# Comparing Students' Reasoning With Function Composition Using a Dynamic Graphical Representation to Traditional Representations

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## Abstract

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As one of the central topics in mathematics education, "function" has received massive attention from scholars. Function composition, on the other hand, as one of the primary operations on functions that has multiple implications in advanced mathematics, such as the chain rule, is still underexplored. In particular, the function composition concept in school curricula is still approached procedurally with substitutions of algebraic equations or discrete points, and research about students' understanding of function composition has also mainly focused on algebraic representation. Very limited research has examined students' reasoning and understanding of graphically depicted function composition. Furthermore, there is a dearth of literature about the impact that dynamic representations can bring when students reason about function composition tasks, while their affordances and limitations have been investigated for some other function topics. This study addressed this gap by exploring how students reason about function composition through different representations with and without a three-dimensional dynamic graphical representation, with particular attention to its affordances and constraints on students' reasoning and understanding.

Through a qualitative multiple case study methodology, fifteen graduate mathematics education students participated in individual task-based interviews that contained three rounds of function composition problems. In the first round, participants sketched composite functions given static graphs of parent functions. In the second round, they worked with algebraic

equations to sketch composite functions. In the third round, participants had access to a dynamic graphical representation demonstrating particular function composition problems alongside static graphs. Data collection included video and audio recordings of participants' processes and their written work. The data analysis examined students' reasoning performance and patterns through: (a) distinct types of reasoning (graphical, algebraic, and discrete point reasoning); (b) levels of understanding as characterized by APOS theory; and (c) integrated versus fragmented reasoning approaches particularly in reasoning for domain and range.

The findings reveal that without a dynamic graphical representation, students employed longer reasoning processes with the need for diverse types of reasoning. The reasoning pattern of alternating between graphical and discrete point approaches proved effective, while solely relying on algebraic or discrete reasoning displayed limited success. In comparison, with the dynamic graphical representation, students demonstrated higher reasoning efficiency with shorter reasoning processes, predominantly using graphical approaches. Their reasoning became more integrated, dynamic, and coherent. The results also indicated that without a dynamic graphical representation, students presented more Action-level understanding of function composition, accompanied by their fragmented reasoning approaches and reliance on explicit equations or discrete points. In contrast, with the dynamic graphical representation, the Process-level understanding moderately increased to be more than the Action-level understanding. The use of the dynamic graphical representation particularly enhanced students' reasoning about domain, range and continuity. While the dynamic graphical representation prompted graphical reasoning and reduced reliance on algebraic equations and discrete points, some students struggled with the three-dimensional model or unfamiliar dynamic presentations, which also illustrated the constraints of this tool.

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## Dedication

To the Wonderful Counselor, Mighty God, Everlasting Father, Prince of Peace.

To my husband Tian Jin, my cornerstone.

To my daughters, Atarah Jin and Aniya Chen, heavenly gifts who remain my deepest motivation to grow and thrive.

# Chapter 1: Introduction

## Need for the Study

Functions have been one of the most popular topics for researchers in mathematics education because functions are one of the central concepts in advanced-level mathematics (Miller et al., 2015). The authors of *Everybody Counts* stated that "If undergraduate mathematics does nothing else, it should help students develop function sense" (National Research Council, 1989, p. 51). Much has been published on the function concept and the teaching and learning of it, including the development of students' understanding of the concept of function (Breidenbach et al., 1992; Mirin et al., 2021; Monk, 1994), the value of functions for the undergraduate curriculum (Thompson, 1994b), and pre-service teachers' perceptions of functions (Chesler, 2012; Paoletti et al., 2018). The abundant literature devoted to the function concept demonstrates the importance of functions in studies of mathematics education.

However, limited studies have focused on the operation of *function composition* [$g \circ f = g\big(f(x)\big)$] – despite the fact that function composition is the primary function operation and plays a key part in function arithmetic, which helps students learn functions effectively (Ratliff & Garofalo, 2006). The National Council of Teachers of Mathematics (NCTM) (2000) advocated that students in grades 9 through 12 be able to understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions. In the *Common Core State Standards for Mathematics*, function composition is also required in the section on functions in high school standards. The standards require that high school students build new functions from existing functions, and, more specifically, verify by composition that one function is the inverse of another (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). However, few studies have paid attention to

composition of functions, and for those that have, it has primarily been from an algebraic perspective (Ayers et al., 1988; Bowling, 2009; Clark et al., 1997; Engelke et al., 2005; Miller et al., 2015).

The small amount of work that has been carried out regarding function composition through graphs has used two-dimensional static (not dynamic) representations and has indicated that student have significant difficulties in this area (Bowling, 2009; Chen et al., 2023; Davis, 2000; Neger & Frame, 2005). In mathematics education, it is often desirable to have not just algebraic but also graphical approaches to solving problems; thus there is a need to focus on graphical approaches to function composition to complement other understandings according to research findings. In addition, given students' difficulties, it is important to try to understand students' reasoning through graphs with function composition, as well as to study possible supports for developing it.

With the advancement of technologies, modern tools like dynamic representations give students new chances to develop transformational reasoning that previous generations of students did not have – which has the possibility to enrich mathematical learning. *Dynamic representations* are mathematical objects that are presented visually on screen and that can be selected, dragged, enlarged, turned, and viewed from different perspectives (Arzarello et al., 2012; Tall, 2003). Students can manipulate these dynamic representations as they would objects, gaining opportunities to construct mathematical concepts, make meaning, and discover relationships as a result of their own actions (Haciomeroglu & Andreasen, 2013; Hanč et al., 2011; Tall, 2003). Researchers have utilized dynamic representations illustrating many mathematical topics (Cory & Garofalo, 2011; Dikovic, 2009; Haciomeroglu & Andreasen, 2013; Wasserman, 2015). Plass et al. (2009) also pointed out the principle of manipulation of content

which suggests that learning from interacting with the content using a dynamic visualization can be effective. Nevertheless, within what I had reviewed, little literature has considered the affordance of a dynamic representation on the topic of understanding function composition through graphs. There is a need to study representations of function composition, as well as dynamic interactions with those representations, and how they help students develop the concept of function composition.

Researchers have studied the understanding of functions through a variety of lenses. *APOS* theory, "action – process – object – schema" theory, has roots in the epistemology of functions and has been adapted by many scholars (Breidenbach et al., 1992; Dubinsky, 1991a; Dubinsky, 1991b). Wasserman (2017) conceptualized function composition through an APOS perspective. Based on studies that explored the understanding levels for function topics, the Action-level understanding on functions is shown as procedural approaches focusing on particular steps using explicit algebraic expressions or specific values. The Process-level understanding shows a more dynamic conception of function as transformations or general mappings between sets of inputs and outputs, compatible with not only algebraic, but also graphical and tabular representations. The Object-level understanding considers the function as a singular object itself that can be examined and transformed. The Schema-level understanding, as the highest level, applies the interactions of all the elements above (actions, processes and objects) as a dynamic and continuous system.

**Graphical Reasoning, Algebraic Reasoning and Discrete Point Reasoning in Function Composition**

Educators in secondary and college math courses typically approach function composition through algebraic equations $(g \circ f)(x) = g\big(f(x)\big)$. For example, if $g(x) = x^2$ and

$f(x) = 2x$, then $g\big(f(x)\big) = (2x)^2$. However, the algebraic substitution of equations does not provide much insight into the concept of function composition when students are provided with graphical representations. Even though there is an algebraic emphasis in school mathematics, some particular properties of the composite function would be the most relevant to graphical representations. For example, the domain, range, symmetry and continuity of $g\big(f(x)\big)$ are directly visible through graphs in ways that they are not with equations.

Reasoning, according to the NCTM (2009), is "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). Therefore, reasoning: (a) makes (multiple) assertions; (b) provides valid justifications for the assertions; and finally (c) draws conclusions. Equations, tables, and graphs are three common representations of numerical functions. Researchers have identified three distinct approaches to reasoning about functions. The first is algebraic reasoning, which focuses on equation-based characteristics of functions (Cuoco et al., 1996; Kriegler, 2008). The second is graphical reasoning, which is based on the graphical characteristics of functions and is likely similar to geometric reasoning (Crowley, 1987; Cuoco et al., 1996). The third is discrete point reasoning, which involves examining functions through a point-wise view, focusing on specific points of functions (Even, 1998).

Graphical reasoning, which is relevant to geometrical reasoning, relies on reasoning about the geometric shapes of function graphs and their properties, including but not limited to reasoning that relies on lines, curves, planes, and surfaces (Cuoco et al., 1996). With functions and function composition, graphical reasoning systematically pays attention to objects – and their properties – which are evident from the Cartesian graph, such as domain, extreme value, continuities, symmetry, and periodicity of the function graph (Cuoco et al., 1996). The interrelationships of properties within and among function graphs is also established during

graphical reasoning (Crowley, 1987). For example, the symmetry of a function, based on graphical reasoning, might be demonstrated by folding the graph over the *y*-axis as a line of symmetry. When utilizing representations of functions, graphical reasoning begins with analyzing function graphs (Cuoco et al., 1996). In this study, *graphical reasoning about function composition* is defined as the (a) assertions, (b) justifications, and (c) conclusions that are based on, and drawn by, geometric objects on graphs and the graphical features of the composite function as well as its original functions.

Algebraic reasoning draws on reasoning that engages with algebraic entities of functions – and their properties – including reasoning that relies on numbers, variables, equations, formulas, and arithmetic binary operations (Cuoco et al., 1996; Kriegler, 2008; Zbiek, 1998). Algebraic reasoning concentrates on generalizing–with those generalizations expressed using variables and arithmetic symbols–as well as manipulating quantities and transforming expressions represented by functions in a relational way (Kaput, 1998; Kieran, 1998; Kieran, 2011). Furthermore, in function composition, algebraic reasoning concerns noticing and formulating patterns of inputs and outputs that hold true in algorithms (Walle, 2007). The symmetry of a function based on algebraic reasoning is identified as $f(x) = f(-x)$. Algebraic reasoning represents information and displays relationships primarily through tables and equations (Herbert & Brown, 1997, pp. 340 - 344). In this study, *algebraic reasoning about function composition* is defined as the (a) assertions, (b) justifications, and (c) conclusions that are based on and drawn from algebraic expressions in equations and the algebraic features of the composite function as well as its original functions.

Discrete point reasoning, as a unique reasoning type under function topics, originally was described as pointwise understanding of functions (Even, 1998; Monk, 1994). Discrete point

reasoning centers on reading, plotting, or dealing with particular numerical values of independent and dependent variables, as well as the location and behavior of the ordered pairs on function graphs. The information from specific discrete points can be derived from algebraic, graphical, and tabular representations. In this study, *discrete point reasoning about function composition* is defined as the (a) assertions, (b) justifications, and (c) conclusions that are based on and drawn from numerical values or ordered pair locations of specific points in the composite function as well as its original functions.

**Purpose of the Study**

The purpose of this study is to: (1) analyze students' reasoning about graphically and algebraically depicted function composition with and without a dynamic graphical representation; (2) explore the depth of students' understanding about function composition with and without a dynamic graphical representation; (3) identify and examine any differences among students' distinct types of reasoning with respect to the topics of domain, range, symmetry, and continuity of function composition; and (4) identify and examine any differences between students' reasoning with and without a dynamic graphical representation with respect to the topics of domain, range, symmetry, and continuity of function composition.

**Research Questions**

This research addresses the following questions:

1. How do students reason about function composition with and without a dynamic graphical representation?
2. What level of understanding, in terms of APOS theory, characterizes students' reasoning about function composition with and without a dynamic graphical representation?

3. On the topics of domain and range of function composition, how do students reason differently with:
    a. distinct reasoning types?
    b. a dynamic graphical representation and without it?
4. On the topics of symmetry and continuity of function composition, how do students reason differently with:
    a. distinct reasoning types?
    b. a dynamic graphical representation and without it?

**Procedure of the Study**

This research employed a qualitative case study through face-to-face and one-on-one task-based and semi-structured interviews between the researcher and participants (Stake, 2013; Yin, 2014). During the interview, participants were asked to sketch the graph of the composite function and to share their thoughts by solving problems designed by the researcher. Each interview lasted for about two hours. In order to collect rich and informative data, it was preferable for participants to have a basic understanding of function composition, so that it would be possible for them to be able to reason during the interview. Participants also needed to be capable of understanding and utilizing a three-dimensional dynamic graphical representation, which can be challenging. Therefore, participants in this study were $n = 15$ students purposively sampled from a graduate mathematics education program. This sample had experience in solving function composition problems and most likely with dynamic graphical representations in applications such as GeoGebra.

The task-based interview contained three rounds. In the first round participants sketched the composite functions graphs given static graphs of the parent functions $f(x)$ and $g(x)$; in the

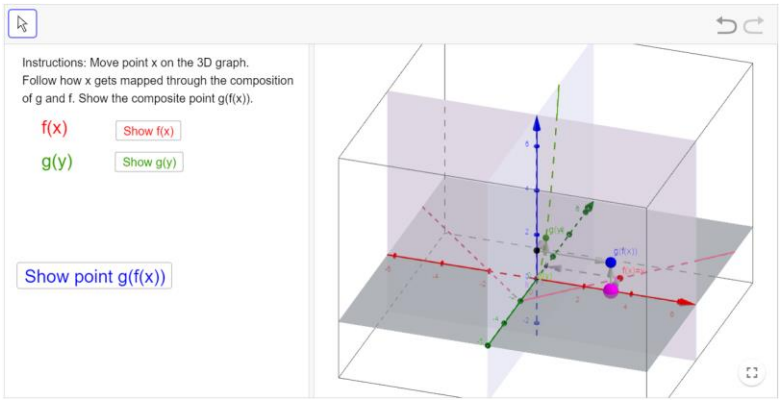
second round participants sketched the composite function given equations of parent functions; and in the third round participants explored a dynamic graphical representation and sketched the composite function given graphs of parent functions. Each interview round consisted of four function composition problems involving diverse types of numerical functions and their properties (Appendix B). There were in total four sets of similar and comparable problems, which are further explained in the methods chapter. In each round, participants were asked to think aloud by speaking any thoughts which came to their minds in real time while sketching the composite functions. Basic scientific calculator use was allowed across three rounds.

The dynamic graphical representation employed in this study (displayed in Figure 1), authorized for use by the applet's author, was a three-dimensional simulation applet of function composition in GeoGebra. The applet dynamically shows the corresponding relations amongst $x$, $f(x)$, $g(x)$, and $g(f(x))$ graphically. The graph of $f(x)$ was displayed on the $xy$-plane, the graph of $g(x)$ was displayed on the $yz$-plane, and the graph of the composition $g(f(x))$ was displayed on the $xz$-plane. The user could drag the point along the $x-axis$, while the corresponding value of $f(x)$ was plotted on the $y-axis$ by an arrow simultaneously, which is also the input of $g(f(x))$. Meanwhile, the traced path of the value of $g(f(x))$ was shown on the $z-axis$ along with the movement of the point on the $x-axis$. The entire model of the three-dimensional Cartesian Coordinate System could be enlarged, turned, and viewed from different angles. During this study, the applet was modified to hide the function equations and the graph of $g(f(x))$. This particular non-standard three-dimensional representation of function composition utilized dynamic technology to deliver the concept of function composition in a different way from how this concept is typically introduced in school. The goal was to explore the affordance and constraints of this three-dimensional dynamic graphic representation on students' reasoning,

studying the possible learning opportunities as well as obstacles when using this representation for solving function composition problems.

**Figure 1**

*The Dynamic Graphical Representation for This Study in GeoGebra*




*Note.* From *Graphical Function Composition: Demo* by N. Wasserman, 2023, GeoGebra (https://www.geogebra.org/m/pyfnv99g). Copyright 2023 by N. Wasserman. Reprinted with permission.

The applet was introduced to participants after the first two rounds and before the third round. The researcher provided a short tutorial on the dynamic representation with a demo problem (see Figure 1) to show the dynamic graphs of $x$, $f(x)$, $g(x)$, and the points of $g(f(x))$. In this final round, students solved composition problems with access to a dynamic representation (similar to Figure 1) that displayed the relevant parent functions for each specific problem.

Qualitative data were collected via on-site researcher-participant interviews. During the interviews, audio of participants' think-aloud processes and conversations with the researcher, as well as videos of participants' workspaces including graphing paper and screens, were recorded. Video and audio data were transcribed. The qualitative data sources included participants'

written responses, students' actions when solving the problems, and the transcription of their oral responses.

For each question participants solved, transcripts of participants' reasoning processes were sorted into a sequence of multiple reasoning steps, where each reasoning step was defined as: (1) a complete reasoning step with a full cycle of making an assertion, providing justifications, and drawing an interim conclusion; or (2) an incomplete reasoning step where participants failed to justify the assertion, therefore could not draw a conclusion.

To address the first research question, each reasoning step was deductively coded according to the distinct reasoning types adopted and adjusted from literature (graphical, algebraic, and discrete point reasoning). The distribution of the length of each reasoning process, the distribution of distinct reasoning types through three rounds, as well as illustrative examples of participants' reasoning performance, were demonstrated.

To answer the second research question, each reasoning step was deductively coded based on what level of understanding it indicated, in terms of APOS theory that was reviewed and described in the literature review chapter. The distribution of reasoning steps that reflected Action-level, Process-level, and Object-level understanding was presented, accompanied by the level of understanding characterized in illustrative examples of participants' reasoning performance.

To address the third and fourth research questions, the codes developed for the first research question regarding distinct reasoning types were used to analyze the reasoning steps that addressed domain, range, symmetry, and continuity. Based on several sets of similar questions through three rounds, distributions of reasoning types were presented, and distinct ways of considering domain, range, symmetry, and continuity from some participants were identified and

illustrated. In addition, each reasoning step for domain and range was inductively coded as a fragmented step or an integrated step based on whether the single step reached the comprehensive conclusion of domain and range of the composite function.

# Chapter 2: Literature Review

## Introduction

Function has been one of the most essential topics in mathematics education studies, yet the study of function composition remains relatively underexplored. The development of the technology of digital tools, particularly virtual graphical representations, offers new opportunities to enhance students' understanding of function-related concepts. This chapter reviews the extant mathematics education research literature on function and function composition, examining the mathematical reasoning processes involved in function themes as well as the evolution of mathematical representations.

## Function

### *The Notion of Function and the Significance of This Concept*

**Function Defined in a Broader Context.** The notion of function on a broader scale has three distinct, yet interrelated aspects (Niss, 2020). First, it represents an essential mathematical entity: a correspondence in which every element in a given domain is associated with a unique element in another domain, called the codomain. Secondly, functions play critical roles where other mathematical objects or theories can be demonstrated or connected, for instance, when maximizing the area of a rectangle given a fixed perimeter. Thirdly, functions are in crucial positions when applying mathematics to extra-mathematical modeling, for example, when the best function approximation is found for a set of experimental data points. These three aspects, based on Niss (2020), make the notion of function one of the most important and fundamental in mathematics as well as mathematics education.

The Dirichlet-Bourbaki definition of a function, used in most texts today, describes it as a relation between two sets A and B such that each value in the set A is associated with exactly

one value in the set B. Historically, the general concept of function was synthesized by a series of mathematicians and researchers, starting with the proposal of a function as a program for the study of motion by Galileo (Malik, 1980). With the development of calculus, mathematicians first thought of functions geometrically – as curves on a graph. Later, they began expressing functions algebraically using formulas (Boyer & Merzbach, 2011; Kleiner, 1989). Eventually, mathematicians generalized the concept of function as the relationship between numbers, variables, two sets, and ordered pairs (Atkinson, 2002). The history of how the notion of function evolved shows the necessity and importance of this concept in mathematics and mathematics education.

**Function Defined in Secondary and Collegiate Curricula.** The modern definition of function seems to be more abstract and difficult for students (Markovits et al., 1986). Sierpinska (1992) pointed out that when the definition is taken literally, Dirichlet's definition of function involves strange functions—ones that cannot be represented by a curve drawn by a free hand or ones that are continuous and yet nowhere differentiable. Students have to be sufficiently mature mathematically to accept these examples as functions. In conclusion, Sierpinska (1992) stated that introducing the general definition of functions early in education is ineffective, as students tend to either ignore or misunderstand it. Studies have shown that students and even teachers can provide a correct formal definition of a function but struggle to correctly identify functions and non-functions (Chesler, 2012; Tabach & Nachlieli, 2015).

In secondary and collegiate curricula, function is defined in various ways to show different characteristics of this concept. Some definitions present functions as taking elements from one set to another set (Holliday et al., 2005; Saxon, 2003), other definitions consider

functions as sets of ordered pairs such that no two pairs have the same first entry but different second entries (e.g., Interactive Mathematics Program, 2000).

**Significance of Teaching and Learning Functions.** As the understanding of functions grew in the mathematics field, the significance of teaching and learning functions also attracted massive attention. Schoenfeld et al. (2019, p. 63) argued that functions form "extremely rich and multifaceted mathematical concept." Building on the three aspects Niss (2020) articulated, functions are essential to mathematics education due to the following reasons. First of all, as a crucial concept bridging algebra and undergraduate mathematics and forming the foundation for modern mathematics, particularly calculus, function played a unifying role in secondary and college mathematics curricula (Carlson & Oehrtman, 2005; Selden & Selden, 1992). While algebra courses are often considered the *gatekeeper* to advanced mathematics and opportunities, functions represent the core component of algebraic understanding (Martinez et al., 2016; Moses, 2002). Some educators have even developed function-based curricula specifically to support struggling students (Martinez et al., 2016). At the same time, algebraic knowledge helps students better understand functions, which in turn contribute to students' success in future mathematics courses (Kriegler, 2008). In this way, understanding functions can be seen as the key that unlocks the gateway to higher mathematics. Along with the key role played on the path from secondary to collegiate mathematics, function is critical in establishing understanding of many other theories and contents such as mathematical modeling. Additionally, beyond the pure mathematics in school and research, background knowledge of function is essential in multiple fields like science, engineering and business (Carlson & Oehrtman, 2005). For instance, mathematical functions can be used to analyze growth in poultry (Kuhi et al., 2010) and to support engineering applications (Hirtz et al., 2002).

Given the significance elaborated above, the NCTM (2000) advocates teaching functions across all age groups from prekindergarten through grade 12 to enable all students to understand patterns, relations, and functions. The NCTM guidelines outline a progression of understanding: students in grades 3-5 should develop the concept of variable as a useful for representing unknown quantities and identify and describe relationships between two quantities that vary together, and students in grades 6-8 should represent a variety of relations and functions. For students in grades 9-12, concepts of function are more directly related and formal in the curriculum as the NCTM (2000) recommends that high school students understand relations and functions select, convert flexibly among, and use various representations for them. Function is also one of the four topics in the *Essential Understanding* books published by the NCTM, which shows that the area of function is difficult to learn, challenging to teach, and critical for students' success in school as well as in their future lives and careers (Lloyd et al., 2010).

### *Studies About Function*

Different types of functions have been studied, such as textbook comparisons on the topic of linear functions (Wang, 2017); students' understanding of trigonometric functions (Weber, 2005); students' learning of linear, quadratic, and piecewise functions (Kohli et al., 2015); and the effect of learning exponential and logarithmic functions in computer-supported collaborative learning environments (Birgin & Acar, 2022). Researchers have also explored advanced aspects of functions, including ideas in calculus like derivatives and integrals of functions, inverse functions, and function properties (De Bock et al., 2017; Jones, 2013; Orhun, 2012; Wasserman, 2017).

Many studies in the literature have demonstrated the difficulties students and teachers encounter with function topics. In general, U.S. students continue to perform poorly in the area

of function in assessments (National Assessment of Educational Progress [NAEP]) especially with the impact of the COVID-19 pandemic (Halloran, 2021). The results for function-related questions from NAEP tests through years of 1990 to 2000 showed that approximately no more than half of students were able to answer them correctly. Approximately 10% of respondents correctly answered questions which involved graphing a system of two linear equations (Kloosterman & Lester, 2004). In addition, numerous researchers have found that students in calculus experience difficulties due to their weak understanding of the function concept (Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994a; Zandieh, 2000).

Researchers have revealed students' difficulties in understanding the concept of function from several perspectives. Firstly, students' "blurred concepts" led them to view a function graph as a picture of an event, rather than a representation of how two quantities change together. For example, Monk (1992) displayed students' responses that interpreted a distance versus time graph of a bicyclist going up and down a hill as incorporating a picture of the hill. Furthermore, Monk (1992) has suggested that students' "blurred concepts" may enable students to answer "pointwise questions" fairly well, where they can apply particular information at some specific point or part of the function. By comparison, students' view of the function as a whole is static and monolithic, as they tend to think (or hope) that there exists a simple governing rule with explicit patterns. Another challenge commonly identified by researchers is students' weak understanding of functions, which often focuses solely on manipulating algebraic equations and procedural techniques, as these are the tasks students are typically asked to accomplish (Carlson, 1998; Carlson & Oehrtman, 2005). This strong procedural emphasis generates all kinds of misconceptions about functions. Thompson (1994b) indicated that students tend to view functions simply as two expressions separated by an equal sign. Breidenbach et al. (1992) found

that some students believe that all functions should be definable by a single algebraic formula. Schwarz and Hershkowitz (1999) showed that many students also tend to assume that functions are either linear or quadratic, and they usually expect any u-shaped graph to be a parabola. To sum up, one important aspect that research has demonstrated is students' difficulty with visualization and graphs of functions, which is evident in their frustration when establishing connections among different representations of functions, especially between the algebraic and the graphical representations (Selden & Selden, 1992). A detailed review of function representations will be presented later in this chapter.

## Function Composition

### *Function Composition Defined in Secondary and Collegiate Curricula*

**Definition of Function Composition**. Based on the NCTM book *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12,* for two functions $f$ and $g$, "the composition of $f$ and $g$, denoted as $f \circ g$, is the function that takes an input $a$ to the output $f(g(a))$" (Lloyd et al., 2010, p. 72). The symbolic way can be indicated as "$a \rightarrow g(a) \rightarrow f(g(a))$". In collegiate precalculus and calculus curricula, both Abramson (2021) (precalculus) and Strang and Herman (2016) (calculus) present function composition within "Combining Functions" sections with other algebraic operations of functions. Abramson (2021) describes composition as combining functions so that the output of one function becomes the input of another. Strang and Herman (2016) emphasize domain and range in the definition: "Consider the function $f$ with domain $A$ and range $B$, and the function $g$ with domain $D$ and range $E$. If $B$ is a subset of $D$, then the composite function $(g \circ f)(x)$ is the function with domain $A$ such that $(g \circ f)(x) = g(f(x))$".

**The Delivery of Function Composition in Curricula.** As shown in the definition from Strang and Herman (2016), determining domain and range of function composition has been considered critical. Abramson (2021) provides mapping relations among the domains and ranges of functions $f$, $g$ and $g \circ f$ and explained: "The domain of a composite function $f(g(x))$ is the set of those input $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$." Lloyd et al. (2010, p. 73) also highlighted in their NCTM guidebook that: "Under appropriate conditions, functions can be composed." The book presented it symbolically as "range of $g \subseteq$ domain of $f$ for $f \circ g$."

Function composition problems in the New York State Common Core Mathematics Curriculum's *Precalculus and Advanced Topics* concentrate on algebraic substitution and evaluating composite function at specific points. For example, when given $f(x) = x^2 - x$ and $g(x) = 1 - x$, students are asked to solve $f \circ g$, $g \circ f$, $g \circ g$, $f \circ f$, $f(g(2))$, and $g(f(-1))$ (Great Minds, 2019). These types of problems are common across other textbooks.

The NCTM guidebook and the textbooks demonstrate that function composition is an indispensable topic in learning functions and, more broadly, in studying calculus (e.g., the chain rule). Moreover, function composition provides students with opportunities to develop a comprehensive understanding of function concepts, as it encompasses various types of functions with diverse properties and dynamically presents the co-varying relationships between variables of inner and outer functions. However, the introduction and problems presented in current curricula and textbooks indicate that students are still primarily experiencing function composition algebraically and procedurally.

### *Studies About Function Composition*

Compared to the extensive literature on teaching and learning functions, function composition has received limited attention, with studies revealing students' generally procedural

understanding. Early research on function composition focused primarily on algebraic perspectives with students mainly practicing repetitive algebraic formula substitution. Ayers et al. (1988) attempted to use computer instantiations for students to practice function composition problems with formulas. The authors found that that the usage of computers seemed to be beneficial as the computer groups received better results on average than the traditional group in the posttest. Sfard (1992) identified students' difficulties in solving composition problems where explicit formulas are not given. The researcher believed that student's misunderstandings and struggles with function composition arise because it is difficult to visualize. Meel (2003) pointed out the limitation of algebraic approaches since rule-based interpretation negatively affected participants' ability to solve composite function problems. Furthermore, Engelke et al. (2005) conducted a study to examine precalculus students' understandings of function composition when given algebraic formulas, graphs, tables and words through the *Precalculus Concept Assessment* (PCA) (Carlson, 1998). Students performed poorly overall, being most successful with the algebraic *plug and chug* approach. Miller et al. (2015) looked into students' misconceptions of function composition. The authors found that when given functions by formulas: (1) students tend to mistakenly reverse the order to compute, for example calculating $f(g(x))$ while they should compute $g(f(x))$ (*reverse*); (2) students treat $f(g(x))$ as $f(x)g(x)$ (*multiplication*); (3) students may incorrectly square part of the expressions, or reverse and incorrectly square (*square/reverse*). Bowling (2009) also revealed students' weak covariational and quantitative reasoning when solving novel function composition problems, specifically noting their poor procedural understanding of function input and output and difficulty reasoning about function behavior over intervals.

In conclusion, function composition is an indispensable part of function when learning algebra and calculus. Studies about function composition have shown students' understanding typically relied on simply algebraic substitutions. Recent research concerning various representations of function composition beyond algebraic equations will be discussed in the next section.

## Representations

### *Mathematical Representations*

**Definition and Categories of Mathematical Representation.** A mathematical representation is a sign or a configuration of signs, characters, or objects that stand for (symbolize, depict, encode, represent, denote, or exemplify) mathematical concepts (Goldin & Shteingold, 2001; Perkins & Unger, 1994). Based on Kaput (1987), the concept of representation involves a representational entity, the entity it represents, the particular aspects of the representational entity, and the correspondence between the two entities. For each mathematical concept as an entity, multiple entities exist to represent it.

It is important to distinguish between two types of mathematical representations. *External mathematical representations* are manifestations of mathematical concepts that "act as stimuli on the senses" and help us understand these concepts. *Internal mathematical representations* are abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience (Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001). The mathematical representations discussed here and examined in this study are *external mathematical representations*. Goldin and Shteingold explained that representations have a two-way nature, where the representing relation – depiction, encoding, or symbolization – can work in both directions. For example, a graph on a Cartesian plane can provide a graphical representation of

an equation in two variables. Alternatively, an equation relating $x$ and $y$ could express an algebraic symbolization of a Cartesian graph.

Common mathematical representations include: (1) verbal representations like written or spoken descriptions; (2) numerical or symbolic representations such as numbers, variables and operators; (3) visual or graphical representations like Cartesian graphs, geometric figures, and diagrams; (4) tabular representations; (5) algebraic representations like equations and expressions; and (6) concrete representations such as physical manipulatives (Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001).

**Relationships Among Mathematical Representations: Affordance and Constraints.** The various representations emphasize different aspects of complex ideas (Kaput, 1989). For example, graphical representations clearly visualize patterns and trends, algebraic representations concisely represent abstract concepts, and tabular representations compare data in an organized way across categories (Pape & Tchoshanov, 2001). However, there is inevitably some ambiguity present in representations which may also be a source of cognitive obstacles when using representations. Goldin and Shteingold (2001) explained that this ambiguity often can be resolved by referring to the contextual information. Besides the ambiguity issue, from a learner's perspective, some mathematical representations (e.g., algebraic expressions) are precise yet typically presented in abstract or symbolic forms and, therefore, harder to grasp for beginners. Some representations may focus more on localized specific information while having limits in showing the global trends over time (e.g., tables), and others may sacrifice precision for visual clarity (e.g., graphs). Mathematical concepts are constructed through the interweaving of multiple representational forms rather than being embedded in individual representations (Smith, 1994). A series of studies have identified benefits in leveraging the connectivity between

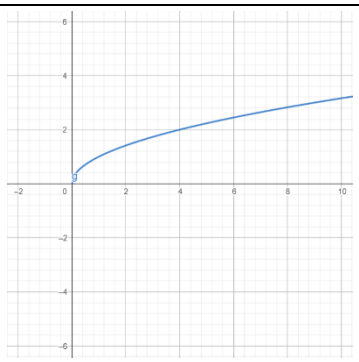
representational systems (Resnick et al., 1994; Schwarz & Bruckheimer, 1988; Zbiek, 1998). Two research teams (Pape & Tchoshanov, 2001; Goldin & Shteingold, 2001) noted that the cognitive capacity of the human brain more closely resembles multiple representational patterning which combines concrete, visual, and abstract information; thus, developing mathematical thinking and problem-solving skills requires a multiple representational approach.

***Representations of Functions***

Tables, graphs, and equations are the major and most widely used representations of functions (see examples in Table 1).

**Table 1**

*Representations of a Function Example*

| Algebraic Representation – Equation | Graphical Representation – Graph | Tabular Representation – Input and Output Table |
|---|---|---|
| $y = \sqrt{x}$ $(x \geq 0)$ |  | $x$ \| $y$<br>0 \| 0<br>1 \| 1<br>4 \| 2<br>9 \| 3<br>16 \| 4 |

Tables show functions as sequences. According to Kaput (1989), tables introduce less cognitive strain as they show only a finite selection of discrete ordered pairs, but less information is certain. This aligns with Sierpinska's (1992) viewpoint that one needs to believe that interpolation methods give exact values of functions at intermediate points to identify tables of functions with the functions themselves.

Sierpinska (1992, p. 50) described the graphical representation of a function as “a curve given mechanically as the path of a moving point or as the intersection of surfaces in space, by

the quantitative relationships." Graphs are also a popular way of representing functions, especially functions from the reals to the reals, as they provide an immediately accessible pictorial image useful in explaining many properties of functions, including monotonicity, concavity, extrema, continuity, and periodicity (Selden & Selden, 1992). Researchers have also pointed out many downsides to the graphical representations of functions. Sierpinska (1992) argued that a graph represents the function in an indirect, symbolic way that does not show directly how and where a particular point has been displaced. The author viewed graphs as static representations that hide the dynamism of functions. Related to this, Kaput (1989) shared the concern that a graph can be misleading without appropriate selection of values shown on the coordinate plane, also known as the *window*. However, these drawbacks of graphical representation might no longer apply with the development of technologies for dynamic representations of functions (Artigue, 1992).

To connect Sierpinska's (1992) demonstration of graphs with equations of functions, the equation of a curve (function graph) describes the relationship between the coordinates of an arbitrary point on the curve. The equation of a function, or its analytic formula, often conveys the relationship between dependent and independent variables (Selden & Selden, 1992; Sierpinska, 1992). Many studies have shown the predominant role of the algebraic (equation) approach throughout most of the development of function theory and function content in curricula (Artigue, 1992; Carlson & Oehrtman, 2005).

Confrey and Smith (1992) emphasized the importance of the building and understanding of functions through multiple representations. Similarly, Ferrara et al. (2006) argued that much of the learning effort under the topic of functions involves assimilating these three types of representations into a meaningful and coherent whole. Among the three representations,

Hoffkamp (2011) considered graphs to be a pivotal form of representation within the function concept. Correspondingly, an increasing number of studies have criticized the predominant focus on the algebraic approach in function topics and called for more attention to graphical perspectives, claiming that multiple lenses of representation might contribute to better conceptualization of the notion of function (Ainsworth, 1999; Artigue, 1992; Van & White, 2004).

### *Representations of Function Compositions*

**Algebraic and Tabular Representations of Function Compositions.** As reviewed in previous section, the majority of function compositions are represented by algebraic equations in school curriculums (e.g., Great Minds, 2019). The tabular representation of function composition follows the path "$x \to f(x) \to g(f(x))$," and by definition, the ordered pair of the composite function should be $(x, g(f(x))$ instead of $(f(x), g(f(x))$ (see Table 2).

**Table 2**

*Algebraic and Tabular Representations of Function Composition Example*

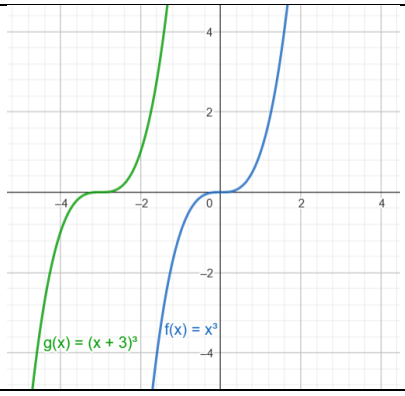
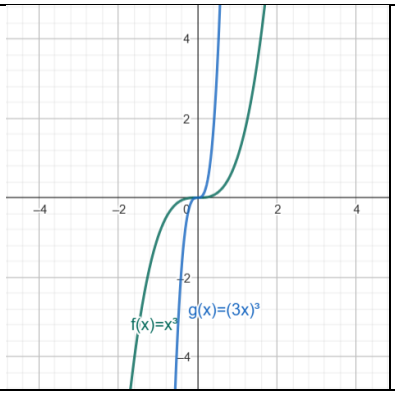
<table>
<tr><th>Parent Functions</th><th>Algebraic Representation – Equation</th><th colspan="4">Tabular Representation – Input and Output Table</th></tr>
<tr><td rowspan="7">$f(x) = 2x$<br>$g(x) = x^2$</td><td rowspan="7">$g(f(x)) = (2x)^2 = 4x^2$</td><td>$x$</td><td>$f(x)$</td><td>$g(f(x))$</td><td>Composite Ordered Pair</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>$(0, 0)$</td></tr>
<tr><td>1</td><td>2</td><td>4</td><td>$(1, 4)$</td></tr>
<tr><td>2</td><td>4</td><td>16</td><td>$(2, 16)$</td></tr>
<tr><td>3</td><td>6</td><td>36</td><td>$(3, 36)$</td></tr>
<tr><td>4</td><td>8</td><td>64</td><td>$(4, 64)$</td></tr>
<tr><td>5</td><td>10</td><td>100</td><td>$(5, 100)$</td></tr>
</table>

**Graphical Representation of Function Compositions in Curricula and Studies.** There is no standardized graphical representation of function composition. As alternatives, curriculum

writers and researchers have proposed several approaches to graphically represent function composition. Lloyd et al. (2010) addressed function composition graphically by encouraging mathematics educators to understand and teach composition through translating and scaling functions to transform graphs (see Table 3). Despite this promising graphical approach presented in the NCTM guidebook (Lloyd et al., 2010) for high school mathematics educators, the examples and exercise students are experiencing today do not seem to be very different from the conventional algebraic ways. In fact, the *Precalculus and Advanced Topics* textbook from the NYS Common Core Mathematics Curriculum lacks any problems that either incorporate a graphical understanding of function composition or include graphical representations in the function composition section (Great Minds, 2019).

**Table 3**

*Example of Composing with Translating and Scaling Functions to Transform Graphs*

| Composing with Translating | Composing with Scaling |
|---|---|
| Composing $f(x) = x^3$ with $h(x) = x + 3$;<br>$g(x) = f \circ h = f\big(h(x)\big) = (x + 3)^3$;<br>Translate $f(x)$ 2 units to the left. | Composing $f(x) = x^3$ with $h(x) = 3x$;<br>$g(x) = f \circ h = f\big(h(x)\big) = (3x)^3 = 27x^3$;<br>Dilate $f(x)$ by a scale factor of 27. |
|  |  |

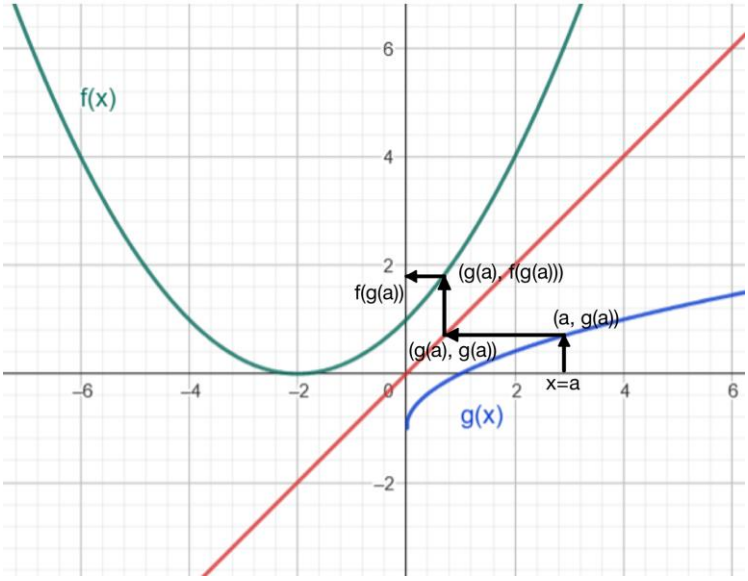
*Note.* Adapted from *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12* (p. 75), by G. Lloyd, S. Beckmann, R. M. Zbiek, and T. Cooney,

2010, National Council of Teachers of Mathematics. Copyright 2010 by National Council of Teachers of Mathematics.

Davis (2000) and Neger and Frame (2005) both employed graphical representations to interpret and teach function composition. Davis (2000) introduced a graphical method utilizing the $y = x$ line to locate points (values) of $f(g(a))$ with the given initial value $x = a$ (see Figure 2). Using the graphical method demonstrated by Davis (2000), students can observe the non-commutative nature of non-inverse composition functions ($f\big(\, g(x)\big) \neq g\big(\, f(x)\big)$), which addresses one of the common mistakes students made from the previous research: reversing the $f\big(\, g(x)\big)$ to get $g\big(\, f(x)\big)$. Similarly, Neger and Frame (2005) utilized the $y = x$ line but focused on finding restricted domain and range of compositions. Instead of concentrating on discrete points, the authors worked with intervals on graphs and used the $y = x$ line to map the restricted domain of $g(x)$ to the restricted range of $f(x)$, and then map again to the restricted domain of $f(x)$ which is also the restricted domain of $g(f(x))$. This graphical approach aligns with the NCTM guidance for high school teachers (Lloyd et al., 2010) and the college calculus curriculum (Strang and Herman, 2016), which emphasize the domain and range issue of function composition. The *graphical iteration* technique Neger and Frame (2005) presented successfully visualizes the statement that the domain of a composite function $g(f(x))$ must be a subset of the domain of $f$ where those input $x$ has $f(x)$ in the domain of $g$.

**Figure 2**

*One Approach to Graphically Represent Function Composition: Locate the Point of $f \circ g$*




*Note.* Adapted from "A graphical method for function composition," by G. J. Davis, 2000, *Teaching Mathematics and its Applications, 19*(4), p. 155. Copyright 2000 by Oxford University Press.

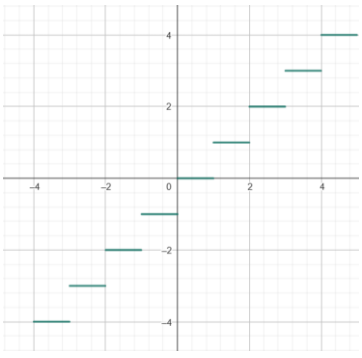
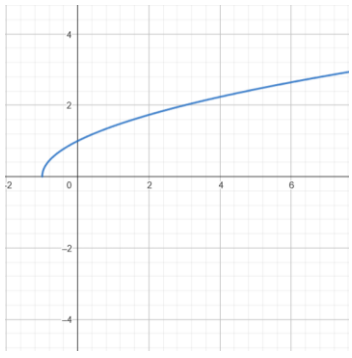
Early studies examining students' understanding of function composition through graphical lenses primarily took graphs as simple representations providing discrete ordered pair values. These studies typically asked students to evaluate specific points of composite functions when given graphs of $f(x)$ and $g(x)$ (Ayers et al., 1988; Engelke et al., 2005; Meel, 2003; Sfard, 1992). Even so, researchers still found that students generally had difficulties and performed poorly on function composition tasks involving graphical representations.

In recent years, a few studies have started to examine to students' graphical understanding of function composition. Chen et al. (2023) and LaPlace et al. (2024) both explored how students respond graphically when composing functions. In their studies, university students were asked to sketch the graph of composite function $(g \circ f)(x)$, primarily based on two given graphs of parent functions. Both studies found that: (1) students experienced significant difficulties in accurately sketching composite functions; and (2) students overly relied

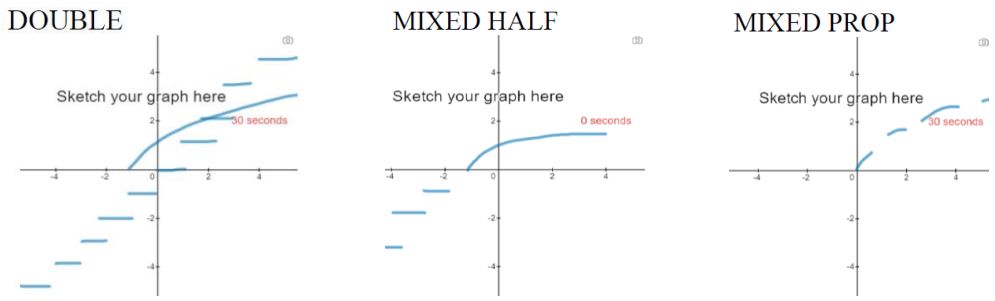
on algebraic equations, which negatively impacted their understanding of function composition, such as ignoring the restricted domain of the composite function when simply substituting the equations. Chen et al. (2023) identified several predominant ways students tried to respond graphically about function composition: (1) they sketched both graphs (double) on the same coordinate plane as the composite function; (2) they sketched just one of the parent function graphs (one) as their response of the composite function; (3) they performed some transformation of one of the parent function graphs and considered it as the composite function graph (one-T); (4) they sketched half of one parent function and half of the other parent function as the graph of the composite function (mixed half); and (5) they applied properties from one parent function to properties of the other parent function to sketch the composite graph (mixed prop) (See Table 4 for examples of three of the typical responses students sketched.)

**Table 4**

*Primary Ways Students Respond to Function Composition Graphically*

| Task: Sketch the composite function $g(f(x))$ given the following function graphs. |
|---|
| $f(x)$  $g(x)$  |

Typical responses:




*Note*. From "Exploring geometric reasoning with function composition" by Y. Chen, N. Wasserman, and T. Paoletti, 2023, in *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (RUME)*, pp. 134-153. Copyright 2023 by RUME. Reprinted with permission.

The growing interest in how students understand and reason with function composition graphically beyond just evaluating points opens promising new directions for research. Future studies can examine more deeply students' graphical reasoning processes when working with function composition tasks. It would also be meaningful to explore potential teaching and learning approaches that can foster rich and deep conceptual understanding of function composition presented by not only equations, but also graphs, tables, words or even other novel representations.

### *Dynamic Representations*

**Dynamic Mathematical Representations.** Dynamic representations, or "dynamic visualizations" as termed by Ploetzner and Lowe (2004), are defined as representations that change their graphical structure during presentation. These representations can be categorized into two types: linear dynamic representations, where changes occur automatically, and interactive dynamic representations that allow learners to have some control over how these changes are displayed (Schwan and Riempp, 2008). Dynamic representations are implemented

through digital technologies that offer various potential improvements to the mathematics teaching and learning experience. First of all, dynamic representations, especially dynamic graphical representations, provide visualization of mathematical concepts, which was reported to be critical in supporting intuition and concept formation (Dreyfus, 1991). Secondly, many studies highlighted that interactive dynamic representations can offer immediate feedback and enable users to adapt the presentation to their individual cognitive needs, create an experimental environment which is conducive to active learning, facilitate the construction of meanings more than symbolic manipulation such as testing conjectures, and provide a foundation for formulating deductive explanations and developing ideas of proof (Cory & Garofalo, 2011; Hanč et al., 2011; Jones, 2000; Schwan & Riempp, 2008). Moreover, scholars have reported that dynamic representations help strengthen students' understandings by connecting and integrating multiple representations (Ainsworth, 1999; Kaput, 1992; Cory & Garofalo, 2011). For example, Cory and Garofalo (2011) found that the participants modified and deepened their conceptions of limit as they integrated the visual representation with the symbolic definition while manipulating a coherent visual representation.

On the other hand, researchers have also raised concerns about the use of dynamic representations. Schwan and Riempp (2008) suggested that managing interactive features may burden the learner with additional cognitive load. Jones' (2000, p. 81) study about students' interpretations when using dynamic geometry software (DGS) indicated that having to learn to use the software would itself present an extra learning load. For instance, when dragging the figures, "students need to gain an appreciation of the notion of the constraint of robustness of a figure under drag as a mathematical feature, rather than as 'mechanical glue'". In addition,

Hanna (1998) has observed that using DGS to confirm conjectures through measurements may reduce the perceived need for deductive proof.

Logo Turtle is another important and broadly explored avenue of technological tools which also offer dynamic representations. It is a programming language that creates a learning environment related to mathematical and functional learning (Solomon et al., 2020). The Logo Turtle, invented and developed by Seymour Papert, initially provided a natural perspective for children to think of what they want the turtle to do from the turtle's viewpoint (Solomon & Papert, 1976). Focused on mathematical function topics, Armon and Leron (1991) compared the more intrinsic turtle representations of functions which complement standard mathematical representations of functions. However, the authors also pointed out the slippery effect on students' intuitions when dealing with turtle representations of more general curves which require knowledge of advanced mathematics. Later on, the Logo Turtle evolved into Turtle Geometry. This advanced form provides transformational and analytic geometry that can, for example, simulate physical properties (Boytchev, 2007; Rowe, 1985; Zoran et al., 1991). Applying Logo Turtle to investigate students' learning of function composition might introduce a significant extra factor for students' performance – their computer programming experience and ability. Besides, although previous studies have used Logo Turtle to explore function understanding, its representations of and relations to function composition remain indirect and may not extend well to more complicated functions and their rich properties (for example, discontinuous or disconnected functions).

**Dynamic Representations of Functions.** Dynamic representations, particularly graphical representations, have been utilized with function-related topics frequently. According to Kaput (1989), the dynamic representations link functions and graphs, eliminating the

transformation constraints of static representations. Goldenberg et al. (1992) analyzed how dynamic visualization might influence the development of the concept of function. The authors argued that with conventional static graphical representations, students must mentally "animate" the graph to perceive the dynamics of covariational relations in functions. In contrast, dynamic representations allow students to directly see variable changes and perceive invariant stability. Furthermore, Goldenberg et al. (1992) considered dynamic graphical representations to be very powerful conceptual tools for understanding functions by permitting the visualization of functions with non-numeric values. Supporting this perspective, Ferrara et al. (2006) concluded that dynamic representations of functions enabled students to more easily use their intuitive knowledge of graphing to make sense of symbolization, rather than the reverse process. In a more recent study, Birgin and Acar (2022) reported students' mathematics achievement in exponential and logarithmic functions using dynamic graphical representation on GeoGebra software. The results revealed several opportunities that GeoGebra created for students: (1) identifying properties of functions, such as graph symmetry; (2) connecting changes in variables in algebraic representations to corresponding changes in graphs; and (3) enhancing conceptual learning of inverse functions, especially exponential and logarithmic functions, through drawing, moving, and observing their graphs.

**Dynamic Representations of Function Composition.** Despite the rich discussion involved with using dynamic representations to learn function topics, very limited research investigated how students understand function composition through dynamic graphical representations. This dissertation study employed a three-dimensional (3-D) dynamic graphical representation (Figure 1) of function composition that was developed and studied by Wasserman (2024). In his work, Wasserman (2024) elaborated the theoretical potential of this graphical

representation for enriching learners' conception of function composition, particularly the opportunities to enrich their understanding of how function composition integrated the graphs of the two parent functions. The author claimed that the 3-D visual representation might foster students' graphical conceptions of $g(f(x))$ such as: (1) horizontal lines in $f$ produce horizontal lines in the composite function; (2) the domain of the composite function consists of those $x$ in domain of $f$ with $f(x)$ in domain of $g$; (3) lines in $f$ produce copies, compressions, or stretches of $g$ in the composite function; (4) symmetry and periodicity in $f$ generate symmetry and periodicity in the composite function. Wasserman pointed out that further work studying this particular 3-D representation as well as the others might help us understand in what ways the representation enriches students' conceptual understanding of function composition.

There exist several other types of dynamic graphical representations of function composition with a majority of them in two dimensions. The simplest one works just like a graphing calculator, asking users to type in $f(x)$ and $g(x)$, then sketching the static graphs of $f(x)$, $g(x)$ and $g(f(x))$ all on the same coordinate plane (e.g., Duncan, 2012). Another type of typical dynamic graphical representation shows the point-by-point process of how an input $x$ can be mapped to the composite function output $g(f(x))$ (e.g., Brzezinski, 2016). The third type of dynamic representation of function composition shares similar features with the tool for this study in Figure 1, where users can interactively drag and move the input point $x$ to see the simultaneous movement of the composite function point $g(f(x))$. However, they either demonstrate the $f(x)$, $g(x)$ and $g(f(x))$ on the same plane with multiple overlapped graphs (e.g., Mhohen, 2011), or miss the simultaneous demonstration of how $f(x)$ moves, which serves as the critical bridge connecting the movement of the input $x$ to the resulting movement of output $g(f(x))$ (Owsiak, 2012).

## Reasoning

### *Mathematical Reasoning*

**Definition.** Reasoning, in its broadest sense, is involved with almost all processes of forming or adjusting beliefs to make inferences, including perceiving, comprehending, categorizing, and decision-making (Hanna, 2020; Rips, 1990). Mathematical reasoning, embedded within broader reasoning concept of cognition, is distinct from everyday reasoning. In order to reach a valid conclusion, mathematical reasoning requires more lucid definition of the rules and structures with a correct chain of inference. The concepts of reasoning, argumentation, and proof have ill-defined boundaries since scholars in mathematics and mathematics education use these words in different ways (Hanna, 2020). Although reasoning is often closely linked to proof in mathematics education research, mathematical reasoning extends well beyond proof in mathematics teaching and learning. In fact, mathematical reasoning is often used interchangeably with mathematical thinking when researchers try to describe how students understand certain concepts.

Adapting the definition from NCTM (2009), *mathematical reasoning* in this study is defined as a thinking process that makes assertions, provide valid justifications, and reach conclusions, to understand mathematical concepts or solve mathematical problems.

**Studies of Mathematical Reasoning.** Researchers have paid close attention to mathematical reasoning from multiple perspectives. Building on general reasoning classifications, mathematical reasoning typically includes several major approaches: deductive, inductive, abductive, and analogical reasoning (Harel & Weber, 2020; Reid & Knipping, 2010; Rips, 1990). When students engage with specific areas of mathematics, distinct reasoning patterns emerge during their learning process. Proportional reasoning, for instance, has attracted

considerable research attention, particularly in studies of function slopes (Cheng et al., 2013; Gonzales et al., 2008; Manouchehri & Sriraman, 2020a). As another example, Saldanha and Thompson (1998) introduced covariational reasoning as the capacity to conceptualize two quantities changing simultaneously. Covariational reasoning has been identified by scholars in different ways established from a variety of mathematical contexts and representational environments (Carlson et al., 2002; Manouchehri & Sriraman, 2020a). Furthermore, Ikram et al. (2020) investigated the role reversible reasoning potentially plays in learners' understanding of inverse function problems. Exploring reasoning commonly involves describing the process of mathematical thinking. Therefore, mathematics education researchers have frequently highlighted the actions taken during the process, including seeking patterns, conducting experiments, making conjectures driven by data or experimental evidence, making generalizations, and looking for the desired result (Ciosek, 2012; Cuoco et al., 1996; Dörfler, 1991; Ellis, 2007; Harel, 2001; Kaput, 2008; ). When it comes to constructing mathematical understanding, especially regarding the notion of functions, the APOS theory has been widely used as an influential model. It is a framework that explains how individuals mentally construct their understanding of mathematical concepts through different stages. This framework will be discussed in detail later.

### *Algebraic Reasoning*

Mathematics education scholars often use the terms *algebraic thinking* and *algebraic reasoning* interchangeably (Herbert & Brown, 1997; Kaput & Blanton, 2005; Walle, 2007; ). Therefore, in this literature review chapter, I refer to both *algebraic reasoning* and *algebraic thinking* depending on the term employed by the scholars in their research. Coincident with the viewpoint raised by Kieran (2011) on the other hand, for the rest of this paper I consider

*algebraic reasoning* to be interpreted in a broader way to encompass not only the classical forms of reasoning (such as deductive, inductive, abductive, or analogical), but also reasoning that goes beyond those and which employs the various and diverse approaches that researchers have studied along with the notion of algebraic thinking. Researchers proposed three different perspectives on algebraic reasoning:

**Kaput and Colleagues: "Algebrafying" the K-12 Curriculum.** Kaput (1998) proposed five forms of algebraic reasoning to support algebra reform across all grades. Building on this foundation, Kaput and other scholars further explored, organized and characterized algebraic reasoning from multiple perspectives (Chimoni et al., 2018; Chimoni & Pitta-Pantazi, 2015; Kaput, 2008; Kaput & Blanton, 2005; Kriegler, 2008).

**Kieran and Colleagues: Principal Activities of Algebra.** Kieran (2004) presented a perspective on algebraic thinking that diverged from Kaput's view. While Kaput emphasized generalization and its progressive symbolization as fundamental, Kieran (2004) argued that algebraic thinking in the earlier grades can be engaged in without the use of the letter as symbol for variable (Kieran, 2011). Kieran (2004) proposed that algebraic thinking in school algebraic activity consists of three components: (1) generational activities, which involve creating algebraic expressions and equations; (2) transformational activities, which involve syntactically-guided manipulation of formalisms such as factoring and solving equations; (3) global/meta-level activities, which involve activities which make use of algebra as a tool, like problem solving and modeling.

**Cuoco and Colleagues: Algebraic Habits of Mind.** Cuoco et al. (1996) considered algebraic thinking as "reasoning like algebraists" through five "habits of mind": reasoning about binary operations, using abstraction, exploring algorithms, decomposing concepts to understand

fundamental structures, and using known structures to study unfamiliar ones. Building on this framework and the NCTM (2000) standards, Driscoll et al. (2003) developed a toolkit for algebraic thinking with three key aspects: *doing/undoing* (working forward and backward), pattern recognition to construct rules, and abstracting system regularities from computations.

**Definition of Algebraic Reasoning for this Study.** Drawing from the frameworks reviewed above, algebraic reasoning can be characterized as the process of generalizing patterns, manipulating symbols, abstracting structures, and modeling relationships. Among these, symbolism and generalization are two essential components. Therefore, in this study, *algebraic reasoning* about functions is defined as a mathematical thinking process that: (a) makes assertions; (b) provides justifications through manipulating algebraic expressions of function equations and their algebraic properties; and (c) draws conclusions.

Specifically, to determine if one reasoned algebraically when solving function composition tasks, the critical criterion is to observe whether algebraic expressions (e.g., function equations, the arithmetic of algebraic expressions, and the properties of the specific expressions) were utilized during the reasoning process when provide justifications through generalizing patterns, abstracting structures, and modeling relationships. Table 5 demonstrates examples of typical algebraic reasoning performance on function composition tasks across different function representations.

**Table 5**

*Examples of Algebraic Reasoning Performance on Function Composition Tasks*

| Representations | Function Composition Tasks | Algebraic Reasoning |
|---|---|---|
| Algebraic Representation | Evaluate $g \circ f$:<br>$f(x) = 2x$<br>$g(x) = x^2$ | Substitute the equations<br>$g(f(x)) = (2x)^2 = 4x^2$ |

<table>
<tr>
<td>Graphical Representation</td>
<td>Evaluate $g \circ f$:<br>$f(x)$ $g(x)$<br>  </td>
<td>Use the graph to identify the equations of $f(x)$ and g$(x)$, then substitute the equations.</td>
</tr>
<tr>
<td>Tabular Representation</td>
<td>Evaluate $g \circ f$:<br>$f(x)$<br><table><tr><th>$x$</th><th>$y$</th></tr><tr><td>-2</td><td>-4</td></tr><tr><td>-1</td><td>-2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>6</td></tr></table>$g(x)$<br><table><tr><th>$x$</th><th>$y$</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>9</td></tr></table></td>
<td>Analyze the input and output relations in $f(x)$ and g$(x)$, and estimate the two function equations, then substitute the equations.</td>
</tr>
</table>

**Algebraic Reasoning Research Under the Topic of Function.** Almost all scholars who work on algebraic reasoning pay close attention to functions and functional thinking. As examples, Wilkie and Clarke (2016) found that generalizing geometric pattern visualization enhances students' functional thinking and understanding of algebraic variables. Using the algebraic habits of mind framework, Magiera et al. (2013) studied how pre-service middle school teachers understood and recognized students' ability to "Build Rules to Represent Functions." Damayanti et al. (2019) identified three main steps in students' algebraic reasoning of quadratic equations: manipulating equations (simplification and standardization), factoring formulas, and finding solutions. Despite this rich literature on algebraic reasoning in function topics, research specifically addressing algebraic reasoning of function composition remains limited.

### *Graphical Reasoning*

**Graphical Reasoning in General.** Graphical reasoning has its roots in several related areas. First of all, it shares some characteristics with geometric reasoning, as they both analyze properties and relationships through dynamic and continuous perspectives (Cuoco et al., 1996),

including the thinking that develops from basic shape recognition to the understanding of complex mathematical systems (Crowley, 1987; Manouchehri & Sriraman, 2020b). However, geometric reasoning's focus on axiomatic systems and shape properties makes it too narrow for understanding function graphs comprehensively, even though geometry plays a connecting role across mathematical domains (Goldenberg et al., 2012). Secondly, visual reasoning comes closer to what the current study is examining, since it involves working with graphs and pictures (Villarreal, 2000; Dreyfus, 1991). It looks at how visual information is used to think analytically, whether the focus is the whole picture or specific details, and whether things are moving or staying still. But as Presmeg (2006) points out, visual reasoning covers a lot more ground – it includes constructing and transforming mental images, working with tables, and dealing with symbolic representations. This makes the concept a bit too broad for specifically understanding function graphs in this study.

Bertin (1983) wrote a foundational work about graphical understanding by defining graphics through three essential elements including the information within the graphic, the properties of the system, and the underlying components that govern and integrate these properties. Mackinlay (1999) advanced this understanding by providing a structured categorization of visual-spatial components and establishing six distinct types of graphical languages in mathematics. Lowrie et al. (2010) further refined the graphical understanding concept by breaking down graphical reasoning into two key processes: (1) “encoding” as how information is delivered through graphical representations; and (2) “decoding” as how meaning is made through interpreting graphics.

**Graphical Reasoning Under the Topic of Functions.** Very limited number of scholars have focused on graphical reasoning within function topics. A few frameworks emerged for

understanding graphical reasoning in functions from early days. Bell and Janvier (1981) focused on graph interpretation in functional situations, emphasizing graphs as tools to reveal non-obvious features from numerical data. The authors defined graphical reasoning as interaction between graph and situation through: (1) point reading on graphs, (2) analysis of global features, and (3) direct translation from situation to graph. Dugdale (1992) identified specific graphical reasoning strategies for trigonometric and polynomial functions, which involve: (1) identifying and understanding special points like turning points, zeros, intercepts, and maximum numbers; (2) analyzing, verifying, and predicting special features and behaviors of the graphs, such as marking undefined values and asymptotes, comparing positive or negative regions, and connecting graph transformation based on exponents and coefficients. During calculus reform, emphasis was placed on connecting algebraic and geometric reasoning through graphical perspectives (Asiala et al., 1997b; Lauten et al., 1994). Rodriguez and Jones (2024) cataloged in detail the knowledge elements that students use during graphical construction and interpretation for one variable as a function of another. The researchers associated graphical form names with graphical patterns and conceptual schemas (e.g., a jump discontinuity on a function graph implies a very sudden change). The authors described graphical forms of reasoning as focusing on the pattern in the graph and assigning a conceptual schema to it.

**Definition of Graphical Reasoning for this Study.** Drawing from the frameworks reviewed above, in this study, *graphical reasoning* about functions is defined as a mathematical thinking process that: (a) makes assertions; (b) provides justifications through reading, interpreting, generalizing, analyzing, and predicting the patterns and properties of graphs in specific detail as well as the whole picture in both static and dynamic ways; and (c) draws conclusions.

Specifically, to determine if one reasoned graphically when solving function composition tasks, the critical criterion is to observe whether graphical patterns and properties (e.g., general shape across the positive and negative regions, symmetry, continuity, maximum points, asymptotes) were used during the reasoning process of reading, interpreting, generalizing, analyzing, and predicting. Table 6 (which shares the same function composition problem as Table 5) demonstrates examples of typical graphical reasoning performance on function composition tasks across different function representations.

**Table 6**

*Examples of Graphical Reasoning Performance on Function Composition Tasks*

| Representations | Graphical Reasoning Performance ($g \circ f$) |
|---|---|
| Graphical Representation | Observe and analyze the graphs of $f(x)$ and g$(x)$: "The line of $f(x)$ rises sharply, which means the output of $f(x)$ increases faster, so $g \circ f$ also rises more sharply than $g(x)$." |
| Algebraic Representation | Sketch the graphs of $f(x)$ and g$(x)$ based on the equations, and then apply the reasoning using the graphical representation. |
| Tabular Representation | Plot the points to sketch $f(x)$ and g$(x)$ based on the tables, and then apply the reasoning using the graphical representation. |

**Graphical Reasoning Research Under the Topic of Functions.** Among studies focused on graphical reasoning, few have examined function topics since the majority of graphical reasoning articles concerned diagrams or geometric shapes. For the limited studies which explored the graphical understanding of functions, scholars recorded students' interpretation of graph features in relation to function properties (Dugdale, 1992; Rodriguez & Jones, 2024). Researchers raised concerns about students' difficulties in interpreting function composition graphically (Ayers et al., 1988; Engelke et al., 2005; Meel, 2003; Sfard, 1992), but rarely explored graphical reasoning about function composition. Therefore, there is a dearth of in-depth investigation into students' graphical reasoning when composing functions.

### *Discrete Point Reasoning*

**Discrete Point Reasoning and Its Research Under the Topic of Functions.** Even (1998) claimed that in the case of functions, it is critical to distinguish between a global approach to function behavior and a point-wise approach. In examining students' understanding of functions in calculus course, Monk (1994) described students' "pointwise understanding" behavior in comparison with "across-time understanding" behavior of function concepts. Students demonstrate pointwise understanding when they treat a function as if it were only a table (regardless of whether the function is given by a table, graph, or formula), meaning students are only reading, plotting or dealing with particular numerical values of the independent variables and their corresponding dependent variables with discrete points (Even, 1998). In contrast, students develop across-time understanding when they are able to answer questions related to covariational thinking or global properties of the functions such as "How does change in one variable lead to change in others?" or " How is the behavior of the output variables influenced by variation in the input value?" Echoing the categorization of pointwise versus across-time understanding, Chen et al. (2023) and LaPlace et al. (2024) identified and differentiated between the approach of utilizing discrete points and general algebraic and graphical responses, as pointwise reasoning is unique compared with other reasoning steps that rely on algebraic expressions or graphs. The information from specific points can be derived from algebraic, graphical, as well as tabular representations, but all are limited to ordered pairs of input and output, and therefore are not reasoned algebraically or graphically. Similarly, in this study, I distinguished the discrete point reasoning from algebraic and graphical reasoning when analyzing students' reasoning processes.

**Definition of Discrete Point Reasoning for this Study.** Building upon the literatures reviewed above, in this study, *discrete point reasoning* about functions is defined as a mathematical thinking process that: (a) makes assertions; (b) provides justifications through reading, plotting, substituting, interpreting, generalizing, and analyzing functions through particular discrete points with numerical values of ordered pairs; and (c) draws conclusions.

Table 7 (which shares the same function composition problem as Table 5) demonstrates examples of typical discrete point reasoning performance on function composition tasks across different function representations.

**Table 7**

*Examples of Discrete Point Reasoning Performance on Function Composition Tasks*

| Representations | Discrete Point Reasoning Performance ($g \circ f$) |
|---|---|
| Graphical Representation | Locate discrete point ordered pair on the graph of $f(x)$, such as $(1, 2)$. Then plot the output of the ordered pair from $f(x)$, in this example 2, into the graph of $g(x)$ to identify the point $(2, 4)$. Conclude the ordered pair $(1, 4)$ in $g \circ f$. |
| Algebraic Representation | Evaluate the ordered pair in $f(x)$ from the equation, such as $(1, 2)$. Then substitute the output of the ordered pair from $f(x)$, in this example 2, into the equation of $g(x)$ to get the point $(2, 4)$. Conclude the ordered pair $(1, 4)$ in $g \circ f$. |
| Tabular Representation | Take the ordered pair in the table of $f(x)$, such as $(1, 2)$. Then find the ordered pair in the table of $g(x)$ with the input equal to the output of the $f(x)$ ordered pair, in this example 2, to identify the point $(2, 4)$. Conclude the ordered pair $(1, 4)$ in $g \circ f$. |

### *APOS Theory*

**Action-Process-Object-Schema (APOS) Theory.** APOS theory is a widely used model for describing how mathematical understanding can be built upon mathematical concepts. Dubinsky (1991b) considered the reasoning underlying mathematical concepts as mental structures (or constructions) which have stages. Based on Piaget's cognitive perspective,

Dubinsky (1991b) discussed the five types of *mental mechanisms*, or *reflective abstraction* – interiorization, coordination, reversal, encapsulation, and generalization – that lead to the structures of APOS that, in turn, explain the mental structures and mechanisms for the construction of mathematical knowledge (Arnon et al., 2014). Through a hierarchical manner and with differences in depth and complexity in each stage, Dubinsky (1991) characterizes the mental construction of mathematical knowledge as a "circular feedback system" which is nonlinear. The following paragraphs summarize the main characteristics of APOS theory and its framework for research and curriculum development reviewed by Arnon et al. (2014).

***Actions*.** At the initial stage, learners usually conceive a mathematical concept as an *action*. An action is external, which means each step of the transformation needs to be explicitly performed and guided by external instructions and the steps can neither be imagined nor be skipped.

***Interiorization and Process.*** Interiorization is the mechanism that makes the mental shift from *action* to *process*. When actions are repeated and reflected on, learners no longer rely on external cues, and they start to have internal control over the transformations. The control shows as the ability to imagine performing the transformations and carrying out the steps without having to execute each step explicitly (Dubinsky et al., 2005). Therefore, processes are interiorized actions.

***Encapsulation and Objects.*** When learners become aware of the process as a totality, where the process as the dynamic structure is now being seen as a static structure, then we say the individual has encapsulated the *process* into a cognitive *object*. Encapsulation occurs when learners apply actions to processes. According to multiple APOS-based studies, the mechanism of encapsulation is the most difficult (Clark et al., 2007; Martínez-Planell, 2010).

***De-encapsulation, Coordination, and Reversal of Processes.*** A process can be encapsulated into an object as well as de-encapsulated back to its underlying process. When necessary, two objects can be de-encapsulated, and their processes can be coordinated. The coordinated process can be encapsulated to form a new object. Some researchers raised the hypothesis that for two processes called $P_A$ and $P_B$, the learner first needs to encapsulate $P_A$ to object as $O_A$, and then apply $P_B$ to it. Then the coordination can continue as either the object of $O_A$ is assimilated and $P_B$ can be applied to it, or $P_B$ is accommodated so that the learner can apply it to $O_A$. However, how the mechanism of coordination carried out mentally is still under investigation. Moreover, a process can be reversed and then be encapsulated to a new object.

***Thematization and Schemas.*** The interactions of the elements outlined above generate *schemas*. Dubinsky (1991) explained a schema as a dynamic and continuous system that constructs and reconstructs a collection of structures (actions, processes, objects, and other schemas) and establishes connections among those structures. Further research is needed to better understand how schemas develop and are applied.

***Genetic Decomposition.*** For research focused on or utilizing APOS theory, a *genetic decomposition* commonly operates as a hypothetical model that describes the mental structures and mechanisms that a learner might need to construct in order to learn a specific mathematical concept. In general, a genetic decomposition consists of descriptions of: (1) the actions that a learner needs to execute on existing mental objects (existing concepts which are considered as prerequisite knowledge); (2) how these actions are interiorized into processes; (3) how the processes can be conceived as totalities, which are encapsulated into mental objects; (4) how the dynamic and continuous constructions of the actions, processes, and objects can be coordinated through encapsulation, de-encapsulation, and reverse processes into schema as a larger mental

structure, and how the schema can be thematized into an object. It is worth noting that a genetic decomposition is neither linear nor exclusive. One mathematical concept could have multiple possible genetic decompositions, and a learner may purse different learning paths to draw on multiple conceptions that have been developed at differing times based on different tasks. When verified empirically, a genetic decomposition can serve as a useful model of cognition and show the efficacy of APOS theory as a tool for describing students' reasoning and understanding of mathematical concepts (Weller et al., 2003).

**The APOS Paradigm for Research.** Research involving APOS theory usually includes theoretical analysis, collection and analysis of data, and sometimes design and implementation of instruction (Asiala et al., 1997a). Arnon et al. (2014) adapted the APOS-based research cycle from Asiala et al. (1997a). Research usually begins with a theoretical analysis of the cognition of the mathematical concept, aiming to establish a preliminary genetic decomposition. Typically, the initial genetic decomposition is based on researchers' own understandings of the concept and their experiences as learners and teachers (Asiala et al., 1997b). In some studies, researchers design activities and exercises to help students construct actions, interiorize them into processes, encapsulate processes into objects, and coordinate two or more processes to construct new processes. Students' responses and learning performance are then collected and analyzed to investigate whether students make the mental constructions described by the genetic decomposition and how well the students learned the mathematical content. The theoretical learning model of the concept is then reconsidered and revised according to the empirical evidence (Asiala et al., 1997b). In other studies, data from students' responses and learning performance are analyzed to help detect particular types of progression of conceptions for establishing mathematical understanding (Wasserman, 2017). For these studies, the APOS

framework provides some pertinent interpretations which are helpful in exploring mathematical understanding of certain topics.

**A Genetic Decomposition for Function and Function Composition.** Students' understanding of functions has been one of the most commonly studied topics using APOS theory (Clark et al., 1997; Carlson et al., 2010; Trigueros & Martínez-Planell, 2010). In addition, function composition is frequently used as an example or serves as the focus of study in APOS-based research on function understanding (Engelke et al., 2005; Miller et al., 2015; Wasserman, 2017). Broadly speaking, students generally hold action conceptions of function and function composition. Particularly when given non-formula representations like tables, graphs, or contexts, students struggle to engage in process-level function thinking. Nevertheless, very few papers have concentrated on the progress of understanding with the use of dynamic graphical representations of function composition through APOS theory, and almost none have been able to describe in depth students' thinking pertaining to mathematical understanding of function composition via graphs.

Referring to Arnon et al. (2014), Dubinsky (1991), Dubinsky et al. (2005), Dubinsky and Harel (1992) and Wasserman (2017), and combining these with my understanding and experiences as a learner and teacher, this study synthesizes a genetic decomposition for function composition (see Table 8) with some special attention to graphical representations.

**Table 8**

*Genetic Decomposition for Function Composition*

| | Characteristics | Examples |
|---|---|---|
| Action | • Dependent on explicit expression;<br>• Dependent on each particular steps;<br>• Focused on specific values;<br>• Limited understanding of procedural tasks; | • Learners require algebraic equations of both functions to compose them; Learners are unable to compose two functions in more general situations, |

| | | |
|---|---|---|
| | • Static understanding of function graphs; | e.g., given only tables or graphs, or have split domains;<br>• Learners compose functions one step at a time and cannot skip any step, while substituting specific values for the variable in the expressions instead of reasoning function composition on entire intervals;<br>• Learners view initial and composed function graphs as being only static curves in the plane. |
| Process | • Construction of mental process for a given function and function composition as dynamic transformations that pairs elements of one set (domain) with elements of a second set (range) without using specified inputs or going through explicit steps.<br>• Dynamic conception of function graphs. | • Learners are able to compose functions without explicit expressions, such as tables or graphs.<br>• Learners are able to compose functions with split or restricted domains; Learners are able to determine domain and range of composite function by operating and reflecting on the set of all possible inputs and outputs;<br>• Learners are able to compose functions by coordinating two input-output processes without checking individual input; Learners are able to view function composition as: input which is processed by one function and its output which is processed by a second function.<br>• Learners are able to recognize that $f(g(x))$ may be different from $g(f(x))$;<br>• Learners view graph of initial and composed functions as general mappings of a set of input values to a set of output values. |
| Object | • Encapsulate function as not just sum of repeatable means, but a singular object itself that can be examined and transformed; Functions are treated as static totalities to which actions can be applied. | • For $g(x) = x + 1$, learners are able to recognize graphically $g(f(x))$ and translate $f(x)$ by 1 unit up;<br>• Learners are able to recognize that $g(f(x))$ is itself an object that, for instance, has an inverse function or can be transformed. |
| Schema | • Apply the interactions of the elements outlined above (actions, processes and objects) as a dynamic and continuous | • To compose two functions $f$ and $g$ to obtain $f(g(x))$ given graphs, learners are able to de-encapsulated the |

| | | |
|---|---|---|
| | system for function composition that can be invoked and used in dealing with new problem situations. | processes that generate the graphs of $f$ and $g$; These processes are then coordinated by applying the process of $f$ to the elements obtained by applying the process of $g$; The resulting process is then encapsulated into a new object $f(g(x))$. |

**How APOS Theory Was Used in This Study.** While many studies have used APOS theory to determine students' overall levels of understanding, this study concentrated on analyzing the detailed progression of students' understanding as demonstrated through their reasoning steps. Specifically, this study investigated what level of understanding, in terms of APOS theory, characterizes students' reasoning in each step of their problem-solving process. As reviewed above, to generate a *schema* level of understanding, the first three levels (*action, process* and *object*) must interact dynamically in a non-linear way. Therefore, this study sought to capture and analyze how these reasoning patterns reflect interactions which occur during the mental construction of function composition concepts. Additionally, the research approach in this study was inspired by Dubinsky and Wilson's (2013) method of analyzing participants' indications of APOS stages during interviews.

**Summary**

With the clear significance and popularity of function topics in secondary and collegiate mathematics curriculum as well as in the research field of mathematics education, function composition is an under-researched area despite its importance in function concepts and in subsequent advanced topics such as the chain rule in calculus. Very few scholars have focused on graphical reasoning leading toward the understanding of function composition, though studies have shown the benefits of incorporating multiple representations into the concept formation and learning process of functions. In addition, almost no research has explored the possibilities and

affordances of technological tools like dynamic graphic representations for learners studying the concept of function composition. This study aimed to address this concern by investigating how students reason about function composition with graphs and a dynamic graphical representation.

# Chapter 3: Methodology

The purpose of this study is to: (a) analyze students' reasoning about graphically and algebraically depicted function composition with and without a dynamic graphical representation; (b) explore the depth of students' understanding about function composition with and without a dynamic graphical representation; (c) identify and examine any differences among students' graphical reasoning, algebraic reasoning and discrete point reasoning, with respect to the topics of domain, range, symmetry, and continuity of function composition; and (4) identify and examine any differences between students' reasoning with and without a dynamic graphical representation, with respect to the topics of domain, range, symmetry, and continuity of function composition.

## Research Questions

This research addresses the following research questions:

1. How do students reason about function composition with and without a dynamic graphical representation?
2. What level of understanding, in terms of APOS theory, characterizes students' reasoning about function composition with and without a dynamic graphical representation?
3. On the topics of domain and range of function composition, how do students reason differently with:
    c. distinct reasoning types?
    d. a dynamic graphical representation and without it?
4. On the topics of symmetry and continuity of function composition, how do students reason differently with:

c. distinct reasoning types?

d. a dynamic graphical representation and without it?

## Research Setting and Participants

### *Research Setting*

The research was conducted in a mathematics education program at a graduate school located in a major city on the eastern coast of the United States. As a multiple case study through individual interviews, this study took place during the fall 2023 and spring 2024 semesters. The research setting consisted of a private conference room on campus. Each interview session lasted between 90 and 120 minutes to ensure sufficient time for the participants to think through their reasoning and solutions.

### *Participants*

The study aimed to thoroughly explore the potential for reasoning about graphically depicted function composition, along with the experience of using a 3D dynamic graphical representation of function composition. Both the tasks and the representation were unconventional and possibly challenging for inexperienced mathematics learners. Therefore, to gather rich, meaningful and in-depth data, participants ($n = 15$) were recruited through purposive sampling mostly ($n = 13$) from the mathematics education graduate program, particularly those with mathematics teaching certificates. The remaining participants ($n = 2$) were also students from the same graduate mathematics education program, but who did not have mathematics teaching certificates. Participants who are future mathematics educators are commonly more experienced with mathematics content as well as mathematical representations, making them desirable for this study. Meanwhile, to ensure that participants contributed from a student's perspective rather than adopting the viewpoint of educators, the researcher recruited

students: (a) who had completed undergraduate degrees in mathematics or related STEM fields in the last two years; and (b) who had no full-time teaching experience. Students interested in participating were given a survey to assess their suitability for the study (Appendix A). Participants for this study needed to: (a) have completed more than five undergraduate and/or graduate mathematics content courses; (b) at least "somewhat agree" with the statement that "I understand the concept of function composition $g(f(x))$"; (c) correctly complete the function composition example question in the survey; and (d) at least "somewhat agree" with the statement that "I have experience in using dynamic graphical representations such as GeoGebra." Of the fifteen participants of this study: (a) all had at least five mathematics content courses in their undergraduate and graduate studies; (b) one "somewhat agreed," four "agreed," and ten "strongly agreed" that they understood the concept of function composition $g(f(x))$; (c) all correctly completed the function composition example question; and (d) seven "somewhat agreed," six "agreed," and two "strongly agreed" that they had experience in using dynamic graphical representations such as GeoGebra.

## Research Design

To explore how students reasoned about function composition through graphical and other lenses, with and without a dynamic graphical representation, this study employed a qualitative multiple case study through face-to-face, one-on-one, task-based, and semi-structured interviews between the researcher and participants. This approach allowed the researcher to explore an in-depth understanding of the participants' reasoning processes and to comprehensively examine the reasoning themes, patterns, similarities and differences across the cases in order to build general explanation (Stake, 2013; Yin, 2014).

### *Interview Design*

A research interview, defined as "a process in which a researcher and participant engage in a conversation focused on questions related to a research study," was applied in this study (deMarrais & Lapan, 2003). Interviews, as essential instruments of case study research, generate detailed and contextual data for understanding complex phenomena like reasoning processes that cannot be fully directly observed (Merriam & Tisdell, 2015). The interviews were conducted face to face for the researcher to accurately capture and collect the non-verbal, such as subjects' gestures for estimating the distance on the graphs or self-correction steps. The one-on-one setting of the interviews ensured a safe environment for participants to share their struggles and challenges and avoided their viewpoint being interfered with the others. Through reasoning and solving function composition tasks during the task-based interviews, participants articulated their thoughts while completing purposefully designed tasks to address the research questions (Creswell & Poth, 2016). Because the interviews were semi-structured, all participants followed a consistent questioning protocol while having the flexibility to demonstrate their reasoning uniquely.

Following the protocol (Appendix C), each interview consisted of three rounds, and in each round there were four problems that asked the students to sketch the composite function ($g(f(x))$) graph based on the given parent functions ($f(x)$ and $g(x)$). Participants were instructed to think aloud throughout the interview, expressing their reasoning as they worked through the problems. To collect rich and informative data, the researcher engaged interactively with participants during the interview, posing follow-up questions about their actions (e.g., "Why did you just erase your sketch on the negative side?"), statements (e.g., "You just mentioned, 'There is a jump here.'. Could you explain more about it?"), and written responses (e.g., "Why did you cross out the $g(f(0))$ value in the table?"). At the onset of each interview, the researcher

explicitly explained that any subsequent follow-up questions would be used exclusively for clarification purposes. These questions were not intended to suggest that the participants were either right or wrong. A scientific calculator was provided to participants to assist them in applying, in particular, discrete point reasoning strategies that involved specific calculations. For example, one participant used the calculator to evaluate $\cos(2)$ in order to sketch the curve at that point on the composite function graph. Based on the protocol, the researcher supplemented interview questions for participants after each round to address any topics not fully explored during the initial discussion. On the other hand, the researcher strategically omitted protocol questions that had already been fully addressed in the round, thereby preventing redundancy and minimizing participants' ability to anticipate subsequent interview questions.

In the first round, students sketched composite functions given only static graphs of $f(x)$ and $g(x)$ printed on paper. The collected data were used to investigate, without dynamic graphical representation: (a) students' reasoning about graphically depicted function composition (research question one); (b) students' level of understanding of graphically depicted function composition through an APOS lens (research question two); and (c) students' reasoning about domain, range, symmetry and continuity of graphically depicted function composition (research questions three and four).

In the second round, students sketched composite functions given only equations of $f(x)$ and $g(x)$ printed on paper. The data served to illustrate: (a) students' reasoning about algebraically represented function composition (research question one); (b) the level of understanding of algebraically depicted function composition through an APOS lens (research question two); and (c) students' reasoning about domain, range, symmetry and continuity of

algebraically depicted function composition (research questions three and four) all in situations devoid of dynamic graphical representation.

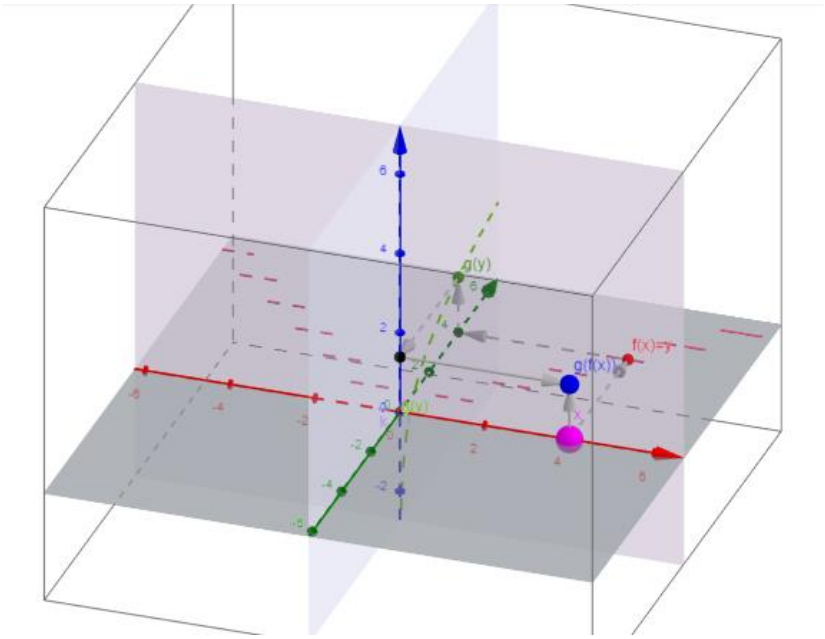
Before the third round, the researcher briefly introduced a dynamic graphical representation to participants with a sample function composition problem (Figure 1). Participants explored the representation and asked questions while working through it. In the third round, students sketched composite functions given a dynamic graphical representation and static graphs of $f(x)$ and $g(x)$ printed on paper. The analysis centered on: with a dynamic graphical representation, (a) students' reasoning about graphically depicted function composition (research question one); (b) the level of understanding of graphically depicted function composition through an APOS lens (research question two); and (c) students' reasoning about domain, range, symmetry and continuity of graphically depicted function composition (research question three and four).

***The Dynamic Graphical Representation Selected for This Study***

This dissertation study employed the three-dimensional (3-D) dynamic graphical representation (Figure 1) of function composition that was developed and studied by Wasserman (2024). Figure 3 shows the set up with $f(x) = |round(x)|$ (red graph in the $xy$-plane) and $g(x) = \ln(x)$ (green graph in the $yz$-plane). By tracking how the $x$ input is transformed first through $f$ (in the $xy$-plane), then through $g$ (in the $yz$-plane), we can observe the resulting height $z$, which traces out points $(x, z)$ in the $xz$-plane that generate the graph of $g \circ f$.

**Figure 3**

*The Dynamic Graphical Representational Tool for Function Composition for This Study*



*Note.* From *Function Composition* by N. Wasserman, 2022, GeoGebra (https://www.geogebra.org/m/cytfe7fm). Copyright 2022 by N. Wasserman. Reprinted with permission.

When dragging the input $x$ (the pink dot in Figure 3), the corresponding $f(x)$ value (the red dot in the $xy$-plane in Figure 3), and the mapped $g(f(x))$ value (the green dot in the $yz$-plane in Figure 3) are moved simultaneously, tracing out the general traced path of the composite function points $(x, z)$ (the blue dot) in the $xz$-plane.

After comparing the representation in Figure 3 with various online dynamic representations of function composition reviewed in the previous chapter, I made the conjecture that the dynamic graphical representation selected for this study has great potential for providing rich information about different aspects of the function composition process and fostering students' conceptual understanding of this topic. Many existing dynamic representations reviewed in previous chapter are either essentially digital versions of static representations where different graphs appear when different parent functions are entered but lack any interaction

between the graphs (e.g., Duncan, 2012), or representations focusing solely on the pointwise transformation of "$x \to f(x) \to g(f(x))$," limiting students' access to the general graphical trends and covariation relationships between parent functions and composite functions (e.g., Brzezinski, 2016). While some dynamic representations attempt to show the general dynamic transformation from the graphs of $f(x)$ and $g(x)$ to the graph of $g(f(x))$ through the traced path of points, most present this in two-dimensions (2-D). This means $f(x)$, $g(x)$, and $g(f(x))$ are either displayed on the same coordinate plane with potentially overlapped graphs or split across two 2-D coordinate planes (e.g., Mhohen, 2011; Owsiak, 2012). As a result, the general transformation of "$x \to f(x) \to g(f(x))$" through graphs of $f(x)$ and $g(x)$ to the graph of $g(f(x))$ cannot be demonstrated directly and covariationally.

***Function Composition Problems Design***

To maximize exposure to diverse scenarios, the interview tasks incorporated various function types that generated distinct function composition situations. Functions were designed with varied properties regarding domain, range, symmetry, continuity and other properties. Each question in a given round was paired with two related questions—one from each of the other rounds. These paired questions were either similar in content or identical but presented in different representations. Table 9 demonstrates the mapping and connections among problems across the three rounds (see problems in Appendix B).

**Table 9**

*Function Composition Problems Design*

| Problem Sets | Properties of $f(x)$ and $g(x)$ |
|---|---|

| | |
|---|---|
| Round 1 Problem 1 (Graph)<br>⇩ Similar<br>Round 2 Problem 1 (Equation)<br>⇩ Identical<br>Round 3 Problem 1 (Graph & Representation) | • Both trigonometry functions in Round 1; Exponential and logarithm function in Rounds 2 & 3<br>• Inverse functions<br>• Restricted domain and range<br>• Both symmetric about origin in Round 1; $f(x)$ and $g(x)$ symmetric about the line $y = x$ in Rounds 2 & 3<br>• Unfamiliar $f(x)$ functions in Round 1 for students |
| Round 1 Problem 2 (Graph)<br>⇩ Identical<br>Round 2 Problem 2 (Equation)<br>⇩ Similar<br>Round 3 Problem 2 (Graph & Representation) | • Constant functions ($f(x)$); Rational functions ($g(x)$) in Rounds 1 & 2; Polynomial functions in Round 3.<br>• Restricted domain and range<br>• $g(x)$ symmetric about a point.<br>• Discontinuity in $f(x)$ in Round 3<br>• Unfamiliar $g(x)$ functions in Rounds 1 & 2 for students |
| Round 1 Problem 3 (Graph)<br>⇩ Similar<br>Round 2 Problem 3 (Equation)<br>⇩ Identical<br>Round 3 Problem 4 (Graph & Representation) | • Radical function ($f(x)$) and a transformed radical function involved with absolute value ($g(x)$) in Round 1; trigonometry functions ($f(x)$) and radical functions (g($x$)) in Rounds 2 & 3<br>• Restricted domain and range<br>• $f(x)$ symmetric about $y$-axis<br>• Disconnection in $g(f(x))$ composite functions due to the restricted domain and range of $f(x)$ and $g(x)$<br>• Unfamiliar $f(x)$ function for students |
| Round 1 Problem 4 (Graph)<br>⇩ Identical<br>Round 2 Problem 4 (Equation)<br>⇩ Similar<br>Round 3 Problem 3 (Graph & Representation) | • Piecewise functions involved with absolute value ($f(x)$) and logarithm functions ($g(x)$) in Rounds 1 & 2; Piecewise function $f(x)$ and trigonometry function g($x$) in Round 3<br>• Restricted domain and range<br>• $f(x)$ symmetric about $y$-axis in Rounds 1 & 2; g($x$) symmetric about $y$-axis in Round 3<br>• Discontinuity inherited from $f(x)$<br>• Unfamiliar $f(x)$ functions for students |

**Data Collection**

Data for this study came from video and audio interview recordings as well as students' written responses. During each interview, a video camera captured both video and audio; it was positioned to record the participants' workspace, including their graphing papers or screens as they explored the dynamic graphical representation or used the scientific calculator. The recordings were transcribed through a two-step process. First, the audio was transcribed into dialogue. Then, the video recordings were analyzed to document participants' actions and gestures, which were integrated into the transcripts to provide a comprehensive account of their problem-solving process. For example, when a participant stated, "I erased this part because I realized that because of that, there is no negatives," the complete transcription including relevant gestural information would be: "I erased this part (pointing to negative values of $f(x)$) because I realized that because of that (circling the positive domain of $g(x)$ with fingers), there is no negatives (referring to the domain restrictions of the composite function $g(f(x))$." Participants' written work served as a secondary data source, including their rough work during the problem-solving process and their sketches of composite function graphs.

## Data Analysis

### *Coding*

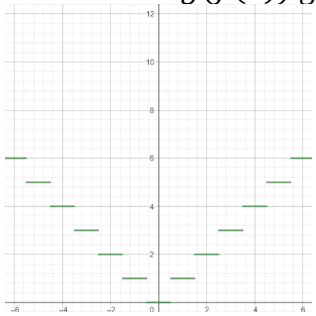
Reasoning in this study was defined as the process of making (multiple) assertions, providing justifications for these assertions, and drawing conclusions. During the reasoning process of composing and sketching $g \circ f$, participants usually conducted multiple rounds of "assertion – justification – interim conclusion" before reaching their final solution. Therefore, this study analyzed participants' reasoning by breaking it down into distinct steps, which were identified under one of the two scenarios. One participant's (Nicole's) reasoning process for Round 1 Problem 4 (Appendix B) is illustrated in Table 10 as an example. First, an incomplete

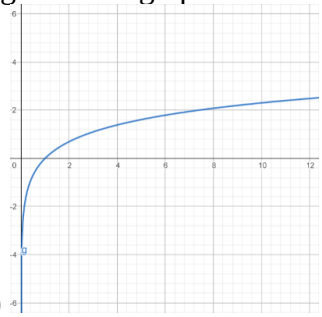
reasoning step was identified when a participant made an assertion and attempted to justify it but failed to reach an interim conclusion (e.g., step 3 in Table 10). In this case, the participant typically realized that the approach wasn't working, so they usually either: (a) initiated a new reasoning step with an entirely new assertion, or (b) developed a new reasoning step that justified the same assertion but from a different approach. In comparison, a complete reasoning step contained a full cycle of "assertion – justification – interim conclusion" (e.g., step 1, 4 and 7 in Table 10). These interim conclusions could be either correct or incorrect. The participant would then begin the next reasoning step either by building upon this conclusion or by exploring a different approach to address another perspective of the task. While participants used informal, spontaneous reasoning rather than formal proofs, a complete reasoning step still required the three key components: assertion, justification, and interim conclusion. These could be either explicitly stated or implied in students' responses. The reasoning process thus became a sequence of distinct reasoning steps, and it ended when the participant reached the final conclusion with a sketch of the composite function.

**Table 10**

*Example of a Reasoning Process Broken Down Into a Sequence of Reasoning Steps*

Round 1 Problem 4:
Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

| Step | Reasoning Process | Recognition as a Reasoning Step |
|---|---|---|

| | | |
|---|---|---|
| 1 | Nicole pointed at $f(x)$ graph: "I knew this was a floor (function)." Next she wrote down $\lfloor x + 0.5 \rfloor$. | Assertion: Based on the graph, $f(x)$ is a floor function.<br>Justification: This certain graph matches the floor function from her previous learning experience.<br>Interim Conclusion: $f(x) = \lfloor x + 0.5 \rfloor$.<br>Note: Complete but incorrect. |
| 2 | Nicole clarified the symbol she wrote down: "It ($f(x)$) is not absolute value (function)." | Assertion: Based on the graph, $f(x)$ does not contain an absolute value operation.<br>Justification: From her learning experience, $f(x)$ displays floor operations properties without absolute value characteristics.<br>Interim Conclusion: $f(x)$ is solely a floor function without any absolute value component.<br>Note: Complete but incorrect. |
| 3 | Nicole pointed at $g(x)$ graph: "There is something going on here in terms of, like, radicals, I'm assuming. Because we are not including zero, assuming exponential, and then…but again, I gave up on that." | Assertion: $g(x)$ is likely a radical or exponential function.<br>Justification: The domain of $g(x)$ does not include zero.<br>Interim Conclusion: Incomplete reasoning step – no conclusion reached.<br>Note: Incomplete. |
| 4 | Nicole indicated various parts on the graph with manual gestures: "We are not interested in, again, the $x$ below zero (pointed at the negative inputs of $f(x)$), because that's not the domain of the composite, (which) is greater than zero (pointed at the domain of $g(x)$, and assume it is the domain of $g(f(x))$." | Assertion: The negative input of $f(x)$ should not be considered.<br>Justification: The domain of $g(x)$ contains only positive values.<br>Interim Conclusion: The composite function $g(f(x))$ must have a positive domain.<br>Note: Complete but incorrect. |
| 5 | Nicole then pointed at the part of the $f(x)$ graph between the interval $(-0.5, 0.5)$ and explained: "So starting from here, (value) is going to be zero, which we have an asymptote (at g$(x)$)…so we are technically starting our domain defined as from 0.5." | Assertion: The composite function's input begins at 0.5.<br>Justification: For the interval of $(-0.5, 0.5)$, $f(x) = 0$, and $g(x)$ has an asymptote at the input of 0, making the function undefined in this interval.<br>Interim Conclusion: The domain of $g(f(x))$ is restricted as $[0.5, +\infty)$.<br>Note: Complete and correct. |

| | | |
|---|---|---|
| 6 | Nicole explained the shape of the composite function: "If you plug in one here (pointed at g$(x)$), you get zero, so you start zero, and then, like you are going up by increments of point five." | Assertion: The output of $g(f(x))$ begins at 0 and increases in increments of 0.5.<br>Justification: The output of $g(f(x))$ begins with $g(f(1)) = 0$. The output values increase by 0.5 because for each input, $f(x)$ increases by 0.5.<br>Interim Conclusion: The graph of $g(f(x))$ is similar to the graph of $f(x)$ increases by 0.5, but begins at $g(f(x)) = 0$.<br>Note: Complete but incorrect. |
| 7 | Nicole corrected her explanation of the shape of the composite function: "So (the input intervals of f$(x)$ are) 0.5 to 1.5, 1.5 to 2.5…exactly as it is in this domain (of $g(f(x))$). (Nicole plotted one point in each interval for several intervals to check their $g(f(x))$ values.) But the outputs, instead of being 2, 4, 6, 8, it's not going to be increments of point five, it's going to be less, going to be getting smaller and smaller." | Assertion: The output of $g(f(x))$ increases in smaller and smaller increments.<br>Justification: From the specific point values identified in different intervals, the differences between consecutive $g(f(x))$ values are getting smaller.<br>Interim Conclusion: The values of $g(f(x))$ increase at a decreasing rate, producing a concave down curve.<br>Note: Complete and correct. |

Each reasoning step was then coded into two major aspects aligned with the research questions: (a) reasoning based on the objects that participants analyzed from (algebraic expressions, graphical features, or discrete points) which produced varying types of reasoning (addressed in research questions 1, 3 and 4); and (b) participants' ways of reasoning as a reflection of levels of understanding according to APOS theory (addressed in research question 2). Additional analysis identified reasoning steps that focused on domain, range, symmetry, and continuity. To develop a comprehensive understanding of participants' performance, the researcher evaluated the accuracy of both their reasoning processes and their composite function sketches.

**Types of Reasoning.** Partially adapting the thematic framework for function composition approaches presented in Chen et al. (2023), this study identified three primary reasoning types:

*graphical*, *algebraic*, and *discrete point reasoning*. This adaptation led to the development of three distinct categories for coding participant responses (as outlined in Table 11) to answer research question 1, 3 and 4. Each reasoning step was systematically coded using one of the three categories, represented by designated color blocks. These color-coded steps were then arranged chronologically into a sequence to facilitate the analysis of participants' reasoning patterns.

**Table 11**

*Reasoning Types Coding Framework*

| Code Names | Color-Codes | Definitions | Examples |
|---|---|---|---|
| Graphical Reasoning | Blue | Reasoning step (particularly in the justification part) that involves the interpretation or application of graphical features and properties. | "The graph of $g(f(x))$ is reflected across the $y$-axis because the graph of $f(x)$ is symmetric about $y$-axis." |
| Algebraic Reasoning | Yellow | Reasoning step (particularly in the justification part) that involves the interpretation or application of equations or algebraic expressions. | "Because $f(x)$ is an absolute value function, which is an even function, that makes $g(f(x))$ also an even function." |
| Discrete Point Reasoning | Green | Reasoning step (particularly in the justification part) that involves working with specific points through plotting, evaluating, or substituting values using either algebraic expressions or graphs. | "I checked that the points $(2,4)$, $(-2,4)$, $(4,8)$, $(-4,8)$, $(5,10)$, and $(-5,10)$ are all points on $g(f(x))$, so I think the graph is reflected across the $y$-axis." |

It is important to explain the rationale for including discrete point reasoning as a separate category in the coding scheme. During preliminary analysis (Chen, 2025) and also from the analysis of Chen et al. (2023), the researcher observed that beyond graphical and algebraic approaches, reasoning through specific points emerged as a common and essential strategy. Participants frequently used graphs to identify coordinate values or employed algebraic equations to generate specific points through substitution. These approaches, while using graphs

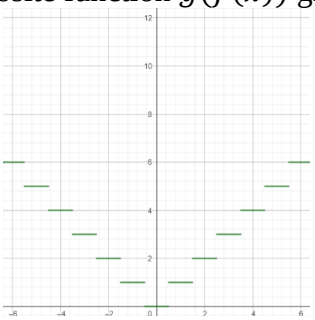
or equations as tools, differed from true graphical or algebraic reasoning because they focused on finding discrete point values rather than applying generalized graphical or algebraic properties and principles. These point-based reasoning steps proved not only prevalent but also instrumental in advancing participants' overall reasoning processes. Therefore, this study established *discrete point reasoning* as a separate category in the coding framework of reasoning types.

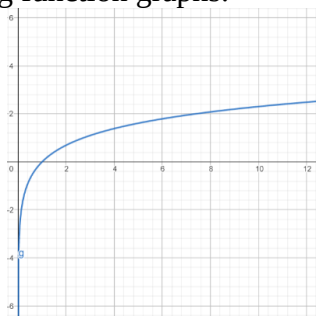
Nicole's reasoning process for Round 1 Problem 4 (Appendix B) is illustrated in Table 12 as an example of the coding focused on reasoning types.

**Table 12**

*Example of Reasoning Types Coding*

Round 1 Problem 4
Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

| Step | Reasoning Process | Type | Coding Justification |
|---|---|---|---|
| 1 | Nicole pointed at $f(x)$ graph: "I knew this was a floor (function)." Next she wrote down $\lfloor x + 0.5 \rfloor$. | Algebraic | Nicole justified the equation of $f(x)$ based on her previous experience with the function's algebraic expressions. |
| 2 | Nicole clarified the symbol she wrote down: "It is not absolute value (function)." | Algebraic | Nicole justified the equation of $f(x)$ based on her previous experience with the function's algebraic expressions. |
| 3 | Nicole pointed at $g(x)$ graph: "There is something going on here in terms of, like, radicals, I'm assuming. Because we are not including zero, assuming exponential, and then…but again, I gave up on that." | Algebraic | Nicole attempted to justify the equation of $f(x)$ but failed. |

| | | | |
|---|---|---|---|
| 4 | Nicole indicated various parts on the graph with manual gestures: "We are not interested in, again, the $x$ below zero (pointed at the negative inputs of $f(x)$), because that's not the domain of the composite, (which) is greater than zero (pointed at the domain of $g(x)$, and assume it is the domain of $g(f(x))$." | Graphical | Nicole justified the restricted domain of the composite function by analyzing domain and range properties presented through graphs. |
| 5 | Nicole then pointed at the part of the $f(x)$ graph between the interval $(-0.5, 0.5)$ and explained: "So starting from here, ($f(x)$ value) is going to be zero, which we have an asymptote (at $g(x)$)…so we are technically starting our domain defined as from 0.5." | Graphical | Nicole justified the restricted domain of the composite function by analyzing domain and range properties presented through graphs. |
| 6 | Nicole explained the shape of the composite function: "If you plug in one here (pointed at $g(x)$), you get zero, so you start (with) zero, and then, like you are going up by increments of point five." | Discrete Point | Nicole justified the general trend of the composite function's shape by plotting several points. |
| 7 | Nicole corrected her explanation of the shape of the composite function: "So (the input intervals of $f(x)$ are) 0.5 to 1.5, 1.5 to 2.5…exactly as it is in this domain (of $g(f(x))$). (Nicole plotted one point in each interval for several intervals to check their $g(f(x))$ values.) But the outputs, instead of being 2, 4, 6, 8, it's not going to be increments of point five, it's going to be less, going to be getting smaller and smaller." | Discrete Point | Nicole justified the general trend of the composite function's shape by plotting several points. |
| | Reasoning Sequence: | | |

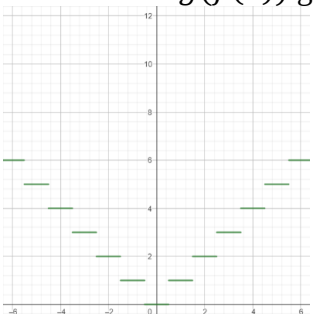
**APOS Level.** While many studies using APOS theory assess students' overall understanding levels, this study explored how students' understanding emerged through their reasoning steps to answer Research Question 2. Based on its performance and approach, each reasoning step was coded as a reflection of level of understanding (Action (A), Process (P), or Object (O)), similar to the method applied by Dubinsky and Wilson (2013). The coding of APOS levels followed the Genetic Decomposition for Function Composition presented in Table 8. The

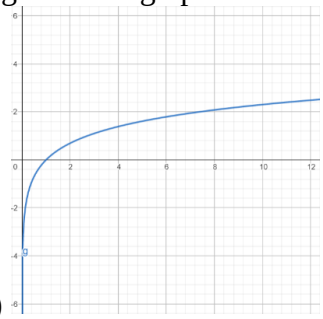
*Schema* level of understanding was not considered as a code here since this level – the dynamic and continuous systematic understanding applied all actions, processes, and objects – cannot be reflected by a single reasoning step. By examining the APOS level of understanding characterized in students' various reasoning steps, the study captured how their understanding differed or shifted across multiple representations. Nicole's reasoning process for Round 1 Problem 4 is illustrated in Table 13 as an example of the coding focused on APOS understanding levels.

**Table 13**

*Example of APOS Understanding Level Coding*

Round 1 Problem 4
Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

| Step | Reasoning Process | Type | Coding Justification |
|---|---|---|---|
| 1 | Nicole pointed at $f(x)$ graph: "I knew this was a floor (function)." Next she wrote down $\lfloor x + 0.5 \rfloor$. | Action | Nicole identified a specific algebraic equation of $f(x)$ which, although it shared the same graphical representations as accurate function, was not the correct equation. Using requires algebraic equations to compose indicates the action level of understanding for this step. |
| 2 | Nicole clarified the symbol she wrote down: "It is not absolute value (function)." | Action | In this step, Nicole further clarified her understanding but mistakenly determined that the algebraic equation of $f(x)$ did not involve an absolute value operation, when it actually did. Similar to the last step, the action level of understanding is appropriate to characterize this reasoning step. |

| | | | |
|---|---|---|---|
| 3 | Nicole pointed at $g(x)$ graph: "There is something going on here in terms of, like, radicals, I'm assuming. Because we are not including zero, assuming exponential, and then…but again, I gave up on that." | Action | Nicole then attempted to identify the specific algebraic equation of $g(x)$ but failed to justify it. Using algebraic equations of both functions to compose indicates the action level of understanding in this reasoning step. |
| 4 | Nicole indicated various parts on the graph with manual gestures: "We are not interested in, again, the $x$ below zero (pointed at the negative inputs of $f(x)$), because that's not the domain of the composite, (which) is greater than zero (pointed at the domain of $g(x)$, and assume it is the domain of $g(f(x))$." | Process | In this step Nicole made assertations about the restricted domain of the composite function and justified them through graphs rather than explicit expressions. Without focusing on explicit point-by-point steps, this indicated the process level of understanding in this reasoning step. |
| 5 | Nicole then pointed at the part of the $f(x)$ graph between the interval $(-0.5, 0.5)$ and explained: "So starting from here, ($f(x)$ value) is going to be zero, which we have an asymptote (at g$(x)$)…so we are technically starting our domain defined as from $0.5$." | Process | In this next step, Nicole made more assertions about the restricted domain of the composite function, employing reasoning similar to that of the previous step, which indicated the process level of understanding in this reasoning step. |
| 6 | Nicole explained the shape of the composite function: "If you plug in one here (pointed at g$(x)$), you get zero, so you start zero, and then, like you are going up by increments of point five." | Process | Nicole made an assertation about the general shape of the composite function, using a dynamic view of incremental increases. This was done through graphs rather than explicit expressions, characterizing the process level of understanding in this reasoning step. |

| | | | |
|---|---|---|---|
| 7 | Nicole corrected her explanation of the shape of the composite function: "So (the input intervals of f($x$) are) $0.5$ to $1.5$, $1.5$ to $2.5$…exactly as it is in this domain (of $g(f(x))$). (Nicole plotted one point in each interval for several intervals to check their $g(f(x))$ values.) But the outputs, instead of being $2$, $4$, $6$, $8$, it's not going to be increments of point five, it's going to be less, going to be getting smaller and smaller." | Process | Nicole corrected her previous conclusion and made a new assertion about the general shape of the composite function through general mappings of values in a dynamic view of increasingly smaller increments, indicating the process level of understanding in this reasoning step.. |

| Reasoning Sequence: | A | A | A | P | P | P | P |
|---|---|---|---|---|---|---|---|

**Fragmented and Integrated Reasoning Approaches.** During preliminary screening and data analysis (Chen, 2025), the comparison between fragmented and integrated reasoning approaches emerged in initial findings, particularly regarding the topic of domain and range in function composition. To substantiate this preliminary observation as part of the analysis for research question 3, reasoning steps specifically focusing on domain and range were coded into two categories: (a) fragmented reasoning steps and (b) integrated reasoning steps, as demonstrated with examples in Table 14. This coding allowed for comparison of participants' reasoning performance with and without a dynamic graphical representation for domain and range. Under the topic of domain and range in function composition, a *fragmented reasoning step* is defined as one that cannot independently reach a complete conclusion about the domain and range of the composite function. In contrast, an *integrated reasoning step* is defined as a comprehensive approach where participants reach a complete conclusion about the domain and range of the composite function in one coherent reasoning process.

**Table 14**

*Examples of Fragmented Versus Integrated Reasoning Coding About Domain and Range*

| | Fragmented Reasoning Step | Integrated Reasoning Step |
|---|---|---|
| Definition | A reasoning step that cannot independently reach a complete conclusion about the domain and range. | A single comprehensive reasoning step that can reach a complete conclusion about the domain and range. |
| Scenario | (1) One reasoning step simply identified the restricted domain and range of $f(x)$ or $g(x)$.<br>(2) One reasoning step identified specific input values that made the composite function undefined, but the subject did not extend this to determine the general domain and range of $g(f(x))$. | One reasoning step simultaneously identified and justified how the restricted domains and ranges of $f(x)$ and $g(x)$ determine the restricted domain and range of their composition $g(f(x))$. |
| Example | (1) In step 3 of Round 1 Problem 3 (Appendix B), Felix stated "I'm going to write the domain and range (of $f(x)$ and $g(x)$)," but instead of continuing reasoning about the domain and range of $g(f(x))$, he focused on identifying the equation of the parent functions in the next step.<br>(2) In step 3 of Round 2 Problem 3 (Appendix B), Olivia found specific input values that made $g(f(x))$ undefined through plotting, but she could not generalize these to determine the restricted domain of the composite function in this reasoning step. She later approached the restricted domain using graphs. | In step 3 of Round 3 Problem 4 (Appendix B), Bree reached the complete conclusion about the restricted domain of $g(f(x))$ by justifying that "Due to the fact that $g(x)$ cannot accept the negative values, that all the (inputs that have the) negative outputs of the $f(x)$ are not going to work out." Therefore, Bree concluded that the restricted domain of $g(f(x))$ consists of the set of inputs $x$ where $f(x) > 0$. |

**Accuracy.** To gain a comprehensive view of students' reasoning results, this study employed evaluation coding to assess the accuracy of students' reasoning in each problem (Saldaña, 2021). Each participant's responses were evaluated using the following criteria: 1 point for solutions demonstrating both accurate sketches of $g(f(x))$ graphs and sound reasoning and 0 points for incorrect solutions that either showed fundamental misconceptions in sketching the composite graphs or contained mostly invalid reasoning steps. A key principle in the accuracy evaluation was the primacy of solid reasoning over correct graphing of $g(f(x))$, particularly in

Round 3. Participants received no credit for simply replicating the traced path of the composite function points from the dynamic graphical representation without proper justification. Additionally, participants could not receive any credit if they showed sound reasoning with incorrect sketches. However, this scenario rarely happened, since incorrect sketches of composite functions were usually accompanied by flawed reasoning.

**Observational Notes for Individual Case's Reasoning Pattern and Quotes.**

Throughout the analysis process, detailed observational notes were maintained to document each participant's distinctive reasoning patterns. These notes captured individual problem-solving behaviors, thinking habits, mathematical perspectives, and significant quotes. This systematic documentation enabled both in-depth analysis of individual cases as well as identification of cross-case patterns, providing authentic evidence to support the research findings.

### *Addressing Research Question 1*

To answer the first research question concerning student reasoning about function composition with and without a dynamic graphical representation, this study analyzed coded data of reasoning types and the observational notes. The problem-solving process was visualized chronologically for each participant and problem using a sequence of color-coded blocks that represented their use of graphical, algebraic, and discrete point reasoning approaches.

This research question was addressed in three aspects. First, the researcher analyzed and demonstrated the number of reasoning steps and their distribution for each participant in each problem across all rounds. This distribution showed the length of reasoning process without a dynamic graphical representation (Rounds 1 and 2) and with it (Round 3), exploring differences in reasoning efficiency. Next, the distribution of steps using each reasoning type in each round was displayed. This included percentage analysis of the proportions of distinct reasoning types to

examine differences and trends in reasoning with and without a dynamic graphical representation. Finally, to provide detailed insights into the varying reasoning patterns, the researcher presented illustrative examples of typical reasoning step sequences with detailed responses and graphs from a set of similar or identical problems.

### *Addressing Research Question 2*

To answer the second research question about what level of understanding characterizes students' reasoning concerning function composition with and without a dynamic graphical representation, this study analyzed coded data of the APOS level indicated in participants' reasoning steps and the observational notes. The problem-solving process was visualized chronologically for each participant and problem using a sequence of blocks labeled with APOS levels (Action (A), Process (P), or Object (O)).

This research question was approached from two aspects. First, the researcher analyzed the distribution of understanding levels across all rounds to identify how participants' reasoning depth changed with and without a dynamic graphical representation. Next, to provide detailed insights into the patterns and trends of understanding levels reflected by the reasoning steps, the researcher presented illustrative examples of typical reasoning performance from a set of similar or identical problems.

### *Addressing Research Question 3*

To answer the third research question regarding how students reason about function composition differently under the topics of domain and range (a) with distinct reasoning types and (b) with a dynamic graphical representation and without it, this study analyzed coded data that specifically focused on domain and range with the observational notes.

This research question was analyzed in three aspects. First, the researcher displayed the distribution of reasoning steps specifically addressing domain and range across rounds. Furthermore, among these reasoning steps, the detailed distribution of specific reasoning types was demonstrated. Next, the distribution of fragmented versus integrated reasoning step involving domain and range across three rounds was presented. Finally, the researcher identified typical examples of approaches to domain and range through distinct reasoning types, examining both fragmented and integrated features with and without a dynamic graphical representation.

### *Addressing Research Question 4*

To answer the fourth research question about how students reason about function composition differently concerning the topics of symmetry and continuity (a) with distinct reasoning types and (b) with a dynamic graphical representation and without it, this study analyzed coded data of reasoning types that specifically focused on symmetry and continuity within the observational notes.

This research question was analyzed in two aspects. First, the researcher demonstrated the distribution of reasoning steps specifically addressing symmetry and continuity across rounds. Then, among these reasoning steps, the detailed distribution of reasoning types was displayed. Next, the researcher identified typical examples of approaches to symmetry and continuity through distinct reasoning types, both with dynamic a graphical representation and without it.

## Chapter 4: Results

This study employed a qualitative, multiple-case study methodology based on participant interviews. The qualitative data of participants' reasoning steps were analyzed through various coding frameworks, including the reasoning types, APOS levels, fragmented versus integrated reasoning features, and accuracy. Participants' individual reasoning performances were also documented through observational notes. This multi-dimensional analysis approach provided rich insights into how students reason about function composition using different representations.

To ensure individuals and their responses remain anonymous, all names used in this study are pseudonyms. All 15 participants' performances are displayed in Table 15 in ascending order of accuracy scores, showing their total accuracy as well as the accuracy for each problem across three rounds (1 point for accurate sketch and reasoning, 0 points for inaccurate sketch or reasoning). Ten out of 15 participants (67%) successfully reasoned about and sketched at least half of the composite functions. As elaborated later in this chapter, Felix and Dorothy did not receive any accuracy credit due to their persistent fundamental misconceptions about function composition throughout all three rounds.

**Table 15**

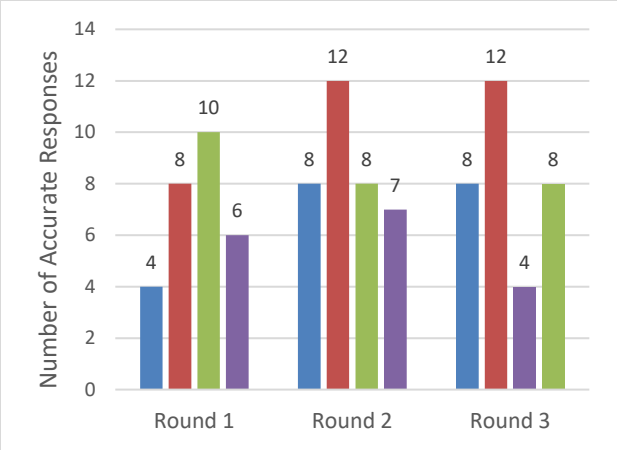
*Participants' Individual Accuracy of Graphs and Reasoning of $g(f(x))$ Across the Rounds*

| | Dorothy | Felix | Caden | Helena | Olivia | Anna | Ivy | Ella | Nicole | Bree | Luna | Morgan | Kylie | Jade | Gia |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **R1P1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **R1P2** | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| **R1P3** | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **R1P4** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| **R2P1** | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| **R2P2** | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **R2P3** | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| **R2P4** | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| **R3P1** | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| **R3P2** | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| **R3P3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| **R3P4** | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| **Total** | **0** | **0** | **2** | **3** | **4** | **6** | **7** | **8** | **8** | **9** | **9** | **9** | **10** | **10** | **10** |

To provide a general sense of students' performance in sketching the function composition tasks, Figure 4 presents participants' overall accuracy across the three rounds. Bars of identical or similar problems in the same problem set are displayed in the same color. Participants performed better with the dynamic graphical representation when given graphs in Round 3 compared to Round 1. Students demonstrated strong accuracy performance with equations in Round 2, even when compared with Round 3 where they were given the dynamic graphical representation. In particular, participants performed better with Round 2 Problem 4 (see the purple bar of Round 2 in Figure 4), in contrast to Round 3 Problem 3 (see the purple bar of Round 3 in Figure 4). This performance difference was possibly due to the more complex composite function in Round 3 Problem 3, though it is similar to the composition in Round 2 Problem 4. However, the accuracy score cannot reveal the detailed and in-depth picture of students' reasoning trends and understanding levels. In fact, many responses not identified as fully accurate actually provided some insightful reasoning approaches, while others were fundamentally incorrect. Although both were marked inaccurate, they provide different data about students' reasoning and understanding. Therefore, instead of accuracy, this study concentrated on exploring participants' reasoning patterns and trends in their reasoning process.

**Figure 4**

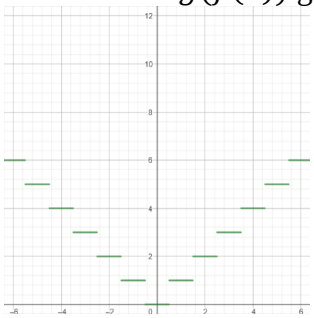
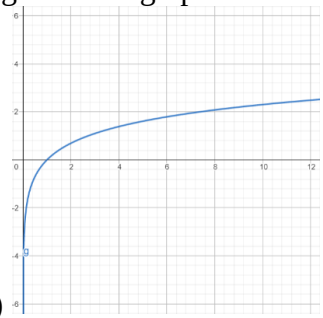
*Participants' Overall Accuracy Across the Rounds*




Participants' reasoning processes were divided into reasoning steps as exemplified in Table 10. Across all 15 participants completing three rounds of function composition problems (four problems per round), a total of 842 reasoning steps were identified. These reasoning steps included both complete and incomplete ones. Complete steps contained a full cycle of "assertion – justification – interim conclusion" and could be either correct or incorrect (see Table 16(b) for an illustrative example of a complete but incorrect reasoning step). Incomplete steps occurred when participants failed to justify the assertion therefore could not draw an interim conclusion (see Table 16(a) for an illustrative example).

**Table 16**

*Examples of Reasoning Step Coding: Nicole's Selected Steps for Round 1 Problem 4*

| Round 1 Problem 4<br>Sketch the composite function $g(f(x))$ given the following function graphs.<br>$f(x)$  $g(x)$  | |
|---|---|
| Reasoning Process | Coding Example |

| | | |
|---|---|---|
| (a) Incomplete Reasoning Step | Nicole pointed at $g(x)$ graph: "There is something going on here in terms of, like, radicals, I'm assuming. Because we are not including zero, assuming exponential, and then…but again, I gave up on that." | Assertion: $g(x)$ is likely a radical or exponential function.<br>Justification: The domain of $g(x)$ does not include zero.<br>Interim Conclusion: Incomplete reasoning step – no interim conclusion reached. |
| (b) Complete Reasoning Step | Nicole indicated various parts on the graph with manual gestures: "We are not interested in, again, the $x$ below zero (pointed at the negative inputs of $f(x)$), because that's not the domain of the composite, (which) is greater than zero (pointed at the domain of $g(x)$, and assuming it is the domain of $g(f(x))$." | Assertion: The negative input of $f(x)$ should not be considered.<br>Justification: The domain of $g(x)$ contains only positive values.<br>Interim Conclusion: The composite function $g(f(x))$ must have a positive domain.<br>Note: Complete but incorrect. |

**Research Question 1: How Do Students Reason About Function Composition With and Without a Dynamic Graphical Representation?**

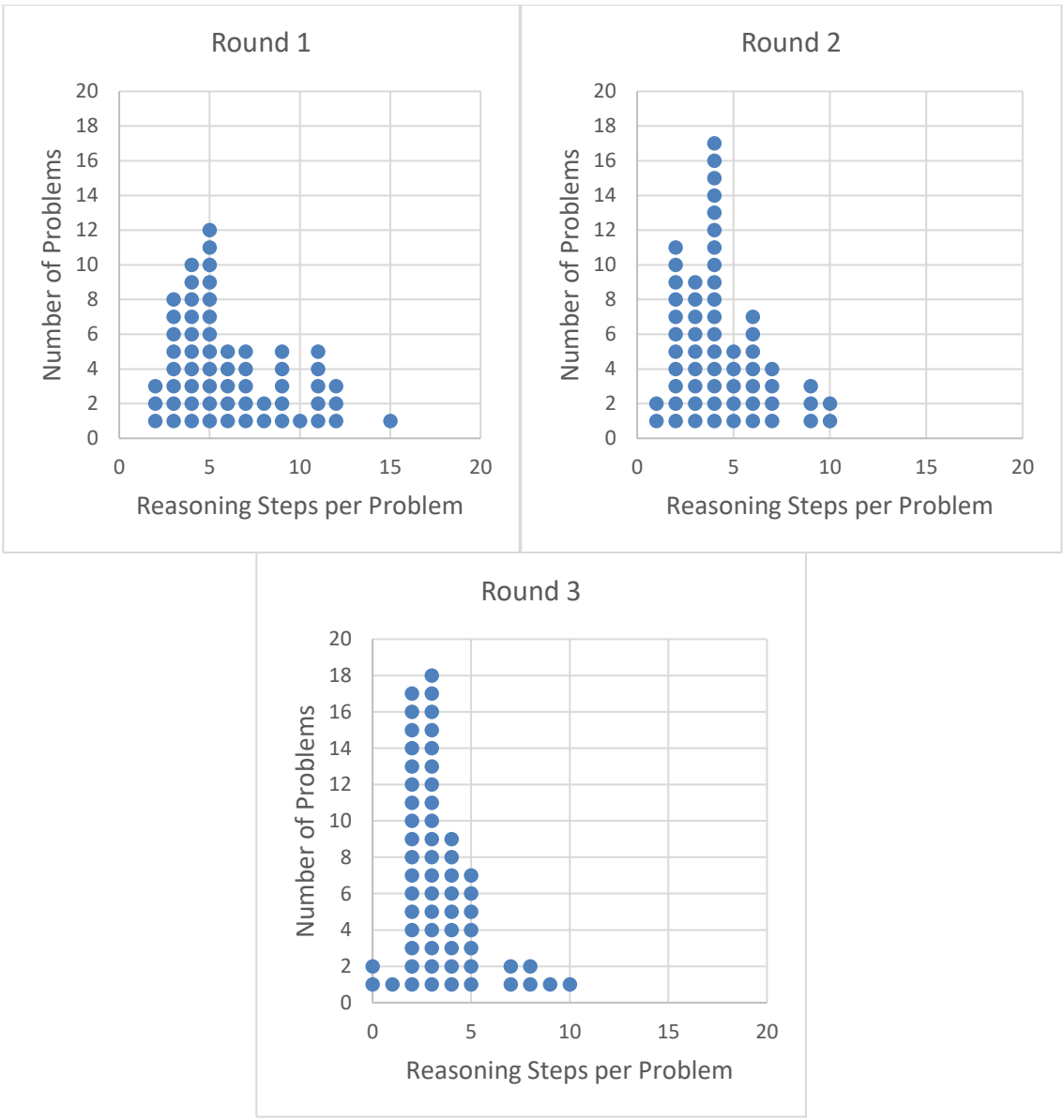
This research question was answered through three lenses. First, the analysis explored the length of participants' reasoning process that revealed the reasoning efficiency. Next, the results identified the predominant types of reasoning used in different rounds. Finally, typical reasoning patterns were presented through illustrative examples from the participants.

***Number of Reasoning Steps***

Figure 5 shows the distribution of reasoning steps across rounds, where each data point represents the number of steps a participant used to solve a problem. With 15 participants each solving 4 problems per round, the dataset comprises 60 data points that capture the reasoning process length.

**Figure 5**

*Number of Reasoning Steps Distribution Across the Rounds*




Out of a total of 842 reasoning steps, there were 372 steps in Round 1, 261 steps in Round 2, and 209 reasoning steps (the least) in Round 3. The distributions reveal distinct patterns across the three rounds. In Round 1, with given graphs, the distribution centers around 4 to 6 reasoning steps per problem, with a wide spread from 2 to 15 steps. In Round 2, with given equations, the distribution centers around 3 to 4 steps, with a narrower spread from 1 to 10 reasoning steps. In comparison, with the dynamic graphical representation in Round 3, the distribution shifts notably left, concentrating around 2 to 3 steps per problem, with most

reasoning lengths between 1 and 6 steps. Density wise, Round 1 exhibits dispersed data with multiple small clusters in the tail region, while Round 2 shows increased concentration. Round 3 demonstrates the highest concentration, with peaks around 2 to 3 steps. The reduction in reasoning steps and the shift from scattered distributions (Rounds 1 and 2) to concentrated distributions (Round 3) indicates improved reasoning efficiency and enhanced reasoning maturity with the dynamic graphical representation. Although at Round 3 the shorter reasoning steps may partly reflect increased familiarity from previous rounds, subsequent analysis of illustrations reveals deeper insights into students' reasoning patterns with and without the dynamic graphical representation. For example, although participants performed better with Round 2 Problem 4 compared to the similar Round 3 Problem 3 due to the difference in complexity level of the tasks, they still presented partially correct reasoning (meaning some steps reached correct interim conclusions, but overall sketches were inaccurate). More importantly, their reasoning in Round 3 Problem 3 was dynamic, coherent, and shorter. More detailed illustrations are shown later in this chapter.

***Distribution of Reasoning Types***

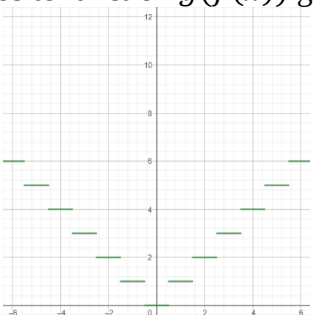
Participants' reasoning steps were categorized into algebraic reasoning (yellow block), graphical reasoning (blue block), or discrete point reasoning (green block), as shown in Table 11. The coding focused primarily on whether the justification in each reasoning step relied on algebraic equations (as exemplified in Table 17(a)), graphical properties (Table 17(b)), or point-wise values of ordered pairs (Table 17(c)). The color-coded reasoning type sequences for all 15 participants are presented in Appendix D.

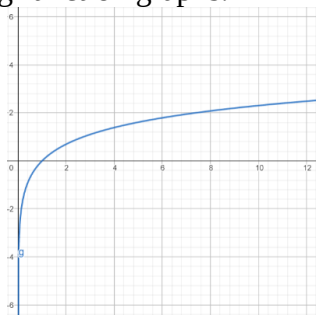
**Table 17**

*Examples of Reasoning Type Coding: Nicole's Selected Steps for Round 1 Problem 4*

Round 1 Problem 4

Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

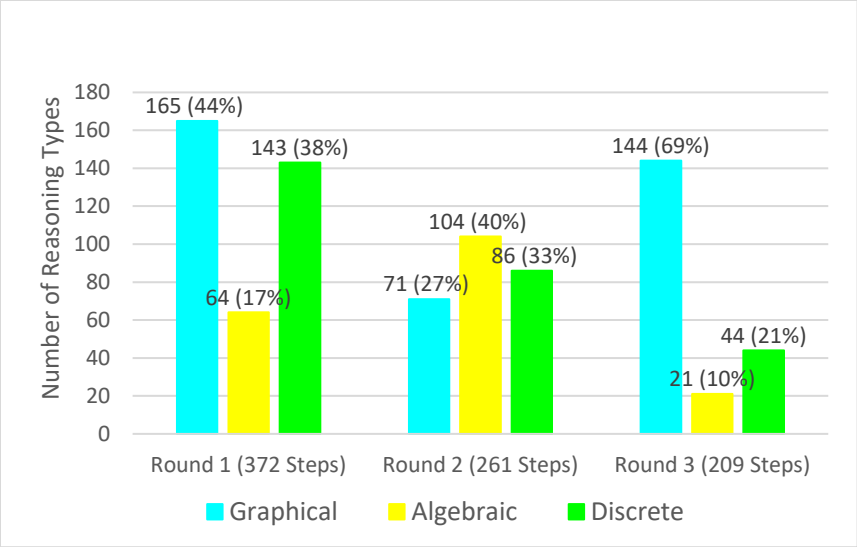
$g(x)$ 

| Step | Reasoning Process | Type | Coding Example |
|---|---|---|---|
| (a) | Nicole pointed at $f(x)$ graph: "I knew this was a floor (function)." Next she wrote down $\lfloor x + 0.5 \rfloor$. | Algebraic | Nicole justified the equation of $f(x)$ based on her previous experience with the function's algebraic expressions. |
| (b) | Nicole then pointed at the part of the $f(x)$ graph between the interval $(-0.5, 0.5)$ and explained: "So starting from here, ($f(x)$ value) is going to be zero, which we have an asymptote (at $g(x)$)…so we are technically starting our domain defined as from 0.5." | Graphical | Nicole justified the restricted domain of the composite function by analyzing domain and range properties presented through graphs. |
| (c) | Nicole explained the shape of the composite function: "If you plug in one here (pointed at $g(x)$), you get zero, so you start zero, and then, like you are going up by increments of point five." | Discrete Point | Nicole justified the general trend of the composite function's shape by plotting several points. |

Figure 6 illustrates the distribution of three reasoning types (graphical, algebraic, and discrete point) across three rounds. Each bar represents the frequency of reasoning types within a specific round, accompanied by their corresponding percentages. For example, the blue bar in Round 1 shows that out of the total 372 reasoning steps, 165 steps (44%) employed graphical reasoning approaches.

**Figure 6**

*Reasoning Type Distribution Across the Rounds*




The distribution in Round 1 implies the leading role of graphical reasoning, which makes sense as participants worked directly with provided static graphical representations. The 17% of algebraic reasoning steps reveals that despite no equations being provided, some students still preferred to use algebraic equations and expressions. Over a third (38%) of the reasoning steps were conducted through discrete points, suggesting that participants needed specific point information to complement their graphical and algebraic reasoning.

When given only equations in Round 2, participants naturally approached function composition problems more algebraically, with 40% of the reasoning steps. About a third (27%) of the reasoning steps were approached graphically regardless of the equations being available, which reveals the importance of graphical reasoning when sketching the composite functions. About another third (33%) of the reasoning steps involved discrete points showing that this type of approach is consistently useful without a dynamic graphical representation.

Unlike the first two rounds, which lacked a dynamic graphical representation, Round 3 was characterized by more graphical reasoning (69%). Both algebraic (10%) and discrete point (21%) reasoning steps took the lowest percentage of their kind in this round. The results demonstrates that the dynamic graphical representation reduced the need for algebraic equations

and manipulations of the algebraic expressions, as well as the need for utilizing specific point information.

In conclusion, without a dynamic graphical representation, no single reasoning type dominated, while graphical reasoning led when subjects were given graphs and algebraic reasoning led when subjects were given equations. Participants used multiple reasoning types to reason and solve the function composition problems. On the other hand, with a dynamic graphical representation in Round 3, students showed strong preference for graphical reasoning, and the use of other two reasoning types dramatically declined. The findings suggest that the dynamic graphical representation reduced the need for other reasoning types, especially reducing the reliance on algebraic reasoning, while strongly promoting graphical reasoning approaches. This conclusion may also apply to other types of dynamic graphical representations, which excel at presenting mathematical contents through interactive and dynamic displays of graphical properties and relationships. Although the dynamic graphical representation demonstrates the traced path of composite function points, which might increase the use of graphical reasoning approaches, the following section presents illustrative examples that reveal the differences in reasoning patterns with and without a dynamic graphical representation, beyond merely copying the traced path of the answer graph.

***Patterns of Reasoning Types Used in Reasoning Process***

This section's analysis provides notable typical reasoning patterns across the three rounds, and furnishes illustrative examples of participants' responses in situations that either contained or did not contain a dynamic graphical representation. The example responses are drawn from Round 1 Problem 3, Round 2 Problem 3, and Round 3 Problem 4 (see Appendix B), which form a set of similar or identical problems (see Table 9). This set of problems is ideal for

illustrating patterns of reasoning types within students' reasoning process. The problems involve rich function properties to consider during composition – including restricted domain and range situations that lead to disconnection, and also symmetry and concavity – which can be approached from multiple perspectives. Participants were provided with static graphs in Round 1, equations in Round 2, and a dynamic graphical representation with static graphs in Round 3 to help them to sketch the graph of the composite function $g(f(x))$. The responses from comparable problems across the three rounds revealed distinct reasoning patterns with and without the dynamic graphical representation.

**Without a Dynamic Graphical Representation.** In Rounds 1 and 2, participants integrated multiple reasoning strategies rather than relying on a single method when sketching and reasoning function composition. Moreover, instead of using a specific reasoning type in clustered steps, students alternated between different approaches (see participants' reasoning sequences in Appendix D). This pattern aligns with students' relatively evenly balanced reasoning type distribution in Rounds 1 and 2, as presented in Figure 6. Many students successfully solved the function composition problems by utilizing all three types of reasoning in the first two rounds (e.g., Round 1 Problem 3 solved by Bree, Kylie, and Nicole; and Round 2 Problem 3 solved by Bree, Jade, Kylie, and Luna). On the contrary, no reasoning sequence that solely used algebraic, graphical, or discrete point reasoning successfully solved the function composition task without a dynamic graphical representation.

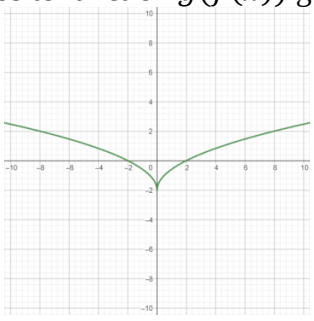
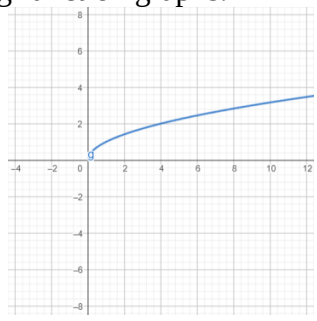
***Round 1.*** In Round 1, when participants were given static graphs, the pattern of alternating between discrete point and graphical reasoning proved especially effective, aside from cases mentioned previously involving all three reasoning types. The sequence "Blue – Green – Blue – Green," representing alternation between these two strategies, revealed the

crucial role of discrete point reasoning. This reasoning type served as an intermediate step bridging students' graphical understanding. Participants typically approached problems in two ways: they either examined function properties through graphical lenses before confirming their understanding with specific points, or they analyzed patterns from discrete points to develop broader insights about the composite function's graphical properties. Participants' performance demonstrated that successfully reasoning about and sketching graphs of composite function is entirely possible without equations. Conversely, all reasoning sequences using only algebraic or discrete point reasoning, or the combination of them, were unsuccessful, highlighting the essential role graphical reasoning played in the solving process.

Gia's reasoning sequence of Round 1 Problem 3 (illustrated in Table 18), along with her work and graph (displayed in Figure 7), present a typical example of the "Blue – Green – Blue – Green" reasoning pattern in Round 1. Steps 2 and 4 were typical bridging steps that analyzed patterns from discrete points to develop broader insights about the composite function's graphical properties in the subsequent reasoning steps.

**Table 18**

*Gia's Reasoning Sequence in Round 1 Problem 3*

| Round 1 Problem 3<br>Sketch the composite function $g(f(x))$ given the following function graphs.<br>$f(x)$  $g(x)$  | | | |
|---|---|---|---|
| Step | Reasoning Process | Type | Analysis of This Step |

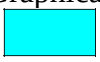
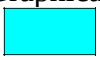
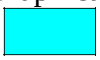
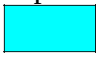
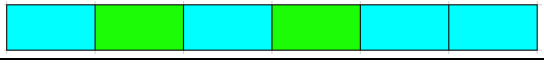
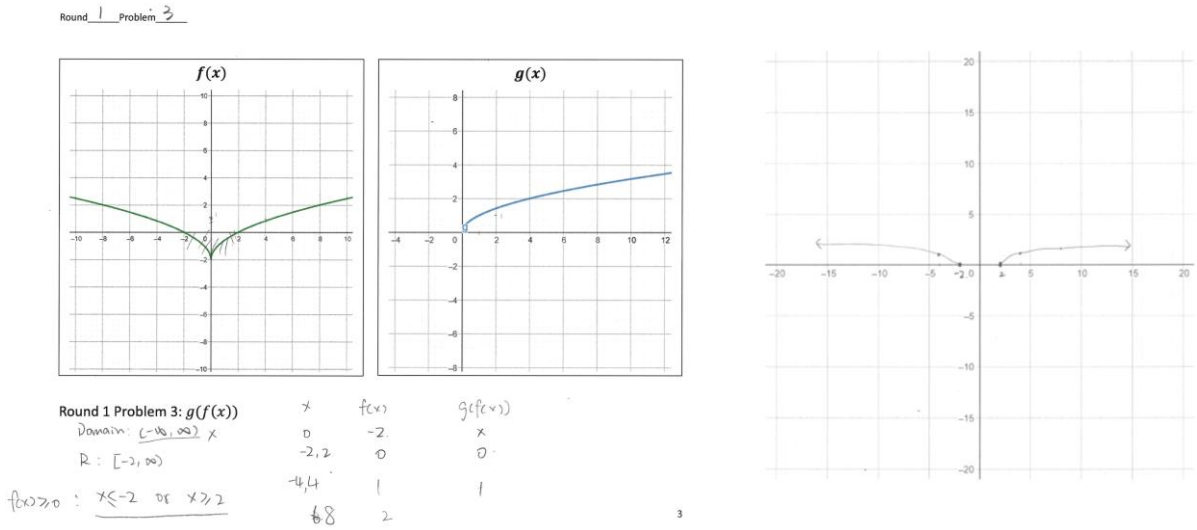
| | | | |
|---|---|---|---|
| 1 | Gia wrote down the domain and range of $f(x)$ by reading the graphs. | Graphical  | Gia justified the domain and range of the parent function $f(x)$ by analyzing the graphical properties of $f(x)$. |
| 2 | Gia located one point $g(f(0))$ by reading the graphs and determined the composite function is undefined with this input. | Discrete Point  | Gia justified the undefined input of $g(f(x))$ by analyzing patterns from specific point. |
| 3 | Gia pointed at the graphs of $f(x)$ and $g(x)$, and explained: "So we have the domain for $g(x)$ can only be greater than zero, which means the range for $f(x)$ has to be greater than or equal to zero...So for our new function, the domain has to be this one (the restricted domain for the part that $f(x) > 0$) instead of this one (pointed at the entire domain of $f(x)$)." | Graphical  | Gia justified the restricted domain of the composite function by analyzing domain and range properties presented through graphs. |
| 4 | Gia plotted a group of points on the $g(f(x))$ graph on the positive side, and then connected the points to sketch the composite function graph on the positive side. | Discrete Point  | Gia justified the shape of her composite function sketch by plotting discrete points. |
| 5 | Gia sketched the graph of the composite function on the negative domain and explained: "So we have the negative (with) the same value…I think we can only look for one side, because this (pointed at the graph of $f(x)$) is symmetry (symmetric)." | Graphical  | Gia justified the reflected sketch of the composite function by analyzing the graphical symmetry property of $f(x)$. |
| 6 | Gia explained the concavity of the composite function graph: "The rate of change is kind of gets slower and slower…(pointed at $f(x)$ and $g(x)$) it's ($f(x)$) increasing, but it's increasing in a slower way, and also the same for this one ($g(x)$). So I guess the composite function of these two is also going to increase, but slower than both." | Graphical  | Gia justified the concavity of her composite function sketch by analyzing the graphical concavity properties of the parent functions. |
| | Reasoning Sequence:  | | |

**Figure 7**

*Gia's Work and Graph of Round 1 Problem 3*



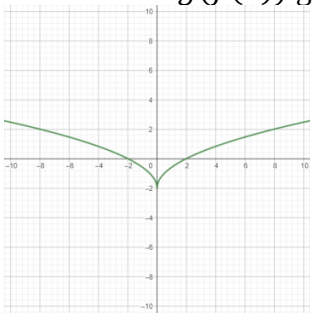

Olivia approached Round 1 Problem 3 through plotting specific points without considering the general features and properties of functions (see Olivia's reasoning sequence in Table 19 and her work with composite function graph in Figure 8). The second step in Olivia's reasoning sequence exemplifies the most common misconception identified in this study about function composition. Students were able to accurately substitute $x$ values to find $f(x)$, and then used the outputs of $f(x)$ as inputs into $g(x)$ to find $g(f(x))$. However, instead of considering $(x, g(f(x)))$ as the proper coordinates of the composite function, students mistakenly plotted $(f(x), g(f(x)))$ on the composite function graph. This misconception appeared in several participants' responses (e.g., Olivia, Dorothy, Helena and Felix), significantly hindering their ability to compose functions and sketch them correctly. The point-wise reasoning strategy alone performed in Olivia's case demonstrated the insufficiency of applying reasoning without a graphical approach when sketching function composition tasks.

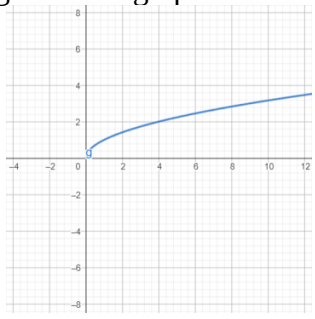
**Table 19**

*Olivia's Reasoning Sequence in Round 1 Problem 3*

Round 1 Problem 3

Sketch the composite function $g(f(x))$ given the following function graphs.

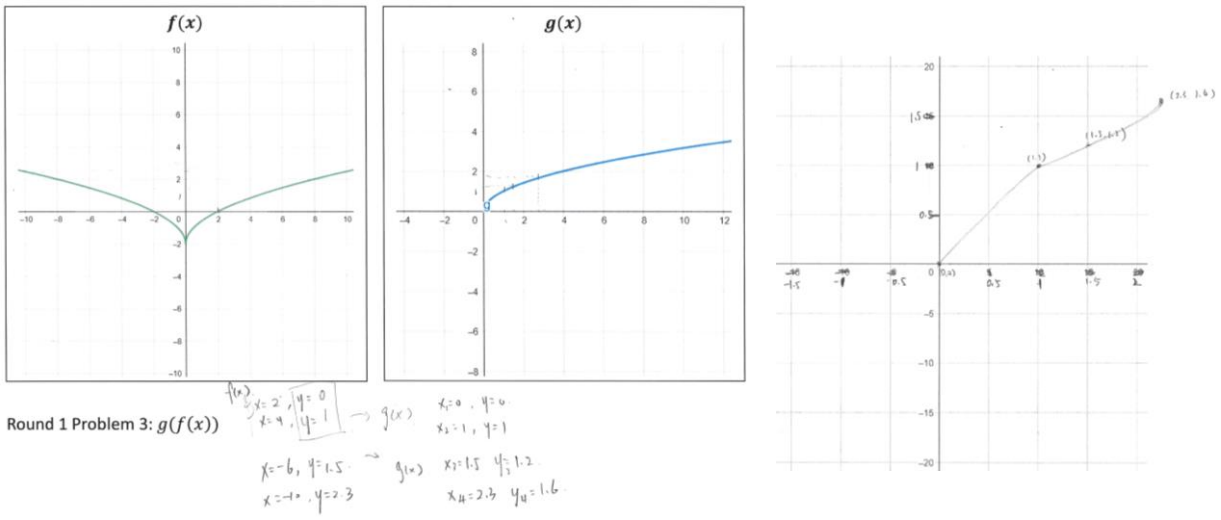
$f(x)$ 

$g(x)$ 

| Step | Reasoning Process | Type | Analysis of This Step |
|---|---|---|---|
| 1 | Olivia substituted multiple specific points to find out point values of $f(x)$ and $g(f(x))$: "So I just pick randomly…pick a number like, $x$ equals two..." | Discrete Point | A discrete point reasoning step, with justification through plotting specific points randomly without systemically considering the restricted domain and range. |
| 2 | Olivia plotted points with the coordinates of $(f(x), g(f(x)))$ and connected them as the sketch of the composite function. | Discrete Point | A discrete point reasoning step that justified the sketch of the composite function through plotting specific points, which demonstrates a common misconception of composite function. |
| | Reasoning Sequence: | | |

**Figure 8**

*Olivia's Work and Graph of Round 1 Problem 3*




***Round 2.*** In Round 2, when participants were given equations, the pattern of alternating between discrete point and algebraic reasoning with supplemental steps from graphical reasoning proved to be effective. The sequence of "Yellow – Green – Yellow – Green" alone was inadequate when sketching the function composition problems, even when equations were provided. This pattern echoes the most evenly split distribution of reasoning types from Round 2 (presented in Figure 6). Moreover, in this equation round, many participants still used "Blue – Green – Blue – Green" reasoning patterns and successfully reasoned the composite function graph (see reasoning sequences of Round 2 Problem 3 from Gia, Jade, Morgan, and Nicole in Appendix D). They typically substituted the equations to have the composite function equation in their first step, then the rest of the reasoning process applied the strategy of alternating graphical and discrete point approaches. The patterns that appeared in Round 2 reinforced the importance of reasoning function composition graphically, as well as how discrete point reasoning served as a transitional steps between different reasoning approaches – for confirming previous understanding or exploring new properties of the composite function graph.

In Round 2 Problem 3, Luna exemplified how participants successfully sketched composite function graphs by alternating between algebraic and discrete point reasoning steps, facilitated by graphical approaches (see Luna's reasoning sequence in Table 20 and her work with a composite function graph in Figure 9).

**Table 20**

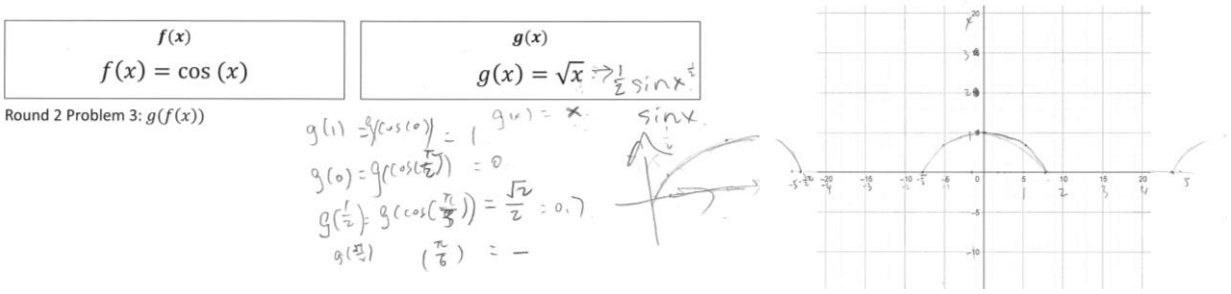
*Luna's Reasoning Sequence in Round 2 Problem 3*

| Round 2 Problem 3<br>Sketch the composite function $g(f(x))$ given the following function equations.<br>$f(x) = \cos(x)$ $g(x) = \sqrt{x}$ | | | |
|---|---|---|---|
| Step | Reasoning Process | Type | Analysis of This Step |
| 1 | Luna first plotted several points on the composite function. For example she wrote down "$g(1) = g(\cos(0)) = 1$" and "$g(0) = g(cos\ (\frac{\pi}{2})\ ) = 0$". | Discrete Point | A discrete point reasoning step, with justification through plotting specific points. |
| 2 | Luna reasoned: "The range of the $\sqrt{x}$ need(s) to be greater (than) or equal to zero, so domain (of the composite function) is the part of $x$ that can make $\cos(x)$ greater (than) or equal to zero, just like $-\frac{\pi}{2}$ to $\frac{\pi}{2}$." | Algebraic | An algebraic reasoning step justifying the restricted domain and range of the composite function by applying the restricted domain and range of parent functions through function equations. |
| 3 | Luna then reasoned: "Also like $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, it's ongoing, and through one branch of this because $\cos(x)$ keeps going." | Algebraic | An algebraic reasoning step justifying the periodicity of the composite function by applying the periodicity of $f(x)$ through the function equation. |
| 4 | Luna plotted a bunch of points and reasoned: "I start from the point of $x = 0$, $g(f(x)) = 1$; and either $x = -\frac{\pi}{2}$ or $x = \frac{\pi}{2}$, $g(f(x)) = 0$…I tried two (other) different points, $x = \frac{\pi}{3}$, and $x = \frac{\pi}{6}$ (used calculator to find the point values)." | Discrete Point | A discrete point reasoning step with justification through plotting specific values to explore the shape of the composite function. |

| | | | |
|---|---|---|---|
| 5 | Luna sketched the graph of $g(x) = \sqrt{x}$ and reasoned based on it: "The slope of $g(x)$ is decreasing as the (value of) $g(x)$ grow(s) bigger and bigger. So it's (pointed at the curve of composite graph she sketched) like increasing first and then decreasing (with slope decreasing)." | Graphical | A graphical reasoning step justifying the concavity of the composite function by applying the concavity of $g(x)$ graphically. |
| 6 | Luna further reasoned the curve of composite function: " The $\cos(x)$ curve will be lower (than the composite function curve) as (pointed at the points she plotted)." | Discrete Point | A discrete point reasoning step justifying the shape details of the composite graph through the location of specific points. |
| | Reasoning Sequence: | | |

**Figure 9**

*Luna's Work and Graph of Round 2 Problem 3*




Helena's reasoning process on the same problem revealed the drawbacks of overreliance on algebraic and discrete point approaches (see Helena's reasoning sequence for Round 2 Problem 3 in Table 21 and her work with a composite function graph in Figure 10). In addition to the misconception of taking $(f(x), g(f(x)))$ as the coordinate of the composite function (step 2 in Table 21), Helena's unsystematic and random selection of points to substitute led her to overlook a key property of the composite function – its restricted domain and range. Moreover, Helena did not realize the calculator sets up the input of the cosine function as degrees instead of radians, for example $\cos(2)$ was degree 2, while on the coordinate plane the input of cosine function is radians. If Helena had used other reasoning approaches to consider the restricted

domain and range issue alongside the periodicity property, or if she had changed the setting of the calculator input to be the radius, she would have found the composite function to be undefined under some of the values, contradicting her estimation and sketch of the composite function.

**Table 21**

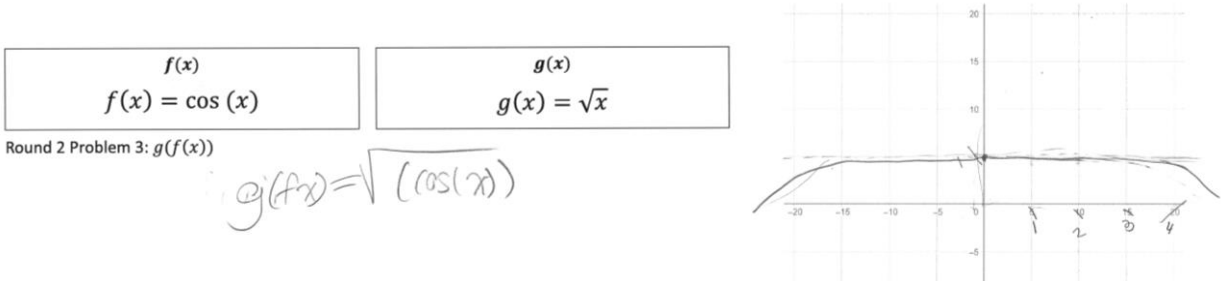
*Helena's Reasoning Sequence in Round 2 Problem 3*

Round 2 Problem 3
Sketch the composite function $g(f(x))$ given the following function equations.
$f(x) = \cos(x)$ $g(x) = \sqrt{x}$

| Step | Reasoning Process | Type | Analysis of This Step |
|---|---|---|---|
| 1 | Helena substituted the equations and wrote down the algebraic expression of the composite function: $\sqrt{\cos(x)}$. | Algebraic | An algebraic reasoning step with justification by substituting function equations. |
| 2 | Helena used a calculator to find values of several specific points and then plotted $(\cos(1), \sqrt{\cos(1)})$, $(\cos(2), \sqrt{\cos(2)})$, $(\cos(3), \sqrt{\cos(3)})$, and $(\cos(4), \sqrt{\cos(4)})$ as the points on the composite function graph. | Discrete Point | A discrete point reasoning step that justified the composite function graph by substituting random and limited number of points, showing the same misconception as Olivia did in Table 19. |
| 3 | Helena substituted $\sqrt{\cos(50)}$ to confirm and adjust the extended curve shape. | Discrete Point | A discrete point reasoning step that justified the general shape of the composite function graph by substituting a random point. |
| 4 | Helena then realized that she hadn't considered the negative side, so she substituted $\sqrt{\cos(-70)}$ to sketch the composite graph on the negative side. | Discrete Point | Similar approach with the previous reasoning step. |
| | Reasoning Sequence: | | |

**Figure 10**

*Helena's Work and Graph of Round 2 Problem 3*




**With A Dynamic Graphical Representation.** With access to a dynamic graphical representation in Round 3, though the "Blue – Green – Blue – Green" pattern still existed in some student's reasoning processes, most participants approached the problems more through graphical lenses with less need for verifying specific points and minimal needs for algebraic analysis. This pattern aligned with the reasoning type distribution of Round 3 (illustrated in Figure 6). More promisingly, several participants (e.g., Ella, Helena, Gia and Jade) successfully reasoned through many function composition tasks solely using graphical approaches, revealing the significant potential of the dynamic graphical representation in understanding function composition concepts.

During the first two rounds, which lacked a dynamic graphical representation, several participants expressed frustration about not being able to determine exact coordinate values. When asked whether small reading errors would significantly affect their understanding of the composite functions' properties and shapes, students acknowledged that these minor discrepancies were not crucial. However, they remained uncomfortable about the uncertainty of the exact point values. The introduction of the dynamic graphical representation in Round 3 made a difference. The dominance of graphical reasoning suggests that participants shifted their attention toward visual and more global understanding of function composition graphs rather than fixating on precise point values. Though the traced path presented by the dynamic graphical representation might have brought advantage to participants, more importantly, they were able to

develop dynamic and more coherent reasoning about function composition. The interactive experience of seeing how the changes of $x$ mapped to the changes of $f(x)$, then further mapped to the changes in $g(f(x))$ provided participant a chance to interpret the graph of the composite function dynamically. For example, participants used descriptions like "the composite function point jumped when the values of $f(x)$ jumped" to explain the covariational relations between the discontinuity of $f(x)$ and the discontinuity of $g(f(x))$.

Furthermore, compared with the more scattered reasoning approaches in the first two rounds where participants tried different approaches with fragmented individual steps, participants offered more coherent and integrated reasoning progression in Round 3. This progression aligned with the previous results for this round, which reflected generally shorter reasoning sequences in this round. The chain of reasoning proved to be more well-structured, as one rich reasoning step in Round 3 might be equivalent to the combination of several steps from previous rounds. These differences were most evident when participants reasoned about the domain and range of the composite function. Without the dynamic graphical representation, some participants took about four steps to identify the restricted domain and range of the composite function while in Round 3 participants were able to construct a complete logical chain in one reasoning step. More detailed steps are demonstrated in the examples answering research questions three.

***Solely Graphical Reasoning Sequence with a Dynamic Graphical Representation – The "All Blue" Pattern.*** The most noticeable progress in Round 3 was the "all blue" reasoning sequence, in which participants were able to reason about the composite function graphically through insightful dynamic observations. Ella's reasoning process for Round 3 Problem 4 serves

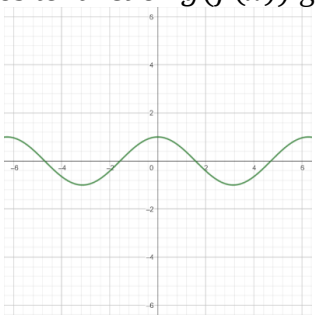
as an example (see Ella's reasoning sequence in Table 22 and her work with dynamic graphical representation in Figure 11).

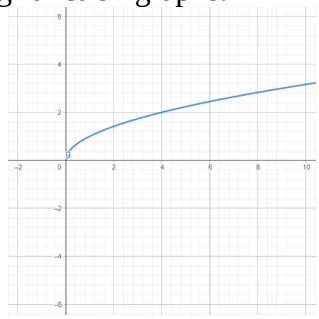
**Table 22**

*Ella's Reasoning Sequence in Round 3 Problem 4*

Round 3 Problem 4

Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

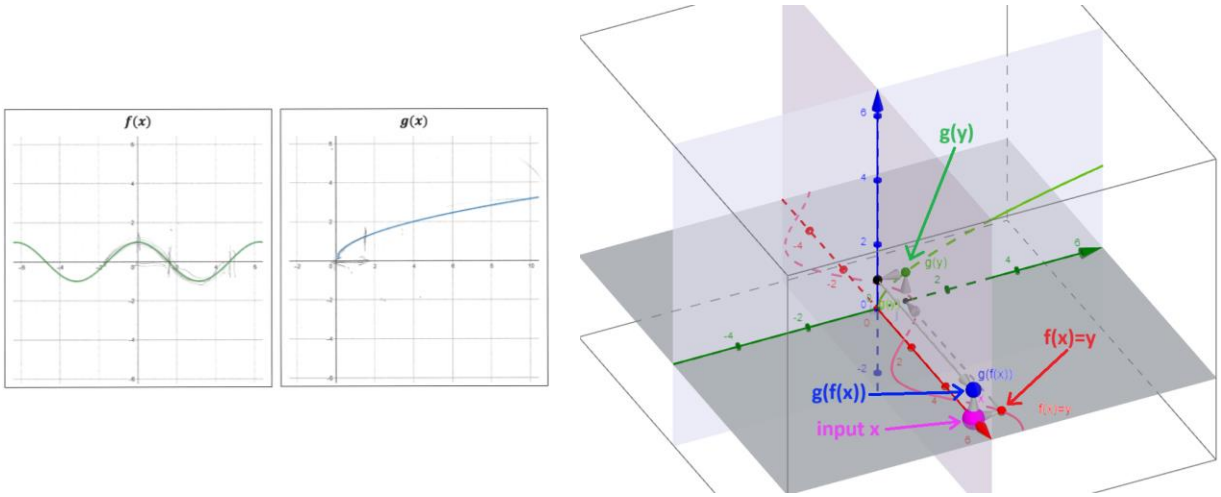
$g(x)$ 

| Step | Reasoning Process | Type | Analysis of This Step |
|---|---|---|---|
| 1 | Ella dragged the input point back and forth and reasoned: "When the $f(x)$ in that interval, the value of $f(x)$ are negative. But for $g(x)$, there's no negative value (for input of $g(x)$), so there's a gap." | Graphical | A graphical reasoning step coherently justified the restricted domain of the composite function through dynamic graphs. |
| 2 | Ella kept moving the input point on dynamic graphical representation and observed: "For that part (the corresponding point of $g(y)$ where $f(x) = y$), the motion point on this line (the $g(x)$ curve), just like from this point this interval (between $g(y) = 0$ and $g(y) = 1$), going back and forth." | Graphical | A graphical reasoning step dynamically and concretely justified the repeating periods of the composite function through dynamic graphs (see Figure 11 on the right, showing the input $x$, the corresponding $f(x) = y$, and the corresponding $g(y) = g(f(x))$). |
| 3 | Ella further explained the restricted range of the composite function: "The range of this ($f(x)$) is (decides) the domain of this part ($g(x)$), so the output of this curve ($g(x)$), just like back and forth from this point (the interval of [0,1]) go up and go down (pointed at the range of $f(x)$ between 0 and 1)." | Graphical | A graphical reasoning step dynamically and concretely justified the restricted range of the composite function through dynamic graphs. |

| Reasoning Sequence: | | | |
|---|---|---|---|

**Figure 11**

*Ella's Work and Dynamic Graphical Representation of Round 3 Problem 4*




***Constraints of the Dynamic Graphical Representation.*** The affordances of the dynamic graphical representation in reasoning through function composition problems were thoroughly elaborated in the previous analysis. However, a few students showed some constraints of this tool. For example, the dynamic graphical representation could not correct some students' persistent and fundamental misconception (such as the one Olivia demonstrated in Table 19). These students (e.g., Felix and Dorothy) were confused by the covariational relations between $x$ and $g(f(x))$ presented by the traced path and even complained that the representation did not directly show the traced path of point $(f(x), g(f(x)))$, which they still mistakenly believed represented the correct coordinates of the composite function. Nevertheless, many participants who held the same misconception in the first two rounds were able to correct their misunderstanding and recognize $x$ as the input of the composite function after using the dynamic graphical representation in Round 3 (e.g., Caden and Helena). Another distinct constraint of this

tool stemmed from students' learning preferences, possibly due to their previous learning experiences and habits. Luna and Anna both showed understanding about function composition demonstrated in Rounds 1 and 2. However, they were resistant to using a dynamic graphical representation in the last round. During the interview they shared three causes for their resistance: (a) the lack of accuracy in specific point values in the dynamic representation; (b) the integrated and dynamic presentation being overwhelming, as they preferred more broken-down and step-by-step reasoning; and (c) the unfamiliarity with the three-dimensional view in the dynamic representation.

***Summary***

To sum up, when reasoning about function composition without a dynamic graphical representation, students applied diverse reasoning strategies, and graphical reasoning proved irreplaceable when algebraic and discrete point reasoning showed their limitations. In comparison, with a dynamic graphical representation, students reasoned more efficiently through shorter reasoning sequences and were able to reason in an integrated and dynamic manner, primarily through graphical approaches.

**Research Question Two: What Level of Understanding, in Terms of APOS Theory, Characterizes Students' Reasoning About Function Composition With and Without a Dynamic Graphical Representation?**

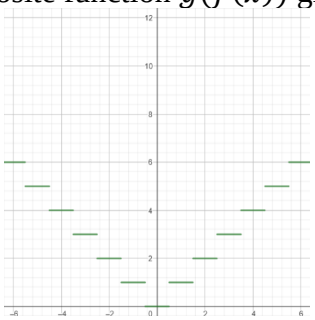
This research question was addressed through two lenses. First, the results demonstrated the distribution of understanding levels that characterize students' reasoning across different rounds in terms of APOS theory. Each reasoning step served as an indicator of the APOS level. Second, typical ways of reasoning that indicated distinct levels of understanding were presented through illustrative examples from the participants.

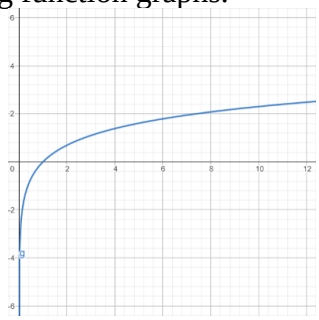
Participants' reasoning steps were coded into levels of understanding the steps indicated using APOS theory: Action (A), Process (P), or Object (O), as defined in Table 8 and shown in Table 13. The coding focused primarily on whether the *justification* in each reasoning step reflected an action level of understanding (as exemplified in Table 23(a)), a process level of understanding (Table 23(b)), or an object level understanding (Table 23(c)). All fifteen participants' letter-coded reasoning sequences are shown in Appendix D.

**Table 23**

*Examples of APOS Levels Coding: Gia's Selected Steps for Round 1 Problem 4*

Round 1 Problem 4
Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

| Step | Reasoning Process | Type | Coding Justification |
|---|---|---|---|
| (a) | Gia set up a table of $x$, $f(x)$, and $g(f(x))$, to look for the coordinates of a set of points on the graph of the composite function. | Action | This reasoning step reflected action level understanding as it justified the composite function by identifying specific values of discrete points, instead of reasoning about the function on entire intervals. |
| (b) | Gia reasoned the restricted domain of the composite function: "When $x$ belongs to the interval of $(-0.5, 0.5)$, $f(x)$ is always 0, where $g(x)$ is going to be an asymptote to negative infinity." | Process | This reasoning step indicated process level understanding when Gia composed functions using graphs rather than explicit expressions. She demonstrated this by operating on sets of inputs across intervals while considering domain restrictions. |

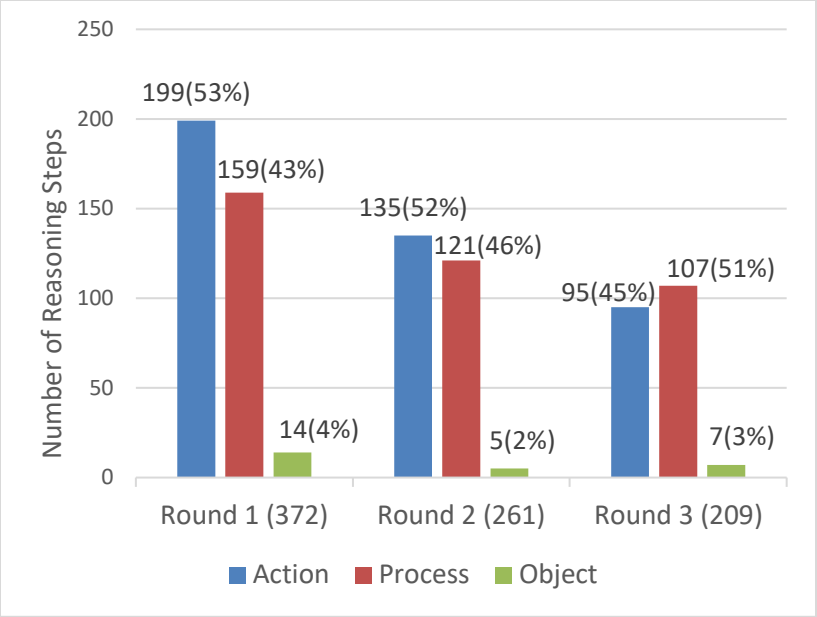
| | | | |
|---|---|---|---|
| (c) | Gia concluded that when composing a straight line with a concave down curve, the composite function must also be concave down. | Object | This reasoning step demonstrated object level understanding as the functions were treated as singular objects during the justification. Specifically, Gia viewed the composition as combining two graphs (as two static totalities) through |

***Distribution of APOS Understanding Level Indicators***

Figure 12 demonstrates the distribution of APOS understanding levels indicated by students' reasoning steps across the three rounds. Each bar represents the frequency of APOS level indicators within a specific round, accompanied by its corresponding percentage. For example, the blue bar in Round 1 shows that out of 372 reasoning steps, the action level of understanding characterized 199 steps (53%). Since object-level understanding proved to be rare throughout the three rounds with similar minimal percentages (all less than 5%), and remained a challenging level to achieve, this study focused on analyzing and comparing the proportion and typical patterns of action-level versus process-level understanding characterized in students' reasoning steps.

**Figure 12**

*Distribution of APOS Understanding Levels Indicators*




In Rounds 1 and 2, without a dynamic graphical representation, the action level of understanding was predominant, characterizing 53% and 52% of the reasoning steps respectively. These two rounds showed quite similar distribution with action-leading understanding, despite employing different approaches when given static graphs (Round 1) or algebraic equations (Round 2), which are illustrated later in detail.

Compared with the first two rounds, which lacked a dynamic graphical representation, there's a notable shift toward process-dominated understanding in Round 3. The proportion of action-level understanding characterized in reasoning steps decreased to 45%, while 51% of the reasoning steps indicated process-level understanding of function composition. More crucially, illustrative examples in this chapter reveal students' unique and dynamic reasoning developed through interaction with the dynamic graphical representation, which promoted a moderately enhanced process-view of function composition.

***Characterizing APOS Understanding Levels: Examples of Reasoning Across the Three Rounds***

To demonstrate how reasoning indicates different APOS levels with and without a dynamic graphical representation, this section presents typical reasoning patterns observed across the three rounds. The examples are drawn from Problem 1 in Rounds 1, 2 and 3 (see Appendix B), which formed a set of similar or identical problems (see Table 9). These problems are ideal for illustrating APOS-characterized reasoning patterns. Their parent functions ($f(x)$ and $g(x)$) are inverse functions, involving rich properties to consider during composition – including restricted domain and range situations as well as symmetry – which can be overlooked, as documented in other studies (Chen et al., 2023). From these comparable problems across the three rounds, Jade's reasoning processes exemplify distinct APOS-level patterns both with and without a dynamic graphical representation. Jade was selected as the example case because her reasoning process demonstrated insightful understanding as well as misconceptions, making her progress in understanding and performance differences across rounds particularly evident.

**Without a Dynamic Graphical Representation.** Without a dynamic graphical representation, exhibited process-level understanding through similar reasoning approaches between Rounds 1 and 2. However, distinct patterns of reasoning approaches indicating action-level understanding of function composition were observed when participants were given static graphs (Round 1) and algebraic equations (Round 2).

***Round 1.*** In Round 1 when provided only with static graphs of $f(x)$ and $g(x)$ to compose, participants generally lacked mature and systematic plans for approaching the task, as this was an unfamiliar way to compose functions. In this situation, one typical approach students showed was attempting to return to their familiar method: composing algebraic equations. The majority of participants tried to look for equations from the graphs of $f(x)$ and $g(x)$. While most participants soon realized the algebraic way was not the most effective, a few insisted on

estimating equations before composition, often resulting in incorrect composite functions due to inaccurate equation estimations. Depending on explicit expressions and requiring them for algebraic composition is a typical characteristics of action-level understanding.

Due to the unfamiliarity of the tasks, particularly when students were provided with unfamiliar function graphs, they tended to show action-level understanding with fragmented reasoning steps that were disconnected from each other. Without a schema for approaching such tasks, students tackled the problem by applying known methods or drawing conclusions from available information without a clear plan for subsequent steps. In Round 1, typical characteristics of action-level understanding in their fragmented reasoning included: (a) identifying the domains and ranges of $f(x)$ and $g(x)$ from graphs, but failing to reason forward about the domain and range of the composite function from there and (b) plotting random specific point values of $g(f(x))$ and just connecting them without justification.

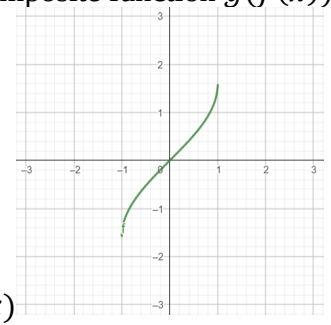
Jade's reasoning sequence for Round 1 Problem 1 (illustrated in Table 24 and accompanied by her work and graph in Figure 13), presents a typical example showing characteristics of action-level understanding of function composition in Round 1. These characteristics include reasoning through specific algebraic equations and points (Steps 1, 2, and 4), as well as fragmented steps that were not helpful for subsequent reasoning (Step 3). It is worth noting that, although demonstrating multiple steps indicating process-level understanding, Jade's conclusion and sketch of the composite function were not accurate. In Steps 10 and 11, despite showing process-level understanding of general mapping across all inputs and outputs, Jade made an incorrect assertion that the shape of the composite function was not a straight line. This conclusion was based on her estimation of the rate of changes from points she read on the graph. Due to reading errors regarding the accurate coordinates of the points, Jade made

inaccurate conclusions about the rate of the change and derivatives. This illustrates the disadvantage of relying on discrete point values.

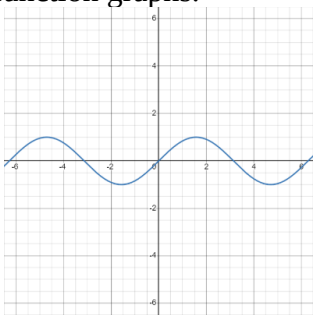
**Table 24**

*Jade's Reasoning Sequence in Round 1 Problem 1*

Round 1 Problem 1
Sketch the composite function $g(f(x))$ given the following function graphs.



$f(x)$



$g(x)$

| Step | Reasoning Process | Type | Analysis of This Step |
|---|---|---|---|
| 1 | Jade identified $g(x)$ as sin $(x)$. | Action | A reasoning step reflected action-level understanding, since the justification required explicit algebraic equations. Note: Jade used sin $(x)$ later in step 4, showing her fragmented reasoning process. |
| 2 | Jade tried to derive the algebraic equation of $f(x)$: "I don' know what is (the function of) this graph $(f(x))$, because this do not have the asymptote, so it cannot be the trigonometry function for the tangent. It is also not the quadratic, or polynomial function…it can be the third power, but it cannot be the quadratic, because quadratic will look more like a parabola." | Action | Unable to determine the algebraic equation of $f(x)$, Jade attempted to eliminate some function equations first, which was not useful for subsequent reasoning steps. Requiring both algebraic equations in the justification of the composition of the function demonstrates a characteristic of action-level understanding. |
| 3 | Jade reasoned: "It $(f(x))$ has the endpoint, so it has the (restricted) domain." | Action | Jade identified the restricted domain of $f(x)$ but did not use it to reason forward to the next step. Justifying the composite function with disconnected steps, one at a time, is a characteristic of action-level understanding. |

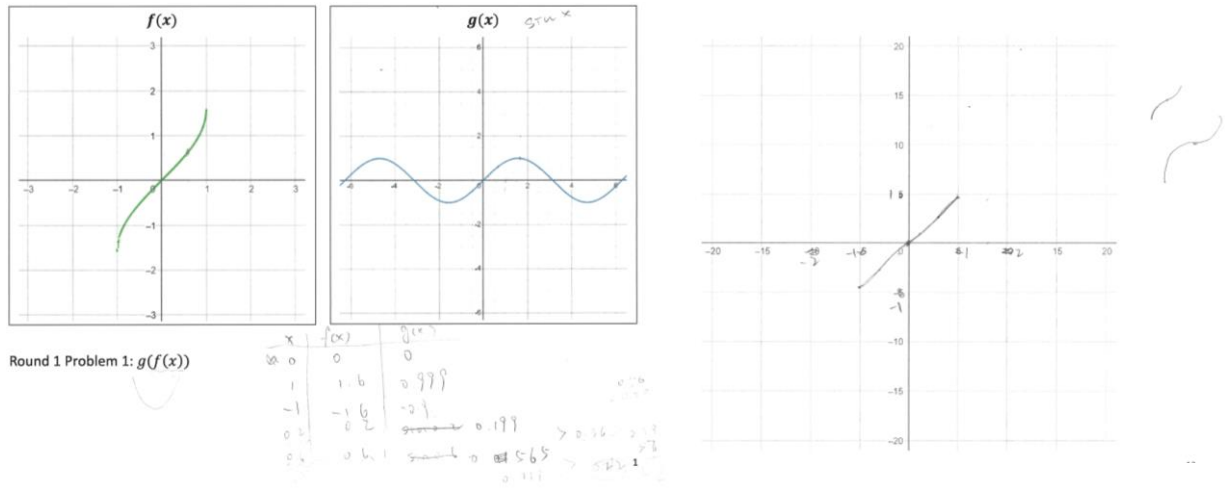
| | | | |
|---|---|---|---|
| 4 | Jade used the calculator and found out values of some specific points including $\sin(0)$, $\sin(0.2)$, $\sin(0.6)$, and $\sin(1.6)$. | Action | Action-level understanding characterized this reasoning step when the justification focused on specific point values. |
| 5 | Jade sketched the graph of the composite function within the domain of the interval $[-1,1]$, and reasoned the restricted domain: "For the composition of the function, the domain combines both $g(x)$ and $f(x)$; and for $f(x)$ the domain is only -1 to 1." | Process | Jade demonstrated process-level understanding by justifying the restricted domain of the composite function through operating and reflecting on the set of all possible inputs and outputs. |
| 6 | Jade sketched the graph in the first quadrant first and reasoned: "So I think those two (composite graph in first quadrant and third quadrant) are just pretty similar because of this graph ($f(x)$). So I only do one side (in first quadrant) and reflects the other side." | Process | Jade displayed Process-level understanding by using the graph instead of algebraic equations to compose functions in her justification and applying the symmetry of $f(x)$ to the composite function with general mapping, bypassing the step-by-step approach with specific points. |
| 7 | Jade reasoned and re-confirmed the symmetry of $g(x)$: "If it (input of $g(x)$) is negative…that's (the output value of $g(x)$) still the opposite (of the corresponding output on the positive side)." | Process | Jade revealed process-level understanding by viewing the symmetry of parent functions' graphs as general mappings of input values (negative input of $g(x)$in domain) to output values (opposite values of the output of $g(x)$ on the positive domain) in her justification. |
| 8 | Jade identified the changing concavity of $f(x)$: "I pick a point here ($f(x)$ graph in the first quadrant) with 0.6 (where the rate of change changed), so I separate this two parts. For the first part (input from 0 to 0.6 on $f(x)$), the derivative, the rate of change is most steep, and then 0.6 to 1." | Action | Jade identified the concavity of $f(x)$. Identifying single property of the function one step at a time in the justification is a characteristic of action-level understanding. |
| 9 | Jade reasoned the concavity of the composite function using the trend of the rate of change among several selected points. | Process | Jade demonstrated process-level understanding by generalizing the composite function's concavity through analyzing the trend of rate of change among points, showing comprehension of general mappings of all possible inputs and outputs in her justification. |

| | | | |
|---|---|---|---|
| 10 | Jade concluded: "The derivative for the composition function, the first part is also greater than the second part like (pointed at the graph of $f(x)$), so it cannot be a straight line." | Process | Based on derivative analysis, Jade showed process-level understanding by generalizing derivatives through all inputs and outputs to conclude the shape of the composite function in her justification. |
| 11 | After a few minutes, Jade further reasoned: "(it is) concave up or concave down (sketched a graph showing the concavity on the side of her graph paper to show the concavity she reasoned)." | Process | Furthermore, Jade displayed process-level understanding by using derivative analysis to generalize trends for all inputs and outputs, determining the concavity of the composite function in the justification. |

| Reasoning Sequence: | A | A | A | A | P | P | P | A | P | P | P |
|---|---|---|---|---|---|---|---|---|---|---|---|

**Figure 13**

*Jade's Work and Graph of Round 1 Problem 1*




***Round 2.*** In Round 2 when students were given algebraic equations, they demonstrated action-level understanding in two ways. First, they algebraically composed the equations and sketched the composite function based on its simplified equation without considering other function properties (see Round 2 Problem 1 in Appendix B). Second, after encountering unfamiliar algebraic equations of composite functions generated by irregular combinations of

parent functions (Round 2 Problems 3 and 4), they employed fragmented reasoning steps similar to their performance in Round 1, as these equations proved not useful.

Round 2 Problem 1 involved a familiar combination that asked students to compose inverse functions $f(x) = \ln(x)$ and $g(x) = e^x$. As presented in Table 25, similar to Jade's steps, many participants algebraically composed the equations and concluded $g(f(x)) = x$, then sketched the line $y = x$ across the entire domain without considering the restricted domain of the composite function. The majority of participants were able to recognize these parent functions as inverse functions, often saying, "This is what my teacher told me in school." However, when presented with follow-up questions such as "Why does the composition of two inverse functions become a linear function?" or "Why do the two curves combine to form a straight line?," students rarely provided insightful answers beyond "Because the equation shows it."

**Table 25**

*Jade's Reasoning Sequence in Round 2 Problem 1*

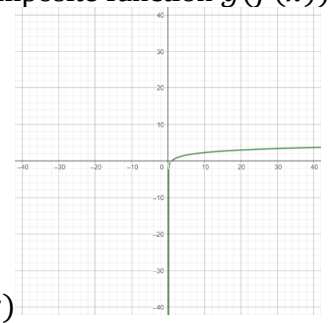
| Round 2 Problem 1<br>Sketch the composite function $g(f(x))$ given the following function equations.<br>$f(x) = \ln(x)$ $g(x) = e^x$ | | | |
|---|---|---|---|
| Step | Reasoning Process | Type | Analysis of This Step |
| 1 | Jade substituted the equations to compose the functions and wrote down:$e^{\ln(x)} = x$. | Action | Jade justified function composition using algebraic equations, indicating action-level understanding. |
| 2 | Jade used the calculator to estimate $e^{\ln(3)} = 3$ to confirm her simplification of algebraic expression from the last step. | Action | Jade demonstrated action-level understanding by justifying the composite function graph through substituting specific point value. |
| 3 | Jade sketched the line $y = x$ across the entire domain. | Action | Jade justified the composite function graph through explicit algebraic equation, indicating action-level understanding. |
| Reasoning Sequence: | **A** | **A** | **A** |

**With a Dynamic Graphical Representation.** In Round 3, with access to a dynamic graphical representation, the percentage of reasoning steps indicating process-level understanding increased. More critically, unique and dynamic reasoning approaches demonstrating process-level understanding emerged. Correspondingly, improved reasoning approaches were observed in areas where participants had previously shown action-level understanding with difficulty. As an example, Table 26 illustrates Jade's reasoning sequence for Round 3 Problem 1 (same functions as Round 2 Problem 1 but in different representation), alongside the dynamic graphical representation of this function composition in Figure 14.

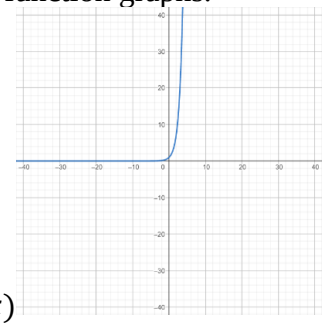
**Table 26**

*Jade's Reasoning Sequence in Round 3 Problem 1*

Round 3 Problem 1
Sketch the composite function $g(f(x))$ given the following function graphs.



$f(x)$



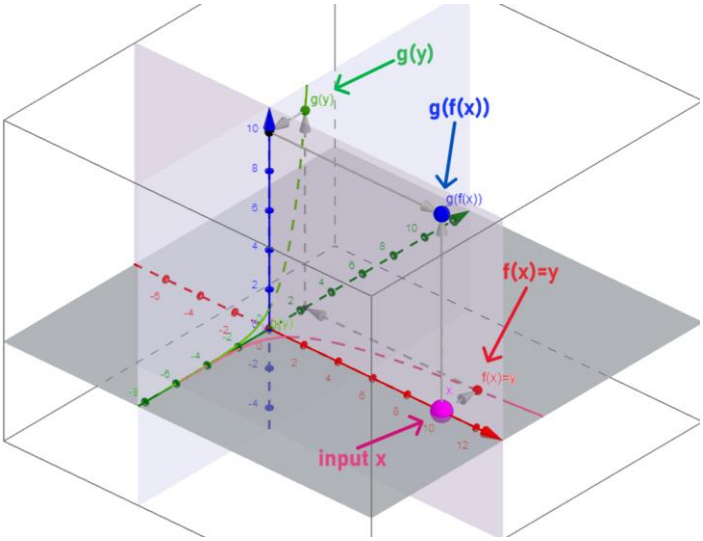
$g(x)$

| Step | Reasoning Process | Type | Analysis of This Step |
|---|---|---|---|
| 1 | Jade sketched the composite graph in the first quadrant and reasoned: "For $x < 0$, it (output of $g(f(x))$) doesn't appear; it (pointed at negative side of $f(x)$) does not exist. So the blue point (output of $g(f(x))$) only exists here (pointed at positive side of $f(x)$, from $(0, 0)$. So (the domain of $g(f(x))$) is greater than 0." | Process | Jade justified the restricted domain of the composite function through general mapping from all possible inputs from $f(x)$ domain to the composite function domain, indicating process-level understanding. |

| | | | |
|---|---|---|---|
| 2 | Jade reasoned the straight line as the composite graph: "I find the trend like (for the composite function), input equals to 2 and output equals to 2; input equals to 4, (then) output equals to 4; and input equals to 6 when output equals to 6, which means inputs and outputs (forms) the linear straight line." | Process | Process-level understanding characterized the reasoning step where Jade generalized the trend of the function shape dynamically, by justifying the input and output relationships. |
| 3 | Jade identified: "They are inverse function (used her finger to trace the symmetric graphs of $f(x)$ and $g(x)$, then used the laptop mouse to locate several points on $f(x)$'s graph and reflect these points to $g(x)$'s graph by the line of $y = x$ she traced using her finger)." Jade continued to explain that "the composition of inverse functions will always be $x$." | Process | Jade recognized the inverse parent functions through the symmetric graphs and further determined the composition of them. This step included identifying function properties in multiple representations, and justifying general mapping of a set of inputs and outputs, showing the characteristics of process-level understanding. |
| 4 | Jade picked a point to confirm her reasoning (see Figure 14): "So for here 0 to 10 (pointed at the input of 10), the output is 0 to 2 (traced the arrow from input mapping to the curve of $f(x)$ output), and for input of $g(x)$ is 2, the output of $g(x)$ is 10. So 0 to 10 (traced the arrow from origin to the $x - axis$ input of 10) with 0 to 10 (traced the arrow from origin to the $g(f(x))$ output on $z - axis$)." | Action | Jade demonstrated action-level understanding by justifying through one specific point to confirm the transformation from 10 to $f(10) = 2$, then to $g(2) = g\big(f(10)\big) = 10$, therefore $(10, 10)$ is a point on the composite function graph. |
| Reasoning Sequence: | **P** | **P** | **P** | **A** |

**Figure 14**

*Dynamic Graphical Representation of Round 3 Problem 1*




When compared to the first two rounds, where students demonstrated reasoning approaches indicating action-level understanding without a dynamic graphical representation, Round 3 showed progress through the use of a dynamic graphical representation, particularly in three areas.

***Beyond Equation Substitution.*** In Rounds 1 and 2, a common yet unproductive strategy involved continued reliance on identifying and substituting algebraic equations, indicating typical action-level understanding of function composition. Participants either struggled with estimating accurate equations from graphs (Round 1) or rushed to simply substitute equations into composition without considering other properties (Round 2). The dynamic graphical representation offered participants a more comprehensive view of functions, revealing properties that were often hidden in algebraic representations, such as restricted domains and ranges. Moreover, the interactive and dynamic representations offered rich information about functions and their composition, which provided participants opportunities to reduce their dependence on algebraic equations and develop more process-level understanding. For example, Jade initially overlooked the restricted domain in Round 2 Problem 1, but when the same functions were

presented through dynamic graphical representation in Round 3 Problem 1, she reasoned about the restricted domain, indicating an improved process-level understanding. More significantly, she did not merely copy the restricted domain presented in the representation stating, for example, "The domain is restricted because the composite function point disappears on the dynamic graph." Instead, she reasoned through it using general mapping, as demonstrated in step 1 of Table 26. This suggests that Jade might have had the capacity to develop a process-level understanding, but the algebraic representation and the procedural approach of simply substituting equations blocked her development and her vision of the comprehensive properties of the composite function, since algebraic equations hide properties (like restricted domain) that need attention. In contrast, graphical representations, especially dynamic ones, naturally presents richer information about functions' properties and, therefore, have potential for triggering or initiating students' process-level understanding.

***Integrated Approaches to Unfamiliar Tasks.*** Participants' reasoning approaches were more coherent and integrated when presented with a dynamic graphical representation, particularly when dealing with unfamiliar function combinations in Round 3. These approaches reflected more logically connected, process-level understanding that enabled students to bypass isolated steps. For instance, Round 3 Problem 3 (see Appendix B) proved challenging for participants since both the graph and equation of $f(x)$ were unfamiliar ($f(x) = x + \lfloor x \rfloor$), and the traced path of the composite function presented by the representation was relatively complicated as well. Jade's reasoning serves as an illustrative example. After initially showing action-level understanding by attempting to identify the function equation of $f(x)$ but failing, Jade returned to the dynamic representations and reasoned based on the provided information through several steps: (a) identified the discontinuous locations and trends of the composite function, then broke

it down into intervals for analysis based on the discontinuities of $f(x)$, (b) observed and analyzed the periodicities of the composite function, which were influenced by the periodicity of $g(x)$, and (c) plotted representative points at both ends of each interval to examine the composite function's shape more accurately. These organized and coherent reasoning steps improved Jades' reasoning and showed the characteristics of process-level understanding through general mapping of inputs and outputs as well as the integration of multiple interrelated function properties.

***Insightful and Dynamic Interpretation.*** With the increased proportion of process-level understanding indicated in their reasoning steps, the participants demonstrated insightful and dynamic interpretation of function composition, and dynamic conception of function graphs is the key characteristic of process-level understanding of functions. Responding to the follow-up question about why two curves composed to be a straight line in Round 3 Problem 1 (see Figure 14), Jade shared:

> For the three-dimensional model, it (the composite function) shares the $x$ (the input) of $f(x)$, and $y$ (the output) of $g(x)$, and they have the same pace. So when the blue ball (point of $g(f(x))$) moves, the pink (point of input $x$) and the green (point of $g(y)$) ball move with the same path.

This explanation indicated Process-level understanding by dynamically going through two processes of mapping of inputs and outputs, revealing how two mappings cancel out to create a straight line. There were several other dynamic interpretations: (a) students described the periodicity of the composite function as "the blue ball (point of $g(f(x))$) bounced back and forth as the red ball (point of $f(x)$) bounced"; (b) students reasoned about the discontinuity of the composite function as "the blue ball (point of $g(f(x))$) jumped when the red ball (point of $f(x)$)

jumped"; (c) students analyzed $\ln(x)$ and $e^x$ composing to be a straight line using the rate of change and concavity: "the red point (value of $f(x)$) grows slower which is the same as the input of $g(x)$; the green point ($g(x)$) grows faster. The faster and slower (rate of increasing) cancel each other out, so it (the composite function graph) is a straight line grows constantly." The fostering of conceptual and covariational reasoning appears to be a unique affordance of the dynamic graphical representation in this study. The conjecture is that not all dynamic graphical representations can effectively convey the in-depth correlations of function composition or other mathematical concepts. As reviewed in the previous chapter, the dynamic representation Brzezinski (2016) designed served as a counter example with its point-by-point demonstration.

***Summary***

In conclusion, without a dynamic graphical representation, action-level understanding was predominant in characterizing students' reasoning, mainly demonstrated by their overreliance on algebraic equations and fragmented reasoning sequences. In comparison, process-level understanding became prevalent with a dynamic graphical representation, indicated by reduced attention to algebraic equations and more coherent and integrated reasoning approaches and accompanied by insightful and dynamic interpretations of functions and their compositions.

**Research Question Three: On the Topics of Domain and Range of Function Composition, How Do Students Reason Differently?**

This research question was examined using three aspects. First, the results demonstrated how different types of reasoning were distributed when analyzing domain and range across rounds. Next, the analysis presented the distribution of fragmented versus integrated reasoning steps about domain and range throughout three rounds. Finally, examples from participants

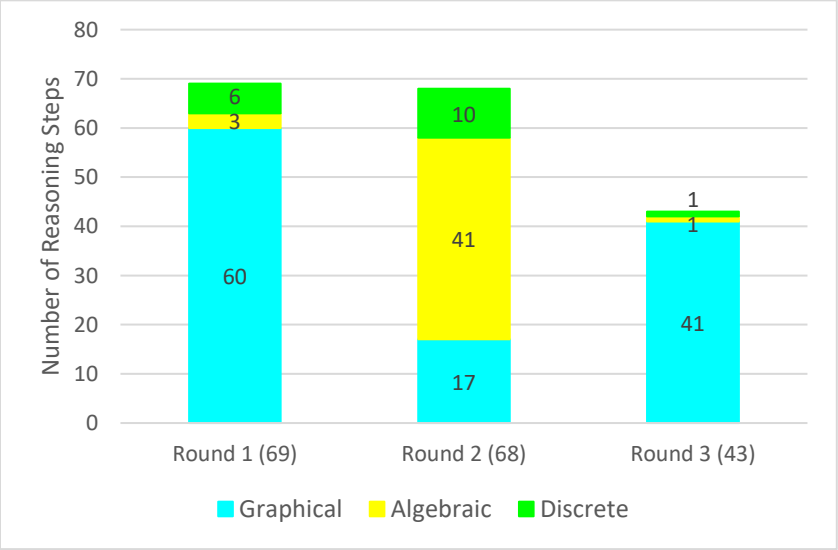
illustrated for domain and range: (a) typical reasoning patterns and (b) the trajectory of changes in reasoning across three rounds from a particular participant, both with and without a dynamic graphical representation. The analysis focused on differences in (a) distinct reasoning types and (b) reasoning with a dynamic graphical representation (Round 3) and without it (Rounds 1 and 2).

### *Distribution of Reasoning Types for Domain and Range*

Figure 15 shows the distribution of reasoning types used in analyzing domain and range across the three rounds. Each bar represents the number of reasoning steps related to domain and range in a specific round. The color-coding matches that used for the first research question: blue sections indicate the number of graphical reasoning steps, yellow sections represent algebraic reasoning steps, and green sections illustrate discrete points reasoning steps employed for domain and range analysis.

**Figure 15**

*Distribution of Reasoning Types for Domain and Range*



**How Do Students Reason Differently With Distinct Reasoning Types?** Across the three rounds, graphical reasoning proved to be the most useful for reasoning about domain and range when sketching composite functions. Graphical reasoning was particularly dominant when working with graphically depicted functions, as evidenced by the 87% (60 out of 69) of reasoning steps in Round 1 and the 95% (41 out of 43) of steps in Round 3 utilizing this approach. Even in Round 2, which presented algebraic equations, about a quarter (17 out of 68) of steps still relied on graphical analysis for domain and range.

Participants approached domain and range primarily (60%, or 41 out of 68) through algebraic reasoning in Round 2 when given equations. However, this reasoning type was rarely used in other rounds, demonstrating minimal effectiveness when algebraic equations were not available.

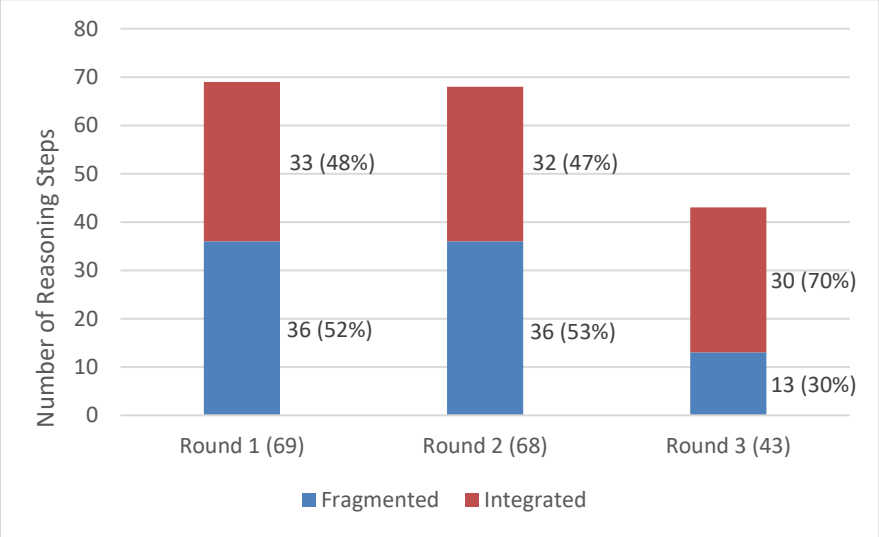
A limited number of participants used discrete reasoning steps to analyze domain and range, with less than 9% in Round 1 (6 out of 69), about 15% in Round 2 (10 out of 68), and only once in Round 3. Nevertheless, the ten discrete points reasoning steps in the second round suggested some value of this reasoning type especially when participants were provided with algebraic equations.

**How Do Students Reason Differently With a Dynamic Graphical Representation and Without It?** The most significant difference in participants' reasoning between Round 3 (with a dynamic graphical representation) and the first two rounds (without it) was the use of integrated versus fragmented reasoning steps. This key distinction was particularly evident when participants reasoned about domain and range of composite functions. In each round, reasoning steps about domain and range were categorized as either fragmented (unable to independently reach a complete conclusion) or integrated (able to reach a complete conclusion in a single step),

as shown in Table 14. Figure 16 displays the distribution of fragmented and integrated reasoning steps for domain and range analysis.

**Figure 16**

*Distribution of Fragmented Versus Integrated Reasoning Steps for Domain and Range*




Without a dynamic graphical representation, Rounds 1 and 2 required a similar number of steps when approaching domain and range (69 and 68 respectively). Reasoning steps in these two rounds were predominantly fragmented (52% and 53% respectively), suggesting that students broke down problems into smaller pieces and took multiple separate steps to reach solutions. Sometimes students used trial-and-error approaches and couldn't even reach valid solutions about domain and range.

In comparison, Round 3 showed a marked reduction to 43 reasoning steps for domain and range, with students demonstrating more comprehensive and effective reasoning. Correspondingly, their approaches were more integrated, with the percentage of integrated steps rising substantially to 70%. Typically, a single step in Round 3 could accomplish what had required three steps in Round 1 or 2 without a dynamic graphical representation.

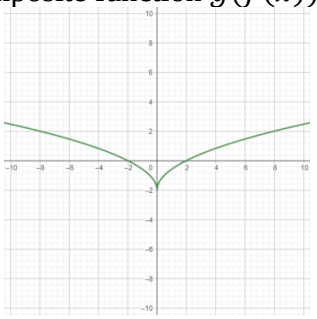
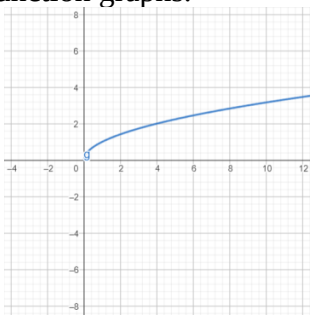
Additionally, both algebraic and discrete point reasoning appeared only once in Round 3, revealing that with a dynamic graphical representation, students no longer needed to return to specific points or algebraic equations. Furthermore, participants in Round 3 showed deeper conceptual understanding of domain and range through dynamic interpretation, demonstrating their grasp of covariational transformations between two mappings from $f(x)$ to $g(x)$ during composition.

***Examples of Reasoning Steps for Domain and Range***

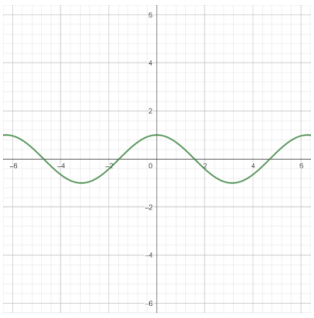
Students' typical approaches to analyzing domain and range in Round 1 Problem 3, Round 2 Problem 3, and Round 3 Problem 4 were presented as examples (see Table 27). This set of similar or identical problems (see Table 9 and Appendix B) provided complex domain and range situations. In particular, these three problems share a specific situation where the restricted domain of $g(x)$ causes the restricted range of $f(x)$, which in turn determines the restricted domain of $f(x)$, which is also the restricted domain of the composite function $g(f(x))$.

**Table 27**

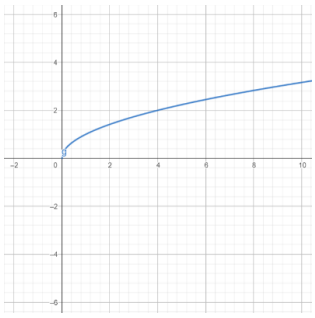
*Example Set of Problems Illustrating Reasoning Steps for Domain and Range*

| |
|---|
| Round 1 Problem 3<br>Sketch the composite function $g(f(x))$ given the following function graphs.<br>$f(x)$  $g(x)$  |
| Round 2 Problem 3<br>Sketch the composite function $g(f(x))$ given the following function equations.<br>$f(x) = \cos(x)$ $g(x) = \sqrt{x}$ |

Round 3 Problem 4



$f(x)$



$g(x)$

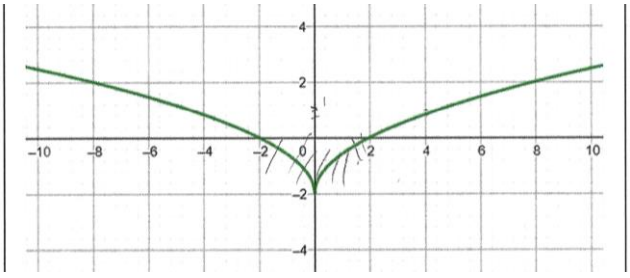
**Typical Domain and Range Approaches Through Different Reasoning Types.**

Graphical reasoning for domain and range involved reading and interpreting domain and range information from graphs and making assertations about restricted domain or range based on the information. In Round 1 Problem 3, a typical approach was (a) reading the graph and identifying the domain of $g(x)$, which should be greater than zero; (b) applying the domain of $g(x)$ to the range of $f(x)$ for the composite function $g(f(x))$, which should also be greater than zero; and (c) reading the graph where $f(x) > 0$, and crossing out the part less than zero to determine the domain (see Figure 17(a)). Graphical reasoning proved to be effective even when algebraic equations were available. In Round 2 Problem 3, several participants approached domain and range graphically to reason the intervals of the restricted domain (see example in Figure 17(b)).

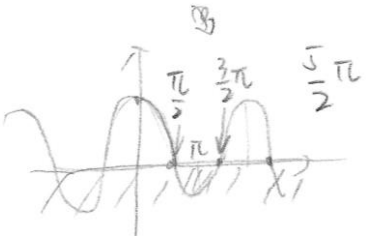
**Figure 17**

*Examples of Graphical Reasoning Steps for Domain and Range*

(a)




(b)



Participants applied algebraic reasoning to analyze domain and range when they were familiar with the function equations and could recall domain and range properties from their prior experience with these equations. For example, Luna demonstrated algebraic reasoning about domain and range in Round 2 Problem 3: "The range of the $\sqrt{x}$ need(s) to be greater (than) or equal to zero, so domain (of the composite function) is the part of $x$ that can make $\cos(x)$ greater (than) or equal to zero, just like $-\frac{\pi}{2}$ to $\frac{\pi}{2}$," (see Table 20). However, fully relying on algebraic approaches sometimes caused participants to neglect hidden and crucial information about domain and range. This was evident in Jade's reasoning (see Table 25), where she simply substituted the algebraic equations to compose $g(f(x)) = e^{\ln(x)} = x$ and overlooked the restricted domain of $f(x) = \ln(x)$.

Discrete point reasoning worked for analyzing domain and range when appropriate points were chosen. When participants used graphs to plot or used equations to substitute points for the composite function, inputs that resulted in non-existing (undefined) outputs suggested to participants that these points and certain domains around them must be eliminated for the restricted domain. However, participants often applied random points and could not always successfully locate the restricted domain (see Olivia's reasoning in Table 19 and Helena's reasoning in Table 21).

**Typical Domain and Range Approaches With a Dynamic Graphical Representation and Without It.** Without a dynamic graphical representation in the first two rounds, many participants identified the domains and ranges of the parent functions but could not reason how these would affect each other in composition. Even when some participants successfully reasoned about domain and range for the composite function, their steps were typically fragmented. For example, Gia took three steps to reason about the restricted domain in Round 1

Problem 3 (see Table 18). In comparison, Ella reasoned about the restricted domain of Round 3 Problem 4 in one integrated step using a dynamic graphical representation (see step 1 in Table 22). On top of that, Ella dynamically interpreted and conceptually reasoned about the restricted range of the composite function in the third step of Table 22, stating,

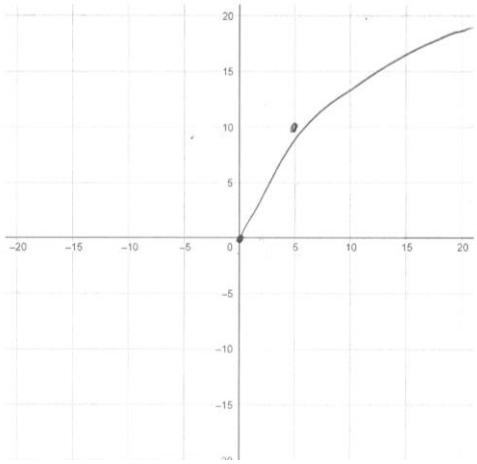
> The range of this ($f(x)$) is (decides) the domain of this part ($g(x)$), so the output of this curve ($g(x)$), just like back and forth from this point (the interval of $[0,1]$) go up and go down (pointed at the range of $f(x)$ between 0 and 1).

**An Illustrative Example: Trajectory of the Reasoning for Domain and Range Across Three Rounds.** To better demonstrate the trajectory of how reasoning changed for domain and range across three rounds, an illustrative example of Helena's reasoning for domain and range in Round 1 Problem 3, Round 2 Problem 3, and Round 3 Problem 4 is presented here.

In Round 1 Problem 3, Helena mainly approached domain and range through plotting points from reading graphs, while believing incorrectly that the input of $g \circ f$ is $f(x)$ instead of $x$. This reasoning approach resulted in Helena's over-simplified and incorrect conclusion concerning the restricted domain of $g(f(x))$. She mistakenly generalized that the domain of $g(f(x))$ follows the domain of $g(x)$. This misunderstanding was also clearly presented by her sketch of Round 1 Problem 3 (see Figure 18), in which the domain of her sketch of $g(f(x))$ is the same as the domain of $g(x)$ which is $x > 0$.

**Figure 18**

*Helena's $g(f(x))$ Sketch of Round 1 Problem 3*




In Round 2 Problem 3, as demonstrated in Table 21, Helena totally ignored the domain and range issues, and sketched the composite function purely by connecting random points from substituting equations.

In contrast, in Round 3 Problem 4, inspired by the dynamic graphical representation, Helena was able to recognize the restricted domain of the composite function. She also intuitively reasoned that the restricted domain could be affected by not only the restricted domain of $g(x)$ (which she mistakenly believed in Round 1 Problem 3) but also the restricted domain of $f(x)$. She stated, "If $f(x)$ has a domain restriction, then it's also going to keep that domain restriction in the composite function." Helena had not yet built up a mature and completely accurate understanding on the restricted domain of function composition even in the third round, mainly due to her fundamental misconceptions around this topic. However, promising signs can be observed with the dynamic graphical representation, which can possibly provide opportunities to trigger and promote students' reasoning approach, leading them to accurate understanding of domain and range.

***Summary***

To sum up, participants most often reasoned about domain and range graphically by reading and interpreting graphs. Algebraic reasoning was useful for domain and range only when

students were familiar with the equations and could cause mistakes if they were not experienced with them. Discrete reasoning could be useful for analyzing domain and range only if appropriate points were chosen.

Students applied fragmented, or even incomplete reasoning steps for domain and range in Rounds 1 and 2 without a dynamic graphical representation. In contrast, more integrated, coherent and dynamic reasoning for domain and range was employed in Round 3, which contained a dynamic graphical representation.

**Research Question Four: On the Topics of Symmetry and Continuity of Function Composition, How Do Students Reason Differently?**

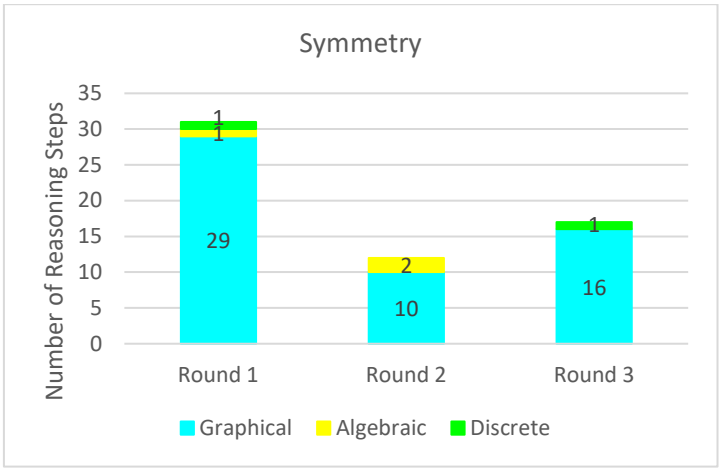
This research question was addressed using two aspects. First, the results demonstrated the distribution of reasoning types employed in analyzing symmetry and continuity across different rounds. Second, examining participants' reasoning relevant to symmetry and continuity showed (a) typical reasoning patterns and (b) the trajectory of changes in reasoning across the three rounds from a particular participant, both with and without a dynamic graphical representation. The analysis focused on differences in (a) distinct reasoning types and (b) reasoning with a dynamic graphical representation (Round 3) and without it (Rounds 1 and 2).

***Distribution of Reasoning Types for Symmetry and Continuity***

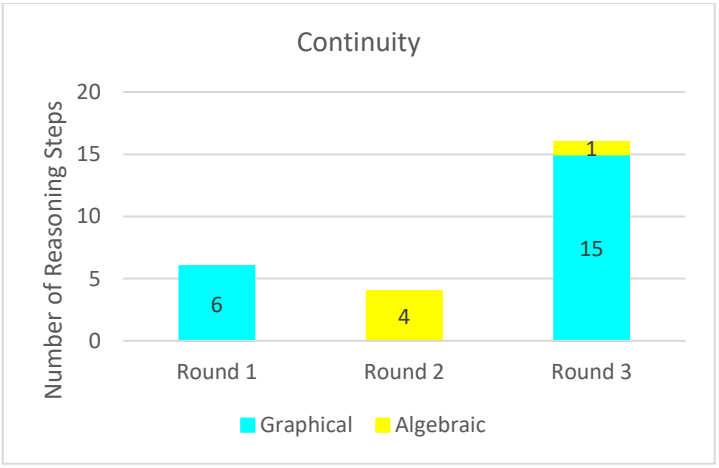
Figure 19 shows the distribution of reasoning types used in analyzing symmetry (Figure 19(a)) and continuity (Figure 19(b)) across the three rounds. Each bar represents the number of reasoning steps related to symmetry or continuity in a specific round. The color-coding is the same as that used for the first and third research question: blue sections indicate the number of graphical reasoning steps, yellow sections represent algebraic reasoning steps, and green section illustrates discrete points reasoning steps employed for symmetry and continuity analysis.

**Figure 19**

*Distribution of Reasoning Types for Symmetry and Continuity*




(a)




(b)

**How Do Students Reason Differently With Distinct Reasoning Types?** Across all rounds, graphical approaches dominated the reasoning steps that specifically targeted symmetry properties, with only one or two instances employing algebraic and discrete point reasoning.

For continuity, no discrete point reasoning was utilized. While Rounds 1 and 3 relied almost exclusively on graphical reasoning, Round 2 was approached solely through algebraic strategies.

**How Do Students Reason Differently With a Dynamic Graphical Representation and Without It?** Without a dynamic graphical representation, Round 1 showed the highest total

number of reasoning steps for symmetry, while Round 2 exhibited the lowest total. These results indicated that students paid less attention to the symmetry property in function composition when graphs were not presented. Even in Round 2, most steps analyzing symmetry were approached through a graphical lens. With a dynamic graphical representation, moderate numbers of steps focused on symmetry, showing both affordance and constraints that are addressed in the following discussion.

Without a dynamic graphical representation in the first two rounds, few reasoning steps focused on continuity, with participants employing exclusively graphical reasoning in Round 1 and exclusively algebraic approaches in Round 2. The number of reasoning steps increased significantly in Round 3 with a dynamic graphical representation, possibly because the representation enhanced the visibility of continuity properties, thereby prompting greater attention to this property during that round.

### *Examples of Reasoning Steps for Symmetry and Continuity*

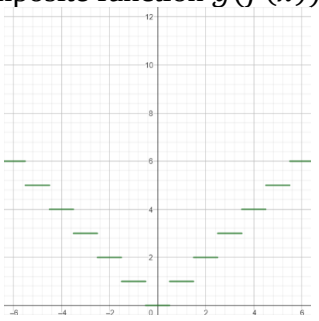
Students' typical approaches to analyzing symmetry and continuity in Round 1 Problem 4, Round 2 Problem 4, and Round 3 Problem 3 were presented as examples (see Table 28). This set of similar or identical problems (see Table 9 and Appendix B) provided complex symmetry and continuity situations, where (a) $f(x)$ is symmetric and discontinued (Round 1 and 2, Problem 4), or $f(x)$ is discontinued and $g(x)$ is symmetric (Round 3 Problem 3), which determined the symmetry and continuity of the composite function and (b) $g(x)$ has both restricted domain or range which affects the intervals of the discontinuity.

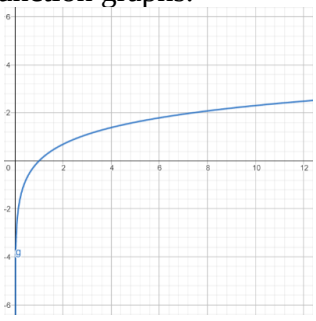
**Table 28**

*Example Set of Problems Illustrating Reasoning Steps for Symmetry and Continuity*

Round 1 Problem 4

Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

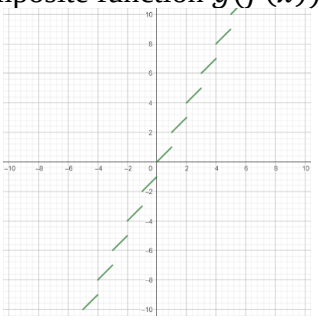
Round 2 Problem 4

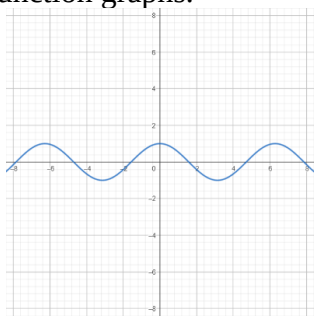
Sketch the composite function $g(f(x))$ given the following function equations.

$f(x) = |round(x)|$ $g(x) = \ln(x)$

Round 3 Problem 3

Sketch the composite function $g(f(x))$ given the following function graphs.

$f(x)$ 

$g(x)$ 

**Typical Symmetry and Continuity Approaches Through Different Reasoning Types.**

***Symmetry.*** Graphical reasoning for symmetry involved identifying the symmetry properties of parent functions through graphs, and applying these properties to composite functions. Participants typically sketched the composite function graph on the positive side with more details, aiming for accuracy, before reflecting their sketch onto the negative side. Most participants did not provide elaborate explanations for their reasoning about symmetry beyond statements such as "The composite graph is symmetric because $f(x)$ is reflected by $y - axis$." In Round 1 Problem 4, Bree provided slightly more detailed reasoning: "The symmetry (of the composite function graph), specifically the symmetry about the $y - axis$, (occurs) because from

$f(x)$ if you plot input as negative and a positive (with same absolute values), the outputs on the graph are the same, so the outputs of $g(f(x))$ (are the same)."

The algebraic reasoning steps for symmetry were very limited and involved a specific function – the absolute value function. In Round 2, Bree and Ivy connected the absolute value operation with even functions and then applied the reflection in their graphs.

***Continuity.*** Special discontinuity cases commonly arose from the scenario that one or both parent functions being discontinuous, which caused the composite function to be discontinuous. This is the main focus of this section, distinct from the scenario where the composite function becomes disconnected due to a restricted domain (e.g., Round 1 Problem 3), which was analyzed previously within the domain and range topic. Graphical reasoning for continuity involved identifying the continuity properties of parent functions through graphs and applying these properties to composite functions. Students typically focused on finding the locations of the starting and ending points of the discontinuous intervals. Similar to the reasoning shown for the symmetry topic, most explanations were simple, such as "The composite function graph has steps/breaks/stairs, because $f(x)$ has them" (the reasoning from Jade, Helena and Ivy in Round 1 Problem 4).

In Round 2 Problem 4, Ivy reasoned about the continuity property through algebraic strategies instead. She analyzed the discontinuity through the meaning and features of $f(x)$, which is the round function. She explained: "There is discontinuity in the composite function because when values run (across) different intervals, the rounding value will also be different, but (within the) same interval, the rounding values are the same." Here, by "intervals," she meant the group of inputs that have the same output after rounding, for example, $[0.5, 1.5)$.

### Typical Symmetry and Continuity Approaches With a Dynamic Graphical Representation and Without It.

***Symmetry.*** Without a dynamic graphical representation, most approaches lacked insight, showing limited understanding of why the symmetric property of $f(x)$ applied to the composite function $g(f(x))$. Even so, when given algebraic equations in Round 2, students successfully employed graphical reasoning as a useful strategy, though they were not particularly insightful. For the unfamiliar function in Round 2 Problem 4 (which used the same functions as Round 1 Problem 4), many students attempted to sketch the graph of $f(x) = |round(x)|$ before they could reason about the reflection of the composite function graph.

Students' performance in reasoning about symmetry revealed both affordance and constraint of the dynamic graphical representation. Students using the representation could reason about symmetric relationships not only within a single function graph but also between parent functions in composition. For example in Round 3 Problem 1, several students recognized and reasoned dynamically about the symmetric relation between the graphs of $f(x)$ and $g(x)$, connecting this symmetry to the straight line of the composite function. As Gia explained: "If you look at these graph (graphs of $f(x)$ and $g(x)$), they are symmetric according to the plane where the composite line lies." However, for complex tasks like Round 3 Problem 3, the traced path without clear point values misled some students. Some incorrectly concluded that the composite function was symmetric about the $y - axis$ like $g(x)$ (e.g., Gia), or symmetric about the origin like $f(x)$ (an incorrect assertation from the beginning since $f(x)$ is not symmetric). Neither conclusion was true.

***Continuity.*** Without a dynamic graphical representation, the reasoning about continuity was simple and showed limited conceptual understanding, especially in Round 1 when

participants used graphical approaches. However, in Round 3 with a dynamic graphical representation, students' graphical reasoning about continuity became robust, featuring dynamic interpretations that covariationally connected the properties of parent functions with the composite function. For example, Helena reasoned Round 3 Problem 3: "There is a lag between shapes of $g(f(x))$, (because) when I am looking at $f(x)$ and tracing the function, I can see that the trace of the composite function jumps when the (output) of $f(x)$ jumps."

**An Illustrative Example: Trajectory of the Reasoning for Symmetry and Continuity Across the Three Rounds.** To better demonstrate the trajectory of how reasoning changed for symmetry and continuity across the three rounds, an illustrative example of Morgan's reasoning for symmetry and continuity in Round 1 Problem 4, Round 2 Problem 4, and Round 3 Problem 3 is presented here. Morgan's reasonings about symmetry and continuity for these problems across three rounds were all accurate and appropriate, but each had a different approach from a distinct perspective, which made Morgan's case representative and informative.

***Symmetry.*** In round 1 Problem 4, Morgan shared that she "did all the negative part of $g(f(x))$ first, and found it is like symmetric." Echoing the common reasoning performance for the symmetry property that was discussed previously, Morgan correctly and intuitively applied the symmetry of the parent function to the symmetry of the composite function without further explanation.

In Round 2 Problem 4, Morgan substituted certain point values into the composite function equations and found that "these two intervals (are) symmetric, one positive and one negative." So when provided with algebraic equations, Morgan used appropriate points as examples to reason about the general symmetric property of the composite function. Her reasoning performance in this round aligned with the results stated previously about discrete

point reasoning being useful for symmetry of function composition, when appropriate points were chosen to exemplify the function property.

In Round 3 Problem 3, some other participants exposed the constraints of the dynamic graphical representation when they relied on the traced path the dynamic representation showed. They mistakenly considered the composite function to be symmetric. However, Morgan avoided this error by returning to static graphical representations and reasoned the symmetry property of $g(f(x))$ through it. Morgan successfully reasoned that the composite function is asymmetric by claiming that "If $f(x)$ is symmetric by the origin then the $g(f(x))$ will be symmetric." She further explained that when $f(x)$ is symmetric by the origin, the opposite inputs of $f(x)$ will result in opposite outputs of $f(x)$, which are opposite inputs of $g(x)$. These opposite inputs of $g(x)$ would finally create the same outputs of $g(x)$ as $g(f(x))$, since $g(x)$ is symmetric by the $y-axis$. However, $f(x)$ is not symmetric by the origin, therefore $g(f(x))$ is not symmetric.

***Continuity.*** In Round 1 Problem 4, Morgan described the discontinuity implicitly by sharing her intuitive reasoning about the "step shape" of the composite function based on the "step shape" of $f(x)$ since "every $x$ belongs to a (discontinued) interval."

In Round 2 Problem 4, Morgan approached it through generalizing the discrete point values by reasoning that points in a certain interval have the same "situation," meaning the same outputs according to the algebraic equations substituted, and then points in the next interval shared a different output than the previous interval. The reasonings for continuity in the first two rounds without a dynamic graphical representation were somewhat indirect and ambiguously implied from graphical features or outcomes of algebraic equations.

In contrast, in Round 3 Problem 3, Morgan built up a global view stating, "It (point of $f(x)$) has a jump here (pointed at the end of one discontinued interval in $f(x)$), so we have also

a jump here (for $g(f(x))$)." Morgan's reasoning about continuity in this round was more direct with dynamic covariational understanding, likely enhanced by the dynamic and covariational presentations of the dynamic graphical representation.

***Summary***

To conclude, students primarily relied on graphical reasoning to analyze symmetry and continuity, while algebraic reasoning proved somewhat useful for continuity analysis when given equations. Participants' reasoning about these two topics remained basic and simple when working without a dynamic graphical representation. In contrast, the dynamic graphical representation enabled participants to recognize and utilize symmetric relations not only within individual functions, but also between functions. However, the representations also had limitations that occasionally misled students. Additionally, students demonstrated more dynamic and robust reasoning about continuity when using a dynamic graphical representation.

# Chapter 5: Conclusion, Discussion and Implications

## Summary

Fifteen graduate students from a mathematics education program in a major city on the eastern coast of the United States were recruited to participate in this research study exploring students' reasoning about function composition using a dynamic graphical representation. The four research questions addressed how students reason and what level of understanding characterizes their reasoning about function composition in terms of APOS theory, both with and without a dynamic graphic representation. Specifically, the research questions addressed how students reason differently regarding domain and range, symmetry, and continuity.

Students interested in participating were given a survey to assess their suitability for the study. The 15 selected participants were recent undergraduate graduates who were experienced with mathematics content as well as mathematical representations. Participants were interviewed individually as they sketched composite functions through three rounds. To sketch $g\big(f(x)\big)$, students were provided with static graphs of $f(x)$ and $g(x)$ in Round 1, equations of them in Round 2, and had access to a dynamic graphical representation of the function composition along with static graphs of them in Round 3. Each round contained four function composition tasks, designed by the researcher and reviewed by colleagues. Each task was designed to be either identical or similar to one task in each of the other rounds, resulting in four sets of comparable tasks throughout the three rounds to examine students' reasoning with and without a dynamic graphical representation. During the semi-structured interviews, students demonstrated their reasoning by solving the function composition tasks while thinking aloud. Data sources used to answer the study's research questions included video and audio interview recordings and participants' written responses.

In order to understand students' reasoning about function composition with and without a dynamic graphical representation, it was necessary to comprehend two aspects: how reasoning is defined and identified, and different ways to describe reasoning. To identify and define reasoning, participants' solving processes were broken down into sequences of distinct reasoning steps, where each step led to either a successful or failed interim conclusion justified from an assertation. This study described each reasoning step from three perspectives: (a) distinct types of reasoning, adapted and adjusted from literature and preliminary study findings, for steps in the entire reasoning process; (b) levels of understanding, in terms of APOS theory, which were indicated in students' reasoning and (c) whether the reasoning steps about domain and range was integrated or fragmented. Conclusions for this study were established through analysis of general proportions and trends of students' reasoning processes, as well as detailed reasoning patterns within them.

## Conclusions

### *Question 1: How Do Students Reason About Function Composition With and Without a Dynamic Graphical Representation?*

Without a dynamic graphical representation in Rounds 1 and 2, students reasoned about function composition through longer reasoning processes. Diverse reasoning approaches were needed, especially with the algebraically depicted functions in Round 2. Students' reasoning patterns revealed that alternating between graphical and discrete point reasoning proved effective, while solely relying on algebraic and discrete point reasoning was limited in effectiveness, especially when participants overlooked the general picture and critical properties of the composite function, such as restricted domain and range.

In comparison, with a dynamic graphical representation in Round 3, the finding of shorter and more concentrated numbers of reasoning steps suggests higher reasoning efficiency. Participants predominantly used graphical reasoning with the lowest percentage of reasoning through algebraic as well as discrete point lenses, indicating that the dynamic graphical representation promoted graphical reasoning while reducing reliance on algebraic and discrete point approaches. Several students developed fully graphical reasoning sequences with a dynamic graphical representation. Moreover, reasoning steps in Round 3 offered more coherent, integrated reasoning sequences with dynamic and insightful interpretations about function composition. Despite the potential and affordance of the dynamic graphical representation, a few students' reasoning experiences revealed some constraints of this particular dynamic graphical representation. These participants, some of whom even performed well in the previous two rounds, either had difficulty reading precise point values or were overwhelmed by unfamiliar dynamic three-dimensional presentations.

***Question 2: What Level of Understanding, in Terms of APOS Theory, Characterizes Students' Reasoning About Function Composition With and Without a Dynamic Graphical Representation?***

Overall, action-level understanding characterized more than half of the reasoning steps in Rounds 1 and 2 without a dynamic graphical representation. In comparison, the number of reasoning steps indicating higher process-level understanding about function composition moderately increased and exceeded the action-level understanding in Round 3, when students had access to a dynamic graphical representation.

Without a dynamic graphical representation, students typically demonstrated action-level understanding about function composition when they relied on looking for and substituting

explicit algebraic equations or plotting random specific points. In contrast, students presented process-level understanding and more integrated approaches when they dynamically reasoned about general mapping among sets of inputs and outputs as well as covariational relations.

***Question 3: On the Topics of Domain and Range of Function Composition, How Do Students Reason Differently With Distinct Reasoning Types and a Dynamic Graphical Representation?***

Graphical reasoning played a critical role when students reasoned about domain and range of function composition across the three rounds, and it was the leading type of reasoning in the first and third rounds with graphically depicted functions. Algebraic reasoning proved to be useful for domain and range when students were familiar with the equations and their compositions. Discrete point reasoning in domain and range analysis was limited, and this approach was only useful when appropriate points were chosen.

Compared with more fragmented reasoning approaches shown in the steps regarding domain and range without a dynamic graphical representation (Rounds 1 and 2), reasoning steps were more integrated and efficient for domain and range with a dynamic graphical representation (Round 3).

***Question 4: On the Topics of Symmetry and Continuity of Function Composition, How Do Students Reason Differently With Distinct Reasoning Types and a Dynamic Graphical Representation?***

Graphical reasoning dominated students' symmetry analysis. On the topic of continuity on the other hand, students solely applied graphical reasoning when given graphs, while approaching it solely algebraically when given equations.

Symmetry analysis was more commonly applied when students were provided with graphs in Round 1. With a dynamic graphical representation, symmetry between graphs was

presented more explicitly for students, but its roughly traced path also misled some students, causing them to determine the symmetry of the composite function incorrectly.

Very little attention was paid to continuity without a dynamic graphical representation, while attention significantly increased in Round 3. Students were able to dynamically and covariationally interpret the discontinuity relations between parent functions and composite functions.

## Discussions

### *The Dynamic Graphical Representation: Primary Driver or Contributing Factor*

An inevitable question for this study is whether students' reasoning progress can be attributed to the dynamic graphical representation itself, or if that progress simply reflected increased familiarity after multiple rounds of solving similar function composition problems. It must be acknowledged that the accumulated reasoning experience brought benefits to students in the third round when they composed functions using the dynamic graphical representation. Therefore, this study concentrated more on certain perspectives of students' reasoning progress that evidently demonstrated the impact of the dynamic graphical representation.

**Dynamic Graphical Representation as the Primary Source of Dynamic Reasoning.** Because the change in reasoning performance was directly and exclusively related to the features of the dynamic graphical representation and that change had never happened before this round, it can be concluded with confidence that the representation was the primary driver of the change in reasoning. If the reasoning stemmed from experience with previous problems, the approaches should have had a trace of the previous successful reasoning performance instead of an entirely fresh approach. As an example, students' reasoning in Round 3 was frequently described as "dynamic," which is clearly linked to the presentation of the dynamic graphical representation.

Descriptions such as "The point of $g(f(x))$ bounced back and forth between 0 and 1 as the input point grows," or "point of $g(f(x))$ jumped when the point of $f(x)$ jumped" were reasonings students demonstrated from their interactions, observations, and interpretations of the presentation shown by the dynamic graphical representation. They had never appeared in the previous two rounds. As another example, when trying to reason about Round 3 Problem 1, students recognized the symmetry between the parent inverse functions presented by the three-dimensional model of the dynamic graphical representation. Furthermore, they understood and noted the linear feature of the composite function when they could see the input and the output points growing at the same pace after two reflected mappings. These dynamic reasoning performances with a global view of function composition of inverse function were innovative and had not been observed in students' reasoning in Problem 1 of Rounds 1 and 2, problems in the comparable set. Most students in Round 2 Problem 1 explained the linear feature of the composite function simply as "That was what I have learned, inverse functions composed to be $y = x$."

**Dynamic Graphical Representation as the Primary Source of Changes in Reasoning Habits.** Since it was apparent that participants' reasoning patterns and habits changed when using the dynamic graphical representation in Round 3, and since this change was not cumulatively observed from the previous rounds as participants were gaining experience, then the conjecture can be raised that the dynamic graphical representation was the primary driver of the reasoning habit changes. The major progress in Round 3 in reasoning habits was the notable decreased need to return to explicit algebraic equations and specific points. However, this change of in reasoning pattern was not apparent during the first two rounds without a dynamic graphical representation. For example, participants approached at least a third of reasoning steps through

discrete point reasoning in both rounds without a dynamic graphical representation. Additionally, Bree's reasoning performance across three rounds can serve as an illustrative example for changes in reasoning habits that had been characterized by a reliance on algebraic equations. Bree demonstrated logical reasoning processes and an understanding of function composition topics in the first two rounds, with a strong reliance on algebraic equations (see Appendix D). This reasoning habit was shown in Round 1 as the urge to figure out the explicit algebraic equations; she persistently tried to test if she had found the accurate equations, but with some unfamiliar functions she was not successful. Therefore, she only accurately reasoned through half of the problems in the first round. In the second round when provided with equations, she was able to successfully reason all of the problems. In contrast, in the third round, although her habit of looking for algebraic equations did not completely disappear, she exhibited much less persistence with the equations. When some graphs represented unknown equations that she could not figure out (e.g., Round 3 Problem 3), she was able to analyze the composition function mostly graphically without fixating on the equations. In Appendix D it can be observed that Bree used far fewer algebraic reasoning steps in Round 3 with the dynamic graphical representation. In fact, she approached the composition algebraically in only two steps in the first problem of this round.

**Clarifying the Contribution of the Dynamic Graphical Representation to Students' Reasoning.** The original task design included an additional round of one problem after Round 3 to assess participants' reasoning and understanding of composing graphically depicted functions without the dynamic graphical representation. This final problem was designed to evaluate whether participants' reasoning and understanding progress was retained after using the dynamic graphical representation. However, due to the length of the interview, this design was not

feasible. Examining students' reasoning without the dynamic graphical representation after they experienced it would provide substantial proof of whether their improved reasoning persisted and could be applied independently. This may be a subject for future research.

### *Affordances and Constraints: Generalizability of the Dynamic Graphical Representation*

Exploring the affordances and constraints of the dynamic graphical representation on students' reasoning about function composition, a question naturally arises: are these affordances and constraints particular to the representation used in this study, or can they be generalized to other dynamic graphical representations? This section discusses the question.

It is difficult to divide the features of the dynamic graphical representation into simply beneficial or non-beneficial for students' reasoning. The same presentation or interactions may yield different reasoning feedbacks depending on the participants.

**Graphical Presentation.** The dynamic graphical representation in this study significantly promoted the use of graphical reasoning. The findings regarding enhanced graphical reasoning through this particular visual approach might to some extent be applied universally to other similar tools of different mathematical contents, since the common feature of dynamic graphical representations is the experience they provide users through intuitive visual interpretation. Through graphical presentations, these displays create opportunities for users to interpret the properties of the mathematical content and justify these properties graphically based on interpretations from the representations.

**Dynamic and Synchronous Interaction.** The representation used in Round 3 of this study enabled students to have dynamic and synchronous interaction with the initial input point they dragged along the $x-axis$, the corresponding $f(x)$ value, and the mapped $g(f(x))$ value. Students observed the dynamic changes in the $f(x)$ and how it affected the final $g(f(x))$ value.

Through the mapping points, they were able to observe (a) the restricted domain when the output points disappeared, (b) the restricted range as points bounced back when they reached a certain point, (c) the periodicity when the points bounced back and forth, (d) the concavity when the corresponding points grew at the same or different pace, and (e) the continuity when the points disappeared and then reappeared or had sudden jumps. All function composition properties that this dynamic graphical representation demonstrated were covariational, showing the relations between parent functions and composite functions and displaying not only the mappings between a single input $x$ to $f(x)$ and then to $g(f(x))$, but also the general two rounds of mappings between sets of inputs and outputs. The dynamic feature likely contributed to students' dynamic interpretation in their reasoning, and the covariational general mapping feature possibly supported students in developing a comprehensive view of the function composition process and building more coherent and integrated reasoning steps.

However, the dynamic feature and the points that kept moving also overwhelmed some students. They shared that they could not process the relationships among three functions all together and preferred to examine them one by one through static graphs.

Similar dynamic graphical representations of function topics that have the feature of showing mapping relations synchronously, especially those with general mappings instead of point-by-point mapping, would possibly also provide similar learning opportunities to students. On the other hand, those dynamic graphical representations probably would share similar constraints and present a learning curve for students.

**Three-Dimensional Model.** Another unique feature of the dynamic graphical representation used in this study is its three-dimensional model compared with many other two-dimensional representations that were reviewed in the second chapter. One possible benefit of

the three-dimensional aspect is the fact that the graph of $f(x)$, $g(x)$, and $g(f(x))$ do not overlap with each other, providing a clear view of how they interact with each other. Furthermore, it provides a different view from which to examine the relationship among functions, such as symmetry among parent functions. Still, the visual presentation of functions in different three-dimensional planes created cognitive challenges for some students. Some participants shared that they were not comfortable with the three-dimensional setting when exploring function concepts.

A conjecture can be made that if any dynamic graphical representation is in a three-dimensional format, it will share similar affordance with the one in this study, and possibly also share similar constraints especially when presenting concepts that are commonly presented in two dimensions.

**Research Contributions**

This study contributes to mathematics education research and practice in several ways. The results of this study can be connected with previous researchers' findings about using dynamic representations with function-related topics. The beneficial outcomes in this research, including improved reasoning accuracy, better composite function sketches, and more integrated, insightful, and dynamic reasoning performance, aligned with previous literature's viewpoint that: (a) dynamic representations with visualization can link function and graphs and influence the development of function concepts (Goldenberg et al., 1992; Kaput, 1989); (b) dynamic representations allow students to directly see variable changes and perceive invariant stability, eliminating the transformation constraints of static representations (Birgin & Acar, 2022; Ferrara et al., 2006; Kaput, 1989); and (c) the dynamic graphical representation create opportunities for students to identify function properties such as graph symmetry, and they enhance conceptual learning of inverse functions (Birgin & Acar, 2022). This study extends these findings by

demonstrating the potential and affordance of the dynamic graphical representation for improving students' understanding of function composition in terms of APOS theory. Students demonstrated more process-level understanding of function composition when using a dynamic graphical representation.

While students' understanding of functions and function composition has been extensively studied using APOS theory (Clark et al., 1997; Carlson et al., 2010; Engelke et al., 2005; Miller et al., 2015; Trigueros & Martínez-Planell, 2010; Wasserman, 2017), this study contributes by revealing the detailed progression of students' understanding when using a dynamic graphical representation with graphically depicted functions. Additionally, this research identified certain constraints of the dynamic graphical representation, supporting previous findings that managing interactive features may burden the learner with additional cognitive load (Jones, 2000; Schwan & Riempp, 2008). These constraints are addressed in the recommendations for further research and educators later in this chapter.

This study addresses a significant gap in mathematics education research by investigating students' graphical reasoning approaches to function composition. Previous research primarily treated graphs as simple representations providing discrete ordered pair values when examining students' understanding of function composition (Ayers et al., 1988; ; Engelke et al., 2005; Meel, 2003; Sfard, 1992). Chen et al. (2022) and LaPlace et al. (2023) advanced the approach by exploring students' intuitive responses to graphically depicted function composition tasks. However, there remains a dearth of research demonstrating what students can accomplish through graphical reasoning and how they approach function composition tasks through graphical lenses. In this study, graphs became the primary representations when students

couldn't easily access algebraic equations, creating opportunities for them to discover unconventional graphical reasoning strategies.

Furthermore, while many studies have analyzed students' understanding of function and function composition with multiple representations (Engelke et al., 2005; Miller et al., 2015), students' responses and reasoning performance were typically recorded as isolated approaches utilizing single representations. This study addresses the gap by showing students' complete and comprehensive reasoning processes about function composition. The findings revealed that in a complete reasoning sequence: (a) distinct types of reasoning (graphical, algebraic, and discrete point) interact and complement each other, for example the pattern of alternating between graphical and discrete reasoning; (b) all three reasoning types can be utilized across algebraic, tabular and graphical representations to successfully reason about function composition; (c) algebraic and discrete reasoning need graphical reasoning support to generate accurate composite function graphs; and (d) there is a potential for using graphical reasoning alone to understand function composition.

This research study also builds on the established consensus regarding students' procedural understanding of functions and function composition, particularly their overreliance on algebraic equation substitution, as identified by Engelke et al. (2005), Meel (2003), Miller et al. (2015) and Sfard (1992). Several participants in this study struggled when algebraic equations were unavailable, persistently seeking equations, which led to either ineffective reasoning or mistakes. This study revealed how critical properties such as restricted domain can be overlooked when fully depending on equation substitutions, echoing findings by Chen et al. (2022) and LaPlace et al. (2023). While this confirmed that such struggles and procedural preferences exist even among mathematically proficient students, our study went beyond this

consensus by revealing how students' reasoning develops when algebraic equations are unavailable.

This study advances the understanding of students' difficulties with function and function composition concepts, even among graduate students in mathematics education. The study explores and demonstrates how students struggle with function composition across multiple representations. By showing the advantages and limitations of point-wise approaches through the analysis and examples of the discrete point reasoning cases, this work extends Monk's (1992) finding that students' preference for point-wise questions can lead to blurred concepts. While the point-wise approach, which is the discrete point reasoning method described in this study, helped students investigate and verify the general shape, symmetry and restricted domain of the composite functions, random point selection without a global view led to inaccurate sketches and incorrect interpretation, demonstrating the limitations of relying solely on discrete point reasoning. This study also revealed a common misconception of function composition that had not been reported from the previous literatures: taking $(f(x), g(f(x)))$ as the coordinates of the composite function.

**Limitations**

While this study provided valuable insights into students' reasoning of function composition using a dynamic graphical representation, several limitations need to be considered. First, the sample populations were graduate mathematics education students with specialized mathematical background and dynamic representation experiences. Their reasoning patterns may provide the potential and possibilities students can reach, but they represent optimal rather than typical students' performance. This sample likely doesn't reflect the struggles general students might encounter.

Another limitation stems from the choice of a specific dynamic graphical representation for this study. Some of the findings and results are subject to the unique design of this particular representation which were discussed previously. While the researcher considers the representation used in this study to be a good example of dynamic graphical representations of function composition that may provide rich opportunities for students' reasoning, different dynamic graphical representations with varying affordances and constraints could yield different findings about students' reasoning and understanding. For example, the function composition tasks in this study were intentionally designed as functions that have single algebraic equations. However, other types of functions might not be suitable for this particular dynamic graphical representation. Additionally, since participants only had access to these representations during a single interview session, the long-term effect on learning and understanding remains unclear.

Furthermore, the think-aloud protocol and the follow-up conversations with the researcher during the interviews may have altered participants' natural reasoning processes. The study also faced challenges with interpreting non-verbal data and capturing all aspects of participants' reasoning. Moreover, the function composition tasks were designed by the researcher based on the judgments about which tasks would offer meaningful learning experiences with complex and diverse function scenarios. Different researchers might have designed other types and combinations of functions.

This study examined only the action, process, and object levels of APOS theory due to research design explained in the methods chapter. The absence of schema-level investigation leaves some questions unanswered. During interviews, several students showed signs of developing schema-level understanding of function composition, which merits further exploration. A representative case is discussed in the recommendations section.

## Recommendations

### *Recommendations for Future Research*

**Case-by-Case Reasoning Progression Trajectory.** This study mainly explored, across 15 cases, students' reasoning about function composition with and without a dynamic graphical representation. However, how individual students progressed distinctively remains an open question for investigation. For future hypotheses, observations about the reasoning progression trajectory of several meaningful cases that merits for some explanations are discussed here. In-depth case studies could be conducted using data from this study, especially the meaningful cases discussed with reasoning trajectories in the previous section.

***Dorothy and Helena: Their Similar Initial Misconceptions and Their Opposite Reasoning Trajectories.*** Dorthy and Helena were among several participants who held the most significant misconception exposed in this study about function composition: considering ordered pairs on the composite function graph as $(f(x), g\big(f(x)\big))$ instead of $(x, g(f(x)))$. Based on this fundamental misconception, in the first two rounds, which lacked a dynamic graphical representation, Dorothy struggled to develop a clear, in-depth and coherent understanding of function composition. She showed some awareness that the domain and range of the composite function are determined by the domain and range situations of $f(x)$ and $g(x)$. However, she was unable to reason about how that effect occurred. In another example, she had some ideas that the concavity of a function relates to the rate of change, but she did not know how to proceed from there to reason about the composite function's concavity. When presented with a dynamic graphical representation, her reasoning did not improve significantly. She demonstrated some dynamic interpretations from her observation, but those observations remained at the surface level due to her shaky and purely procedural understanding of function composition, made worse

by her misconception. She shared that she could not believe what the dynamic graphical representation presented, and she was confused. In comparison, Helena's reasoning processes in the first two rounds also clearly showed that she had the same misconception as Dorothy as well as an overreliance on the discrete point approach (see Table 21 and Figure 18). However, unlike Dorothy's responses, Helena seemed to derive more benefit from the dynamic graphical representation. She did not feel surprised about what the traced path showed regarding the composite function graph; and, more importantly, she was able to begin building some promising interpretations leading towards accurate reasoning about function composition, as mentioned in the illustrative example for research question three.

***Ella: Her Sound Concepts, Her Algebraic Struggles, and Her Dynamic Insights.*** The trajectory of Ella's reasoning performance across three rounds provides another meaningful case. When given static graphs in the first round, Ella displayed correct reasoning and understanding of function composition with three out of four problems solved accurately, especially with the typical "Blue – Green – Blue – Green" pattern alternating graphical and discrete point reasoning. However, in the second round when provided algebraic equations, Ella struggled, with three out of four function compositions incorrect. She mechanically substituted equations and ignored the restricted domain issue as did Jade (see Table 25). She selected random points to substitute into the composite function equation, which resulted in a flawed sketch that overlooked the restricted domain issue. When given algebraic equations, she seemed to suddenly forget to consider the properties of the functions in order to compose them, which she was able to accomplish in the first round, and simply substituted equations and points instead. However, her reasoning performance improved remarkably from the previous rounds when using the dynamic graphical representation, not only compared to the second round, but even to the first round. Her success

rate in the third round was 100%. She was one of the participants who dynamically interpreted the periodicity and restricted range: "The point of $g(f(x))$ bounced back and forth between 0 and 1 as the input point grows," as well as the discontinuity: "point of $g(f(x))$ jumped when the point of $f(x)$ jumped." Ella stated, "I think it (the dynamic graphical representation) provides me a really straightforward way to see what's (the) relationship between $f(x)$ and $g(x)$, for $g(f(x))$."

***Luna: Her Robust Conceptual Foundation and Her Unexpected Challenges with a Dynamic Graphical Representation.*** Luna's reasoning performance changes across three rounds represented another case that is worth exploration. With the "Blue – Green – Blue – Green" pattern in Round 1 when given static graphs and balanced approaches utilizing all three reasoning types in Round 2 with algebraic equations, Luna showed her strong conceptual understanding of function composition with 100% accuracy for both rounds. She did not show any overreliance on any one kind of reasoning type either. However, her reasoning performance declined unexpectedly in the third round when using the dynamic graphical representation, with three out of four responses being inaccurate. She mixed up the impact of $f(x)$ and $g(x)$ in the process of determining the restricted domain of the composite function, which she had clearly reasoned out from the previous rounds. She read the values of some critical points inaccurately whereas she had been able to read and successfully estimate the point locations with the static graph. She inaccurately determined the composite graph to be symmetric without carefully examining the properties of the functions. Luna shared that "Technology definitely make(s) people lazy." She expressed her preference for the traditional static graphs and equations because with those representations she could slow down and take her time to analyze function composition step by step. She also shared that she had difficulty reading exact point values on

the dynamic graphs, which made her feel insecure since inaccurate point reading was how she lost credit during tests when she was in high school.

***Gia: A Case Demonstrating the Schema Stage of Understanding of Function Composition.*** Although this study did not focus on the schema level of understanding in APOS theory, the cases exhibiting this level through reasoning type sequences merit future research. As an example, Gia achieved the highest accuracy of the group with 10 out of 12 correct reasoning processes and sketches. She also showed an object level of understanding of function composition, all through a graphical lens, in 7 steps among the reasoning processes, which is also the highest record. In general, Gia actively applied the interactions of actions, processes and objects levels of understanding as a dynamic and continuous system for function composition in dealing with new problem situations. Her reasoning sequence patterns presented in Appendix D showed the characteristics of having a schema stage of understanding because of: (a) the neat and consistent color (reasoning types) patterns in her reasoning sequences within each round, which showed her systematic structure for approaching function composition problems with different representations; (b) the dominating role of graphical reasoning throughout all rounds with no reliance on either algebraic equations or discrete points and (c) the flexibility of adjusting her reasoning strategy in each round and being coherent within the round with the new adapted strategy.

In the first round of sketching the composite function given static graphs, the patterns of "Blue – Green – Blue – Green" can be observed from Gia's reasoning sequences. Gia systematically approached the problems through the schema of: (a) examining the general shapes of graphs and identifying the domain and range of $f(x)$ and $g(x)$, considering any necessary restrictions to be applied to the domain and range of $g(f(x))$ (primarily exhibiting action and

process level of understanding); (b) plotting key discrete points on the $g(f(x))$ graph to verify the general trend of the composite graph (usually presenting action and process level of understanding); (c) reasoning about detailed features of the composite graph through a more global view on graphical properties such as concavity, continuity, symmetry and periodicity (displaying process and object level reasoning).

When given equations, the pattern of "Yellow – Blue – (Green – Blue – Green – Blue when necessary)" can be found in Gia's reasoning sequences. Gia de-encapsulated the processes she followed in the previous round and adjusted her schema based on the different representations provided. The schema in this round began with Gia substituting the equations to compose functions algebraically. Notably, Gia then sketched the graphs of $f(x)$ and $g(x)$ and visually reasoned about the domain and range restrictions of $g(f(x))$ like the first step in the first round. Subsequently, she returned to her previous schema and continued along that path.

In the third round with the dynamic graphical representation, the reasoning sequences demonstrated Gia's highly effective reasoning of function composition through purely graphical approaches in only a few steps. Of the total 11 reasoning steps in this round, 3 reflected action-level understanding while 7 indicated process-level, suggesting that dynamic graphical representations enabled Gia to reason about function composition coherently through graphical approaches. The representation saved her steps of confirming individual points of the composite function and supported her in reasoning dynamically through graphical features. For example, Gia explained the reason why in Round 3 Problem 1 inverse functions $f(x) = \ln(x)$ and $g(x) = e^x$ can be composed to a straight line, stating:

> So I think the acceleration of $f(x)$ is increasing, it's going to be negative, but the acceleration of $g(x)$ increasing is going to be positive. So that's how they cancel. They

cancel out. The acceleration is…they are same…same absolute value but in different sign.

**Other Future Research Possibilities.** Besides in-depth exploration of individual cases, the contributions and limitations of this study also suggest some other possibilities for further research. First, using the existing data from this study, researchers could expand the investigation of APOS understanding levels by analyzing how different types of reasoning relate to various APOS levels of understanding. Such analysis could deepen the understanding of the relationship between distinct reasoning types and the progression of APOS levels, thereby building up the knowledge about how different reasoning types are affected by the understanding of functions and function composition.

Future research should also explore students' reasoning about function composition using dynamic graphical representations with a broader population. For example, research could recruit undergraduate students from introductory mathematics courses such as pre-calculus, representing more diverse academic backgrounds. Such studies could provide more generalizable data about students' reasoning approaches and understanding levels. Researchers could also extend the study duration to investigate the long-term impact of dynamic graphical representations by having students use these representations throughout the semester. With possible larger sample sizes, researchers could employ mixed methods to examine whether there are statistically significant changes in students' reasoning and understanding.

Research could inspect the effect of various dynamic graphical representations on students' reasoning about function composition. Following this study's structure and methodology, further studies could evaluate the common affordance and constraints of these

representations, as well as examine how specific features of particular representations correlate with students' reasoning performance.

### *Recommendations for Educators*

Findings from this study provide valuable insights for mathematics educators teaching function composition and related concepts. This research showed that graphical reasoning is both efficient and critical for developing students' deep understanding of function and function composition. The findings also showed that discrete point reasoning can be useful when students thoughtfully select their reference points. To move students beyond procedural thinking of simply substituting equations and substituting numbers, classroom teachers, curriculum designers, and textbook writers need to ensure students are exposed to and familiar with multiple representations of function and function composition from the very beginning of their learning experience with function concepts.

Teachers should encourage their students to reason about functions through different representations and diverse reasoning approaches. When introducing new concepts, both lectures and examples should showcase various representations and reasoning styles. Modeling good reasoning sequences in class offers students concrete examples of what thorough reasoning looks like under the topic of functions. Assessments like assignments and tests should also reflect this emphasis on multiple representations and conceptual understanding rather than returning to just procedural substitution and computation. For example, teachers could ask students to explain the domain of the composite function and its reasoning, then have students predict how functions will behave when composed graphically. This approach helps to prevent the kinds of fundamental misconceptions some participants demonstrated in this study.

Some students in this study found the three-dimensional features and the complexity of dynamic representation overwhelming at first, so introducing these tools thoughtfully can make a big differences. Instead of using them just for assessment, educators should incorporate these dynamic graphical representations into group projects where students can explore together. Instructors should have students investigate how parent functions and composite functions behave, and teachers should guide them to look for patterns in how the composite functions relate to their parent functions graphically. The purpose of the project should not be finding "right answers," but rather asking "What can you conclude from your exploration of function composition?" Educators should keep these dynamic representations available throughout the unit so students can get comfortable using them for increasingly challenging problems.

Teachers also need robust professional development to successfully incorporate dynamic graphical representations in their classroom. During the training, teachers should collaborate with each other to strengthen their own understanding and explore the potential of these representations in groups. Furthermore, having teachers design activities using the tool and predict students' misconceptions would also enhance their preparation.

**Closing**

In this study, students' reasoning patterns and performance proved the promising opportunities that a dynamic graphical representation can offer to enhance students' understanding of function composition. By allowing students to visualize and interact with the covariational relationships among functions, the dynamic graphical representation promoted more efficient, integrated, dynamic, and conceptually rich reasoning approaches, thereby reducing reliance on procedural algebraic manipulation and point-wise view on specific values. This study demonstrates the possibility of reasoning about function composition tasks through

primarily graphical lenses when facilitated by a dynamic graphical representation; it presents both students' successful reasoning patterns involving diverse reasoning types as well as unsuccessful reasoning attempts showing limitations of algebraic and discrete point reasoning strategies. With continued research and refinement of teaching approaches in classrooms, these dynamic graphical representations may significantly enhance how students learn and reason about function composition, pointing toward promising directions for studying and implementing more dynamic representations of other topics in mathematics education.

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## Appendix A

## Survey

1. The gender you identify as is_____;
2. The race/ethnicity you identify as is _____;
3. Your undergraduate major is

   _____;
4. How many numbers of undergraduate level math courses have you taken?

   ☐ Less than 5 ☐ 5 to 10 ☐ 10 to 15 ☐ More than 15
5. How many numbers of graduate level math courses have you taken?

   ☐ Less than 5 ☐ 5 to 10 ☐ 10 to 15 ☐ More than 15
6. The highest math course(s) you have taken is/are _____;

   (if many, list one or two as examples).
7. a) Do you agree with the following statement: I understand the concept of function composition $g(f(x))$:

   ☐ Strongly agree ☐ Agree ☐ Somewhat agree

   ☐ Somewhat disagree ☐ Disagree ☐ Strongly disagree

   b) What is the equation of the composite function, $(g \circ f)(x) = g(f(x))$, if $f(x) = 3x$ and $g(x) = 2x^2$?

   _____
8. Do you agree with the following statement: I have experiences in using dynamic graphical representations such as Geogebra:

   ☐ Strongly agree ☐ Agree ☐ Somewhat agree

   ☐ Somewhat disagree ☐ Disagree ☐ Strongly disagree

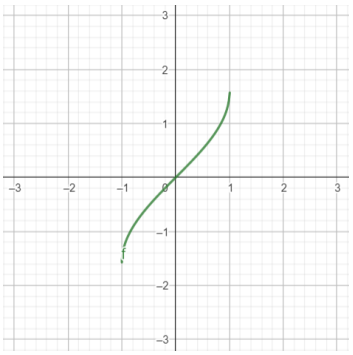
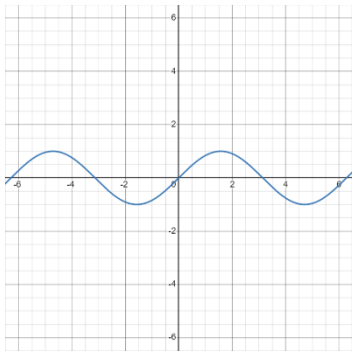
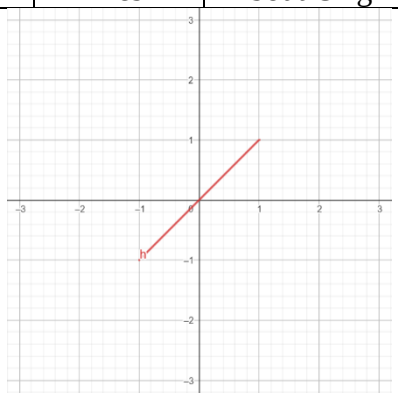
## Appendix B

## Function Composition Problems

(Green graph as $f(x)$. Blue graph as $g(x)$. Red graph as $g(f(x))$.)

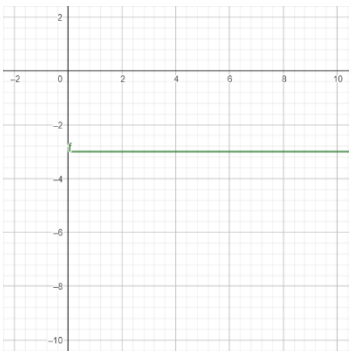
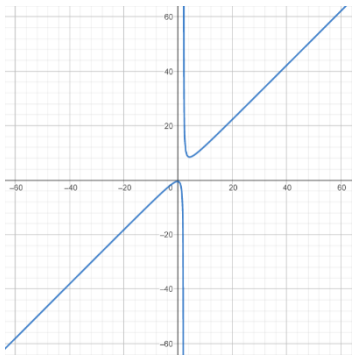
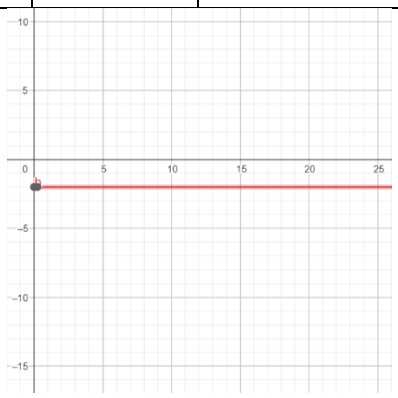
1. Round 1 Problem 1 (in graph form).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = \arcsin(x)$ | $[-1, 1]$ | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ | Yes | About Origin |
| $g(x) = \sin(x)$ | $\mathbb{R}$ | $[-1, 1]$ | Yes | About Origin |
| $g(f(x)) = x$ | $[-1, 1]$ | $[-1, 1]$ | Yes | About Origin |

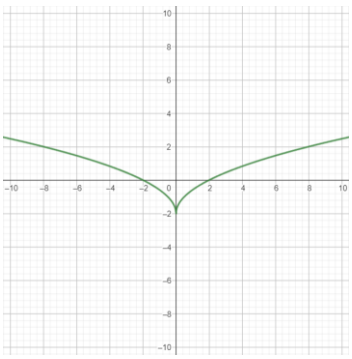
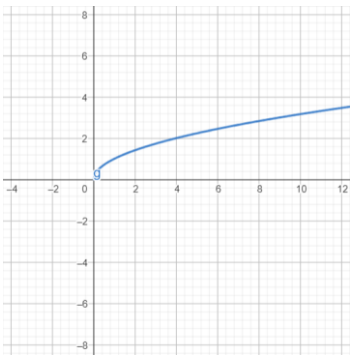
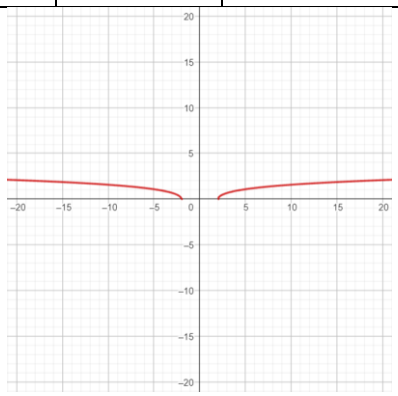
2. Round 1 Problem 2 (in graph form) and Round 2 Problem 2 (in equation form).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = -3$ | $[0, \infty)$ | $-3$ | Yes | No |
| $g(x) = \frac{x^2 + 1}{x - 2}$ | $(-\infty, 2)$ $\cup$ $(2, \infty)$ | $(-\infty, -2\sqrt{5} + 4)$ $\cup$ $(2\sqrt{5} + 4, \infty)$ | No | About the point $(2, 4)$ |
| $g(f(x))$ $= \frac{(-3)^2 + 1}{-3 - 2}$ $= -2$ | $[0, \infty)$ | $-2$ | Yes | No |

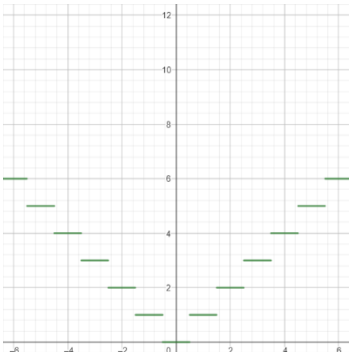
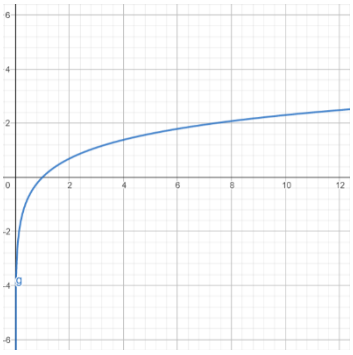
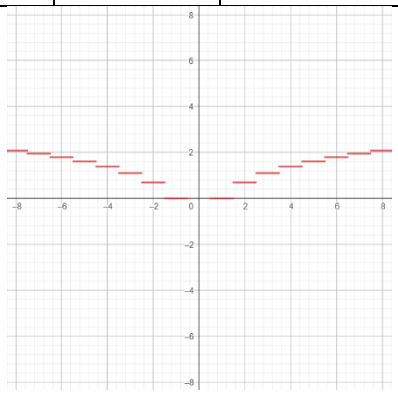
3. Round 1 Problem 3 (in graph form).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = \sqrt{2\|x\|} - 2$ | $\mathbb{R}$ | $[-2, \infty)$ | Yes | About y-axis |
| $g(x) = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ | Yes | No |
| $g(f(x)) = \sqrt{\sqrt{2\|x\|} - 2}$ | $(-\infty, -2]$ $\cup [2, \infty)$ | $[0, \infty)$ | No | About y-axis |

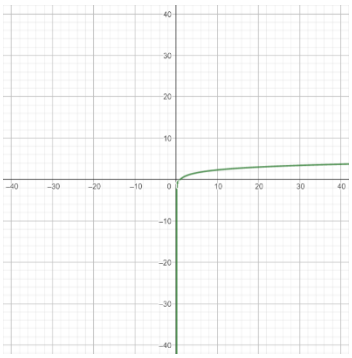
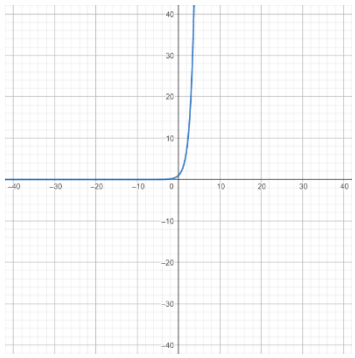
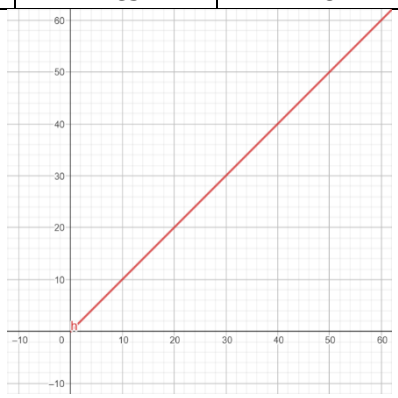
4. Round 1 Problem 4 (in graph form) and Round 2 Problem 4 (in equation form).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x)$ $= \|round(x)\|$ | $\mathbb{R}$ | $k, (k \in \mathbb{Z}, k \geq 0)$ | No | About y-axis |
| $g(x) = \ln(x)$ | $(0, \infty)$ | $\mathbb{R}$ | Yes | No |
| $g(f(x))$ $= \ln \|round(x)\|$ | $(-\infty, -\frac{1}{2}]$ $\cup [\frac{1}{2}, \infty)$ | $\ln(k), (k \in \mathbb{Z}, k \geq 0)$ | No | About y-axis |

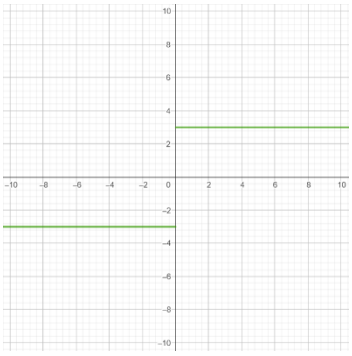
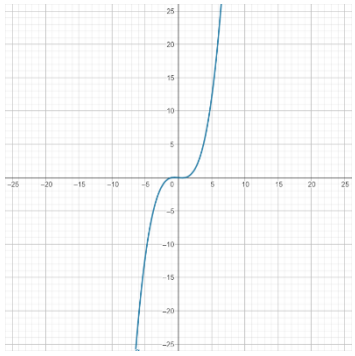
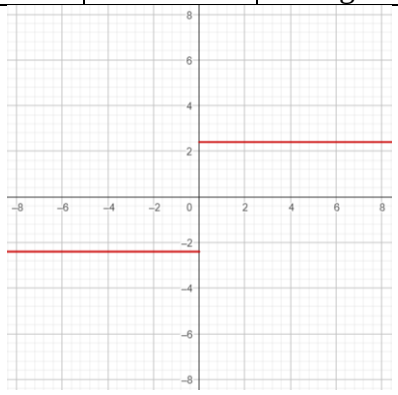
5. Round 3 Problem 1 (in graph form) and Round 2 Problem 1 (in equation form). (Comparable to Round 1 Problem 1).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = \ln(x)$ | $(0, \infty)$ | $\mathbb{R}$ | Yes | No |
| $g(x) = e^x$ | $\mathbb{R}$ | $(0, \infty)$ | Yes | No |
| $g(f(x)) = x$ | $(0, \infty)$ | $(0, \infty)$ | Yes | No |

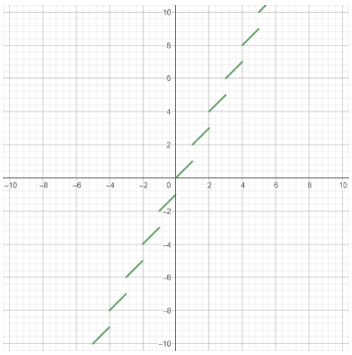
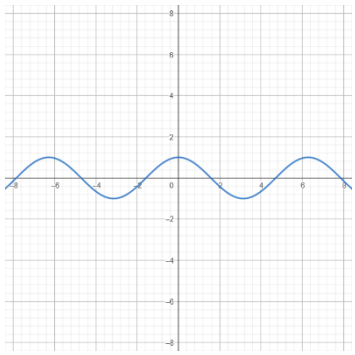
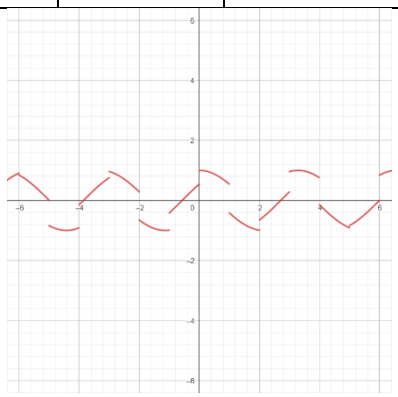
6. Round 3 Problem 2 (Comparable to Round 1 Problem 2).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = \begin{cases} -3: x \leq 0 \\ 3: x > 0 \end{cases}$ | $\mathbb{R}$ | $-3$ *and* $3$ | No | About Origin |
| $g(x) = 0.1x(x+1)(x-1)$ | $\mathbb{R}$ | $\mathbb{R}$ | Yes | About Origin |
| $g(f(x)) = \begin{cases} -2.4: x \leq 0 \\ 2.4: x > 0 \end{cases}$ | $\mathbb{R}$ | $-2.4$ and $2.4$ | No | About Origin |

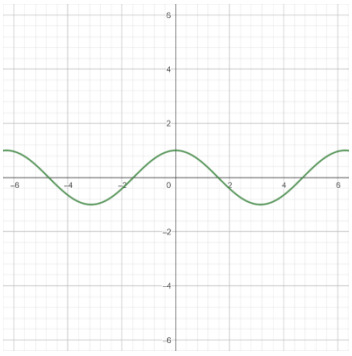
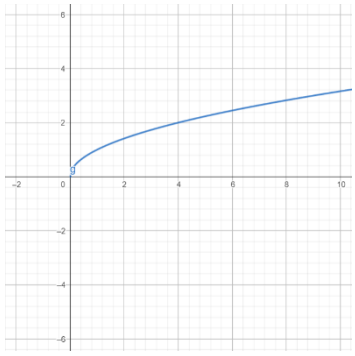
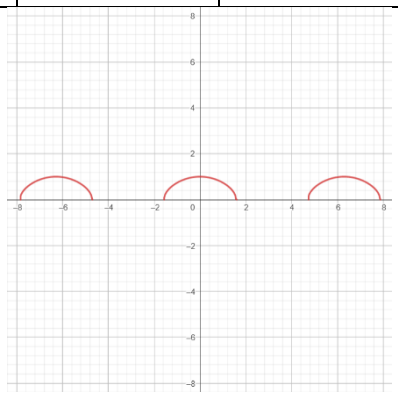
7. Round 3 Problem 3 (Comparable to Round 1 Problem 4).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = x + \lfloor x \rfloor$ | $\mathbb{R}$ | $\ldots \cup [-4,-3] \cup [-2,-1] \cup [0,1] \cup [2,3] \ldots$ | No | No |
| $g(x) = cos\ (x)$ | $\mathbb{R}$ | $[-1,1]$ | Yes | About y-axis |
| $g(f(x)) = \cos\ (x + \lfloor x \rfloor)$ | $\mathbb{R}$ | $[-1,1]$ | No | No |

8. Round 3 Problem 4 (in graph form) and Round 2 Problem 3 (in equation form). (Comparable to Round 1 Problem 3).

| | Domain | Range | Continuity | Symmetry |
|---|---|---|---|---|
| $f(x) = \cos(x)$ | $\mathbb{R}$ | $[-1,1]$ | Yes | About y-axis |
| $g(x) = \sqrt{x}$ | $[0,\infty)$ | $[0,\infty)$ | Yes | No |
| $g(f(x)) = \sqrt{\cos(x)}$ | $\left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right], k \in \mathbb{Z}$ | $[0,1]$ | No | About y-axis |

## Appendix C

## Interview Protocol

### Instructions

*Good morning (afternoon). Thank you for coming. This interview involves three rounds. The first two rounds will be completed with pencil and paper, and in the third round you will have access to a dynamic graphical representations of function composition. A tutorial about dynamic graphical representations will be provided. In each round, you will solve some problems about function composition and be interviewed with open-ended questions about your reasoning through the problem-solving process.*

### Recording Instructions

*If it is okay with you, I will record our conversations. The purpose of this is to make sure all the details could be documented but at the same time I am able to carry on an attentive conversation with you. I assure you that all the responses will be reported with pseudonyms.*

### Round One Interview Questions

1. Tell me about how did you come up with these composition graphs?
2. Could you share more about how the sketches of $g(f(x))$ combines the graphs of $f(x)$ and $g(x)$?
3. Do you see any relation among the three graphs through the problems?
4. Do you see any relation among the domain and range of the three graphs through the problems?
5. Do you see any relation among the three graphs in terms of symmetry or continuities through the problems?

6. Tell me more about your reasoning and thinking process if you haven't had a chance to share about it in the previous questions.
7. Are these tasks challenging to you? If yes, why do you think so? If no, what in your opinion contribute to solving this problem successfully?

**Round Two Interview Questions**

*During this round, participants are welcome to try to graph the functions anytime they request it.*

1. Tell me about how did you come up with these composition equations?
2. What are the domain and range of $f(x)$ and $g(x)$? What are the domain and range of $g(f(x))$? How do the domain and range of $f(x)$ and $g(x)$ affect the domain and range of $g(f(x))$?
3. Are $f(x)$ or $g(x)$ symmetric? How does the symmetry of $f(x)$ and/or $g(x)$ affect the symmetry of $g(f(x))$?
4. Are $f(x)$ or $g(x)$ continuous? How does the continuity of $f(x)$ and/or $g(x)$ affect the continuity of $g(f(x))$?
5. Can you try to sketch $g(f(x))$ based on the equation of the composite functions you have?
6. Tell me about how did you come up with these composition graphs?
7. Tell me more about your reasoning and thinking process if you haven't had a chance to share about it in the previous questions.
8. Are tasks challenging to you? If yes, why do you think so? If no, what in your opinion contribute to solving this problem successfully?

**Dynamic Graphical Representation Tutorial**

*This is dynamic graphical representations we will use in the next round. Before that, let us explore this applet and getting you familiar with this tool. You could input the $f(x)$ and $g(x)$ of your choice here. The graph of $f(x)$ will be displayed on the $xy-plane$, the graph of $g(x)$ will be displayed on the $yz-plane$, and the graph of the composition $g(f(x))$ will be displayed on the $xz-plane$. Here you could drag the point along the $x-axis$, while the corresponding value of $f(x)$ will be pointed on the $y-axis$ by an arrow simultaneously, which is also the input of the $g(f(x))$. Meanwhile, the traced path of the value of $g(f(x))$ will be shown here on the $z-axis$ along with the movement of the point on the $x-axis$. The entire model of the three-dimensional Cartesian Coordinate System can be enlarged, turned and viewed from different perspectives. We could choose to show or hide the graph of $f(x)$, $g(x)$, and $g(f(x))$ at our own will by checking the boxes here. Feel free to try to input the $f(x)$ and $g(x)$ you want and drag along to observe.*

**Round Three Interview Questions**

1. Tell me about how did you come up with these composition graphs?
2. How did you connect graphs of $f(x)$ and $g(x)$ to graphs of $g(f(x))$?
3. What aspects of $f(x)$ and $g(x)$ impact the domain and range of the composite function through the problems?
4. What aspects of $f(x)$ and $g(x)$ impact the symmetry of the composite function through the problems?
5. What aspects of $f(x)$ and $g(x)$ impact the continuity of the composite function through the problems?
6. In your opinion, what information did dynamic graphical representations provided in terms of function composition?

7. Tell me more about your reasoning and thinking process if you haven't had a chance to share about it in the previous questions.
8. Are these tasks challenging to you? If yes, why do you think so? If no, what in your opinion contribute to solving this problem successfully?

# Appendix D

## Reasoning Type Sequences

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | O | A | A | A | P | A |
| R1P2 | A | A | O | A | P | P | | | |
| R1P3 | A | P | A | P | | | | | |
| R1P4 | A | P | P | P | A | P | O | P | P |
| | | | | | | | | | |
| R2P1 | A | A | | | | | | | |
| R2P2 | A | A | | | | | | | |
| R2P3 | A | P | A | P | | | | | |
| R2P4 | A | A | P | P | P | P | A | P | O |
| | | | | | | | | | |
| R3P1 | P | A | A | P | | | | | |
| R3P2 | A | P | A | | | | | | |
| R3P3 | QUIT | | | | | | | | |
| R3P4 | CONFIRMED | | | | | | | | |

Anna

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | O | A | A | A | A | P | | | | | | |
| R1P2 | P | A | A | A | A | A | P | A | | | | | | | |
| R1P3 | A | A | A | A | A | A | A | O | A | A | P | A | P | A | P |
| R1P4 | A | A | A | A | P | P | P | O | P | P | P | P | | | |
| | | | | | | | | | | | | | | | |
| R2P1 | A | A | A | P | | | | | | | | | | | |
| R2P2 | P | P | A | P | | | | | | | | | | | |
| R2P3 | A | A | P | A | A | P | A | A | P | P | | | | | |
| R2P4 | A | A | A | A | P | P | P | P | P | | | | | | |
| | | | | | | | | | | | | | | | |
| R3P1 | P | A | A | A | A | P | P | O | | | | | | | |
| R3P2 | P | P | | | | | | | | | | | | | |
| R3P3 | A | A | P | P | P | | | | | | | | | | |
| R3P4 | A | P | P | | | | | | | | | | | | |

Bree

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | A | P | | | | | |
| R1P2 | A | A | A | P | | | | | | | |
| R1P3 | A | A | A | A | O | A | A | A | P | A | A |
| R1P4 | A | P | P | | | | | | | | |
| | | | | | | | | | | | |
| R2P1 | A | | | | | | | | | | |
| R2P2 | A | P | | | | | | | | | |
| R2P3 | A | A | A | A | P | P | | | | | |
| R2P4 | A | A | P | | | | | | | | |
| | | | | | | | | | | | |
| R3P1 | A | P | A | A | A | A | P | | | | |
| R3P2 | A | P | P | | | | | | | | |
| R3P3 | A | A | A | A | P | | | | | | |
| R3P4 | P | confirm | | | | | | | | | |

Caden

| | | | | |
|---|---|---|---|---|
| R1P1 | P | A | P | P |
| R1P2 | A | P | P | A |
| R1P3 | P | A | P | |
| R1P4 | A | P | P | P |

| | | | | | |
|---|---|---|---|---|---|
| R2P1 | A | A | A | P | |
| R2P2 | P | A | A | A | |
| R2P3 | A | P | P | | |
| R2P4 | P | A | P | P | P |

| | | | | | |
|---|---|---|---|---|---|
| R3P1 | P | A | A | A | A |
| R3P2 | A | P | A | | |
| R3P3 | P | A | Quit | | |
| R3P4 | A | A | Quit | | |

Dorothy

| | | | | | | |
|---|---|---|---|---|---|---|
| R1P1 | A | P | A | P | P | |
| R1P2 | A | A | P | P | P | |
| R1P3 | A | P | A | P | | |
| R1P4 | A | P | P | P | P | |

| | | | | | | |
|---|---|---|---|---|---|---|
| R2P1 | A | | | | | |
| R2P2 | A | A | P | | | |
| R2P3 | A | A | | | | |
| R2P4 | A | P | P | A | P | P |

| | | | | | |
|---|---|---|---|---|---|
| R3P1 | P | P | A | P | |
| R3P2 | A | P | | | |
| R3P3 | A | P | A | P | P |
| R3P4 | P | P | P | | |

Ella

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | A | P | A | A | A | P | P | P |
| R1P2 | P | A | A | A | A | A | A | | | | | |
| R1P3 | A | A | A | A | A | A | A | P | A | P | P | |
| R1P4 | A | A | P | A | A | P | | | | | | |

| | | | | |
|---|---|---|---|---|
| R2P1 | A | A | A | A |
| R2P2 | A | P | A | A |
| R2P3 | A | A | A | P |
| R2P4 | A | A | A | P |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| R3P1 | P | A | P | A | P | P | A | A | P | A |
| R3P2 | P | A | P | | | | | | | |
| R3P3 | A | A | A | A | P | A | A | A | | |
| R3P4 | A | A | P | P | | | | | | |

Felix

| | | | | | | |
|---|---|---|---|---|---|---|
| R1P1 | P | P | A | A | P | |
| R1P2 | A | P | P | P | P | |
| R1P3 | A | A | P | A | P | O |
| R1P4 | A | A | P | P | P | O |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R2P1 | A | A | A | P | | | | | |
| R2P2 | P | P | | | | | | | |
| R2P3 | A | A | P | P | P | A | P | A | O |
| R2P4 | A | A | O | P | | | | | |

| | | | |
|---|---|---|---|
| R3P1 | O | O | P |
| R3P2 | P | P | |
| R3P3 | P | P | O |
| R3P4 | P | P | A |

Gia

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | A | A | P |
| R1P2 | A | A | A | P | A | | |
| R1P3 | A | A | P | | | | |
| R1P4 | A | P | P | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R2P1 | A | P | P | P | A | A | A |
| R2P2 | A | A | A | A | P | P | |
| R2P3 | A | A | A | A | | | |
| R2P4 | A | A | O | A | A | P | P |

| | | | | |
|---|---|---|---|---|
| R3P1 | P | A | P | P |
| R3P2 | P | A | | |
| R3P3 | A | P | P | |
| R3P4 | P | O | | |

Helena

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R1P1 | A | P | O | A | P | A | P |
| R1P2 | P | P | | | | | |
| R1P3 | P | A | P | O | P | | |
| R1P4 | P | P | P | P | P | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| R2P1 | A | P | | | | |
| R2P2 | P | P | P | | | |
| R2P3 | A | A | P | P | P | O |
| R2P4 | P | P | P | P | | |

| | | | | |
|---|---|---|---|---|
| R3P1 | A | A | P | |
| R3P2 | P | P | | |
| R3P3 | P | P | | |
| R3P4 | A | P | O | |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | P | P | P | A | P | P | P |
| R1P2 | A | A | P | P | | | | | | | |
| R1P3 | A | P | P | P | | | | | | | |
| R1P4 | A | P | P | P | P | P | P | P | | | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| R2P1 | A | A | A | | | | | | | |
| R2P2 | P | A | P | | | | | | | |
| R2P3 | A | A | A | A | P | P | P | A | P | P |
| R2P4 | P | P | A | P | P | | | | | |

| | | | | | |
|---|---|---|---|---|---|
| R3P1 | P | P | P | A | |
| R3P2 | O | A | | | |
| R3P3 | A | P | P | P | P |
| R3P4 | P | A | P | P | A | P | A |

Ivy

Jade

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | A | A | A | A | A | P | P |
| R1P2 | A | P | P | | | | | | | | |
| R1P3 | A | A | A | A | O | P | P | A | P | A | A | P |
| R1P4 | A | A | A | A | A | P | A | A | P | P | A | |

| | | | | | |
|---|---|---|---|---|---|
| R2P1 | P | A | | | |
| R2P2 | P | P | | | |
| R2P3 | A | A | P | A | P |
| R2P4 | P | P | P | P | |

| | | | | | |
|---|---|---|---|---|---|
| R3P1 | A | P | A | | |
| R3P2 | A | P | | | |
| R3P3 | A | P | P | A | P |
| R3P4 | A | P | P | A | |

Kylie

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | A | A | A | A | P | A | P | P |
| R1P2 | P | P | P | A | | | | | | |
| R1P3 | P | P | P | | | | | | | |
| R1P4 | P | P | P | P | P | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R2P1 | P | A | P | A | | | |
| R2P2 | P | P | P | | | | |
| R2P3 | A | P | P | A | A | A | |
| R2P4 | A | P | A | P | P | P | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R3P1 | A | P | | | | | | | |
| R3P2 | P | P | | | | | | | |
| R3P3 | A | A | P | P | P | A | A | A | P |
| R3P4 | P | P | P | P | | | | | |

Luna

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | P | P | A | A | A | A | P |
| R1P2 | A | P | A | A | A | | | | |
| R1P3 | A | P | P | A | P | | | | |
| R1P4 | P | A | P | P | P | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| R2P1 | A | A | A | A | A | |
| R2P2 | P | P | | | | |
| R2P3 | A | P | P | A | P | P |
| R2P4 | A | P | P | A | | |

| | | | | | |
|---|---|---|---|---|---|
| R3P1 | A | A | A | | |
| R3P2 | P | A | | | |
| R3P3 | P | P | A | A | P |
| R3P4 | P | P | P | | |

Morgan

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| R1P1 | A | A | P | O | A | A | P | | |
| R1P2 | P | P | P | | | | | | |
| R1P3 | A | A | A | O | P | A | P | A | P |
| R1P4 | A | A | A | P | P | P | P | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R2P1 | A | P | | | | | |
| R2P2 | P | A | P | | | | |
| R2P3 | A | A | A | P | P | P | P |
| R2P4 | A | A | P | P | P | A | P |

| | | | |
|---|---|---|---|
| R3P1 | P | A | A |
| R3P2 | A | P | |
| R3P3 | A | P | |
| R3P4 | A | P | P |

Nicole

| | | | | | |
|---|---|---|---|---|---|
| R1P1 | A | A | A | | |
| R1P2 | P | A | A | P | |
| R1P3 | A | A | | | |
| R1P4 | A | A | | | |

| | | | | | |
|---|---|---|---|---|---|
| R2P1 | A | A | A | A | P |
| R2P2 | P | A | | | |
| R2P3 | A | A | A | P | |
| R2P4 | A | P | P | | |

| | | | | |
|---|---|---|---|---|
| R3P1 | A | A | A | A |
| R3P2 | A | A | | |
| R3P3 | A | A | P | P |
| R3P4 | A | P | P | |

Olivia