Cellular **dg**-categories and their applications to homotopy theory of A_∞-categories

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2020

Abstract

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We introduce the notion of cellular **dg**-categories mimicking the properties of topological CW-complexes. We study the properties of such categories and provide various examples corresponding to the well-known geometrical objects. We also show that these categories are suitable for encoding coherence conditions in homotopy theoretical constructs involving A_∞-categories. In particular, we formulate the notion of a homotopy coherent monoid action on an A_∞-category which can be used in constructions involved in Homological Mirror Symmetry.

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Acknowledgements

I am very grateful to my advisor Mohammed Abouzaid for suggesting the problem for this thesis and for his guidance and support during all stages of preparation of this work. I am also grateful to Andrew Blumberg, Mikhail Khovanov, Francesco Lin and Zachary Sylvan for helpful discussions.

I also thank my advisor Dmitri Orlov for helping me in transitioning to Columbia University, my math high school teacher Mikhail Yakir for encouraging me to continue education in the world's best schools, my mentor Timothy Brown for helping me to stay focused while completing a Ph.D., and my extended family, friends, colleagues and well-wishers for encouraging and supporting me on this journey.

Especially, I want to thank my mother Galyna Petrova for her love and caring throughout my life. Without her sacrifices in raising and educating me, I would not be able to make it so far.

Dedication

Dedicated to Sergiy Ovsienko (1953–2016), my first academic advisor who introduced me to the world of A_∞-categories. (See also [Bav+17].)

Chapter 0: Introduction

0.1 Motivation

The Homological Mirror Symmetry conjectures in general try to establish equivalences between algebraic data given by the derived categories of coherent sheaves on algebraic varieties on one side, and symplectic data given by Fukaya categories on the corresponding symplectic manifolds on the other side. In particular cases, we also may experience different variations of these notions. For example, Fukaya categories might be replaced by partially wrapped Fukaya categories, symplectic manifolds and algebraic varieties might be additionally equipped with a Landau-Ginzbur potential and/or a group action, and derived categories of coherent sheaves might be replaced by categories of singularities (categories of matrix factorizations). While the data on the algebraic side can be nicely described in terms of **dg**-categories, the data on the symplectic side naturally fall into the realm of A_∞-categories. This makes the symplectic side essentially more complex as it is based both on the notions of differential topology and Fredholm theory of J-holomorphic curves (needed just in order to define the A_∞-operations), and on the abstract formalism of A_∞-categories, all of which are non-trivial theories lying on the different sides of algebra/geometry/analysis-division of mathematics.

Our work was motivated by the attempt to define the notion of a semidirect product of a Fukaya category with a monoid acting on this category appearing in certain examples of equivariant mirror symmetry. However, eventually our main focus switched into developing the foundations required for these notions to be properly defined. In particular, we realized that the notion of a (homotopy coherent) action on A_∞-categories is not so obvious to explicitly formulate itself, even though the idea behind it is clear.

A natural approach for defining a homotopy coherent action of a monoid Γ on an A_∞-category $\mathcal{A}$ is to start from the classical definition of a strict action as a monoid morphism $\rho\colon \Gamma \to \mathsf{A}_\infty(\mathcal{A}, \mathcal{A})$, where $\mathsf{A}_\infty(\mathcal{A}, \mathcal{A})$ denotes the class of all A_∞-endofunctors of $\mathcal{A}$ forming a monoid under the composition of functors, and try to replace the equalities arising in this definition by the A_∞-equivalences. Namely, we should firstly provide the set of A_∞-functors

$$\rho(g)\colon \mathcal{A} \to \mathcal{A} \ \text{ for } \ g \in \Gamma,$$

then we should provide the set of natural transformations/equivalences

$$\eta_{g,h}\colon \rho(gh) \longrightarrow \rho(g)\rho(h) \ \text{ for } \ g, h \in \Gamma,$$

measuring the failure of ρ to commute with the compositions. Then, we also need to provide the higher coherence data for the above functors and transformations in order to get a workable definition. Namely, for $n \geq 3$, we just need to specify one homotopy for each choice of $g_1, \ldots, g_n \in \Gamma$ between certain A_∞-transformations $\rho(g_1 \ldots g_n) \to \rho(g_1) \ldots \rho(g_n)$ defined in terms of previous components. However, even though this idea is clear, it becomes less and less obvious of how to write down these coherence conditions as n increases. For example, it is not clear of how to unambiguously combine the homotopies appearing in a multi-dimensional diagram into a single algebraic expression so that to define the boundary conditions for each subsequent homotopy. The problems even worsen as the composition of transformations in the A_∞-setting is not associative.

0.2 Idea and implementation

It is well-known that we generally can present the coherence conditions on the objects of some category as "functors" from "diagrams" into this category. For example, the A_∞-transformations of two A_∞-functors $f, g\colon \mathcal{A} \to \mathcal{B}$ can be encoded as the functors from $(* \to *)$ to the category of A_∞-functors $\underline{\mathsf{A}_\infty}(\mathcal{A}; \mathcal{B})$. Such diagrams should be also equipped with the differentials, so that we could encode the homotopies between A_∞-transformations. Similarly, there should be a way to compose the morphisms in our diagrams to account for the compositions of transformations.

In this thesis, we introduce the notion of cellular **dg**-categories as the objects able to encode the above diagrams. They can be thought as just "free categories on diagrams equipped with a differential". Moreover, we notice that the diagrams involved in the definition of a homotopy coherent action can be described in terms of polyhedra, similarly to how the A_∞-categories themselves can be described in terms of Stasheff associahedra. Due to this, our categories are designed to mimic the topological objects such as CW-complexes rather than more abstract diagrams.

In Chapter 2, we provide the precise definition and study the general properties of cellular **dg**-categories by comparing them to CW-complexes. In particular, we describe various examples of cellular **dg**-categories corresponding to well-known geometric objects. We also suggest in Conjecture 2.37 that cellular **dg**-categories form invariants of CW-complexes. (This chapter does not involve A_∞-formalism.)

In Section 3.1, we prove our main statements (Theorem 3.1 and Corollary 3.4) showing that cellular **dg**-categories indeed form a good model for the diagrams involved in describing coherence conditions in A_∞-setting. Namely, we show that every A_∞-functor from a cellular **dg**-category $\mathcal{A}$ into an A_∞-category $\mathcal{B}$ can be described by the values on the objects and cells of $\mathcal{A}$ (which can be thought as the vertices and arrows in the diagram) satisfying the boundary conditions defined from the structure of the differential in $\mathcal{A}$. We also note that such functor can be uniquely reconstructed from the corresponding values up to a certain type of equivalence called *s-homotopy* which was introduced by Seidel in [Sei08] under the name of homotopy. We dedicate Appendix B to formulating the notion of an s-homotopy and for describing its various properties and implications needed for us.

Remark 0.1. An important feature of the above facts is that the coherence conditions in A_∞-setting can be encoded by **dg**-categories, which means that one might be able to formulate different homotopy theoretic notions in A_∞-categories without preforming too many computations with A_∞-structures per se. For example, we ourselves rarely need to perform any sign computations in this work, even though a significant part of it is formulated in A_∞-terms.

In Section 3.2, we apply the above ideas and suggest the precise definition of a homotopy coherent monoid action on A_∞-categories. We notice that such action can be described in terms of A_∞-categorical analogues of topological A_∞-maps of strict monoids (similarly to how a strict action can be described as a strict map of monoids). The components of the topological A_∞-maps are parameterized by ordinary n-dimensional cubes (see §1.3), and their A_∞-categorical counterparts are parameterized by the corresponding **dg**-n-cubes defined in §2.5.2. In Appendix A, we describe two examples of A_∞-categorical monoids needed in order to formulate the definition of a homotopy coherent action.

Chapter 1: Preliminaries

We use the following notations and conventions:

- We are working in the $\Bbbk$-linear setting, where $\Bbbk$ is a commutative (unital) ring not assumed to be a field.
- Our differentials have degree 1 and are denoted by ∂. The chain complexes are denoted by $X^\bullet$. The shift of complexes is denoted by $sX^\bullet := X^\bullet[1]$. The corresponding operation $s\colon X^\bullet \to sX^\bullet$ has degree -1.
- The grading always means $\mathbb{Z}$-grading.
- We use notation $(-)^n := (-1)^n$.
- We compose functions in the traditional right-to-left order and write the arguments on the right of the functions as in $(f \circ g)(x) = f(g(x))$. Similarly, we compose morphisms and paths from right-to-left.
- We usually denote composition of functions/morphisms/paths/etc. by juxtaposition.
- We denote the units/identities in categories and **dg**-categories as 1_X for an object X.
- We denote the units and strict units in A_∞-categories as $\mathbf{i}_X$ and $\mathbf{i}^{\mathrm{su}}_X$ respectively for an object X.
- We use two kinds of A_∞-operations m_n and b_n when dealing with A_∞-categories.
- We mostly follow notations and conventions of [Man07] when discussing A_∞-categories.

Remark 1.1. Even though half of this thesis is built on the notion of A_∞-categories, the nature of our results is such, that we rarely need to perform any sign computations with A_∞-categories.

1.1 Chain complexes and dg-categories

A *chain complex* $X^\bullet$ is a $\mathbb{Z}$-graded space of $\Bbbk$-modules $(X^n)_{n\in\mathbb{Z}}$ equipped with a differential $\partial_X \colon X^\bullet \to X^\bullet$ which is a graded map of degree 1 such that $(\partial_X)^2 = 0$. We will often omit the subscript and will write $\partial(x)$ instead of $\partial_X(x)$ as soon as we know which complex the element x belongs to.

Throughout this work, whenever we talk about the grading, we always mean $\mathbb{Z}$-grading.

If $x \in X^n$, we call x an *element of* $X^\bullet$ *of degree* n. For an element $x \in X^\bullet$, we denote its degree by $|x|$. An element x of $X^\bullet$ is called a *cycle* if $\partial(x) = 0$, and it is called a *boundary* if $x = \partial(y)$ for some element y. In the latter case, we will also say that x is a boundary of y. Any boundary is automatically a cycle.

A *graded map* $f \colon X^\bullet \to Y^\bullet$ *of degree* n of two chain complexes is a sequence of $\Bbbk$-linear maps $f_k \colon X^k \to Y^{k+n}$ for $k \in \mathbb{Z}$. It is called a *chain map*, if it has degree 0 and commutes with the differentials: $f \circ \partial_X = \partial_Y \circ f$.

We define the *shifted complex* $sX^\bullet = X^\bullet[1]$ by its components as follows: $(sX^\bullet)^n := X^{n+1}$. For each $x \in X^\bullet$ of degree n, we will denote by sx the same object considered as an element of $sX^\bullet$ of degree $n-1$. Thus, we define the *shift map* $s \colon X^\bullet \to sX^\bullet$, $x \mapsto sx$ which is a graded map of degree -1 whose components are basically the identity maps. We define the differential on $sX^\bullet$ as the negation of the differential on $X^\bullet$:

$$\partial_{sX}(sx) := -s\big(\partial_X(x)\big), \text{ for any } x \in X^\bullet.$$

Remark 1.2. The sign in the definition of ∂_{sX} is chosen in such a way that the shift map s becomes itself a cycle of degree -1, that is $\partial_{\mathrm{Hom}}(s) = 0$ in the corresponding Hom-space.

Definition 1.3 (Leibniz rule)**.** We say that a bilinear operation $\star$ of degree 0 on two complexes $X^\bullet$ and $Y^\bullet$ satisfies the *Leibniz rule* if the following equation holds for all elements $x \in X^\bullet$ and $y \in Y^\bullet$:

$$\partial(x \star y) = \partial(x) \star y + (-)^{|x|}\, x \star \partial(y).$$

Definition 1.4. A ***dg**-category* $\mathcal{A}$ is a category whose sets of morphisms have structures of chain complexes such that the units are 0-cycles and the operations of composition satisfy Leibniz rule. In such case, the differential on the morphisms is denoted by $\partial_{\mathcal{A}}$.

The chain complexes form a closed monoidal category, meaning that the tensor products and inner Hom-spaces also form chain complexes. The differentials in the latter chain complexes are defined in such a way that the Leibniz rule is satisfied for either of the operations of the tensor product of morphisms, of the function evaluation and the function composition.

Definition 1.5. The differential $\partial_{X\otimes Y}$ on the tensor product $X^\bullet \otimes Y^\bullet$ of two chain complexes $X^\bullet$ and $Y^\bullet$ is defined via the Leibniz rule:

$$\partial_{X\otimes Y}(x \otimes y) := \partial_X(x) \otimes y + (-)^{|x|} x \otimes \partial_Y(y).$$

Definition 1.6. For two chain complexes $X^\bullet$ and $Y^\bullet$, the graded maps $f\colon X^\bullet \to Y^\bullet$ of various degrees form a chain complex $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ with the *inner differential* ∂_{Hom} defined by the following formula:

$$\partial_{\mathrm{Hom}}(f) := \partial_Y \circ f - (-)^{|f|} f \circ \partial_X.$$

This definition can be also interpreted in terms of Leibniz rule as follows. Applied to an element $x \in X^\bullet$, it gives us the equality

$$\big(\partial_{\mathrm{Hom}}(f)\big)(x) = \partial_Y\big(f(x)\big) - (-)^{|f|} f\big(\partial_X(x)\big),$$

which is equivalent to the following:

$$\partial_Y\big(f(x)\big) = \big(\partial_{\mathrm{Hom}}(f)\big)(x) + (-)^{|f|} f\big(\partial_X(x)\big).$$

The latter is precisely the Leibniz rule for the function application.

The definition of ∂_{Hom} implies that the Leibniz rule holds also for the compositions of the graded maps $f\colon Y^\bullet \to Z^\bullet$ and $g\colon X^\bullet \to Y^\bullet$:

$$\partial(f\circ g) = \partial(f)\circ g + (-)^{|f|} f\circ\partial(g),$$

where the first differential is taken in the complex $\mathrm{Hom}^\bullet(X^\bullet, Z^\bullet)$, the second in $\mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet)$ and the third in $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ respectively.

Remark 1.7. The above definitions imply that the composition map

$$\circ_{X,Y,Z}\colon\ \mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet)\otimes\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)\to\mathrm{Hom}^\bullet(X^\bullet, Z^\bullet),\quad f\otimes g\mapsto f\circ g,$$

and the evaluation map

$$\mathrm{ev}_{X,Y}\colon \mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)\otimes X^\bullet\to Y^\bullet,\quad f\otimes x\mapsto f(x),$$

are chain maps of the corresponding complexes, that is 0-cycles in the corresponding complexes of graded maps.

Remark 1.8. The definition of the inner differential implies that the differential $\partial_X\colon X^\bullet\to X^\bullet$ considered as an element of $\mathrm{Hom}^\bullet(X^\bullet, X^\bullet)$ is itself a cycle of degree 1, that is $\partial_{\mathrm{Hom}}(\partial_X) = 0$.

We will also need to use the general principle called the **Koszul sign rule**. It states that in any meaningful construction or definition we need to use signs of type $(-)^{|x||y|}$ whenever the symbols x and y appear in the "wrong" order.

It is used in the following two definitions.

Definition 1.9. For two graded maps $f\colon X^\bullet\to Z^\bullet$ and $g\colon Y^\bullet\to T^\bullet$, we consider the tensor product $f\otimes g$ as a graded map from $X^\bullet\otimes Y^\bullet$ to $Z^\bullet\otimes T^\bullet$ via the following definition:

$$(f\otimes g)(x\otimes y) := (-)^{|g||x|} f(x)\otimes g(y),$$

for any $x\in X^\bullet$ and $y\in Y^\bullet$.

Definition 1.10. The composition of the tensor products of graded maps is defined as follows:

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) := (-)^{|g_1||f_2|}(f_1 \circ f_2) \otimes (g_1 \circ g_2).$$

Leibniz rule and Koszul sign rule allow us to compute the differentials of more complicated expressions in a straightforward manner as follows.

Definition 1.11. Consider any expression which is formed out of tensor products, function applications, function compositions and a finite sequence of elements x_1, …, x_n participating in this expression in this precise order. We will call it a *standard expression of* $(x_1, \ldots, x_n)$.

Example 1.12. The expression $(f \otimes f)(x \otimes y)$ is a standard expression of (f, f, x, y). The expression $(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)$ is a standard expression of $(f, \mathrm{id}, \mathrm{id}, g)$.

The inductive application of the Leibniz rule immediately implies the following.

Lemma 1.13 (Generalized Leibniz Rule)**.** *The value of the differential on any standard expression* $expr(x_1, \ldots, x_n)$ *can be computed from the following formula:*

$$\partial(expr(x_1, \ldots, x_n)) = \sum_{i=1}^{n} (-)^{|x_1|+\ldots+|x_{i-1}|} expr(x_1, \ldots, \partial(x_i), \ldots, x_n),$$

where the differentials are taken in the corresponding chain complexes.

This in particular means that if x_i belongs to the complex $X_i^\bullet$ for $1 \leq i \leq n$, and the value of a standard expression $expr(x_1, \ldots, x_n)$ belongs to the complex $Y^\bullet$, then $expr$ defines a chain map of complexes:

$$expr\colon X_1^\bullet \otimes \ldots \otimes X_n^\bullet \to Y^\bullet, \;\; x_1 \otimes \ldots \otimes x_n \mapsto expr(x_1, \ldots, x_n).$$

Remark 1.14. The sign in the generalized Leibniz rule can be thought of as coming from the Koszul sign rule where the sign $(-)^{|x|}$ appears from permutation of x and ∂, the latter having degree 1.

Lemma 1.15. *If the elements x_1, …, x_n are cycles in the corresponding chain complexes, then the value of any standard expression of $(x_1, \dots, x_n)$ also forms a cycle. Moreover, if one of x_i is also a boundary, then the whole expression also becomes a boundary.*

Remark 1.16. Similar statements hold when we allow the permutations of symbols in the expressions. In such case, one just need to add extra-signs according to Koszul sign rule.

1.2 A_∞-categories

In this section, we list the basic notions from the theory of A_∞-categories that will be used in Section 3. We omit some of the technical details which are not relevant to our work and refer the reader to [Man07] and to other works by Manzyuk and Lyubashenko for more details whenever needed.

Remark 1.17 (Conventions)**.** Though we will mostly follow the notations and conventions of the above mentioned authors, we will differ from them in the following ways. The first difference is that we use the more traditional approach of composing functions from right to left, and of writing the argument after the function, as in $(f \circ g)(x) = f(g(x))$, while the above authors compose the functions from left to right and write the argument of the function before the function itself. Most definitions however stay the same and the only difference occurs in some of the signs due to Koszul sign rule. We try to indicate such differences whenever possible.

Another difference is that our units $\mathbf{i}_X$ and $\mathbf{i}_X^{\mathrm{su}}$ in A_∞-categories will have degree 0, while the mentioned authors define the corresponding shifted elements $s\mathbf{i}_X$ and $s\mathbf{i}_X^{\mathrm{su}}$ of degree -1 as their units. We introduce this difference so that to match the notions of units in **dg**- and A_∞-settings. Similarly, we want natural transformations to have degree 0 and not -1, so we separate the notions of A_∞-transformations and the corresponding coderivations via the extra-shift, while the above authors identify A_∞-transformations with the coderivations.

A *graded quiver* $\mathcal{A}$ consists of a set of objects $\operatorname{Ob}\mathcal{A}$ and a $\mathbb{Z}$-graded $\Bbbk$-module $\mathcal{A}(X, Y)$ of morphisms from X to Y for each $X, Y \in \operatorname{Ob}\mathcal{A}$. We also consider the shifted quivers $s\mathcal{A}$ and

$s^{-1}\mathcal{A}$ which have the same objects as $\mathcal{A}$ and whose spaces of morphisms are shifted as in the definition of shifted complexes.

A *graded map of quivers* $f\colon \mathcal{A} \to \mathcal{B}$ of degree d is given by a map $\operatorname{Ob} f\colon \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ on the objects, and by the corresponding graded maps $f_{X,Y}\colon \mathcal{A}(X,Y) \to \mathcal{B}(f(X), f(Y))$ of degree d on the morphisms. If $\operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{A}$, then we also consider graded maps of quivers over $\operatorname{Ob}\mathcal{A}$ which are the graded maps $f\colon \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_{\operatorname{Ob}\mathcal{A}}$. The *shift map* $s\colon \mathcal{A} \to s\mathcal{A}$ is a graded map of quivers over $\operatorname{Ob}\mathcal{A}$ given by the shift maps on the spaces of morphisms.

For a graded quiver $\mathcal{A}$, we will denote by $\Bbbk 1_{\mathcal{A}}$ the quiver which has the same objects as $\mathcal{A}$ and whose morphisms are defined as follows:

$$\Bbbk 1_{\mathcal{A}}(X,Y) := 0, \quad \text{if } X \neq Y,$$
$$\Bbbk 1_{\mathcal{A}}(X,X) := \Bbbk 1_X,$$

where 1_X just denotes the generator of the corresponding one-dimensional space.

For a graded quiver $\mathcal{A}$, we define the *n-fold tensor product* $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ for $n \geq 0$ as the graded quiver which has the same objects as $\mathcal{A}$ and whose morphisms are as follows:

$$T^n\mathcal{A}(X_0, X_n) := \bigoplus_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{A}} \mathcal{A}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{A}(X_0, X_1),$$

for any $X_0, X_n \in \operatorname{Ob}\mathcal{A}$. For example, $T^0\mathcal{A} = \Bbbk 1_{\mathcal{A}}$ and $T^1\mathcal{A} = \mathcal{A}$. Then we define the *tensor coalgebra* $T\mathcal{A}$ as

$$T\mathcal{A} := \bigoplus_{n\geq 0} T^n\mathcal{A}.$$

We will also denote by

$$\mathrm{pr}_n\colon T\mathcal{A} \to T^n\mathcal{A}$$

the projection map from $T\mathcal{A}$ onto its n-th component, where $n \geq 0$.

Remark 1.18. Note that in the above definition we differ from Manzyuk in that we multiply the factors $\mathcal{A}(X_i, X_{i+1})$ in the opposite order so that to match our convention of composing the morphisms from right to left.

Definition 1.19. An A_∞*-category* is a graded quiver $\mathcal{A}$ equipped with the graded maps of quivers $b_n = b_n^{\mathcal{A}} \colon T^n s\mathcal{A} \to s\mathcal{A}$ of degree 1 over $\operatorname{Ob}\mathcal{A}$ for $n \geq 1$, called A_∞*-operations*, satisfying the famous A_∞-relations:

$$\sum_{\substack{p+k+q=m \\ p,q\geq 0,\, k\geq 1}} b_{p+1+q}(1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) = 0 : \; T^m s\mathcal{A} \to s\mathcal{A}, \text{ for each } m \geq 1. \tag{1.1}$$

The A_∞-operations b_n always come together with their more traditional counterparts m_n of degrees $2-n$, which we will call *traditional* A_∞*-operations*, defined as follows:

$$m_n := s^{-1} b_n s^{\otimes n} \colon \mathcal{A}^{\otimes n} \to \mathcal{A}. \tag{1.2}$$

These operations satisfy the following set of A_∞-conditions:

$$\sum_{\substack{p+k+q=m \\ p,q\geq 0,\, k\geq 1}} (-)^{p+kq}\, m_{p+1+q}(1^{\otimes p} \otimes m_k \otimes 1^{\otimes q}) = 0 : \; T^m\mathcal{A} \to \mathcal{A}, \quad m \geq 1. \tag{1.3}$$

Remark 1.20. The equivalence of A_∞-relations (1.1) and (1.3) can be checked directly by plugging in the definition (1.2) into (1.3) and rearranging the terms s and b_k after that. The signs will all cancel out due to the Koszul sign rule, and we will get exactly the expression in (1.1) multiplied by s^{-1} on the left and by $s^{\otimes m}$ on the right.

Remark 1.21. Our sign in (1.3) differs from the ones in the works of Lyubashenko and Manzyuk, because they use the left-to-right order of function composition.

Example 1.22. Every (not necessarily unital) **dg**-category $(\mathcal{A}, \partial_{\mathcal{A}})$ can be treated as an A_∞-category such that $m_1^{\mathcal{A}} = \partial_{\mathcal{A}}$, $m_2^{\mathcal{A}}$ is given by the composition of morphisms, and $m_n = 0$ for $n \geq 3$.

When we discuss the A_∞-categories in general, we prefer to use operations b_n because the corresponding A_∞-relations and subsequent computations all have straightforward signs. However, in some cases, it is more convenient to use traditional operations m_n. For example, we prefer to use m_n in the following two situations: 1) when we are talking about **dg**-categories; and, 2) when we are transferring the customary notions of the traditional category theory, such as units, to the A_∞-setting.

Example 1.23. Transitioning between b_n and m_n on the morphisms of $T^n\mathcal{A}$ can be done via the following formula:

$$b_n(sx_1 \otimes \ldots \otimes sx_n) = (-)^{\sum_i (n-i)|x_i|} sm_n(x_1 \otimes \ldots \otimes x_n),$$

for $x_1 \otimes \ldots \otimes x_n \in \mathcal{A}^{\otimes n}$. This follows from definition (1.2) and from the fact that $s^{\otimes n}(x_1 \otimes \ldots \otimes x_n) = (-)^{\sum_i (n-i)|x_i|}(sx_1 \otimes \ldots \otimes sx_n)$ by Koszul sign rule. For example, for $n = 1$ we have $b_1(sx) = sm_1(x)$ from definition and for $n = 2$ we have:

$$b_2(sx \otimes sy) = (-)^{|x|} b_2(s \otimes s)(x \otimes y) = (-)^{|x|} sm_2(x \otimes y).$$

Remark 1.24. The usual interpretation of the traditional A_∞-operations is as follows. The first A_∞-relation

$$m_1^2 = 0$$

allows to interpret the first operation m_1 as the differential on $\mathcal{A}$ thus making it into a **dg**-quiver. We will thus denote it by

$$\partial_{\mathcal{A}} := m_1^{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}.$$

The second A_∞-relation

$$m_1 m_2 - m_2(m_1 \otimes 1) - m_2(1 \otimes m_1) = 0$$

is equivalent to the condition that $m_2 \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the chain map of **dg**-quivers which is equivalent to the equality $\partial_{\mathrm{Hom}}(m_2) = 0$, where the differential ∂_{Hom} on the space of graded maps from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}$ is defined in terms of the differential $\partial_{\mathcal{A}} = m_1^{\mathcal{A}}$ on $\mathcal{A}$. Hence, m_2 satisfies the Leibniz rule and can be interpreted as the "composition" of morphisms in $\mathcal{A}$. This composition fails to satisfy the strict associativity relation. However, the third A_∞-relation

$$m_1 m_3 + m_2(m_2 \otimes 1) - m_2(1 \otimes m_2) + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) = 0$$

is equivalent to

$$\partial_{\mathrm{Hom}}(m_3) = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1),$$

which means that m_2 is associative up to the homotopy given by the third operation m_3. Similarly, the higher operations m_n then provide the higher homotopies for transitioning between the different ways of "composing" n morphisms.

Remark 1.25. The quiver $Ts\mathcal{A}$ has a natural structure of an augmented graded coalgebra and the A_∞-operations b_n uniquely extend to a $(\mathrm{id}_{Ts\mathcal{A}}, \mathrm{id}_{Ts\mathcal{A}})$-coderivation $b \colon Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1 such that $b^2 = 0$ and $b|_{T^0 s\mathcal{A}} = 0$. (See more details in [Man07].) The corresponding matrix coefficients $b_{n;m}$ can be then found from the following formulas:

$$b_{n;m} = \sum_{\substack{p+k+q=n \\ p+1+q=m \\ p,q \geq 0,\, k \geq 1}} 1^{\otimes p} \otimes b_k \otimes 1^{\otimes q} \colon\ T^n s\mathcal{A} \to T^m s\mathcal{A}. \tag{1.4}$$

1.2.1 A_∞-functors and transformations

An A_∞-*functor* $f \colon \mathcal{A}_1, \dots, \mathcal{A}_k \to \mathcal{B}$ with multiple inputs is a morphism of augmented **dg**-coalgebras $f \colon\ \boxtimes^i Ts\mathcal{A}_i \to Ts\mathcal{B}$. Any such functor is uniquely determined by its components

$$f_{(n_i)_i} \colon\ \boxtimes^i T^{n_i} s\mathcal{A}_i \to s\mathcal{B}$$

for $n_i \geq 0$, satisfying certain A_∞-conditions. Throughout this work, we won't be using the exact form of these A_∞-conditions for the functors of multiple inputs, and we refer the reader to [Man07] for the precise definition.

Example 1.26. In a case of a single input, an A_∞-functor $f\colon \mathcal{A} \to \mathcal{B}$ is defined by its components $f_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ for $n \geq 0$ of degree 0, such that $f_0 = 0$. The A_∞-conditions in this case become as follows:

$$\sum_{r+k+t=n} f_{r+1+t}(1^{\otimes r} \otimes b_k^{\mathcal{A}} \otimes 1^{\otimes t}) = \sum_{i_1+\ldots+i_p=n} b_p^{\mathcal{B}}\left(f_{i_1} \otimes \ldots \otimes f_{i_p}\right) : T^n s\mathcal{A} \to s\mathcal{B}, \tag{1.5}$$

for each $m \geq 1$.

Remark 1.27. The matrix coefficients of a functor $f\colon \mathcal{A} \to \mathcal{B}$ are defined from its components $f_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ via the following formula:

$$f_{n;m} = \sum_{\substack{i_1+\ldots+i_p=n \\ i_1,\ldots,i_p \geq 1}} f_{i_1} \otimes \ldots \otimes f_{i_p}\colon\; T^n s\mathcal{A} \to T^m s\mathcal{B}. \tag{1.6}$$

Then, the components $f_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ satisfy conditions (1.5) if and only if the function

$$f\colon Ts\mathcal{A} \to Ts\mathcal{B}$$

defined by its matrix coefficients $f_{m;n}$ from (1.6) satisfies the following equation:

$$f\, b^{\mathcal{A}} \;=\; b^{\mathcal{B}}\, f\colon\; Ts\mathcal{A} \to Ts\mathcal{B},$$

where $b^{\mathcal{A}}$ and $b^{\mathcal{B}}$ are defined via components $b_k^{\mathcal{A}}$ and $b_k^{\mathcal{B}}$ by (1.4).

Remark 1.28. The conditions (1.5) admit a well-known form expressed in terms of the components m_i rather than b_i.

Definition 1.29. For two functors $f, g\colon \mathcal{A}_1, \ldots, \mathcal{A}_k \to \mathcal{B}$ with multiple arguments, we define the graded space $\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_k;\mathcal{B})(f,g)$ of A_∞-*transformations* to be the back-shifted space of graded (g,f)-coderivations $\boxtimes^i Ts\mathcal{A}_i \to Ts\mathcal{B}$:

$$\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots\mathcal{A}_k;\mathcal{B})(f,g) := s^{-1}\Big\{(g,f)\text{-coderivations } \boxtimes^i Ts\mathcal{A}_i \to Ts\mathcal{B}\Big\}.$$

In other words, an A_∞-*transformation* $s^{-1}r$ of degree k from f to g is defined by the corresponding (g,f)-coderivation $r\colon\ \boxtimes^i\ Ts\mathcal{A}_i \to Ts\mathcal{B}$ of degree $k-1$.

An A_∞-transformation $s^{-1}r$ of degree k is uniquely determined by the components

$$r_{(n_i)_i}\colon\ \boxtimes^i\ T^{n_i}s\mathcal{A}_i \to s\mathcal{B}$$

for $n_i \geq 0$ of the corresponding coderivation r of degree $k-1$.

Example 1.30. In the case of a single input $\mathcal{A}$, the matrix coefficients of r are determined by the components $r_i\colon T^i s\mathcal{A} \to s\mathcal{B}$ via the following formula:

$$r_{n;m} = \sum g_{j_1}\otimes\dots\otimes g_{j_q}\otimes r_t\otimes f_{i_1}\otimes\dots\otimes f_{i_p}\colon\ T^n s\mathcal{A} \to T^m s\mathcal{B}, \tag{1.7}$$

where the summation is taken over all partitions

$$j_1+\dots+j_q+t+i_1+\dots+i_p = n, \quad q+1+p = m,$$

where $j_1,\dots,j_q \geq 1$; $i_1,\dots,i_p \geq 1$ and $t,p,q \geq 0$.

1.2.2 Multicategory of A_∞-categories

The A_∞-functors from $(\mathcal{A}_1,\dots,\mathcal{A}_k)$ to $\mathcal{B}$ themselves form an A_∞-category denoted by

$$\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_k;\mathcal{B}),$$

in which the graded spaces of morphisms are given by the corresponding spaces of A_∞-transformations and the A_∞-operations B_n are defined by explicit formulas. (See [Man07; Lyu03] for the precise definition; see also [Fuk02; Sei08] for the definition of A_∞-categories $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ given by A_∞-functors with a single input.)

Let us describe the first A_∞-operation B_1 in the case of a single input. For two functors $f, g\colon \mathcal{A} \to \mathcal{B}$, we define

$$B_1\colon s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g),$$

which maps a (g, f)-coderivation r of degree $|r|$ to a (g, f)-coderivation $B_1(r)$ of degree $|r| + 1$, by the following formula:

$$B_1(r) \ := \ br - (-)^{|r|}rb\colon \ Ts\mathcal{A} \to Ts\mathcal{B}.$$

Component-wisely, we can write is as follows:

$$\begin{aligned}(B_1(r))_n = &\sum b_{l+1+k}(g_{j_1} \otimes \ldots \otimes g_{j_l} \otimes r_p \otimes f_{i_1} \otimes \ldots \otimes f_{i_k}) \\ &- (-)^{|r|} \sum r_{l+1+k}(1^{\otimes l} \otimes b_q \otimes 1^{\otimes k}) : T^n s\mathcal{A} \to s\mathcal{B},\end{aligned} \tag{1.8}$$

where the first sum is taken over the decompositions $n = j_1 + \ldots + j_l + p + i_1 + \ldots + i_k$ for $l \geq 0$, $k \geq 0$, $j_m \geq 1$, $i_m \geq 1$ and $p \geq 0$, and the second sum is taken over the decompositions $n = l + q + k$ for $l \geq 0$, $k \geq 0$, $q \geq 1$.

Definition 1.31. Let $\eta = s^{-1}r\colon f \to g\colon \mathcal{A} \to \mathcal{B}$ be an A_∞-transformation given by a (g, f)-coderivation r. We call η a *natural* transformation, if it is a 0-cycle in $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g)$, that is if r has degree -1 and $B_1(r) = 0$.

The categories $\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \ldots, \mathcal{A}_k;\mathcal{B})$ satisfy many nice properties and together form an object $\underline{\mathsf{A}_\infty}$ called a *closed multicategory of A_∞-categories*. We do not list all of these properties here, but we will describe a few useful constructs arising from this concept. For more details on the multicategory $\underline{\mathsf{A}_\infty}$ and the subsequent notions, we refer the reader to [Man07]. For the general definition of a *multicategory*, we refer the reader to [Man07; Lei04; EM06].

There is an *evaluation* A_∞-functor

$$\mathrm{ev}^{\mathsf{A}_\infty} = \mathrm{ev}^{\mathsf{A}_\infty}_{\mathcal{A}_1,\dots,\mathcal{A}_k;\mathcal{B}} \colon \underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_k;\mathcal{B}), \mathcal{A}_1,\dots,\mathcal{A}_k \to \mathcal{B},$$

which maps the tuple $(f, X_1, \dots, X_k)$ consisting of an A_∞-functor $f\colon \mathcal{A}_1,\dots\mathcal{A}_k \to \mathcal{B}$ and objects $X_i \in \mathcal{A}_i$ for $1 \le i \le k$ to the object $f(X_1,\dots,X_k) \in \mathcal{B}$. The only non-vanishing components of $\mathrm{ev}^{\mathsf{A}_\infty}$ are: $\mathrm{ev}^{\mathsf{A}_\infty}_{0,n_1,\dots,n_k}$ for $\sum_i n_i \neq 0$ which evaluates to the components $f_{n_1,\dots,n_k}$ of the functor f provided by the first input, and $\mathrm{ev}^{\mathsf{A}_\infty}_{1,n_1,\dots,n_k}$ which evaluates to the component $r_{n_1,\dots,n_k}$ of the coderivation r provided by the first input.

For example, in a case of a single input $k = 1$, we will have:

$$\begin{aligned}
&\mathrm{ev}^{\mathsf{A}_\infty}_{\mathcal{A};\mathcal{B}} \colon \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\ \mathcal{A} \to \mathcal{B},\quad (f, X) \mapsto f(X),\\
&\mathrm{ev}^{\mathsf{A}_\infty}_{0,n} \colon \Bbbk_{\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})} \otimes T^n s\mathcal{A} \to s\mathcal{B},\quad 1_f \otimes (x_1 \otimes \dots \otimes x_n) \mapsto f_n(x_1,\dots,x_n),\\
&\mathrm{ev}^{\mathsf{A}_\infty}_{1,n} \colon s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \otimes T^n s\mathcal{A} \to s\mathcal{B},\quad r \otimes (x_1 \otimes \dots \otimes x_n) \mapsto r_n(x_1,\dots,x_n).
\end{aligned}$$

The result of [Man07, §3.3.2] says that there exists an isomorphism of A_∞-categories

$$\varphi^{\mathsf{A}_\infty} \colon \underline{\mathsf{A}_\infty}\Big(\mathcal{A}_1,\dots,\mathcal{A}_k;\ \underline{\mathsf{A}_\infty}(\mathcal{B}_1,\dots,\mathcal{B}_l;\ \mathcal{C})\Big) \to \underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_k,\ \mathcal{B}_1,\dots,\mathcal{B}_l;\ \mathcal{C}),$$

defined on the objects by mapping an A_∞-functor

$$f \colon \mathcal{A}_1,\dots,\mathcal{A}_k \to \underline{\mathsf{A}_\infty}(\mathcal{B}_1,\dots,\mathcal{B}_l;\ \mathcal{C})$$

to the functor

$$\varphi^{\mathsf{A}_\infty}(f) = \mathrm{ev}^{\mathsf{A}_\infty}_{\mathcal{B}_1,\dots,\mathcal{B}_l;\mathcal{C}} \circ (f, \mathrm{id}_{\mathcal{B}_1},\dots,\mathrm{id}_{\mathcal{B}_l}).$$

There exists a *composition* A_∞-functor

$$M = M_{\mathcal{A},\mathcal{B},\mathcal{C}} \colon\ \underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}),\ \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C})$$

which is a unique functor that makes the following diagram commutative

$$\begin{tikzcd}
\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}),\ \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\ \mathcal{A} \arrow[r, "{M, \mathrm{id}}"] \arrow[d, "{\mathrm{id}, \mathrm{ev}^{\mathsf{A}_\infty}}"'] & \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}),\ \mathcal{A} \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\
\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}),\ \mathcal{B} \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \mathcal{C}
\end{tikzcd}$$

This condition means that $\varphi^{\mathsf{A}_\infty}(M) := \mathrm{ev}^{\mathsf{A}_\infty} \circ (M, \mathrm{id}) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (\mathrm{id}, \mathrm{ev}^{\mathsf{A}_\infty})$, hence the existence and uniqueness of such functor M follows from the fact that $\varphi^{\mathsf{A}_\infty}$ is a bijection on the objects. On the objects, the composition functor acts as a regular composition of A_∞-functors:

$$\mathrm{Ob}\, M_{\mathcal{A},\mathcal{B},\mathcal{C}} \colon (f \colon \mathcal{B} \to \mathcal{C}),\ (g \colon \mathcal{A} \to \mathcal{B}) \mapsto (f \circ g \colon \mathcal{A} \to \mathcal{C}).$$

1.2.3 Unitality and A_∞-equivalences

Definition 1.32. An A_∞-category $\mathcal{A}$ is called *strictly unital* if there exist 0-cycles $\mathbf{i}^{\mathrm{su}}_X \colon X \to X$ for each object $X \in \mathcal{A}$ such that $m_2(\mathbf{i}^{\mathrm{su}}_Y, x) = x = m_2(x, \mathbf{i}^{\mathrm{su}}_X)$ for any morphism $x \colon X \to Y$, and $m_n(\dots, \mathbf{i}^{\mathrm{su}}, \dots) = 0$ for $n \geq 3$. In this case, the cycles $\mathbf{i}^{\mathrm{su}}_X$ are called the *strict units* of $\mathcal{A}$.

Example 1.33. If $m_n = 0$ for $n \geq 3$, the previous notion gives us the definition of a regular **dg**-category.

Remark 1.34. In terms of b_n and $s\mathcal{A}$, the condition for $\mathcal{A}$ of being strictly unital means that there are (-1)-cycles $s\mathbf{i}^{\mathrm{su}}_X \in s\mathcal{A}(X, X)$ for each X such that $b_2(s\mathbf{i}^{\mathrm{su}}_Y, x) = x$ and $b_2(x, s\mathbf{i}^{\mathrm{su}}_X) = (-)^{|x|+1}x$ for any $x \in s\mathcal{A}(X, Y)$, and $b_n(\dots, s\mathbf{i}^{\mathrm{su}}, \dots) = 0$ for $n \geq 3$.

Most categories we encounter in applications (especially in symplectic topology) are not strictly unital but admit a weaker version of unitality as follows.

Definition 1.35 (Lyubashenko, see [LM08] and [Lyu03]). An A_∞-category $\mathcal{A}$ is called *unital* if there are 0-cycles $\mathbf{i}_X\colon X \to X$ for each $X \in \mathcal{A}$, called the *units*, such that the chain maps $m_2(\mathbf{i}_Y \otimes 1), m_2(1 \otimes \mathbf{i}_X)\colon \mathcal{A}(X,Y) \to \mathcal{A}(X,Y)$ are homotopic to the identity map.

Remark 1.36. In terms of b_n and $s\mathcal{A}$, the condition of being unital means that there are (-1)-cycles $s\mathbf{i}_X \in s\mathcal{A}(X,X)$ for each X such that $b_2(s\mathbf{i}_Y \otimes 1)$ and $-b_2(1 \otimes s\mathbf{i}_X)$ are homotopic to the identity map on $s\mathcal{A}(X,Y)$.

Remark 1.37. It follows from the definition, that if $\mathcal{A}$ is unital, then its units $\mathbf{i}_X$ are determined uniquely up to homotopy.

We will also use an equivalent definition which is due to Fukaya. In order to formulate it, we will need to consider the extended quiver $\mathcal{A}^+ = \mathcal{A} \oplus \Bbbk \mathbf{i}^{\mathrm{su}}_{\mathcal{A}} \oplus \Bbbk \mathbf{j}_{\mathcal{A}}$ for an A_∞-category $\mathcal{A}$. This quiver is obtained from $\mathcal{A}$ by attaching to each object X in $\mathcal{A}$ the new morphisms $\mathbf{i}^{\mathrm{su}}_X\colon \mathcal{A} \to \mathcal{A}$ of degree 0 and $\mathbf{j}_X\colon \mathcal{A} \to \mathcal{A}$ of degree -1.

Definition 1.38 (Fukaya, see [Fuk02] and also [LM08]). An A_∞-category $\mathcal{A}$ is called *homotopy unital* if the extended quiver $\mathcal{A}^+ = \mathcal{A} \oplus \Bbbk \mathbf{i}^{\mathrm{su}}_{\mathcal{A}} \oplus \Bbbk \mathbf{j}_{\mathcal{A}}$ admits an A_∞-structure m^+_n with the following properties:

(1) $m^+_n|_{\mathcal{A}} = m_n$, that is m^+_n is an extension of m_n;

(2) $\mathbf{i}^{\mathrm{su}}_X$ are the strict units of $\mathcal{A}^+$;

(3) there are $\mathbf{i}_X \in \mathcal{A}(X,X)$ such that $m^+_1(\mathbf{j}_X) = \pm(\mathbf{i}^{\mathrm{su}}_X - \mathbf{i}_X)$;

(4) $\mathrm{Im}(m^+_n|_{\mathcal{A}\oplus\Bbbk\mathbf{j}}) \subseteq \mathcal{A}$.

Remark 1.39. It was proved in [LM08] that the different notions of unitality are equivalent. In particular, every unital structure $\mathbf{i}_{\mathcal{A}}$ on $\mathcal{A}$ can be extended to the homotopy unital structure $(\mathcal{A}^+, m^+_n)$. This becomes handy when we want to prove some statement for unital categories in general, but want to simplify computations involving the units. In such case, we can firstly prove the statement for the strictly unital category $\mathcal{A}^+$ and then try to transfer the result via the A_∞-equivalence between $\mathcal{A}$ and $\mathcal{A}^+$ obtained in Lemma B.21.

Remark 1.40. The homotopy unital structure can be thought as the external strictification of the unital structure as follows. The elements $\mathbf{j}_X$ can be thought as formal homotopies between the actual units $\mathbf{i}_X$ and the external strict units $\mathbf{i}_X^{\mathrm{su}}$. Defining the extended A_∞-operations on $\mathcal{A} \oplus \Bbbk\mathbf{j}_{\mathcal{A}}$ can be thought as providing the explicit unit homotopies on $\mathcal{A}$. For example, the restrictions of the maps $m_2^+(\mathbf{j}_Y \otimes 1)$ and $m_2^+(1 \otimes \mathbf{j}_X)$ on $\mathcal{A}(X, Y)$ provide the homotopies of $m_2(\mathbf{i}_Y \otimes 1)$ and $m_2(1 \otimes \mathbf{i}_X)$ with the identity map, called the *right* and *left unit homotopies* respectively. This implies that we have two canonical homotopies between $m_2(\mathbf{i}_X, \mathbf{i}_X)$ and $\mathbf{i}_X$, one given by the right unit homotopy $m_2^+(\mathbf{i}_X, \mathbf{j}_X)$ and another given by the left unit homotopy $m_2^+(\mathbf{j}_X, \mathbf{i}_X)$. The element $m_2^+(\mathbf{j}_X, \mathbf{j}_X)$ then provides the higher homotopy between these two choices. The higher operations involving $\mathbf{j}$ may be given a similar interpretation as providing the higher coherence conditions between the right and left unit homotopies.

In the unital A_∞-categories, the isomorphisms between the objects can be correctly defined as follows.

Definition 1.41. Let $\mathcal{A}$ be a unital A_∞-category. We say that a morphism $x\colon X \to Y$ in $\mathcal{A}$ is an *isomorphism* (or an *equivalence*), if x is a 0-cycle which has a two-sided homotopic inverse. In other words, there is a 0-cycle $y\colon Y \to X$, the *inverse* (or the *inverse equivalence*), such that $m_2(y, x)$ is homotopy equivalent to the unity on X and $m_2(x, y)$ is homotopy equivalent to the unity on Y. In this situation, we also say that objects X and Y are *isomorphic* (or *equivalent*).

The assumption of unitality is essential if we want to transfer the well-known notions of equivalency from regular category theory into the A_∞-setting. For example, this assumption allows us talk about the equivalences of A_∞-functors and A_∞-categories as follows.

Lemma 1.42 ([Man07, Proposition 3.4.10])**.** *For any A_∞-categories $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ and a unital A_∞-category $\mathcal{B}$, the category $\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B})$ is unital as well.*

Definition 1.43. Let $f, g\colon \mathcal{A}_1, \dots, \mathcal{A}_k \to \mathcal{B}$ be two A_∞-functors into a unital A_∞-category $\mathcal{B}$. We will call a natural A_∞-transformation $\eta\colon f \to g$ a *natural equivalence*, if η is an isomorphism from f to g considered as the objects in the corresponding unital A_∞-category of functors $\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \ldots, \mathcal{A}_k; \mathcal{B})$.

By using the above notion, we can compare two A_∞-functors if their output is a unital A_∞-category. Namely, we can say that $f, g \colon \mathcal{A}_1, \ldots, \mathcal{A}_k \to \mathcal{B}$ are *naturally equivalent* (or simply, *equivalent*) if there exists a natural equivalence between them. Another way of comparing A_∞-functors not requiring the output A_∞-category $\mathcal{B}$ of being unital was introduced by Seidel in [Sei08]. It will be quite useful for our work and we dedicate Appendix B to this notion and its properties.

1.3 Topological A_∞-maps

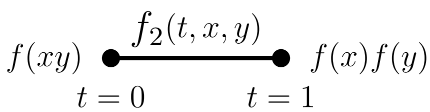
Definition 1.44. A *topological monoid* (X, m, e) is a topological space X endowed with a continuous map $m \colon X \times X \to X$, called "multiplication" and an element $e \in X$, called "the unit", satisfying the associativity property $m(m(x, y), z) = m(x, m(y, z))$ for all $x, y, z \in X$ and the unity properties $m(e, x) = x = m(x, e)$ for all $x \in X$. (We will often denote the result of multiplication by the juxtaposition. In the latter notation, the given conditions can be written as $(xy)z = x(yz)$ and $ex = x = xe$.)

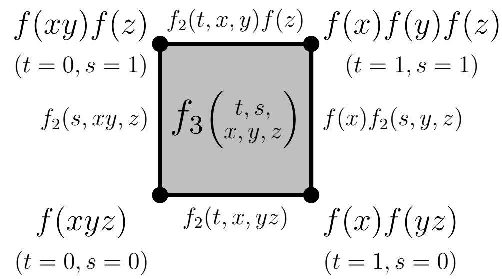
Definition 1.45. An A_∞*-map* f between the two topological monoids (X, m_X, e_X) and (Y, m_Y, e_Y) is given by the collection of maps $f_n \colon I^{n-1} \times X^n \to Y$ for each $n \geq 1$, where $I = [0, 1]$ is the unit interval and I^{n-1} is the $(n-1)$-dimensional unit cube. The maps $(f_n)_{n \geq 1}$ should satisfy the following boundary conditions for each $1 \leq i \leq n-1$:

$$f_n(t_1, \ldots, t_{n-1},\, x_1, \ldots, x_n) = \begin{cases} f_{n-1}(t_1, \ldots, \hat{t}_i, \ldots, t_{n-1},\, x_1, \ldots,\, x_i x_{i+1}\,, \ldots, x_n) & \text{if } t_i = 0 \\ f_i(t_1, \ldots, t_{i-1},\, x_1, \ldots, x_i)\, f_{n-i}(t_{i+1}, \ldots, t_{n-1},\, x_{i+1}, \ldots, x_n) & \text{if } t_i = 1, \end{cases}$$

where $0 \leq t_j \leq 1$ and $x_j \in X$ for all j. (See [BV73, Definition 1.14] and also [Sug60].) We also require f to be compatible with the units by providing a path f_e between $f_1(e_X)$ and e_Y inside Y. (In other words, we provide a map $f_e \colon I \to Y$ such that $f_e(0) = f_1(e_X)$ and $f_e(1) = e_Y$.)

Schematically, we can depict the conditions for f_2 and f_3 as follows, where we denote $f = f_1$ for convenience:





Here, f_2 is the homotopy between the maps $(x, y) \mapsto f(xy)$ and $(x, y) \mapsto f(x)f(y)$, where $x, y \in X$. Such homotopy gives rise to the two homotopies connecting $f(xyz)$ to $f(x)f(y)f(z)$, one via $f(xy)f(z)$ and another via $f(x)f(yz)$. Then f_3 provides a homotopy between the latter two homotopies and so on.

Chapter 2: Cellular dg-categories

In this section we introduce and study the notion of cellular **dg**-categories. These are the **dg**-categories of special type designed to mimic the properties of CW-complexes. Such categories allow us to transfer some notions and constructs from the homotopy theory of topological spaces into the homotopy theory of **dg**/A_∞-categories.

Before formulating the precise definition, let us describe an idea of how we can map the topological notions into the world of **dg**-categories:

$$
\begin{array}{rcl}
\text{topological spaces} & \longrightarrow & \mathbf{dg}\text{-categories} \\[1ex]
\text{points} & \longrightarrow & \text{objects} \\[1ex]
\text{paths between points} & \longrightarrow & \text{morphisms which are } 0\text{-cycles} \\[1ex]
\text{composition of paths} & \longrightarrow & \text{composition of morphisms} \\[1ex]
\text{homotopy of paths } h\colon p_1 \sim p_2 & \longrightarrow & \left[\begin{array}{c} \text{homotopy of morphisms} \\[1ex] \partial(h) = p_2 - p_1 \end{array}\right] \\[1ex]
\left[\begin{array}{c} \text{composition of homotopies} \\[1ex] h_1\colon p_1 \sim p_2 \text{ and } h_2\colon p_2 \sim p_3 \end{array}\right] & \longrightarrow & \text{sum of homotopies } h_1 + h_2 \\[1ex]
\text{higher homotopies} & \longrightarrow & \text{higher homotopies}
\end{array}
$$

Our observation is that if the topological space has some kind of a cellular structure (for example, is a CW-complex), then on the **dg**-side, the homotopic properties of this space can be expressed by only using the images of the cells and applying the operations of composition and differentiation inside the **dg**-category. Thus, all the needed data is essentially contained in the subcategory generated by the cells. This motivates our definition of a cellular **dg**-category as a linear category freely generated by cells and equipped with the differential.

In §2.1, we recall the notions related to free or path categories on graphs. In §2.2, we use this notion in order to define cellular **dg**-categories. In §2.3, we provide some geometric examples of cellular **dg**-categories mimicking the well-known geometric objects. In §2.4, we compare cellular **dg**-categories with CW-complexes and conjecture that there exists a more straightforward correspondence producing cellular **dg**-categories as invariants of CW-complexes. Finally, in §2.5, we show that, even though we might not yet be able to construct a cellular **dg**-category associated to every possible CW-complex, we can still produce some meaningful non-trivial examples, such as **dg**-cubes, by using the notion of a cellular product of cellular **dg**-categories.

2.1 Preliminaries on path categories

A *(directed) graph* Λ is given by a set Λ_0 of *vertices*, by a set Λ_1 of *arrows*, and by two maps $S, T\colon \Lambda_1 \to \Lambda_0$ called *source* and *target* respectively. For any arrow $a \in \Lambda_1$, we will say that a is an arrow *from* its source *to* its target, and we will denote this situation as $a\colon X \to Y$, where $X = S(a)$ and $Y = T(a)$.

A *morphism of graphs* $f\colon \Lambda \to \Lambda'$ is given by a map $f_0\colon \Lambda_0 \to \Lambda'_0$ on the vertices, and by a map $f_1\colon \Lambda_1 \to \Lambda'_1$ on the arrows which is compatible with f_0 in a way that $S(f_1(a)) = f_0(S(a))$ and $T(f_1(a)) = f_0(T(a))$ for any $a \in \Lambda_1$.

A *marked graph* Λ_Σ is given by a graph Λ and a subset $\Sigma \subseteq \Lambda_1$ of its arrows marked as *invertible*. A morphism of marked graphs $f\colon \Lambda_\Sigma \to \Lambda'_{\Sigma'}$ is a morphism of graphs $f\colon \Lambda \to \Lambda'$ such that $f(\Sigma) \subseteq \Sigma'$.

After this section, we will usually denote marked graphs simply by Λ, especially when we don't need to compare them with the underlying unmarked graph.

Remark 2.1. Any category can be considered as a graph by treating all of its objects as the vertices and all of its morphisms as the arrows. This allows us to talk about the *subgraphs of categories*. There is also a natural choice of the marking on a category given by marking of all invertible morphisms. When we talk about the *marked subgraphs of categories*, we assume that the category is equipped with such marking, that is that invertible arrows of this subgraph are also invertible morphisms in the category.

A *path* p in Λ is a sequence

$$(X_n, a_n, X_{n-1}, \ldots, X_1, a_1, X_0)$$

of alternating vertices X_i and arrows a_i, such that $S(a_i) = X_{i-1}$ and $T(a_i) = X_i$ for $1 \leq i \leq n$, where $n \geq 0$. (So, the path goes from right to left.) We define the *length* $l(p)$ of a path p as the number n of arrows in the corresponding sequence. We also set $S(p) := X_0$ and $T(p) := X_n$ and say that p is a path from X_0 to X_n. If $n > 0$, then we will also say that p starts with a_1 and ends with a_n. For any vertex X, the sequence (X), which is a path of length 0, will be called an *empty path on* X. We will call paths of positive length *non-empty*. We will also treat any arrow $a\colon X \to Y$ as a path (Y, a, X) of length 1.

We say that two paths $p = (X_n, a_n, \ldots, a_1, X_0)$ and $q = (Y_m, b_m, \ldots, b_1, Y_0)$ are *composable* if $X_0 = Y_m$. In such situation, we define their *concatenation* $pq = p \circ q$ to be a path

$$(X_n,\ a_n, \ldots, a_1,\ X_0 = Y_m,\ b_m, \ldots, b_1,\ Y_0).$$

This operation is clearly associative and has identities given by the empty paths.

Any non-empty path $p = (X_n, a_n, \ldots, a_1, X_0)$ can be unambiguously recovered from the sequence of its arrows $(a_n, \ldots, a_1)$, since the vertices X_i are determined as being the sources and targets of the corresponding arrows. Thus, the non-empty paths can be thought of as the concatenations of non-empty sequences of composable arrows. In particular, this allows us to denote any non-empty path $p = (X_n, a_n, \ldots, a_1, X_0)$ simply by a juxtaposition $a_n a_{n-1} \cdots a_1$ of its arrows. We will also denote an empty path on X by 1_X for any vertex X.

A *path category* $\Omega[\Lambda]$ is a category formed by the vertices and paths of Λ:

$$\operatorname{Ob}\Omega[\Lambda] := \Lambda_0,$$
$$\Omega[\Lambda](X,Y) := \{\text{paths in } \Lambda \text{ from } X \text{ to } Y\},$$

where the composition is given by the concatenation of paths, and the identities are given by the empty paths.

For a marked graph Λ_Σ, we also consider the graph $\Lambda \cup \Sigma^{-1}$ formed by adding formal inverse arrows to Λ. Namely, for each arrow $a\colon X \to Y$ in Σ, we add a new arrow $a^{-1}\colon Y \to X$ to the graph Λ. We say that a path in $\Lambda \cup \Sigma^{-1}$ is *reduced*, if it does not have subpaths of type aa^{-1} and $a^{-1}a$ for $a \in \Sigma$. We say that two paths in $\Lambda \cup \Sigma^{-1}$ are *equivalent* if they can be connected by a sequence of elementary equivalences of types $paa^{-1}q \sim pq$ and $pa^{-1}aq \sim pq$, where p, q are paths (maybe empty), $a \in \Sigma$, and the concatenations are well-defined. Every path p is equivalent to some reduced path obtained by consecutive removal of subpaths of type aa^{-1} and $a^{-1}a$ until there are none left. The resulted path does not depend on the order of removal and we will call it the *reduction* of p. Also, the equivalent paths have the same reduction. (These statements are proved in the same way as in the theory of free groups. See [MKS76, §1.4] for more details.)

We define a *path category* $\Omega[\Lambda_\Sigma]$ of a marked graph Λ_Σ as follows:

$$\operatorname{Ob}\Omega[\Lambda_\Sigma] := \Lambda_0,$$
$$\Omega[\Lambda_\Sigma](X,Y) := \{\text{reduced paths in } \Lambda \cup \Sigma^{-1} \text{ from } X \text{ to } Y\},$$

where the composition is given by the reduced concatenation of paths, and the identities are given by the empty paths. Another way to describe the category $\Omega[\Lambda_\Sigma]$ is as the category of fractions $\Omega[\Lambda][\Sigma^{-1}]$ which is produced from $\Omega[\Lambda]$ by formally inverting the arrows in Σ. (See [Bor94, §5] for more details on the path categories and categories of fractions.)

For the sake of convenience, we will define the *paths in a marked graph* Λ_Σ as the reduced paths in $\Lambda \cup \Sigma^{-1}$, and the concatenation of paths in Λ_Σ as the reduced concatenation.

Remark 2.2. If $\Sigma = \Lambda_1$, that is if all arrows in Λ are marked invertible, then the corresponding category $\Omega[\Lambda_\Sigma]$ coincides with the free groupoid on Λ. Then for any other choice of Σ, the category $\Omega[\Lambda_\Sigma]$ can be thought as a subcategory of this free groupoid generated by all arrows of Λ and by the inverses of the arrows in Σ.

We will denote by $\Bbbk\Omega[\Lambda]$ the linearized version of $\Omega[\Lambda]$. It has the same objects as $\Omega[\Lambda]$, and its morphisms from X to Y are the formal $\Bbbk$-linear combinations of different paths from X to Y in Λ. Similarly, we will denote by $\Bbbk\Omega[\Lambda_\Sigma]$ the linearized version of $\Omega[\Lambda_\Sigma]$.

A *grading* (or $\mathbb{Z}$-*grading*) on a graph Λ is given by assigning to each arrow $a \in \Lambda_1$ an integer value $|a| \in \mathbb{Z}$ called the *degree* of a. Any grading on Λ gives rise to the gradings on the corresponding path categories $\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda]$, in which the degree of a non-empty path $a_n \dots a_1$ is given by the sum of degrees of a_i, and empty paths have zero degree: $|a_n \dots a_1| = \sum_i |a_i|$, $|1_X| = 0$. Similarly, for the marked graphs, we obtain the gradings on $\Omega[\Lambda_\Sigma]$ and $\Bbbk\Omega[\Lambda_\Sigma]$ if we also define the degrees of the inverse arrows as $|a^{-1}| := -|a|$ for $a \in \Sigma$.

A graph Λ is naturally included into the path categories $\Omega[\Lambda]$, $\Bbbk\Omega[\Lambda]$, $\Omega[\Lambda_\Sigma]$ and $\Bbbk\Omega[\Lambda_\Sigma]$ treated as graphs. These inclusions then satisfy the following universal properties:

- any morphism of graphs from Λ to a category $\mathcal{A}$ uniquely extends to a functor from $\Omega[\Lambda]$ to $\mathcal{A}$;

- any morphism of graphs from Λ to a $\Bbbk$-linear category $\mathcal{A}$ uniquely extends to a $\Bbbk$-linear functor from $\Bbbk\Omega[\Lambda]$ to $\mathcal{A}$;

- any morphism of graphs from Λ to a category $\mathcal{A}$, which maps arrows in Σ to invertible morphisms, uniquely extends to a functor from $\Omega[\Lambda_\Sigma]$ to $\mathcal{A}$;

- any morphism of graphs from Λ to a $\Bbbk$-linear category $\mathcal{A}$, which maps arrows in Σ to invertible morphisms, uniquely extends to a $\Bbbk$-linear functor from $\Bbbk\Omega[\Lambda_\Sigma]$ to $\mathcal{A}$.

We also see that if Λ and $\mathcal{A}$ are graded, and the given morphism from Λ to $\mathcal{A}$ preserves the grading, then the corresponding functors preserve the grading as well.

An inclusion of (marked) graphs $\Lambda \subseteq \Lambda'$ gives rise to the inclusions of the corresponding path categories: $\Omega[\Lambda] \subseteq \Omega[\Lambda']$, $\Bbbk\Omega[\Lambda] \subseteq \Bbbk\Omega[\Lambda']$. Similarly, any morphism of (marked) graphs $f\colon \Lambda \to \Lambda'$ induces the morphisms of the corresponding path categories: $\Omega[f]\colon \Omega[\Lambda] \to \Omega[\Lambda']$, $\Bbbk\Omega[f]\colon \Bbbk\Omega[\Lambda] \to \Bbbk\Omega[\Lambda']$.

2.1.1 Differentials on path categories

Let us formulate a few observations about the **dg**-structures on path categories. For that, let us firstly recall a few facts about derivations.

Definition 2.3. A graded linear map ∂ on a graded category $\mathcal{A}$ is called a *derivation* if it satisfies the Leibniz rule for the composition:

$$\partial(xy) = \partial(x)y + (-)^{|\partial||x|}x\partial(y).$$

Thus, a differential is a derivation of degree 1 whose square is zero.

Lemma 2.4. *Let ∂ be a derivation on $\mathcal{A}$ of degree 1.*

(1) For any object X in $\mathcal{A}$, we have: $\partial(1_X) = 0$.

(2) For any sequence $(x_i)_{1\leq i\leq n}$ of composable morphisms in $\mathcal{A}$, we have:

$$\partial(x_1 \dots x_n) = \sum_i (-)^{|x_1|+\dots+|x_{i-1}|}x_1 \dots \partial(x_i) \dots x_n.$$

(3) If a morphism x has a two-sided inverse x^{-1}, then

$$\partial(x^{-1}) = (-)^{|x|+1}\, x^{-1}\, \partial(x)\, x^{-1}.$$

Proof. To prove the first statement, let us note that

$$\partial(1_X) = \partial(1_X \cdot 1_X) = \partial(1_X) \cdot 1_X + 1_X \cdot \partial(1_X) = 2\partial(1_X),$$

hence $\partial(1_X) = 0$. The second statement follows from induction on n, similarly to Lemma 1.13. The third statement follows from

$$0 = \partial(1) = \partial(xx^{-1}) = \partial(x)x^{-1} + (-)^{|x|}x\partial(x^{-1})$$

after we multiply this equation by x^{-1} from the left. □

Now, let us consider a graded marked graph Λ and the corresponding $\Bbbk$-linear path category $\Bbbk\Omega[\Lambda]$.

Lemma 2.5. *Any derivation ∂ of degree 1 on $\Bbbk\Omega[\Lambda]$ can be uniquely determined from its values on the arrows of Λ. Moreover, such derivation exists for any given set of values $\partial(a) \in \Bbbk\Omega[\Lambda](X, Y)$ for the arrows $a\colon X \to Y$ in Λ such that $|\partial(a)| = |a| + 1$.*

Proof. Lemma 2.4 shows that the value of ∂ on any path in Λ can be uniquely determined from the values on its arrows.

To prove the existence, we can just use both parts of Lemma 2.4 in order to define the values of ∂ on arbitrary paths in Λ. In particular, the Leibniz rule for the compositions of arrows will hold by definition. Then it is a trivial check that the Leibniz rule will also hold for the compositions of paths of arbitrary length. □

Lemma 2.6. *Let ∂ be a derivation of degree 1 on $\Bbbk\Omega[\Lambda]$. Then ∂ is a differential if and only if $\partial^2(a) = 0$ for every arrow a in Λ.*

Proof. We need to prove that if the equation $\partial^2 = 0$ holds on the arrows of Λ, then it holds on all paths as well. To check this, let us just use Lemma 2.4. For an empty path 1_X, we have

$\partial^2(1_X) = \partial(0) = 0$. For a non-empty path $x_1 \dots x_n$, we have:

$$\begin{aligned}
\partial^2(x_1 \dots x_n) &= \sum_i (-)^{|x_1|+\ldots+|x_{i-1}|} \partial\big(x_1 \dots \partial(x_i) \dots x_n\big) \\
&= \sum_{j<i} (-)^{|x_j|+\ldots+|x_{i-1}|} x_1 \dots \partial(x_j) \dots \partial(x_i) \dots x_n \\
&\quad + \sum_i x_1 \dots \partial\big(\partial(x_i)\big) \dots x_n \\
&\quad + \sum_{j>i} (-)^{1+|x_i|+\ldots+|x_{j-1}|} x_1 \dots \partial(x_i) \dots \partial(x_j) \dots x_n.
\end{aligned}$$

The second sum is zero, because x_i is an arrow, hence $\partial(\partial(x_i)) = 0$ by assumption. The terms in the first and third sums cancel with each other due to extra 1 in the sign inside the third sum. Thus $\partial^2(x_1 \dots x_n) = 0$. Hence, we conclude the proof. □

Remark 2.7. The previous two lemmas show that in order to define a differential ∂ on a path category $\Bbbk\Omega[\Lambda]$, it is enough to specify its values $\partial(a)$ on all arrows a of Λ such that $|\partial(a)| = |a|+1$ and then to verify that $\partial^2(a) = 0$ holds for all these arrows and for the resulting derivation ∂.

Remark 2.8. Similarly to the previous remark, if we have an inclusion $\Lambda \subseteq \Lambda'$ of the (marked) graphs and the corresponding inclusion of the path categories $\Bbbk\Omega[\Lambda] \subseteq \Bbbk\Omega[\Lambda']$, then in order to extend a differential ∂ from $\Bbbk\Omega[\Lambda]$ to $\Bbbk\Omega[\Lambda']$, we just need to specify its values on the new arrows (by Lemma 2.5) in such a way that the equation $\partial^2 = 0$ holds on these arrows (by Lemma 2.6).

Also, we will use the following lemma.

Lemma 2.9. *Let $\Lambda \subseteq \Lambda'$ be an inclusion of graphs and $\Bbbk\Omega[\Lambda] \subseteq \Bbbk\Omega[\Lambda']$ be the corresponding inclusion of path categories. If ∂ is a differential on $\Bbbk\Omega[\Lambda']$, then the subcategory $\Bbbk\Omega[\Lambda]$ is closed under ∂ if and only if $\partial(a) \in \Bbbk\Omega[\Lambda]$ for all arrows $a \in \Lambda$.*

Proof. Follows from Lemma 2.4 and from the fact that every morphism in $\Bbbk\Omega[\Lambda]$ can be expressed as the linear combination of the empty paths and the compositions of arrows in Λ and their inverses.

□

2.2 Definition of a cellular dg-category

In this section, we formulate our main notion of a cellular **dg**-category and state its basic properties.

Definition 2.10. A *cellular graph* is a graded marked graph such that all of its arrows have non-positive degrees and the invertible arrows have degree zero. In such graph, we assign a positive number to each arrow, called *dimension*, via the formula $\dim a := 1 - |a|$. We say that the vertices of this graph are its 0-*cells*, and the arrows of dimension i are its i-*cells*.

Remark 2.11. By definition, the invertible arrows in a cellular graph are all 1-cells.

Remark 2.12. We can build a cellular graph from a non-graded graph by assigning positive dimensions to its arrows and by marking some of the arrows of dimension 1 as invertible. Then we can define the grading by the inverse formula $|a| := 1 - \dim a$.

As any graded marked graph, a cellular graph Λ gives rise to a graded $\Bbbk$-linear path category $\Bbbk\Omega[\Lambda]$.

Definition 2.13. A **dg**-category $\mathcal{A}$ is called *cellular*, if it is equipped with a cellular subgraph $\Lambda_{\mathcal{A}}^{\mathrm{cell}}$ such that the inclusion $\Lambda_{\mathcal{A}}^{\mathrm{cell}} \subseteq \mathcal{A}$ gives rise to an isomorphism $\Bbbk\Omega[\Lambda_{\mathcal{A}}^{\mathrm{cell}}] \xrightarrow{\sim} \mathcal{A}$ of graded $\Bbbk$-linear categories. In such situation, we call $\Lambda_{\mathcal{A}}^{\mathrm{cell}}$ a *cellular structure* on $\mathcal{A}$, and we will say that the i-cells of $\Lambda_{\mathcal{A}}^{\mathrm{cell}}$ are the i-*cells* of $\mathcal{A}$. We will also say that the inverses of invertible arrows of $\Lambda_{\mathcal{A}}^{\mathrm{cell}}$ are the *inverse 1-cells* of $\mathcal{A}$.

Remark 2.14. Due to the isomorphism $\mathcal{A} \cong \Bbbk\Omega[\Lambda_{\mathcal{A}}^{\mathrm{cell}}]$, all objects of $\mathcal{A}$ are its 0-cells, and every morphism of $\mathcal{A}$ can be represented as the linear combination of reduced paths on cells of $\mathcal{A}$ and their inverses in a unique way.

Remark 2.15. Essentially, a cellular **dg**-category is just a path category of a cellular graph equipped with a differential.

Definition 2.16. We define the k*-skeleton* $\mathrm{sk}_k(\Lambda)$ of a cellular graph Λ to be the subgraph of Λ consisting of cells of dimension less or equal than k. (If $k \geq 1$ and a was an invertible 1-arrow in Λ, we keep it as an invertible arrow in $\mathrm{sk}_k(\Lambda)$ as well.)

We have the tower of skeletons as follows:

$$\Lambda_0 = \mathrm{sk}_0(\Lambda) \subseteq \mathrm{sk}_1(\Lambda) \subseteq \mathrm{sk}_2(\Lambda) \subseteq \cdots \subseteq \Lambda = \bigcup_{k \geq 0} \mathrm{sk}_k(\Lambda).$$

Definition 2.17. We define the k*-skeleton* $\mathrm{sk}_k(\mathcal{A})$ of a cellular **dg**-category $\mathcal{A}$ to be the minimal **dg**-subcategory containing all cells of dimension less or equal than k and their inverses.

Similarly to the previous, we have the tower of skeletons of $\mathcal{A}$ as well:

$$\Bbbk 1_{\mathcal{A}} = \mathrm{sk}_0(\mathcal{A}) \subseteq \mathrm{sk}_1(\mathcal{A}) \subseteq \mathrm{sk}_2(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{A} = \bigcup_{k \geq 0} \mathrm{sk}_k(\mathcal{A}),$$

where $\Bbbk 1_{\mathcal{A}}$ denotes the subcategory of $\mathcal{A}$ consisting of all its objects and the scalar multiples of the identity morphisms.

Remark 2.18. For a cellular **dg**-category $\mathcal{A}$, the path subcategory on the k-skeleton of $\Lambda_{\mathcal{A}}^{\mathrm{cell}}$ is automatically contained in the k-skeleton of $\mathcal{A}$, that is

$$\Bbbk\Omega[\mathrm{sk}_k(\Lambda_{\mathcal{A}}^{\mathrm{cell}})] \subseteq \mathrm{sk}_k(\mathcal{A}).$$

Lemma 2.19 (Properties of cellular **dg**-categories)**.** *Let $\mathcal{A}$ be a cellullar **dg**-category with a cellular structure $\Lambda = \Lambda_{\mathcal{A}}^{\mathrm{cell}}$. Then the following statements hold:*

(1) All morphisms of $\mathcal{A}$ are situated in nonpositive degrees.

(2) If c is a 1-cell, then $\partial(c) = 0$. Moreover, if c is an invertible 1-cell, then $\partial(c^{-1}) = 0$.

(3) Any morphism x in $\mathcal{A}$ of degree d belongs to $\Bbbk\Omega[sk_{1-d}(\Lambda)]$, hence to $sk_{1-d}(\mathcal{A})$.

(4) If $i > 1$ and c is an i-cell, then $\partial(c) \in \Bbbk\Omega[sk_{i-1}(\Lambda)] \subseteq sk_{i-1}(\mathcal{A})$.

(5) *The k-skeleton of $\mathcal{A}$ coincides with the path subcategory on the k-skeleton of Λ, that is $sk_k(\mathcal{A}) = \Bbbk\Omega[sk_k(\Lambda)]$.*

Proof. (1) By definition, Λ is cellular, hence all its arrows have nonpositive degrees. Also, the invertible arrows have zero degree, hence their formal inverses have zero degree as well. Thus, the degree of every path in Λ is also nonpositive, since it is the sum of the degrees of its arrows and their inverses. The paths constitute the basis for the morphisms in $\Bbbk\Omega[\Lambda] \cong \mathcal{A}$, hence the proof.

(2) Due to the formula $|c| = 1 - \dim c$, all 1-cells have degree 0. Hence, $\partial(c)$ should have degree 1. However, (1) implies that there are no nontrivial morphisms of degree 1 in $\mathcal{A}$, hence $\partial(c) = 0$. The statement for invertible c then follows from Lemma 2.4(3).

(3) Let us firstly prove the statement for the cells and their inverses. If c is a cell of degree d, then we have $\dim c = 1-d$, so $c \in \mathrm{sk}_{1-d}(\Lambda) \subseteq \Bbbk\Omega[\mathrm{sk}_{1-d}(\Lambda)]$ by definition. If c is invertible, then $|c| = 0$ and we have $c^{-1} \in \Bbbk\Omega[\mathrm{sk}_1(\Lambda)]$ by definition as well. Now since any morphism x is a linear combination of paths in Λ of degree d, it is enough to prove the statement only for the paths.

For the empty path 1_X, the statement is true, because $d = |1_X| = 0$ and $1_X \in \mathrm{sk}_0(\mathcal{A}) \subseteq \mathrm{sk}_1(\mathcal{A})$. Any non-empty path p however is the composition $x_1 \ldots x_n$ of cells and their inverses, so we have $|p| = \sum_i |x_i|$ where $|x_i| \leq 0$ by (1). Hence, $|p| \leq |x_i|$ for every i. If $|p| = d$, then $d' := |x_i| \geq d$. Since we already proved the statement for cells and their inverses, we have $x_i \in \Bbbk\Omega[\mathrm{sk}_{1-d'}(\Lambda)]$. The latter category is however contained in $\Bbbk\Omega[\mathrm{sk}_{1-d}(\Lambda)]$ since $1 - d' \leq 1 - d$. Thus, p also belongs to $\Bbbk\Omega[\mathrm{sk}_{1-d}(\Lambda)]$, since all x_i do.

(4) Indeed, $|c| = 1 - i$, hence $|\partial(c)| = |c| + 1 = 2 - i$. Now by (3), we have that $\partial(c) \in \Bbbk\Omega[\mathrm{sk}_{i-1}(\Lambda)]$.

(5) Indeed, we already know that $\Bbbk\Omega[\mathrm{sk}_k(\Lambda)] \subseteq \mathrm{sk}_k(\mathcal{A})$, where $\Bbbk\Omega[\mathrm{sk}_k(\Lambda)]$ is the minimal *graded subcategory* of $\mathcal{A}$ containing all cells of dimension $\leq k$, and $\mathrm{sk}_k(\mathcal{A})$ is the mini-

mal ***dg**-subcategory* of $\mathcal{A}$ containing the same cells. Hence, in order to prove that these two subcategories coincide, we just need to show that $\Bbbk\Omega[\mathrm{sk}_k(\Lambda)]$ is closed under the differential. By Lemma 2.9, in order to prove the latter, we just need to check that $\partial(c) \in \Bbbk\Omega[\mathrm{sk}_k(\Lambda)]$ for every cell c in $\mathrm{sk}_k(\Lambda)$ of positive dimension. Let us do this. If $\dim c = 1$, then by (2), we have $\partial(c) = 0 \in \Bbbk\Omega[\mathrm{sk}_k(\Lambda)]$. If $\dim c > 1$, then by (4), we have $\partial(c) \in \Bbbk\Omega[\mathrm{sk}_{i-1}(\Lambda)] \subseteq \Bbbk\Omega[\mathrm{sk}_k(\Lambda)]$, where $i = \dim c \leq k$ by definition. Hence the proof.

□

2.3 Examples

It follows from Remarks 2.12, 2.15 and 2.7, that we can construct a cellular **dg**-category by performing the following steps:

1) specify a graph Λ;

2) assign positive dimensions to the arrows of Λ and define the corresponding grading by setting $|a| := 1 - \dim a$;

3) mark some arrows of dimension 1 as invertible;

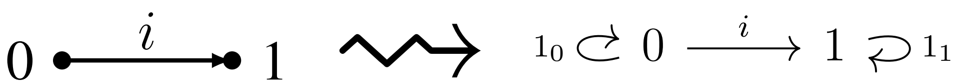
4) consider Λ with the above data as a marked graded graph and form the corresponding path category $\Bbbk\Omega[\Lambda]$;

5) define a derivation ∂ of degree 1 on $\Bbbk\Omega[\Lambda]$ by choosing the values $\partial(a)$ on all arrows a;

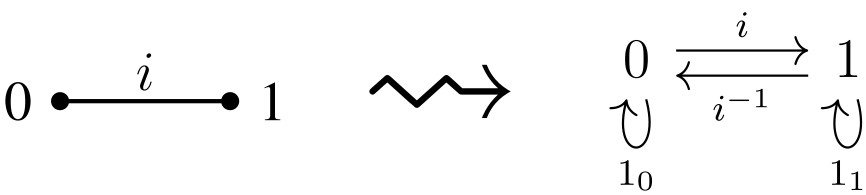
6) verify that ∂ is a differential by checking that $\partial^2(a) = 0$ holds on all arrows a.

Such procedure produces a cellular **dg**-category $(\Bbbk\Omega[\Lambda], \partial)$ whose cellular structure is Λ. In this section, we follow this procedure and construct several examples of cellular **dg**-categories mimicking the well-known topological and combinatorial objects.

Example 2.20 (Directed interval)**.** Let us consider a graph $I^{\rightarrow}$ having two vertices denoted 0 and 1, and one arrow $i\colon 0 \to 1$ between them of dimension 1, that is of degree $|i| = 0$. The only paths in this graph are 1_0, 1_1 and i all having degree 0. These paths consitute the basis for the morphisms in the category $\Bbbk\Omega[I^{\rightarrow}]$. The differential ∂ on this category is necessarily trivial due to Lemma 2.19(2). This situation can be depicted as follows:



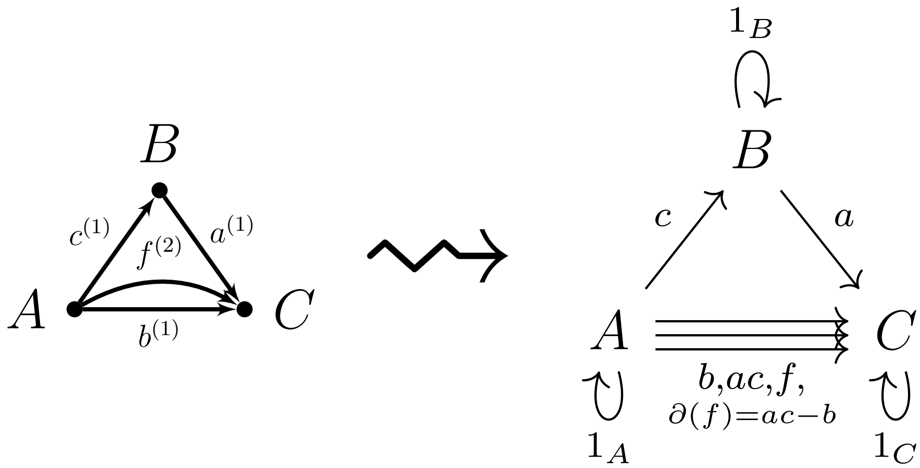
Example 2.21 (Undirected interval)**.** Similarly to the previous example, let us consider a graph I which is formed by two vertices 0 and 1, and one *invertible* arrow $i\colon 0 \to 1$ of dimension 1. The paths in I (which are reduced paths on symbols i and i^{-1}) will be 1_0, 1_1, i and i^{-1} of degree 0. The differential in $\Bbbk\Omega[I]$ will be again trivial. This situation can be depicted as follows:



Example 2.22 (Directed triangle)**.** Consider a filled triangle $\triangle ABC$ with vertices A, B, C. We can build a cellular **dg**-category corresponding to it as follows. The graph will consist of: three vertices A, B and C; three arrows $c\colon A \to B$, $a\colon B \to C$ and $b\colon A \to C$ of dimension 1, corresponding to the edges of the triangle; and one arrow $f\colon A \to C$ of dimension 2 corresponding to the interior of the triangle. The only paths in this graph will be 1_A, 1_B, 1_C, a, b, c and ac, all having degree 0, and f of degree -1.

The differential ∂ on 1-cells a, b and c should be trivial by Lemma 2.19(2). This implies that $\partial(ac) = 0$ as well. Thinking of f as the homotopy connecting two paths b and ac from A to C, we define the differential on f by setting $\partial(f) := ac - b$. The condition $\partial^2(f) = 0$ holds since

$\partial(ac - b) = 0$.



Example 2.23 (Loop)**.** We can also start with a graph with a single vertex $*$ and a single arrow $\gamma\colon * \to *$ of dimension 1, hence of degree 0. The corresponding $\Bbbk$-linear path category will have a single object and its space of morphisms will be either $\Bbbk[\gamma]$, if we do not invert γ, or $\Bbbk[\gamma, \gamma^{-1}]$, if we invert γ. The differential will be necessarily trivial.

Example 2.24 (n-sphere)**.** For $n \geq 2$, we consider a graph with a single vertex $*$ and a single arrow $a\colon * \to *$, similarly to the loop example. However, in this case, we assign dimension n to our arrow, hence it will have degree $1 - n < 0$, so cannot be inverted. The morphisms in the corresponding path category will be given by a graded ring $\Bbbk[a]$, and we define the differential ∂ by setting $\partial(a) := 0$, so it will be again trivial.

Example 2.25 (n-simplex)**.** For any $n \geq 0$, we can define a **dg**-categorical version of an n-simplex as follows. Let us denote the vertices of our simplex by the integers $0, \ldots, n$. Also, let us denote by $[i_0, \ldots, i_k]$ the k-dimensional face of the simplex on the vertices $i_0 < \ldots < i_k$. We define the corresponding cellular graph Λ as follows. It will have $n+1$ vertices labeled by the integers $0, \ldots,$ n corresponding to the vertices of the simplex. The k-faces of our simplex, where $k \geq 1$, will give rise to the arrows of Λ as follows. For each k-face $[i_0, \ldots, i_k]$ we define the corresponding arrow $e_{i_0 \ldots i_k}$ of dimension k from i_0 to i_k. So, the degrees of arrows are $|e_{i_0 \ldots i_k}| = 1 - k$. Then, we define the differential on $\Bbbk\Omega[\Lambda]$ by its values on arrows as follows:

$$\partial(e_{i_0\dots i_k}) := \sum_{0<j<k} (-)^{k-j} \left(e_{i_0\dots\hat{i}_j\dots i_k} - e_{i_j i_{j+1}\dots i_k} \cdot e_{i_0 i_1\dots i_j} \right). \tag{2.1}$$

For example, for 1-, 2- and 3-cells, we get respectively:

$$\begin{aligned}
\partial(e_{01}) &= 0, \\
\partial(e_{012}) &= -(e_{02} - e_{12}e_{01}) \quad = e_{12}e_{01} - e_{02}, \\
\partial(e_{0123}) &= -(e_{023} - e_{123}e_{01}) + (e_{013} - e_{23}e_{012}) \\
&= e_{123}e_{01} + e_{013} - e_{023} - e_{23}e_{012}.
\end{aligned}$$

Then we can verify that $\partial^2 = 0$ holds on arrows by using the above definition and the Leibniz rule as follows:

$$\begin{aligned}
\partial^2(e_{i_0\dots i_k}) &= \sum_{0<j<k} (-)^{k-j} \Big(\partial(e_{i_0\dots\hat{i}_j\dots i_k}) - \partial(e_{i_j\dots i_k}) e_{i_0\dots i_j} \\
&\qquad\qquad - (-)^{1-(k-j)} e_{i_j\dots i_k} \partial(e_{i_0\dots i_j}) \Big) \\
&= \Bigg[\sum_{0<l<j<k} (-)^{(k-j)+(k-1-l)} \big(\underline{e_{i_0\dots\hat{i}_l\dots\hat{i}_j\dots i_k}} - \underline{\underline{e_{i_l\dots\hat{i}_j\dots i_k} e_{i_0\dots i_l}}} \big) \\
&\qquad + \sum_{0<j<l<k} (-)^{(k-j)+(k-l)} \big(\underline{e_{i_0\dots\hat{i}_j\dots\hat{i}_l\dots i_k}} - \uwave{e_{i_l\dots i_k} e_{i_0\dots\hat{i}_j\dots i_l}} \big) \Bigg] \\
&\quad + \sum_{0<j<l<k} (-)^{(k-j+1)+(k-l)} \big(\underline{\underline{e_{i_j\dots\hat{i}_l\dots i_k}}} - \dashuline{e_{i_l\dots i_k} e_{i_j\dots i_l}} \big) e_{i_0\dots i_j} \\
&\quad + \sum_{0<l<j<k} (-)^{j-l} e_{i_j\dots i_k} \big(\uwave{e_{i_0\dots\hat{i}_l\dots i_j}} - \dashuline{e_{i_l\dots i_j} e_{i_0\dots i_l}} \big) \quad = \quad 0\,,
\end{aligned}$$

where the cancelling terms are underscored respectively. This cancellation works, because each term appears twice, once with the sign $(-)^{j+l}$ and another time with the sign $(-)^{j+l+1}$.

Remark 2.26. The previous examples of directed/undirected interval and directed triangle are particular cases of an example of n-simplexes for $n = 1, 2$, where in the undirected case we also invert all 1-arrows.

Example 2.27 (Simplicial complex)**.** Using the previous example, we can build a cellular **dg**-category associated to arbitrary simplicial complex K given by its set of *vertices* $V(K)$ and a set $S(K)$ of *simplices*, which are finite non-empty subsets of $V(K)$ such that every subset of a simplex is a simplex and every vertex $v \in V(K)$ forms a simplex $\{v\} \in S(K)$. In order to build an associated cellular **dg**-category, we firstly order the vertices $V(K)$ in arbitrary way and then proceed in the same way as in Example 2.25. Namely, we define a cellular graph Λ which has the same vertices as K and whose arrows correspond to the simplices of positive dimension in K. Then we equip $\Bbbk\Omega[\Lambda]$ with the differential defined by (2.1) and obtain the desired cellular **dg**-category. (Alternatively, we can consider K as a subcomplex of a full simplex on its vertices $V(K)$ and just take the corresponding **dg**-subcategory of the category constructed in Example 2.25.)

Remark 2.28. In the previous example, there is an ambiguity in choosing the order on the vertices of a simplicial complex. However, if we invert all 1-vertices in the associated cellular graph Λ, then it is easy to check that the resulting **dg**-category won't depend on the choice of the ordering, hence it defines an *invariant of a simplicial complex*.

Namely, we can notice that any reordering of vertices is a composition of elementary reorderings of types $i \leftrightarrow i+1$. Then, we can check that these elementary reorderings indeed induce isomorphisms of the corresponding **dg**-categories, even though the cellular structures themselves might be different. For example, a 1-arrow $e_{i,i+1}$ in the original category will correspond to the inverse arrow $(e'_{i+1,i})^{-1}$ in the category for which the vertices i and $i+1$ are ordered in the opposite way. Similarly, a 2-simplex $e_{i,i+1,j}$, where $j > i+1$, will correspond to a morphism $-e'_{i+1,i,j} \cdot (e'_{i+1,i})^{-1}$ in the reordered category, and similar correspondences can be written for the higher simplices as well.

2.4 Cell attachment

A useful feature of CW-complexes is that they can be constructed inductively via a consecutive cell attachment starting from an empty CW-complex. Here, we define different kinds of elementary extensions of cellular **dg**-categories analogous to cell attachments of CW-complexes, and analyze how the morphisms in these categories change under such kinds of extensions. We also prove that every cellular **dg**-category can be inductively constructed via elementary extensions starting from the empty category, similarly to the analogous result for CW-complexes. At the end, we conjecture that the inductive construction procedure for arbitrary CW-complex can be unambiguously transferred into cellular **dg**-setting, thus producing a cellular **dg**-category which is an invariant of the original CW-complex.

Definition 2.29. For a given cellular graph Λ, we define the following types of *elementary extensions* of Λ:

1) **(adding a vertex)** For some new symbol X, we define $\Lambda \sqcup X$ as a cellular graph obtained from Λ by adding an extra-vertex labelled by X.

2) **(adding an arrow)** Given $k \geq 1$, two vertices X and Y of Λ, and a new symbol a denoting an arrow $a\colon X \to Y$ of dimension k, we define $\Lambda \cup a$ as a cellular graph obtained from Λ by adding an extra-arrow labelled by a which has the given source, target and dimension, and which is *not* marked as invertible.

3) **(adding an invertible 1-arrow)** Given two vertices X and Y of Λ, and a new symbol a denoting an arrow $a\colon X \to Y$, we define $\Lambda \cup a^{\pm}$ as a cellular graph obtained from Λ by adding an extra-arrow labelled by a of dimension 1 which has the given source and target and is marked as invertible.

4) **(inverting an existing 1-arrow)** Given a non-invertible 1-arrow a of Λ, we define $\Lambda \cup a^{-1}$ as a cellular graph obtained from Λ by marking a as invertible.

Remark 2.30. The third operation can be thought as the composition of the second and fourth operations.

The following lemma is obvious.

Lemma 2.31. *Any cellular graph can be obtained from the empty graph $\varnothing$ via an inductive sequence of elementary extensions of types 1), 2) and 3), or of types 1), 2) and 4). Moreover, if a cellular graph does not have invertible arrows, then only extensions of types 1) and 2) are necessary.*

Now, let us describe how the elementary extensions of graphs affect the sets of paths.

Lemma 2.32. *Let Λ' be one of the elementary extensions of Λ. Then the paths in Λ' are related to the paths in Λ as follows:*

(1) If $\Lambda' = \Lambda \sqcup X$, then the paths in Λ' are just the paths in Λ plus an empty path on X.

(2) If $\Lambda' = \Lambda \cup a$, then every path in Λ' has a unique representation of the form

$$p_0\, a\, p_1\, a\, \ldots\, a\, p_n\,,$$

where $n \geq 0$ and p_i are paths in Λ (possibly empty).

(3) If $\Lambda' = \Lambda \cup a^{\pm}$, then every path in Λ' has a unique representation of the form

$$p_0\, a^{\varepsilon_1}\, p_1\, a^{\varepsilon_2}\, \ldots\, a^{\varepsilon_n}\, p_n\,,$$

where $n \geq 0$, p_i are paths in Λ, and $\varepsilon_i = \pm 1$, such that if $\varepsilon_{i+1} = -\varepsilon_i$ then the intermediary path p_i is non-empty.

(4) If $\Lambda' = \Lambda \cup a^{-1}$ for a in Λ, then every path in Λ' has a unique representation of the form

$$p_0\, a^{-1}\, p_1\, a^{-1}\, \ldots\, a^{-1}\, p_n\,,$$

where $n \geq 0$ and p_i are paths in Λ, such that p_i does not start with a for $i < n$ and does not end with a for $i > 0$. (Remember that paths go from right to left.)

Proof. Follows from the definitions of paths and reduced paths in graphs. □

By Remark 2.8, if we have an elementary extension $\Lambda \subset \Lambda'$ of cellular graphs, then in order to extend a differential from $\Bbbk\Omega[\Lambda]$ to $\Bbbk\Omega[\Lambda']$, we just need to specify its value on a new positive cell and check that $\partial^2 = 0$ still holds on this cell, if we are adding any new cell at all. This allows us to define the elementary extensions of cellular **dg**-categories as follows.

Definition 2.33. For a given cellular **dg**-category $\mathcal{A}$ with a cellular structure Λ, we define the *elementary extensions* of $\mathcal{A}$ by extending the graph Λ and extending the values of the differential from $\mathcal{A}$ to the new arrows as follows:

1) **(attaching a 0-cell/adding an object)** For a new object X, we define a cellular **dg**-category $\mathcal{A} \sqcup X$ via its cellular graph $\Lambda \sqcup X$.

2) **(attaching a positive cell)** Given $k \geq 1$, two objects X and Y of $\mathcal{A}$, a new symbol c denoting a cell $c\colon X \to Y$ of dimension k, and a chosen morphism $b \in \mathcal{A}^{2-k}(X, Y)$ of degree $2 - k$ such that $\partial(b) = 0$, we define a cellular **dg**-category $\mathcal{A} \cup_b c$ by its cellular graph $\Lambda \cup c$ and by setting the value of a differential to $\partial(c) = b$. We will also call b the *boundary condition* for c.

   If $k = 1$, then we always choose $b = 0$ and write $\mathcal{A} \cup c$ instead of $\mathcal{A} \cup_0 c$.

3) **(attaching an invertible 1-cell)** Given two objects X and Y of $\mathcal{A}$ and a new symbol c denoting a cell $c\colon X \to Y$ of dimension 1, we define a cellular **dg**-category $\mathcal{A} \cup c^{\pm}$ by its cellular graph $\Lambda \cup c^{\pm}$ and by setting $\partial(c) = 0$.

4) **(inverting an existing 1-cell)** For a 1-cell c in $\mathcal{A}$ not marked as invertible, we define a cellular **dg**-category $\mathcal{A} \cup c^{-1}$ by its cellular graph $\Lambda \cup c^{-1}$.

Similarly to Lemma 2.31, we have the following statement for cellular **dg**-categories.

Proposition 2.34. *Every cellular* **dg***-category $\mathcal{A}$ can be obtained from an empty* **dg***-category $\varnothing$ via an inductive sequence of elementary extensions of types 1), 2) and 3), or of types 1), 2) and 4). Moreover, if $\mathcal{A}$ does not have invertible 1-cells, then only extensions of types 1) and 2) are necessary.*

Proof. Let Λ denotes a cellular structure of $\mathcal{A}$. Then by Lemma 2.31 we can obtain Λ from an empty graph by a sequence of elementary extensions of the required types:

$$\varnothing = \Lambda^{(0)} \subset \Lambda^{(1)} \subset \cdots \subset \Lambda^{(n)} = \Lambda.$$

Moreover, we can choose this sequence in such a way, that we firstly add all 0-cells/vertices of Λ, then we add all 1-cells (and their inverses), then all 2-cells etc.

Now, let us prove that the path subcategories $\Bbbk\Omega[\Lambda^{(i)}] \subseteq \mathcal{A}$ are all closed under the differential ∂ of $\mathcal{A}$ for all i, hence they are cellular **dg**-categories themselves with the cellular structures $\Lambda^{(i)}$. We will use induction on i. The statement is obvious for the empty category. Now, let us assume that it holds for $\Bbbk\Omega[\Lambda^{(i)}]$ and consider the elementary extension $\Lambda^{(i)} \subset \Lambda^{(i+1)}$. By Lemma 2.9, in order to prove that $\Bbbk\Omega[\Lambda^{(i+1)}]$ is closed under ∂, we just need to check that $\partial(c) \in \Bbbk\Omega[\Lambda^{(i+1)}]$ for every arrow in $\Lambda^{(i+1)}$. However, this property already holds for every arrow in $\Lambda^{(i)}$ by assumption, hence we need to check it only for the new arrow, if we add any on this step. Let c be this arrow. If c has dimension 1, then we automatically have $\partial(c) = 0$ by Lemma 2.19(2), so the statement is true. If c has dimension $k > 1$, then its differential $\partial(c)$ belongs to the $(k-1)$-skeleton $\Bbbk\Omega[\mathrm{sk}_{k-1}(\Lambda)]$ of $\mathcal{A}$ by Lemma 2.19(4). However, by construction, since we are adding a cell of dimension k at this step, all cells of lesser dimension were already added to the graph before, so the $(k-1)$-skeleton is contained in $\Bbbk\Omega[\Lambda^{(i)}]$, hence $\partial(c) \in \Bbbk\Omega[\Lambda^{(i)}]$. Thus, the statement is true in this case as well. This concludes the induction.

Finally, let us note that each inclusion $\Bbbk\Omega[\Lambda^{(i)}] \subset \Bbbk\Omega[\Lambda^{(i+1)}]$ is an elementary extension of cellular **dg**-categories. Indeed, if an extension $\Lambda^{(i)} \subset \Lambda^{(i+1)}$ has type 1), 3), 4), or 2) with $k = 1$, then the inclusion $\Bbbk\Omega[\Lambda^{(i+1)}]$ can be expressed as an elementary extension of $\Bbbk\Omega[\Lambda^{(i)}]$ in the same way as $\Lambda^{(i+1)}$ can be expressed via $\Lambda^{(i)}$, since in all of these cases the differential extends from $\Bbbk\Omega[\Lambda^{(i)}]$ to $\Bbbk\Omega[\Lambda^{(i+1)}]$ in a unique way. For example, if $\Lambda^{(i+1)} = \Lambda^{(i)} \sqcup X$, then $\Bbbk\Omega[\Lambda^{(i+1)}] = \Bbbk\Omega[\Lambda^{(i)}] \sqcup X$; and, if $\Lambda^{(i+1)} = \Lambda^{(i)} \cup *$, where $* \in \{c, c^{\pm}, c^{-1}\}$, then $\Bbbk\Omega[\Lambda^{(i+1)}] = \Bbbk\Omega[\Lambda^{(i)}] \cup *$. However, if an extension $\Lambda^{(i)} \subset \Lambda^{(i+1)}$ has type 2) with $k > 1$, then we also need to note that $\partial(c)$ belongs to $\Bbbk\Omega[\Lambda^{(i)}]$ by the previous argument, so we can express $\Bbbk\Omega[\Lambda^{(i+1)}]$ as $\Bbbk\Omega[\Lambda^{(i)}] \cup_{\partial(c)} c$. $\square$

Now, let us express how the morphisms in the extended cellular **dg**-categories are related to the original ones by using Lemma 2.32. For that, let us introduce a few auxiliary notations as follows:

$$
\begin{aligned}
\Omega^*[\Lambda](X,Y) &:= \{\text{non-empty paths in } \Lambda \text{ from } X \text{ to } Y\},\\
\Omega_{\breve{a}..}[\Lambda](X,Y) &:= \{\text{paths in } \Lambda \text{ from } X \text{ to } Y \text{ whose first letter is } \textit{not } a\},\\
\Omega_{..\breve{a}}[\Lambda](X,Y) &:= \{\text{paths in } \Lambda \text{ from } X \text{ to } Y \text{ whose last letter is } \textit{not } a\},\\
\Omega_{\breve{a}..\breve{a}}[\Lambda](X,Y) &:= \{\text{paths whose first and last letters are not } a\}.
\end{aligned}
$$

Remark 2.35. In most cases, the above sets contain all paths from X to Y. For example, we have:

- $\Omega^*[\Lambda](X,Y) = \Omega[\Lambda](X,Y)$ if $X \neq Y$,
- $\Omega_{\breve{a}..}[\Lambda](X,Y) = \Omega[\Lambda](X,Y)$ if $Y \neq T(a)$;
- $\Omega_{..\breve{a}}[\Lambda](X,Y) = \Omega[\Lambda](X,Y)$ if $X \neq S(a)$;
- $\Omega_{\breve{a}..\breve{a}}[\Lambda](X,Y) = \Omega[\Lambda](X,Y)$ if $Y \neq T(a)$ and $X \neq S(a)$.

Using the above notations, we can reformulate Lemma 2.32 for cellular **dg**-categories as follows.

Lemma 2.36. *Let $\mathcal{A} \subset \mathcal{A}'$ be an elementary extension of* **dg***-categories with cellular structures Λ and Λ' respectively. Then the categories are related as follows:*

(1) If $\mathcal{A}' = \mathcal{A} \sqcup X$, then we have:

$$\begin{aligned}
\operatorname{Ob}\mathcal{A}' &= \operatorname{Ob}\mathcal{A} \sqcup \{X\},\\
\mathcal{A}'(X,X) &= \Bbbk 1_X,\\
\mathcal{A}'(X,Y) &= 0 \ \textit{for} \ Y \in \operatorname{Ob}\mathcal{A},\\
\mathcal{A}'(Y,X) &= 0 \ \textit{for} \ Y \in \operatorname{Ob}\mathcal{A},\\
\mathcal{A}'(Y,Z) &= \mathcal{A}(Y,Z) \ \textit{for} \ Y, Z \in \operatorname{Ob}\mathcal{A}.
\end{aligned}$$

(2) If $\mathcal{A}' = \mathcal{A} \cup_b c$, then $\operatorname{Ob}\mathcal{A}' = \operatorname{Ob}\mathcal{A}$ and the morphisms in $\mathcal{A}'$ are as follows:

$$\begin{aligned}
\mathcal{A}'(X,Y) \;=\; & \mathcal{A}(X,Y) \oplus \\
& \bigoplus_{n\geq 1} \quad \mathcal{A}(T(c),Y) \otimes \Bbbk c \otimes \mathcal{A}(T(c),S(c)) \\
& \qquad\qquad \otimes \Bbbk c \otimes \ldots \otimes \Bbbk c \otimes \\
& \qquad\qquad\qquad \mathcal{A}(T(c),S(c)) \otimes \Bbbk c \otimes \mathcal{A}(X,S(c)),
\end{aligned}$$

where the term $\Bbbk c$ appears n times inside the direct sum.

(3) If $\mathcal{A}' = \mathcal{A} \cup c^{\pm}$, then $\operatorname{Ob}\mathcal{A}' = \operatorname{Ob}\mathcal{A}$ and the morphisms in $\mathcal{A}'$ are as follows:

$$\begin{aligned}
\mathcal{A}'(X,Y) \;=\; & \mathcal{A}(X,Y) \oplus \\
& \bigoplus_{\substack{n\geq 1\\ \varepsilon_1,\ldots,\varepsilon_n = \pm 1}} \mathcal{A}(T(c^{\varepsilon_1}),Y) \otimes \Bbbk c^{\varepsilon_1} \otimes \bar{\mathcal{A}}(T(c^{\varepsilon_2}),S(c^{\varepsilon_1})) \\
& \qquad\qquad \otimes \Bbbk c^{\varepsilon_2} \otimes \cdots \otimes \Bbbk c^{\varepsilon_{n-1}} \otimes \\
& \qquad\qquad\qquad \bar{\mathcal{A}}(T(c^{\varepsilon_n}),S(c^{\varepsilon_{n-1}})) \otimes \Bbbk c^{\varepsilon_n} \otimes \mathcal{A}(X,S(c^{\varepsilon_n})),
\end{aligned}$$

where

$$\bar{\mathcal{A}}(T(c^{\varepsilon_{i+1}}),S(c^{\varepsilon_i})) = \begin{cases} \mathcal{A}(T(c^{\varepsilon_{i+1}}),S(c^{\varepsilon_i})) & \textit{if } \varepsilon_{i+1} = \varepsilon_i, \\ \Bbbk\Omega^*[\Lambda](T(c^{\varepsilon_{i+1}}),S(c^{\varepsilon_i})) & \textit{if } \varepsilon_{i+1} = -\varepsilon_i. \end{cases}$$

(4) If $\mathcal{A}' = \mathcal{A} \cup c^{-1}$, then $\operatorname{Ob}\mathcal{A}' = \operatorname{Ob}\mathcal{A}$ and the morphisms in $\mathcal{A}'$ are as follows:

$$\begin{aligned}\mathcal{A}'(X,Y) \;=\; & \mathcal{A}(X,Y) \oplus \\ & \bigoplus_{n\geq 1} \quad \Bbbk\Omega_{..\check{c}}[\Lambda](S(c),Y) \otimes \Bbbk c^{-1} \otimes \Bbbk\Omega_{\check{c}..\check{c}}[\Lambda](S(c),T(c)) \\ & \qquad\qquad \otimes \Bbbk c^{-1} \otimes \cdots \otimes \Bbbk c^{-1} \otimes \\ & \qquad\qquad\qquad \Bbbk\Omega_{\check{c}..\check{c}}[\Lambda](S(c),T(c)) \otimes \Bbbk c^{-1} \otimes \Bbbk\Omega_{\check{c}..}[\Lambda](X,T(c)),\end{aligned}$$

where the term $\Bbbk c^{-1}$ appears n times inside the direct sum.

2.4.1 Invariant of CW-complexes

We know that every CW-complex can be inductively constructed by applying consecutive cell attachment. Namely, in order to attach a new n-cell to a CW-complex X, we need to define an attachment map $f : S^{n-1} \to X$ from the boundary of the n-dimensional ball, which is an $(n-1)$-dimensional sphere, into X. Up to homotopy equivalence, such map can be thought of as a composite sphere inside X patched from its $(n-1)$-cells. Our idea here is that if X is modelled by some cellular **dg**-category $\mathcal{A}$, then such patched sphere in X corresponds to a well-defined element in $\mathcal{A}$ which will define the boundary conditions for the corresponding elementary extension of $\mathcal{A}$. This gives rise to the following conjecture.

Conjecture 2.37. *(1) Any CW-complex X gives rise to a cellular **dg**-category $\mathcal{A}$ which has the same set of cells as X, where the values of the differential on the cells of $\mathcal{A}$ correspond to the attachment maps of cells of X, and in which all 1-arrows are invertible. Moreover, such category $\mathcal{A}$ is defined uniquely up to a **dg**-isomorphism, though the choices made in the reconstruction process might give rise to different cellular structures in $\mathcal{A}$.*

*(2) The above correspondence extends to a functor from CW-complexes to **dg**-categories, where cellular maps of CW-complexes give rise to the **dg**-functors of the corresponding **dg**-categories and the homotopy equivalences give rise the isomorphic categories.*

Remark 2.38. Let us consider a CW-complex X and a suggested associated cellular **dg**-category $\mathcal{A}$ which is an invariant of X. The category $\mathcal{A}$ in fact captures many nice homotopical properties of X. For example, the connected components of X are in one-to-one correspondence with the classes of isomorphic objects in $\mathcal{A}$, thus $\mathcal{A}$ fully captures $\pi_0(X)$. We can also observe that $H^0(\mathcal{A}(x,x)) = \Bbbk[\pi_1(X,x)]$ for any point/0-cell x in X.

2.5 Cellular product

In this section, we define a notion of a cellular product of cellular **dg**-categories mimicking the construction of the product of CW-complexes. We also use this notion in order to explicitly construct **dg**-categorical counterparts of n-dimensional cubes by representing them as the powers of the interval category from Examples 2.20 and 2.21.

2.5.1 Definition

Before defining the cellular product for **dg**-categories, let us define an analogous notion for the graphs as follows.

Definition 2.39. Let Λ and Λ' be two cellular graphs. We define their *cellular product* $\Lambda \circledast \Lambda'$ as a cellular graph having one $(i+j)$-cell $a * b$ for each pair of an i-cell a in Λ and a j-cell b in Λ'. More precisely, its vertices are

$$(\Lambda \circledast \Lambda')_0 := \Lambda_0 \times \Lambda'_0,$$

and its arrows are

$$(\Lambda \circledast \Lambda')_1 := (\Lambda_0 \times \Lambda'_1) \sqcup (\Lambda_1 \times \Lambda'_0) \sqcup (\Lambda_1 \times \Lambda'_1).$$

The source and target maps are defined as follows:

$$
\begin{aligned}
S(X * b) &:= X * S(b), & T(X * b) &:= X * T(b), \\
S(a * Y) &:= S(a) * Y, & T(a * Y) &:= T(a) * Y, \\
S(a * b) &:= S(a) * S(b), & T(a * b) &:= T(a) * T(b),
\end{aligned}
$$

for any vertices X and Y and arrows a and b in Λ and Λ' respectively, and the dimensions are:

$$\dim(X * b) = \dim b, \quad \dim(a * Y) = \dim a, \quad \dim(a * b) = \dim a + \dim b.$$

If a is an invertible arrow, then we invert $a * Y$ for all vertices Y in Λ'. Similarly, if b is invertible, then we invert $X * b$ for all vertices X in Λ. (We do not invert any arrows of type $a * b$, where a and b are arrows, as they have dimension ≥ 2.)

Remark 2.40. For the vertices X and Y and arrows a and b in Λ and Λ' respectively, it follows from the formulas for dimensions that the degrees of the corresponding cellular products are as follows:

$$|X * b| = |b|, \qquad |a * Y| = |a|, \qquad |a * b| = |a| + |b| - 1.$$

Example 2.41. For example, if $\Lambda = \Lambda' = I^{\rightarrow}$ is the interval graph from Example 2.21, then $\Lambda \boxast \Lambda'$ will be as follows:

$$
\left(0 \xrightarrow{i} 1\right) \boxast \left(0 \xrightarrow{i} 1\right) \quad = \quad
\begin{tikzcd}
0 * 1 \arrow[r, "i * 1"] & 1 * 1 \\
0 * 0 \arrow[u, "0 * i"] \arrow[ur, "i * i"] \arrow[r, "i * 0"'] & 1 * 0 \arrow[u, "1 * i"']
\end{tikzcd}
\quad ,
$$

where the arrows $0 * i$, $i * 1$, $i * 0$ and $1 * i$ have dimension 1, and the arrow $i * i$ has dimension 2.

In order to define a cellular product for cellular **dg**-categories we just need to take the product of their cellular graphs and then to set up the differential in the right way. Before doing that, let us introduce the following auxiliary operation on the objects and morphisms of two path categories $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$.

Definition 2.42. For two cellular graphs Λ and Λ', let us consider $X * b$, $a * Y$ and $a * b$ as the operations whose inputs are arrows a and b and whose outputs are arrows in $\Lambda \circledast \Lambda'$ (for the specified vertices X and Y in Λ and Λ' respectively). Let us define extensions of these operations to graded $\Bbbk$-linear operations $X * (-)$ and $(-) * Y$ of degree 0, and to a graded $\Bbbk$-bilinear operation $(-) * (-)$ of degree -1, whose inputs are arbitrary morphisms in the respective path categories $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ and whose outputs are morphisms in $\Bbbk\Omega[\Lambda \circledast \Lambda']$ as follows.

1) For a vertex X of Λ, we define the operation $X * (-)$ on paths in Λ' as follows:

$$\begin{aligned} X * 1_Y &:= 1_{X*Y}, \\ X * b^{-1} &:= (X * b)^{-1}, \\ X * (y_1 \dots y_n) &:= (X * y_1) \cdot \ldots \cdot (X * y_n), \end{aligned}$$

where Y is a vertex in Λ', b is an invertible arrow in Λ', and y_i are arrows or their inverses in Λ'. Then we extend to all morphisms of $\Bbbk\Omega[\Lambda']$ via $\Bbbk$-linearity.

2) For a vertex Y of Λ', we define the operation $(-) * Y$ on paths in Λ as follows:

$$\begin{aligned} 1_X * Y &:= 1_{X*Y}, \\ a^{-1} * Y &:= (a * Y)^{-1}, \\ (x_1 \dots x_n) * Y &:= (x_1 * Y) \cdot \ldots \cdot (x_n * Y), \end{aligned}$$

where X is a vertex in Λ, a is an invertible arrow in Λ, and x_i are arrows or their inverses in Λ. Then we extend to all morphisms of $\Bbbk\Omega[\Lambda]$ via $\Bbbk$-linearity.

3) For an arrow b in Λ', we define the operation $(-) * b$ on paths in Λ as follows:

$$1_X * b \;:=\; 0\colon X * S(b) \to X * T(b),$$
$$a^{-1} * b \;:=\; -\big(a * T(b)\big)^{-1}(a * b)\big(a * S(b)\big)^{-1},$$
$$(x_1 \dots x_n) * b \;:=\; \sum_{1\le i\le n} (-)^{(\sum_{j>i}|x_j|)\cdot(|b|+1)} \Big(\prod_{j<i}(x_j * T(b))\Big)(x_i * b)\Big(\prod_{j>i}(x_j * S(b))\Big),$$

where X is a vertex in Λ, a is an invertible arrow in Λ, and x_i are arrows or their inverses in Λ.

4) Finally, we define the operation $(-) * (-)$ on all paths in Λ and Λ' as follows:

$$p * 1_Y \;:=\; 0\colon S(p) * Y \to T(p) * Y,$$
$$p * b^{-1} \;:=\; -\big(T(p) * b)\big)^{-1}(p * b)\big(S(p) * b\big)^{-1},$$
$$p * (y_1 \dots y_n) \;:=\; \sum_{1\le i\le n} (-)^{(|p|+1)\cdot(\sum_{j<i}|y_j|)} \Big(\prod_{j<i}(T(p) * y_j)\Big)(p * y_i)\Big(\prod_{j>i}(S(p) * y_j)\Big),$$

where p is a path in Λ, Y is a vertex in Λ', b is an invertible arrow in Λ', y_i are arrows or their inverses in Λ', and $p * b$ was defined in the previous step. Then we extend to all morphisms of $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ via $\Bbbk$-linearity.

Remark 2.43. Equations used to define $(x_1 \dots x_n) * b$ and $p * (y_1 \dots y_n)$ hold if we replace x_i and y_i by arbitrary paths in Λ and Λ' respectively.

Remark 2.44. The definition is asymmetric with respect to its two arguments, as in order to define the composition $(x_1 \dots x_n) * (y_1 \dots y_m)$, we firstly need to express the second argument via $(x_1 \dots x_n) * y_j$ by step 4, and then to use the definitions for the latter expressions from step 3.

Definition 2.45. Given the derivations ∂ and ∂' of degree 1 on the path categories $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ respectively, let us use Lemma 2.5 to define a derivation $\partial \circledast \partial'$ of degree 1 on $\Bbbk\Omega[\Lambda \circledast \Lambda']$ by its values on the arrows of $\Lambda \circledast \Lambda'$ as follows:

1) For an arrow a in Λ and a vertex Y in Λ', we define the value on $a * Y$ as

$$(\partial \circledast \partial')(a * Y) \; := \; \partial(a) * Y. \tag{2.2}$$

2) For a vertex X in Λ and an arrow b in Λ', we define the value on $X * b$ as

$$(\partial \circledast \partial')(X * b) \; := \; X * \partial'(b). \tag{2.3}$$

3) For two arrows a and b in Λ and Λ' respectively, we define the value on $a * b$ as

$$(\partial \circledast \partial')(a * b) \; := \; \partial(a) * b - (-)^{|a|}\big(a * \partial'(b)\big) - (-)^{|a|}(a \diamond b), \tag{2.4}$$

where

$$a \diamond b \; := \; (a * T(b))(S(a) * b) - (-)^{|a|\cdot|b|}(T(a) * b)(a * S(b)).$$

Remark 2.46. The correction term $a \diamond b$ in the above definition is crucial as otherwise the cellular product of two 1-cells would be trivial (as for 1-cells we have $\partial(a) = \partial(b) = 0$). This term provides exactly what we want, that is a boundary of the square formed by arrows $a*S(b)$, $a*T(b)$, $S(a)*b$ and $T(a)*b$. In fact, if we would have the fully linearized picture in which we would allow subtraction of the objects and would define $\partial(a) = T_a - S_a$ for 1-cells a (instead of $\partial(a) = 0$) and would replace the composition of paths by their formal sum (instead of product), then the term $a \diamond b$ would appear naturally among the first two summands of $(\partial \circledast \partial')(a * b)$.

Lemma 2.47. *For any derivations ∂ and ∂' on $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ respectively, the following equations hold for arbitrary paths p and q in Λ and Λ' respectively:*

$$(\partial \circledast \partial')(p * Y) = \partial(p) * Y,$$

$$(\partial \circledast \partial')(X * q) = X * \partial'(q).$$

Proof. Immediately follows from definitions (2.2) and (2.3), from definitions of $X * q$ and $p * Y$, and from Leibniz rule applied for ∂, ∂' and $\partial \circledast \partial'$. $\square$

Lemma 2.48. *If $a = 1_X$ or $b = 1_Y$, then $a \diamond b = 0$.*

Proof.

$$\begin{aligned}
1_X \diamond b &= (1_X * T(b))(X * b) - (X * b)(1_X * S(b)) \\
&= (1_{X*T(b)}) \cdot (X * b) - (X * b) \cdot (1_{X*S(b)}) = X * b - X * b = 0, \\
a \diamond 1_Y &= (a * Y)(S(a) * 1_Y) - (T(a) * 1_Y)(a * Y) \\
&= (a * Y) \cdot (1_{S(a)*Y}) - (1_{T(a)*Y}) \cdot (a * Y) = a * Y - a * Y = 0.
\end{aligned}$$

$\square$

Lemma 2.49. *Operation $p \diamond q$ satisfies Leibniz rule for arbitrary paths p and q in Λ and Λ' respectively:*

$$(\partial \circledast \partial')(p \diamond q) = \partial(p) \diamond q + (-)^{|p|}\big(p \diamond \partial'(q)\big).$$

Proof. Immediately follows from Lemma 2.47 and Leibniz rule for derivation $\partial \circledast \partial'$. $\square$

Lemma 2.50. *If ∂ and ∂' are derivations on $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ respectively, and $\partial' = 0$, then Equation (2.4) holds when the arrow a is replaced by arbitrary path p in Λ:*

$$(\partial \circledast 0)(p * b) \;=\; \partial(p) * b - (-)^{|p|}(p \diamond b).$$

Proof. When p is an arrow in Λ, the equation holds by definition. Let us prove that this equation also holds when $p = a^{-1}$ is an inverse of an arrow in Λ. In this case, we have $|a| = 0$, $\partial(a) = 0$ and $\partial(a^{-1}) = 0$, since a is invertible, hence

$$(\partial \circledast 0)(a * b) \;=\; \partial(a) * b - a \diamond b \;=\; 0 * b - a \diamond b \;=\; -a \diamond b,$$

and we need to prove that

$$(\partial \circledast 0)(a^{-1} * b) \;=\; -(a^{-1}) \diamond b.$$

We can do it as follows:

$$\begin{aligned}
(\partial \boxast 0)(a^{-1} * b) \;&=\; \boxed{\text{Definition of } a^{-1} * b} \\
&= (\partial \boxast 0)\left(-\big(a * T(b)\big)^{-1}(a * b)\big(a * S(b)\big)^{-1}\right) \\
&= \boxed{\text{Leibniz rule for } \partial \boxast 0 \text{ using that } \partial(a * Y) = \partial(a) * Y = 0} \\
&= -\,\big(a * T(b)\big)^{-1} \cdot \big((\partial \boxast 0)(a * b)\big) \cdot \big(a * S(b)\big)^{-1} \\
&= \big(a * T(b)\big)^{-1} \cdot (a \diamond b) \cdot \big(a * S(b)\big)^{-1} \\
&= \boxed{\text{definition of } a \diamond b} \\
&= \cancel{\big(a * T(b)\big)}^{-1}\cancel{\big(a * T(b)\big)}\,\big(S(a) * b\big)\,\big(a * S(b)\big)^{-1} \\
&\qquad -\,\big(a * T(b)\big)^{-1}\big(T(a) * b\big)\,\cancel{\big(a * S(b)\big)}\,\cancel{\big(a * S(b)\big)}^{-1} \\
&= \big(S(a) * b\big)\big(a * S(b)\big)^{-1} - \big(a * T(b)\big)^{-1}\big(T(a) * b\big) \\
&= \big(S(a) * b\big)\big(a^{-1} * S(b)\big) - \big(a^{-1} * T(b)\big)\big(T(a) * b\big) \\
&= \boxed{\text{definition of } a^{-1} \diamond b} \;\;=\;\; -(a^{-1}) \diamond b.
\end{aligned}$$

The statement also holds when $p = 1_X$ is an empty path, because in this case, both sides become zero due to the definition of $1_X * b$ and Lemma 2.48:

$$(\partial \boxast 0)(1_X * b) = (\partial \boxast 0)(0) = 0,$$

$$\partial(1_X) * b - (-)^{|1_X|}(1_X \diamond b) = 0 * b - 0 = 0.$$

Now let us prove that if the equation holds for x_1 and x_2, then it holds for $x_1 x_2$. Namely, assume that it holds for x_1 and x_2:

$$\begin{aligned}(\partial \circledast 0)(x_1 * b) &= \partial(x_1) * b - (-)^{|x_1|}(x_1 \diamond b),\\ (\partial \circledast 0)(x_2 * b) &= \partial(x_2) * b - (-)^{|x_2|}(x_2 \diamond b).\end{aligned}$$

After that, let us prove it for x_1x_2 by using Leibniz rule and Remark 2.43 as follows:

$$\begin{aligned}
(\partial \circledast 0)(x_1x_2 * b) = &\ \boxed{\text{Remark 2.43 for } x_1x_2 * b}\\
&= (\partial \circledast 0)\Big((-)^{|x_2|\cdot(|b|+1)}(x_1 * b)(x_2 * S(b)) + (x_1 * T(b))(x_2 * b)\Big)\\
&= (-)^{|x_2|\cdot(|b|+1)}(\partial \circledast 0)\Big((x_1 * b)(x_2 * S(b))\Big)\\
&\qquad + (\partial \circledast 0)\Big((x_1 * T(b))(x_2 * b)\Big)\\
&= \boxed{\text{Leibniz rule for } \partial \circledast 0}\\
&= (-)^{|x_2|\cdot(|b|+1)}\Big((\partial \circledast 0)(x_1 * b)\Big) \cdot (x_2 * S(b))\\
&\quad + (-)^{|x_2|\cdot(|b|+1)+|x_1|+|b|+1}(x_1 * b)\Big((\partial \circledast 0)(x_2 * S(b))\Big)\\
&\quad + \Big((\partial \circledast 0)(x_1 * T(b))\Big) \cdot (x_2 * b)\\
&\quad + (-)^{|x_1|}(x_1 * T(b)) \cdot \Big((\partial \circledast 0)(x_2 * b)\Big)\\
&= \boxed{\text{assumptions for } x_1 \text{ and } x_2}\\
&= (-)^{|x_2|\cdot(|b|+1)}\Big(\underline{\partial(x_1) * b} - (-)^{|x_1|}(x_1 \diamond b)\Big) \cdot (x_2 * S(b))\\
&\quad + (-)^{(|x_2|+1)(|b|+1)+|x_1|}\underline{\underline{(x_1 * b)\big(\partial(x_2) * S(b)\big)}}\\
&\quad + \underline{\big(\partial(x_1) * T(b)\big)(x_2 * b)}\\
&\quad + (-)^{|x_1|}(x_1 * T(b)) \cdot \Big(\underline{\underline{\partial(x_2) * b}} - (-)^{|x_2|}(x_2 \diamond b)\Big)\\
&= \boxed{\text{Remark 2.43 for } \partial(x_1)x_2 * b \text{ and } x_1\partial(x_2) * b}\\
&= \underline{\big(\partial(x_1)x_2\big) * b} + (-)^{|x_1|}\underline{\underline{\big(x_1\partial(x_2)\big) * b}}\\
&\quad + (-)^{|x_2|\cdot(|b|+1)+|x_1|+1}(x_1 \diamond b)\big(x_2 * S(b)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ (-)^{|x_1|+|x_2|+1}\big(x_1 * T(b)\big)(x_2 \diamond b) \\
&= \boxed{\text{Leibniz rule for } \partial \text{ and definitions of } x_1 \diamond b \text{ and } x_2 \diamond b} \\
&= \big(\partial(x_1x_2)\big) * b + (-)^{|x_2|\cdot(|b|+1)+|x_1|+1}\Big(\cancel{(x_1 * T(b))(S(x_1) * b)} \\
&\qquad\qquad - (-)^{|x_1|\cdot|b|}(T(x_1) * b)(x_1 * S(b))\Big) \cdot \big(x_2 * S(b)\big) \\
&+ (-)^{|x_1|+|x_2|+1}\big(x_1 * T(b)\big) \cdot \Big((x_2 * T(b))(S(x_2) * b) \\
&\qquad\qquad - (-)^{|x_2|\cdot|b|}\cancel{(T(x_2) * b)(x_2 * S(b))}\Big) \\
&= \big(\partial(x_1x_2)\big) * b + (-)^{(|x_1|+|x_2|)\cdot(|b|+1)}\big(T(x_1) * b\big)\big(x_1 * S(b)\big)\big(x_2 * S(b)\big) \\
&\qquad\qquad + (-)^{|x_1|+|x_2|+1}\big(x_1 * T(b)\big)\big(x_2 * T(b)\big)\big(S(x_2) * b\big) \\
&= \boxed{\text{Lemma 2.47 for } x_1x_2 * S(b) \text{ and } x_1x_2 * T(b)} \\
&= \big(\partial(x_1x_2)\big) * b + (-)^{(|x_1|+|x_2|)\cdot(|b|+1)}\big(T(x_1x_2) * b\big)\big(x_1x_2 * S(b)\big) \\
&\qquad\qquad + (-)^{|x_1|+|x_2|+1}\big(x_1x_2 * T(b)\big)\big(S(x_1x_2) * b\big) \\
&= \boxed{\text{definition of } x_1x_2 \diamond b} \;=\; \big(\partial(x_1x_2)\big) * b - (-)^{|x_1|+|x_2|}\big(x_1x_2 \diamond b\big).
\end{aligned}
$$

Now, we can conclude that the needed equation holds for arbitrary paths p in Λ, since every path is either empty or is a product of arrows and their inverses for which the equation holds. Hence the proof. $\square$

Lemma 2.51. *If ∂ and ∂' are differentials on $\Bbbk\Omega[\Lambda]$ and $\Bbbk\Omega[\Lambda']$ respectively, and $\partial' = 0$, then the derivation $\partial \circledast \partial'$ is a differential on $\Bbbk\Omega[\Lambda \circledast \Lambda']$.*

Proof. Indeed, by Lemma 2.6, we just need to check that $(\partial \circledast \partial')^2 = 0$ on arrows of $\Lambda \circledast \Lambda'$. This equation holds on the arrows of type $a * Y$, since

$$
\begin{aligned}
(\partial \circledast \partial')^2(a * Y) &= (\partial \circledast \partial')\big((\partial \circledast \partial')(a * Y)\big) \\
&= (\partial \circledast \partial')(\partial(a) * Y) = \partial\big(\partial(a)\big) * Y = 0 * Y = 0
\end{aligned}
$$

by Lemma 2.47. Similarly, it holds on the arrows of type $X * b$ by the same Lemma. On the arrows of type $a * b$, we can check it as follows:

$$\begin{aligned}
(\partial \circledast 0)^2(a * b) &= (\partial \circledast 0)\big((\partial \circledast 0)(a * b)\big) \\
&= \boxed{\text{definition of } \partial \circledast 0} \\
&= (\partial \circledast 0)\,\big(\partial(a) * b - (-)^{|a|}(a \diamond b)\big) \\
&= (\partial \circledast 0)\big(\partial(a) * b\big) - (-)^{|a|}(\partial \circledast 0)(a \diamond b) \\
&= \boxed{\text{Lemma 2.50 for } \partial(a) * b} \\
&= \big(\partial\big(\partial(a)\big) * b - (-)^{|\partial(a)|}\partial(a) \diamond b\big) - (-)^{|a|}(\partial \circledast 0)(a \diamond b) \\
&= \boxed{\text{Lemma 2.49}} \\
&= 0 * b - (-)^{|a|+1}\cancel{\partial(a) \diamond b} - (-)^{|a|}\,\big(\cancel{\partial(a) \diamond b} + (-)^{|a|}\,a \diamond 0\big) \\
&= 0.
\end{aligned}$$

□

Remark 2.52. The condition $\partial' = 0$ is technical here. In general case, a similar statement should work, though one need to add an extra-correction term to (2.4) handling the asymetricity in the definition of $p * q$ (see Remark 2.44).

Finally, we are ready to define a cellular product for **dg**-categories.

Definition 2.53. Let $(\mathcal{A}, \partial_{\mathcal{A}})$ and $(\mathcal{B}, \partial_{\mathcal{B}})$ be cellular **dg**-categories with cellular structures $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{B}}$ respectively, such that $\partial_{\mathcal{B}} = 0$. We define a *cellular product* $\mathcal{A} \circledast \mathcal{B}$ as a cellular **dg**-category whose cellular structure is $\Lambda_{\mathcal{A}} \circledast \Lambda_{\mathcal{B}}$ and whose differential is $\partial_{\mathcal{A}} \circledast \partial_{\mathcal{B}}$.

Remark 2.54. For any object X in $\mathcal{A}$, the operation $X * (-)$ on the objects and morphisms of $\mathcal{B}$ gives rise to an inclusion **dg**-functor $\mathcal{B} \xrightarrow{\subseteq} \mathcal{A} \circledast \mathcal{B}$. Similarly, for any object Y in $\mathcal{B}$, the operation $(-) * Y$ defines an inclusion **dg**-functor from $\mathcal{A}$ into $\mathcal{A} \circledast \mathcal{B}$.

Remark 2.55. If $\mathcal{A} \subseteq \mathcal{A}'$ is the inclusion of cellular **dg**-subcategories corresponding to the inclusion of their cellular graphs, then $\mathcal{A} \circledast \mathcal{B}$ naturally includes into the category $\mathcal{A}' \circledast \mathcal{B}$ (as soon as these products are well-defined). Similarly, if $\mathcal{B} \subseteq \mathcal{B}'$ is such inclusion, then $\mathcal{A} \circledast \mathcal{B}$ includes into $\mathcal{A} \circledast \mathcal{B}'$. The images of $\mathcal{A} \circledast \mathcal{B}$ under these inclusions will coincide with the **dg**-subcategories generated by cells of type $x * y$ where x is a cell in $\mathcal{A}$ and y is a cell in $\mathcal{B}$.

Conjecture 2.56. *Let $\mathcal{A}$ and $\mathcal{B}$ be cellular **dg**-categories constructed by the procedure of Conjecture 2.37 from two CW-complexes X and Y respectively. Then the cellular **dg**-product $\mathcal{A} \circledast \mathcal{B}$ is isomorphic to the CW-category constructed from a direct product CW-complex $X \times Y$.*

2.5.2 **dg**-cubes

Let us denote by $\vec{\mathcal{I}}$ the cellular **dg**-category obtained from the directed interval graph $I^{\to}$ in Example 2.20, and by $\mathcal{I}$ the corresponding undirected category from Example 2.21. We will call them as *directed* and *undirected interval categories*, respectively. Also, whenever we are talking about just an interval category, we will mean that the statement applies to both directed and undirected cases.

Both interval categories have trivial differential, so we can multiply them by themselves using Definition 2.53 as follows.

Definition 2.57. We define a directed and undirected ***dg**-n-cubes* $\vec{\mathcal{I}}^{\circledast n}$ and $\mathcal{I}^{\circledast n}$ as the cellular n-fold products of the corresponding interval categories:

$$\vec{\mathcal{I}}^{\circledast n} := \Big((\vec{\mathcal{I}} \circledast \vec{\mathcal{I}}) \circledast \dots \Big) \circledast \vec{\mathcal{I}} \quad (n \text{ times});$$
$$\mathcal{I}^{\circledast n} := \Big((\mathcal{I} \circledast \mathcal{I}) \circledast \dots \Big) \circledast \mathcal{I} \quad (n \text{ times}).$$

Since an interval category contains three cells 0, 1 and i, the corresponding **dg**-n-cube contains exactly 3^n cells corresponding to the elements of the power set $\{0, 1, i\}^n$. Let us denote by $e_{\alpha_1 \dots \alpha_n}$ the cell $\alpha_1 * \dots * \alpha_n$ for any $\alpha_j \in \{0, 1, i\}$. An **dg**-n-cube has exactly $2^{n-k}\binom{n}{k}$ cells of dimension k which correspond to the tuples $(\alpha_1, \dots, \alpha_n)$ in which exactly k elements are equal to i and the rest $n-k$ elements are 0 or 1. In terms of the corresponding cellular graph, it has 2^n vertices/0-cells $e_{\alpha_1 \dots \alpha_n}$ where $\alpha_j \in \{0, 1\}$. Arrows are the cells $e_{\alpha_1 \dots \alpha_n}$ in which at least one α_j is equal to i.

For an arrow $e_{\alpha_1 \ldots \alpha_n}$, its source is given by a vertex $e_{\beta_1 \ldots \beta_n}$ where $\beta_j = \alpha_j$ if $\alpha_j \in \{0,1\}$ and $\beta_j = S(i) = 0$ if $\alpha_j = i$. Similarly, the target is given by a vertex $e_{\beta_1 \ldots \beta_n}$ where $\beta_j = \alpha_j$ if $\alpha_j \in \{0,1\}$ and $\beta_j = T(i) = 1$ if $\alpha_j = i$.

Example 2.58. A **dg**-1-cube is just an interval (see Examples 2.21 and 2.20).

Example 2.59. A **dg**-2-cube is a square. The graph for the square was described in Example 2.41. The differential is defined by setting $\partial(0 * i) = \partial(1 * i) = \partial(i * 0) = \partial(i * 1) = 0$ and $\partial(i * i) = -i \diamond i = (1 * i)(i * 0) - (i * 1)(0 * i)$ by (2.4) which means that $i * i$ denotes a 2-cell homotopy connecting two paths from $0 * 0$ to $1 * 1$ (see also the picture in Example 2.41).

Example 2.60. A **dg**-3-cube is a regular cube. The differential of an interior cell $i * i * i$ is equal to

$$\begin{aligned}\partial(i * i * i) = &- (i * 1 * i)(0 * i * 0) - (i * 1 * 1)(0 * i * i) + (1 * i * i)(i * 0 * 0) \\ &+ (1 * i * 1)(i * 0 * i) + (i * i * 1)(0 * 0 * i) - (1 * 1 * i)(i * i * 0),\end{aligned}$$

which has precisely six terms corresponding to the six square faces as expected.

Now, since all of our constructions work equally in directed and undirected cases, let us use the uniform notation and denote by $\mathcal{C}_n$ the **dg**-n-cube $\vec{\mathcal{I}}^{\circledast n}$ or $\mathcal{I}^{\circledast n}$ for the rest of this section. Thus the corresponding directed or undirected interval will be denoted by $\mathcal{C}_1$. Also, we can denote by $\mathcal{C}_0$ the one-point category $\{*\}$ with a single 0-cell.

Let us now describe the faces in **dg**-cubes. Namely, for a cube $\mathcal{C}_n$, let us denote by $d_k^{\varepsilon}(\mathcal{C}_n)$ the subcategory of $\mathcal{C}_n$ generated by cells of type $\alpha_1 * \ldots * \alpha_n$, where $\alpha_k = \varepsilon \in \{0,1\}$ and $1 \leq k \leq n$. These $2n$ subcategories for different k and ε correspond to the $(n-1)$-faces of a regular n-cube.

Lemma 2.61. *The faces of cubes are cubes themselves. Namely, for any $n \geq 1$, any $\varepsilon = 0, 1$, and any $1 \leq k \leq n$, there exist an inclusion functor*

$$\iota_{n,k}^{\varepsilon} \colon \ \mathcal{C}_{n-1} \xrightarrow{\cong} d_k^{\varepsilon}(\mathcal{C}_n) \xrightarrow{\subseteq} \mathcal{C}_n,$$

*isomorphically mapping a **dg**-$(n-1)$-cube $\mathcal{C}_{n-1}$ onto the face $d_k^\varepsilon(\mathcal{C}_n)$ of a **dg**-n-cube $\mathcal{C}_n$, and acting on cells as follows:*

$$\iota_{n,k}^{\varepsilon} \colon \ \alpha_1 * \ldots * \alpha_{n-1} \ \longmapsto \ \alpha_1 * \ldots \alpha_{k-1} * \varepsilon * \alpha_k * \ldots \alpha_{n-1}.$$

Proof. Let us construct these functors as follows. If $n = 1$, then the inclusion functor is just the mapping of a one-point category $\{*\} = \mathcal{C}_0$ onto the object $\varepsilon = 0, 1$ of the interval category $\mathcal{C}_1$. If $n > 1$, we will use the inductive definition of $\mathcal{C}_n$ as $\mathcal{C}_{n-1} \circledast \mathcal{C}_1$ and will define our functors inductively by $n - k$.

If $k = n$, then we just consider the product of $\mathcal{C}_{n-1}$ with the inclusion $\{\varepsilon\} \subseteq \mathcal{C}_1$ of a single object $\varepsilon = 0$ or $\varepsilon = 1$ into the interval category $\mathcal{C}_1$ as in Remark 2.55:

$$\iota_{n,n}^{\varepsilon} \colon \ \mathcal{C}_{n-1} \xrightarrow{\cong} \mathcal{C}_{n-1} \circledast \{\varepsilon\} \xrightarrow{\subseteq} \mathcal{C}_{n-1} \circledast \mathcal{C}_1 = \mathcal{C}_n.$$

If $k < n$, then we can inductively express $\iota_{n,k}^{\varepsilon}$ via $\iota_{n-1,k}^{\varepsilon}$ as follows:

$$\iota_{n,k}^{\varepsilon} \colon \ \mathcal{C}_{n-1} = \mathcal{C}_{n-2} \circledast \mathcal{C}_1 \xrightarrow{\iota_{n-1,k}^{\varepsilon} \circledast \mathcal{C}_1} \mathcal{C}_{n-1} \circledast \mathcal{C}_1 = \mathcal{C}_n,$$

where the functor $\iota_{n-1,k}^{\varepsilon} \circledast \mathcal{C}_1$ is defined via Remark 2.55. $\square$

Chapter 3: Applications to homotopy theory of A_∞-categories

3.1 A_∞-functors from cellular dg-categories

Here, we consider unital A_∞-functors $f\colon \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is a cellular **dg**-category and $\mathcal{B}$ is a unital A_∞-category. We also assume that $\mathcal{A}$ does not have invertible cells. We show that every such functor is essentially determined by the images of cells of $\mathcal{A}$ in $\mathcal{B}$. In order for such functor to exist, these cell images should also satisfy boundary conditions compatible with the differential in $\mathcal{A}$.

Since every cellular **dg**-category without invertible cells can be obtained from an empty category by either adding new objects or adding new cells, the above fact follows from the following theorem.

Theorem 3.1. *Let $\mathcal{A}' = \mathcal{A} \cup_b c$ be a cellular **dg**-category obtained from a cellular **dg**-category $\mathcal{A}$ by attaching a single cell $c\colon X \to Y$ of dimension k with the boundary condition $\partial(c) = b$ where $b \in \mathcal{A}^{2-k}(X,Y)$ and $\partial(b) = 0$. Then for any unital A_∞-functor $f\colon \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to a unital A_∞-category $\mathcal{B}$, and an element $c' \in \mathcal{B}(f(X), f(Y))$ such that $b_1(sc') = f_1(sb)$, there exists a functor $f'\colon \mathcal{A}' \to \mathcal{B}$ such that $f'|_{\mathcal{A}} = f$ and $f'_1(sc) = sc'$. Moreover, such functor f' is uniquely defined up to an s-homotopy (see Definition B.1).*

Remark 3.2. Note that the boundary condition $b_1(sc') = f_1(sb)$ for the morphism c' is a necessary requirement in the theorem. Indeed, for any functor f' with the given value $f'_1(sc) = sc'$ such that $f'|_{\mathcal{A}} = f$, the following A_∞-condition should hold on sc:

$$f'_1 b_1(sc) = b_1 f'_1(sc).$$

However, here the right-hand-side is $b_1 f'_1(sc) = b_1(sc')$ and the left-hand-side is exactly $f'_1 b_1(sc) =$

$f_1'(s\partial(c)) = f_1'(sb) = f_1(sb)$ by our assumptions, hence we obtain the necessary condition.

Remark 3.3. A similar statement holds when we restrict ourselves to the **dg**-world. Namely, if $\mathcal{B}$ is a **dg**-category, f is a **dg**-functor and c' satisfies the same boundary conditions, then there exists a *unique* **dg**-functor f' extending f with the given value on c. Theorem 3.1 can be thought as a homotopy version of this statement.

Corollary 3.4. *If $\mathcal{A}$ is a cellular **dg**-category without invertible arrows, then any untial A_∞-functor $f\colon \mathcal{A} \to \mathcal{B}$ to a unital A_∞-category $\mathcal{B}$ is uniquely determined (up to an s-homotopy) by the images $f(X)$ of the objects in $\mathcal{A}$ and by the images $f_1(sc)$ of the positive-dimensional cells c in $\mathcal{A}$, satisfying boundary conditions of type $b_1(f_1(sc)) = f_1(s\partial(c))$, where $f_1(s\partial(c))$ is described in terms of the images of cells of lower dimension.*

The rest of this section is dedicated to proof of Theorem 3.1. The idea of the proof is quite straightforward and is as follows.

By definition, an A_∞-functor is given by its values on the composable tuples of morphisms in the input category satisfying A_∞-conditions (1.5). In our setting of the input category being a cellular **dg**-category $\mathcal{A}$ with an underlying cellular graph Λ, the $\Bbbk$-linear basis in the set of the composable tuples of morphisms in $\mathcal{A}$ is formed by the composable tuples $[p_1, \dots, p_n]$ of paths in Λ. Thus, for any map of quivers $f\colon Ts\mathcal{A} \to s\mathcal{B}$, the condition for it of being an A_∞-functor can be expressed as the set of equations on the values $f_n(sp_1 \otimes \dots \otimes sp_n)$ of f on such tuples.

In Theorem 3.1, we are trying to construct a functor $f'\colon \mathcal{A}' \to \mathcal{B}$ satisfying certain conditions. Let Λ' denote the underlying cellular graph in $\mathcal{A}'$. Then, as in the above discussion, we will treat the values $f_n'(sp_1 \otimes \dots \otimes sp_n)$ of f' on the composable tuples of paths in Λ' as our *variables*. In the following, we will analyze the equations on these variables in large detail. In particular, we will show that the variables can be nicely classified into the independent and dependent ones and that the required functor can be explicitly constructed by arbitrary assignment of the values to the independent variables and by deducing the values of the dependent variables from A_∞-conditions.

In §3.1.1, we classify the tuples of paths according to our discussion of independent and dependent variables. In §3.1.2, we construct f' as a map of quivers from $Ts\mathcal{A}$ to $s\mathcal{B}$ by assigning arbitrary values to the independent variables and by deducing the values of the rest of the variables from a subset of A_∞-conditions. In §3.1.3, we are checking that f' satisfies the rest of A_∞-conditions, hence is a well-defined A_∞-functor. Finally, in §3.1.4, we show that any two possible extensions are s-homotopic, thus concluding the proof of Theorem 3.1.

Example 3.5. Let us consider the directed interval category $\vec{\mathcal{I}} = \{0 \xrightarrow{i} 1\}$ from Example 2.20. Let us add a new object to it labelled as 2 and denote the resulting category by $\mathcal{A}$. Ignoring the unitality issues, an A_∞-functor f from $\mathcal{A}$ to an A_∞-category $\mathcal{B}$ is uniquely determined by the images of objects 0, 1 and 2, and by a 0-cycle $x\colon f(0) \to f(1)$ in $\mathcal{B}$ which will be the image of i under f up to the shifts, that is $f_1(si) = sx$:

$$\mathcal{A} = \left(\begin{tikzcd} 0 \arrow[r, "i"] & 1 & 2 \end{tikzcd} \right) \xrightarrow{f} \mathcal{B}$$

Then, let us add a new 1-cell $j\colon 1 \to 2$ to $\mathcal{A}$ and denote the resulting category by $\mathcal{A}'$. Thus $\mathcal{A}' = \mathcal{A} \cup j$, where $\partial(j) = 0$. Now, let us see how Theorem 3.1 works in this situation.

In order to define an A_∞-functor $f'\colon \mathcal{A}' \to \mathcal{B}$, we need to specify the values of $f'_1(si)$, $f'_1(sj)$, $f'_1\big(s(ji)\big)$ and $f'_2(sj \otimes si)$:

$$\mathcal{A}' = \left(\begin{tikzcd} 0 \arrow[r, "i"] \arrow[rr, bend right, "ji"'] & 1 \arrow[r, "j"] & 2 \end{tikzcd} \right) \xrightarrow{f'} \mathcal{B}$$

By assumption of our theorem, f' extends f, so the value $f'_1(si) = f_1(si)$ is predefined. Also, the value of $f'_1(sj)$ is also specified in the assumptions of our theorem as the value on the new cell. Only the other two values $f'_1\big(s(ji)\big)$ and $f'_2(sj \otimes si)$ will be our actual variables. Since $b_1(si) = s\partial(i) = 0$, $b_1(sj) = s\partial(j) = 0$ and $b_2(sj \otimes si) = s(ji)$, the A_∞-conditions (1.5) for the functor f' reduce to the following three equations on these variables:

$$\begin{aligned} 0 &= b_1 f_1'(si), \\ 0 &= b_1 f_1'(sj), \\ f_1'\big(s(ji)\big) &= b_2\big(f_1'(sj) \otimes f_1'(si)\big) + b_1 f_2'(sj \otimes si). \end{aligned}$$

The first equation is satisfied automatically, since it was true for the original functor f. The second equation is just the boundary condition for the image of the new cell j, so it holds by initial assumption on $f_1'(sj)$. The third equation, however, provides a non-trivial relation between our variables. Moreover, we see that as soon as we know the values of $f_1'(si)$, $f_1'(sj)$ and $f_2'(sj \otimes si)$, we can immediately recover the value of $f_1'\big(s(ji)\big)$. Thus in this case, we can choose $f_2'(sj \otimes si)$ as an independent variable, and $f_1'\big(s(ji)\big)$ as a dependent variable. (Extending this language, we can also call $f_1'(si)$ and $f_1'(sj)$ as predefined constants.) In particular, we can set $f_2'(sj \otimes si) := 0$ which will imply that $f_1'\big(s(ji)\big) = b_2\big(f_1'(sj) \otimes f_1'(si)\big)$.

3.1.1 Types of tuples of paths

Let us denote the underlying cellular graph of $\mathcal{A}$ by Λ and the corresponding cellular graph of $\mathcal{A}'$ by $\Lambda' = \Lambda \cup c$. Let us classify the composable tuples of paths in Λ' as follows.

Definition 3.6. For tuples $[p_1, \dots, p_n]$ of composable paths p_i in Λ' we define the following four their *types*:

(1) Tuples that have at least one empty path in them, that is $p_i = 1_X$ for some i and some object X.

(2) (a) Tuples that solely consist of paths in the old graph Λ;

(b) a 1-tuple consisting of a new cell c.

(3) (a) Tuples such that $p_1 = c$ and $n \geq 2$;

(b) tuples whose k-th element p_k is the path with first letter c and whose preceding elements are paths in the old graph Λ, where $k > 1$.

(4) (a) Tuples such that $p_1 = cq$ for some non-empty path q in Λ';

(b) tuples whose k-th element is the path whose word starts with qc where q is a non-empty path in the old graph Λ and whose previous elements are paths in the old graph Λ, where $k \geq 1$.

Remark 3.7. The above types have the following significance. Type (2) consists of the tuples on which the extended functor f' has predefined values (*constants*). Type (3) consists of the tuples on which we will be able to define f' in arbitrary way (*independent variables*), and type (4) consists of the tuples on which the value of f' will be uniquely determined from the previous values and from the A_∞-equations (*dependent variables*). Also, type (1) serves a technical role of simplifying constructions involving unitality properties.

Schematically, we can describe the above types as follows. Let us use the following notations:

$$\blacksquare := \{\text{paths in } \Lambda\}, \qquad \square := \{\text{non-empty paths in } \Lambda\},$$
$$\bigstar := \{\text{paths in } \Lambda'\}, \qquad \text{☆} := \{\text{non-empty paths in } \Lambda'\}.$$

Then the above types are as follows:

(1) $[\dots, 1, \dots]$;

(2) (a) $[\underbrace{\blacksquare, \dots, \blacksquare}_{\geq 1}]$, (b) $[c]$;

(3) (a) $[c, \underbrace{\bigstar, \dots, \bigstar}_{\geq 1}]$, (b) $[\underbrace{\blacksquare, \dots, \blacksquare}_{\geq 1}, \ c\bigstar, \ \bigstar, \dots, \bigstar]$;

(4) (a) $[c\text{☆}, \ \bigstar, \dots, \bigstar]$, (b) $[\blacksquare, \dots, \blacksquare, \ \square c\bigstar, \ \bigstar, \dots, \bigstar]$;

where the number of objects in the dotted sequences $\blacksquare, \dots, \blacksquare$ and $\bigstar, \dots, \bigstar$ might be zero, unless they are denoted as "≥ 1".

Lemma 3.8. *Every tuple $[p_1, \dots, p_n]$ of composable paths in Λ' for $n \geq 1$ belongs to a single type among types (2), (3) and (4) where all symbols ■, □, ★ and ☆ have uniquely determined values.*

Proof. Indeed, by Lemma 2.32(2), we know that every path in Λ' is either an old path in Λ, that is ■, or a path of type ■ c★ where c denotes the first occurrence of letter c and ■ and ★ are uniquely defined. Thus for the given tuple of paths, we can decide which type it belongs to by performing the following algorithm:

- Does the tuple have at least one occurrence of a new cell c?

  No ⇒ type (2a). Yes ⇒ proceed further.

- Denote by k the position in the tuple where the first occurence of c is located. That is, $p_i =$ ■ for $i < k$ and $p_k =$ ■ c★.

- If $k = 1$, then $p_1 =$ ■ c★. If ■ is non-empty, then $p_1 =$ □ c★ and the tuple has type (4b) (with zero elements ■ in it). Otherwise, $p_1 = c$★. If ★ is empty, then $p_1 = c$ and the tuple has type (2b) if $n = 1$ or type (3a) if $n > 1$. Otherwise, $p_1 = c$☆ and the tuple has type (4a).

- If $k > 1$, then $p_k =$ ■ c★. If ■ is empty, then the tuple has type (3b). Otherwise, the tuple has type (4b).

□

3.1.2 Construction of the functor extension

Here, we will assume for simplicity of the argument that $\mathcal{B}$ is a strictly unital category with strict units $\mathbf{i}_Y^{\mathrm{su}}$ for $Y \in \operatorname{Ob} \mathcal{B}$, and that the original functor $f \colon \mathcal{A} \to \mathcal{B}$ is strictly unital in a sense that $f_1(s1_X) = s\mathbf{i}^{\mathrm{su}}_{f(X)}$ and $f_i(\dots, s1_X, \dots) = 0$ for $i > 1$. The extended functor will then also satisfy the same condition.

Remark 3.9. We can assume that $\mathcal{B}$ is strictly unital, since we can always embed a unital A_∞-category into an equivalent strictly unital category $\mathcal{B}^+$ by Remark 1.39. Then after we are done with extending the functor, we can project the result back onto $\mathcal{B}$ (see also Lemma B.21). We can further assume that the original functor is strictly unital. Otherwise, we can deform it to a strictly unital one by Lemma B.11. After that we can extend it via the procedure described below. Finally, we can deform the result back by using the same s-homotopic deformation, thus obtaining the extension of the original functor.

We will define the extended functor on different types of input tuples as follows.

If the tuple has type (1), then we will define it via the above strict unitality property regardless of whether it belongs to type (2), (3) or (4).

If the tuple has type (2a), then we define f' as having the same value as f, and if it has type (2b), then we define f' by the given value of $f'(sc) = sc'$. (By assumption, the original functor satisfies strict unitality property, so this definition is compatible with the definition for the tuples of type (1).)

On each tuple of type (3) but not of type (1), we define f' in arbitrary way. (Here, "arbitrarily" still means meaningfully, that is we remember from the definition of an A_∞-functor which degree the value of f' should have, and what are the source and the target of the output morphism.)

On the tuples of type (4) but not of type (1), we define f' from the A_∞-conditions using the previously defined values. We also do it inductively on the number of occurrences of c in the tuple. We do it as follows. We start by noticing that f' was already defined as f on the tuples with no occurrences of c. To perform inductive step, let us assume that we have already defined the value of f' on the tuples with the number of occurrences of c less than m, where $m \geq 1$. Consider now a tuple $[p_1, \ldots, p_n]$ which has exactly m occurrences of c, where $n \geq 1$. Let us consider the cases of subtypes (4a) and (4b) separately.

If $[p_1, \ldots, p_n] = [\,c\star,\ \bigstar, \ldots, \bigstar\,]$ has type (4a), where $n \geq 1$, then let us consider a new tuple $[p'_1, \ldots, p'_{n+1}]$ obtained from $[p_1, \ldots, p_n]$ by splitting p_1 into c and $\star$:

$$[p'_1, \ldots, p'_{n+1}] \;:=\; [\,c,\ \star,\ \bigstar, \ldots, \bigstar\,].$$

This tuple itself has type (3a) and all of its subtuples $[p'_i, \ldots, p'_j]$, where $1 \leq i \leq j \leq n+1$, are either of types (2b) or (3a) if $i = 1$, or they have less than m occurrences of c if $i > 1$, since they do not contain the first c in p'_1. Now, let us rewrite an A_∞-condition (1.5) in a simplified notation (without indices):

$$\sum f'(1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes \ldots \otimes 1) = \sum b(f' \otimes \ldots \otimes f').$$

Let us apply this equation to the tuple $[p'_1, \ldots, p'_{n+1}]$, that is to the element $sp'_1 \otimes \ldots \otimes sp'_{n+1}$. Each term on the right-hand-side will have the form $b_l(x_1 \ldots x_l)$ where x_i are the values of f' on the sub-tuples of $[p'_1, \ldots, p'_{n+1}]$ which are all predefined by the previous steps and by inductive assumption. On the left-hand-side, our b can be either b_1 or b_2, since the input category is a **dg**-category. If it is b_1, then it is basically the differential (up to the shifts), and if it is b_2, then it is a path composition. The terms with b_1 are basically the values of f' on the tuples of type $[p'_1, \ldots, \partial(p'_i), \ldots, p'_{n+1}]$ where $1 \leq i \leq n+1$. If $i = 1$, then this becomes $[\partial(c), p'_2, \ldots, p'_{n+1}]$ which has less than m occurrences of c, because $\partial(c) \in \mathcal{A} = \Bbbk\Omega[\Lambda]$ is a linear composition of old paths by definition. If $i > 1$, then this becomes $[c, \ldots, \partial(p'_i), \ldots, p'_{n+1}]$ which is a linear combination of paths of type (3a). Thus the terms with b_1 on the left-hand-side have predefined values from the previous steps or from the inductive assumption. Now, let us consider the terms on the left-hand-side with b_2. These terms correspond to the values of f' on the tuples of type $[p'_1, \ldots, p'_i p'_{i+1}, \ldots, p'_{n+1}]$, where $1 \leq i \leq n$. If $i = 1$, then this becomes our original tuple $[p_1, \ldots, p_n]$. If $i > 1$, then the result has type (3a) which has the predefined value of f'. Summarizing, all terms in our equation applied to $[p'_1, \ldots, p'_{n+1}]$ have predefined values except for a single term which is $\pm$ of the value of f' on the original tuple $[p_1, \ldots, p_n]$. Thus, we extend f' onto this tuple from this linear equation.

A similar argument can be applied in the situation when

$$[p_1, \ldots, p_n] = [\blacksquare, \ldots, \blacksquare,\ \square\, c\bigstar,\ \bigstar, \ldots, \bigstar]$$

has type (4b), where $n \geq 1$. Let us denote by k the position of the term $\square\, c\bigstar$ in this expression, where $1 \leq k \leq n$. In this case, we consider the tuple

$$[p'_1, \dots, p'_{n+1}] \;:=\; [\blacksquare, \dots, \blacksquare, \square, c\bigstar, \;\bigstar, \dots, \bigstar]$$

of type (3b) obtained from $[p_1, \dots, p_n]$ by splitting $p_k = \square\, c\bigstar$ into $p'_k = \square$ and $p'_{k+1} = c\bigstar$. By analyzing the A_∞-condition on this tuple, we similarly see that on the right-hand-side all terms are expressed in terms of the values of f' on the subtuples $[p'_i, \dots, p'_j]$ of $[p'_1, \dots, p'_{n+1}]$. If such subtuple includes the term $p'_{k+1} = c\bigstar$, that is when $i \leq k+1 \leq j$, then it either has the same type (3b) when $i \leq k$ or has one of types (2b), (3a) or (4a) when $i = k+1$. Otherwise, it has less occurrences of c. Similarly, on the left hand side, the terms with b_1 are the linear combinations of the values of f' on the tuples of type $[\blacksquare, \dots, \partial(\blacksquare), \dots, \blacksquare, \square, c\bigstar, \;\bigstar, \dots, \bigstar]$, $[\blacksquare, \dots, \blacksquare, \partial(\square), c\bigstar, \;\bigstar, \dots, \bigstar]$, $[\blacksquare, \dots, \blacksquare, \square, c\bigstar, \;\bigstar, \dots, \partial(\bigstar), \dots, \bigstar]$ which all are linear combinations of tuples of type (3b), and $[\blacksquare, \dots, \blacksquare, \square, \partial(c\bigstar), \;\bigstar, \dots, \bigstar]$ which by Leibniz rule becomes a sum of

$$[\blacksquare, \dots, \blacksquare, \square, \partial(c)\bigstar, \;\bigstar, \dots, \bigstar]$$

which has less occurrences of c and $[\blacksquare, \dots, \blacksquare, \square, c\partial(\bigstar), \;\bigstar, \dots, \bigstar]$ which is a linear combination of paths of types (3b). When we have b_2 on the left side, then we similarly either obtain the original tuple $[p_1, \dots, p_n]$ or the tuples of the same type (3b). Thus, as in the case of (4a), all terms except for the needed one have predefined values, which allows us to define the value of f' on the original tuple $[p_1, \dots, p_n]$ in a unique way from the linear equation.

This concludes our construction of the functor f'.

3.1.3 Checking A_∞-correctness

In the previous section, we defined the map $f' \colon Ts\mathcal{A}' \to s\mathcal{B}$ by applying the A_∞-conditions to certain tuples of paths. However, in order to prove that f' is a well-defined A_∞-functor, we also need to check that these A_∞-conditions hold on all other possible tuples. Let us do it.

Proposition 3.10. *The map f' defined above is an A_∞-functor from $\mathcal{A}'$ to $\mathcal{B}$.*

In order to prove the above proposition, let us firstly examine how the validity of A_∞-conditions on different tuples are related to each other.

Definition 3.11. We say that a map $f\colon Ts\mathcal{A} \to s\mathcal{B}$ *satisfies* A_∞*-conditions on a tuple* $[x_1, \dots, x_n]$ of composable morphisms in $\mathcal{A}$, if the equation (1.5) holds on the input $sx_1 \otimes \dots \otimes sx_n$.

Definition 3.12. Let $[x_1, \dots, x_n]$ be a tuple of composable morphisms in $\mathcal{A}$, where $n \geq 1$. We will call tuples of kind $[x_i, \dots, x_j]$, where $1 \leq i \leq j \leq n$, the *subtuples of* $[x_1, \dots, x_n]$. We will also call tuples of kinds $[x_1, \dots, \partial(x_i), \dots, x_n]$, where $1 \leq i \leq n$, and $[x_1, \dots, x_i x_{i+1}, \dots, x_n]$, where $1 \leq i < n$, the *reductions of* $[x_1, \dots, x_n]$.

Lemma 3.13. *Let* $[x_1, \dots, x_n]$ *be a tuple of composable morphisms in* $\mathcal{A}$*, where* $n \geq 1$*. If a map* $f\colon Ts\mathcal{A} \to s\mathcal{B}$ *satisfies* A_∞*-conditions on this tuple, all its subtuples, and all its reductions, except for a single reduction, then it satisfies* A_∞*-conditions on the other reduction as well.*

Proof. Let us denote $\pi := \mathrm{pr}_1\colon Ts\mathcal{B} \to s\mathcal{B}$. Then A_∞-conditions (1.5) are equivalent to equation $\pi b f = \pi f b$, where f and b are defined from the components f_i and b_i via (1.4) and (1.6). By assumption, it holds on the given tuple $[x_1, \dots, x_n]$, hence we can apply b_1 to both sides of this equation and obtain that $b_1 b f = b_1 f b$ holds on $[x_1, \dots, x_n]$ as well. Similarly, we get the equation $b_m b f = b_m f b$ on $[x_1, \dots, x_n]$ for any $m \leq n$, since $\pi b f = \pi f b$ holds on all subtuples of $[x_1, \dots, x_n]$. This implies that $\pi b b f = \pi b f b$ on $[x_1, \dots, x_n]$. However, $\pi b b f = 0$, since $bb = 0$ in $\mathcal{B}$. Thus $\pi b f b = 0$ on $[x_1, \dots, x_n]$. Similarly, we have $\pi f b b = 0$ on $[x_1, \dots, x_n]$ since $bb = 0$ in $\mathcal{A}$. Thus $\pi(bf)b = 0 = \pi(fb)b$ on $[x_1, \dots, x_n]$. However, by definition, the latter equation is equal to the sum of equations $\pi b f = \pi f b$ on all possible reductions of the tuple $[x_1, \dots, x_n]$. By assumption, all except for one of these equations hold, hence the other one holds as well. $\square$

Now, we are ready to prove our statement.

Proof of Proposition 3.10. In order to prove the statement, we need to check that f' satisfies A_∞-conditions on all possible tuples $[p_1, \dots, p_n]$ of composable paths in Λ'. Let us check it type by type.

If the tuple has type (1), then the A_∞-conditions are satisfied by definition and strict unitality condition.

If the tuple has type (2a), then the A_∞-conditions are satisfied because f' coincides with f in this case by definition, and f is an A_∞-functor.

If the tuple has type (2b), then the A_∞-conditions take the form of

$$f'_1 b_1(sc) = b_1 f'_1(sc)$$

which holds by definition of $f'_1(sc)$.

If the tuple $[p_1, \dots, p_n]$ has type (3a) but not (1), where $n \geq 2$, then $[p_1, \dots, p_n] = [\, c, ☆, \dots, ☆\,]$, since all the elements should be non-empty. Then the A_∞-conditions on this tuple are satisfied because they were used in order to define the value of f' on the corresponding reduced tuple $[\, c☆, \dots, ☆\,]$ of type (4a).

If the tuple $[p_1, \dots, p_n]$ has type (3b) but not (1), where $n \geq 2$, then

$$[p_1, \dots, p_n] = [\, \square, \dots, \square,\ c★,\ ☆, \dots, ☆\,],$$

since all the elements should be non-empty, and there is at least one element $\square$ before $c★$. Then the A_∞-conditions on this tuple are satisfied because they were used in order to define the value of f' on the corresponding reduced tuple $[\, \square, \dots, \square\, c★, ☆, \dots, ☆\,]$ of type (4b).

If the tuple $[p_1, \dots, p_n]$ has type (4) but not (1), we will prove this statement by induction on the number of occurrences of c in the tuple by using and partially the arguments we used in constructing the values of the functor f' on such tuples. We will start by noticing that the A_∞-conditions are satisfied when there are no occurrences of c, since this is the type (2a) situation which was already covered. Now, let us assume that the statement is true for tuples which have less than m occurrences of c, where $m \geq 1$, and prove that the statement holds for the tuples with m occurrences of c as well. Let us do this by considering two subtypes (4a) and (4b) separately.

If our tuple has type (4a), where $n \geq 1$, then $[p_1, \dots, p_n] = [\, c☆,\ ★, \dots, ★\,]$. Now, let us consider the corresponding tuple

$$[p'_1, \dots, p'_{n+1}] := [\, c, \star, \bigstar, \dots, \bigstar \,]$$

obtained by splitting $p_1 = c\star$ into $p'_1 = c$ and $p'_2 = \star$. This tuple has type (3a), all its subtuples have either type (2b), or type (3a), or have less occurrences of c. Also, all its reductions are either our original tuple $[p_1, \dots, p_n]$, or tuples of type (3a), or the tuple $[\, \partial(c), \star, \star, \dots, \star \,]$ which has less occurrences of c. We previously proved or assumed that the A_∞-conditions are satisfied on all of these tuples except for the original one, hence by Lemma 3.13, they are satisfied on the original tuple as well.

We perform similar argument for a tuple $[p_1, \dots, p_n] = [\blacksquare, \dots, \blacksquare, \, \square\, c\bigstar, \, \bigstar, \dots, \bigstar]$ of type (4b), where $n \geq 1$. In this case we consider the tuple

$$[p'_1, \dots, p'_{n+1}] := [\blacksquare, \dots, \blacksquare, \, \square, c\bigstar, \, \bigstar, \dots, \bigstar]$$

of type (3b). All its subtuples have either type (2b), or (3a), or (3b), or (4a), or have less occurrences of c. Also, all its reductions are either the original tuple $[p_1, \dots, p_n]$, or tuples of type (3b), or tuples with less occurrences of c. Again, we notice that for all these tuples, except for the original one, we have already proved the validity of A_∞-conditions. Thus, we can apply Lemma 3.13 to conclude the result on the original tuple as well. This concludes the proof of the whole statement as well, since we have covered all possible types of tuples. $\square$

3.1.4 Building an s-homotopy between two extensions

Proposition 3.14. *Any two extensions f' and f'' of the original functor $f \colon \mathcal{A} \to \mathcal{B}$ satisfying the assumptions of Theorem 3.1 are s-homotopic.*

Before proving the previous statement, let us firstly establish the following result which is analogous to Lemma 3.13.

Lemma 3.15. *For the linear maps $r_n\colon T^n s\mathcal{A} \to s\mathcal{B}$, where $n \geq 1$, and two functors $f, g\colon \mathcal{A} \to \mathcal{B}$, let us consider Equation (B.1) stating that r defines an s-homotopy from f to g. Then, if this equation holds on a tuple $[x_1, \dots, x_n]$ of composable morphisms in $\mathcal{A}$, on all its subtuples, and on all its reductions, except for a single reduction, then it holds on the latter reduction as well.*

Proof. The proof is analogous to the proof of Lemma 3.13. For $\pi := \mathrm{pr}_1\colon Ts\mathcal{B} \to s\mathcal{B}$, we note that (B.1) is equivalent to $\pi(g - f) = \pi(rb + br)$, where r and b denote the maps $Ts\mathcal{A} \to Ts\mathcal{B}$ defined in terms of the components r_i, b_i, f_i and g_i via (1.4) and (1.7). We can apply b_1 to this equation and get $b_1(g - f) = b_1(rb + br)$. Similarly, we get $b_m(g - f) = b_m(rb + br)$ for $m > 1$, since (B.1) holds on all subtuples of $[x_1, \dots, x_n]$. Thus $b(g-f) = b(rb+br)$ holds on $[x_1, \dots, x_n]$. However, $bbr = 0$, since $bb = 0$ in $\mathcal{B}$. Also, $bg = gb$ and $bf = fb$, since f and g are functors (see Remark 1.27). Thus we have $gb - fb = brb + bbr = brb$. We also have $rbb = 0$, since $bb = 0$ in $\mathcal{A}$. Thus $(g - f)b = gb - fb = brb = brb + rbb = (br + rb)b$ holds on $[x_1, \dots, x_n]$. By definition, the latter equation is decomposed into the sum of equations (B.1) applied to all reductions of $[x_1, \dots, x_n]$. Since all of those equations except for one hold by assumption, so is the latter one. □

Now we are ready to prove our main statement.

Proof of Proposition 3.14. The process for constructing the required s-homotopy is very similar to the process of defining the extension functor. Namely, we consider the values of this unknown s-homotopy on the composable tuples of paths as the variables, and then we determine them from the relation that this is indeed an s-homotopy. We also continue assuming for simplicity that f, f' and f'' are all strictly unital, as the general case reduces to this one by replacing $\mathcal{B}$ by $\mathcal{B}^+$, and by deforming f' and f'' to strictly unital functors as in Remark 3.9.

Let us denote by $r_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ the unknown components of an s-homotopy r between f' and f'', where $n \geq 1$. The definition of such an s-homotopy can be expressed as the set of equations on the values of r_n on all possible composable tuples of paths in Λ'. Omitting the indices, we can write down these equations in the following simplified form:

$$
\begin{aligned}
f'' - f' \ &= \sum r\,(1 \otimes \ldots \otimes 1 \otimes b^{\mathcal{A}} \otimes 1 \otimes \ldots \otimes 1) \\
&\qquad + \sum b^{\mathcal{B}}\,(f'' \otimes \ldots \otimes f'' \otimes r \otimes f' \otimes \ldots \otimes f').
\end{aligned} \tag{3.1}
$$

Now, let us define the values of r_n as follows. On tuples of types (1), (2) and (3), we set $r = 0$. On tuples $[p_1, \ldots, p_n]$ of type (4) but not (1), we define r similarly to how we defined f' in §3.1.2. Namely, we do apply induction on the number of occurrences of c in our tuple. Then, we consider the same tuple $[p'_1, \ldots, p'_{n+1}]$ as in §3.1.2 and apply (3.1) to this tuple. This will become a linear equation for the term $r_n(sp_1 \otimes \ldots \otimes sp_n)$ in which all other terms are expressed via the previously defined values.

Thus we have constructed the map of quivers $r\colon Ts\mathcal{A} \to s\mathcal{B}$ by requiring it to satisfy (3.1) on the given subset of tuples. Now, we need to prove that this equation also holds on the rest of tuples. We can do it similarly to how we have done it in the proof of Proposition 3.10. Namely, we see that the equation is satisfied on tuples of type (1), (2) and (3) automatically. Indeed, for tuples of type (1), this follows from strict unitality assumption. For tuples of type (2), this follows from the definition of our functors f' and f'' as the extensions of f with the given value on c. And for tuples of type (3) but not (1), this follows from the definition of r on the corresponding tuples of type (4). Now, for tuples of type (4) but not (1), we can prove the statement by induction on the number of occurrences of c in the same way as in the proof of Proposition 3.10. Namely, the argument is identical, except that we are using Lemma 3.15 instead of Lemma 3.13.

□

3.2 Homotopy coherent monoid actions on A_∞-categories

Here, we demonstrate how we can use the machinery of cellular **dg**-categories in order for transferring various topological notions into an A_∞-setting. In particular, our main impetus for this work was to provide the correct and workable definition of a homotopy coherent monoid action on A_∞-categories which would have various applications to Homological Mirror Symmetry.

3.2.1 A_∞-categorical A_∞-maps

Here, our aim is to transfer Definition 1.45 of a topological A_∞-map into the setting of A_∞-categories. Namely, we will just replace all topological notion involved in the definition by their A_∞-categorical counterparts as follows.

Let us denote $L_n := I^{n-1}$ which are the $(n-1)$-dimensional cubes parameterizing the components of topological A_∞-maps. In the A_∞-categorical version, these spaces will be replaced by the corresponding **dg**-cubes

$$\mathcal{L}_n := \vec{\mathcal{I}}^{\circledast(n-1)}$$

constructed in Definition 2.57. The topological monoids will be replaced by monoids in $\underline{\mathsf{A}_\infty}$ (see Definition A.1), and continuous functions of multiple arguments (whose input is the direct product of spaces) will be replaced by the corresponding A_∞-functors of multiple inputs.

The boundary conditions for topological A_∞-maps tell us how the restriction of the n-th component on the faces of the corresponding parameterizing cube L_n can be expressed in terms of the previous components. The A_∞-categorical version can be formulated similarly and will involve the definition of faces of **dg**-cubes from §2.5.2 and the corresponding inclusion functors:

$$\iota^{\varepsilon}_{n-1,k} \colon \mathcal{L}_{n-1} \xrightarrow{\cong} d^{\varepsilon}_k(\mathcal{L}_n) \xrightarrow{\subseteq} \mathcal{L}_n,$$

where $1 \le k \le n-1$ and $\varepsilon = 0, 1$.

We will also need to use another A_∞-functor $\pi_{k,l} \colon \vec{\mathcal{I}}^{\circledast k}, \vec{\mathcal{I}}^{\circledast l} \to \vec{\mathcal{I}}^{\circledast(k+l)}$ defined by mapping the pair of cells $(\alpha_1 * \ldots * \alpha_k,\ \beta_1 * \ldots * \beta_l)$ into the cell $\alpha_1 * \ldots * \alpha_k * \beta_1 * \ldots * \beta_l$, where α_i and β_j are among the cells $\{0, 1, i\}$ of the interval $\vec{\mathcal{I}}$. Then we define the corresponding A_∞-functors

$$\jmath^{\varepsilon}_{n-1,k} \colon \mathcal{L}_k,\ \mathcal{L}_{n-k} \xrightarrow{\pi_{k-1,n-k-1}} \mathcal{L}_{n-1} \xrightarrow{\iota^{\varepsilon}_{n-1,k}} \mathcal{L}_n,$$

where $1 \le k \le n-1$ and $\varepsilon = 0, 1$, which map the pair of cells $(\alpha_1 * \ldots * \alpha_{k-1},\ \alpha_{k+1} * \ldots * \alpha_{n-1})$ into $\alpha_1 * \ldots * \alpha_{k-1} * \varepsilon * \alpha_{k+1} * \ldots * \alpha_{n-1}$.

Now, we are ready to formulate the definition.

Definition 3.16. Let $(\mathcal{X}, M_{\mathcal{X}}, E_{\mathcal{X}})$ and $(\mathcal{Y}, M_{\mathcal{Y}}, E_{\mathcal{Y}})$ be two monoids in the multicategory $\underline{\mathsf{A}_\infty}$ of A_∞-categories (see Definition A.1). An *A_∞-categorical A_∞-map* $F\colon \mathcal{X} \to \mathcal{Y}$ consists of the set of A_∞-functors

$$F_n\colon \mathcal{L}_n, \underbrace{\mathcal{X}, \dots, \mathcal{X}}_{n \text{ times}} \longrightarrow \mathcal{Y},$$

for $n \geq 1$, satisfying the following boundary conditions:

$$F_n \circ \iota^0_{n-1,k} = F_{n-1} \circ (\mathrm{id}_{\mathcal{X}}, \dots, \underbrace{M_{\mathcal{X}}}_{k\text{-th place}}, \dots, \mathrm{id}_{\mathcal{X}})\colon\ \mathcal{L}_{n-1}, \underbrace{\mathcal{X}, \dots, \mathcal{X}}_{n \text{ times}} \longrightarrow \mathcal{Y},$$

$$F_n \circ \jmath^1_{n-1,k} = M_{\mathcal{Y}} \circ (F_k, F_{n-k})\colon\ \mathcal{L}_k, \underbrace{\mathcal{X}, \dots, \mathcal{X}}_{k \text{ times}}, \mathcal{L}_{n-k}, \underbrace{\mathcal{X}, \dots, \mathcal{X}}_{n-k \text{ times}} \longrightarrow \mathcal{Y},$$

where $n \geq 2$ and $1 \leq k \leq n-1$, which are analogous to the boundary conditions in Definition 1.45. We also equip F with an additional functor

$$F_e\colon \vec{\mathcal{I}} \longrightarrow \mathcal{Y}$$

such that $F_e(0) = F_1(E_{\mathcal{X}})$ and $F_e(1) = E_{\mathcal{Y}}$.

Remark 3.17. The functors F_n can be viewed as the functors from $\mathcal{L}_n$ to $\underline{\mathsf{A}_\infty}(\mathcal{X}, \dots, \mathcal{X}; \mathcal{Y})$ via equivalences $\varphi^{\mathsf{A}_\infty}$ of §1.2.2. From this viewpoint, we can say that the boundary conditions predefine the values of the functors F_n on the boundary of the cube $\mathcal{L}_n = \vec{\mathcal{I}}^{\boxtimes(n-1)}$ generated by all its cells except for the interior $(n-1)$-cell $i * \ldots * i$. Hence, by Theorem 3.1, such functor is determined uniquely up to s-homotopy by its image on the interior cell.

Remark 3.18. With the above notion, we can imitate various constructs involving topological A_∞-maps in the A_∞-setting. For example, we can define a composition of two A_∞-categorical A_∞-maps similarly to the composition of two topological A_∞-maps.

3.2.2 Homotopy coherent actions

Now, we can define a homotopy coherent monoid action on A_∞-categories as follows.

Definition 3.19. A *homotopy coherent action* of a monoid Γ on an A_∞-category $\mathcal{A}$ is defined to be an A_∞-categorical A_∞-map ρ from the **dg**-category $\Bbbk_\Gamma$ to the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A})$, where both categories are equipped with the structures of monoids in $\underline{\mathsf{A}_\infty}$ as described in Examples A.2 and A.3.

In other words, our definition says that a homotopy coherent action is given by the set of A_∞-functors

$$\rho_n \colon \vec{\mathcal{I}}^{\boxtimes(n-1)}, \underbrace{\Bbbk_\Gamma, \dots, \Bbbk_\Gamma}_{n \text{ times}} \longrightarrow \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A}),$$

satisfying the boundary conditions of Definition 3.16. More precisely, the first component is given by specifying the A_∞-functor

$$\rho_1 \colon \Bbbk_\Gamma \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A}).$$

The second component is given by specifying the A_∞-functor

$$\rho_2 \colon \vec{\mathcal{I}}, \Bbbk_\Gamma, \Bbbk_\Gamma \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A})$$

whose restrictions on $0, 1 \in \operatorname{Ob}\vec{\mathcal{I}}$ coincide with the functors $\rho_1 \circ M_{\Bbbk_\Gamma}$ and $M_{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A})} \circ (\rho_1, \rho_1)$ respectively. The third component is given similarly by the A_∞-functor

$$\rho_3 \colon \vec{\mathcal{I}}^{\boxtimes 2}, \Bbbk_\Gamma, \Bbbk_\Gamma, \Bbbk_\Gamma \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A})$$

with an analogous boundary condition on the boundary of the square $\vec{\mathcal{I}}^{\boxtimes 2}$.

As follows from Remark 3.17 and Theorem 3.1, any homotopy coherent action ρ can be uniquely recovered (up to an s-homotopy) from a quite small subset of its data. Namely, the first component of the corresponding A_∞-map is determined by specifying the collection of A_∞-functors

$$\rho(g)\colon \mathcal{A} \to \mathcal{A} \ \text{ for } \ g \in \Gamma.$$

The second component (measuring the failure of the first component to commute with the multiplications) is determined by the choice of natural transformations

$$\eta_{g,h}\colon \rho(gh) \longrightarrow \rho(g)\rho(h) \ \text{ for } \ g, h \in \Gamma,$$

which can be assumed to be natural equivalences, if needed. The higher components are determined by the choice of higher coherence data on $\eta_{g,h}$. Namely, for $n \geq 3$ we just need to specify one homotopy for each choice of $g_1, \dots, g_n \in \Gamma$ between certain A_∞-transformations $\rho(g_1 \dots g_n) \to \rho(g_1) \dots \rho(g_n)$ defined in terms of previous components.

Remark 3.20. The above description suggests that we could define a homotopy coherent action by just providing the functors $\rho(g)$, the transformations $\eta_{g,h}$ and the higher coherence data satisfying the respective coherence conditions. Actually, one may question the need for introducing the notion of cellular **dg**-categories and A_∞-categorical A_∞-maps and try to write down such coherence data and conditions explicitly. However, this is not such an obvious task, as these conditions become exponentially more complicated at each step. In fact, the problem of explicitly describing these conditions requires performing a procedure equivalent to the one described in §2.4.1 for n-cubes which is not trivial.

Example 3.21. If we have two equivalent A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ with the equivalences given by $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, then we can transfer monoid actions from $\mathcal{A}$ to $\mathcal{B}$ and vice versa. For example, if ρ is an action of Γ on $\mathcal{A}$, that is an A_∞-map from $\Bbbk_\Gamma$ to $\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$, then we can define the corresponding action on $\mathcal{B}$ by composing this A_∞-map with another naturally defined A_∞-map $\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}) \to \underline{\mathsf{A}_\infty}(\mathcal{B}, \mathcal{B})$ whose first component is given by the functor

$$\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}) \to \underline{\mathsf{A}_\infty}(\mathcal{B}, \mathcal{B}), \quad H \mapsto F \circ H \circ G.$$

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Appendix A: Monoids in $\underline{\mathsf{A}_\infty}$

The category $\underline{\mathsf{A}_\infty}$ of the A_∞-categories does not have a natural symmetric monoidal structure. In other words, there is no natural notion of a tensor product of two A_∞-categories compatible with all the data. However, it still makes sense to talk about the monoids in $\underline{\mathsf{A}_\infty}$ by using its natural structure of the multicategory. Here, we recall this definition and provide two examples of monoids in $\underline{\mathsf{A}_\infty}$ used in our construction in §3.2.1.

Definition A.1 (see also [Lei04, Ex. 2.1.14])**.** A *monoid* $(\mathcal{X}, M, E)$ in the multicategory $\underline{\mathsf{A}_\infty}$ is an A_∞-category $\mathcal{X}$ equipped with a multiplication $M\colon \mathcal{X}, \mathcal{X} \to \mathcal{X}$, which is an A_∞-functor of two arguments, and a unit $E\colon () \to \mathcal{X}$, which is an A_∞-functor of no arguments defined by picking an object of $\mathcal{X}$, satisfying the associativity law $M \circ (M, \mathrm{id}_\mathcal{X}) = M \circ (\mathrm{id}_\mathcal{X}, M)$ and the right and left unity laws $M \circ (\mathrm{id}_\mathcal{X}, E) = \mathrm{id}_\mathcal{X} = M \circ (E, \mathrm{id}_\mathcal{X})$, where $\mathrm{id}_\mathcal{X}$ is the identity A_∞-functor of $\mathcal{X}$.

Example A.2 (Composition monoid)**.** The A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$ of A_∞-endofunctors of a given A_∞-category $\mathcal{A}$ has a natural structure of a monoid in $\underline{\mathsf{A}_\infty}$. Namely, the multiplication is given by the composition functor

$$M\colon \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}),\ \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}) \to \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}),$$

and the unit

$$E\colon () \to \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$$

is defined by picking the identity functor $\mathrm{id}_\mathcal{A}\colon \mathcal{A} \to \mathcal{A}$ considered as the object in $\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$. In §A.1, we check that so-defined multiplication and unit indeed satisfy the associativity condition $M \circ (M, \mathrm{id}) = M \circ (\mathrm{id}, M)$ and the unity conditions $M \circ (\mathrm{id}, E) = \mathrm{id} = M \circ (E, \mathrm{id})$, where $\mathrm{id} = \mathrm{id}_{\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})}$.

Example A.3 (Discrete monoid)**.** Consider an ordinary monoid, that is a set Γ equipped with a multiplication map $m\colon \Gamma \times \Gamma \to \Gamma$ and a unit $e \in \Gamma$, satisfying associativity and unity conditions. We can define a discrete **dg**-category $\Bbbk_\Gamma$ corresponding to Γ as follows:

$$\begin{aligned} \operatorname{Ob} \Bbbk_\Gamma &:= \Gamma, \\ \Bbbk_\Gamma(x,x) &:= \Bbbk 1_x, \\ \Bbbk_\Gamma(x,y) &:= 0 \ \text{ if } \ x \neq y. \end{aligned}$$

This category trivially inherits the structure of a monoid in $\underline{\mathsf{A}_\infty}$ from that in Γ. The unit in $\Bbbk_\Gamma$ is given by the object e and the multiplication is given by a strict A_∞-functor $\Bbbk_m\colon \Bbbk_\Gamma, \Bbbk_\Gamma \to \Bbbk_\Gamma$ which acts as $(x,y) \mapsto m(x,y)$ on the objects. The functor $\Bbbk_m$ satisfies the associativity and unity properties with respect to the unit e, thus endowing $\Bbbk_\Gamma$ with the structure of monoid in $\underline{\mathsf{A}_\infty}$.

A.1 Properties of the composition functors

Here, we check that composition functors in A_∞-categories satisfy the associativity and the unity conditions, thus verifying that the composition monoid described in Example A.2 is well-defined.

For the sake of brevity, we will be denoting the identity functors by 1. Also, we note that the proofs becomes easier to read if we consider compositions of arbitrary functors $\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})$ rather than only of endofunctors $\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})$, thus we consider different categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, instead of just $\mathcal{A}$.

By definition from §1.2.2, for the A_∞-categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the composition A_∞-functor

$$M\colon \underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}), \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{C})$$

is defined as the unique functor satisfying the following commutative diagram:

$$
\begin{tikzcd}
\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}),\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A} \arrow[r, "M{,}1"] \arrow[d, "1{,}\mathrm{ev}^{\mathsf{A}_\infty}"'] & \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{C}),\mathcal{A} \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\
\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}),\mathcal{B} \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \mathcal{C}
\end{tikzcd}
$$

We can use this property in order to prove that M satisfies associativity and unity properties even without invoking the lengthy explicit formulas for the components of M.

Lemma A.4 (Associativity property)**.** *For A_∞-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, the following holds:*

$$M \circ (1, M) \;=\; M \circ (M, 1)\colon\; \underline{\mathsf{A}_\infty}(\mathcal{C},\mathcal{D}),\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}),\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{D}).$$

Proof. Indeed, consider the following commutative diagram:

$$
\begin{tikzcd}
\boxed{\begin{matrix}\underline{\mathsf{A}_\infty}(\mathcal{C},\mathcal{D}),\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}),\\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A}\end{matrix}} \arrow[r, "M{,}1{,}1"] \arrow[rr, bend left, "M\circ(M{,}1){,}1"] \arrow[d, "1{,}M{,}1"] \arrow[dr, "1{,}1{,}\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[dd, bend right=60, "M\circ(1{,}M){,}1"'] & \boxed{\begin{matrix}\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{D}),\\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A}\end{matrix}} \arrow[r, "M{,}1"] \arrow[dr, "1{,}\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[d, phantom, "2"] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{D}),\mathcal{A}} \arrow[dr, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[d, phantom, "1"] & \\
\boxed{\begin{matrix}\underline{\mathsf{A}_\infty}(\mathcal{C},\mathcal{D}),\\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{C}),\mathcal{A}\end{matrix}} \arrow[d, "M{,}1"] \arrow[dr, "1{,}\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[r, phantom, "1"] & \boxed{\begin{matrix}\underline{\mathsf{A}_\infty}(\mathcal{C},\mathcal{D}),\\ \underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{C}),\mathcal{B}\end{matrix}} \arrow[r, "M{,}1"] \arrow[d, "1{,}\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[dr, phantom, "1"] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{D}),\mathcal{B}} \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \mathcal{D} \arrow[dl, equal] \\
\boxed{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{D}),\mathcal{A}} \arrow[dr, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[r, phantom, "1"] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{C},\mathcal{D}),\mathcal{C}} \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \mathcal{D} \arrow[dl, equal] & \\
& \mathcal{D} & &
\end{tikzcd}
$$

In this diagram, the squares labelled by 1 commute due to the described property of M applied for the different triples of categories among $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. The square 2 commutes because both paths from its top-left to the bottom-right give $(M, \mathrm{ev}^{\mathsf{A}_\infty})$. Hence, the whole diagram is commutative and we get the equality

$$\mathrm{ev}^{\mathsf{A}_\infty} \circ (1, M \circ (M, 1)) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (1, M \circ (1, M)). \tag{A.1}$$

As we mentioned in §1.2.2, the function

$$\varphi^{\mathsf{A}_\infty} \colon \mathsf{A}_\infty\Big(\underline{\mathsf{A}_\infty}(\mathcal{C}, \mathcal{D}), \underline{\mathsf{A}_\infty}(\mathcal{B}, \mathcal{C}), \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{B});\ \ \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{D})\Big)$$
$$\longrightarrow \mathsf{A}_\infty\Big(\underline{\mathsf{A}_\infty}(\mathcal{C}, \mathcal{D}), \underline{\mathsf{A}_\infty}(\mathcal{B}, \mathcal{C}), \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{B}),\ \mathcal{A};\ \ \mathcal{D}\Big),$$

defined by mapping an A_∞-functor

$$f \colon \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{B}), \underline{\mathsf{A}_\infty}(\mathcal{B}, \mathcal{C}), \underline{\mathsf{A}_\infty}(\mathcal{C}, \mathcal{D}) \to \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{D})$$

to the functor $\varphi^{\mathsf{A}_\infty}(f) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (f, 1)$, is bijective (here, $1 = \mathrm{id}_{\mathcal{A}}$). However, in terms of $\varphi^{\mathsf{A}_\infty}$, we can rewrite Equation (A.1) as

$$\varphi^{\mathsf{A}_\infty}\big(M \circ (1, M)\big) = \varphi^{\mathsf{A}_\infty}\big(M \circ (M, 1)\big),$$

which implies that $M \circ (M, 1) = M \circ (1, M)$ due to bijectivity of $\varphi^{\mathsf{A}_\infty}$. Hence the proof. $\square$

For proving the unity property, we will need the following lemma.

Lemma A.5. *For any A_∞-category $\mathcal{A}$, and the functor $E \colon () \to \underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$ defined by picking the object $\mathrm{id}_{\mathcal{A}}$ in $\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A})$, the following diagram of A_∞-functors commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "{E,1}"] \arrow[dr, "1"'] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{A}, \mathcal{A}), \mathcal{A}} \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\ & \mathcal{A} \end{tikzcd}$$

Proof. The proof follows from a simple computation of the components of the composition

$$\mathrm{ev}^{\mathsf{A}_\infty} \circ (E, 1)$$

by using the explicit expressions for the components of $\mathrm{ev}^{\mathsf{A}_\infty}$ and $\mathrm{id}_{\mathcal{A}}$. □

Lemma A.6 (Right unity property)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary A_∞-categories. Then the composition functor*

$$M\colon\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}), \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})$$

and the unity functor

$$E\colon\ () \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A}), \quad * \mapsto \mathrm{id}_{\mathcal{A}}$$

satisfy the right unity property as follows:

$$M \circ (1, E) \ =\ 1\colon\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),$$

where $1 = \mathrm{id}_{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})}$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A} \arrow[r, "{1,E,1}"] \arrow[rr, bend left, "{M\circ(1,E),1}"] \arrow[dr, "{1,1}"'] & \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{A}),\mathcal{A} \arrow[r, "{M,1}"] \arrow[d, "{1,\mathrm{ev}^{\mathsf{A}_\infty}}"] \arrow[dl, phantom, "3"] \arrow[dr, phantom, "1"] & \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A} \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\
& \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A} \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \mathcal{B}
\end{tikzcd}$$

In this diagram, the square 1 is commutative by the above definition of M, while the triangle 3 is commutative due to Lemma A.5. Hence, the whole diagram commutes, and we get

$$\mathrm{ev}^{\mathsf{A}_\infty} \circ (1, M \circ (E, 1)) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (1, 1).$$

Similarly to the proof of Lemma A.4, let us consider the bijective function

$$\varphi^{\mathsf{A}_\infty} \colon \mathsf{A}_\infty\Big(\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})\Big) \to \mathsf{A}_\infty\Big(\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A};\ \mathcal{B}\Big),$$

defined by $\varphi^{\mathsf{A}_\infty}(f) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (f,1)$, where $1 = \mathrm{id}_{\mathcal{A}}$. From above, we get

$$\varphi^{\mathsf{A}_\infty}\Big(M \circ (E,1)\Big) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (M \circ (1,E),1) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (1,1) = \varphi^{\mathsf{A}_\infty}(1),$$

hence $M \circ (E,1) = 1$ due to bijectivity of $\varphi^{\mathsf{A}_\infty}$. □

Lemma A.7 (Left unity property)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary A_∞-categories. Then the composition functor*

$$M \colon \underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{B}), \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})$$

and the unity functor

$$E \colon\ () \to \underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{B}), \quad * \mapsto \mathrm{id}_{\mathcal{B}}$$

satisfy the left unity property as follows:

$$M \circ (E,1) \ = \ 1 \colon\ \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}) \to \underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),$$

where $1 = \mathrm{id}_{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B})}$.

Proof. Consider the following diagram:

$$
\begin{tikzcd}
\boxed{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A}} \arrow[r, "{E,1,1}"] \arrow[rr, bend left, "{M\circ(E,1),1}"] \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[dr, phantom, "2"] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A}} \arrow[r, "{M,1}"] \arrow[d, "{1,\mathrm{ev}^{\mathsf{A}_\infty}}"] \arrow[dr, phantom, "1"] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{A},\mathcal{B}),\mathcal{A}} \arrow[dd, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\
\mathcal{B} \arrow[r, "{E,1}"] \arrow[ddr, phantom, "3"] \arrow[drr, bend right, "1"'] & \boxed{\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{B}),\mathcal{B}} \arrow[dr, "\mathrm{ev}^{\mathsf{A}_\infty}"] & \\
 & & \mathcal{B}
\end{tikzcd}
$$

In this diagram, 1 commutes by definition of M, 2 commutes since both sides produce $(E, \mathrm{ev}^{\mathsf{A}_\infty})$, and 3 commutes by Lemma A.5. Thus, the whole diagram is commutative. Similarly to the proof of Lemma A.6, we get

$$\varphi^{\mathsf{A}_\infty}\Big(M \circ (E,1)\Big) = \mathrm{ev}^{\mathsf{A}_\infty} \circ (M \circ (E,1),1) = \mathrm{ev}^{\mathsf{A}_\infty} = \varphi^{\mathsf{A}_\infty}(1),$$

where $\varphi^{\mathsf{A}_\infty}$ is the same bijective map as in Lemma A.6, and the middle equality holds due to the commutativity of our diagram. Thus $M \circ (E,1) = 1$, hence the proof. □

Appendix B: s-homotopies and deformations of A_∞-functors

Seidel in [Sei08] introduced a way of comparing A_∞-functors $f, g\colon \mathcal{A} \to \mathcal{B}$ that does not require the output category $\mathcal{B}$ to be unital but requires the functors to have the same images on the objects, that is $\operatorname{Ob} f = \operatorname{Ob} g$.

Before formulating the definition, let us just notice that if two functors $f, g\colon \mathcal{A} \to \mathcal{B}$ have the same images on the objects, then we can formally substract the corresponding maps of graded quivers $f, g\colon Ts\mathcal{A} \to Ts\mathcal{B}$. It is easy to check that the difference $g - f\colon Ts\mathcal{A} \to Ts\mathcal{B}$ then becomes a (g, f)-coderivation of degree 0.

Definition B.1 (Seidel)**.** Let two functors $f, g\colon \mathcal{A} \to \mathcal{B}$ have the same images on the objects. We will call them *s-homotopic*, if there is a (g, f)-coderivation $r\colon Ts\mathcal{A} \to Ts\mathcal{B}$ of degree -1 such that $r_0 = 0\colon T^0 s\mathcal{A} \to s\mathcal{B}$ and $g - f = B_1(r)$ is the equality of (g, f)-coderivations. Such r is called an *s-homotopy* between f and g. We will denote this situation as $r\colon f \overset{s}{\sim} g$.

Remark B.2. In [Sei08], the functors from the above definition are called just *homotopic*.

We can use explicit formula for B_1 in (1.8) in order to express the condition that $r\colon f \overset{s}{\sim} g$ as the following set of equations on the components of f, g and r:

$$\begin{aligned} g_n - f_n = &\sum b_{l+1+k}\,(g_{j_1} \otimes \ldots \otimes g_{j_l} \otimes r_p \otimes f_{i_1} \otimes \ldots \otimes f_{i_k}) \\ &+ \sum r_{l+1+k}\,(1^{\otimes l} \otimes b_q \otimes 1^{\otimes k}) : T^n s\mathcal{A} \to s\mathcal{B}, \end{aligned} \tag{B.1}$$

where $n \geq 1$, the first sum is taken over the decompositions $n = j_1 + \ldots + j_l + p + i_1 + \ldots + i_k$ for $l \geq 0$, $k \geq 0$, $j_m \geq 1$, $i_m \geq 1$ and $p \geq 1$, and the second sum is taken over the decompositions $n = l + q + k$ for $l \geq 0$, $k \geq 0$, $q \geq 1$. (The summations here are the same as in (1.8), except that $p \neq 0$ due to assumption that $r_0 = 0$.)

Remark B.3. Note that the right-hand-side of (B.1) depends only on the components f_i and g_i for $i < n$, and r_i for $i \le n$.

B.1 Deformation Lemma and applications

The following result is a useful tool for deforming various A_∞-functors.

Lemma B.4 (Deformation Lemma)**.** *Let $f\colon \mathcal{A} \to \mathcal{B}$ be an arbitrary functor and $t_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ be the morphisms of quivers for $n \ge 1$ of degrees -1 that act identically with f on the objects. Then the following statements hold:*

(1) There exist a functor $g\colon \mathcal{A} \to \mathcal{B}$ s-homotopic to f together with an s-homotopy $r\colon f \overset{s}{\sim} g$ such that $r_n = t_n$. Such g and r are uniquely defined by f and t. We will denote them as $g = \phi_t(f)$, and $r = t_{f\to}$.

(2) There exist a functor $g\colon \mathcal{A} \to \mathcal{B}$ s-homotopic to f together with an s-homotopy $r\colon g \overset{s}{\sim} f$ such that $r_n = t_n$. Such g and r are uniquely defined by f and t. We will denote them as $g = \phi_t^{-1}(f)$ and $r = t_{\to f}$.

Remark B.5. We can describe the situations of this lemma by using its new notations as follows:

$$t_{f\to}\colon f \overset{s}{\sim} \phi_t(f), \qquad t_{\to f}\colon \phi_t^{-1}(f) \overset{s}{\sim} f.$$

Proof of Lemma B.4. Let us prove only the first part, since the proof of the second part is identical. The proof follows from (B.1) and Remark B.3. Indeed, we can use (B.1) in order to inductively define g_n from the components f_i for $i \le n$, g_i for $i < n$, and $r_i = t_i$ for $i \le n$. For example, the first two components of g will be defined as follows:

$$\begin{aligned} g_1 =& f_1 + b_1 r_1 + r_1 b_1, \\ g_2 =& f_2 + b_1 r_2 + b_2(g_1 \otimes r_1 + r_1 \otimes f_1) + r_1 b_2 + r_2(1 \otimes b_1 + b_1 \otimes 1). \end{aligned}$$

These components g_n satisfy A_∞-conditions, hence g is a well-defined A_∞-functor which is s-homotopic to f by construction, where the s-homotopy has the given components. $\square$

Lemma B.6. *Let $f\colon \mathcal{A} \to \mathcal{B}$ and $t_n\colon T^n s\mathcal{A} \to s\mathcal{B}$ be as above, and $g = \phi_t(f)$.*

1. *If $t_i = 0$ for $i < k$, then $g_i = f_i$ for $i < k$ and $g_k = f_k \pm \partial(t_k)$.*

2. *If $t_i = 0$ for $i < k$ and $\partial(t_k) = 0$, then $g_i = f_i$ for $i \le k$.*

Lemma B.7. *Let $\mathcal{A}$ is a unital A_∞-category, $\mathcal{A}^+$ be a compatible homotopy unital structure, and $\iota\colon \mathcal{A} \to \mathcal{A}^+$ be a corresponding strict inclusion functor. There is an A_∞-functor $\pi\colon \mathcal{A}^+ \to \mathcal{A}$ such that $\pi\iota = \mathrm{id}_\mathcal{A}$ and $\iota\pi \overset{s}{\sim} \mathrm{id}_{\mathcal{A}^+}$.*

Proof. We can build π by deforming the identity functor $\mathrm{id}_{\mathcal{A}^+}$ via the deformation lemma. We define $t_1|_{\mathcal{A}\oplus\Bbbk\mathbf{j}} = 0$, $t_1(\mathbf{i}_X^{\mathrm{su}}) = \pm\mathbf{j}_X$ and $t_i = 0$ for $i \ge 2$. Then the image of $\phi_t^{\pm}(\mathrm{id}_{\mathcal{A}^+})$ will lie inside $\mathcal{A}$, hence we can pass it through $\mathcal{A}$ to get π. $\square$

Remark B.8. We will see later that the above lemma also implies that the functor π is an A_∞-equivalence inverse to ι.

Lemma B.9. *If $g^i\colon \mathcal{A} \to \mathcal{B}$ is a sequence of functors identical on the objects such that the components g_k^i stabilize to g_k for each $k \ge 1$ for large enough i, then the components g_k form a well-defined functor $g = \lim_i g^i\colon \mathcal{A} \to \mathcal{B}$.*

Proof. Indeed, we just need to check A_∞-conditions for the components of g. Since the n-th condition depends only on the components g_k of g for $k \le n$, let us take i large enough so that the first n components of g^i have already stabilized to g, that is $g_k^i = g_k$ for $k \le n$. Now, we see that g satisfies the n-th A_∞-condition because g^i does. $\square$

Lemma B.10. *Let g^i be a sequence of functors converging to $g = \lim_i g^i$ in the sense of the previous lemma, and $r^i\colon f \overset{s}{\sim} g^i$ be a sequence of corresponding s-homotopies such that the components r_k^i stabilize to r_k for each $k \ge 1$ for large enough i. Then the components $(r_k)_k$ define an s-homotopy $r = \lim_i r^i\colon f \to g = \lim_i g^i$.*

Proof. Similarly to the proof of the previous lemma, for each $n \geq 1$, let us notice that the n-th component of the condition $g^i - f = B_1(r^i)$ can be expressed in terms of only the first n components of g^i, f and r^i. Thus as soon as all these components stabilize to the first n components of g, f and r respectively, we get the equality $(g - f)_n = (B_1(r))_n$. Hence, $g - f = B_1(r)$ as well. $\square$

Lemma B.11. *Let $\mathcal{A}$ be a cellular* **dg***-category and $\mathcal{B}$ be a strictly unital A_∞-category. Then any unital functor from $\mathcal{A}$ to $\mathcal{B}$ is s-homotopic to a strictly unital one.*

Proof. Idea of the proof is to inductively deform our functor so that to make it stricter at each step and then to take the inductive limit. Namely, at n-th step we ensure that the strict unitality condition holds for all $i \leq n$ and then we choose the right deformation in order to make the functor strict at the level $n + 1$ as well. $\square$

B.2 Strict unit transformations

Here, we consider the functors from an arbitrary A_∞-category $\mathcal{A}$ to a strictly unital A_∞-category $\mathcal{B}$ having the strict units $\mathbf{i}^{\mathrm{su}}_X$ for each $X \in \mathcal{B}$.

Definition B.12. For any two A_∞-functors $f, g\colon \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \operatorname{Ob} g$, let us define a *strict unit transformation* $\mathbf{i}^{\mathrm{su}}_{f\to g}\colon f \to g\colon \mathcal{A} \to \mathcal{B}$ of degree 0 by specifying the components of the corresponding (g, f)-coderivation $s\mathbf{i}^{\mathrm{su}}_{f\to g}$ of degree -1 as follows:

$$\left(s\mathbf{i}^{\mathrm{su}}_{f\to g}\right)_{0,X} = s\mathbf{i}^{\mathrm{su}}_{f(X)}, \text{ for } X \in \mathcal{A},$$

$$\left(s\mathbf{i}^{\mathrm{su}}_{f\to g}\right)_n = 0\colon T^n s\mathcal{A} \to s\mathcal{B}, \text{ for } n \geq 1.$$

Lemma B.13. *For any strict unit transformation $\mathbf{i}^{\mathrm{su}}_{f\to g}\colon f \to g\colon \mathcal{A} \to \mathcal{B}$, we have $B_1(s\mathbf{i}^{\mathrm{su}}_{f\to g}) = f - g$ and $\partial(\mathbf{i}^{\mathrm{su}}_{f\to g}) = s^{-1}(g - f)$.*

Proof. Indeed, the direct computation shows that

$$\big(B_1(s\mathbf{i}^{\mathrm{su}}_{f\to g})\big)_1 = b_2(g_1 \otimes s\mathbf{i}^{\mathrm{su}}) + b_2(s\mathbf{i}^{\mathrm{su}} \otimes f_1) = (f-g)_1 \colon s\mathcal{A} \to s\mathcal{B},$$

and similarly for higher n. In other words, $B_1(s\mathbf{i}^{\mathrm{su}}_{f\to g}) = f - g$ and $\partial(\mathbf{i}^{\mathrm{su}}_{f\to g}) = s^{-1}(g-f)$. □

Remark B.14. If $f = g$, then $\mathbf{i}^{\mathrm{su}}_{f\to g} = \mathbf{i}^{\mathrm{su}}_{f\to f}$ becomes a natural transformation, since $B_1(s\mathbf{i}^{\mathrm{su}}_{f\to f}) = f - f = 0$.

Any (g, f)-coderivation $r\colon Ts\mathcal{A} \to Ts\mathcal{B}$ can be uniquely determined by its components $r_n\colon T^n s\mathcal{A} \to s\mathcal{B}$, for $n \geq 0$, where the higher matrix components of r depend both on r_n and on the components of f and g. It will be useful for us to vary the functors f and g while keeping the components r_n the same. For this purpose, we introduce the following notation.

Definition B.15. Let $f', g'\colon \mathcal{A} \to \mathcal{B}$ be the functors which act on the objects in the same way as $f, g\colon \mathcal{A} \to \mathcal{B}$ respectively. For any (g, f)-coderivation r, we will denote by $r_{f'\to g'}$ the (g', f')-coderivation which has the same components as r, that is $(r_{f'\to g'})_n = r_n$ for $n \geq 0$.

Lemma B.16. *The strict unit transformations satisfy the following properties, where $h\colon \mathcal{A} \to \mathcal{B}$ is another functor acting on the objects in the same way as f and g:*

(1) For any (h, g)-coderivation r, we have $B_2(r, s\mathbf{i}^{\mathrm{su}}_{f\to g}) = (-)^{|r|+1} r_{f\to h}$.

(2) For any (g, f)-coderivation r, we have $B_2(s\mathbf{i}^{\mathrm{su}}_{g\to h}, r) = r_{f\to h}$.

(3) In particular, $B_2(s\mathbf{i}^{\mathrm{su}}_{g\to h}, s\mathbf{i}^{\mathrm{su}}_{f\to g}) = s\mathbf{i}^{\mathrm{su}}_{f\to h}$.

(4) $B_n(\dots, s\mathbf{i}^{\mathrm{su}}_{f\to g}, \dots) = 0$.

Proof. This follows from the trivial computations. □

The immediate corollary is the following.

Lemma B.17. *The transformations $\mathbf{i}^{\mathrm{su}}_{f\to f}$ for the functors $f\colon \mathcal{A} \to \mathcal{B}$ provide a set of strict units on the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A}; \mathcal{B})$.*

B.3 s-homotopies as functor equivalences

Seidel noted that if $\mathcal{B}$ is strongly unital, then any s-homotopy $r\colon f \overset{s}{\sim} g$ gives rise to a natural transformation $(\mathbf{i}^{\mathrm{su}}_{f\to g} + s^{-1}r)\colon f \to g$, since $B_1(s\mathbf{i}^{\mathrm{su}}_{f\to g} + r) = (f-g) + (g-f) = 0$. In the next lemma, we show that this tranformation is in fact an equivalence. After that, we also prove an analogous result in a more general case when $\mathcal{B}$ is non-strongly unital.

Lemma B.18. *Let $r\colon f \overset{s}{\sim} g$ be an s-homotopy between the functors from $\mathcal{A}$ to $\mathcal{B}$ where $\mathcal{B}$ is strictly unital. Then $(\mathbf{i}^{\mathrm{su}}_{f\to g} + s^{-1}r)\colon f \to g$ is a natural equivalence.*

Proof. To prove that the given natural transformation is an equivalence, we need to find both right and left homotopy inverse to it. In fact, due to $\mathcal{B}$ being strict, we can explicitly build the strict inverses to the given transformation. These strict inverses will be also defined via the s-homotopies.

Let us construct the *right inverse* s-homotopy $r^{\vee}\colon g \overset{s}{\sim} f$ which satisfies the following equation:

$$B_2(s\mathbf{i}^{\mathrm{su}}_{f\to g} + r,\ s\mathbf{i}^{\mathrm{su}}_{g\to f} + r^{\vee}) = s\mathbf{i}^{\mathrm{su}}_{g\to g}.$$

By opening the brackets on the left side, applying Lemma B.16 and cancelling $s\mathbf{i}^{\mathrm{su}}_{g\to g}$, we get the following equivalent equation:

$$r_{g\to g} + r^{\vee}_{g\to g} + B_2(r, r^{\vee}) = 0. \tag{B.2}$$

Now, let us note that if we set $\left(r^{\vee}_{g\to g}\right)_0 = 0$, then the n-th component of $B_2(r, r^{\vee})$ only depends on f, g, and on the components r_i and $r^{\vee}_i$ for $i < n$. Thus, the n-th component $r^{\vee}_n$ can be uniquely defined from (B.2) in terms of r_n and of the previous components of r and $r^{\vee}$.

For example, the first three components of $r^\vee$ are defined as follows:

$$\begin{aligned}
r_1^\vee &= -\, r_1, \\
r_2^\vee &= -\, r_2 - b_2(r_1 \otimes r_1^\vee) = -r_2 + b_2(r_1 \otimes r_1), \\
r_3^\vee &= -\, r_3 - b_2(r_1 \otimes r_2^\vee) - b_2(r_2 \otimes r_1^\vee) \\
&\quad - b_3(g_1 \otimes r_1 \otimes r_1^\vee) - b_3(r_1 \otimes f_1 \otimes r_1^\vee) - b_3(r_1 \otimes r_1^\vee \otimes g_1).
\end{aligned}$$

Thus $\mathbf{i}^{\mathrm{su}}_{f\to g} + s^{-1}r$ has a right strict inverse natural transformation given by $\mathbf{i}^{\mathrm{su}}_{g\to f} + s^{-1}(r^\vee)\colon g \to f$. Similarly, it has a left strict inverse natural transformation $\mathbf{i}^{\mathrm{su}}_{g\to f} + s^{-1}({}^\vee r)\colon g \to f$ for some other homotopy ${}^\vee r\colon g \overset{s}{\sim} f$ whose components are also uniquely defined by those of r, f and g from the following equation:

$$B_2(s\mathbf{i}^{\mathrm{su}}_{g\to f} + {}^\vee r,\, s\mathbf{i}^{\mathrm{su}}_{f\to g} + r) = s\mathbf{i}^{\mathrm{su}}_{f\to f}.$$

For example, the first two components of ${}^\vee r$ will be the same as of $r^\vee$. This concludes the proof that $\mathbf{i}^{\mathrm{su}}_{f\to g} + s^{-1}r$ is an equivalence. □

Remark B.19. The equation (B.2) makes sense for a non-unital category $\mathcal{B}$ as well. In [Sei08], Seidel used it to define the inverse s-homotopies. He also defined the composition $q \overset{s}{\circ} r$ of two s-homotopies $r\colon f \overset{s}{\sim} g$ and $q\colon g \overset{s}{\sim} h$ from the analogous formula

$$B_2(s\mathbf{i}^{\mathrm{su}}_{g\to h} + q,\, s\mathbf{i}^{\mathrm{su}}_{f\to g} + r) = s\mathbf{i}^{\mathrm{su}}_{f\to h} + q \overset{s}{\circ} r.$$

Now, let us generalize the previous result to the case of an arbitrary unital category $\mathcal{B}$.

Lemma B.20. *If $\mathcal{B}$ is unital and the functors $f, g\colon \mathcal{A} \to \mathcal{B}$ are s-homotopic, then they are also naturally equivalent.*

Proof. Consider some s-homotopy $r\colon f \overset{s}{\sim} g$. Let us compose it with the inclusion $\iota_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}^+$. It gives rise to an s-homotopy $\bar r\colon \bar f \overset{s}{\sim} \bar g$ of the correspoding functors to the category $\mathcal{B}^+$. Since $\mathcal{B}^+$ is strongly unital, the previous lemma provides us a natural equivalence $(\mathbf{i}^{\mathrm{su}}_{\bar f\to \bar g} + s^{-1}\bar r)\colon \bar f \to$

$\bar{g}\colon \mathcal{A} \to \mathcal{B}^+$ according to the previous lemma. Now, let us compose this equivalence with the projection functor $\mathcal{B}^+ \to \mathcal{B}$ constructed before. The result will give us an equivalence from f to g as well. $\square$

In particular, this implies the following.

Lemma B.21. *The functors $\iota\colon \mathcal{A} \to \mathcal{A}^+$ and $\pi\colon \mathcal{A}^+ \to \mathcal{A}$ are A_∞-equivalences of unital A_∞-categories.*