LMN maps in Equivariant K-theory and Applications

Davis Michael Lazowski

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Abstract

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The goal of this thesis is to explain various Weyl/braid group actions on categories and characters associated to finite type quantum affine algebras geometrically.

Specifically, we define an interesting action of a braid group on equivariant K-theory of a finite type ADE Nakajima variety, which we prove is related to the classical Lusztig and Chari braid group actions.

We then combine this braid group action with Henry Liu's study of asymptotic modules of quantum affine algebras [1].

As a result, we get a geometric perspective on various formulas and conjectures about TQ relations obtained in the extremely interesting recent works of Frenkel, Hernandez and Wang [2], [3], [4]. All results of this work involving asymptotic algebras rely on conjectural properties of the critical K-theory of Nakajima varieties. We explain the assumptions further in the main text.

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Acknowledgements

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Chapter 1: Introduction

1.1 ADE quiver varieties at different stability conditions

Given a quiver $\vec{Q}$, there is an associated Nakajima variety $\mathcal{M}_{\theta,\vec{Q}}(w) := \sqcup_v \mathcal{M}_{\theta,\vec{Q}}(v,w)$. This is a symplectic variety which is contructed as the semistable locus, with respect to a stability condition θ, of a certain symplectic quotient stack $[\mu^{-1}(0)/G]$. Its localised equivariant K-theory $K_T\mathcal{M}_\theta(w)_{loc}$ is naturally a module for a quantum affine group via the construction of certain wrong-way maps called *stable envelopes*[5], [6], [7].

If θ, η are 'general' stability conditions, meaning not on a wall in the space of stability conditions, one may wonder about the relation between $\mathcal{M}_\theta(w)$ and $\mathcal{M}_\eta(w)$. If $\vec{Q}$ is a type ADE quiver, there are two different interesting relations.

1. There is a Lagrangian Steinberg correspondence between $\mathcal{M}_\theta(v,w)$ and $\mathcal{M}_\eta(v,w)$, induced by the explicit isomorphism $\mathcal{M}_{\theta,t\zeta}(v,w) \simeq \mathcal{M}_{\eta,t\zeta}(v,w)$ for general ζ and any $t \neq 0$.

2. There is a quiver Weyl group, W_Q which acts on type ADE quivers and their associated data. For an ADE quiver, it is more or less the Weyl group of the attached root system. Then, there is a so-called Lusztig-Maffei-Nakajima (LMN) symplectomorphism:

$$\Phi_{LMN}(s_i) : \mathcal{M}_\theta(v,w) \to \mathcal{M}_{s_i\theta}(s_i v, w)$$

These maps provide two different interesting isomorphisms of equivariant K-theory

$$K_T\mathcal{M}_\theta(w) \simeq K_T\mathcal{M}_{s_i\theta}(w)$$

and also, therefore, an interesting automorphism

$$K_T\mathcal{M}_\theta(w) \xrightarrow{\Phi^\star_{LMN}} K_T\mathcal{M}_{s_i\theta}(w) \xrightarrow{Steinberg} K_T\mathcal{M}_\theta(w)$$

The key technical tool of this thesis is the study of how these maps interact with geometric representation theory.

1.1.1 Relations to geometric representation theory

We show that the LMN maps are twisted equivariant and commute with stable envelopes. Therefore we show

Theorem 1.1. Let θ be a generic stability condition. Then the operators $\Phi_{LMN}(\sigma)^\star$, for $\sigma \in W_{\vec{Q}}$ the quiver Weyl group, form a representation of the Weyl group on

$$\oplus_{\sigma\in W} K_T\mathcal{M}_{\sigma\theta}(w)$$

We also study the braid group action of the composite automorphisms $\Phi^\star_{s_i} \in Aut(K_T\mathcal{M}_\theta(w))_{loc}$. We show

Theorem 1.2. The composite automorphism

$$\Phi^\star_{s_i} \in End\, K_T(\mathcal{M}_\theta(w))_{loc}$$

induces Chari's braid group action on q-characters.

We also show, in type A_1, that our braid group action agrees with Lusztig's.

1.2 Potential applications to the study of TQ relations

Henry Liu's thesis [1] uses finitary quasimaps to associates to a Nakajima variety $\mathcal{M}_\theta(v, w)$ an associated Hernandez-Jimbo category $\mathcal{O}_{HJ}$ for a specific stability condition θ_-.

The category $\mathcal{O}_{HJ}$ depends on the choice of a subalgebra of our quantum group $\mathcal{B}$ called a *quantum affine Borel* and a triangular decomposition. Roughly,

$\mathcal{O}_{HJ} =$ category of representations of $\mathcal{B}$ which are 'nice' w.r.t a triangular decomposition

Stable envelopes, hence the representations of quantum groups we construct, depend on the stability parameter θ alongside a choice of chamber C (either + or −), a polarisation $T^{1/2}$ and a slope $s \in Pic_{eq}(\mathcal{M}_\theta(w))$.

The choice of $\mathcal{B}$ depends on the choice of chamber $\mathfrak{C}$ and the slope, whilst the triangular decomposition depends on the stability parameter θ.

One important point about $\mathcal{O}_{HJ}$ is that it is the simplest category that contains representations whose q-characters correspond to the so-called T, Q operators. Again very roughly,

1. $T_i(u) =$ generalised eigenvalues of the simplest finite dimensional representations in $K_0\mathcal{O}_{HJ}$;
2. $Q_j(u) =$ generalised eigenvalues of the simplest highest weight representations in $K_0\mathcal{O}_{HJ}$;

These operators were first defined in the pioneering work Hans Bethe [8] in the study of quantum integrable systems. They satisfy important relations, which one may view as relations between tensor products of various representations in $\mathcal{O}_{HJ}$.

Imprecise Definition 1. A *TQ-relation* is a relation of the schematic form

$$Q_j(u)T_i(u) = \sum_k c_k Q_k(\hbar^{I_k}u)$$

A *QQ-relation*[1] is a relation of the schematic form

$$c_1\tilde{Q}_i(u\hbar)Q_j(u) - c_2\tilde{Q}_i(u)Q_j(u\hbar) = \prod_k Q_k(\hbar^{I_k}u)$$

[1] In the main text, we say nothing more about QQ-relations, only introducing them here to further explain how the W-action is crucial to a full understanding of the properties of Q-operators, and to note a possible relation to [9] here.

Here, the operators $\tilde{Q}$ are operators closely related to Q. It was the insight of Frenkel, Hernandez and Wang [2], [4] that these operators $\tilde{Q}$ are best thought of as

$$\tilde{Q}_j(u) = \text{ class of simplest highest weight representations in } K_0\mathcal{O}^w_{HJ}$$

taken under a certain isomorphism $K_0\mathcal{O}^w_{HJ} \to K_0\mathcal{O}_{HJ}$ [2],

wherein

$\mathcal{O}^w_{HJ} =$ category of representations of $\mathcal{B}$ which are 'nice' w.r.t a **different** triangular decomposition

We prove the equivalence of a subcategory of $\mathcal{O}^w_{HJ}$ and a subcategory of $\mathcal{O}_{HJ}$ in the main text, see 29 4.6, and provide further connections.

For example, one of the most basic properties of the categores $\mathcal{O}^w_{HJ}$ is the existence of a Weyl group action on

$$\oplus_{w\in W} K_0\mathcal{O}^w_{HJ}$$

which has the property that it leaves the classes of finite dimensional representations invariant. This would follow immediately from extending the Weyl group action we define (Theorem 2.1) to all of $\mathcal{O}^w_{HJ}$.

Also assuming some conjectures on the asymptotic construction of representations of quantum affine algebras, we prove the extended TQ relation conjecture of [2] in the generality of finite type ADE.

1.3 Motivations for the work and future directions

It is a longstanding conjecture of Nekrasov and Shatashvili [10], [11] that Q-operators are related to quantum multiplication by tautological bundles. Previous work [12] has proven this conjecture in finite type A_1.

The original motivation of this work was to give a full geometric understanding of the representation-

theoretic approach to Q operators, with the hope that this would allow a more systematic approach to the Nekrasov-Shatashvili conjecture. This aim is very far from being achieved, but we hope that our work will be a step in the right direction.

A separate motivation is to better understand the dependence of a quantum affine algebra on its presentation. Much recent work has given a clearer understanding of the slope dependence [5], [13]. We hope adding an understanding of the braid group action will help futher clarify this issue.

Outside of type ADE, the LMN and Steinberg maps continue to exist, and work of Yaochen Wu [14] provides an anlogue of the Weyl group. Therefore the abstract framework underlying this thesis mostly extends. However for practical computations of the action on R-matrices and quantum groups we rely on a technical fact about type ADE quiver varieties (see 1). It is unclear to the author how to avoid this.

Chapter 2: Introduction

The goal of this thesis is to explain various Weyl/braid group actions on categories and characters associated to finite type quantum affine algebras geometrically.

Specifically, we define an interesting action of a braid group on equivariant K-theory of a finite type ADE Nakajima variety, which we prove is related to the classical Lusztig and Chari braid group actions.

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2.0.1 ADE quiver varieties at different stability conditions

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Chapter 3: LMN maps

3.1 LMN maps

For an ADE quiver variety $\vec{Q}$, let $\sigma \in W_{\vec{Q}}$, the quiver Weyl group (same as the Weyl group for the underlying root system). Then there are so-called LMN, Lusztig-Maffei-Nakajima, isomorphisms: [15]

$$\Phi_{LMN}(\sigma) : \mathcal{M}_\theta(v, w) \to \mathcal{M}_{\sigma\theta}(\sigma v, w)$$

This map is a symplectomorphism, and has the property that

$$\Phi^\star_{LMN}(s_i)^\star Taut_\chi = Taut_{s_i^\star \chi}$$

Claim 1. Yaochen Wu [14] and McGerty-Nevins [16] have computed that

- Stability chambers correspond to Weyl chambers in type ADE.[1]
- The map $\Phi_{LMN}(s_i)$ is equivariant with respect to $\prod_{i\neq j} GL(W_j)$ and sends a character a_i of $GL(W_i)$ to its symplectic dual $\hbar^{-1}a_i^{-1}$.
- $\Phi_{LMN}(s_i)^\star Taut_\chi = Taut_{s_i^\star \chi}$
- $\Phi^\star_{LMN}(\sigma)\Phi^\star_{LMN}(\sigma') = \Phi^\star_{LMN}(\sigma\sigma')$.

Theorem 3.1. Let $\Phi^\star_{LMN}(s_i)$ be the LMN isomorphism. Then there is a commutative diagram

$$\begin{array}{ccc} K_T(\mathcal{M}_{s_i\theta}(s_i v, w)) & \xrightarrow{\Phi_{LMN}(s_i)^\star} & K_T(\mathcal{M}_\theta(v, w)) \\ {\scriptstyle Stab_{s_i\mathbb{C}, s_i T^{1/2}, s_i s}}\uparrow & & \uparrow{\scriptstyle Stab_{\mathbb{C}, T^{1/2}, s}} \\ K_T(\mathcal{M}_{s_i\theta}(s_i v, w)^A) & \xrightarrow{\Phi_{LMN}(s_i)^\star} & K_T(\mathcal{M}_\theta(v, w)^A) \end{array}$$

[1] To be precise, for any fixed quiver data, the walls are a subset of the walls of the root system walls. If the quiver data are too small, some Weyl group elements might act trivially.

where the horizontal maps are equivariant only as above.

Proof. Follows by uniqueness of stable envelopes. The diagonal condition and degree bound are clear. The support condition follows since we invert the attracting and repelling directions at i when we send $u \to u^{-1}\hbar^1$, so compensate by $\mathfrak{C} \to s_i\mathfrak{C}$. □

We can trade the shift in polarisation for a shift in slope. Let

$$\delta_i = det^{1/2}(T^{1/2} - s_i T^{1/2})$$

well defined since the difference of two polarisations is a balanced class.

The normalisation of the stable envelope shifts by δ_i and so, to satisfy the degree bound, the slope shifts inversely.

$$Stab_{s_i\mathfrak{C},s_iT^{1/2},s_is} = \delta_i Stab_{s_i\mathfrak{C},T^{1/2},s_is-\delta_i}$$

The overall normalisation shift will disappear in R-matrices.

Example 1. In type A_1, the claim is that a certain R-matrix for stability $\theta > 0$ intertwines the standard R-matrix. Explicitly, this matrix is

$$R^{-1,T}_{-\mathfrak{C},\hbar^{-1}T^{1/2,\vee},s}(\hbar^{-1}u^{-1}) = R_{\mathfrak{C},T^{1/2},s}(\hbar u)$$

Remark 3.1. The LMN operators give a representation of the Weyl group on

$$\bigoplus_{\sigma\in W} K_T\mathcal{M}_{\sigma\theta}(w)$$

Which is strongly suggestive of a relation to the Frenkel-Hernandez Weyl group action on $\oplus_{w\in W}K_0O^w_{HJ}$.

Their W-action satisfies the property that it leaves the q-character of a representation invariant iff it is finite dimensional.

Recall. The **q-character** of a representation V is the equivalence class $\chi_q(V)$, where we say

$\chi_q(V) \sim \chi_q(W)$ if the generalised eigenspaces of R_∞, together with their eigenvalues, are the same.

Formally, we can interpret the target of the q-character maps as

$$Char = \mathbb{Z}[[\Psi]]$$

where Ψ ranges over all possible generalised eigenvalues for all matrix elements R_∞.

One may show that the q-character map factors through K-theory, for a suitable notion of category $\mathcal{O}$.

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\chi_q} & Char \\ \downarrow & \nearrow & \\ K_0\mathcal{O} & & \end{array}$$

Since our LMN operators permute bases for an abstractly isomorphic representations, they also preserve q-characters, further evidence for the conjecture.

3.2 The composite braid group action

Now consider conjugation of R-matrices by the composite automorphism

$$K_T(\mathcal{M}_\theta(w)) \xrightarrow{\Phi^\star_{LMN}(s_i)} K_T(\mathcal{M}_{s_i\theta}(w)) \xrightarrow{Steinberg} K_T(\mathcal{M}_\theta(w))$$
$$K_T(\mathcal{M}_\theta(w)) \xrightarrow{\Phi^\star_{s_i}} K_T(\mathcal{M}_\theta(w))$$

Theorem 3.2. The automorphisms $\Phi^\star_{s_i}$ satisfy braid group relations.

Proof. The LMN maps are defined for general moment map parameter. Losev and Bezrukavnikov [15] show that the LMN maps act on the moment map parameters for deformed Nakajima varieties $\mathcal{M}_{\theta,\zeta}(w)$ by a shifted action:

$$\Phi_{LMN}(s_i) : \mathcal{M}_{\theta,\zeta}(v,w) \to \mathcal{M}_{\theta,s_i\cdot^\nu\zeta}(s_iv,w)$$

The Steinberg map is defined by specialisation in equivariant K-theory [17] for generic ζ and the deformed variety $\mathcal{M}_{\theta,t\zeta}(w)$.

So it suffices to show that

$$s_j \cdot^{s_i v} (s_i \cdot^{v} \zeta) = s_i \cdot^{s_j v} (s_j \cdot^{v} \zeta)$$

when $i \neq j$.

By definition, with C the equivariant Cartan matrix,

$$\rho(v) = \frac{Cv - w}{2}$$
$$s_i \cdot^{v} \zeta = s_i\zeta + \rho(s_i v) - \rho(v)$$

The claim then follows by explicitly computing

$$s_j(\rho(s_i v) - \rho(v)) - \rho(s_i v) = s_i(\rho(s_j v) - \rho(v)) - \rho(s_j v)$$

□

The Steinberg map is equivariant, but possibly in a complicated way. We henceforth make this assumption:

Assumption 1. We always assume the torus $(1, u)$ with which we define the R-matrix acts on framing vectors by $w = w' + un\delta_j$.

Factorisation of R-matrices implies that any R-matrix factors into ones of this form, so this assumption is harmless.[2]

Under this assumption, the composite map by reflection at j is twisted equivariant for the action sending $u \to u^{-1}\hbar^{-1}$. One could use unitarity of the R-matrix

$$R_{21}(u^{-1})^{-1} = R(u)$$

[2]Thanks to Tommaso Maria Botta for this point.

to consider the R-matrix across the composite isomorphism as an R-matrix for the parameter $\hbar u$.

This shift of parameters $u \longrightarrow \hbar u$ or $u \longrightarrow \hbar^{-1}u^{-1}$, depending on perspective taken, is at the heart of why the composite map induces a braid, rather than Weyl, group action on quantum affine groups.

The only property we use of the Steinberg map is that it preserves weight, i.e. it sends $\mathcal{M}_{s_i\theta}(v,w) \longrightarrow \mathcal{M}_{\theta}(v,w)$. We are henceforth only interested in R-matrices, and so we view the Steinberg map as simply giving an identification of the fixed points of $\mathcal{M}^A_{s_i\theta}$ with those of $\mathcal{M}^A_{\theta}$.

Example 2. Consider the example of $\mathcal{M}_{\theta>0,A_1}(1) \simeq T^\star\mathbb{P}^1$. Fix the positive chamber, antiample slope and polarisation by the base. We have an explicit formula for the R-matrix [1] (omitting trivial components):

$$R(u) = R_{\theta>0,\mathbb{C},T^{1/2},s}(u) = \begin{pmatrix} \hbar^{1/2}\frac{1-u}{\hbar-u} & u\frac{\hbar-1}{\hbar-u} \\ \frac{\hbar-1}{\hbar-u} & \hbar^{1/2}\frac{1-u}{\hbar-u} \end{pmatrix}$$

Tracing definitions, the composite map $\Phi^\star_{LMN}$ should conjugate this to the R-matrix $R_{\theta<0,-\mathbb{C},\hbar^{-1}T^{1/2,\vee},-s}(u^{-1}\hbar^{-1})$ $R(\hbar u)^{-1}$.

The braid group action on the associated geometric quantum group is the map

$$< i|R(u)|j > \longrightarrow < i|R(\hbar u)^{-1}|j >$$

3.3 Chari's braid group action and q-characters

Elements of R_∞ are given by a geometric formula [5]:

$$< F_1, F_2|R_\infty|F_1, F_2 >= \widehat{S}((1-\hbar^{-1})N_+)$$

Herein,

Definition 3.1.

$$\widehat{S^\bullet}(X) = det(V)^{1/2} \prod_{w \text{ weights of} X} \frac{1}{1-w}$$

extended by linearity. For example, if U has weight u,

$$\widehat{S}((1-\hbar^{-1})u) = s^{1/2}\frac{1-\hbar^{-1}u}{1-u}$$

This may not live in $K_T(X)$ but possibly a completion.

We can write N_+ uniformly on the stack of quiver representations in terms of the quiver data of the fixed points:

$$N_+ = \sum_{i=1}^{\ell} W_i^\star V_i' + \hbar \sum_{i=1}^{\ell} W_i' V_i^\star - \sum_{i\to j} C_{ij} V_i (V_j')^\star$$

The partial ordering of components read

$$\begin{aligned} Z_\eta = \mathcal{M}(\eta, w') \times \mathcal{M}(v-\eta', w'') \subset \mathcal{M}(v,w) \\ Z_\eta > Z_{\eta'} \iff \theta\cdot\eta > \theta\cdot\eta' \end{aligned} \tag{3.1}$$

Assume $\theta = \theta_-$. Then the partial ordering implies that the highest weight vector corresponds to $\eta = 0$.

So the highest weight vector is given by

$$< v_0 | R_\infty | v_0 >= \widehat{S}((1-\hbar^{-1})\hbar W \cdot Taut^\star)$$

Any lower vector in the order will be at least of the form

$$\mathcal{M}(\eta+\delta_i, w') \times \mathcal{M}(v-\eta-\delta_i, w'')$$

The difference in normal bundles contributed by such a change is

$$\delta N_+^i = \hbar (W')_i Taut_i^\star - W_i^\star Taut_i' - \sum_{i\to j} C_{ij} Taut_i (Taut_j')^\star$$

Assuming the equivariant weights of W, W' are $1, u$ respectively, the expression simplifies:

$$\widehat{S}((1-\hbar^{-1})\delta N_+^i) = \hbar^{-rk(Taut_i)}\widehat{S}^\bullet_{\hbar u}((1-\hbar^{-1})Taut_i^\star)^{-1}S^\bullet_u((1-\hbar^{-1})Taut_i^\star)^{-1} \times \prod_{i\to j} S^\bullet_{\hbar^{1/2}u}((1-\hbar^{-1})(-C_{ij})Taut_j^\star)$$

where we use the lemma

Lemma 3.1. Let X a K-theory class. Then

$$\widehat{S}((1-\hbar^{-1})\hbar X^\star) = \hbar^{-rk(X)}\widehat{S}((1-\hbar)X) = \hbar^{-rk(X)}\widehat{S}((1-\hbar^{-1})\hbar X)^{-1}$$

Proof. Let w_i be the weights of X. Then, the weights of $\hbar X^\star$ are w_i^{-1}. So

$$\begin{aligned}
&\widehat{S}((1-\hbar^{-1})\hbar X^\star)\\
&= \prod_{w_i} \hbar^{1/2}\frac{1-\hbar^{-1}w_i^{-1}}{1-w_i^{-1}}\\
&= \prod_{w_i} \hbar^{-1/2}\frac{\hbar w_i - 1}{w_i - 1}\\
&= \hbar^{-rk(X)}\prod_{w_i} \hbar^{1/2}\frac{1-\hbar w_i}{1-w_i}\\
&= \hbar^{-rk(X)}\widehat{S}((1-\hbar)X)
\end{aligned}$$

□

Definition 3.2. Let

$$Y_{i,a} = \widehat{S}^\bullet_u((1-\hbar^{-1})W_i^\star Taut_i)$$

And

$$A_{i,\hbar^{-1/2}u} = (\widehat{S}^\bullet_u(1-\hbar^{-1})\delta N_+^i)^{-1}$$

Remark 3.2. By definition of the A, the spectrum of R_∞ is given by expressions of the form

$$\prod_{i\in I} Y_{i,u}(1 + \sum \prod A^{-1}_{j_{11},b_{11}} \dots A^{-1}_{j_{1m_1},b_{1m_1}})$$

We caution the reader that what we call $A_{i,u}, Y_{i,u}$ only agree with what is typically called $A_{i,u}, Y_{i,u}$ in the literature if the module we are in is a defining representation.

Now, consider changing $\theta_- \to s_i\theta_-$. The new highest weight is given by $Y_{i,u}A^{-1}_{i,u\hbar^{-1/2}}$, by 3.1.

Nothing here was special to the case $\theta = \theta_-$ except the explicit formulas. In general, define

$$Y_{i,u} =< v_0|R_\infty|v_0 >_{|i}$$

– the support of this K-theory class at the vertex i.

One may recognise the map $Y_{i,u} \to Y_{i,u}A^{-1}_{i,u\hbar^{-1/2}}$ as exactly Chari's braid group action [18] in simply laced type. So we conclude

Theorem 3.3. Chari's action of the braid group on q-characters is given by mapping the highest weight of a representation $K_T\mathcal{M}_\theta(v)$ to the highest weight of the induced representation $\Phi^\star_{s_i}K_T\mathcal{M}_\theta(w)$.

3.4 Slope R-matrices and Lusztig's braid group action

Recall. The factorisation around ∞

$$R^s(u) = (\overleftarrow{\prod}_{<0} R^-_{w_i})R_\infty(\overleftarrow{\prod}_{\geq 0} R^+_{w_i})$$

depends on the stability parameter θ, in that we take a product of slope R-matrices going to ∞ in the ample cone defined by θ for which R^+ is θ-strictly upper triangular.

There is a parallel factorisation around 0:

$$R^s(u) = \overrightarrow{\prod}_{i\geq 0}(R^-_{w_i})^{-1}R_\infty\overrightarrow{\prod}_{i<0}(R^+_{w_i})^{-1}$$

For $F_1 > F_2$ in the ample partial order (meaning that of 3.1), and $\mathcal{L}_w$ a line bundle on the wall w, we have

$$R_w^+ \sim u^{<\mu(F_2)-\mu(F_1)|\mathcal{L}_w>}$$

The slope factorisation implies that we can write a quantum affine group as the algebra generated by all matrix elements of all wall R-matrices appearing in the slope factorisation.

Classical formulas for the braid group action on quantum affine algebras are typically given in a presentation of the quantum group closely related to this fact. To start, we explain the simplest example.

Example 3. Take the case of $\mathcal{M} = T^\star\mathbb{P}^1$. Recall that in this case, our braid group action sends

$$R(u) \to R(\hbar u)^{-1}$$

Recall further [5] that walls correspond to integers and

$$R_n^+ = \mathcal{O}(n)R_0^+\mathcal{O}(-n)$$
$$R_n^- = \mathcal{O}(n)R_0^-\mathcal{O}(-n)$$

The slope factorisation of $R(\hbar u)^{-1}$ around ∞ is almost the slope factorisation of $R(u)$ around 0 inverted under the map $u \to \hbar u$:

$$R(\hbar u)^{-1} = \prod_{n<0}^{\leftarrow} R_n^{+,-1} R_\infty^{-1} \prod_{n\geq 0}^{\leftarrow} R_n^{-,-1}$$

So the braid group action sends, for example $R_0^+ \to (R_0^-)^{-1}$. More explicitly, we can write

$$\prod\nolimits^{\leftarrow}_{n\geq 0} R_n^{-,-1} = \prod\nolimits^{\leftarrow}_{n\geq 0} \mathcal{O}(1-n)R_1^{-,-1}\mathcal{O}(n-1)$$

The terms telescope, and

$$\mathcal{O}(1)R_\infty^{-1}\mathcal{O}(-1) = R_\infty^{-1}$$

so defining generators of the quantum affine algebra and using that our equivariant parameter is inverted and so hence too is $O(1)$,

$$E = R_0^+$$

$$F = R_1^-$$

$$K = O(1)$$

$$H = R_\infty$$

this is the map

$$F \to K^{-1}E^{-1}$$

$$E \to F^{-1}K$$

$$K \to K^{-1}$$

$$R_\infty \to R_\infty^{-1}$$

which is the same as the braid group action clasically defined [19].

Chapter 4: Weyl group actions and TQ relations

4.1 Category $\mathcal{O}$ for quantum affine algebras

Theorem 4.1. (*Gauss factorisation*, [20]) Let T an $N \times N$ matrix with coefficients in an arbitrary unital ring. Suppose the submatrices of T formed by the first m rows and first m columns are invertible. Then there exists unique matrices E, H, F so that

- E is lower triangular with 1s along the diagonal;
- F is upper triangular with 1s along the diagonal;
- H is diagonal;
- $T = EHF$.

Definition 4.1. The **quantum affine Borel**, $\mathcal{B}^+$, is generated by all coefficients of all matrix elements of

$$E, H, F$$

in the Gauss factorisation of all R around $u = \infty$.

Remark 4.1. The slope factorisation provides slope subalgebras isomorphic to $U_{\hbar^{1/2}}\mathfrak{g}$. Choose the slope 0 part. The triangular decomposition of the quantum affine Borel, in particular, gives a triangular decomposition of this algebra,

$$U_{\hbar^{1/2}}\mathfrak{g} = U_{\hbar^{1/2}}\mathfrak{g}^- \otimes U_{\hbar^{1/2}}\mathfrak{h} \otimes U_{\hbar^{1/2}}\mathfrak{g}^+$$

This choice of triangular decomposition endows the root lattice of $\mathfrak{g}$ with a notion of positive roots.

Definition 4.2. Let $\mathcal{B}^+ \subset U_q\widehat{\mathfrak{g}}$ be a quantum affine algebra with a choice of quantum Borel. Category $\mathcal{O}^{\mathcal{B}^+}$, or just **category** $\mathcal{O}$, is the full subcategory of finite type representations, M, of $\mathcal{B}^+$ on which

- $\mathfrak{h}$ acts locally finitely and semisimply on M;
- The weights of $\mathfrak{h}$ on M are all lower than a finite set of weights, with respect to the triangular decomposition above.

Remark 4.2. The slope factorisation around $u = \infty$

$$R = (\prod_{w<0} R_w^-) R_\infty (\prod_{w\geq 0} R_w^+)$$

is finer than the Gauss factorisation. So in terms of slope, the positive half just means nonnegative slope.

4.1.1 Category $\mathcal{O}_{HJ}^w$

Keyu Wang [4] defined a version of this category inspired by the W-action on q-characters of [3],

Definition 4.3. Let W the Weyl group of $\mathfrak{g}$. Let $w \in W$. The w-twisted category $\mathcal{O}_{HJ}$, $\mathcal{O}_{HJ}^w$, for the choice of Borel $\mathcal{B}^+$, is the full subcategory of finite type representations, M, of $\mathcal{B}^+$ on which

- $\mathfrak{h}$ acts locally finitely and semisimply on M;
- The weights of $\mathfrak{h}$ on M are all lower than a finite set of weights, with respect to the ordering given by the triangular decomposition twisted by w. That is,

$$wt(M) \subset \cup_{i=1}^n \{\omega_i + w\Lambda_-\}$$

where Λ_- is the negative half of the root lattice.

Fix θ^+ to be generic. In other words, $\mathcal{O}_{HJ}^w$ is the category of representations generated by matrix elements of $w\theta^+$-upper triangular R-matrices of positive slope.

So, $\mathcal{O}_{HJ}^w$ is the category of representations for the quantum affine Borel generated by $K_T\mathcal{M}_{s_i\theta}(\vec{w})$. As an immediate consequence, we get an isomorphism between finite dimensional representations of $\mathcal{O}_{HJ}^w$ and $\mathcal{O}_{HJ}$.

Claim 2. The categories $\mathcal{O}_{HJ}^{w,\,\text{fin dim}_{HJ}}$ are isomorphic for all $w \in W$. The isomorphism is given by conjugation by the LMN maps.

This does not imply that the categories $\mathcal{O}_{HJ}^w$ are isomorphic for all $w \in W$.

4.2 Frenkel-Hernandez Weyl group action on q-characters

Let $A_{i,u}, Y_{i,u}$ be defined as before in the case of a defining representation of our quantum affine algebra. Frenkel-Hernandez define a Weyl group action on $\oplus_{w\in W} K_0\mathcal{O}_{HJ}^w$:

$$\Theta_i : K_0\mathcal{O}_{HJ}^w \to K_0\mathcal{O}_{HJ}^{ws_i}$$
$$Y_{i,u} \to Y_{i,u}A_{i,\hbar^{1/2}u}^{-1}\frac{\Sigma_{i,u\hbar^{3/2}}^{ws_i}}{\Sigma_{i,u\hbar^{1/2}}^{ws_i}}$$

and Θ_i for $i \neq j$ leaves the elements $Y_{j,u}$ invariant.

Here, $\Sigma_{i,u\hbar^{3/2}}^{ws_i}$ is the unique element in $K_0\mathcal{O}^{ws_i}$ which solves the difference equation

$$\Sigma_{i,u}^{ws_i} = 1 + A_{i,u}^{-1}\Sigma_{i,u\hbar}^{ws_i}$$

We point out that following the logic of our proof of Chari's braid group up until our use of the Steinberg map gives a geometric formula for a Weyl group action on highest ℓ-weights for any finite dimensional representation $\oplus_{\sigma\in W} K_T\mathcal{M}_{\sigma\theta}(\vec{w})$,

$$\widehat{S_i^\bullet}((1-\hbar^{-1})\widehat{Taut_i}) \to \widehat{S_i^\bullet}((1-\hbar^{-1})\widehat{Taut_i})(\widehat{S_u^\bullet}(1-\hbar^{-1})\delta N_+^i))$$

where $\widehat{Taut_i}$ denotes the appropriate modification of the tautological bundle depending on whether we have removed the trivial W-bundles or not, and

$$\delta N_+^i = s_i N^+ - N^+$$

Conjecture 1. This geometric formula gives the same W-action as that of Frenkel-Hernandez.

4.3 Shifted geometric quantum affine algebras and quasimaps

This section reviews the geometric construction of shifted quantum affine algebras in terms of finitary quasimaps developed in [1]. We define finitary quasimaps and explain how shifted quantum affine algebras arise via FRT reconstruction on moduli spaces of finitary quasimaps.

4.3.1 Finitary quasimaps

Recall. Recall that we can present Nakajima varieties as critical loci for the moment map.

Let

$$M_Q = (T^\star Rep(\vec{Q}))^{\theta-ss} \times Lie(GL(\vec{v}))$$

with coordinates

$$\vec{a}, \vec{a}^\star, \vec{b}, \vec{b}^\star, \xi$$

where $\vec{b}, \vec{b}^\star$ labels a map in the unframed quiver, $\vec{a}$ a map from the quiver to its framing, and $\xi \in Lie(GL(\vec{v}))$. Then setting

$$\phi_Q(\vec{a}, \vec{a}^\star, \vec{b}, \vec{b}^\star, \xi) := tr(aa^\star\xi) + tr([b, b^\star]\xi)$$

We have

$$\mathcal{M}_Q(v, w) = crit(\phi_Q)$$

Setup 1. Fix a quiver $\vec{Q}$, a framing vector $\vec{w}$, and an integer L.

Define a new framing bundle $\overline{W} = W \otimes \mathbb{C}_{i=0}^{L-1}\mathbb{C}u_k$.

Choose a framing torus $GL(\vec{w})$, and let $\mathbb{G}_{m,\hbar}$ act on u_k with weight k.

Fix a map

$$\Xi : \overline{W} \to \overline{W}$$

$$\Xi u_k = u_{k+1}$$

$$\Xi u_{L-1} = 0$$

Define

$$\vec{\phi}_Q := \phi_Q - tr(a\Xi a^\vee)$$
$$\vec{\mathcal{M}}_Q^{>,L}(w) := crit(\vec{\phi}_Q)$$

Remark 4.3. Equivalently, $\vec{\mathcal{M}}_Q^{>,L}(w)$ is the moduli of quiver representations for $\vec{Q}$ with framing vector $\overline{w}$ which are $\mathbb{C}[z]$-equivariant, where z acts as (ξ, Ξ).

We will often view a point $M \in \vec{\mathcal{M}}_Q{}^{>,L}(w)$ as a sequence of quiver representations M_i endowed with maps inbetween,

$$z : M_i \to M_{i+1}$$

Stability for finitary quasimaps

It is not true that every individual $M_i \leftleftarrows W$ is θ-semistable as a representation of $\vec{Q}$.

The only two conditions we have are that

- The entire representation

$$\oplus_{i=0}^{L-1} M_i \leftleftarrows W$$

  is θ-semistable.

- The diagrams

$$\begin{array}{ccc} M_i & \longrightarrow & M_{i+1} \\ \Uparrow\Downarrow & & \Uparrow\Downarrow \\ W & \xrightarrow{=} & W \end{array} \tag{4.1}$$

commute.

Recall. ([21]): Let $\theta_+ = (1, \ldots, 1)$. A representation of a doubled framed loopfree quiver variety $V \leftrightarrows W$ is θ_+-semistable if and only if the map $V \to W$ is injective. Conversely a representation is θ_--semistable if and only if the map $W \to V$ is surjective.

Claim 3. Let $\vec{M}$ be a quiver representation corresponding to a point in $\mathcal{M}^L_{\theta^+}$. Then:

1. The maps $M_i \to M_{i+1}$ are injective.

2. Each $M_i \leftrightarrows W$ is θ_+-semistable.

Proof. Commutativity of 4.1 implies 1). 2) follows from the fact that the direct sum of maps is injective if and only if each summand is injective. □

Remark 4.4. Therefore, $\mathcal{M}^L_{\vec{Q},\theta^+}$ is alternately described as the moduli space of length L increasing flags of θ_+-semistable doubled framed representations of $\vec{Q}$.

In particular, there are maps, for $i \in \{0, \ldots, L-1\}$,

$$\begin{aligned} ev_i : \mathcal{M}^L_{\vec{Q},\theta^+} &\to \mathcal{M}_{\vec{Q},\theta^+} \\ \vec{M} &\to (M_i \leftrightarrows W) \end{aligned} \tag{4.2}$$

Totally dually, $\vec{\mathcal{M}}^L_{\vec{Q},\theta^-}$ is the moduli space of length L decreasing flags of θ_--semistable doubled framed representations of $\vec{Q}$.

Conjecture 2. (Henry Liu, [1]): There is an injection $K_T\vec{\mathcal{M}}^L_{\vec{Q},\theta}(w) \to K_T\mathcal{M}(Lw)$ which makes $K_T\vec{\mathcal{M}}^L$ a submodule for the action of the quantum affine group.

Further, let $\lim^{asymptotic}$ denote the leading-order $\hbar$ asymptotic, taken separately in every irreducible submodule.

Let W a fixed finite dimensional module, and $\vec{R}^L(u)_W \in End(W \otimes K_T(\mathcal{M}(Lw)))$. Then

$$\vec{R}(u) := \overset{asymptotic}{\lim_{L\to\infty}} \vec{R}^L(u)$$

exists for some submodule $\vec{V}(u)$ corresponding to the limit of $K_T\vec{\mathcal{M}}^L$ in the limit $L \to \infty$. In this setting, we call $\vec{V}(u)$ an *asymptotic module* and label its irreducible submodule $\vec{V}(u)_v$, over some set v.

This conjecture can be made much more concrete for $\theta = \theta^+$ or $\theta = \theta^-$. In those cases, Henry Liu proves the conjecture in types $A_1, \widehat{A}_1$.

Assuming the conjecture, further,

Theorem 4.2. [1] for finite type ADE quivers $\vec{Q}$, the FRT reconstructed algebra from $\vec{R}_{\vec{V},W}(u)$ is a shifted quantum algebra $U_q^\mu(\widehat{\mathfrak{g}})$.

In the case of θ_+/θ_-, the correct irreducible submodules to take are the submodules of infinite ascending/descending flags which after finitely many steps become constant, and so v above labels the quiver representation on which they eventually stabilise.

The issue is that for general θ, there is no concrete description of a point of $\mathcal{M}_\theta^L$.

Remark 4.5. For each finite L, the LMN map and Steinberg maps make sense. Therefore, we have a well-defined twisted equivariant braiding map

$$\Phi_{s_i}^\star : K_T(\vec{\mathcal{M}}^L_{\vec{Q},\theta^-}) \to K_T(\vec{\mathcal{M}}^L_{\vec{Q},\theta^-})$$

and hence, for finite L, a braided R-matrix

$$\vec{R}^L(u) \to \Phi_{s_i}^\star(\vec{R}^L)(\hbar u)$$

Then we are in the setting of [1], which shows immediately

Theorem 4.3. (Assuming conjecture 2) The FRT reconstructed algebra from $\Phi^{\star}_{s_i}(\vec{R}^L)(\hbar u)$ is a shifted quantum affine algebra of same type as as the underlying quiver. Highest weight modules are highest weight with respect to the $s_i\theta$ -triangular decomposition.

The key idea of this theorem, namely replacing the LMN map with a braid group action to do an asymptotic construction, is essentially present in [4].

Remark 4.6. The above theorem implies the subcategory of $\mathcal{O}^w_{HJ}$ formed by all w-twisted asymptotic modules is isomorphic to the subcategory of $\mathcal{O}_{HJ}$ formed by all untwisted asymptotic modules.

4.4 TQ-relations

A very concrete and classical [22] definition of the Q-operator, and consequent proof of TQ relations, essentially works verbatim in the geometric setting.

Write

$$R^L(u) = R(u)R^{L-1}(\hbar u)$$

Taking the leading order asymptotic as $L \to \infty$ in an appropriate irreducible submodule $\vec{V}(u)_v$,

$$\vec{R}(u) = R(u)\vec{R}(\hbar u)$$

And so also (by triangularity of the coproduct, see [1])

$$\vec{R}_\infty(u) = R_\infty(u)\vec{R}_\infty(\hbar u)$$

The TQ relation is just an equation for certain diagonal matrix elements of this operatorial equation.

Let i label a quiver representation. Define

$$T_i(u) =< i|R_\infty(u)|i >$$

$$Q_i(u) =< iii\ldots|\vec{R}_\infty(u)|iii\cdots >$$

where, $|iii\cdots>$ labels the constant flag of quiver representations which is i at every step.

Then the above equality implies

$$Q_i(u) = T_i(u)Q_i(\hbar u)$$

This is Bethe's *TQ-relation*.

Remark 4.7. Applying the above computation to the twisted asymptotic modules of the previous section implies the extended TQ-relation conjecture of [2] (still assuming the conjecture 2).

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Appendices

Appendix A: An introduction to reflection for quiver representations

The main text assumes familiarity with classical results about reflection functors on quiver representations. This appendix serves as a brief introduction.

A.1 Quiver Weyl group

Definition A.1. A *quiver* $\vec{Q}$ is the data of a set of vertices, $Vert(\vec{Q})$, and directed edges between vertices $Edges(\vec{Q})$.

Example 4. Consider  This quiver has two vertices and two arrows $1 \to 2$.

Remark A.1. While quivers often have more than one edge between two vertices, we will often denote an edge a quiver by the vertices they connect, i.e. as $i \to j$. In formulas where this notation appears, e.g. something like

$$\sum_{i \to j}$$

if multiple arrows connect $i \to j$, it is understood that we are summing over all of them.

Definition A.2. Given a quiver $\vec{Q}$ and a vertex i, there is another quiver denoted $s_i\vec{Q}$ which is obtained by reversing the direction of all arrows leaving or entering i.

Example 5. Consider the quiver $\vec{A}_3$,

$$1 \to 2 \to 3$$

where we have given the vertices labels 1,2,3.

Then the quiver $s_3\vec{A}_3$ is the quiver

$$1 \to 2 \leftarrow 3$$

Definition A.3. The set of reflections s_i gives an action on the set of orientations on the underlying graph of $\vec{Q}$. The group generated by the s_i is called the *quiver Weyl group* of $\vec{Q}$, denoted $W_{\vec{Q}}$.

Claim 4. This action is transitive for loopfree quivers with finitely many vertices.

Proof. Use induction. The case of a quiver with one vertex is trivial. Now proceed inductively. The hypotheses imply that there exists a vertex v connected to exactly one other vertex by one arrow. Use transitivity of the action on $\vec{Q} - v$ together with s_v to fix the direction of the unique edge touching v. □

Example 6. The claim is very false when you add loops. For example, the action is not transitive for the underlying graph of the quiver

For reflection at either 1 or 2 reverse all arrows, so there is no element that sends the class where two arrows point in opposite directions to the class where they point in the same direction.

A.2 Quiver representations

Definition A.4. A quiver $\vec{Q}$ defines a category, also denoted $\vec{Q}$. It is the category with objects the vertices of $\vec{Q}$, and morphisms generated by the arrows of $\vec{Q}$.

Definition A.5. A *quiver representation* is a functor from a quiver category to the category of vector spaces, $\vec{Q} \to Vect_k$. The category $Rep(\vec{Q})$ is the category of such functors.

Remark A.2. For example, a functor $V : \vec{A}_3 \to Vect_k$ is the data of three vector spaces V_1, V_2, V_3 and linear maps $e_{1\to 2} : V_1 \to V_2$ and $e_{2\to 3} : V_2 \to V_3$. A *morphism* of $\vec{A}_3$ quiver representations is

a natural transformation of functors, equivalently a commutative diagram

$$\begin{array}{ccccc} V_1 & \xrightarrow{e_{V,1\to 2}} & V_2 & \xrightarrow{e_{V,2\to 3}} & V_3 \\ \downarrow & & \downarrow & & \downarrow \\ W_1 & \xrightarrow{e_{W,2\to 3}} & W_2 & \xrightarrow{e_{W,2\to 3}} & W_3 \end{array}$$

Remark A.3. Since the category of functors from any (small) category to an abelian category is itself abelian, the category $Rep(\vec{Q})$ is obviously abelian. As a consequence, it must be the category of modules for some k-algebra.

Definition A.6. There is a standard representative of such a k-algebra, $k\vec{Q}$, the *path algebra* of $\vec{Q}$. It has generators

- For every vertex $i \in Vert(\vec{Q})$, an idempotent e_i;
- For every arrow $i \to j$, a generator $m_{i\to j}$.

Relations in the path algebra are generated by

- All idempotents e_i are orthogonal and their sum is 1;
- $m_{i\to j}m_{u\to v}$ is zero unless $v = i$
- $e_i m_{u\to v}$ is zero unless $v = i$, in which case it is $m_{u\to v}$
- $m_{u\to v}e_i$ is zero unless $u = i$, in which case it is $m_{u\to v}$

A.3 Classical reflection functors

In general, there is no functor between the categories $\vec{Q}$ and $s_i\vec{Q}$. Nonetheless, the quivers $\vec{Q}$ and $s_i\vec{Q}$ are related in some sense and we might hope to define a functor between their categories of representations.

Definition A.7. A vertex i of a quiver is a **sink** if there are no arrows leaving i, and a **source** if there are no arrows entering i.

One may consider a derived version of the category of quiver representations, which we take to be the dg-category or ∞-category of dg/∞-functors from $\vec{Q}$ to $Vect_k$. We denote this category as $D^b Rep(\vec{Q})$.

Remark A.4. The reason to use a slightly more enriched derived category is that in these settings, given a map $A \to B$, there are maps $B \to cofib(A \to B)$ and $fib(A \to B) \to A$. Furthermore, $A \to cofib(fib(A \to B) \to A)$ is equivalent (in the proper sense) to $A \to B$.

The reader unfamiliar with these notions may view cofibre/fibre as a functorial version of cone/cocone or even just as a derived version of cokernel/kernel. The advantage of using this slightly more refined notion is that functoriality removes much of the necessary checking that one must do when using cones/cocones.

Remark A.5. Let $F : \vec{Q} \to Vect_k$ be a derived quiver representation. Let i be a sink. Then there is a quiver representation

$$s_i F : s_i\vec{Q} \to Vect_k$$

defined as follows.

- For any vertex $j \neq i$, $s_i F(j) = F(j)$.
- $s_i F(i) = fib(\oplus_{j\to i} F(j) \to F(i))$.
- The maps $s_i F(i) \to s_i F(j)$ are the canonical maps $fib \oplus_{j\to i} F(j) \to F(i)) \to F(j)$.

Functoriality of fibre implies immediately

Claim 5. Let i be a sink. The above construction gives an equivalence of categories

$$\Phi_{BGP}(s_i) : Rep(\vec{Q}) \to Rep(s_i\vec{Q})$$

Furthermore, its inverse is given by a functor constructed the same way, also denoted $\Phi_{BGP}(s_i)$, but this time defined for i a source using cofibre.

This functor is called a **Bernstein-Gel'fand-Ponomarev** (BGP) **reflection functor**, after its original inventors.

Definition A.8. Given a bounded chain complex of vector spaces, let $\dim V^\bullet := \sum_i (-1)^i \dim V^i$.

Claim 6. This definition of dimension is additive along exact triangles, so

$$\dim(s_i F(i)) = \sum_{j \to i} \dim F(j) - \dim F(i)$$

Example 7. Consider $\vec{A}_2$. The second vertex, 2, is a sink. There is a representation

$$0 \to \mathbb{C}$$

The BGP reflection functor at 2 sends this to the representation

$$0 \leftarrow \mathbb{C}[1]$$

This sends a dimension $(0, 1)$-representation to a dimension $(0, -1)$-representation.

Definition A.9. The **dimension** of a representation of a quiver is the vector $(\dim F(i))_{i \in Vert(\vec{Q})}$.

Let $\vec{v} \in \mathbb{R}^{Vert(\vec{Q})}$. Define

$$s_i \vec{v}$$

to be the vector, for $i \neq j$, $(s_i \vec{v})_j = \vec{v}_j$, and

$$(s_i \vec{v})_i = \sum_{j \text{ adjacent to } i} \vec{v}_j - \vec{v}_i$$

Whenever $\vec{v}$ is really the dimension of a representation, $s_i \vec{v}$ will be the dimension of the representation obtained by BGP reflection.

A.4 Stability and orientation

For a loopfree quiver, the derived categories $D^b Rep(\vec{Q})$ are equivalent for *any* choice of orientation on the underlying graph of $\vec{Q}$.

As Example 7 shows, the BGP reflection functor does not preserve the heart of this derived category. In fact, the BGP reflection functors more-or-less are tilting functors between choices of heart. This point will not be pursued in further detail, but it already suggests that there is some connection between stability conditions and orientations on $\vec{Q}$.

A.4.1 Classical stability conditions on quivers and orientation

Definition A.10. Fix $\theta \in \mathbb{R}^{Vert(\vec{Q})}$, and $V \in Rep(\vec{Q})$. A subrepresentation V is *θ-semistable* if, for all subrepresentations S,

$$\theta \cdot \dim(S) \leq \theta \cdot \dim(V)$$

and it is *θ-stable* if the above inequality is strict for all proper subrepresentations.

This simplest example of a stability condition will facilitate computations we'll need later.

Remark A.6. Let's compute

$$\begin{aligned} \theta \cdot \dim(s_i V) &= \theta \cdot (\dim V - \delta_i (\dim V)_i + \sum_{j \to i} \delta_i (\dim V)_j) \\ &= \theta \cdot \dim V - \theta_i (\dim V)_i + \sum_{j \to i} \theta_i (\dim V)_j \\ &= (\theta - \delta_i \theta_i + \sum_{j \to i} \theta_i \delta_j) \cdot \dim V \end{aligned} \tag{A.1}$$

So, stability for subrepresentations of $s_i V$ is equivalent to stability for subrepresentations of V with respect to a different stability condition. We make a definition out of this computed equivalence,

Definition A.11. Let

$$s_i \theta = (\theta - \delta_i \theta_i + \sum_{j \text{ adjacent to } i} \theta_i \delta_j)$$

Herein, δ_j is the vector which is 1 at j and 0 elsewhere.

A.5 Preprojective algebras and the Crawley-Boevey trick

One may endow the space of quiver representations of $\vec{Q}$ with a symplectic form and take a Hamiltonian reduction. We let $T^\star\vec{Q}$ denote the quiver with one arrow $j \to i$ for every arrow $i \to j$ in $\vec{Q}$; we keep track of which arrows were in $\vec{Q}$ by denoting them a with a subscript, and arrows which are doubled are denoted as $a^\vee$.

Definition A.12. The **deformed preprojective algebra** $\Pi_\lambda\vec{Q}$, for $\lambda \in \mathbb{R}^{\vec{Q}}$, is the quotient

$$\Pi_\lambda\vec{Q} := kT^\star\vec{Q}/\sum_{i\to j}[a_{i\to j}, a^\vee_{i\to j}] - \lambda_i e_i$$

Remark A.7. A module for the deformed preprojective algebra is the same thing as a quiver representation of $T^\star\vec{Q}$ satisfying the moment map condition $\sum[a, a^\vee] = \lambda_i e_i$.

Definition A.13. Fix $\theta \in \mathbb{R}^{Vert(\vec{Q})}$.

A $\Pi_\lambda\vec{Q}$-module M is θ-*semistable* if, for any subrepresentation $S \subset M$, $\theta\cdot\dim(S) \leq \theta\cdot\dim(M)$, and it is *stable* if the above inequality is strict for all proper subrepresentations.

Example 8. Consider the quiver $\vec{A}_2$ with one arrow $1 \to 2$. Then a $\Pi_\lambda\vec{A}_2$-module is the data of two vector spaces V, W, a map $a : V \to W$ and a map $a^\vee : W \to V$ so that

$$aa^\vee = \lambda_W id_W$$
$$a^\vee a = -\lambda_V id_V$$

We will often ask for $\lambda_V = 0$,

A.5.1 The Crawley-Boevey trick

Setup 2. One often wants to add a *framing* to a quiver representation, meaning for every vertex i, an additional vertex f_i with an arrow $f_i \to i$, and wants to take a Hamiltonian reduction with

respect to only the unframed part of the quiver.

Suppose we want the framing to have dimension vector $\vec{w}$. We can alternately add a single node, ∞, and add $\vec{w}_i$-many arrows from ∞ to i. Then modules with dimension 1 at ∞ are equivalent to modules with dimension $\vec{w}$ at the framed vertices.

Definition A.14. By abuse of notation, if we add a framing vector $\vec{w}$ we modify the quiver as above and denote the algebra $\Pi_\lambda^{\vec{w}}\vec{Q}$. We extend stability conditions by setting $\theta_\infty = -\sum \vec{v}_i\theta_i$.

Definition A.15. Fix a framing dimension vector $\vec{w}$. The moduli space of θ-semistable $\Pi_\lambda^{\vec{w}}\vec{Q}$-modules with dimension 1 at ∞ is a *Nakajima quiver variety* denoted $\mathcal{M}_{\lambda,\theta,\vec{Q}}(\vec{w})$.

Remark A.8. $\mathcal{M}$ admits a natural stratification by dimension vector $\vec{v}$,

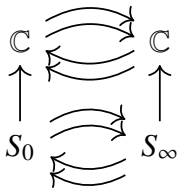
$$\mathcal{M}_{\lambda,\theta,\vec{Q}}(\vec{w}) = \sqcup_{\vec{v}}\mathcal{M}_{\lambda,\theta,\vec{Q}}(\vec{v},\vec{w})$$

Each individual $\mathcal{M}_{\lambda,\theta,\vec{Q}}(\vec{v},\vec{w})$ is a symplectic variety which is the semistable locus of a quotient stack. Namely,

$$\mathcal{M}_{\lambda,\theta,\vec{Q}}(\vec{v},\vec{w}) = \mu^{-1}(\lambda)^{\theta-ss}/GL(\vec{v}) \subset [\mu^{-1}(\lambda)/GL(\vec{v})] \subset [Rep_{T^\vee\vec{Q}}(\vec{v},\vec{w})/GL(\vec{v})]$$

Example 9. Let's explain very wordily, how to compute the θ-stable representations in the simplest example. Fix $\vec{Q} = \vec{A}_1$, $v = 1w = 2$, $\lambda = 0$ and $\theta > 0$. A representation of the Crawley-Boevey modification of our quiver is two maps $i_1, i_2 : \mathbb{C} \to \mathbb{C}$, and two maps $i_1^\vee, \ldots, i_2^\vee : \mathbb{C} \to \mathbb{C}$. The moment map imposes

A subrepresentation of this is a commutative diagram with vertically injective maps of vector spaces



Slope stability reads

$$\theta \dim(S_0) - \theta \dim(S_\infty) \leq 0 \implies \theta \dim(S_0) \leq \theta \dim(S_\infty)$$

By our assumption that $\theta > 0$, stable subrepresentations are therefore those of dimension vectors $(0,0)$ or $(0,1)$.

To understand when that is true more-or-less reduces to understanding what subrepresentations arise in $\vec{A}_2$-quiver representations, on which we give a quick reminder.

Namely, consider the $\mathbb{C} \to \mathbb{C}$ representation of an $\vec{A}_2$-quiver. It has a subrepresentation $0 \to \mathbb{C}$,

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

since this diagram commutes.

However, the diagram

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

does **not** commute, and so $\mathbb{C} \to 0$ is not a subrepresentation.

The upshot of this discussion is that if we want a representation of our above quiver to only have subrepresentations with $\dim(S_\infty) = 1$, we must require that at least one arrow $0 \to \infty$ is nonzero. Therefore, the vector $(i_1, i_2) \in \mathbb{C}^2$ is nonzero. So the data of a $\theta > 0$-stable representation of dimension vector 1 is

- A nonzero vector $(i_{1,2}) \in \mathbb{C}^2$;
- An orthogonal (possibly zero) vector $(i_1^\vee, i_2^\vee) \in \mathbb{C}^2$.

Remark A.9. What about for dimension vector $\vec{v} \neq 1$ and $\theta > 0$? The stability condition implies that any subrepresentation satisfied $\dim(S_\infty) = 1$. So necessarily $ker(i_1) \cap ker(i_2) = 0$. By rank-

nullity, this is impossible if $\dim(V) > 2$ and $\dim(V) = 2$ has a unique solution up to isomorphism labelled by the identity map. $\dim(V) = 0$ again has a unique solution, the zero map.