33] has proved that the genus of the symmetric group $S_5$ is 4. White [40] gives an upper bound for the genus of symmetric groups.

**Problem 2.6.** Determine the genus of the symmetric groups.

More information about the genus of finite groups is given by White [40–42]. Levinson [30] has proved that the genus of an infinite group is either zero or infinite, but the present knowledge of the genus of infinite groups is otherwise scanty.

## 3. EFFECTS OF GRAPH OPERATIONS ON GENUS

Let $G$ and $H$ be graphs, and let $f : T \rightarrow U$ be an isomorphism between a subgraph $T$ of $G$ and a subgraph $U$ of $H$. The *amalgamation* $G *_f H$ is the graph obtained from $G$ and $H$ by identifying the subgraphs $T$ and $U$ according to the isomorphism. If the subgraphs are both isomorphic to $K_1$, then the genus of $G *_f H$ is the sum of the genera of $G$ and $H$, as proved by Battle et al. [4]. Decker et al. [8] have recently calculated the genus of an amalgamation on two nonadjacent points. Harary and Kodama [25] have studied amalgamations on larger sets of mutually nonadjacent points. Stahl and Beineke [38] have proved that for one-vertex amalgamations, the nonorientable genus is not additive.

If the graphs to be amalgamated are the complete graphs $K_m$ and $K_n$, and if $J$ is the isomorphism type of the amalgamating subgraphs, then $K_m *_J K_n$ denotes the isomorphism type of the amalgamation. Alpert [1] studied the genus of $K_m *_{K_t} K_n$. He obtained an essentially complete answer for the case with arbitrary values of $m$ and $n$ and the value $t = 2$, and he made substantial progress for larger values of $t$, especially $t = 3$, 4, and 5.

**Problem 3.1.** What is the genus of the amalgamation $K_m *_{K_t} K_n$?

Let $f : G \rightarrow S$ be a 2-cell embedding of a graph in a closed surface. The *dual embedding* $f^* : G^* \rightarrow s$ is obtained by placing a dual vertex $c^*$ at the center of each primal face $c$ and a dual edge $x^*$ crossing each primal edge $x$ at its midpoint so that $x^*$ runs between the dual vertices $c^*$ and $d^*$ at the center of whichever faces $c$ and $d$ meet at edge $x$. Since it is possible that $d = c$, it is possible that the *dual graph* $G^*$ is not a graph, but a pseudograph.

A graph embedding $f : G \rightarrow S$ is called *minimum* if it realizes the genus of $G$. Unpublished examples due to White (orientable case) and Haggard (nonorientable case) show that the dual embedding of a minimum embedding need not be minimum.

**Problem 3.2.** Let $f : G \rightarrow S$ be a minimum embedding of a graph $G$ in a surface $S$ whose dual graph is a graph. Under what circumstances is the dual embedding $f^* : G^* \rightarrow S$ also minimum?

The *join* $G + H$ (also called the *suspension*) of the graphs $G$ and $H$ is obtained from the disjoint union $G \cup H$ by adding an edge from each vertex of $G$ to each vertex of $H$.

**Problem 3.3.** Describe the genus of $G + K_1$ in terms of the genus of $G$ and other properties of $G$.

**Problem 3.4.** Calculate the genus of $G+K_2$ (or more generally, of $G+K_n$).

Ringel [35] has calculated the genus of most cases of the cartesian product $K_n \times K_2$. A special case of Theorem 4 of White [43] is that the genus of $K_{n,n} \times K_2$ is asymptotic to $2 \times \text{genus}\,(K_{n,n})$, where $K_{n,n}$ denotes the $n$-regular complete bipartite graph. These results suggest the following problem:

**Problem 3.5.** Calculate the genus of $G \times K_2$ (or more generally, of $G \times K_n$).

Both the Kuratowski graphs $K_5$ and $K_{3,3}$ are known to have planar covering spaces. For instance, assigning voltage 0 modulo 2 to the edges of one 2-factor of $K_5$ and voltage 1 modulo 2 to the edges of the other yields a planar derived graph (see Gross and Tucker [19]). Also, as Angluin [2] has observed, assigning voltage 0 modulo 2 to the edges of one 1-factor of $K_{3,3}$ and voltage 1 modulo 2 to the edges of the other two 1-factors yields a planar derived graph, isomorphic to $C_6 \times K_2$.

**Problem 3.6.** Does every 3-regular graph have a planar covering space?

**Problem 3.7.** Does every 4-regular graph have a planar covering space?

## 4. THICKNESS, COARSENESS, AND CROSSING NUMBERS

For complete bipartite graphs, it is known from Beineke et al. [7] and Beineke [5] that

$$\theta(K_{m,n}) = \lceil mn/2(m+n-2) \rceil \quad \text{unless } m<n,\ mn \text{ is odd, and there exists an integer } k \text{ such that } n = \lfloor 2k(m-2)/(m-2k) \rfloor,$$

where the notations $\lfloor x \rfloor$ and $\lceil x \rceil$ (one says "floor" and "ceiling") mean the greatest integer $\le x$ and the least integer $\ge x$, respectively.

**Problem 4.1.** Finish the calculation of the thickness of the complete bipartite graphs.

Guy and Beineke [20] have calculated most of the values of the coarseness of $K_p$. However, they are not known for $p = 9r+7$ ($r \le 1$) or for $p = 13, 18, 21, 24$, or $27$.

**Problem 4.2.** Finish the calculation of the coarseness of the complete graphs.

The crossing number $\nu(G)$ has not as yet been determined for the complete graphs $K_p$ or for the complete bipartite graphs $K_{m,n}$. We conjecture that $\upsilon(K_p)$ and $\nu(K_{m,n})$ equal their well-known upper bounds, as given by Harary [22, p. 122].

**Problem 4.3.** Calculate the exact value of the crossing number $\nu(K_p)$.

**Problem 4.4.** Calculate the exact value of the crossing number $\nu(K_{m,n})$.

Aside from the facts that $\nu(G) = 0$ when the graph $G$ is planar and that $\nu(G) = 1$ for the Möbius ladders (Guy and Harary [21]), there are very few exact crossing number results. It has been observed by Harary et al. [24] that there are toroidal graphs, in particular, cartesian products $C_m \times C_n$ of two cycles, with arbitrarily large crossing number, so that $\nu(G)$ and $\gamma(G)$ are independent topological invariants. Beineke and Ringeisen [7] have calculated the crossing numbers for cartesian products of certain graphs.

**Problem 4.5.** Calculate the exact values of the crossing number $\nu(G)$ for other interesting graphs $G$.

The rectilinear crossing number $\bar{\nu}(G)$, introduced by Harary and Hill [23], is the minimum number of crossings required when the graph $G$ is drawn in the plane so that each edge is a straight line segment.

**Problem 4.6.** Determine the rectilinear crossing number $\bar{\nu}(G)$ for some interesting graphs $G$.

## 5. ALGORITHMS

The values of some functions can be computed by substituting numbers into a formula. However, the values of other functions cannot be calculated so easily (if at all). This section is concerned with algorithms to compute the values of functions important to topological graph theory.

By dualizing Heffter's method [27] for describing graph embeddings, Edmonds [9] obtained an algorithm to calculate the genus of a graph. Unfortunately, this algorithm requires approximately $(p!)^p$ steps, where $p$ is the number of vertices. To decide whether a graph can be embedded in certain particular surfaces, faster algorithms are known. Hopcroft and Tarjan [28] have obtained an algorithm to decide whether a graph is planar, whose running time is bounded by a linear function of the number of vertices. Filotti [10] has constructed an algorithm to decide whether a cubic graph is toroidal, whose running time is bounded by a polynomial in the number of vertices. Moreover, Filotti and Miller [11] have generalized this result to all graphs and to all other orientable surfaces. Conceivably, however, as the genus increases, the degree of the polynomial bound might increase, leaving no overall polynomial bound.

**Problem 5.1.** Is there an algorithm to compute the genus of a graph, whose running time is bounded by a polynomial in the number of vertices?

Questions about polynomial running time also apply to thickness and coarseness.

**Problem 5.2.** Is there an algorithm to compute the thickness of a graph, whose running time is bounded by a polynomial in the number of vertices? Conjecture: No.

**Problem 5.3.** Is there an algorithm to compute the coarseness of a graph, whose running time is bounded by a polynomial in the number of vertices? Conjecture: No.

It is apparently not known whether the crossing number is computable, in the usual sense of recursive function theory.

**Problem 5.4.** Is the crossing number of a graph computable? Conjecture: No.

Glover and Huneke [12] have proved that the set of irreducible graphs for the projective plane is finite, an analogue to Kuratowski's theorem that the only two irreducible graphs for the plane are $K_5$ and $K_{3,3}$. Glover et al. [13] have proved that there are at least 103 irreducible graphs for the projective plane. Hundreds of irreducible graphs for the torus have been listed by Haggard (unpublished) and by Glover and Huneke (unpublished). Although an algorithm based on a finite complete list of irreducible graphs for the torus would not be fast, the existence of such a list would surely be interesting.

**Problem 5.5.** Is the set of irreducible graphs for the torus finite?

## 6. EMBEDDING OF 2-COMPLEXES IN SURFACES

Gross and Rosen [17, 18] have proved that it can be decided in linear running time whether a simplicial 2-complex is planar. Their algorithm makes use of a local planarity criterion derived by Harary and Rosen [26].

**Problem 6.1.** Can it be decided within polynomial time what is the smallest genus surface in which a 2-complex can be embedded?

Gross and Rosen [17] have also proved that a locally planar simplicial 2-complex is embeddable in the sphere iff the 1-skeleton of its first barycentric subdivision is planar, which motivates the last problem in this collection.

**Problem 6.2.** Let $C$ be a locally planar simplicial 2-complex. Is the least genus of any surface in which $C$ can be embedded equal to the genus of the 1-skeleton of the first barycentric subdivision of $C$?

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