

On Wrapped Fukaya Category and loop space of Lagrangians in a Liouville Manifold

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Abstract

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We introduce an A_∞ map from the singular complex of based loop space of Lagrangian submanifolds with legendrian boundary of a Liouville Manifold $C_{n-*}(\Omega_L \mathcal{L}ag)$ to wrapped floer cohomology of Lagrangian submanifold $C\mathcal{W}^*(L, L)$. In the case of a cotangent bundle and a Lagrangian cofiber, the composition of our map with the map from $C\mathcal{W}^*(L, L) \rightarrow C_{n-*}(\Omega_q Q)$ as defined in [5] shows that this map is split surjective.

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Chapter 1: Introduction

Originally arising from attempts to understand the geometry of Hamiltonian mechanics, symplectic geometry is the study of even dimensional manifolds that are equipped with a non-degenerate 2-form. The canonical model of a symplectic manifold is the cotangent bundle of a smooth manifold, where the symplectic form arises from pairing tangent and cotangent vectors. Lagrangian submanifolds, which are half dimensional submanifolds on which the symplectic form vanishes, are key to the study of symplectic structures. This is illustrated in the Weinstein neighborhood theorem [26], which roughly states that every Lagrangian locally looks like the zero section of a cotangent bundle. In the case of a cotangent bundle T^*Q , the zero section and all cofibers T_q^*Q are Lagrangian submanifolds. The study of symplectic geometry underwent a revolution after Gromov [16] introduced pseudo-holomorphic curves. Soon after, Floer introduced his famous homology theory to solve Arnold's conjecture on fixed points of Hamiltonian systems and Fukaya changed the understanding of symplectic structures by introducing Fukaya Categories, whose objects are compact Lagrangians and whose morphisms are generated by their intersections; one key discovery is that the operations in this category are encoded in holomorphic discs.

One of the motivating problems of symplectic topology is Arnold's nearby Lagrangian conjecture: any closed exact Lagrangian in a cotangent bundle is Hamiltonian isotopic to the zero section. Work related to this conjecture has spanned over two decades: in [23],[24] Viterbo proved that the Floer homology of the cotangent bundle of a closed manifold (a symplectic object) is isomorphic to the cohomology of the loop space of the zero section (a topological object). Abbondandolo and Schwarz showed the existence of a chain level isomorphism from the Morse homology of the based loop space of a closed manifold $H_*(\Omega_q Q)$ to the Lagrangian version of Hamiltonian Floer homology of the cotangent bundle T^*Q [1], and that this map preserves the ring structures in that there is a correspondence between the Chas-Sullivan loop product and pair-of-pants product. This

version of Floer homology is now known as the wrapped Floer homology, which was developed by Abouzaid and Seidel, who used wrapping to give control at infinity. This approach enlarged the Fukaya category to allow certain non-compact exact Lagrangians, such as the cotangent fibers T_q^*Q , where the morphisms given by two Lagrangians are given by Floer Homology modified by a large Hamiltonian Perturbation at infinity; Abouzaid further showed that the wrapped Fukaya category $C\mathcal{W}(T^*Q)$ is generated by a single Lagrangian cofiber T_q^*Q [4]. Building on the correspondence between the Morse homology of loop space and symplectic cohomology of cotangent bundles, in [5] Abouzaid showed that $C\mathcal{W}^*(T_q^*Q, T_q^*Q)$ is quasi-isomorphic as an A_∞ -algebra to $C_{-*}(\Omega_q Q, \eta)$, the chains of based loop space of the zero section twisted by a canonical local system, which is a dg-algebra under the concatenation of loops. This is the inverse to the map constructed in [1] on the level of homology. Recently Kragh also showed homotopy equivalence by studying directly the geometry of an exact Lagrangian through intersection with cotangent fiber [17].

We would like to understand the relations between the wrapped Floer homology of any open Liouville manifold and the loop space of exact Lagrangians inside. Before we state the results, let us briefly review the geometric setup.

Consider a Liouville manifold with contact boundary $(M^{in}, \partial M^{in}, \omega)$. By definition, there is a one-form θ such that $\omega = d\theta$ is the symplectic form on M^{in} ; there is also a Liouville vector field Z such that $i_Z \omega = \theta$, which strictly points outward along ∂M^{in} . The restriction of θ to the boundary ∂M^{in} gives a contact form α :

$$\alpha = \theta|_{\partial M^{in}}. \tag{1.1}$$

There is a collar neighborhood of ∂M^{in} which is parametrized by a map $\varphi : \partial M^{in} \times (0, 1] \rightarrow M^{in}$ such that $\varphi|_{\partial M^{in} \times \{1\}} = Id$. Consider the completion

$$M = M^{in} \cup_{\partial M^{in}} ([1, \infty) \times \partial M^{in}). \tag{1.2}$$

Denote radial coordinate by r . M is canonically equipped with an extension $\hat{\theta}$ of θ , a symplectic

form $\hat{\omega}$ which extends ω and a Liouville vector field \hat{Z} that extends Z . On $\partial M^{in} \times (0, \infty)$:

$$\hat{\theta} = r(\theta|_{\partial M^{in}}), \hat{\omega} = d(\hat{\theta}), \hat{Z} = r\partial_r.$$

Moreover, we have a distinguished class of almost complex structures on the symplectization

$$\partial M^{in} \times (0, \infty)$$

as follows: we write

$$T(\partial M^{in} \times (0, \infty)) = \mathbb{R}^2 \oplus \kappa, \quad (1.3)$$

where κ is the contact distribution on ∂M^{in} where the contact form α vanishes and the first factor is spanned by $r\partial_r$ and $(0, Y)$ where Y is a Reeb vector field associated with α . On the cylindrical ends $\partial M^{in} \times (0, \infty)$, the complex structure sends $r\partial_r$ to $(0, Y)$. Any almost complex structure on the symplectization satisfying the above condition is called *of contact type*. Any such structure is invariant under the flow of $r\partial_r$.

Thus, on cylindrical ends, an almost complex structure J of contact type on M satisfies the following condition,

$$dr \circ J = -\hat{\theta}. \quad (1.4)$$

In later sections, we define a perturbation of the almost complex structures to vary within small neighborhood of J_0 inside the space of smooth almost complex structures on M of contact type.

In the collar neighborhood $\partial M^{in} \times (0, 1)$ of ∂M^{in} , we let M^o denote the complement of $\partial M \times [\frac{1}{2}, 1]$, thus M^o is a compact Liouville manifold with contact boundary.

In order to understand the topological structure of the space \mathcal{Lag} of exact Lagrangian submanifolds with Legendrian boundary of a Liouville manifold, whose homotopic information can be obtained from its based loop space, we construct an A_∞ functor between the *Pontryagin category* of decorated Lagrangians to the wrapped Fukaya category of the Liouville Manifold.

Here the Pontryagin category of a topological space X is a DG category whose objects are the

points of X and morphisms between two points are given by cubical chains on the space of paths in X between the points. The associative product structure is given by concatenation of paths. The exact definition of Pontryagin Category of a path connected space is in Section (2.0.7).

Consider exact Lagrangian $\iota : L \hookrightarrow M$ that is the completion of one with Legendrian boundary near cylindrical ends. Let $b \in H^2(M; \mathbb{Z}/2)$ be a class whose pullback under ι is the second Stiefel-Whitney class of L :

$$\iota^* b = w_2(L). \quad (1.5)$$

Let $\mathcal{L}ag_b$ denote the space of Lagrangians that satisfy the above equation. We may define a relative Pin structure on L relative to M from b , which is a free $H^1(L; \mathbb{Z}/2)$ -torsor. Therefore we consider the space $\widetilde{\mathcal{L}ag}_b$ of decorated Lagrangian $\mathcal{L} = (L, \beta)$ such that $L \in \mathcal{L}ag_b$ and $\beta \in H^1(L; \mathbb{Z}/2)$; in other words, it is the space of all possible relative Pin structures on a Lagrangian.

$$\begin{array}{ccc} H^1(L; \mathbb{Z}/2) & \longrightarrow & \widetilde{\mathcal{L}ag}_b \\ & & \downarrow \pi \\ & & \mathcal{L}ag_b \end{array}$$

Let $\widetilde{\mathcal{L}ag}$ denote the space of all decorated Lagrangians, we get the main theorem of this paper.

Theorem 1.0.1. *For any open Liouville manifold (M, ω) with cylindrical ends, there exists an A_∞ functor from the Pontryagin category of decorated Lagrangians to the wrapped Fukaya category of the ambient symplectic manifold:*

$$\mathcal{P}(\widetilde{\mathcal{L}ag}) \xrightarrow{\mathcal{F}} \mathcal{W}^*(M). \quad (1.6)$$

Remark 1.0.2. The degree 0 map of the A_∞ morphism is a chain map \mathcal{F}^1 :

$$C_*(\Omega_{\mathcal{L}_0, \mathcal{L}_1} \widetilde{\mathcal{L}ag}) \xrightarrow{\mathcal{F}^1} C\mathcal{W}^{-*}(\mathcal{L}_0, \mathcal{L}_1). \quad (1.7)$$

When $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}$, this induces a map from the homology of the based loop space of Lagrangians

with Legendrian boundary to the wrapped Floer homology of the base point Lagrangian.

$$H_*(\Omega_{\mathcal{L}} \widetilde{\mathcal{L}ag}) \xrightarrow{H(\mathcal{F})} \mathcal{HW}^{-*}(\mathcal{L}). \quad (1.8)$$

If $M = T^*Q$ and $\mathcal{L} = T_q^*Q$ this is surjective as shown by the theorem below, but not necessarily surjective in the general case.

The above theorem implies that we can use Wrapped Floer homology to detect nontrivial families of Lagrangians. As an example for computation, let the Liouville manifold M be the cotangent bundle T^*Q of a compact manifold Q . In this case, M. Abouzaid constructed an A_∞ functor, which, in the case of a cotangent fiber $\mathcal{L} = T_q^*Q$, gives a map on morphism spaces $f : \mathcal{CW}^*(\mathcal{L}, \mathcal{L}; H) \rightarrow C_{-*}(\Omega_q Q)$ [5]. Here each cotangent fibre T_q^*Q is contractible, thus the map $\widetilde{\mathcal{L}ag}_b \rightarrow \mathcal{L}ag_b$ is an isomorphism. By taking the above A_∞ functor on the level of automorphisms of a single Lagrangian and composing with the chain level homomorphism from f mentioned above, we obtain the following homotopy commutative diagram:

Theorem 1.0.3. *With $L = T_q^*Q$, the map $(f \circ \mathcal{F} \circ \eta)_*$ in the diagram below equals the identity map on $H_{-*}(\Omega_q Q)$.*

$$\begin{array}{ccc} C_{-*}(\Omega_q Q) & \xrightarrow{\eta} & C_{-*}(\Omega_L \mathcal{L}ag) \\ \downarrow id & \searrow \text{dotted} & \downarrow \mathcal{F} \\ C_{-*}(\Omega_q Q) & \xleftarrow{f} & \mathcal{CW}^*(\mathcal{L}, \mathcal{L}; H) \end{array}$$

Here the map η sends every loop in the base to the corresponding loop of cotangent fibers in the space of Lagrangian embeddings.

Remark 1.0.4. Recently in a series of papers[13, 15, 14], Ganatra, Pardon and Shende used the theory of microlocal sheaves to study the symplectic structure of Liouville manifolds and it is very likely that one can also use their machinery to achieve the results of this paper.

Our paper first reviews the theory of wrapped Fukaya category and Pontryagin category, then construct the moduli spaces that give rise to the functor. Afterwards we take the special case of a

cotangent bundle of a compact manifold as an example of this A_∞ functor. Finally we attach brief explanations of relative Pin structures, signed operations, C^0 estimates and gluing theorem in the appendix.

Chapter 2: Wrapped Fukaya Category

2.0.1 Review of Floer theory in open Liouville manifold

In [7] Abouzaid and Seidel constructed the wrapped Floer cohomology on Liouville domains in analogy with the Lagrangian Floer cohomology of a closed symplectic manifold. They defined $HW^*(L; H) \cong \varinjlim_w HF^*(L; wH)$ from a direct system of continuation maps from Lagrangian Floer cohomology with Hamiltonians of increasing slope at infinity. Later in [5], Abouzaid used Hamiltonians that are quadratic at infinite ends to define wrapped Floer homology. We are going to follow this definition.

Geometric Setup

From now on we work on open Liouville manifold (M, ω) with Liouville 1-form θ and infinite cylindrical ends. Namely there is a compact Liouville domain (M^{in}, ω) with boundary, together with a one-form θ such that $\omega = d\theta$ is symplectic, and the dual Liouville vector field Z , uniquely determined by the requirement that $i_Z\omega = \theta$ that points strictly outwards along ∂M^{in} . This implies that $\theta|_{\partial M^{in}}$ is a contact one-form. By flowing inwards from the boundary along Z , one obtains a collar embedding $(0, 1] \times \partial M^{in} \hookrightarrow M^{in}$. One can then complete M^{in} to obtain M by attaching the big half of symplectization:

$$M = M^{in} \cup_{\partial M^{in}} ([1, \infty) \times \partial M^{in}). \quad (2.1)$$

The piece $[1, \infty) \times \partial M^{in}$ is called the *infinite cone*. The completion carries a natural one-form $\hat{\theta}$ such that $d\hat{\theta} = \hat{\omega}$ is symplectic and an associated Liouville vector field \hat{Z} . On $M^{in} \subset M$ these restrict to the previous data, while on the overlapping piece $(0, \infty) \times \partial M^{in}$, $\hat{\theta} = r(\theta|_{\partial M^{in}})$ and $\hat{Z} = r\partial_r$, where r is the variable on $(0, \infty)$.

Let $L \in M^{in}$ be a Lagrangian intersecting ∂M^{in} transversally, and which has the following property

$$\begin{aligned} \theta|_L \text{ is exact, } \theta|_L = df. \text{ Moreover, } \theta|_L \text{ vanishes near the boundary } \partial L = \\ L \cap \partial M^{in}. \end{aligned} \tag{2.2}$$

The second part implies that ∂L is a Legendrian submanifold of ∂M^{in} . Furthermore, by attaching an infinite cone to this boundary, one can extend L to a smooth noncompact Lagrangian submanifold

$$\hat{L} = L \cup_{\partial L} ([1, \infty) \times \partial L) \subset M. \tag{2.3}$$

In the remainder of this paper, we only use θ, ω, L to denote their extensions (resp. completions) in M since there won't be any confusion.

We use ψ^ρ to denote the image of the Liouville flow for time $\log(\rho)$ outwards along the cylindrical end. Namely,

$$\psi^\rho(r, x) = (\rho \cdot r, x).$$

Let $\mathcal{H}(M) \subseteq C^\infty(M, \mathbb{R})$ denote the set of smooth functions H such that

$$H(q, r) = r^2 \tag{2.4}$$

for any (q, r) (where $q \in \partial M^{in}$) on the cylindrical end and $r \gg 1$. Namely H is quadratic with respect to the radial coordinate away from some compact subset of M . We fix such a function for the purpose of defining Floer homology, and let X_H denote the Hamiltonian flow of H defined by the equation

$$i_{X_H} \omega = -dH. \tag{2.5}$$

We denote by $\mathcal{L}ag$ the space of all Lagrangians with vanishing Maslov index that also satisfies Condition (2.64), namely those that are equipped with Legendrian boundary and the restriction of

the one form θ on L is exact,

$$\theta|_L = df, \tag{2.6}$$

for some $f \in C^\infty(L, \mathbb{R})$. When L moves in $\mathcal{L}ag$, we shall choose a primitive f_L for the restriction of θ to each such Lagrangian that is continuous with respect to L .

For each pair of Lagrangians $L_0, L_1 \in \mathcal{L}ag$, we define $\mathcal{X}(L_0, L_1)$ to be the set of time-1 flow lines of X_H that start from L_0 and stop at L_1 , namely a map $x : [0, 1] \rightarrow M$ such that

$$\begin{cases} x(0) \in L_0 \\ x(1) \in L_1 \\ \frac{dx}{dt} = X_H \end{cases}$$

In order to get well-defined moduli spaces, we assume that

$$\text{all time-1 Hamiltonian chords of } H \text{ with boundaries on } L_0 \text{ and } L_1 \text{ are non-degenerate.} \tag{2.7}$$

It is convenient to point out that the Hamiltonian chords above correspond bijectively to the intersection points of L_1 with the time-1 flow $\phi_H^1(L_0)$ of L_0 under the Hamiltonian flow of H . In particular, the non-degeneracy of the chords is the same as transversal intersection of L_1 and $\phi_H^1(L_0)$. Condition (2.69) is therefore true after we do a generic Hamiltonian perturbation on the Lagrangians that intersect transversally with ∂M^{in} . Since Lagrangian Floer Homology is invariant under Hamiltonian perturbation, the above assumption is valid.

Brief discussion of relative Pin and brane structures

We briefly introduce the definition of Lagrangian branes for the sake of a proper definition of Lagrangian Floer homology. Let (M, ω, J) be symplectic with $2c_1(M) = 0$, equipped with a non-vanishing quadratic volume form η_M^2 , namely a global section of the line bundle $(\Omega^{n,0}M)^{\otimes 2} = (\wedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$. Likewise, given a symplectic vector bundle $\varphi : E \rightarrow B$ of rank r such that $2c_1(E) = 0$,

we have a global non-vanishing complex volume form $\eta_E^2 \in H^0(B, (\wedge_{\mathbb{C}}^r E)^{\otimes 2})$. Let $Gr(E)$ be the set of Lagrangian sub-bundles of E , and v_1, \dots, v_r be a basis of E_x for any $x \in B$. We define the squared phase map of E as follows:

Definition 2.0.1. *Given a symplectic vector bundle $\varphi : E \rightarrow B$ of rank r such that $2c_1(E) = 0$, and any Lagrangian sub bundle $\phi : F \subset E$, consider the following squared phase map of E :*

$$\alpha_E : Gr(E) \times B \rightarrow S^1,$$

$$\alpha_E(F, x) := \frac{\eta_x^2(v_1 \wedge v_2 \wedge \dots \wedge v_r)}{|\eta_x^2(v_1 \wedge v_2 \wedge \dots \wedge v_r)|}.$$

Remark 2.0.2. A simple linear algebra exercise shows that α_E is independent of the choices of the set of basis; thus it is well defined.

Moreover, if the Maslov class of F vanishes, namely $\alpha_E(F, \cdot)_* : H_1(B) \rightarrow H_1(S^1) \simeq \mathbb{Z}$ is zero, we lift the map $\alpha_E(F, \cdot)$ to a map $\alpha^\#$ from B to the universal cover \mathbb{R} of S^1 . In this way we obtain the following definition:

Definition 2.0.3. *For a Lagrangian sub-bundle F with vanishing Maslov class inside a symplectic fibration $E \rightarrow B$, an associated grading $\alpha^\#$ on F is a function:*

$$\alpha^\# : B \rightarrow \mathbb{R}, \tag{2.8}$$

$$\text{such that } \exp(2\pi i \alpha^\#(x)) := \alpha_E(F, x). \tag{2.9}$$

Remark 2.0.4. The choice of grading is not a canonical one, in fact, it is a free torsor of $H^0(B; \mathbb{Z})$.

We further assume that for some $b \in H^2(B; \mathbb{Z}/2)$, the second Stiefel-Whitney class $w_2(\phi) \in H^2(B; \mathbb{Z}/2)$ is the same as b , namely $b = w_2(\phi)$. Fixing a triangulation of B , we can choose an oriented vector bundle over the three skeleton $B_{[3]}$, $\psi : E_b \rightarrow B_{[3]}$ such that $w_2(\psi) = b|_{B_{[3]}}$. Then our assumption on $w_2(\phi)$ gives

$$w_2(\phi|_{B_{[3]}} \oplus \psi) = 0. \tag{2.10}$$

From the standard literature([22, Section 11i], [21]), we know that second Stiefel-Whitney class is precisely the obstruction to the existence of Pin structures:

Definition 2.0.5. *A relative Pin structure on a Lagrangian sub-bundle F is the choice of a Pin structure on the vector bundle $F|_{B_{[3]}} \oplus E_b$.*

Definition 2.0.6 (brane structure). *Given a symplectic vector bundle $E \rightarrow B$ and $b \in H^2(B; \mathbb{Z}/2)$, consider a Lagrangian sub-bundle $F \rightarrow B$ such that $2\mu(F) = 0$, $w_2(F) + b = 0$. A brane structure with respect to b consists of a pair $(\alpha^\#, P^\#)$, where $\alpha^\#$ satisfies the equation (2.71) and $P^\#$ is a relative Pin structure on F .*

Remark 2.0.7. The obstruction to the existence of a grading is the Maslov class $\mu(F) \in H^1(B; \mathbb{Z})$ represented by the map $\alpha_E(F, \cdot) : B \rightarrow S^1$; if $b = 0$, then we have a Pin-structure on F in the sense of [22], whose obstruction is the second Stiefel-Whitney class $w_2(E) \in H^2(B, \mathbb{Z}/2)$. Here we take the direct sum with E_b so as to “kill” the obstruction given by possibly non-zero b .

Now in our situation, the base B of the fibration is an exact Lagrangian sub manifold $L \in M$ with Legendrian boundary, the sub-bundle is just $TL \subset TM|_L$ whose Maslov class $\mu(TL)$ vanishes.

Therefore, to each Hamiltonian chord $x \in \mathcal{X}(L_0, L_1)$, we can assign a grading $\mu(x)$ from the gradings on L_0, L_1 for given transverse brane structures on L_0, L_1 . Here we define $\mu(x) := \lfloor \alpha_1^\# - \alpha_0^\# \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function that lies in \mathbb{Z} . We also define the orientation space $o_x = Hom(P_0^\#, P_1^\#) \otimes TL_1^{\otimes \mu(x)}$. For details we refer to [22, Section 11].

We denote the space of exact Lagrangians with Legendrian boundary condition with a relatively Pin structure with respect to the background class $b \in H^2(M)$ by

$$\mathcal{L}ag_b \tag{2.11}$$

Namely Lagrangians with vanishing $w_2(L) + b$. We know that the isomorphism classes of Pin structures on TL is a free torsor of $H^1(L; \mathbb{Z}/2)$ by examining the long exact sequence

$$\dots \rightarrow H^1(L; \mathbb{Z}/2) \rightarrow H^1(L; Pin(n)) \rightarrow H^1(L, O(n)) \rightarrow H^2(L; \mathbb{Z}/2) \rightarrow \dots \tag{2.12}$$

that arises from the exact sequence of classifying spaces:

$$B\mathbb{Z}/2 \rightarrow BPin(n) \rightarrow BO(n) \rightarrow B^2\mathbb{Z}/2 \rightarrow \dots \quad (2.13)$$

Thus for each $L \in \mathcal{L}ag_b$, together with a Pin structure $P^\#$ on TL and any class $\beta \in H^1(L; \mathbb{Z}/2)$, we may twist $P^\#$ by β as follows: take a line bundle ξ on L whose first Stiefel-Whitney class is β and its associated double cover $S(\xi)$. The Pin structure $P^\#$ is a $Pin(n)$ -principal bundle $P^\# \rightarrow L$ with an isomorphism $P^\# \times_{Pin(n)} \mathbb{R}^n \cong TL$. Let $\pm e$ denote the kernel of the double cover $Pin(n) \rightarrow O(n)$. We take the product $S(\xi) \times_{\mathbb{Z}/2} P^\#$ where $\mathbb{Z}/2$ acts on $S(\xi)$ by ± 1 and on $P^\#$ by $\pm e$. This gives a new $Pin(n)$ -structure on TL .

A brane structure on L , by Definition (2.0.30), consists of a grading $\alpha^\# : L \rightarrow \mathbb{R}$ and Pin structure $P^\#$; the gradings $\alpha^\#$ are free torsors of $H^0(L, \mathbb{Z})$. Notice in our situation we don't require that $w_2(L) = 0$, yet all the Lagrangians we consider have to satisfy $w_2(TL) = i_L^* b$ for some background class $b \in H^2(M, \mathbb{Z}/2)$. Denoting by $\widetilde{\mathcal{L}ag}_b$ the space of Lagrangians equipped with a specific Pin structure $(L, P^\#)$ with a fixed background class b (we may omit this in later section and just denote the space $\widetilde{\mathcal{L}ag}$ when it is clear from the context), we obtain a fibration:

$$\begin{array}{ccc} H^1(L; \mathbb{Z}/2) & \longrightarrow & \widetilde{\mathcal{L}ag}_b \\ & & \downarrow \pi \\ & & \mathcal{L}ag_b \end{array}$$

Note that for Lagrangians of different first cohomology groups, this fibration gives different fibers, thus in practice we only look at a component of any given L such that the fibration π is surjective with the same fiber $H^1(L; \mathbb{Z}/2)$.

2.0.2 Almost complex structures

In this paper we work with almost complex structures of contact type, namely those satisfy Equation (1.4). Let $\mathcal{J}(M)$ denote the space of such almost complex structures on M which are also compatible with the symplectic form on M . This is an infinite-dimensional manifold. Its

tangent space $T\mathcal{J}(M)$ at any point J consists of endomorphisms K of TM which anti-commute with J , and which have the property that $\omega(\cdot, K\cdot)$ is symmetric. We call these endomorphisms *infinitesimal deformations* of the almost complex structure. One can get an actual deformation by exponentiating K , namely setting

$$J' = J \exp(-JK). \quad (2.14)$$

As in [7, Section 3], for any disc S with punctures on the boundary that are equipped with a set of strip-like ends $\{\epsilon^k\} : Z_{\pm} \rightarrow S$, and weights $\{w^k\}$ there exists a contractible choice of almost complex structures parametrized by S such that for each strip-like end $\epsilon^k : Z_{\pm} \rightarrow S$ with (s, t) denoting the coordinates of Z_{\pm} , the almost complex structure $J_{\epsilon^k(s,t)}$ is only dependent on t for $|s| \gg 0$. In other words, if we go sufficiently far along any end, J becomes independent of s , hence compatible with translations in the s direction.

2.0.3 Moduli spaces

We consider the space of maps from the strip $Z = \mathbb{R} \times [0, 1]$ to a symplectic manifold M satisfying certain equations. First of all, we need to fix some data for the equations: a family $J_t \in \mathcal{J}(M)$ of almost complex structures parametrized by $t \in [0, 1]$; a closed one-form γ on Z such that $\gamma(s, t) = dt$ when $|s| \gg 0$ and a Hamiltonian $H : M \rightarrow \mathbb{R}$ that satisfies Condition (2.69). Given a pair $x_0, x_1 \in X(L_0, L_1)$, we have the following definition:

Definition 2.0.8. $\tilde{\mathcal{M}}(x_0; x_1)$ is the moduli space of maps

$$u : Z = \mathbb{R} \times [0, 1] \rightarrow M,$$

satisfying the following conditions

$$\left\{ \begin{array}{l} u(\mathbb{R} \times \{1\}) \subseteq L_1, \\ u(\mathbb{R} \times \{0\}) \subseteq L_0, \\ \lim_{s \rightarrow -\infty} u(s, \cdot) = x_0(\cdot), \\ \lim_{s \rightarrow \infty} u(s, \cdot) = x_1(\cdot). \end{array} \right.$$

such that u solves the Cauchy-Riemann equation

$$(du - X_H \otimes \gamma)^{0,1} = 0 \tag{2.15}$$

which can also be written explicitly with the family of almost complex structure J_t :

$$J_t(\partial_t u - X_H \cdot \gamma) - (\partial_t u - X_H \cdot \gamma) \circ j = 0.$$

Since equation (2.77) is invariant under translation of the variable s , there is an action of \mathbb{R} on $\tilde{\mathcal{M}}(x_0; x_1)$. After taking the quotient by the free \mathbb{R} -action, we get $\mathcal{M}(x_0; x_1)$; and let it be empty if the action is not free. Standard arguments of the gluing theorem such as those in Appendix (6.3) of [3] give the following:

Proposition 2.0.9. *The moduli space $\mathcal{M}(x_0; x_1)$ is regular for a generic family of almost complex structures J_t and has dimension $\mu(x_0) - \mu(x_1) - 1$. The Gromov compactification $\overline{\mathcal{M}}(x_0; x_1)$ is a topological manifold with boundary. The boundary is stratified into topological manifolds with the closure of codimension 1 strata given by the images of embeddings*

$$\overline{\mathcal{M}}(x_0; y) \times \overline{\mathcal{M}}(y; x_1) \rightarrow \overline{\mathcal{M}}(x_0; x_1), \tag{2.16}$$

for all possible y in $\mathcal{X}(L_0, L_1)$. For each pair (y_1, y_2) that gives a boundary stratum of higher

codimension, we obtain a commutative diagram:

$$\begin{array}{ccc}
\overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; y_2) \times \overline{\mathcal{M}}(y_2; x_1) & \longrightarrow & \overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; x_1) \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}(x_0; y_2) \times \overline{\mathcal{M}}(y_2; x_1) & \longrightarrow & \overline{\mathcal{M}}(x_0; x_1).
\end{array} \tag{2.17}$$

□

In particular, the codimension k boundary strata consist of disjoint unions over all possible sequences of products of moduli spaces of the following form:

$$\overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; y_2) \times \cdots \times \overline{\mathcal{M}}(y_k; x_1). \tag{2.18}$$

Since the Lagrangians L_0, L_1 have infinite ends, we need to show that $\overline{\mathcal{M}}(x_0; x_1)$ is compact.

Note that the Hamiltonian chords $x \in \mathcal{X}(L_0, L_1)$ are critical points of an action functional \mathcal{A} on the space $\Lambda(L_0, L_1)$ of smooth paths in M with end points on L_0, L_1 . It is defined as

$$\mathcal{A}(x) = - \int_0^1 x^* \theta + \int H(x(t)) dt + f_{L_1}(x(1)) - f_{L_0}(x(0)),$$

where f_{L_i} is the primitive function given by Equation (2.68).

Condition (2.66) for the Hamiltonian and Equation (2.67) imply that for $r \gg 1$, any Hamiltonian chord that intersects some slice $\partial M^{in} \times \{r\}$ is necessarily contained inside this slice because X_H neither points outwards nor inward on cylindrical ends. The action of x for any $x \in \mathcal{X}(L_0, L_1)$ is given by

$$\mathcal{A}(x) = - \int_0^1 x^* \theta + H(x(t)) dt + f_1(x(1)) - f_0(x(0)). \tag{2.19}$$

From the above equation we realize that \mathcal{A} is a proper map from $\mathcal{X}(L_0, L_1)$ to \mathbb{R} as there are only finitely many non-degenerate Hamiltonian chords between L_0 and L_1 in any compact subset of M , and the above equation implies that the action of a sequence of chords that escapes any compact subset must go to ∞ .

Lemma 2.0.10. *Given any solution of Equation (1.3) that converges to $x_0(t)$ at the negative end and to $x_1(t)$ at the positive end, the following inequality holds*

$$\mathcal{A}(x_0) \geq \mathcal{A}(x_1). \quad (2.20)$$

Furthermore, all such solutions lie within some compact subset of M that is dependent only on x_0 and x_1 .

Remark 2.0.11. This is the Integrated Maximum Principle as shown in Lemma 7.2 of [7]. We give a brief discussion of the proof here.

Proof. First we define the topological energy of a solution of Equation (2.77) to be:

$$E^{top}(u) = \int_Z u^* \omega - d(u^* H dt) \quad (2.21)$$

$$= \int_Z (u^* \omega - u^* dH \wedge \gamma) - \int_Z (u^* H \cdot d\gamma) \quad (2.22)$$

$$= \int_Z \frac{1}{2} \|du - X_H \otimes dt\|^2 - \int_Z u^* H \wedge d\gamma \quad (2.23)$$

$$\geq 0. \quad (2.24)$$

Let s be large enough such that $u(-s, t) = x_0(t)$ and $u(s, t) = x_1(t)$. Denote the subspace $[-s, s] \times [0, 1] \subset Z$ by Z_s . We use Stoke's theorem to show:

$$\begin{aligned} \mathcal{A}(x_0) - \mathcal{A}(x_1) &= - \int_0^1 x_0^* \theta + H_0(x_0) dt + \int_0^1 x_1^* \theta - H_1(x_1) dt \\ &\quad + f_1(x_0(1)) - f_1(x_1(1)) - (f_0(x_0(0)) - f_0(x_1(0))) \\ &= - \int_0^1 x_0^* \theta + \int_0^1 x_1^* \theta + \int_{x_0(1)}^{x_1(1)} df_1 - \int_{x_0(0)}^{x_1(0)} df_0 + \int H_0(x_0) dt - \int H_1(x_1) dt \\ &= \int_{\partial Z_s} u^* \theta - \int_{\partial Z_s} u^* H dt \\ &= \int_{Z_s} u^* \omega - \int_{Z_s} d(u^* H dt) \\ &= E^{top}(u) \geq 0. \end{aligned}$$

The third equality above uses the fact that $\theta|_{L_i} = df_i$ and that dt vanishes on the upper and lower boundaries of Z_s . Thus the inequality (2.82) holds by non-negativity of topological energy.

To prove compactness, consider a hypersurface $\partial M^{in} \times \{r\}$ where $r \gg 0$ such that the radial coordinates of the two Hamiltonian chords $x_0(t), x_1(t)$ don't exceed r . If a solution of Equation (2.77) escapes the compact subset $M^{in} \cap \partial M^{in} \times [1, r]$, we may find a compact subset Σ of the domain $Z = \mathbb{R} \times I$ whose boundary is mapped into $\partial M^{in} \times \{r\}$. Depending on whether Σ intersects with the upper or lower boundary of Z , its own boundary $\partial \Sigma$ might contain components that are mapped into the Lagrangians L_0 and L_1 with Legendrian boundary, which are the images of the intersection of $\partial \Sigma$ and the upper and lower boundaries of Z . We denote the union of such boundary components of $\partial \Sigma$ by $\partial_l \Sigma$. The remaining boundary components of Σ are mapped into the slice $\partial M^{in} \times \{r\}$ and we denote them by $\partial_r \Sigma$.

The one-form θ is exact when restricted to the Lagrangians as required by (2.64). We shall also specify conditions on the one-form γ such that it vanishes on the Lagrangians with Legendrian boundary as in Definition (2.0.41). The topological energy of the map restricted to Σ is non-negative:

$$\begin{aligned}
0 \leq E^{top}(u|_\Sigma) &= \int_\Sigma u^* \omega - dH \otimes \gamma \\
&= \int_{\partial \Sigma} u^* \theta - H \otimes \gamma \\
&= \int_{\partial_r \Sigma} u^* \theta - H \otimes \gamma + \int_{\partial_l \Sigma} u^* \theta - H \otimes \gamma \\
&= \int_{\partial_r \Sigma} \theta(du - X_H \otimes \gamma) + 0 \\
&= \int_{\partial_r \Sigma} (\theta \circ J_z) \circ (du - X_H \otimes \gamma) \circ j.
\end{aligned}$$

Note in the above equations, $\int_{\partial_l \Sigma} H \otimes \gamma$ vanishes because γ vanishes on $\partial_l \Sigma$; $\int_{\partial_l \Sigma} u^* \theta$ vanishes because θ is exact on the Lagrangians, namely $\theta|_{L_i} = df_i$ and the end points of the union of all boundary components on either Lagrangian L_i end on $L_i \cap \partial M^{in} \times \{r\}$.

Since H is quadratic with respect to radial coordinate r as in Equation (2.66), dH is a multiple of dr . Equation (1.4) implies that $\theta(J_z X_H) = -dr(X_H)$ is a multiple of $dH(X_H)$, and therefore

vanishes. The above inequality becomes

$$0 \leq \int_{\partial_r \Sigma} \theta \circ J \circ du \circ j. \quad (2.25)$$

Meanwhile, if ξ is a tangent vector compatible with orientation of $\partial_r \Sigma$, then $j\xi$ points towards the interior of Σ , thus $du(j\xi)$ points towards the direction of growing radial coordinate. Equation (1.4) implies $\theta(J_z \circ du \circ j\xi) < 0$. This is a contradiction. \square

The above lemma together with the observation that \mathcal{A} is a proper map above implies that the image of all possible $\mathcal{M}(x_0; x_1)$ for a fixed $x_1 \in \mathcal{X}(L_0; L_1)$ are contained in a compact subset of M . Therefore we may apply the usual arguments using Gromov's compactness and gluing results to conclude:

Corollary 2.0.12. *For each chord $x_1(t)$, the moduli space $\overline{\mathcal{M}}(x_0; x_1)$ is empty for all but finitely many choices of x_0 , and is a compact manifold with boundary of dimension $\mu(x_0) - \mu(x_1) - 1$ whenever J_t is a generic family of almost complex structure. Moreover, the boundary is covered by the images of the natural inclusions.*

$$\overline{\mathcal{M}}(x_0; y) \times \overline{\mathcal{M}}(y; x_1) \rightarrow \overline{\mathcal{M}}(x_0; x_1).$$

\square

From now on through the end of this section, we fix the almost complex structure J_t for which the above corollary holds.

2.0.4 Wrapped Floer complex

We define the graded vector space underlying the Floer chain complex to be the direct sum

$$C\mathcal{W}_b^*(L_0; L_1) := \bigoplus_{x \in \mathcal{X}(L_0; L_1)} |o_x|, \quad (2.26)$$

where o_x is the determinant line bundle explained in Appendix A. We may simply assume that it is just the vector space freely generated by the sets of chords unless one wants to worry about signs.

The differential is a count of the solutions of rigid moduli spaces of equation (2.77), namely those with $\mu(x_0) = \mu(x_1) + 1$. Every element u of the moduli space $\mathcal{M}(x_0, x_1)$ is rigid and thus defines an isomorphism:

$$o_{x_0} \cong o_{x_1}.$$

Thus an orientation of o_{x_0} is induced from one on o_{x_1} , and the map on the orientation lines is denoted μ_u . We define the map:

$$\mu_1 : \mathcal{CW}_b^i(L_0; L_1) \rightarrow \mathcal{CW}_b^{i+1}(L_0, L_1) \quad (2.27)$$

$$[x_1] \mapsto (-1)^i \sum_u \mu_u([x_1]). \quad (2.28)$$

The proof that this count is a well defined differential of chain complexes is standard (see eg. [8]). Moreover, Corollary (2.0.36) shows that each chord can be the input of only finitely many solutions to (2.77), thus we know that on the right hand side of (2.89) there are only finitely many summands.

Remark 2.0.13. The graded vector space underlying $\mathcal{CW}_b^*(L_0; L_1)$ depends on L_0, L_1, H, ω , meanwhile the differential depends on J_t . When we want to distinguish them as in the next lemma, we write $\mathcal{CW}_b^*(L_0, L_1; \omega, H, J_t)$.

Lemma 2.0.14. *If $\psi : M \rightarrow M$ satisfies $\psi^*(\omega) = \rho\omega$ for some non-zero constant ρ , then we have a canonical isomorphism*

$$\mathcal{CW}(\psi) : \mathcal{CW}_b^*(L_0, L_1; \omega, H, J_t) \cong \mathcal{CW}_b^*(\psi(L_0), \psi(L_1), \frac{H \circ \psi}{\rho}, \psi^*(J_t)). \quad (2.29)$$

This was shown as [5, Lemma 3.4], which I rewrite here for the convenience of the reader.

Proof. Given a diffeomorphism ψ , we obtain an isomorphism of chain complexes

$$C\mathcal{W}_b^*(L_0, L_1; \omega, H, J) \cong C\mathcal{W}_b^*(\psi(L_0), \psi(L_1), \psi^*\omega, \psi^*H, \psi^*(J)).$$

This comes from the composition of each chord from L_0 to L_1 with ψ and the composition of each solution to equation (2.77) with ψ ; thereby obtaining an equation with ψ^*J_t . Since $\psi^*\omega = \rho\omega$, the Hamiltonian flow of $H \circ \psi$ with respect to $\psi^*\omega$ agrees with the flow of $\frac{H \circ \psi}{\rho}$ with respect to ω . Therefore we obtain an identification:

$$C\mathcal{W}_b^*(\psi(L_0), \psi(L_1); \psi^*\omega, H \circ \psi, \psi^*J_t) \cong C\mathcal{W}_b^*(\psi(L_0), \psi(L_1), \omega, \frac{H \circ \psi}{\rho}, \psi^*J_t). \quad (2.30)$$

From now on, we omit the Floer data and define

$$C\mathcal{W}_b^*(\psi(L_0), \psi(L_1)) := C\mathcal{W}_b^*(\psi(L_0), \psi(L_1); \omega, \frac{H \circ \psi}{\rho}, \psi^*J_t)$$

□

2.0.5 Product Structure

We would like to study the product operation on wrapped Floer complexes,

$$C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(L_0, L_2; H). \quad (2.31)$$

However, the naturally defined map would take value in $C\mathcal{W}_b^*(L_0, L_2; 2H)$. And the usual continuation map from $C\mathcal{W}_b^*(L_0, L_2; 2H)$ to $C\mathcal{W}_b^*(L_0, L_2; H)$ is not well defined. To solve this problem, we appeal to the rescaling trick in [3]. We briefly sketch the main idea here.

First, we define a map

$$\mu_2^{\psi^2} : C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(\psi^2(L_0), \psi^2(L_2)), \quad (2.32)$$

where ψ^2 is the time-log(2) Liouville flow. Secondly we compose with the inverse of equation (2.92) to get

$$\mu_2 : C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(L_0, L_2; H). \quad (2.33)$$

The map (2.94) counts solutions to the following version of the Cauchy-Riemann equation:

$$(du - X_S \otimes \gamma_S)^{0,1} = 0, \quad (2.34)$$

where S is the surface obtained by removing 3 points ξ^0, ξ^1, ξ^2 from the boundary of \mathbb{D}^2 . In the above equation, γ_S is a closed 1-form on S while X_S is the Hamiltonian vector field of H_S on M , which depends on S . We require that the map

$$H_S : M \rightarrow \mathcal{H}(M) \quad (2.35)$$

restrict to H near ξ^1, ξ^2 and to $\frac{H}{4} \circ \psi^2$ near ξ^0 . Note that $\frac{H}{4} \circ \psi^2 \in \mathcal{H}(M)$ since $H(q, r) = r^2$ for r large enough in the cylindrical ends.

To specify the remaining data required for equation (2.96), we can pick strip like ends for S , namely Z_\pm for the positive (respectively, negative) half strips in Z and embeddings

$$\epsilon_0 : Z_- \rightarrow S, \quad (2.36)$$

$$\epsilon_1 : Z_+ \rightarrow S, \quad (2.37)$$

$$\epsilon_2 : Z_+ \rightarrow S, \quad (2.38)$$

which map ∂Z_\pm to ∂S and converge to the respective marked points ξ_k for $k = 0, 1, 2$.

For the closed 1-form γ_S , it should vanish on ∂S and satisfy

$$\epsilon_0^*(\gamma_S) = 2dt, \quad (2.39)$$

$$\epsilon_k^*(\gamma_S) = dt, k = 1, 2. \quad (2.40)$$

In addition, we choose a family of almost complex structures $J_S : S \rightarrow \mathcal{J}(M)$ parametrized by the Riemann surface S whose compositions with ϵ_k agrees with J_t if $k = 1, 2$ and $(\psi^2)^*J_t$ if $k = 0$.

The Lagrangian boundary conditions near ξ^1 should be given by (L_0, L_1) , similarly near ξ^2 given by (L_1, L_2) and near ξ^0 given by $(\psi(L_0), \psi(L_2))$. For the Lagrangians along the two segments of ∂S that converge to ξ^0 , we cannot have a constant Lagrangian condition, instead we need a moving boundary Lagrangian condition. In order to pick a simple moving Lagrangian condition we fix a map ρ_S from ∂D^2 to $[1, 2]$ such that

$$\rho_S(z) \equiv 1 \text{ if } z \text{ is near } \xi^1 \text{ or } \xi^2 \text{ and } \rho_S(z) \equiv 2 \text{ if } z \text{ is near } \xi^0. \quad (2.41)$$

Definition 2.0.15. *The moduli space $\mathcal{M}(x_0; x_1, x_2)$ is the space of solutions to equation (2.96) with boundary conditions*

$$\begin{cases} u(z) \in \psi^{\rho_S(z)}(L_1) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_2 \\ u(z) \in \psi^{\rho_S(z)}(L_2) & \text{if } z \in \partial S \text{ lies between } \xi_2 \text{ and } \xi_0 \\ u(z) \in \psi^{\rho_S(z)}(L_0) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_0 \end{cases} \quad (2.42)$$

and such that the image of u converges to x_1 and x_2 at the corresponding incoming strip-like ends, and to $\psi^2(x_0)$ at the outgoing strip-like end.

From our construction, the pullback of equation (2.96) by the strip-like ends ξ^k is given by

$$(du - X \otimes dt)^{0,1} = 0 \text{ with respect to } J_t \text{ if } k = 1, 2,$$

$$(du - 2X_{\frac{H}{4} \circ \psi^2} \otimes dt)^{0,1} = 0 \text{ with respect to } (\psi^2)^* J_t \text{ if } k = 0,$$

with Lagrangian boundary conditions (L_0, L_1) near ξ^1 , (L_1, L_2) near ξ^2 , and $(\psi^2(L_0), \psi^2(L_2))$ near ξ^0 . Near the incoming punctures ξ^1, ξ^2 , the inhomogeneous equation is the same as Equation (2.77), while near the outgoing puncture ξ^0 the Hamiltonian becomes $\frac{H}{2} \circ \psi^2$. We conclude that the Gromov compactification of the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is obtained by adding the strata

$$\coprod_{y \in \mathcal{X}(L_0, L_1)} \mathcal{M}_2(x_0; y, x_2) \times \overline{\mathcal{M}}(y; x_1), \quad (2.43)$$

$$\coprod_{y \in \mathcal{X}(L_1, L_2)} \mathcal{M}_2(x_0; x_1, y) \times \overline{\mathcal{M}}(y; x_2), \quad (2.44)$$

$$\coprod_{y \in \mathcal{X}(L_0, L_2)} \overline{\mathcal{M}}(x_0; y) \times \mathcal{M}_2(y; x_1, x_2). \quad (2.45)$$

The above three cases correspond to the breaking of a holomorphic strip at each strip-like end of the the punctured disc. Note that since we are working with exact Lagrangians, there is no bubbling.

Lemma 2.0.16. *For any fixed pair (x_1, x_2) , the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is empty for all but finitely many x_0 , and for a generic family of almost complex structures J_S and Hamiltonians H_S , $\overline{\mathcal{M}}_2(x_0; x_1, x_2)$ is a compact manifold of dimension $\mu(x_0) - \mu(x_1) - \mu(x_2)$, the boundary of which is the union of the codimension 1 strata listed in equations (2.105)-(2.107).*

Transversality is given by standard Sard-Smale arguments going back to [11] in the case of Hamiltonian Floer cohomology; compactness of the moduli spaces is given by similar arguments of Lemma (2.0.36), and is explicitly explained in [3, Lemma 3.2]. Whenever the dimension of the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is 0, it consists of finitely many elements, each of which defines an

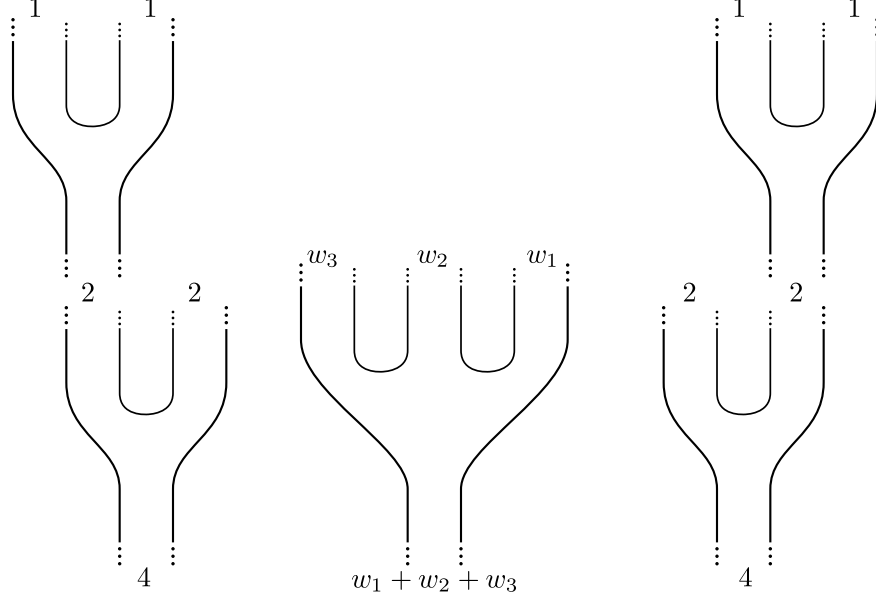


Figure 2.1:

isomorphism

$$o_{x_1} \otimes o_{x_2} \rightarrow o_{\psi^2(x_0)}. \quad (2.46)$$

We also denote by μ_u this isomorphism induced on orientation lines; now we are set to define a product on the wrapped Floer complex as follows:

$$\begin{aligned} \mu_2^{\psi^2} : \mathcal{CW}_b^*(L_2, L_1) \otimes \mathcal{CW}_b^*(L_1, L_0) &\rightarrow \mathcal{CW}_b^*(\psi^2(L_2), \psi^2(L_0)) \\ \mu_2^{\psi^2}([x_1], [x_2]) &= \sum_{\substack{|x_0|=|x_1|+|x_2| \\ u \in \mathcal{M}_2(x_0; x_1, x_2)}} (-1)^{\mu(x_1)} \mu_u([x_2], [x_1]). \end{aligned} \quad (2.47)$$

2.0.6 A_∞ structure

We want to equip each strip-like end with Floer data that would facilitate the definition of an A_∞ structure on the wrapped Fukaya category. First, denote by \mathcal{R}_d the moduli space of disks that is equipped with d incoming boundary punctures $\{\xi^k\}_{k=1}^d$, ordered clockwise and a outgoing boundary puncture ξ^0 . Denote by $\overline{\mathcal{R}}_d$ the Deligne-Mumford compactification of this moduli space; as in [22, Section 9], we fix a *universal and consistent* choice of strip-like ends. In other words,

for each surface S and each puncture, there exists a map

$$\epsilon^k : Z_{\pm} \rightarrow S,$$

whose source is Z_- if it is the outgoing puncture, and Z_+ otherwise, and such a choice varies smoothly in the interior of $\overline{\mathcal{R}}_d$. Each boundary stratum σ of $\overline{\mathcal{R}}_d$ has the following form:

$$\sigma = \mathcal{R}_{d_1} \times \mathcal{R}_{d_2} \times \cdots \times \mathcal{R}_{d_j}.$$

A neighborhood of σ looks like gluing of strip-like ends chosen on these lower dimensional moduli spaces:

$$\sigma \times [1, \infty)^{j-1} \rightarrow \mathcal{R}_d.$$

Let us assign to each end ξ^i of a punctured disc a weight w_i , which is a positive real number, with the convention that any $\bar{\partial}$ operator on such a disc must pull back under ϵ^i to the Cauchy Riemann equation (2.77) up to applying ψ^{w_i} . In the product case where there are two incoming ends and one outgoing end, the inputs have weights 1 while the output has weight 2.

If we have three inputs and one output, the moduli space of all such possible configurations $\overline{\mathcal{R}}_3$ is actually an interval whose two ends are represents by nodal discs obtained by gluing the two possible different ways as shown in Figure (2.0.12), two discs with two incoming ends of weight 1 and the other with weight 2. Recall from the previous section (2.0.11) where we defined the product (2.95) to be the composition of (2.94) with rescaling. Instead of rescaling first and then take composition, one pulls back by ψ^2 of all the data used to define $\mathcal{M}(x_0; x_1, x_2)$, and thus obtain a new $\bar{\partial}$ operator where the weights are now 2 at the incoming ends and 4 at the outgoing end. Then one composes $C\mathcal{W}_b^*(\psi^2(L_3), \psi^2(L_2))$ with $C\mathcal{W}_b^*(\psi^2(L_2), \psi^2(L_0))$ after applying (2.94) with L_0, L_1, L_2 . In this way we define the product with three inputs directly:

$$\mu_3 : C\mathcal{W}_b^*(L_3, L_2) \otimes C\mathcal{W}_b^*(L_2, L_1) \otimes C\mathcal{W}_b^*(L_1, L_0) \rightarrow C\mathcal{W}_b^*(L_3, L_0). \quad (2.48)$$

However, we immediately encounter the problem of ordering, namely when we multiply the first two terms then with the third, the result is different from that of multiplying the last two terms first on the chain level on the right hand side. In other words, μ_2 is not associative at the chain level.

Yet these two give the same element on the cohomology level by looking at the following diagram as shown in [5].

$$\begin{array}{ccc}
CW_b^*(L_2, L_3) \otimes CW_b^*(L_1, L_2) \otimes CW_b^*(L_0, L_1) & \xrightarrow{\mu_2^{\psi^2} \otimes CW(\psi^2)} & CW_b^*(\psi^2(L_1), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_1)) \\
\downarrow CW(\psi^2) \otimes \mu_2^{\psi^2} & & \downarrow \mu_2^{\psi^4} \\
CW_b^*(\psi^2(L_2), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_2)) & \xrightarrow{\mu_2^{\psi^4}} & CW_b^*(\psi^4(L_0), \psi^4(L_3)). \\
& & \downarrow \cong \\
& & CW_b^*(L_0, L_3)
\end{array} \tag{2.49}$$

In order to show the existence of a homotopy of maps that links the former two orders of taking product, we need a moduli space of maps whose sources are disks with 4 punctures on its boundary and arbitrary conformal structure. We take its compactification $\overline{\mathcal{R}}_3$ as in Figure (2.0.12). Whatever equation we define on $\overline{\mathcal{R}}_3$ has to restrict to equation (2.96) on each component of the boundary. In particular, the Cauchy-Riemann equation imposed on a disc whose conformal equivalence class is close to the boundary of $\overline{\mathcal{R}}_3$ is obtained by gluing the $\bar{\partial}$ operators associated to equation (2.96) at the node. This only works if the restriction of these $\bar{\partial}$ operators restrict to the two strip-like ends at the node agree, which is not true yet, since even the Lagrangian boundary conditions do not yet agree. However, after applying the time log-2 Liouville flow ψ^2 , they agree. Thus we can glue two solutions to equation (2.96) after applying ψ^2 to one of them.

We also notice that the weights on the inputs of the three punctured disc have to be allowed to change with its modulus, as they are given by (1, 1, 2) at one of the endpoints of $\overline{\mathcal{R}}_3$ and (2, 1, 1) at the other endpoint. We now define auxiliary data of families of Hamiltonians and almost complex structures on M , and of 1-forms on elements of $\overline{\mathcal{R}}_3$ which would then help define an operation

$$C\mathcal{W}^*(L_3, L_2) \otimes C\mathcal{W}^*(L_2, L_1) \otimes C\mathcal{W}^*(L_1, L_0) \rightarrow C\mathcal{W}^*(L_3, L_0), \tag{2.50}$$

based on the homotopy in diagram (2.111). We note that this would only work with product with three inputs; actually we need arbitrarily many inputs, thus we proceed to the following general construction originally from [3].

Definition 2.0.17. *A Floer datum D_S on a stable disc $S \in \overline{\mathcal{R}}_d$ consists of the following choices:*

1. *Weights: A positive integer $w_{k,S}$ assigned to the k th end such that*

$$w_{0,S} = \sum_{1 \leq k \leq d} w_{k,S}$$

2. *Moving conditions: A map $\rho_S : \partial \bar{S} \rightarrow [1, +\infty)$ which agrees with w_k near the k th end.*

3. *Basic 1-form: A closed 1-form γ_S whose restriction to the boundary vanishes and whose pullback under ϵ^k agrees with $w_{k,S} dt$.*

4. *Hamiltonian perturbations: A map $H_S : S \rightarrow \mathcal{H}(M)$ which agrees with $\frac{H \circ \psi^{w_{k,S}}}{w_{k,S}^2}$ near the k th end.*

5. *Almost complex structure: A map $J_S : S \rightarrow \mathcal{J}(M)$ whose pullback under ϵ^k agrees with $(\psi^{w_{k,S}})^* J_t$.*

If we write X_S for the Hamiltonian flow of H_S , then these data allow us to write down a Cauchy-Riemann equation

$$(du - X_S \otimes \gamma_S)^{0,1} = 0, \tag{2.51}$$

where the $(0, 1)$ part is taken with respect to J_S .

The long list of definition (2.0.41) guarantees the following:

Lemma 2.0.18. *The pullback of equation (2.113) under ϵ^k is*

$$(du \circ \epsilon^k - X_{\frac{H \circ \psi^{w_{k,S}}}{w_{k,S}}} \otimes dt)^{0,1} = 0. \tag{2.52}$$

It agrees with equation (2.77) up to conformal rescaling with $\psi^{w_{k,S}}$. □

At the boundary of the moduli space \mathcal{R}_d , we require that Floer data be given by the choices performed on smaller dimensional moduli spaces. When $d = 3$, we already observed that the choice of Floer data at the boundary can't be given precisely by the Floer data for \mathcal{R}_2 on each component, since these Floer data can't be glued. We shall consider the following notion of equivalence among Floer data which is weaker than equality:

Definition 2.0.19. *We say that a pair $(\rho_S^1, \gamma_S^1, H_S^1, J_S^1)$ and $(\rho_S^2, \gamma_S^2, H_S^2, J_S^2)$ of Floer data on a surface S are conformally equivalent if there exists a constant C such that*

$$\begin{aligned}\rho_S^2 &= C\rho_S^1, \\ \gamma_S^2 &= C\gamma_S^1, \\ J_S^2 &= \psi^{C*}J_S^1, \\ H_S^2 &= \frac{H_S^1 \circ \psi^C}{C^2}.\end{aligned}$$

The idea that rescaling by ψ^2 in Equation (2.91) gives an identification of moduli spaces can be generalized as follows:

Lemma 2.0.20. *Composition with ψ^C defines a bijective correspondence between solutions to equation (2.96) for conformally equivalent Floer data. \square*

We may now state the desired compatibility between Floer data at the boundary of the moduli spaces $\overline{\mathcal{R}}_d$:

Definition 2.0.21. *A universal and conformally consistent choice of Floer data \mathbf{D}_μ for the A_∞ structure, is a choice of Floer data for every element of $\overline{\mathcal{R}}_d$ and every integer $d \geq 2$, which varies smoothly over the interior of the moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from lower dimensional moduli spaces, and which near such a boundary stratum agrees to infinite order, in the coordinates (2.79), with the Floer data obtained by gluing.*

Note that conformal consistency fixes, up to a constant depending on the modulus, the choice of Floer data on $\partial\overline{\mathcal{R}}_d$ once Floer data on each irreducible component has been chosen. For example, the choice of Floer data for $d = 2$ above determines the Floer data on the boundary of $\overline{\mathcal{R}}_3$ as used in the previous section, where we chose to use the original Floer data on the disc that does not contain the original end, and rescale it by ψ^2 on the other disc (see Figure 2.0.12). After introducing a perturbations of this Floer data which vanishes to infinite order at the boundary, we extend it from a neighbourhood of the boundary to the remainder of \mathcal{R}_3 . One may proceed inductively to construct a universal datum \mathbf{D}_μ using the fact that the boundary of $\overline{\mathcal{R}}_d$ is covered by the images of codimension 1 inclusions

$$\overline{\mathcal{R}}_{d_1} \times \overline{\mathcal{R}}_{d-d_1+1} \rightarrow \partial\overline{\mathcal{R}}_d \quad (2.53)$$

and the contractibility of the space of Floer data on a given surface. The contractibility of this space also implies that we may extend any Floer data chosen on a given surface to universal data as in [3, Lemma 3.8]:

Lemma 2.0.22. *The restriction map from the space of universal and conformally consistent Floer data to the space of Floer data for a fixed connected stable surface S is surjective.*

Now we construct the A_∞ structure on the wrapped Fukaya category which is given by higher products

$$C\mathcal{W}^*(L_d, L_{d-1}) \otimes C\mathcal{W}^*(L_{d-1}, L_{d-2}) \otimes \cdots \otimes C\mathcal{W}^*(L_1, L_0) \rightarrow C\mathcal{W}^*(L_d, L_0), \quad (2.54)$$

which comes from the count of solutions of Equation (2.96).

Given a sequence of chords $\vec{x} = x_k \in \mathcal{X}(L_{k-1}, L_k)$ for $1 \leq k \leq d$ and $x_0 \in \mathcal{X}(L_0, L_d)$, we define $\mathcal{M}_d(x_0; \vec{x})$ to be the space of solutions to Equation (2.96) whose source is an arbitrary element $S \in \mathcal{R}_d$ with marked points (ξ^0, \dots, ξ^d) such that

$$\lim_{s \rightarrow \pm\infty} u \circ \epsilon^k(s, \cdot) = \psi^{w_{k,S}} x_k, \quad (2.55)$$

and with boundary conditions

$$u(z) \in \psi^{\rho_S(z)}(L_k) \text{ if } z \in \partial S \text{ lies between } \xi^k \text{ and } \xi^{k+1}. \quad (2.56)$$

Note that Lemma (2.0.42) shows that Equation (2.96) restricts to Equation (2.77) up to applying $\psi^{w_{k,S}}$.

If we don't take into account the strip-breaking phenomenon at the ends, the virtual codimension 1 strata of the Gromov compactification $\overline{\mathcal{M}}_d(x_0; \vec{x})$ lie over the codimension 1 strata of $\overline{\mathcal{R}}_d$. The consistency condition on \mathbf{D}_μ implies that whenever a disc breaks, each component is a solution to equation (2.96) for the Floer data \mathbf{D}_μ up to applying ψ^C for some constant C that depends on the modulus in $\overline{\mathcal{R}}_d$. Such rescaling identifies the solutions of rescaled equation with moduli space of the solutions to the original equation, so we conclude that for each integer k between 0 and $d - d_2$ and chord $y \in \mathcal{X}(L_{k+1}, L_k)$, we get a natural inclusion:

$$\overline{\mathcal{M}}_{d_1}(x_0; \vec{x}^1) \times \overline{\mathcal{M}}_{d_2}(y; \vec{x}^2) \rightarrow \overline{\mathcal{M}}_d(x_0; \vec{x}) \quad (2.57)$$

where the sequences of inputs in the respective factors are given by $\vec{x}^2 = (x_{k+1}, \dots, x_{k+d_2})$ and $\vec{x}^1 = (x_1, \dots, x_k, y, x_{k+d_2+1}, \dots, x_d)$.

Thus we apply the same arguments for Lemma (2.0.40) to get the following lemma.

Lemma 2.0.23. *The moduli space $\overline{\mathcal{M}}_d(x_0; \vec{x})$ is compact and is empty for all but finitely many x_0 once the inputs \vec{x} are fixed. For a generic choice of data \mathbf{D}_μ , they form manifolds of dimension*

$$|x_0| + d - 2 - \sum_{1 \leq k \leq d} |x_k|,$$

whose boundary is covered by the images of inclusions (2.119). □

Whenever $|x_0| = x - d + \sum_{1 \leq k \leq d} |x^k|$, there are therefore only finitely many elements of $\overline{\mathcal{M}}_d(x_0; \vec{x})$. Via the procedure described in Appendix (A.0.1.) every such element $u \in \overline{\mathcal{M}}_d(x_0; \vec{x})$:

$S \rightarrow M$ induces an isomorphism:

$$\mathcal{O}_{\psi^{w_d, S, x_d}} \otimes \cdots \otimes \mathcal{O}_{\psi^{w_1, S, x_1}} \rightarrow \mathcal{O}_{\psi^{w_0, S, x_0}}. \quad (2.58)$$

Let us write μ_u for the induced map on orientation lines, omitting composition with $C\mathcal{W}^*(\psi^{w_k, S})$ or its inverse from the notation. We define the d th higher product:

$$\mu_d : CW_b^*(L_{d-1}, L_d) \otimes \cdots \otimes CW_b^*(L_1, L_2) \otimes CW_b^*(L_0, L_1) \rightarrow CW_b^*(L_0, L_d) \quad (2.59)$$

Explicitly,

$$\mu_d([x_d], \dots, [x_1]) = \sum_{\substack{|x_0|=2-d+\sum_{1 \leq k \leq d} |x_k| \\ u \in \mathcal{M}_d(x_0; \vec{x})}} (-1)^\dagger \mu_u([x_d], \dots, [x_1]), \quad (2.60)$$

where the sign is given by

$$\dagger = \sum_{k=1}^d k |x_k|. \quad (2.61)$$

This is just saying we count the elements of $\overline{\mathcal{M}}_d(x_0; \vec{x})$ with the right signs.

If we now consider the dimension 1 moduli spaces, Lemma (2.0.47) claims that their boundaries are the strata in equation (2.119) which are rigid and thus correspond to the composition of operations μ_d . Taking into account of signs, we get the following proposition:

Proposition 2.0.24. *The operations μ_d define an A_∞ structure on the category $C\mathcal{W}^*(M)$. Explicitly, we have*

$$\sum_{\substack{d_1+d_2=d+1 \\ 0 \leq k < d_1}} (-1)^{\mathfrak{X}_1^k} \mu_{d_1}(x_d, \dots, x_{k+d_2+1}, \mu_{d_2}(x_{k+d_2}, \dots, x_{k+1}), x_k, \dots, x_1) = 0, \quad (2.62)$$

where the sign is given by

$$\mathfrak{X}_1^k = k + \sum_{1 \leq j \leq k} |x_j|.$$

□

2.0.7 Review of Floer theory in open Liouville manifold

In [7] Abouzaid and Seidel constructed the wrapped Floer cohomology on Liouville domains in analogy with the Lagrangian Floer cohomology of a closed symplectic manifold. They defined $HW^*(L; H) \cong \varinjlim_w HF^*(L; wH)$ from a direct system of continuation maps from Lagrangian Floer cohomology with Hamiltonians of increasing slope at infinity. Later in [5], Abouzaid used Hamiltonians that are quadratic at infinite ends to define wrapped Floe homology. We are going to follow this definition.

Geometric Setup

From now on we work on open Liouville manifold (M, ω) with Liouville 1-form θ and infinite cylindrical ends. Namely there is a compact Liouville domain (M^{in}, ω) with boundary, together with a one-form θ such that $\omega = d\theta$ is symplectic, and the dual Liouville vector field Z , uniquely determined by the requirement that $i_Z\omega = \theta$ that points strictly outwards along ∂M^{in} . This implies that $\theta|_{\partial M^{in}}$ is a contact one-form. By flowing inwards from the boundary along Z , one obtains a collar embedding $(0, 1] \times \partial M^{in} \hookrightarrow M^{in}$. One can then complete M^{in} to obtain M by attaching the big half of symplectization:

$$M = M^{in} \cup_{\partial M^{in}} ([1, \infty) \times \partial M^{in}). \quad (2.63)$$

The piece $[1, \infty) \times \partial M^{in}$ is called the *infinite cone*. The completion carries a natural one-form $\hat{\theta}$ such that $d\hat{\theta} = \hat{\omega}$ is symplectic and an associated Liouville vector field \hat{Z} . On $M^{in} \subset M$ these restrict to the previous data, while on the overlapping piece $(0, \infty) \times \partial M^{in}$, $\hat{\theta} = r(\theta|_{\partial M^{in}})$ and $\hat{Z} = r\partial_r$, where r is the variable on $(0, \infty)$.

Let $L \in M^{in}$ be a Lagrangian intersecting ∂M^{in} transversally, and which has the following property

$$\begin{aligned} \theta|_L \text{ is exact, } \theta|_L = df. \text{ Moreover, } \theta|_L \text{ vanishes near the boundary } \partial L = \\ L \cap \partial M^{in}. \end{aligned} \quad (2.64)$$

The second part implies that ∂L is a Legendrian submanifold of ∂M^{in} . Furthermore, by attaching an infinite cone to this boundary, one can extend L to a smooth noncompact Lagrangian submanifold

$$\hat{L} = L \cup_{\partial L} ([1, \infty) \times \partial L) \subset M. \quad (2.65)$$

In the remainder of this paper, we only use θ, ω, L to denote their extensions (resp. completions) in M since there won't be any confusion.

We use ψ^ρ to denote the image of the Liouville flow for time $\log(\rho)$ outwards along the cylindrical end. Namely,

$$\psi^\rho(r, x) = (\rho \cdot r, x).$$

Let $\mathcal{H}(M) \subseteq C^\infty(M, \mathbb{R})$ denote the set of smooth functions H such that

$$H(q, r) = r^2 \quad (2.66)$$

for any (q, r) (where $q \in \partial M^{in}$) on the cylindrical end and $r \gg 1$. Namely H is quadratic with respect to the radial coordinate away from some compact subset of M . We fix such a function for the purpose of defining Floer homology, and let X_H denote the Hamiltonian flow of H defined by the equation

$$i_{X_H} \omega = -dH. \quad (2.67)$$

We denote by \mathcal{Lag} the space of all Lagrangians with vanishing Maslov index that also satisfies Condition (2.64), namely those that are equipped with Legendrian boundary and the restriction of the one form θ on L is exact,

$$\theta|_L = df, \quad (2.68)$$

for some $f \in C^\infty(L, \mathbb{R})$. When L moves in \mathcal{Lag} , we shall choose a primitive f_L for the restriction of θ to each such Lagrangian that is continuous with respect to L .

For each pair of Lagrangians $L_0, L_1 \in \mathcal{Lag}$, we define $\mathcal{X}(L_0, L_1)$ to be the set of time-1 flow

lines of X_H that start from L_0 and stop at L_1 , namely a map $x : [0, 1] \rightarrow M$ such that

$$\begin{cases} x(0) \in L_0 \\ x(1) \in L_1 \\ \frac{dx}{dt} = X_H \end{cases}$$

In order to get well-defined moduli spaces, we assume that

$$\text{all time-1 Hamiltonian chords of } H \text{ with boundaries on } L_0 \text{ and } L_1 \text{ are non-degenerate.} \quad (2.69)$$

It is convenient to point out that the Hamiltonian chords above correspond bijectively to the intersection points of L_1 with the time-1 flow $\phi_H^1(L_0)$ of L_0 under the Hamiltonian flow of H . In particular, the non-degeneracy of the chords is the same as transversal intersection of L_1 and $\phi_H^1(L_0)$. Condition (2.69) is therefore true after we do a generic Hamiltonian perturbation on the Lagrangians that intersect transversally with ∂M^{in} . Since Lagrangian Floer Homology is invariant under Hamiltonian perturbation, the above assumption is valid.

Brief discussion of relative Pin and brane structures

We briefly introduce the definition of Lagrangian branes for the sake of a proper definition of Lagrangian Floer homology. Let (M, ω, J) be symplectic with $2c_1(M) = 0$, equipped with a non-vanishing quadratic volume form η_M^2 , namely a global section of the line bundle $(\Omega^{n,0}M)^{\otimes 2} = (\wedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$. Likewise, given a symplectic vector bundle $\varphi : E \rightarrow B$ of rank r such that $2c_1(E) = 0$, we have a global non-vanishing complex volume form $\eta_E^2 \in H^0(B, (\wedge_{\mathbb{C}}^r E)^{\otimes 2})$. Let $Gr(E)$ be the set of Lagrangian sub-bundles of E , and v_1, \dots, v_r be a basis of E_x for any $x \in B$. We define the squared phase map of E as follows:

Definition 2.0.25. *Given a symplectic vector bundle $\varphi : E \rightarrow B$ of rank r such that $2c_1(E) = 0$,*

and any Lagrangian sub bundle $\phi : F \subset E$, consider the following squared phase map of E :

$$\alpha_E : Gr(E) \times B \rightarrow S^1,$$

$$\alpha_E(F, x) := \frac{\eta_x^2(v_1 \wedge v_2 \wedge \cdots \wedge v_r)}{|\eta_x^2(v_1 \wedge v_2 \wedge \cdots \wedge v_r)|}.$$

Remark 2.0.26. A simple linear algebra exercise shows that α_E is independent of the choices of the set of basis; thus it is well defined.

Moreover, if the Maslov class of F vanishes, namely $\alpha_E(F, \cdot)_* : H_1(B) \rightarrow H_1(S^1) \simeq \mathbb{Z}$ is zero, we lift the map $\alpha_E(F, \cdot)$ to a map $\alpha^\#$ from B to the universal cover \mathbb{R} of S^1 . In this way we obtain the following definition:

Definition 2.0.27. For a Lagrangian sub-bundle F with vanishing Maslov class inside a symplectic fibration $E \rightarrow B$, an associated grading $\alpha^\#$ on F is a function:

$$\alpha^\# : B \rightarrow \mathbb{R}, \tag{2.70}$$

$$\text{such that } \exp(2\pi i \alpha^\#(x)) := \alpha_E(F, x). \tag{2.71}$$

Remark 2.0.28. The choice of grading is not a canonical one, in fact, it is a free torsor of $H^0(B; \mathbb{Z})$.

We further assume that for some $b \in H^2(B; \mathbb{Z}/2)$, the second Stiefel-Whitney class $w_2(\phi) \in H^2(B; \mathbb{Z}/2)$ is the same as b , namely $b = w_2(\phi)$. Fixing a triangulation of B , we can choose an oriented vector bundle over the three skeleton $B_{[3]}$, $\psi : E_b \rightarrow B_{[3]}$ such that $w_2(\psi) = b|_{B_{[3]}}$. Then our assumption on $w_2(\phi)$ gives

$$w_2(\phi|_{B_{[3]}} \oplus \psi) = 0. \tag{2.72}$$

From the standard literature([22, Section 11i], [21]), we know that second Stiefel-Whitney class is precisely the obstruction to the existence of Pin structures:

Definition 2.0.29. A relative Pin structure on a Lagrangian sub-bundle F is the choice of a Pin structure on the vector bundle $F|_{B_{[3]}} \oplus E_b$.

Definition 2.0.30 (brane structure). *Given a symplectic vector bundle $E \rightarrow B$ and $b \in H^2(B; \mathbb{Z}/2)$, consider a Lagrangian sub-bundle $F \rightarrow B$ such that $2\mu(F) = 0$, $w_2(F) + b = 0$. A brane structure with respect to b consists of a pair $(\alpha^\#, P^\#)$, where $\alpha^\#$ satisfies the equation (2.71) and $P^\#$ is a relative Pin structure on F .*

Remark 2.0.31. The obstruction to the existence of a grading is the Maslov class $\mu(F) \in H^1(B; \mathbb{Z})$ represented by the map $\alpha_E(F, \cdot) : B \rightarrow S^1$; if $b = 0$, then we have a Pin-structure on F in the sense of [22], whose obstruction is the second Stiefel-Whitney class $w_2(E) \in H^2(B, \mathbb{Z}/2)$. Here we take the direct sum with E_b so as to “kill” the obstruction given by possibly non-zero b .

Now in our situation, the base B of the fibration is an exact Lagrangian sub manifold $L \in M$ with Legendrian boundary, the sub-bundle is just $TL \subset TM|_L$ whose Maslov class $\mu(TL)$ vanishes.

Therefore, to each Hamiltonian chord $x \in \mathcal{X}(L_0, L_1)$, we can assign a grading $\mu(x)$ from the gradings on L_0, L_1 for given transverse brane structures on L_0, L_1 . Here we define $\mu(x) := \lfloor \alpha_1^\# - \alpha_0^\# \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function that lies in \mathbb{Z} . We also define the orientation space $\mathcal{o}_x = Hom(P_0^\#, P_1^\#) \otimes TL_1^{\otimes \mu(x)}$. For details we refer to [22, Section 11].

We denote the space of exact Lagrangians with Legendrian boundary condition with a relatively Pin structure with respect to the background class $b \in H^2(M)$ by

$$\mathcal{Lag}_b \tag{2.73}$$

Namely Lagrangians with vanishing $w_2(L) + b$. We know that the isomorphism classes of Pin structures on TL is a free torsor of $H^1(L; \mathbb{Z}/2)$ by examining the long exact sequence

$$\dots \rightarrow H^1(L; \mathbb{Z}/2) \rightarrow H^1(L; Pin(n)) \rightarrow H^1(L, O(n)) \rightarrow H^2(L; \mathbb{Z}/2) \rightarrow \dots \tag{2.74}$$

that arises from the exact sequence of classifying spaces:

$$B\mathbb{Z}/2 \rightarrow BPin(n) \rightarrow BO(n) \rightarrow B^2\mathbb{Z}/2 \rightarrow \dots \tag{2.75}$$

Thus for each $L \in \mathcal{L}ag_b$, together with a Pin structure $P^\#$ on TL and any class $\beta \in H^1(L; \mathbb{Z}/2)$, we may twist $P^\#$ by β as follows: take a line bundle ξ on L whose first Stiefel-Whitney class is β and its associated double cover $S(\xi)$. The Pin structure $P^\#$ is a $\text{Pin}(n)$ -principal bundle $P^\# \rightarrow L$ with an isomorphism $P^\# \times_{\text{Pin}(n)} \mathbb{R}^n \cong TL$. Let $\pm e$ denote the kernel of the double cover $\text{Pin}(n) \rightarrow O(n)$. We take the product $S(\xi) \times_{\mathbb{Z}/2} P^\#$ where $\mathbb{Z}/2$ acts on $S(\xi)$ by ± 1 and on $P^\#$ by $\pm e$. This gives a new $\text{Pin}(n)$ -structure on TL .

A brane structure on L , by Definition (2.0.30), consists of a grading $\alpha^\# : L \rightarrow \mathbb{R}$ and Pin structure $P^\#$; the gradings $\alpha^\#$ are free torsors of $H^0(L, \mathbb{Z})$. Notice in our situation we don't require that $w_2(L) = 0$, yet all the Lagrangians we consider have to satisfy $w_2(TL) = i_L^* b$ for some background class $b \in H^2(M, \mathbb{Z}/2)$. Denoting by $\widetilde{\mathcal{L}ag}_b$ the space of Lagrangians equipped with a specific Pin structure $(L, P^\#)$ with a fixed background class b (we may omit this in later section and just denote the space $\widetilde{\mathcal{L}ag}$ when it is clear from the context), we obtain a fibration:

$$\begin{array}{ccc} H^1(L; \mathbb{Z}/2) & \longrightarrow & \widetilde{\mathcal{L}ag}_b \\ & & \downarrow \pi \\ & & \mathcal{L}ag_b \end{array}$$

Note that for Lagrangians of different first cohomology groups, this fibration gives different fibers, thus in practice we only look at a component of any given L such that the fibration π is surjective with the same fiber $H^1(L; \mathbb{Z}/2)$.

2.0.8 Almost complex structures

In this paper we work with almost complex structures of contact type, namely those satisfy Equation (1.4). Let $\mathcal{J}(M)$ denote the space of such almost complex structures on M which are also compatible with the symplectic form on M . This is an infinite-dimensional manifold. Its tangent space $T\mathcal{J}(M)$ at any point J consists of endomorphisms K of TM which anti-commute with J , and which have the property that $\omega(\cdot, K\cdot)$ is symmetric. We call these endomorphisms *infinitesimal deformations* of the almost complex structure. One can get an actual deformation by

exponentiating K , namely setting

$$J' = J \exp(-JK). \quad (2.76)$$

As in [7, Section 3], for any disc S with punctures on the boundary that are equipped with a set of strip-like ends $\{\epsilon^k\} : Z_{\pm} \rightarrow S$, and weights $\{w^k\}$ there exists a contractible choice of almost complex structures parametrized by S such that for each strip-like end $\epsilon^k : Z_{\pm} \rightarrow S$ with (s, t) denoting the coordinates of Z_{\pm} , the almost complex structure $J_{\epsilon^k(s,t)}$ is only dependent on t for $|s| \gg 0$. In other words, if we go sufficiently far along any end, J becomes independent of s , hence compatible with translations in the s direction.

2.0.9 Moduli spaces

We consider the space of maps from the strip $Z = \mathbb{R} \times [0, 1]$ to a symplectic manifold M satisfying certain equations. First of all, we need to fix some data for the equations: a family $J_t \in \mathcal{J}(M)$ of almost complex structures parametrized by $t \in [0, 1]$; a closed one-form γ on Z such that $\gamma(s, t) = dt$ when $|s| \gg 0$ and a Hamiltonian $H : M \rightarrow \mathbb{R}$ that satisfies Condition (2.69). Given a pair $x_0, x_1 \in X(L_0, L_1)$, we have the following definition:

Definition 2.0.32. $\tilde{\mathcal{M}}(x_0; x_1)$ is the moduli space of maps

$$u : Z = \mathbb{R} \times [0, 1] \rightarrow M,$$

satisfying the following conditions

$$\left\{ \begin{array}{l} u(\mathbb{R} \times \{1\}) \subseteq L_1, \\ u(\mathbb{R} \times \{0\}) \subseteq L_0, \\ \lim_{s \rightarrow -\infty} u(s, \cdot) = x_0(\cdot), \\ \lim_{s \rightarrow \infty} u(s, \cdot) = x_1(\cdot). \end{array} \right.$$

such that u solves the Cauchy-Riemann equation

$$(du - X_H \otimes \gamma)^{0,1} = 0 \quad (2.77)$$

which can also be written explicitly with the family of almost complex structure J_t :

$$J_t(\partial_t u - X_H \cdot \gamma) - (\partial_t u - X_H \cdot \gamma) \circ j = 0.$$

Since equation (2.77) is invariant under translation of the variable s , there is an action of \mathbb{R} on $\tilde{\mathcal{M}}(x_0; x_1)$. After taking the quotient by the free \mathbb{R} -action, we get $\mathcal{M}(x_0; x_1)$; and let it be empty if the action is not free. Standard arguments of the gluing theorem such as those in Appendix (6.3) of [3] give the following:

Proposition 2.0.33. *The moduli space $\mathcal{M}(x_0; x_1)$ is regular for a generic family of almost complex structures J_t and has dimension $\mu(x_0) - \mu(x_1) - 1$. The Gromov compactification $\overline{\mathcal{M}}(x_0; x_1)$ is a topological manifold with boundary. The boundary is stratified into topological manifolds with the closure of codimension 1 strata given by the images of embeddings*

$$\overline{\mathcal{M}}(x_0; y) \times \overline{\mathcal{M}}(y; x_1) \rightarrow \overline{\mathcal{M}}(x_0; x_1), \quad (2.78)$$

for all possible y in $\mathcal{X}(L_0, L_1)$. For each pair (y_1, y_2) that gives a boundary stratum of higher codimension, we obtain a commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; y_2) \times \overline{\mathcal{M}}(y_2; x_1) & \longrightarrow & \overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; x_1) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(x_0; y_2) \times \overline{\mathcal{M}}(y_2; x_1) & \longrightarrow & \overline{\mathcal{M}}(x_0; x_1). \end{array} \quad (2.79)$$

□

In particular, the codimension k boundary strata consist of disjoint unions over all possible

sequences of products of moduli spaces of the following form:

$$\overline{\mathcal{M}}(x_0; y_1) \times \overline{\mathcal{M}}(y_1; y_2) \times \cdots \times \overline{\mathcal{M}}(y_k; x_1). \quad (2.80)$$

Since the Lagrangians L_0, L_1 have infinite ends, we need to show that $\overline{\mathcal{M}}(x_0; x_1)$ is compact.

Note that the Hamiltonian chords $x \in \mathcal{X}(L_0, L_1)$ are critical points of an action functional \mathcal{A} on the space $\Lambda(L_0, L_1)$ of smooth paths in M with end points on L_0, L_1 . It is defined as

$$\mathcal{A}(x) = - \int_0^1 x^* \theta + \int H(x(t)) dt + f_{L_1}(x(1)) - f_{L_0}(x(0)),$$

where f_{L_i} is the primitive function given by Equation (2.68).

Condition (2.66) for the Hamiltonian and Equation (2.67) imply that for $r \gg 1$, any Hamiltonian chord that intersects some slice $\partial M^{in} \times \{r\}$ is necessarily contained inside this slice because X_H neither points outwards nor inward on cylindrical ends. The action of x for any $x \in \mathcal{X}(L_0, L_1)$ is given by

$$\mathcal{A}(x) = - \int_0^1 x^* \theta + \int H(x(t)) dt + f_1(x(1)) - f_0(x(0)). \quad (2.81)$$

From the above equation we realize that \mathcal{A} is a proper map from $\mathcal{X}(L_0, L_1)$ to \mathbb{R} as there are only finitely many non-degenerate Hamiltonian chords between L_0 and L_1 in any compact subset of M , and the above equation implies that the action of a sequence of chords that escapes any compact subset must go to ∞ .

Lemma 2.0.34. *Given any solution of Equation (1.3) that converges to $x_0(t)$ at the negative end and to $x_1(t)$ at the positive end, the following inequality holds*

$$\mathcal{A}(x_0) \geq \mathcal{A}(x_1). \quad (2.82)$$

Furthermore, all such solutions lie within some compact subset of M that is dependent only on x_0 and x_1 .

Remark 2.0.35. This is the Integrated Maximum Principle as shown in Lemma 7.2 of [7]. We give a brief discussion of the proof here.

Proof. First we define the topological energy of a solution of Equation (2.77) to be:

$$E^{top}(u) = \int_Z u^* \omega - d(u^* H dt) \quad (2.83)$$

$$= \int_Z (u^* \omega - u^* dH \wedge \gamma) - \int_Z (u^* H \cdot d\gamma) \quad (2.84)$$

$$= \int_Z \frac{1}{2} \|du - X_H \otimes dt\|^2 - \int_Z u^* H \wedge d\gamma \quad (2.85)$$

$$\geq 0. \quad (2.86)$$

Let s be large enough such that $u(-s, t) = x_0(t)$ and $u(s, t) = x_1(t)$. Denote the subspace $[-s, s] \times [0, 1] \subset Z$ by Z_s . We use Stoke's theorem to show:

$$\begin{aligned} \mathcal{A}(x_0) - \mathcal{A}(x_1) &= - \int_0^1 x_0^* \theta + H_0(x_0) dt + \int_0^1 x_1^* \theta - H_1(x_1) dt \\ &\quad + f_1(x_0(1)) - f_1(x_1(1)) - (f_0(x_0(0)) - f_0(x_1(0))) \\ &= - \int_0^1 x_0^* \theta + \int_0^1 x_1^* \theta + \int_{x_0(1)}^{x_1(1)} df_1 - \int_{x_0(0)}^{x_1(0)} df_0 + \int H_0(x_0) dt - \int H_1(x_1) dt \\ &= \int_{\partial Z_s} u^* \theta - \int_{\partial Z_s} u^* H dt \\ &= \int_{Z_s} u^* \omega - \int_{Z_s} d(u^* H dt) \\ &= E^{top}(u) \geq 0. \end{aligned}$$

The third equality above uses the fact that $\theta|_{L_i} = df_i$ and that dt vanishes on the upper and lower boundaries of Z_s . Thus the inequality (2.82) holds by non-negativity of topological energy.

To prove compactness, consider a hypersurface $\partial M^{in} \times \{r\}$ where $r \gg 0$ such that the radial coordinates of the two Hamiltonian chords $x_0(t), x_1(t)$ don't exceed r . If a solution of Equation (2.77) escapes the compact subset $M^{in} \cap \partial M^{in} \times [1, r]$, we may find a compact subset Σ of the domain $Z = \mathbb{R} \times I$ whose boundary is mapped into $\partial M^{in} \times \{r\}$. Depending on whether Σ intersects

with the upper or lower boundary of Z , its own boundary $\partial\Sigma$ might contain components that are mapped into the Lagrangians L_0 and L_1 with Legendrian boundary, which are the images of the intersection of $\partial\Sigma$ and the upper and lower boundaries of Z . We denote the union of such boundary components of $\partial\Sigma$ by $\partial_l\Sigma$. The remaining boundary components of Σ are mapped into the slice $\partial M^{in} \times \{r\}$ and we denote them by $\partial_r\Sigma$.

The one-form θ is exact when restricted to the Lagrangians as required by (2.64). We shall also specify conditions on the one-form γ such that it vanishes on the Lagrangians with Legendrian boundary as in Definition (2.0.41). The topological energy of the map restricted to Σ is non-negative:

$$\begin{aligned}
0 \leq E^{top}(u|_\Sigma) &= \int_\Sigma u^* \omega - dH \otimes \gamma \\
&= \int_{\partial\Sigma} u^* \theta - H \otimes \gamma \\
&= \int_{\partial_r\Sigma} u^* \theta - H \otimes \gamma + \int_{\partial_l\Sigma} u^* \theta - H \otimes \gamma \\
&= \int_{\partial_r\Sigma} \theta(du - X_H \otimes \gamma) + 0 \\
&= \int_{\partial_r\Sigma} (\theta \circ J_z) \circ (du - X_H \otimes \gamma) \circ j.
\end{aligned}$$

Note in the above equations, $\int_{\partial_l\Sigma} H \otimes \gamma$ vanishes because γ vanishes on $\partial_l\Sigma$; $\int_{\partial_l\Sigma} u^* \theta$ vanishes because θ is exact on the Lagrangians, namely $\theta|_{L_i} = df_i$ and the end points of the union of all boundary components on either Lagrangian L_i end on $L_i \cap \partial M^{in} \times \{r\}$.

Since H is quadratic with respect to radial coordinate r as in Equation (2.66), dH is a multiple of dr . Equation (1.4) implies that $\theta(J_z X_H) = -dr(X_H)$ is a multiple of $dH(X_H)$, and therefore vanishes. The above inequality becomes

$$0 \leq \int_{\partial_r\Sigma} \theta \circ J \circ du \circ j. \quad (2.87)$$

Meanwhile, if ξ is a tangent vector compatible with orientation of $\partial_r\Sigma$, then $j\xi$ points towards the interior of Σ , thus $du(j\xi)$ points towards the direction of growing radial coordinate. Equation (1.4) implies $\theta(J_z \circ du \circ j\xi) < 0$. This is a contradiction. \square

The above lemma together with the observation that \mathcal{A} is a proper map above implies that the image of all possible $\mathcal{M}(x_0; x_1)$ for a fixed $x_1 \in \mathcal{X}(L_0; L_1)$ are contained in a compact subset of M . Therefore we may apply the usual arguments using Gromov's compactness and gluing results to conclude:

Corollary 2.0.36. *For each chord $x_1(t)$, the moduli space $\overline{\mathcal{M}}(x_0; x_1)$ is empty for all but finitely many choices of x_0 , and is a compact manifold with boundary of dimension $\mu(x_0) - \mu(x_1) - 1$ whenever J_t is a generic family of almost complex structure. Moreover, the boundary is covered by the images of the natural inclusions.*

$$\overline{\mathcal{M}}(x_0; y) \times \overline{\mathcal{M}}(y; x_1) \rightarrow \overline{\mathcal{M}}(x_0; x_1).$$

□

From now on through the end of this section, we fix the almost complex structure J_t for which the above corollary holds.

2.0.10 Wrapped Floer complex

We define the graded vector space underlying the Floer chain complex to be the direct sum

$$C\mathcal{W}_b^*(L_0; L_1) := \bigoplus_{x \in \mathcal{X}(L_0; L_1)} |o_x|, \quad (2.88)$$

where o_x is the determinant line bundle explained in Appendix A. We may simply assume that it is just the vector space freely generated by the sets of chords unless one wants to worry about signs.

The differential is a count of the solutions of rigid moduli spaces of equation (2.77), namely those with $\mu(x_0) = \mu(x_1) + 1$. Every element u of the moduli space $\mathcal{M}(x_0, x_1)$ is rigid and thus defines an isomorphism:

$$o_{x_0} \cong o_{x_1}.$$

Thus an orientation of o_{x_0} is induced from one on o_{x_1} , and the map on the orientation lines is

denoted μ_u . We define the map:

$$\mu_1 : C\mathcal{W}_b^i(L_0; L_1) \rightarrow C\mathcal{W}_b^{i+1}(L_0, L_1) \quad (2.89)$$

$$[x_1] \mapsto (-1)^i \sum_u \mu_u([x_1]). \quad (2.90)$$

The proof that this count is a well defined differential of chain complexes is standard (see eg. [8]). Moreover, Corollary (2.0.36) shows that each chord can be the input of only finitely many solutions to (2.77), thus we know that on the right hand side of (2.89) there are only finitely many summands.

Remark 2.0.37. The graded vector space underlying $C\mathcal{W}_b^*(L_0; L_1)$ depends on L_0, L_1, H, ω , meanwhile the differential depends on J_t . When we want to distinguish them as in the next lemma, we write $C\mathcal{W}_b^*(L_0, L_1; \omega, H, J_t)$.

Lemma 2.0.38. *If $\psi : M \rightarrow M$ satisfies $\psi^*(\omega) = \rho\omega$ for some non-zero constant ρ , then we have a canonical isomorphism*

$$C\mathcal{W}(\psi) : C\mathcal{W}_b^*(L_0, L_1; \omega, H, J_t) \cong C\mathcal{W}_b^*(\psi(L_0), \psi(L_1), \frac{H \circ \psi}{\rho}, \psi^*(J_t)). \quad (2.91)$$

This was shown as [5, Lemma 3.4], which I rewrite here for the convenience of the reader.

Proof. Given a diffeomorphism ψ , we obtain an isomorphism of chain complexes

$$C\mathcal{W}_b^*(L_0, L_1; \omega, H, J) \cong C\mathcal{W}_b^*(\psi(L_0), \psi(L_1), \psi^*\omega, \psi^*H, \psi^*(J)).$$

This comes from the composition of each chord from L_0 to L_1 with ψ and the composition of each solution to equation (2.77) with ψ ; thereby obtaining an equation with ψ^*J_t . Since $\psi^*\omega = \rho\omega$, the Hamiltonian flow of $H \circ \psi$ with respect to $\psi^*\omega$ agrees with the flow of $\frac{H \circ \psi}{\rho}$ with respect to ω .

Therefore we obtain an identification:

$$C\mathcal{W}_b^*(\psi(L_0), \psi(L_1); \psi^* \omega, H \circ \psi, \psi^* J_1) \cong C\mathcal{W}_b^*(\psi(L_0), \psi(L_1), \omega, \frac{H \circ \psi}{\rho}, \psi^* J_1). \quad (2.92)$$

From now on, we omit the Floer data and define

$$C\mathcal{W}_b^*(\psi(L_0), \psi(L_1)) := C\mathcal{W}_b^*(\psi(L_0), \psi(L_1); \omega; \frac{H \circ \psi}{\rho}, \psi^* J_1)$$

□

2.0.11 Product Structure

We would like to study the product operation on wrapped Floer complexes,

$$C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(L_0, L_2; H). \quad (2.93)$$

However, the naturally defined map would take value in $C\mathcal{W}_b^*(L_0; L_2; 2H)$. And the usual continuation map from $C\mathcal{W}_b^*(L_0; L_2; 2H)$ to $C\mathcal{W}_b^*(L_0; L_2; H)$ is not well defined. To solve this problem, we appeal to the rescaling trick in [3]. We briefly sketch the main idea here.

First, we define a map

$$\mu_2^{\psi^2} : C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(\psi^2(L_0), \psi^2(L_2)), \quad (2.94)$$

where ψ^2 is the time-log(2) Liouville flow. Secondly we compose with the inverse of equation (2.92) to get

$$\mu_2 : C\mathcal{W}_b^*(L_1, L_2; H) \otimes C\mathcal{W}_b^*(L_0, L_1; H) \rightarrow C\mathcal{W}_b^*(L_0, L_2; H). \quad (2.95)$$

The map (2.94) counts solutions to the following version of the Cauchy-Riemann equation:

$$(du - X_S \otimes \gamma_S)^{0,1} = 0, \quad (2.96)$$

where S is the surface obtained by removing 3 points ξ^0, ξ^1, ξ^2 from the boundary of \mathbb{D}^2 . In the above equation, γ_S is a closed 1-form on S while X_S is the Hamiltonian vector field of H_S on M , which depends on S . We require that the map

$$H_S : M \rightarrow \mathcal{H}(M) \quad (2.97)$$

restrict to H near ξ^1, ξ^2 and to $\frac{H}{4} \circ \psi^2$ near ξ^0 . Note that $\frac{H}{4} \circ \psi^2 \in \mathcal{H}(M)$ since $H(q, r) = r^2$ for r large enough in the cylindrical ends.

To specify the remaining data required for equation (2.96), we can pick strip like ends for S , namely Z_{\pm} for the positive (respectively, negative) half strips in Z and embeddings

$$\epsilon_0 : Z_- \rightarrow S, \quad (2.98)$$

$$\epsilon_1 : Z_+ \rightarrow S, \quad (2.99)$$

$$\epsilon_2 : Z_+ \rightarrow S, \quad (2.100)$$

which map ∂Z_{\pm} to ∂S and converge to the respective marked points ξ_k for $k = 0, 1, 2$.

For the closed 1-form γ_S , it should vanish on ∂S and satisfy

$$\epsilon_0^*(\gamma_S) = 2dt, \quad (2.101)$$

$$\epsilon_k^*(\gamma_S) = dt, k = 1, 2. \quad (2.102)$$

In addition, we choose a family of almost complex structures $J_S : S \rightarrow \mathcal{J}(M)$ parametrized by the Riemann surface S whose compositions with ϵ_k agrees with J_t if $k = 1, 2$ and $(\psi^2)^* J_t$ if $k = 0$.

The Lagrangian boundary conditions near ξ^1 should be given by (L_0, L_1) , similarly near ξ^2

given by (L_1, L_2) and near ξ^0 given by $(\psi(L_0), \psi(L_2))$. For the Lagrangians along the two segments of ∂S that converge to ξ^0 , we cannot have a constant Lagrangian condition, instead we need a moving boundary Lagrangian condition. In order to pick a simple moving Lagrangian condition we fix a map ρ_S from ∂D^2 to $[1, 2]$ such that

$$\rho_S(z) \equiv 1 \text{ if } z \text{ is near } \xi^1 \text{ or } \xi^2 \text{ and } \rho_S(z) \equiv 2 \text{ if } z \text{ is near } \xi^0. \quad (2.103)$$

Definition 2.0.39. *The moduli space $\mathcal{M}(x_0; x_1, x_2)$ is the space of solutions to equation (2.96) with boundary conditions*

$$\begin{cases} u(z) \in \psi^{\rho_S(z)}(L_1) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_2 \\ u(z) \in \psi^{\rho_S(z)}(L_2) & \text{if } z \in \partial S \text{ lies between } \xi_2 \text{ and } \xi_0 \\ u(z) \in \psi^{\rho_S(z)}(L_0) & \text{if } z \in \partial S \text{ lies between } \xi_1 \text{ and } \xi_0 \end{cases} \quad (2.104)$$

and such that the image of u converges to x_1 and x_2 at the corresponding incoming strip-like ends, and to $\psi^2(x_0)$ at the outgoing strip-like end.

From our construction, the pullback of equation (2.96) by the strip-like ends ξ^k is given by

$$\begin{aligned} (du - X \otimes dt)^{0,1} &= 0 \text{ with respect to } J_t \text{ if } k = 1, 2, \\ (du - 2X_{\frac{H}{4} \circ \psi^2} \otimes dt)^{0,1} &= 0 \text{ with respect to } (\psi^2)^* J_t \text{ if } k = 0, \end{aligned}$$

with Lagrangian boundary conditions (L_0, L_1) near ξ^1 , (L_1, L_2) near ξ^2 , and $(\psi^2(L_0), \psi^2(L_2))$ near ξ^0 . Near the incoming punctures ξ^1, ξ^2 , the inhomogeneous equation is the same as Equation (2.77), while near the outgoing puncture ξ^0 the Hamiltonian becomes $\frac{H}{2} \circ \psi^2$. We conclude that

the Gromov compactification of the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is obtained by adding the strata

$$\coprod_{y \in \mathcal{X}(L_0, L_1)} \mathcal{M}_2(x_0; y, x_2) \times \overline{\mathcal{M}}(y; x_1), \quad (2.105)$$

$$\coprod_{y \in \mathcal{X}(L_1, L_2)} \mathcal{M}_2(x_0; x_1, y) \times \overline{\mathcal{M}}(y; x_2), \quad (2.106)$$

$$\coprod_{y \in \mathcal{X}(L_0, L_2)} \overline{\mathcal{M}}(x_0; y) \times \mathcal{M}_2(y; x_1, x_2). \quad (2.107)$$

The above three cases correspond to the breaking of a holomorphic strip at each strip-like end of the the punctured disc. Note that since we are working with exact Lagrangians, there is no bubbling.

Lemma 2.0.40. *For any fixed pair (x_1, x_2) , the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is empty for all but finitely many x_0 , and for a generic family of almost complex structures J_S and Hamiltonians H_S , $\overline{\mathcal{M}}_2(x_0; x_1, x_2)$ is a compact manifold of dimension $\mu(x_0) - \mu(x_1) - \mu(x_2)$, the boundary of which is the union of the codimension 1 strata listed in equations (2.105)-(2.107).*

Transversality is given by standard Sard-Smale arguments going back to [11] in the case of Hamiltonian Floer cohomology; compactness of the moduli spaces is given by similar arguments of Lemma (2.0.36), and is explicitly explained in [3, Lemma 3.2]. Whenever the dimension of the moduli space $\mathcal{M}_2(x_0; x_1, x_2)$ is 0, it consists of finitely many elements, each of which defines an isomorphism

$$o_{x_1} \otimes o_{x_2} \rightarrow o_{\psi^2(x_0)}. \quad (2.108)$$

We also denote by μ_u this isomorphism induced on orientation lines; now we are set to define a product on the wrapped Floer complex as follows:

$$\begin{aligned} \mu_2^{\psi^2} : \mathcal{CW}_b^*(L_2, L_1) \otimes \mathcal{CW}_b^*(L_1, L_0) &\rightarrow \mathcal{CW}_b^*(\psi^2(L_2), \psi^2(L_0)) \\ \mu_2^{\psi^2}([x_1], [x_2]) &= \sum_{\substack{|x_0|=|x_1|+|x_2| \\ u \in \mathcal{M}_2(x_0; x_1, x_2)}} (-1)^{\mu(x_1)} \mu_u([x_2], [x_1]). \end{aligned} \quad (2.109)$$

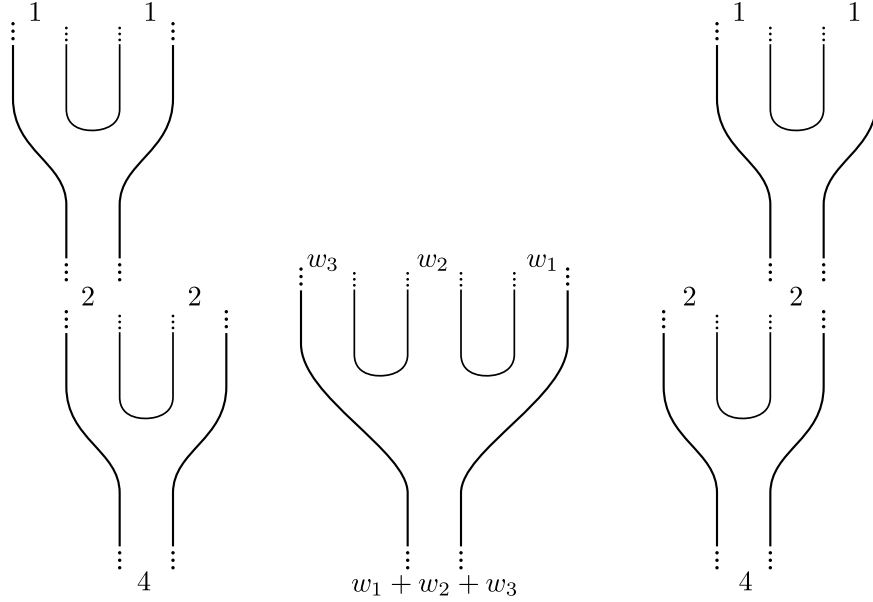


Figure 2.2:

2.0.12 A_∞ structure

We want to equip each strip-like end with Floer data that would facilitate the definition of an A_∞ structure on the wrapped Fukaya category. First, denote by \mathcal{R}_d the moduli space of disks that is equipped with d incoming boundary punctures $\{\xi^k\}_{k=1}^d$, ordered clockwise and a outgoing boundary puncture ξ^0 . Denote by $\overline{\mathcal{R}}_d$ the Deligne-Mumford compactification of this moduli space; as in [22, Section 9], we fix a *universal and consistent* choice of strip-like ends. In other words, for each surface S and each puncture, there exists a map

$$\epsilon^k : Z_\pm \rightarrow S,$$

whose source is Z_- if it is the outgoing puncture, and Z_+ otherwise, and such a choice varies smoothly in the interior of $\overline{\mathcal{R}}_d$. Each boundary stratum σ of $\overline{\mathcal{R}}_d$ has the following form:

$$\sigma = \mathcal{R}_{d_1} \times \mathcal{R}_{d_2} \times \cdots \times \mathcal{R}_{d_j}.$$

A neighborhood of σ looks like gluing of strip-like ends chosen on these lower dimensional moduli spaces:

$$\sigma \times [1, \infty)^{j-1} \rightarrow \mathcal{R}_d.$$

Let us assign to each end ξ^i of a punctured disc a weight w_i , which is a positive real number, with the convention that any $\bar{\partial}$ operator on such a disc must pull back under ϵ^i to the Cauchy Riemann equation (2.77) up to applying ψ^{w_i} . In the product case where there are two incoming ends and one outgoing end, the inputs have weights 1 while the output has weight 2.

If we have three inputs and one output, the moduli space of all such possible configurations $\bar{\mathcal{R}}_3$ is actually an interval whose two ends are represents by nodal discs obtained by gluing the two possible different ways as shown in Figure (2.0.12), two discs with two incoming ends of weight 1 and the other with weight 2. Recall from the previous section (2.0.11) where we defined the product (2.95) to be the composition of (2.94) with rescaling. Instead of rescaling first and then take composition, one pulls back by ψ^2 of all the data used to define $\mathcal{M}(x_0; x_1, x_2)$, and thus obtain a new $\bar{\partial}$ operator where the weights are now 2 at the incoming ends and 4 at the outgoing end. Then one composes $C\mathcal{W}_b^*(\psi^2(L_3), \psi^2(L_2))$ with $C\mathcal{W}_b^*(\psi^2(L_2), \psi^2(L_0))$ after applying (2.94) with L_0, L_1, L_2 . In this way we define the product with three inputs directly:

$$\mu_3 : C\mathcal{W}_b^*(L_3, L_2) \otimes C\mathcal{W}_b^*(L_2, L_1) \otimes C\mathcal{W}_b^*(L_1, L_0) \rightarrow C\mathcal{W}_b^*(L_3, L_0). \quad (2.110)$$

However, we immediately encounter the problem of ordering, namely when we multiply the first two terms then with the third, the result is different from that of multiplying the last two terms first on the chain level on the right hand side. In other words, μ_2 is not associative at the chain level.

Yet these two give the same element on the cohomology level by looking at the following

diagram as shown in [5].

$$\begin{array}{ccc}
CW_b^*(L_2, L_3) \otimes CW_b^*(L_1, L_2) \otimes CW_b^*(L_0, L_1) & \xrightarrow{\mu_2^{\psi^2} \otimes CW(\psi^2)} & CW_b^*(\psi^2(L_1), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_1)) \\
\downarrow CW(\psi^2) \otimes \mu_2^{\psi^2} & & \downarrow \mu_2^{\psi^4} \\
CW_b^*(\psi^2(L_2), \psi^2(L_3)) \otimes CW_b^*(\psi^2(L_0), \psi^2(L_2)) & \xrightarrow{\mu_2^{\psi^4}} & CW_b^*(\psi^4(L_0), \psi^4(L_3)) \\
& & \downarrow \cong \\
& & CW_b^*(L_0, L_3)
\end{array} \tag{2.111}$$

In order to show the existence of a homotopy of maps that links the former two orders of taking product, we need a moduli space of maps whose sources are disks with 4 punctures on its boundary and arbitrary conformal structure. We take its compactification $\overline{\mathcal{R}}_3$ as in Figure (2.0.12). Whatever equation we define on $\overline{\mathcal{R}}_3$ has to restrict to equation (2.96) on each component of the boundary. In particular, the Cauchy-Riemann equation imposed on a disc whose conformal equivalence class is close to the boundary of $\overline{\mathcal{R}}_3$ is obtained by gluing the $\bar{\partial}$ operators associated to equation (2.96) at the node. This only works if the restriction of these $\bar{\partial}$ operators restrict to the two strip-like ends at the node agree, which is not true yet, since even the Lagrangian boundary conditions do not yet agree. However, after applying the time log-2 Liouville flow ψ^2 , they agree. Thus we can glue two solutions to equation (2.96) after applying ψ^2 to one of them.

We also notice that the weights on the inputs of the three punctured disc have to be allowed to change with its modulus, as they are given by (1, 1, 2) at one of the endpoints of $\overline{\mathcal{R}}_3$ and (2, 1, 1) at the other endpoint. We now define auxiliary data of families of Hamiltonians and almost complex structures on M , and of 1-forms on elements of $\overline{\mathcal{R}}_3$ which would then help define an operation

$$C\mathcal{W}^*(L_3, L_2) \otimes C\mathcal{W}^*(L_2, L_1) \otimes C\mathcal{W}^*(L_1, L_0) \rightarrow C\mathcal{W}^*(L_3, L_0), \tag{2.112}$$

based on the homotopy in diagram (2.111). We note that this would only work with product with three inputs; actually we need arbitrarily many inputs, thus we proceed to the following general construction originally from [3].

Definition 2.0.41. A Floer datum D_S on a stable disc $S \in \overline{\mathcal{R}}_d$ consists of the following choices:

1. *Weights:* A positive integer $w_{k,S}$ assigned to the k th end such that

$$w_{0,S} = \sum_{1 \leq k \leq d} w_{k,S}$$

2. *Moving conditions:* A map $\rho_S : \partial \overline{S} \rightarrow [1, +\infty)$ which agrees with w_k near the k th end.

3. *Basic 1-form:* A closed 1-form γ_S whose restriction to the boundary vanishes and whose pullback under ϵ^k agrees with $w_{k,S} dt$.

4. *Hamiltonian perturbations:* A map $H_S : S \rightarrow \mathcal{H}(M)$ which agrees with $\frac{H \circ \psi^{w_{k,S}}}{w_{k,S}^2}$ near the k th end.

5. *Almost complex structure:* A map $J_S : S \rightarrow \mathcal{J}(M)$ whose pullback under ϵ^k agrees with $(\psi^{w_{k,S}})^* J_t$.

If we write X_S for the Hamiltonian flow of H_S , then these data allow us to write down a Cauchy-Riemann equation

$$(du - X_S \otimes \gamma_S)^{0,1} = 0, \tag{2.113}$$

where the $(0, 1)$ part is taken with respect to J_S .

The long list of definition (2.0.41) guarantees the following:

Lemma 2.0.42. *The pullback of equation (2.113) under ϵ^k is*

$$(du \circ \epsilon^k - X_{\frac{H \circ \psi^{w_{k,S}}}{w_{k,S}^2}} \otimes dt)^{0,1} = 0. \tag{2.114}$$

It agrees with equation (2.77) up to conformal rescaling with $\psi^{w_{k,S}}$. □

At the boundary of the moduli space \mathcal{R}_d , we require that Floer data be given by the choices performed on smaller dimensional moduli spaces. When $d = 3$, we already observed that the choice of Floer data at the boundary can't be given precisely by the Floer data for \mathcal{R}_2 on each component,

since these Floer data can't be glued. We shall consider the following notion of equivalence among Floer data which is weaker than equality:

Definition 2.0.43. *We say that a pair $(\rho_S^1, \gamma_S^1, H_S^1, J_S^1)$ and $(\rho_S^2, \gamma_S^2, H_S^2, J_S^2)$ of Floer data on a surface S are conformally equivalent if there exists a constant C such that*

$$\begin{aligned}\rho_S^2 &= C\rho_S^1, \\ \gamma_S^2 &= C\gamma_S^1, \\ J_S^2 &= \psi^{C*}J_S^1, \\ H_S^2 &= \frac{H_S^1 \circ \psi^C}{C^2}.\end{aligned}$$

The idea that rescaling by ψ^2 in Equation (2.91) gives an identification of moduli spaces can be generalized as follows:

Lemma 2.0.44. *Composition with ψ^C defines a bijective correspondence between solutions to equation (2.96) for conformally equivalent Floer data. \square*

We may now state the desired compatibility between Floer data at the boundary of the moduli spaces $\overline{\mathcal{R}}_d$:

Definition 2.0.45. *A universal and conformally consistent choice of Floer data \mathbf{D}_μ for the A_∞ structure, is a choice of Floer data for every element of $\overline{\mathcal{R}}_d$ and every integer $d \geq 2$, which varies smoothly over the interior of the moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from lower dimensional moduli spaces, and which near such a boundary stratum agrees to infinite order, in the coordinates (2.79), with the Floer data obtained by gluing.*

Note that conformal consistency fixes, up to a constant depending on the modulus, the choice of Floer data on $\partial\overline{\mathcal{R}}_d$ once Floer data on each irreducible component has been chosen. For example, the choice of Floer data for $d = 2$ above determines the Floer data on the boundary of $\overline{\mathcal{R}}_3$ as

used in the previous section, where we chose to use the original Floer data on the disc that does not contain the original end, and rescale it by ψ^2 on the other disc (see Figure 2.0.12). After introducing a perturbations of this Floer data which vanishes to infinite order at the boundary, we extend it from a neighbourhood of the boundary to the remainder of \mathcal{R}_3 . One may proceed inductively to construct a universal datum \mathbf{D}_μ using the fact that the boundary of $\overline{\mathcal{R}}_d$ is covered by the images of codimension 1 inclusions

$$\overline{\mathcal{R}}_{d_1} \times \overline{\mathcal{R}}_{d-d_1+1} \rightarrow \partial\overline{\mathcal{R}}_d \quad (2.115)$$

and the contractibility of the space of Floer data on a given surface. The contractibility of this space also implies that we may extend any Floer data chosen on a given surface to universal data as in [3, Lemma 3.8]:

Lemma 2.0.46. *The restriction map from the space of universal and conformally consistent Floer data to the space of Floer data for a fixed connected stable surface S is surjective.*

Now we construct the A_∞ structure on the wrapped Fukaya category which is given by higher products

$$C\mathcal{W}^*(L_d, L_{d-1}) \otimes C\mathcal{W}^*(L_{d-1}, L_{d-2}) \otimes \cdots \otimes C\mathcal{W}^*(L_1, L_0) \rightarrow C\mathcal{W}^*(L_d, L_0), \quad (2.116)$$

which comes from the count of solutions of Equation (2.96).

Given a sequence of chords $\vec{x} = x_k \in \mathcal{X}(L_{k-1}, L_k)$ for $1 \leq k \leq d$ and $x_0 \in \mathcal{X}(L_0, L_d)$, we define $\mathcal{M}_d(x_0; \vec{x})$ to be the space of solutions to Equation (2.96) whose source is an arbitrary element $S \in \mathcal{R}_d$ with marked points (ξ^0, \dots, ξ^d) such that

$$\lim_{s \rightarrow \pm\infty} u \circ \epsilon^k(s, \cdot) = \psi^{W_{k,S}} x_k, \quad (2.117)$$

and with boundary conditions

$$u(z) \in \psi^{\rho_S(z)}(L_k) \text{ if } z \in \partial S \text{ lies between } \xi^k \text{ and } \xi^{k+1}. \quad (2.118)$$

Note that Lemma (2.0.42) shows that Equation (2.96) restricts to Equation (2.77) up to applying $\psi^{w_{k,S}}$.

If we don't take into account the strip-breaking phenomenon at the ends, the virtual codimension 1 strata of the Gromov compactification $\overline{\mathcal{M}}_d(x_0; \vec{x})$ lie over the codimension 1 strata of $\overline{\mathcal{R}}_d$. The consistency condition on \mathbf{D}_μ implies that whenever a disc breaks, each component is a solution to equation (2.96) for the Floer data \mathbf{D}_μ up to applying ψ^C for some constant C that depends on the modulus in $\overline{\mathcal{R}}_d$. Such rescaling identifies the solutions of rescaled equation with moduli space of the solutions to the original equation, so we conclude that for each integer k between 0 and $d - d_2$ and chord $y \in \mathcal{X}(L_{k+1}, L_k)$, we get a natural inclusion:

$$\overline{\mathcal{M}}_{d_1}(x_0; \vec{x}^1) \times \overline{\mathcal{M}}_{d_2}(y; \vec{x}^2) \rightarrow \overline{\mathcal{M}}_d(x_0; \vec{x}) \quad (2.119)$$

where the sequences of inputs in the respective factors are given by $\vec{x}^2 = (x_{k+1}, \dots, x_{k+d_2})$ and $\vec{x}^1 = (x_1, \dots, x_k, y, x_{k+d_2+1}, \dots, x_d)$.

Thus we apply the same arguments for Lemma (2.0.40) to get the following lemma.

Lemma 2.0.47. *The moduli space $\overline{\mathcal{M}}_d(x_0; \vec{x})$ is compact and is empty for all but finitely many x_0 once the inputs \vec{x} are fixed. For a generic choice of data \mathbf{D}_μ , they form manifolds of dimension*

$$|x_0| + d - 2 - \sum_{1 \leq k \leq d} |x_k|,$$

whose boundary is covered by the images of inclusions (2.119). □

Whenever $|x_0| = x - d + \sum_{1 \leq k \leq d} |x^k|$, there are therefore only finitely many elements of $\overline{\mathcal{M}}_d(x_0; \vec{x})$. Via the procedure described in Appendix (A.0.1.) every such element $u \in \overline{\mathcal{M}}_d(x_0; \vec{x})$:

$S \rightarrow M$ induces an isomorphism:

$$\mathcal{O}_{\psi^{w_d, S, x_d}} \otimes \cdots \otimes \mathcal{O}_{\psi^{w_1, S, x_1}} \rightarrow \mathcal{O}_{\psi^{w_0, S, x_0}}. \quad (2.120)$$

Let us write μ_u for the induced map on orientation lines, omitting composition with $C\mathcal{W}^*(\psi^{w_k, S})$ or its inverse from the notation. We define the d th higher product:

$$\mu_d : CW_b^*(L_{d-1}, L_d) \otimes \cdots \otimes CW_b^*(L_1, L_2) \otimes CW_b^*(L_0, L_1) \rightarrow CW_b^*(L_0, L_d) \quad (2.121)$$

Explicitly,

$$\mu_d([x_d], \dots, [x_1]) = \sum_{\substack{|x_0|=2-d+\sum_{1 \leq k \leq d} |x_k| \\ u \in \mathcal{M}_d(x_0; \vec{x})}} (-1)^\dagger \mu_u([x_d], \dots, [x_1]), \quad (2.122)$$

where the sign is given by

$$\dagger = \sum_{k=1}^d k|x_k|. \quad (2.123)$$

This is just saying we count the elements of $\overline{\mathcal{M}}_d(x_0; \vec{x})$ with the right signs.

If we now consider the dimension 1 moduli spaces, Lemma (2.0.47) claims that their boundaries are the strata in equation (2.119) which are rigid and thus correspond to the composition of operations μ_d . Taking into account of signs, we get the following proposition:

Proposition 2.0.48. *The operations μ_d define an A_∞ structure on the category $C\mathcal{W}^*(M)$. Explicitly, we have*

$$\sum_{\substack{d_1+d_2=d+1 \\ 0 \leq k < d_1}} (-1)^{\mathfrak{X}_1^k} \mu_{d_1}(x_d, \dots, x_{k+d_2+1}, \mu_{d_2}(x_{k+d_2}, \dots, x_{k+1}), x_k, \dots, x_1) = 0, \quad (2.124)$$

where the sign is given by

$$\mathfrak{X}_1^k = k + \sum_{1 \leq j \leq k} |x_j|.$$

□

Chapter 3: The Pontryagin Category

Let \mathcal{L} be any path connected topological space. Consider the topological category with objects the points of \mathcal{L} , and morphisms from L^0 to L^1 is given by the Moore path space

$$\Omega(L^0, L^1) \equiv \{\gamma : [0, R] \rightarrow \mathcal{L} \mid \gamma(0) = L^0, \gamma(1) = L^1\} \quad (3.1)$$

Here R is allowed to vary between 0 and infinity including infinity. The composition is given by concatenation:

$$\begin{aligned} \Omega(L^0, L') \times \Omega(L', L^1) &\rightarrow \Omega(L^0, L^1), \\ (\gamma_1, \gamma_2) &\rightarrow \gamma_2 \cdot \gamma_1(l) \equiv \begin{cases} \gamma_1(l) & \text{if } 0 \leq l \leq R_1 \\ \gamma_2(l - R_1) & \text{if } R_1 \leq l \leq R_1 + R_2, \end{cases} \end{aligned} \quad (3.2)$$

where γ_i has domain $[0, R_i]$.

This formula defines an associative composition of paths. In order for this operation to induce the structure of a DGA on chains, we use cubical chains as in [5] instead of singular chains. As in [18] we define a map from a cube to a topological space to be *degenerate* if it factors through the projection to a face. The underlying vector space for the cubical chain complex is:

$$C_i(X) = \frac{\mathbb{Z}[\text{Map}([0, 1]^i, X)]}{\mathbb{Z}[\text{degenerate maps}]} \quad (3.3)$$

We write $\delta_{k,\epsilon}$ to denote the inclusion of the codimension -1 face where the k th coordinate is equal

to $\epsilon \in \{0, 1\}$. The differential of $C_*(X)$ is defined as

$$\partial\sigma = \sum_{k=1}^i \sum_{\epsilon=0,1} \partial_{k,\epsilon}\sigma = \sum_{k=1}^i \sum_{\epsilon=0,1} (-1)^{k+\epsilon} \sigma \circ \partial_{k,\epsilon}. \quad (3.4)$$

Since the product of cubes is still a cube, it is easy to define the following map

$$C_*(X) \times C_*(Y) \rightarrow C_*(X \times Y), \quad (3.5)$$

which may easily be checked to be associative. Now we apply this to the path space $\Omega_{\mathcal{L}}$, and obtain a DG category. Let

$$\mathcal{P}(\mathcal{L}) \quad (3.6)$$

denote the differential graded category whose objects are points of \mathcal{L} and morphism spaces

$$\text{Hom}_*(L^0, L^1) = C_*(\Omega(L^0, L^1)). \quad (3.7)$$

The differentials and the products are given by

$$\mu_1^P \sigma = \partial\sigma, \quad (3.8)$$

$$\mu_2^P(\sigma_2, \sigma_1) = (-1)^{|\sigma_1|} \sigma_2 \cdot \sigma_1. \quad (3.9)$$

We may consider this as an A_∞ category by defining $\mu_d^P = 0$ for $d > 2$. Notice morphisms in this category satisfy A_∞ relations:

$$\sum_{m,n} (-1)^{\mathfrak{X}_n} \mu_{d-m+1}^P(\sigma_d, \dots, \sigma_{n+m+1}, \mu_m^P(\sigma_{n+m}, \dots, \sigma_{n+1}), \sigma_n, \dots, \sigma_1) = 0, \quad (3.10)$$

namely

$$\mu_1(\mu_2(\sigma_2, \sigma_1)) + \mu_2(\sigma_2, \mu_1(\sigma_1)) + (-1)^{|\sigma_1|+1} \mu_2(\mu_1(\sigma_2), \sigma_1) = 0$$

in this situation.

Remark 3.0.1. Since we assumed that the space \mathcal{L} is path connected, every object is quasi-isomorphic to any other object. In particular, if $L \in \mathcal{L}$ we may fully faithfully embed the A_∞ category with the single object L and morphism $C_*(\Omega(L, L))$ to the Pontryagin category:

$$(L, C_*(\Omega(L, L))) \rightarrow \mathcal{P}(\mathcal{L}). \quad (3.11)$$

Remark 3.0.2. In the next section \mathcal{L} is the space $\widetilde{\mathcal{L}ag}_b$ of exact Lagrangians with Pin structure relative to b as in Equation (2.73); we obtain the differential graded category $\mathcal{P}(\widetilde{\mathcal{L}ag})$ from the above constructions, from which we would like to construct a functor to the wrapped Fukaya category of the ambient symplectic manifold in the following sections.

To understand the composition in the commutative diagram in the case of a cotangent bundle in Theorem 1.0.3, we need to understand the functor constructed in [5]. The category $\mathcal{P}(\mathcal{L})$ is not necessarily triangulated and it was enlarged to its triangulated envelope (see section (3j) of [22]), which is the triangulated closure of the image of $\mathcal{P}(\mathcal{L})$ in its category of modules under the Yoneda embedding. We would like to study a practical model of twisted complexes [22, Section 3]. Here we follow the model of [5] or [22, Remark 3.26]. First enlarge $\mathcal{P}(\mathcal{L})$ by allowing shifts of all objects by any integer and define

$$\mathrm{Hom}_*(L^0[m_0], L^1(m_1)) \equiv \mathrm{Hom}_*(L^0, L^1)[m_1 - m_0]. \quad (3.12)$$

Differentials are given by the usual μ_1^P as defined above. For μ_2^P , given a triple

$$(L^0[m_0], L^1[m_1], L^2[m_2]),$$

and morphisms $\sigma_i \in \mathrm{Hom}_*(L^{i-1}, L^i)$, we define

$$\mu_2^P(\sigma_2[m_2 - m_1], \sigma_1[m_1 - m_0]) = (-1)^{(\deg(\sigma_2)+1)(m_1-m_0)} \mu_2^P(\sigma_2, \sigma_1)[m_2 - m_0]. \quad (3.13)$$

From the equations above a twisted complex is defined:

Definition 3.0.3. *A twisted complex consists of the following data:*

A finite collection of objects $\{L^i\}_{i=1}^r$ and integers m_i , together with a collection of morphisms $\{\delta_{i,j}\}_{i < j}$ of degree 1 in $\text{Hom}_(L^i[m_i], L^j[m_j])$, such that*

$$\mu_1^P \delta_{i,j} + \sum_k \mu_2^P(\delta_{k,j}, \delta_{i,k}) = 0. \quad (3.14)$$

Writing D for the matrix $(\delta_{i,j})$ and $T = (\oplus L^i[m_i], D)$ for such a twisted complex, and note that the above equation can be rewritten as

$$\mu_1^P(D) + \mu_2^P(D, D) = 0. \quad (3.15)$$

Given two such complexes $T^1 = (\oplus L_1^i[m_i^1], D^1)$ and $T^2 = (\oplus L_2^i[m_i^2], D^2)$, we define the space of morphisms between them as

$$\text{Hom}_*(T^1, T^2) = \bigoplus_{i_1, i_2} C_*(\Omega_{L_1^{i_1}, L_2^{i_2}})[m_{i_1}^1 - m_{i_2}^2]. \quad (3.16)$$

The differential of an element $S = \{\sigma_{i_1, i_2}\}$ in this space is given by an elementary matrix:

$$\sigma_{i_1, i_2} \mapsto \mu_1^P(\sigma_{i_1, i_2}) + \sum_{k < i_1} \mu_2^P(\sigma_{i_1, i_2}, \delta_{k, i_1}^1) + \sum_{k > i_2} \mu_2^P(\delta_{i_2, k}^2, \sigma_{i_1, i_2}), \quad (3.17)$$

or in terms of matrix multiplication as

$$\mu_1^{T^w(\mathcal{P})} S = \mu_1^P(S) + \mu_2^P(S, D^1) + \mu_2^P(D^2, S). \quad (3.18)$$

Composition is also defined in terms of matrix multiplication

$$\mu_2^{T^w(\mathcal{P})}(S_2, S_1) = \mu_2^P(S_2, S_1). \quad (3.19)$$

Remark 3.0.4. $\mathcal{P}(\mathcal{L})$ embeds fully faithfully into $\text{Tw}(\mathcal{P}(\mathcal{L}))$ by viewing each object in $\mathcal{P}(\mathcal{L})$ as its own twisted complex in grading 0.

Bondal and Kapranov in [9] essentially proved the following result:

Lemma 3.0.5. *Twisted complexes form a triangulated A_∞ category.*

□

Chapter 4: Construction of the functor

Suppose we are given a open Liouville manifold (M, ω) that comes from the completion of a Liouville domain M^{in} , with cylindrical end $\partial M^{in} \times (0, \infty)$. From the two previous sections, we obtain its wrapped Fukaya category $C\mathcal{W}^*(M)$, with operations μ_d satisfying the A_∞ relations (2.0.48), and the differential graded category $\mathcal{P}(\widetilde{\mathcal{L}ag})$ of exact Lagrangians with Legendrian boundaries with morphisms $\mu_1^{\mathcal{P}}, \mu_2^{\mathcal{P}}$. Our goal in this section is to construct a functor from $\mathcal{P}(\widetilde{\mathcal{L}ag})$ to $C\mathcal{W}^*(M)$ that also preserves the A_∞ structures. This is just Theorem (1.0.1) in the introduction. We write it here for the reader's convenience.

Theorem 4.0.1. *For any open Liouville manifold (M, ω) with cylindrical ends, there exists an A_∞ functor \mathcal{F} from the triangulated closure of the Pontryagin category $\mathcal{P}(\widetilde{\mathcal{L}ag})$ of exact Lagrangians with Legendrian boundaries decorated with a Pin structure to the wrapped Fukaya category $C\mathcal{W}^*(M)$ of the Liouville manifold:*

$$\mathcal{P}(\widetilde{\mathcal{L}ag}) \xrightarrow{\mathcal{F}} C\mathcal{W}^*(M). \quad (4.1)$$

On the level of objects, the functor sends every graded Lagrangian with Pin structure $\mathcal{L} = (L, P^\#)$ to itself; on the level of homomorphisms, we have:

$$\begin{aligned} \text{Mor}(\mathcal{P}(\widetilde{\mathcal{L}ag}_b)) &\xrightarrow{\mathcal{F}} C\mathcal{W}(M), \\ C_*(\Omega_{\mathcal{L}_1, \mathcal{L}_0} \mathcal{L}ag_b) &\mapsto C\mathcal{W}^{-*}(\mathcal{L}_1, \mathcal{L}_0; H, J) \end{aligned}$$

where the pair (H, J) is a generic choice of Floer data that makes the moduli space of solutions to Floer equations regular.

In the above theorem we have suppressed the notation of $(L, P^\#)$ of Pin structures in the nota-

tions above when it is clear.

We shall give a proof of this theorem in the following sections.

Remark 4.0.2. The above theorem generalizes the functors in [1] and [5] to the based loop space of Lagrangian embeddings with Legendrian boundary.

4.0.1 Chain map

Before proving Theorem (4.0.1), let us consider the geometric set up of the domain. We work over the upper half plane \mathbb{H} with 2 marked points z_1, z_2 on the boundary, which is bi-holomorphic to $U = \mathbb{D}^2 \setminus \{1 \text{ point on } \partial D^2\}$, let ξ denote the punctured point on the boundary of U . The group of automorphisms of \mathbb{H} that preserve the point at infinity is the intersection of $PSL(2; \mathbb{R})$ and the affine transformations $\text{Aff}(\mathbb{R}^2)$, namely $z \mapsto az + b$ where $a \in \mathbb{R}_{>0}, b \in \mathbb{R}$. Thus after an automorphism, we may assume $z_1 = (0, 0), z_2 = (1, 0)$.

To have a proper definition of a moduli space of pseudo-holomorphic curves, we work over the space $\mathcal{J}(M)$ of almost complex structures of contact type along cylindrical ends of M defined in Section (2.0.8). Furthermore, we allow more freedom by taking a k -dimensional family of almost complex structures so we consider the set $\mathcal{J}_{U,k}(M)$, which denotes the space of families of smooth almost complex structures of contact type on M parametrized by $U \times I^k$ and such that for any $\tau \in I^k$,

$$J_{\tau,z} \in \mathcal{J}(M)$$

only depends on t when $s \rightarrow \infty$ at the strip-like end $\epsilon^0 : Z_- = (-\infty, 0] \times [0, 1] \rightarrow U$ near the puncture ξ parametrized by the s, t coordinates and that these almost complex structures near the strip-like end ϵ^0 is independent of τ ; in other words, there is a 1-parameter family of almost complex structure J_t parametrized by $t \in [0, 1]$ such that for every τ ,

$$J_{\tau, \epsilon^0(s,t)} = J_t \text{ when } s \ll 0. \tag{4.2}$$

We also require that this k -family of almost complex structures satisfies the following: for any

family $\{u_\tau\} : U \rightarrow M$ that are solutions of the Cauchy-Riemann Equation (2.77) such that u_τ sends the boundary of U between z_1 and z_2 to Lagrangians in the path given by $\sigma(\tau)$ (the exact condition is (4.4) below), the corresponding linearized $\bar{\partial}$ operator for $J_{\tau,z}$ is surjective. This requirement can be achieved inductively on k (the dimension of the cube): for a single point we just pick a generic J_z by standard arguments of transversality as in [19, chapter 3]; for higher k suppose we have picked $J_{\tau,z}$ for one $k - 1$ chain in the boundary strata of the family parametrized by I^k , we may simply extend it smoothly to other $k - 1$ chains of $\partial\sigma$, extend to a neighborhood of $\partial\sigma \simeq \partial I^k$ and then extend to the whole k -chain by contractibility of $\mathcal{J}(M)$. Note that during the extensions we can modify the almost complex structures $J_{\tau,z}$ via a family of infinitesimal deformations $\{K_\tau\}$ where $K_\tau \in \text{End}(TM, J_\tau, \omega)$ by taking almost complex structures of the form $J_\tau \exp(-J_\tau K_\tau)$ as in Equation (2.76) to make the linearized operator surjective.

The above space of almost complex structures has a subspace that is a Banach manifold as we can give a local neighborhood of every element $J \in \mathcal{J}_{U,k}$ by taking the exponential map of a small infinitesimal deformation in a suitable Banach space: $K \in C_{\epsilon,M}^\infty(I^k, \text{End}(TM, J, \omega))$ of all sections such that the norm $\|K_\tau\|_{\bar{\epsilon}}$ is finite; here the above norm $\|\cdot\|_{\bar{\epsilon}}$ is defined in [10] as

$$\|K_\tau\|_{\bar{\epsilon}} := \sum_{l=1}^{\infty} \epsilon_l \|K_\tau\|_{C^l}. \quad (4.3)$$

When the sequence ϵ_l converges to zero sufficiently rapidly, there exists sections $K \in C_{\epsilon,M}^\infty(I^k, \text{End}(TM, J, \omega))$ which are supported in arbitrarily small neighborhood of any point $x \in M$ and $\tau \in I^k$ and take arbitrary small values.

We denote a k family of U -parametrized almost complex structures that satisfies the above conditions by $J_{\tau,z}$ or J_τ if there is no confusion.

We would like to consider a map:

$$H : U \rightarrow \mathcal{H}(M),$$

satisfying quadratic growth at infinity as in (B.15). Let X_U denote the Hamiltonian flow of H , and

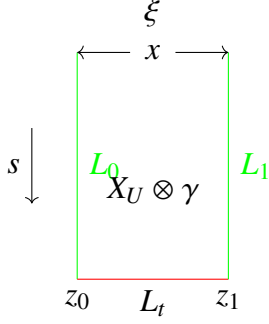


Figure 4.1: Upper half plane

consider the following Cauchy-Riemann equation:

$$J_\tau(du - X_U \otimes \gamma) - (du - X_U \otimes \gamma) \circ j_U = 0 \quad (4.4)$$

for maps $u : U \rightarrow M$.

Given σ representing an element in $C_k(\Omega_{L_0, L_1} \widetilde{\mathcal{L}ag})$, namely $\sigma : [0, 1]^k \rightarrow \Omega_{L_0, L_1} \widetilde{\mathcal{L}ag}$, for any $\tau \in I^k$, where $\sigma(\tau)$ is a path of Lagrangians $[0, R] \rightarrow \mathcal{P}_{L_0, L_1} \widetilde{\mathcal{L}ag}$ between L_0, L_1 for every $\tau \in I^k$, let $|\sigma(\tau)|$ denote the length of this path; we also fix $x \in \mathcal{X}(\mathcal{L}_0, \mathcal{L}_1)$.

Definition 4.0.3 (Moduli space of upper half plane). *We define the moduli space of pseudo-holomorphic maps with moving Lagrangian boundary $\mathcal{M}(x, \sigma, H, J_\tau)$ to be pairs (u, τ) where $\tau \in I^k$, and $u : U \rightarrow M$ satisfying the following condition:*

$$\left. \begin{array}{l} (du - X_U \otimes \gamma) \circ j_U - J_{\tau, z}(du - X_U \otimes \gamma) = 0, \\ u(z) \in L_0 \text{ if } z \in \partial U \text{ lies on the segment between } \xi \text{ and } z_1, \\ u(z) \in L_1 \text{ if } z \in \partial U \text{ lies on the segment between } \xi \text{ and } z_2, \\ u(z) \in L_{\sigma(\tau)(\frac{t}{|\sigma(\tau)|})} \text{ if } z = (t, 0) \in \partial U \text{ lies between } z_1 \text{ and } z_2, \\ \lim_{s \rightarrow +\infty} u(\epsilon^0(s, t)) = x(t), \text{ (asymptotic condition)} \end{array} \right\} \quad (4.5)$$

Standard arguments involving Sard-Smale and transversality results in Floer theory give:

Lemma 4.0.4. *For generic choices of Hamiltonians satisfying Equation (2.66) and a generic family*

of almost complex structures $J_\tau \in \mathcal{J}(M)$, $\mathcal{M}(x, \sigma, H, J_\tau)$ is a smooth manifold of dimension $\mu(x) + |\sigma|$, where $|\sigma| = k$ denote the dimension of the cubical chain.

From the standard transversality results in Floer theory we realize that the dimension of this moduli space is independent of H, J_τ when it is regular, thus we can omit it and write $\mathcal{M}(x, \sigma)$ when there is no confusion.

Fixing a choice of H, J_τ , we would like to study the boundary of the Gromov compactification of $\mathcal{M}(x, \sigma, H, J_\tau)$. Since there are three special points on the boundary of the domain (two markings and one puncture with a strip-like end), there is no modulus. If τ is fixed, the only strata that we need to add to the Gromov-compactification for the moduli space with boundary condition given by $\sigma(\tau)$ are obtained by considering breakings of strips at the ends. This gives a map:

$$\mathcal{M}(\sigma; y) \times \mathcal{M}(y; x) \rightarrow \overline{\mathcal{M}}(\sigma; x), \quad (4.6)$$

for all possible $y \in \mathcal{X}(\mathcal{L}_0, \mathcal{L}_1)$.

On the other hand, we also need to take into consideration of the boundary strata $\sigma' \subseteq \partial\sigma$ of the k -dimensional cubical chain σ , which gives:

$$\mathcal{M}(\sigma'; x) \rightarrow \overline{\mathcal{M}}(\sigma; x). \quad (4.7)$$

The standard compactness result gives the components of the boundary strata of the compactification $\overline{\mathcal{M}}(\sigma; x)$ in the following proposition.

Proposition 4.0.5. *The Gromov compactification $\overline{\mathcal{M}}(x; \sigma)$ is a compact manifold whose boundary is covered by the images of the following strata.*

$$\coprod_{y \in \mathcal{X}(L_0, L_1)} \mathcal{M}(y; \sigma) \times \mathcal{M}(x; y), \quad (4.8)$$

$$\coprod_{\sigma' \subseteq \partial \sigma} \mathcal{M}(x; \sigma'). \quad (4.9)$$

□

In addition, we shall need to compare the product orientations with the natural orientation of the strata $\partial \overline{\mathcal{M}}_3(\sigma; x)$. The proof is delayed until Appendix (A.0.1).

Lemma 4.0.6. *The difference between the orientation induced as a boundary of $\overline{\mathcal{M}}(\sigma; x)$ and the product orientation is given by signs whose parity is*

$\mu(x) + \mu(\sigma)$ for the strata (4.8) if $\mathcal{M}(x, y)$ is rigid,

0 for the strata (4.9)

4.0.2 The linear term.

Consider the following evaluation map:

$$ev : \overline{\mathcal{M}}(x; \sigma) \rightarrow \mathcal{X}(L_0, L_1), \quad (4.10)$$

which takes every punctured disc u to its infinite end x , a Hamiltonian chord between L_0 and L_1 .

If we consider the boundary strata (4.8), we obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(y; \sigma) \times \mathcal{M}(x; y) & \longrightarrow & \overline{\mathcal{M}}(x; \sigma) \\ \downarrow & & \downarrow_{ev} \\ \mathcal{M}(x; y) & \longrightarrow & \mathcal{X}(L_0, L_1). \end{array} \quad (4.11)$$

The first left vertical arrow is projection to the second factor and the top map is the inclusion of a boundary stratum.

Meanwhile, if we consider the boundary strata (4.9), we get the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{M}(x; \sigma) & \longrightarrow & \overline{\mathcal{M}}(x; \sigma) \\
 & \searrow & \downarrow \\
 & & \mathcal{X}(L_0, L_1).
 \end{array} \tag{4.12}$$

Lemma 4.0.7. *There exist fundamental cycles $[\overline{\mathcal{M}}(x; \sigma)] \in C_*(\overline{\mathcal{M}}(x; \sigma))$ such that the assignment*

$$\mathcal{F}^1(\sigma) = \bigoplus_x (-1)^\dagger ev_*([\overline{\mathcal{M}}(x; \sigma)]) \tag{4.13}$$

defines a chain map in (4.0.1).

Consider moduli spaces $\overline{\mathcal{M}}(x; \sigma)$ whose boundary only has codimension 1 strata, which must be products of closed manifolds. By taking the product of the fundamental chains of the factors in each boundary stratum, we obtain a chain in $C_*(\overline{\mathcal{M}}(x; \sigma))$:

$$(-1)^{\mu(x)+\mu(\sigma)} \sum_{x' \in \mathcal{X}(L_0, L_1)} [\mathcal{M}(x'; \sigma)] \quad \times \quad [\mathcal{M}(x, x')] \quad + \quad \sum_{\sigma' \subseteq \partial\sigma} [\mathcal{M}(x; \sigma')]. \tag{4.14}$$

We now define the fundamental chain

$$[\overline{\mathcal{M}}(x; \sigma)]$$

to be any chain whose boundary is the sum of chains represented by equation (4.14).

Proof of Lemma 4.0.7. Consider $\sigma : I^k \rightarrow \Omega_{\mathcal{L}_0, \mathcal{L}_1} \widetilde{\mathcal{L}ag}$, fix $x \in C\mathcal{W}^{-k}(\mathcal{L}_1, \mathcal{L}_0; H, J)$ for some Hamiltonian H that is quadratic with respect to the radial coordinate $|p|$ at cylindrical ends of M outside a compact set. We know that, up to signs which we shall ignore, proving that \mathcal{F}^1 is a chain map is equivalent to proving that

$$\mu_{C\mathcal{W}}^1 \circ \mathcal{F}^1(\sigma) = \mathcal{F}^1 \circ \mu_1^{Tw}(\sigma). \tag{4.15}$$

From the two previous sections we know $\mu_1^{Tw}(\sigma)$ is just $\bigcup_{\sigma' \subseteq \partial\sigma} \sigma'$ and $\mu_{C\mathcal{W}}^1(x)$ is a sum of Hamil-

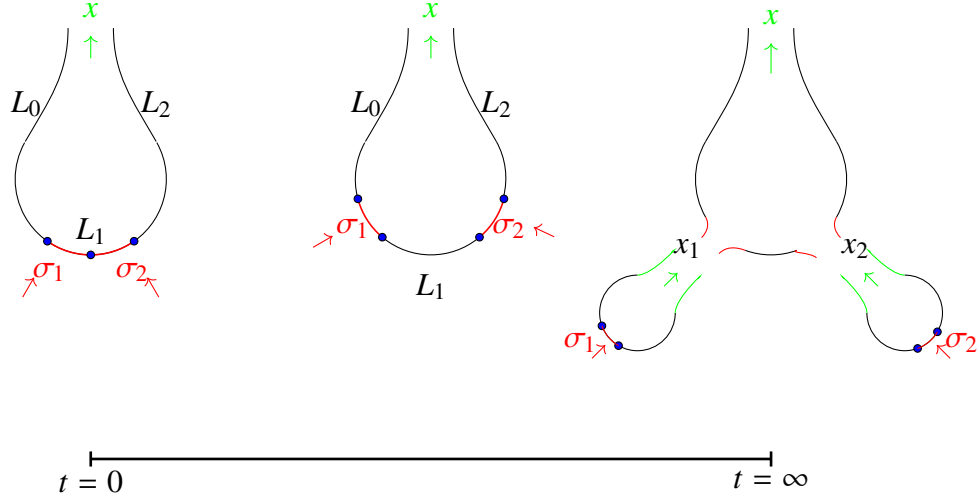


Figure 4.2: boundary strata of one-punctured discs with 4 boundary marked points

tonian chords of Maslov index 1 higher than that of x as shown in Equation (2.89). Going through the stratification of $\partial \overline{\mathcal{M}}(x; \sigma)$, we find that the strata (4.8) for rigid moduli spaces $\mathcal{M}(x; y)$ correspond to $\mu_{CW}^1 \circ \mathcal{F}^1(\sigma)$ and that the strata (4.9) correspond to $\mathcal{F}^1 \circ \mu_1^P(\sigma)$. Thus \mathcal{F}^1 satisfies the first A_∞ relationship, namely it is a chain map.

□

4.0.3 Homotopy between compositions

Having constructed the chain map \mathcal{F}^1 , we would like to check it preserves product structures on the two chains, on the source it is given by the concatenation on the based loop space, namely the concatenation of paths; on the target it is the pair of pants product. As in the failure of pair of pants product to be associative, the preservation of product structure induced by \mathcal{F}^1 is only observed at the level of homology. At the level of chains, there is a homotopy between the $\mathcal{F}^1(\mu_2^P(\sigma_1, \sigma_2))$ and $\mu_{CW}^2(\mathcal{F}^1(\sigma_1), \mathcal{F}^1(\sigma_2))$; these two compositions are represented by the two outermost diagrams in Figure (4.0.2). We shall therefore need to introduce a one-dimensional moduli space $\mathcal{Z}_4^{l_1, l_2}$ for each pair of positive l_1, l_2 representing the lengths of the path of moving Lagrangians in M , namely given $\sigma_1 \in C_{k_1}(\Omega_{\mathcal{L}_0, \mathcal{L}_1} \text{Lag}), \tau_i \in I^{k_1}$, then $l_1 = |\sigma_1(\tau_1)|$ and similarly for l_2 . The boundary of this moduli space is represented by the broken curve and the collapsing of two adjacent marked points as

shown in Figure (4.0.2). As in the proof of the homotopy associativity of the product in the Fukaya category, we shall define a family of Cauchy-Riemann equations on this abstract moduli space, interpolating between the equations on the two boundary figures. The family of moduli spaces of solutions to this family of equations, with their corresponding boundary conditions, shall define the desired homotopy. For higher homotopies that is incorporated in A_∞ functor, the constructions are similar and we describe them in one piece through Section (4.0.4) and Section (4.0.8).

4.0.4 Abstract moduli spaces of upper half planes

Given a tuple of positive real numbers l_1, \dots, l_d write $\mathcal{Z}_{2d}^{l_1, l_2, \dots, l_d}$ for the moduli space of upper half planes with $2d$ consecutive boundary marked points z_1, \dots, z_{2d} , and that the distance $\tilde{l}_i = |z_{2i+1} - z_{2i}|$ satisfy $\frac{\tilde{l}_i}{l_j} = \frac{l_i}{l_j}$. This is bi-holomorphic to a disc with a puncture and $2d$ marked points on the boundary with fixed ratios. In addition, we fix an orientation on the moduli space $\mathcal{Z}_{2d}^{l_1, l_2, \dots, l_d}$ using the conventions for Stasheff polyhedra [22]. In fact, we have the following lemma:

Lemma 4.0.8. *The space $\mathcal{Z}_{2d}^{l_1, l_2, \dots, l_d}$ is isomorphic to the moduli space \mathcal{R}_{d+1} of discs with $d + 2$ boundary marked points.*

$$\mathcal{Z}_{2d}^{l_1, l_2, \dots, l_d} \cong \mathcal{R}_{d+1} \subseteq \mathcal{R}_{2d}. \quad (4.16)$$

Proof. We take the boundary points on the source indexed by odd numbers $(z_1, z_3, \dots, z_{2d-1})$ to the first d incoming marked points on \mathcal{R}_{d+1} and z_2 to the $d + 1$ -th incoming marked point (actually we may take any $z_{2k} \forall k \in \{1, \dots, d\}$ instead of z_2 as the $(d + 1)$ th incoming marked point); the outgoing marked point for \mathcal{R}_{d+1} is mapped from the puncture at $\infty \in \bar{\mathbb{H}}$. Note that the other even numbered marked points on the boundary of the upper half plane is fixed as long as we pick z_2 because the ratios between different $|z_{2k} - z_{2k-1}|$ are fixed. \square

Given the space of all configurations $\{(z_1, \dots, z_{2d})\}$ where $\{z_i\}$ lie on the boundary of \mathbb{H} , to achieve stability, we need to take the quotient of the action of the automorphism of \mathbb{H} on the configurations $\{z_1, \dots, z_{2d}\}$ which fixes the ratio of different lengths $\frac{l_i}{l_j}$. This automorphism group $Aut_{2d}^{\vec{l}}$ turns out to be $PSL(2; \mathbb{R}) \cap \text{Aff}(2; \mathbb{R})$, thus we have the following equivalence relation on

the space of configurations $\{(z_1, \dots, z_{2d})\}$ on $\partial\mathbb{H}$:

$$(z_1, \dots, z_{2d}) \simeq (z'_1, \dots, z'_{2d}) \iff z'_i = az_i + b \text{ for some } a \in \mathbb{R}_+, b \in \mathbb{R} \text{ and all } i, \quad (4.17)$$

Therefore the dimension of the moduli space $\mathcal{Z}_{2d}^{l_1, \dots, l_d}$ is:

$$2d - (d - 1) - 2 = d - 1. \quad (4.18)$$

The first term $2d$ is the sum of $2d$ marked points on $\partial\mathbb{H}$; the term $(d - 1)$ is due to the fact that as soon as we fix z_2 , then the ratio $l_1 : \dots : l_d$ would fix the remaining $d - 1$ marked points z_4, \dots, z_{2d} ; the last term 2 is to mod out the automorphism group $\text{Aut}_{2d}^{\vec{l}}$.

Yet the compactification $\overline{\mathcal{Z}}_{2d}^{l_1, \dots, l_d}$ of $\mathcal{Z}_{2d}^{l_1, \dots, l_d}$ inside $\overline{\mathcal{R}}_{2d}$ is not quite the same as that of $\overline{\mathcal{R}}_{d+1}$ as we shall see below.

From now on, if there is no confusion, we shall use \mathcal{Z}_{2d} or $\mathcal{Z}_{2d}^{\vec{l}}$ to abbreviate for $\mathcal{Z}_{2d}^{l_1, \dots, l_d}$.

There are two scenarios to be taken into account:

1. If two consecutive marked points z_{2k}, z_{2k+1} come together, as in the picture on the left hand side of figure (4.0.7), the topological type is determined by sequences $z_1, \dots, z_{2k}, z_{2k+1}, \dots, z_d$, thus we obtained a map:

$$\mathcal{Z}_{2d-1}^{l_1, \dots, l_k+l_{k+1}, \dots, l_d} \rightarrow \overline{\mathcal{Z}}_{2d}^{l_1, \dots, l_d}, \quad (4.19)$$

where we regard the original two markings z_{2k}, z_{2k+1} as a single marked point z'_{2k} .

2. If two consecutive marked points indexed by z_{2k-1}, z_{2k} come close together, then by the property of *fixed ratio* of $z_{2i} - z_{2i-1}$ and $z_{2k} - z_{2k-1} \forall i, k$, all two consecutive markings with such configurations have to come close; thus there are multiple bubbles breaking off with corresponding gluing parameters such that the ratio is still fixed. This phenomenon corresponds to the right hand side of Figure (4.0.7). Note the topological type is determined by sequences $\{1, \dots, d_1\}, \dots, \{d_1 + \dots + d_{r-1} + 1, \dots, d\}$, a partition of d into r sets of

consecutive integers. We get a map:

$$\mathcal{Z}_{2d_1} \times \cdots \times \mathcal{Z}_{2d_r} \times \mathcal{R}_r \rightarrow \overline{\mathcal{Z}}_{2d}, \quad (4.20)$$

Remark 4.0.9. The dimension of the left hand side of 4.20 is

$$\sum_{i=1}^r (d_i - 1) + (r + 1 - 3) = \left(\sum d_i \right) - 2 = d - 2,$$

which is 1 less than that of $\mathcal{Z}_{2d}^{\vec{1}}$. Here the second term $(r + 1 - 3)$ is the dimension of the moduli space of abstract discs \mathcal{R}_r with r incoming marked points and a single outgoing marked point on the boundary.

We shall denote the unique disc representing \mathcal{R}_r by level 1 disc, and call the other discs representing \mathcal{Z}_{2d_i} level 2 discs.

4.0.5 Floer data for half planes

The following definition is a somewhat indirect extension of Definition (2.0.41) to the moduli space of upper half planes. Here we have the outgoing end at the puncture ξ_0 as before.

Definition 4.0.10. *A Floer datum D_U on a stable upper half plane $U \in \mathcal{Z}_{2d}^{\vec{1}}$ consists of the following choices on each component:*

1. *Weights: a positive real number $w_{k,U}$, where $k \in \{0, \dots, r\}$ associated to each end for the 2nd type of boundary strata (4.20) where the principal holomorphic disc is broken into a level 1 disc with r incoming ends and a outgoing end and r level 2 discs each equipped with a single outgoing ends, namely $\mathcal{Z}_{2s_1} \times \cdots \times \mathcal{Z}_{2s_r} \times \mathcal{R}_r \subseteq \overline{\mathcal{Z}}_{2d+1}$ w_i satisfy $1 = w_0 \geq \sum_1^r w_{k,U}$. For U inside the open strata or first type of boundary strata where there is no breaking of holomorphic discs (4.19), the weight is just 1.*
2. *time shifting maps: for the curve breaking boundary (4.20), map $\rho_U : \partial\bar{U} \rightarrow [1, +\infty)$ which*

agrees with $w_{k,U}$ near ξ^k where ξ^k denote the infinite ends and equals 1 for open strata and first type of boundary strata (4.19).

3. *Hamiltonian perturbation:* a map $H_U : U \rightarrow \mathcal{H}(M)$ on each surface such that the restriction of H_U to a neighborhood of each segment between z_{2i+1} and z_{2i+2} for $1 \leq i \leq d$ takes value in $\mathcal{H}(M)$, satisfying Condition (2.66), and whose value near ξ^k for $0 \leq k \leq r$ is :

$$\frac{H \circ \psi^{w_{k,U}}}{w_{k,U}^2} \quad (4.21)$$

for the boundary strata represented by (4.20) and equals H near ξ^0 for open strata and first type boundary strata (4.19).

4. *Basic one-form:* a sub-closed one-form γ_U whose restriction to the complement of a neighborhood of the intervals between z_{2i-1} and z_{2i} ($1 \leq i \leq d$) vanishes; and whose pullback under ϵ^k for $0 \leq k \leq r$ agrees with $w_{k,U} dt$.
5. *Almost complex structure:* a family $J_{\tau,z} \in \mathcal{J}(M)$ of almost complex structures parametrized by U and $\tau \in I^k$ for any positive integer k , whose pullback under ϵ_k agrees with $\psi^{w_{k,U}} d\tau$.

As before we write X_U for the Hamiltonian flow of H_U and consider the differential equation (4.4):

$$(du - X_U \otimes \gamma_U)^{0,1} = 0$$

with respect to $J_{\tau,z}$. Note that the pullback of equation (4.4) under the ends ϵ^k agrees with

$$(du \circ \xi^k - X_{\frac{H \circ \psi^{w_{k,U}}}{w_{k,U}}} \otimes dt)^{0,1} = 0. \quad (4.22)$$

In order for the count of solutions to equation (4.4) to define the desired homotopies, we must choose its restriction to the boundary strata of the moduli spaces in a compatible way. In the case where $d = 2$, the moduli space is an interval, whose two endpoints may be identified with the product $\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{R}_2$ and \mathcal{Z}_3 (see figure 4.0.2).

Our discussion in sections (2) and (4.0.1) fixed the Floer data, on the unique elements of \mathcal{R}_2 and \mathcal{Z}_2 . In the case of $\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{R}_2$ the equations on the out-coming ends of the level 2 discs and the incoming ends of the level 1 discs agree up to some rescaling. Thus we obtain Floer data in a neighborhood of $\partial\overline{\mathcal{Z}}_4$, which may be extended to the interior of the moduli space. Having fixed this choice, we proceed inductively for the rest of the moduli spaces:

Definition 4.0.11. *A universal and conformally consistent choice of Floer data for the homomorphism \mathcal{F} , is a choice $\mathbf{D}_{\mathcal{F}}$ of such Floer data for every integer $d \geq 1$ and every element of \mathcal{Z}_{2d} which varies smoothly over this compactified moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from either \mathbf{D}_{μ} or lower dimensional moduli spaces $\overline{\mathcal{Z}}_{d_i}$ and which near such a boundary agrees to infinite order with the Floer data obtained by gluing.*

The consistency condition implies that each irreducible component of a curve representing a point in the stratum (4.20) carries the restriction of the data \mathbf{D}_{μ} if it comes from the factor $\overline{\mathcal{R}}_r$, and the restriction of the datum $\mathbf{D}_{\mathcal{F}}$ if it comes from \mathcal{Z}_{d_i} , up to conformal equivalence.

4.0.6 Moduli space of upper half planes

In order to extend \mathcal{F}^1 to an A_{∞} functor $\{\mathcal{F}^d\}$, we need to construct corresponding moduli spaces. First of all, let us briefly review the moduli space constructed for \mathcal{F}^1 : Consider the upper half plane \mathbb{H} with 2 marked points z_1, z_2 on the boundary \mathbb{R} , with length $l = z_2 - z_1$.

Remark 4.0.12. The domain is bi-holomorphic to a unit disc with a puncture at $(0, i) \in \mathbb{C}$ and 2 marked points on the boundary where $\infty \in \overline{\mathbb{H}}$ is identified with the puncture $(0, i)$.

For $\sigma \in C_k(\Omega_{\mathcal{L}_0, \mathcal{L}_1}(\mathcal{L}ag))$ and $x \in \mathcal{X}(\mathcal{L}_0, \mathcal{L}_1)$, we associated in Definition (4.0.3) a moduli space $\mathcal{M}_3(\sigma; x)$ of upper half planes with boundary marked points with moving Lagrangian boundary conditions given by σ . It can be viewed as a family of spaces over I^k such that $\forall \tau \in I^k$, the member of the family over τ is $\mathcal{M}_3(\sigma(\tau); x)$, the space of holomorphic maps from the upper half space with two boundary markings $z_1 = (0, 0), z_2 = (1, 0)$ such that the path of Lagrangian submanifolds between them is dictated by $\sigma(\tau)$.

Note in the definition of $\mathcal{M}(\sigma, x)$ and $\mathcal{M}(\sigma(\tau), x)$, we already fixed the action of automorphism of \mathbb{H} with two boundary markings. We can study the pair (u, τ) satisfying Condition (4.5), with the small modification that instead of the strict requirement that $z_1 = (0, 0), z_2 = (1, 0)$ and the equation denoting the moving boundary to be

$$u(z) \in L_{\sigma(\tau)\left(\frac{t}{|\sigma(\tau)|}\right)} \text{ if } z = (t, 0) \in \partial U \text{ lies between } z_1 \text{ and } z_2, \quad (4.23)$$

we just let z_1, z_2 be two boundary markings and the component of the boundary of \mathbb{H} satisfies

$$u(z) \in L_{\sigma(\tau)(t)} \text{ if } z = (t, 0) \in \partial U \text{ lies between } z_1 \text{ and } z_2 \quad (4.24)$$

This gives us a new space which we denote $\tilde{\mathcal{M}}_3(\sigma, x)$, whose quotient by the automorphism group $\mathrm{PSL}(2; \mathbb{R}) \cap \mathrm{Aff}(2, \mathbb{R})$ on the domain is $\mathcal{M}_3(\sigma, x)$. Similarly we have a space $\tilde{\mathcal{M}}_3(\sigma(\tau), x)$ for any τ .

The above construction gives rise to following projection:

$$\begin{array}{ccc} \tilde{\mathcal{M}}_3(\sigma(\tau); x) & \longrightarrow & \tilde{\mathcal{M}}_3(\sigma; x) \\ \downarrow & & \downarrow \pi \\ \{\tau\} & \xrightarrow{\epsilon} & I^k. \end{array} \quad (4.25)$$

The automorphism group acts on $\tilde{\mathcal{M}}_3(\sigma(\tau); x)$ and $\tilde{\mathcal{M}}_3(\sigma; x)$ simultaneously and gives the following projection:

$$\begin{array}{ccc} \mathcal{M}_3(\sigma(\tau); x) & \longrightarrow & \mathcal{M}_3(\sigma; x) \\ \downarrow & & \downarrow \pi \\ \{\tau\} & \xrightarrow{\epsilon} & I^k. \end{array} \quad (4.26)$$

We now set out to generalize the constructions to multiple markings in order to give the higher operations \mathcal{F}^d of the A_∞ functor. For any integer $d \geq 1$, we consider the upper half plane \mathbb{H} with $2d$ boundary markings, which is bi-holomorphic to the unit disc with 1 puncture on the boundary ξ^0 and $2d$ markings, such that lengths $l_i = z_{2i} - z_{2i-1}$ denote the length of the moving Lagrangian

boundary condition. Let

$$t_i = z_{2i+1} - z_{2i}$$

represent the lengths of the boundary component of \mathbb{H} that lies in L_i , namely constant boundary condition. As in section (4.0.1), we work with a Hamiltonian H_U that is quadratic with respect to the radial coordinate near cylindrical ends as in condition (4.0.10); also similar to the discussion of families of almost complex structures, we consider the space $\mathcal{J}_{U,\vec{k}}$ of almost complex structures parametrized by the product of I^{k_i} with domain U such that for any $\vec{\tau} \in (I^{k_1}, \dots, I^{k_d})$, $J_{\vec{\tau},z} \in \mathcal{J}(M)$ is independent of $\vec{\tau}$ when we go far enough in the strip-like ends near the puncture ξ as in Equation (4.2). Denote such a family of almost complex structure by $J_{\vec{\tau}}$ if there is no confusion.

Given $\sigma_i \in C_{k_i}(\Omega_{L_i, L_{i+1}} \mathcal{L}ag)$ and $\tau_i \in I^{k_i}$, let $|\sigma_i(\tau_i)|$ denote the length of this path; fixing $x \in \mathcal{X}(\mathcal{L}_i, \mathcal{L}_{i+1})$, we define the space of pseudo-holomorphic maps with multiple moving Lagrangian boundary condition as follows:

Definition 4.0.13. *The space of perturbed pseudo-holomorphic maps with multiple moving Lagrangian boundary $\tilde{\mathcal{M}}(\sigma_1, \dots, \sigma_d, x, H, J_{\vec{\tau}})$ is the space of pairs $(u, \vec{\tau})$ consisting of a map $u : U \rightarrow M$ and $\tau_i \in I^{k_i}$ satisfying*

$$\left. \begin{array}{l} (du - X_{H_U} \otimes dt) \circ j_U - J_{\vec{\tau},z}(du - X_{H_U} \otimes dt) = 0, \\ u(z) \in \psi^{\rho(z)}(L_0) \text{ if } z \in \partial U \text{ lies on the segment between } \xi^0 \text{ and } z_1, \\ u(z) \in \psi^{\rho(z)}(L_d) \text{ if } z \in \partial U \text{ lies on the segment between } \xi^0 \text{ and } z_{2d}, \\ u(z) \in L_{\sigma_i(\tau_i)(t)} \text{ if } z = (t, 0) \in \partial U \text{ lies between } z_{2i-1} \text{ and } z_{2i}, \\ \lim_{s \rightarrow +\infty} u(\epsilon^0(s, t)) = x(t) \text{ (asymptotic condition).} \end{array} \right\} \quad (4.27)$$

We shall write $\tilde{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)$ to abbreviate it if there is no confusion.

Fixing τ_1, \dots, τ_d , we define $\tilde{\mathcal{M}}_{2d+1}(\sigma_1(\tau_1), \dots, \sigma_d(\tau_d); x)$ to be the space of solutions to Equa-

tion (4.27) then $\forall \tau_i \in I^{k_i}$ we get the following projection:

$$\begin{array}{ccc} \tilde{\mathcal{M}}_{2d+1}(\sigma_1(\tau_1), \dots, \sigma_d(\tau_d); x) & \longrightarrow & \tilde{\mathcal{M}}_{2d+1}(\sigma_1, \dots, \sigma_d; x) \\ \downarrow & & \downarrow \pi \\ \{\tau_1, \dots, \tau_d\} & \longrightarrow & I^{k_1 + \dots + k_d}. \end{array}$$

Note that the domain of the maps in any given fiber $\tilde{\mathcal{M}}_{2d+1}(\sigma_1(\tau_1), \dots, \sigma_d(\tau_d); x)$ is \mathbb{H} with $2d$ markings z_1, \dots, z_{2d} such that the lengths l_i between z_{2i-1} and z_{2i} are equal to $|\sigma(\tau_i)|$. Let $\vec{\sigma}(\tau)$ denote $\sigma_1(\tau_1), \dots, \sigma_d(\tau_d)$. We mod out the automorphism group $Aut_{2d}^{\vec{l}}$ on the space of maps $\tilde{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)$ with moving Lagrangian boundary conditions $\sigma_i(\tau_i)$ between z_{2i-1} and z_{2i} to achieve stability of the domain. Thus we get the moduli space of pseudo-holomorphic curves with moving Lagrangian conditions given by multiple paths $\sigma_i(\tau_i) \in \Omega_{\mathcal{L}_{i-1}, \mathcal{L}_i} \mathcal{Lag}$ on an element $\mathcal{Z}_{2d+1}^{\vec{l}}$.

Similarly, the space $\tilde{\mathcal{M}}_{2d+1}(\sigma_1, \dots, \sigma_d; x)$ also allows the action of $Aut_{2d}^{\vec{l}}$ and the *moduli space of pseudo-holomorphic curves with moving Lagrangian boundary conditions* $\mathcal{M}_{2d+1}(\sigma_1, \dots, \sigma_d; x)$ is defined as the quotient of $Aut_{2d}^{\vec{l}}$ action. As in the discussion after Definition (4.0.3), $\mathcal{M}_{2d+1}(\vec{\sigma}; x)$ is a smooth manifold of dimension $\sum |\sigma| + \mu(x) + d - 1$ for generic choice of Floer data and its Gromov compactification is a stratified topological manifold with corners.

There is the diagram similar to Equation (4.26):

$$\begin{array}{ccc} \mathcal{M}_{2d+1}(\vec{\sigma}(\tau); x) & \longrightarrow & \mathcal{M}_{2d+1}(\vec{\sigma}; x) \\ \downarrow & & \downarrow \\ \vec{\tau} & \longrightarrow & I^{\sum k_i}. \end{array}$$

In order to show that $\{\mathcal{F}^d\}_{d=1}^{\infty}$ is a map of A_{∞} categories, we need to study the relevant strata of the Gromov compactification of $\mathcal{M}_{2d+1}(\vec{\sigma}; x)$.

First of all, we study the boundary strata of the compactification of the fibers $\mathcal{M}_{2d+1}(\vec{\sigma}(\tau); x)$ of the fibration (4.0.6). From the discussion in Section (4.0.4) and the standard gluing arguments similar to that for Proposition (4.0.5) we have the following lemma:

Lemma 4.0.14. *The boundary strata of $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)$ consist of strata from the following sce-*

narios:

1. (The strata corresponding to Case (1) of the boundary strata of $\overline{\mathcal{Z}}_{2d+1}$.) Let $\sigma_i(\tau_{i,j+1})$ denote the concatenation of two adjacent paths $\sigma_i(\tau_i)$ and $\sigma_{i+1}(\tau_{i+1})$ in $\mathcal{L}ag$; in other words, z_{2i} and z_{2i+1} come together. This gives a moduli space

$$\overline{\mathcal{M}}_{2d}(\sigma_1(\tau_1), \dots, \sigma_i(\tau_{i,j+1}), \dots, \sigma_d(\tau_d); x) \quad (4.28)$$

of pseudo holomorphic maps from upper half plane with $2d - 1$ boundary markings.

2. (The strata corresponding to Case (2) of $\partial\overline{\mathcal{Z}}_{2d+1}$.) When z_{2i-1} and z_{2i} come close together, the upper half plane admits simultaneous bubble breaking with configuration given by $\{1, \dots, s_1\}, \dots, \{s_1 + \dots + s_{r-1} + 1, \dots, d\}$, a partition of d into r sets of consecutive integers. This gives rise to a boundary strata represented by

$$\mathcal{M}_{d_1, \dots, d_r}(\vec{\sigma}(\tau); x_1, \dots, x_r, x) := \mathcal{M}_{2d_1+1}(\sigma_1(\tau_1), \dots, \sigma_{d_1}(\tau_{d_1}); x_1) \times \dots \times \quad (4.29)$$

$$\mathcal{M}_{2d_r+1}(\sigma_{\sum_{i=1}^{r-1} d_i+1}(\tau_{\sum_{i=1}^{r-1} d_i+1}), \dots, \sigma_d(\tau_d); x_r) \times \mathcal{M}(x_1, \dots, x_r, x). \quad (4.30)$$

□

Standard arguments from gluing similar to the argument for Proposition (4.0.5) give the strata of the Gromov compactification $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)$ of the total space.

Lemma 4.0.15. *The Gromov compactification $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)$ is a stratified topological manifold with corners that consists of the following pieces:*

1. The interior $\mathcal{M}_{2d+1}(\vec{\sigma}; x)$.
2. The union of the closures of strata representing maps of upper half plane with $2d - 1$ boundary marked points corresponding to Case (1). For each member $\{\tau_1, \dots, \tau_d\}$ of the family parametrized by $I^{\Sigma d_i}$, the intersections of such strata with $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)$ are the unions

of boundary strata (4.28) where i ranges from 1 to d . We write $\overline{\mathcal{M}}_{2d}(\vec{\sigma}_i; x)$ to denote such a stratum whose intersections with $\{\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)\}_{\vec{\tau} \in I^{\Sigma k_i}}$ is the stratum represented by Equation (4.28).

3. The strata

$$\mathcal{M}_{d_1, \dots, d_r}(\vec{\sigma}; x_1, \dots, x_r, x) := \mathcal{M}_{2d_1+1}(\sigma_1, \dots, \sigma_{d_1}; x_1) \times \dots \times \quad (4.31)$$

$$\mathcal{M}_{2d_r+1}(\sigma_{\sum_{i=1}^{r-1} d_i+1}, \dots, \sigma_d; x_r) \times \mathcal{M}(x; x_1, \dots, x_r). \quad (4.32)$$

whose intersection with $\{\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}(\tau); x)\}_{\vec{\tau} \in I^{\Sigma k_i}}$ are the boundary strata (4.29). This corresponds to the multiple bubbling in Equation (4.20) and left side of Figure (4.0.7).

4. $\overline{\mathcal{M}}_{2d+1}(\sigma_1, \dots, \sigma', \dots, \sigma_d; x)$ where $\sigma' \subset \partial\sigma_i$, this stratum is from the boundary of i -th cube in the cubical chain.

□

4.0.7 Compatible choices of fundamental chains

In order to define operations using the moduli space $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)$ we must compare the orientation of the boundary of this moduli space with that of the interior. As with other sign related issues, the proof of this result is delayed to Appendix (A.0.1).

Lemma 4.0.16. *The product orientation of the strata (4.31) differs from the boundary orientation by the following equation:*

$$\sum_{k=1}^r d_k \cdot \left[\sum_{i=1}^{d_{k-1}} \mu(\sigma_i) + \sum_{j=1}^{k-1} \mu(x_j) \right] + \sum_{k=1}^{r-1} \mu(x_k) \cdot [\Sigma_k] + \sum_{k=1}^r \left(\sum_1^k d_i \right) \left(\sum_1^{k-1} d_i \right), \quad (4.33)$$

where Σ_k denote the sum of all $\mu(\sigma_k)$ for k in the last k tuples of the partition $\{1, \dots, d_1\}, \dots, \{d_1 + \dots, d_{r-1} + 1, \dots, d\}$ for d .

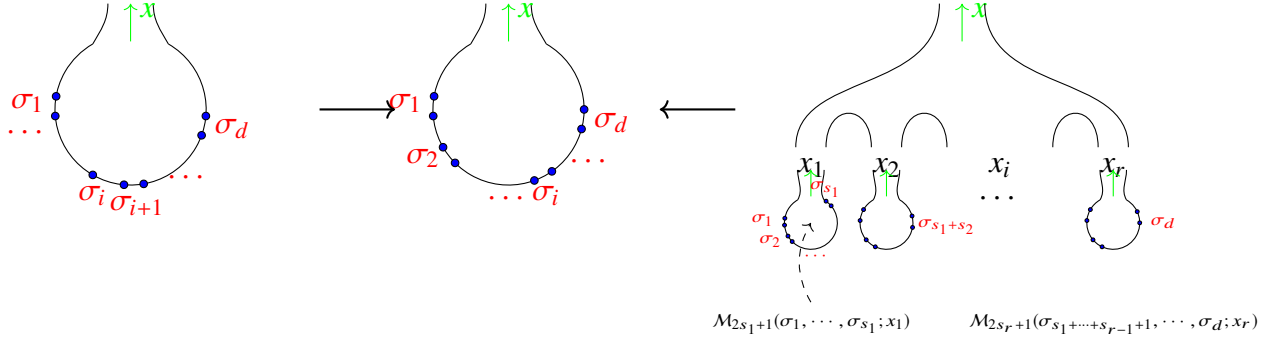


Figure 4.3: boundary strata of one-punctured discs with $2d$ boundary marked points

The difference between the orientation on the stratum $\overline{\mathcal{M}}_{2d}$ corresponding to (4.28) and the boundary orientation is

$$2d + 1. \quad (4.34)$$

4.0.8 Proof of main theorem (4.0.1)

We now extend \mathcal{F}^1 to a sequence of multi-linear maps of all orders $d \geq 1$:

$$\mathcal{F}^d : C_*(\Omega_{\mathcal{L}_d, \mathcal{L}_{d-1}} \widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_*(\Omega_{\mathcal{L}_1, \mathcal{L}_0} \widetilde{\mathcal{L}ag}) \rightarrow C\mathcal{W}_b^*(\mathcal{L}_d, \mathcal{L}_0)[1-d]$$

To be more precise,

$$\mathcal{F}^d : C_{k_d}(\Omega_{\mathcal{L}_d, \mathcal{L}_{d-1}} \widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_{k_1}(\Omega_{\mathcal{L}_1, \mathcal{L}_0} \widetilde{\mathcal{L}ag}) \rightarrow C\mathcal{W}_b^{-|\vec{k}|-d+1}(\mathcal{L}_d, \mathcal{L}_0), \quad (4.35)$$

where $|\vec{k}| = \sum_{i=1}^d k_i$. The collection $\{\mathcal{F}^d\}$ should satisfy the following equations:

$$\begin{aligned} \sum_r \sum_{s_1, \dots, s_r} \mu_{C\mathcal{W}}^r(\mathcal{F}^{s_r}(\sigma_d, \dots, \sigma_{d-s_r+1}), \dots, \mathcal{F}^{s_1}(\sigma_{s_1}, \dots, \sigma_1)) \\ = \sum_{m,n} (-1)^{\mathfrak{X}_n} \mathcal{F}^{d-m+1}(\sigma_d, \dots, \sigma_{n+m+1}, \mu_m^P(\sigma_{n+m}, \sigma_n, \dots, \sigma_{n+1}), \dots, \sigma_1), \end{aligned} \quad (4.36)$$

where the left hand side is the sum over all $r \geq 1$ and partitions $s_1 + \dots + s_r = d$, the right hand sum is over all possible $1 \leq m \leq d, 0 \leq n \leq d - m$, and \mathfrak{X}_n is defined in the same way as in (2.0.48). Note that in our definition of the Pontryagin category $\mathcal{P}(\widehat{\mathcal{L}ag})$, the only nontrivial A_∞ operations are μ_1^P, μ_2^P . Thus the above equation (4.36) can be rewritten as

$$\sum_r \sum_{\vec{s}} \mu_{CW}^r(\mathcal{F}^{s_r}(\dots), \dots, \mathcal{F}^{s_1}(\dots)) \quad (4.37)$$

$$= (-1)^{\mathfrak{X}_n} \mathcal{F}^d(\sigma_d, \dots, \sigma_{n+2}, \mu_1^P(\sigma_{n+1}), \sigma_n, \dots, \sigma_1) \quad (4.38)$$

$$+ (-1)^{\mathfrak{X}_n} \mathcal{F}^{d-1}(\sigma_d, \dots, \sigma_{n+3}, \mu_2^P(\sigma_{n+2}, \sigma_{n+1}), \sigma_n, \dots, \sigma_1). \quad (4.39)$$

Following the same strategy as the proof of Lemma (4.0.7), we may equip these moduli spaces with appropriate cubical fundamental chains:

Lemma 4.0.17. *There exists a family of fundamental chains:*

$$[\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)] \in C_*(\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)), \quad (4.40)$$

such that

$$\begin{aligned} \partial[\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)] &= \sum (-1)^{2d+1} [\overline{\mathcal{M}}_{2d}(\sigma_d \dots \sigma_{i \cup (i+1)}, \dots, \sigma_1; x)] \\ &+ \sum (-1)^\dagger [\overline{\mathcal{M}}_{2d_r+1}(\sigma_d, \dots, \sigma_{\sum_{i=1}^{r-1} d_i+1}; x_r)] \times \dots \times [\overline{\mathcal{M}}_{2d_1+1}(\sigma_{d_1}, \dots, \sigma_1; x_1)] \times [\overline{\mathcal{M}}_{r+1}(x_r, \dots, x_1; x)] \\ &\quad + \sum_{i=1}^d \sum_{\sigma' \subset \partial \sigma_i} [\overline{\mathcal{M}}_{2d+1}(\sigma_d, \dots, \sigma', \dots, \sigma_1; x)], \end{aligned} \quad (4.41)$$

where \dagger is an expression such that Equation (4.33) holds.

Similar to the definition of evaluation map of $\overline{\mathcal{M}}(\sigma; x)$ in (4.0.1), we also define the evaluation on $\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)$ as follows:

$$ev : \overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x) \rightarrow \mathcal{X}(L_0, L_1), \quad (4.42)$$

which takes the punctured disc to its infinite end x . We obtain commutative diagrams similar to (4.11) and (4.12).

With the evaluation map, we are set to define the higher operations \mathcal{F}^d :

Definition 4.0.18. *The operations \mathcal{F}^d for higher homotopies are given by the following equation:*

$$\mathcal{F}^d : C_{k_d}(\Omega_{\mathcal{L}_d, \mathcal{L}_{d-1}} \widetilde{\mathcal{L}ag}) \otimes \cdots \otimes C_{k_1}(\Omega_{\mathcal{L}_1, \mathcal{L}_0} \widetilde{\mathcal{L}ag}) \rightarrow C\mathcal{W}_b^{-|\vec{k}|-d+1}(\mathcal{L}_d, \mathcal{L}_0) \quad (4.43)$$

$$\sigma_d \otimes \cdots \otimes \sigma_1 \mapsto \sum_{\dim(\mathcal{M}(\sigma_d, \dots, \sigma_1; x))=0} (-1)^\dagger ev_*([\overline{\mathcal{M}}_{2d+1}(\vec{\sigma}; x)]) \quad (4.44)$$

Now we may proceed to give the proof of Theorem 4.0.1 based on Lemma 4.0.7 and 4.0.17.

Proof of Theorem 4.0.1. From the description of the moduli spaces $\overline{\mathcal{Z}}_{2d}$ and their boundary strata from section (4.0.6) and the definition above; we realize that this is quite similar to the proof of Lemma (4.0.7). We just need to identify the boundaries of the moduli space (4.40) and that of the equations of the A_∞ relations:

1. the left hand side of Equations (4.37) corresponds to the 2nd terms on the right hand side of Equation (4.40) (when $r \geq 2$) and the left hand side of Equation (4.40) (when $r = 1$).
2. the 2nd terms on the right hand side of Equation (4.37) corresponds to the Left hand side of Equation (4.40).
3. the first summands on the right hand side of Equation (4.37) corresponds to the 3rd summands of Equation (4.40).

□

Chapter 5: Proof of Homotopy Equivalence on cotangent bundles

In this section we work with the cotangent bundle T^*Q of a closed manifold Q . It can be realized at the completion of the unit disc bundle and the boundary ∂M^{in} is the unit circle bundle; the radial coordinate r in this case is $|p|$ where p denotes the cotangent vectors. Note that there is a natural map $\eta : C_*(\Omega_{q_0, q_1}Q) \rightarrow C_*(\Omega_{T_{q_0}^*Q, T_{q_1}^*Q} \mathcal{L}ag)$ by taking a path in the zero section to the path of cotangent fibers in the space of Lagrangians with Legendrian boundary. We shall construct a chain homotopy equivalence between $Id_{C_*(\Omega_q Q)}$ and $f \circ F \circ \eta$ in the following commutative diagram:

$$\begin{array}{ccc}
 C_*(\Omega_q Q) & \xrightarrow{\eta} & C_*(\Omega_{T_q^*Q} \mathcal{L}ag) \\
 & \searrow \text{dotted} & \downarrow \mathcal{F} \\
 & & C\mathcal{W}^{-*}(L, L; H) \\
 & \swarrow f & \\
 & &
 \end{array}$$

We use the concatenation of paths on the zero section Q and evaluation of moduli spaces of pseudo-holomorphic discs with moving Lagrangian boundary condition defined below to show the homotopy equivalence in the later part of this section.

5.0.1 Gluing with moduli space of half-strips

From [5, Section 4] we have a moduli space $\mathcal{H}(x; q_0, q_1)$ of pseudo holomorphic maps from the upper half space with 2 punctures on the boundary to T^*Q such that the two punctures are mapped to $q_0, q_1 \in Q$ and the boundary component of the upper half plane between the two punctures is mapped into Q . The evaluation map from the compactification $\tilde{\mathcal{H}}(q_0, q_1, x) \rightarrow \Omega(q_0, q_1)$ gives the chain map $f^1 : C\mathcal{W}_b^k(T_{q_0}^*Q, T_{q_1}^*Q) \rightarrow C_k(\Omega_{q_0, q_1}Q)$.

Let Z denote the Riemann surface that is a disk with two punctures ξ_0, ξ_1 on its boundary, which is bi-holomorphic to a strip $\mathbb{R} \times I$.

We denote the domain of the moduli spaces $\{\mathcal{H}(x, q_0, q_1)\}$ by S_+ for any $x \in \mathcal{X}(T_{q_0}^*Q, T_{q_1}^*Q)$.

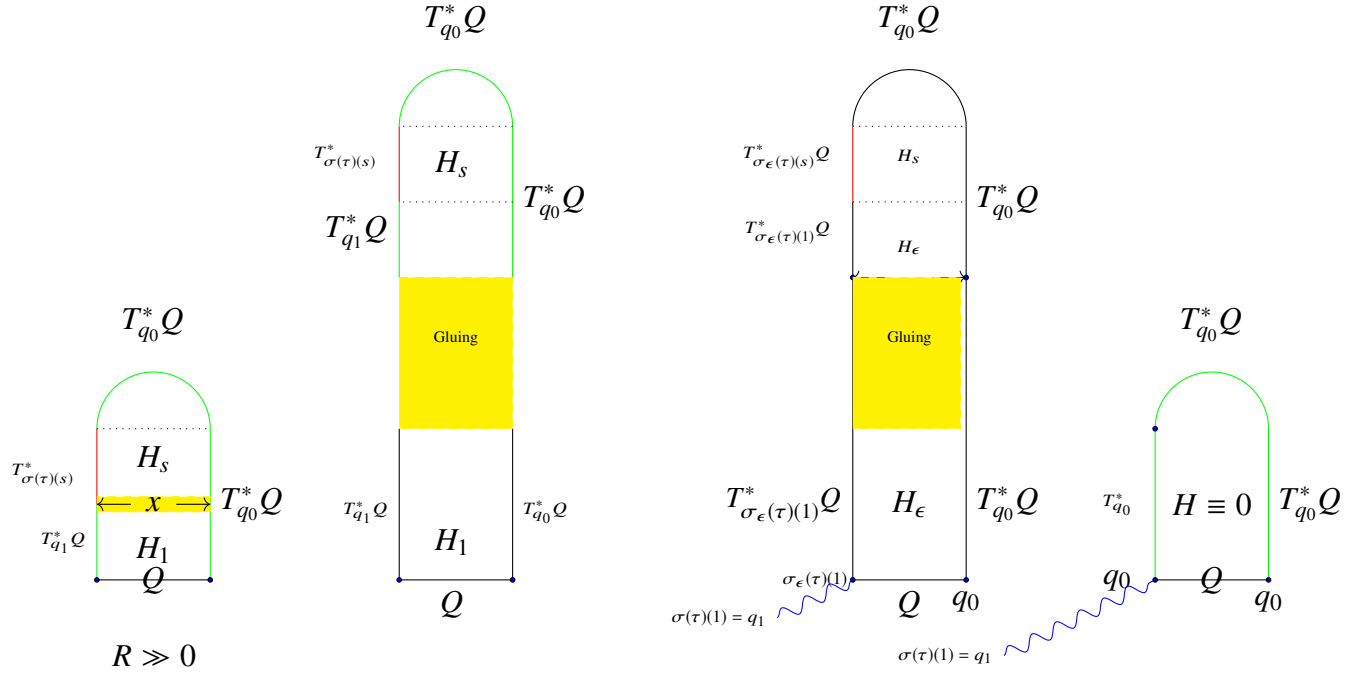


Figure 5.1: Homotopy between Id and $f \circ F \circ \eta$

S_+ is equipped with three strip-like ends $\epsilon_{\pm} : Z_{\pm} \rightarrow S_+$, where the positive ends are near the incoming points $q_0, x(t)$ and negative ends near q_1 as defined in [5, Section4]; the domain S_- of $\{\mathcal{M}(\sigma, x, H, J_{\tau})\}$ has a negative strip-like end for $x(t)$. For any gluing parameter $\delta \in (0, 1)$, we set the associated gluing length to be $R = -\log(\delta)$. One then defines another domain

$$S_R = S_+ \#_R S_- \quad (5.1)$$

by first removing $\epsilon_+([R, +\infty) \times [0, 1])$ in S_+ and $\epsilon_-((-\infty, -R] \times [0, 1])$ in S_- , both corresponding to strip-like ends near the puncture x , from $S_+ \sqcup S_-$, and then identifying the remaining finite pieces of the ends through the formula $\epsilon_+(s, t) \sim \epsilon_-(s - R, t)$. The positive and negative ends at infinity in S are the ones inherited from S_+ .

We had imposed conditions on Hamiltonians (2.66) and the behavior of almost complex structures near strip-like ends (4.2); by [22, chapter 9], there is a consistent choice of Floer data and strip like-ends that for small values of the gluing parameter δ . Therefore we may impose a new inhomogeneous Cauchy-Riemann equation on this new Riemann surface S_R , and this equation

agrees with the original equations in the definition of the two moduli spaces when restricted to $S_+ \setminus \epsilon_+([R, +\infty) \times [0, 1])$ and $S_- \setminus \epsilon_-((-\infty, -R] \times [0, 1])$ respectively and interpolates in the overlapping region of gluing. For every $R \in [1, \infty)$, let us denote the moduli space of the solutions to the new equation on S_R by $\mathcal{G}(\sigma; R)$.

Note that the domain S_R is bi-holomorphic to the strip Z with one component of the boundary mapped to the zero section Q , a constant Lagrangian. It is illustrated in the second picture of Figure (5). The component that is mapped to Q is drawn as the bottom interval in the picture, and the arc in the picture forms the other component.

When $R \rightarrow \infty$, the gluing theorem (eg. [12]) implies that the moduli space is homeomorphic to the union over all orbits x of the Cartesian product of the moduli space of upper half space with 2 punctures $\mathcal{H}(x; q_0, q_1)$ and that of our moduli space $\mathcal{M}(\sigma, x)$:

$$\mathcal{G}(\sigma, R) \simeq \bigsqcup_{x \in \mathcal{X}(T_{q_0}^*Q, T_{q_1}^*Q)} \mathcal{M}(\sigma, x) \times \mathcal{H}(x; q_0, q_1). \quad (5.2)$$

Therefore, we may also define $\mathcal{G}(\sigma, \infty)$ to be $\bigsqcup_{x \in \mathcal{X}(T_{q_0}^*Q, T_{q_1}^*Q)} \mathcal{M}(\sigma, x) \times \mathcal{H}(x; q_0, q_1)$ for all given $\sigma \in C_*(\Omega_{(q_0, q_1)}Q)$.

We may pick a smooth monotone map $\nu : [1/2, 1] \rightarrow [1, \infty]$ such that ν maps a small neighborhood of $\frac{1}{2}$ in this half interval constantly to 1. We are going to work with this half interval from now on and still name its parameter R . Gluing induces a family of maps parametrized by $\tau \in I^k$ on the boundary of Z

$$\Phi : \partial Z \times [1/2, 1] \times I^k \rightarrow \mathcal{Lag}(T^*Q) \quad (5.3)$$

such that for each τ , $\Phi_\tau : \partial Z \times [1/2, 1]$ is locally constant near $1/2$ by our construction. Therefore, Equation (5.2) holds when R is close to 1 under this new parametrization.

5.0.2 Moduli space of family of moving Lagrangian boundary conditions

Given $\sigma : [0, 1]^k \rightarrow \Omega_{(q_0, q_1)}Q$, namely $[0, 1]^k \times [0, 1] \rightarrow Q$, a chain of paths on the compact manifold Q , we will define $\sigma_\epsilon : [0, 1]^k \times [0, 1] \rightarrow Q$ such that $\sigma_\epsilon(\tau)$ moves from q_0 to q_ϵ smoothly,

where $q_\epsilon = \sigma(\tau)(\epsilon)$. We want to define a moduli space $\mathcal{M}(\sigma_\epsilon, q_0, H)$ of J-holomorphic curves with boundary conditions given by the above one parameter family. As for the Floer data, we are concerned with the moving boundary conditions and Hamiltonian perturbation as we require smoothness.

We use linear Hamiltonian flow to generate the moving Lagrangian co-fibers. Consider the following mollifier function:

$$\phi : \mathbb{R} \rightarrow [0, 1], \phi = \begin{cases} 2e \cdot \exp(-\frac{1}{1-(2s)^2}) & s \in (-1, 1) \\ 0 & |s| \geq 1 \end{cases}$$

Notice $\phi|_I$ has compact support in $(-1/2, 1/2)$ and it is C^∞ in this open interval. By convolution of the characteristic function χ of the unit ball B_1 inside \mathbb{R} with the mollifier ϕ defined above, we obtain the function $\psi_{1/2}$:

$$\psi_{1/2}(s) = \chi * \phi(s) = \int_{\mathbb{R}} \chi(s-t)\phi(t)dt = \int_{B_{1/2}} \chi(s-t)\phi(t)dt$$

which is smooth and equal to 1 inside $B_{1/2}$ and is supported inside $B_{3/2}$. This is observed with the fact that if $|s| \leq 1/2$ and $|t| \leq 1/2$, then $|s-t| \leq 1$, hence for $|s| \leq 1/2$,

$$\int_{B_{1/2}} \chi_{1/2}(s-t)\phi(t)dt = \int_{B_{1/2}} \phi dt = 1.$$

Definition 5.0.1. $\psi(s) := \psi_{1/2}(s - \frac{1}{2})$ for $s \in [0, 1]$.

So far, we have constructed a smooth function ψ which is 1 at $s = 0$, 0 at $s = 1$ and all order right (resp. left) differentials vanish at that these two points. We extend ψ by letting $\psi(s) = 1$ for $s < 0$ and $\psi(s) = 0$ for $s > 1$ and get a smooth function $\mathbb{R} \rightarrow [0, 1]$ that is a smooth diffeomorphism restricted to $[0, 1]$.

Now for any σ, τ , $\sigma(\tau)$ is a loop in Q . In other words, it is a map $I \rightarrow Q$ where both ends are mapped to the same point and that differentials of all orders on both ends are exactly opposite. If

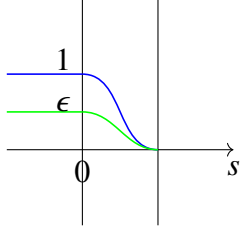


Figure 5.2: ψ and ψ_ϵ

we compose it with the inverse of ψ we still get a loop $\sigma(\tau) \circ \psi^{-1}$ in Q that is smooth. Thus when τ traverses I^k , we get another cubical chain of based loops in Q . Let us denote it by $\tilde{\sigma}$. From the above construction we may write σ as $\tilde{\sigma} \circ \psi$ to abbreviate for $\sigma(\tau)(s) = \tilde{\sigma}(\tau)(\psi(s))$ for any given $\tau \in I^k$.

Definition 5.0.2. *The boundary condition $\sigma_\epsilon : I^k \times \mathbb{R} \rightarrow [0, 1]$ is given by:*

$$\sigma_\epsilon = \tilde{\sigma} \circ (\epsilon \cdot \psi)$$

namely:

$$\sigma_\epsilon(\tau)(s) := \tilde{\sigma}(\tau)(\epsilon \cdot \psi(s))$$

where ψ is defined as above.

From the construction above, we get the smoothness of σ_ϵ with respect to s . It is easy to see that it is also smooth with respect to ϵ as both $\tilde{\sigma}(\tau)$ and ψ are smooth. Therefore we have the following lemma:

Lemma 5.0.3. *σ_ϵ is smooth with respect to both s and ϵ .*

Now that we have dealt with smoothness, we are set to define the moduli space of the family of moving Lagrangian boundary conditions.

First we require the Hamiltonian function H to be quadratic in the distance to the zero section and large enough slope with respect to s as in (B.15), namely $H = c(s) \cdot |p|^2$ for some smooth function $c(s)$ that vanishes outside $[-1, 2]$. The reason for such a choice is for the result of a C^0

$$\begin{array}{c}
Q \\
\hline
(du - X_H \otimes \gamma)^{0,1} = 0 \\
\hline
T_{q_\epsilon}^* Q \quad T_{\sigma_\epsilon(\tau)(s)}^* Q \quad T_{q_0}^* Q
\end{array}$$

bound of the pseudo holomorphic curves given by the solutions of the inhomogeneous equations, the details are in Appendix B.

As for the family of almost complex structures $J_{\tau,z,\epsilon}$, they satisfy the condition laid out in Section 4.0.1 and they are also parametrized by ϵ for the Cauchy-Riemann equation in the following definition.

Definition 5.0.4. *Given Floer data (H, J_τ, γ) , we define the moduli space of the family of pseudo-holomorphic maps with moving Lagrangian boundary condition $\mathcal{M}(\sigma_\epsilon, q_0, H, J_\tau)$ to be pairs (u, τ) consisting of maps $u : Z = \mathbb{R} \times [0, 1]$ (parametrized by (s, t)) to T^*Q and $\tau \in I^k$ satisfying the following conditions:*

$$\left. \begin{array}{l}
(du - X_H \otimes \gamma)^{0,1} = 0, \\
u(z) \in Q \text{ if } z \in \mathbb{R} \times \{1\} \subseteq \partial Z \text{ lies on the upper boundary of } Z, \\
u(z) \in T_{q_0}^* Q \text{ if } z \in [1, \infty] \times \{0\}, \\
u(z) \in T_{\sigma_\epsilon(\tau)(s)}^* Q \text{ if } z = (s, 0) \in \partial Z \text{ lies between } (0, 0) \text{ and } (1, 0), \\
u(z) \in T_{q_\epsilon} \text{ if } z \in [-\infty, 0] \times \{0\}, \\
\lim_{s \rightarrow -\infty} u(s, \cdot) = q_\epsilon, \lim_{s \rightarrow \infty} u(s, \cdot) = q_0, \text{ (asymptotic condition)}
\end{array} \right\} \quad (5.4)$$

In particular, when $\epsilon = 0$, we have the moduli space of curves $\mathcal{M}(\sigma_0, q_0)$ with constant boundary condition in Q and $T_{q_0}^* Q$, which is a copy of I^k .

A picture of the boundary condition for the above Equation (5.4) is illustrated in Figure (5.0.2).

For the discussion of later sections we want to re-parametrize the interval by $[0, 1/2]$ instead of $[0, 1]$; this can be done easily by picking a smooth function $g : [1/2] \rightarrow [0, 1]$ such that g is locally constant near $1/2$. We will use R to parametrize this half unit interval.

Definition 5.0.5. *The family $\{\sigma_\epsilon\}_{\epsilon \in [0,1]}$ also induces a map $\Phi : \partial Z \times [0, 1/2] \times I^k \rightarrow \mathcal{L}ag(T^*Q)$.*

Note this Φ can be extended with that in Equation (5.3) to define a family of smooth maps on the whole unit interval.

5.0.3 Moduli spaces with partial moving Lagrangian boundary condition

We have constructed two families of moduli spaces of pseudo holomorphic curves in T^*Q , the first being the glued curve of $\mathcal{G}(\sigma, R)$, and the second being the family $\mathcal{M}(\sigma_\epsilon, q_0)$. We take the disjoint union of these two families and show that this gives a moduli space of J-holomorphic curves with partial Lagrangian moving boundary condition which gives the cobordism connecting

$$\bigsqcup_{x \in \mathcal{X}(T_{q_0}^*Q, T_{q_1}^*Q)} \mathcal{H}(x, q_0, q_1) \times \mathcal{M}(\sigma, x),$$

to the moduli space with constant boundary condition $\mathcal{M}(\sigma_0, q_0)$.

Definition 5.0.6 (Moduli space with partial moving Lagrangian boundary conditions). *For a generic family J_τ and a Hamiltonian H on T^*Q , the moduli space $\mathcal{N}(\sigma, J_\tau, H)$ of pseudo-holomorphic curves with partial Lagrangian moving boundary condition is the disjoint union of tuples (u, τ, R) where u is a map from Z to T^*Q that satisfies the following conditions:*

$$\left. \begin{array}{l} (du - X_H \otimes \gamma)^{0,1} = 0, \text{ for closed 1-form } \gamma \in \Omega(Z), \text{ such that } \gamma = dt \text{ when } s \rightarrow \pm\infty \\ u(z) \in Q \text{ for all } R \text{ and if } z \in \mathbb{R} \times \{1\} \subseteq \partial Z \text{ lies on the upper boundary of } Z, \\ u(z) \in T_{q_0}^*Q \text{ for all } R \text{ and if } z \in [1, \infty] \times \{0\}, \\ u(z) \in \Phi((s, 0), \tau, R) \text{ if } z = (s, 0) \in \partial Z \text{ lies between } (0, 0) \text{ and } (1, 0), \\ u(z) \in T_{\sigma_R(\tau)(1)}^*Q \text{ if } z \in [-\infty, 0] \times \{0\}, \\ \lim_{s \rightarrow -\infty} u(s, \cdot) = \sigma_R(\tau)(1), \lim_{s \rightarrow \infty} u(s, \cdot) = q_0, \text{ (asymptotic condition)} \end{array} \right\} \quad (5.5)$$

To show that $\mathcal{N}(\sigma, J_\tau, H)$ is a manifold with corners of the appropriate dimension for a generic family J_τ , we first show that its interior $\mathcal{N}^*(\sigma, J_\tau, H)$ is the zero set of a section of an infinite

dimensional vector bundle whose differential is a Fredholm operator that is regular for generic J_τ . It is a standard argument similar to that in [19, Chapter 3]. However, we need to consider a different bundle from that discussed over there. We briefly describe the argument in the following part of this section.

First we define the base of this infinite dimensional vector bundle in the context of Definition 5.0.5.

Definition 5.0.7. *For a family of smooth maps $\Phi : \partial Z \times [0, 1] \times I^k = \mathbb{R} \times \{0, 1\} \times [0, 1] \times I^k \rightarrow \mathcal{Lag}(T^*Q)$ and $\tau \in I^k$, and any k, p such that $kp > 2$, let $\mathcal{B}_\Phi^{k,p}$ denote the space of tuples (u, τ, R) where $R \in [0, 1]$, $\tau \in I^k$ and $u \in W^{k,p}(Z, T^*Q)$ such that $u(s, k) \in \Phi(s, k, R, \tau)$ where $k \in \{0, 1\}$. In other words, $\mathcal{B}_\Phi^{k,p}$ is a family of maps from the Riemann Surface Z to T^*Q with boundary lying in the Lagrangians given by $\Phi(\cdot, R, \tau)$.*

We argue that the above space is a Banach bundle over unit interval I parametrized by R . The fiber at each R is \mathcal{B}_{Φ_R} , the space of pairs (u, τ) such that $u|_{\partial Z} \in L_{\Phi_R, \tau}$. The tangent space of the fiber \mathcal{B}_{Φ_R} over $R \in I$ at $(u, \tau, R) \in \mathcal{B}_\Phi$ is the space:

$$T_{(u, \tau, R)}\mathcal{B}_{\Phi_R} = W^{k,p}(Z \times I^k, u^*T(T^*Q))$$

of all families of smooth vector fields $\xi(z) \in T_{u(z)}T^*Q$ along u parametrized by I^k .

The trivialization of the bundle over any small neighborhood of (R, τ) goes as follows:

For any $\sigma \in C_k(\Omega_{q_0, q_1} Q)$, we may realize the path of Lagrangians from $T_{q_0}^*Q$ to $T_{q_1}^*Q$ as the result of a flow of Hamiltonians W_τ parametrized by I^k such that for any $\tau \in I^k$, the Hamiltonian W_τ has linear bounds with respect to the distance given by a certain metric from the zero section outside of a compact set. In other words, for given σ , there exists large enough constants r_0, C such that

$$|W_\tau(q, p)| \leq C \cdot |p| \text{ for } |p| \geq r_0. \quad (5.6)$$

Thus the Lagrangian boundary condition $T_{q_s}^*Q = \tilde{\sigma}(\tau)(s)$ is just the time- s flow $\phi_{W_\tau}^s(T_{q_0}^*Q)$ of $T_{q_0}^*Q$ of this Hamiltonian W_τ . Lemma 5.0.3 and the definition of σ_ϵ shows that for each τ , the path of

Lagrangians given by the extra parameter R ,

$$\Phi_\tau : \partial Z \times (0, 1/2) \rightarrow \mathcal{L}ag(T^*Q),$$

maps $((s, 0), R)$ to $\phi_{W_\tau}^{R \cdot \psi(s)}(T_{q_0}^*Q)$, where ψ was defined in Definition 5.0.1. Therefore, we shall define the local trivialization map using a time- δ flow of W_τ . Note that such a diffeomorphism is further parametrized by $s \in \mathbb{R}$, where s denote the infinite coordinate of the strip $Z = \mathbb{R} \times I$.

Definition 5.0.8 (local trivialization). *For any $R \in (0, 1)$ and small enough δ such that $R + \delta \in (0, 1)$, the local trivialization map $\phi_{R,\tau,s}(\delta)$ on the space of pseudo-holomorphic maps with Lagrangian moving boundary conditions is given by a time- δ flow of the Hamiltonian W_τ :*

$$\phi_{R,\tau,s}(\delta) : \mathcal{B}_{\Phi_{R,\tau}} \rightarrow \mathcal{B}_{\Phi_{R+\delta,\tau}}, u(s, t) \mapsto \phi_{W_\tau}^{\delta \cdot \psi(s)}(u(s, t)),$$

where ψ is the function defined in Definition 5.0.1.

The following lemma guarantees the above definition is valid.

Lemma 5.0.9. $\phi_{R,\tau,s}(\delta)$ sends $\mathcal{B}_{\Phi_{R,\tau}}$ to $\mathcal{B}_{\Phi_{R+\delta,\tau}}$.

Proof. For any $u \in \mathcal{B}_{\Phi_{R,\tau}}$, Definition 5.0.2 gives the boundary condition $u(s, 0) \in \tilde{\sigma}(\tau)(R \cdot \psi(s))$, which, from the discussion above Definition 5.0.8, is just $\phi_{W_\tau}^{R \cdot \psi(s)}(T_{q_0}^*Q)$. We need to show that $\phi_{R,\tau,s}(\delta)$ maps the above path of Lagrangians into the path given by $\phi_{W_\tau}^{(R+\delta) \cdot \psi(s)}(T_{q_0}^*Q)$. By Definition 5.0.8, the following equation holds for any $u \in \mathcal{B}_{\Phi_{R,\tau}}$

$$\begin{aligned} \phi_{R,\tau,s}(\delta)(u(s, 0)) &= \phi_{W_\tau}^{\delta \cdot \psi(s)}(u(s, 0)) \\ &\in \phi_{W_\tau}^{\delta \cdot \psi(s)} \circ \phi_{W_\tau}^{R \cdot \psi(s)}(T_{q_0}^*Q) = \phi_{W_\tau}^{(\delta+R) \cdot \psi(s)}(T_{q_0}^*Q) = \Phi_{R+\delta}(s). \end{aligned}$$

Thus $\phi_{R,s}(\delta)$ maps $\mathcal{B}_{\Phi_{R,\tau}}$ into $\mathcal{B}_{\Phi_{R+\delta,\tau}}$. Since Hamiltonian flow is a diffeomorphism, this gives an identification of the above two spaces. \square

Therefore, $\phi_{R,\tau,s}$ gives a local trivialization of $\mathcal{B}_{\Phi_{R,\tau}}$ over $(R - \delta, R + \delta)$.

Remark 5.0.10. B_Φ can further be realized as a vector bundle over $I^k \times I$ whose fiber consists of maps u satisfying $u|_{\partial Z} \in L_{\Phi_{R,\tau}}$.

5.0.4 A brief discussion of transversality

To show that $\mathcal{N}^*(\sigma, J_\tau)$, the interior of $\mathcal{N}(\sigma, J_\tau)$, is a manifold of the appropriate dimension for a generic family J_τ , we need to show that it is the zero section of a Fredholm operator of an infinite dimension vector bundle.

Recall that in Section 4.0.1, we defined a Banach manifold $\mathcal{J}_{Z,k}(T^*Q)$. It denotes a choice of k -family of smooth almost complex structures of contact type outside of a compact set of T^*Q (here we may regard M^{in} as the unit cotangent bundle S^*Q .) that is also parametrized by Z and such that for any $\tau \in I^k$, the corresponding Z -dependent almost complex structure

$$J_{\tau,z} \in \mathcal{J}(T^*Q)$$

only depends on t when $s \rightarrow \pm\infty$ near the two infinite ends ξ^1, ξ^2 of Z parametrized by the s, t coordinates and that they agree. The choice of such a k family follows from the construction given in Section 4.0.1; therefore, it also satisfies the regularity criterion.

Following the standard argument given in [19, Section 3.2] and that in [10], we consider an infinite dimensional vector bundle \mathcal{E} over $\mathcal{B}_\Phi \times \mathcal{J}_{Z,k}(T^*Q)$. The fiber of \mathcal{E} at (u, τ, R, J_τ) is the space

$$\mathcal{E}_{(u,\tau,R,J_\tau)} = W^{k-1,p}(Z \times I^{k+1}, \Lambda^{0,1} \otimes_{J_\tau} u^*T(T^*Q)),$$

a family parametrized by τ and R of J_τ -antilinear 1-forms on Z with values in $u^*T(T^*Q)$.

Lemma 5.0.11. \mathcal{E} is a Banach manifold and infinite dimensional vector bundle over $\mathcal{B}_\Phi \times \mathcal{J}_{Z,k}(T^*Q)$.

sketch of proof. The Banach manifold structure of \mathcal{E} can be constructed as follows. Fix any smooth metric on T^*Q to identify a neighborhood $\mathcal{N}(u_{R,\tau})$ of an element $u_{R,\tau} : Z \rightarrow T^*Q$ in $\mathcal{B}_{\Phi_{R,\tau}}^{k,p}$ with a neighborhood of zero in the Banach space $W^{k,p}(Z, u_{R,\tau}^*T(T^*Q))$ via $\xi \mapsto \exp_{u_{R,\tau}}(\xi)$. This gives coordinate charts on $\mathcal{B}_{\Phi_{R,\tau}}^{k,p}$ with smooth transition maps. Note that we identified \mathcal{B}_Φ as a vector

bundle over I^{k+1} with fiber $\mathcal{B}_{\Phi_{R,\tau}}^{k,p}$ and base I^{k+1} in the previous section, thus there is a coordinate chart $\mathcal{N}(u, \tau, R)$ by taking the product of a local coordinate chart $\mathcal{N}(\tau, R)$ with $\mathcal{N}(u_{R,\tau})$. We trivialize the bundle \mathcal{E} over such a coordinate chart by using a connection $\tilde{\nabla}$ that preserves the family of almost complex structure J_τ . This constructs local trivializations over open neighborhoods $\mathcal{N}(u, \tau, R)$ in $\mathcal{B}_\Phi \times \{J_\tau\}$. To extend this over neighborhoods of the form $\mathcal{N}(u, \tau, R) \times \mathcal{N}(J_\tau)$ first trivialize over a neighborhood $\{(u, \tau, R)\} \times \mathcal{N}(J_\tau)$ via the isomorphism:

$$\Lambda^{0,1} \otimes_{J_\tau} u_R^* T(T^*Q) \rightarrow \Lambda^{0,1} \otimes_{J_{\tau'}} u_R^* T(T^*Q) : \alpha \mapsto \frac{1}{2}(\alpha + J_{\tau'} \circ \alpha \circ j)$$

and then extend this trivialization over each slice $\mathcal{N}(u, \tau, R) \times \mathcal{N}(J_\tau)$ using parallel transport as before. Here $\mathcal{N}(J_\tau)$ can be obtained by taking small perturbations in $End(T(T^*Q))$ satisfying the conditions in Section (2.0.8) and then taking exponentials with respect to these perturbations around J_τ . \square

From now on we use \mathcal{U} to denote the map from $Z \times I^{k+1} \rightarrow T^*Q$, namely $\mathcal{U}(z, \tau, R) = u(z)$ for $u \in \mathcal{B}_{\Phi_{R,\tau}}$.

The complex antilinear part of du defines a section $\mathcal{S} : \mathcal{B}_\Phi \rightarrow \mathcal{E}$ of this vector bundle:

$$\mathcal{S}(\mathcal{U}) = (\mathcal{U}, \bar{\partial}_{J_\tau}(u)),$$

which is a Fredholm operator as we explain below. From the definition of $\mathcal{N}(\sigma)$ above, we realize that its interior $\mathcal{N}^*(\sigma)$ is just the zero set of this section \mathcal{S} .

$$\begin{array}{ccc} & & \mathcal{E} \\ & \nearrow \bar{\partial} & \downarrow \\ \mathcal{N}(\sigma) \hookrightarrow & & \mathcal{B}_\Phi \\ & & \downarrow p \\ & & I \end{array}$$

In order to show that $\mathcal{N}^*(\sigma)$ is a manifold of appropriate dimension, we must show that \mathcal{S} is

transverse to the zero section. This means that the image of the differential $d\mathcal{S}(\mathcal{U}) : T_{\mathcal{U}}\mathcal{B}_{\Phi} \rightarrow T_{((\mathcal{U},0))}\mathcal{E}$ is complementary to the tangent space $T_{\mathcal{U}}\mathcal{B}_{\Phi}$ of the zero section for every $\mathcal{U} \in \mathcal{N}(\sigma)$. Given \mathcal{U} , we denote by

$$D_{\mathcal{U}} := W^{k,p}(Z \times I^{k+1}, u^*T(T^*Q)) \rightarrow W^{k-1,p}(Z \times I^{k+1}, \Lambda^{0,1} \otimes u^*T(T^*Q))$$

the composition of the differential $d\mathcal{S} : T_{(u,\tau,R)}\mathcal{B} \rightarrow T_{(u,\tau,R,0)}\mathcal{E}$ with the projection

$$\pi_u : T_{(u,0)}\mathcal{E} = T_{\mathcal{U}}\mathcal{B}_{\Phi} \oplus \mathcal{E}_{\mathcal{U}} \rightarrow \mathcal{E}_{\mathcal{U}}.$$

The operator $D_{\mathcal{U}}$ is called the vertical differential of the section \mathcal{S} of the section \mathcal{S} at \mathcal{U} . The exact definition is as follows:

$D_{\mathcal{U}}$ can be extended to general maps $Z \times I^{k+1} \rightarrow T^*Q$, regardless of J_{τ} -holomorphic or not. However, it depends on a choice of splitting of the tangent space $T_{(\mathcal{U},\bar{\delta}_J(u))}\mathcal{E}$ into horizontal and vertical subspaces. Such a splitting corresponds to a connection on $T(T^*Q)$. We assume the connection preserves the almost complex structure so that the fibers of \mathcal{E} are invariant under point wise parallel transport. The connection

$$\tilde{\nabla}_v X := \nabla_v X - \frac{1}{2}J_{\tau}(\nabla_v J_{\tau})X \quad (5.7)$$

induced by the connection ∇ of the metric works. Given $\xi \in W^{k,p}(Z \times I^{k+1}, u^*T(T^*Q))$, let

$$\Psi_{\mathcal{U}}(\xi) : \mathcal{U}^*T(T^*Q) \rightarrow \exp_{\mathcal{U}}(\xi)^*T(T^*Q) \quad (5.8)$$

denote the complex bundle isomorphism, given by parallel transport with respect to $\tilde{\nabla}$ along the

geodesics $s \mapsto \exp_{\mathcal{U}(z,\tau,R)}(s\xi_{z,\tau,R})$. We define the map

$$\mathcal{F} : W^{k,p}(Z \times I^{k+1}, u^*T(T^*Q)) \rightarrow W^{k-1,p}(Z \times I^{k+1}, \Lambda^{0,1} \otimes u^*T(T^*Q)) \quad (5.9)$$

$$\mathcal{F}_{\mathcal{U}}(\xi) := \Psi_{\mathcal{U}}(\xi)^{-1} \bar{\partial}_{J_{\tau}}(\exp_{\mathcal{U}}(\xi)). \quad (5.10)$$

For general map $\mathcal{U} : Z \times I^{k+1} \rightarrow T^*Q$, $D_{\mathcal{U}}$ is defined as

$$D_{\mathcal{U}}\xi := D\mathcal{F}_{\mathcal{U}}(0)\xi. \quad (5.11)$$

Similar to [19, Proposition (3.1.1)], we see that

$$D_{\mathcal{U}}\xi = \frac{1}{2}(\nabla\xi + J_{z,\tau}\nabla\xi \circ j_Z) - \frac{1}{2}J_{z,\tau}(\nabla_{\xi}J)(\mathcal{U})\partial_J(\mathcal{U}), \quad (5.12)$$

where j_Z acts on any tangent vector $\zeta = (\zeta_z, \zeta_{\tau}, \zeta_R) \in T_{(z,\tau,R)}(Z \times I^{k+1})$ by $(j_Z(\zeta_z), \zeta_{\tau}, \zeta_R)$. The above equation implies that $D_{\mathcal{U}}$ is a real linear Cauchy-Riemann operator and hence a Fredholm operator as explained in [19, Appendix C] or [7, Section 8]. For a generic infinitesimal perturbation $K_{\tau,z} \in \text{End}(TM, J_{\tau,z}, \omega)$, The resulting linearized operator at $J_{\tau,z} \exp(-J_{\tau,z}K_{\tau,z})$ with the expanded dense Banach subspace \mathcal{K} of infinitesimal perturbations given by

$$\mathcal{K} \oplus W^{k,p}(Z \times I^{k+1}, \mathcal{U}^*T(T^*Q)) \rightarrow W^{k-1,p}(Z \times I^{k+1}, \mathcal{U}^*(T^*Q)) \quad (5.13)$$

$$(K, \xi) \mapsto \frac{1}{2}K_{z,\mathcal{U}(z,\tau,R)} \circ (d\mathcal{U} - \xi \otimes \gamma) \circ j_Z + D_{\mathcal{U}}\xi, \quad (5.14)$$

is also Fredholm. Thus we arrive at the following proposition:

Proposition 5.0.12. *For a generic family of almost complex structure J_{τ} , \mathcal{F} is a Fredholm operator of index $\mu(q_0) - \mu(q_1) + |\sigma| + 1$ and the interior $\mathcal{N}^*(\sigma, J_{\tau})$ is a smooth manifold of same dimension.*

Remark 5.0.13. Transversality would come from the surjectivity of $D_{\mathcal{U}}$ for every $\mathcal{U} \in \mathcal{N}^*(\sigma)$ for a given regular J_{τ} as a $W^{k,p}$ -neighborhood of zero in $\mathcal{F}_{\mathcal{U}}^{-1}(0)$ is diffeomorphic to a $W^{k,p}$ -neighborhood of (u, τ, R) via the map $\xi \mapsto \exp_{\mathcal{U}}(\xi)$ and that $\mathcal{F}_{\mathcal{U}}^{-1}(0)$ intersects a small neigh-

neighborhood of zero in a smooth submanifold due to the surjectivity of $D\mathcal{U}$ and standard Sard-Smale argument. The details of this argument is similar to the proof of transversality shown in [19, Chapter 3].

5.0.5 Homotopy equivalence

To show the homotopy equivalence of the identity functor Id and $f \circ F \circ \eta$ on $C_*(\Omega_{q_0, q_1} \mathcal{Q})$, we construct maps from the cubical chains of maps in \mathcal{Q} to itself:

$$Hp : C_*(\Omega_{q_0, q_1} \mathcal{Q}) \rightarrow C_{*+1}(\Omega_{(q_0, q_1)} \mathcal{Q}), \quad (5.15)$$

induced by $\mathcal{N}(\sigma)$ such that $(f \circ F \circ \eta - Id)_* = d \circ Hp + Hp \circ d$ in $C_*(\Omega_{q_0, q_1} \mathcal{Q})$.

Remark 5.0.14. The geometric idea is that for every path in σ , concatenating $\mathcal{M}(\sigma_\epsilon)$ with the remainder of the path after the ϵ -th part, we can homotope to the original $(f \circ F \circ \eta)_*(\sigma)$ that is the set $\bigcup_{x \in \mathcal{X}(T_{q_0}^* \mathcal{Q}, T_{q_1}^* \mathcal{Q})} \mathcal{M}(\sigma, x) \times \mathcal{H}(x, q_0, q_1)$, represented by the first picture in Figure (5).

From the previous subsection and Sard-Smale argument we know that $\mathcal{N}^*(\sigma, J_\tau)$ is a smooth manifold of dimension $|\sigma| + \mu(q_0) - \mu(q_1) + 1$ for a generic family of almost complex structures J_τ . We list the boundary components of its compactification $\mathcal{N}(\sigma, J_\tau)$ below. For any $\sigma' \subseteq \partial\sigma$, we obtain a map:

$$\mathcal{N}(\sigma', J_\tau) \rightarrow \mathcal{N}(\sigma, J_\tau). \quad (5.16)$$

Also, by gluing a half disc with output x and input σ to a half disc with input x , we get the map

$$\mathcal{M}(\sigma, x) \times \mathcal{H}(q_0, x, q_1) \rightarrow \mathcal{N}(\sigma, J_\tau). \quad (5.17)$$

Maps with the above form for all possible $x \in \mathcal{X}(T_{q_0}^* \mathcal{Q}, T_{q_1}^* \mathcal{Q})$ is identified with the inverse image of the boundary at $R = 1$ of the unit interval $[0, 1]$ that gives the 1-parameter family of holomorphic maps in the definition of $\mathcal{N}(\sigma, J_\tau)$.

Similarly, taking the inverse image of the boundary at $R = 0$, we obtain the following map:

$$\mathcal{M}(Id_{q_0}, q_0) \rightarrow \mathcal{N}(\sigma, J_\tau). \quad (5.18)$$

Proposition 5.0.15. *Fixing the Floer data and a generic choice of family of almost complex structures J_τ , $\mathcal{N}(\sigma, J_\tau)$ is a smooth manifold with corners and its codimension 1 boundary components consist of the images of the inclusion maps given by equation (5.16), (5.17), (5.18), namely*

$$\bigsqcup_x \mathcal{M}(\sigma, x) \times \mathcal{H}(q_0, x, q_1), \quad (5.19)$$

$$\mathcal{M}(\sigma_0, q_0) = \mathcal{M}(Id_{q_0}, q_0), \quad (5.20)$$

$$\mathcal{N}(\sigma') \text{ for } \sigma' \subseteq \partial\sigma. \quad (5.21)$$

□

Consider the family of evaluation map:

$$ev_{\tau, R} : \mathcal{N}(\sigma)_{\tau, R} \rightarrow \Omega(q_0, q_1), \quad (5.22)$$

which for every R, τ takes every pseudo-holomorphic strip $u : Z \rightarrow T^*Q$ to the path along Q between q_0 and q_R obtained by restricting to the component of ∂Z that stays within Q and the concatenation with the remainder of the original path from q_R to q_1 that is given by σ . As a whole this gives a map:

$$ev : \mathcal{N}(\sigma) \rightarrow C^\infty(I^{k+1}, \Omega(q_0, q_1))$$

parametrized by R, τ and induce a map $ev_* : C_*(\mathcal{N}(\sigma)) \rightarrow C_{*+k+1}(\Omega_{(q_0, q_1)}Q)$.

The map in equation (5.17) gives a commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}(\sigma, x) \times \mathcal{H}(q_0, x, q_1) & \longrightarrow & \mathcal{N}(\sigma) \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
C_k(\Omega_{(q_0, q_1)}\mathcal{Q}) & \longrightarrow & C_{k+1}(\Omega_{(q_0, q_1)}\mathcal{Q}),
\end{array} \tag{5.23}$$

where the above horizontal map is the inclusion of a boundary stratum and the left vertical map is projection onto the second factor and the remaining two maps are evaluations.

Similarly, the map in equation (5.18) gives the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}(Id_{q_0}, q_0) & \longrightarrow & \mathcal{N}(\sigma) \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
C_k(\Omega_{(q_0, q_1)}\mathcal{Q}) & \longrightarrow & C_{k+1}(\Omega_{(q_0, q_1)}\mathcal{Q}),
\end{array} \tag{5.24}$$

where the first horizontal map is the inclusion of the boundary stratum $\mathcal{M}(Id_{q_0}, q_0)$; the left vertical map is also the evaluation map on the component of the boundary of Z that stays inside \mathcal{Q} ; the bottom horizontal map is concatenation with the path given by $\sigma(\tau)$.

The following commutative diagram is obtained by equation (5.16) in a similar fashion:

$$\begin{array}{ccc}
\mathcal{N}(\sigma') & \longrightarrow & \mathcal{N}(\sigma) \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
C_k(\Omega_{(q_0, q_1)}\mathcal{Q}) & \longrightarrow & C_{k+1}(\Omega_{(q_0, q_1)}\mathcal{Q}).
\end{array} \tag{5.25}$$

Notice that $\mathcal{N}(\sigma, J_\tau)$ is by definition a family of moduli spaces parametrized by $R \in [0, 1]$. Taking $R = 1$ we get the boundary strata represented by Equation (5.17), given by standard gluing techniques which are reviewed in Appendix (C); when $R = 0$, this is the boundary given by equation (5.18); we also incorporate the boundary component that arise from the boundary chains $\partial\sigma$ of chains σ as in Equation (5.16). Thus the following proposition.

Lemma 5.0.16. *There exist fundamental chains $[\mathcal{N}(\sigma)] \in C_*(\mathcal{N}(\sigma))$ such that the assignment*

induced by the evaluation map

$$Hp(\sigma) = ev_*([\mathcal{N}(\sigma)])$$

defines the homotopy described in equation (5.15).

Proof. The strategy is to build up from chains of lower dimensions, starting with the fundamental chains $[\mathcal{N}(\sigma')]$ for $\sigma' \subseteq \partial\sigma$. Let us consider moduli spaces $\mathcal{N}(\sigma)$ whose boundary only has codimension 1 strata, which must therefore be products of closed manifolds. By taking the product of the fundamental chains of the factors in each boundary stratum, we obtain a chain in $C_*(\mathcal{N}(\sigma))$ with appropriate signs as before:

$$\sum_{x \in \mathcal{X}(T_{q_0}^* \mathcal{Q}, T_{q_1}^* \mathcal{Q})} (-1)^\dagger [\mathcal{M}(\sigma, x)] \times [\mathcal{H}(q_0, x, q_1)] + \sum_{\sigma' \subseteq \partial\sigma} [\mathcal{N}(\sigma')] + [\mathcal{M}(\sigma_0, q_0)]. \quad (5.26)$$

Lemma (5.0.16) implies that the above equation is a cycle. We now define the fundamental chain

$$[\mathcal{N}(\sigma)]$$

to be any chain whose boundary is equation (5.26).

By definition of homotopy on chains, we realize that up to signs which we shall ignore, proving Hp gives a homotopy on $C_*(\Omega_{(q_0, q_1)} \mathcal{Q})$ between Id and $f \circ F \circ \eta$, is equivalent to showing that $(f \circ F \circ \eta - Id)(\sigma) = \partial \circ Hp(\sigma) + Hp(\partial\sigma) \in C_*(\Omega_{(q_0, q_1)} \mathcal{Q})$.

We note $\partial \circ Hp(\sigma)$ consists of the image of the boundary components listed in equation (5.19), the first of which corresponds to $f \circ F \circ \eta$, the second corresponds to Id_* and the third corresponds to $\sum_{\sigma' \subset \partial\sigma} ev_*([\mathcal{N}(\sigma')]) = Hp(\partial\sigma)$. □

Signs and orientations

Consider a pair L and K of graded exact Lagrangians in a symplectic manifold M , and a time 1 non-degenerate Hamiltonian chord y starting at L and ending on K . Let U be a Riemann surface obtained by removing a point from the boundary of D^2 and equipped with a negative end near this puncture. We pick a projection map

$$t : U \rightarrow [0, 1], \tag{A.1}$$

which extends the projection to the interval along the strip-like end, and a one form γ on U which agrees with the pullback of dt along the strip-like end. A choice of almost complex structure J_t on the tangent space fo M allows us to define a Cauchy-Riemann operator

$$(du - X \otimes \gamma)^{0,1} = 0 \tag{A.2}$$

on sections of $U \times M$. The linearization of this operator has source sections of $(y \circ t)^*(TM)$, and the data of gradings on L, K define a path of Lagrangians Λ_y in the restriction of this Symplectic vector bundle to the boundary, uniquely determined up to homotopy by the requirement that it lift to a path of graded Lagrangians interpolating between the graded structures on $T_{y(0)}L$ and $T_{y(1)}K$ on either side of the puncture.

Definition A.0.1. *(Lemma 11.11 of [22]) The vector space o_y is the determinant bundle of the linearization of the operator (A.2) with boundary conditions Λ_y .*

The Floer chain complexes we consider are direct sums of the orientation lines of o_y ; namely the free abelian group generated by the two possible orientations of o_y with the relations that their sum vanishes. In the case of exact Lagrangian sections of cotangent bundles, such an orientation is the same as an orientation of the descending manifolds of the corresponding critical point, and

explicit signs in Morse theory require choices of orientations on such descending manifolds.

Similarly, in the case of a k -family of path of graded Lagrangians $\sigma : I^k \rightarrow \Omega_{L_0, L_1} \mathcal{L}ag$, any two paths in this family are homotopic and thus have same relative *Pin* structures.

Definition A.0.2. *The vector space \mathfrak{o}_σ is the determinant bundle of the linearization of the operator (A.2) with boundary conditions given by $\sigma(\tau)$ for some τ in I^k .*

Imposing an additional condition on the choices made in Section (2.0.7), we require that our chosen triangulation satisfy the following condition:

Every Hamiltonian time-1 chord between elements of $Ob(\mathcal{W}_b(M))$ and every intersection point between such a Lagrangian and Q is contained in the 3-skeleton of M . (A.3)

Note that while there are, a priori, infinitely many such chords, only finitely many of them intersect any given compact subset, which implies the existence of such a triangulation of M . Given a chord y with endpoints on K and L (and letting E_b denote the vector bundle fixed in Section 2.0.7), we define a *relative Pin structure* on Λ_y to be a *Pin* structure on $y^*(E_b) \oplus \Lambda_y$ whose restriction to an end of the chord agrees with the relative *Pin* structure on either K or L . By thinking of an intersection point between L and Q as a time-1 chord for the zero Hamiltonian, we also associate a path Λ_q to such a point, on which a relative *Pin* structure may be chosen.

We now consider the following situation: (L_0, \dots, L_d) is a collection of objects of $\mathcal{W}_b(M)$, $\vec{x} = (x_1, \dots, x_d)$ a sequence of chords with x_i starting at L_{i-1} and ending at L_i and $\vec{\sigma} = (\sigma_1, \dots, \sigma_d)$ a sequence of cubical chains with every element $\sigma_i(\tau_i)$ starting at L_{i-1} and ending at L_i . In addition, we have either a chord $x_0 \in \mathcal{X}(L_0, L_d)$ or intersection points q_0 and q_d of L_0 and L_d with Q . We shall only need the following Lemma which is a minor generalization of the standard results required to define operations in Lagrangian Floer theory away from characteristic 2. In our notation below, λ stands for the top exterior power of the tangent space of the appropriate manifold.

Lemma A.0.3. *Choices of relative Pin structures on the Lagrangians L_k and on the paths Λ_{x_k} for*

each element of \vec{x} and for x_0 induce a canonical up to homotopy isomorphism

$$\lambda(\mathcal{M}_d(\vec{x}; x_0)) \otimes \mathfrak{o}_{x_1} \otimes \cdots \otimes \mathfrak{o}_{x_d} \cong \lambda(\mathcal{R}_d) \otimes \mathfrak{o}_{x_0}, \quad (\text{A.4})$$

whenever the moduli space is regular. Similarly, choices on the families of paths σ_k for each element of $\vec{\sigma}$ and for x_0 induce a canonical up to homotopy isomorphism

$$\lambda(\mathcal{M}_{2d+1}(\vec{\sigma}; x_0)) \otimes \mathfrak{o}_{\sigma_1} \otimes \cdots \otimes \mathfrak{o}_{\sigma_d} \cong \lambda(\mathcal{Z}_{2d}) \otimes \mathfrak{o}_{x_0}. \quad (\text{A.5})$$

Sketch of Proof: Equation (12.8) of [22] is the special case where the background class b vanishes and Proposition 4.2.2 of [25] (see, in particular Equation (38)) covers the relatively *Spin* case. We shall therefore limit ourselves to a discussion of the relevance of *Pin* structures to sign considerations; we shall see that they are introduced in order to resolve an ambiguity coming from the existence of homotopically distinct isotopies between a fixed pair of Fredholm operators.

Given a curve $u \in \mathcal{M}_d(x_0; \vec{x})$ with source S we have a canonical up to homotopy isomorphism

$$\lambda(\mathcal{M}_d(x_0; \vec{x})) \cong \lambda(\mathcal{R}_d) \otimes \det(D_u), \quad (\text{A.6})$$

where $\det(D_u)$ is the determinant line of D_u , defined as

$$\det(D_u) = \lambda^\vee(\text{Coker}(D_u)) \otimes \lambda(\ker(D_u)). \quad (\text{A.7})$$

The next step is to deform $u = u_0$ through a family of smooth curves $\{u_r\}_{r \in [0,1]}$ with the same asymptotic and boundary conditions to a curve u_1 mapping ∂S to the 3-skeleton of M . Because any homotopy of homotopies can be deformed so that the boundary still maps to the 3-skeleton of M , the construction we give below does not depend on the choice of $\{u_r\}$ (see the analogous argument in Chapter 8 of [12]). From the homotopy, we obtain an isomorphism

$$\det(D_u) \cong \det(D_{u_1}). \quad (\text{A.8})$$

The asymptotic conditions satisfied by D_{u_1} near the positive punctures imply that we may glue to it the operators D_{x_d}, \dots, D_{x_1} and obtain an operator on a disc with one negative puncture corresponding to x_0 which we denote:

$$D_{u_1} \# D_{x_d} \# \dots \# D_{x_1}. \quad (\text{A.9})$$

Two salient fact about this operator will be relevant to us: First, gluing kernel and cokernel elements gives a natural identification

$$\det(D_{u_1} \# D_{x_d} \# \dots \# D_{x_1}) \cong \det(D_{u_1}) \otimes \mathfrak{o}_{x_d} \otimes \dots \otimes \mathfrak{o}_{x_1}, \quad (\text{A.10})$$

as discussed, e.g. in Section (11c) of [22]. Next, the fact that all Lagrangians are *graded* implies that the boundary condition of this glued operator are isotopic to those of the operator D_{x_0} through an isotopy which is constant near the end. A *choice* of isotopy defines an isotopy of Fredholm operators on the disc, and hence using Equation (A.10), an isomorphism

$$\det(D_{u_1}) \otimes \mathfrak{o}_{x_d} \otimes \dots \otimes \mathfrak{o}_{x_1} \cong \mathfrak{o}_{x_0}, \quad (\text{A.11})$$

which, together with Equations (A.6) and (A.8), gives an isomorphism between the right spaces.

Note that the choice of isotopy is parametrized, up to homotopy, by the fundamental group of the space of Lagrangian paths between two fixed points, which is a group of order 2 in dimensions greater than 2; the two different choices give different signs in the above isomorphism and we must be able to make a coherent choice in order to obtain consistent operations away from characteristic 2 (see, e.g. Lemma 11.17 of [22]). The choice of relative *Pin* structures enters only in this choice of homotopy.

Indeed, Definition 2.0.30 required that we choose a *Pin* structure on the vector bundle $TL|_{L_{[3]}} \oplus E_b|_{L_{[3]}}$ over the 3-skeleton of every object L of $\mathcal{W}(M)$. This choice determines a *Pin* structure on (1) the direct sum of the boundary condition of the operator D_{x_0} with the trivial vector bundle with

fibre $E_b|_{x_0}$ and (2) the direct sum of the boundary condition for $D_{u_1} \# D_{x_d} \# \cdots \# D_{x_1}$ with the vector bundle

$$u_1^*(E_b)|_{\partial S} \# (x_d \circ t)^* E_b|_{\partial T} \# \cdots \# (x_1 \circ t)^* E_b|_{\partial T} \quad (\text{A.12})$$

over the boundary of the once-punctured disc obtained by gluing the pull-back of E_b to ∂S with the pullbacks of E_b to the one punctured disc T under the compositions of the chords x_d, \dots, x_1 with the projection $t : T \rightarrow [0, 1]$ fixed in Equation (A.1). Note that the pullback of E_b by u_1 , and by $x_k \circ t$, extends the vector bundle (A.12) to the entire Riemann surface, hence induces a canonical up to homotopy isomorphism between this glued vector bundle and the trivial vector bundle with fibre $E_b|_{x_0}$.

With this in mind, we now fix the path between the glued operator and D_{x_0} to be the unique one, up to homotopy, such that direct sum of the boundary conditions with the trivial bundle $E_b|_{x_0}$ admits a *Pin* structure which restricts, at both ends, to the one fixed above.

The situation for Equation (A.5) is similar. Since the vector space o_σ is determined by any path $\sigma(\tau)$ by contractibility of cubes, we repeat the argument for Equation (A.4).

□

Remark A.0.4. Whenever $d = 2$ in Equation (A.4) (or 1 in Equation A.5), it is useful to think of \mathcal{M}_1 or \mathcal{Z}_0 as a manifold of dimension -1 whose tangent space is “generated” by translation on the strip.

Let us now consider the special case where M is the cotangent bundle of Q , and the Lagrangians L_k agree with a cotangent fiber T_q^*Q . We choose a vector bundle E_b on a 3-skeleton for T^*Q with Stiefel-Whitney class $b \in H^*(T^*Q, \mathbb{Z}_2)$, and fix a submanifold $N_b \subset T^*Q$ Poincaré dual to b , and which is disjoint from T_q^*Q as well as from all time-1 Hamiltonian chords of H with endpoints on this cotangent fiber.

Since it is contractible, the cotangent fiber T_q^*Q is an object of the wrapped Fukaya category of T^*Q for all possible twistings, and we would like to compare the isomorphisms (A.4) for an arbitrary class b , with the one coming from the trivial class. In order to do this, we choose a *Pin*

structure on the Lagrangian path Λ_x associated to any chord $x \in \mathcal{X}(T_q^*Q, T_q^*Q)$. Then, noting that the restriction of E_b to the complement of N_b in $T^*Q_{[3]}$ admits a *Pin* structure, we choose such a structure, and equip

$$\Lambda_{x_k} \oplus x^*(E_b), \tag{A.13}$$

with the associated direct sum of *Pin* structures.

Lemma A.0.5. *Given a curve $u \in \mathcal{M}_d(x_0; \vec{x})$, the two isomorphisms*

$$\det(D_u) \otimes \mathfrak{o}_{x_d} \otimes \cdots \otimes \mathfrak{o}_{x_1} \cong \mathfrak{o}_{x_0}, \tag{A.14}$$

*obtained from the sign conventions associated to T_q^*Q either as a *Pin* submanifold of T^*Q , or as a relatively *Pin* submanifold with respect to the background class $b \in H^*(T^*Q, \mathbb{Z}_2)$ differ by $(-1)^{N_b \cdot u}$.*

Proof. Fix an isotopy of operators which induces the isomorphism (A.14) in the untwisted case, i.e. such that the associated family of Lagrangian boundary conditions admits a *Pin* structure restricting at the two endpoints of the isotopy to the *Pin* structure on Λ_{x_0} , and on the glued boundary conditions. By definition, this given isotopy induces the desired isomorphism for the twisted case if and only if the direct sum with the pullback of E_b admits a *Pin* structure, i.e. if and only if the pullback of E_b under u_1 is *Pin*. The obstruction is precisely the intersection number of u_1 with N_b , which, because u and u_1 have the same asymptotic and boundary conditions, agrees with the intersection number of u with N_b . □

A.0.1 Signed operations

To obtain operations from Equations (A.4) and (A.5), one must choose orientations on the abstract moduli space \mathcal{R}_d and \mathcal{Z}_{2d} , or in the exceptional cases, a trivialization of the \mathbb{R} action on the moduli spaces of strips. The orientations on \mathcal{R}_d is fixed by identifying it with the configuration space of d ordered points on an interval, which correspond to the marked points ξ^1, \dots, ξ^d ; this also fixes an orientation on \mathcal{Z}_{2d} via the isomorphism of Equation (4.16). In the case of a strip, we

shall use ∂_s as the generator of the action of \mathbb{R} . For simplicity, we shall also fix trivializations of \mathfrak{o}_q for every intersection point between a Lagrangian $L \in \text{Ob}(\mathcal{W}_b(M))$ and Q .

We shall use the following conventions in determining signs: Every orientation on a manifold X induces an orientation on its boundary via the canonical isomorphism

$$\lambda(X) \cong \nu \otimes \lambda(\partial X), \quad (\text{A.15})$$

where ν is the normal bundle of ∂X which is canonically trivialized by the outwards normal vector. Whenever X is a moduli space, say for specificity $\mathcal{M}(x_0; \vec{x})$, Lemma A.0.3 gives an orientation *relative* the orientations lines associated to the inputs and outputs (and an orientation of the abstract moduli space). In all the cases we consider, the boundary of the moduli space is a product, and hence inherits two relative orientations, one coming from Equation (A.15), the other from taking the product of the two relative orientations coming from Lemma A.0.3. The discrepancy between these two signs must be computed. We illustrate the computation by providing the proof of some sign-related results. The first is a Lemma from Section 2.0.7:

Proof of Lemma 4.0.6. By rearranging the terms in Equation A.5, we obtain an isomorphism

$$\lambda(\mathcal{M}(\sigma; x)) \cong \langle \partial_s \rangle \otimes \mathfrak{o}_\sigma \otimes \mathfrak{o}_x^\vee. \quad (\text{A.16})$$

Taking the tensor product of these isomorphisms for the pairs $(\sigma; y)$ and $(y; x)$, we obtain

$$\lambda(\mathcal{M}(\sigma; y)) \otimes \lambda(\mathcal{M}(y; x)) \cong \mathfrak{o}_\sigma \otimes \mathfrak{o}_y^\vee \otimes \langle \partial_s \rangle \otimes \mathfrak{o}_y \otimes \mathfrak{o}_x^\vee.$$

If we identify the translation vector field with the normal vector ν , reorder the terms, and cancel the copy of $\mathfrak{o}_y^\vee \otimes \mathfrak{o}_y$ we arrive at

$$\lambda(\mathcal{M}(\sigma; y)) \otimes \lambda(\mathcal{M}(y; x)) \cong \nu \otimes \mathfrak{o}_\sigma \otimes \mathfrak{o}_x^\vee. \quad (\text{A.17})$$

with a Koszul sign equal to

$$(-1)^{\mu(y)+\mu(\sigma)}. \quad (\text{A.18})$$

The tensor product from the pair $(\sigma'; x)$ is

$$\lambda(\mathcal{M}(\sigma'; x)) \cong \langle \partial_s \rangle \otimes o_{\sigma'} \otimes o_x^\vee. \quad (\text{A.19})$$

Since σ' is a boundary stratum of σ , any path in σ can be homotoped to a path on σ' . Therefore they have the same sign. \square

Next, we prove Lemma (4.0.16)

Proof of Lemma 4.0.16. When breaking takes places on the incoming segments, Equation (A.5) implies one factor of the product orientation is given by an isomorphism

$$\lambda(\mathcal{M}_{2d+1}(\sigma_1, \dots, \sigma_d; x)) \cong \lambda(\mathcal{R}_d) \otimes o_{\sigma_1} \otimes \dots \otimes o_{\sigma_d} \otimes o_x^\vee. \quad (\text{A.20})$$

Therefore we have the following isomorphism:

$$\begin{aligned} & \lambda(\mathcal{M}_{2d_1+1}(\sigma_1, \dots, \sigma_{d_1}; x_1)) \otimes \dots \otimes \lambda(\mathcal{M}_{2d_r+1}(\sigma_{\sum_{i=1}^{r-1} d_i+1}, \dots, \sigma_d; x_r)) \otimes \lambda(\mathcal{M}(\vec{x}; x)) \\ & \cong \lambda(\mathcal{R}_{d_1}) \otimes o_{\sigma_1} \otimes \dots \otimes o_{\sigma_{d_1}} \otimes o_{x_1}^\vee \otimes \dots \otimes \lambda(\mathcal{R}_{d_r}) \otimes o_{\sigma_{\sum_{i=1}^{r-1} d_i+1}} \otimes \dots \otimes o_{\sigma_{d_1}} \otimes o_{x_r}^\vee \\ & \otimes \lambda(\mathcal{R}_r) \otimes o_{x_1} \otimes \dots \otimes o_{x_r} \otimes o_x^\vee. \end{aligned}$$

The Koszul signs given by moving $\lambda(\mathcal{R}_{d_2}), \dots, \lambda(\mathcal{R}_{d_r})$ to be adjacent to $\lambda(\mathcal{R}_{d_1})$ and shuffling o_{x_1}, \dots, o_{x_r} to the back together with the difference between the product orientation of $\prod \mathcal{R}_{2d_i} \times \mathcal{R}_d$ and that of \mathcal{R}_{2d} gives Equation (4.33).

For the other strata, we only need to take into consideration the difference of orientations $\mathcal{R}_{2d} \hookrightarrow \partial \mathcal{R}_{2d}$ from the argument above as there is no shuffling in the expression similar to the above equation.

\square

C^0 -estimates

In this section we denote by M^{in} a Liouville domain with contact boundary and M its completion as in (1.2), namely $M = M^{in} \cup \partial M^{in} \times [1, \infty)$ with a collar neighborhood of ∂M^{in} inside M^{in} modeled by $\partial M^{in} \times (0, 1]$. Let us use r to denote the radial coordinate of the completion and use $\alpha = \theta|_{\partial M^{in}}$ to denote the contact form on ∂M^{in} as in (1.1).

In order to show that our moduli space of pseudo-holomorphic curves inside M is compact, we need to prove Gromov compactness by restricting to a compact subset of M , which comes from a C^0 bound on $\rho = r \circ u$. Here u is the pseudo-holomorphic curve from $\mathbb{D} \setminus \{\xi_1, \xi_2\}$ to M , where \mathbb{D} is the unit disc. We denote this domain of u by Σ in this section, and it's easy to see that this is a strip

$$\Sigma \simeq \mathbb{R} \times [0, 1].$$

Let $z = (s, t)$ denote the coordinate for $\mathbb{R} \times I$.

A C^0 -bound is achieved if ρ is constraint within a compact subset of in M . Suppose the moduli space of pseudo-holomorphic strip has boundary conditions given by a family of moving Lagrangians L_s for $s \in [0, 1]$ in the lower boundary of Σ ; this family of Lagrangians in M has Legendrian boundaries, namely the infinite ends of each L_s is a product of Legendrian submanifolds of ∂M^{in} and an open interval:

$$L_s \cap \partial M^{in} \times (0, \infty) = \Lambda_s \times (0, \infty),$$

near ∂M^{in} , where Λ_s is a family of Legendrian submanifolds generated by a Hamiltonian H_0^s parametrized by s on ∂M^{in} . In later discussions, we just write H_0 if there is no confusion. Thus

there exists a vector field X_{H_0} in ∂M^{in} such that

$$i_{X_{H_0}} d\alpha = -dH_0,$$

and Λ_s is the time- s flow of X_{H_0} in ∂M^{in} . A calculation similar to that of exercise 3.48 in [20] gives

$$X_{H_0} = H_0 \cdot Y + Z, \tag{B.1}$$

where Y is the Reeb vector field on ∂M^{in} and Z lies in the contact distribution $\kappa = \ker(\alpha)$ such that

$$i_Z d\alpha|_{\kappa} = dH_0|_{\kappa}.$$

Now consider a Hamiltonian H^l on M that is linear with respect to the radius on cylindrical ends $\partial M^{in} \times (1, \infty)$, given by $H^l(q, r) := r \cdot H_0(q)$ for $q \in \partial M^{in}$. This gives rise to a vector field on the conical end

$$X_{H^l}(q, r) = (X_{H_0}(q), -r dH_0(Y)\partial_r) \text{ when } r \gg 0. \tag{B.2}$$

Remark B.0.1. This Hamiltonian H^l is the one denoted by W in Section (5.0.3).

As for the properties of the Hamiltonian that enters in the definition of the wrapped Fukaya category, we consider functions H on M parametrized by Σ such that:

$$\begin{aligned} &H \text{ is positive everywhere. Moreover, } H(r, q) = c(s) \cdot r^2 \text{ on the infinite cone} \\ &\text{for some smooth } c : \mathbb{R} \rightarrow \mathbb{R}_+. \end{aligned} \tag{B.3}$$

Let X_H be the Hamiltonian vector field of H . On the infinite cone it is a multiple of the Reeb vector field:

$$X_H = 2c(s)r \cdot (0, Y),$$

where Y is the Reeb vector field associated to the contact form $\alpha = \theta|_{\partial M^{in}}$ and the first coordinate corresponds to the component in the direction of ∂_r . By restriction on ∂M^{in} , which corresponds to $r = 1$, one sees that X_H satisfies $X_H|_{\partial M^{in}} = c(1)Y$.

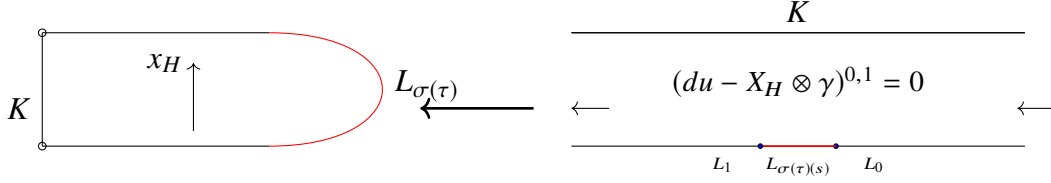


Figure B.1: Maximum Principle for glued disc

On both infinite ends of the domain Σ there are asymptotic Hamiltonian chords x_0 and x_1 and the curves satisfy the in-homogeneous Cauchy Riemann equation (2.77), which we write here for the reader's convenience:

$$\begin{aligned} \text{The map } u : \Sigma \rightarrow M \text{ satisfies } (du - X_H \otimes \gamma)^{0,1} = 0, \text{ where } u(s, 1) \text{ lies in a} \\ \text{fixed Lagrangian } K, u(s, 0) \in L_0 \text{ when } s \gg 0, u(s, 0) \in L_1 \text{ when } s \ll 0 \text{ and} \\ u(s, t) \in L_s \text{ for } s \in [0, 1] \text{ and } \gamma \text{ is a closed 1-form on } \Sigma \text{ such that } \gamma = dt \text{ near} \\ \text{both infinite ends and that } \gamma = dt \text{ in a small neighborhood of } [0, 1] \times \{0\}. \end{aligned} \quad (\text{B.4})$$

Namely we restrict the moving boundary Lagrangian condition to an interval on the lower boundary of Σ .

With the above geometric settings, we state the following C^0 estimate for moduli space of pseudo-holomorphic curves with moving Lagrangian boundary with Legendrian ends inside a Liouville manifold.

Theorem B.0.2. *Given a 2-dimensional disk Σ with punctures $\{\xi_1, \xi_2\}$ on the boundary, every element in the moduli space of pseudo-holomorphic curves in an open Liouville manifold M from the domain Σ and moving Lagrangian boundary conditions generated by the flow of a Hamiltonian with linear growth rate at the cylindrical ends has a priori C^0 bound. Namely, there is a choice of Hamiltonian data H, J and closed 1-form γ on Σ satisfying the conditions of B.4 such that all solutions of the inhomogeneous pseudo-holomorphic equation*

$$(du - X_H \otimes \gamma)^{0,1} = 0 \quad (\text{B.5})$$

stay within a compact subset of M .

Proof. We use the Hopf maximum principle on a function that is monotone and bijective with ρ to show that when the radial coordinate is too large, the maximum can't live on the moving part of the boundary of the domain Σ nor the interior, thus forcing any solution of the equation to be within a chosen compact space inside M . The proof consists of showing non-existence of local maximum at the moving boundary and in the interior.

First, we work with an almost complex structure J of contact type which satisfies Equation (1.4) near the cylindrical ends and $\gamma = dt$:

$$\begin{aligned}
\partial_s \log(\rho) &= \partial_s(\log(r \circ u)) \\
&= \frac{dr}{r}(\partial_s u) = \frac{dr}{r} \circ (-J)(\partial_t u - X_H) \\
&= \frac{\theta}{r}(\partial_t u - 2c\rho X_\rho) = \alpha(\partial_t u) - 2c\rho\alpha(X_\rho) \\
&= \alpha(\partial_t u) - 2c(s)\rho.
\end{aligned}$$

The third equality comes from the J -holomorphic curve equation; the fourth equality comes from the condition (1.4) and the last equality is due to the fact that $X_\rho = Y$. Similarly,

$$\begin{aligned}
\partial_t \log(\rho) &= \frac{dr}{r} \cdot (-J) \cdot J \cdot (\partial_t u) = \alpha(J\partial_t u) \\
&= \alpha(-\partial_s u + JcX_\rho) = \alpha(-\partial_s u)
\end{aligned}$$

Since H_0 is smooth on the compact set ∂M^{in} we have a bound on H_0 , let μ denote twice the maximum value,

$$\mu := 2 \cdot \max H_0. \tag{B.6}$$

Consider the following auxiliary function:

$$h_\mu(s, t) = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot (\frac{t^2}{2} - t)\mu & s \in (-1, 0) \\ (\frac{t^2}{2} - t)\mu & s \in [0, 1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot (\frac{t^2}{2} - t)\mu & s \in (1, 2) \\ 0 & \textit{otherwise} \end{cases} \quad (\text{B.7})$$

Remark B.0.3. The above construction has 2 components, the first part is $(\frac{t^2}{2} - t)\mu$ with the property that $d^c h(\partial_s) \geq \mu$ on all of the lower boundary of Σ with $d^c h(\partial_s)$ equals μ when $s \in [0, 1]$ (the portion of moving boundary illustrated in red in figure (B)) and $d^c h(\partial_s)$ vanishes on the upper boundary of Σ which, is mapped into a constant Lagrangian K . The second part is just a mollifier function so that $h_\mu(s, t)$ vanishes near infinite ends.

We want to show that there is a C^0 -bound on the function $\log(\rho) - h_\mu$ to give a C^0 -bound on ρ since h_μ itself is bounded above on Σ , thus an upper bound on $\log(\rho) - h_\mu$ would imply an upper bound on ρ ; to this end we apply the Hopf maximum principle to $\log(\rho) - h_\mu$. To this end we present the following two lemmas:

Lemma B.0.4. *Suppose there is an auxiliary function h_μ defined in Equation (B.7) and a pseudo-holomorphic strip $u : \Sigma \rightarrow M$ with moving boundary Lagrangian conditions on a Liouville domain M satisfying the condition in Theorem B.0.2. When the radial coordinate r is large enough, the maximum value of $\log(\rho) - h_\mu$ does not lie on the boundary of the domain Σ .*

Lemma B.0.5. *Suppose we are equipped with a pseudo-holomorphic strip as in Lemma B.0.4, when r is large enough, the maximum value of $\log(\rho) - h_\mu$ does not lie on the interior of the domain Σ .*

Thus, from the above two lemmas, there is an upper bound on the radial coordinate of the image of the strip and stays within a compact set.

Proof of Lemma B.0.4. We first look at the partial derivatives of h_μ with respect to s and t .

$$\partial_t h_\mu = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot (t-1)\mu & s \in (-1, 0) \\ (t-1)\mu & s \in [0, 1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot (t-1)\mu & s \in (1, 2) \\ 0 & \text{otherwise} \end{cases}$$

$$\partial_s h_\mu = \begin{cases} e^{\frac{-s^2}{1-s^2}} \cdot \frac{-2s}{(1-s^2)^2} \left(\frac{t^2}{2} - t\right)\mu & s \in (-1, 0) \\ 0 & s \in [0, 1] \\ e^{\frac{-(s-2)^2}{1-(s-2)^2}} \cdot \frac{-2(s-2)}{(1-(s-2)^2)^2} \left(\frac{t^2}{2} - t\right)\mu & s \in (1, 2) \\ 0 & \text{otherwise} \end{cases}$$

Thus we obtain the following equations:

$$\begin{aligned} d^c \log(\rho) &= d \log(\rho) \circ j \\ &= -\partial_s \log(\rho) dt + \partial_t \log(\rho) ds \\ &= (-\alpha(\partial_t u) + 2c(s)\rho) dt - \alpha(\partial_s u) ds \\ &= -u^* \alpha + 2c(s)\rho \cdot dt \\ d^c h_\mu &= -\partial_s h_\mu dt + \partial_t h_\mu ds \end{aligned} \tag{B.8}$$

Let ξ denote $\partial_s u(s, 0)$, the vector field that is mapped to M from the lower boundary of Σ , note ξ is tangent to the Hamiltonian flow of L_s when $s \in [0, 1]$. We may decompose ξ into three components:

$$\xi = a\partial_r + X_{H_0} + Z', \tag{B.9}$$

where the first component is a radial vector, X_{H_0} is the Hamiltonian vector field on ∂M^{in} generated by H_0 and Z' is a vector in the tangent space of the Legendrian given by the intersection of the moving Lagrangian and ∂M^{in} at $u(s, 0)$, namely $Z' \in T\Lambda_s \subseteq \kappa = \ker(\theta)$. Recall from equation

(B.1) that $X_{H_0} = H_0 \cdot Y + Z$ where $Z \in \kappa$ also.

From Equations (B.8) and (B.9), it is easy to see:

$$\begin{aligned}
(d^c \log \rho - d^c h_\mu)(\partial_s) &= -\alpha(\xi) + 2c(s)dt(\partial_s) - \partial_s h_\mu dt(\partial_s) - \partial_t h_\mu \\
&= -\alpha(a \cdot \partial_r + X_{H_0} + Z') + 0 + 0 - \partial_t h_\mu \\
&= -\alpha(H_0 Y) - \partial_t h_\mu \\
&= -H_0 - \partial_t h_\mu = -H_0 + (1-t)\mu \cdot f(s).
\end{aligned}$$

Note $0 \leq f(s) \leq 1$, and $H_0(q)$ is strictly less than μ for any $q \in \partial M^{in}$ by definition of μ in equation (B.6), we get $(d^c \log \rho - d^c h_\mu)(\partial_s) > 0$ when $t = 0$; in other word, the gradient of $\log \rho - h_\mu$ paired with a inner normal vector on the lower part of $\partial \Sigma$ where the Lagrangian boundary moves from L_0 to L_1 is positive.

If the maximum value appears on $(s, 0)$ where $s \notin [0, 1]$, say $s < 0$. Then the maximum value appears on the fixed Lagrangian L_0 . We may apply the inverse Hamiltonian flow of H^l on the lower boundary to obtain a strip with constant Lagrangian boundary condition:

$$\tilde{u}(s, t) = \begin{cases} u(s, t) & \text{when } s < 0, \\ \phi_{-H^l}^{(1-t)s}(u(s, t)) & s \in [0, 1], \\ \phi_{-H^l}^{1-t}(u(s, t)) & \text{when } s > 1. \end{cases} \quad (\text{B.10})$$

From the C^0 -estimate on the radial coordinate derived by energy estimates given by Lemma 2.0.34, we know that the maximum value of $\log(\rho) - h_\mu$ is bounded a priori.

When $t = 1$, if K is compact, C^0 -bound of ρ is automatic; otherwise K is a Lagrangian with Legendrian boundary near the cylindrical ends, thus $(d^c \log \rho - d^c h_\mu)(-\partial_s) \equiv 0$ as $u(s, 1) \in K$ and $K \cap \partial M \times [1, \infty) = \Lambda \times [1, \infty)$ for some Λ Legendrian on ∂M . Since $H_0 < \mu$, the maximum of $\log \rho - h_\mu$ can't occur on the boundary of Σ by the Hopf Lemma, which claims that if the maximum occurs at the boundary for $\log \rho - h_\mu$, then the partial derivative on the outward normal direction

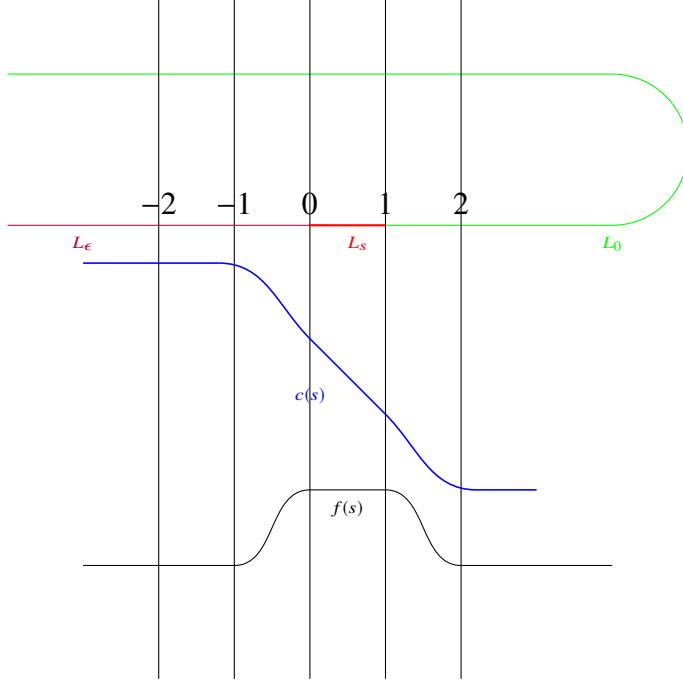


Figure B.2: The blue curve in the middle illustrates the slope of coefficient of r^2 as we asked the Hamiltonian to be $H = c(s) \cdot r^2$; the bottom curve illustrates the bump function who vanishes outside a neighborhood of the moving Lagrangian.

would be strictly positive. □

Proof of Lemma B.0.5. We now use the Laplacian to show that the maximum can't lie in the interior:

$$\begin{aligned}
\Delta(\log \rho - h_\mu) &= -dd^c(\log \rho - h_\mu)(\partial_s, \partial_t) = u^* d\alpha - \partial_s[2c(s)\rho]ds \wedge dt - \Delta h_\mu \\
&= d\alpha(\partial_s u, \partial_t u) - 2c'(s)\rho - 2c(s)\partial_s \rho - \Delta h_\mu \\
&= d\alpha(\partial_s u, J\partial_s u + cX_\rho) - 2c'(s)\rho - 2c(s)\partial_s \rho - \Delta h_\mu \\
&= d\alpha(\partial_s u, J\partial_s u) - 2c'(s)\rho - 2c(s)\partial_s \rho - \Delta h_\mu
\end{aligned} \tag{B.11}$$

Note we had the decomposition (1.3), thus if we let $\partial_s u = a \cdot Y + b\partial_r + Z$ where $Z \in \kappa$ as before.

Then $J\partial_s u = -ar\partial_r + \frac{b}{r} \cdot Y + J \cdot Z$. Also $\omega = d(r \cdot \alpha) = dr \wedge \alpha + r d\alpha$. Therefore

$$\begin{aligned} d\alpha(\partial_s u, J\partial_s u) &= d\alpha(a \cdot Y + b\partial_r + Z, -ar\partial_r + \frac{b}{r} \cdot Y + J \cdot Z) \\ &= d\alpha(Z, J \cdot Z) = \frac{(\omega - dr \wedge \alpha)(Z, J \cdot Z)}{r} \\ &= \frac{\omega(Z, J \cdot Z)}{r} = \frac{|Z|_J^2}{r} \end{aligned} \quad (\text{B.12})$$

From the above equation we get

$$\begin{aligned} \Delta(\log \rho - h_\mu) &= d\alpha(\partial_s u, J\partial_s u) - 2c'(s)\rho - 2c(s)\partial_s \rho - \Delta h_\mu \\ &= \frac{|Z|_J^2}{r} - 2c'(s)\rho - 2c(s)\partial_s \rho - \Delta h_\mu \end{aligned} \quad (\text{B.13})$$

We notice that in applying the Hopf maximum principle, we can ignore the term $\partial_s \rho$ at the maximum point. Therefore we only care about $\frac{|Z|_J^2}{r} - 2c'(s)\rho - \Delta h_\mu$.

When $s \in [0, 1]$, $\Delta h_\mu = \mu$, thus the above term becomes

$$\frac{|Z|_J^2}{r} - 2c'(s)\rho - \mu.$$

When $c'(s)$ is very negative such that $2c'(s) + \mu < 0$, if $\rho \geq 1$, the maximum of $\log \rho - h_\mu$ can't be in the interior, since in that case at the interior maximum point, $\Delta(\log \rho - h_\mu) < 0$, yet $0 > \Delta(\log \rho - h_\mu) + 2c'(s) + \mu = |Z|_J^2 \geq 0$ a contradiction. Thus $\log \rho - h_\mu$ on the image of the pseudo holomorphic strip lies inside M^{in} .

When $s \in [-1, 0]$,

$$\Delta(\log \rho - h) = |Z|^2 - c'(s) - \left(1 + \frac{4s^2 - 2(1-s^2)^2 + 8s^2(s^2-1)}{(1-s^2)^4} \left(\frac{t^2}{2} - t\right)\right) e^{\frac{-s^2}{1-s^2}} \mu.$$

Some Calculation shows that $\frac{4s^2 - 2(1-s^2)^2 + 8s^2(s^2-1)}{(1-s^2)^4} \cdot e^{\frac{-s^2}{1-s^2}} \in [-200, 200]$ for $s \in [-1, 0]$; while $-\frac{1}{2} \leq (\frac{t^2}{2} - t) \leq 0$. Similarly for $s \in [1, 2]$.

Thus if we impose the condition $c'(s) \leq -100|\max_Q H_0|$, we get upper bound of ρ on all of Σ .

We denote the negative constant on the right hand side of our inequality by $-A$.

Therefore, we may pick a constant C large enough such that $C > 2A$.

Let $f(s)$ be a bump function that vanishes outside $[-1, 2]$ and is equal to 1 between 0 and 1 as shown in Figure (B). For example we can require $f(s)$ to be:

$$f(s) = \begin{cases} 0 & s \in (-\infty, -1] \\ \exp(-\frac{s^2}{1-s^2}) & s \in (-1, 0) \\ 1 & s \in [0, 1] \\ \exp(-\frac{(s-2)^2}{1-(s-2)^2}) & s \in [1, 2] \\ 0 & s \in [2, \infty), \end{cases} \quad (\text{B.14})$$

which happens to be the same function in the definition of h_μ . Thus we may define the Hamiltonian H_s to be any smooth function whose restriction to the cylindrical end $\partial M^{in} \times (0, \infty)$ is of the following form:

$$H_s(q, r) = (C - A \cdot s)r^2 \cdot f(s), \quad (\text{B.15})$$

whenever $r \in [1, \infty)$. Namely the function $c(s)$ in condition (B.3) is given by $(C - A \cdot s^2) \cdot f(s)$, thus $c'(s)$ would be $-A$ when $s \in [0, 1]$ and vanishes outside $[-1, 2]$. This definition makes sure that for the moving Lagrangian boundary, namely when $s \in [0, 1]$, the Hamiltonian H on cylindrical end is of the form $c(s) \cdot r^2$ for $c'(s)$ sufficiently negative.

The Hamiltonian defined in equation (B.15) satisfies the requirement of condition (B.3) and the previous calculations would follow through. \square

In this way we have shown that the maximum of $\log(\rho) - h_\mu$ on the pseudo-holomorphic strip can't lie on the boundary of Σ for large enough r and can't lie in the interior of Σ either whenever the radial coordinate is larger than 1. Since when $s \rightarrow \pm\infty$, the ends of the strip converges to some intersection points of different (constant) Lagrangians with Legendrian boundary and thus they have to be within M^{in} . We get a C^0 bound that depends on H_0 on the radial coordinate of any

holomorphic strip satisfying equation (B.4).

□

Gluing

We sketch the proof of proposition (5.0.15). This is a standard consequence of a gluing theorem which can be found in [6] or [12]. First, the moduli space $\mathcal{N}(\sigma)$ is by definition the disjoint union of all pseudo holomorphic curves given by different Lagrangian boundary conditions which satisfy equation (5.4). An application of the Sard-Smale theorem in the appropriate Banach space with boundary implies that $\mathcal{N}(\sigma)$ is a smooth manifold with boundary near $R = 0$ of the appropriate dimension for generic choices of perturbation data. For any $x \in C\mathcal{W}^*(T_{q_0}^*Q, T_{q_1}^*Q)$, the difference of the dimensions between $\mathcal{M}(\sigma, x) \times \mathcal{H}(x, q_0, q_1)$ and $\mathcal{N}(\sigma)$ is 1. Let $\vec{u} = (u^0, u^1)$ denote such a stratum. We must show that we can choose a neighborhood U of \vec{u} in this product of lower dimensional manifolds so that there is local homeomorphism (as a result of gluing theorem)

$$G : U \times [0, \epsilon) \rightarrow \mathcal{N}(\sigma).$$

We focus on the interior of $[0, \epsilon)$, assume without loss of generality that each map u^i has non-vanishing derivative at some point on the boundary of their respective domains, call them z_0^0, z_0^1 . We fix the local coordinates s, t near z_0, z_1 such that t denotes the normal vector at the boundary pointing to the interior of the domain. In addition, writing d_0, d_1 for the dimension of $\mathcal{M}(\sigma, x)$ and $\mathcal{H}(x, q_0, q_1)$, we choose points $(z_1^0, \dots, z_{d_0}^0 \in \mathbb{H}^{d_0})$ and $(z_1^1, \dots, z_{d_1}^1 \in Z_+^{d_1})$ which are regular points of u^0, u^1 and such that the derivative of the evaluation map at these points

$$ev^0 = (ev_{z_1^0}, \dots, ev_{z_{d_0}^0}) : \mathcal{M}(\sigma, x) \rightarrow M^{d_0},$$

and

$$ev^1 = (ev_{z_1^1}, \dots, ev_{z_{d_1}^1}) : \mathcal{H}(x, q_0, q_1) \rightarrow M^{d_1},$$

are isomorphism at u^i . We fix a product of hypersurfaces:

$$N_l = N_{1,l} \times \cdots \times N_{d_l+1,l} \subset M^{d_l+1,l}$$

for $l \in \{0, 1\}$ which is transverse to the image of

$$D_{\vec{u}} e v^l \oplus D_{z_0^l} u^l | \partial_t$$

and such that the inverse image of each factor under u^l is transverse to the lines obtained by fixing the t -coordinates near the points z_0^0, z_0^1 .

Given a sequence \vec{v} of holomorphic maps close to \vec{u} , there is a unique parametrization v^l which is C^∞ close to u^l and $z_0^l \in (v^l)^{-1}(N_{d_l+1,l})$. Having fixed this parametrization, the inverse image under v^l of the remaining factors of N_l determines a unique sequence of points $(z_1^l(v^l), \cdots, z_{d_l}^l(v^l))$ which are C^0 -close to the sequence $(z_1^l, \cdots, z_{d_l}^l)$ and such that the t -coordinates agree. Thus fixing a sufficiently small neighborhood U_l of v^l in $\mathcal{H}(x, q_0, q_1)$ or $\mathcal{M}(\sigma, x)$, we obtain an embedding

$$U_0 \rightarrow \mathbb{H}^{d_0}$$

and

$$U_1 \rightarrow Z_+^{d_1}.$$

An application of the implicit function theorem implies that the composition of the above maps with the projection to the s -coordinate defines a local diffeomorphism $U_l \rightarrow \mathcal{R}^{d_l}$.

Summarizing the above discussion, we have chosen a product of hypersurfaces such that the moduli space of holomorphic strip (disc with 2 boundary punctures) near u_1 and the moduli space of holomorphic upper half plane (discs with one boundary puncture) is parametrized by the s -coordinates of the unique inverse image of $N_{k,l}$ whose t -coordinate agrees with t -coordinates of the original markings z_k^l .

The choice of N_l determines a family of right inverses to the linearization $\bar{\partial}$ operator

$$C^\infty(u_l^*TM) \rightarrow C^\infty(u_l^*TM \times \Lambda^{0,1}).$$

This operator becomes an isomorphism when restricted to those sections whose value at z_r^l lies in $TN_{r,l}$. The inverse of this isomorphism gives a right inverse to the linearization of the $\bar{\partial}$ operator in a Sobolev completion of the space of smooth sections. To be precise, we work with functions satisfying an exponential decay condition along the ends of the domain.

Now we define a pre-gluing map

$$\text{preG} : U \times (0, \epsilon) \rightarrow C^\infty(Z, M)$$

by

$$(v^1, v^2, \epsilon) \mapsto v^1 \#_\epsilon v^2,$$

obtained by removing the half-infinite strips

$$[2/\epsilon, \infty) \times [-1, 1] \text{ and } (-\infty, 2/\epsilon] \times [-1, 1]$$

respectively from the domains of the curves (v^1, v^2) and gluing the maps along the finite strips $[1/\epsilon, 2/\epsilon] \times [-1, 1]$ and $[-2/\epsilon, -1/\epsilon] \times [-1, 1]$ with a cut off function. This makes sense because these curves exponentially converge along the appropriate end to the same points in M . We assume that ϵ is small enough that the regions along which we glue the curves are disjoint from all marked points on the domains of u_l and u_{l-1} .

There is an \mathbb{R} ambiguity in the parametrization of $\text{preG}(v^1, v^2, \epsilon)$. We resolve this ambiguity by making origin of the domain of preG agree with the marked points z_1^0 coming from the domain of v^1 and glue the domains of the remaining curves to it. In particular, the pre-glued curve is equipped with a marked point $z_0(\epsilon)$ coming from the marked point z_0^0 of the domain of v^0 . Moreover, there

are $d_0 + d_1$ marked points

$$z_1^0(\vec{v}, \epsilon), \dots, z_{d_0}^0(\vec{v}, \epsilon), z_1^1(\vec{v}, \epsilon), \dots, z_{d_1}^1(\vec{v}, \epsilon)$$

coming from the marked points on the curve v^0, v^1 . Letting d stand for the dimension of $\mathcal{N}(\sigma)$, we obtain a sequence of d marked points

$$z_1(\vec{v}, \epsilon), \dots, z_d(\vec{v}, \epsilon)$$

on the domain of $\text{preG}(\vec{v}, \epsilon)$ with the convention that the last marked point correspond to the original marking z_0^1 of domain of v^1 and set $z_0(\vec{v}, \epsilon)$ to be the original as before. Note that at each such marked point the image of $\text{preG}(\vec{v}, \epsilon)$ intersects a distinguished hypersurface N_k transversely; these hypersurfaces are the ones we chose earlier to rigidify v^0 and v^1 .

As preG is not holomorphic, there is no natural linearization of the $\bar{\partial}$ operator acting on the tangent to the space of smooth maps at $\text{preG}(\vec{v}, \epsilon)$. To obtain such a linearization, we choose for each curve v^l , a family of metrics on M parametrized by Z which is constant and such that $N_{l,k}$ is totally geodesic with respect to a metric corresponding to the marked point $z_k^l(\vec{v})$. Note that for ϵ sufficiently small, this choice naturally induces a choice of metrics in M parametrized by the domain of $\text{preG}(\vec{v}, \epsilon)$ such that N_k is totally geodesic with respect to the metric corresponding to $z_k(\vec{v}, \epsilon)$.

Let us write

$$C_N^\infty(\text{preG}(\vec{v}, \epsilon)^*(TM))$$

for those sections of $\text{preG}(\vec{v}, \epsilon)^*(TM)$ whose value at the k th marked point lies in the tangent space to N_k . Note one must impose an appropriate decay condition at the ends in order to obtain a well-behaved elliptic problem.

Lemma C.0.1. *For ϵ sufficiently small, the linearization of $\bar{\partial}$ operator becomes an isomorphism when restricted to $C_N^\infty(\text{preG}(\vec{v}, \epsilon)^*(TM))$. Moreover, in an appropriate Sobolev completion of the*

space of smooth functions, the following quantities are uniformly bounded with respect to ϵ .

1. The norm of $\bar{\delta}(\text{pre}G(\vec{v}, \epsilon))$.
2. The norm of the right inverse to the linearization of the $\bar{\delta}$ operator whose image is $C_{\bar{N}}^{\infty}(\text{pre}G(\vec{v}, \epsilon)^*(TM))$
3. The difference between linearization of the $\bar{\delta}$ operator at $\text{pre}G(\vec{v}, \epsilon)$ and at points obtained by exponentiating vector fields of a fixed norm.

Remark C.0.2. We need to find a choice of a Sobolev completion in which all these properties hold at once. Following [12], we would prove this result by choosing a family of weights depending on the parameter ϵ . The first property is an elementary estimate while the second can be proved by constructing a pre-gluing map at the level of tangent vector field following [12]. In general, one cannot explicitly describe the result of pre-gluing tangent vectors in the image of the chosen right inverse along the component curves of \vec{v} . However, for our explicit choices of right inverses, one can check that the result is the Sobolev completion of $C_{\bar{N}}^{\infty}(\text{pre}G(\vec{v}, \epsilon)^*TM)$.

With the previous lemma we can apply implicit function theorem to achieve the following consequence:

Corollary C.0.3. *There exists a unique vector in $C_{\bar{N}}^{\infty}(\text{pre}G(\vec{v}, \epsilon)^*(TM))$, which is bounded with respect to the preferred Sobolev norm, and whose image under the exponential map is an honest solution to the $\bar{\delta}$ equation.*

□

The gluing map

$$G : U \times (0, \epsilon) \rightarrow \mathcal{N}^*(\sigma) \tag{C.1}$$

is defined by exponentiating this vector field at every curve obtained by pre-gluing. The proof that this map extends continuously upon letting the gluing parameter go to 0 is a direct consequence of Gromov compactness, and is omitted. It remains to explain why this map is a homeomorphism onto the intersection of a neighborhood of \vec{u} in $\mathcal{N}(\sigma)$ with $\mathcal{N}^*(\sigma)$.

To show this, we observe that as a consequence of Gromov compactness, there is a neighborhood W of \vec{u} in $\mathcal{N}^*(\sigma)$ consisting of curves w which admit a unique point $z_k(w)$ close to $z_k(\vec{u}, \epsilon)$ for some ϵ such that the t -coordinates of $z_k(w)$ and $z_k(\vec{u}, \epsilon)$ agree, and N_k intersects the image of w transversely at $z_k(w)$.

In particular, taking the s -coordinate of each point z_k defines a map

$$R : W \rightarrow \mathbb{R}^d.$$

As the right inverse to the $\bar{\delta}$ operator consists of vector fields which, at $z_k(\vec{v}, \epsilon)$ point in the direction of N_k and as N_k is totally geodesic with respect to the chosen metric, we conclude that $G(\vec{v}, \epsilon)$ intersects N_k at $z_k(\vec{v}, \epsilon)$. This implies

Lemma C.0.4. *The map R is a right inverse to G . Namely we have a commutative diagram*

$$\begin{array}{ccc} U \times (0, \epsilon) & \xrightarrow{G} & W \\ & \searrow & \swarrow R \\ & \mathbb{R}^d & \end{array}$$

where the map $U \times (0, \epsilon) \rightarrow \mathbb{R}^d$ is the codimension 0 embedding which takes (\vec{v}, ϵ) to the s -coordinates of $z_k(\vec{v}, \epsilon)$.

Corollary C.0.5. *The gluing map G is injective.*

□

The existence of this a priori right inverse to the gluing map also helps in proving surjectivity. Indeed, if the images of $w \in W$ and $G(\vec{v}, \epsilon)$ under R agree, then w can be expressed as the exponential of a vector field along $\text{pre}G(\vec{v}, \epsilon)$ lying in $C_N^\infty(\text{pre}G(\vec{v}, \epsilon)^*TM)$. By the uniqueness part of the implicit function theorem, w and $G(\vec{v}, \epsilon)$ must agree if we can prove that this vector field is bounded in the appropriate Sobolev norm. For w sufficiently close to \vec{u} , this follows from Gromov compactness, and a standard estimate on the energy of long strips (see e.g. Section 5d of [2]). This completes our sketch of the proof of Proposition 5.0.15.

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