Admissible subcategories of del Pezzo surfaces

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Abstract

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Admissible subcategories are building blocks of semiorthogonal decompositions. Many examples of them are known, but few general properties have been proved, even for admissible subcategories in the derived categories of coherent sheaves on basic varieties such as projective spaces. We use a relation between admissible subcategories and anticanonical divisors to study admissible subcategories of del Pezzo surfaces. We show that any admissible subcategory of the projective plane has a full exceptional collection, and since all exceptional objects and collections for the projective plane are known, this provides a classification result for admissible subcategories. We also show that del Pezzo surfaces of degree at least three do not contain so-called *phantom* subcategories. These are the first examples of varieties of dimension larger than one that have some nontrivial admissible subcategories, but provably do not contain phantoms.

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Chapter 1: Introduction

The derived category of coherent sheaves on an algebraic variety is a large and complicated invariant. It contains a lot of information about the variety, and many other invariants may be extracted out of the derived category. Working with this huge invariant directly is difficult, and thus an important notion in this field is the notion of a semiorthogonal decomposition. This is a particular way of decomposing the derived category into smaller pieces. Those pieces are called admissible subcategories.

We know many examples of semiorthogonal decompositions. For example, a full exceptional collection is nothing but a semiorthogonal decomposition of a category such that each component of the decomposition is equivalent to the derived category of vector spaces. The first example of such a decomposition was given in [Bei78] for projective spaces. Full exceptional collections are also known for Grassmannians, del Pezzo surfaces, and other varieties (see, for example, [Kuz14]). Later *mutations* were introduced in [Gor89; BK90], which are operations that transform a given semiorthogonal decomposition into other semiorthogonal decompositions. There are other tools of various complexity to produce new semiorthogonal decompositions, such as Orlov's blow-up formula [Orl93] or Kuznetsov's homological projective duality [Kuz07]. More methods and examples may be found in [Kuz14].

Despite a large number of examples of semiorthogonal decompositions, we do not have a good understanding of the structure of an arbitrary semiorthogonal decomposition. Some things are known, but mostly negative ones. For instance, counterexamples for the Jordan–Hölder property for semiorthogonal decompositions are given in [BBS14; Kuz13]. Another somewhat pathological behavior is the existence of so-called *phantom subcategories*, shown in [GO13; Böh+15], which are admissible subcategories which behave as zero subcategories on the level of K-theory. Among the positive constraints on the structure of admissible subcategories, perhaps the strongest one is proved in [KO15]: admissible subcategories are closed under small deformations of objects.

Nevertheless, it is expected that for sufficiently nice varieties, e.g., for projective spaces, none of the pathologies should occur. For example, it is conjectured in [KP16, Rem. 1.7] that there are no phantom subcategories in homogeneous spaces. It is surprisingly hard to check these expectations for any variety which is more complicated than $\mathbb{P}^1$. There are open questions about admissible subcategories which are not known even for $\mathbb{P}^2$.

Even the semiorthogonal decompositions of the simplest kind, the ones coming from full exceptional collections, are quite mysterious. In this case we can push a little bit further than just $\mathbb{P}^1$. The paper [GR87] by Gorodentsev–Rudakov about $\mathbb{P}^2$ and further work by Kuleshov–Orlov [KO94] show that any exceptional object on a del Pezzo surface fits into some full exceptional collection, and any full exceptional collection can be obtained from a standard one by a sequence of mutations. However, already for $\mathbb{P}^3$ things are more complicated, and not everything is known. See [Pol11] for some results.

In this paper we study arbitrary admissible subcategories of del Pezzo surfaces over an algebraically closed field of characteristic zero. We have two major results. The first one is about projective plane. On $\mathbb{P}^2$ we are able to produce a full classification of admissible subcategories. They all turn out to be of the special kind described above, i.e., they are generated by exceptional collections:

4.1.1. Theorem. *Any admissible subcategory in $D^b_{\mathrm{coh}}(\mathbb{P}^2)$ has a full exceptional collection.*

A category with a full exceptional collection is never a phantom subcategory. Thus we obtain a corollary:

4.1.2. Corollary. *There are no phantom subcategories in $D^b_{\mathrm{coh}}(\mathbb{P}^2)$.*

To the best of author's knowledge, this is the first example of a variety of dimension larger than one which admits some nontrivial semiorthogonal decompositions, but provably does not contain any phantom subcategories. In Section 5 we produce more examples in

Corollary 5.3.5, where we show that blow-ups of distinct points on surfaces with globally generated canonical bundles also do not contain any phantom subcategories.

For more complicated del Pezzo surfaces we do not have a classification statement. However, our methods are sufficient to show the non-existence of phantoms in del Pezzo surfaces of degree at least 3, which is our second main result:

6.4.6. Theorem. *Let Y be a del Pezzo surface of degree at least 3. Then there are no phantom subcategories in $D^b_{\mathrm{coh}}(Y)$.*

Remark. The reason for the degree restriction is that in the argument we need to construct many convenient smooth anticanonical divisors. In an ongoing project I apply the results from the paper [BKl06] to perform a relatively similar argument relying purely on irreducibility instead of smoothness, and this leads to the non-existence of phantoms in arbitrary del Pezzo surfaces. This is work in progress. Unfortunately, it appeared too late to be written up as a part of this thesis, but the methods build directly upon the arguments in Chapter 6.

The main technical tool that allows us to prove these results is a relation between admissible subcategories and autoequivalences of derived categories of various anticanonical divisors. It is a generalization to arbitrary admissible subcategories of an observation that a restriction of an exceptional object to an anticanonical divisor is a so-called spherical object, and thus defines a certain autoequivalence of the derived category of sheaves on that divisor. This relation was discovered by Addington [Add16, Prop. 2.1] in 2011, but it seems that the consequences of this relation for the study of semiorthogonal decompositions have not yet been fully explored. Addington's result gives exceptionally strong structural constraint on the semiorthogonal decomposition of surfaces, which we discuss in Proposition 3.1.3.

The thesis is structured as follows. In Chapter 2 we state and prove miscellaneous lemmas about derived categories of coherent sheaves and admissible subcategories. Chapter 3 is the technical core of the paper, describing the constraints that anticanonical divisors put on admissible subcategories. In Chapter 4 we prove the classification of admissible subcategories in the derived category of the projective plane $\mathbb{P}^2$. Chapter 5 contains a local classification of admissible subcategories which are supported on a single (-1)-curve in a surface, and an application for phantom subcategories in some blow-ups. The classification result is used in Chapter 6, where we prove that there are no phantom subcategories in del Pezzo surfaces of degree at least three.

Chapter 2: Preliminaries

2.1 Conventions and notation

We work over an algebraically closed field $\Bbbk$ of characteristic zero. All varieties and triangulated categories in this paper are assumed to be over $\Bbbk$. All functors are assumed to be derived, and a subcategory of a triangulated category is assumed to be a triangulated subcategory.

For an algebraic variety X we denote by $D^b_{\mathrm{coh}}(X)$ the bounded derived category of coherent sheaves on X. We denote by $\mathrm{Perf}(X)$ the triangulated category of perfect complexes on X. When X is smooth, these two categories coincide.

For an algebraic variety X and an object $F \in D^b_{\mathrm{coh}}(X)$ we denote by $\mathcal{H}^i(F)$ the i'th cohomology sheaf of F. We also use the *canonical truncation* $\tau_{\leq i}F$ which has the same cohomology sheaves as F in degrees $\leq i$ and zero cohomology sheaves in degrees strictly greater than i. If an object $F \in D^b_{\mathrm{coh}}(X)$ is represented as a complex of sheaves, $\tau_{\leq i}F$ can be represented as a subcomplex. We define $\tau_{>i}F$ similarly.

2.2 Exceptional objects and semiorthogonal decompositions

In this subsection we fix the notation and cite several standard results about triangulated categories, exceptional objects, and semiorthogonal decompositions. These notations, definitions, and results are used throughout the paper. For a more detailed introduction, see, for example, [BK90].

Until the end of this subsection, we work with an idempotent-complete triangulated category T.

For any two objects $A, B \in T$ we denote by $\mathrm{RHom}(A, B)$ the graded vector space

$\oplus_{i\in\mathbb{Z}}\mathrm{Hom}_T(A, B[i])$. The graded components are referred as $\mathrm{R}^i\mathrm{Hom}(A, B)$ or $\mathrm{Ext}^i(A, B)$. Similarly, the symbol $\mathrm{REnd}(A)$ denotes $\mathrm{RHom}(A, A)$ and its graded components are referred to as $\mathrm{End}^i(A)$. Given any graded vector space $V^\bullet = \oplus_{i\in\mathbb{Z}}V^i$ and an object $F \in T$, the tensor product $V^\bullet \otimes F$ is an object of T defined to be the direct sum of shifts $\bigoplus_{i\in\mathbb{Z}} F^{\oplus \dim V^i}[-i]$.

For an arbitrary object $F \in T$ we denote by $\langle F\rangle$ the smallest strictly full triangulated subcategory which contains F and is closed under taking direct summands. We say that an object F is a *classical generator* of T if $\langle F\rangle = T$. For any quasi-compact and quasi-separated scheme the category of perfect complexes has a classical generator [BB03, Cor. 3.1.2].

2.2.1. Definition. An object $E \in T$ is called *exceptional* if $\mathrm{REnd}(E) \cong \Bbbk[0]$. A sequence of exceptional objects $E_1, \dots, E_n$ is called an *exceptional collection* if $\mathrm{RHom}(E_j, E_i) = 0$ for any $j > i$. An exceptional collection is *full* if the smallest strictly full triangulated subcategory containing every E_i is all of T.

2.2.2. Definition. For a full subcategory $\mathcal{A} \subset T$ we define the left and right *orthogonal subcategories*:

$$^\perp\mathcal{A} := \{F \in T \mid \forall t \in \mathcal{A}\ \ \mathrm{RHom}(F, t) = 0\},$$
$$\mathcal{A}^\perp := \{F \in T \mid \forall t \in \mathcal{A}\ \ \mathrm{RHom}(t, F) = 0\}.$$

2.2.3. Lemma ([BB03]). *If $G \in \mathcal{A}$ is a classical generator, then $F \in {}^\perp\mathcal{A}$ if and only if $\mathrm{RHom}(F, G) = 0$, and similarly for $\mathcal{A}^\perp$.*

2.2.4. Definition. A *semiorthogonal decomposition* of a triangulated category T is a sequence of strictly full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of T such that $\mathcal{A}_i \subset \mathcal{A}_j^\perp$ for any $i < j$ and the smallest strictly full triangulated subcategory containing every $\mathcal{A}_i$ is T. We denote this using angle brackets, i.e., by writing $T = \langle \mathcal{A}_1, \dots, \mathcal{A}_n\rangle$.

The key property of semiorthogonal decompositions is that any object of T has a filtration whose associated graded components belong to the component subcategories $\mathcal{A}_i \subset T$. In this paper we work mostly with semiorthogonal decompositions into two components, so to avoid introducing complicated notation, we only state this result for semiorthogonal decompositions like that.

2.2.5. Definition ([BK90]). Let $T = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. For any object $F \in T$ there exists a unique *projection triangle* in T:

$$R_{\mathcal{B}}(F) \to F \to L_{\mathcal{A}}(F) \to R_{\mathcal{B}}(F)[1] \tag{2.2.5.1}$$

such that the object $R_{\mathcal{B}}(F)$ lies in $\mathcal{B}$ and $L_{\mathcal{A}}(F)$ lies in $\mathcal{A}$. Moreover, the projection triangle is functorial in F, thus we obtain two functors: the *right projection functor* $R_{\mathcal{B}} \colon T \to \mathcal{B}$ which is a right adjoint functor to the inclusion $\mathcal{B} \hookrightarrow T$, and the *left projection functor* $L_{\mathcal{A}} \colon T \to \mathcal{A}$ which is a left adjoint functor to the inclusion $\mathcal{A} \hookrightarrow T$.

2.2.6. Corollary. *Let $T = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Let $F \in T$ be any object. The composition with the projection map $R_{\mathcal{B}}(F) \to F$ from Definition 2.2.5 induces an isomorphism* $\mathrm{REnd}(R_{\mathcal{B}}(F)) \xrightarrow{\sim} \mathrm{RHom}(R_{\mathcal{B}}(F), F)$.

Proof. This follows from the fact that the functor $R_{\mathcal{B}}$ is an adjoint functor to the inclusion functor $\mathcal{B} \hookrightarrow T$. Alternatively, we can deduce the statement from semiorthogonality: an application of the functor $\mathrm{RHom}(R_{\mathcal{B}}(F), -)$ to the triangle (2.2.5.1) results in the triangle

$$\mathrm{REnd}(R_{\mathcal{B}}(F)) \to \mathrm{RHom}(R_{\mathcal{B}}(F), F) \to \mathrm{RHom}(R_{\mathcal{B}}(F), L_{\mathcal{A}}(F))$$

in the derived category of vector spaces. Since $\mathcal{A}$ is semiorthogonal to $\mathcal{B}$, the graded vector space $\mathrm{RHom}(R_{\mathcal{B}}(F), L_{\mathcal{A}}(F))$ vanishes. Therefore the first arrow is an isomorphism. $\square$

Exceptional collections may be used to construct many examples of semiorthogonal decompositions. A common abuse of notation in this context is to write an exceptional object E as a component in the semiorthogonal decomposition, having in mind the triangulated subcategory $\langle E \rangle \subset T$ generated by that object.

2.2.7. LEMMA ([BK90]). *Let* $\langle E_1, \dots, E_n\rangle$ *be an exceptional collection in* T. *Suppose that for any two objects* $F, G \in T$ *the graded vector space* $\mathrm{RHom}_T(F, G)$ *has finite total dimension.*

- *Let* $\mathcal{A}$ *be the right orthogonal subcategory* $\langle E_1, \dots, E_n\rangle^\perp$. *Then the sequence*

  $$\langle \mathcal{A}, E_1, \dots, E_n\rangle$$

  is a semiorthogonal decomposition of T. *If the exceptional collection consists of one object* $E \in T$, *then the projection functor* R_E *is given by* $F \mapsto E \otimes \mathrm{RHom}_T(E, F)$ *and the projection triangle for* $T = \langle \mathcal{A}, E\rangle$ *is a cone of the evaluation morphism*

  $$E \otimes \mathrm{RHom}_T(E, F) \xrightarrow{\mathrm{ev}} F \to L_{E^\perp}(F).$$

- *Let* $\mathcal{A}$ *be the left orthogonal subcategory* ${}^\perp\langle E_1, \dots, E_n\rangle$. *Then the sequence*

  $$\langle E_1, \dots, E_n, \mathcal{A}\rangle$$

  is a semiorthogonal decomposition of T. *If the exceptional collection consists of one object* $E \in T$, *then the projection functor* L_E *is given by* $F \mapsto \mathrm{RHom}_T(F, E)^\vee \otimes E$ *and the projection triangle for* $T = \langle E, \mathcal{A}\rangle$ *is a fiber of the coevaluation morphism*

  $$R_{{}^\perp E}(F) \to F \xrightarrow{\mathrm{coev}} \mathrm{RHom}_T(F, E)^\vee \otimes E.$$

Remark. The projection triangles for longer exceptional collections may also be written explicitly, in terms of *dual exceptional collections*, as in [Kap88]. We omit this since this is not necessary for our paper.

2.3 Derived categories of coherent sheaves

We continue with several miscellaneous lemmas, mostly related to homological algebra. We will use the following definitions throughout the paper.

2.3.1. Definition. Let X be an algebraic variety, and let $E \in D^b_{\mathrm{coh}}(X)$ be an object. The (set-theoretic) *support* $\mathrm{supp}(E)$ of the object E is the union $\cup_{i \in \mathbb{Z}} \mathrm{supp}(\mathcal{H}^i(E))$ of supports of cohomology sheaves.

2.3.2. Lemma ([Huy06, Ex. 3.30]). *Let X be an algebraic variety, and let $E \in D^b_{\mathrm{coh}}(X)$ be an object. Then a point $p \in X$ lies in $\mathrm{supp}(E)$ if and only if $\mathrm{RHom}(E, \mathcal{O}_p) \neq 0$, where $\mathcal{O}_p$ is the skyscraper sheaf at the point p.*

2.3.3. Lemma ([Huy06, Lem. 3.9]). *Let X be an algebraic variety, and let $E \in D^b_{\mathrm{coh}}(X)$ be an object. Suppose that $\mathrm{supp}(E)$ is a disjoint union $Z_1 \sqcup Z_2$ of two closed subsets of X. Then there exists a unique decomposition $E \simeq E_1 \oplus E_2$ into a direct sum such that $\mathrm{supp}(E_1) = Z_1$ and $\mathrm{supp}(E_2) = Z_2$.*

2.3.4. Definition. Let X be an algebraic variety. An object $E \in D^b_{\mathrm{coh}}(X)$ is called *locally free* if all cohomology sheaves of E are locally free. Similarly, it is called a *torsion object* if all cohomology sheaves are torsion sheaves.

There are multiple ways to define locally free objects in a derived category. In the following lemma we show some equivalent characterizations. The lemma is well-known, but we include the proof due to the lack of a convenient reference.

2.3.5. Definition. The *length* of a graded vector space $V^\bullet$ is the number $l(V^\bullet) := \sum_i \dim V^i$. The length of a complex of vector spaces is the length of its cohomology viewed as a graded vector space.

2.3.6. Lemma. *Let X be a smooth algebraic variety, and let $E \in D^b_{\mathrm{coh}}(X)$ be an object. The following are equivalent:*

1. *E is a locally free object.*

2. *For any point $x \in X$ there exists a Zariski-neighborhood $U \subset X$ containing x such that the restriction $E|_U$ is isomorphic to $\mathcal{O}_U \otimes V^\bullet$ for some graded vector space $V^\bullet$.*

3. *The length of the derived fiber of E at each point $x \in X$ is the same.*

Proof. It is clear that the condition (2) implies both (1) and (3). It is enough to show that (1) implies (2), and that (3) implies (1).

(1) $\implies$ (2): Let $U \subset X$ be an affine open neighborhood of $x \in X$ such that each cohomology sheaf of E becomes trivial. Such a neighborhood exists since E has only finitely many nonzero cohomology sheaves. Then each cohomology sheaf of $E|_U$ is a direct sum of several copies of the structure sheaf $\mathcal{O}_U$. Since U is affine, there are no higher Ext's between copies of the structure sheaf, and hence the complex $E|_U$ is formal, i.e., quasiisomorphic to a direct sum of its cohomology sheaves.

(3) $\implies$ (1): For each point $x \in X$ denote by $\iota_x \colon \operatorname{Spec} \Bbbk \hookrightarrow X$ the inclusion morphism. For any $k \in \mathbb{Z}$ the dimension of the k'th derived pullback functor $L_k\iota_x^*(E)$ is an upper semicontinuous function, so the total length of the object $\iota_x^*(E)$ is constant if and only if the dimension of each $L_k\iota_x^*(E)$ is constant as a function of $x \in X$. Assume that some cohomology sheaf $\mathcal{H}^i(E)$ is not locally free. Without loss of generality we may assume that each cohomology sheaf $\mathcal{H}^j(E)$ with $j > i$ is locally free. Consider the spectral sequence for the derived pullback ι_x^* [Huy06, (3.10)]:

$$E_2^{p,q} = L_{-q}\iota_x^*(\mathcal{H}^p(E)),\ d_r^{p,q} \colon E_r^{p,q} \to E_r^{p-r+1,q+r} \quad \Rightarrow \quad \mathcal{H}^{p+q}(\iota_x^* E).$$

For any $j > i$ by assumption we know that $L_q\iota_x^*(\mathcal{H}^j(E)) = 0$ for $q > 0$. This implies that the cell $E_2^{i,0} = L_0\iota_x^*(\mathcal{H}^i(E))$ survives to E_∞. In particular, $L_i\iota_x^*(E) \simeq L_0\iota_x^*(\mathcal{H}^i(E))$ for any point $x \in X$. Since $\mathcal{H}^i(E)$ is not locally free, its (nonderived) rank $L_0\iota_x^*(\mathcal{H}^i(E))$ is not a constant function, but then the dimension of $L_i\iota_x^*(E)$ is also not constant, a contradiction. □

2.3.7. Lemma. *Let X be an algebraic variety, and let $F \in D^b_{\mathrm{coh}}(X)$ be an object concentrated in nonpositive cohomology degrees. Then for any coherent sheaf $\mathcal{F}$ on X there is a canonical* *isomorphism*

$$\mathrm{R}^0\mathrm{Hom}(\mathcal{H}^0(F), \mathcal{F}) \xrightarrow{\sim} \mathrm{R}^0\mathrm{Hom}(F, \mathcal{F}).$$

Proof. Let $\tau_{\leq -1}F$ denote the canonical truncation of the complex F. There exists a truncation triangle

$$\tau_{\leq -1}F \to F \to \mathcal{H}^0(F) \to (\tau_{\leq -1}F)[1].$$

The application of the cohomological functor $\mathrm{R}^0\mathrm{Hom}(-, \mathcal{F})$ together with the fact that there are no negative Ext's between coherent sheaves finishes the proof of the lemma. $\square$

2.3.8. Lemma. *Let Y be a variety, and let $j\colon D \hookrightarrow Y$ be an embedding of a Cartier divisor. Let $F \in \mathrm{Perf}(Y)$ be an object. Then for every $i \in \mathbb{Z}$:*

1. *there exists a short exact sequence*

$$0 \to L_0 j^* \mathcal{H}^i(F) \to \mathcal{H}^i(j^* F) \to L_1 j^* \mathcal{H}^{i+1}(F) \to 0.$$

2. $\mathrm{supp}(\mathcal{H}^i(F)) \cap D \subset \mathrm{supp}\, \mathcal{H}^i(j^* F)$*;*
3. *If $\mathcal{H}^i(j^*F) = 0$, then the support of $\mathcal{H}^i(F)$ does not intersect D.*

Proof. Consider the spectral sequence converging to the cohomology sheaves of the derived pullback j^*F:

$$E_2^{p,q} = L_{-q} j^* \mathcal{H}^p(F),\ d_r^{p,q}\colon E_r^{p,q} \to E_r^{p-r+1,q+r} \quad \Longrightarrow \quad \mathcal{H}^{p+q}(j^*F).$$

Since $j\colon D \hookrightarrow Y$ is an inclusion of a Cartier divisor, the E_2-page of that spectral sequence has only two rows, and therefore it degenerates at the second page by dimension reasons, producing a collection of short exact sequences as in the statement. The other two claims in the statement easily follow from this observation. $\square$

The derived categories of coherent sheaves on curves and surfaces have some special convenient properties, which makes them easier to deal with than the derived categories for higher-dimensional varieties. We recall some of the properties in the following several well-known lemmas, and include the sketches of proofs for completeness.

2.3.9. LEMMA. *Let C be a smooth curve, and let $W \in D^b_{\mathrm{coh}}(C)$ be an object.*

1. *There is a decomposition $W \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(W)[-i]$ into a direct sum of shifts of cohomology sheaves.*

2. *There is a direct sum decomposition $W \simeq T \oplus V$ where T is a torsion object and V is a locally free object.*

Proof. The first claim follows from the fact that the category of coherent sheaves on a smooth curve has homological dimension one, see, e.g., [Huy06, Cor. 3.15]. Consequently, it is enough to prove the second claim for coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on C. Denote by $\mathcal{T} \subset \mathcal{F}$ the torsion subsheaf. Then there is a short exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{F}/\mathcal{T} \to 0.$$

The quotient sheaf $\mathcal{F}/\mathcal{T}$ is torsion-free on a smooth curve, so it is locally free. Then the space $\mathrm{Ext}^1(\mathcal{F}/\mathcal{T}, \mathcal{T})$ vanishes and the extension splits. □

2.3.10. LEMMA. *Let C be a curve, and let W be a coherent sheaf on C supported at a smooth point $p \in C$.*

1. *There is a direct sum decomposition $W \simeq \bigoplus_k \left(\mathcal{O}_C/\mathfrak{m}^k\right)^{\oplus w_k}$, where $\mathfrak{m}$ is the ideal sheaf of the point p and $\{w_k\}$ is some set of multiplicities.*

2. *If $W \simeq \mathcal{O}_C/\mathfrak{m}^n$ and $W' \simeq \mathcal{O}_C/\mathfrak{m}^m$ are two indecomposable torsion coherent sheaves on C supported at point p, then $\dim \mathrm{Hom}_C(W, W') = \dim \mathrm{Ext}^1_C(W, W') = \min(m, n)$.*

Proof. A local ring of C at a smooth point p is a discrete valuation ring [Eis95, Prop. 11.1]. In particular it is a principal ideal domain. The classification of finitely generated modules over a PID establishes the first claim. If $f \in \mathfrak{m}$ is a generator, then the sheaf $\mathcal{O}_C/\mathfrak{m}^n$ has a two-term locally free resolution

$$0 \to \mathcal{O}_C \xrightarrow{f^n} \mathcal{O}_C \to \mathcal{O}_C/\mathfrak{m}^n \to 0,$$

which lets us compute Hom and Ext for the second part of the statement. □

2.3.11. Lemma. *Let S be a smooth surface. A choice of an object $M \in D^b_{\mathrm{coh}}(S)$ up to an isomorphism is the same as a choice of the following two pieces of information:*

1. *a collection of cohomology sheaves $F^i := \mathcal{H}^i(M)$;*
2. *a collection of glueing maps $\xi_i \in \mathrm{Ext}^2(F^i, F^{i-1})$.*

Remark. In general, the glueing data for an object in the derived category also includes additional information related to higher Ext's, and it is not easy to describe explicitly. On a smooth surface all Ext's of degree larger than two between coherent sheaves vanish, and this gives us a simpler description.

Proof. If M is concentrated in a single cohomological degree, this is clear. Assume that the claim is proved for complexes concentrated in at most n degrees, and let M be an object concentrated in exactly $n+1$ different cohomological degrees. Let i be the largest integer such that $\mathcal{H}^i(M) \neq 0$. Consider the truncation triangle

$$\tau_{\leq i-1} M \to M \to \mathcal{H}^i(M)[-i] \to (\tau_{\leq i-1} M)[1].$$

The object M is determined up to an isomorphism by its truncation $\tau_{\leq i-1} M$ and the glueing map $\xi \in \mathrm{Ext}^1(\mathcal{H}^i(M)[-i], \tau_{\leq i-1} M)$. By induction the lemma holds for the truncation. Thus it remains to show that $\mathrm{Ext}^1(\mathcal{H}^i(M)[-i], \tau_{\leq i-1} M) \simeq \mathrm{Ext}^2(\mathcal{H}^i(M), \mathcal{H}^{i-1}(M))$.

Consider the truncation triangle for $\tau_{\leq i-1} M$:

$$\tau_{\leq i-2} M \to \tau_{\leq i-1} M \to \mathcal{H}^{i-1}(M)[-i+1] \to (\tau_{\leq i-2} M)[1].$$

An application of the functor $\mathrm{Ext}^\bullet(\mathcal{H}^i(M)[-i], -)$ leads to a long exact sequence of vector spaces. Since the homological dimension of a smooth surface is two, both vector spaces $\mathrm{Ext}^1(\mathcal{H}^i(M)[-i], \tau_{\leq i-2}M)$ and $\mathrm{Ext}^1(\mathcal{H}^i(M)[-i], (\tau_{\leq i-2}M)[1])$ vanish by dimension reasons. Thus the induction step is established. $\square$

2.3.12. Lemma. *Let S be a smooth surface, and let $\mathcal{F}$ be a torsion-free coherent sheaf on S. Then there exists a unique up to a unique isomorphism short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{E}$ is locally free, and $\mathcal{Q}$ is a torsion sheaf supported on a zero-dimensional subset.

Proof. Any morphism from $\mathcal{F}$ to a locally free sheaf factors through the double dual coherent sheaf $\mathcal{F}^{\vee\vee}$. On a smooth surface the double dual is locally free [OSS11, Lem. 2.1.1.10]. The morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism on an open set where $\mathcal{F}$ is locally free. The complement to that open set has codimension two [OSS11, Lem. 2.1.1.8], so the quotient is a zero-dimensional torsion sheaf. Uniqueness follows from the universal property of the double dual. $\square$

2.3.13. Lemma. *Let S be a smooth surface, and let $\mathcal{F}$ be a torsion-free coherent sheaf on S. For any divisor $j\colon D \hookrightarrow S$ the derived restriction $j^*\mathcal{F} \in D^b_{\mathrm{coh}}(D)$ is concentrated only in degree 0.*

Proof. The object $j_*j^*\mathcal{F} \in D^b_{\mathrm{coh}}(S)$ can be represented as a cone of a morphism

$$\mathcal{F} \otimes \mathcal{O}(-D) \to \mathcal{F}.$$

Since $\mathcal{F}$ is torsion-free, this map is injective, and hence $j_*j^*\mathcal{F}$ is concentrated in degree zero. Since the pushforward j_* is an exact functor, this implies that $j_*(L_1 j^*\mathcal{F}) = 0$. A pushforward of a nonzero coherent sheaf along the closed embedding is nonzero, so in fact $L_1 j^*\mathcal{F} = 0$, as claimed. $\square$

2.4 Spectral sequences for Ext-groups

In the rest of the paper we often compute Ext's between objects in the derived category. In this subsection we describe two useful spectral sequences which aid these computations. The first one is a spectral sequence which computes the self-Ext's of an object in the derived category in terms of the Ext's between its cohomology sheaves. It is a special case of the spectral sequence for Ext's between two objects admitting lifts to a filtered derived category, constructed in [BBD82, (3.1.3)]. We work with the usual derived category, however any object has a canonical filtration whose associated graded factors are quasiisomorphic to the cohomology sheaves. We describe the resulting spectral sequence in this case explicitly for convenience.

2.4.1. Lemma. *Let X be a smooth algebraic variety, and let $F \in D^b_{\mathrm{coh}}(X)$ be an arbitrary object. There exists a E_1-spectral sequence with*

$$E_1^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^{2p+q}(\mathcal{H}^i(F), \mathcal{H}^{i-p}(F)) \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

which converges to $\mathrm{Ext}^{p+q}(F,F)$. The d_1 differential is given by pre- and post-compositions with glueing maps $\xi_{i+1} \in \mathrm{Ext}^2(\mathcal{H}^{i+1}(F), \mathcal{H}^i(F))$ and $\xi_{i-p} \in \mathrm{Ext}^2(\mathcal{H}^{i-p}(F), \mathcal{H}^{i-p-1}(F))$.

Proof. Since F is a bounded complex and smooth varieties have finite homological dimension, it is possible to find an injective resolution for F which is a bounded complex equipped with a decreasing filtration whose associated graded factors are injective resolutions for the cohomology sheaves $\mathcal{H}^i(F)$ such that the filtration in each degree is split. The resolution with this filtration represents an object in the filtered derived category. The spectral sequence in [BBD82, (3.1.3.4)] computing $\mathrm{Ext}(F,F)$ in the usual derived category is the spectral sequence claimed in the statement. $\square$

2.4.2. Corollary. *Let S be a smooth surface, and let $F \in D^b_{\mathrm{coh}}(S)$ be an object in the derived* *category. Then*

$$\dim \operatorname{Ext}^1(F,F) \geq \sum_{i\in\mathbb{Z}} \dim \operatorname{Ext}^1(\mathcal{H}^i(F), \mathcal{H}^i(F)).$$

Proof. Consider the spectral sequence from Lemma 2.4.1. Note that

$$E_1^{0,1} = \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^1(\mathcal{H}^i(F), \mathcal{H}^i(F)).$$

On a smooth variety of dimension n the spectral sequence degenerates at E_n for dimension reasons, thus on a surface the only nonzero differential is d_1. Consider the cell $E_1^{0,1}$ and the d_1-differentials starting and ending on that cell:

$$\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^{-1}(\mathcal{H}^i(F), \mathcal{H}^{i+1}(F)) \xrightarrow{d_1} \boxed{\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^1(\mathcal{H}^i(F), \mathcal{H}^i(F))} \xrightarrow{d_1} \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^3(\mathcal{H}^i(F), \mathcal{H}^{i-1}(F)).$$

On a smooth surface both Ext^{-1} and Ext^3 between coherent sheaves are always zero, thus the vector space in $E_1^{1,0}$ survives to E_∞ and is a subquotient of $\operatorname{Ext}^1(F,F)$. This implies the inequality for dimensions of those vector spaces. $\square$

Another useful spectral sequence is the following one. It lets us compute Ext's between cones of maps in $D^b_{\mathrm{coh}}(X)$. An important class of cones to keep in mind is the ones coming from short exact sequences of coherent sheaves on X.

2.4.3. Lemma. *Let X be a smooth algebraic variety. Suppose that there are two distinguished triangles in $D^b_{\mathrm{coh}}(X)$:*

$$A_1 \to B_1 \to C_1 \to A_1[1] \qquad A_2 \to B_2 \to C_2 \to A_2[1].$$

There exists a E_1-spectral sequence which degenerates at E_3 and converges to $\operatorname{Ext}^*(C_1, C_2)$*:*

$$E_1^{p,q} = \begin{cases} \operatorname{Ext}^q(B_1, A_2), & p = -1; \\ \operatorname{Ext}^q(A_1, A_2) \oplus \operatorname{Ext}^q(B_1, B_2), & p = 0; \\ \operatorname{Ext}^q(A_1, B_2), & p = 1; \\ 0, & \textit{otherwise.} \end{cases}$$

with differential $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. The differential d_1 is given by compositions with the morphisms $A_1 \to B_1$ and $A_2 \to B_2$.

The key observation is that both C_1 and C_2 lift to objects in the filtered derived category. There are various notions of a filtration on an object in the triangulated category $D^b_{\mathrm{coh}}(X)$, and most of them do not allow lifting the object to the filtered category, but the two-step filtrations arising from the distinguished triangles are always sufficient.

Proof. Choose injective resolutions for A_1 and B_1. Then the morphism $A_1 \to B_1$ in the derived category may be represented as an actual map of complexes. The cone of this map of complexes is a complex representing the object $C_1 \in D^b_{\mathrm{coh}}(X)$. This cone is equipped with a filtration whose associated graded components are quasiisomorphic to A_1 and B_1 respectively. A similar procedure applied to C_2 lets us conclude by invoking [BBD82, (3.1.3.4)] again. □

We may use the spectral sequences from Lemmas 2.4.1 and 2.4.3 to obtain the following property of objects on smooth surfaces.

2.4.4. Lemma. *Let S be a smooth surface, and let $p \in S$ be a point. Assume that $F \in D^b_{\mathrm{coh}}(S)$ is an object which is locally free away from p, but not locally free at p. Then* $\dim \operatorname{Ext}^1(F, F) \geq 2$.

Remark. For some surfaces such as $\mathbb{P}^2$ there is a geometric argument for this inequality. Consider a two-dimensional family of automorphisms of $\mathbb{P}^2$ which moves the point p around. The pullbacks of F with respect to that family form a deformation of F over a two-dimensional base. It may be checked that, in characteristic zero, the first-order deformation along any direction of the two-dimensional base is nontrivial, and therefore $\dim \operatorname{Ext}^1(F, F) \geq 2$.

Proof. If F is not locally free at p, by definition this means that at least one cohomology sheaf of F is not locally free at p. By Corollary 2.4.2 it is enough to prove the inequality for the dimension of self-Ext^1 of that cohomology sheaf. So suppose that $\mathcal{F}$ is a coherent sheaf which is not locally free at p, but locally free on the complement $S \setminus \{p\}$. The inequality for coherent sheaves is related to the inequalities in [Muk87, Cor. 2.11 and 2.12], but we include a direct proof for completeness. We consider several cases to prove the inequality.

Suppose first that $\mathcal{F}$ is a torsion sheaf supported at p. Then the Euler characteristic $\chi(\mathcal{F}, \mathcal{F})$ is zero since it stays constant in flat families and the sheaf $\mathcal{F}$ may be deformed by moving the point p in a flat family. Since we are on a smooth surface we may use Serre duality to find the following expression for Euler characteristic. Note that the canonical bundle is trivial in a neighborhood of the point p, so:

$$\chi(\mathcal{F}, \mathcal{F}) = 2 \cdot \dim \operatorname{Hom}(\mathcal{F}, \mathcal{F}) - \dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}).$$

The sheaf $\mathcal{F}$ is nonzero, so $\dim \operatorname{Hom}(\mathcal{F}, \mathcal{F}) \geq 1$. Therefore $\dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \geq 2$.

Suppose now that $\mathcal{F}$ is a torsion-free sheaf which is not locally free at p. Then by Lemma 2.3.12 there exists a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{E}$ is locally free and $\mathcal{Q}$ is a nonzero torsion sheaf supported at the point p. Consider the spectral sequence from Lemma 2.4.3 which computes $\operatorname{Ext}^*(\mathcal{F}, \mathcal{F})$ in terms of that short exact sequence. The E_1 page contains the following fragment:

$$\operatorname{Ext}^1(\mathcal{Q}, \mathcal{E}) \xrightarrow{d_1} \operatorname{Ext}^1(\mathcal{Q}, \mathcal{Q}) \oplus \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \xrightarrow{d_1} \operatorname{Ext}^1(\mathcal{E}, \mathcal{Q}).$$

Since $\mathcal{E}$ is locally free and $\mathcal{Q}$ is supported on a zero-dimensional set, it is easy to see that

$\mathrm{Ext}^\bullet(\mathcal{E},\mathcal{Q})$ is concentrated only in degree zero and $\mathrm{Ext}^\bullet(\mathcal{Q},\mathcal{E})$ is concentrated only in degree two. Thus the vector space in $E_1^{0,1}$-cell, which contains $\mathrm{Ext}^1(\mathcal{Q},\mathcal{Q})$ as a subspace, survives to E_2. By dimension reasons there are no nonzero differentials on the E_2 page which start or end at $E_2^{0,1}$. Hence

$$\dim \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \geq \dim \mathrm{Ext}^1(\mathcal{Q},\mathcal{Q}).$$

From the previous case we considered we know that the right hand side is at least two, which confirms the claim for torsion-free sheaves.

It remains to consider the case where $\mathcal{F}$ has a nonzero torsion subsheaf with a nonzero torsion-free quotient:

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{G} \to 0.$$

Consider again the spectral sequence from Lemma 2.4.3 which computes $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$. The E_1-page contains the following fragment:

$$\mathrm{Hom}(\mathcal{T},\mathcal{G}) \xrightarrow{d_1} \mathrm{Ext}^1(\mathcal{T},\mathcal{T}) \oplus \mathrm{Ext}^1(\mathcal{G},\mathcal{G}) \xrightarrow{d_1} \mathrm{Ext}^2(\mathcal{G},\mathcal{T}).$$

Since $\mathcal{G}$ is torsion-free, $\mathrm{Hom}(\mathcal{T},\mathcal{G}) = 0$. Using the embedding from Lemma 2.3.12 it is easy to see that $\mathrm{Ext}^2(\mathcal{G},\mathcal{T})$ is also zero. Thus the $E_1^{0,1}$-cell survives to E_2, and similarly to the previous case purely by dimension reasons it survives to E_∞. Therefore the lemma is proved for all coherent sheaves, and hence for all objects in the derived category as well. □

2.5 Admissible subcategories and their properties

We begin with several general observations about admissible subcategories and also consider their consequences for admissible subcategories of projective spaces, especially $\mathbb{P}^2$.

2.5.1. Definition. Let X be an algebraic variety. A strictly full triangulated subcategory $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ is an *admissible subcategory* if the inclusion functor admits both left and right adjoint functors. We denote the left adjoint by $L_\mathcal{A}$ and the right adjoint by $R_\mathcal{A}$.

For smooth and proper varieties an admissible subcategory is essentially the same thing as a semiorthogonal decomposition with two components, and the choice of notation for adjoints is compatible with Definition 2.2.5. More precisely, we have the following statement.

2.5.2. Lemma ([BK90]). *Let X be a smooth and proper algebraic variety. If $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then both $\mathcal{A}$ and $\mathcal{B}$ are admissible subcategories of $D^b_{\mathrm{coh}}(X)$. Conversely, if $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ is an admissible subcategory, then both $\langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ and $\langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$ are semiorthogonal decompositions of $D^b_{\mathrm{coh}}(X)$.*

The main property of admissible subcategories in the geometric situation is the fact that they are closed under small deformations of objects in the following sense:

2.5.3. Proposition ([KO15, Cor. 3.12]). *Let X be a smooth proper algebraic variety. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. For any smooth variety Y with a chosen point $y \in Y$, and any object $R \in D^b_{\mathrm{coh}}(X \times Y)$ such that the derived restriction $R|_{X \times \{y\}} \in D^b_{\mathrm{coh}}(X)$ is in $\mathcal{A}$, there exists a Zariski open neighborhood $U \subset Y$ of the point y such that $R|_{X \times \{u\}} \in \mathcal{A}$ for any $u \in U$. Moreover, $\mathcal{A}$ is invariant under the action of the connected automorphism group $\mathrm{Aut}^o(X)$.*

In general, it is very difficult to control even the basic behavior of admissible subcategories. For example, the following question is still open:

2.5.4. Conjecture ([Kuz09]). *Let X be a smooth projective variety. If*

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset D^b_{\mathrm{coh}}(X)$$

is an infinite increasing chain of admissible subcategories, then it stabilizes at some finite step.

Several results of this paper are related to phantom subcategories:

2.5.5. Definition. Let X be a smooth and proper variety, and let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. It is called a *phantom subcategory* if $K_0(\mathcal{A}) = 0$ and $\mathcal{A} \neq 0$.

It is not easy to construct examples of phantom subcategories. It is expected that they do not exist for nice varieties, such as homogeneous spaces [KP16, Rem. 1.7] or varieties admitting a full exceptional collection [Kuz14, Conj. 1.10]. In this paper we confirm this expectation for del Pezzo surfaces of degree at least three.

The invariance of admissible subcategories under the connected automorphism group, shown in Proposition 2.5.3, has several important implications.

2.5.6. LEMMA. *Let X be a smooth proper variety. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory, and let $F \in D^b_{\mathrm{coh}}(X)$ be an object. Consider the projection triangle as in Definition 2.2.5:*

$$B \to F \to A$$

with $A \cong L_{\mathcal{A}}(F) \in \mathcal{A}$ and $B \in {}^{\perp}\mathcal{A}$. If F is invariant under the action of some subgroup $G \subset \mathrm{Aut}^o(X)$, then both A and B are also invariant under the action of G.

Proof. Pick an automorphism $g \in G$. By Proposition 2.5.3 the pullbacks g^*A and g^*B lie in the subcategories $\mathcal{A}$ and ${}^{\perp}\mathcal{A}$ respectively. Thus the pullback of the projection triangle is another decomposition of $F \simeq g^*F$ into components from $\mathcal{A}$ and ${}^{\perp}\mathcal{A}$. Such a decomposition is unique, thus $g^*A \simeq A$ and $g^*B \simeq B$. $\square$

2.5.7. COROLLARY. *Every admissible subcategory of $D^b_{\mathrm{coh}}(\mathbb{P}^n)$ has a $\mathrm{PGL}(n+1)$-invariant classical generator.*

Proof. The category $D^b_{\mathrm{coh}}(\mathbb{P}^n)$ has a $\mathrm{PGL}(n+1)$-invariant classical generator $G = \bigoplus_{0 \leq i \leq n} \mathcal{O}(i)$. Let $L_{\mathcal{A}}$ be the projection functor to $\mathcal{A}$ as in Definition 2.2.5. Then $L_{\mathcal{A}}(G)$ is a classical generator of $\mathcal{A}$ which is $\mathrm{PGL}(n+1)$-invariant by Lemma 2.5.6. $\square$

2.5.8. LEMMA. *Let X be a smooth variety. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory, and let $A \in \mathcal{A}$ be an object. Let $\mathcal{O}_p$ be a skyscraper sheaf at some point $p \in X$. If there exists a morphism $\mathcal{O}_p \to A[a]$ for some shift $a \in \mathbb{Z}$ which induces a nonzero map on the zeroth* *cohomology sheaves, then any object of the subcategory* ${}^\perp\mathcal{A}$ *is set-theoretically supported on the complement to the point* $X \setminus \{p\}$.

Proof. Let $B \in {}^\perp\mathcal{A}$ be any object. Suppose that at least one of its cohomology sheaves is not zero at p. Without loss of generality we may assume that the support of $\mathcal{H}^0(B)$ contains p, while the supports of $\mathcal{H}^i(B)$ for $i > 0$ do not. It is easy to check that

$$\mathrm{R}^0\mathrm{Hom}(B, \mathcal{O}_p) \cong \mathrm{R}^0\mathrm{Hom}(\mathcal{H}^0(B), \mathcal{O}_p) \neq 0.$$

Pick any nonzero map $f\colon B \to \mathcal{O}_p$. Then the composition $B \to \mathcal{O}_p \to A[a]$ induces a nonzero map on the zeroth cohomology sheaves, but this contradicts semiorthogonality. Therefore any object in ${}^\perp\mathcal{A}$ is supported away from p. $\square$

2.5.9. Corollary. *Let X be a smooth algebraic variety such that the connected automorphism group* $\mathrm{Aut}^o(X)$ *acts transitively on X. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory, and let $A \in \mathcal{A}$ be an object. If there exists a morphism $\mathcal{O}_p \to A[a]$ from a skyscraper sheaf at some point $p \in X$ to a shift of A which induces a nonzero map on the zeroth cohomology sheaves, then $\mathcal{A} = D^b_{\mathrm{coh}}(X)$.*

Proof. By Lemma 2.5.8 any object of the orthogonal subcategory ${}^\perp\mathcal{A}$ is not supported at p. For any element $g \in \mathrm{Aut}^o(X)$, the pullback $g^*(\mathcal{O}_p \to A[a])$ lets us conclude similarly that any object of ${}^\perp\mathcal{A}$ is not supported anywhere along the orbit of p under $\mathrm{Aut}^o(X)$. Therefore ${}^\perp\mathcal{A} = 0$ and $\mathcal{A} = D^b_{\mathrm{coh}}(X)$. $\square$

2.6 Projections of skyscraper sheaves

To study admissible subcategories, in this paper we often consider the projections of skyscraper sheaves into them. The following several lemmas prove some important properties of the projections of skyscrapers.

2.6.1. Lemma. *Let $D^b_{\mathrm{coh}}(\mathbb{P}^n) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Consider a projection triangle for a skyscraper sheaf $\mathcal{O}_p$ at some point $p \in \mathbb{P}^n$:*

$$B \to \mathcal{O}_p \to A \to B[1].$$

If $\mathcal{B} \neq 0$, then the morphism $\mathcal{H}^0(B) \to \mathcal{O}_p$ is surjective.

Proof. If the map $\mathcal{H}^0(B) \to \mathcal{O}_p$ is not surjective, then it is zero, and by the long exact sequence of cohomology this would imply that $\mathcal{O}_p \to A$ induces a nonzero map on $\mathcal{H}^0$. The result follows from Corollary 2.5.9. □

2.6.2. Lemma. *Let $D^b_{\mathrm{coh}}(\mathbb{P}^n) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Consider a decomposition of a skyscraper sheaf at a point $p \in \mathbb{P}^n$ into the components:*

$$B \to \mathcal{O}_p \to A \to B[1].$$

1. *If $n > 1$, at least one of A and B is not a locally free object at the point p.*
2. *Both A and B are invariant under the action of $\mathrm{Stab}(p) \subset \mathrm{PGL}(n+1)$.*
3. *If B is set-theoretically supported at the point p, then $\mathcal{A} = 0$ and $\mathcal{B} = D^b_{\mathrm{coh}}(\mathbb{P}^n)$.*

Proof. If $\mathcal{B} = 0$, all properties are clear. So we assume that $\mathcal{B}$ is a nonzero admissible subcategory.

(1): consider the fragment of the long exact sequence of cohomology sheaves associated to the projection triangle:

$$0 \to \mathcal{H}^{-1}(A) \to \mathcal{H}^0(B) \to \mathcal{O}_p \to \mathcal{H}^0(A).$$

Since $\mathcal{B} \neq 0$, by Corollary 2.5.9 we see that the morphism $\mathcal{O}_p \to \mathcal{H}^0(A)$ vanishes. Then the fragment above produces a short exact sequence

$$0 \to \mathcal{H}^{-1}(A) \to \mathcal{H}^0(B) \to \mathcal{O}_p \to 0.$$

If both $\mathcal{H}^{-1}(A)$ and $\mathcal{H}^0(B)$ are locally free at p, this produces a locally free resolution of $\mathcal{O}_p$ of length one, which is impossible by homological dimension reasons if the dimension n is greater than one. Thus at least one of those two cohomology sheaves is not locally free.

(2) is an immediate consequence of Lemma 2.5.6.

(3) If B is supported at p, then there is a nonzero morphism from a skyscraper sheaf $\mathcal{O}_p$ to the leftmost cohomology sheaf of B. By Corollary 2.5.9 this is equivalent to $\mathcal{B} = D^b_{\mathrm{coh}}(\mathbb{P}^n)$ and hence $\mathcal{A} = 0$. □

2.7 Fourier–Mukai transforms

We recall some material about Fourier–Mukai transforms. For a more detailed exposition, see, for example, the book [Huy06, Ch. 5].

2.7.1. Definition. Let X and Y be two smooth and proper varieties. Let π_X, π_Y be the projection maps from $X \times Y$ to X and Y respectively. Let $K \in D^b_{\mathrm{coh}}(X \times Y)$ be any object. Then the *Fourier–Mukai transform with kernel* K is the functor $\Phi_K \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ given by the formula $\Phi_K(-) := \pi_{Y*}(\pi_X^*(-) \otimes K)$.

Most natural functors between derived categories of sheaves are Fourier–Mukai transforms. The identity functor on $D^b_{\mathrm{coh}}(X)$ is given by a Fourier–Mukai transform with respect to the structure sheaf of the diagonal $\mathcal{O}_{\Delta_X} \in D^b_{\mathrm{coh}}(X \times X)$. See [Huy06, Ex. 5.4] for many other examples.

2.7.2. Proposition ([Huy06, Prop. 5.9]). *Let X and Y be smooth and proper varieties. For any object $K \in D^b_{\mathrm{coh}}(X \times Y)$ the functor Φ_K has both left and right adjoint functors, and they are also Fourier–Mukai transforms.*

2.7.3. Proposition ([Kuz11, Th. 7.1]). *Let X be a smooth and proper variety, and let $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then the projection functors $R_{\mathcal{B}}$ and $L_{\mathcal{A}}$ from Definition 2.2.5 are Fourier–Mukai transforms. The kernels of those functors, which we also denote by $R_{\mathcal{B}}$ and $L_{\mathcal{A}}$, fit into a distinguished triangle*

$$R_{\mathcal{B}} \to \mathcal{O}_{\Delta_X} \to L_{\mathcal{A}}$$

of objects in $D^b_{\mathrm{coh}}(X \times X)$.

2.7.4. Lemma. *Let X be a smooth variety, and let $f\colon Y \to X$ be a proper morphism. Let $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition, with the right projection functor defined by the Fourier–Mukai kernel $R_{\mathcal{B}} \in D^b_{\mathrm{coh}}(X \times X)$. Then the Fourier–Mukai transform along the object $(f,f)^* R_{\mathcal{B}} \in D^b_{\mathrm{coh}}(Y \times Y)$ is the functor $f^* \circ R_{\mathcal{B}} \circ f_* \colon D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(Y)$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
 & & Y \times Y \arrow[dl] \arrow[dr] & & \\
 & Y \times X \arrow[r] \arrow[dl, "\pi_1"] & X \times X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & X \times Y \arrow[l] \arrow[dr, "\pi_2"] & \\
Y \arrow[r] & X & & X & Y \arrow[l]
\end{tikzcd}$$

All three commutative squares in this diagram are Cartesian and are easily seen to be Tor-independent ([Stacks, Tag 08IA]). The claimed formula follows by diagram chasing using the projection formula and the base change theorem for Tor-independent squares (see, for example, [Stacks, Tag 08IB]). $\square$

2.8 Serre functors

Given a morphism $f\colon M \to N$ in some category T and another object $L \in T$, we can use composition with f on either side to get two morphisms:

$$\mathrm{Hom}(L, M) \xrightarrow{f\circ -} \mathrm{Hom}(L, N), \qquad \mathrm{Hom}(N, L) \xrightarrow{-\circ f} \mathrm{Hom}(M, L).$$

Those two morphisms are quite different. Using Serre duality we may find a different, but similar, pair of morphisms, also given by pre-composition and post-composition with f, which are closely related to each other.

2.8.1. Definition ([BK90, §3]). Let T be a triangulated category with finite-dimensional Hom's. An endofunctor $S\colon T \to T$ is called a *Serre functor* if for each $M, N \in T$ we are given an isomorphism

$$\mathrm{RHom}(M, N) \xrightarrow{\sim} \mathrm{RHom}(N, S(M))^\vee \tag{2.8.1.1}$$

which is functorial in both arguments.

The naturality of the morphism (2.8.1.1) lets us give another description of the duality. Since $\mathrm{RHom}(M, S(M))^\vee \simeq \mathrm{REnd}(M)$, there is a canonical functional tr_M on $\mathrm{RHom}(M, S(M))$, corresponding to the identity map on M. Given any object N, the pairing in (2.8.1.1) is the trace of the composition:

$$\mathrm{RHom}(M, N) \otimes \mathrm{RHom}(N, S(M)) \xrightarrow{-\circ-} \mathrm{RHom}(M, S(M)) \xrightarrow{\mathrm{tr}_M} \Bbbk \tag{2.8.1.2}$$

We will need the following well-known lemma for which we could not find a reference.

2.8.2. Lemma. *Let T be a triangulated category admitting a Serre functor S. Let $f\colon M \to N$ be a morphism in T, and let $L \in T$ be an object. Consider the following diagram, where the vertical maps are isomorphisms given by Serre duality* (2.8.1.1)*:*

$$
\begin{tikzcd}
\mathrm{RHom}(L, M) \arrow[r, "f\circ -"] & \mathrm{RHom}(L, N) \arrow[d, "\sim"] \\
\mathrm{RHom}(M, S(L))^\vee \arrow[u, "\sim"] & \mathrm{RHom}(N, S(L))^\vee
\end{tikzcd}
\tag{2.8.2.1}
$$

The composition defines a map $\mathrm{RHom}(M, S(L))^\vee \to \mathrm{RHom}(N, S(L))^\vee$*. By dualizing it corresponds to a unique morphism* $\mathrm{RHom}(N, S(L)) \to \mathrm{RHom}(M, S(L))$*. Then this map, up to a sign, is given by the composition* $(- \circ f)$ *with the morphism* $f\colon M \to N$*.*

Proof. We will ignore the sign changes induced by shifts of complexes since we are only interested in the answer up to a sign. Let $\varphi \in \mathrm{Ext}^i(L, M)$ be a class in $\mathrm{RHom}(L, M)$. Using the description (2.8.1.2) of Serre duality, we see that the image of φ in the graded vector space $\mathrm{RHom}(N, S(L))^\vee$ is a functional defined as follows:

$$
g \in \mathrm{Ext}^j(N, S(L)) \;\mapsto\; \mathrm{tr}_L \left(L \xrightarrow{\varphi} M[i] \xrightarrow{f[i]} N[i] \xrightarrow{g[i]} S(L)[i+j] \right).
$$

Similarly, the image of φ in the graded vector space $\mathrm{RHom}(M, S(L))^\vee$ is a functional defined as follows:

$$
h \in \mathrm{Ext}^j(M, S(L)) \;\mapsto\; \mathrm{tr}_L \left(L \xrightarrow{\varphi} M[i] \xrightarrow{h[i]} S(L)[i+j] \right).
$$

It is clear from those formulas that setting the bottom horizontal map in the diagram (2.8.2.1) to be a pre-composition with $f\colon M \to N$ makes the diagram commute when evaluated on any element $\varphi \in \mathrm{Ext}^\bullet(L, M)$. Since the vertical arrows are isomorphisms of graded vector spaces, this implies the statement of the lemma. □

Chapter 3: Semiorthogonal decompositions and anticanonical divisors

Many standard examples of semiorthogonal decompositions arise from exceptional objects. Restriction to an anticanonical divisor is an important tool for studying exceptional objects. It has been used, for example, in [Zub90], to prove the stability of exceptional vector bundles on $\mathbb{P}^3$. Given an arbitrary semiorthogonal decomposition which does not arise from an exceptional collection, it is more difficult to apply this approach. It is not even clear what exactly should we restrict to the divisor. In this section we collect several statements that allow us to use anticanonical divisors to study admissible subcategories. More precisely, we show in Theorem 3.1.1 that a choice of an admissible subcategory induces an autoequivalence of the derived category of sheaves on an anticanonical divisor. This statement and its consequences form the technical core of the paper. The strongest results are obtained in the surface case, where an anticanonical divisor is a curve.

Almost all results in this section follow from a theorem by Nicolas Addington [Add16, Prop. 2.1] about the relation between so-called *spherical functors* and admissible subcategories. This particular statement and the whole idea that spherical functors are useful for questions about admissible subcategories became known to the author of this thesis only in the very late stages of preparing the manuscript. Originally, this section contained a direct proof of Proposition 3.1.3, which later was replaced by a direct proof of more general Theorem 3.1.1. Only after this proof had been mostly written up did the author discover Addington's work.

In this section we state Theorem 3.1.1, explain briefly why it follows from Addington's result, and show how to deduce Proposition 3.1.3, which is a key statement for the rest of this paper. After this, we have included a direct proof of Theorem 3.1.1 which does not rely on the notion of a spherical functor. Some lemmas in this proof may be of independent interest.

3.1 Admissible subcategories and autoequivalences

The following theorem is essentially due to Addington.

3.1.1. Theorem ([Add16]). *Let X be a smooth proper variety. Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory, and let $R_{\mathcal{B}} \in D^b_{\mathrm{coh}}(X \times X)$ be a Fourier–Mukai kernel for the right projection functor to $\mathcal{B}$, equipped with the morphism $\varphi_{\mathcal{B}} \colon R_{\mathcal{B}} \to \mathcal{O}_{\Delta_X}$ as in Proposition 2.7.3.*

Let $j \colon D \hookrightarrow X$ be an inclusion morphism of a smooth anticanonical divisor. Consider the composition of the restricted morphism $\varphi_{\mathcal{B}}|_{D\times D} \colon R_{\mathcal{B}}|_{D\times D} \to \mathcal{O}_{\Delta_X}|_{D\times D}$ with the tautological map $\mathcal{O}_{\Delta_X}|_{D\times D} \to \mathcal{O}_{\Delta_D}$. Take the cone of this composition to obtain a distinguished triangle in $D^b_{\mathrm{coh}}(D \times D)$:

$$R_{\mathcal{B}}|_{D\times D} \to \mathcal{O}_{\Delta_D} \to T. \tag{3.1.1.1}$$

Then the Fourier–Mukai transform with respect to the object $T \in D^b_{\mathrm{coh}}(D \times D)$ is an autoequivalence of $D^b_{\mathrm{coh}}(D)$.

To deduce this from Addington's paper, we need to recall a notion of a spherical functor. Roughtly speaking, a functor $F \colon T_1 \to T_2$ between two triangulated categories which admits a left adjoint L and a right adjoint R is called *spherical* if the endofunctors obtained as cones of the unit natural transformations $\mathrm{Id}_{T_1} \Rightarrow R \circ F$ and $\mathrm{Id}_{T_2} \Rightarrow F \circ L$ are both autoequivalences, of T_1 and T_2, respectively. Of course, this definition only makes in settings where taking a cone of a natural transformation is a meaningful operation. This is not really possible in the realm of triangulated subcategories, and we need either dg-enhancements or stable $(\infty, 1)$-categories to make this into a rigorous definition ([AL17]; however, see [Kuz19, Def. 2.8] for an alternative approach). There exist several other equivalent definitions.

Proof of Theorem 3.1.1. It is a standard fact that the restriction $j^* \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(D)$ to any divisor is a spherical functor (see, e.g., [Add16, 2.2 (4)]). In our case D is an anticanonical divisor, in particular its canonical bundle is trivial, and hence its Serre functor is just a shift.

Then [Add16, Prop. 2.1] implies that the composition $\mathcal{B} \hookrightarrow D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(D)$ is also a spherical functor. Compare [Add16, 2.2 (4′)]. Spherical functors are associated with many endofunctors, and one may check that a so-called *spherical twist* in this context is exactly a Fourier–Mukai transform along the object $T \in D^b_{\mathrm{coh}}(D \times D)$. Spherical twists are always autoequivalences [Add16, Th. 2.3]. □

3.1.2. Corollary. *In the notation of Theorem 3.1.1, for any object $F \in D^b_{\mathrm{coh}}(D)$ there exists a distinguished triangle $j^* R_{\mathcal{B}}(j_* F) \to F \to T(F)$.*

Proof. The only thing to check is that the Fourier–Mukai transform of an object F with respect to the kernel $R_{\mathcal{B}}|_{D \times D}$ is isomorphic to $j^* R_{\mathcal{B}}(j_* F)$, but this is true by Lemma 2.7.4. □

When the ambient variety is a surface, we can deduce from Theorem 3.1.1 a strong structural result that lets us control the behavior of arbitrary admissible subcategories. This is used to classify admissible subcategories of $\mathbb{P}^2$ in Theorem 4.1.1 and show the non-existence of phantom subcategories for del Pezzo surfaces of degree at least three in Theorem 6.4.6.

3.1.3. Proposition. *Let S be a smooth proper surface, let $j \colon E \hookrightarrow S$ be an anticanonical divisor, and let $p \in E$ be a smooth point of E. Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(S)$ be an admissible subcategory. Denote by $B := R_{\mathcal{B}}(\mathcal{O}_p)$ the (right) projection of a skyscraper sheaf $\mathcal{O}_p$ to the subcategory $\mathcal{B}$.*

If $\mathrm{supp}(j^* B) \subset E$ is a nonempty zero-dimensional subset and each point of this subset is a smooth point of E, then $j^* B \in D^b_{\mathrm{coh}}(E)$ is isomorphic to one of the following options:

1. *$j^* B \simeq \mathcal{O}_p[0] \oplus \mathcal{O}_q[a]$ for a smooth point $q \in E$ which may coincide with p, and $a \in \mathbb{Z}$;*
2. *$j^* B \simeq \mathcal{O}_{2p}[0]$, where $\mathcal{O}_{2p} \in \mathrm{Coh}(E)$ is a quotient of $\mathcal{O}_E$ by the square of the maximal ideal of the point $p \in E$;*

If E is a connected smooth curve and $\mathrm{supp}(j^* B) = E$, then $j^* B \in D^b_{\mathrm{coh}}(E)$ is isomorphic to one of the following options:

3. *$j^* B \simeq \mathcal{O}_p[0] \oplus M[b]$ for some simple vector bundle M on E.*

4. $j^*B \simeq \widetilde{M}[0]$*, where* $\widetilde{M}$ *is a vector bundle on* E *which fits into a short exact sequence*

$$0 \to M \to \widetilde{M} \to \mathcal{O}_p \to 0$$

where M *is a simple vector bundle on* E*.*

In any case, the support $\operatorname{supp}(j^*B)$ *has at most two connected components.*

Proof. The object B is by definition isomorphic to $R_{\mathcal{B}}(j_*\mathcal{O}_p)$. By Corollary 3.1.2 the derived pullback $j^*B \in D^b_{\mathrm{coh}}(E)$ fits into a triangle

$$j^*B \to \mathcal{O}_p \to C \to j^*B[1] \tag{3.1.3.1}$$

where $C := T(\mathcal{O}_p) \in D^b_{\mathrm{coh}}(E)$ is some object. By Theorem 3.1.1 the functor T is an autoequivalence, thus

$$\mathrm{REnd}_E(C) \simeq \mathrm{REnd}_E(\mathcal{O}_p) \simeq \Bbbk[0] \oplus \Bbbk[-1].$$

In particular, C is a simple object, i.e., its endomorphisms of degree zero are spanned by the identity map. Note that simple objects are automatically indecomposable, and on curves many objects split into direct sums via Lemmas 2.3.9 and 2.3.10. Using those lemmas, we may essentially classify all possible options for C, and hence for j^*B, as follows.

Observe first that by Lemma 2.3.3 the support of the indecomposable object C is connected. Thus the triangle (3.1.3.1) implies that $\operatorname{supp}(j^*B)$ has at most two connected components. This confirms the last claim of the statement.

Assume that $\operatorname{supp}(j^*B) \subset E$ is zero-dimensional, nonempty, and consists of smooth points of E. Then the same holds for $\operatorname{supp}(C) \subset E$. In particular, C is supported on a smooth part of the curve E. Then we may apply Lemma 2.3.9 to see that C is a torsion coherent sheaf shifted to some degree, and by Lemma 2.3.10 any such simple C is necessarily isomorphic to a shift $\mathcal{O}_q[a]$ of a skyscraper sheaf at some point $q \in E$.

If q is not the same point as p, then the map $\mathcal{O}_p \to \mathcal{O}_q[a]$ from (3.1.3.1) is necessarily zero, and hence $j^*B \simeq \mathcal{O}_p[0] \oplus \mathcal{O}_q[a-1]$. If $q = p$ and the map $\mathcal{O}_p \to \mathcal{O}_p[a]$ from (3.1.3.1) is nonzero, there are two cases. Either $a = 0$ and the map is an isomorphism, but then the cone j^*B is zero, which contradicts the assumption that $\operatorname{supp}(j^*B)$ is nonempty. Or $a = 1$ and the map is a nonzero element of $\operatorname{Ext}^1_E(\mathcal{O}_p, \mathcal{O}_p) \simeq \Bbbk$, in which case the object j^*B is isomorphic to the unique nontrivial extension of a skyscraper sheaf by itself, i.e., $j^*B \simeq \mathcal{O}_{2p}[0]$.

Assume now that we are in the second situation, i.e., the curve E is smooth and j^*B is not a torsion object. Then the triangle (3.1.3.1) shows that C is also not a torsion object. By Lemma 2.3.9 we see that any simple object C on a smooth curve which is not a torsion object is a shift of a simple vector bundle, $C \simeq M[a]$ for some $a \in \mathbb{Z}$.

If the morphism $\mathcal{O}_p \to M[a]$ in (3.1.3.1) is zero, then $j^*B \simeq \mathcal{O}_p[0] \oplus M[a-1]$. On a smooth curve $\operatorname{Ext}^a_E(\mathcal{O}_p, M)$ is nontrivial only when $a = 1$, so any nonzero map in (3.1.3.1) arises from some short exact sequence

$$0 \to M \to \widetilde{M} \to \mathcal{O}_p \to 0,$$

and for those maps in $\operatorname{Ext}^1_E(\mathcal{O}_p, M)$ we have an isomorphism $j^*B \simeq \widetilde{M}[0]$ in (3.1.3.1). Thus the list of possible isomorphism classes of j^*B in the statement is exhaustive. □

The description of j^*B in the proposition above implies an interesting property for restrictions of the object B to arbitrary anticanonical divisors. Suppose that we are in a situation where j^*B is isomorphic to a direct sum of two skyscraper sheaves. Consider a different anticanonical divisor, $j' \colon E' \hookrightarrow S$, which does not necessarily pass through the point $p \in S$. If E' is in some sense "close" to the divisor E, it is reasonable to expect that j'^*B is also a torsion object, and by semicontinuity it should not be significantly more complicated than two skyscrapers. This imprecise intuitive picture may be improved to a rigorous statement. We state it as Lemma 3.4.1 and prove it later in this section, since the proof uses a technical lemma from our direct approach to Theorem 3.1.1.

3.2 Admissible subcategories and pushforwards

In the previous section we have given a short proof of Theorem 3.1.1 based on the properties of spherical functors. The rest of this chapter is an alternative, more direct proof of this theorem. We start with an observation about admissible subcategories and objects pushed forward along some map.

Let X be an algebraic variety and let $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. Suppose $E \in D^b_{\mathrm{coh}}(X)$ is an object supported on a closed subvariety $Z \subset X$. Does the projection of E to $\mathcal{B}$ know anything about Z? At the first glance, there is no relation. Certainly the support of the projection does not have to be Z. However, when E is not just supported on Z, but happens to be a pushforward from $D^b_{\mathrm{coh}}(Z)$, there is a nontrivial relationship. In this section we explain it in Lemma 3.2.4.

3.2.1. Setting. We fix a smooth proper variety X and a proper morphism $f\colon Y \to X$ of varieties. Let $\mathcal{I}$ be the (shift of the) cone of the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$:

$$\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to \mathcal{I}[1].$$

We also fix a semiorthogonal decomposition $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. Recall that for any object $E \in D^b_{\mathrm{coh}}(X)$ there exists a projection triangle as in Definition 2.2.5:

$$R_{\mathcal{B}}(E) \to E \to L_{\mathcal{A}}(E).$$

We use the symbols $R_{\mathcal{B}}$ and $L_{\mathcal{A}}$ also for the Fourier–Mukai kernels of the projection functors, as in Proposition 2.7.3.

3.2.2. Setting. In some lemmas we are interested only in the case where the map $f\colon Y \to X$ from Setting 3.2.1 is an embedding of an anticanonical divisor of X, and in this case we use the notation $j\colon D \hookrightarrow X$ instead.

When the map $f\colon Y \to X$ from Setting 3.2.1 is a closed embedding, the object $\mathcal{I}$ is just the ideal sheaf. Objects pushed forward from Y have the property that they are annihilated by the ideal sheaf. When f is an arbitrary morphism, a similar property holds.

3.2.3. LEMMA. *Let $f\colon Y \to X$ and $\mathcal{I} \in D^b_{\mathrm{coh}}(X)$ be as in Setting 3.2.1. Let $F \in D^b_{\mathrm{coh}}(Y)$ be an arbitrary object. Then the morphism*

$$f_*F \otimes (\mathcal{I} \to \mathcal{O}_X) : f_*F \otimes \mathcal{I} \to f_*F \tag{3.2.3.1}$$

in the category $D^b_{\mathrm{coh}}(X)$ is a zero map.

Proof. The morphism (3.2.3.1) may be extended to a distinguished triangle:

$$f_*F \otimes \mathcal{I} \to f_*F \to f_*F \otimes f_*\mathcal{O}_Y.$$

It is enough to prove that the map $f_*F \to f_*F \otimes f_*\mathcal{O}_Y$ is a split monomorphism. We construct the splitting explicitly. Consider the commutative diagram:

$$\begin{tikzcd}
f_*F \otimes \mathcal{O}_X \arrow[r, "f_*F\otimes -"] \arrow[d, "\sim"] & f_*F \otimes f_*f^*\mathcal{O}_X \arrow[d, "\sim"] & \\
f_*(F \otimes f^*\mathcal{O}_X) \arrow[r, "f_*(F\otimes f^*(\mathrm{unit}_{\mathcal{O}_X}))"] & f_*(F \otimes f^*f_*f^*\mathcal{O}_X) \arrow[r, "f_*(F\otimes \mathrm{counit}_{f^*\mathcal{O}_X})"] & f_*(F \otimes f^*\mathcal{O}_X)
\end{tikzcd}$$

Here the vertical isomorphisms are given by the projection formula, the square commutes by the construction of projection formula, the morphism $\mathrm{unit}_{\mathcal{O}_X}$ is the natural transformation $\mathrm{Id} \Rightarrow f_*f^*(-)$ applied the the object $\mathcal{O}_X$, and similarly for the counit. The composition in the bottom row is an identity map by the definition of adjoint functors. Thus it provides a splitting. Therefore the lemma is proved. $\square$

Remark. More generally, for any object $G \in D^b_{\mathrm{coh}}(X)$ define $T(G)$ to be the fiber of the unit morphism $G \to f_*f^*G$. Then the same vanishing occurs for $f_*F \otimes (T(G) \to G)$.

Semiorthogonal decomposition is a global notion. The projection of an object usually changes its support in a difficult to control way. However, some analogue of the vanishing in

Lemma 3.2.3 still holds for the projections.

3.2.4. Lemma. *Let $f\colon Y \to X$, $\mathcal{I} \in D^b_{\mathrm{coh}}(X)$, and $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be as in Setting 3.2.1. Let $F \in D^b_{\mathrm{coh}}(Y)$ be any object. Consider the morphism*

$$\iota_F\colon R_{\mathcal{B}}(f_*F) \otimes \mathcal{I} \to R_{\mathcal{B}}(f_*F)$$

induced by the map $\mathcal{I} \to \mathcal{O}_X$.

1. *The morphism $R_{\mathcal{B}}(\iota_F)\colon R_{\mathcal{B}}(R_{\mathcal{B}}(f_*F) \otimes \mathcal{I}) \to R_{\mathcal{B}}(f_*F)$ is a zero morphism in $\mathcal{B}$.*

2. *For any object $B \in \mathcal{B}$ the morphism of graded vector spaces*

$$\mathrm{RHom}(B, R_{\mathcal{B}}(f_*F) \otimes \mathcal{I}) \xrightarrow{\iota_F \circ -} \mathrm{RHom}(B, R_{\mathcal{B}}(f_*F)) \tag{3.2.4.1}$$

given by composition with ι_F, is a zero map.

Proof. Consider the commutative square of projections:

$$\begin{array}{ccc} R_{\mathcal{B}}(f_*F) \otimes \mathcal{I} & \longrightarrow & R_{\mathcal{B}}(f_*F) \\ \big\downarrow & & \big\downarrow \\ f_*F \otimes \mathcal{I} & \longrightarrow & f_*F \end{array}$$

Note that the bottom arrow is zero by Lemma 3.2.3. An application of the projection functor $R_{\mathcal{B}}(-)$ to this commutative diagram leads to a commutative square where the bottom horizontal map is still zero, while the right vertical map is an isomorphism. This establishes the first claim of the Lemma. The second follows from it by definition of the right projection functor $R_{\mathcal{B}}$. □

3.2.5. Lemma. *Let $f\colon Y \to X$, $\mathcal{I} \in D^b_{\mathrm{coh}}(X)$ and $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be as in Setting 3.2.1. Let $B \in \mathcal{B}$ be an arbitrary object.*

1. *Let $G \in D^b_{\mathrm{coh}}(X)$ be an arbitrary object. Then*

$$\dim \mathrm{Ext}^i_Y(f^*B, f^*R_{\mathcal{B}}(G)) \leq \dim \mathrm{Ext}^i_X(B, G) + \dim \mathrm{Ext}^{i+1}_X(B, R_{\mathcal{B}}(G) \otimes \mathcal{I}). \tag{3.2.5.1}$$

2. *Suppose additionally that $G \simeq f_* F$ for some object $F \in D^b_{\mathrm{coh}}(Y)$. Then the inequality* (3.2.5.1) *becomes an equality.*

3. *Suppose additionally that $f \colon Y \to X$ is an embedding of an anticanonical divisor, and that $B \simeq R_{\mathcal{B}}(f_* E)$ for some $E \in D^b_{\mathrm{coh}}(Y)$. Then*

$$\begin{aligned}\dim \operatorname{Ext}^i_Y(f^* R_{\mathcal{B}}(f_* E), f^* R_{\mathcal{B}}(f_* F)) = &\dim \operatorname{Ext}^i_Y(f^* R_{\mathcal{B}}(f_* E), F) + \\ &+ \dim \operatorname{Ext}^i_Y(E, f^* R_{\mathcal{B}}(f_* F)).\end{aligned} \tag{3.2.5.2}$$

Proof. Consider the triangle

$$R_{\mathcal{B}}(G) \otimes \mathcal{I} \to R_{\mathcal{B}}(G) \to f_* f^* R_{\mathcal{B}}(G).$$

An application of the functor $\operatorname{RHom}(B, -)$ leads to a long exact sequence of Ext-groups. Note that $\operatorname{RHom}(B, R_{\mathcal{B}}(G)) \cong \operatorname{RHom}(B, G)$ by definition of the right projection functor. This proves the inequality (3.2.5.1).

The second point follows from Lemma 3.2.3: the first arrow in the triangle above becomes zero when $G \simeq f_* F$.

The last point is a consequence of Serre duality. In this case $\mathcal{I}$ is the canonical line bundle on X. Note that $\operatorname{RHom}(R_{\mathcal{B}}(f_* F), R_{\mathcal{B}}(f_* E))$ is isomorphic to $\operatorname{RHom}(R_{\mathcal{B}}(f_* F), f_* E)$ and then the adjunction transforms the expression in (3.2.5.1) into a symmetric expression from the last point. ∎

3.3 Proof of autoequivalence

3.3.1. Lemma. *Let $f \colon Y \to X$ and $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be as in Setting 3.2.1. Let $F \in D^b_{\mathrm{coh}}(Y)$ be an arbitrary object. Consider the morphism*

$$c_F \colon f^* R_{\mathcal{B}}(f_* F) \to F$$

corresponding by adjunction to the projection map $\pi\colon R_{\mathcal{B}}(f_*F) \to f_*F$*. Then the morphism*

$$R_{\mathcal{B}}(f_*(c_F))\colon R_{\mathcal{B}}(f_*f^*R_{\mathcal{B}}(f_*F)) \to R_{\mathcal{B}}(f_*F)$$

is a split epimorphism. For any object $B \in \mathcal{B}$*, the application of* $\mathrm{RHom}(f^*B, -)$ *to* c_F *leads to a degree-wise surjective map of graded vector spaces:*

$$\mathrm{RHom}(f^*B, c_F)\colon \mathrm{RHom}(f^*B, f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{c_F \circ -} \mathrm{RHom}(f^*B, F).$$

Proof. Recall that the adjunction between f_* and f^* produces a unit and counit natural transformations. Consider the following diagram:

$$\begin{tikzcd}
R_{\mathcal{B}}(f_*F) \arrow[r, "\mathrm{unit}_{R_{\mathcal{B}}(f_*F)}"] \arrow[d, "\pi"] & f_*f^*R_{\mathcal{B}}(f_*F) \arrow[d, "f_*f^*(\pi)"] \arrow[dr, bend left, "f_*(c_F)"] & \\
f_*F \arrow[r, "\mathrm{unit}_{f_*F}"] \arrow[rr, bend right, "\mathrm{id}"'] & f_*f^*f_*F \arrow[r, "f_*(\mathrm{counit}_F)"] & f_*F
\end{tikzcd}$$

Here the square commutes since $\mathrm{unit}_{(-)}\colon \mathrm{Id} \Rightarrow f_*f^*(-)$ is a natural transformation. The lower triangle commutes by definition of adjoint functors. The right-side triangle commutes by definition of c_F via adjuncation and π.

An application of the functor $R_{\mathcal{B}}$ to this diagram produces a commutative diagram in $\mathcal{B}$. Note that $R_{\mathcal{B}}(\pi)$ becomes an identity morphism. Thus $R_{\mathcal{B}}(f_*(c_F))$ is a split epimorphism.

The second claim in the statement follows by definition of the right projection functor $R_{\mathcal{B}}$ and the adjunction between f_* and f^*. $\square$

3.3.2. Corollary. *Let* $f\colon Y \to X$ *and* $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ *be as in Setting* 3.2.1*. Suppose that* Y *is a smooth and proper variety with trivial canonical bundle. Let* $E, F \in D^b_{\mathrm{coh}}(Y)$ *be two objects. Consider the morphisms*

$$c_E\colon f^*R_{\mathcal{B}}(f_*E) \to E \qquad c_F\colon f^*R_{\mathcal{B}}(f_*F) \to F$$

as in Lemma 3.3.1. Then the complex

$$\mathrm{RHom}(E, f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{-\circ c_E} \mathrm{RHom}(f^*R_{\mathcal{B}}(f_*E), f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{c_F\circ -} \mathrm{RHom}(f^*R_{\mathcal{B}}(f_*E), F).$$

of graded vector spaces, given by compositions with c_E and c_F, has cohomology only in the middle. If $f\colon Y \to X$ is an embedding of an anticanonical divisor, then this complex is exact.

Proof. By Lemma 3.3.1 the map of graded vector spaces

$$\mathrm{RHom}(f^*R_{\mathcal{B}}(f_*E), f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{c_F\circ -} \mathrm{RHom}(f^*R_{\mathcal{B}}(f_*E), F)$$

is degree-wise surjective. Note that the map

$$\mathrm{RHom}(f^*R_{\mathcal{B}}(f_*F), f^*R_{\mathcal{B}}(f_*E)) \xrightarrow{c_E\circ -} \mathrm{RHom}(f^*R_{\mathcal{B}}(f_*F), E)$$

is also surjective, for the same reason. We will deduce the injectivity claimed in the statement using Serre duality. Recall that the derived category of a smooth and proper variety Y has a Serre functor, and since Y has trivial canonical bundle, its Serre functor is just a shift. Thus by Lemma 2.8.2 the graded surjection above is a shift of the graded dual to the map

$$\mathrm{RHom}(E, f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{-\circ c_E} \mathrm{RHom}(f^*R_{\mathcal{B}}(f_*E), f^*R_{\mathcal{B}}(f_*F)),$$

which therefore is degree-wise injective.

To deal with the last statement, note that Lemma 3.2.5 shows that the graded dimension in the middle equals the sum of graded dimensions, and therefore the complex is exact. □

Now we are ready to give an alternative proof of Theorem 3.1.1. We repeat the statement of this theorem for the ease of reading.

THEOREM. *Let X be a smooth proper variety. Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory, and let $R_{\mathcal{B}} \in D^b_{\mathrm{coh}}(X \times X)$ be a Fourier–Mukai kernel for the right projection functor to $\mathcal{B}$,* *equipped with the morphism* $\varphi_{\mathcal{B}}\colon R_{\mathcal{B}} \to \mathcal{O}_{\Delta_X}$ *as in Proposition 2.7.3.*

Let $j\colon D \hookrightarrow X$ *be an inclusion morphism of a smooth anticanonical divisor. Consider the composition of the restricted morphism* $\varphi_{\mathcal{B}}|_{D\times D}\colon R_{\mathcal{B}}|_{D\times D} \to \mathcal{O}_{\Delta_X}|_{D\times D}$ *with the tautological map* $\mathcal{O}_{\Delta_X}|_{D\times D} \to \mathcal{O}_{\Delta_D}$*. Take the cone of this composition to obtain a distinguished triangle in* $D^b_{\mathrm{coh}}(D\times D)$*:*

$$R_{\mathcal{B}}|_{D\times D} \to \mathcal{O}_{\Delta_D} \to T.$$

Then the Fourier–Mukai transform with respect to the object $T \in D^b_{\mathrm{coh}}(D\times D)$ *is an autoequivalence of* $D^b_{\mathrm{coh}}(D)$*.*

Proof. Abusing the notation a little, we use the symbol T not only for an object in $D^b_{\mathrm{coh}}(D\times D)$, but also for the functor $D^b_{\mathrm{coh}}(D) \to D^b_{\mathrm{coh}}(D)$ given by the Fourier–Mukai transform with that kernel. First, observe that it is enough to show that T is a fully faithful functor. Indeed, any Fourier–Mukai transform has both left and right adjoints by Proposition 2.7.2, so by definition the image of T would be an admissible subcategory. However, since D is an anticanonical divisor, its canonical bundle is trivial, and therefore $D^b_{\mathrm{coh}}(D)$ does not have any nontrivial admissible subcategories [KO15, Th. 1.2]. Thus in this case T is automatically essentially surjective, i.e., it is an autoequivalence.

Let $E, F \in D^b_{\mathrm{coh}}(D)$ be two arbitrary objects. Consider the triangles

$$j^*R_{\mathcal{B}}(j_*E) \to E \to T(E) \qquad j^*R_{\mathcal{B}}(j_*F) \to F \to T(F).$$

We may use the spectral sequence from Lemma 2.4.3 to compute $\mathrm{RHom}(T(E), T(F))$ in terms of other spaces. The first page is the following complex of graded vector spaces, where the parentheses mean RHom for brevity:

$$(E, f^*R_{\mathcal{B}}(f_*F)) \xrightarrow{-\circ c_E} (f^*R_{\mathcal{B}}(f_*E), f^*R_{\mathcal{B}}(f_*F)) \oplus (E, F) \xrightarrow{c_F\circ -} (f^*R_{\mathcal{B}}(f_*E), F).$$

By Corollary 3.3.2 the first arrow is injective and the second arrow is surjective. Moreover, the complex is quasiisomorphic to $\mathrm{RHom}(E,F)$ by the last claim of that corollary. Therefore the natural map

$$\mathrm{RHom}(E,F) \to \mathrm{RHom}(T(E),T(F))$$

is an isomorphism for each pair E, F. Thus the functor T is fully faithful and hence an autoequivalence. $\square$

3.4 Skyscrapers and arbitrary anticanonical divisors

Proposition 3.1.3 concerns the projections of the skyscraper sheaves for points lying on some anticanonical divisor in a surface. After the proof of the proposition, we have remarked that there are some implications even for skyscraper sheaves at the points not on that divisor. Now we are ready to make the imprecise statement mentioned in that remark into a lemma.

3.4.1. Lemma. *Let S, $j\colon E \hookrightarrow S$, $p \in E$ and $B \in \mathcal{B} \subset D^b_{\mathrm{coh}}(S)$ be as in Proposition 3.1.3. Suppose that E is smooth. Let $j'\colon E' \hookrightarrow S$ be another smooth anticanonical divisor, not necessarily passing through the point $p \in S$. Suppose that the support of j^*B consists of two distinct points, and suppose that j'^*B is a torsion object. Then j'^*B is isomorphic to one of the following options:*

1. *$j'^*B = 0$;*
2. *$j'^*B \simeq \mathcal{O}_q[a]$ for some point $q \in E'$ and a shift $a \in \mathbb{Z}$;*
3. *$j'^*B \simeq \mathcal{O}_q[a] \oplus \mathcal{O}_r[b]$ for some points $q, r \in E'$ and shifts $a, b \in \mathbb{Z}$;*
4. *$j'^*B \simeq \mathcal{O}_{2q}[a]$ for some point $q \in E'$ and a shift $a \in \mathbb{Z}$, where $\mathcal{O}_{2q}$ is the quotient of the structure sheaf $\mathcal{O}_E$ by the square of the maximal ideal of the point $q \in E$.*

Proof. Consider a restriction triangle for the object $B \in D^b_{\mathrm{coh}}(S)$ to the divisor $E \subset S$:

$$B \otimes K_S \to B \to j_*j^*B$$

An application of the functor $\mathrm{RHom}(B, -)$ produces a triangle of graded vector spaces

$$\mathrm{RHom}(B, B \otimes K_S) \to \mathrm{RHom}(B, B) \to \mathrm{REnd}(j^* B). \tag{3.4.1.1}$$

From Proposition 3.1.3 we know that $j^* B$ is isomorphic to a direct sum of two distinct skyscrapers. Then the length of the graded vector space $\mathrm{REnd}(j^* B)$ is equal to four. Since by definition B is the projection $R_{\mathcal{B}}(j_* \mathcal{O}_p)$, Lemma 3.2.4 implies that the first arrow in the triangle (3.4.1.1) is zero, and thus $\mathrm{REnd}(j^* B)$ is isomorphic to a direct sum of the other two terms. Therefore we get

$$\ell(\mathrm{RHom}(B, B \otimes K_S)) + \ell(\mathrm{RHom}(B, B)) = 4.$$

By a similar procedure we obtain a triangle of graded vector spaces corresponding to the restriction to the divisor $j' \colon E' \hookrightarrow S$:

$$\mathrm{RHom}(B, B \otimes K_S) \to \mathrm{RHom}(B, B) \to \mathrm{REnd}(j'^* B).$$

The length of the cone is bounded from above by the sum of lengths of the first two terms. This produces an inequality:

$$\ell(\mathrm{REnd}(j'^* B)) \leq \ell(\mathrm{RHom}(B, B \otimes K_S)) + \ell(\mathrm{RHom}(B, B)) = 4.$$

Using Lemmas 2.3.9 and 2.3.10 it is easy to see that a torsion object $j'^* B$ on a smooth curve with $\ell(\mathrm{REnd}(j'^* B)) \leq 4$ is isomorphic to one of the four options listed above. $\square$

Chapter 4: Classification of admissible subcategories of $\mathbb{P}^2$

4.1 Overview

The goal of this chapter is to prove the following result about admissible subcategories in the derived category $D^b_{\text{coh}}(\mathbb{P}^2)$ of coherent sheaves on $\mathbb{P}^2$. Since exceptional objects and exceptional collections in $D^b_{\text{coh}}(\mathbb{P}^2)$ have been classified in [GR87], this theorem may be described as a classification of admissible subcategories.

4.1.1. Theorem. *Any admissible subcategory in $D^b_{\text{coh}}(\mathbb{P}^2)$ has a full exceptional collection.*

This classification immediately implies the following.

4.1.2. Corollary. *There are no phantom subcategories in $D^b_{\text{coh}}(\mathbb{P}^2)$.*

Proof. For any category $\mathcal{A}$ with a full exceptional collection of length n the Grothendieck group $K_0(\mathcal{A})$ is a free abelian group on n generators. Thus by Theorem 4.1.1 an admissible subcategory of $D^b_{\text{coh}}(\mathbb{P}^2)$ is either a zero category, or has non-vanishing K_0. $\square$

As mentioned in Lemma 2.5.2, any admissible subcategory $\mathcal{A} \subset D^b_{\text{coh}}(\mathbb{P}^2)$ leads to a semiorthogonal decomposition of that category, $D^b_{\text{coh}}(\mathbb{P}^2) = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$. Since the length of any full exceptional collection in $D^b_{\text{coh}}(\mathbb{P}^2)$ is three, the result above implies that in any nontrivial decomposition at least one of the subcategories $\mathcal{A}$ and ${}^\perp\mathcal{A}$ is generated by a single exceptional object. In fact, in the proof of Theorem 4.1.1 we do not construct nontrivial exceptional collections directly, but rather recognize which of the subcategories $\mathcal{A}$ and ${}^\perp\mathcal{A}$ is a simpler one. More precisely, Theorem 4.1.1 is implied by the following statement:

4.1.3. Theorem. *Let $D^b_{\text{coh}}(\mathbb{P}^2) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition with $\mathcal{A} \neq 0$ and*

$\mathcal{B} \neq 0$. Pick a point $p \in \mathbb{P}^2$. Consider a projection triangle for the skyscraper sheaf $\mathcal{O}_p$:

$$B \to \mathcal{O}_p \to A \to B[1]$$

with $B \cong R_{\mathcal{B}}(\mathcal{O}_p) \in \mathcal{B}$ and $A \cong L_{\mathcal{A}}(\mathcal{O}_p) \in \mathcal{A}$. Assume that B is not locally free at p. Then the subcategory $\mathcal{A} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ is generated by a single exceptional vector bundle.

The strategy of the proof of Theorem 4.1.3 is discussed in Section 4.2. First we show how to deduce Theorem 4.1.1 from this statement.

Proof of the implication (4.1.3) $\implies$ (4.1.1). Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ be an arbitrary admissible subcategory. Denote the orthogonal subcategory ${}^{\perp}\mathcal{A} \subset D^b_{\mathrm{coh}}(\mathbb{P}^2)$ by $\mathcal{B}$. Then $D^b_{\mathrm{coh}}(\mathbb{P}^2) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition. If either $\mathcal{A}$ or $\mathcal{B}$ is a zero subcategory, there is nothing to prove, so we assume that the decomposition is nontrivial. Let $p \in \mathbb{P}^2$ be a point. Consider a projection triangle

$$B \to \mathcal{O}_p \to A \to B[1]$$

of the skyscraper sheaf. By parts (1) and (2) of Lemma 2.6.2 we know that at least one of projections A and B is not locally free at p.

Suppose B is not locally free. Then Theorem 4.1.3 implies that there is an exceptional vector bundle $E \in D^b_{\mathrm{coh}}(\mathbb{P}^2)$ such that $\mathcal{A} = \langle E \rangle$, confirming Theorem 4.1.1 in this case.

Suppose now that A is not locally free. Observe that the dualized and shifted triangle

$$A^{\vee}[2] \to \mathcal{O}_p \to B^{\vee}[2] \to A^{\vee}[3]$$

is the projection triangle of the skyscraper $\mathcal{O}_p$ corresponding to the dual semiorthogonal decomposition $D^b_{\mathrm{coh}}(\mathbb{P}^2) = \langle \mathcal{B}^{\vee}, \mathcal{A}^{\vee} \rangle$. Note that A is locally free if and only if $A^{\vee}[2]$ is. By the same argument as above we see that $\mathcal{B}^{\vee} = \langle E \rangle$ for some exceptional bundle $E \in D^b_{\mathrm{coh}}(\mathbb{P}^2)$. This implies that $\mathcal{B}$ is generated by a single exceptional bundle $E^{\vee}$.

By [GR87, Th. 5.10] an exceptional vector bundle $E^{\vee}$ on $\mathbb{P}^2$ may be extended to a full exceptional collection $\langle E', E'', E^\vee \rangle$. Therefore the category $\mathcal{A} = \mathcal{B}^\perp$ is equal to the subcategory $\langle E', E'' \rangle$. Thus Theorem 4.1.1 holds in this case as well. □

4.2 Strategy of the proof

The proof of Theorem 4.1.3 relies on the properties of the restriction of B to a cubic curve passing through the point $p \in \mathbb{P}^2$. The proof is split into several parts. First in Section 4.3 we use the results from Section 3 to study the restriction of the object B to a cubic curve, i.e., to an anticanonical divisor. We use the classification from Proposition 3.1.3 to deduce strong constraints on the object B itself. For instance, in that subsection we show that B is concentrated in at most two cohomology degrees. Then in Section 4.4 we prove that the zeroth cohomology sheaf of B is a skyscraper sheaf $\mathcal{O}_p$ and the minus first cohomology sheaf is locally free. Finally, in Section 4.5 we conclude that A is a direct sum of several copies of a single exceptional vector bundle, which lets us finish the proof by Lemma 4.5.2.

4.3 Restricting projections of a skyscraper to a cubic curve

4.3.1. Setting. From here on we fix the data involved in Theorem 4.1.3, namely a semiorthogonal decomposition $D^b_{\mathrm{coh}}(\mathbb{P}^2) = \langle \mathcal{A}, \mathcal{B} \rangle$ with $\mathcal{A} \neq 0$ and $\mathcal{B} \neq 0$, a point $p \in \mathbb{P}^2$, and the projection triangle for the skyscraper sheaf

$$B \to \mathcal{O}_p \to A \to B[1]$$

with $B \in \mathcal{B}$ and $A \in \mathcal{A}$, such that B is not locally free at p. We also fix a smooth cubic curve $j \colon E \to \mathbb{P}^2$ cut out by an equation $s \in \Gamma(\mathbb{P}^2, \mathcal{O}(3))$ which passes through p.

Remark. In our approach to the proof of Theorem 4.1.3 we often use the fact that PGL(3), the automorphism group of $\mathbb{P}^2$, acts doubly transitively on $\mathbb{P}^2$. For example, this implies that the stabilizer subgroup $\mathrm{Stab}(p) \subset \mathrm{PGL}(3)$ of the point $p \in \mathbb{P}^2$, which acts on the projections of the skyscraper sheaf by Lemma 2.6.2 (2), has only two orbits in $\mathbb{P}^2$. It is possible to avoid most instances of relying on symmetry by using general cubic curves instead of fixing the curve E in Setting 4.3.1. We use a strategy like that in some parts of Section 6, where we deal with del Pezzo surfaces. However, for Theorem 4.1.3 we need some global geometric properties of $\mathbb{P}^2$ in any case, so there is no immediate benefit from circumventing the arguments based on symmetry.

4.3.2. Lemma. *Let B be as in Setting 4.3.1. Then the support* $\operatorname{supp}(B)$ *is* $\mathbb{P}^2$.

Proof. The object B is invariant under the action of the group $\operatorname{Stab}(p) \subset \operatorname{PGL}(3)$ by Lemma 2.6.2, so $\operatorname{supp}(B)$ is a closed $\operatorname{Stab}(p)$-invariant subset of $\mathbb{P}^2$. Thus it is either $\mathbb{P}^2$, or a point p.

Assume that B is an object set-theoretically supported only at the point $p \in \mathbb{P}^2$. Pick the smallest integer $i \in \mathbb{Z}$ such that $\mathcal{H}^i(B) \neq 0$. Then there exists a morphism $\mathcal{H}^i(B)[-i] \to B$ in the derived category inducing the identity map on the i'th cohomology sheaves. Since $\mathcal{H}^i(B)$ is a nonzero torsion sheaf supported at a point p, there exists a inclusion $\mathcal{O}_p \hookrightarrow \mathcal{H}^i(B)$ of sheaves. The composition $\mathcal{O}_p[-i] \to \mathcal{H}^i(B)[-i] \to B$ is a map inducing a nonzero morphism on the i'th cohomology sheaves, so by Corollary 2.5.9 this implies that $\mathcal{B} = D^b_{\mathrm{coh}}(\mathbb{P}^2)$ and $\mathcal{A} = 0$. This is a contradiction with the assumption that $\mathcal{A} \neq 0$. □

4.3.3. Lemma. *Let B be as in Setting 4.3.1. For any smooth cubic curve $j\colon E \to \mathbb{P}^2$ which passes through p, the derived restriction j^*B is isomorphic to $\mathcal{O}_p[0] \oplus M[a]$ for some simple vector bundle M on the curve E and some shift $a \in \mathbb{Z}$.*

Proof. Note that we are exactly in the situation of Proposition 3.1.3: we restrict a projection of a skyscraper to a smooth anticanonical divisor on a surface. It only remains to rule out all options except $\mathcal{O}_p[0] \oplus M[a]$.

The object B is $\operatorname{Stab}(p)$-invariant by Lemma 2.6.2. There are only two orbits of $\operatorname{Stab}(p)$ on $\mathbb{P}^2$, the point p and the complement $\mathbb{P}^2 \setminus \{p\}$. Thus if B is not locally free at p, by Lemma 2.3.6 the length of the derived fiber at p is strictly larger than at any other point of $\mathbb{P}^2$. This implies that the restriction j^*B to E is also not locally free at $p \in E$ since the (derived) restriction does not change the lengths of derived fibers. By Lemma 4.3.2 the support of j^*B is the curve E, so the pullback j^*B is not a torsion object. Among the options listed in Proposition 3.1.3, only one is an object which is not torsion and not locally free at p, and therefore $j^*B \simeq \mathcal{O}_p[0] \oplus M[a]$ for a simple vector bundle M on E, as claimed in the statement. ☐

4.3.4. Lemma. *Let B and $j\colon E \to \mathbb{P}^2$ be as in Setting 4.3.1. If $\mathcal{H}^i(j^*B) = 0$, then $\mathcal{H}^i(B) = 0$.*

Proof. Since $j\colon E \to \mathbb{P}^2$ is an inclusion of a (Cartier) divisor, by Lemma 2.3.8 the vanishing of $\mathcal{H}^i(j^*B)$ implies that $\operatorname{supp}(\mathcal{H}^i(B)) \cap E = \emptyset$. By Lemma 2.6.2 the object B is $\operatorname{Stab}(p)$-invariant, hence $\mathcal{H}^i(B)$ is also $\operatorname{Stab}(p)$-invariant. Since E passes through p and $\operatorname{Stab}(p)$ acts transitively on $\mathbb{P}^2 \setminus \{p\}$, we obtain that the nonderived restriction of $\mathcal{H}^i(B)$ to any point of $\mathbb{P}^2$ is zero, but this implies $\mathcal{H}^i(B) = 0$. ☐

4.3.5. Corollary. *Let B be as in Setting 4.3.1. Then B has at most two nonzero cohomology sheaves, and at most one of them is not a torsion sheaf supported at p.*

Proof. Pick an elliptic curve $j\colon E \to \mathbb{P}^2$ which passes through p. Then Lemmas 4.3.3 and 4.3.4 imply that B has at most two nonzero cohomology sheaves. Moreover, we see that the (derived) restriction of B to some point $q \in E$ which is distinct from p is concentrated in a single degree. Since B is $\operatorname{Stab}(p)$-invariant, it is locally free away from p and thus only one of cohomology sheaves is nonzero around the point q. ☐

4.4 The structure of B

4.4.1. Lemma. *Let $\mathcal{F}$ be a nonzero coherent sheaf on a smooth surface S supported at a single point $p \in S$. Then for any curve $j\colon C \hookrightarrow S$ passing through p we have $L_1 j^*\mathcal{F} \neq 0$ and $L_0 j^*\mathcal{F} \neq 0$. Moreover, those two zero-dimensional sheaves have the same length.*

Proof. We may work locally and assume that S is a spectrum of a local ring. Let $\mathfrak{m} \subset \mathcal{O}_S$ be the ideal sheaf of the point p. The curve C is given by $f = 0$ for some $f \in \mathfrak{m}$. The derived pullback $j^*\mathcal{F}$ is computed by the complex $\mathcal{F} \xrightarrow{f\cdot -} \mathcal{F}$. Since $\mathcal{F}$ is set-theoretically supported at the point p, for some $n \gg 0$ we have $f^n \in \mathrm{Ann}(\mathcal{F})$. The multiplication by f thus cannot be an automorphism of $\mathcal{F}$. Since $\mathcal{F}$ is a vector space of finite dimension, this means the kernel and cokernel of the multiplication map are both nonzero and have the same dimension. □

4.4.2. Lemma. *Let $\mathcal{F}$ be a nonzero coherent sheaf on a smooth surface S supported at a single point $p \in S$. Assume that for any tangent direction at p there exists a smooth curve $j\colon C \hookrightarrow S$ passing through p with that tangent direction such that the torsion sheaf $L_0 j^*\mathcal{F}$ has length one. Then $\mathcal{F}$ is isomorphic to a skyscraper sheaf $\mathcal{O}_p$ on S.*

Proof. Let $A := \mathcal{O}_{S,p}$ be the local ring of the point $p \in S$, and denote by $\mathfrak{m}$ the maximal ideal of A. Let $C \subset S$ be one of the curves from the statement, and let $f \in \mathfrak{m}$ be an equation of the curve C. Then the nonderived restriction $L_0 j^*\mathcal{F}$ is isomorphic to $\mathcal{F}/f\mathcal{F}$. Note that the quotient $\mathcal{F}/\mathfrak{m}\mathcal{F}$ is nonzero since $\mathcal{F}$ is a nonzero sheaf. Since the length of $\mathcal{F}/f\mathcal{F}$ is one, this implies that $\mathcal{F}/\mathfrak{m}\mathcal{F}$ is an one-dimensional vector space. By Nakayama's lemma $\mathcal{F}$ is a cyclic module, i.e., $\mathcal{F} \simeq A/I$ for some ideal $I \subset A$ contained in $\mathfrak{m}$.

Let I_p be the image of $I \subset \mathfrak{m}$ in the cotangent space $T_p^\vee := \mathfrak{m}/\mathfrak{m}^2$. If $I_p = T_p^\vee$, then by Nakayama's lemma $I = \mathfrak{m}$ and then $\mathcal{F} \simeq A/\mathfrak{m} \simeq \mathcal{O}_p$, so the lemma is proved. Assume now that I_p is a proper subset of $T_p^\vee$. For an equation $f \in \mathfrak{m}$ of a curve C as in the statement let $[f] \in T_p^\vee$ denote its class in $T_p^\vee$. If I_p is a nonzero subspace, choose a curve $C = \{f = 0\}$ such that $[f] \in I_p$, and if I_p is zero, choose an arbitrary C. The assumption on the length of $L_0 j^*\mathcal{F}$ implies that $(I, f) = \mathfrak{m}$. But by the choice of f the image of the ideal (I, f) in the cotangent space $T_p^\vee$ is a proper subset of $T_p^\vee$, a contradiction. Thus $I = \mathfrak{m}$ is the only option. □

4.4.3. Lemma. *Let B be as in Setting 4.3.1. At least one cohomology sheaf $\mathcal{H}^i(B)$ has torsion.*

Proof. Assume that all cohomology sheaves are torsion-free. By Corollary 4.3.5 the object B has only one nonzero cohomology sheaf. Moreover, by Lemma 2.6.1 the sheaf $\mathcal{H}^0(B)$ is not zero. Hence $B \simeq \mathcal{F}[0]$ for some $\mathrm{Stab}(p)$-invariant torsion-free coherent sheaf $\mathcal{F}$ on $\mathbb{P}^2$. By

Lemma 2.3.13 the derived restriction $j^*\mathcal{F}$ is concentrated in degree zero. From Lemma 4.3.3 we conclude that $L_0 j^*\mathcal{F} \simeq \mathcal{O}_p \oplus M$ for a vector bundle M on the curve E, and $L_1 j^*\mathcal{F} = 0$.

Since $\mathcal{F}$ is a torsion-free sheaf on a surface, we may consider the short exact sequence from Lemma 2.3.12:

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0 \tag{4.4.3.1}$$

where $\mathcal{E}$ is locally free and $\mathcal{Q}$ is a torsion sheaf. By $\mathrm{Stab}(p)$-invariance of $\mathcal{F}$ and the uniqueness of the short exact sequence the torsion sheaf $\mathcal{Q}$ is supported only at the point p.

Consider the long exact sequence of derived pullbacks $L_\bullet j^*$ induced by the short exact sequence (4.4.3.1):

$$0 \to L_1 j^*\mathcal{Q} \to L_0 j^*\mathcal{F} \to L_0 j^*\mathcal{E} \to L_0 j^*\mathcal{Q} \to 0.$$

The sheaf $L_1 j^*\mathcal{Q}$ is a nonzero torsion sheaf by Lemma 4.4.1. Since the torsion part of $L_0 j^*\mathcal{F}$ is isomorphic to a skyscraper $\mathcal{O}_p$, this implies that $L_1 j^*\mathcal{Q} \simeq \mathcal{O}_p$. By Lemma 4.4.1 the nonderived pullback $L_0 j^*\mathcal{Q}$ is also isomorphic to a skyscraper at p. Since $\mathcal{Q}$ is $\mathrm{Stab}(p)$-invariant, the same holds for cubic curves passing through p in any direction. By Lemma 4.4.2 this implies that $\mathcal{Q} \simeq \mathcal{O}_p$. Then one easily computes that

$$\mathrm{Ext}^1(B, \mathcal{O}_p) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_p) \simeq \mathrm{Ext}^2(\mathcal{Q}, \mathcal{O}_p) \simeq \Bbbk.$$

Since the object B is the projection of a skyscraper sheaf, by Corollary 2.2.6 the vector space $\mathrm{Ext}^1(B, \mathcal{O}_p)$ is isomorphic to $\mathrm{Ext}^1(B, B)$. On the other hand, $\mathcal{F}$ is not locally free at a single point $p \in \mathbb{P}^2$, so $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ is at least two-dimensional by Lemma 2.4.4. This is a contradiction, so at least one cohomology sheaf of B is not torsion-free. □

Remark. The first part of the argument in Lemma 4.4.3 shows that if B is a single coherent sheaf, then it is a torsion-free sheaf which is a kernel of a map between a vector bundle and a skyscraper. Sheaves like that do actually arise in semiorthogonal decompositions

$D^b_{\mathrm{coh}}(\mathbb{P}^2) = \langle \mathcal{A}, \mathcal{B} \rangle$ as *left* projections $L_{\mathcal{A}}(\mathcal{O}_p)$ of a skyscraper sheaf when the subcategory $\mathcal{B}$ is generated by a single exceptional vector bundle. For example, when $\mathcal{B} = \langle \mathcal{O} \rangle$, the projection triangle is

$$\mathcal{O} \to \mathcal{O}_p \to \mathcal{I}_p[1].$$

Here the ideal sheaf $\mathcal{I}_p$ is exactly the sheaf described by the first part of the argument in Lemma 4.4.3. Thus the second part of the argument may be considered as a way to distinguish the left projection and the right projection of a skyscraper sheaf.

4.4.4. Lemma. *Let B be as in Setting 4.3.1. Then B is concentrated in degrees $[-1;0]$, $\mathcal{H}^0(B)$ is isomorphic to $\mathcal{O}_p$, the sheaf $\mathcal{H}^{-1}(B)$ is locally free, and the projection triangle*

$$B \to \mathcal{O}_p \to A$$

from Setting 4.3.1 is isomorphic to a truncation triangle of B, with $A \simeq \mathcal{H}^{-1}(B)[2]$.

Proof. By Lemma 4.4.3 we know that there exists some $i \in \mathbb{Z}$ such that the sheaf $\mathcal{H}^i(B)$ is not torsion-free. Let $T \subset \mathcal{H}^i(B)$ be the torsion subsheaf. It is $\mathrm{Stab}(p)$-invariant, so it is supported only at the point p. Consider the short exact sequence

$$0 \to T \to \mathcal{H}^i(B) \to \mathcal{H}^i(B)/T \to 0.$$

Consider the long exact sequence of derived pullbacks $L_\bullet j^*$ induced by that short exact sequence. The quotient $\mathcal{H}^i(B)/T$ is a torsion-free sheaf on a smooth surface, so using Lemma 2.3.13 we see $L_1 j^*(\mathcal{H}^i(B)/T) = 0$, and hence $L_1 j^* \mathcal{H}^i(B) \simeq L_1 j^* T$. This space is nonzero by Lemma 4.4.1. We also see that $L_0 j^* \mathcal{H}^i(B)$ contains the nonzero torsion subsheaf isomorphic to $L_0 j^* T$.

The relation between cohomology sheaves of $j^* B$ and derived pullbacks $L_\bullet j^* \mathcal{H}^i(B)$ is described in Lemma 2.3.8. In particular, this lemma implies that $\mathcal{H}^{i-1}(j^* B)$ has a quotient isomorphic to $L_1 j^* \mathcal{H}^i(B)$, and $\mathcal{H}^i(j^* B)$ has a subsheaf isomorphic to $L_0 j^* \mathcal{H}^i(B)$. Thus both

i'th and $(i-1)$'th cohomology sheaves of j^*B are nonzero, and moreover $\mathcal{H}^i(j^*B)$ has a nonzero torsion subsheaf.

By Lemma 4.3.3 this implies that $i = 0$, the object j^*B is concentrated in degrees $[-1; 0]$, the sheaf $\mathcal{H}^0(j^*B)$ is isomorphic to a skyscraper sheaf $\mathcal{O}_p$, and the sheaf $\mathcal{H}^{-1}(j^*B)$ is locally free. By Lemma 4.3.4 the cohomology sheaves of the complex B are also zero outside of the range $[-1; 0]$. Since in this case $L_0 j^* \mathcal{H}^0(B) \simeq \mathcal{H}^0(j^*B) \simeq \mathcal{O}_p$, and the sheaf $\mathcal{H}^0(B)$ is $\mathrm{Stab}(p)$-invariant, by Lemma 4.4.2 this implies that $\mathcal{H}^0(B) \simeq \mathcal{O}_p$.

Since the sheaf $\mathcal{H}^{-1}(j^*B)$ is locally free on a curve, its subsheaf $L_0 j^* \mathcal{H}^{-1}(B)$ is also locally free. The sheaf $\mathcal{H}^{-1}(B)$ is $\mathrm{Stab}(p)$-invariant, and the curve $j \colon E \to \mathbb{P}^2$ passes through p, so the nonderived rank of the sheaf $\mathcal{H}^{-1}(B)$ is constant over $\mathbb{P}^2$. Therefore $\mathcal{H}^{-1}(B)$ is locally free.

Thus B is concentrated in degrees -1 and 0, with $\mathcal{H}^0(B) \simeq \mathcal{O}_p$ and $\mathcal{H}^{-1}(B)$ locally free. Using Lemma 2.3.7 it is easy to compute that $\mathrm{Hom}(B, \mathcal{O}_p)$ is one-dimensional. Any nonzero map is proportional to the truncation morphism $B \to \tau_{\geq 0}(B) \simeq \mathcal{O}_p[0]$, and the cone of this map is isomorphic to $\mathcal{H}^{-1}(B)[2]$. This confirms the last claim of the statement. □

4.5 Full description of A and B

4.5.1. Lemma. *Let X be a smooth and proper variety, and let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. Let $E \in \mathcal{A}$ be an exceptional object and suppose that for any point $p \in X$ the projection $L_{\mathcal{A}}(\mathcal{O}_p) \in \mathcal{A}$ lies in the subcategory $\langle E \rangle \subset \mathcal{A}$. Then $\mathcal{A} = \langle E \rangle$.*

Proof. By Lemma 2.2.7 the subcategory $\langle E \rangle \subset \mathcal{A}$ is admissible in $\mathcal{A}$. Consider the induced semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}', E \rangle$. Let $L_{\mathcal{A}'} \colon D^b_{\mathrm{coh}}(X) \to \mathcal{A}'$ be the left projection functor. It is equal to the composition of the left projection functor $L_{\mathcal{A}}$ and the left projection to $\mathcal{A}'$ inside $\mathcal{A}$. Thus the condition $L_{\mathcal{A}}(\mathcal{O}_p) \in \langle E \rangle$ implies that $L_{\mathcal{A}'}(\mathcal{O}_p) = 0$ for all skyscrapers. Since $L_{\mathcal{A}'}$ is the left adjoint for the inclusion functor $\mathcal{A}' \hookrightarrow D^b_{\mathrm{coh}}(X)$, for any object $A \in \mathcal{A}'$ we have

$$\mathrm{RHom}_X(A, \mathcal{O}_p) \cong \mathrm{RHom}(A, L_{\mathcal{A}'}(\mathcal{O}_p)) = 0.$$

This is true for all points $p \in X$, so the support of any object $A \in \mathcal{A}'$ is empty. Therefore the subcategory $\mathcal{A}'$ is a zero subcategory, which means that $\mathcal{A} = \langle E \rangle$, as claimed. □

4.5.2. Lemma. *Let A, B be as in Setting 4.3.1. Then $\mathcal{A}$ is generated by a single exceptional vector bundle.*

Proof. By Lemma 4.4.4 we know that the object A in the projection triangle $B \to \mathcal{O}_p \to A$ is isomorphic to $N[2]$ for some vector bundle N on $\mathbb{P}^2$. Semiorthogonality of A and B implies that

$$\mathrm{RHom}(N, N) \cong \mathrm{RHom}(N[2], N[2]) \cong \mathrm{RHom}(\mathcal{O}_p, N[2]).$$

Since N is locally free, the space $\mathrm{RHom}(\mathcal{O}_p, N[2])$ is concentrated in degree 0. Therefore $\mathrm{Ext}^*(N, N)$ is also concentrated in degree zero. Thus the bundle N is rigid. By [Dre86, Cor. 7] all rigid vector bundles on $\mathbb{P}^2$ are direct sums of exceptional bundles. Suppose that N is not a direct sum of copies of the same exceptional bundle. Then N has two non-isomorphic direct summands R_0 and R_1, which are both exceptional bundles. It is known that an exceptional vector bundle on $\mathbb{P}^2$ is uniquely determined by its slope [DLP85, Lem. 4.3], so without loss of generality we may assume that the slope of R_0 is strictly smaller than the slope of R_1.

Since every exceptional bundle on $\mathbb{P}^2$ is stable [GR87, Th. 4.1], the inequality of slopes implies that

$$\mathrm{R}^0\mathrm{Hom}(R_1, R_0) = 0.$$

Then the pair R_0, R_1 is semiorthogonal: indeed, $\mathrm{Ext}^*(N, N) = \mathrm{Ext}^0(N, N)$, so there are no higher Exts between the direct summands of N, and there are no R^0Homs from R_1 to R_0 by semistability.

The category $\mathcal{A}$ is closed under direct summands, so both R_0 and R_1 lie in $\mathcal{A}$. The orthogonal subcategory $\mathcal{B} = {}^\perp\mathcal{A}$ is contained inside ${}^\perp\langle R_0, R_1 \rangle$. By [GR87, Th. 5.10] the orthogonal to an exceptional pair on $\mathbb{P}^2$ is generated by a single exceptional vector bundle. In particular, this would imply that any object in $\mathcal{B}$ is locally free, but we assumed from the very beginning in Setting 4.3.1 that $B \in \mathcal{B}$ is not a locally free object. This contradiction shows that $A \simeq N[2] \simeq (N')^{\oplus n}[2]$ is a direct sum of several copies of an exceptional vector bundle N'.

All exceptional bundles on $\mathbb{P}^2$ are rigid and therefore PGL(3)-invariant. Thus by Proposition 2.5.3 we know that the pullback of the projection triangle

$$B \to \mathcal{O}_p \to A$$

along some element $g \in \mathrm{PGL}(3)$ is a projection triangle for a skyscraper at the point $g^{-1}(p)$. Thus the projection of any skyscraper to $\mathcal{A}$ is isomorphic to $(N')^{\oplus n}[2]$. By Lemma 4.5.1 we see that the subcategory $\mathcal{A}$ is generated by an exceptional vector bundle N'. This establishes the second part of the statement. $\square$

This lemma is the final step in the proof of Theorem 4.1.3, and hence it also establishes the main theorem of this chapter, Theorem 4.1.1.

Chapter 5: Admissible subcategories supported on (-1)-curves

In the previous section we showed that any admissible subcategory of $D^b_{\mathrm{coh}}(\mathbb{P}^2)$ is one of the examples we know. We would like to generalize this statement for some other varieties. The next natural case would be to study del Pezzo surfaces. They are closely related to $\mathbb{P}^2$, and there are strong structural results [KO94] about exceptional objects and exceptional collections on del Pezzo surfaces.

The case of $\mathbb{P}^2$ is quite special. For instance, on other surfaces we may have admissible subcategories supported set-theoretically on closed subsets. A simple example is that the structure sheaf of an exceptional divisor for a blow-up of a smooth point is an exceptional object. Thus it is interesting to understand and potentially classify those kinds of subcategories.

The main result of this chapter is Proposition 5.3.4, where we prove that any admissible subcategory supported on a smooth (-1)-curve in a surface is a standard subcategory, i.e., it is generated by a twist of the structure sheaf of that (-1)-curve. It is possible and not too dificult to give a proof along the lines of Theorem 4.1.3: restricting the projections of a skyscraper sheaf to various anticanonical divisors, using Proposition 3.1.3 to understand the possibilities, and then proceed with reductions similar to the ones in Section 4.4. However, we use a different, perhaps more conceptual approach, following a suggestion by Kuznetsov. It uses the additivity of Hochschild homology in the form proved in [Kuz09].

One application of this local classification result is given in Corollary 5.3.5, where we prove the non-existence of phantom subcategories in some blow-ups of surfaces. Note that any nontrivial blow-up has a nontrivial semiorthogonal decomposition [Orl93], so the non-existence of phantoms is interesting.

We start with an outline of a direct proof for the classification result in Section 5.1. We continue with a reminder on Hochschild homology and its interaction with semiorthogonal decompositions, following [Kuz09], in Section 5.2. We complete the classification of possible admissible subcategories supported on (-1)-curves in Section 5.3.

5.1 A sketch of a direct proof for the classification

As mentioned above, the actual proof for the classification is given in Section 5.3, but here we give an outline for a direct argument. We omit many details in this rough sketch.

Let S be a smooth and proper surface, and let $E \subset S$ be a smooth (-1)-curve. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(S)$ be an admissible subcategory. Assume that any object of $\mathcal{A}$ is set-theoretically supported on E. For each point $p \in E$ let A_p be the left projection $L_{\mathcal{A}}(\mathcal{O}_p)$ of the skyscraper sheaf at p into the subcategory $\mathcal{A}$. The idea behind our argument is that admissible subcategories are closed under small deformations (Proposition 2.5.3), but most complicated objects supported on a curve E may be deformed away from E. Since all objects in $\mathcal{A}$ are assumed to be supported on E, the object $A_p \in \mathcal{A}$ cannot be deformed away from E, and this is a strong constraint on that object.

Moreover, there are other constraints arising from Proposition 3.1.3. It gives us a list of possible options for the restriction of A_p to an anticanonical divisor of S. It is not hard to check that the only property of anticanonical divisors that is used in Proposition 3.1.3 (and Theorem 3.1.1 as well, on which the proposition is based) is their relation with Serre duality. If $j\colon C \to S$ is a curve which intersects the (-1)-curve E in a single point, transversely, then in a Zariski open neighborhood of E the curve C is equivalent to an anticanonical divisor. Since we work only with objects in $\mathcal{A}$, and they all are supported on E, the list of options in Proposition 3.1.3 applies as well to the restriction j^*A_p to the curve C. So it is a torsion object of length two supported at the point p. We ought to be careful about the fact that the type of an object j^*A_p may, in principle, depend on the choices of the point p and the curve C, but for the purpose of this outline we ignore this difficulty.

If the restriction j^*A_p is a direct sum of skyscrapers $\mathcal{O}_p[0] \oplus \mathcal{O}_p[a]$ with $a \neq 0$, then by an argument similar to the ones from Section 4.4 we may check that the object A_p has only two cohomology sheaves, and both of them are pushforwards of line bundles from E. Since $E \simeq \mathbb{P}^1$, we know all line bundles, and we know all complexes made out of their pushforwards. This lets us enumerate all possibilities for A_p, and there is only one case which does not lead to a contradiction by deforming the object A_p away from the (-1)-curve (Proposition 2.5.3) and does not lead to a contradiction with the universal property of the projection functors (Corollary 2.2.6 for the morphism $\mathcal{O}_p \to L_{\mathcal{A}}(\mathcal{O}_p)$). This unique possible case is where $A_p \simeq \mathcal{O}_E(n)[0] \oplus \mathcal{O}_E(n)[-1]$ for some integer $n \in \mathbb{Z}$. Then we may show using Lemma 4.5.1 that $\mathcal{A} = \langle \mathcal{O}_E(n) \rangle$, as expected.

If the restriction $j^* A_p$ is a direct sum of two skyscrapers, both in degree zero, then we may check that A_p has only one cohomology sheaf, and it is a pushforward from $E \simeq \mathbb{P}^1$. An argument similar to the one we employ below in Proposition 5.3.4 shows that there are no admissible subcategories where projections of skyscraper look like that, so this situation is impossible.

If the restriction $j^* A_p$ is a torsion coherent sheaf of length two, then we may show that the object A_p is an extension of two pushforwards of line bundles from $E \simeq \mathbb{P}^1$. Again, we know all the line bundles on E, and we know all extensions between their pushforwards, so we once again may show that this situation is impossible.

5.2 Reminder on Hochschild homology

The material below is taken from [Kuz09]. See the reference for additional details and the proofs.

Let X be a smooth and proper variety, and let $\langle \mathcal{A}, \mathcal{B} \rangle = D^b_{\mathrm{coh}}(X)$ be a semiorthogonal decomposition. Let $R_{\mathcal{B}}$ and $L_{\mathcal{A}}$ denote the projection functors from $D^b_{\mathrm{coh}}(X)$ to $\mathcal{B}$ and $\mathcal{A}$, right and left respectively. By Proposition 2.7.3 there exist Fourier–Mukai kernels in $D^b_{\mathrm{coh}}(X \times X)$ representing those functors, and we denote the kernels with the same symbols. The kernels for the projection functors fit into a triangle in $D^b_{\mathrm{coh}}(X \times X)$:

$$R_{\mathcal{B}} \to \Delta_* \mathcal{O}_X \to L_{\mathcal{A}}.$$

The graded vector space $\mathrm{RHom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_* K_X[\dim X])$ is called the *Hochschild homology* of X, denoted by $\mathrm{HH}_\bullet(X)$. It is a straightforward consequence of this definition that there exists an isomorphism $\mathrm{HH}_{-\dim X}(X) \cong H^0(X, K_X)$. There exists an interpretation of the entire Hochschild homology in terms of Hodge decomposition, known as Hochschild–Kostant–Rosenberg theorem, but we do not need it.

For objects in $D^b_{\mathrm{coh}}(X \times X)$ there is a binary operation $-\circ-$, called convolution, which corresponds to the composition of Fourier–Mukai transforms. The structure sheaf $\Delta_*\mathcal{O}_X$ of the diagonal is an identity element for this operation, and thus there is an isomorphism $\Delta_* K_X[\dim X] \simeq \Delta_*\mathcal{O}_X \circ \Delta_* K_X[\dim X]$. It is proved in [Kuz09, Prop. 5.5] that any morphism $\varphi \in \mathrm{HH}_m(X)$ can be uniquely extended to a morphism of triangles:

$$\begin{tikzcd}
R_{\mathcal{B}} \arrow[r] \arrow[d, "\gamma_{\mathcal{A}}(\varphi)"] & \Delta_*\mathcal{O}_X \arrow[r] \arrow[d, "\varphi"] & L_{\mathcal{A}} \arrow[d, "\gamma_{\mathcal{B}}(\varphi)"] \\
R_{\mathcal{B}} \circ \Delta_* K_X[m+\dim X] \arrow[r] & \Delta_* K_X[m+\dim X] \arrow[r] & L_{\mathcal{A}} \circ \Delta_* K_X[m+\dim X]
\end{tikzcd} \tag{5.2.0.1}$$

The spaces $\mathrm{RHom}(R_{\mathcal{B}}, R_{\mathcal{B}} \circ \Delta_* K_X[\dim X])$ and $\mathrm{RHom}(L_{\mathcal{A}}, L_{\mathcal{A}} \circ \Delta_* K_X[\dim X])$ are called Hochschild homology spaces $\mathrm{HH}_\bullet(\mathcal{B})$ and $\mathrm{HH}_\bullet(\mathcal{A})$ respectively. Thus the uniqueness and existence of the extension (5.2.0.1) of the map φ to a morphism of triangles using certain maps $\gamma_{\mathcal{A}}(\varphi)$ and $\gamma_{\mathcal{B}}(\varphi)$ produces a map

$$\mathrm{HH}_\bullet(X) \xrightarrow{(\gamma_{\mathcal{A}}, \gamma_{\mathcal{B}})} \mathrm{HH}_\bullet(\mathcal{A}) \oplus \mathrm{HH}_\bullet(\mathcal{B}).$$

Theorem 7.3 in [Kuz09] shows that this map is an isomorphism, i.e., Hochschild homology is additive for semiorthogonal decompositions.

5.3 Local classification on (-1)-curves

Let S be a smooth proper surface. Suppose that $\mathcal{A} \subset D^b_{\mathrm{coh}}(S)$ is an admissible subcategory supported set-theoretically on some smooth (-1)-curve in S. In this subsection we first show that $\mathrm{HH}_{-2}(\mathcal{A}) = 0$ in Lemma 5.3.1, and then in Lemmas 5.3.2 and 5.3.3 we deduce from the vanishing of this homology group the fact that any object of $\mathcal{A}$ is a pushforward of some object from the derived category $D^b_{\mathrm{coh}}(\mathbb{P}^1)$ of sheaves on the (-1)-curve. Finally, in Proposition 5.3.4 we complete the classification.

5.3.1. Lemma. *Let X be a smooth and proper variety, and let $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ be an admissible subcategory. Assume that $\mathcal{A}$ is supported on a proper closed subset $Z \subset X$. Then the bottom Hochschild homology $\mathrm{HH}_{-\dim X}(\mathcal{A})$ vanishes.*

Remark. In other conventions this space may be called $\mathrm{HH}_{\dim X}(\mathcal{A})$. See [Kuz09, Rem. 2.2].

Proof. Let $\mathcal{B} := {}^{\perp}\mathcal{A}$ be the orthogonal subcategory in $D^b_{\mathrm{coh}}(X)$. Let $\gamma_{\mathcal{B}} \colon \mathrm{HH}_\bullet(X) \to \mathrm{HH}_\bullet(\mathcal{B})$ be the restriction morphism defined in Section 5.2. By the additivity of Hochschild homology [Kuz09, Th. 7.3] the kernel of the map $\mathrm{HH}_{-\dim X}(X) \to \mathrm{HH}_{-\dim X}(\mathcal{B})$ is isomorphic to $\mathrm{HH}_{-\dim X}(\mathcal{A})$. Thus it is enough to prove that this map is injective.

Using the definition given in Section 5.2 it is easy to compute that the vector space $\mathrm{HH}_{-\dim X}(X)$ is isomorphic to $H^0(X, K_X)$. Suppose $s \in H^0(X, K_X)$ is a nonzero section such that $\gamma_{\mathcal{B}}(s)$ is a zero class in $\mathrm{HH}_{-\dim X}(\mathcal{B})$. Pick a point p in the open subset $X \setminus Z$ such that s does not vanish at p. By assumption the skyscraper sheaf $\mathcal{O}_p$ is orthogonal to every object in $\mathcal{A}$, and hence $\mathcal{O}_p \in \mathcal{B}$. The morphism of triangles (5.2.0.1) of objects in $D^b_{\mathrm{coh}}(X \times X)$ for the class $s \in \mathrm{HH}_{-\dim X}(X)$ produces the following morphism of triangles in $D^b_{\mathrm{coh}}(X)$ via a Fourier–Mukai transform of the skyscraper sheaf $\mathcal{O}_p$:

$$\begin{tikzcd}
R_{\mathcal{B}}(\mathcal{O}_p) \simeq \mathcal{O}_p \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma_{\mathcal{B}}(s)(\mathcal{O}_p)"] & \mathcal{O}_p \arrow[r] \arrow[d, "s(p)"] & 0 \arrow[d] \\
R_{\mathcal{B}}(\mathcal{O}_p \otimes K_X) \simeq \mathcal{O}_p \otimes K_X \arrow[r, "\mathrm{id}"] & \mathcal{O}_p \otimes K_X \arrow[r] & 0
\end{tikzcd} \tag{5.3.1.1}$$

If the class $\gamma_{\mathcal{B}}(s) \in \mathrm{HH}_{-\dim X}(\mathcal{B})$ is zero, then the natural transformation obtained by the Fourier–Mukai transform along $\gamma_{\mathcal{B}}(s)$ vanishes on every object, hence the leftmost vertical morphism is also necessarily zero. But by the commutativity of the diagram it vanishes if and only if the section s vanishes at the point p. However, by the choice of p this does not happen. Thus the morphism $\gamma_{\mathcal{B}}$ is injective on $\mathrm{HH}_{-\dim X}$ and the lemma is proved. $\square$

5.3.2. Lemma. *Let S be a smooth and proper surface, and let $C \subset S$ be a smooth (-1)-curve. Let $c \in \Gamma(S, \mathcal{O}_S(C))$ be a section cutting out the curve C. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(S)$ be an admissible subcategory supported on C. Then for every object $A \in \mathcal{A}$ the morphism $A \to A(C)$ in the derived category given by the multiplication with a section c is a zero morphism.*

Proof. Let $S \to S'$ denote the contraction of the (-1)-curve C to a point $p \in S'$. Let $U' \subset S'$ denote a Zariski neighborhood of the point p on which the canonical bundle $K_{S'} \simeq \Omega^2_{S'}$ is trivial. Denote by $U \subset S$ its preimage in S. A pullback of a constant section of K_S on U to a section $s \in \Gamma(U, K_S|_U)$ vanishes exactly along $C \subset U$ with multiplicity one. By construction in a neighborhood of $C \subset S$ the line bundle K_S is isomorphic to the line bundle $\mathcal{O}_S(C)$ with the sections s and c corresponding to each other. Since any object $A \in \mathcal{A}$ is supported on a subset $C \subset U$, the tensor multiplication with the section s produces a map $A \xrightarrow{s} A \otimes K_S$ well-defined on the whole surface S, and the claim in the statement of this lemma is equivalent to the fact that this map is zero.

We want to study the multiplication by s as a natural transformation using Hochschild homology methods. If s were a global section of K_S, then it would by definition give a class in Hochschild homology $\mathrm{HH}_{-2}(S)$. Since s is only defined in a neighborhood of the (-1)-curve, we can only construct a class in $\mathrm{HH}_{-2}(\mathcal{A})$ by a more careful procedure. To do this, note that $s \in H^0(U, K_S|_U)$ produces a morphisms $\mathcal{O}_U \to K_S \otimes \mathcal{O}_U$ of quasicoherent sheaves on S. We define the following morphism of quasicoherent sheaves on $S \times S$, where $\Delta \colon S \to S \times S$ denotes the diagonal inclusion:

$$\varphi_s \colon \Delta_* \mathcal{O}_U \to \Delta_*(K_S \otimes \mathcal{O}_U).$$

Let $L_{\mathcal{A}} \in D^b_{\mathrm{coh}}(S \times S)$ denote the Fourier–Mukai kernel for the left projection functor from $D^b_{\mathrm{coh}}(S)$ to the subcategory $\mathcal{A}$.

Claim. *1. $L_{\mathcal{A}}$ is supported on the closed subset $C \times C \subset S \times S$;*

2. the convolution $L_{\mathcal{A}} \circ \Delta_* \mathcal{O}_U$ is isomorphic to $L_{\mathcal{A}}$;

3. the convolution $L_{\mathcal{A}} \circ \Delta_*(K_S \otimes \mathcal{O}_U)$ is isomorphic to $L_{\mathcal{A}} \circ \Delta_* K_S$.

Proof of the claim. Since the projection of any skyscraper to $\mathcal{A}$ is by assumption an object supported on C, we see that $L_{\mathcal{A}}$ is set-theoretically supported on $S \times C$. Moreover, any skyscraper at a point in $S \setminus C$ is orthogonal to each object of $\mathcal{A}$, and hence projects to zero via $L_{\mathcal{A}}$, which implies that $L_{\mathcal{A}}$ is supported set-theoretically on $C \times C$. This directly implies that the convolution $L_{\mathcal{A}} \circ \Delta_* \mathcal{O}_U$ is isomorphic to the convolution $L_{\mathcal{A}} \circ \Delta_* \mathcal{O}_S$ with the structure sheaf of the diagonal, and this convolution is isomorphic to $L_{\mathcal{A}}$. The last statement is proved similarly. $\square$

Consider now the convolution $L_{\mathcal{A}} \circ \varphi_s$. By the claim above it may be considered as the following morphism in $D^b_{\mathrm{coh}}(S \times S)$:

$$L_{\mathcal{A}} \circ \varphi_s \colon L_{\mathcal{A}} \to L_{\mathcal{A}} \circ \Delta_* K_S.$$

By definition, this morphism is a class in $\mathrm{HH}_{-2}(\mathcal{A})$. By Lemma 5.3.1 this group vanishes. Therefore the morphism is zero in the derived category $D^b_{\mathrm{coh}}(S \times S)$, and the natural transformation between the Fourier–Mukai functors is also zero on every object.

Let $A \in \mathcal{A}$ be an arbitrary object. Then $L_{\mathcal{A}}(A) \cong A$ by definition. Then the projection of the morphism $A \otimes K_S^\vee \xrightarrow{s} A$ to the subcategory $\mathcal{A}$ fits into the following commutative square:

$$\begin{tikzcd}
A \otimes K_S^\vee \arrow[r, "s"] \arrow[d] & A \arrow[d, "\mathrm{id}"] \\
L_{\mathcal{A}}(A \otimes K_S^\vee) \arrow[r] & L_{\mathcal{A}}(A) \cong A
\end{tikzcd}$$

The bottom horizontal morphism is zero since it is given by the natural transformation arising from the zero morphism $L_{\mathcal{A}} \circ \varphi_s$. Since the right vertical arrow is an isomorphism, the top horizontal map $A \otimes K_S^\vee \to A$ is also zero, which is equivalent to the vanishing of the morphism $A \to A \otimes K_S$, and this is exactly what we wanted to show. $\square$

5.3.3. Lemma. *Let S be a smooth and proper surface, and let $C \subset S$ be a smooth curve. Pick a section $s \in \Gamma(S, \mathcal{O}_S(C))$ cutting out the curve C. Assume that $A \in D^b_{\mathrm{coh}}(S)$ is an object such that the morphism $A \xrightarrow{\cdot s} A(C)$ is zero in the derived category. Then A is isomorphic to a pushforward of an object from $D^b_{\mathrm{coh}}(C)$.*

Remark. A stronger result valid in arbitrary dimension was recently proved in [LO20, Th. 3.2]. The two-dimensional case is significantly easier than the general statement, so we include the direct proof.

Proof. Denote by $j\colon C \hookrightarrow S$ the inclusion morphism. Consider the restriction triangle for A:

$$A(-C) \xrightarrow{s} A \to j_* j^* A.$$

The first morphism in this triangle vanishes by assumption, thus the morphism $A \to j_* j^* A$ is a split monomorphism, i.e., an inclusion of a direct summand. Note that the derived pullback $j^* A \in D^b_{\mathrm{coh}}(C)$ in the derived category of a smooth curve is automatically formal, i.e., it is a direct sum of shifts of cohomology sheaves. Then the pushforward $j_* j^* A$ is also formal, and it is easy to show that any direct summand of a formal complex is formal, given by a choice of a direct summand in each cohomology sheaf. Thus $A \simeq \oplus \mathcal{H}^i(A)[-i]$, and each cohomology sheaf $\mathcal{H}^i(A)$ is a direct summand of a sheaf $j_* \mathcal{H}^i(j^* A)$. Any direct summand of the pushforward sheaf $j_* \mathcal{H}^i(j^* A)$ is a pushforward of some direct summand of $\mathcal{H}^i(j^* A)$. Thus A is isomorphic to a pushforward of an object in $D^b_{\mathrm{coh}}(C)$. $\square$

Now we can prove the main result of this section.

5.3.4. Proposition. *Let S be a smooth surface, and let $j\colon E \hookrightarrow S$ be an embedding of a smooth (-1)-curve. Let $\mathcal{A} \subset D^b_{\mathrm{coh}}(S)$ be a nonzero admissible subcategory supported on E. Then $\mathcal{A}$ is generated by an exceptional sheaf $j_* \mathcal{O}_E(k)$ for some $k \in \mathbb{Z}$.*

Proof. By Lemmas 5.3.2 and 5.3.3 any object of the subcategory $\mathcal{A}$ is a pushforward of some object from $D^b_{\mathrm{coh}}(E)$. Let $G \in D^b_{\mathrm{coh}}(E)$ be an object such that $j_* G$ is a generator of $\mathcal{A}$. Note that $\mathcal{A}$ contains pushforwards of all objects in $\langle G \rangle \subset D^b_{\mathrm{coh}}(\mathbb{P}^1)$. Suppose that G generates the entire derived category of E. Then $\mathcal{A}$ contains a skyscraper sheaf at some point of E. On the surface S this skyscraper sheaf may be deformed into a skyscraper sheaf at some point away from $E \subset S$. Since admissible subcategories are closed under small deformations (Proposition 2.5.3), this is a contradiction with the assumption that $\mathcal{A}$ is supported only on E. Therefore $G \in D^b_{\mathrm{coh}}(\mathbb{P}^1)$ cannot be a generator.

On $\mathbb{P}^1$ any object of the derived category splits into a direct sum of shifts of torsion sheaves and line bundles. It is easy to check that an object $G \in D^b_{\mathrm{coh}}(\mathbb{P}^1)$ is not a generator only in two cases: either G is a torsion object, or G is a direct sum of several copies of the same line bundle. The object G cannot be torsion by the same argument as above. Thus G is a direct sum of shifts of copies of $\mathcal{O}_{\mathbb{P}^1}(k)$ for some fixed k, and the subcategory $\mathcal{A}$ generated by its pushforward $j_* G$ can also be generated by $j_* \mathcal{O}_E(k)$, as claimed. $\square$

This classification implies that there are no phantom subcategories supported on a smooth (-1)-curve. Using the properties of Hochschild homology we may deduce from this the non-existence of phantom subcategories in some surfaces. This enlarges the list of surfaces that admit nontrivial semiorthogonal decompositions but provably do not have phantom subcategories from just the plane $\mathbb{P}^2$, checked in Theorem 4.1.1, to many other examples. The reduction to the local classification of admissible subcategories supported on (-1)-curves is essentially due to Kuznetsov (private communication). This idea was the starting point for the approach we used to prove the classification in this section.

5.3.5. Corollary. *Let S be a surface with a globally generated canonical bundle, and let*

$\pi\colon S' \to S$ be a blow-up of several distinct points. Then $D^b_{\mathrm{coh}}(S')$ does not contain any phantom subcategories.

Proof. Assume that $\mathcal{A} \subset D^b_{\mathrm{coh}}(S')$ is a phantom subcategory. Then by definition $\mathrm{HH}_{-2}(\mathcal{A}) = 0$. Let $s \in \Gamma(S, K_S)$ be any nonzero section. Its pullback $\pi^* s \in \Gamma(S', K_{S'})$ is a class in $\mathrm{HH}_{-2}(S')$ which necessarily restricts to a zero class in $\mathrm{HH}_{-2}(\mathcal{A})$. Similarly to the proof of Lemma 5.3.2 we see that for any object $A \in \mathcal{A}$ the morphism $A \to A(K_{S'})$ given by the multiplication with the section $\pi^* s$ vanishes. If the support of A contains a point $p \in S'$ such that $(\pi^* s)(p) \neq 0$, then at that point the multiplication with $\pi^* s$ is an isomorphism, hence nonzero. Therefore any object of $\mathcal{A}$ is set-theoretically supported on the vanishing locus of $\pi^*(s)$. This is a union of the preimage of the vanishing locus of s together with all exceptional divisors for the morphism π.

Since the same holds for an arbitrary section $s \in \Gamma(S, K_S)$ of the globally generated line bundle K_S, the conclusion is that any object $A \in \mathcal{A}$ is supported on the union of exceptional divisors. This is a disjoint union of several (-1)-curves. Objects supported on different (-1)-curves are completely orthogonal to each other. Thus $\mathcal{A}$ splits into a completely orthogonal sum of subcategories supported on each (-1)-curve separately. The options for each summand are classified in Proposition 5.3.4, and there are no phantom subcategories among them. $\square$

Chapter 6: On phantoms in del Pezzo surfaces

From the classification of admissible subcategories of $D^b_{\mathrm{coh}}(\mathbb{P}^2)$ given by Theorem 4.1.1 we easily see that there are no phantom subcategories in $\mathbb{P}^2$. In fact, the full classification is not necessary, and it is not hard to come to the same conclusion right after Lemma 4.3.3. In this section we explore other situations where the strong structural result given by Proposition 3.1.3 is sufficient to rule out the possibility of phantom subcategories. In Theorem 6.4.6 we show that on a del Pezzo surface of degree at least three there are no phantoms.

It seems difficult to give a meaningful classification of admissible subcategories in the derived category of a del Pezzo surface, or even to show that any admissible subcategory is generated by an exceptional collection. However, a key observation in Lemma 6.4.1 shows that many complicated admissible subcategories are not phantoms. The remaining options are easier to deal with. In some situations the result of this lemma is strong enough to imply that any phantom subcategory must be supported on a union of some (-1)-curves. For del Pezzo surfaces of degree greater or equal to three we can improve this result to non-existence of phantom subcategories.

The proof of Theorem 6.4.6 is based upon the notion of a *point-support* of a semiorthogonal decomposition at some point. This notion is introduced in Section 6.1. We also need some additional lemmas about objects set-theoretically supported on curves in surfaces. We study them by pulling them back along curves transverse to the support in Section 6.2. After establishing several probably well-known statements about smooth anticanonical divisors in del Pezzo surfaces in Section 6.3, we finish the proof of the main theorem in Section 6.4.

6.1 Point-supports of semiorthogonal decompositions

6.1.1. Definition. Let X be an algebraic variety. Pick a point $p \in X$. The *point-support* at p of a semiorthogonal decomposition $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a set defined as follows. Consider the projection triangle of the skyscraper sheaf at p:

$$B \to \mathcal{O}_p \to A \to B[1].$$

Then the point-support is the set-theoretic support of $A \oplus B$.

Question. Suppose $\mathcal{A} \subset D^b_{\mathrm{coh}}(X)$ is an admissible subcategory. Is it true that for any point $p \in X$ the supports of the two projections $L_{\mathcal{A}}(\mathcal{O}_p)$ and $R_{\mathcal{A}}(\mathcal{O}_p)$ coincide? If this is true, it would be more convenient to define the point-support of an admissible subcategory instead of the point-support of a semiorthogonal decomposition.

6.1.2. Lemma. *Let X be an algebraic variety, let $p \in X$ be a point, and let $D^b_{\mathrm{coh}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Let S_p be the point-support of this decomposition at p.*

1. *S_p is a connected closed subset which contains p.*

2. *If $S_p = \{p\}$, then either $\mathcal{O}_p \in \mathcal{A}$, or $\mathcal{O}_p \in \mathcal{B}$.*

3. *If $S_p \neq \{p\}$, then* $\mathrm{supp}(A) = \mathrm{supp}(B) = S_p$.

Proof. Let A_p and B_p denote the projections of the skyscraper. Consider the subset $\mathrm{supp}(B_p)$. Suppose that it has a nonempty connected component which does not contain p. Then by Lemma 2.3.3 the object B_p has a nonzero direct summand B' which is not supported at p. Then the map $B_p \to \mathcal{O}_p$ factors through the projection to B_p/B', and therefore its cone, A_p, has a direct summand isomorphic to $B'[1]$. But then $\mathrm{RHom}(B_p, A_p) \neq 0$, which is a contradiction with semiorthogonality. Thus the support of B_p (and by a similar argument the support of A_p as well) is either empty, or a connected subset containing p. The union of two connected subsets both containing p is also connected, and this proves part (1).

Suppose now that $S_p = \{p\}$. Assume that both A_p and B_p are nonzero objects supported at a single point p. Let $a \in \mathbb{Z}$ be the smallest number such that $\mathcal{H}^a(A_p) \neq 0$ and let $b \in \mathbb{Z}$ be the largest number such that $\mathcal{H}^b(B_p) \neq 0$. Pick nonzero morphisms $\mathcal{H}^b(B_p) \twoheadrightarrow \mathcal{O}_p$ and $\mathcal{O}_p \hookrightarrow \mathcal{H}^a(A_p)$, which always exist for coherent sheaves supported at one point. Then the composition

$$B_p \to \mathcal{H}^b(B_p)[-b] \twoheadrightarrow \mathcal{O}_p[-b] \hookrightarrow \mathcal{H}^a(A_p)[-b] \to A_p[a-b]$$

with truncation morphisms is a morphism $B_p \to A_p[a-b]$ which by construction is nonzero on cohomology sheaves. This contradicts semiorthogonality, and thus at least one of A_p and B_p must be a zero object when $S_p = \{p\}$, so the part (2) is proved.

To deal with the last part, note that the long exact sequence of cohomology sheaves proves that $\operatorname{supp}(B_p) \subset \operatorname{supp}(\mathcal{O}_p) \cup \{p\}$ and similarly for $\operatorname{supp}(A_p)$. Since both of those supports are either empty or contain the point p, part (3) follows. $\square$

6.1.3. Lemma. *Let Y be a smooth variety, $S_1, S_2 \subset Y$ two closed subsets whose set-theoretic intersection $S_1 \cap S_2$ contains an isolated point. Let $F_1, F_2 \in \operatorname{Perf}(Y)$ be objects whose set-theoretical supports are S_1, S_2 respectively. Then* $\operatorname{RHom}(F_1, F_2) \neq 0$.

Proof. We can compute the RHom-space by the dualization:

$$\operatorname{RHom}(F_1, F_2) \cong \mathrm{R}\Gamma\,(Y, F_1^\vee \otimes F_2).$$

The support of the tensor product $F_1^\vee \otimes F_2$ is the intersection $S_1 \cap S_2$. It contains an isolated point, so by Lemma 2.3.3 the object $F_1^\vee \otimes F_2$ has a nonzero direct summand supported only at a single point. Any object with zero-dimensional support has a nonvanishing (hyper)cohomology class given by a nonzero global section of the lowest degree cohomology sheaf, so the lemma is proved. $\square$

6.1.4. Lemma. *Let Y be a smooth variety, $p, q \in Y$ distinct points. Let $D^b_{\mathrm{coh}}(Y) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Denote by S_p, S_q the point-supports of the decomposition at p*

and q respectively. Then the set-theoretic intersection $S_p \cap S_q$ does not contain any isolated points.

Remark. This lemma is most useful on surfaces, where all nontrivial point-supports are curves, and curves usually intersect along finitely many points.

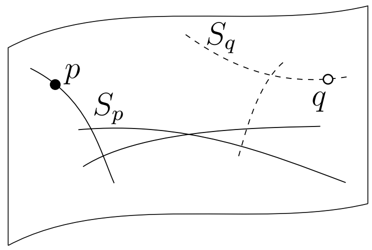

Figure 6.1: An impossible situation

Proof. Without loss of generality assume S_p is larger than just $\{p\}$. Consider first the situation where $S_q = \{q\}$. In this case by Lemma 6.1.2 there are two options:

- $\mathcal{O}_q \in \mathcal{A}$. Since $q \in S_p$ by assumption, we have $\mathrm{RHom}(R_{\mathcal{B}}(\mathcal{O}_p), \mathcal{O}_q) \neq 0$, but this contradicts semiorthogonality.

- or $\mathcal{O}_q \in \mathcal{B}$. Note that $\mathrm{supp}(L_{\mathcal{A}}(\mathcal{O}_p)) = S_p$ contains q by assumption, hence the graded space $\mathrm{RHom}(\mathcal{O}_q, L_{\mathcal{A}}(\mathcal{O}_p))$ is nonzero, but this also contradicts semiorthogonality.

Thus we may assume that $S_q \neq \{q\}$. Then by the same lemma $\mathrm{supp}(L_{\mathcal{A}}(\mathcal{O}_q)) = S_q$. Consider the space $\mathrm{RHom}(R_{\mathcal{B}}(\mathcal{O}_p), L_{\mathcal{A}}(\mathcal{O}_q))$. If the intersection $S_p \cap S_q$ contains an isolated point, then by Lemma 6.1.3 this space is not zero, but this again is impossible by semiorthogonality of $\mathcal{A}$ and $\mathcal{B}$. $\square$

Point-support subsets cannot be entirely arbitrary. For example, on surfaces we can show that their intersections with anticanonical divisors are relatively simple.

6.1.5. Lemma. *Let S be a smooth proper surface, and let $D^b_{\mathrm{coh}}(S) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Let $p \in S$ be a point, denote by S_p the point-support of the decomposition at p.*

Assume that S_p has dimension one. Let $j\colon D \hookrightarrow S$ be an anticanonical divisor of S passing through p. Suppose that the point p and some additional point $q \neq p$ satisfy the following:

1. *both p and q are isolated points in the set-theoretic intersection $S_p \cap D$;*
2. *both p and q are smooth points of D.*

Then $S_p \cap D = \{p, q\}$. Moreover, let B denote the (right) projection of the skyscraper sheaf at the point p to $\mathcal{B}$. Then $j^*B \simeq \mathcal{O}_p[0] \oplus \mathcal{O}_q[d_q]$ for some shift $d_q \in \mathbb{Z}$.

Proof. Consider a projection triangle $B \to \mathcal{O}_p \to A$ of the skyscraper at the point p. By Lemma 6.1.2 we know that $\mathrm{supp}(B) = S_p$, so Lemma 2.3.8 implies that $\mathrm{supp}(j^*B) = S_p \cap D$. Since B is a projection of a skyscraper and j is an inclusion of an anticanonical divisor into a surface, we may apply Proposition 3.1.3. By the last claim of the proposition the support of the object j^*B has at most two connected components. Since the points p and q are isolated in the intersection $S_p \cap D$, this implies that $\mathrm{supp}(j^*B) = \{p\} \sqcup \{q\}$. Now we may apply the classification result from Proposition 3.1.3. Among the options listed in the lemma there is only one whose support is two distinct points, and it is a direct sum of two skyscrapers in some degrees. $\square$

6.2 Cutting lemmas

This subsection contains a few observations about objects in the derived categories of surfaces which are set-theoretically supported on curves.

6.2.1. Definition. Let S be a smooth surface, $F \in D^b_{\mathrm{coh}}(S)$ an object. Assume that the set-theoretic support of F is a reduced curve $C \subset S$. A *slice* of F at a point $p \in C$ is the derived pullback j^*F to a curve $j\colon D \to S$ which is smooth at the point p and does not intersect C anywhere else.

Note that an alternative way to state the definition would be to let D intersect C at some other points, but replace the derived pullback by the largest direct summand supported at the point p. This is equivalent to replacing D with an open neighborhood of p in D.

6.2.2. LEMMA. *Let U be a smooth surface, let $i\colon C \hookrightarrow U$ be a curve, and let $p \in C$ be a point. Let $F \in \mathrm{Perf}(U)$ be an object whose set-theoretic support is C. Suppose that there exists a slice of F at p which is a torsion object of length one. Then C equipped with a reduced scheme structure is smooth at the point p, and, after possibly replacing U by a Zariski-neighborhood of p, the object F is isomorphic to $i_*\mathcal{O}_C[a]$ for some shift $a \in \mathbb{Z}$.*

Proof. Let $j\colon D \to U$ be a smooth curve such that the derived pullback $j^*F \in D^b_{\mathrm{coh}}(D)$ is a torsion object of length one, i.e., it is isomorphic to a shift of a skyscraper sheaf $\mathcal{O}_p[a]$ for some $a \in \mathbb{Z}$. By Lemma 2.3.8 the isomorphism $j^*F \simeq \mathcal{O}_p[a]$ implies that in a neighborhood of the point p the object F has only one nonzero cohomology sheaf, $F \simeq \mathcal{F}[a]$ for some coherent sheaf $\mathcal{F} \in \mathrm{Coh}(U)$. By shrinking U we may assume that $U \simeq \mathrm{Spec}\,(A)$ for some ring A, the coherent sheaf $\mathcal{F}$ corresponds to a module M over the ring A, and the smooth curve $D \subset U$ is defined by an equation $\{d = 0\}$ for an element $d \in A$. The assumption that M/dM is isomorphic to a skyscraper sheaf at the point p implies that $M/\mathfrak{m}_pM$ is also a skyscraper sheaf, so by Nakayama's lemma M is locally isomorphic to a cyclic module, i.e., the quotient $M \simeq A/I$ for some ideal $I \subset A$.

Since $(A/I)/d = (A/d)/I$ has length one, and $A/(d)$ is a discrete valuation ring, this means that the image of I in the quotient ring $A/(d)$ is generated by one regular element $\tilde{f} \in A/(d)$ such that $\tilde{f}$ generates the maximal ideal of $A/(d)$. Pick a preimage $f \in A$ of $\tilde{f}$ in the ideal I. We will show that f generates I. Consider the short exact sequence

$$0 \to I/fA \to A/fA \to A/I \to 0.$$

The derived pullback to the smooth curve $j\colon D \to U$ produces a long exact sequence of modules over the quotient ring A/d. Consider the following fragment:

$$L_1j^*(A/I) \to L_0j^*(I/fA) \to L_0j^*(A/fA) \to L_0j^*(A/I) \to 0$$

Since $L_1j^*(A/I) \simeq L_1j^*M = 0$ by the assumption of the theorem, this is in fact a short exact sequence. Note that $L_0 j^*(A/fA)$ by the definition is isomorphic to the quotient $A/(d,f)$. Since f is equal to $\tilde{f}$ modulo (d), this quotient is isomorphic to $(A/d)/\tilde{f}$, which by the choice of $\tilde{f}$ is isomorphic to $(A/d)/I$. Thus the last two terms of the short exact sequence are both torsion sheaves of length one. Therefore $L_0 j^*(I/fA)$ can only be zero. In particular, I/fA is not supported at the point $p \in U$.

Thus the inclusion $(f) \subset I$ is an isomorphism at the point p, so after shrinking U we can assume that $I = (f)$, so the module $M \simeq A/fA$ is the structure sheaf of the curve $\{f = 0\}$. Note additionally that since $\tilde{f} \in A/d$ has valuation 1, the curve $C = \{f = 0\}$ is smooth at the point p. This finishes the proof of the lemma. $\square$

This local description may be improved to a global one if we consider the slices at all points of the curve instead of a single point.

6.2.3. Lemma. *Let S be a smooth surface, and let $i\colon C \hookrightarrow S$ be a connected curve. Let $F \in \mathrm{Perf}(S)$ be an object whose set-theoretic support is C. Suppose that at each point $p \in C$ there exists a slice of F which is a torsion object of length one. Then the curve C is smooth, and the object F is isomorphic to a pushforward $i_*(L)[a]$ for some line bundle $L \in \mathrm{Pic}\,(C)$ and a shift $a \in \mathbb{Z}$.*

Remark. If C is not connected, the pushforwards of line bundles from different connected components may have different shifts, but otherwise the conclusion is the same.

Proof. By Lemma 6.2.2, applied at all points of $C = \mathrm{supp}(B)$, the object B is a shift of some coherent sheaf $\mathcal{F} \in \mathrm{Coh}(Y)$. Moreover, the scheme-theoretic support of $\mathcal{F}$ is equal to the reduced scheme structure on the curve C. Therefore $\mathcal{F}$ is a pushforward of a coherent sheaf $\mathcal{F}' \in \mathrm{Coh}(C)$. Locally the sheaf $\mathcal{F}'$ is isomorphic to the structure sheaf of C, thus $\mathcal{F}'$ is in fact a line bundle on C. $\square$

6.2.4. Definition. Let U be a smooth surface, and let $F \in D^b_{\mathrm{coh}}(U)$ be an object whose set-theoretic support is a reduced curve $C \subset U$. We say that the object F is *thin at the point*

$p \in C$ if there exists a slice of F at the point p which is a torsion object of the length equal to the multiplicity of the curve C at p.

6.2.5. Lemma. *Let U be a smooth surface, and let $F \in D^b_{\mathrm{coh}}(U)$ be an object whose set-theoretic support is a reduced curve $C \subset U$. Let $j\colon D \hookrightarrow U$ be a smooth curve which intersects C at a single point $p \in C$. Denote by d the multiplicity of the curve C at p.*

1. *For the length of the torsion object $j^*F \in D^b_{\mathrm{coh}}(D)$ we have an inequality $\ell(j^*F) \geq d$.*
2. *If the tangent vector to the curve D at p lies in the tangent cone of the curve C at p, then $\ell(j^*F) > d$.*
3. *If $F \simeq \mathcal{F}[0]$ is a single coherent sheaf, then the bounds above hold for $\ell(L_0 j^*\mathcal{F})$.*

Proof. Since $D \subset U$ is a Cartier divisor, by Lemma 2.3.8 we know that

$$\ell(j^*F) = \sum_{n\in\mathbb{Z}} \ell(j^*\mathcal{H}^n(F)).$$

Thus we may replace the object F with the coherent sheaf $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}^n(F)$ without changing the lengths of the slices. Thus it is enough to prove the bounds for the length of the nonderived pullback $L_0 j^*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ on U.

In the proof we use the notion of a (zeroth) Fitting ideal of a coherent sheaf. Recall the definition: given a finitely generated module M over a Noetherian ring A, pick an arbitrary free presentation:

$$A^k \xrightarrow{Q} A^n \to M \to 0.$$

The Fitting ideal $\mathrm{Fit}(M)$ is defined to be the ideal of A generated by the $(n \times n)$-minors of the matrix Q. This construction globalizes to coherent sheaves. The Fitting ideal is contained in the annihilator ideal, and the formation of Fitting ideals is compatible with arbitrary base change (see, e.g., [Stacks, Tag 07Z6]).

Consider the coherent sheaf $L_0 j^*\mathcal{F}$ on a curve D. It is supported at a single point $p \in D$. Let $\mathfrak{m} \subset \mathcal{O}_D$ be the maximal ideal sheaf of the point p. Using Lemma 2.3.10 it is easy to compute that for any coherent sheaf on a smooth curve D supported at the point p the Fitting ideal is equal to $\mathfrak{m}^\ell$, where ℓ is the length of the torsion sheaf. Thus in order to bound the length of $L_0 j^* \mathcal{F}$ it is enough to understand the Fitting ideal of this sheaf.

By passing to an étale neighborhood of the point $p \in U$ we may assume that U is the affine plane $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and p is the origin. Let $f \in \Bbbk[x, y]$ be the reduced equation of the curve C. Since the set-theoretic support of $\mathcal{F}$ is C, we know that the annihilator ideal of the sheaf $\mathcal{F}$ is contained in the ideal (f). The Fitting ideal $\operatorname{Fit}(\mathcal{F})$ is contained in the annihilator ideal, so $\operatorname{Fit}(\mathcal{F}) \subset (f)$.

Since Fitting ideals are compatible with base change, we know that $\operatorname{Fit}(L_0 j^* \mathcal{F})$ is contained in the restriction of the ideal (f) to the curve D. The pullback of (f) is contained in $\mathfrak{m}^d$, where d is the multiplicity of C at p, which is the lowest degree of a monomial occuring in f with nonzero coefficient. Thus $\ell(L_0 j^* \mathcal{F}) \geq d$. Moreover, if the tangent vector to D lies in the tangent cone of C at the point p, by definition this means that the degree-d part of the polynomial f restricts to zero in the quotient $\mathfrak{m}^d / \mathfrak{m}^{d+1}$. Thus in this case the pullback of (f) to the curve D is contained in $\mathfrak{m}^{d+1}$, and then $\ell(L_0 j^* \mathcal{F}) > d$, as claimed. $\square$

6.2.6. Lemma. *Let U be a smooth surface, let $C \subset S$ be a reduced curve, and let $p \in C$ be a point. Let $\mathcal{F}$ be a coherent sheaf on U whose set-theoretic support is $C \subset S$. Suppose that $\mathcal{F}$ is thin at the point $p \in C$. Then, after possibly replacing U by a Zariski neighborhood of $p \in U$, the sheaf $\mathcal{F}$ is a pushforward of a torsion-free rank one sheaf $\mathcal{F}'$ on C.*

Proof. The foundational case is when the multiplicity of the curve C at the point p is equal to one. Then by definition $\mathcal{F}$ is thin at p if and only if there exists a slice of $\mathcal{F}$ of length one. This case is proved in Lemma 6.2.2. Otherwise, let $d > 1$ be the multiplicity of the curve C at the point p.

By shrinking U we may assume that U is affine. Let $f \in \mathrm{H}^0(\mathcal{O}_U)$ be the equation for the reduced scheme structure on the curve C. Since the characteristic of the base field is zero, by further shrinking U we may assume that all points in $C \setminus \{p\}$ are smooth in the curve C.

Let $j\colon D \hookrightarrow S$ be a smooth curve passing through p such that the derived pullback $j^*\mathcal{F}$ is a torsion object of the length d.

We first show that $\mathcal{F}$ has no point-torsion at p. We know that the length

$$\ell(j^*\mathcal{F}) := \ell(L_0 j^*\mathcal{F}) + \ell(L_1 j^*\mathcal{F})$$

is equal to d. By Lemma 6.2.5 the summand $\ell(L_0 j^*\mathcal{F})$ is greater or equal to d. Thus $L_1 j^*\mathcal{F}$ has length zero, so it is a zero object. Consider the subsheaf $\mathcal{T} \subset \mathcal{F}$ spanned by sections supported only at the point p. Consider the short exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{F}/\mathcal{T} \to 0.$$

Since j is an inclusion of a Cartier divisor, $L_2 j^*(-)$ vanishes at every argument. Thus the long exact sequence of derived pullbacks along $j\colon D \hookrightarrow S$ shows that $L_1 j^*\mathcal{F}$ has a subsheaf isomorphic to $L_1 j^*\mathcal{T}$. If $\mathcal{T}$ is a nonzero sheaf, then by Lemma 4.4.1 the sheaf $L_1 j^*\mathcal{T}$ is also nonzero, but this leads to a contradiction with the fact that $\mathcal{F}$ is thin at p. Thus $\mathcal{T} = 0$, i.e., the sheaf $\mathcal{F}$ has no point-torsion.

Let $g \in \mathrm{H}^0(\mathcal{O}_U)$ be the equation of the smooth curve $D \subset U$. Consider a family of inclusions $j_t\colon D_t \hookrightarrow U$, where the curve D_t is given by the equation $\{g = t\}$. By Lemma 6.2.5 the tangent vector of D at the point p does not lie in the tangent cone of C, and thus for a general value of t the curve D_t intersects C transversely in exactly d distinct points (see, e.g., [Mum95, §5A]; we use the assumption of characteristic zero here). Thus, after possibly shrinking U, by semicontinuity we may assume that at each point of $C \setminus \{p\}$ the sheaf $\mathcal{F}$ has a slice which is a torsion object of length one.

In particular, by Lemma 6.2.3 this implies that on $U \setminus \{p\}$ the sheaf $\mathcal{F}$ is a direct sum of pushforwards of line bundles from the irreducible components of $C \setminus \{p\}$. Assume that on the unpunctured surface U the sheaf $\mathcal{F}$ is not a pushforward from the curve C. Since U is affine, this is equivalent to the fact that the equation $f \in \mathrm{H}^0(\mathcal{O}_U)$ does not annihilate $\mathcal{F}$.

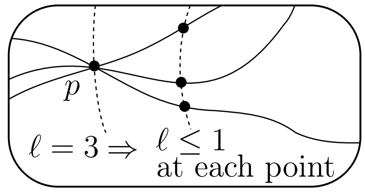

Figure 6.2: Semicontinuity and slices

Then there exists a section $s \in \mathrm{H}^0(\mathcal{F})$ such that $f \cdot s$ is not zero. The equation f annihilates any section on the open set $U \setminus \{p\}$. Therefore $f \cdot s$ is a section of $\mathcal{F}$ which is supported only at a single point p. But we proved that $\mathcal{F}$ has no zero-dimensional torsion, a contradiction. Thus the scheme-theoretic support of the sheaf $\mathcal{F}$ is equal to C. □

6.2.7. Lemma. *Let S be a smooth surface, and let $F \in \mathrm{Perf}(S)$ be an object whose set-theoretic support is a reduced curve $C \subset S$. Suppose that F is thin at every point of C. Then F is a formal complex, and each cohomology sheaf $\mathcal{H}^n(F)$ is isomorphic to a pushforward of a torsion-free rank one sheaf from some subcurve $C_n \subset C$.*

Proof. Let $p \in C$ be a point, and let $j\colon D \hookrightarrow S$ be a smooth curve passing through p such that j^*F is a torsion object of length equal to the multiplicity of C at p. By Lemma 2.3.8 we know that $\ell(j^*F) = \sum_n \ell(j^*\mathcal{H}^n(F))$. Let $C_n := \mathrm{supp}(\mathcal{H}^n(F)) \subset C$ be the set-theoretic support of the n'th cohomology sheaf. Denote by $I \subset \mathbb{Z}$ the subset of those indices $n \in \mathbb{Z}$ such that C_n contains p and p is not an isolated point in C_n. For $n \in I$, let m_n be the multiplicity of C_n at the point p. By Lemma 6.2.5 we have $\ell(j^*\mathcal{H}^n(F)) \geq m_n$ for each $n \in I$. Let m be the multiplicity of the curve C at p. Since $C = \cup_{n\in I} C_n$ near the point p set-theoretically, we have $\sum_{n\in I} m_n \geq m$. Taking all this information into account, we get a chain of inequalities:

$$\ell(j^*F) = \sum_{n\in\mathbb{Z}} \ell(j^*\mathcal{H}^n(F)) \geq \sum_{n\in I} \ell(j^*\mathcal{H}^n(F)) \geq \sum_{n\in I} m_n \geq m.$$

The assumption that F is thin at p implies that each inequality is in fact an equality. Note that this holds for any point $p \in C$. Thus we conclude that:

1. For any $n \in \mathbb{Z}$ such that $\mathcal{H}^n(F) \neq 0$, the subset $C_n = \mathrm{supp}(\mathcal{H}^n(F))$ is a curve, and the

sheaf $\mathcal{H}^n(F)$ is thin at any point of its support C_n.

2. For any two distinct $n, n' \in \mathbb{Z}$ the intersection $C_n \cap C_{n'}$ is a zero-dimensional set.

Consider a nonzero cohomology sheaf $\mathcal{H}^n(F)$. It is thin at every point of the curve C_n, so by Lemma 6.2.6 the sheaf $\mathcal{H}^n(F)$ is isomorphic to a pushforward of a torsion-free rank one sheaf from C_n. It only remains to show that F is a formal complex.

By Lemma 2.3.11 the glueing data for F consists of classes in $\mathrm{Ext}^2(\mathcal{H}^n(F), \mathcal{H}^{n-1}(F))$ for each $n \in \mathbb{Z}$. Since the supports $C_n \cap C_{n-1}$ intersect along a zero-dimensional set, the Ext-group may be computed locally at each intersection point, i.e.

$$\mathrm{Ext}^2(\mathcal{H}^n(F), \mathcal{H}^{n-1}(F)) = \mathrm{H}^0(\mathcal{E}xt^2(\mathcal{H}^n(F), \mathcal{H}^{n-1}(F)).$$

Let $p \in S$ be any point. Since $\mathcal{H}^n(F)$ is a pushforward of a torsion-free sheaf from a curve via an inclusion $C_n \hookrightarrow S$, it has no point-torsion at p. By definition this means that the depth of the coherent sheaf $\mathcal{H}^n(F)$ at $p \in S$ is not zero. Since S is a smooth surface, by Auslander–Buchsbaum formula this implies that the projective dimension of $\mathcal{H}^n(F)$ over the local ring $\mathcal{O}_{S,p}$ is at most one. Since this true for every point $p \in S$, the local Ext-sheaf $\mathcal{E}xt^2(\mathcal{H}^n(F), -)$ vanishes for any second argument. Therefore $\mathrm{Ext}^2(\mathcal{H}^n(F), \mathcal{H}^{n-1}(F)) = 0$. This shows that complex F splits into a direct sum of its cohomology sheaves, and the lemma is proved. □

Recall that a curve is called nodal if it is smooth away from finitely many ordinary double points. Torsion-free rank one sheaves on nodal curves are well-understood. To deal with disconnected curves it is convenient to use the following definition.

6.2.8. Definition. A *line bundle object* on a curve C with connected components $C = \sqcup_{i \in I} C_i$ is an object $F \in \mathrm{Perf}(C)$ which is isomorphic to a direct sum $\oplus_{i \in I} L_i[a_i]$ for some line bundles $L_i \in \mathrm{Pic}\,(C_i)$ and shifts $a_i \in \mathbb{Z}$.

6.2.9. Lemma. *Let S be a smooth surface, and let $F \in \mathrm{Perf}(S)$ be an object whose set-theoretic* *support is a nodal curve $C \subset S$. Suppose that F is thin at every point of C. Then F is a pushforward of a line bundle object from some partial normalization of C.*

Proof. Denote by C_n the set-theoretic support of $\mathcal{H}^n(F)$. By Lemma 6.2.7 we know that $F \simeq \oplus \mathcal{H}^n(F)[-n]$ and that any nonzero cohomology sheaf $\mathcal{H}^n(F)$ is a pushforward of a torsion-free rank one sheaf on C_n. Since C_n is a subcurve of a nodal curve $C \subset S$, it is also nodal, and hence by [OS79, Prop. 10.1] any torsion-free rank one sheaf on C_n is a pushforward of a line bundle L_n on some partial normalization $C'_n \to C_n$. Then F is isomorphic to a pushforward of a line bundle object on the partial normalization $\sqcup C'_n \to \cup C_n = C$. □

6.3 del Pezzo lemmas

In this subsection we collect some facts about smooth anticanonical divisors on del Pezzo surfaces.

6.3.1. Definition. Let Y be a del Pezzo surface. The *skeleton* of Y, denoted by $Y^{\mathrm{sk}} \subset Y$, is a union of all (-1)-curves contained in Y.

6.3.2. Lemma. *Let Y be a del Pezzo surface. Let $\pi\colon Y' \to Y$ be a blow-up of several distinct points such that Y' is also a del Pezzo surface. If $D' \subset Y'$ is a smooth anticanonical divisor, then the image $\pi(D') \subset Y$ is also a smooth anticanonical divisor passing through the blown-up points.*

Proof. An anticanonical divisor intersects each (-1)-curve with multiplicity one, and since the divisor D' is smooth, it is also irreducible, so the intersection is indeed a transverse intersection at a single point. Therefore the image of D' under the blow-down map is also smooth. □

6.3.3. Lemma. *Let Y be a del Pezzo surface of degree ≥ 2. For any point $p \in Y$ there exist infinitely many smooth anticanonical divisors passing through p.*

Proof. By Lemma 6.3.2 it is enough to prove this on a degree 2 del Pezzo surface. In this case the anticanonical linear system defines a morphism $\pi\colon Y \to \mathbb{P}^2$ which is a degree two covering branched along a smooth quartic curve $C \subset \mathbb{P}^2$. A preimage of any line $L \subset \mathbb{P}^2$ which is not tangent to the curve C is a smooth anticanonical divisor of Y. Among the lines in $\mathbb{P}^2$ passing through the point $\pi(p)$ only finitely many are tangent to the smooth curve C. Thus there are infinitely many lines whose preimages are smooth anticanonical divisors in Y passing through p. $\square$

6.3.4. LEMMA. *Let Y be a del Pezzo surface of degree ≥ 2. Let $Z \subset Y$ be the subset of singular points of the skeleton, i.e., points which lie on more than one (-1)-curve. For any point $p \in Y \setminus Z$ there exist infinitely many smooth anticanonical divisors passing through p which do not intersect Z.*

Proof. It is enough to work with a degree 2 del Pezzo surface. The anticanonical linear system defines a degree two morphism $\pi\colon Y \to \mathbb{P}^2$ branched along a smooth quartic $C \subset \mathbb{P}^2$. From the explicit description of the 56 (-1)-curves in Y as irreducible components of preimages of bitangent lines to C we see that if $p \in Y$ is not in Z, then the other point in the fiber of π over $\pi(p)$ also does not lie in Z. Therefore the linear system of lines in $\mathbb{P}^2$ separates $\pi(p)$ from the finitely many points in the image $\pi(Z)$. By Lemma 6.3.3 there exist infinitely many lines through $\pi(p)$ whose preimages are smooth anticanonical divisors passing through p and avoiding Z. $\square$

6.3.5. LEMMA. *Let Y be a del Pezzo surface of degree ≥ 3. Let $p, q \in Y$ be two points. Assume that at least one of them lies in $Y \setminus Y^{\mathrm{sk}}$. Then there exist infinitely many smooth anticanonical divisors passing through p and q.*

Proof. Without loss of generality suppose that $p \in Y \setminus Y^{\mathrm{sk}}$. Then the blow-up $\pi\colon Y' \to Y$ of the point p is a del Pezzo surface of degree ≥ 2. By Lemma 6.3.3 in Y' there exist infinitely many smooth anticanonical divisors passing through the point $\pi^{-1}(q)$. Then by Lemma 6.3.2 their images are smooth anticanonical divisors of Y passing through both p and q. $\square$

6.4 On phantoms in del Pezzo surfaces of degree at least 3

The goal of this section is to show that there are no phantom subcategories in del Pezzo surfaces of degree greater or equal to 3. The proof is split into two stages. In the first stage, culminating in Lemma 6.4.3, we show that any phantom subcategory is necessarily supported on the union of (-1)-curves, in other words, on the skeleton of the del Pezzo surface. In the second stage we classify possible admissible subcategories supported on the skeleton. In Lemma 6.4.5 we show that the only option is to choose several disjoint (-1)-curves, pick a line bundle on each one, and span the subcategory by their pushforwards. In particular, there are no phantoms of this form, and hence no phantoms in del Pezzo surfaces of degree at least 3.

Until the end of this section, we fix a del Pezzo surface Y with degree greater or equal to 3, and a semiorthogonal decomposition $D^b_{\mathrm{coh}}(Y) = \langle \mathcal{A}, \mathcal{B} \rangle$. Recall that Y^{sk} denotes the union of (-1)-curves in Y.

6.4.1. Lemma. *Let $p \in Y$ be a point. Denote by $S_p \subset Y$ the point-support of the decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$ at p. If $S_p = Y$, then neither $\mathcal{A}$ nor $\mathcal{B}$ is a phantom subcategory.*

Proof. Consider the projection triangle $B \to \mathcal{O}_p \to A$ of the skyscraper sheaf at the point p. One of the objects A and B is not locally free at the point p by Lemma 2.6.2 (1). For simplicity of notation assume that it is B. Consider the subset $R \subset Y$ of points where the graded dimension of the derived fiber of B is the same as the graded dimension of $B|_{\{p\}}$. In a Zariski neighborhood of the point p the subset R is closed by semicontinuity. Moreover, R does not contain any neighborhood of p since B is not locally free at p. By Lemma 6.3.3 there exist infinitely many distinct smooth anticanonical divisors passing through p. Since the dimension of R is at most one, we may choose a smooth anticanonical divisor $j\colon D \to S$ passing through p which is not contained in R. This implies that the derived restriction $j^*B \in \mathrm{Perf}(D)$ is not locally free at the point $p \in D$. By Lemma 6.1.2 we know that $\mathrm{supp}(B) = S_p = Y$, so $\mathrm{supp}(j^*B) = D$.

The object j^*B is a restriction to an anticanonical divisor on a surface, so we may use the classification from Proposition 3.1.3. From the discussion above we see that the only possible option is for $j^*(B)$ to be isomorphic to $\mathcal{O}_p[a] \oplus M[b]$ for some nonzero vector bundle M. In particular, the alternating sum of generic ranks of cohomology sheaves of j^*B is, up to a sign, a rank of M. This alternating sum is well-defined for classes in K_0, so in particular the class of j^*B in $K_0(D)$ is nonzero. Since the pullback induces a homomorphism $K_o(Y) \to K_0(D)$ of abelian groups, this implies that the class of B in $K_0(Y)$ is nonzero, and therefore $\mathcal{B}$ is not a phantom subcategory.

Note that the derived pullback $j^*(\mathcal{O}_p)$ is zero in the Grothendieck group, hence the class of j^*A in $K_0(D)$ is the opposite to the class of j^*B, in particular also nonzero, so $\mathcal{A}$ is not a phantom subcategory. □

6.4.2. Lemma. *Let $p \in Y \setminus Y^{\mathrm{sk}}$ be a point. Denote by $S_p \subset Y$ the point-support of the decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$ at p. If S_p is a curve, then neither $\mathcal{A}$ nor $\mathcal{B}$ is a phantom subcategory.*

Proof. Consider the projection triangle $B \to \mathcal{O}_p \to A$ of the skyscraper sheaf at p. Choose an irreducible component $C \subset S_p$ containing p, and then choose a point q on the curve C. Since $p \in Y \setminus Y^{\mathrm{sk}}$, by Lemma 6.3.5 there exist infinitely many smooth anticanonical divisors of Y passing through p and q. Since S_p is a curve, we may choose among them a divisor $j\colon D \hookrightarrow Y$ which is not contained in S_p. Then the intersection $D \cap S_p$ contains p and q as isolated points. By Lemma 6.1.5 the restriction j^*B is isomorphic to $\mathcal{O}_p[a] \oplus \mathcal{O}_q[b]$ in the derived category of the curve D.

If $a = b$, then the class of j^*B in $K_0(D)$ is nonzero, and similarly to Lemma 6.4.1 we conclude that neither $\mathcal{A}$ nor $\mathcal{B}$ is a phantom subcategory. Assume now that $a \neq b$. Since j^*B is isomorphic to $\mathcal{O}_p[a]$ in a neighborhood of p, by Lemma 6.2.2 the curve S_p is smooth at p, and in a neighborhood of p the object B is isomorphic to a pushforward of a line bundle shifted to degree $-a$. In particular, the support of $\mathcal{H}^{-a}(B)$ contains an open neighborhood of the point p in the irreducible component $C \subset S_p$. Since the support of any coherent sheaf is a closed subset, this implies that $\mathrm{supp}(\mathcal{H}^{-a}(B)) \supset C$. In particular, $q \in \mathrm{supp}(\mathcal{H}^{-a}(B))$.

By Lemma 2.3.8 this implies that $q \in \operatorname{supp}(\mathcal{H}^{-a}(j^*B))$, but this is a contradiction with the description of j^*B. $\square$

Remark. It seems possible that a slight variation of the argument will prove that S_p is a smooth curve and B is a pushforward of a line bundle from it. This situation probably never happens.

6.4.3. Lemma. *Assume that $\mathcal{B} \subset D^b_{\mathrm{coh}}(Y)$ is a phantom subcategory. Then $\mathcal{B}$ is supported on the skeleton.*

Proof. Let $p \in Y \setminus Y^{\mathrm{sk}}$ be any point. Let $S_p \subset Y$ be the point-support of the semiorthogonal decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$. It is a connected closed subset in a surface, so it is equidimensional. From Lemmas 6.4.1 and 6.4.2 we know that the only option is for S_p to have dimension zero. By Lemma 6.1.2 this implies that $S_p = \{p\}$ and the skyscraper sheaf $\mathcal{O}_p$ lies in either $\mathcal{A}$ or $\mathcal{B}$. Since $\mathcal{B}$ is a phantom subcategory, $\mathcal{O}_p \in \mathcal{A}$. This holds for a skyscraper sheaf at each point of $Y \setminus Y^{\mathrm{sk}}$. Then semiorthogonality of $\mathcal{A}$ and $\mathcal{B}$ implies that any object of $\mathcal{B}$ is supported on the skeleton Y^{sk}. $\square$

6.4.4. Lemma. *Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(Y)$ be an admissible subcategory. Assume that it is supported on the skeleton. Let p be a smooth point of the skeleton, and let S_p be the point-support of the decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$. Then S_p is either a point $\{p\}$ or the unique (-1)-curve containing p.*

Proof. Let B denote the right projection of the skyscraper sheaf $\mathcal{O}_p$ to the subcategory $\mathcal{B}$. The support of B is a closed subset of the skeleton. By Lemma 6.1.2 the subset S_p is connected, and if it is not a single point $\{p\}$, then $S_p = \operatorname{supp}(B)$ is a connected union of several (-1)-curves. By Lemma 6.3.3 there exists a smooth anticanonical divisor $j\colon D \hookrightarrow Y$ passing through p. It intersects each (-1)-curve with multiplicity one, and since D is irreducible, it does not contain any (-1)-curves, and hence intersects each of them transversely at a single point. By Lemma 6.3.4 we may assume that D does not pass through any singular point of the skeleton. In particular, the intersection $D \cdot S_p = |D \cap S_p|$ is equal to the number of the irreducible components in S_p. By Proposition 3.1.3 we know that $|D \cap S_p| = |\operatorname{supp}(j^*B)|$ is at most two. Thus there are at most two irreducible components.

Assume that S_p has exactly two irreducible components, $S_p = C_1 \cup C_2$. Then according to Proposition 3.1.3 the restriction j^*B is isomorphic to a direct sum of two skyscrapers. Then for an arbitrary smooth anticanonical divisor $j'\colon D' \hookrightarrow Y$, not necessarily passing through p, Lemma 3.4.1 shows that j'^*B is a torsion object of length at most two.

Since the degree of the del Pezzo surface Y is at least three, (-1)-curves can be realized as lines on a cubic surface, hence any two distinct intersecting (-1)-curves on Y intersect in at most one point, and the intersection is transverse. Let $z = C_1 \cap C_2$ be the intersection point of the two (-1)-curves. By Lemma 6.3.3 there exists a smooth anticanonical divisor $j'\colon D' \hookrightarrow Y$ passing through z. The restriction $j'^*B \in \operatorname{Perf}(D')$ is a transverse slice of B at the point z, and it has length at most two by the observation above. Moreover, if $z_1 \in S_p$ is any point which is different from z, then by Lemma 6.3.3 there exists a smooth anticanonical divisor passing through z_1, and it also intersects the other irreducible component of S_p at some point z_2. Since the total length of the restriction to that anticanonical divisor is at most two, the transverse slice at z_1 (and at z_2) has length one. Therefore B satisfies the assumptions of Lemma 6.2.9.

Since C_1 and C_2 are smooth curves intersecting in a single point, the only nontrivial partial normalization of the union $S_p = C_1 \cup C_2$ is the disjoint union $C_1 \sqcup C_2$. The conclusion of Lemma 6.2.9 is that there are two possibilities for B. The first option is that B is a direct sum of two (shifts of) sheaves, one of them a pushforward of a line bundle from C_1, and the other a pushforward of a line bundle from C_2. Suppose that we are in the first type of situation, and B is a direct sum of two pushforwards from distinct curves. Consider the projection triangle $B \to \mathcal{O}_p \to A$. Since the point p lies on only one of the components of the curve $S_p = C_1 \cup C_2$, the map $B \to \mathcal{O}_p$ necessarily vanishes on one of the direct summands of B. Therefore A and B have isomorphic nonzero direct summands, which contradicts the semiorthogonality of B and A. So this option does not happen.

The other possibility is for B to be a shift of a pushforward of a line bundle in $\mathrm{Pic}\,(C_1 \cup C_2)$.

Claim. *Any line bundle on the reducible curve $C_1 \cup C_2$ extends to a line bundle on the del Pezzo surface Y.*

Proof of the claim. Both components of $C_1 \cup C_2$ are smooth rational curves, so the Picard group $\mathrm{Pic}\,(C_1 \cup C_2)$ is isomorphic to $\mathbb{Z}^{\oplus 2}$, and a line bundle is determined by the degrees of its restrictions to the two irreducible components.

Choose a blow-down map $\pi\colon Y \to \mathbb{P}^2$ such that C_1 is one of the exceptional divisors. The curve C_2 intersects C_1, so C_2 cannot be an exceptional divisor of this map. Hence C_2 is a proper transform of some curve in $\mathbb{P}^2$. Since the intersection $C_1 \cap C_2$ is transverse, C_2 passes through the point $\pi(C_1)$ with multiplicity one, but since C_2 is a (-1)-curve and the self-intersection of any curve in $\mathbb{P}^2$ is at least 1, it necessarily passes through some other blown-up point $s \in \mathbb{P}^2$. The exceptional divisor $\pi^{-1}(s)$ is a (-1)-curve on Y which by construction intersects C_2 and is disjoint from C_1. The line bundle $L_2 = \mathcal{O}(\pi^{-1}(s))$ on Y has degree 0 on C_1 and degree 1 on C_2. The same method produces a line bundle L_1 with degree 1 on C_1 and degree 0 on C_2. Tensor products of powers of those bundles L_1 and L_2 show that the morphism $\mathrm{Pic}\,(Y) \to \mathrm{Pic}\,(C_1 \cup C_2)$ is surjective. □

The claim shows that in this case $B \simeq \mathcal{O}_{C_1 \cup C_2} \otimes L[a]$ for some shift $a \in \mathbb{Z}$ and some line bundle $L \in \mathrm{Pic}\,(Y)$. The curve $C_1 \cup C_2 \subset Y$ has self-intersection zero, so this curve can be deformed into a smooth curve which does not lie in the skeleton. Since the object B is a twist of the structure sheaf of $C_1 \cup C_2$, it can also be deformed into an object whose support does not lie in the skeleton of Y. But admissible subcategories are closed under small deformations (Proposition 2.5.3), so this is a contradiction with the assumption that $\mathcal{B}$ is supported on the skeleton. Thus S_p cannot be a reducible curve, and the lemma is proved. □

6.4.5. Lemma. *Let $\mathcal{B} \subset D^b_{\mathrm{coh}}(Y)$ be an admissible subcategory. Assume that it is supported on the skeleton and that for any smooth point p in the skeleton the point-support S_p is irreducible.*

Then there is a pairwise disjoint set of (-1)*-curves* $\{C_j\}_{j\in J}$ *in* Y *and a set of twists* $\{n_j\}_{j\in J}$ *such that* $\mathcal{B} = \oplus_{j\in J}\langle \mathcal{O}_{C_j}(n_j)\rangle$.

Proof. Let C be a (-1)-curve in Y. Let C^{o} be the open subset of points in C which are smooth in the skeleton. For a point $p \in C^{\mathrm{o}}$ we have only two choices for the point-support S_p to be irreducible. It is either a single point p or the whole curve C. We say that the (-1)-curve C is *included* in $\mathcal{B}$ if there exists a point $p \in C^{\mathrm{o}}$ such that $S_p = C$.

Note that if C is not included in $\mathcal{B}$, then for any $p \in C^{\mathrm{o}}$ we have $S_p = \{p\}$. By Lemma 6.1.2 this implies that the skyscraper sheaf $\mathcal{O}_p$ belongs to either $\mathcal{A}$ or $\mathcal{B}$. It does not lie in $\mathcal{B}$ since admissible subcategories are closed under small deformations and $\mathcal{O}_p$ may be deformed into a skyscraper at a point outside of the skeleton. Thus in this case the skyscraper sheaf of every point in C^{o} lies in $\mathcal{A}$. By Lemma 2.5.8 the semiorthogonality of $\mathcal{A}$ and $\mathcal{B}$ implies that the support of any object of $\mathcal{B}$ does not intersect C^{o}. Thus any object of $\mathcal{B}$ is supported on the union of the included (-1)-curves and possibly some of the finitely many points which are singular in the skeleton. Any object supported at a point on a smooth surface may be deformed away from the skeleton, so there are in fact no isolated points in the support of $\mathcal{B}$, only the union of the included curves.

Let C, C' be two distinct (-1)-curves which are included in $\mathcal{B}$. Pick the points $p \in C^{\mathrm{o}}$ and $p' \in (C')^{\mathrm{o}}$ such that the point-supports are $S_p = C$, $S_{p'} = C'$. If the curves C and C' are not disjoint, then they intersect at a single point. But point-supports of distinct points cannot have zero-dimensional intersections by Lemma 6.1.4.

We conclude that $\mathcal{B}$ is supported on the disjoint union of included (-1)-curves. The objects supported at disjoint curves are completely orthogonal to each other, so $\mathcal{B}$ splits into an orthogonal sum of subcategories, where for each included (-1)-curve C we have a nonzero subcategory $\mathcal{B}_C$ supported on C. The possible options for subcategories $\mathcal{B}_C$ are classified in Proposition 5.3.4. This finishes the proof. $\square$

6.4.6. Theorem. *Let* Y *be a del Pezzo surface of degree at least* 3*. Then there are no phantom* *subcategories in* $D^b_{\mathrm{coh}}(Y)$*.*

Proof. Assume $\mathcal{B} \subset D^b_{\mathrm{coh}}(Y)$ is a phantom subcategory. Then by Lemma 6.4.3 it is supported at the skeleton. By Lemma 6.4.4 the point-supports of the semiorthogonal decomposition $\langle \mathcal{B}^\perp, \mathcal{B} \rangle$ for smooth points of the skeleton are always irreducible. However, the subcategories with this property are classified in Lemma 6.4.5 and there are no phantom subcategories among the listed options. $\square$

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