Assortment Planning From A Large Universe

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy under the Executive Committee of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2020
Abstract
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Discrete choice models and the assortment optimization problem are the fundamental aspects of the broader field of revenue management, which now spans a broad array of industries such as airlines, hotels and online advertising. The main focus here is to first study the consumer preferences and their substitution behavior when they are faced with multiple options, explain those observed behaviors with mathematical models and then identify an optimal set of options to offer to maximize revenues. This dissertation enriches the choice models and assortment optimization fields by studying the setting when such options are available in multitude, either to the sellers or to the consumers to choose from.

The first half of this dissertation focuses on the situation when sellers have access to a vast array of features to be chosen for products they want to offer. The second half of the dissertation focuses on the situation when customers are faced with a lot of options to choose from. This dissertation formulates concrete mathematical discrete choice models to tackle those situations, then studies the assortment optimization problem of maximizing the expected revenue resulting from these newly introduced choice models, and finally also designs efficient algorithms to solve them.

Chapter 1 explores discrete choice models which capture consumer behavior and choices when faced with a set of different alternatives, and the resulting assortment optimization problem along with the different existing algorithms for solving them as well as the existing challenges therein. Chapter 2 models and solves the problem when the sellers have access to a vast array of inventory of products. Chapter 3 models dynamic preferences of consumers and the choice overload phenomenon when the customers are faced with a lot of options, and solves the ensuing optimization problem. Chapter 4 showcases the applicability and effectiveness of such models and approaches on high dimensional data from a field experiment on Flipkart, the largest e-commerce firm in India.
Table of Contents

List of Tables ........................................................................................................ v

List of Figures ......................................................................................................... vi

Acknowledgments ................................................................................................... viii

Dedication ................................................................................................................ xi

Introduction ............................................................................................................. 1

Chapter 1: Choice Model and Assortment Optimization ........................................ 3

1.1 Choice Models .................................................................................................. 3

1.1.1 Random Utility Models ............................................................................. 4

1.1.2 Markov Chain Choice Model ..................................................................... 6

1.2 Assortment Optimization .................................................................................. 7

1.2.1 Existing Results for Various Choice Models .............................................. 8

1.3 Existing Challenges in Choice Modeling and Assortment Optimization ......... 9

1.4 Summary of Contributions of Chapters 2, 3 and 4 ......................................... 10

1.4.1 Assortment Optimization Over a Dense Universe ................................... 10

1.4.2 Modeling Choice Overload and Dynamic Preferences ............................ 13

1.4.3 Empirical Evaluation of Effectiveness of Generalized MNL Model ........ 15
Chapter 2: Assortment Optimization Over a Dense Universe

2.1 Problem Formulation

2.1.1 Discrete Finite-Item Model

2.1.2 Continuous Product Space

2.2 Continuous-Space Assortment Optimization Problem

2.2.1 One Dimensional Case

2.2.2 Higher Dimensions

2.3 Performance Bounds of Nested-by-Revenue Policies in Discrete Models via Continuous Approximation

2.3.1 A Tighter Bound

2.4 Continuous Approximation of Discrete Models – Probabilistic Error Bounds

2.4.1 Discussion on the Bound

2.5 Numerical Results

2.6 Conclusion

Chapter 3: A Generalized Markov Chain Choice Model to Capture Choice Overload and Dynamic Preferences

3.1 Generalized Markov Chain Model and Notation

3.1.1 Customer Substitution Behavior in the Markov Chain Model

3.1.2 Substitution Behavior in Our Model

3.1.3 Assortment Optimization Problem

3.2 Computation of Choice Probabilities

3.2.1 Choice Probabilities

3.2.2 Examples
3.3 Generalized Multinomial Logit Model ........................................... 59
  3.3.1 Choice Probability ................................................................. 61
  3.3.2 Example ................................................................................. 62
  3.3.3 Parameter Estimation for Generalized MNL Model .................. 63
3.4 Assortment Optimization for the Generalized Multinomial Logit Model . 65
  3.4.1 NP-Hardness of the Assortment Optimization Problem ............ 66
  3.4.2 Our Algorithm for the FPTAS .................................................. 67
3.5 Generalized Markov Chain Model With Low Rank Matrix .................. 69
  3.5.1 Assortment Optimization and FPTAS ....................................... 70
  3.5.2 FPTAS for Generalized Markov Chain Model With Low Rank Matrix 71
3.6 Numerical Results ........................................................................ 74
3.7 Conclusion ................................................................................... 76

Chapter 4: Empirical Evaluation of Generalized MNL Model: Evidence From Flipkart 78
4.1 Introduction ............................................................................... 79
  4.1.1 Background ........................................................................... 80
  4.1.2 Problem Description ............................................................... 81
4.2 Data Description ......................................................................... 82
4.3 Multinominal Logit Model and Logistic Regression Model .................. 85
  4.3.1 Logistic Regression Model ....................................................... 85
  4.3.2 MNL Choice Model ................................................................. 86
  4.3.3 Results: Fit of Logistic Regression vs Fit of MNL Model ........... 87
  4.3.4 Discussion .............................................................................. 88
4.4 Generalized MNL Model ..................................................... 89
4.4.1 Results: Fit of the Generalized MNL Model ..................... 90
4.4.2 Discussion ................................................................. 91
4.5 Conclusion ................................................................. 92
Epilogue ............................................................... 93
References ............................................................... 95
Appendix A: Assortment Optimization Over a Dense Universe .......... 100
A.1 Proof of Lemma 2.2.1 ..................................................... 100
A.2 Proof of Lemma 2.2.2 ..................................................... 101
A.3 Proof of Lemma 2.2.3 ..................................................... 103
A.4 Proof of Lemma 2.2.4 ..................................................... 104
A.5 Proof of Lemma 2.2.5 ..................................................... 106
A.6 Other Lemmas Used ...................................................... 108
Appendix B: Generalized Markov Chain Choice Model ............... 111
B.1 Proof of Lemma 3.3.1 ..................................................... 111
B.2 Proof of Theorem 3.4.1 ................................................... 112
B.3 Proof of Theorem 3.4.2 ................................................... 118
B.4 Proof of Lemma 3.5.1 ..................................................... 120
B.5 Proof of Lemma 3.5.2 ..................................................... 121
B.6 Proof of Lemma 3.5.3 ..................................................... 125
B.7 Proof of Theorem 3.5.1 ................................................... 126
List of Tables

2.1 Approximation Ratio ......................................................... 47

4.1 Description of User Attributes ............................................ 82

4.2 Description of Widget Attributes ......................................... 83

4.3 Description of Other Attributes ........................................... 84
List of Figures

2.1 First part of Step 1. After splitting out the rightmost region, we focus on $b$ and show that the optimal $b = 1$. ................................................................. 26

2.2 Second part of Step 1. We show that gaps within $[a, 1]$ can be filled without decreasing expected revenue. ................................................................. 27

2.3 Illustration of Step 2. We show that gaps within $[a_0, a]$ can be filled without decreasing expected revenue. ................................................................. 28

2.4 Illustration of the setup for Step 3. We show that we can either fill in the "gap" or remove the rectangle $[a, b] \times [c, d]$ without decreasing the expected revenue. ................................................................. 29

2.5 Increasing Approximation Ratio With Increasing Total Items, $n$ (95% confidence bounds) ................................................................. 48

3.1 Example of A 4-Vertex Graph With $S = \{3, 4\}$ ................................................................. 55

3.2 Probability of No Purchase ................................................................. 58

3.3 Estimate of $\alpha$ vs number of iterations for the estimation algorithm. The feature vector was 4 dimensional and we had 10 products, i.e., $d = 4, n = 10$. We can see convergence after a few iterations of the alternating projection algorithm. ................................................................. 66

3.4 ROC Curves for MNL and Generalized MNL Models ................................................................. 76

4.1 (Left) Example of Flipkart’s Homepage. (Right) The enlarged widget, containing group of products. The widget on the top has products that is being pushed by the sales team with discounts, while the widget below has pre-selected smartphones. ................................................................. 80

4.2 Example of a homepage displaying widgets of similar theme ................................................................. 82
4.3 Fit of Logistic Regression and MNL Choice Model on Flipkart’s consumer click data. ................................................................. 87
4.4 Fit of Logistic Regression and Generalized MNL Choice Model on Flipkart’s consumer click data. .................................................. 91
Acknowledgements

When I look back into the five years of my PhD at Columbia, it just amazes me how recent it feels. I had no idea what I was getting into when I joined the program but needless to say it has been an awesome experience filled with a lot of ups and very few downs.

First and foremost, I’d like to express my sincere gratitude to my advisors Vineet and Henry. Vineet has been a tremendous support throughout my PhD stay and has been an equally wonderful person to get to know. I have learnt so much from him about not only research in general, including how to think about the problems, how to write papers and how to discuss and present ideas, but also about other things apart from research. He has always been very patient with me and I am grateful to him for believing in me. I really looked forward to the weekly meetings with him and thoroughly enjoyed talking about a lot of other things like politics, sports and science. I also want to give huge thanks to Henry. I still remember when we were looking for new ideas on our research problem and had asked him to join the project which he agreed to very enthusiastically. He was always available for discussing different ideas and his patience during the research was extremely encouraging and helpful. He literally had an open door policy and I fondly remember our late evening meetings when we discussed ideas and proofs in details which I really enjoyed as he was also a great person to talk to.

I would also like to thank Garud and Martin with whom I worked on my first research project. That was my initial exposure to independent research and I have learnt so much from them, most importantly thinking independently and presenting ideas on a new problem.
Special thanks to other members of my thesis committee: Adam, Omar and Van-Anh for their valuable feedback and suggestions on improving the thesis. Adam has been on the committee since my thesis proposal and his suggestions since then have been extremely helpful.

I have always enjoyed taking different courses and learning about new things and I have had the privilege of doing the same during my stay here with some outstanding and amazing teachers. I want to give special thanks to Prof. Stein, Prof. Yao and Prof. Haugh, whose courses I thoroughly enjoyed. I have learnt so many new things not only about the topics which were definitely helpful in my research, but also about how to teach and present them. I also want to thank Prof. Yao and Prof. Derman for giving me the opportunity to be the TA for their courses throughout my PhD. Teaching has always been very close to my heart and the various interactions and experience with students during this were a tremendous source of enjoyment for me during the same and had a very fulfilling effect on me.

An enormous thanks to all the support staff at the IEOR department. I truly believe that we have the best staff among all departments at Columbia. Huge thanks to Kristen, Liz, Carmen, Yosimir, Jaya, Shi and others for making our stay so smooth and for organizing so many unique and fun events to keep us sane during our time here.

A big thanks to all of my fellow PhDs in the department. I used to spend so much time in and around my office and have always found someone or the other to talk to about research and other general things. I believe our cohort was one of the most cohesive groups. I have made so many close friends throughout my stay and I really feel all of you were great. Thanks to Omar, Wenbo, Randy, Ryan, Allen, Min, Xuan, Mali, Mike, Nouri, Vashist, Rajan, Tom and ALDM. Special thanks to Vlada, Xingyu and Apurv for always being around and for discussing about research as well as a lot of stuff apart from it. A special mention to Raghav, who not only started in the program together with me, but we had even met during our campus visit to Columbia. We have spent so much time discussing about almost everything we could and I am so happy now that we are graduating within a few weeks of each other.
Lastly, I would like to express my sincere thanks from the bottom of my heart to my parents who are everything to me. The fact that I’m currently here is all because of them. I would also like to thank my sisters, other family members and my close friends for always being there for support and for always believing in me. I love you all.
To my sisters
Introduction

The electronic commerce is currently a multi-trillion dollar industry affecting over a billion people worldwide. Such an explosive growth has generated revived attention to the field of revenue management. Most of the big players in this industry such as Amazon run their operations on high volume and low margins and hence even small relative increases in revenues can result in high profits. Thus a critical aspect of their business is formulating predictive models to capture consumer behavior and then solving the resulting optimization problem to transpire a better decision making process, which ultimately helps in maximizing the firm’s revenue growth. The scope of revenue management currently also encompasses industries such as airlines, hotels and online advertising wherein by applying mathematical modeling and analytics, firms have enriched their decision making process and achieved better growth in revenues.

My dissertation contributes to the broader field of revenue management by focusing primarily on developing new discrete choice models and solving the resulting assortment optimization problem. The central theme of this dissertation is studying the situations when we have a multitude of options available to us. The first half of this dissertation focuses on the particular situation when sellers have access to a vast array of features to be chosen for products they want to offer; and the decision they have to make is which features to be chosen for those products to offer to the customers to maximize expected revenue. We formulate a new discrete choice model to tackle this setting and then also study and solve the ensuing assortment optimization problem of maximizing the expected revenue.
The second half of the dissertation focuses on the situation when customers are faced with a lot of options to choose from. We develop a new discrete choice model which naturally captures the dynamic preferences of the customers and their observed behaviors such as choice overload under such a setting. Once again, we also design efficient algorithms to solve the resulting assortment optimization problem of maximizing the expected revenue.

Chapter 1 explores discrete choice models which capture consumer behavior and choices when faced with a set of different alternatives, and the resulting assortment optimization problem along with the different existing algorithms for solving them. We also discuss the existing challenges in the choice modeling and assortment optimization. Chapter 2 models and solves the problem when the sellers have access to a vast array of inventory of products. Chapter 3 models dynamic preferences of consumers and the choice overload phenomenon, and solves the ensuing optimization problem. Chapter 4 showcases the applicability and effectiveness of such models and approaches on high dimensional data from a field experiment on Flipkart, the largest e-commerce firm in India.
Chapter 1: Choice Model and Assortment Optimization

1.1 Choice Models

Choice is prevalent in our day-to-day life. Which of these shirts on Amazon should I purchase? Where should I eat and which sushi should I choose? Which movie should I watch on Netflix? We are always making choices multiple times a day. Hence, trying to model how we choose amongst possible offered options has been a fundamental topic of research in many different academic fields including but not limited to transportation, marketing, economics, psychology and operations management. In most of the applications, our choice heavily depends on the options available to us. Underlying our choice is the fundamental phenomenon of substitution effect: when our most preferred option is not available we substitute to another option. Modeling this phenomenon is at the core of the theory of discrete choice modeling. Since the focus is on revenue management applications, we will refer to these options as products or items. Choice models make the demand for each product a function of the entire offer set.

Mathematically, a choice model specifies customer preferences in the form of a probability distribution over products in an offer set. More precisely, the choice model is defined by the following choice probabilities:

\[
\pi(j; S) = Pr(\text{customer selects product } j \text{ from offer set } S);
\]

where we assume that we have a universe \( \mathcal{N} \) consisting of \( n \) products such that \( j \in \mathcal{N} \) and \( S \subseteq \mathcal{N} \). We refer to \( \pi(j; S) \) as a choice probability. Such a model allows us to model the substitution behavior of the customers. It is easy to see that since the choice model
determines the demand and therefore the expected revenue, it is extremely important.

For example, having \( \pi(j; S) > \pi(j; S \cup \{i\}) \) captures a cannibalization of product \( j \) by product \( i \): when \( i \) is also offered along with \( j \), the demand for product \( j \) decreases. However, this flexibility comes at a cost. Indeed, note that such a model needs to specify the demand of each product for each of the \( 2^n \) possible subset \( S \subseteq N \). The theory of discrete choice modeling provides more parsimonious descriptions of these models by imposing some assumptions on the form of the choice probabilities. Many choice models have been studied in the literature (see [5] for a survey). Their considerations are often based on a balance between accuracy of the model (in approximating the reality) and tractability of the resulting parameter estimation and assortment optimization problems. In this dissertation, we study and enrich random utility models and Markov chain based choice model. These models addresses the modeling concerns of different customer preferences in a distinct fashion which we introduce and discuss. But first, we give a brief literature review for each of these types of choice models.

1.1.1 Random Utility Models

Classical economic theory posits that customers select an alternative by assigning a utility to each available option and then selecting the alternative with the maximum utility. This is the fundamental basis for the family of random utility maximization (RUM) based choice models. Under this framework, the (random) utility of product \( j \) for any customer is given by

\[
U_j = u_j + \xi_j,
\]

where \( u_j \) is the deterministic utility depending on various product attributes and \( \xi_j \) is the random component that also captures the idiosyncratic customer choice. A choice model based on RUM is specified by the choice of the deterministic utility \( u_j \) and the distribution of \( \xi_j \) and then we have

\[
\pi(j, S) = Pr \left( U_j \geq U_i, \forall i \in S \right).
\]
One of the most popular choice model in practice is the Multinomial Logit (MNL) model. The MNL model was introduced independently by [35] and [41], and later studied by [36]. In this model, the random component of the utility is assumed to be i.i.d. according to the standard Gumbel distribution. For the MNL model, the probability that a customer purchases the product $i \in S$ when the subset $S$ is offered is given by

$$
\pi(i, S) = \frac{e^{\mu_i}}{\sum_{j \in S} e^{\mu_j} + e^{\mu_0}} 1_{i \in S} =: \frac{v_i}{\sum_{j \in S} v_j + v_0} 1_{i \in S},
$$

where $v_j = e^{\mu_j}$.

However, this model suffers from certain simplifying assumptions, such as the IIA - Independence of Irrelevant Alternatives property (see [4] and [38]), which limit its applicability in many practical settings.

Therefore, more complex choice models have been developed to capture a richer class of substitution behaviors including the Nested Logit model and the mixture of Multinomial Logit model (see [37] and [44]). In the Nested Logit model (see [53] and [11]), the products are partitioned into nests.

In the mixture of Multinomial Logit model, we consider the population to be a mixture of several segments, each of which is given by a MNL. The probability of selecting product $i \in S$ when the set $S$ is offered is given by

$$
\pi(i, S) = \sum_{k=1}^{K} \alpha_k \frac{v^k_i}{\sum_{j \in S} v^k_j + v^k_0},
$$

where $K$ is the number of segments, $\alpha_k$ for all $k \in [K]$ denotes the probability that a random customer belongs to segment $k$ (hence $\alpha_1 + \ldots + \alpha_K = 1$), and $v^k \in Q^+_n$ for all $k \in [K]$ denote the MNL parameters for segment $k$.

This model is significantly more general than a single MNL, in that it can approximate arbitrarily closely any choice model arising from the theory of random utility maximization (RUM) (see [37]).
1.1.2 Markov Chain Choice Model

More recently, different approaches coming from the operations literature have emerged. [8] present a Markov chain based model where the substitutions are modeled via transitions in a Markov chain. The Markov chain based model was first considered in [54] and several variants of this model have since been considered (see for instance [14] and [40]). The main idea motivating the Markov chain (MC) model is to model a customer’s choice by explicitly modeling the substitution behavior. Here, the substitution is captured by a Markov chain, where each product corresponds to a state of the Markov chain, and substitutions are modeled using transitions in the Markov chain. Given an offer set, the states corresponding to the offered products become absorbing. A random customer arrives to each product according to some arrival probabilities. Upon arrival, the customer chooses the product if offered. Otherwise, the customer then substitutes according to the underlying transition probabilities of the Markov chain and continues to do so until they reach another offer product, at which point, they simply choose that product. In other words, in order to determine the chosen product for some random customer, we perform a random walk on the Markov chain and stop when we first hit one of the absorbing states. The corresponding product is chosen. Under this model, we can reformulate the choice probabilities as:

$$\pi(j; S) = Pr(\text{customer gets absorbed in state } j \text{ when subset } S \text{ of nodes is absorbing}).$$

Furthermore, [8] show that the Markov chain model captures the Multinomial Logit model exactly and provides a good approximation for any random utility based model under some fairly general assumptions.

In addition, ranking-based choice models have also been studied in the literature (see [30], [21], [19], [31], [45]). With the whole spectrum of choice models, finding the right model to capture customer preferences in a particular application is a challenging problem. This is especially true since we only observe sales and not the complete preferences of the customers.
1.2 Assortment Optimization

Assortment optimization is an important problem that arises in various industries such as online advertising, retailing and airline ticketing. In this problem, the aim of a seller is to select a subset of products from a universe of substitutable products to offer to customers in order to maximize the expected revenue. The objective is to find such an optimal subset to offer which maximizes expected revenue. More specifically, suppose we are given a universe of $n$ substitutable products, $\mathcal{N} = \{1, \ldots, n\}$ with exogenous prices $p_1, \ldots, p_n$. Once we fix a discrete choice model to be used, the choice probabilities $\pi(i, S)$ are given according to that particular model. Hence, the expected revenue of an offer set $S \subseteq \mathcal{N}$ can be calculated as:

$$R(S) = \sum_{i \in S} p_i \cdot \pi(i, S),$$

The assortment optimization problem which aims to maximize the expected revenue can be then simply formulated as

$$\max_{S \subseteq \mathcal{N}} R(S) \quad (1.1)$$

An important thing to make a note of here is the presence of an outside option which models the fact that a customer decides not to purchase anything. We denote it by product or item 0 and it is always included as a possible product in all discrete choice models and always assumed to be an option available to the customers for each assortment $S$. Note that the optimization in (1.1) is a combinatorial problem in general and hence simply trying all the possible $2^n$ assortments is not a viable solution. However particular structure in the choice probabilities and the expected revenue expressions might help us to formulate efficient algorithms to solve the optimization problem depending on the particular choice model used.
1.2.1 Existing Results for Various Choice Models

Multinomial Logit (MNL) model is by far the most popularly used model, thanks to both its convenience in the resulting parameter estimation, and simplicity in its assortment optimization. [47] show that both the estimation and the assortment optimization under the MNL model are tractable. More precisely, the optimal assortment is nested by price order, i.e. it is composed of the top $k$ priced products for some $k \leq n$. Several algorithms including greedy, local search and linear programming based methods are known to solve the assortment optimization problem efficiently under the MNL model ([47], [24], [17], and [30]). Both estimation and assortment optimization problems become more challenging under the Nested Logit model. For instance, the assortment optimization problem under this model is NP-hard in general [16]. [16] show that under specific assumptions this problem can become polynomially time solvable. Even for a mixture of MNL model with only two segments, [44] show that assortment optimization problem is NP-hard. Moreover, [18] show that under reasonable complexity assumptions, there is no polynomial time algorithm that gives an approximation for this model within a factor better than $\Omega(n^{1-\epsilon})$ for any $\epsilon > 0$ in general. So the mixture of MNL model is quite intractable.

Consequently, various approximation algorithms have been proposed, including [13] and [44]. For example, [13] show that the assortment optimization under a mixture of MNLs can be formulated as a mixed integer program. They demonstrate that a greedy heuristic performs quite well when compared with the optimal solutions obtained by the mixed integer program. However, it is difficult to evaluate the optimality gap of the greedy heuristic for large problem instances and there is no good upper bound on the optimal expected revenue. [44] analyze how well simple nested-by-revenue assortments (i.e., simple threshold policies based only on ordering the product revenue) perform against the optimal in the mixture of MNLs case. These guarantees and error bounds are mainly of two flavors. In one result, the approximation ratio scales with either the number of support points for the mixing distribution (hence it is only applicable to discrete mixing distributions) or with the
number of products (where the simple threshold policy is not asymptotically optimal). In the other result, the approximation guarantee scales logarithmically with the ratio of largest and smallest product revenues, with the optimality gap being arbitrarily large with respect to the number of products. The latter result is also discussed as a special case by [6], where they give bounds for general discrete choice models and improve the bound for mixture of MNLs, although the error bound has the same flavor.

Furthermore, [8] show that the assortment optimization under the Markov chain choice model can be solved efficiently; thereby, providing good balance between predictive power and tractability.

1.3 Existing Challenges in Choice Modeling and Assortment Optimization

This thesis elaborates and tackles important challenges which exist in choice modeling and the resulting assortment optimization literature. As discussed earlier, since mixture of Multinomial Logit model is the most general one within the class of RUM models (as it can approximate any RUM based choice model arbitrarily closely), it has been well studied and numerous attempts have been made to solve the resulting assortment optimization problem (which is NP-hard in general). Since the current approximation algorithms and the bounds provided by them are not general enough, we aim to study an alternate continuous space model, which we motivate in detail in Chapter 2 from natural settings and discuss the implications of solving the assortment optimization problem.

Given the multitude of options available to customers during this era of e-commerce and streaming applications, it becomes imperative to study how does this affect the customer choices and how do their preferences change. However, all standard random utility based models and distribution over ranking models suffer from two serious limitations in practice, namely, i) the models assume that customer preferences are static and exogenous to the set of products offered by the seller, and ii) the total probability of buying any product always increases (not necessarily strictly) if the seller adds more products to the assortment. In
many settings, these properties are not satisfied. In particular, the customers may form their preferences based on the offered set of products and also, the purchase probability might decrease when the seller adds more products to the assortment. This phenomenon which is well documented empirically is termed as choice overload and discussed in detail in Chapter 3. However, there is no choice model which can capture them in a natural and parsimonious manner. We present a new choice model to do the same and give efficient algorithms to solve the ensuing assortment optimization problem.

1.4 Summary of Contributions of Chapters 2, 3 and 4

As mentioned before, Chapter 2 focuses on assortment optimization over a dense universe ([26]) and Chapter 3 proposes a new choice model based on the Markov chain choice model to capture choice overload and the dynamic preferences of customers ([27]). Here, we summarize the main contributions of these chapters.

1.4.1 Assortment Optimization Over a Dense Universe

Chapter 2 makes the following contributions regarding the main insights:

Assortment Optimization over a Dense Universe is Easy. Our main assertion is that the optimal solution to the assortment problem in this continuous feature space, which models a dense collection of products, is an assortment region resulting from a simple nested-by-revenue policy. More precisely, we show that, among any possible assortment schemes which can be topologically represented by open sets containing assorted features, there exists a nested-by-revenue policy that is optimal. Importantly, this characterization holds regardless of the mixing distribution (including both discrete as well as continuous distributions) and the dimension of the feature space. Our result also implies that nested-by-revenue assortment is essentially optimal in this continuous-space model for any choice model arising from RUM, since mixed MNL can approximate any such model arbitrarily closely ([37]).
Near Optimality of Nested-by-Revenue in Discrete but Diverse Products Setting.

We translate our insights from the continuous-space model to assert that nested-by-revenue is close to optimal in settings with discrete but many diverse products. More precisely, we show that the optimality gap between the discrete-item revenue attained by assortment resulting from the best nested-by-revenue policy, and that obtained by the assortment from the global best policy, shrinks to zero under suitable scaling as the number of products increases. This optimality gap bound depends on the maximum revenue, an exponential moment of the mixing distribution, and a "similarity" measure that captures the number of items sharing the same revenue. This optimality gap that shows the near-optimality of nested-by-revenue in large problem instances is orthogonal to all previous works on the hardness of the assortment problem under mixed MNL.

We also derive another error bound that takes the form of a probabilistic concentration inequality on the gap between the (discrete-item) revenue attained by the nested-by-revenue policy and the global best policy, among a chosen policy class with a "complexity" measure expressible in terms of the problem dimension and the policy structure (e.g., tree-based). Although the implication of this bound is less desirable than the first deterministic bound due to the assumptions therein and the bound being probabilistic in nature, which we discuss in detail in the relevant section, the technique used to derive it is new. The probability in the bound is under the randomness of the product features which can be interpreted in the continuous-space model as "random vectors", and the gap holds with high probability under this randomness. As a key characteristic of this bound, the optimality gap in using nested-by-revenue policy decreases (with a constant high probability) with the number of products, ultimately achieving optimality when there are infinite products. In other words, the optimality gap shrinks to zero in probability. Thus, for an arbitrary large problem instance (in the sense of sampling of many products from the dense feature space), using nested-by-revenue is essentially the best possible. This insight is also orthogonal to all previous works on the hardness of the assortment problem under mixed MNL.
To achieve the above mentioned insights, we create the following three main technical contributions in our developments, which are the first within the assortment optimization literature as far as we know.

**Analytical Technique based on Set Perturbation.** We introduce a new analytical technique in showing the optimality of nested-by-revenue policy, via a perturbation analysis on the boundaries of open sets. A key element is the investigation of the "sign independence" of the derivatives on the mixture parameters, which allows us to reshape the assortment set in directions towards simple structures. This then invokes an iterative argument that sequentially reduces the number of connected sets in the assortment region via reshaping, ultimately leading to a nested-by-revenue policy that is optimal. This analysis leverages the continuous-space properties of the features and the assortment schemes, and applies to both continuous and discrete mixtures. Our proof of optimality is constructive, via the aforementioned iteration starting from any suboptimal set. This proof method is first in the assortment optimization context to our best knowledge and is significantly different from the combinatorial arguments proposed by all the previous works.

**"Super-Optimality Gap" from Continuous to Discrete Model.** Our second main technical contribution is the derivation of an optimality gap in using the best nested-by-revenue assortment, which comprises an optimal policy class in the continuous model, in the finite discrete-item setting. Deriving this gap uses a key perspective to view the continuous-space model as a discrete-item model with a relaxation on the integrality of products. In other words, by allowing a fractional selection of products, we obtain a super-optimal policy for the discrete-item model that is precisely nested-by-revenue. We then use this "super-optimality gap" to upper bound the optimality gap for the discrete-item problem, the former being readily analyzable thanks to the simplicity of the nested-by-revenue policy.

As mentioned earlier, for the probabilistic bound, although being a less desirable bound, we use the following new technique to derive the same.
Probabilistic approximation bound for the discrete case. Our third main technical contribution is the derivation of the probabilistic approximation bounds in using the nested-by-revenue policy obtained from a continuous-model in discrete-item settings. We utilize tools from the empirical process literature by viewing the distribution of features in the discrete-item setting as an empirical distribution in a "pre-limit", and the continuous-space model as the "population limit". These allow us to derive probabilistic concentration bounds in terms of the functional complexity of the policy class, encoded by the Vapnik-Chervonenkis (VC) dimension ([50]). Such types of bounds are also the first to appear in the assortment optimization literature as far as we know.

Finally, from the work of [37], it is well known that any choice model arising from the theory of random utility maximization (RUM) can be approximated arbitrarily closely by an appropriate mixture of MNLs. Observing this fact, we can extend our result of existence of a simple threshold policy for finding the optimal assortment (i.e., nested by revenue) to any choice model which originates from RUM theory, with some error.

1.4.2 Modeling Choice Overload and Dynamic Preferences

The main goal of Chapter 3 is to develop a model for substitutions that addresses the existing challenges discussed in Section 1.3 and lead to a more practical framework for choice modeling and assortment optimization. We propose a generalization of the Markov chain model introduced in [8], where we consider a Markovian comparison based choice process instead of one that is only based on Markovian substitutions. We decide to extend and generalize this choice model as we believe the Markov chain choice model presents a natural framework to capture these limitations as the substitution behavior of the customers are well modeled due to the way the customers make the choice (by doing a random walk on the offered products). Hence this is also naturally amenable to capture the search cost as well which in turn can model the customer preferences. Our model addresses the limitations of the random utility and rank-based choice models in the following sense:
**Dynamic Preferences.** Our model captures dynamic preferences of the customers. In practice, the substitution behavior of the customers in our model can be different for different assortments. More specifically, the implied distribution over preferences depends on the assortment offered and there may not be any single distribution over preferences that is consistent with choices for all assortments. Our model captures this and to the best of our knowledge, this is the first systematic approach to capture dynamic preferences.

**Choice Overload Phenomenon.** An important consequence of our model is capturing the choice overload phenomenon. In particular, the probability of purchase does not necessarily increase (which happens with all choice models arising from the theory of random utility maximization) if the seller includes more products in the assortment. More specifically, consider assortments $S, T \subseteq \{1, \ldots, n\}$ such that $S \subseteq T$. Then it is not necessarily true that $\pi(0, S) \geq \pi(0, T)$, where $\pi(0, S)$ denotes the probability of no purchase when the set of products offered is $S$. We present several examples illustrating this.

**Generalized Multinomial Logit Model** We consider the special case of the Generalized Markov chain model where the underlying Markov chain has rank one and name it as the Generalized Multinomial Logit model. [8] show that the Multinomial Logit model can be exactly captured by a Markov chain model where the transition probability matrix has rank one.

We study the assortment optimization problem under the above Generalized MNL model. Through multiple examples, we demonstrate that an optimal assortment in this new model balances between too few and too many choices and favors cluster centers. We show that the assortment optimization is NP-hard in general by a reduction from the partition problem. On the positive side, we present a fully polynomial time approximation scheme (FPTAS) for the assortment optimization problem. Our algorithm for the FPTAS is based on exploiting the structure of the choice probability expression and consequently the expected revenue function. In particular, we show that the choice probabilities for any given assortment exhibits a nice rational functional form. While the problem of revenue maximization is not
convex, we show that we can obtain a convex approximation of the objective function by guessing the values of a small number of linear functions for the optimal assortment. Our algorithm is a dynamic programming based algorithm that adapts ideas from the dynamic programming algorithm for the knapsack problem.

We also present a parameter estimation algorithm for the Generalized MNL model from a given choice dataset. We first show that the maximum likelihood estimation problem is not convex and then give an alternate projection algorithm which breaks the optimization in two parts and alternates between estimating the two different sets of parameters.

**Generalized Mixture of MNLs Model** We also consider the case where the underlying Markov chain has a low but fixed rank greater than one. This case resembles the Mixture of MNLs model and we call it as the Generalized Mixture of MNLs model. Although the choice probabilities do not admit a closed form anymore, we are still able to get an expression for the objective function for the assortment optimization problem (which is the expected revenue) in terms of a system of linear equations which can be solved efficiently for any given assortment $S$.

Our algorithm for the fully polynomial time approximation scheme (FPTAS) for the assortment optimization problem in this case of low rank transition matrix is then based on exploiting this linear system of equations’ structure of the expected revenue function. We guess the values of only a polynomial number of linear functions for the optimal assortment and show that our guesses are not too far away from the actual values. Our algorithm is once again a dynamic programming based algorithm.

1.4.3 Empirical Evaluation of Effectiveness of Generalized MNL Model

In Chapter 4, we present evidence of empirical gains from using Generalized MNL model to optimize product recommendations on Flipkart, the largest Indian e-commerce firm. First, we study whether the use of discrete choice models like MNL choice model can capture consumer preferences over an assortment better than the traditional models which consider
each item independently. In particular, in the MNL choice model, every item is described by a given set of attributes and the mean utility of a product is linear in the values of these attributes, and then the chosen item depends on the utilities of the offered items only (hence it will vary with the offered set accordingly). We compare this model against a Logistic regression based model with the same set of attributes but which considers each item in the same way irrespective of the offered set. We find out that the performance of both the Logistic regression based model as well as the MNL choice model are quite similar and discuss the possible reasons behind the same.

However, since the setting of product recommendations in Flipkart is done via the use of widgets and the performance is measured through the click through rates, the choice overload phenomenon is expectedly prevalent in such conditions. Hence, we also study the Generalized MNL model we develop in Chapter 3 under the same setting and show that this new choice model has a higher predictive power in capturing customer preferences in such settings.
Chapter 2: Assortment Optimization Over a Dense Universe

Given the challenges in previous approximation schemes for the assortment optimization problem in the mixture of MNLs model case as discussed in Section 1.3, in this chapter we investigate an alternate, continuous-space approximation to the assortment optimization problem under mixed MNL model with a general mixing distribution. More specifically, we consider situations where product space is large, i.e., the case where assortment optimization problem is most challenging, and which appears frequently in industries where sellers have access to a wide range of products with multiple features (e.g., these features include color, size, fabric type etc. in fashion, and memory size, processor speed etc. in laptops). The main intuition is that, in a dense product universe, the requirement of selecting an "integral" product can be viewed as insignificant, and instead we allow a "fractionalization" of products in our selection. We manifest this intuition via our "continuous space" model where all products are mathematically distributed under a continuous density in the feature space. This model, which removes the integrality of item selection, turns out to be very tractable by using a new analytical technique that we develop. Moreover, through linking this continuous-space model to the discrete finite-item setting, we reveal rigorous optimality gaps in the simple policy that we derive.

Outline Section 2.1 presents the details and relations between the discrete and continuous-space problems. Section 2.2 presents our main results and analyses on the continuous-space assortment optimization problem. Section 2.3 studies the optimality gap in using nested-by-revenue for the discrete model by leveraging our continuous-model results. Section 2.4 studies and derives a probabilistic error bound for the optimality gap in the same setting. Section 2.5 shows some numerical results to support our findings. Finally, Section 2.6 concludes.
2.1 Problem Formulation

2.1.1 Discrete Finite-Item Model

We first state a model with discrete-item formulation that will motivate our subsequent continuous approximation. In this model we have a set of $n$ distinct substitutable products. Suppose each product $j$ is characterized by a feature vector $x_j \in \mathbb{R}^d$, where typically one can choose the first component of $x_j$ as the price. We also denote $x_0$ as the no purchase option. Let $\mathcal{N} = \{x_1, \ldots, x_n\}$. For any offer set $S \subseteq \mathcal{N}$, we denote $\pi(j, S)$ as the choice probability of product $j$, for any $x_j \in S$.

Under the standard MNL model, given $\beta$,

$$\pi(j, S|\beta) = \frac{v_j}{v_0 + \sum_{x_i \in S} v_i},$$

where $v_j = e^{\beta^T x_j}, \beta \in \mathbb{R}^d$.

Under the mixed MNL model, we have a mixture distribution $G(\cdot)$ on $\beta$, which is a general distribution and can be either discrete or continuous:

$$\pi(j, S) = \int_\beta \frac{e^{\beta^T x_j}}{v_0 + \sum_{x_i \in S} e^{\beta^T x_i}} dG(\beta)$$  \hspace{1cm} (2.1)

Let $r_j$ be the revenue obtained from product $j$. Then the expected revenue for the offer set $S$ is given by:

$$R(S) = \sum_{x_j \in S} r_j \pi(j, S).$$

Hence the assortment optimization problem becomes:

$$\max_{S \subseteq \mathcal{N}} R(S) = \max_{S \subseteq \mathcal{N}} \int_\beta \frac{\sum_{x_i \in S} r_i v_i}{v_0 + \sum_{x_i \in S} v_i} dG(\beta)$$  \hspace{1cm} (2.2)

where $v_i = e^{\beta^T x_i}$. 

18
2.1.2 Continuous Product Space

Instead of using (2.2), we consider a continuous-space mixed MNL model where the feature vectors are regarded as densely distributed vectors on a continuous space. Namely, we write the assortment optimization problem as

$$\max_{S \subseteq [0, 1]^d} \int_{\beta} \mathbb{E}_x [r(x)v(x); x \in S] \frac{1}{1 + \mathbb{E}_x [v(x); x \in S]} dG(\beta),$$

(2.3)

where $v(x) = e^{\beta^T x}$ and $x$ is the vector of all the features. The revenue function, $r(x)$, depends only on the price, denoted by $x$ and is without loss of generality set as the first component of $x$ for convenience. The feature space that contains $x$ is assumed compact, which for simplicity we will set as $[0, 1]^d$, with $d$ the dimension of $x$, throughout this chapter. $\mathbb{E}_x [\cdot]$ denotes the expectation with respect to $x$, defined as

$$\mathbb{E}_x [\phi(x)] = \int_{[0, 1]^d} \phi(x) dF(x),$$

for any function $\phi(\cdot)$ integrable with respect to the probability distribution function $F(\cdot)$. We assume that $F(\cdot)$ has a density given by $f(\cdot)$, i.e., we can write

$$\mathbb{E}_x [\phi(x)] = \int_{[0, 1]^d} \phi(x) f(x) dx,$$

for any integrable function $\phi(\cdot)$. Note that, under the assumption that the product space is compact, the use of $[0, 1]^d$ is without loss of generality, as we can simply set the density function as zero for the region where no feature is located.

The model arising in (2.3) is motivated as follows. Suppose the number of products $n$ in (2.1) is big, then the objective function of (2.2), which can be rewritten as

$$\int_\beta \frac{1}{n} \sum_{i \in S} r_i v_i dG(\beta),$$

(2.4)
is intuitively close to
\[ h(S) = \int_{\beta} \frac{\mathbb{E}_x[r(x)v(x); x \in S]}{1 + \mathbb{E}_x[v(x); x \in S]} dG(\beta) \]  
(2.5)
where we have \( v(x) = e^{\beta^T x} \), and \( \mathbb{E}_x[r(x)v(x); x \in S] \) and \( \mathbb{E}_x[v(x); x \in S] \) are continuous approximations to \( (1/n) \sum_{x_i \in S} r_i v_i \) and \( (1/n) \sum_{x_i \in S} v_i \) respectively. Here we assume that the no-purchase option in the continuous-space model scales as \( \lim_{n \to \infty} = \frac{1}{n} \) where the constant 1 here is without loss of generality. Note that we can also intuit the above alternately by viewing \( x \) as random vectors in the space \([0, 1]^d\), where we use the law of large numbers to approximate (2.4) by (2.5). Regardless of how we think about this approximation, we will show rigorously in Section 2.3 how to translate our results for the continuous model in (2.3) into results for the discrete model in (2.2).

2.2 Continuous-Space Assortment Optimization Problem

Our goal in this section is to show that the nested-by-revenue policy is optimal for (2.3). We make the following assumptions on \( r(\cdot) \) and \( v(\cdot) \):

**Assumption 2.2.1** The revenue function \( r(\cdot) \) is non-negative and increasing in price \( x \), with \( r(0) = 0 \).

**Assumption 2.2.2** The revenue function \( r(\cdot) \) is bounded over the space of \( x \in [0, 1] \), i.e., \( \sup_{x \in [0,1]} r(x) = R_{\max} < \infty \).

**Assumption 2.2.3** The derivative of the revenue function \( r(\cdot) \) exists and is bounded over the space of \( x \in [0, 1] \), i.e., \( \sup_{x \in [0,1]} r'(x) = R'_{\max} < \infty \).

**Assumption 2.2.4** The function \( e^{2\|\beta\|} \) is integrable with respect to the distribution of \( \beta \), i.e., \( \mathbb{E}_\beta [e^{2\|\beta\|}] < \infty \) (here \( \| \cdot \| \) is the standard Euclidean norm).

Assumption 2.2.1 is standard on the revenue function in the literature. In fact, in most cases, we just have \( r(x) = x \). Assumptions 2.2.2-2.2.4 are used to guarantee the interchange
of derivatives and integrals that would be needed throughout our developments. The bounds $R_{\text{max}}$ and $R'_{\text{max}}$ can be arbitrarily large and are used only as technical conditions to ensure the interchange arguments. Assumption 2.2.4 is also needed in our optimality gap analysis for the discrete-item model later.

To ensure the problem is well-defined topologically, throughout our developments we consider $S$ in the collection of all open sets, denoted $O$. Along our way, we will also frequently use the collection of a finite union of rectangles, namely

$$\mathcal{R} = \left\{ S : S = \bigcup_{i=1}^{m} \mathcal{U}_i \text{ for some } m, \mathcal{U}_i = \prod_{j=1}^{d} [a_{ij}, b_{ij}], 0 \leq a_{ij} < b_{ij} \leq 1 \right\}$$

Note that the closure of $\mathcal{R}$ is $O$. Moreover, in our developments, since the density $f$ of $x$ exists, any lines have measure zero with respect to $F$, and we will not explicitly distinguish the occurrence of these measure-zero sets in our arguments to avoid technical nuisance.

### 2.2.1 One Dimensional Case

We start with the simplest case when the feature space is one-dimensional.

**Theorem 2.2.1** With Assumptions 2.2.1-2.2.4, among class of assortment regions $O$, an optimal assortment region for (2.3) results from a simple threshold policy, i.e., the optimal region is a single interval of the form $x \geq a^*$ for some $a^*$ depending upon the distribution of $\beta$ and the revenue function $r(\cdot)$.

Our proof uses a perturbation argument by showing that any assortment region not in the form $x \geq a^*$ can be perturbed and ultimately reshaped to a set in the latter form that has at least the same total revenue. Note that it suffices to consider sets inside $\mathcal{R}$ and conclude with taking the closure on these sets. The main argument consists of two steps. First we show that, for any $S \in \mathcal{R}$, the highest interval among the union has to include the highest revenue product. This result is reminiscent of the standard MNL choice model as well as mixed MNL models which are already known ([43] and [44]). In the second step, we then
argue that either filling in the "gap" between any two intervals or removing an interval would always lead to at least the same revenue, thus concluding that there exists an optimal assortment region in the form $x \geq a^*$. This second step is achieved by scrutinizing the sign of the derivative when perturbing the boundaries of each interval.

**Proof** Recall that we assume $x \in [0,1]$. We start with an arbitrary offer set in $R$. Let $[a, b]$ be the rightmost interval from this set, and write this offer set as $[a, b] \cup S$, where $\max_{x \in S} x < a$. So the objective function (expected revenue) in (2.3) becomes:

$$h((a, b), S) = \frac{\mathbb{E}_x [r(x)v(x); x \in (a, b) \cup S]}{1 + \mathbb{E}_x [v(x); x \in (a, b) \cup S]} dG(\beta)$$

(2.6)

**Step 1:** We first show that we can choose $b$ to be 1 in the set $[a, b] \cup S$ depicted above. This is summarized in the following lemma:

**Lemma 2.2.1** Keeping the remaining set $S$ (as defined above) fixed, the expected revenue is maximized when the interval $[a, b]$ considered is of the form $[a^*, 1]$.

**Proof** We show that the function $h$ in (2.6) is increasing in $b$, by considering its partial derivative with respect to $b$. For this, we abbreviate the function $h((a, b), S)$ as $h(b)$, with the terms not depending on $b$ as constants. By Lemma A.1.1 from the appendix, we get that

$$\frac{\partial h}{\partial b} \geq 0,$$

and hence optimal $b = 1$. □

**Step 2:** Now once we have the result from Step 1, we can assume that the total offer set is $S' = \{S \cup [a, b] \cup [a^*, 1]\}$, where $S \in R$, $b < a^*$ and $\max_{x \in S} x < a$. Keeping $S$ fixed, we want to characterize an optimal choice among the intervals $[a, b], [a^*, 1]$. We have the following result:
Lemma 2.2.2 Keeping $S$ (as defined above) fixed, the expected revenue is maximized when the pair of intervals $[a, b]$ and $[a^*, 1]$ considered above is a single interval of the form $[a^{**}, 1]$ for some $a^{**}$ such that $\max_{x \in S} x < a^{**}$.

Proof Considering $h$ as a function of $a, b$ and $a^*$, we consider the partial derivatives with respect to $a, b$ and in Lemma A.2.1 in the appendix we show that

$$\text{either } \frac{\partial h}{\partial a} \geq 0 \forall a \in [0, b], \text{ or } \frac{\partial h}{\partial b} \geq 0 \forall b \in [a, a^*].$$

We can see that for the first case, removing the interval $[a, b]$ leads to an objective value at least as good. Similarly, for the second case, merging the interval $[a, b]$ with $[a^*, 1]$, which reduces to one interval only, again leads to an objective value at least as good. Thus, in either case we would end up having one interval as an optimal region. □

Iterating the argument in Lemma 2.2.2, we see that we can keep reducing the number of considered intervals by one and ultimately, we are in the one-interval case, implying that an optimal offer set is ultimately of the form $[a^*, 1]$.

Finally, note that by Assumptions 2.2.1-2.2.4 the objective function is continuous (in set difference). Since any assortment region in $R$ has a revenue at most that of the nested-by-revenue, and that $R$ is dense in $O$, by taking the closure we conclude that any assortment region in $O$ also has a revenue at most that of the nested-by-revenue, the latter also inside $O$. This completes the proof of Theorem 2.2.1. □

As an aside, we also note that calculating the $a^*$ in this one-dimensional case, when we have only one feature, is just a line search (as we have to optimize over only one parameter, the revenue threshold). This line search can be efficiently done as the expected revenue for any threshold can be calculated as a simple one dimensional integral over the distribution of $\beta$. For an illustration, if $\beta \sim \mathcal{N}(0, 1)$, then we get $a^* = 0.280$. 

23
2.2.2 Higher Dimensions

In this case, we have multiple features: $x$ and $y$, where $x$ denotes the price and $y$ denotes the vector of all additional features, so that $x = [x \ y]^{\top}$. We extend the structural result from the one-dimensional case. Note that in the multi-dimensional problem, $R$ now consists of the union of all open hyper-rectangles. Let $y = [y_1 \ y_2 \ \ldots \ y_d]^{\top}$.

**Theorem 2.2.2** With Assumptions 2.2.1-2.2.4, among class of assortment regions $O$, an optimal assortment region $S \in O$ for problem (2.3), for general $d$, is a hyper-rectangle $[a^*, 1] \times [0, 1]^d$ for some $a^*$ depending upon the distribution of $\beta$ and the revenue function $r(\cdot)$.

**Proof** The proof is constructive and is presented in a few steps. Like in the one-dimensional case, we focus on $R$ and use a closure argument at the end to translate our conclusion to $O$. First, we show that all the products having maximum revenue (to be more precise, a nonzero-measure hyper-rectangle touching the entire maximum-revenue hyperplane $x = 1$) should be included in the optimal assortment region. This resembles and generalizes the one-dimensional case. Then, we show by construction that any region connected to such a hyper rectangle covering the maximum revenue can be "grown" to include everything on the $y$-dimension. This assures that the "right-most" hyper rectangle on the revenue axis always covers everything on the $y$-axis. In the next step, we show that when we encounter a region which is not connected to the hyper-rectangle covering the maximum revenue, we can either drop that region or extend it to connect it to the latter hyper-rectangle, thereby reducing the number of connected components by one. Lastly, by iterating the last two steps, each time reducing the number of connected components by one and expanding the size of the rightmost hyper-rectangle, we finally show that an optimal assortment region is connected and includes everything on the $y$-axis beyond a certain threshold on the $x$-axis.

While the iterative idea is similar in spirit to the second step of the one-dimensional case, the sequential growth of the region along the $y$-direction and the argument in reducing the number of connected components requires new developments in this higher-dimensional case.
We also demonstrate the individual steps involved in the proof via simple diagrams in two-dimensional setting, which capture the essence of those steps. The horizontal x-axis is revenue.

**Step 1** We first present the following result. As discussed in the proof of Theorem 2.2.1, the optimal assortment set always includes the highest revenue product in the one-dimensional case. This is true for higher dimensional feature space as well.

**Lemma 2.2.3** Let $R_{\text{max}}$ be the highest revenue among all products available in the universe (which without loss of generality we have assumed to be $R_{\text{max}} = 1$). Among the class of assortment regions $R$, an optimal assortment region includes a hyper-rectangle that touches the hyperplane $x = R_{\text{max}}$ ($x = 1$) and covers everything on the y-axis, i.e., it is of the form $[a, 1] \times [0, 1]^d$ for some $a < 1$.

A very similar result is already known in the literature for the finite product case ([44] and [43]). For the continuous-product case, this result asserts that we have all the products in a neighborhood of nonzero measure of $R_{\text{max}}$ in an optimal assortment region. We prove part of this statement via a similar technique as that of Lemma 2.2.1.

**Proof** Let us write a given assortment region (which is a union of hyper-rectangles) after picking out the rightmost rectangle, i.e., $S' = S \cup \{[a, b] \times [c, d]\}$, with the notation $[c, d] := [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_d, d_d]$ where $\max_{(x,y) \in S} x \leq b$. Now we find the optimal $b$ keeping other values fixed by a perturbation analysis. We have the following objective function:

$$h(b) = \int_\beta^1 \frac{\int_a^b \int_c^d r(x)v(x,y)dF(x,y) + c_1}{1 + \int_a^b \int_c^d v(x,y)dF(x,y) + c_2}dG(\beta),$$

where $c_1 = \mathbb{E}_x[r(x)v(x,y); (x,y) \in S]$ and $c_2 = \mathbb{E}_x[v(x,y); (x,y) \in S]$. By Lemma A.3.1 from the appendix, we get that

$$\frac{\partial h}{\partial b} \geq 0,$$

and hence optimal $b = 1$. This part is illustrated in Figure 2.1 (in the case where y is one-dimensional for the sake of visualization).
Figure 2.1: First part of Step 1. After splitting out the rightmost region, we focus on $b$ and show that the optimal $b = 1$.

Thus we show the expected revenue is not deteriorated if we lengthen the rightmost hyper-rectangle to include the highest revenue. Considering this hyper-rectangle, we can now re-write the assortment region $S'$ generally as the following union of disjoint sets: $S' = \{[a, 1] \times [c_1, d_1]\} \cup \cdots \cup \{[a, 1] \times [c_m, d_m]\} \cup S$, where $\max_{(x, y) \in S} x \leq a$ and $m \geq 1$. This turns out to be a special case of the setting of Lemma 2.2.4, the proof of which is presented later on. Using the result, we would get that the expected revenue is dominated by the region: $S' = \{[a, 1] \times [0, 1]^{d}\} \cup S$. We illustrate this part in Figure 2.2.

This completes the proof that we can include a nonzero-measure hyper-rectangle around the maximum-revenue hyperplane in an optimal set.

\[\square\]

**Step 2:** In the next step, we show that if there are hyper-rectangle(s) connected to the hyper-rectangle that touches the maximum-revenue hyperplane $x = 1$ and covers everything on the $y$-dimension (i.e., the hyper-rectangle obtained from Lemma 2.2.3 in **Step 1**), we can grow those hyper-rectangle(s) to include everything on the $y$-dimension and not decrease the expected revenue. To show this, we start with a structure of assortment region obtained in Step 1 and build on that.
Figure 2.2: Second part of Step 1. We show that gaps within $[a, 1]$ can be filled without decreasing expected revenue.

More precisely, consider the assortment region in the form of the union of three pieces: a "rightmost" hyper-rectangle, $m$ disjoint hyper rectangles that touch this hyper rectangle, and the remainder, i.e., $S' = S \cup \{[a_0, a] \times [c_1, d_1]\} \cup \cdots \cup \{[a_0, a] \times [c_m, d_m]\} \cup \{[a, 1] \times [0, 1]^d\}$, where $S \in \mathcal{R}$, $a_0 < a \leq 1$, $\max_{x \in S} x \leq a_0$ and $m \geq 1$. Keeping $S$ fixed, we want to characterize an optimal choice for $c_j$ and $d_j$. We have the following result:

**Lemma 2.2.4** Keeping $S$ as defined above fixed, the expected revenue is maximized when $\{[a_0, a] \times [c_1, d_1]\} \cup \cdots \cup \{[a_0, a] \times [c_m, d_m]\} \cup \{[a, 1] \times [0, 1]^d\}$ considered above collapses to become one hyper-rectangle extending from 0 to 1 on the $y$-axis, i.e., $S' = S \cup \{[a^*, 1] \times [0, 1]^d\}$ for some $a^*$.

**Proof** We use perturbation analysis to find the optimal values of $\{c_{jk}\}_{k=1}^d$ and $\{d_{jk}\}_{k=1}^d$ for $1 \leq j \leq m$. First, let us fix a dimension on the $y$-axis to be $k$. Then we have an ordering of $c_j$ and $d_j$ for this $k$-th dimension and without loss of generality, we can assume that the ordering is

$$c_{1k} < d_{1k} < \ldots c_{mk} < d_{mk}.$$
We show in Lemma A.4.1 in the appendix that for this case, we have:

\[ \frac{\partial h}{\partial c_{jk}} \leq 0 \text{ and } \frac{\partial h}{\partial d_{jk}} \geq 0. \]

This gives that the optimal values are \( c^*_j = d_{(j-1),k} \) and \( d^*_j = c_{(j+1),k} \), i.e., we can merge these two "adjacent" levels and not decrease the expected revenue. Hence the number of levels considered for the \( k \)-th dimension reduces by one. Iterating this, we get that we can merge all the levels and include everything from 0 to 1 on the \( y \)-axis for the \( k \)-th dimension while not decreasing the expected revenue. This argument is demonstrated in Figure 2.3.

![Figure 2.3: Illustration of Step 2. We show that gaps within \([a_0, a]\) can be filled without decreasing expected revenue.](image)

Lastly, since \( k \) was arbitrary, we repeat this argument for other dimensions. We note that the ordering might change for this newly considered dimension, but we can use Lemma A.4.1 with this new ordering and repetitively reduce the number of levels by one at each iteration, again without deteriorating the expected revenue. Ultimately, we would get that the optimal region becomes: \( S' = \{[a^*, 1] \times [0, 1]^d\} \cup S \). □

We note here that if we have \( a = 1 \) in this setting, it becomes the same case as required in the second part of the proof of Lemma 2.2.3 (with \( a \) in that setting being \( a_0 \) here).
Step 3: Next we show that if we have a region that is not connected to the hyper-rectangle covering the maximum revenue, i.e., the hyper-rectangle obtained from Lemma 2.2.4 in Step 2 or Lemma 2.2.3 in Step 1, we can either drop that region or extend it to connect it to the latter hyper-rectangle, thereby reducing the number of connected components by one. Formally, consider an assortment region \( S' = S \cup \{[a, b] \times [c, d]\} \cup \{[a^*, 1] \times [0, 1]^d\} \), where \( S \in \mathcal{R}, b < a^* \) and \( \max_{x \in S} x \leq b \). Keeping \( S \) fixed, we want to characterize an optimal choice among the hyper-rectangles \( \{[a, b] \times [c, d]\} \) and \( \{[a^*, 1] \times [0, 1]^d\} \). This setup is illustrated in Figure 2.4. We have the following result:

![Figure 2.4: Illustration of the setup for Step 3. We show that we can either fill in the "gap" or remove the rectangle \([a, b] \times [c, d]\) without decreasing the expected revenue.](image)

Lemma 2.2.5 Keeping \( S \) as defined above fixed, the expected revenue is maximized when the pair of hyper-rectangles \([a, b] \times [c, d]\) and \([a^*, 1] \times [0, 1]^d\) considered above is a single hyper-rectangle of the form \([a^{**}, 1] \times [0, 1]^d\) for some \( a^{**}\).

Proof The proof is very similar in spirit as that of Lemma 2.2.2. Since now we have to work with multiple dimensions instead of only one, we have to be more careful with the extra bookkeeping. Although the key ideas are similar to the one dimensional case, there are a few differences which are discussed next.
Viewing $h$ as a function of $a, b$ and $a^*$, we consider the partial derivatives with respect to $a, b$ and in Lemma A.5.1 in the appendix we show that

$$\text{either } \frac{\partial h}{\partial a} > 0 \forall a \in [0, b], \text{ or } \frac{\partial h}{\partial b} > 0 \forall b \in [a, a^*].$$

For the first case, removing the hyper-rectangle $[a, b] \times [c, d]$ leads to an objective value at least as good. Similarly, for the second case, merging the hyper-rectangle $[a, b] \times [c, d]$ with $[a^*, 1] \times [0, 1]^d$ leads to an objective value at least as good. Thus, we would either obtain an assortment region consisting of only a full hyper-rectangle of the form $[a^*, 1] \times [0, 1]^d$ (in the case if removing the disconnected smaller hyper-rectangle does not deteriorate the expected revenue) or the hyper-rectangle $[a^*, 1] \times [0, 1]^d$ joint with another hyper-rectangle connected to it but not necessarily covering everything on the $y$-axis, i.e., $[a, a^*] \times [c, d]$. We observe that the latter case reduces to the setting of Step 2 and we use the result of Lemma 2.2.4 to argue that we can include everything on the $y$-axis so that the region becomes $[a, 1] \times [0, 1]^d$. This completes the proof of the lemma.

**Step 4:** The final step iterates Step 2 and Step 3. Starting from the rightmost rectangle obtained from Step 1 that touches the maximum-revenue hyperplane and covers everything along the $y$-axis (obtained by Lemma 2.2.3), we look for either the first hyper-rectangle(s) that touch this hyper-rectangle, or the first disconnected hyper-rectangle(s). In the first case, we use Lemma 2.2.4 in Step 2 to "grow" a bigger rightmost hyper-rectangle; while in the second case we use Lemma 2.2.5 in Step 3 to reduce the number of connected sets, and moreover obtain a new rightmost rectangle by further using Lemma 2.2.4 again if needed. We iterate this process until eventually we get a single connected region as an optimal assortment set which covers everything from 0 to 1 on the $y$-dimension, i.e., $S' = [a^*, 1] \times [0, 1]^d$ for some $a^*$.  

30
Lastly, as in the one-dimensional case, note that by Assumptions 2.2.1-2.2.4, the objective function is continuous (in set difference). Since any assortment region in $\mathcal{R}$ has a revenue at most that of the nested-by-revenue, and from the fact that $\mathcal{R}$ is dense in $\mathcal{O}$, by taking the closure, we conclude that any assortment region in $\mathcal{O}$ also has a revenue at most that of the nested-by-revenue, the latter also inside $\mathcal{O}$. This completes the proof of Theorem 2.2.2. $\square$

2.3 Performance Bounds of Nested-by-Revenue Policies in Discrete Models via Continuous Approximation

In this section, we translate our insights from the optimal assortment in continuous models in Section 2.2 to discrete finite-item settings. In particular, we derive an explicit bound on the optimality gap (compared to the global optimum) in using the best nested-by-revenue assortment, which comprises an optimal policy class in continuous models, in the discrete setting.

Suppose we have $n$ products in the feature space $[0, 1]^d$ (or any bounded region; the $[0, 1]^d$ space is without loss of generality). We denote $S^*$ to be an optimal assortment for the finite-item problem in (2.7) among the class of all nested-by-revenue policies. Correspondingly, we denote $S^*_{nbr}$ to be optimal assortment for the finite-item problem in (2.7) among all the possible policies. Hence, we can see that the class considered to derive $S^*$ is potentially much bigger than that for $S^*_{nbr}$.

We consider the difference

$$Z(S^*) - Z(S^*_{nbr}),$$
which is the optimality gap between the best nested-by-revenue assortment versus the global best assortment for the model, measured by the achieved expected revenue. Our main result is the following bound:

**Theorem 2.3.1** Suppose Assumptions 2.2.2 and 2.2.4 hold. Then we have

\[
Z(S^*) - Z(S^*_{nbr}) \leq R_{\text{max}} \frac{n - m + 1}{v_0} E_\beta [e^{\|\beta\|}],
\]

where \( m \) denotes the total number of distinct revenue values for the \( n \) items and \( v_0 \) is the parameter representing the no-purchase option in (2.7).

We discuss some insights from Theorem 2.3.1. The bound (2.8) depends on the number of products \( n \), the number of products with distinct revenue values \( m \), the "no purchase" parameter \( v_0 \), the exponential moment of the mixing distribution for \( \beta \) in the MNL and the maximum revenue \( R_{\text{max}} \). Our bound works best when the products are "spread out" on the revenue axis, and when the no-purchase parameter \( v_0 \) is relatively big. The former means that the number of distinct revenue values \( m \) is equal or close to \( n \). In this case, \( n - m + 1 \) is a small number (as small as 1). For the latter, note that our bound is reciprocal in \( v_0 \). If \( v_0 \) is a large number, then the gap \( Z(S^*) - Z(S^*_{nbr}) \) is small. This case is the most prominent in practice as we expect the no-purchase probability to be significant. On the other hand, if \( v_0 \) is small compared to \( n \), say as small as 0, then it would mean that the optimal strategy is simply to put only the highest-revenue product into the assortment set. Our bound cannot well capture such an approximation. However, we can tighten the bound with the expense of more computation, which is stated in Theorem 2.3.2. To sum up, (2.8) is most useful in regimes where the approximation of discrete with continuous models applies as in Section 2.1.2. In particular, in the regime where \( v_0 \to \infty \) as \( n \to \infty \) and the products have different revenue values, the gap \( Z(S^*) - Z(S^*_{nbr}) \) converges to 0. This convergence scales linearly with both \( R_{\text{max}} \) and \( E_\beta [e^{\|\beta\|}] \).
We also discuss the main insights on how to derive Theorem 2.3.1. It uses two main arguments. First is that we can view the continuous-space model analyzed in Section 2.2 as a discrete finite-item model but with a relaxation on the integrality of products. This means that we allow a "fractional" selection of products in the assortment rather than restricting to selecting a whole item each time. In some sense, it is not the "continuity" of the space, but rather the "fractionality" in selecting products that is the key in driving our results in Section 2.2. Second, as Section 2.2 reveals, this fractional relaxation of the discrete finite-item model possesses an optimal assortment entailed by the nested-by-revenue policy. Because of the relaxation, this policy is "super-optimal", meaning that it attains a total revenue higher than the global best on the original discrete model. At the same time, this policy is easy to analyze. Therefore this gives us a tractable super-optimality gap to bound \( Z(S^*) - Z(S^*_{nbr}) \).

**Proof** We divide the proof into two steps according to the discussion above:

**Step 1.** We first argue that Theorem 2.2.2 applies to the fractional relaxation of (2.7) as follows. By a fractional relaxation we mean to allow selecting a \( W_i \) proportion of each product \( i \), where \( 0 \leq W_i \leq 1 \), so that

\[
Z(\tilde{S}) = \int_\beta \gamma_i r(x_i) v(\beta, x_i) dG(\beta),
\]

where we use \( \tilde{S} \) to denote a policy parameterized by

\[
\{\gamma_i, i = 1, \ldots, n : 0 \leq \gamma_i \leq 1\},
\]

and we abuse notation slightly to continue using \( Z(\cdot) \) as the objective function. We argue that an optimal policy for (2.9) is in the form

\[
\begin{cases}
\gamma_i = 0 & \text{for } i \text{ such that } r(x_i) < a \\
0 \leq \gamma_i \leq 1 & \text{for } i \text{ such that } r(x_i) = a \\
\gamma_i = 1 & \text{for } i \text{ such that } r(x_i) > a
\end{cases}
\]

(2.10)
for some threshold \( a \).

To this end, we consider a modification of the discrete model by placing, for each item \( i \), a uniform density on an \( \epsilon \)-sized ball centered at that item, denoted \( N(x_i; \epsilon) \) (if the item is at the boundary of the feature space, we put the uniform density on the truncated ball, which would not affect our subsequent argument). The distribution

\[
\sum_{i=1}^{n} \frac{1}{n} \text{Uniform}(N(x_i; \epsilon))
\]

defines an expectation \( E_x[\cdot] \) in

\[
\max_{S \subseteq [0,1]^d} \int_{\beta} \frac{E_x[r(x)v(\beta, x); x \in S]}{c + E_x[v(\beta, x); x \in S]} dG(\beta), \tag{2.11}
\]

which is the same as (2.3) except that \( c = v_0/n \) is used instead of 1.

On the other hand, (2.9) can be rewritten as

\[
Z(S^*) = \int_{\beta} \frac{\sum_{i=1}^{n} \frac{1}{n} \gamma_i r(x_i)v(\beta, x_i)}{c + \sum_{i=1}^{n} \frac{1}{n} \gamma_i v(\beta, x_i)} dG(\beta) \tag{2.12}
\]

Comparing (2.11) and (2.12), we see that, as \( \epsilon \to 0 \), any assortment region \( S \) for (2.11) reduces to \( \tilde{S} \) in (2.12). It is straightforward to check that all our arguments and results in Section 2.2 continue to hold when the constant 1 in (2.3) is replaced by \( c \). Thus, nested-by-revenue is optimal for (2.11), and in turn also optimal for (2.12) (seen via a standard "\( \epsilon \)-small" argument), which reduces to the form (2.10).

**Step 2.** We denote by \( \tilde{S}^* \) an optimal assortment for problem (2.9), which comprises of \( \gamma_i \)'s in the form (2.10). On the other hand, recall that we denote by \( S^*_{nbr} \) and \( S^* \) an optimal nested-by-revenue assortment and a globally optimal assortment for the original problem (2.7). We have the following relation:

\[
Z(S^*_{nbr}) \leq Z(S^*) \leq Z(\tilde{S}^*). \tag{2.13}
\]
The first inequality holds by the definition of global optimum for $S^*$. The second inequality holds since $Z(\tilde{S})$ relaxes the integral item selection in $Z(S)$ to fractional selection. Note that $\tilde{S}^*$ may not be implementable in reality, but it provides a tool for us to analyze the optimality gap on the implementable nested-by-revenue policy $S_{nbr}^*$.

Next we construct an assortment $\tilde{S}'$ that differs from $\tilde{S}^*$ by, for each fractional product (i.e., index $i$ where $0 < \gamma_i < 1$), either dropping or taking it completely (ideally, the complete dropping or taking is done in a way that gives the highest expected revenue, but this is not required). We say that $\tilde{S}'$ is the "de-fractioned" assortment of $\tilde{S}^*$. Since $S_{nbr}^*$ is the best nested-by-revenue assortment for (2.7), we have

$$Z(\tilde{S}') \leq Z(S_{nbr}^*).$$

(2.14)

Hence, combining the inequalities from (2.13) and (2.14), we get

$$Z(S^*) - Z(S_{nbr}^*) \leq Z(\tilde{S}^*) - Z(S_{nbr}^*) \leq Z(\tilde{S}^*) - Z(\tilde{S}').$$

(2.15)

Notice that both $\tilde{S}^*$ and $\tilde{S}'$ in the last difference of (2.15) are nested-by-revenue assortments, which we can explicitly analyze to approximate their difference. This provides a bound on our target optimality gap $Z(S^*) - Z(S_{nbr}^*)$ on the left hand side of (2.15).

To this end, consider the revenues of the $n$ products $r(x_1), \ldots, r(x_n)$, each of which take one of the $m$ possible revenue values: $r_1 > r_2 > \ldots > r_m$ (hence we must have $m \leq n$). Then there must exist a $k$ such that $\tilde{S}^* = \{\gamma_i, i = 1, \ldots, n : 0 \leq \gamma_i \leq 1\}$ consists of $\gamma_i = 0$ for all items $i$ such that $r(x_i) < r_k$, $\gamma_i = 1$ for all items $i$ such that $r(x_i) > r_k$, and $0 \leq \gamma_i \leq 1$ for all items $i$ such that $r(x_i) = r_k$. Then the de-fractioned assortment $\tilde{S}' = \{\gamma'_i, i = 1, \ldots, n : 0 \leq \gamma'_i \leq 1\}$ is also a nested-by-revenue assortment that has all $\gamma'_i = \gamma_i$ in $\tilde{S}^*$, except that $\gamma'_i = 0$ or 1 for all items $i$ such that $r(x_i) = r_k$. In the following, we take $\gamma'_i = 0$ for all these $i$'s.
We have

\[
Z(\tilde{S}^*) - Z(\tilde{S}') = \int \frac{1}{n} \sum_{i} \gamma_i r(x_i) \nu(\beta, x_i) dG(\beta) - \int \frac{1}{n} \sum_{i} \gamma_i' r(x_i) \nu(\beta, x_i) dG(\beta)
\]

\[
= \int \frac{1}{n} \sum_{j=1}^{k-1} r_j \sum_{i: r(x_i) = r_j} \nu(\beta, x_i) + \frac{r_k}{n} \sum_{i: r(x_i) = r_k} \gamma_i \nu(\beta, x_i)
\]

\[
- \int \frac{1}{n} \sum_{j=1}^{k-1} r_j \sum_{i: r(x_i) = r_j} \nu(\beta, x_i) + \frac{1}{n} \sum_{i: r(x_i) = r_k} \gamma_i \nu(\beta, x_i)
\]

\[
= \int f \left( t_1 + \frac{r_k}{n} \sum_{i: r(x_i) = r_k} \gamma_i \nu(\beta, x_i), \quad y_1 + \frac{1}{n} \sum_{i: r(x_i) = r_k} \gamma_i \nu(\beta, x_i) \right) dG(\beta) - \int f(t_1, y_1) dG(\beta)
\]

\[
\leq \int \frac{R_{\text{max}}}{nc} \sum_{i: r(x_i) = r_k} \nu(\beta, x_i) dG(\beta).
\]

In the above, we define

\[
f(t, y) = \frac{t}{c + y}, \quad c = \frac{v_0}{n}
\]

and

\[
t_1 = \frac{1}{n} \sum_{j=1}^{k-1} r_j \sum_{i: r(x_i) = r_j} \nu(\beta, x_i) \quad \text{and} \quad y_1 = \frac{1}{n} \sum_{j=1}^{k-1} \sum_{i: r(x_i) = r_j} \nu(\beta, x_i).
\]

The quantity \(\alpha\) in the gradient expression is a scalar between 0 and 1, and the dot in (2.16) is the standard dot product. The equality in (2.16) comes from applying the two-dimensional
mean value theorem on $f(t, y)$. Note that the gradient of $f(t, y)$ is given by

$$
\left( \frac{1}{c + y}, \frac{-t}{(c + y)^2} \right) .
$$

(2.19)

Hence, the dot product expression inside the integral in (2.16) becomes

$$
\frac{1}{c + y} \cdot \frac{r_k}{n} \sum_{i: r(x_i) = r_k} \gamma_i v(\beta, x_i) - \frac{t}{(c + y)^2} \cdot \frac{1}{n} \sum_{i: r(x_i) = r_k} \gamma_i v(\beta, x_i)
$$

where now $t$ and $y$ are given by

$$
t = t_1 + \alpha \frac{r_k}{n} \sum_{i: r(x_i) = r_k} \gamma_i v(\beta, x_i), \quad y = y_1 + \alpha \frac{1}{n} \sum_{i: r(x_i) = r_k} \gamma_i v(\beta, x_i); \quad 0 \leq \alpha \leq 1.
$$

Moreover, note that $t$ and $y$ are related by

$$
yR_{\min} \leq t \leq yR_{\max},
$$

where $R_{\max} = \sup_{x \in [0, 1]} r(x)$ as defined in Assumption 2.2.2 and $R_{\min} = \inf_{x \in [0, 1]} r(x)$. Hence, the dot product can be bounded by

$$
\left( \frac{R_{\max}}{c + y} - \frac{R_{\min} y}{(c + y)^2} \right) \frac{1}{n} \sum_{i: r(x_i) = r_k} \gamma_i v(\beta, x_i)
$$

Finally, a derivative calculation shows that the following function in $y$:

$$
\left( \frac{R_{\max}}{c + y} - \frac{R_{\min} y}{(c + y)^2} \right)
$$

is monotonically decreasing for $y \geq 0$ and hence is upper bounded by $\frac{R_{\max}}{c}$, giving us the bound in (2.17).
Hence from (2.15) and (2.17), we have

\[ Z(S^*) - Z(S^*_{nbr}) \leq \int \frac{R_{\text{max}}}{nc} \sum_{i : r(x_i) = r_k} v(\beta, x_i) dG(\beta) \]
\[ = \frac{R_{\text{max}}}{v_0} \int \sum_{i : r(x_i) = r_k} v(\beta, x_i) dG(\beta) \]
\[ \leq R_{\text{max}} \frac{n - m + 1}{v_0} \mathbb{E}_\beta [e^{\|\beta\|}] \]

The last step follows as the size of the set \{i : r(x_i) = r_k\} is bounded above by \((n - m + 1)\).

Observing that \(E_\beta [v(\beta, x_i)] \leq E_\beta [e^{\|\beta\|}]\), we get the required result. \(\square\)

### 2.3.1 A Tighter Bound

We close this section with a tighter bound on the optimality gap of nested-by-revenue policy for the considered discrete finite-item setting, directly using a super-optimality gap:

**Theorem 2.3.2** Suppose Assumptions 2.2.2 and 2.2.4 hold. We have

\[ Z(S^*) - Z(S^*_{nbr}) \leq Z(\tilde{S}^*) - Z(\tilde{S}'^*), \tag{2.20} \]

where \(\tilde{S}^*\) is an optimal policy for the fractionally relaxed problem in (2.9). This policy, parameterized by \(\{\gamma_i, i = 1, \ldots, n : 0 \leq \gamma_i \leq 1\}\), is nested-by-revenue in the form given by (2.10). On the other hand, \(\tilde{S}'^*\) is the best de-fractioned nested-by-revenue policy, which is obtained from \(\tilde{S}^*\) by optimally taking \(\gamma_i = 0\) or \(1\) for each \(i\) such that \(r(x_i) = a\), from the definition (2.10) for the policy \(\tilde{S}^*\).

The bound (2.20) is extracted from (2.15) in the proof of Theorem 2.3.1, except that \(\tilde{S}'\), obtained from setting \(\gamma_i = 1\) or \(0\) arbitrarily for any \(i\) satisfying \(r(x_i) = a\) in (2.10), is replaced by \(\tilde{S}'^*\), which chooses these \(\gamma_i\) to be \(1\) or \(0\) optimally. This bound is a super-optimality gap as it uses policy \(\tilde{S}^*\) that applies only for the hypothetical fractional relaxation of the problem. It performs tighter than the bound in Theorem 2.3.1, albeit with the expense of more computation. The improvement is foreseen to be more significant especially when
$v_0$ is small (e.g., 0), in which case $a$ in the definition (2.10) is foreseen to be close to highest revenue, so that (2.20) works reasonably while (2.8) is loose. Nonetheless, when $v_0$ scales properly with $n$, which is the norm situation, then (2.8) performs sufficiently tightly (see the experimental results in Section 2.5). The proof of Theorem 2.3.2 follows exactly the first half of that of Theorem 2.3.1, and is thus skipped.

Computing $Z(\tilde{S}^*)$ can be done by first setting the policy threshold $a$ at each possible revenue value $r_j$, $j = 1, \ldots, m$, and then solving the respective $\gamma_i$’s for each $i$ that has $r(x_i) = r_j$ (the possible "fractional" products). Then, $Z(\tilde{S}^*)$ and the best $r_j$ can be obtained by outputting the maximum and the corresponding argument over all these resulting expected revenues for $j = 1, \ldots, m$. Suppose the best threshold thus obtained is $r_k$ for some $1 \leq k \leq m$. Then, to compute $Z(\tilde{S}^*)$, we simply "de-fraction" the assortment $\tilde{S}^*$ by doing a brute-force search that sets $\gamma_i$ to be either 0 or 1 for all items $i$ with $r(x_i) = r_k$ (the "fractional" products).

Therefore, obtaining $Z(\tilde{S}^*)$ involves solving $m$ nonlinear optimization problems, each problem with a decision dimension equal to the number of products with $r(x_i) = r_j$. On the other hand, obtaining $Z(\tilde{S}^*)$ then involves solving one integer program with the same number of products. When there are few products that share the same revenue values, both $Z(\tilde{S}^*)$ and $Z(\tilde{S}^*)$ can be readily computed. For instance, if $m = n$ (i.e., no two products share the same revenue value), then $Z(\tilde{S}^*)$ requires $n$ line searches, and $Z(\tilde{S}^*)$ requires only a simple comparison between two values.

### 2.4 Continuous Approximation of Discrete Models – Probabilistic Error Bounds

In this section, we continue studying the optimality gap obtained when we apply the threshold policy concluded in Section 2.2 to (2.2). The error bound we derive is a probabilistic Chernoff-styled bound, where the randomness is with respect to the sampling of the many product features. We caution that these bounds are not deterministic worst-case bounds like that of the previous section and which have appeared in the past for assortment optimization problems resulting from other choice models.
To explain, let us denote

\[ Z_n(S) = \int_{\beta} \frac{\sum_{x_i \in S} r(x_i) v(\beta, x_i)}{v_0 + \sum_{x_i \in S} v(\beta, x_i)} dG(\beta) \quad (2.21) \]

as the objective function in (2.2), where we have used the notation \( v(\beta, x) = e^{\beta^T x} \) to highlight the dependence of each \( v_i \) on both \( \beta \) and \( x_i \). On the other hand, we denote

\[ Z(S) = \int_{\beta} \frac{\mathbb{E}_x [r(x) v(\beta, x); x \in S]}{1 + \mathbb{E}_x [v(\beta, x); x \in S]} dG(\beta) \quad (2.22) \]

as the objective function for the continuous-space model, where we again use the notation \( v(\beta, x) \) to highlight the dependence on \( \beta \). We denote by \( S^* \) as an optimal assortment region for the continuous-space optimization problem introduced in (2.3), which we have shown in Section 2.2 to be nested-by-revenue. Correspondingly, we denote \( S_n \) as an optimal assortment region for the finite-item problem (2.21). In particular, we will focus on \( S \) that belongs to a particular policy class \((\tilde{R})\), which has a finite VC-dimension, such as a collection of halfspaces or a union of open rectangles (both in the discrete and continuous-space settings).

We want to analyze

\[ Z_n(S_n^*) - Z_n(S^*), \]

which is the error in using the solution from the continuous-space model for a finite-item problem. We have the following approximation error:

**Theorem 2.4.1** Suppose Assumptions 2.2.2 and 2.2.4 hold. We consider the scaling that as \( n \to \infty, v_0/n = 1 \) and \( x_i \) are i.i.d. following a distribution that generates an expectation \( \mathbb{E}_x[\cdot] \). Suppose we optimize over \( S \in \tilde{R} \) in both the objective functions (2.21) and (2.22). Then we have

\[ \Pr(Z_n(S_n^*) - Z_n(S) > t) \leq 2 \left( \frac{c^* \sqrt{n}}{2R_{\max} \sqrt{V}} \right)^{2V-2} e^{-n^2/2R_{\max}^2} \]

where \( c^* \) is a universal constant and \( V \) is the VC-dimension of the policy class \( \tilde{R} \) capturing
its complexity.

**Proof** Consider

\[ Z_n(S_n^*) - Z_n(S^*) = [Z_n(S_n^*) - Z(S_n^*)] + [Z(S_n^*) - Z(S^*)] + [Z(S^*) - Z_n(S^*)] \] (2.23)

The middle term above is bounded from above by zero since \( S^* \) is the maximizer of \( Z(\cdot) \) by definition. Thus (2.23) can be further bounded from above by

\[ \sup_{S_n^* \in \tilde{R}} |Z_n(S_n^*) - Z(S_n^*)| + |Z(S^*) - Z_n(S^*)| \]

or simply

\[ 2 \sup_{S_n^* \in \tilde{R}} |Z_n(S_n^*) - Z(S_n^*)| \] (2.24)

We show a probabilistic bound for (2.24).

Note that the class of indicator functions

\[ \{I(\cdot \in S) : \mathbb{R}^d \to \mathbb{R} | S \in \tilde{R}\}, \]

where the collection \( \tilde{R} \) is a VC-class of sets, is a VC-class of functions (also called simply as VC-class) with the same index (see Exercise 9 in Chapter 6, [49]). Next, we can also see that, since \( \mathbb{E}_\beta[v(\beta, \cdot)] \) is finite (as mentioned as an assumption in Theorem 2.4.1) and using the fact that \( \tilde{R} \) resides in a compact space,

\[ \{\mathbb{E}_\beta[v(\beta, \cdot)]I(\cdot \in S) : \mathbb{R}^d \to \mathbb{R} | S \in \tilde{R}\} \] (2.25)

is also a VC-class with the same index (using Theorem 2.6.18 from [49]).

Next, note that \( r(\cdot) \) is monotone and uniformly bounded (from Assumption 2.2.2). Then,
again by Theorem 2.6.18 from [49], we conclude that

\[ \{ r(\cdot) \mathbb{E}_\beta [\nu(\beta, \cdot)] I(\cdot \in S) : \mathbb{R}^d \to \mathbb{R} | S \in \tilde{R} \} \] (2.26)

is also a VC-class with the same index as that for (2.25).

We then use Theorem 2.14.9 in [48], which we present here for completeness and so that we can detail out the constants appearing in the bound. We have

**Theorem 2.4.2** Consider the empirical process \( G_n = \sqrt{n}(\mathbb{P}_n - \mathbb{P}) \) where \( \mathbb{P}_n \) denotes the empirical measure for \( \mathbb{P} \). Let \( \mathcal{F} \) be a class of measurable functions \( f : X \to [0, 1] \) that satisfies the condition of having a polynomial covering number (which holds true for a VC-class of functions with index \( V \) by Theorem 2.6.7 from [49]). Then, for every \( t > 0 \),

\[ Pr^*(\|G_n\|_\mathcal{F} > t) \leq \left( \frac{Dt}{\sqrt{2V - 2}} \right)^{2V-2} e^{-2t^2}, \]

where \( D \) is a constant depending only on the VC-dimension of class \( \mathcal{F} \), denoted by \( V \).

We discuss in Lemma A.6.1 in the appendix about the exact dependence of \( D \) on \( V \) and show that the above bound reduces to

\[ \left( \frac{c^* t}{\sqrt{V}} \right)^{2V-2} e^{-2t^2}, \]

where \( c^* \) is a universal constant. Using the above result in our setting, we have

\[ Pr \left( \sup_{S \in \tilde{R}} \left| \frac{1}{n} \sum_{X_i \in S} \mathbb{E}_\beta [\nu(\beta, X_i)] - \mathbb{E}[\mathbb{E}_\beta [\nu(\beta, X)]; X \in S] \right| > t \right) \leq \left( \frac{c^* t \sqrt{n}}{\sqrt{V}} \right)^{2V-2} e^{-2nt^2} \quad (2.27) \]

and similarly

\[ Pr \left( \sup_{S \in \tilde{R}} \left| \frac{1}{n} \sum_{X_i \in S} r(X_i) \mathbb{E}_\beta [\nu(\beta, X_i)] - \mathbb{E}[r(X) \mathbb{E}_\beta [\nu(\beta, X)]; X \in S] \right| > t \right) \leq \left( \frac{c^* t \sqrt{n}}{\sqrt{V}} \right)^{2V-2} e^{-2nt^2} \quad (2.28) \]
Now consider
\[
\left| \int \frac{\sum_{x \in S} r(X) \nu(\beta, X)}{1 + \sum_{x \in S} \nu(\beta, X)} dG(\beta) - \int \frac{\mathbb{E}[r(x) \nu(\beta, x); x \in S]}{1 + \mathbb{E}[\nu(\beta, x); x \in S]} dG(\beta) \right|
\]
\[
= \left| \int \frac{\mathbb{E}[r(x) \nu(\beta, x); x \in S] + \left( \frac{1}{n} \sum_{x \in S} r(X) \nu(\beta, X) - \mathbb{E}[r(x) \nu(\beta, x); x \in S] \right)}{1 + \mathbb{E}[\nu(\beta, x); x \in S]} \right| dG(\beta)
\]
\[
= \left| \int f \left( t_1 + \frac{1}{n} \sum_{x \in S} r(X) \nu(\beta, X) - \mathbb{E}[r(x) \nu(\beta, x); x \in S], \ y_1 + \frac{1}{n} \sum_{x \in S} \nu(\beta, X) - \mathbb{E}[\nu(\beta, x); x \in S] \right) dG(\beta)
\]
\[
- \int f(t_1, y_1) dG(\beta)
\]
\[
= \int \nabla f \left( t_1 + \frac{1}{n} \sum_{x \in S} r(X) \nu(\beta, X) - \mathbb{E}[r(x) \nu(\beta, x); x \in S], \ y_1 + \frac{1}{n} \sum_{x \in S} \nu(\beta, X) - \mathbb{E}[\nu(\beta, x); x \in S] \right)
\]
\[
\cdot \left( \frac{1}{n} \sum_{x \in S} r(X) \nu(\beta, X) - \mathbb{E}[r(x) \nu(\beta, x); x \in S], \ \frac{1}{n} \sum_{x \in S} \nu(\beta, X) - \mathbb{E}[\nu(\beta, x); x \in S] \right) dG(\beta)
\]
\[
\leq \int R_{\max} \left( \frac{1}{n} \sum_{x \in S} \nu(\beta, X) - \mathbb{E}[\nu(\beta, X)] \right) dG(\beta). \tag{2.29}
\]

In the above, we have used the same definition of the function $f(t, y)$ as defined earlier in 2.18. Using the same arguments of first applying the two-dimensional mean value theorem on $f(t, y)$ and then bounding the resultant dot product involving its gradient, we get the inequality in (2.29).

From (2.29), we have
\[
\sup_{S \in \mathbb{R}} \left| \int \frac{\sum_{x \in S} r(X) \nu(\beta, X)}{1 + \sum_{x \in S} \nu(\beta, X)} dG(\beta) - \int \frac{\mathbb{E}[r(x) \nu(\beta, x); x \in S]}{1 + \mathbb{E}[\nu(\beta, x); x \in S]} dG(\beta) \right|
\]
\[
\leq \sup_{S \in \mathbb{R}} \int R_{\max} \left( \left| \frac{1}{n} \sum_{x \in S} \nu(\beta, X) - \mathbb{E}[\nu(\beta, X)] \right| \right) dG(\beta)
\]

43
so that we can apply (2.27) and (2.28) to get a uniform error bound of the form

\[
Pr \left( \sup_{S \in \tilde{R}} \left| \sum_{X_i \in S} r(X) v(\beta, X) \right| n + \sum_{X_i \in S} v(\beta, X) dG(\beta) - \int \left[ \mathbb{E}[r(x) v(\beta, x); x \in S] + \mathbb{E}[v(\beta, x); x \in S] dG(\beta) \right] > t \right)
\]

\[
\leq Pr \left( \sup_{S \in \tilde{R}} \int R_{\max} \left( \left| \frac{1}{n} \sum_{X_i \in S} v(\beta, X) - \mathbb{E}[v(\beta, X)] \right| dG(\beta) > t \right) \right)
\]

\[
\leq \left( \frac{c^* t \sqrt{n}}{R_{\max} \sqrt{V}} \right)^{2V-2} e^{-2n^2/R_{\max}^2}
\]

where the last inequality follows from (2.27) and (2.28). Using the definitions of \(Z_n(\cdot)\) and \(Z(\cdot)\) in (2.21) and (2.22), we have

\[
Pr (Z_n(S_n^*) - Z_n(S^*) > t) \leq Pr (2 \sup_{S_n^* \in \tilde{R}} |Z_n(S_n^*) - Z(S_n^*)| > t)
\]

\[
\leq 2 \left( \frac{c^* t \sqrt{n}}{2R_{\max} \sqrt{V}} \right)^{2V-2} e^{-n^2/2R_{\max}^2}
\]

This proves the theorem. \(\square\)

### 2.4.1 Discussion on the Bound

The implication of this bound is less desirable as compared to the earlier deterministic bound due to the assumptions on the distribution of features of products in the continuous space that would not necessarily capture many realistic settings as sellers typically don’t draw their products at random from the space of possible features. However when such a situation is possible, the bound can be used as follows. Theorem 2.4.1, which is a Chernoff styled probabilistic bound, stipulates that the optimality gap in using a nested-by-revenue policy that targets at a continuous-space model, for the discrete-item setting, shrinks to zero in probability as the number of sampled products increases. Alternately, we see that the optimality gap can be kept small with a constant probability. In particular, this gap is of order \(\Theta(\sqrt{\frac{V}{n}})\) where \(V\) is the complexity (as measured by its VC-dimension) of the policy class \(\tilde{R}\). This implies we need \(n = \Omega(V)\) products to control the approximation error in
using nested-by-revenue. If the policy class has a polynomial VC-dimension, say \( V = \Theta(d) \), then having only a polynomial number of products suffices. Note that more products is always beneficial for using nested-by-revenue, as the optimality gap only gets better when \( n \) increases.

Here, we present a few examples of policy classes \( \tilde{\mathcal{R}} \) with their respective VC-dimensions.

1. Halfspaces: Let \( \mathcal{H} \) denote the family of policies defined by halfspaces, i.e.,

\[
\mathcal{H} = \{ S_{a,b} : a \in \mathbb{R}^d, b \in \mathbb{R} \}, \quad \text{where} \quad S_{a,b} = \{ \beta : \beta^\top a \leq b \}.
\]

By the definition of VC-dimension, we get that \( \text{VC}(\mathcal{H}) = d + 2 \).

2. Collection of \( k \) halfspaces: Let \( \mathcal{H}(k) \) denote the family of policies defined by a collection (either union or intersection) of \( k \) different halfspaces, i.e.,

\[
\mathcal{H}(k) = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k, \quad \text{or} \quad \mathcal{H}(k) = \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_k,
\]

where \( \mathcal{H}_i \) is a halfspace as defined above. [15] show that \( \text{VC}(\mathcal{H}(k)) = \Theta(dk \log k) \).

3. Union of \( k \) axis-aligned rectangles: Let \( \mathcal{S}(k) \) denote the family of policies defined by a union of \( k \) different axis-aligned hyper-rectangles in \( \mathbb{R}^d \), i.e.,

\[
\mathcal{S}(k) = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_k, \quad \text{where} \quad \mathcal{U}_i = \prod_{j=1}^d [a_{ij}, b_{ij}].
\]

From the definition of VC-dimension and [15], we get that \( \text{VC}(\mathcal{S}(k)) = \Theta(dk \log k) \).

Hence, even when we have a policy class which consists of unions of an exponential number of axis-aligned hyper-rectangles, the latter already a very rich policy class, its VC dimension would be \( \Theta(2^d) \). In this case, having \( n = \Omega(2^d) \) products suffices to control the optimality gap.
2.5 Numerical Results

In this section we present numerical results on real data. We use the popular sushi choice dataset from [32], which is publicly available, for this experiment. The main goal here is to study the optimality gap when we use a nested-by-revenue assortment to demonstrate the bound obtained in Theorem 2.3.1. As mentioned earlier, in general, we neither know the optimal assortment nor do we have any handle on its properties. But in this case, we have extra information on the choice ranking which helps us get the optimal assortment via an alternate optimization that will be discussed momentarily.

**Setup.** The sushi choice data has the rank preferences of 5000 different people (collected via a questionnaire) where they rank their 10 most preferred sushi out of a list of 100 available sushi. The data also has a list of features for each sushi and there are a total of 7 features available in the dataset like type of sushi, oiliness in taste, price, frequency of consumption etc.

Since we have rank information for each person, we can easily find the expected revenue for an offered assortment: each person chooses the first sushi available to them according to the preference list; if none of the preferred sushi is offered, they do not choose anything (the no-purchase). For this setting, the optimal expected revenue and optimal assortment can be found by solving a mixed integer linear program by using the rank-based assortment optimization formulation given in [7]. We find the best nested-by-price assortment and compare its expected revenue against the optimal. Note that here we have implicitly assumed the choices follow a choice model based on RUM, which can be expressed as a mixture of MNLs ([37]). Hence, we do not need to fit a separate mixture of MNLs model and estimate their parameters.

To demonstrate the result of Theorem 2.3.1, we study how the optimality gap between using the best nested-by-revenue assortment and the global optimal assortment changes with number of products $n$. For this, we take $n$ products (sushi) out of the 100 available, find the
optimal expected revenue by the mixed integer formulation described above, and compare it with the best expected revenue among the nested-by-revenue assortments. According to the result, the gap should decrease with increasing $n$.

**Results.** For every $n$, we repeat 30 times, each time by randomly subsampling a portion of the 100 products, and compare the best expected revenue from the threshold policy and the optimal expected revenue by solving the corresponding mixed integer linear program. We calculate the approximation ratio, namely the ratio of these two expected revenues (so this is at most 1 and a higher value is better).

We observe that the optimality gap in expected revenue is small and the approximation ratio is about 0.99 when $n$ is 50 or above. The optimality gap decreases with increasing $n$ as evident in Table 2.1 and Figure 2.5. We also report the 95% confidence intervals for the values to illustrate that our estimates are all sufficiently accurate.

<table>
<thead>
<tr>
<th>Total Items ($n$)</th>
<th>Approximation Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.9795 ± 0.003</td>
</tr>
<tr>
<td>30</td>
<td>0.9843 ± 0.002</td>
</tr>
<tr>
<td>40</td>
<td>0.9890 ± 0.003</td>
</tr>
<tr>
<td>50</td>
<td>0.9906 ± 0.004</td>
</tr>
<tr>
<td>60</td>
<td>0.9908 ± 0.004</td>
</tr>
<tr>
<td>70</td>
<td>0.9917 ± 0.003</td>
</tr>
<tr>
<td>80</td>
<td>0.9925 ± 0.003</td>
</tr>
<tr>
<td>90</td>
<td>0.9926 ± 0.002</td>
</tr>
</tbody>
</table>

Another interesting observation is that the optimal nested-by-revenue assortment is smaller in size (which has 17 products) than the globally optimal assortment set (which has 21 products) for $n = 100$ case. This is also better as a smaller offer set is usually preferable in practice.
2.6 Conclusion

In this chapter we study assortment optimization under mixed MNL model, a common class of choice models that can arbitrarily closely approximate any choice model based on RUM, but whose resulting optimization is known to be generally NP-hard to approximate within any reasonable factor. We show that a simple nested-by-revenue policy is optimal under mixed MNL model when products are dense, regardless of the mixture distribution and the feature space dimension. We argue this using a continuous-space model where the product features are densely distributed. We also further translate our insight from the continuous-space model to the discrete finite-item setting by establishing an optimality gap.
bound on the nested-by-revenue policy in the discrete case, which works tightly in the regime of many diverse products where a continuous approximation is suitable.

Our results thus advocate the use of nested-by-revenue policies when facing a multitude of products, which is frequently encountered in the industries where sellers have access to products with features from a continuum.

To show the optimality of nested-by-revenue policy, we introduce a new technique based on perturbation analysis on the offer set and an iterative construction via sequential set reshaping. To our best knowledge, these techniques are new and distinct from all previous works in the assortment optimization literature.

On the other hand, to analyze the discrete-item optimality gap, we consider a nested-by-revenue "super-optimal" policy on a fractional relaxation of the problem, which is closely linked to the continuous-space model. This super-optimality gap technique also appears new in the literature. For future directions, we can study constrained assortment optimization problem using our new analysis approaches and framework where the product space is dense, starting from simple cardinality constraints to more general capacity constraints.
Chapter 3: A Generalized Markov Chain Choice Model to Capture Choice Overload and Dynamic Preferences

As discussed in Section 1.3, all standard random utility based models and distribution over ranking models suffer from two serious limitations in practice, namely, \( i \) the models assume that customer preferences are static and exogenous to the set of products offered by the seller, and \( ii \) the total probability of buying any product always increases (not necessarily strictly) if the seller adds more products to the assortment. In many settings, these properties are not satisfied. In particular, the customers may form their preferences based on the offered set of products and also, the purchase probability might decrease when the seller adds more products to the assortment. The latter is referred to as the choice overload phenomenon and has been observed empirically in practice (see [29] and [46]). None of the random utility models in their fundamental and standard form can capture this. However, at least in principle, some modifications of existing methods might be able to achieve that. For example, one could potentially estimate a different logit or mixed logit model for each assortment offered to consumers, which would allow the preference parameters to vary with assortment. This seems to be a flexible way of letting the estimated demand functions depend on the products offered and thus should be able to capture choice overload. But in practice, this approach will not be viable if the number of observations per assortment is small or the number of products itself is large. Hence more parsimonious models are better. [51] propose a model that incorporates an explicit search cost for users and can capture choice overload in some settings. [52] also consider a choice model with endogenous network effects that capture dynamic preferences in some settings, mainly, where the utility of product for a customer depends on the number of customers interested in that product.
Outline The rest of the chapter is organized as follows. In Section 3.1, we present the Generalized Markov Chain model and our notation. In Section 3.2, we present choice probability computations and we also discuss some examples. In Section 3.3, we present the properties of the Generalized Multinomial Logit model, a special case of the previous model along with an algorithm for its parameter estimation. Section 3.4 discusses the assortment optimization under the Generalized MNL model and shows it is NP-hard. We also present a fully polynomial-time approximation scheme (FPTAS) for the assortment optimization problem. In Section 3.5, we present results on another case of the Generalized Markov chain model, where the initial transition matrix is of low rank, and show that this generalizes the mixture of MNLs model. These results allow us to present an FPTAS for this model as well. Finally, we include some numerical results on real-life data in Section 3.6 and conclude in Section 3.7.

3.1 Generalized Markov Chain Model and Notation

In this section we present the Generalized Markov Chain Model and the notation that we use for the rest of the chapter. We assume that we are given a universe of $n$ substitutable products indexed from 1 to $n$: $\mathcal{N} = \{1, \ldots, n\}$. Given this set, we first construct a directed graph $\mathcal{G}$ and give the customers’ substitution behavior on this graph.

3.1.1 Customer Substitution Behavior in the Markov Chain Model

We model the customer substitution behavior using transitions on a Markov chain on $(n + 1)$ states, where there is a state corresponding to each product and a state 0 for the no-purchase alternative.

We first describe the Markov chain choice model introduced in [8]. In this Markov chain model, the customer substitutions are modeled using a Markov chain over $(n + 1)$ states, $\mathcal{N}_+ = \{0, 1, \ldots, n\}$: there is one state for each of the $n$ substitutable products and a state 0 for the no-purchase alternative. Let $\mathcal{S} \subseteq \mathcal{N}$ be a subset of offered products, let $\mathcal{S}_+ := \mathcal{S} \cup \{0\}$. 

51
The model is specified by the parameters \( \lambda_i, i \in [n] \) and \( \rho_{ij} \), for all \( i \in [n] \) and \( j \in \{0, \ldots, n\} \).

- \( \lambda_i \) denotes the arrival probability at state \( i \): a customer starts at the state corresponding to product \( i \) with probability \( \lambda_i \),

- \( \rho_{ij} \) denotes the transition probability from state \( i \) to state \( j \) if \( i \) is unavailable.

In the model considered in [8], when \( S \) is offered, all the states corresponding to \( S \) are absorbing: if the random walk of any customer reaches state \( i \in S \), then they select product \( i \) with probability one regardless of what else is being offered. However, the assumption that the customer will buy product \( i \in S \) with probability one if they reach it, implies that the model suffers from a certain limitation: indeed in this model, the customer preferences do not depend on the offered set. Therefore, if the seller offers more products, the customer will more likely reach a state in the offered set as now there are more absorbing states, thus decreasing the probability of reaching the no-purchase alternative. But as we pointed out above, this is not true in practice. In practice, when there are too many options, it is more difficult to make a decision, and therefore it is more probable to choose the no-purchase alternative. This is why we consider a Markovian model that captures customers’ preferences.

3.1.2 Substitution Behavior in Our Model

In our model, we use the Markovian framework as above to model substitution behavior. However, we introduce a stopping probability function \( \mu(i, S) \).

**Stopping probability function.** For any \( i \in S \), \( \mu(i, S) \) denotes the probability that a customer selects product \( i \), given that they are currently already in state \( i \) of the Markov chain. In the model considered by [8], this probability is equal to 1. In this chapter, we aim to capture the following fundamental component of customer choice, namely, that customer preferences and eventual selection depend on comparisons among the offered products. To capture this behavior, we model \( \mu(i, S) \) as a decreasing function of \( \sum_{j \in S} \rho_{ij} \) and consider the
following formulation for $\mu(i,S)$

$$
\mu(i,S) = \exp\left(-\alpha \sum_{j \in \mathcal{N}_s} \rho_{ij} x_j\right), \quad \forall i \in S,
$$

where $x_j = 1$ if $j \in S_+$ and 0 otherwise. Also, we have $\mu(i,S) = 0 \forall i \notin S$. Note that if a large number of products similar to $i$ (i.e. with large $\rho_{ij}$) are offered in the assortment, then the stopping probability is small. This reflects the scenario that it is difficult for a customer to select a product if a large number of similar options are available. Similarly, if we include more products in the assortment, $\mu(i,S)$ decreases. This models the fact that customers need more comparisons and time (which is captured by the number of transitions) to select the best product. This can be interpreted as the higher search cost for finding the best products if many similar items are offered.

We refer to this model as the Generalized Markov Chain Model. Here $\alpha$ is a new scale parameter that amplifies the comparison effect. In our proposed model, we have $\alpha \geq 0$. A large value of $\alpha$ implies a very picky and risk-averse customer. We would like to note that our model generalizes the model introduced in [8]. In particular, we recover the model in [8] by assuming $\alpha = 0$.

Note that the choice of exponential function is fairly arbitrary here. We use this as it is a parsimonious choice to model transition probability. But any other function that is decreasing in $\sum_{j \in \mathcal{N}_s} \rho_{ij} x_j$ with range in $[0,1]$ could have worked as well.

To model the eventual choice of the product $i$ by the customer, we add a new vertex $i'$, which denotes an absorbing state in the Markov chain. A directed edge joins the vertex $i$ to this newly added vertex $i'$ with weight $\mu(i,S)$, which represents the probability of buying the product $i$ when the customer is at the vertex $i$. This probability is equal to 0 if and only if the product is not in the offered set, i.e., $i \notin S$. We denote $\mathcal{N}_+'$ to be the set of all absorbing states: $\{i' \mid i \in [n] \} \cup \{0\}$. Hence, after a certain time $t$, for $t$ large enough, we can see that the customer is either in a certain state $i'$, with $i \in S$, or in the no-purchase state 0.
Modified transition probabilities. Since the sum of the probabilities of getting out of $i$ has to be equal to 1, we change $\rho_{ij}$ to $\tilde{\rho}_{ij}$ defined as follows:

$$\tilde{\rho}_{ij} = (1 - \mu(i, S))\rho_{ij}.$$

Customer substitution behavior on the new graph. Let us summarize how a customer behaves on the new graph given a certain set of products $S \subseteq \mathcal{N}$ to sell under the Generalized Markov Chain model:

- The customer arrives with probability $\lambda_i$ at the vertex $i$.
- If the product $i$ is in $S$ then the customer, currently at the vertex $i$,
  - either selects it with probability $\mu(i, S)$, arrives at the vertex $i'$ and then stops,
  - or goes to another vertex $j$ with probability $\tilde{\rho}_{ij}$.
- If $i \notin S$, the customer cannot purchase $i$, so with probability $\tilde{\rho}_{ij}$ they go to another vertex $j$.
- If $i = 0$, then the customer has decided not to purchase any product, and they stop.

We then proceed recursively.

Example We consider the following 4-vertex graph (see Figure 3.1), where we have chosen to offer the subset $S = \{3, 4\}$. Each product $i$ has a state $i$ as well as a state $i'$ where the latter is the absorbing state denoting that the customer buys product $i$.

We can see that since the offered set has products 3 and 4, the transition probabilities of going from node 3 or node 4 to any other node now gets modified, while other transition probabilities (i.e., from nodes 1 and 2 are unchanged). Firstly, nodes 3 and 4 are not absorbing anymore and hence the probability to go to nodes 3' and 4' which are now absorbing are $\mu(3, S)$ and $\mu(4, S)$ respectively. Secondly, this also changes the transition probabilities from nodes 3 and 4 to other nodes accordingly.
3.1.3 Assortment Optimization Problem

Let $\pi(i, S)$ be the probability of buying the product $i$ when the subset $S$ is offered, and $p_i$ be the price of the product $i$. We finally assume that the seller has a fixed capacity of goods, therefore the cost price does not depend on our decision. The assortment optimization is the following

$$\max_{S \subseteq N} \sum_{i \in S} \pi(i, S) \cdot p_i.$$ 

Now the objective is to compute $\pi(i, S)$ under our model.

3.2 Computation of Choice Probabilities

Given the parameters $\lambda_i$, $\rho_{ij}$ and $\mu(i, S)$ for all $i \in N$ and $j \in N$, we can compute the choice probabilities for any $S \subseteq N$ in a very similar way as [8]. Our assumption is that a customer arrives at the state $i$ with probability $\lambda_i$, and continues to transition according to probabilities $\rho_{ij}$ until they decide to buy a product $i$ with probability $\mu(i, S)$ when they are at the vertex $i$, or decide not to buy any product and end at the no-purchase vertex 0. We therefore assume that any customer buys at most one product.
3.2.1 Choice Probabilities

Let $\rho(N, N)$ be the transition probability matrix from states $N$ to $N$. We recall that since the total probability of exiting a vertex $i$ is 1, and since there is a probability $\mu(i, S)$ of buying the product represented by the vertex $i$, the transition probabilities are given by:

$$\hat{\rho}_{ij} = (1 - \mu(i, S))\rho_{ij}. \text{ Consequently,}$$

$$\rho(N, N) = \text{Diag}((1 - \mu(i, S))) \times \rho,$$

where $\rho$ is the initial transition probability matrix, $\rho = (\rho_{ij});_{i,j}[n]$, and $\text{Diag}((1 - \mu(i, S)))$ is the diagonal matrix with $(1 - \mu(i, S))$ on its diagonal. Also recall that we have $\mu(i, S) = 0 \forall i \notin S$.

After a certain time, every customer will be in an absorbing state. In order to compute $\pi(i, S)$, we have to know the probability that a customer arrives at the vertex $i$. For $i \in [n]$ we have:

$$\pi(i, S) = \lim_{q \to \infty} \lambda^T(P(S))^q e_i^{2n+1},$$

where $P(S)$ is the transition probability matrix in the graph when the subset $S$ is offered and is of the form:

$$P(S) = \begin{bmatrix}
\rho(N', N') & \rho(N', N) \\
\rho(N, N') & \rho(N, N)
\end{bmatrix} = \begin{bmatrix}
I_{n+1} & 0 \\
\Pi(S) & \text{Diag}((1 - \mu(i, S)))\rho
\end{bmatrix},$$

and $\Pi(S)$ is the following matrix:

$$\Pi(S) = \rho(N, N') = \begin{bmatrix}
\mu(1, S) & 0 & \ldots & 0 & \bar{\rho}_{10} \\
0 & \mu(2, S) & \ldots & 0 & \bar{\rho}_{20} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \mu(n, S) & \bar{\rho}_{n0}
\end{bmatrix} = \begin{bmatrix}
\text{Diag}(\mu(i, S)) & \bar{\rho}_0
\end{bmatrix},$$

56
and $e_i^{2n+1} \in \{0,1\}^{2n+1}$ has 0 on each component except on its $i^{th}$ component. Since we have $i \in [n]$, $e_i^{2n+1}$ will always have its last $n+1$ components equal to 0. $\rho(N'_+, N'_+) = 1_{n+1}$ because all the states in $N'_+$ are absorbing, which also implies that $\rho(N'_+, N) = 0$.

For $q \in \mathbb{N}$, we have:

$$P(S)^q = \begin{bmatrix} I_{n+1} & 0 \\
\sum_{k=0}^{q} (\text{Diag}((1 - \mu(i,S)))\rho)^k \Pi(S) & (\text{Diag}((1 - \mu(i,S)))\rho)^q \end{bmatrix}.$$ 

Therefore, if we assume that the spectral radius of $\rho(N, N) = \text{Diag}((1 - \mu(i,S)))\rho$ is strictly less than 1:

$$\lim_{q \to \infty} P(S)^q = \begin{bmatrix} I_{n+1} & 0 \\
(I_n - \text{Diag}((1 - \mu(i,S)))\rho)^{-1}\Pi(S) & 0 \end{bmatrix},$$

and hence

$$\pi(i, S) = \lambda^T (I_n - \text{Diag}((1 - \mu(i,S)))\rho)^{-1}\Pi(S)e_i,$$

where $e_i \in \{0,1\}^{n+1}$. Lastly, if we want the probability of no purchase, we can simply compute $\lambda^T (I_n - \text{Diag}((1 - \mu(i,S)))\rho)^{-1}\Pi(S)e_0$, where $e_0 = (0, \ldots, 0, 1) \in \{0,1\}^{n+1}$.

### 3.2.2 Examples

We now provide a couple of examples to show that our model is better at capturing the choice overload phenomenon than any other random utility based choice models.

**Example 1 (Homogeneous Graph).** We first consider a complete graph with $n$ vertices with homogeneous transition probabilities, $\rho_{ij} = \frac{1}{n+1}$ for all $i \in N$ and $j \in N_+$, and also homogeneous probabilities of arrival, $\lambda_i = \frac{1}{n+1}$ for all $i \in N_+$. We suppose that all products have same price $p$. The symmetry of this example implies that we only need to find the number $k$ of vertices to offer that would maximize revenue, and then randomly take a subset of $k$ elements. For any random utility based choice model presented in the first section, the optimal set to maximize revenue will be the entire set of products. Indeed, in all these models,
the product with the highest price is always in the optimal assortment. Since all the products have the same price, they will all be in the set. Furthermore, the probability of no purchase always decreases when we add more products into the offered set. However this is not true in our model, and the probability of no purchase depends on the parameter $\alpha$ chosen. We have chosen a universe of $n = 15$ substitutable products, and given the assumptions above, we compute the probability of purchase under our model when $k$ products are in the offered set, for all $k \in [n]$ and we obtain the graph in Figure 3.2:

We see that, depending on the value of $\alpha$, the probability of no purchase may increase when we add more items into the offered assortment. Especially, a high $\alpha$ implies a sooner (in terms of the number of products) increase in the probability of no-purchase.

We present another example, the star graph, which shows that our model will favour the cluster centers as items for the assortment set, unlike the Markov chain model.

**Example 2 (Star Graph).** We consider the following star graph with $n$ vertices. We suppose that vertex 1 is linked to all other vertices in the graph, but other vertices are only linked to 1 and to no purchase vertex 0. We suppose that the transition probabilities are homogeneous. Therefore, they are given by: $\rho_{1i} = \frac{1}{n} \forall i \in N_{+}\backslash\{1\}$ and $\rho_{11} = \rho_{i0} = \frac{1}{2} \forall i \in$
\(\mathcal{N}\{1\}\). We suppose that the arrival probabilities are all equal \(\lambda_i = \frac{1}{n} \forall i \in \mathcal{N}\). Finally we suppose that all products, except 1, have a price \(P\), and product 1 has a smaller price \(p < P\).

For any random utility based choice model considered in the literature, since the product with the highest price is always in the optimal offered set, then \(\{2, ..., n\} \subseteq S^*\) where \(S^*\) is the optimal set. And this is true, for any \(n\), even for very large \(n\).

However, our model considers that selling only the product 1 will give a higher revenue, when \(\alpha\) is large enough. For \(\alpha \geq 9\), the optimal set will always be \(\{1\}\). And this result is closer to the reality. Indeed, it seems more logical in practice for the seller to only offer the product which is similar to many other products, even if this product is slightly less expensive than the others.

In order to get a generalization of the MNL model, we will now suppose that the initial transition probability matrix, \(\rho = (\rho_{ij})_{i,j \in [n]}\) is of rank one, and show that with such an assumption, the optimization problem is NP-hard.

### 3.3 Generalized Multinomial Logit Model

In this section we suppose that the transition probability matrix \(\rho\) is of rank one. Given this assumption, we refer to our model as the Generalized Multinomial Logit model, which is a special case of the Generalized Markov chain model. We remind that \(\mathcal{N}\) represents all the vertices in the graph, \(\mathcal{N}_*\) is the union of all the vertices and the vertex 0 which represents the no-purchase, and \(\rho(\mathcal{N}, \mathcal{N}_*)\) is the transition probability matrix from \(\mathcal{N}\) to \(\mathcal{N}_*\). In this model, we suppose that there exists \(v = (v_i)_{i \in [n+1]} \in [0,1]^{n+1} = [v_0 \ v_*^T]\), such that \(\sum_{i=0}^n v_i = 1\) and:

\[
\rho(\mathcal{N}, \mathcal{N}_*) = \begin{bmatrix}
1 - \mu(1, S) & 0 & \cdots & 0 \\
0 & 1 - \mu(2, S) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 - \mu(n, S)
\end{bmatrix}
\begin{bmatrix}
v_0 \\
v_1 \\
\vdots \\
v_n
\end{bmatrix}
\]

Since there is a probability \(\mu(i, S)\) that the customer goes from vertex \(i\) to vertex \(i'\), the probability that the customer goes from vertex \(i\) to vertex \(j\), with \(j \in \mathcal{N}_*\), has to be
(1 − \(\mu(i, S)\))v_j, therefore the probability of exiting from vertex \(i\) is 1. We also suppose in this model that \(\lambda = v\), therefore the probability of arriving at a vertex \(i\) is proportional to \(v_i\). Finally we suppose that the probability of buying the product \(i\) while being at vertex \(i\) is given by:

\[
\mu(i, S) := e^{-\alpha \sum_{j \in S} v_j}.
\]

Additionally, we have \(\mu(i, S) = 0 \text{ } \forall i \notin S\). With the notation given in Section 3.2.1:

\[
\mathcal{P}(S) = \begin{bmatrix}
    \rho(N'_s, N'_s) & \rho(N'_s, N) \\
    \rho(N, N'_s) & \rho(N, N)
\end{bmatrix} = \begin{bmatrix}
    I_{n+1} & 0 \\
    \Pi(S) & D(S)\rho_v
\end{bmatrix},
\]

where

\[
\Pi(S) = e^{-\alpha \sum_{j \in S} v_j} \times \begin{bmatrix}
    1_{1 \in S} & 0 & (e^{\alpha \sum_{j \in S} v_j} - 1_{1 \in S})v_0 \\
    \vdots & \vdots & \vdots \\
    0 & 1_{n \in S} & (e^{\alpha \sum_{j \in S} v_j} - 1_{n \in S})v_0
\end{bmatrix}
\]

and

\[
D(S)\rho_v = \begin{bmatrix}
    1 - \mu(1, S) & 0 \\
    \vdots & \vdots \\
    0 & 1 - \mu(n, S)
\end{bmatrix} \times \begin{bmatrix}
    v_1 & \cdots & v_n \\
    \vdots & \vdots & \vdots \\
    v_1 & \cdots & v_n
\end{bmatrix} = \left(1 - e^{-\alpha \sum_{j \in S} v_j}\right) \times \begin{bmatrix}
    v_1 & \cdots & v_n \\
    \vdots & \vdots & \vdots \\
    v_1 & \cdots & v_n
\end{bmatrix}.
\]

We summarize here the assumptions of the Generalized Multinomial Logit model:

**Generalized Multinomial Logit Model** In this model we make the following assumptions:

- the initial transition probability matrix within \(N_+\) is of rank one, i.e. there exists \(v = (v_i)_{i \in N_+} = \begin{pmatrix} v_0 \\ v_* \end{pmatrix} \in [0, 1]^{n+1}\) such that \(\rho(N, N) = Diag((1 - \mu(i, S)))v_*^T\) and \(\sum_{j \in N} v_j + v_0 = 1\),

- given a subset \(S \subseteq N\), for all \(i \in N\) we have \(\mu(i, S) = e^{-\alpha \sum_{j \in S} v_j}\),

- and for all \(j \in N_+\), \(\lambda_j = v_j\).
As we show in Section 3.2.1, the assortment optimization problem under our model is given by:

\[
\max_{S \subseteq N} v^T (I_n - D(S)\rho_v)^{-1} \Pi(S)p.
\]

We now give an exact formulation of choice probability and see why it generalizes the MNL model.

### 3.3.1 Choice Probability

We can compute the probability of choosing a product \(i\) in our model as follows.

**Lemma 3.3.1** The probability of purchasing a product \(i\) given a chosen subset \(S \subseteq N\) under the Generalized Multinomial Logit model is given by:

\[
\pi(i, S) = \frac{v_i}{\sum_{k \in S} v_k + v_0 e^{\alpha \sum_{j \in S^c} v_j}} \mathbb{1}_{i \in S}.
\]

The proof follows from Section 3.2.1 applied to this particular case and is presented in detail in Appendix B.1.

**Our model is a generalization of the MNL model.** We recall that under the MNL model, the probability of buying the product \(i \in S\) when the set \(S\) is offered is given by

\[
\pi^{\text{MNL}}(i, S) = \frac{v_i}{\sum_{j \in S} v_j + v_0} \mathbb{1}_{i \in S}.
\]

Therefore, our model can be considered as a generalization of the MNL model where the no purchase probability is not constant as is the case in MNL, but depends on the assortment \(S\) as \(v_0 e^{\alpha \sum_{j \in S^c} v_j}\), which increases the utility of the no-purchase alternative as compared to the MNL model.

Suppose that, instead of choosing, \(\mu(i, S) = e^{-\alpha \sum_{j \in S^c} v_j}\), we had chosen a different function, say, \(\mu(i, S) = \frac{1}{\sum_{j \in S^c} v_j}\), which would also convey the idea that \(\mu(i, S)\) is a decreasing function of \(\sum_{j \in S^c} v_j\). Then, the probability of purchasing the product \(i\) given a set \(S\) of offered products
would have been:

\[
\pi(i, S) = \frac{v_i}{\sum_{j \in S} v_j + v_0 (\sum_{j \in S} v_j)} = \frac{v_i}{(1 + v_0) \sum_{j \in S} v_j + v_0^2}.
\]

which in a ratio scale, is exactly the choice probability of MNL model. Therefore, such a function would have given a nesting by price order and therefore an optimization problem solvable in polynomial time, just as MNL. However, this model would not have given sufficient weights to the \( v_j \)'s for \( j \in S \) on the no-purchase option, which is what we want to capture, namely the choice overload phenomenon. This is why we want to emphasize the importance of choice of the function \( \mu(i, S) \) in our model, and why the choice of \( e^{\alpha \sum_{j \in S} v_j} \) meets the requirements of our model.

3.3.2 Example

Let us revisit the example of a homogeneous Markov chain from Section 3.2.2 in this context.

Example (Homogeneous Graph). We recall the assumptions in this model. Consider the case of the complete graph with \( n \) vertices with homogeneous transition probabilities, \( \rho_{ij} = \frac{1}{n+1} \) for all \( i \in \mathcal{N} \) and \( j \in \mathcal{N}_i \), and homogeneous probabilities of arrival, \( \lambda_i = \frac{1}{n+1} \) for all \( i \in \mathcal{N}_i \). We suppose that all the products have the same price \( p \).

For any random utility based choice model presented in the first section, the optimal set to maximize our revenue will be the entire set of products, as we explained before. This example is a particular case where the initial transition probability is of rank one. Therefore under the Generalized MNL model, the assortment optimization problem for the homogeneous graph is:

\[
\max_{S \subseteq \mathcal{N}} \frac{|S|}{n+1} P = \max_{k \in [n]} \frac{kp}{k + e^{\alpha \frac{|S|}{n+1}}}.
\]

A simple computation shows that the optimal number of products in the offered set is \( k^* = \frac{n+1}{\alpha} \). Therefore, if \( \alpha < \frac{n+1}{n} \) then the optimal assortment set will be the entire universe.
However, if we take $\alpha$ large enough, then $\frac{n+1}{\alpha} \leq n - 1$ and there will be less products in the optimal offered set. This also highlights the meaning of $\alpha$: $\alpha$ amplifies the comparison effect. A large value of $\alpha$ implies risk-averse customer, and therefore a strategy where the seller should offer less products.

### 3.3.3 Parameter Estimation for Generalized MNL Model

Recall that the choice probabilities for the Generalized MNL model are given by:

$$
\pi(i, S) = \frac{V_i}{v_0 e^{\alpha \sum_{j \in S^+} v_j + \sum_{j \in S} v_k}}, \quad \forall i \in S
$$

$$
\pi(0, S) = \frac{v_0 e^{\alpha \sum_{j \in S^+} v_j}}{v_0 e^{\alpha \sum_{j \in S^+} v_j + \sum_{j \in S} v_k}},
$$

where $v_j = e^{\beta^T x_j}$.

Given a choice dataset: $D = \{j_t, S_t\}_{t=1}^T$, where $S_t$ is the assortment set offered at time $t$ and $j_t$ is the choice made at time $t$ (which could be the outside option of no purchase), the log-likelihood can be formulated as:

$$
\ell(D, \beta, \alpha) = \sum_{\mathbf{j} \in \mathbf{D}_0} \beta^T x_{j_t} + \sum_{\mathbf{j} \in \mathbf{D}_0} (\beta^T x_0 + \alpha \sum_{j \in S^+} e^{\beta^T x_j}) \sum_{t=1}^T \log \left( e^{\beta^T x_0} e^{\alpha \sum_{j \in S^+} e^{\beta^T x_j}} + \sum_{k \in S_t} e^{\beta^T x_k} \right),
$$

where $\mathbf{D}_0$ is defined as the subset of $\mathbf{D}$ where there was no purchase.

While $\ell(\beta, \alpha)$ is not jointly concave in $(\beta, \alpha)$, we present an alternate algorithm based on searching for $\alpha$. In particular, we have the following lemma.

**Lemma 3.3.2** For a given value of $\alpha$, the maximization problem over $\beta$ can be reformulated as a convex optimization problem.

**Proof** The partial maximization problem over $\beta$ when $\alpha$ is known is the following:

$$
\max_{\beta} \sum_{\mathbf{j} \in \mathbf{D}_0} \beta^T x_{j_t} + \sum_{\mathbf{j} \in \mathbf{D}_0} (\beta^T x_0 + \alpha \sum_{j \in S^+} e^{\beta^T x_j}) \sum_{t=1}^T \log \left( e^{\beta^T x_0} e^{\alpha \sum_{j \in S^+} e^{\beta^T x_j}} + \sum_{k \in S_t} e^{\beta^T x_k} \right).
$$
We introduce the following new variables:

\[ z_t = \beta^T x_0 + \alpha \sum_{j \in S_t} e^{\beta^T x_j}, \quad t = 1, \ldots, T \]

Then we can re-write the above maximization as:

\[
\begin{aligned}
\max_{\beta, z_t} & \sum_{t \in D_0} \beta^T x_t + \sum_{t \in D_0} z_t - \sum_{t=1}^T \log \left( e^{z_t} + \sum_{k \in S_t} e^{\beta^T x_k} \right) \\
\text{s.t.} & \beta^T x_0 + \alpha \sum_{j \in S_t} e^{\beta^T x_j} - z_t = 0, \quad t = 1, \ldots, T
\end{aligned}
\] (3.1)

The objective function is now jointly concave in \((\beta, z_t)\) as it is a sum of linear functions of \(\beta\) and \(z_t\) and the negative of log-sum-exp function. Also, the equality constraints are convex functions in \((\beta, z_t)\). Hence we can solve this optimization problem in (3.1) efficiently. We also note that we are only introducing \(T\) new variables and constraints. \(\Box\)

**Lemma 3.3.3** For a given value of \(\beta\), the log-likelihood function is strictly concave in \(\alpha\) and hence it is unimodal. So the maximization problem over \(\alpha\) can be solved.

**Proof** For a given value of \(\beta\), the partial maximization problem over \(\alpha\) is given by:

\[
\max_{\alpha} \sum_{t \in D_0} \alpha \sum_{j \in S_t} v_j - \sum_{t=1}^T \log \left( v_0 e^{\alpha \sum_{j \in S_t} v_j} + \sum_{k \in S_t} v_k \right)
\]

Defining \(c_t := \sum_{k \in S_t} v_k = \sum_{k \in S_t} e^{\beta^T x_k}\), we can re-write this as

\[
\max_{\alpha} \sum_{t \in D_0} \alpha (v_0 + c_t) - \sum_{t=1}^T \log \left( v_0 e^{\alpha (v_0 + c_t)} + c_t \right)
\] (3.2)

A simple derivative calculation shows the above function is strictly concave in \(\alpha\). Hence the maximization problem in (3.2) is also easy to solve. \(\Box\)

In accordance with the above results, an iterative algorithm for maximizing the log-likelihood is to keep maximizing over \(\alpha\) and \(\beta\) alternatively until convergence. This will lead
to a local maximum. Since this type of alternative maximization algorithm is dependent on the initial point, a good initial point could be $\beta_{MLE}^{MNL}$ which is the maximum likelihood estimate for the MNL model (which can be easily found as the MLE for MNL model is a convex optimization problem). The details are given in Algorithm 1. We found Algorithm 1 to converge after a few iterations only, as evident from Figure 3.3.

**Algorithm 1** Parameter Estimation for Generalized Multinomial Logit model

```plaintext
def ParamEstGENMNL(D):
    Let $\beta_{MLE}^{MNL}$ be the maximum likelihood estimate from given data D for the MNL model
    Set $\beta^{(0)} = \beta_{MLE}^{MNL}$
    Set $\alpha^{(0)} = 0$
    for $k = 1, 2, \ldots$ do
        Solve the optimization in (3.2) with $\beta = \beta^{(k-1)}$ and set $\alpha^{(k)}$ to be the optimal solution
        Solve the optimization problem in (3.1) with $\alpha = \alpha^{(k)}$ and set $\beta^{(k)}$ to be the optimal solution
        Stop until convergence is achieved
    end for
    Let $K$ be the index after convergence at the end of for loop
    Set $\beta_{MLE}^{GMNL} = \beta^{(K)}$
    Set $\alpha_{MLE}^{GMNL} = \alpha^{(K)}$
    return $\beta_{MLE}^{GMNL}$ and $\alpha_{MLE}^{GMNL}$ as the estimates
end procedure
```

3.4 Assortment Optimization for the Generalized Multinomial Logit Model

In this section we consider the assortment optimization problem under the Generalized MNL model. Unlike the MNL model, even unconstrained assortment optimization under Generalized MNL model is NP-hard. Under the Generalized Multinomial Logit model, using the expression of choice probability we derived in Lemma 3.3.1, the assortment optimization problem can be written as

$$
\max_{S \subseteq N} R(S) := \max_{S \subseteq N} \frac{\sum_{i \in S} v_i p_i}{\sum_{i \in S} v_i + v_0 e^{\alpha \sum_{i \in S} v_i}}.
$$

(3.3)
Figure 3.3: Estimate of $\alpha$ vs number of iterations for the estimation algorithm. The feature vector was 4 dimensional and we had 10 products, i.e., $d = 4, n = 10$. We can see convergence after a few iterations of the alternating projection algorithm.

3.4.1 NP-Hardness of the Assortment Optimization Problem

In particular, we prove the following result.

**Theorem 3.4.1** The assortment optimization problem under the Generalized MNL model in (3.3) is NP-hard.

We use a reduction from the partition problem to prove this result and the details of the proof are presented in Appendix B.2.

Given that the assortment optimization problem in (3.3) is NP-hard, we can only hope to get an approximation. We present the best possible approximation in the form of a fully polynomial time approximation scheme (FPTAS).

Our algorithm for the FPTAS is based on the structure of the revenue function that
depends on a single linear function of the assortment $S$, namely

$$V(S) = \sum_{j \in S} v_j.$$ 

In particular, for any assortment $S \subseteq [n]$, $R(S)$ is completely determined by $V(S)$. Hence if we can guess the value of $V(S^*)$ corresponding to an optimal assortment $S^*$, and find an assortment $S$ with $V(S) \approx V(S^*)$, we can use a dynamic programming based algorithm similar to the knapsack problem to construct such an approximately optimal assortment. This technique has been used in the past (see [28] and [42]). One of the most closely related works is [18] which presents algorithms for constrained assortment optimization under many parametric models where the revenue function satisfies this linear structural property.

### 3.4.2 Our Algorithm for the FPTAS

In this section, we now present a fully polynomial time approximation scheme (FPTAS) for the assortment optimization under the Generalized Multinomial Logit model discussed in (3.3). For the FPTAS, we first consider different guesses for $V(S^*)$ in increasing powers of $(1 + \epsilon)$.

Let $v$ (resp. $V$) be the minimum (resp. maximum) value of the transition probabilities. We can assume that $v > 0$. For any given $\epsilon > 0$, we use the following set of guesses for $V(S^*)$:

$$V_\epsilon = \{v(1 + \epsilon)^l, \ l = 0, ..., L\},$$

where $L = O(\log(nV/v)/\epsilon)$. Hence, the number of guesses is polynomial in the number of products and $1/\epsilon$.

Then for each guess $h \in V_\epsilon$, we consider discretized values of $v_j$ and try to construct an assortment $S$ such that:

$$h(1 - \epsilon) \leq V(S) \leq h(1 + \epsilon),$$

67
using a knapsack like dynamic programming. In particular, for given guess \( h \in V_\epsilon \), we try to find the best revenue possible with
\[
\sum_{j \in S} v_j \leq h,
\]
by a dynamic program.

We consider the following discretized values of \( v_j \) in multiples of \( \epsilon h/n \):
\[
\forall j \in N \quad \bar{v}_j = \left\lfloor \frac{v_j}{\epsilon h/n} \right\rfloor.
\]

Let \( I = [n/\epsilon] + n \). For each \((i, k) \in [I] \times [n]\), let \( R(i, k) \) be the maximum revenue of any subset \( S \subseteq \{1, \ldots, k\} \) such that
\[
\sum_{j \in S} \bar{v}_j \leq i.
\]
We compute \( R(i, k) \) using the following dynamic program
\[
R(i, 1) = \begin{cases} 
  v_1 p_1 & \text{if } \bar{v}_1 \leq i \\
  0 & \text{if } i \geq 0 \\
  -\infty & \text{otherwise}
\end{cases}
\]
\[
R(i, k + 1) = \max\{v_{k+1} p_{k+1} + R(i - \bar{v}_{k+1}, k), R(i, k)\}.
\]

Let \( S_h \) be the subset corresponding to \( R(I, n) \), that is, the assortment \( S_h \) that maximizes the sum \( \sum_{j \in S} v_j p_j \) such that the inequality is verified for \( i = I \). We then construct a set of candidate assortments \( S_h \) for all guesses \( h \), and return the best revenue that we get from all the candidates in the set. Algorithm 2 presents the details for the FPTAS.

**Theorem 3.4.2** Algorithm 2 returns an assortment which is \((1 - O(\epsilon))\)-optimal solution to the assortment optimization problem in (3.3). The running time is \( O\left(\frac{n^2}{\epsilon^2} \log(nV/v)\right)\).

We present the complete proof in Appendix B.3.
Algorithm 2 FPTAS for the Generalized Multinomial Logit model

procedure FPTASGenMNL($\epsilon, v$)
    for $h \in V_e$
        Compute the discretized coefficients $\bar{v}_j = \left\lceil \frac{v_j}{\epsilon h/n} \right\rceil$
        Compute $R(i, k)$ for all $(i, k) \in [I] \times [n]$ using the dynamic program above
        Let $S_h$ be the subset corresponding to $R(I, n)$
    end for
    Let $C = \bigcup_{h \in V_e} S_h$
    return the set $S^* \in C$ that has the best revenue
end procedure

3.5 Generalized Markov Chain Model With Low Rank Matrix

In this section, we consider a general model, when the initial transition matrix is of low rank. In particular, we assume that the rank is some constant $K < n$ and the initial transition matrix $\rho$ is given by

$$\rho(N, N_0) = \left( \sum_{k \in [K]} u_{ik}v_{jk} \right)_{i \in [n], j \in N_0} = \sum_{k \in [K]} u_kv_k^T,$$

where $\forall i \in [n], \sum_{j=0}^n \sum_{k=1}^K u_{ik}v_{jk} = 1$. Given this initial transition probability matrix, we have that the probability of purchasing the product $i$ while being at vertex $i$ when $S$ is offered is

$$\mu^{LR}(i, S) = e^{-\alpha(\sum_{j \in S} \sum_{k \in [K]} u_{ik}v_{jk})} = e^{-\alpha(\sum_{k \in [K]} u_{ik}V_k(S))},$$

where we define

$$V_k(S) := \sum_{j \in S} v_{jk} \quad \forall k \in [K].$$

We also make the following assumptions:

- we suppose that $\forall j \in [n], \forall k \in [K] u_{jk}v_{jk} \leq \frac{1}{n}$ (by this, we mean that the probability of staying at the state $j$ without buying the product $j$ cannot be too high);

- we also suppose that $\alpha$ is not too large compared to $n$, more precisely, $\alpha \leq \log n$. 

69
First assumption is natural as it stipulates that a customer either buys the product or moves to another state. The second assumption is a technical assumption which makes sure that the spectral radius of the following matrix is bounded away from 1.

**Lemma 3.5.1** Let $S \subseteq \mathcal{N}$, we define the following matrix $\mathbf{M}_K(\mathbb{R})$:

$$
\mathbf{M} = \mathbf{U} \mathbf{V}(S) = \left( \sum_{j=1}^{n} (1 - \mu^{LR}(j, S)) u_{jm} v_{jm} \right)_{k,m \in [K]}.
$$

Then the spectral radius of $\mathbf{M}$,

$$
\rho(\mathbf{M}) \leq 1 - \frac{1}{n^2}.
$$

We present the proof in Appendix B.4.

### 3.5.1 Assortment Optimization and FPTAS

We present an FPTAS for the case of the rank being a constant $K$, with the running time of the algorithm being exponential in $K$. In particular, we first show that the expected revenue of an assortment $S$, $R(S)$ depends on $O(K)$ linear functions of $S$. Therefore if we can guess the values of these $O(K)$ linear functions for an optimal assortment $S^*$, and then find an assortment that approximately matches these values, we can compute the approximately optimal assortment.

Unlike other problems in literature (e.g., see [23], [33], [20] and [39]), the revenue function depends on a system of equations where the coefficients depend on the linear function values. Therefore to control the error in $R(S)$, we need to control the error in the estimates of the solution to the system of equations and not just the linear function values. This is one of the main challenges we address while constructing the algorithm for the FPTAS.

In particular, we choose the linear functions to guess more carefully, which allows us to give a theoretical bound on the error in solution of the system of equations.

We first compute the expected revenue $R(S)$ of an assortment $S$ and give the following decomposition.
Lemma 3.5.2 Under the Generalized Markov chain model with the rank of transition matrix being $K$, the expected revenue that we get from offering the assortment $S$ is

$$R^{LR}(S) = \sum_{i \in [n]} \lambda_i (1 - \mu^{LR}(i, S)) \left( \sum_{j \in S} p_j \mu^{LR}(j, S) u_i^T (I - UV(S))^{-1} v_j \right) + \sum_{i \in S} \lambda_i \mu^{LR}(i, S) p_i$$

where the matrix $UV(S) \in M_K(\mathbb{R})$ is defined by

$$UV(S) = \left( \sum_{j=1}^n (1 - \mu^{LR}(j, S)) u_{jk} v_{jm} \right)_{k,m \in [K]}.$$

The proof builds from the general choice probability expression from Section 3.2.1. We present the details in Appendix B.5.

For any $i$ and $S$, let

$$f(i, S) := \sum_{j \in S} p_j \mu^{LR}(j, S) u_i^T (I - UV(S))^{-1} v_j. \quad (3.4)$$

The assortment optimization problem under the Generalized Markov chain model can then be formulated as

$$\max_{S \subseteq \mathcal{N}} R^{LR}(S) := \max_{S \subseteq \mathcal{N}} \sum_{i \in [n]} \lambda_i (1 - \mu^{LR}(i, S)) f(i, S) + \sum_{i \in S} \lambda_i \mu^{LR}(i, S) p_i. \quad (3.5)$$

3.5.2 FPTAS for Generalized Markov Chain Model With Low Rank Matrix

We guess the following $K$ linear functions of $S$:

$$V_k(S) = \sum_{j \in S_v} v_{jk} \quad \forall k \in [K].$$

We show the following result which stipulates that if our guesses are within $(1 \pm \epsilon)$ of the optimal values, the error in solution of the system of equations is also within $(1 \pm O(\epsilon))$. 

71
Lemma 3.5.3 Let $S \subseteq \mathcal{N}$. Suppose that $\exists H \in \mathcal{M}_K(\mathbb{R})$ and $\tilde{v} \in \mathbb{R}$ such that

$$(1 - O(\epsilon))H \leq UV(S) \leq (1 + O(\epsilon))H \quad \text{and} \quad (1 - O(\epsilon))\tilde{v} \leq v_j \leq (1 + O(\epsilon))\tilde{v}. $$

Then we have that

$$(1 - O(\epsilon))[I - H]^{-1}\tilde{v} \leq [I - UV(S)]^{-1}v_j \leq (1 + O(\epsilon))[I - H]^{-1}\tilde{v}. $$

The proof builds from Lemma 3.5.1. We present the details in Appendix B.6.

Now we are ready to present the FPTAS. We first describe the guesses. Let $v^k$ (resp. $v^k$) be the minimum (resp. maximum) value of \{v^{ik}\}_{i \in \mathcal{N}} for all $k \in [K]$. We can assume that $v^k > 0$. For any given $\epsilon > 0$, we use the following sets of guesses:

$$W^k_{\epsilon} = \{v^k(1 + \epsilon)^t, t = 0, ..., T^k\}, \text{ for all } k \in [K],$$

where $T^k = O(\log(nv^k/v^k)/\epsilon)$. A guess $h$ belongs in the set

$$W_{\epsilon} = W^1_{\epsilon} \times ... \times W^K_{\epsilon}.$$ 

The number of guesses is polynomial in the input size and $1/\epsilon$. For given guess $h = (h_1, ..., h_K) \in W_{\epsilon}$, we try to find the best revenue possible with

$$h_k \leq \sum_{j \in \mathcal{S}_s} v^k_{j} \leq h_k(1 + \epsilon), \text{ for all } k \in [K],$$

using a dynamic program which we describe momentarily.

In particular, consider the following discretized values of $v^k_{j}$ in multiples of $\epsilon h_k/n$:

$$\forall k \in [K], \quad \forall j \in \mathcal{N}, \quad \bar{v}^k_{j} = \left\lfloor \frac{v^k_{j}}{\epsilon h_k/n} \right\rfloor.$$ 

72
We denote by \( \bar{v}_j \) the vector \( \bar{v}_j := (\bar{v}_{j1}, ..., \bar{v}_{jK}) \). Let \( L = [n/\epsilon] \), and \( U = [n/\epsilon] + n \). We use a dynamic program to maximize the total expected revenue. For each \( (l, u, m) \in [L]^K \times [U]^K \times [n] \), let \( R^{DP}(l, u, m) \) be the maximum revenue of any subset \( S \subseteq \{1, ..., m\} \) such that

\[
  l_k \leq \sum_{j \in S} \bar{v}_{jk} \leq u_k \quad \forall k \in [K].
\]

For each guess \( h \), let

\[
  \mu_i(h) := e^{-\alpha(\sum_{k \in [K]} h_k u_{ik})} \quad \forall i \in \mathcal{N}.
\]

Therefore, \( \mu(h) = (\mu_1(h), ..., \mu_n(h)) \) is an estimate of the value of the \( \mu^{LR}(i, S) \)'s. Given this, we also use the following estimation \( H \) of the matrix \( UV(S) \) defined before:

\[
  H(h) := \left( \sum_{i=1}^n (1 - \mu_i(h)) u_{ik} v_{k'} \right)_{k, k' \in [K]}.
\]

Finally, let us also consider for each \( i \in \mathcal{N} \) and each \( S \), the following estimate for \( f(i, S) \) defined in equation (3.4)

\[
  f_i(h, S) = \sum_{j \in S} p_j \mu_j(h) u_i^T [I - H(h)]^{-1} v_j.
\]

For each guess \( h \), we define the approximate revenue of the subset \( S \) as

\[
  R^{DP}(h, S) = \sum_{i \in S} \lambda_i \mu_i(h) p_i + \sum_{i=1}^n \lambda_i (1 - \mu_i(h)) f_i(h, S).
\]

We compute \( R^{DP}(l, u, m) \) using the following dynamic program

\[
  R^{DP}(l, u, 1) = \begin{cases} 
    \lambda_1 \mu_1(h) p_1 + \sum_{i=1}^n \lambda_i (1 - \mu_i(h)) p_1 \mu_1(h) u_i^T [I - H(h)]^{-1} v_1 & \text{if } l \leq \bar{v}_1 \leq u \\
    0 & \text{if } l \leq 0 \text{ and } u \geq 0 \\
    -\infty & \text{otherwise}
  \end{cases}
\]
\[ R^{DP}(l, u, m) = \max \left\{ \lambda_m \mu_m(h) p_m + \sum_{i=1}^{n} \lambda_i (1 - \mu_i(h)) p_i \mu_m(h) \mathbf{u}_i^T [I - H(h)]^{-1} \mathbf{v}_m \right. \\
\left. + R^{DP}(l - \bar{v}_m, u - \bar{v}_m, m - 1), R^{DP}(l, u, m - 1) \right\}. \]

Note that the number of states in the above dynamic program is \( O\left(\left(\frac{n}{\epsilon}\right)^{2K} n^2\right) \). Each step of the dynamic program requires a summation that can be done in \( O(nK^2) \) time. This results in a total running time of \( O\left(\left(\frac{n}{\epsilon}\right)^{2K} n^2 K^2 \right) \). We construct a set of candidate assortments \( S_h \) for all guesses \( h \), and return the best revenue that we get from all the candidates in the set.

Algorithm 3 presents the details for the FPTAS.

**Algorithm 3** FPTAS for the Generalized Markov chain model with Low rank matrix

```plaintext
procedure FPTASGENMIXTMNL(\( \epsilon, \mathbf{u}_k, \mathbf{v}_k \))
for \( h \in W_\epsilon \) do
    Compute the discretized coefficients \( \bar{v}_{jk} = \left\lceil \frac{v_{jk}}{\epsilon h_k} \right\rceil \)
    Compute \( R^{DP}(l, u, m) \) for all \( (l, u, m) \in [L]^K \times [U]^K \times [n] \) using the dynamic program
    Let \( S_h \) be the subset corresponding to \( R^{DP}(L, U, n) \)
end for
Let \( C = \cup_{h \in W_\epsilon} S_h \)
return the set \( S^* \in C \) that has the best revenue
end procedure
```

**Theorem 3.5.1** Algorithm 3 returns an assortment which is \( (1 - O(\epsilon)) \)-optimal solution to the assortment optimization problem in (3.5). The running time is \( O\left(\frac{n^{2K} v^2}{\epsilon^3 K^2} K^2 \log(nV/v)^K \right) \).

We present the proof in Appendix B.7. Note that the running time is exponential in the fixed rank \( K \) of the transition probability matrix.

### 3.6 Numerical Results

In this section, we present numerical results on real data. We use the publicly available "Related Article Recommendation Dataset" from [2] for performing this experiment.

**Description.** The dataset contains information on about 57.4 million recommendations that were displayed in the form of an ordered-list to the users of the digital library Sowiport. Information includes details such as which recommendation algorithms were used to order
the list (one out of content-based filtering, stereotype, most popular and random) and also
the date and time when those recommendations were requested, delivered and clicked.

From the digital library’s point of view, the decisions to be made are which articles,
how many of them and in which order should they be displayed when a request is received.
The objective is to maximize the overall click-through rate (CTR), which is the ratio of
clicked recommendations to those delivered. This dataset has also been used by [3] to study
empirical evidence of the choice overload phenomenon. That study finds out that higher
numbers of recommendations for a request lead to lower click-through rates.

Setup. Since the choice models assume that at most one product is selected from the
offered assortment, we first filter out a few recommendations which had multiple clicks.
After that, we do feature engineering and build a few features which would be used to train
the data on both Multinomial Logit (MNL) and Generalized MNL models. Then we fit
both the models and estimate their respective parameters: \( \beta \in \mathbb{R}^d \) for the MNL model and
\( \beta \in \mathbb{R}^d, \alpha > 0 \) for Generalized MNL model. Since the parameter estimation for the MNL
model is a convex optimization problem, we use the popular gradient descent method. We
use the parameter estimation algorithm discussed in Section 3.3.3 for the Generalized MNL
model, i.e., Algorithm 1.

Once we have estimated the parameters for both the models, we use them to predict the
click probabilities on a separate held-out dataset. After getting these click probabilities, we
order the recommended articles for each request made, according to these values. Since the
objective is to generate clicks on the recommended articles, we compare the predictions from
both the models against actually clicked articles.

Results. The standard metric used in the literature when the objective is to maximize CTR
is the area under the Receiver Operating Characteristic curve (the ROC AUC value, which
lies between 0 and 1, with a higher value being preferable). We find out that Generalized
MNL model improves the ROC AUC value over MNL model by 7%. The plots of the ROC
curves are shown in Figure 3.4.
Another important observation is on the $a$ estimated from the data. We get a high value for the estimate of $a$ ($a \approx 20$) which suggests that the choice overload phenomenon is prominent in such situations and hence Generalized MNL model would be able to capture them much better as compared to the MNL model.

3.7 Conclusion

Our main contribution in this chapter is to build upon the existing Markov chain based choice model presented by [8] and present a generalized model that addresses two major significant limitations of existing random utility maximization and rank-based choice models in capturing dynamic preferences and the choice overload phenomenon.

The Generalized Markov chain model attempts to capture both dynamic preferences and choice overload phenomenon by considering a modified choice or selection process, where a customer stops at a state corresponding to an offered product with some probability that depends on the particular set of offered products. This implicitly models the search cost.
in the selection process and therefore, captures both dynamic preferences and the choice overload phenomenon. Therefore, we present a novel framework to overcome the limitations in the existing choice models.

Considering the special cases when the transition matrix of the Markov chain is of rank one as well as when it has a low but fixed rank, we show that the corresponding assortment optimization problem under these models is NP-hard. We also present a fully polynomial time approximation scheme (FPTAS) for both settings. The first model generalizes MNL model while the second model generalizes Mixture of MNLs model. We also present the effectiveness of Generalized MNL model on real data.
Chapter 4: Empirical Evaluation of Generalized MNL Model: Evidence From Flipkart

In this chapter, we present evidence of empirical gains from employing richer choice models to capture customer preferences and then to optimize product recommendations based on those models on Flipkart, the largest e-commerce firm in India. This chapter builds on the work of [1]. We first present a brief background on the Flipkart’s homepage optimization problem in Section 4.1. We talk about Flipkart and in particular describe our collaboration with Flipkart’s homepage optimization team, where we consider the problem of improving product recommendations on the homepage while accounting for substitution patterns and adjusting the recommendations. We describe the data available and used for this study in detail in Section 4.2. Then, in Section 4.3 we discuss the Logistic regression based model which considers each item independently, where every item is described by a set of attributes and the mean utility of a product is linear in the values of attributes, which is the current model used at Flipkart. We then study choice models like MNL which capture consumer preferences over an assortment of products with the same set of attributes. We find out that the fit of this stylized MNL model is similar to that based on Logistic regression. We present a discussion on what could be the potential reasons for the same and how could we enrich and use a better choice model. In Section 4.4, we then present empirical evidence using click data from Flipkart to show that we can improve by using the Generalized MNL model developed in Section 3.3. In particular, we argue that Generalized MNL choice model can capture the customer preferences well and has better predictive power too. Finally, we conclude in Section 4.5.
4.1 Introduction

Flipkart is an Indian e-commerce firm that has been founded in 2007 and has grown rapidly since then to capture 39% of the total Indian e-commerce market ([10]). It deals with a diverse range of products, serving more than 15 million active monthly consumers ([34]) who have collectively helped Flipkart to generate a revenue of US $7 billion in 2017 ([9]). Most of Flipkart’s consumer base access Flipkart using a mobile app or a browser on the mobile phone, providing the firm with an unprecedented access in tracking consumer behavior on their app/site and using this information for future decision making.

One fundamental problem that concerns Flipkart is that of identifying the relevant set of products to display to a user. However, the challenges involved in identifying such an optimal set of products to display are multi-fold. In a setting like that of Flipkart, demand trends constantly change and the inventory is also regularly updated with new items. Hence, one has to constantly learn consumer preferences while concurrently attempting to maximize revenue. This problem is further compounded by the fact that we have a large selection of product categories but can only show a small number of products from that and consumer preferences for a product depend on the overall set displayed (cf. the substitution effect). Moreover, apart from selecting the set of items to display we also need to decide how to bundle the individual items and where to display them (the location is a key factor when dealing with mobile devices). Motivated by this apparent need for a structural framework to recommend relevant set of items to consumers, in this chapter, we restrict ourselves to consider the important problem of identifying the optimal configuration of products on the homepage while accounting for substitution patterns, once the individual products have been grouped together by different teams externally.
4.1.1 Background

On Flipkart, when a consumer visits the homepage they are displayed a wide range of products (see Figure 4.1 for an example). The standard practice at Flipkart is to group a selected set of products that follow a common theme or serve a common sales purpose as a "widget" and display an assortment of these widgets to that consumer. For example, in Figure 4.1, we can observe that there are 6 widgets (on the left image) and the products in each widget follow a common theme. More specifically, the widget titled "Deals of the Day" consists of products which are currently having an ongoing discount offer, while the widget titled "Smartphones You Love" exclusively contains a pre-decided set of smartphones.

To help manage such a large number of products that could be possibly displayed on its homepage, Flipkart follows the following mechanism in generating and selecting the widgets to be displayed. There are several units/teams within Flipkart, with each generating content (widgets) that serve their team’s business function. For example, a sales team creates widgets consisting of products that are being offered with discounts. A merchandising team generates
widgets consisting of specific brands that they want to advertise on the home page. Similarly, a recommendation team generates widgets consisting of products that the team perceives are ideal fit for the consumer on whom they have collected the data before. Then, whenever a consumer visits the homepage, all the teams (either automatically or algorithmically) generate their respective contents in the form of individual widgets and send the widget requests to a centralized team, referred to as the homepage optimization team, which is then tasked to identify the optimal combination of widgets to be displayed for the consumer along with their respective positions.

4.1.2 Problem Description

On an average, the homepage optimization team receives around 30 – 40 widget requests from various teams for each user, after which it has to decide on the order in which the suggested widgets should be displayed. Typically, consumers only interact with around 3 – 5 widgets depending on the screen space and hence, it is essential for the homepage optimization team to also optimize the rankings of the individual widgets to be displayed once they are selected, so that the most relevant widgets are displayed in the most visible segments. Moreover widgets generated by different teams usually have an overlap in the theme of products and this leads to substitution among the widgets displayed. For example, in Figure 4.2, we can observe that widgets "Six Day Super Savers," "Offers for You," and "Discounts for You" which are all similar in spirit being displayed to a consumer. Consumers who are looking for a good deal on these items would be equally interested in all the three widgets in contrast to the case where only one of the widget pertains to such an offer. Hence, if one estimates the popularity of the widgets individually without accounting for substitution patterns, then the estimates might be significantly different in the two scenarios mentioned above. Therefore, to ensure the optimal configuration of the widgets, it is essential to consider a framework that accounts for substitution among the available alternatives.

We first briefly describe the data available for the study.
4.2 Data Description

**User Attributes** As mentioned before, Flipkart’s customer base predominantly interacts with the firm either via the mobile app or a mobile browser. This makes it easy for Flipkart to track user attributes and personalize the widgets for that specific user. We provide the details of user attributes available in Table 4.1. However, due to an actively growing user base, there are still a considerable number of users for whom the personal attributes are unknown. For these users, Flipkart typically displays widgets assuming the average value for the unknown attributes.

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td>Binary: male/female</td>
</tr>
<tr>
<td>RFM</td>
<td>Categorical, indicating the status: {Platinum, Bronze, Gold, Silver}</td>
</tr>
<tr>
<td>Recency</td>
<td>Categorical, indicating the activity: {1,2,3,4,5}</td>
</tr>
<tr>
<td>Frequency</td>
<td>Categorical, indicating the activity: {1,2,3,4,5}</td>
</tr>
<tr>
<td>Monetary</td>
<td>Categorical, indicating spending power: {1,2,3,4,5}</td>
</tr>
<tr>
<td>Is Parent</td>
<td>Binary: true/false</td>
</tr>
<tr>
<td>Is Student</td>
<td>Binary: true/false</td>
</tr>
<tr>
<td>Single Category Customer</td>
<td>Binary: whether only interested in single category</td>
</tr>
<tr>
<td>Pincode</td>
<td>Categorical, indicating the pincode</td>
</tr>
<tr>
<td>City</td>
<td>Categorical, indicating the city</td>
</tr>
<tr>
<td>State</td>
<td>Categorical, indicating the state</td>
</tr>
</tbody>
</table>

Table 4.1: Description of User Attributes
**Widget Attributes** As discussed earlier, every time a consumer interacts with Flipkart’s app or the homepage, different business units generate new contents in widgets and request the homepage optimization team to display their content to the customer. The homepage optimization team, in order to predict which widgets will be more relevant for the user, keeps track of certain widget attributes including the business unit that generated the widget, the content in the widget, the theme of the widget, at what position (rank) and with what layout has it been displayed. Table 4.2 provides the detailed descriptions of the widget attributes.

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Widget Title</td>
<td>Categorical (1000+ types) - indicating the title of widget, such as &quot;Offers for You&quot;, &quot;Discounts for You&quot;</td>
</tr>
<tr>
<td>Widget Type</td>
<td>Categorical (13 types) - indicating the type of widget, for example if it is an advertisement/product card/deal card</td>
</tr>
<tr>
<td>Content Type</td>
<td>Categorical (14 types) - indicating the content and generator of the widget, for example personalized recommendation card based on past purchases</td>
</tr>
<tr>
<td>Is Pinned</td>
<td>Binary - whether the widget is forced to be displayed by one of the business unit</td>
</tr>
<tr>
<td>View Type</td>
<td>Categorical (12 types) - indicating the display configuration of the widget</td>
</tr>
<tr>
<td>Rank</td>
<td>Position/Rank of the widget displayed. There are 40 unique rank/positions</td>
</tr>
<tr>
<td>Store Categories</td>
<td>Product categories grouped in the widget. On an average there are 2 product categories for every widget. Over all there are 1483 unique product categories.</td>
</tr>
<tr>
<td>Store Null</td>
<td>A dummy feature to indicate product categories information is not available.</td>
</tr>
</tbody>
</table>

Table 4.2: Description of Widget Attributes
**Other Attributes** Apart from the user and widget related attributes described so far, we also have a few special attributes. Table 4.3 provides the detailed descriptions of all those attributes.

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page Type</td>
<td>Categorical - which page the user is currently on, homepage or a search page or a product page</td>
</tr>
<tr>
<td>Channel</td>
<td>Categorical - indicating the channel, whether desktop website or mobile website or mobile app</td>
</tr>
<tr>
<td>OS Family</td>
<td>Categorical - indicating os of the mobile channel, android or ios</td>
</tr>
<tr>
<td>Hour</td>
<td>Categorical, indicating the hour</td>
</tr>
<tr>
<td>Day of Week</td>
<td>Categorical, indicating the particular weekday</td>
</tr>
<tr>
<td>Week of Month</td>
<td>Categorical, indicating the month</td>
</tr>
<tr>
<td>Cross Features</td>
<td>Other features obtained by taking a set product of existing features</td>
</tr>
</tbody>
</table>

Table 4.3: Description of Other Attributes

We first standardize the feature data and then convert all the categorical attributes described in Table 4.1 to Table 4.3 to attributes with binary values by adding dummy attributes (for example each of the 13 widget types is considered as a different attribute that can take values 1 or 0) resulting in a very high dimensional feature space, with $d \approx 40k$ attributes. Now every widget $i$ is associated with an attribute vector $x_i \in \{0,1\}^d$. We focus on consumer click data over the last week in the month of January in 2019. The click rate for individual widgets (out of all displayed widgets) is about 10%, while the click rate for the homepage itself (i.e., at least one of the widget is clicked) is around 35%. Interestingly, around 8% of the users have clicked on multiple widgets. Since random utility based choice models do not allow for the possibility of clicking multiple items (a maximum of only one choice), we assume that only one of these widgets is clicked and randomly select a widget (out of the clicked ones) to be the clicked widget. In the next section, we now discuss the fit of the Logistic regression and the MNL model on this data set.
4.3 Multinomial Logit Model and Logistic Regression Model

We now discuss the use of choice models to capture the substitution behavior first starting with the MNL model. More specifically, we hypothesise that the MNL model which accounts for presence of similar alternatives, might have a better predictive power than the Logistic regression model, which is the current model used for estimating the popularity of individual widgets by the homepage optimization team at Flipkart.

4.3.1 Logistic Regression Model

In the Logistic regression based model, every item’s demand is estimated independently of the offer set. More specifically, under the Logistic regression based model, the probability that a consumer will click on a widget whose attributes are defined by the vector \( x \in \{0, 1\}^d \) and the attribute weight vector \( \beta_{\text{LogReg}} \) is given by

\[
P_{\text{click}}(\beta_{\text{LogReg}}, x) = \mathbb{P}
(\text{click} | \beta_{\text{LogReg}}) = \frac{e^{\beta_{\text{LogReg}} \cdot x}}{1 + e^{\beta_{\text{LogReg}} \cdot x}}.
\]

We utilize the click information on each widget offered and then leverage the maximum likelihood estimate \( \beta_{\text{LogReg}}^{\text{MLE}} \) for \( \beta_{\text{LogReg}} \) to estimate the click through rate of the offered widgets and study the fit of the Logistic regression model for the estimated \( \beta_{\text{LogReg}}^{\text{MLE}} \). In particular, we compute \( \beta_{\text{LogReg}}^{\text{MLE}} \) that maximizes the following regularized log-likelihood

\[
\beta_{\text{LogReg}}^{\text{MLE}} = \arg\max_{\beta} \sum_{t=1}^{T} \log P_{\text{click}}(\beta, x_t) - \lambda \|\beta\|_2.
\]

The objective function in the preceding optimization problem is convex and therefore we can use any of the standard convex optimization techniques to obtain the estimate \( \beta_{\text{LogReg}}^{\text{MLE}} \) (see [12].) We obtain the estimates using the popular stochastic gradient descent method with the tuning parameter \( \lambda \) chosen via cross-validation.
4.3.2 MNL Choice Model

In the MNL choice model, we assume that the mean utility of a product is linear in the values of attributes. More specifically, the mean utility of widget \( i \), with attribute vector \( x_i \) is given by the inner product

\[
u_i = \beta_{\text{MNL}} \cdot x_i \quad \forall i,
\]

where \( \beta_{\text{MNL}} \in \mathbb{R}^d \) is some fixed but initially unknown attribute weight vector. Under this model, the probability that a consumer clicks on widget \( i \) when offered an assortment of widgets \( S \subset \{1, \ldots, N\} \) is assumed to be,

\[
Pr(i, S) = \begin{cases} 
\frac{e^{\beta_{\text{MNL}} \cdot x_i}}{1 + \sum_{j \in S} e^{\beta_{\text{MNL}} \cdot x_j}} & \text{if } i \in S \cup \{0\} \\
0 & \text{otherwise},
\end{cases}
\]

(4.1)

where 0 denotes no-click (the no-choice or outside option). Here, again, we utilize the click information for each user request and then leverage the maximum likelihood estimate \( \beta_{\text{MNL}}^{\text{MLE}} \) for \( \beta_{\text{MNL}} \) to estimate the click through rate of the offered widgets and study the fit of the MNL choice model for the estimated \( \beta_{\text{MNL}}^{\text{MLE}} \). In particular, we compute \( \beta_{\text{MNL}}^{\text{MLE}} \) that maximizes the following regularized log-likelihood

\[
\beta_{\text{MNL}}^{\text{MLE}} = \arg \max_{\beta} \sum_{i=1}^{T} \sum_{i \in S \cup \{0\}} 1 \text{ (widget } i \text{ is clicked)} \cdot \log Pr_i(\beta, x_i) - \lambda \| \beta \|_2.
\]

(4.2)

The objective function in the preceding optimization problem is also convex and therefore we can use any of the standard convex optimization techniques to obtain the estimate, \( \beta_{\text{MNL}}^{\text{MLE}} \) (see [12].) We obtain the estimates using the popular stochastic gradient descent method with the tuning parameter \( \lambda \) chosen via cross-validation.
4.3.3 Results: Fit of Logistic Regression vs Fit of MNL Model

For both Logistic regression based and MNL choice models, we perform a 10-fold cross validation with 70-30% train and test split. In particular, we randomly split the consumer click data into training and testing sets with 70% of the data in training segment and the remaining 30% of the data in the testing segment. We repeat this 10 times and compute the average of the 10 results for a more robust comparison. In Figure 4.3, we plot the ROC curves of both the fits on the test data. We can observe that the fit corresponding to the MNL model is of the same order as compared to the fit corresponding to the Logistic regression based model with the corresponding area under the curve (AUC) being about 60% for both of them. In fact, the Logistic regression based model performs slightly better than the MNL choice model.

Figure 4.3: Fit of Logistic Regression and MNL Choice Model on Flipkart’s consumer click data.
4.3.4 Discussion

We would have expected that working with a choice model such as MNL which accounts for substitution patterns will provide significantly better handle on understanding consumer preferences and thereby give superior results as compared to a simple Logistic regression based model. However, we do not see that to be the case here. We think there could be a lot of possible reasons for this but here we discuss some of them which we think are more important.

- Non-homogeneity: the complete data is highly heterogeneous in terms of the users. MNL choice models usually assume a homogeneous set of users to better capture the substitution behavior. Indeed, during an initial limited experiment when we restricted the data to a homogeneous group of users and only considered a particular consumer segment (by fixing the user attributes such as gender, frequency, recency, monetary etc. to one particular value each), the MNL choice model did outperform the Logistic regression based model (see [1]).

- Revenue information: the usual objective for an assortment optimization problem based on a choice model is to maximize expected revenue and generally most of the items have different revenues too. Thus when we use the revenue data to fit an MNL choice model, it tends to capture the substitution behavior better. However, in this case, since the objective is to maximize the clicks, we do not maximize the expected revenue and information about widgets’ revenue are also not captured in the attributes. This could limit the predictive power of the MNL thereby resulting in a not so superior fit. This is also evident in the field experiment by [22].

Hence we look to use an enriched choice model which could give us a better fit and thereby better results. An important aspect while deciding which choice model to use is the click through rates in the data. The low click through rates (which is the case for any data where only clicks are recorded) translates to a high "no-purchase" value in the choice model
terminology. This can be thought of as the presence of choice overload phenomenon and hence we can use the Generalized MNL choice model we developed in Section 3.3. Recall that in the numerical results we presented in Section 3.6, we had similar observations where the data had the click information and a low value of the click through rate. The Generalized MNL choice model outperformed the MNL choice model in that case as well.

We do make a note here that another possibility could have been to fit a separate MNL choice model for each customer segment to tackle the heterogeneity aspect. However that would have created two complications. Firstly, since the total number of homogeneous groups is significantly high (due to multiple user features having multiple values), this would have meant more high dimensional parameters to estimate. Secondly, these decisions are to be made in real time and hence we need a model which naturally ties in all the data together without expanding the set of parameters by much. Hence, the Generalized MNL choice model which adds only one more parameter ($\alpha$) to the MNL choice model, is quite appealing.

4.4 Generalized MNL Model

Here, we briefly describe the Generalized MNL choice model from Section 3.3 once again. Apart from the same settings of the MNL model, in this model we have an additional parameter $\alpha > 0$, which determines the importance we put on the "no-purchase" option. So, as in MNL model, the mean utility of widget $i$, with attribute vector $x_i$ is still given by the inner product

$$u_i = \beta^\text{GMNL} \cdot x_i \quad \forall i,$$

where $\beta^\text{GMNL} \in \mathbb{R}^d$ is some fixed but initially unknown attribute weight vector. But under this model, the probability that a consumer clicks on widget $i$ when offered an assortment
of widgets \( S \subset \{1, \ldots, N\} \) is given by,

\[
Pr(i, S) = \begin{cases} 
\frac{e^{\beta_{GMNL} x_i}}{e^{a \sum_{j \in S} e^{\beta_{GMNL} x_j}} + \sum_{j \in S} e^{\beta_{GMNL} x_j}} & \text{if } i \in S \cup \{0\} \\
0 & \text{otherwise,}
\end{cases}
\] (4.3)

where 0 denotes no-click (the no-choice or outside option). Here, again, we utilize the click information for each user request and then leverage the maximum likelihood estimates \( \alpha_{MLE} \) and \( \beta_{MLE}^{GMNL} \) for \( \alpha \) and \( \beta_{MLE}^{GMNL} \) to estimate the click through rate of the offered widgets and study the fit of the Generalized MNL choice model for the estimated \( \alpha_{MLE} \) and \( \beta_{MLE}^{GMNL} \).

In particular, we compute \( \alpha_{MLE} \) and \( \beta_{MLE}^{GMNL} \) that maximize the following regularized log-likelihood

\[
\alpha_{MLE}, \beta_{MLE}^{GMNL} = \arg \max_{\alpha, \beta} \sum_{i=1}^{T} \sum_{j \in S_i \cup \{0\}} \mathbf{1} (\text{widget } i \text{ is clicked}) \cdot \log Pr_i(\alpha, \beta, x_i) - \lambda \| \beta \|_2.
\] (4.4)

The objective function in the preceding optimization problem is not convex and hence to obtain the estimates \( \alpha_{MLE} \) and \( \beta_{MLE}^{GMNL} \), we use Algorithm 1 that we developed in Section 3.3.3.

### 4.4.1 Results: Fit of the Generalized MNL Model

Once again, the setup for the experiment is the same as before. We perform a 10–fold cross validation with 70-30 % train and test split by randomly splitting the consumer click data into training and testing sets with 70% of the data in training segment and the remaining 30% of the data in the testing segment. We repeat this 10 times and compute the average of the 10 results for a more robust comparison. In Figure 4.4, we plot the ROC curves of the fits on the test data. We can observe that the fit corresponding to the Generalized MNL model is better as compared to the fit corresponding to the Logistic regression based model with the corresponding area under the curve (AUC) being about 62% for the Generalized MNL choice model.
4.4.2 Discussion

The Generalized MNL model captures the consumer preferences well and performs better than Logistic regression model as well as the MNL choice model. As mentioned before, it does that well because it models the choice overload phenomenon.

Another interesting observation is that the $\alpha$ estimated from the data is significant (we get $\alpha \approx 10$) which reaffirms the hypothesis of prevalent choice overload which we postulated from low click through rates.

We do mention that if we modify our objective function to include some information about the conversion rate (the fraction of people who ended up buying a product from the clicked widget), a choice model like MNL can also perform well. This is because the conversion rate would record the revenue information and hence we would be maximizing the expected revenue. Of course, this will require an elaborate set up and logging in and extracting a lot of data apart from just homepage visits to get revenue information about all the individual products in the widgets.
4.5 Conclusion

Our main contribution in this chapter is to advocate the use of richer class of discrete choice models which capture consumer preferences and substitution behavior better in the presence of multiple items as compared to other models which view each item independently on their own. We study a rich dataset from a large scale e-commerce company and show that although a simple MNL choice model might not be able to achieve a better fit of the data when we are trying to maximize only the click through rate, we can use Generalized MNL choice model, which models and captures the choice overload phenomenon prevalent in this situation, to do the same and thereby get better results.
Epilogue

This dissertation contributes significantly to the field of revenue management by developing new discrete choice models which help capture the prevalent settings currently observed in the era of e-commerce and big data. The continuous-space based discrete choice model developed in the first half of the dissertation naturally captures the setting when sellers have access to a multitude of features for the products they want to offer. This model is closely related to the mixture of MNLs model which is the most general random utility maximization (RUM) based choice model as it can approximate any RUM choice model arbitrarily closely. By solving the assortment optimization problem under this new continuous-space model, we also reveal important connections to the assortment optimization under the mixture of MNLs model.

The Generalized Markov chain choice model developed in the second half of this dissertation is a natural choice model for capturing the observed customer behaviors, most prominently among them, the choice overload phenomenon, when they are faced with a lot of options to choose from. This model also captures the dynamic preferences of the customers. Although we show that the assortment optimization problem under this new choice model becomes NP-hard and hence we can only solve it approximately; on the positive side, we also develop efficient algorithms to find suboptimal assortments which give an expected revenue value arbitrarily close to the optimal value. In particular, we develop FPTAS for the assortment optimization problem under reasonable assumptions. We also study in detail a special case which generalizes the popular MNL choice model and captures choice overload.
Finally, through a field experiment on high dimensional big data from a large scale e-commerce firm, we also demonstrate the applicability of using such choice models, in particular the Generalized MNL model we develop in the second half of the dissertation, to capture the observed consumer preferences and behavior in a better way compared to traditional models.
References


[18] Antoine Désir, Vineet Goyal, and Jiawei Zhang. “Near-optimal algorithms for capacity constrained assortment optimization”. In: *Available at SSRN 2543309* (2014).


[25] Paul Glasserman. “Performance Continuity and Differentiability in Monte Carlo Op-
timization”. In: Proceedings of the 20th Conference on Winter Simulation. WSC ’88.

[26] Kumar Goutam, Vineet Goyal, and Henry Lam. “Assortment Optimization over Dense
Universe is Easy”. In: Available at SSRN 3649233 (2020).

Model to Capture Dynamic Preferences and Choice Overload”. In: arXiv preprint

random item sizes”. In: Operation Research Letter (2009).

[29] Sheena S Iyengar and Mark R Lepper. “When choice is demotivating: Can one desire
too much of a good thing?” In: Journal of personality and social psychology 79.6 (2000),
p. 995.

[30] S Jagabathula and P Rusmevichientong. A two-stage model of consideration set and

[31] Srikanth Jagabathula and Gustavo Vulcano. “A Partial-Order-Based Model to Esti-
mate Individual Preferences using Panel Data”. In: Management Science (2017).

[32] Toshihiro Kamishima. “Nantonac Collaborative Filtering: Recommendation Based on
Order Responses”. In: Proceedings of the Ninth ACM SIGKDD International Confer-
ence on Knowledge Discovery and Data Mining. KDD ’03. Washington, D.C.: ACM,

[33] Eugene L Lawler. “Fast approximation algorithms for knapsack problems”. In: Mathe-

[34] Livemint.“https://www.livemint.com/Industry/a8wtqtRj8duaLThltkUBI/Flipkart-to-
look-beyond-gross-sales-numbers-Kalyan-Krishnam.html”. In: ()

[35] R Duncan Luce. Individual choice behavior: A theoretical analysis. Courier Corpora-
tion, 2005.

[36] Daniel McFadden. “Modeling the choice of residential location”. In: Transportation
Research Record 1.673 (1978).

onlinelibrary.wiley.com/doi/pdf/10.1002/1099-1255%28200009%2915%3A5%3C447
%3EAI\%3AAID-JAE570%3E3.0.CO%3B2-1.


Appendix A: Assortment Optimization Over a Dense Universe

A.1 Proof of Lemma 2.2.1

Lemma A.1.1 For

\[ h(b) = \int_{\beta}^{b} \frac{r(x)v(x)dF(x) + c_1}{1 + \int_{\beta}^{b} v(x)dF(x) + c_2} dG(\beta), \]

where \( c_1 = \mathbb{E}_x [r(x)v(x); x \in S] \) and \( c_2 = \mathbb{E}_x [v(x); x \in S] \), we have

\[ \frac{\partial h}{\partial b} \geq 0. \]

Proof To take the partial derivative of \( h(\cdot) \) with respect to \( b \), we observe that its integrand is differentiable in \( b \) over a compact space and has bounded derivative upper bounded by \( e^{||\beta||}(R^\prime_{max} + ||\beta||R_{max} + R^\prime_{max}e^{||\beta||}) \). Combining these facts and using Assumption 2.2.4, we can appeal to Lemma 1 from [25], to interchange the derivative and the integral. Hence we get

\[ \frac{\partial h}{\partial b} = \int_{\beta}^{b} \frac{r(b)v(b)f(b)[1 + \int_{a}^{b} v(x)dF(x) + c_2] - v(b)f(b)\int_{a}^{b} r(x)v(x)dF(x) + c_1}{(1 + \int_{a}^{b} v(x)dF(x) + c_2)^2} dG(\beta) \]

which we can simplify to

\[ \frac{\partial h}{\partial b} = \int_{\beta}^{b} \frac{v(b)f(b)[r(b) + \int_{a}^{b} (r(b) - r(x))v(x)dF(x) + r(b)c_2 - c_1]}{(1 + \int_{a}^{b} v(x)dF(x) + c_2)^2} dG(\beta) \]

\[ = \int_{\beta}^{b} \frac{v(b)f(b)[r(b) + \int_{a}^{b} (r(b) - r(x))v(x)dF(x) + \mathbb{E}_x [(r(b) - r(x))v(x); x \in S]]}{(1 + \int_{a}^{b} v(x)dF(x) + c_2)^2} dG(\beta) \]
Since $r$ is increasing in $x$ and also recall that $\max_{x \in S} x < a$, we have

$$v(x) = e^{\beta x} \geq 0 \ \forall \ x \text{ and } r(b) \geq r(x) \ \forall \ x \in [a, b] \cup S$$

and hence

$$\frac{\partial h}{\partial b} \geq 0. \ \Box$$

### A.2 Proof of Lemma 2.2.2

**Lemma A.2.1** For

$$h(a, b, a^*) = \int_{\beta} \frac{\int_a^b r(x)v(x)dF(x) + \int_a^1 r(x)v(x)dF(x) + c_1}{1 + \int_a^b v(x)dF(x) + \int_a^1 v(x)dF(x) + c_2} dG(\beta),$$

where $c_1 = \mathbb{E}_x [r(x)v(x); x \in S]$ and $c_2 = \mathbb{E}_x [v(x); x \in S]$, we have

either $\frac{\partial h}{\partial a} \geq 0 \ \forall a \in [0, b],$

or $\frac{\partial h}{\partial b} \geq 0 \ \forall b \in [a, a^*].$

**Proof** Considering $h$ as a function of $a, b$ and $a^*$, we consider the partial derivatives with respect to $a, b$ and $a^*$:

$$\frac{\partial h}{\partial a} =$$

$$\int_{\beta} v(a)f(a)\left[ \int_a^b (r(x) - r(a))v(x)dF(x) + \int_a^1 (r(x) - r(a))v(x)dF(x) - r(a) + c_1 - r(a)c_2 \right] dG(\beta),$$

$$\frac{\partial h}{\partial b} =$$

$$\int_{\beta} v(b)f(b)\left[ \int_a^b (r(b) - r(x))v(x)dF(x) + \int_a^1 (r(b) - r(x))v(x)dF(x) + r(b) + r(b)c_2 - c_1 \right] dG(\beta),$$

$$\frac{\partial h}{\partial b} =$$

$$\int_{\beta} v(x)dF(x) + \int_a^1 v(x)dF(x) + 2 v(x)dF(x).$$
and

\[
\frac{\partial h}{\partial a^*} = \int_\beta v(a^*) f(a^*) \left[ \int_{a^*}^1 (r(x) - r(a^*)) v(x) dF(x) + \int_a^{b} (r(x) - r(a^*)) v(x) dF(x) - r(a^*) + c_1 - r(a^*)c_2 \right] \\
\frac{1}{(1 + \int_a^{b} v(x) dF(x) + \int_{a^*}^1 v(x) dF(x))^2} dG(\beta),
\]

where we define \(c_1 = \mathbb{E}_x [r(x)v(x); x \in S]\) and \(c_2 = \mathbb{E}_x [v(x); x \in S]\), and that we have also used Assumption 2.2.4 and Lemma 1 from [25] for interchanging the derivatives and the integrals.

Since we have \(v(\cdot), f(\cdot) \geq 0\), we focus our attention on the expression inside the big brackets in the numerator. Let us define:

\[
g_1(a) = \int_a^{b} (r(x) - r(a)) v(x) dF(x) + \int_{a^*}^1 (r(x) - r(a)) v(x) dF(x) - r(a) + c_1 - r(a)c_2,
\]

\[
g_2(b) = \int_a^{b} (r(b) - r(x)) v(x) dF(x) + \int_{a^*}^1 (r(b) - r(x)) v(x) dF(x) + r(b) + r(b)c_2 - c_1,
\]

and

\[
g_3(a^*) = \int_{a^*}^1 (r(x) - r(a^*)) v(x) dF(x) + \int_a^{b} (r(x) - r(a^*)) v(x) dF(x) - r(a^*) + c_1 - r(a^*)c_2.
\]

We have the following expressions:

\[
\frac{\partial g_1}{\partial a} = -\int_a^{b} r'(a) v(x) dF(x) - \int_{a^*}^1 r'(a) v(x) dF(x) - r'(a) - r'(a)c_2,
\]

\[
\frac{\partial g_2}{\partial b} = \int_a^{b} r'(b) v(x) dF(x) + \int_{a^*}^1 r'(b) v(x) dF(x) + r'(b) + r'(b)c_2,
\]

and

\[
\frac{\partial g_3}{\partial a^*} = -\int_{a^*}^1 r'(a^*) v(x) dF(x) - \int_a^{b} r'(a^*) v(x) dF(x) - r'(a^*) - r'(a^*)c_2.
\]

Hence, using the fact that \(r(x)\) is increasing in \(x\), i.e., \(r'(x) > 0\) (Assumption 2.2.1), we get
that:

\[ \frac{\partial g_1}{\partial a} \leq 0, \quad \frac{\partial g_2}{\partial b} \geq 0, \quad \frac{\partial g_3}{\partial a^*} \leq 0 \]

Also, on the boundary points (recall that \( x \in [0, 1] \)) we have the following relations (again using Assumption 2.2.1 for \( r(0) = 0 \)):

\[ g_1(0) > 0, \quad g_2(b) = \int_a^b (r(b) - r(x))v(x)dF(x) - g_1(b) \]

Combining the above two facts, we have two cases.

1. In the first case, both \( g_1(0) > 0 \) and \( g_1(b) > 0 \).

2. In the other case, if \( g_1(b) < 0 \), we get \( g_2(b) > 0 \) and \( g_2(a^*) > 0 \).

We note that the assumption of \( r(0) = 0 \) is without loss of generality as \( r(0) < 0 \) also gives us \( g_1(0) > 0 \); whereas if \( r(0) > 0 \) and if this does lead to \( g_1(0) < 0 \), since \( g_1(\cdot) \) is non-increasing, we would also get \( g_1(b) < 0 \), which is the second case again.

The first case gives us

\[ \frac{\partial h}{\partial a} \geq 0 \quad \forall a \in [0, b], \]

and the second case gives us

\[ \frac{\partial h}{\partial b} \geq 0 \quad \forall b \in [a, a^*]. \quad \square \]

### A.3 Proof of Lemma 2.2.3

**Lemma A.3.1** For

\[ h(b) = \int \int \int_{\mathbb{R}^2} r(x)v(x,y)dF(x,y) + c_1 \int_{\mathbb{R}} dG(\beta), \]

where \( c_1 = \mathbb{E}_x[r(x)v(x,y); (x,y) \in \mathcal{S}] \) and \( c_2 = \mathbb{E}_x[v(x,y); (x,y) \in \mathcal{S}] \), we have

\[ \frac{\partial h}{\partial b} \geq 0. \]

103
Proof. We again use Assumption 2.2.4 and Lemma 1 from [25] to help interchange the derivatives and the integrals. Hence we get (after doing some simplifications)

\[
\frac{\partial h}{\partial b} = \int_{\beta} \frac{\int_0^d (b, y) f(b) dF(y)}{(1 + \int_0^b \int_0^d v(x, y) dF(x, y) + c_2)^2} \left[ r(b) + \int_0^b \int_0^d (r(b) - r(x)) v(x, y) dF(x, y) + r(b) c_2 - c_1 \right] dG(\beta)
\]

\[
= \int_{\beta} \frac{\int_0^d (b, y) f(b) dF(y)}{(1 + \int_0^b \int_0^d v(x, y) dF(x, y) + c_2)^2} \left[ r(b) + \int_0^b \int_0^d (r(b) - r(x)) v(x, y) dF(x, y) + \mathbb{E}_x [(r(b) - r(x)) v(x, y); (x, y) \in S] \right] dG(\beta)
\]

Since

\[v(x, y) \geq 0 \quad \forall \quad x \quad \text{and} \quad r(b) \geq r(x) \quad \forall \quad (x, y) \in \{(a, b) \times [c, d]) \cup S\]

(as \(r\) is increasing in \(x\) and also recall that \(\max_{(x, y) \in S} x < b\)), we get that

\[
\frac{\partial h}{\partial b} \geq 0. \quad \Box
\]

A.4 Proof of Lemma 2.2.4

Lemma A.4.1 Let \(S' = S \cup \{(a_0, a] \times [c_1, d_1]\} \cup \cdots \cup \{(a_0, a] \times [c_m, d_m]\} \cup \{(a, 1) \times [0, 1]^d\}, where \(S \in R\), \(a_0 < a \leq 1\) and \(\max_{x \in S} x \leq a_0\). Keeping \(S\) fixed, we want to characterize an optimal choice for \(c_j\) and \(d_j\). Then, fixing the dimension \(k\), for any \(j\), the optimal \(c_{(j)}, k\) is achieved at the lower boundary point \(d_{(j-1), k}\) and the optimal \(d_{(j), k}\) is achieved at the upper boundary point \(c_{(j+1), k}\). In other words, for

\[
h(c_j, d_{jk}) = \int_{\beta} \frac{\int_{a_0}^a \int_{[c_1, d_1] \cup \cdots \cup [c_m, d_m]} r(x) v(x, y) dF(x, y) + \int_a^1 \int_{[0,1]^d} r(x) v(x, y) dF(x, y) + c_1}{1 + \int_{a_0}^a \int_{[c_1, d_1] \cup \cdots \cup [c_m, d_m]} v(x, y) dF(x, y) + \int_a^1 \int_{[0,1]^d} v(x, y) dF(x, y) + c_2} dG(\beta),
\]

104
Similarly, we also get the following expression after doing some simplifications

We can observe that

These expressions are valid for each $1 \leq j \leq m$ and hence we get the required result.  \hfill \square

**Proof** Using Assumption 2.2.4 and Lemma 1 from [25] to interchange the derivatives and the integrals and simplifying, we get the following expression:

\[
\frac{\partial h}{\partial c_{jk}} = \mathbb{E}_\beta \left[ \frac{-v(c_{jk})f_y(c_{jk})}{(1 + \int_{a_0}^a \int_{[c_1,d_1] \cup \cdots \cup [c_m,d_m]} v(x,y) dF(x,y) + \int_0^1 \int_{[0,1]^d} v(x,y) dF(x,y) + c_2)^2} \right] \\
\int_{a_0}^a \int_{[c_1,d_1] \cup \cdots \cup [c_m,d_m]} r(x)v(x,y) dF(x,y) \\
- \int_{a_0}^a \int_{c_{-jk}}^{d_{-jk}} v(x,y_{-jk}) dF(x,y_{-jk}) \\
+ \int_{a_0}^a \int_{c_{-jk}}^{d_{-jk}} r(x)v(x,y_{-jk}) dF(x,y_{-jk}) \right] \\
\]

which simplifies to

\[
\frac{\partial h}{\partial c_{jk}} = \mathbb{E}_\beta \left[ \frac{-v(c_{jk})f_y(c_{jk}) \int_{a_0}^a \int_{c_{-jk}}^{d_{-jk}} r(x)v(x,y_{-jk}) dF(x,y_{-jk})}{(1 + \int_{a_0}^a \int_{[c_1,d_1] \cup \cdots \cup [c_m,d_m]} v(x,y) dF(x,y) + \int_0^1 \int_{[0,1]^d} v(x,y) dF(x,y) + c_2)^2} \right] dG(\beta).
\]

Similarly, we also get the following expression after doing some simplifications

\[
\frac{\partial h}{\partial d_{jk}} = \mathbb{E}_\beta \left[ \frac{v(d_{jk})f_y(d_{jk}) \int_{a_0}^a \int_{c_{-jk}}^{d_{-jk}} r(x)v(x,y_{-jk}) dF(x,y_{-jk})}{(1 + \int_{a_0}^a \int_{[c_1,d_1] \cup \cdots \cup [c_m,d_m]} v(x,y) dF(x,y) + \int_0^1 \int_{[0,1]^d} v(x,y) dF(x,y) + c_2)^2} \right] dG(\beta).
\]

We can observe that

\[
\frac{\partial h}{\partial c_{jk}} \leq 0 \quad \text{and} \quad \frac{\partial h}{\partial d_{jk}} \geq 0.
\]

These expressions are valid for each $1 \leq j \leq m$ and hence we get the required result.
A.5 Proof of Lemma 2.2.5

Lemma A.5.1 For

\[
\begin{align*}
h(b) &= \int_{[a,b]} \int_{c}^{d} \frac{r(x)v(x,y)dF(x,y)}{1 + \int_{[a,b]} \int_{c}^{d} v(x,y)dF(x,y) + \int_{[a*,1]} \int_{[c]}^{d} v(x,y)dF(x,y) + c_1} dG(\beta), \\
&= \int_{[a,b]} \int_{c}^{d} \frac{v(a,y)f(a)dF(y)}{1 + \int_{[a,b]} \int_{c}^{d} v(x,y)dF(x,y) + \int_{[a*,1]} \int_{[c]}^{d} v(x,y)dF(x,y) + c_1} dG(\beta), \\
&= \int_{[a,b]} \int_{c}^{d} \frac{v(b,y)f(b)dF(y)}{1 + \int_{[a,b]} \int_{c}^{d} v(x,y)dF(x,y) + \int_{[a*,1]} \int_{[c]}^{d} v(x,y)dF(x,y) + c_1} dG(\beta),
\end{align*}
\]

where \( c_1 = \mathbb{E}_x[r(x)v(x,y); (x,y) \in S] \) and \( c_2 = \mathbb{E}_x[v(x,y); (x,y) \in S] \), we have

either \( \frac{\partial h}{\partial a} > 0 \ \forall a \in [0, b] \),

or \( \frac{\partial h}{\partial b} > 0 \ \forall b \in [a, a^*] \).

Proof We consider the following partial derivatives with respect to \( a, b \) and \( a^* \):

\[
\begin{align*}
\frac{\partial h}{\partial a} &= \int_{[a,b]} \int_{c}^{d} \frac{v(a,y)f(a)dF(y)}{1 + \int_{[a,b]} \int_{c}^{d} v(x,y)dF(x,y) + \int_{[a*,1]} \int_{[c]}^{d} v(x,y)dF(x,y) + c_1} \left[ \int_{[a,b]} \int_{c}^{d} (r(x) - r(a))v(x,y)dF(x,y) \\
&+ \int_{[a*,1]} \int_{[c]}^{d} (r(x) - r(a))v(x,y)dF(x,y) - r(a) + c_1 - r(a)c_2 \right] dG(\beta),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial h}{\partial b} &= \int_{[a,b]} \int_{c}^{d} \frac{v(b,y)f(b)dF(y)}{1 + \int_{[a,b]} \int_{c}^{d} v(x,y)dF(x,y) + \int_{[a*,1]} \int_{[c]}^{d} v(x,y)dF(x,y) + c_1} \left[ \int_{[a,b]} \int_{c}^{d} (r(b) - r(x))v(x,y)dF(x,y) \\
&+ \int_{[a*,1]} \int_{[c]}^{d} (r(b) - r(x))v(x,y)dF(x,y) + r(b) + r(b)c_2 - c_1 \right] dG(\beta),
\end{align*}
\]
and

\[
\frac{\partial h}{\partial a^*} = \int_{\beta} \left[ \int_c^d v(a^*, y) f(a^*) dF(y) \right] \left( 1 + \int_a^b \int_c^d v(x, y) dF(x, y) + \int_{[0,1]^d} v(x, y) dF(x, y) \right)^2 \\
\int_{a^*}^1 \int_{[0,1]^d} (r(x) - r(a^*)) v(x, y) dF(x, y) \\
+ \int_a^b \int_c^d (r(x) - r(a^*)) v(x, y) dF(x, y) - r(a^*) + c_1 - r(a^*) c_2 \right] dG(\beta),
\]

where we define \( c_1 = \mathbb{E}_x [r(x) v(x, y); (x, y) \in S] \) and \( c_2 = \mathbb{E}_x [v(x, y); (x, y) \in S] \), and that we have also used Assumption 2.2.4 and Lemma 1 from [25] to interchange the derivatives and the integrals.

Since we have \( v(\cdot), f(\cdot) \geq 0 \), we focus our attention on the expression inside the big brackets in the numerator. Let us define:

\[
g_1(a) = \int_a^b \int_c^d (r(x) - r(a)) v(x, y) dF(x, y) \\
+ \int_{a^*}^1 \int_{[0,1]^d} (r(x) - r(a)) v(x, y) dF(x, y) - r(a) + c_1 - r(a) c_2,
\]

\[
g_2(b) = \int_a^b \int_c^d (r(b) - r(x)) v(x, y) dF(x, y) \\
+ \int_{a^*}^1 \int_{[0,1]^d} (r(b) - r(x)) v(x, y) dF(x, y) + r(b) + r(b) c_2 - c_1,
\]

and

\[
g_3(a^*) = \int_{a^*}^1 \int_{[0,1]^d} (r(x) - r(a^*)) v(x, y) dF(x, y) \\
+ \int_a^b \int_c^d (r(x) - r(a^*)) v(x, y) dF(x, y) - r(a^*) + c_1 - r(a^*) c_2.
\]
Using the fact that $r(x)$ is increasing in $x$, i.e., $r'(x) > 0$, we easily get:

$$\frac{\partial g_1}{\partial a} < 0, \quad \frac{\partial g_2}{\partial b} > 0, \quad \frac{\partial g_3}{\partial a^*} < 0$$

Also, on the boundary points we have the following:

$$g_1(0) > 0, \quad g_2(b) = \int_a^b \int_c^d (r(b) - r(x)) v(x, y) dF(x, y) - g_1(b)$$

Combining the above two facts, we have two cases.

1. In the first case, both $g_1(0) > 0$ and $g_1(b) > 0$.

2. In the other case, if $g_1(b) < 0$, we get $g_2(b) > 0$ and $g_2(a^*) > 0$.

The first case gives us

$$\frac{\partial h}{\partial a} > 0 \ \forall a \in [0, b],$$

and the second case gives us

$$\frac{\partial h}{\partial b} > 0 \ \forall b \in [a, a^*]. \quad \Box$$

### A.6 Other Lemmas Used

**Lemma A.6.1** Using the exact dependence of $D$ on $V$, the probability bound is given by

$$\left(\frac{c^* t}{\sqrt{V}}\right)^{2V-2} e^{-2r^2},$$

where $c^*$ is a universal constant.

**Proof** The $D$ term is defined by the following expression:

$$D > \frac{1}{\log K},$$
where $K$ depends on $V$ via:

$$\log K = \frac{\log V + c_0 V + c_1}{2V - 2},$$

where $c_0$ and $c_1$ are universal constants independent of everything else ($c_0 = \log(16\epsilon)$). The above two expressions are detailed in the proof of Theorem 2.14.9 of [48]. Hence we have

$$D > \frac{2V - 2}{\log V + c_0 V + c_1}.$$

To get the best upper bound for the theorem above, we take the appropriate bound for $D$. Putting everything together, the bound in the theorem becomes

$$\left( \frac{2(V - 1)t^2}{(c_0 V + \log V + c_1)^2} \right)^{V-1} e^{-2t^2}.$$

Since $V$ dominates the $\log V$ term, the above expression would reduce to the required form.

□

**Lemma A.6.2** For the function

$$f(x, y) = \frac{x}{v_0/n + y},$$

the norm of the gradient is bounded above by

$$\frac{(R_{\text{max}} + 1)^2}{4R_{\text{max}}} \frac{n}{v_0}. $$

**Proof** We have

$$||\nabla(f)|| \leq \frac{1}{\frac{v_0}{n} + y\beta} + \frac{x}{(\frac{v_0}{n} + y\beta)^2},$$
where \( x \leq R_{\text{max}}y_\beta \), and \( y_\beta = \sum_{i \in S} \nu(\beta, x_i) \) which is positive. Hence,

\[
||\nabla(f)|| \leq \frac{1}{\nu_0/n + y_\beta} + \frac{R_{\text{max}}y_\beta}{(\nu_0/n + y_\beta)^2}
\]

\[
= \frac{\nu_0/n + (1 + R_{\text{max}}y_\beta)}{(\nu_0/n + y_\beta)^2}
\]

A simple calculation shows that the above bound, as a function of \( y_\beta \) is first increasing in the argument and then decreasing, giving us an absolute bound of

\[
\frac{(R_{\text{max}} + 1)^2 n}{4R_{\text{max}} \nu_0}.
\]
Appendix B: Generalized Markov Chain Choice Model

B.1 Proof of Lemma 3.3.1

Let $S \subseteq \mathcal{N}$, and $(I_n - D(S)\rho)\psi = (x_{ij})_{i,j \in \mathcal{N}}$, then the assortment optimization problem becomes:

$$\max_{S \subseteq \mathcal{N}} \lambda^T (I_n - D(S)\rho)^{-1} \Pi(S)p = \sum_{i \in S} \left( \sum_{k \in \mathcal{N}} \lambda_k x_{ki} \right) e^{-\alpha \sum_{j \in S} v_j} p_i.$$ 

Let $i \in S$ we want to compute

$$\pi(i, S) = \left( \sum_{k \in \mathcal{N}} \lambda_k x_{ki} \right) e^{-\alpha \sum_{j \in S} v_j}.$$ 

We can show that:

$$\forall k, j \in \mathcal{N}, \begin{cases} 
\frac{x_{kj}}{v_j (1 - \mu(k,S))} = \frac{1}{\sum_{s \in \mathcal{N}} v_s \mu(s,S) + v_0} & \text{if } k \neq j, \\
\frac{x_{kk} - 1}{v_k (1 - \mu(k,S))} = \frac{1}{\sum_{s \in \mathcal{N}} v_s \mu(s,S) + v_0} & \text{otherwise.}
\end{cases}$$

Let

$$\pi_S = e^{-\alpha \sum_{j \in S} v_j}$$

and

$$x = \frac{1}{\sum_{k \in \mathcal{N}} v_k \mu(k,S) + v_0} = \frac{1}{\pi_S \sum_{k \in S} v_k + v_0},$$

then we have:
\[
\sum_{k \in \mathcal{N}} \lambda_k x_{ki} = \lambda_i x_{ii} + \sum_{k \neq i} \lambda_k x_{vi}(1 - \mu(k, S)) \\
= \lambda_i x_{ii} + x_{vi} \sum_{k \notin S} \lambda_k + x_{vi}(1 - \pi_S) \sum_{k \in S \setminus \{i\}} \lambda_k \\
= \lambda_i x_{ii} + x_{vi}(1 - \lambda_i - \lambda_0) - x_{vi} \pi_S \sum_{k \in S \setminus \{i\}} \lambda_k \\
= \lambda_i(1 + (1 - \pi_S)x_{vi}) + x_{vi}(1 - \lambda_i - \lambda_0) - x_{vi} \pi_S \sum_{k \in S \setminus \{i\}} \lambda_k \\
= \lambda_i + x_{vi} \left(1 - \lambda_0 - \pi_S \sum_{k \in S} \lambda_k\right)
\]

Since we supposed that \(\lambda_j = v_j\) for all \(j \in \mathcal{N}\), then the probability of buying the product \(i\) becomes:

\[
\pi(i, S) = \left(\sum_{k \in \mathcal{N}} \lambda_k x_{ki}\right) e^{-\sigma \sum_{j \in S^c} v_j} \\
= \pi_S v_i \left(1 + \frac{1 - v_0 - \pi_S \sum_{k \in S} v_k}{\pi_S \sum_{k \in S} v_k + v_0}\right) \\
= \pi_S v_i \left(\frac{1}{\pi_S \sum_{k \in S} v_k + v_0}\right) \\
\pi(i, S) = \frac{v_i}{\sum_{k \in S} v_k + v_0 e^{\sigma \sum_{j \in S^c} v_j}}
\]

And if \(i \notin S\), we have of course \(\pi(i, S) = 0\) which finishes the proof.

**B.2 Proof of Theorem 3.4.1**

In this section, we present the details of the proof of the NP-hardness of the assortment optimization problem discussed in (3.3). We distinguish two different cases: \(\alpha \leq 1\) and \(\alpha > 1\) based on a nice structural property of an optimal solution, when \(\alpha \leq 1\).

**Lemma B.2.1** In the Generalized Multinomial Logit model with \(\alpha \leq 1\), the product with the highest price is always in the optimal set, i.e. if \(p_1 > p_2 \geq \ldots \geq p_n\), for all subset \(S \subseteq \mathcal{N}\setminus\{1\}\),
we have
\[ R(S \cup \{1\}) \geq R(S). \]

**Proof** If \( N \setminus \{1\} = \emptyset \) then the result is trivial since \( R(\emptyset) = 0 \). Now, suppose that there is at least one other product than 1 in \( N \). Let \( S \subseteq N \setminus \{1\} \). We use the following notation:

\[
V(S) = \sum_{j \in S} v_j \quad \text{and} \quad VP(S) = \sum_{j \in S} v_j p_j.
\]

Therefore we have

\[
R(S \cup \{1\}) - R(S) \geq 0 \iff \frac{VP(S) + v_1 p_1}{V(S) + v_1 + v_0 e^{a V(S)}} - \frac{VP(S)}{V(S) + v_0 e^{a V(S)}} \geq 0
\]
\[
\iff v_1(p_1 V(S) - VP(S)) + v_0 e^{a V(S)}(v_1 p_1 - (e^{a V_1} - 1) VP(S)) \geq 0
\]

Since 1 has the highest price, \( p_1 V(S) \geq VP(S) \) and

\[
(e^{a v_1} - 1) VP(S) \leq (e^{a v_1} - 1) V(S) p_1 = (e^{a v_1} - 1)(1 - v_1 - \beta(S)) p_1,
\]

where \( \beta(S) = \sum_{k \in N \setminus \{1\} \cup S} v_k \). Moreover, since

\[
(e^{a v_1} - 1)(1 - v_1 - \beta(S)) - v_1 = e^{a v_1}(1 - v_1 - \beta(S)) - 1 + \beta(S),
\]

we want to prove that \( g : x \mapsto e^{ax}(1 - x - \beta(S)) - 1 \) is a negative function on \((0, 1)\). Indeed, it is a strictly decreasing function on \((0, 1)\):

\[
g'(x) = e^{ax}(\alpha - ax - \alpha \beta(S) - 1) < 0 \iff 1 - \beta(S) - x < \frac{1}{\alpha}.
\]

And \( 1 - \beta(S) - x = V(S) \leq 1 - v_0 < \frac{1}{\alpha} \) since we assumed \( \alpha \leq 1 \). Moreover, \( g(0) = 0 \), therefore \( g \) is negative on \((0, 1)\) and we have \( R(S \cup \{1\}) - R(S) \geq 0 \).

Once we have this result, we now make a reduction from the partition problem. Consider
the following instance of the partition problem: we are given \( n \) integers \( c_1, \ldots, c_n \) and the goal is to decide whether there is a subset \( S \subseteq \{1, \ldots, n\} \) such that \( \sum_{i \in S} c_i = \sum_{i \in (1, \ldots, n) \setminus S} c_i \).

Let \( T = \frac{1}{2} \sum_{i=1}^n c_i \), then \( \sum_{i \in S} c_i = \sum_{i \in (1, \ldots, n) \setminus S} c_i \) if and only if \( \sum_{i \in S} c_i = T \). We can suppose without loss of generality that \( c_i > 0 \) for all \( i \in [n] \). We construct an instance of our problem as follows:

\[
v_i = \begin{cases} 
\frac{c_i}{2T+1} & \text{if } i \geq 1, \\
1 - \sum_{i=1}^n v_i = \frac{1}{2T+1} & \text{if } i = 0.
\end{cases}
\]

and let \( c_0 := v_0 e^{\alpha v_0} > 0 \), we define the prices as follows

\[
p_i = \begin{cases} 
\frac{1}{(2T+1)c_0} + \frac{e^{\alpha T} - 1}{T} + \frac{1}{c_i} & \text{if } i = 1, \\
\frac{1}{(2T+1)c_0} + \frac{e^{\alpha T} - 1}{T} & \text{otherwise}.
\end{cases}
\]

Finally we set the target revenue as \( K = \frac{T + (2T+1)c_0 e^{\alpha T}}{T + (2T+1)c_0 e^{\alpha T}} \).

First, we can note that 1 is necessarily in the optimal set. Indeed 1 has the highest price in \( \mathcal{N} \), so the previous lemma implies that 1 is necessarily in the optimal set (we can note that the choice of 1 is random and we could have chosen any \( i \) in \( \mathcal{N} \)).

In this case, our problem becomes

\[
\max_{S \subseteq \{1, \ldots, n\}} R(S) := \max_{S \subseteq \{2, \ldots, n\}} R(S \cup \{1\}) \\
= \max_{S \subseteq \{2, \ldots, n\}} \left( \sum_{i \in S} v_i p_i + v_1 p_1 \right) \\
= \max_{S \subseteq \{2, \ldots, n\}} \left( \sum_{i \in S \cup \{1\}} v_i + c_0 e^{\alpha T} \sum_{i \in S \cup \{1\}} v_i \right) \\
= \max_{S \subseteq \{1, \ldots, n\}} \left( \sum_{i \in S} c_i + (2T + 1)c_0 e^{\alpha T} \sum_{i \in S} c_i \right) \\
= \max_{S \subseteq \{1, \ldots, n\}} \left( \sum_{i \in S} c_i \right)
\]

where

\[
F : [0, 2T] \rightarrow \mathbb{R}_+, \\
x \mapsto \frac{h(x)}{x + (2T+1)c_0 e^{\alpha T}}
\]
and
\[ h(x) = \left( \frac{1}{(2T + 1)c_0} + \frac{e^{\frac{aT}{2T + 1}} - 1}{T} \right) x + 1. \]

\( F \) is increasing at \( x \) if and only if
\[
F'(x) \geq 0 \iff \frac{h'(x)(x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}) - h(x)(1 + c_0ae^{\frac{aT}{2T + 1}})}{(x + (2T + 1)c_0e^{\frac{aT}{2T + 1}})^2} \geq 0
\]
\[
\iff h'(x) - \frac{1 + c_0ae^{\frac{aT}{2T + 1}}}{x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}} h(x) \geq 0
\]
\[
\iff \frac{h'(x)}{h(x)} \geq \frac{1 + c_0ae^{\frac{aT}{2T + 1}}}{x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}} > 0 \quad \text{since} \quad h > 0 \quad \text{on} \quad [0, 1]
\]
\[
\iff \ln(h(x)) - \ln(h(0)) \geq \ln(x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}) - \ln((2T + 1)c_0)
\]
\[
\iff h(x) \geq \frac{1}{(2T + 1)c_0} (x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}) \quad \text{since} \quad h(0) = 1
\]
\[
\iff h(x) - g(x) \geq 0
\]

where \( g : x \mapsto \frac{1}{(2T + 1)c_0} (x + (2T + 1)c_0e^{\frac{aT}{2T + 1}}) \). We note that \( g \) is a strictly increasing function on \([0, 2T]\) such that \( h(0) = g(0) = 1 \) and \( h(2T) < g(2T) \). Indeed,
\[
h(2T) - g(2T) = 2(e^{\frac{aT}{2T + 1}} - 1) + 1 - e^{\frac{aT}{2T + 1}} = -(e^{\frac{aT}{2T + 1}} - 1)^2 < 0.
\]

Therefore, since \( h \) is a line with a positive slope, there exists a unique \( x^* \in (0, 2T) \) such that for all \( 0 < x < x^* \), \( h(x) - g(x) > 0 \), \( h(x^*) - g(x^*) = 0 \) and for all \( 2T \geq x > x^* \), \( h(x) - g(x) < 0 \). But \( h(T) = g(T) \). So \( x^* = T \). So this proves that \( F \) is strictly increasing on \([0, T)\) then strictly decreasing on \((T, 2T]\). So \( F \) has a unique maximum at \( T \) on \((0, 2T)\). Hence,

\[
\max_{S \subseteq \{1, \ldots, n\}} R(S) = \max_{S \subseteq \{1, \ldots, n\}} \frac{\sum_{i \in S} c_i}{T} \leq F(T) = \frac{\frac{T}{(2T + 1)c_0} + e^{\frac{aT}{2T + 1}}}{T + (2T + 1)c_0e^{\frac{aT}{2T + 1}}} = K.
\]

So there exists an assortment \( S \subseteq \{1, \ldots, n\} \) whose expected revenue is at least \( K \) if and only if the chain of inequalities hold as equalities. For this to happen we need to have \( \sum_{i \in S'} c_i = T \) for some assortment \( S' \subseteq \{1, \ldots, n\} \). Therefore there exists an assortment \( S \subseteq \{1, \ldots, n\} \) whose
expected revenue is at least \( K \) if and only if there exists an assortment \( S' \subseteq \{1,\ldots,n\} \) that satisfies \( \sum_{i \in S'} c_i = T \).

Although when \( \alpha > 1 \), we no longer have any nice structure in the optimal assortment and the product with the highest price may not necessarily be in an optimal assortment set, we can still prove that the assortment optimization problem is NP-hard as long as \( \alpha > 2 \) (recall that we are interested in the high \( \alpha \) case).

To prove the NP-hardness in this case, we once again make a reduction from the partition problem. Consider the following instance of the partition problem: we are given \( n \) integers \( c_1,\ldots,c_n \) and the goal is to decide whether there is a subset \( S \subseteq \{1,\ldots,n\} \) such that \( \sum_{i \in S} c_i = \sum_{i \in \{1,\ldots,n\}\setminus S} c_i \).

Let \( T = \frac{1}{2} \sum_{i=1}^{n} c_i \), then \( \sum_{i \in S} c_i = \sum_{i \in \{1,\ldots,n\}\setminus S} c_i \) if and only if \( \sum_{i \in S} c_i = T \). We can suppose without loss of generality that \( c_i > 0 \) for all \( i \in [n] \). We construct an instance of our problem as follows:

\[
v_i = \begin{cases} \frac{c_i}{T \alpha^a} & \text{if } i \geq 1, \\ 1 - \sum_{i=1}^{n} v_i = 1 - \frac{2}{\alpha} & \text{if } i = 0, \end{cases}
\]

and let \( c_0 := v_0 e^{a v_0} > 0 \). We note that \( v_0 > 0 \) because we have supposed \( \alpha > 2 \). We define the prices as follows

\[
\forall i \in [n] \quad p_i = 1.
\]

Finally we set the target revenue as \( K = \frac{1}{1 + \alpha c_0} \). In this case, our problem becomes

\[
\max_{S \subseteq \{1,\ldots,n\}} R(S) := \max_{S \subseteq \{1,\ldots,n\}} \frac{\sum_{i \in S} v_i p_i}{\sum_{i \in S} v_i + c_0 e^a \sum_{i \in S} v_i} = \max_{S \subseteq \{1,\ldots,n\}} \frac{1}{T \alpha} \sum_{i \in S} c_i + c_0 e^a \sum_{i \in S} c_i \\
= \max_{S \subseteq \{1,\ldots,n\}} F \left( \sum_{i \in S} c_i \right) ,
\]
where
\[
F : [0, 2T] \rightarrow \mathbb{R}_+, \quad x \mapsto \frac{x}{x + T \alpha c_0 e^{\frac{x}{T}}}.
\]

$F$ is increasing at $x$ if and only if
\[
F'(x) \geq 0 \iff \frac{x + T \alpha c_0 e^{\frac{x}{T}} - x(1 + \alpha c_0 e^{\frac{x}{T}})}{(x + T \alpha c_0 e^{\frac{x}{T}})^2} \geq 0
\implies x \leq T.
\]

Therefore $F$ is strictly increasing on $[0, T)$ then strictly decreasing on $(T, 2T]$. So $F$ has a unique maximum at $T$ on $(0, 2T)$. Hence,

\[
\max_{S \subseteq \{1, \ldots, n\}} R(S) = \max_{S \subseteq \{1, \ldots, n\}} F \left( \sum_{i \in S} c_i \right) \leq F(T) = \frac{1}{1 + \alpha c_0 e} = K.
\]

So there exists an assortment $S \subseteq \{1, \ldots, n\}$ whose expected revenue is at least $K$ if and only if the chain of inequalities hold as equalities. For this to happen we need to have $\sum_{i \in S'} c_i = T$ for some assortment $S' \subseteq \{1, \ldots, n\}$. Therefore there exists an assortment $S \subseteq \{1, \ldots, n\}$ whose expected revenue is at least $K$ if and only if there exists an assortment $S' \subseteq \{1, \ldots, n\}$ that satisfies $\sum_{i \in S'} c_i = T$.

We make a special note here about $1 < \alpha \leq 2$. We point out that we are more interested in the “picky customer” case, i.e., when $\alpha$ is large enough, because this is when we actually characterize choice overload (see the homogeneous graph example in Section 3.3.2, or the numerical results in Section 3.6). Hence this is an uninteresting case, although we do believe that assortment optimization problem is still NP-hard for this particular setting as well (and we have examples to verify this claim).
B.3 Proof of Theorem 3.4.2

Let $S^*$ be the optimal solution to the assortment optimization problem. There exists $l$ such that

$$v(1 + \epsilon)^{l-1} \leq \sum_{j \in S^*} v_j \leq v(1 + \epsilon)^l.$$ 

Let $h = v(1 + \epsilon)^l$. Then

$$\sum_{j \in S} \frac{v_j}{\epsilon h/n} \leq \frac{n}{\epsilon h} = \frac{n}{\epsilon},$$

and rounding up gives us

$$\sum_{j \in S^*} \bar{v}_j \leq \left\lceil \frac{n}{\epsilon} \right\rceil + n = I.$$

Thus $S^*$ belongs to the set of assortments such that inequality (1) is verified for $I$. Let $S_h$ be the assortment corresponding to $R(I, n)$ for the guess $h$, that is the one that maximizes $\sum_{j \in S} v_j p_j$ subject to (1). Then since $S^*$ satisfies (1)

$$\sum_{j \in S_h} v_j p_j \geq \sum_{j \in S^*} v_j p_j.$$

Moreover,

$$\sum_{j \in S_h} v_j \leq \epsilon h/n \sum_{j \in S_h} \bar{v}_j \leq h(1 + \epsilon + \epsilon/n) \leq h(1 + 2\epsilon).$$

Since

$$x \mapsto \frac{1}{x + v_0 e^{\alpha(v_0 + x)}}$$

is a decreasing function and

$$R(S_h) = \sum_{j \in S_h} \frac{v_j p_j}{\sum_{k \in S_h} v_k + v_0 e^{\alpha(v_0 + \sum_{j \in S_h} v_j)}} \geq \frac{\sum_{j \in S_h} v_j p_j}{v(1 + \epsilon)^l(1 + 2\epsilon) + v_0 e^{\alpha(v_0 + v(1 + \epsilon)^l(1 + 2\epsilon))}}.$$
Let us first show that there exists $\beta > 0$ such that

\[ v(1 + \epsilon)^{l-1} + v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)} \geq (v(1 + \epsilon)^l(1 + 2\epsilon) + v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)}) \times (1 - 2\epsilon). \]  

(B.1)

Indeed,

\[
 v(1 + \epsilon)^{l-1} \geq v(1 + \epsilon)^l(1 + 2\epsilon)(1 - \beta\epsilon)
\]

\[ \iff 1 \geq (1 + \epsilon)(1 + 2\epsilon)(1 - \beta\epsilon) = 1 + (3 - \beta)\epsilon + (2 - 3\beta)\epsilon^2 - \beta\epsilon^3, \]

which is clearly true at least for $\beta \geq 3$. Moreover,

\[
 v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)} \geq v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)} \times (1 - 2\epsilon)
\]

\[ \iff e^{\sigma(1 + \epsilon)^{l-1}(1 - (1 + \epsilon)(1 + 2\epsilon))} \geq 1 - 2\epsilon. \]

Note that $e^{\sigma(1 + \epsilon)^{l-1}(1 - (1 + \epsilon)(1 + 2\epsilon))} = 1 - 3\alpha \epsilon + o(\epsilon)$. Therefore if we take $\beta \geq \max(3, 3\alpha \nu)$, then the inequality (2) is verified. Consequently

\[
 \frac{1}{v(1 + \epsilon)^l(1 + 2\epsilon) + v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)}} \geq \frac{1 - \beta\epsilon}{v(1 + \epsilon)^l + v_0e^{\sigma(v_0 + \epsilon)(1 + 2\epsilon)}} \times \frac{1 - \beta\epsilon}{1 - \beta\epsilon}
\]

\[ \geq \sum_{k \in S'} v_k + v_0e^{\sigma(v_0 + \sum_{k \in S'} v_k)}. \]

by definition of $l$. Therefore,

\[
 R(S_h) \geq (1 - \beta\epsilon) \frac{\sum_{j \in S_h} v_j p_j}{\sum_{k \in S'} v_k + v_0e^{\sigma(v_0 + \sum_{k \in S'} v_k)}},
\]

\[
 R(S_h) \geq (1 - \beta\epsilon)R(S^*), \]

where in the last inequality we used that $\sum_{j \in S_h} v_j p_j \geq \sum_{j \in S'} v_j p_j$. This proves that our algorithm returns a $(1 - O(\epsilon))$-optimal solution to the assortment problem.

**Running time** We try a total of $L = O(\log(nV/v)/\epsilon)$ guesses $h$, and for each guess we
formulate a dynamic programming with $O(n^2/\varepsilon)$ steps. Consequently the running time of
the algorithm is $O\left(\frac{n^2}{\varepsilon^2} \log(nV/v)\right)$ which is polynomial in the input size $n$ and $\frac{1}{\varepsilon}$.

B.4 Proof of Lemma 3.5.1

Let $\|\cdot\|$ be the usual Euclidean norm, and $\langle \cdot, \cdot \rangle$ its associated scalar product. We have that

$$\rho(UV(S)) = \max_{x, \|x\|=1} \langle UV(S)x, x \rangle.$$ 

Let $k \in [K]$,

$$\langle UV(S)e_k, e_k \rangle = \sum_{l=1}^{n} (1 - \mu^L(l, S))u_{lk}v_{lk} \leq \frac{1}{n} \sum_{l=1}^{n} (1 - \mu^L(l, S)),$$

$$\leq \frac{1}{n} \left( n - |S| + \sum_{l \in S} (1 - e^{-\alpha \sum_{k \in [K]} u_{lk}v_k(S)}) \right),$$

where the first inequality follows from the assumption made above on the coefficients $u_{lk}v_{lk}$.

Furthermore, since $\alpha \leq \log n$,

$$e^{-\alpha \sum_{k \in [K]} u_{lk}v_k(S)} \geq e^{-\alpha \sum_{k \in [K]} u_{lk} \sum_{j \in N^e} v_{jk}} = e^{-\alpha} \geq \frac{1}{n}.$$ 

Hence

$$\langle UV(S)e_k, e_k \rangle \leq \frac{1}{n} \left( n - |S| + |S| \left( 1 - \frac{1}{n} \right) \right) = 1 - \frac{|S|}{n^2} \leq 1 - \frac{1}{n^2}.$$ 

The inequality holds for all $e_i, i \in [n]$, and therefore $\forall x \in \mathbb{R}^K$ such that $\|x\| = 1$. Hence,

$$\rho(UV(S)) \leq 1 - \frac{1}{n^2}.$$
B.5 Proof of Lemma 3.5.2

Let \( S \subseteq N \) be the chosen subset of products, and let \( i \in S \). We recall from Section 3.2.1, that

\[
\pi(i, S) = \lambda^T (I_n - \text{Diag}((1 - \mu(i, S))) \rho(N, N))^{-1} \Pi(S) e_i. \tag{B.2}
\]

This result still holds as the Generalized Multinomial Logit model with Low rank matrix has the same assumptions as the one needed to get this result. However, there is some difference with the assumptions made in Section 3.3. On the contrary of the Generalized Markov chain model presented in Section 3.3, we now have that

\[
\mu^{LR}(i, S) = e^{-\alpha x(\sum_{k \in [K]} u_k v_k(S))}
\]

and

\[
\rho(N, N) = \sum_{k \in [K]} u_k v_k^T,
\]

where \( v_k = (v_{1k}, ..., v_{nk}) \). Therefore we have to compute the coefficients of the matrix

\[
\left( I_n - \text{Diag}((1 - \mu^{LR}(i, S))) \rho(N, N) \right)^{-1}
\]

under this new model. We know that

\[
\left( I_n - \text{Diag}((1 - \mu^{LR}(i, S))) \rho(N, N) \right)^{-1} = \sum_{l=0}^{\infty} \left( \text{Diag}((1 - \mu^{LR}(i, S))) \rho(N, N) \right)^l. \tag{B.3}
\]

We will use the following notation:

- let \( M := \text{Diag}((1 - \mu^{LR}(i, S))) \rho(N, N) = ((1 - \mu^{LR}(i, S)) \rho_{ij})_{i,j \in [n]} \),
- let \( U(V(S) = (\sum_{l=1}^{n} (1 - \mu^{LR}(l, S)) u_{lk} v_{lm})_{k,m \in [K]} \),
• for \( l \in \mathbb{N} \), we note \((UV(S)^l)_{k'k}\) the coefficient of indices \( k', k \) of the matrix \( UV(S)^l \), ie of the matrix \( UV(S) \) elevated to the power of \( l \).

First, we show by induction that

\[
\forall l \geq 1 \quad M^l = \left( (1 - \mu^{LR}(i, S)) \sum_{k, k' \in [K]} u_{ik} v_{jk'} \left( (UV(S)^{l-1})_{k'k} \right) \right)_{i, j \in [n], k' \in \mathbb{N}_*}
\]

**Initiation** For \( l = 1 \), we use the definition of \( M \) and \( \rho(N, N) \)

\[
M = \left( (1 - \mu^{LR}(i, S)) \rho_{ij} \right)_{i, j \in [n]} = \left( (1 - \mu^{LR}(i, S)) \sum_{k \in [K]} u_{ik} v_{jk} \right)_{i, j \in [n]}
\]

Since \( UV(S)^0 = I_K \), we have that \((UV(S)^0)_{k'k} = 1_{k' = k}\). Therefore

\[
M = \left( (1 - \mu^{LR}(i, S)) \sum_{k, k' \in [K]} u_{ik} v_{jk'} \left( (UV(S)^0)_{k'k} \right) \right)_{i, j \in [n]}
\]

The result holds for \( l = 1 \).

**Inductive step** Suppose that the result holds for a certain \( l \in \mathbb{N}^* \). We compute the coefficient of indices \( i, j \) of \( M^{l+1} \):

\[
\left( M^{l+1} \right)_{ij} = \sum_{q=1}^{n} \left( M^l \right)_{iq} (1 - \mu^{LR}(q, S)) \rho_{qj}
\]
Using the induction hypothesis,

\[
(M^{l+1})_{ij} = \sum_{q=1}^{n} \left( (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{qk'} (UV(S)^{l-1})_{k'k} \right) (1 - \mu^{LR}(q, S)) \sum_{k'' \in [K]} u_{qk''} v_{jk''} \\
= (1 - \mu^{LR}(i, S)) \sum_{k,k'' \in [K]} u_{ik} v_{jk''} \left( \sum_{q \in [n]} (1 - \mu^{LR}(q, S)) u_{qk''} v_{qk'} \right) (UV(S)^{l-1})_{k'k} \\
= (1 - \mu^{LR}(i, S)) \sum_{k,k'' \in [K]} u_{ik} v_{jk''} \sum_{k' \in [K]} (UV(S))_{k''k'} (UV(S)^{l-1})_{k'k} \\
= (1 - \mu^{LR}(i, S)) \sum_{k,k'' \in [K]} u_{ik} v_{jk''} (UV(S)^{l})_{k''k}. 
\]

Therefore the result holds for \( l + 1 \).

**Conclusion** For all \( l \geq 1 \), we have that

\[
M^l = \left( (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{jk'} (UV(S)^{l-1})_{k'k} \right)_{i \in [n], j \in \mathbb{N}^+} 
\]

Now that once we have this result, we can compute the coefficient of individual indices \( i, j \) of the matrix \((I_n - \text{Diag}((1 - \mu^{LR}(i, S))\rho(N, N))^{-1} = (I_n - M)^{-1}\) using (2):

\[
\left( (I_n - \text{Diag}((1 - \mu^{LR}(i, S))\rho(N, N))^{-1} \right)_{ij} = \sum_{l=0}^{\infty} (M^l)_{ij} \\
= \mathbb{1}_{i=j} + \sum_{l=1}^{\infty} (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{jk'} (UV(S)^{l-1})_{k'k} \\
= \mathbb{1}_{i=j} + (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{jk'} \sum_{l=1}^{\infty} (UV(S)^{l-1})_{k'k} \\
= \mathbb{1}_{i=j} + (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{jk'} (I_K - UV(S))^{-1}_{k'k}. 
\]

The last inequality holds since \( \rho(UV(S)) < 1 \), as shown in Lemma 3.5.1. Injecting it in
(1), and computing $\sum_{i \in S} \pi^{LR}(i, S)p_i$, we have

$$R^{LR}(S) = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \left( 1_{i=j} + (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} v_{jk'} \left((I_K - UV(S))^{-1}\right)_{k'k} \right) \mu^{LR}(j, S)p_j$$

$$= \sum_{i=1}^{n} \lambda_i \mu^{LR}(i, S)p_i + \sum_{i=1}^{n} \lambda_i (1 - \mu^{LR}(i, S)) \sum_{j=1}^{n} \left( \sum_{k,k' \in [K]} u_{ik} v_{jk'} \left((I_K - UV(S))^{-1}\right)_{k'k} \right) \mu^{LR}(j, S)p_j$$

$$= \sum_{i=1}^{n} \lambda_i \mu^{LR}(i, S)p_i + \sum_{i=1}^{n} \lambda_i (1 - \mu^{LR}(i, S)) \sum_{k,k' \in [K]} u_{ik} \left((I_K - UV(S))^{-1}\right)_{k'k} \sum_{j=1}^{n} v_{jk'} \mu^{LR}(j, S)p_j$$

$$= \sum_{i=1}^{n} \lambda_i \sum_{k,k' \in [K]} u_{ik} \left((I_K - UV(S))^{-1}\right)_{k'k} \sum_{j=1}^{n} v_{jk'} \mu^{LR}(j, S)p_j$$

$$+ \sum_{i \in S} \lambda_i \mu^{LR}(i, S)p_i - \sum_{k,k' \in [K]} u_{ik} \left((I_K - UV(S))^{-1}\right)_{k'k} \sum_{j=1}^{n} v_{jk'} \mu^{LR}(j, S)p_j \right).$$

Since $\mu^{LR}(j, S) = 0, \forall j \notin S$, after reorganizing the terms, we can rewrite this as

$$R^{LR}(S) = \sum_{i \in [n]} \lambda_i (1 - \mu^{LR}(i, S)) \sum_{k,k' = 1}^{K} u_{ik} (I - UV(S))^{-1}_{k'k} \left( \sum_{l \in S} v_{lk'} \mu^{LR}(l, S)p_l \right)$$

$$+ \sum_{i \in S} \lambda_i \mu^{LR}(i, S)p_i.$$

Finally, writing the sum over $k, k'$ in matrix form and using the notation $u_i = \{u_{ik}\}_{k=1}^{K}$ and $v_j = \{v_{jk}\}_{k=1}^{K}$, we get

$$R^{LR}(S) = \sum_{i \in [n]} \lambda_i (1 - \mu^{LR}(i, S)) \left( \sum_{j \in S} p_j \mu^{LR}(j, S) u_i^T (I - UV(S))^{-1} v_j \right) + \sum_{i \in S} \lambda_i \mu^{LR}(i, S)p_i,$$

which concludes the proof.
B.6 Proof of Lemma 3.5.3

From Lemma 3.5.1, we know that \( \rho(US) \leq 1 - \frac{1}{n^2} < 1 \). Therefore we can write

\[
[I - UV(S)]^{-1} = \sum_{l=0}^{\infty} (UV(S)^l) .
\]

We first show one side of the inequality. Let \( \alpha_1 = (1 - O(\epsilon)) \). Furthermore, let \( L = [I - UV(S)]^{-1}v_j \) and \( L_\epsilon = [I - \alpha_1 UV(S)]^{-1}\alpha_1 v_j \). Hence, we have

\[
L - L_\epsilon = \left( [I - UV(S)]^{-1} - \alpha_1 [I - \alpha_1 UV(S)]^{-1} \right) v_j
= \left( \sum_{l=0}^{\infty} (UV(S)^l) - \alpha_1 \sum_{l=0}^{\infty} (\alpha_1 UV(S)^l) \right) v_j
= \left( \sum_{l=0}^{\infty} (1 - \alpha_1^{l+1}) (UV(S)^l) \right) v_j
\geq \left( \sum_{l=0}^{\infty} (1 - \alpha_1)^{l+1} (UV(S)^l) \right) v_j \quad \text{(since } 0 < \alpha_1 < 1) \\
= (1 - \alpha_1) \left( \sum_{l=0}^{\infty} (1 - \alpha_1)^l (UV(S)^l) \right) v_j
= (1 - \alpha_1)[I - (1 - \alpha_1)UV(S)]^{-1}v_j
\geq O(\epsilon)L
\]

which completes the proof. The other side of the inequality is similarly proved using the fact that for \( \alpha_1 > 1 \) (when we set \( \alpha_1 = (1 + O(\epsilon)) \), we have \( \alpha_1^l - 1 \geq (\alpha_1 - 1)^l \) for any \( l \geq 0 \).
B.7 Proof of Theorem 3.5.1

Let $S^*$ be the optimal solution to the assortment optimization problem. There exist $t_1, \ldots, t_K$ such that for all $k \in [K]$

$$v^k(1 + \epsilon)^{t_k} \leq \sum_{j \in S^*} v_{jk} =: V_k(S^*) \leq v^k(1 + \epsilon)^{t_k + 1}.$$ 

Let $h = (v^1(1 + \epsilon)^{t_1}, \ldots, v^K(1 + \epsilon)^{t_K})$. Choose the set $S_h$ that maximizes the dynamic program defined above. Then by definition of $\bar{v}_{jk}$, for each $k \in [K]$

$$V_k(S_h) := \sum_{j \in S_h} v_{jk} \leq \frac{\epsilon h_k}{n} \sum_{j \in S_h} \bar{v}_{jk} \leq \frac{\epsilon h_k}{n} U = \frac{\epsilon h_k}{n} ([n/\epsilon] + n) \leq h_k(1 + 2\epsilon),$$

and

$$V_k(S_h) := \sum_{j \in S_h} v_{jk} \geq \frac{\epsilon h_k}{n} \sum_{j \in S_h} (\bar{v}_{jk} - 1) \geq \frac{\epsilon h_k}{n} (L - |S_h|) \geq \frac{\epsilon h_k}{n} ([n/\epsilon] - n) \geq h_k(1 - \epsilon).$$

First of all, since for all $k \in [K]$, $h_k \leq V_k(S^*) \leq h_k(1 + \epsilon)$,

$$e^{-\alpha \sum_{k=1}^{K} h_k u_{jk}(1+\epsilon)} \leq \mu^{LR}(j, S^*) = e^{-\alpha \sum_{k=1}^{K} V_k(S^*) u_{jk}} \leq e^{-\alpha \sum_{k=1}^{K} h_k u_{jk}} = \mu_j(h).$$

Thus, if we note $\sum_{k=1}^{K} h_k u_{jk} =: \langle h, u_j \rangle$,

$$\mu_j(h)(1 - \epsilon \alpha \langle h, u_j \rangle) \leq \mu^{LR}(j, S^*) \leq \mu_j(h), \quad (B.4)$$

and using the same arguments for $h_k(1 - \epsilon) \leq V_k(S_h) \leq h_k(1 + 2\epsilon)$, we can show that

$$\mu_j(h)(1 - 2\epsilon \alpha \langle h, u_j \rangle) \leq \mu^{LR}(j, S_h) \leq \mu_j(h)(1 + \epsilon \alpha \langle h, u_j \rangle).$$
Therefore

\[ UV(S^*)_{kk'} = \sum_{i \in [n]} (1 - \mu^{LR}(i, S^*))u_{ik}v_{ik'} \leq \sum_{i \in [n]} (1 - \mu_i(h))u_{ik}v_{ik'} \left( 1 + \frac{\mu_i(h)}{1 - \mu_i(h)} \epsilon \alpha \langle h, u_i \rangle \right) \]
\[ \leq \sum_{i \in [n]} (1 - \mu_i(h))u_{ik}v_{ik'} \left( 1 + \epsilon \alpha \max_{i \in [n]} \left\{ \frac{\mu_i(h)}{1 - \mu_i(h)} \langle h, u_i \rangle \right\} \right) \]

Hence we get for all \( k, k' \in [K] \)

\[ H(h)_{kk'} \leq UV(S^*)_{kk'} \leq H(h)_{kk'}(1 + \delta), \tag{B.5} \]

where

\[ \delta := \epsilon \alpha \max_{i \in [n]} \left\{ \frac{\mu_i(h)}{1 - \mu_i(h)} \langle h, u_i \rangle \right\}, \]

and likewise

\[ H(h)_{kk'}(1 - \delta) \leq UV(S_h)_{kk'} \leq H(h)_{kk'}(1 + 2\delta). \]

Since \( \delta = O(\epsilon) \), we can use the result from Lemma 3.5.3. We want to prove the following

\[ R^{LR}(S_h) \geq (1 - O(\epsilon))R^{LR}(S^*). \]

Recall the expression of optimal revenue

\[ R^{LR}(S^*) = \sum_{i \in S^*} \lambda_i \mu^{LR}(i, S^*)p_i + \sum_{i=1}^{n} \lambda_i (1 - \mu^{LR}(i, S^*))f(i, S^*). \]

Let us now compare \( f_i(h, S^*) \) and \( f(i, S^*) := \sum_{j=1}^{n} p_j \mu^{LR}(j, S^*)u_j^T [I - UV(S^*)]^{-1} v_j \) using bounds from (B.4) and the result from Lemma (3.5.3). We have

\[ f_i(h, S^*)(1 - \delta_2) \leq f(i, S^*) := \sum_{j=1}^{n} p_j \mu^{LR}(j, S^*)u_j^T [I - UV(S^*)]^{-1} v_j \leq f_i(h, S^*)(1 + O(\epsilon)), \]

127
where

\[ \delta_2 := \epsilon \alpha \max_{i \in [n]} \{ h, u_i \} = O(\epsilon). \]

Using the bounds for \( \mu_{LR}(j, S_h) \) and \( f(i, S_h) \), we also have

\[ f_i(h, S_h) (1 - O(\epsilon)) (1 - 2\delta_2) \leq f(i, S_h) \leq f_i(h, S_h) (1 + O(\epsilon)) (1 + \delta_2). \]

Using these upper and lower bounds for \( f(i, S^*) \) (resp. \( f(i, S_h) \)) and the corresponding lower and upper bounds for \( \mu_{LR}(j, S^*) \) (resp. \( \mu_{LR}(j, S_h) \)) from equation (B.4), we have

\[
R_{LR}^{LR}(S^*) \leq \sum_{i \in S^*} \lambda_i \mu_i(h) p_i + \sum_{i=1}^{n} \lambda_i (1 - \mu_i(h)) (1 + \delta) f_i(h, S^*) (1 + O(\epsilon))
\]

\[ \leq (1 + \delta) (1 + O(\epsilon)) R_{DP}^{DP}(h, S^*), \]

and

\[
R_{LR}^{LR}(S_h) \geq \sum_{i \in S^*} \lambda_i (1 - 2\delta_2) \mu_i(h) p_i + \sum_{i=1}^{n} \lambda_i (1 - \mu_i(h)) (1 - \delta) (1 - O(\epsilon)) (1 - 2\delta_2) f_i(h, S_h)
\]

\[ \geq (1 - \delta) (1 - O(\epsilon)) (1 - 2\delta_2) R_{DP}^{DP}(h, S_h). \]

Next, by the definition of \( S_h \) being the assortment that maximizes the dynamic program for the given guess \( h \), we also have \( R_{DP}^{DP}(h, S^*) \leq R_{DP}^{DP}(h, S_h) \). Combining this with the bounds derived above, we get

\[
R_{LR}^{LR}(S^*) \leq \frac{(1 + \delta)(1 + O(\epsilon))}{(1 - \delta)(1 - O(\epsilon))(1 - 2\delta_2)} R_{LR}^{LR}(S_h). \tag{B.6}
\]

Hence we get the following lower bound for the expected revenue obtained by the assortment \( S_h \):

\[
\frac{(1 - \delta)(1 - O(\epsilon))(1 - 2\delta_2)}{(1 + \delta)(1 + O(\epsilon))} R_{LR}^{LR}(S^*) \leq R_{LR}^{LR}(S_h).
\]
Finally, to argue that this gives us \((1 - O(\epsilon))\) solution, to conclude the proof, we need to show that the quantity
\[
\frac{(1 - \delta)(1 - O(\epsilon))(1 - 2\delta_2)}{(1 + \delta)(1 + O(\epsilon))}
\]
is not too far from 1. This holds since,
\[
\delta := O(\epsilon) \quad \text{and} \quad \delta_2 := O(\epsilon).
\]
Thus
\[
\frac{(1 - \delta)(1 - O(\epsilon))(1 - 2\delta_2)}{(1 + \delta)(1 + O(\epsilon))} \geq 1 - O(\epsilon),
\]
We have finally proven that our algorithm returns an assortment \(S_h\) that is a \((1 - O(\epsilon))\) optimal solution to our assortment problem.

\[
R^{LR}(S^*)(1 - O(\epsilon)) \leq R^{LR}(S_h) \leq R^{LR}(S^*) \quad \text{(B.7)}
\]

**Running time** We try a total of \(\prod_{k \in [K]} O(\log(nV^k/v^k)/\epsilon) = O((\log(nV/v)/\epsilon)^K)\) guesses for \(h\) (where \(V := \max_{k \in [K]} V^k\) and \(v := \min_{k \in [K]} v^k\)). For each guess we formulate a dynamic programming with \(O\left(\left(\frac{n}{\epsilon}\right)^2 n^2 K^2\right)\) run-time. Consequently the running time of the algorithm is \(O\left(\frac{n^{2(K+1)}K^2}{\epsilon v^k} \log^K (nV/v)\right)\) which is polynomial in the input size \(n\) and \(\frac{1}{\epsilon}\).