Essays in Decision Theory

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Abstract

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When a choice model fails, the standard economics exercise is to weaken one assumption at a time to study what has changed. This is often accompanied by the understanding that future work will relax multiple assumptions simultaneously in order to explain actual behavior. This dissertation does exactly that, and by studying seemingly independent behavioral anomalies as related to one another we obtain new insights about why behavior departs from standard models.

Chapter 1 studies how violations of structural assumptions like expected utility and exponential discounting can be connected to reference dependent preferences with set-dependent reference points, even if behavior conforms with these assumptions when the reference is fixed. This is done with the introduction of a unified framework under which both general rationality (WARP) and domain-specific structural postulates (e.g., Independence for risk preference, Stationarity for time preference) are jointly relaxed using a systematic reference dependence approach. The framework allows us to study risk, time, and social preferences collectively, where behavioral departures from WARP and structural postulates are explained by a common source—changing preferences due to reference dependence. In our setting, reference points are given by a linear order that captures the relevance of each alternative in becoming the reference point and affecting preferences. In turn, they determine the domain-specific preference parameters for the underlying choice problem (e.g., utility functions for risk, discount factors for time).

Chapter 2, a joint work with Silvio Ravaioli, conducts an empirical test for one of the models in Chapter 1. It studies how the introduction of a very safe or very risky option affects risk attitude. In a laboratory experiment, we find that adding safer options increases displayed risk aversion, and it does so even when the added options are not chosen. This finding is robust across participants and treatments (e.g., degenerate and non-degenerate safe options). By contrast, we find that the addition of risky options does not result in a detectable change in risk attitude. Our results are in line with Chapter 1's Avoidable Risk Expected Utility model.

Chapter 3 studies choices over time, which allows us to study anomalies "at a given time" and "across time" as related to one another. This is achieved by studying how past choices affect future choices in the framework of attention. Limited attention has been proposed as an explanation for the failure of "rationality", where better options are not chosen because the decision maker has failed to consider them. We investigate this idea in a setting where (1) the observable are sequences of choices and (2) the decision makers are aware of the alternatives they chose in the past when they face future choice sets. This provides a link between two kinds of rationality violations: those that occur in a cross section of one-shot decisions and those that occur within a sequence of realized choices. Unlike the former, the frequency of the latter is naturally bounded, and their occurrence helps pin down preferences whenever a standard model of limited attention cannot.

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Acknowledgments

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Everyone claims that their advisors are the best, but mine must be the global maxima, without uniqueness only because there are two of them. They taught me how to be an *actual researcher*, for which I am forever grateful even though “it’s their job”. What was not part of their job, and that I had done without permission, was to secretly learn from them about how to be a *better person*, by observing how they treat their family, friends and students. I have grown by knowledge and by inspirations. Thank you, Mark (Dean) and Pietro (Ortoleva).

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Dedication

To those who have relentlessly pursued their passions, enabled others to do so, or given up their passions so that their children don't have to.

CHAPTER 1: ORDERED REFERENCE DEPENDENT CHOICE

1.1 Introduction

The standard model of choice in economics faces two separate strands of empirical challenges. First, structural assumptions such as the *expected utility form* (Independence) and *exponential discounting* (Stationary) are violated in simple choice experiments, most notably the Allais paradox and present bias. Second, and separately, studies have shown that choices are affected by reference points, resulting in "*non-rational*" behavior that violates the weak axiom of revealed preferences (WARP). With few exceptions, these two classes of prominent departures from standard models have been studied separately, and independently for each domain of choice, propelling models that seek to explain one phenomenon in isolation of the others.[1]

In this paper, we propose a unified framework in which failures of WARP and violations of structural assumptions, across the risk, time, and social domains, are jointly explained by reference dependency. This allows us to study different types of documented departure from standard models as related to one another, and in doing so suggests new empirical directions.

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[1] For reference dependence, see for example Kahneman & Tversky (1979), Kőszegi & Rabin (2006), Masatlioglu & Ok (2005), Masatlioglu & Ok (2013), Ok et al. (2015), and Dean et al. (2017). For models weakening the expected utility form see Quiggin (1982), Bell (1982); Loomes & Sugden (1982), Chew (1983); Fishburn (1983); Dekel (1986), Gul (1991), and Cerreia-Vioglio et al. (2015). For models weakening the discounted utility form see Loewenstein & Prelec (1992), Laibson (1997) and Frederick et al. (2002). For models that use reference dependency to explain violations of structural assumptions, see for example Kőszegi & Rabin (2007), Ortoleva (2010). An exception where both WARP and structural assumptions are relaxed is Bordalo et al. (2012).

The intuition comes from a simple observation: If decision makers have preferences (e.g., utility functions, discount factors) that depend on a reference point, then even if they are otherwise standard and maximize exponentially discounted expected utility, they would still violate both WARP and structural assumptions like the Independence and Stationarity axioms from time to time—when reference points change.

Working with choice behavior, we provide the axiomatic foundation for a set of four models—generic choice, risk preference, time preference, and social preference—in which behavioral anomalies are explained by a common source: changing preferences due to reference dependence. In these models, reference points are endogenously determined by *reference orders*, which rank each alternative by their relevance in becoming the reference point and affecting preferences.

To illustrate, consider a decision maker who contradicts expected utility theory by exhibiting increased risk aversion in the presence of safer options. This narrative is consistent with a myriad of anomalous choice documented in Herne (1999); Wakker & Deneffe (1996); Andreoni & Sprenger (2011), and prominently Allais (1953)'s paradox.[2] This behavior can be explained without fully rejecting the expected utility form—that decision makers maximize the expectation of some utility function for each choice problem—but by allowing for reference dependent utility functions. We propose, in the risk domain, that a decision maker's utility function depends on the safest available alternative, which reflects changing risk aversion. When the safest alternative is fixed, standard expected utility holds. But when reference point changes, then the safer the reference, the more concave the utility function. We then

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[2] In the Allias paradox, a decision maker is drawn to the safe option when it is available, contradicting the irrelevance of common consequence assumption in standard expected utility theory. We will (re)introduce the Allais paradox and discuss the application of our model in Section 1.3. Herne (1999); Wakker & Deneffe (1996); Andreoni & Sprenger (2011) document other behaviors consistent with our risk model, discussed in Section 1.3.

show that the same concept can be applied to time preference and social preference. Hence, we have a "unified framework".

The framework we propose has two persistent components: (i) a complete and transitive binary relation that determines which is the reference point and (ii) preference parameters (e.g., utility functions, discount rates) that depend on the reference point.

Our first step is a general representation theorem for choices in a generic domain: *Ordered Reference Dependent Utility (ORDU)* (Section 1.2). In this model, the decision maker uses a reference order to identify the reference alternative of a choice problem. In turn, this determines a utility function that she maximizes. Hence, it is as if that alternatives are ranked by their relevance in affecting preferences, and the underlying preference is determined by the alternative ranked highest in this order among those that are available.

The key behavioral postulate underlying the model is *Reference Dependence* (RD): it posits that if we fix the reference point, WARP holds. Since we do not know which is the reference point, we scarcely posit that there is one option in every choice problem such that if we keep said option when taking subsets, WARP holds. To illustrate, consider two choice sets $B \subset A$ such that WARP is violated; for instance, when an alternative is available in both A and B but is only chosen from A. Our axiom RD makes the behavioral assumption that the reference alternative of A is not present in B, causing a change in reference and WARP violation. Hence for any choice set A, RD demands that choices from subsets of A satisfy WARP as long as a certain (reference) alternative remains present. A formal definition is provided in Section 1.2. This axiom, along with a standard continuity assumption when X is infinite, characterizes the ORDU representation.

Next, we consider the special case of risk preference in Section 1.3. Here the postulate becomes: preserving one of the safest alternatives in a choice set preserves WARP *and* the

Independence condition. This follows the same intuition we had above: maintaining the reference point, normative postulates hold; for risk preference, this includes Independence. We call this *Risk Reference Dependence*. A second axiom, *Monotone Risk Aversion*, requires that if we add alternatives to a choice problem, choices can only become more risk averse, since an even safer reference increases risk aversion. Together with standard continuity and first order stochastic dominance we obtain the *Avoidable Risk Expected Utility* (AREU) representation (formally presented in Section 1.3), in which a decision maker's utility function depends on the safest alternative available, and safer references lead to more concave utility functions.

We then turn to the time domain in Section 1.4. The standard model for time preferences is Exponentially Discounted Utility, yet it is routinely challenged in empirical studies in which consumers tend to exhibit less patience in short-term decisions, or *present bias*.[3] We propose a model in which the decision maker has a single utility function, maximizes exponentially discounted utility, but uses a discount factor indexed by the earliest available payment in a choice problem. The availability of a sooner alternative makes the decision maker impatient. The key axiom, *Reference Dependent Stationarity*, is the counterpart of RD, where now we require WARP *and* Stationarity to hold only when we preserve the earliest alternative. The reference effect is characterized by the axiom *Increasing Patience*, which simply posits that symmetrically advancing the options can only increase delay aversion. The resulting model gives rise to the well-known violation of dynamic consistency, in which the same delay between consumption is tolerable in the future but not in the present.

The application for social preference is studied in Section 1.5. Often viewed as a desire to be *fair*, subjects in economics and psychology experiments display behavior consistent with

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[3] See for example Laibson (1997), Frederick et al. (2002), and Benhabib et al. (2010).

increased altruism when a more equitable split of payment is available.[4] In this setup, an alternative is an allocation for the decision maker and another individual. A natural measure of equity is the (normalized) ratio between the incomes, and attainable equity is therefore the maximum of such measure in a choice problem. As is standard for choices involving money, we use quasi-linear utility as foundation, but introduce the innovation that utility for money shared is increasing in attainable equity. This modification reflects our unified framework adapted to this setting—the presence of certain alternatives, as given by a reference order, affects the underlying preference for sharing. Like before, the main axiom *Reference Dependent Social Preference* posits the conformity with WARP and quasi-linear preferences when we preserve the most-equal option. A second axiom, *Increasing Altruism*, posits that decision makers are weakly more willing to share when more options are added to a choice problem, which can only increase attainable equity. In addition to capturing changing altruism, the model also explains increased sharing when splitting a fixed pie due to the availability of a more equitable division, as well as increased tendency to forgo a larger pie in favor of sharing a smaller one.

In our applications, failures of WARP and violations of structural assumptions are tightly linked. For example in the risk domain, adding full WARP to our model implies full compliance with Independence, and vice-versa. Therefore, our model departs from standard expected utility *only when* both WARP and Independence fail. Equivalent results obtain for the time and social domains. These findings separate our work from models that weaken a structural assumption but maintains WARP—the equivalence of a stable preference, it also suggests that necessary violation of WARP in our models is the behavioral manifestation of changing preferences. It hence provides a new perspective to study classic paradoxes like

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[4] See for example Ainslie (1992), Rabin (1993), Nelson (2002), Fehr & Schmidt (2006), and Sutter (2007).

Allais and present bias.

In relation to the existing literature, we first note that reference points are not exogeneously observed in our models. This strikes a fundamental difference in primitives/datasets to prospect theory by Kahneman & Tversky (1979), the endowment effect by Kahneman et al. (1991), and models of status quo bias led by Masatlioglu & Ok (2005).[5] Our models belong to a separate set of literature built on *endogenous reference*, where reference points are neither part of the primitive nor directly observable, such as in Kőszegi & Rabin (2006) and Ok et al. (2015). Unlike these models, our reference alternatives are given by a *reference order*. With this added structure, reference points—albeit unobservable—can be easily pinned down. Subsection 1.2.3 discusses in details the differences between ORDU and these two work, whereas Subsection 1.2.2 discusses the identification of our reference points and the consequent out-of-sample predictions via the reference order.

Our most general model, in which choices are over generic alternatives, is most similar in spirit and concurrent to Kıbrıs et al. (2018) albeit having different axiomatimization. Their paper focuses on choices over generic alternatives and contains no counterpart to our applications in the risk, time, and social domains. Their axiom depicts a conspicuity ranking between any two alternatives: if dropping x in the presence of y results in a WARP violation, then dropping y in the presence of x does not. Our approach is different and more involved, as it requires comparison between multiple choice problems differing by more than one alternative. However, this allows us to accommodate a wide range of behavioral postulates (in addition to WARP), such as the Independence and Stationarity conditions, with which we deliver reference-dependent expected utility and reference-dependent exponential discounting respectively. Moreover, their model is limited to a finite set of alternatives, whereas we allow

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[5] For other models of status quo bias, see Masatlioglu & Ok (2013) and Dean et al. (2017). Ortoleva (2010) extends this idea to preferences under uncertainty.

the set to be any separable metric space. This is not (just) a technical contribution, as the added generality is indispensable for choices over lotteries.

We compare our applications in risk, time, and social preferences to existing models in their respective sections. However our main contribution is, instead of a single model that captures a specific departure from standard theory, a unified framework. The closest work that resembles a unified framework is *salience*, pioneered by Bordalo et al. (2012, 2013), in which options are evaluated differently depending on which attribute is salient. We are different in that our framework comprises of a systematic reference dependence approach of weakening normative postulates, with which we apply universally to the risk, time, and social domains. This approach allows us to study reference dependence in risk as related to reference dependence in time, and failure of WARP as related to failure of structural assumptions. Indispensable to this innovation is the use of choice correspondences as opposed to preference relations as primitive *and* foregoing WARP—the conventional "rationality" assumption increasingly scrutinized by empirical evidence. Otherwise, behavior is summarized by binary comparisons, leaving on the table useful information about how people make decisions in real-life situations where they choose from more than two alternatives. This richer scope allows us to utilize behavior from non-binary choice problems to study and explain anomalies traditionally found in binary choice.

The remainder of the paper is organized as follows. In Section 1.2, we provide the axioms and the representation theorem for a generic ordered-reference dependent utility representations. Later in that section, we introduce a companion result to incorporate the accommodation of properties other than WARP, and a template for additional structure in the reference order R. Section 1.3, Section 1.4, and Section 1.5 each provides a representation theorem under this unified framework for the risk, time, and social preference settings respectively, discusses the model's implications, as well as compares it to related models in the literature.

1.2 Ordered-Reference Dependence

We start with most general model, in which a decision maker chooses from generic alternatives.

1.2.1 Reference Dependent Choice

We introduce a reference-based approach of imposing a standard behavioral postulate. In this section, said postulate is WARP.

Let X be an arbitrary set of alternatives, $\mathcal{A}$ the set of all finite and nonempty subsets of X, and $c : \mathcal{A} \rightarrow \mathcal{A}$, $c(A) \subseteq A$, a choice correspondence. Recall that c satisfies WARP if for all choices problems A, B such that $B \subset A$, $c(A) \cap B \neq \emptyset$ implies $c(A) \cap B = c(B)$.[6]

Even though choices may violate WARP, it may still be the case that they comply with it among a subset of all choice problems $\mathcal{S} \subset \mathcal{A}$. We define this notion formally.

Definition 1.1. Let $c : \mathcal{A} \rightarrow \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *WARP* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$,

$$B \subset A,\ c(A) \cap B \neq \emptyset \Rightarrow c(A) \cap B = c(B).$$

WARP is hence equivalent to the statement "c satisfies WARP over $\mathcal{A}$."

Our first axiom is a reference-based generalization of WARP.

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[6] For an arbitrary $\mathcal{A}$, this definition of WARP is weaker than another popular version: $x \in c(A)$, $y \in c(B)$, and $x, y \in A \cap B$ implies $x \in c(B)$. They are equivalent whenever $\mathcal{A}$ contains all doubletons and tripletons subsets of X.

Axiom 1.1 (Reference Dependence (RD))**.** *For every choice problem* $A \in \mathcal{A}$*, there exists an alternative* $x \in A$ *such that* c *satisfies WARP over* $\mathcal{S} = \{B \subseteq A : x \in B\}$.

Note that this axiom generalizes WARP, since "c satisfies WARP over $\mathcal{A}$" implies "c satisfies WARP over $\mathcal{S}$" for any $\mathcal{S} \subseteq \mathcal{A}$.

We explain the intuition of Axiom 1.1. Suppose choices between choice problems A and $B\,(\subset A)$ violate WARP; for example, $y \in c\,(A)$ but $y \in B \backslash c\,(B)$. We postulate that this is due to a change in reference point. Specifically, that the reference alternative of A must have been removed when take subset B of A, that is, it is in the set $A \backslash B$. Then, a natural limitation of WARP violations arise: have we *not* removed the reference alternative of A when taking an arbitrary subset B of A, choices would have complied with WARP. To put it differently, suppose that when taking subsets of A, if by preserving some alternative x in this process choices from these subsets comply with WARP. x is hence an endogenous candidate for "the reference alternative of A".[7] Axiom 1.1 demands that every choice problem contains (at least) one candidate alternative that achieves this.

Next we provide an example of compliance. Consider the following choice correspondence for $X = \{a, b, c, d\}$, where the notation $\{a, \underline{b}, c, d\}$ means b is chosen from the choice problem $\{a, b, c, d\}$.

$$\boxed{\begin{array}{c} \{a, \underline{b}, c, d\} \\ \{a, \underline{b}, c\} \quad \{a, \underline{b}, d\} \quad \{a, c, \underline{d}\} \quad \{\underline{b}, \underline{c}, d\} \\ \{a, \underline{b}\} \quad \{\underline{a}, c\} \quad \{a, \underline{d}\} \quad \{\underline{b}, c\} \quad \{\underline{b}, d\} \quad \{\underline{c}, d\} \end{array}}$$

This choice correspondence does not satisfy WARP globally (there are three instances of WARP violations: (i) between $\{a, \underline{b}, c, d\}$ and $\{\underline{b}, \underline{c}, d\}$, (ii) $\{\underline{b}, \underline{c}, d\}$ and $\{\underline{b}, c\}$, and (iii) $\{a, c, \underline{d}\}$ and $\{\underline{c}, d\}$). Yet WARP is satisfied from choice sets that contain a. To reconcile

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[7] Using the language in Ok et al. (2015), this alternative can be called a *potential reference alternative* of A.

with Axiom 1.1, when $S = X$, a is a candidate reference alternative. This is also true for any choice set S that contains a. Likewise, for $S = \{b, c, d\}$, d is a candidate reference, and this is true for any choice set S that contains d but not a. The only choice set left to be checked is $S = \{b, c\}$, but since the only non-singleton subset of $\{b, c\}$ is itself, WARP is trivial.

Although Axiom 1.1 allows for WARP violations, it is falsifiable as long as $|X| \geq 3$ (i.e., as soon as WARP is non-trivial). For example, the following choice correspondence violates Axiom 1.1.

$$\boxed{\left\{a, \underline{b}, c\right\} \quad \left\{\underline{a}, b\right\} \quad \left\{b, \underline{c}\right\} \quad \left\{\underline{a}, c\right\}}$$

In this example, instances of WARP violations are (i) between $\left\{a, \underline{b}, c\right\}$ and $\left\{\underline{a}, b\right\}$ and (ii) between $\left\{a, \underline{b}, c\right\}$ and $\left\{b, \underline{c}\right\}$. So when $A = \{a, b, c\}$, a does not preserve WARP since the first instance is not excluded, b does not preserve WARP since neither instance is excluded, and c does not preserve WARP since the second instance is not excluded. Hence the axiom does not hold.

Another way of "measuring" falsifiability is to count the number of observations (choice problems) required to falsify an axiom. For standard WARP that number is 2: for example, when WARP is violated between $\left\{a, \underline{b}, c\right\}$ and $\left\{\underline{a}, b\right\}$. Whereas for Axiom 1.1, a weakening of WARP, that number is 3: for example $\left\{a, \underline{b}, c\right\}$, $\left\{\underline{a}, b\right\}$, and $\left\{\underline{a}, c\right\}$, since the reference of $\{a, b, c\}$ is in $\{a, b\}$ and/or $\{a, c\}$, but WARP is violated both between $\left\{a, \underline{b}, c\right\}$, $\left\{\underline{a}, b\right\}$ and between $\left\{a, \underline{b}, c\right\}$, $\left\{\underline{c}, b\right\}$.[8] Thus reference dependence makes Axiom 1.1 harder to reject relative to WARP by one additional observation.

When X is infinite, we also assume Continuity. Say (X, d) is a metric space.

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[8] This can be generalized: Axiom 1.1 is falsified when there are WARP violations between A, B_1 and between A, B_2 such that $B_1 \cup B_2 = A$, where $A, B_1, B_2 \in \mathcal{A}$.

Axiom 1.2 (Continuity)**.** *We say* $c : \mathcal{A} \to \mathcal{A}$ *satisfies Continuity if it has a closed-graph (with respect to the Hausdorff distance):* $x_n \to_d x$*,* $A_n \to_H A$*, and* $x_n \in c(A_n)$ *for every* $n = 1, 2, \ldots$ *implies* $x \in c(A)$*.*[9]

1.2.2 Ordered-Reference Dependent Utility Functions

Let R be a complete and transitive binary relation, $\arg\max_{x \in A} R$ denotes the set $\{x \in A : xRy \, \forall y \in A\}$.

Definition 1.2 (Ordered-Reference Dependent Utility)**.** c admits an Ordered Reference Dependent Utility (ORDU) representation if there exist a complete, transitive, and antisymmetric reference order R on X and a set of reference-indexed utility functions $\{u_x : X \to \mathbb{R}\}_{x \in A}$ such that

$$c(A) = \arg\max_{y \in A} u_{r(A)}(y),$$

where $r(A) = \arg\max_{x \in A} R$.

Proposition 1.1.

1. *Let* X *be a finite set.* c *satisfies RD if and only if it admits an ORDU representation.*

2. *Let* X *be a separable metric space.* c *satisfies RD and Continuity if and only if it admits an ORDU representation where* $c(A) = \arg\max_{y \in A} u_{r(A)}(y)$ *has a closed-graph.*

ORDU represents a special type of context-dependent preferences. A decision maker's preference may change with the choice set, but depends only on its reference alternative, characterized by *reference-dependent utilities*. Reference-dependent utilities are more restrictive

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[9] By $\to_H$ we mean convergence in the Hausdorff distance, defined by $d_H(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} d_2(x, y), \sup_{y \in Y} \inf_{x \in X} d_2(x, y)\right\}$.

than *set-dependent utilities*, where each choice problem has its own utility function.[10] When $|X|$ is finite, there are at most $|X|$ distinct utility functions but around $2^{|X|}$ choice problems, and this difference increases exponentially in $|X|$. Furthermore, a linear order, called *reference order*, uniquely pins down the reference point for each choice problem.

The reference order has natural interpretations in richer settings, as we demonstrate in the risk, time, and social preference sections. When the setting is choices over generic alternatives, an interpretation of the reference order is a subjective salience ranking of alternatives. The most salient alternative determines the underlying preference used with the problem. In this setting, it is as if that the decision maker's attention is drawn to a certain salient alternative, and her preference ranking depends on that alternative. It is the fact that her attention is not always drawn to the same (reference) alternative that gives rise to WARP violations. But when she has the same reference alternative for a set of choice problems, her choices are consistent with a stable preference ranking.

The suggestion that certain salient component of a choice problem affect choices is not new, for example Bordalo et al. (2012, 2013). In their model, alternatives have attributes, and depending on which alternatives are being compared certain attributes are more salient than others, and weighted differently, from one choice problem to another. This is the source of WARP violations in their model. In ORDU, attributes are not part of the primitive/model, allowing for a different characterization of salience when the modeler either does not observe attributes or do not know the relevant attributes that play in role in decision making.

Combining *reference-dependent utilities* with a *reference order* yields out-of-sample predictions. For example, when the reference alternative in choice problem A is present in the choice problem $B \subset A$, that alternative is still the reference, and the preference ranking re-

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[10] Set-dependent utilities, that each choice problem A has a utility function $u_A(x)$ that is maximized, puts no restriction on behavior, since we can simply set $u_A(c(A)) = 1$ and $u_A(x) = 0$ for all $x \neq c(A)$.

mains the same. This is a feature of the reference order. So once we have identified the reference alternative r of A, we know that subsets of A that contain r use the same utility function.

Furthermore, reference alternatives are identified whenever we observe WARP violation upon removing them, since it is *only* through changes in references points that WARP violations arise. For example if WARP is violated between A and B, and $B = A \setminus \{x\}$, then x is the reference of A. Hence we infer the presence of reference points, and pin them down uniquely, through inconsistency or incoherence in choice with respect to WARP. Instead, if a decision maker complies with WARP, the idea that preferences are reference dependent cannot be substantiated.

Together, the model allows us to make out-of-sample predictions between two worlds: On one end, the decision maker's reference alternatives are not identifiable with choice data precisely because she satisfies WARP and maximizes a single preference ranking; on the other end, the decision maker's choices are reference dependent and result in WARP violations, which allows us to identify reference points and subsequently make predictions using the reference order.

The idea behind ORDU has natural applications. The rest of the paper demonstrates it in the risk, time, and social preference domains. In each setting, a domain-specific interpretation is given to the reference order. In the risk setting, the minimum amount of risk the decision maker must take, as measured by the safest alternative in a choice problem, may influence risk aversion. In the time setting, the earliest available payment characterizes the imminence of consumption, which may affect the decision maker's patience. In the social preference setting, how much a decision maker is allowed to share may affect how generous she is, where greater attainable equity increases generosity. We formally these models in

Section 1.3, Section 1.4, and Section 1.5 respectively.

1.2.3 Comparison with other non-WARP models

We conclude this section with a summary discussion of ORDU as compared with other axiomatized utility representations that model WARP violations. Technical details and complete characterizations of the examples used are deferred to Appendix B.

Under comparable setups, ORDU neither nests nor is nested by any of the following models: (i) Ok et al. (2015)'s *revealed (p)reference*, (ii) Kőszegi & Rabin (2006)'s *reference-dependent preferences* (*personal equilibrium*), (iii) Manzini & Mariotti (2007)'s *rational shortlist method*, and (iv) Masatlioglu et al. (2012)'s *choice with limited attention*.[11] Our focus will be the former two, as they also use reference formation to explain choices. The latter two are models in which reference formation is not used, but the addition (or removal) of alternatives directly contribute to WARP violations.

A key observation separates ORDU from other non-WARP models. In ORDU, reference points are given by a reference order and choices maximize reference-dependent utilities. Thus, either $c(\{a,b\})$ or $c(\{b,c\})$ must agree with $c(\{a,b,c\})$ in terms of being consistent with the maximization of a single utility function. This is because the reference point of $\{a,b,c\}$ is either in $\{a,b\}$ or in $\{b,c\}$ (or in both). In fact, since the reference point of A is in every subset of A that contains it, this condition generalizes into Remark 1.1.

Remark 1.1. Suppose c admits an ORDU representation. Take any finite collection of choice problems $A_1, ..., A_n$. For some $x \in A_1 \cup A_2 \cup ... \cup A_n$, choices from the collection of choice problems $\{A_i : x \in A_i\}$ must comply with standard utility maximization.

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[11] That is, for each of the four external models we consider, there are choice correspondences that admit ORDU but not the eternal model and vice versa. Complete specifications of the choice correspondences used are provided in Appendix B.

Recall that notation "$\{\underline{a},b,c\}$" means choice problem $\{a,b,c\}$, from which a is chosen.

In Ok et al. (2015)'s (endogeneous) *reference dependent choice*, the decision maker maximizes a single utility function, but only chooses from alternatives that are better than the reference in all (endogenously determined) attributes, where references do not necessarily come from an order. In their model, a *decoy* d may block the choice of a in $\{a,\underline{b},d\}$ and $\{a,\underline{c},d\}$ due to the *attraction effect,* where b and c are *elevated* since they are better than d in all attributes, but a isn't. However, since reference formation is more flexible, as contrasted with the use of a reference order in ORDU, d need not be the decoy in $\{\underline{a},b,c,d\}$, resulting in the choice of a. Remark 1.1 excludes this behavior from ORDU. Conversely, intransitive behavior in binary choice problems, such as $\{\underline{a},b\},\{\underline{b},c\},\{\underline{c},a\}$, can be explained by ORDU but are ruled out by Ok et al. (2015), since the absence of a third alternative impedes their decoy-effect from taking place. Hence the two models are not nested.

Kőszegi & Rabin (2006)'s *reference-dependent preferences* is another related model. Gul et al. (2006) provided the axiomatic foundation for *personal equilibrium* (PE), in which a decision maker has a joint utility function $v : X \times X \to \mathbb{R}$ and chooses $PE(A) = \{x : v(x|x) \geq v(y|x)\ \forall y \in A\}$. That is, the choice maximizes a reference-dependent utility function, and the reference point is itself the eventually chosen alternative (hence "equilibrium"). This permits the following behavior: b is not chosen in $\{\underline{a},b,c,d\}$, but is chosen in the subset $\{\underline{a},\underline{b},c\}$, in which d—an alternative better than b under $v(\cdot|b)$—was removed. Suppose further that d is chosen in $\{\underline{a},\underline{d}\}$ for the same reason—c is better than d under $v(\cdot|d)$, but now c is removed. Remark 1.1 concludes that this behavior cannot be accommodated by ORDU, a consequence of the reference order, where the reference point of $\{\underline{a},b,c,d\}$ and $\{\underline{a},\underline{d}\}$ must both be d under ORDU. Hence a WARP violation between these two choice problems rules out ORDU. Conversely, an immediate implication PE is, if $x \in c(A)$ and $x \in B \subset A$, then $x \in c(B)$. A simple intransitive choice pattern $\{\underline{a}, b\}, \{\underline{b}, c\}, \{\underline{c}, a\}$ is hence admissible by ORDU but not PE.[12] We conclude that the two models are not nested.

Non-nestedness between ORDU and Manzini & Mariotti (2007)'s *rational shortlist method*, as well as between ORDU and Masatlioglu et al. (2012)'s *choice with limited attention*, are shown and explained in Appendix B.

1.2.4 A unified framework for structural anomalies

Reference Dependence (Axiom 1.1) weakened WARP by demanding that WARP is satisfied among choice problems that share a reference alternative (as opposed to all choice problems). This method of generalizing an axiom is not only applicable to WARP, but also a wide range of behavioral properties defined on choice behavior. For example, we can demand compliance with Independence in a similar way, where Independence is not necessarily satisfied between every two choices, but is complied with whenever the choices come from choice problems that have the same reference point.

This reference dependence approach of weakening an arbitrary postulate serves as the starting point of our models in the risk, time, and social domains. In their respective sections, we adapt Axiom 1.1 to postulates of the form "For every choice problem A, there exists an alternative $x \in A$ such that c satisfies $\mathcal{T}$ over $\mathcal{S} = \{B \subseteq A : x \in B\}$". $\mathcal{T}$ is "WARP and Independence" for the risk domain, "WARP and Stationarity" for the time domain, and "WARP and quasi-linearity" for the social domain.

The result is as anticipated—ordered-reference expected utility, ordered-reference exponentially discounted utility, and ordered-reference quasi-linear utility. In fact, the represen-

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[12] Gul et al. (2006) shows that Kőszegi & Rabin (2006)'s *personal equilibrium* is equivalent to the maximization of a complete (but not necessarily transitive) preference relation.

tation theorems for all four models in the present paper start with a quintessential result in Appendix A, Lemma 1.2, which demonstrates the wide applicability of our approach by accommodating a class of behavioral postulates we call *finite properties*, of which WARP, Independence, Stationarity, quasi-linearity, transitivity, convexity, monotonicity, stochastic dominance, etc. are examples. Then, complemented with additional structure on reference orders, we obtain the reference-dependent versions of the corresponding utility representations.

The next three sections are natural applications of this approach.

1.3 Risk Preferences

We now turn to an application in the domain of risk, where we provide a utility representation, with axiomatic foundation, that explains increased risk aversion when safer options are present than when they are not. Consider a decision maker whose willingness to take risk depends on how much of it is avoidable, as measured by the safest alternative among those that are available. This depends on the underlying choice set: Sometimes, we have the option to fully avoid risk by keeping our asset in cash or by buying an insurance policy, and so the safest option is quite safe. In other situations, all options are risky and we are forced to take some risk, and so the safest option is quite risky. The premise of our model, in the risk setting, is that a decision maker's risk aversion may differ between these two types of choice problems in a particular way: she could be more risk averse when risk is avoidable than when it is not.

Suggestive evidence for this behavior is present in the literature. In the well-known paradox introduced by Allais (1953), when one choice problem contains a safe option and the other does not, subjects tend to chose the safer option in the former. This observation is consistent with increased risk aversion when safer options are present. We provide a quick recap of the Allais paradox and its relevance as pertain to our model when we discuss applications. We will also present a result that shows that Allais-type behavior is the consequence of changing utility functions by concave transformations, which characterizes greater risk aversion under the expected utility form.

In a separate setting meant to test for the *compromise effect*, Herne (1999) showed that the presence of a safer option results in WARP violations in the direction of more risk averse behavior. Wakker & Deneffe (1996) introduced the *tradeoff method* to elicit risk aversion without using a sure prize and showed that the estimated utility functions are in general less concave relative to the standard certainty equivalent / probability equivalent methods.[13] Andreoni & Sprenger (2011) reinforces this observation when the safest option is close to certainty.

1.3.1 Preliminaries

Consider a finite set of prizes $X \subset \mathbb{R}$. Let $\Delta(X)$ be the set of all lotteries over X endowed with the Euclidean metric d_2. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of $\Delta(X)$. We call $A \in \mathcal{A}$ a choice problem. We take as primitive a choice correspondence $c : \mathcal{A} \longrightarrow \mathcal{A}$ that gives, for each choice problem A, a subset $c(A) \subseteq A$. We make the following notational simplifications: Per convention, δ_x denotes the lottery that gives prize $x \in X$ with probability 1. For $p, q \in \Delta(X)$ and $\alpha \in [0,1]$, we denote by $p^{\alpha} q$ the convex combination $\alpha p + (1-\alpha) q \in \Delta(X)$. Let $b := \max_{\geq} X$ and $w := \min_{\geq} X$ denote the highest and lowest prizes respectively.

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[13] *Certainty equivalent method* finds the value of a sure prize such that a subject is indifferent to a fixed lottery. *Probability equivalent metho*d fixes the sure prize and alters the probability of a lottery until the subject is indifferent. *Tradeoff method* finds the indifferent point between two lotteries by varying one of the prizes.

We denote by $q(x)$ the probability lottery q gives prize $x \in X$.

1.3.2 Risk Reference Dependent Choice

Recall that in Section 1.2 we defined what it means for WARP to hold on an arbitrary set of choice problems. We now do the same for *Independence.*

Definition 1.3. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *Independence* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$ and $\alpha \in (0,1)$,

1. $p \in c(A)$, $q \in A$, $q^{\alpha}s \in c(B)$ and $p^{\alpha}s \in B \Rightarrow p^{\alpha}s \in c(B)$, and
2. $p^{\alpha}s \in c(A)$, $q^{\alpha}s \in A$, $q \in c(B)$ and $p \in B \Rightarrow p \in c(B)$.

In standard expected utility, c satisfies both WARP and Independence over $\mathcal{A}$.

Now, we are interested in the behavior where changes in the safest available alternatives affect risk aversion, but WARP and Independence are complied with whenever the safest alternatives of a collection of choice problems are the same.

First we define what "the safest available alternative" means through the use of two partial orders. A *mean-preserving spread* (MPS) is clearly not safest, this is our first order. However, mean-preserving spread is a (very) partial order, and many lotteries are left unranked, making it hard to predict when should WARP and Independence hold.

To account for this limitations we also deem riskier any lottery that is an *extreme spread*, our second risk order, which we now define. We call p is an extreme spread of q ($p\mathrm{ES}q$) if $p = \beta q + (1-\beta)(\alpha(\delta_b) + (1-\alpha)(\delta_w))$ for some $\beta \in [0,1]$ and $\alpha \in (q(b), 1-q(w))$. This captures lotteries that assigned more probability to extreme prizes while being proportionally identical for intermediate prizes. Extreme spread shares the core intuition of Aumann & Serrano (2008)'s risk index (which in their paper only applies to gain-loss prospects), where lotteries are deemed safer than another in the "economics sense"—if more-risk-averse decision makers prefer them whenever less-risk-averse decision makers do.[14]

The two risk orders are compatible with each other but are not nested. Now we characterize the set of alternatives that are not risky by these two measures. Let $MPS(A) = \{p \in A : \exists q \in A \text{ s.t. } p\text{MPS}q\}$ and $ES(A) = \{p \in A : \exists q \in A \text{ s.t. } p\text{ES}q\}$ denote the mean-preserving spreads and extreme spreads in A respectively.

Definition. Define the least risky set of A by $\Psi(A) := A \setminus (MPS(A) \cup ES(A))$.

We now replace Reference Dependence from Section 1.2 with a stronger axiom that demands (i) reference-dependent compliance with both WARP *and* Independence and that (ii) the reference is a least risky lottery.

Axiom 1.3 (Risk Reference Dependence (RRD))**.** *For every choice problem $A \in \mathcal{A}$, there exists $p \in \Psi(A)$ such that c satisfies WARP and Independence over $\mathcal{S} = \{B \subseteq A : p \in B\}$.*

Axiom 1.3 identifies a candidate reference for choice problem A. If $\Psi(A) = \{p\}$, and B_1 and B_2 are subsets of A containing p, then neither a violation of WARP nor a violation of Independence is produced between $c(B_1)$ and $c(B_2)$.

Like Reference Dependence (Axiom 1.1 in Section 1.2), Risk Reference Dependence postulates that there is a reference point, in the sense that WARP holds in its presence. But it additionally postulates that Independence also holds, and that this reference point is in $\Psi(A)$—a least risky lottery.[15]

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[14] For every q, the set of extreme spreads of q is small and lives entirely within the probability triangle containing q, δ_b, and δ_w. In this probability triangle, it is exactly the set of lotteries such that a more risk loving decision maker would prefer (over q) whenever a more risk averse one does, under the framework of standard expected utility. In particular, it is a superset of mean-preserving spreads in this triangle. The intuition behind this notion is that, when probabilities are allocated to the most extreme prizes, even if mean is not preserved, we should still deem the resulting lottery riskier. Note that an extreme spread need not be a mean-preserving spread, and vice versa.

[15] This is where the decision maker's subjectivity enters the model: For two lotteries not ranked by objective

Finally, note that Axiom 1.3 weakens the axioms of standard expected utility, which demands compliance of WARP and Independence over the entire $\mathcal{A}$.

1.3.3 Monotone Risk Attitude

Our motivation is that the decision maker's choices vary only in terms of magnitude of risk aversion. Moreover, increases in risk aversion are due to the presence of a safer alternative. Consider the following axiom.

Axiom 1.4 (Monotone Risk Attitude)**.** *For any choice problems $A, B \in \mathcal{A}$ such that $B \subset A$,*

$$\delta^{\alpha} r \in c(B),\ p^{\alpha} r \in B, p^{\beta} q \in c(A),\ \delta^{\beta} q \in A \ \Rightarrow\ \delta^{\beta} q \in c(A),$$

where $p, q, r \in \Delta(X)$, δ is a degenerate lottery, and $\alpha, \beta \in [0,1]$.

It is standard that $\succsim_1$ is deemed *more risk averse* than $\succsim_2$ if for any degenerate alternative δ and lottery p, $\delta \succsim_2 p \Rightarrow \delta \succsim_1 p$. Here we extend this definition to lotteries that are not entirely riskless, but differ by a degenerate and (possibly) non-degenerate components: $\delta^{\alpha} q$ and $p^{\alpha} q$ (where δ is a degenerate alternative).[16] Under standard expected utility this extension is without loss, i.e., the two notions coincide.[17] It is precisely because we depart from the standard expected utility model that we require this extended definition—the choice between δ and p does not pin down the choice between $\delta^{\alpha} q$ and $p^{\alpha} q$ due to changing risk aversion.

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notions of risk, one individual may deem one lottery riskier, whereas another individual disagrees. The axiom demands that a reference point exists and is a least risky alternative, but in instances where $|\Psi(A)| > 1$, the decision maker's choices determine which lottery in $\Psi(A)$ is the reference.

[16] It is straightforward to show that $p^{\alpha} q$ is obtained from $\delta^{\alpha} q$ by moving probabilities from one prize to one or more prizes. We hence deem $\delta^{\alpha} q$ safer than $p^{\alpha} q$, and say that a *more risk averse* decision maker prefers $\delta^{\beta} q$ to $p^{\beta} q$ whenever a less risk averse decision maker prefers $\delta^{\alpha} r$ to $p^{\alpha} r$.

[17] This is the consequence of the Independence axiom of standard expected utility, in which $\delta^{\alpha} q$ is chosen over $p^{\alpha} q$ if and only if $\delta^{\beta} r$ is chosen over $p^{\beta} r$ if and only if δ is chosen over p.

Axiom 1.4 postulates that as a choice problem expands, the decision maker is *not more risk loving*. Our intuition is that the introduction of new alternatives can only reduce minimum risk / increase avoidable risk, and consequently the decision maker views risk less favorably and becomes more risk averse in the choices she makes. This is the source of Independence violation in our model, but only one type of violation is allowed: that choices become *more risk averse*, where other channels remain shut.

Note that standard expected utility satisfies this axiom trivially—an expected utility maximizer can neither be more risk loving nor more risk averse between any two choice sets, a consequence of the Independence axiom. Therefore, our departure from expected utility is to permit increased risk aversion when new alternatives are added to a choice set.

Final two axioms are standard: that choice is continuous (defined in Section 1.2) and abides by first order stochastic dominance.

Axiom 1.5 (FOSD)**.** *For any $p, q \in \Delta(X)$ such that $p \neq q$ and p first order stochastically dominates q, $p \in A$ implies $q \notin c(A)$.*

1.3.4 Representation Theorem

We now introduce the utility representation.

Definition. We say an order R is risk-consistent if, whenever (i) p is a mean-preserving spread of q or (ii) p is an extreme spread of q (or both), we have qRp.

Definition 1.4. c admits an Avoidable Risk Expected Utility (AREU) representation if there exist (i) a complete, transitive, and antisymmetric reference order R on $\Delta(X)$ and (ii) a set of strictly increasing utility functions $\left\{u_p : X \to [0,1]\right\}_{p\in\Delta(X)}$, such that

$$c\left(A\right) = \operatorname*{arg\,max}_{p\in A} \mathbb{E}_p u_{r(A)}\left(x\right),$$

where

- $r\left(A\right) = \operatorname*{arg\,max}_{q\in A} R$,
- R is risk-consistent,
- qRp implies $u_q = f \circ u_p$ for some concave $f{:}[0,1] \to [0,1]$, and
- $\arg\max_{p\in A} \mathbb{E}_p \left[u_{r(A)}\left(x\right)\right]$ has a closed-graph.

Proposition 1.2. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Risk Reference Dependence, Monotone Risk Attitude, FOSD and Continuity.*
2. *c admits an AREU representation.*

Furthermore in every AREU representation, given R, u_p is unique for all $p \neq \left(\delta_b\right)^{\alpha}\left(\delta_w\right)$.

When choices admit an AREU representation, it is as if the decision maker goes through the following decision making process: Facing a choice problem, she first looks for the safest alternative using R, which is risk consistent—it ranks safer alternatives higher. This determines the (Bernoulli) utility function for the choice problem and she proceeds to choose the option that maximizes expected utility. Moreover, the safer the reference, a more concave utility function is used, resulting in weakly more risk averse choices. This generalizes the standard model where a decision maker chooses the option that maximizes expected utility using a single utility function. It departs from standard expected utility by allowing greater risk aversion when alternatives are added to a choice set, but prohibits any other types of preference changes.

Note that utility functions in AREU are generically unique (up to an affine transformation). This property guarantees that their relationships by concave transformations are not arbitrary, and choices manifest changing risk aversions. Here, each utility function u_p is used to evaluate options for a set of choice problems that deem p as the safest alternative. When $p \neq (\delta_b)^{\alpha}(\delta_w)$, there are many of these choice problems in which p is not the chosen alternative, making u_p non-arbitrary.

1.3.5 Applications of AREU

We now show that AREU is compatible with the Allais paradox.

In experimental settings, subjects tend to choose the degenerate lottery $p_1 = \delta_{3000}$ over the lottery $p_2 = 0.8\delta_{4000} + 0.2\delta_0$, but choose $q_2 = 0.2\delta_{4000} + 0.8\delta_0$ over $q_1 = 0.25\delta_{3000} + 0.75\delta_0$. Note that the second pair of options are derived from the first pair using a common mixture, $q_1 = 0.2p_1 + 0.8\delta_0$ and $q_2 = 0.2p_2 + 0.8\delta_0$. Under expected utility theory, those who prefer p_1 to p_2 should prefer q_1 to q_2, and vice versa. Hence choices of p_1 and q_2 is a direct contradiction of expected utility theory. This is called the Allais paradox (and in particular the *common ratio effect*), a prominent "anomaly" in the study of choices under uncertainty, began with Allais (1953).[18]

AREU is compatible with this phenomenon. Given a reference order R that deems the safest alternative in the first choice problem—in which a sure prize is available—as safer, a decision maker is more risk averse and uses a more concave utility function. That is, where $A = \{p_1, p_2\}$ and $B = \{q_1, q_2\}$, we have $u_{r(A)} = f \circ u_{r(B)}$ for some concave transform f.[19] It

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[18] This example is taken from Starmer (2000). Camerer (1995) and Starmer (2000) provide an in-depth survey.

[19] The choice in the first problem can be explained by a (Bernoulli) utility function u_A if and only if, after

is because of this change in utility function characterizing increased risk aversion that makes p_1, the safe option, appealing in the first choice problem. More generally, AREU captures the favoring of risk-free options whenever they are available, similar to Kahneman & Tversky (1979)'s *certainty effect.*

However, AREU is incompatible with choices of p_2 and q_1—violation of expected utility theory in the opposite direction. Since the decision maker is more risk averse in the first choice problem, if instead p_2 is chosen over p_1, then q_2 must be chosen over q_1 in the second choice problem since a less concave utility function will make q_2 more appealing. Analogously, a choice of q_1 implies a choice of p_1. To summarize, behaviors compatible with AREU are: (p_1, q_1), (p_2, q_2), and (p_1, q_2). The opposite behavior in which the decision maker in more risk loving in the first choice problem, (p_2, q_1), is ruled out.[20] This stipulates a specific type of departure from standard expected utility—more risk aversion in the presence of safer alternatives.

AREU's compatibility with the Allais paradox is not limited to the above specification. Moreover, it captures both the *common ratio effect* and the *common consequence effect*. We generalize the previous arguments when $|X| := |\text{supp}(\{\delta, p, q\})| = 3$ in the following statement:

Fact. *Consider a degenerate lottery δ and a lottery p such that neither of them first order stochastically dominates another. Consider lotteries $\delta' = \delta^\alpha q$ and $p' = p^\alpha q$ for any $\alpha \in (0,1)$ and lottery (degenerate or otherwise) q. Suppose $|X| = 3$, then*

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normalization ($u_A(0) = 0$ and $u_A(4000) = 1$), $u_A(3000) > 0.8$. Similarly, the choice in the second problem can be explained by a normalized utility function u_B if and only if $u_b(3000) < 0.8$. This is from solving $u_A(3000) > 0.8u_A(4000) + 0.2u_A(\$0)$ and $0.25u_B(3000) + 0.75u_B(0) > 0.2u_B(4000) + 0.8u_B(0)$ for $u_A(3000)$ and $u_B(3000)$. Furthermore, as long as $u_A(3000) > u_B(3000)$, we have $u_A = f \circ u_B$ for some concave $f : [0,1] \to [0,1]$.

[20] This is consistent with the behavioral postulate referred to as *Negative Certainty Independence* in Dillenberger (2010); Cerreia-Vioglio et al. (2015).

1. *If* $\delta \in c(\{\delta, p\})$ *and* $p' \in c(\{\delta', p'\})$*, then*
    a) *For all* $u_1, u_2 : X \rightarrow \mathbb{R}$ *such that* u_1 *explains the first choice and* u_2 *explains the second,* $u_1 = f \circ u_2$ *for some concave function* $f : \mathbb{R} \rightarrow \mathbb{R}$*.*
    b) *Moreover, the choices admit an AREU representation such that* $r(\{\delta, p\}) R r(\{\delta', p'\})$*.*
2. *If the choices* $c(\{\delta, p\})$*,* $c(\{\delta', p'\})$ *admit an AREU representation, then*
    a) *If* $p \in c(\{\delta, p\})$*, then* $p \in c(\{\delta', p'\})$*.*
    b) *If* $\delta' \in c(\{\delta', p'\})$*, then* $\delta \in c(\{\delta, p\})$*.*

Last we consider one other application of AREU. A known phenomenon in behavioral finance is *reaching for yield*, in which investors invest less when the risk-free rate is higher, which is at odds with the standard expected utility model with commonly used specifications such as those that exhibit constant relative risk aversion. Lian et al. (2017) shows that this behavior is at odds with utility functions exhibiting constant or decreasing absolute risk aversion, capturing a large class of utility functions typically used in behavioral finance. The authors also provided evidence of this behavior.

AREU is consistent with this observation, and more specifically that the addition of a *better* sure prize increases risk aversion. Consider a choice set A which contains a sure prize of \$5 and choice set B which is A with an added option: a sure prize of \$7. Although risk is fully avoidable in both choice problems due to the availability of sure prizes, it is intuitive that a decision maker may display greater risk aversion in B. AREU captures this behavior using the specification $\delta_x R \delta_y$ whenever $x > y$, in which the decision maker maximizes expected utility with a more concave utility function when a *better* sure prize is present. This extends the intuition of increased risk aversion from the addition of safer alternative to the addition of safer *and* better alternatives.

1.3.6 Linkage between violations of WARP and violations of Independence

Risk Reference Dependence (Axiom 1.3) demands WARP and Independence over certain subsets of all choice problems. Given an AREU choice correspondence, we now state the consequences of imposing each of Transitivity, WARP, and Independence over the (entire) set of all choice problems $\mathcal{A}$. It turns out adding any one of these assumptions bring us back to standard expected utility, giving us a formal separation of AREU from the wide range of non-expected utility models in which WARP / complete preference ranking is maintained.

First, we adapt Transitivity, typically defined on a preference relation, to the framework of choice.

Definition 1.5. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. We say c satisfies *Transitivity* over $\mathcal{S} \subseteq \mathcal{A}$ if for any $\{p,q\},\{q,s\},\{s,p\} \in \mathcal{S}$,

$$p \in c(\{p,q\}) \text{ and } q \in c(\{q,s\}) \Rightarrow p \in c(\{q,s\}).$$

Transitivity on c is analogous to the standard definition of Transitivity on a preference relation (where $p \succsim q$ and $q \succsim s \Rightarrow p \succsim s$). In the framework of choice, c may satisfy Transitivity over the entire collection of choice problems $2^Y \setminus \{\emptyset\}$ but violate WARP. For example: $x \in c(\{x,y\})$, $y \in c(\{y,z\})$, $x \in c(\{x,z\})$, but $z \in c(\{x,y,z\})$. However, the reversed implication is true: If c satisfies WARP over $2^Y \setminus \{\emptyset\}$, then c satisfies Transitivity over $2^Y \setminus \{\emptyset\}$.[21]

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[21] When $S \neq 2^Y \setminus \{\emptyset\}$, Transitivity and WARP are not nested. For instance take $\mathcal{S} = \{\{x,y\},\{y,z\},\{x,z\}\}$, the choice correspondence $x \in c(\{x,y\})$, $y \in c(\{y,z\})$, $z \in c(\{x,z\})$ satisfies WARP (over $\mathcal{S}$) but not Transitivity.

We say c admits a utility representation if there exists $U : \Delta(X) \to \mathbb{R}$ such that $c(A) = \arg\max_{p \in A} U(p)$.

Proposition 1.3. *Suppose c admits an AREU representation. The following are equivalent:*

1. *c satisfies Transitivity (over $\mathcal{A}$).*
2. *c satisfies WARP (over $\mathcal{A}$).*
3. *c satisfies Independence (over $\mathcal{A}$).*
4. *c admits an expected utility representation.*
5. *c admits a utility representation.*

This result states that AREU cannot independently accommodate Transitivity, WARP, or Independence. Put it differently, although AREU weakens multiple postulates (WARP and Independence), it is in fact a "tight" deviation from standard expected utility in that the re-instiution of either postulate recovers the expected utility model.[22] In particular, the intersection of AREU and models maintaining WARP, or equivalently a complete and transitive preference relation, is expected utility.

AREU is a model that explains non-expected utility behavior as a consequence of changing (risk) preferences. To this end, Proposition 1.3 shows that AREU leaves no extra explanatory power in explaining the violations separately. Instead, the violations are inextricably linked to one another, and resolving either one will bring us back to standard expected utility. When

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[22] Transitivity and WARP do not imply one another. While obtaining Transitivity from WARP requires little additional assumptions, the other direction is typically difficult to achieve without explicitly assuming WARP. A classic example of a choice correspondence that satisfies Transitive but not WARP is $c(\{a,b\}) = \{a\}$, $c(\{b,c\}) = \{b\}$, $c(\{a,c\}) = a$, $c(\{a,b,c\}) = b$. In models of context-dependent choice, violations of WARP are sometimes accommodated alongside violations of Transitivity, and in other cases they are accommodated whilst keeping Transitivity imposed.

a choice correspondence admits a utility representation, choices are interpreted as the consequence of a stable preference ranking. Proposition 1.3 formally states that AREU is in line with this interpretation—standard expected utility ensues when preferences over lotteries are stable, and failure of expected utility is due to changing preferences.

This sets us apart from the majority of non-EU models where Independence is weakened under the assumption of WARP. In those cases, a single utility function is maximized, but it doesn't take the expected utility form. In our case, choices come from utility functions that conform with the expected utility form, but there are many of them, which characterize changing preferences. This is the result of a joint weakening of both WARP and Independence, the core idea of the unified framework we propose. In the time and social domain, we show that the same results and arguments hold.

1.3.7 Comparison to other non-expected utility models

In this section, we investigate the consequence of adding various restrictions to AREU, and use them to explore the relationship between AREU and other non-expected utility models of risk preferences.

1.3.7.1 AREU with Transitivity in a probability triangle

While Proposition 1.3 provides a strong separation between AREU and many non-EU models, we can more meaningfully recover the extent to which AREU is related to other models by imposing Transitivity "partially". To this end, we turn our attention to Marschak-Machina triangles (also "probability triangle") for the next part of our analysis.

We will show that AREU is very close to Betweenness, a well-known property first introduced (on preference relations) by Chew (1983); Fishburn (1983); Dekel (1986). Like expected utility, models of betweenness preferences have the characteristic of linear indifference curves (or in higher dimensions, hyperplanes). Their departure from standard expected utility is that the indifference curves need not have the same slopes (resp. gradients). Since Betweenness is typically defined on a preference relation, we first proceed to define Betweenness on a choice correspondence.

Definition 1.6. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. We say c satisfies *Betweenness* over $\mathcal{S} \subseteq \mathcal{A}$ if for any $\{p,q\}, \{p, p^{\alpha}q\}, \{p^{\alpha}q, q\} \in \mathcal{S}$ and $\alpha \in (0,1)$,

1. $c(\{p,q\}) = \{p\} \Rightarrow c(\{p,p^{\alpha}q\}) = \{p\}$ and $c(\{p^{\alpha}q,q\}) = \{p^{\alpha}q\}$,
2. $c(\{p,q\}) = \{p,q\} \Rightarrow c(\{p,p^{\alpha}q\}) = \{p,p^{\alpha}q\}$ and $c(\{p^{\alpha}q,q\}) = \{p^{\alpha}q,q\}$.

We are ready for the first result when restricting attention to a Marschak-Machina triangle. For any three prizes $\{a,b,c\} \subseteq X$, consider the set of all lotteries induced by them, $\Delta(\{a,b,c\})$. Let $\mathcal{B}_{a,b,c}$ denote the set of all finite and nonempty subsets of $\Delta(\{a,b,c\})$. Note that $\mathcal{B}_{a,b,c} \subseteq \mathcal{A}$. Going forward, we omit subscripts and use the notation $\mathcal{B}$. We begin with a few standard definitions. Let p be a mean-preserving spread of q. We say c is *weakly risk averse* (resp. *risk loving*) *over* $\mathcal{B}$ if $\{p,q\} \in \mathcal{B}$ implies $q \in c(A)$ (resp. $p \in c(A)$). If $c(\{p,q\}) = \{p,q\}$ whenever $\{p,q\} \in \mathcal{B}$, we say c is *risk neutral.* We say that indifference curves fan out (resp. fan in) if they become weakly steeper (resp. flatter) in the first order stochastic dominance direction.

Proposition 1.4. *Suppose c admits an AREU representation. If c satisfies Transitivity over $\mathcal{B}$, then:*

1. *c satisfies Betweenness over $\mathcal{B}$.*
2. *c is either weakly risk averse over $\mathcal{B}$, weakly risk loving over $\mathcal{B}$, or risk neutral over $\mathcal{B}$.*

3. *Indifference curves fan out if c is weakly risk averse.*

4. *Indifference curves fan in if c is weakly risk loving.*

Proposition 1.4 connects AREU to linear indifferent curves—a property of standard expected utility. Moreover, Proposition 1.4 also pins down the set of admissible indifferent curves. Even though AREU allows a decision maker to have varying magnitudes of risk aversion, compliance with Transitivity would bound the increase in risk aversion such that choices are either exclusively risk averse or exclusively risk loving (in this probability triangle). In each case, a particular direction of fanning is also prescribed (Figure 1.3.1). These results provide testable predictions for AREU, and separates it from other models, which we discuss next.

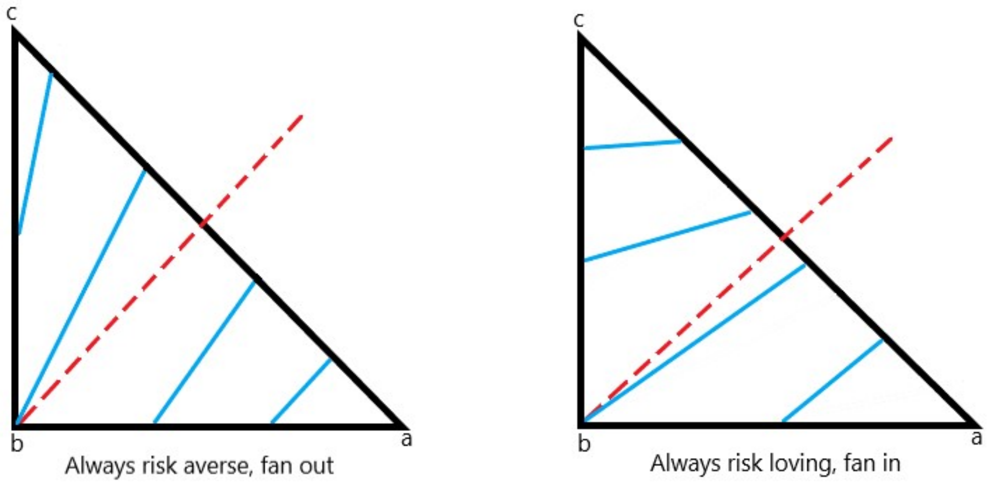

Figure 1.3.1: Fanning Out and Mixed Fanning
Let $a < b < c$. Dotted (red) lines are the mean-preserving spread lines. Solid (blue) lines are indifferent curves. Referring to Proposition 1.4, the picture on the left corresponds to point 3 and the picture on the right corresponds to point 4.

1.3.7.2 Comparison to other non-expected utility models

Various alternatives to expected utility were introduced by Quiggin (1982), Chew (1983); Fishburn (1983); Dekel (1986), Bell (1985); Loomes & Sugden (1986), Gul (1991), Kőszegi & Rabin (2007), and Cerreia-Vioglio et al. (2015). We now use Proposition 1.3 and Proposition 1.4 to study their relationship to AREU.

The AREU model has a close relationship with *betweenness preferences* introduced by Chew (1983); Fishburn (1983); Dekel (1986). Although the two intersect only at expected utility, a direct application of Proposition 1.3, the two make similar predictions for binary choices when Transitivity is added to AREU in a probability triangle.

Among models of betweenness preferences, Gul (1991)'s *disappointment aversion* is closest in spirit to AREU, but the two predict different behavior. In *disappointment aversion*, the set of possible outcomes of each lottery is decomposed into elevation prizes and disappointment prizes, and the utilities from disappointment prizes are discounted using a function of the probability of disappointment. An implication of disappointment aversion is the property of mixed fanning, in which indifference curves first fan in and then fan out, for example. AREU cannot accommodate mixed fanning, a direct application of Proposition 1.4, and so the two models differ in their coverage of non-expected utility behavior.

For the same reason, AREU and Cerreia-Vioglio et al. (2015)'s *cautious expected utility* put fourth different behavioral predictions. In their model, a decision maker evaluate each lottery as its worst certainty equivalence under a set of (Bernoulli) utility functions. The result is a behavior that resembles cautiousness. A property resembling mixed fanning is an implication of their model, where indifferent curves are steepest in the middle, a consequence of the axiom *Negative Certainty Independence*: $p \gtrsim \delta$ implies $p^{\alpha}q \gtrsim \delta^{\alpha}q$.

Like AREU, Kőszegi & Rabin (2007)'s *reference-dependent risk preferences* uses reference points to explain non-expected utility behavior. However, both the identification of reference points and the consequence of changing reference points differ. In AREU, reference alternatives are given by the safest alternatives in choice problems, and they serve as a proxy for changing risk preferences. In Kőszegi & Rabin (2007), a decision maker is subjected to gain-loss utility relative to a reference point, where the reference point is the lottery she expects to receive. We focus on *choice-acclimating personal equilibrium* (CPE), in which reference points are endogenously set as the eventually-chosen alternatives. Masatlioglu & Raymond (2016) shows that when a CPE specification satisfies first order stochastic dominance, the implied behavior can be explained by the *quadratic utility* functionals of Machina (1982); Chew et al. (1991). Yet, Chew et al. (1991) demonstrates that quadratic functionals intersect with betweenness preferences only at expected utility, and hence the CPE model of Kőszegi & Rabin (2007) intersects with AREU only at expected utility.[23]

The model closest to AREU, to my knowledge, is the *context-dependent gambling effect* by Bleichrodt & Schmidt (2002). In their model, a decision maker's preferences are explained by two (Bernoulli) utility functions, one for comparisons that involve a riskless option and another for the rest. Unlike AREU, their model only applies to binary decisions, which results in different axioms and applicability. Furthermore, when a degenerate lottery is slightly perturbed into a non-degenerate one, it produces a choice reversal, which seems implausible. Their model also does not accommodate violations of expected utility in choice problems without a riskless option, such as variations of the Allais paradox. Finally, while their axioms are separately imposed on binary decisions involving and not involving riskless options, our axioms are imposed on the choice correspondence without such discrimination.

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[23] Similar conclusions of non-intersection with AREU (other than expected utility) can be made for Quiggin (1982)'s *rank dependent utility* (see Chew & Epstein (1989)) and Bell (1985); Loomes & Sugden (1986)'s *disappointment theory*. Some of these results, and a comprehensive summary, are provided by Masatlioglu & Raymond (2016).

1.4 Time Preferences

In this section, we provide an application of our unified framework for choices over delayed consumption. The canonical model for this setting is Discounted Utility, axiomatized by Fishburn & Rubinstein (1982), in which a decision maker evaluates each payment-time pair (x,t) by $\delta^t u(x)$. However, Discounted Utility has routinely failed experimental tests as subjects violate Stationarity: the choice between two payments changes when the decision is made in advance, typically favoring the later option for the long-term decision.[24] To accommodate this violation, we weaken the axioms of Fishburn & Rubinstein (1982) using an approach analogous to Section 1.2's Reference Dependence. The outcome is a utility representation in which choices maximize exponentially discounted utilities using a discount factor that depends on the timing of the earliest payment.

1.4.1 Preliminaries

Let $X = [a,b] \subset \mathbb{R}_+$ be an interval of non-negative payments and let $T = [1,\bar{t}] \subset \mathbb{R}_+$ be an interval of non-negative time points. $X \times T$ is the set of alternatives, where $(x,t) \in X \times T$ denotes a payment of x at time t. We endow $X \times T$ with the standard Euclidean metric. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of $X \times T$. Finally, let $c : \mathcal{A} \to \mathcal{A}$, $c(A) \subseteq A$, be a choice correspondence.

We maintain the following standard axioms for time preference, that higher payments and sooner payments are better.

Axiom 1.6.

1. *Outcome Monotonicity: if* $x > y$*, then* $c(\{(x,t),(y,t)\}) = \{(x,t)\}$.

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[24] See for example Laibson (1997), Frederick et al. (2002), and Benhabib et al. (2010).

2. *Impatience: if* $t < s$*, then* $c\left(\{(x,t),(x,s)\}\right) = \{(x,t)\}$.

1.4.2 Reference Dependent Patience

Time consistency in choice is captured by a well-known behavioral property called Stationarity. Under Stationarity, a decision maker's preference between two future payments is consistent regardless of when the decision is made. For this reason, Stationarity is often deemed a normative postulate in economic analysis.

Similar to what we did to WARP and Independence in previous sections, we first define what it means for a choice correspondence c to satisfy Stationarity over a subset of all choice problems $\mathcal{S} \subseteq \mathcal{A}$.

Definition 1.7. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *Stationarity* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$, $a > 0$,

$$(x,t) \in c(A),\ (y,q) \in A,\ (y,q+a) \in c(B), \text{ and } (x,t+a) \in B \Rightarrow (x,t+a) \in c(B).$$

Supplied with Axiom 1.6, a direct adaption of Fishburn & Rubinstein (1982) into the framework of choice gives that c satisfies WARP and Stationarity over $\mathcal{A}$ if and only if it admits a (exponential) Discounted Utility representation.

A choice correspondence that exhibits time inconsistency fails to satisfy Stationarity over $\mathcal{A}$. However, the choice correspondence may still satisfy Stationary over some subsets of $\mathcal{A}$. Consider the following axiom, which states that Stationarity is satisfied between any two choice problems that share an earliest payment.

Axiom 1.7 (Reference Dependent Patience (RDP))**.** *For any $A, B \in \mathcal{A}$, if A and B share an earliest payment, then c satisfies WARP and Stationary over $\{A, B\}$.*

The axiom posits that a violation of WARP and Stationarity between two choice problems can only occur if they do not share an earliest payment. If we interpret compliance with Stationarity as having a stable level of patience, the axiom proposes that patience may depend on how soon any payment can be attained. This allows us to capture behavior in which compliance with WARP and Stationarity is not necessarily upheld between long-term and short-term choice problems, such as those exhibited in time consistency experiments.

Note that this postulate can be rewritten in the style of Reference Dependence (Axiom 1.1) and Risk Reference Dependence (Axiom 1.3) from previous sections, stated formally in the following lemma.

Lemma 1.1. *Fix a choice correspondence c, the following are equivalent.*

1. *c satisfies Axiom 1.7.*
2. *For every choice problem $A \in \mathcal{A}$ and every earliest payment (x, t) in it, c satisfies WARP and Stationarity over $\{B \subseteq A : (x, t) \in B\}$.*

Albeit straightforward, the lemma reassures us that the unified method of weakening standard postulates proposed in this paper is not dissimilar to demanding compliance between pairs of choice problems.

In fact, their equivalence in this setting is due to two details. First, unlike our general model (Section 1.2) and application in the risk domain (Section 1.3), in which the reference order is either fully or partly subjective, the reference points in the present setting is completely objective—the earliest payments in the choice sets. Because of this objectivity, the reference order is pinned down axiomatically, and the axiom does not involve an existential statement that allows for subjectivity in determining reference points. Second, WARP and Stationarity are properties between pairs of choices (and not more). This is not the case for all postulates. For example, Transitivity is an axiom that is trivially satisfied between any pair of choices, but a violation can be found when more choices are considered. Identifying this equivalence, and the reasons thereof, allows us to design more efficient tests of the axioms in our unified framework.

1.4.3 Increasing Patience

We postulate that patience (may) increase when options are postponed.

Consider prizes $x_1 < x_3$ arriving at time $t_1 < t_3$ respectively. We posit that by *postponing* the options by $a > 0$, the decision maker is (weakly) more patient and will choose $(x_3, t_3 + a)$ over $(x_1, t_1 + a)$ if she chose (x_3, t_3) over (x_1, t_1). The postulate differs from Stationarity as it allows for the choice of (x_1, t_1) over (x_3, t_3) but $(x_3, t_3 + a)$ over $(x_1, t_1 + a)$, or *present bias*. To summarize, it allows for violation of Stationarity in one direction but not the other.

However, this falls short of capturing changes in patience. Difference in delay aversion between individuals cannot be directly categorized into difference in discounting and difference in consumption utility, an issue discussed in Ok & Benoît (2007). Just because a decision maker chooses a sooner option, and another a later one, it is not conclusive that the first decision maker discounts more. It could be that there is difference in consumption utility, where the first decision maker's marginal utility for money is a lot lower than that of the second decision maker inducing the choice of a sooner but smaller prize.

We introduce a technique that allows us to "fix" consumption utilities and only allow discounting to change. This can be used to characterize a set of individuals whose consumption utility is (as-if) the same, and differ only in their patience level. For this paper, we use it to restrict the choice behavior of a single individual to those that can be explained by a single consumption utility function while allowing patience to vary.

Consider $c(\{(x,t),(y,q),(z,s)\}) = \{(x,t),(y,q),(z,s)\}$, where (x,t) gives the smallest payment but arrives earliest, (z,s) gives the largest payment but arrives latest, and (y,q) is intermediate in both. Now consider the new choice problem $\{(x,\lambda t),(y,\lambda q),(z,\lambda s)\}$, where $\lambda < 1$; that is, all payments will now arrive at a (common) fraction of time. Under Stationarity, a decision maker only cares about the delay between the alternatives and would now strictly prefer the latest option since the time-difference between any two options is smaller. Yet it is ambiguous how a decision maker of the present model would behave. On one hand, an earlier choice problem causes the decision maker to choose more impatiently; on the other hand, delays between alternatives have decreased, which favor later options. The competing forces render the choice ambiguous. The same competing forces occur when $\lambda > 1$: the decision maker is more patient, but delays between options are larger. In these situations, we restrict the decision maker's behavior in the following way: if the decision maker chooses both the earliest and the latest alternatives after such a transformation (and recall that he was indifferent between all three before), then he also chooses the intermediate option in the new choice problem.

The same restriction is imposed when the transformation is of the form $\{(x,\lambda t-a),(y,\lambda q-a),(z,\lambda s-a)\}$. Note that only λ changes the delay between the options, and a symmetrically shifts arrival time. Also, the only instance this restriction is non-trivial is when the aforementioned competing forces are present: when "$\lambda < 1$ and $\lambda t - a < t$", where the decision maker becomes less patient but she no longer has to wait as long for a better payment, and when "$\lambda > 1$ and $\lambda t - a > t$", where she is more patient but also has to wait longer for a better payment.

This gives rise to the following axiom.

Axiom 1.8 (Increasing Patience)**.** *For all* $t_1 < t_2 < t_3$*,* $A = \{(x_1,t_1),(x_2,t_2),(x_3,t_3)\}$*, and* $A' = \{(x_1,\lambda t_1 + a),(x_2,\lambda t_2 + a),(x_3,\lambda t_3 + a)\}$*,*

1. $c(\{(x_1,t_1),(x_3,t_3)\}) = \{(x_3,t_3)\} \Rightarrow c(\{(x_1,t_1+a),(x_3,t_3+a)\}) = \{(x_3,t_3+a)\}$ *for all* $a > 0$*,*
2. $c(A) = A$ *and* $(x_1,\lambda t_1 + a),(x_3,\lambda t_3 + a) \in c(A') \Rightarrow (x_2,\lambda t_2 + a) \in c(A')$ *for all* $\lambda, a \in \mathbb{R}$*.*

This postulate is trivially satisfied by a decision maker whose behavior fully complies with Stationarity, since she can neither be more patient nor less patient when options are symmetrically postponed.

1.4.4 Representation Theorem

We are ready for the utility representation and representation theorem.

Definition 1.8. c admits a Present-Biased Discounted Utility representation (PBDU) if there exist a strictly increasing and continuous utility function $u : X \longrightarrow \mathbb{R}$ and a set of time-indexed discount factors $\{\delta_t\}_{t\in T}$ such that

$$c(A) = \underset{(x,t)\in A}{\arg\max}\, \delta_{r(A)}^{t} u(x),$$

where

- $r(A) = \min\{t : (x,t) \in A\}$,
- $t < t' \Rightarrow \delta_t \leq \delta_{t'}$,
- δ_t is continuous on $[1,\bar{t}]$.

Proposition 1.5. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Reference Dependent Patience, Increasing Patience, Outcome Monotonicity, Impatience, and Continuity.*
2. *c admits a PBDU representation.*

Furthermore, in every PBDU representation, discount factors δ_t are unique given u.

In this model, it is *as if* the decision maker maximizes exponentially discounted utility, but with discount factors that depend on the timing of the earliest available payment. When the earliest available payment arrives sooner in one choice problem than another, then the decision maker uses a lower discount factor in the former. Since discount factors are often interpreted as a measure of *patience*, our model can be viewed as one in which the decision maker's patience changes systematically across choice problems, where she is less patient when an earlier payment is available.

This model conforms with present bias, the empirically prevalent failure of dynamic consistency in which decision makers exhibit less delay aversion for long-term decisions. Take for example the classic observation of present bias, where \$$x$ today is preferred to \$$y$ tomorrow (choice problem A) but the opposite decision is made when both payments are postponed by a year (choice problem B). The model we propose explains the behavior with the simple interpretation that, since the earliest alternative for A arrives sooner than that for B (i.e., $r(A) < r(B)$), the decision maker is less patient in the former (i.e., $\delta_{r(A)} < \delta_{r(B)}$).

Moreover, even though the choice between \$$x$ at time t and \$$y$ a day later is not consistent across the time horizon t, the model predicts that as we gradually postpone both options with s, the choice can only switch from $(x, t+s)$ to $(y, t+1+s)$. That is, if there is a point in time at which the decision maker becomes sufficiently patient to choose \$$y$ over \$$x$, she must continue to do so as we further postpone both options. This "single switching" property is closely connected to the ordered-reference nature of PBDU and provides testable predictions.

Finally, the underpinning of PBDU is the simultaneous weakening of WARP and Stationarity in a reference-dependent approach. Reminiscent of the observation made in Proposition 1.3, WARP and Stationarity are interconnected in our model: neither of which can be independently weakened, formally stated in the following result.

Proposition 1.6. *Suppose c admits a PBDU representation. Then the following are equivalent:*

1. *c satisfies WARP (over $\mathcal{A}$).*
2. *c satisfies Stationarity (over $\mathcal{A}$).*
3. *c admits an exponential discounting utility representation.*
4. *c admits a utility representation.*

1.4.5 Related models of time preferences

The biggest difference between Present-Biased Discounted Utility (PBDU) and *hyperbolic discounting,* a class of models in which future options are discounted disproportionately less, is that PBDU (when non-trivial) necessitates WARP violations and hyperbolic discounting models satisfy WARP. Furthermore, unlike models of hyperbolic discounting, PBDU evaluates all alternatives in a choice problem using a single discount factor.[25] However, the empirically informed intuition that discount factors vary across time is shared between models of hyperbolic discounting and PBDU, albeit implemented differently. For our model,

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[25] See for instance Loewenstein & Prelec (1992) and Laibson (1997).

PBDU, discount rate changes at the choice problem level, whereas for hyperbolic discounting it changes at the alternative level. The difference is stark when we consider choice problems that contain more than two alternatives. In hyperbolic discounting, the preference between any two options stays the same regardless of what choice problems they appear in, hence WARP is never violated. This is not the case for PBDU, where a sooner option may become superior to a later one from the introduction of a third (but not necessarily chosen) alternative, and results in WARP violations in PBDU.

Exponential discounting has advantageous properties in economic applications, propelling Laibson (1997)'s well-known *quasi-hyperbolic discounting*. In their model, behavior complies with Stationarity as long as the choice is between two future payments, and present bias only arises when an immediate payment is involved. This is not the case in PBDU, as the switch from choosing the earlier payment to choosing the later one can occur at any time as we gradually shift both payments into the future. Another implication of quasi-hyperbolic discounting in our setting is the failure of continuity, where an instantaneous change in choice occurs when the earlier payment arrives at time 1 ("today"). Our model complies with continuity of choice, and instead forgoes WARP to explain dynamic inconsistency.

We now turn to two other models that both explain dynamic inconsistency and can explain WARP violations.

Lipman et al. (2013) provides an explanation of dynamic inconsistency that builds on Gul & Pesendorfer (2001)'s introduction of *temptation*. In Gul & Pesendorfer (2001), a decision maker has commitment utilities and temptation utilities, and chooses a menu (a choice problem) taking into account both. The result is that a larger menu may be inferior, a departure from the conventional understanding that more options should never be worse. Lipman et al. (2013) extends this to the setting of time preference and proposes that a decision maker assess current consumption using temptation utility and future consumption using commitment utility. When making decisions in advance, it is as if the decision maker is choosing between singleton menus for her future self, and the absence of temptation utility allows her to make a more patient decision relative to choices over immediate consumption. Like *quasi-hyperbolic discounting*, present-bias is restricted to immediate consumption, whereas PBDU allows present-bias to kick in at time frame and as long as its effect is persistent when both options are further postponed—the "single switching" property discussed earlier.

More recently, Freeman (2016) introduced a framework in which WARP is weakened, and reversals are explained by time-inconsistent preferences. In their model, a decision maker chooses when to complete a task, and may exhibit choice reversal when additional opportunities for completions are introduced (a expansion of the choice set). In particular, in response to the addition of an opportunity for completion, a sophisticated decision maker may choose to complete the task earlier (and never later) in anticipation that allowing her future self to make that decision would result in an eventual completion time that is worse than completing the task now. A naive decision maker, however, could only end up completing later.

While our model, PBDU, allows for choice reversal in the direction of choosing an earlier option when the choice set expands, it is in fact incompatible with Freeman (2016)'s decision makers' behavior (other than WARP-conforming behavior). In PBDU, a reversal can only occur when the discount rate changes, which only happens when an alternative earlier than any other already available is added. However, in Freeman (2016)'s model, either that this added alternative is chosen (which is not a reversal) or the decision problem becomes identical to before and so WARP is complied with. Indeed, the necessary conditions of their model, Irrelevant Alternatives Delay (for a naive agent) and Irrelevant Alternatives Expedite (for a sophisticated agent) only hold in PBDU if WARP holds.

1.5 Social Preference

We now turn to our last application.

Consider a decision maker who has a particular type of set-dependent social preference—she shares more when greater equity is attainable. Experiments in economics and psychology have shown that, instead of being fully selfish and maximize monetary payment to oneself, people are often willing to share their wealth. This leads to models of *other-regarding preferences* and *inequality aversion,* first introduced by Fehr & Schmidt (1999); Bolton & Ockenfels (2000); Charness & Rabin (2002). Furthermore, one's desire to share, or inequality aversion, may be affected by the options they have, often in the direction where the availability of more equitable options results in greater sharing. One explanation for this behavior is outcome-based, where a decision maker becomes more inequality averse in the presence of more equitable distributions. Another explanation is intention-based, where the decision maker seeks to be perceived as fair.[26] Our model does not distinguish between these two causes for increased altruism, we refer interested readers to surveys by Fehr & Schmidt (2006); Kagel & Roth (2016) for the vast evidence and suggested explanations.

To illustrate, suppose a decision maker is endowed with \$10 and is given a number options to share it with another individual. When she is asked to choose between giving \$2 and giving \$3, giving \$2 may seem reasonable. However, when the choice is between giving \$2, \$3 or \$5, the decision maker may opt for \$3 (and keeping \$7) instead. The pair of choices (over income distributions) $c\left(\{(\$8,\$2),(\$7,\$3)\}\right)=\{(\$8,\$2)\}$ and $c\left(\{(\$8,\$2),(\$7,\$3),(\$5,\$5)\}\right)=\{(\$7,\$3)\}$ violates WARP. Hence the assumption of utility maximization, even if the utility function captures other-regarding preferences and inequality aversion, is incapable of explaining this behavior.

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[26] See for example Ainslie (1992), Rabin (1993), Nelson (2002), and Sutter (2007).

Using our unified framework, where both WARP and a standard postulate are weakened using reference-dependence, we provide a model in which a decision maker's degree of inequality aversion increases when equitable options are added to a choice problem.

1.5.1 Preliminaries

Let $X = \mathbb{R}_+ \times \mathbb{R}_+$ be the set of all pairs of non-negative monetary payments. We call a pair $(x, y) \in X$ an income distribution, where x is the dollar amount for the decision maker and y for a second individual. We endow X with the standard Euclidean metric. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of X and $c : \mathcal{A} \to \mathcal{A}$, $c(A) \subset A$ a choice correspondence.

The first axiom is standard, an income distribution that gives everyone weakly more, and at least one person strictly more, is strictly preferred.

Axiom 1.9 (Monotonicity)**.** $c(\{(x, y), (x', y')\}) = \{(x, y)\}$ *whenever* $x \geq x'$, $y \geq y'$, *and* $(x, y) \neq (x', y')$.

1.5.2 Fairness Dependence

Our first axiom for this section is a specialization of Axiom 1.1 from Section 1.2. It characterizes behavior in which choices from choice problems containing the same amount of attainable equity conform with quasi-linear preferences. The use of quasi-linear preferences for choices involving money is common in the economics. Since our model introduces reference-dependent utility functions, using quasi-linear utilities when preferences are stable provides meaningful restrictions to choices.

Definition 1.9. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *quasi-linearity* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$ and $a \in \mathbb{R} \setminus \{0\}$,

$$(x,y) \in c(A),\ (x',y') \in A,\ (x'+a,y') \in c(B), \text{ and } (x+a,y) \in B \Rightarrow (x+a,y) \in c(B).$$

In order to characterize attainable equity, we first need a measure of equity. A nature candidate is the ratio between x, y for any income distribution (x,y). We define the equity index of (x,y) as $e_{(x,y)} := \min\left\{\frac{x}{y}, \frac{y}{x}\right\}$. The use of min is to treat income distributions (a,b) and (b,a) indiscriminately in terms of measuring equity. Note that $e_{(x,y)}$ is always weakly less than 1, and values closer to 1 correspond to greater equity. The index captures how "close" are the two payments within an income distribution (a,b), adjusting for scale, and is ordinally equivalent to the form taken by the Gini coefficient $|a-b|/a$. The cardinality of this index plays no role in our analysis.

Analogous to our approach in previous sections, we demand that choices comply with WARP and quasi-linearity when the fairest income distribution of a choice problem is unchanged. Departing from complete compliance with WARP and quasi-linearity, we allow the decision maker to choose different income distributions when the fairest income distribution is dropped, potentially violating WARP. Formally, we impose a weakening of WARP and quasi-linearity in the following way:

Definition 1.10. For any set of income distributions $A \in \mathcal{A}$, we call $\Psi(A) := \left\{(x,y) \in A : e_{(x,y)} \geq e_{(x',y')} \forall (x',y') \in A\right\}$ the set of *fairest* income distributions in A.

Axiom 1.10 (Fairness Dependence (FD))**.** *For every choice problem $A \in \mathcal{A}$ and any fairest distribution $(x,y) \in \Psi(A)$, c satisfies WARP and quasi-linearity over $\{B \subseteq A : (x,y) \in B\}$.*

1.5.3 Increasing Altruism

We study choices that exhibit increased sharing when greater equity is attainable. Consider the following postulate. Suppose in a choice problem an income distribution (x, y) is chosen over $(x', 0)$, a distribution where the decision maker does not give at all. We postulate that by adding any other income distributions into the choice set, since this can only (weakly) increase attainable equity, she does not switch to not sharing, $(x', 0)$. Additionally, we extend this postulate to cases where the comparison is between (x, y) and (x', y') such that $y' < y$. Effectively, this restriction imposes a direction on which willingness to share changes—the decision maker is weakly more altruistic when more options are available, which weakly increases attainable equity. Formally:

Axiom 1.11 (Increasing Altruism)**.** *For any* $A, B \in \mathcal{A}$ *such that* $A \subset B$ *and* $(x, y), (x', y') \in A$ *such that* $y > y'$. *If* $(x, y) \in c(A)$ *and* $(x', y') \notin c(A)$, *then* $(x', y') \notin c(B)$.

1.5.4 Representation Theorem

Consider the following utility representation in which utility from receiving the amount \$$x$ is always evaluated consistently but utility from giving amount \$$y$ depends on how much equity is attainable from the choice problem.

Definition 1.11. c admits a Fairness-based Social Preference Utility representation (FSPU) if there exists a set of strictly increasing utility functions $\{v_r : \mathbb{R}_+ \longrightarrow \mathbb{R}\}_{r \leq 1}$ such that

$$c(A) = \arg\max_{(x,y) \in A} x + v_{r(A)}(y),$$

where

- $r(A) = \max_{(x,y)\in A} e_{(x,y)}$,
- $r > r' \Rightarrow v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$,
- $\arg\max_{(x,y)\in A} x + v_{r(A)}(y)$ has a closed-graph.

Proposition 1.7. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Fairness Dependence, Increasing Altruism, Monotonicity, and Continuity.*
2. *c admits a FSPU representation.*

Furthermore, in every FSPU representation, v_r is unique for all r.

In this model, the decision maker's utility from giving dollar amount y, $v_{r(A)}(y)$, depends on how much equity is attainable in the underlying choice problem, as measured by $r(A)$. Recall that $e_{(x,y)}$ is weakly less than 1 and a number closer to 1 represents greater equity. Hence, attainable equity from choice set A is simply the highest value $e_{(x,y)}$ among available income distributions $(x,y) \in A$, or $r(A) = \max\left\{e_{(x,y)} : (x,y) \in A\right\}$. When $r(A)$ is greater, the decision maker values any given shared amount y more. Consequently, even if income distribution (x,y) is chosen over (x',y') in some choice problem, where $y' > y$, adding a very fair option could cause the decision maker to switch to (x',y').

This model accommodates increased willingness to give when distributing a fixed pie with different splitting options. To illustrate, suppose a decision maker must allocate a fixed amount of money, say \$100, between her and another individual, but she is not allowed to split the amount however she likes. Instead, there is a set of feasible distributions characterized by $D \subset [0,1]$; she can choose to allocate $\alpha \cdot \$100$ to herself if and only if $\alpha \in D$. By specifying two different sets of feasible distributions, D and D', we have effectively specified two choice problems in our setup. Say $D = \{0.5, 0.6, 0.7\}$ and $D' = \{0.6, 0.7\}$. If $\alpha = 0.7$ is chosen in D' (the decision maker keeps \$70 for herself and \$30 is given to the other individual), she might choose to keep less in D due to increased altruism from greater attainable equity. However, if she chose $\alpha = 0.6$ in D', then she must not choose $\alpha = 0.7$ in D; this is a testable prediction.

In FSPU, altruism is maximal when a perfectly balanced income distribution is available. In particular, note that increased altruism is not the result of opportunity to *give* more; instead, it is attainable equity that drives altruism. To illustrate the difference, consider the same example but with $D = \{0.5, 0.3, 0.2\}$ and $D' = \{0.3, 0.2\}$. Even though D contains alternatives that achieve greater equity, the decision maker's ability to give is the same across the two choice problems. Yet, since the feasible allocations are always unfavorable to her (she can never keep more than half), higher attainable equity results from her ability to *take* more. In this setting, our decision maker can be interpreted as being less altruistic when the world is unfair to her, and she becomes more altruistic when fairer options are added.

Lastly, FSPU allows for willingness to forgo a greater surplus in favor of giving more. Suppose the decision maker must choose between $(\$30, \$20)$ and $(\$60, \$0)$. The second option is appealing in that the total amount of money extracted is greater, whereas the first option sacrifices both surplus and payment to oneself in favor of providing a share to the other individual. Suppose $(\$60, \$0)$ is chosen. The model allows for the behavior in which the addition of $(\$25, \$25)$ to the choice set causes the decision maker to switch from $(\$60, \$0)$ to $(\$30, \$20)$ due to increased generosity. While this behavior is reasonable, it cannot be accommodated by any model that complies with WARP.

The familiar linkage between WARP violation and violation of standard postulate, in this case quasi-linearity, is summarized in the following statement.

Proposition 1.8. *Suppose c admits a PBDU representation. Then the following are equiva-*

lent:

1. *c satisfies WARP (over $\mathcal{A}$).*
2. *c satisfies quasi-linearity (over $\mathcal{A}$).*
3. *c admits a quasi-linear utility representation.*
4. *c admits a utility representation.*

1.5.5 Related Literature

Other-regarding preferences have been extensively studied, and well-known models are introduced by Fehr & Schmidt (1999); Bolton & Ockenfels (2000). However, the primary focus of these models is to capture the characterization of *inequality aversion* using functional forms. In particular, a single and persistent preference ranking of income distributions is assumed throughout these models. Charness & Rabin (2002) introduced a departure that allows for *reciprocity* using a term that lowers utility from giving when the other player is deemed to have "misbehaved".

FSPU, departs from these models by introducing preferences over income distributions that may change from one choice problem to another. In particular, utility from giving depends on how much equity is attainable in the underlying choice set. The vast literature on distributional preferences provides suggestive evidence of this behavior. List (2007); Bardsley (2008); Korenok et al. (2014) showed that in a dictator game, adding (or increasing) the option to take from the receiver significantly reduces a dictator's willingness to give, and in some cases result in choice reversals (WARP violations). However, although the narratives are related, the design of their experiments does not provide a complete test for the predictions of FSPU, as additions of less equitable distributions do not affect preferences in FSPU.

The study of *audience effect* also provides empirical evidence that decision makers care about how others perceive their choices. In Dana et al. (2006), dictators were given the option exit (avoid) a \$10 dictator game and receive \$9, a option that leaves the receiver with nothing. Since a payoff of \$9 (and \$10) can be achieved by going through with the dictator game, exiting is interpreted as a costly effort to avoid the dictator game. 28% of the subjects chose to exit. When the game is conducted such that the decision to exit or not is completely veiled from the receivers, only 4% chose to exit.

In a separate study, Dana et al. (2007) provides dictators a costless opportunity to find out how much the receivers will receive from each of their two options, $(6,1)$ and $(5,5)$, before making a choice (payoffs to themselves, the first number in each pair, are always displayed). 44% of dictators chose not to find out, and among them 86% chose "$(6,?)$" over "$(5,?)$". Only 47% of dictators chose to reveal the payoffs and subsequently chose $(5,5)$ over $(6,6)$. In the baseline, in which all payoffs are displayed by default, 74% of subject chose $(5,5)$ over $(6,6)$. Based on subjects' apparent exploitation of this "moral wiggle room", the authors conclude that fair behavior is primarily motivated by the desire to appear fair, either to themselves or to others.

In game theoretic settings, Rabin (1993)'s pioneering work introduced intention based reciprocity through a notion of *kindness*. In their model, kindness is measured using the set of payoffs an opponent *could* induce. A player's kindness depends on how kind the opponent is, due to the desire to be *fair*, and vice versa, leading to the solution concept term *fairness equilibrium*. Since kindness is measured using the set of available actions, the Rabin (1993)'s model and FSPU share some conceptual similarity. However, since FSPU is built on a decision theoretic framework, it is unable to capture the type of reciprocity concerns depicted in Rabin (1993). The same argument separates FSPU from related models in game theory.

To my knowledge, Cox et al. (2016) is the only other paper, with a decision theoretic setup, that introduces a model to explain WARP violations of this kind. Unlike FSPU, they take *endowment* into account, which allows for the study of *giving* versus *taking*. This is different to the approach in FSPU, where only income distributions are relevant and endowments are not part of the primitive. Based on an intuition related to FSPU, Cox et al. (2016) uses *moral reference points* to explain changes in dictator's willingness to allocate, where a moral reference point more favorable to the dictator (and/or less favorable to the receiver) results in allocating more to the dictator herself. However, unlike FSPU, their reference points are not alternatives, but instead a vector of reference payoffs that depend on multiple allocations within the feasible set as well as the endowment. Consequently, there are many choice problems in which the addition of a more equitable alternative cannot result in choice reversal in their model, since it does not affect the moral reference point, yet preference reversals as a result of adding more equitable alternatives is precisely the behavioral tenet in FSPU.[27] Although the two models are different in many ways, they both seek to capture the increasingly evident intuition that social preference depends on the set of feasible allocations, which results in WARP-violating behavior.

1.6 Conclusion

This paper presents a unified framework for ordered-reference dependence choice. The framework, a reference-oriented weakening of standard postulates, is adaptable to suitably accommodate a wide range of reference orders and reference effects. We demonstrate this

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[27] For example, if a choice problem contains income distributions $(0,1)$ and $(1,0)$, then adding (x,x) for any $x \in (0,1)$ will not change Cox et al. (2016)'s moral reference point, and their model demands compliance with WARP.

universality by providing (axiomatized) utility representations in the context of risk, time, and social preference, where we use reference dependent preferences to account for well-known behavioral anomalies. The resulting models are akin to their standard counterparts, inheriting many of the standard models' properties while explaining non-conforming behavior through intuitive changes in specifications. This is possible primarily due to the use of choice correspondences as primitive (instead of preference relations) along with the weakening of WARP. Together, they allow us to prescribe a new way of weakening standard postulates using behavior in non-binary choice problems.

A natural question is the generality of this exercise—does every choice theoretic model have an ordered-reference dependence version by simply having their axioms weakened using an adapted version of Axiom 1.1? We are able to provide some answers to this question.

In Appendix A, we provide a sufficient condition for an arbitrary standard postulate to be accommodated by our method. We call these standard postulates *finite properties*, they are axioms that are satisfied whenever a violation fails to be substantiated with just finitely many observations. For example, WARP is a finite property, since it is inherently a property between a pair (hence "finite") of choices. To put it differently, if a choice correspondence fails WARP, a violation can be substantiated with just two observations. A non-example is continuity, since a choice correspondence can fail continuity whilst a violation can never be substantiated with finitely many observations. However, at this level of generality, we are only able to achieve a result for ordered-reference dependent choice, and not for ordered-reference dependent utility representations. We formalize and discuss this limitation in Appendix A.

1.7 Appendix A: Toward a Unified Framework

1.7.1 Classifying Axioms

In this technical section, we state a companion result to Proposition 1.1 that allows for (i) either fully or partially prescribing the reference order R and (ii) expanding the accommodated property from WARP to a much larger class. For the latter, we call them *finite properties*, which we will now define.

Let X be an arbitrary set of alternatives, $\mathcal{A}$ the set of all finite and nonempty elements of 2^X. For any $\mathcal{B} \subseteq \mathcal{A}$, we call $c : \mathcal{B} \to \mathcal{A}$, where $c(A) \subseteq A$ for all $A \in \mathcal{B}$, a choice correspondence. Let C be the set of all choice correspondences one can possibly observe from X and $\mathcal{A}$. Formally,

$$C := \{c : \mathcal{B} \to \mathcal{A} \text{ s.t. } \mathcal{B} \subseteq \mathcal{A}\}.$$

A property imposed on a choice correspondence can be viewed as a subset of C that is itself closed under subset operations (where each choice correspondence, a member of C, is viewed as a set of pairs). For instance, the set of all choice correspondences satisfying WARP form a collection of choice correspondences defined by the WARP property. We use this notation to characterize an arbitrary property, formally:

Definition 1.12. We call $\mathcal{T} \subseteq C$ a *property* if for all $c, \hat{c} \in C$ such that $\hat{c} \subset c$, $c \in \mathcal{T}$ implies $\hat{c} \in \mathcal{T}$.

We use "c satisfies $\mathcal{T}$" and "$c \in \mathcal{T}$" interchangeably.

In decision theoretic terms, what we call properties here are features of a choice correspondence that are more likely satisfied when we have less observations (i.e. instead of observing c, we only observe $\hat{c}$). For example, WARP ($A \subset B$ and $c(A) \cap B \neq \emptyset \Rightarrow c(A) \cap B = c(B)$) is a property defined on a pair of a choice sets and their corresponding choices. If the statement of WARP is satisfied for some $c : \mathcal{B} \to \mathcal{A}$, that is, all pairs of choice sets and their corresponding choices satisfy WARP, and $\hat{c} : \mathcal{B}' \to \mathcal{A}$ is where $\mathcal{B}' \subset \mathcal{B}$ and $\hat{c}(B) = c(B)$, then the statement of WARP is also satisfied for $\hat{c}$.

1.7.2 Features of Finite Properties

Fact. *The intersection of properties is a property.*[28]

Now, we consider a subset of all properties:

Definition 1.13. Let $\mathcal{T}$ be a property. We call $\mathcal{T}$ a *finite property* if for all $c \in C$, $c \notin \mathcal{T}$ if and only if there exists a finite set of choice sets $A_1, ..., A_n \in \mathrm{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$, is not in $\mathcal{T}$.

In words, a finite property is (defined as) a property in which non-compliance can be concluded with finitely many observations (i.e. choices from finitely many choice sets). The majority of decision theoretic axioms are finite properties.

Fact. *When X is finite, any property is a finite property.*[29]

When X is infinite, examples of finite properties include Convexity (either $a^{\alpha}b \in c(\{a^{\alpha}b, a\})$ or $a^{\alpha}b \in c(\{a^{\alpha}b, b\})$), Monotonicity ($c(\{a,b\}) = \{a\}$ if $a > b$), Transitivity ($a \in c(\{a,b\})$ and $b \in c(\{b,d\})$ implies $a \in c(\{a,d\})$), von Neumann-Morgenstern (vNM) Independence ($p \in c(\{p,q\})$ if and only if $\alpha p + (1-\alpha) r \in c(\{\alpha p + (1-\alpha) r, \alpha q + (1-\alpha) r\})$), Betweenness, Stationarity, and Separability, to name a few.

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[28] To see this: Consider any $c \in C$ such that $c \in \mathcal{T}_1 \cap \mathcal{T}_2$. So $c \in \mathcal{T}_1, \mathcal{T}_2$. And since $\mathcal{T}_1, \mathcal{T}_2$ are properties, we have $\hat{c} \in \mathcal{T}_1, \mathcal{T}_2$, and hence $\hat{c} \in \mathcal{T}_1 \cap \mathcal{T}_2$, for all $\hat{c} \in C$ and $\hat{c} \subset c$.

[29] To see this: Fix any c. Sufficiency is a direct result of the definition of a property. Necessity is also straightforward: Let $A_1, ..., A_n = \mathrm{dom}(c)$, then $\hat{c} = c$, so $c \notin \mathcal{T}$ completes the proof.

Non-examples of finite properties (that are nonetheless properties) include various versions of continuity (e.g., $x_n \in c(A_n)$, $x_n \to x$, $A_n \to A$ implies $x \in c(A)$) and infinite acyclicity ($a_i \in c(\{a_i, a_{i+1}\})$ for $i = 1, 2, ..., \sigma$, where σ is an ordinal number, implies $a_1 \in c(\{a_1, a_\sigma\})$). Usually, the determination of whether a property is a finite property is immediate when a property is defined algorithmically (as in the axioms in this paragraph) as opposed to defined as an arbitrary subset of $\mathcal{C}$.[30]

Fact. *The intersection of finite properties is a finite property.*[31]

For instance, let $\mathcal{T}_1$ be the subset of all choice correspondences that satisfy WARP and $\mathcal{T}_2$ the subset of all choice correspondences that satisfy vNM Independence. These are both finite properties. We can define "WARP and vNM Independence" as a single finite property $\mathcal{T}_1 \cap \mathcal{T}_2$. It characterizes the set of all choice correspondences that satisfies *both* WARP and Independence.

Fact. *The intersection of finite properties and properties that are not finite properties may or may not be finite properties.*[32]

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[30] The empirical falsifiability of a property (that with finitely many observations the property can be falsified) is not sufficient to establish that it is a finite property. Consider the combination of WARP and continuity, there is no reason why this cannot be defined as a single property. It is empirically falsifiable, since WARP needs only two observations to falsify. Yet in the absence of a violation of WARP, a choice correspondence can very well violate the continuity portion, rendering the property unsatisfied but not falsified with finitely many observations. Conversely, if a property is empirically non-falsifiable, then it is a finite property if and only if it is always trivially satisfied.

[31] To see this: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are both finite properties, $\mathcal{T}_1 \cap \mathcal{T}_2$ is a property. We check Definition 1.13 that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a finite property. Fix any $c \in \mathcal{C}$. Suppose $c \notin \mathcal{T}_1 \cap \mathcal{T}_2$. Then without loss of generality say $c \notin \mathcal{T}_1$, take the choice sets $A_1, ..., A_n \in \text{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$. Since $\hat{c} \notin \mathcal{T}_1$, so $\hat{c} \notin \mathcal{T}_1 \cap \mathcal{T}_2$, and the rest is straightfoward. Now suppose there exist $A_1, ..., A_n \in \text{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$, is not in $\mathcal{T}_1 \cap \mathcal{T}_2$. Without loss of generality say $\hat{c} \notin \mathcal{T}_1$, so $c \notin \mathcal{T}_1$, so $c \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

[32] We provide examples. Take $X = [0, 1]$. The intersection of WARP and Continuity is clearly not a finite property, since WARP can hold whereas Continuity will (trivially) hold for any set of choices from finitely many choice sets, but fails to hold in general. The intersection of Monotonicity (that $x > y \Leftrightarrow y \notin c(A)$ for all $A \ni x, y$) and Continuity, on the other hand, is a finite property; essentially, Monotonicity is so strong that Continuity holds whenever Monotonicity does, and since Monotonicity is a finite property (in fact, it is

1.7.3 A Systematic Generalization of Models

Let $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ be a correspondence with $\Psi(A) \subseteq A$ such that $a \in B \subset A$ and $a \in \Psi(A)$ implies $a \in \Psi(B)$.

Definition. We say that a linear order (R, X) is Ψ-*consistent* if $y \in A \backslash \Psi(A)$ implies xRy for some $x \in \Psi(A)$.

Lemma 1.2. *Consider a choice correspondence $c : \mathcal{A} \rightarrow \mathcal{A}$, a finite property $\mathcal{T}$, and a correspondence Ψ. The following are equivalent:*

1. *For every finite $A \in \mathcal{A}$, there exists $x \in \Psi(A)$ such that the choice correspondence $\tilde{c} : \{B : B \subset A \text{ and } x \in B\} \rightarrow \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is in $\mathcal{T}$.*

2. *There exists a complete, transitive, antisymmetric, and Ψ-consistent binary relation (R, X) such that for all $x \in X$, the choice correspondence $\tilde{c} : \left\{B : \arg\max_{y \in B} R = x\right\} \rightarrow \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is in $\mathcal{T}$.*

First, consider the case in which $\Psi(A) := \text{id}(A) = A$. The first condition in Lemma 1.2 is satisfied when, for each choice problem, an alternative serves as an anchor that guarantees compliance with finite property $\mathcal{T}$ among choices from subsets of the choice problem containing this anchor. Like before, this anchor is a potential reference alternative with which desirable properties of c hold. When Ψ is not the identity function, we are demanding that at least one alternative in a restricted set of each choice problem (restricted according to Ψ) is a potential reference alternative.

This lemma is the backbone of the models in Section 1.3, Section 1.4, and Section 1.5. For now, we present a simple demonstration. Consider again the wine example, but now

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one where a violation can be detected with just the choice from one choice set), their intersection is a finite property.

the set of all alternatives X contains multiple entries of the same wine at different prices. Each alternative is hence a wine-price pair (x, p). Like before, a decision maker was seen choosing a more expensive wine over a cheaper one, but sometimes the reverse (at the exact same prices). The economist postulates that for each choice problem, it is either the cheapest or the most expensive wine that the consumer's underlying preference depends on. Given this postulate, let $\Psi(A)$ be the set of cheapest and most expensive wine-price pairs in A. Furthermore, in addition to WARP, the economist would like to postulate that for a fixed reference, if the decision maker chooses wine x at price p over wine y at price q, then he would also choose wine x at price p over wine y at price $q' > q$; we will call this property "Money is Good". This is an example of a finite property on c.

Lemma 1.2 establishes that, for a choice correspondence that satisfies these postulates, a reference order (R, X) can be built such that WARP and Money is Good are satisfied among choice sets that share the same $R-$maximal element. Furthermore, for any three wine-price pairs, the intermediate-in-price option is either reference dominated by the more expensive option, the cheaper option, or both. A prediction follows: If a wine-price pair (x, p) reference dominates another wine-price pair (y, q), then all wine-price pairs (z, s) such that $s \in [\min\{p, q\}, \max\{p, q\}]$ are reference dominated by (x, p). That is, even if the economist hasn't fully pinned down this partially subjective R, she can conclude that among choice sets that contain (x, p) and (z, s), where s is between p and q, choices satisfy WARP and Money is Good.

If instead the economist makes the weaker postulate that some reference alternative exists (i.e., $\Psi = id$), then no structure on R can be guaranteed (other than it is a linear order). Conversely, if the economist makes the stronger postulate that the cheapest wine is exactly the reference alternative, then for any two wine-price pairs, the cheaper option reference dominates the other. This demonstrates the flexibility Ψ provides in the trade-off between explanation and prediction. If $\Psi(A)$ is a very restrictive set, such as a singleton, then the model is easy to test and provides strong predictions. If $\Psi(A)$ is very nonrestrictive, such as $\Psi(A) = A$, then the model is harder to test but accommodates more behavior.

To summarize, we expanded the result of 1.1 to include (i) how properties of R can be axiomatically introduced and (ii) what kind of properties, beyond WARP, of a choice correspondence, can be accommodated in this unified framework of ordered-reference dependent choice. Both are crucial for our specialized models.

Lemma 1.2 falls short of achieving a utility representation. Indeed, the underlying difficulty is related to the literature on limited datasets, in which one observes choices from a strict subset of all choice problems. de Clippel & Rozen (2014) points out that, in this case, even if observed choices are consistent with behavioral postulates, it need not be sufficient for a corresponding utility representation. In our case, even though we started with an exhaustive dataset ($c : \mathcal{A} \to \mathcal{A}$), we have effectively created a partition such that each part contains only a subset of all choice problems. Nevertheless, as demonstrated in Section 1.3, ordered-reference dependent expected utility can be achieved with normative restrictions on Ψ.

1.8 Appendix B: Additional Materials

1.8.1 ORDU vs other non-WARP models

To simplify notation, we use "$\{\underline{a}, b, c\}$" for $c(\{a, b, c\}) = \{a\}$.

In Ok et al. (2015)'s (endogeneous) *reference dependent choice*, the decision maker maximizes a single utility function, but only chooses from alternatives that are better than the reference in all (endogenously given) attributes. Consider the following choices accommodated by their model but not ORDU.[33]

$$\boxed{\boldsymbol{\{\underline{a},b,c,d\}} \quad \boldsymbol{\{a,\underline{b},d\}} \quad \boldsymbol{\{a,\underline{c},d\}}}$$

An interpretation by Ok et al. (2015) is that a gives the highest utility but a *decoy* d blocks the choice of a from $\{a,b,d\}$ and $\{a,c,d\}$, the *attraction effect.* Yet their model does not require the decoy to function for $\{a,b,c,d\}$. On the contrary, ORDU requires that one of the reference points of $\{a,b,d\}$ and $\{a,c,d\}$ is the reference of $\{a,b,c,d\}$, a consequence of the reference order that excludes this behavior from ORDU. Finally, an intransitive choice correspondence $\{\underline{a},b\},\{\underline{b},c\},\{\underline{c},a\}$ may be explained by ORDU, but is never accommodated by Ok et al. (2015). Hence the two models are not nested.

Kőszegi & Rabin (2006)'s *reference-dependent preferences* is another related model. In *personal equilibrium* (PE), decision makers has a joint utility function $v : X \times X \to \mathbb{R}$ and chooses $PE(A) = \{x : v(x|x) \geq v(y|x)\ \forall y \in A\}$; that is, the choice maximizes a reference-dependent utility function, and the reference point is itself the eventually chosen alternative. This permits the type of behavior where x is not chosen in a set but is chosen in the subset—when the alternatives x fails to beat are removed. While ORDU also allows for $x \in c(B) \setminus c(A)$ where $B \subset A$, it does so with two implications: (i) an alternative $y \in A \setminus B$ must had been the reference point of A and so (ii) for some $y \in A \setminus B$, choices are consistent between $c(A)$ and $c(T)$ for all $T \subset A$ that contains y. Consider the following example.

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[33] The complete choice correspondence is

$$\boxed{\begin{array}{cccccc} \boldsymbol{\{\underline{a},b,c,d\}} & \{\underline{a},b,c\} & \boldsymbol{\{a,\underline{b},d\}} & \boldsymbol{\{a,\underline{c},d\}} & \{\underline{b},c,d\} & \{\underline{a},b\} \\ \{\underline{a},c\} & \{\underline{a},d\} & \{\underline{b},c\} & \{\underline{b},d\} & \{\underline{c},d\} & \end{array}}$$

Using an Ok et al. (2015) specification where $u(a) > u(b) > u(c) > u(d)$. $r(\{a,b,d\}) = r(\{a,c,d\}) = r(\{b,d\}) = r(\{c,d\}) = d$, $r(A) = \diamond$ otherwise, and $\mathcal{U} = \{U\}$ where $U(b) > U(c) > U(d) > U(a)$.

$$\boxed{\boldsymbol{\{\underline{a},b,c,d\}} \quad \boldsymbol{\{\underline{a},\underline{b},c\}} \quad \boldsymbol{\{\underline{a},\underline{d}\}}}$$

Since $c(\{a,b,c,d\})$ and $c(\{a,b,c\})$ does not come from standard utility maximization, the reference of $\{a,b,c\}$ is d, and so $c(\{a,b,c,d\})$ and $c(\{a,d\})$ must maximize the same utility function. But this is not the case either, so this choice pattern is incompatible with ORDU. It is, however, compatible with PE.[34]An immediate implication PE is: $x \in c(A)$ and $x \in B \subset A$, then $x \in c(B)$. A simple intransitive choice pattern $\{\underline{a},b\},\{\underline{b},c\},\{\underline{c},a\}$, admissible by ORDU, concludes that the two models are not nested.

Manzini & Mariotti (2007) proposes a non-WARP model without a reference point interpretation. In *rational shortlist method* (RSM), decision makers are endowed with two asymmetric relations P_1 and P_2. Facing a choice problem A, she first creates a shortlist by eliminating inferior alternatives according to P_1 (eliminate x if yP_1x for some $y \in A$), and then choose from this shortlist according to P_2. WARP violation appears when an alternative x is eliminated in a set S, but not in the subset $T \subset S$, where it subsequently chosen over the best alternative of S. An example of this behavior is displayed by the first of the following choices.

$$\boxed{\boldsymbol{\{\underline{a},b,c,d\}} \quad \boldsymbol{\{a,b,\underline{d}\}} \quad \boldsymbol{\{b,\underline{c},d\}} \quad \boldsymbol{\{\underline{b},c\}}}$$

For ORDU to reconcile, c must be deemed the reference of $\{a,b,c,d\}$, but then choices from $\{b,c,d\}$ and $\{b,c\}$ must comply with standard utility maximization. Since this is not the

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[34] The complete choice correspondence is

$$\boxed{\begin{array}{cccccc} \boldsymbol{\{\underline{a},b,c,d\}} & \boldsymbol{\{\underline{a},\underline{b},c\}} & \{\underline{a},b,\underline{d}\} & \{\underline{a},c,d\} & \{b,\underline{c},d\} & \{\underline{a},\underline{b}\} \\ \{\underline{a},c\} & \boldsymbol{\{\underline{a},\underline{d}\}} & \{\underline{b},\underline{c}\} & \{b,\underline{d}\} & \{\underline{c},d\} & \end{array}}$$

Gul et al. (2006) shows that PE is equivalent to choices maximizing a complete (but not necessarily transitive) preference relation. This choice correspondence is explained by $a \sim b$, $a \succ c$ $a \sim d$, $b \sim c$, $d \succ b$, $c \succ d$.

case, ORDU does not nest RSM.[35] While ORDU is constrained by fixed reference points, the model is more flexible than RSM when reference points do change, since no restriction is put on the new utility function. RSM, however, cannot accommodate a choice that makes the shortlists in a small and large set but not an intermediate one. The result is the following behavior accommodated by ORDU but not RSM.[36]

$$\boxed{\boldsymbol{\{\underline{a},b,c,d\}} \quad \boldsymbol{\{a,\underline{b},d\}} \quad \boldsymbol{\{\underline{a},b\}}}$$

We conclude that ORDU and RSM are not nested.

Last, we compare ORDU to Masatlioglu et al. (2012)'s *choice with limited attention* (CLA). A decision maker has a complete and transitive ranking $\succ_{CLA}$ of alternatives and an attention filter that limits choices to a subset of each choice problems, the "consideration set". When another choice problem is derived by removing choices not in the consideration set, the consideration set remains the same. Although a single ranking is used (as opposed to ORDU's many utility functions), flexibility in constructing consideration sets easily allows for behavior not accommodated by ORDU.

$$\boxed{\boldsymbol{\{a,\underline{b},c\}} \quad \boldsymbol{\{\underline{a},b\}} \quad \boldsymbol{\{b,\underline{c}\}} \quad \{\underline{a},c\}}$$

CLA is provided under the framework of choice functions (no indifference), and with that

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[35] The complete choice correspondence is

$$\boxed{\begin{array}{cccccc} \boldsymbol{\{\underline{a},b,c,d\}} & \{\underline{a},b,c\} & \boldsymbol{\{b,\underline{c},d\}} & \{\underline{a},c,d\} & \boldsymbol{\{a,b,\underline{d}\}} & \{\underline{a},b\} \\ \{\underline{a},c\} & \{a,\underline{d}\} & \boldsymbol{\{\underline{b},c\}} & \{b,\underline{d}\} & \{\underline{c},d\} & \end{array}},$$

induced by $(aP_1b,\ aP_1c,\ cP_1d,\ dP_1b)$ and $(aP_2b,\ aP_2c,\ dP_2a,\ bP_2c,\ dP_2b,\ cP_2d)$.

[36] The complete choice correspondence is

$$\boxed{\begin{array}{cccccc} \boldsymbol{\{\underline{a},b,c,d\}} & \{a,\underline{b},c\} & \boldsymbol{\{a,\underline{b},d\}} & \{\underline{a},c,d\} & \{\underline{b},c,d\} & \boldsymbol{\{\underline{a},b\}} \\ \{\underline{a},c\} & \{\underline{a},d\} & \{\underline{b},c\} & \{\underline{b},d\} & \{\underline{c},d\} & \end{array}}.$$

This is explained by the ORDU specification: $bRaRcRd$, $u_i(a) > u_i(b) > u_i(c) > u_i(d)$ when $i \in \{a,b,d\}$, and $u_c(b) > u_c(a) > u_c(c) > u_c(d)$.

restriction ORDU is nested by CLA.[37] However, the two models make different predictions under a comparable setup. For the analysis, we modify CLA by allowing for indifferences in the ranking of alternatives (replacing $\succ_{CLA}$ with $\succsim_{CLA}$), but preserve in entirety the attention filter / consideration set component. The following behavior is accommodated by ORDU but not CLA.[38]

$$\boxed{\boldsymbol{\{\underline{a},\underline{b},c,d\}} \quad \boldsymbol{\{a,\underline{b},\underline{c}\}} \quad \boldsymbol{\{\underline{b},c\}}}$$

When indifferences are allowed, the single ranking limitation of CLA becomes the bottleneck in explaining behavior. The two models are hence not nested under comparable setups.

1.9 Appendix C: Proofs

1.9.1 Proof of Lemma 1.2

Lemma 1.3. *Let Z be a set, and $\mathbb{Z}$ be the set of all finite and nonempty subsets of Z. Let $\mathcal{R}$ be a self map on $\mathbb{Z}$, $\mathcal{R}(S) \subseteq S$, such that*

(i) For all $S \in \mathbb{Z}$, $\mathcal{R}(S) \neq \{\emptyset\}$, and

(ii) α - for all $T, S \in \mathbb{Z}$, $x \in T \subseteq S$, if $x \in \mathcal{R}(S)$, then $x \in \mathcal{R}(T)$.

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[37] Consider any choice function c that admits an ORDU representation, define CLA's parameters as follows: attention filter $\Gamma(A) := \left\{\min(A,R), \arg\max_{x \in A} u_{\min(A,R)}(x)\right\}$ (singleton if $\min(A,R) = \arg\max_{x \in A} u_{\min(A,R)}(x)$) and CLA's preference $x \succ y$ if xRy.

[38] The complete choice correspondence is

$$\boxed{\begin{array}{cccccc} \boldsymbol{\{\underline{a},\underline{b},c,d\}} & \boldsymbol{\{a,\underline{b},\underline{c}\}} & \{\underline{a},\underline{b},d\} & \{\underline{a},c,d\} & \{\underline{b},c,d\} & \{a,\underline{b}\} \\ \{a,\underline{c}\} & \{\underline{a},d\} & \boldsymbol{\{\underline{b},c\}} & \{\underline{b},d\} & \{\underline{c},d\} & \end{array}}.$$

This is explained by the ORDU specification: $cRbRaRd$, $u_d(a) = u_d(b) > u_d(c) > u_d(d)$, $u_a(b) = u_a(c) > u_a(a)$, $\underline{u_b(b) > u_b(c)}$. Now we show non-compliance with CLA (with the indifference extension): Since $a \in c(\{a,b,c,d\}) \setminus c(\{a,b,c\})$, CLA reconciles this by setting the consideration sets $\Gamma(\{a,b,c,d\}) = \{a,b,d\}$ and $\Gamma(\{a,b,c\}) = \{b,c\}$, so a is not considered in the smaller set. However, the property of consideration sets then requires $\Gamma(\{b,c\}) = \{b,c\}$, and $\{a,c\}$

Then, there exist $\mathcal{R}^* \subseteq \mathcal{R}$ *such that*

(i) *For all* $S \in \mathbb{Z}$, $\mathcal{R}^*(S) \neq \{\emptyset\}$,

(ii) α *- for all* $T, S \in \mathbb{Z}$, $x \in Z$ *such that* $x \in T \subseteq S$, *if* $x \in \mathcal{R}^*(S)$, *then* $x \in \mathcal{R}^*(T)$, *and*

(iii) β *- for all* $T, S \in \mathbb{Z}$, $x, y \in Z$ *such that* $x, y \in T \subseteq S$, *if* $x \in \mathcal{R}^*(T)$ *and* $y \in \mathcal{R}^*(S)$, *then* $x \in \mathcal{R}^*(S)$.

Proof. We prove by construction.

1. We say $\mathcal{R}' \subseteq \mathcal{R}$ if $\mathcal{R}'(S) \subseteq \mathcal{R}(S)$ $\forall S \in \mathbb{Z}$. Assume and invoke Zermelo's theorem to well-order the set of all doubletons in the domain of $\mathcal{R}$ (there may be uncountable many of them, hence Zermelo's theorem). Now we start the transfinite recursion using this order.

2. In the zero case, we have $\mathcal{R}_0 = \mathcal{R}$. This correspondence satisfies α and is nonempty-valued. Suppose $\mathcal{R}_\sigma$ satisfies α and is nonempty-valued.

3. For the successor ordinal $\sigma + 1$, we take the corresponding doubleton $B_{\sigma+1}$ and take $x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$. Suppose such an x does not exist, then for both $x, y \in B_{\sigma+1}$, there are $S_x \supset B_{\sigma+1}$ and $S_y \supset B_{\sigma+1}$ such that $\mathcal{R}_\sigma(S_x) = \{x\}$ and $\mathcal{R}_\sigma(S_y) = \{y\}$ since $\mathcal{R}_\sigma$ is nonempty-valued. Consider $S_x \cup S_y \in \mathbb{Z}$. Since $\mathcal{R}_\sigma$ is nonempty-valued, $\mathcal{R}_\sigma(S_x \cup S_y) \neq \emptyset$. But since $\mathcal{R}_\sigma$ satisfies α, it must be that $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \mathcal{R}_\sigma(S_x) \cup \mathcal{R}_\sigma(S_y)$, hence $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \{x, y\}$. Suppose without loss $x \in \mathcal{R}_\sigma(S_x \cup S_y)$, then due to α again and that $x \in B_{\sigma+1} \subset S_y$, it must be that $x \in \mathcal{R}_\sigma(S_y)$, which contradicts $\mathcal{R}_\sigma(S_y) = \{y\}$. (That is, we showed that with nonempty-valuedness and α, no two elements can each have a unique appearance in the $\mathcal{R}_{(\cdot)}$-image of a set containing those two elements.) Hence, $\exists x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$.

Define $\mathcal{R}_{\sigma+1}$ from $\mathcal{R}_{\sigma}$ in the following way: $\forall S \supset B_{\sigma+1}, \mathcal{R}_{\sigma+1}(S) := \mathcal{R}_{\sigma}(S) \setminus \{x\}$. Note: (i) Since x is deleted from $\mathcal{R}_{\sigma}(T)$ only if it is also deleted (if it is in it at all) from $\mathcal{R}_{\sigma}(S)$ $\forall S \supset T$, we are preserving α, and (ii) since x is never the unique element in $\mathcal{R}_{\sigma}(S)$ $\forall S \supset B_{\sigma+1}$, we preserve nonempty-valuedness.

4. For a limit ordinal λ, define $\mathcal{R}_{\lambda} = \cap_{\sigma<\lambda}\mathcal{R}_{\sigma}$. Note that since $\mathcal{R}_{\sigma'} \subset \mathcal{R}_{\sigma''}$ $\forall \sigma' > \sigma''$, $\cap_{\sigma \leq \bar{\sigma}} = \mathcal{R}_{\bar{\sigma}}$. Furthermore, for any $\sigma < \lambda$, $\mathcal{R}_{\sigma}$ is constructed such that α and nonempty-valuedness are preserved. Hence $\mathcal{R}_{\lambda}$ satisfies α and is nonempty-valued.

5. Note that this process terminates when all the doubletons have been visited, for we would otherwise have constructed an injection from the class of all ordinals to the set of all doubletons in $\mathbb{Z}$, which is impossible.

6. Finally, we check that $|\mathcal{R}_{\lambda}(S)| = 1$ for all $S \in \mathbb{Z}$, hence β is satisfied trivially. Suppose it is not a function, hence $\exists S \in \mathbb{Z}$ such that $x, y \in \mathcal{R}_{\lambda}(S)$. Then by α we have that $x, y \in \mathcal{R}_{\lambda}(\{x, y\})$, which is not possible as the recursion process has visited it and deleted something from $\mathcal{R}(\{x, y\})$.

7. Set $\mathcal{R}_{\lambda} = \mathcal{R}^{*}$.

□

We state the following observation. Let $c : \mathcal{A} \rightarrow \mathcal{A}$ be a choice correspondence. Recall that $\mathcal{A}$ is the set of all finite and nonempty subsets of X. For $S \subseteq Y$ and $x \in S$, define $\mathbb{A}_{S}^{x} := \{A \subseteq S : A \in \mathcal{A} \text{ and } x \in A\}$. We use the notation $\left(c, \mathbb{A}_{A}^{x}\right)$ to denote the choice correspondence $\tilde{c} : \mathbb{A}_{A}^{x} \rightarrow \mathcal{A}$ where $\tilde{c}(B) = c(B)$. In other words, $\left(c, \mathbb{A}_{A}^{x}\right)$ is a subset of c where the domain is restricted to $\mathbb{A}_{A}^{x}$– the set of all subsets of A containing x.

Remark 1.2. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{T}$ a finite property as defined in Definition 1.13. Define $\Gamma(S) := \left\{x \in S : \left(c, \mathbb{A}_A^x\right) \in \mathcal{T}\right\}$.

1. If $y \in \Gamma(A)$, then $y \in \Gamma(B)$ whenever $B \subset A$.
2. If $y \in \Gamma(A)$ for all finite $A \subseteq D$, then $y \in \Gamma(D)$.

We call x a reference alternative for S if $x \in \Gamma(S)$. Remark 1.2 states that if x is a reference alternative for some choice problem A, i.e., $\left(c, \mathbb{A}_A^x\right) \in \mathcal{T}$, then x is also a reference alternative for $B \subseteq A$. This is an immediate consequence of the definition of a property (and the fact that $\mathbb{A}_B^x \subseteq \mathbb{A}_A^x$ whenever $B \subseteq A$). In words, if a violation is undetected with more observations, then it cannot be detected with less. Furthermore, if x is a reference alternative for all finite subsets of an arbitrary set of alternatives D, then x is also a reference alternative for D; this, is due to $\mathcal{T}$ being a finite property. Otherwise, take a finite set of choice problems $\mathcal{S} = A_1, \dots, A_n$, each of which a subset of D containing x, such that a finite property is violated, i.e., $\tilde{c} : \mathcal{S} \to \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is not in $\mathcal{T}$. Since this is a finite tuple of finite choice problems, consider the finite set $A := \cup_i A_i$. Clearly, $x \notin \Gamma(A)$, but A is a finite subset of D, hence a contradiction. Intuitively, if x is not a reference alternative for some arbitrary set of alternatives D, then violation of a finite property would have been detected in a finite subset of D, rendering x not a reference alternative for some choice problem $A \subseteq D$.

Now, let $\mathcal{R}' : \mathcal{A} \to \mathcal{A} \cup \{\emptyset\}$ be a set valued function that picks out, for each choice problem $A \in \mathcal{A}$, the set of all reference alternatives $\mathcal{R}'(A) \subseteq A$; formally, $\mathcal{R}'(A) := \left\{x \in S : \left(c, \mathbb{A}_A^x\right) \in \mathcal{T}\right\}$. Since $\mathcal{T}$ is a finite property, by Remark 1.2, $\mathcal{R}'$ satisfies property α (as defined in Lemma 1.3). Furthermore, the hypothesis in Lemma 1.2 gives that $\mathcal{R}'(A) \cap \Psi(A)$ is nonempty for all $A \in \mathcal{A}$. Finally, define $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{R}(A) := \mathcal{R}'(A) \cap \Psi(A)$. Since both $\mathcal{R}'(A)$ and $\Psi(A)$ satisfy property α, $\mathcal{R}(A)$ satisfies property α.

Putting our $\mathcal{R}$ through Lemma 1.3, we get a set-valued function $\mathcal{R}^* : \mathcal{A} \to \mathcal{A}$ that is now a singleton everywhere (i.e., $|\mathcal{R}^*(A)| = 1$ for all $A \in \mathcal{A}$). Furthermore, this function satisfies property α, and satisfies property β trivially. With this, we build the order (R, Y) by setting xRy if $\{x\} = \mathcal{R}^*(\{x, y\})$, and xRx. The result is a complete, transitive, and antisymmetric binary relation.

Lemma 1.4. *For an (R, Y) constructed according to the the aforementioned procedure, $y \in A \backslash \Psi(A) \Rightarrow xRy$ for some $x \in \Psi(A)$ (i.e. R is Ψ-consistent).*

Proof. Suppose not, say $y \in A \backslash \Psi(A)$ but yRx for all $x \in \Psi(A)$. Consider $\{\{x, y\} : x \in \Psi(A)\}$. Since this is a finite set of doubletons, suppose without loss of generality $\{x^*, y\}$ is the last one (in $\{\{x, y\} : x \in \Psi(A)\}$) visited by the procedure in Lemma 1.3, and denote the step corresponding to $\{x^*, y\}$ by the ordinal $\sigma_{\{x^*, y\}}$. Since yRx for all $x \in \Psi(A)$ such that $x \neq x^*$, $\mathcal{R}_{\sigma_{\{x^*, y\}}}(A) \cap \Psi(A) = \{x^*\}$. Since $\mathcal{R}_\sigma \subseteq \mathcal{R}_0 := \mathcal{R}' \cap \Psi$ for all σ, $\mathcal{R}_{\sigma_{\{x^*, y\}}}(A) = \{x^*\}$. Hence x^* uniquely appears in the image of $\mathcal{R}_{\sigma_{\{x^*, y\}}}$ evaluated at some superset of $\{x^*, y\}$, and the recursion procedure sets, ultimately, $\mathcal{R}^*(\{x^*, y\}) = \{x^*\}$. But this implies x^*Ry, a contradiction. □

Finally, consider the set $R^{\downarrow}(x) := \{y \in X : xRy\}$. This is a set of alternatives that are, according to our binary relation R, reference dominated by x. For any finite subset $A \subseteq R^{\downarrow}(x)$ such that $x \in A$, we have $x \in \mathcal{R}^*(A) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}'(A)$, which by definition implies x is a reference alternative of A. Using point 2 in Remark 1.2, we conclude that x is reference alternative for $R^{\downarrow}(x)$, which need not be finite.

To summarize, we have effectively created a partition of $\mathcal{A}$ where the parts are characterized by $\left\{\mathbb{A}^x_{R^{\downarrow}(x)}\right\}_{x \in X}$. To see this, take any $A \in \mathcal{A}$, since R is a linear order, there is a unique $z \in A$ such that zRy for all $y \in A$, and so $A \in \mathbb{A}^z_{R^{\downarrow}(z)}$ and $A \notin \mathbb{A}^y_{R^{\downarrow}(y)}$ for any $y \neq z$. Furthermore for each part $\mathbb{A}^x_{R^{\downarrow}(x)}$, $\left(c, \mathbb{A}^x_{R^{\downarrow}(x)}\right)$ is in $\mathcal{T}$. Since $\left\{B \in \mathcal{A} : \arg\max_{y \in B} R = z\right\}$ is simply $\mathbb{A}^z_{R^{\downarrow}(z)}$, the proof is complete.

1.9.2 Proof of Proposition 1.1, Part 1 (without the use of Lemma 1.2)

Suppose X is finite. We provide an independent proof that a choice correspondence c that satisfies RD (Axiom 1.1) has an ORDU representation.

1. Let $\Gamma(A)$ be the set of reference alternatives for A. We create a list of alternatives in the following way; list $\Gamma(X)$ with an arbitrary order. Since $X \backslash \Gamma(X)$ is again finite, list $\Gamma(X \backslash \Gamma(X))$ with an arbitrary order; and continue until all $x \in X$ are listed. Finally, let i_x denote the position of x in the list. For any $x, y \in X$, construct xRy if $i_x > i_y$ and xRx.

2. We now construct $\succsim_x$ for each $x \in X$. Consider the set $R^{\downarrow}(x) := \{y : xRy\}$. Consider c on $\mathbb{A}^x_{R^{\downarrow}(x)} := \left\{A \subseteq R^{\downarrow}(x) \cap \mathcal{A} : x \in A\right\}$, which by construction satisfies WARP.

3. First we set $x \succsim_x x$ for all $x \in X$.

4. Using the doubletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$, all of which would contain x, we set, for all $y \in R^{\downarrow}(x)$, either $y \succsim_x x$, or $x \succsim_x y$, or both, according to $c(\{x, y\})$.

5. Now for all $y_1, y_2 \succsim_x x$, we set either $y_1 \succsim_x y_2$, or $y_2 \succsim_x y_1$, or both, according to $c(\{x, y_1, y_2\})$, using tripletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$. Due to WARP (of c on $\mathbb{A}^x_{R^{\downarrow}(x)}$), $\succsim_x$ is now complete on the set $\mathbb{P}^x := \{y : y \succsim_x x\}$, which we call the *prediction set* of x, containing alternatives that are both reference dominated by x (i.e. xRy) and are weakly better than x in binary comparison (i.e. $y \in c(\{y, x\})$).

6. Now consider $X \backslash \mathbb{P}^x = \{y : yRx \text{ or } x \succ_x y\}$. We set $y_1 \sim_x y_2$ for all $y_1, y_2 \in X \backslash \mathbb{P}^x$, and $y_1 \succ_x y_2$ for all $y_1 \in \mathbb{P}^x$, $y_2 \in X \backslash \mathbb{P}^x$. Our constructed $\succsim_x$ is now complete (on X).[39]

Using quadrupletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$, we show that $\succsim_x$ constructed above is transitive: Suppose $y_1 \succsim_x y_2$ and $y_2 \succsim_x y_3$, and that $y_1, y_2, y_3 \in \mathbb{P}^x$ (if any of them is in $X \backslash \mathbb{P}^x$ then the argument is straightforward by $\sim_x$), hence $y_1 \in c(\{x, y_1, y_2\})$ and $y_2 \in c(\{x, y_2, y_3\})$. Furthermore, since $y_1, y_2, y_3 \in \mathbb{P}^x$, we have $\{x, y_1, y_2, y_3\} \in \mathbb{A}^x_{R^{\downarrow}(x)}$, and c on $\mathbb{A}^x_{R^{\downarrow}(x)}$ satisfies WARP implies $y_1 \in c(\{x, y_1, y_2, y_3\})$, and hence $y_1 \in c(\{x, y_1, y_3\})$, which implies $y_1 \succsim_x y_3$.

Finally, we show that (R, X) and $\{(\succsim_x, X)\}_{x \in X}$ explain c. Take any $A \in \mathcal{A}$, since A is finite, and R is antisymmetric, there is a unique $R-$maximizer $x \in A$ (i.e., xRy for all $y \in A$), hence $A \subseteq R^{\downarrow}(x)$. Suppose for contradiction $c(A) \nsubseteq \{y \in A : y \succsim_x z \forall z \in A\}$; so for some $a \in c(A)$, $a' \succ_x a$ for some $a' \in A$. Then $a \notin c(\{x, a', a\})$. Since $\{x, a', a\}$ is a subset of A, and both choice problems are elements of $\mathbb{A}^x_{R^{\downarrow}(x)}$, this is a violation of the statement c satisfies WARP on $\mathbb{A}^x_{R^{\downarrow}(x)} := \left\{A \subseteq R^{\downarrow}(x) \cap \mathcal{A} : x \in A\right\}$, hence a contradiction. Suppose for contradiction $c(A) \nsupseteq \{y \in A : y \succsim_x z \forall z \in A\}$, so for some $a \in A$, $a \succsim_x z$ for all $z \in A$, but $a \notin c(A)$. Take $a' \in c(A)$; since $a \succsim_x a'$, $a \in c(\{x, a', a\})$. Since $\{x, a', a\}$ is a subset of A, and both choice problems are elements of $\mathbb{A}^x_{R^{\downarrow}(x)}$, a contradiction of the statement c satisfies WARP on $\mathbb{A}^x_{R^{\downarrow}(x)}$ is reached. Hence $c(A) = \{y \in A : y \succsim_x z \forall z \in A\}$.

It remains to show that for each alternative-indexed preference relation defined, we can construct a utility function representing it. Since X is finite, and each $\succsim_x$ is a complete and transitive preference relation, this is standard.

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[39] That is, for any $y_1, y_2 \in X$, either $y_1 \succsim_x y_2$, or $y_2 \succsim_x y_1$, or both.

1.9.3 Proof of Proposition 1.1, Part 2

We invoke Lemma 1.2 to prove the intermediary result that, if c satisfies Reference Dependence (Axiom 1.1) and Continuity (Axiom 1.2), then there exists a linear order (R, X) and a set of complete preference relations $\{(\succsim_x, X)\}_{x\in X}$ such that for all $A \in \mathcal{A}$, we have $c(A) = \left\{y \in A : y \succsim_{r(A)} x \forall x \in A\right\}$, where $r(A) = \underset{z\in A}{\arg\max} R$:

Using the terminology developed in Appendix A, define $\mathcal{T}$ as the property WARP. By Lemma 1.2, there exists a Ψ-consistent linear order (R, X) such that c on $\left\{S \in \mathcal{A} : \underset{z\in A}{\arg\max} S = x\right\}$ satisfies $\mathcal{T}$ for all $x \in X$. Notice that $\left\{S \in \mathcal{A} : \underset{z\in A}{\arg\max} S = x\right\} = \mathbb{A}^x_{R\downarrow(x)}$, and so we conclude that for all $T, S \in \mathbb{A}^x_{R\uparrow(x)}$, $c(S) \cap T = c(T)$ whenever $T \subset S \subseteq A$ and $c(S) \cap T \neq \emptyset$. We proceed to build $\{(\succsim_x, X)\}_{x\in X}$ using the method outlines in the special case proof above, which gives us a complete and transitive $\succsim_x$ for all x, as well as $c(A) = \left\{y \in A : y \succsim_{r(A)} z \forall z \in A\right\}$ where $r(A) = \underset{x\in A}{\arg\max} S := \{x \in A : xRy \forall y \in A\}$.

It remains to show that for each alternative-indexed preference relation defined, we can construct a utility function representing it. [To be completed, but essentially just Efe Ok Order 9 Pg 18]

1.9.4 Proof of Proposition 1.2

Remark. (Notational) Currently, this older version of the proof reverses, without loss, the order R. That is, $r(A) = \underset{p\in A}{\arg\min} R$ as opposed to $\arg\max$. The proof remains valid, and readers are advised to simply, at the very end, “reverse” the order R constructed here.

We define $\Delta(X)$ as a $|X| - 1$ dimensional simplex, as is conventional, and hence *full-dimensional* means $|X| - 1$ dimensional. First, we split $r \in \Delta(X)$ into two groups, $E = \{r \in \Delta(X) : r = (\delta_b)^{\alpha}(\delta_w)\}$, and $I = \Delta(X) \setminus E$, the “exterior” and “interior” sets. Set $\Psi(A) = A \setminus \Phi(A)$, it is easy to check that $a \in \Psi(B)$ if $a \in B \subseteq A$ and $a \in \Psi(A)$. Applying Lemma

1.2, we get a linear order $(R, \Delta(X))$ that gives a partition of $\mathcal{A}$, $\left\{\mathbb{A}^r_{R^\uparrow(r)}\right\}_{r\in\Delta X}$, such that c on $\mathbb{A}^r_{R^\uparrow(r)}$ satisfies WARP and Independence for all $r \in \Delta(X)$. Furthermore, since R is Ψ-consistent, or $\min(A,R) \in A\backslash\Phi(A)$, we have pRq for all $p \in \Phi(\{p,q\})$.

Lemma 1.5. *For $r \in I$ and any open ball B_r around r, $B_r \cap R^\uparrow(r)$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Take $r \in I$. By definition, $r(x) \neq 0$ for some $x \neq b, w$ ($r(x)$ is the probability that lottery r gives prize x). Consider all mean-preserving spread of r, $MPS(r) \subseteq \Delta A$, this is a $|X-2|$ dimensional convex set. Since $q \in MPS(r)$ implies $q \in \Phi(\{r,p\})$, we have that qRr and hence $MPS(r) \subseteq R^\uparrow(r)$. Consider the set $\mathbb{S}(r) := \cup_{q\in MPS(r)\cup\{r\}} \{e \in \Delta X : e \text{ is an extreme spread of } q\}$, this is an interval on the line connecting δ_b and δ_w. Consider the convex hull $\mathbb{C}(r) := \text{conv}(MPS(r) \cup \mathbb{S}(r) \cup \{r\})$. Clearly, $\mathbb{C}(r)$ is a convex set. Furthermore, since $\mathbb{S}(r)$ is not contained in $MPS(r) \cup \{r\}$ (otherwise lotteries in $\mathbb{S}$ has the same mean, but this is not possible), $\mathbb{C}(r)$ is full dimensional. Since $e \in \mathbb{S}(r)$ only if e is an extreme spread of q for some $q \in MPS(r) \cup \{r\} \subseteq R^\uparrow(r)$, and $e \in \Phi(\{e,q\})$, we have $eRqRr$, hence $\mathbb{S}(r) \subseteq R^\uparrow(r)$. Finally, for $p \in \mathbb{C}(r) \setminus (MPS(r) \cup \mathbb{S}(r) \cup \{r\})$, it must be that $p = e^\alpha q$ for some $q \in MPS(r) \cup \{r\}$ and e an extreme spread of q, hence again $p \in \Phi(\{p,q\})$, so $pRqRr$, so $\mathbb{C}(r) \subseteq R^\uparrow(r)$. Since B_r is also a full-dimensional and convex set, $B_r \cap \mathbb{C}(r)$ is a full-dimensional convex set in $B_r \cap R^\uparrow(r)$. □

Define for each $r \in \Delta X$ the *strict prediction set* $\mathbb{P}^+_r := \left\{p \in R^\uparrow(r) : r \notin c(\{p,r\})\right\}$. There are lotteries that are both reference dominated by r and is chosen over r in a binary decision.

Lemma 1.6. *For $r \in I$, $\mathbb{P}^+_r$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Take $r \in I$. Suppose for contradiction $r \in c(\{e,r\})$ for all e an extreme spread of r; then since the lottery $(\delta_w)^{r(w)}(\delta_b)$ is in the closure of the extreme spread of r, continuity of

c implies $r \in c\left(\left\{r, (\delta_w)^{r(w)}(\delta_b)\right\}\right)$, which is a violation of FOSD. Hence there is an extreme spread of r, e, such that $r \notin c(\{r, e\})$. Since $r^{\alpha}e \in R^{\uparrow}(r)$ and c on $\mathbb{A}^{r}_{R^{\uparrow}(r)}$ satisfies Independence, we can find $p := r^{\alpha}e \in \mathbb{P}^{+}_{r}$ where $\alpha \in (0,1)$, hence $p \in I$. By continuity of c, there exists an open ball B_p around p such that $r \notin c(\{r, q\})$ for all $q \in B_p$. By Lemma 1.5, $B_p \cap R^{\uparrow}(p)$ contains a full-dimensional convex subset of $\Delta(X)$. Since pRr, $B_p \cap R^{\uparrow}(p) \subseteq B_p \cap R^{\uparrow}(r)$, hence $\mathbb{P}^{+}_{r}$ contains a full-dimensional convex subset of $\Delta(X)$. $\square$

Immediately, this implies that for $r \in I$, we can build an increasing $u_r : X \to \mathbb{R}$, unique up to an affine transformation, such that $c(A) = \arg\max_{p\in A} \mathbb{E}_p u_r(x)$ if $A \in \mathbb{A}^{r}_{R^{\uparrow}(r)}$. The technique is standard. Let $\mathbb{P}$ be a full-dimensional convex subset of $\mathbb{P}^{+}_{r}$. First, notice that for all $p, q \in \mathbb{P}$, we have $\{r, p, q\} \in \mathbb{A}^{r}_{R^{\uparrow}(r)}$ and $r \notin c(\{r, p, q\})$. Recall that c on $\mathbb{A}^{r}_{R^{\uparrow}(r)}$ satisfies WARP and Independence. By define $p \gtrsim_r q$ if $p \in c(\{r, p, q\})$, we get a binary relation $(\gtrsim_r, \mathbb{P})$ that is complete, transitive, continuous, and satisfies independence. Since $\mathbb{P}$ is full-dimensional and convex, it contains a subset that is essentially a linear transformation of a $|X| - 1$ dimensional simplex. Since $(\gtrsim_r, \mathbb{P})$ satisfies FOSD, an increasing utility function $u_r : X \to \mathbb{R}$, unique up to an affine transformation, such that $c(A) = \arg\max_{p\in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^{r}_{\mathbb{P}}$. We normalize this function to $u_r : X \longrightarrow [0,1]$, where $u_r(w) = 0$ and $u_b(b) = 1$.

We now show that this utility function works for $\mathbb{A}^{r}_{R^{\uparrow}(r)}$. First, for any two lotteries $p, q \in \Delta X$, there exist $p', q' \in \mathbb{P}$ such that $p' = (p)^{\alpha} s$ and $q' = (q)^{\alpha} s$ for some $s \in \Delta X$ and $\alpha \in [0,1]$; we call p', q' $\mathbb{P}$-*common mixtures* of p, q. This can be done by using an arbitrary point $s \in \operatorname{int}(\mathbb{P})$ and take α small enough until both p' and q' enter $\mathbb{P}$. Take any $p \in R^{\uparrow}(r)$ and let r', p' be $\mathbb{P}$-common mixtures of r, p. Since c on $\mathbb{A}^{r}_{R^{\uparrow}(r)}$ satisfies Independence, $i' \in c(\{r, r', p'\})$ if and only if $i \in c(\{r, p\})$, for $i = r, p$. Now take any $p, q \in R^{\uparrow}(r)$ such that $p \in c(\{r, p\})$ and $q \in c(\{r, q\})$, then again by Independence on $\mathbb{A}^{r}_{R^{\uparrow}(r)}$, $p' \in c(\{r, p', q'\})$ if and only if $p \in c(\{r, p, q\})$, where p', q' are $\mathbb{P}$-common mixtures of p, q.

We have thus shown that $c(\{r,p\}) = \arg\max_{s\in\{r,p\}} \mathbb{E}_s u_r(x)$ for all $\{r,p\} \in \mathbb{A}^r_{R^\uparrow(r)}$ and $c(\{r,p,q\}) = \arg\max_{s\in\{r,p,q\}} \mathbb{E}_s u_r(x)$ for all $\{r,p,q\} \in \mathbb{A}^r_{R^\uparrow(r)}$ where $p \in c(\{r,p\})$ and $q \in c(\{r,q\})$. Since c on $\mathbb{A}^r_{R^\uparrow(r)}$ satisfies WARP, showing $c(A) = \arg\max_{p\in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^\uparrow(r)}$ is straightforward from here.

Corollary 1.1. *For $r \in \Delta(X)$ and $p \in R^\uparrow(r) \cap I$ such that $r \notin c(\{r,p\})$, there exists $q \in R^\uparrow(r) \cap I$ such that $\{q\} = c(\{r,p,q\})$. Furthermore, $\mathbb{P}^{+p}_r := \left\{q \in R^\uparrow(r) : \{q\} = c(\{r,p,q\})\right\}$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. The proof utilizes techniques in the proofs of 1.5 and 1.6. First, we show the existence of $q \in R^\uparrow(r) \cap I$ such that $\{q\} = c(\{r,p,q\})$. Consider the set of extreme spread of of p, we know that this set is a subset of $R^\uparrow(p)$, and is hence a subset of $R^\uparrow(r)$. Notice that $r \notin c(\{r,p,e\})$ for any extreme spread e of p since c on $\mathbb{A}^r_{R^\uparrow(r)}$ satisfies WARP and $r \notin c(\{r,p\})$. Using the technique in the proof of Lemma 1.6, it must be that for some extreme spread e^* of p, we have $p \notin c(\{r,p,e^*\})$, otherwise by continuity of c we have $p \in c\left(\left\{r,p,(\delta_w)^{p(w)}(\delta_b)\right\}\right)$, a violation of FOSD. Take any non-trivial convex combination $p^\alpha e^*$, this is in $R^\uparrow(p) \subseteq R^\uparrow(r)$, in I, and $\{p^\alpha e^*\} = c(\{r,p,p^\alpha e^*\})$, so let $q = p^\alpha e^*$. Finally, by continuity of c, take an open ball B_q such that $q' \in B_q$ implies $\{q'\} = c(\{r,p,q'\})$. By Lemma 1.5, $B_q \cap R^\uparrow(q)$ contains a full-dimensional convex subset of $\Delta(X)$. Moreover, $B_q \cap R^\uparrow(q) \subseteq B_q \cap R^\uparrow(r) \subseteq \mathbb{P}^{+p}_r$. So $\mathbb{P}^{+p}_r$ contains a full-dimensional convex subset of $\Delta(X)$. □

Lemma 1.7. *Consider $r_1, r_2 \in I$ and $r_2 R r_1$. Then $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing $f : [0,1] \to [0,1]$.*

Proof. This proof uses Axiom 1.4. Take any $r_1, r_2 \in I$ such that $r_2 R r_1$. u_{r_1} and u_{r_2} are defined above, let $\bar{f}$ be defined on the utility numbers $u_{r_2}(x), x \in X$, such that $u_{r_1}(x) = \bar{f} u_{r_2}(x)$. Since

u_{r_1} and u_{r_2} are strictly increasing, $\bar{f}$ is strictly increasing in its domain. We show that for any $x_1, x_2, x_3 \in X$ such that $x_1 < x_2 < x_3$, we have $\bar{f}(\alpha u_2(x_1) + (1-\alpha) u_2(x_3)) \geq \alpha \bar{f}(u_2(x_1)) + (1-\alpha)\bar{f}(u_2(x_3))$, where $\alpha u_2(x_1) + (1-\alpha) u_2(x_3) = u_2(x_2)$. Suppose not, then for some $\beta < \alpha$, we have $\bar{f}(\alpha u_2(x_1) + (1-\alpha) u_2(x_3)) < \beta \bar{f}(u_2(x_1)) + (1-\beta)\bar{f}(u_2(x_3)) < \alpha \bar{f}(u_2(x_1)) + (1-\alpha)\bar{f}(u_2(x_3))$. Consider lotteries $\delta = \delta_{x_2}$ and $p = (\delta_{x_1})^{\beta}(\delta_{x_3})$. The previous equation shows that $\mathbb{E}_{\delta} u_{r_1}(x) < \mathbb{E}_{p} u_{r_1}(x)$ and $\mathbb{E}_{\delta} u_{r_2}(x) > \mathbb{E}_{p} u_{r_2}(x)$. Let δ_1, p_1 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}_{r_1}^{+r_2}$ if $r_1 \notin c(\{r_1, r_2\})$, and of $\mathbb{P}_{r_1}^{+}$ otherwise. Let δ_2, p_2 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}_{r_2}^{+}$. Since u_{r_1} and u_{r_2} are Bernoulli utility functions for r_1 and r_2 respectively, we have $\{p_1\} = c(\{r_1, \delta_1, p_1\})$ and $\{\delta_2\} = c(\{r_2, \delta_2, p_2\})$. Notice that $A := \{r_1, r_2, \delta_1, \delta_2, p_1, p_2\} \in \mathbb{A}^{r_1}_{R^{\uparrow}(r_1)}$, so $c(A) = \arg\max_{q \in A} \mathbb{E}_{q} u_{r_1}(x)$. We established that $\mathbb{E}_{r_1} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, $\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, and $\mathbb{E}_{\delta_i} u_{r_1}(x) < \mathbb{E}_{p_i} u_{r_1}(x)$ for $i = 1, 2$, so $c(\{r_1, r_2, \delta_1, \delta_2, p_1, p_2\}) \subseteq \{p_1, p_2\}$. But this and $\{\delta_2\} = c(\{r_2, \delta_2, p_2\})$ violates Axiom 1.4. Finally, it is straightforward that one can extend $\bar{f}$ to a concave function $f : [0,1] \to [0,1]$ (for example, by connecting the points with straight lines). $\square$

At this point we are almost done with proving the representation, less $r \in E$.

Lemma 1.8. *For $r \in E$ and $p \in R^{\uparrow}(r)$, $p \neq r$, either p first order stochastically dominates r or the converse.*

Proof. Take $r \in E$ and $p \in R^{\uparrow}(r)$, $p \neq r$. Let $\alpha = r(b)$, then $r(w) = 1 - \alpha$. If $p(b) < \alpha$ and $p(w) < (1-\alpha)$, then r is an extreme spread of p and rRp, so $p \notin R^{\uparrow}(r)$. Furthermore, it is not possible that $p(b) \geq \alpha$ and $p(w) \geq (1-\alpha)$ if $p \neq r$. Hence either $p(b) \geq \alpha$ and $p(w) \leq (1-\alpha)$ with at least one strict inequality, or $p(b) \leq \alpha$ and $p(w) \geq (1-\alpha)$ with at least one strict inequality. If the earlier, p FOSD r; if the later r FOSD p. $\square$

With this observation in mind, we construct u_r for $r \in E$. Define $E_1 := \left\{r \in E : r \notin c\left(\{r, p\}\right) \text{ for some } p \in R^{\uparrow}\left(r\right) \cap I\right\}$. and $E_2 := E \backslash E_1$. For $r \in E_1$, $\mathbb{P}^{+}_{r}$ contains a full-dimensional convex subset of $\Delta\left(X\right)$, and so we will build u_r using the same method we used to build u_r for $r \in I$. We will construct u_r for $r \in E_2$ after the following result.

Corollary 1.2. *Consider $r_1, r_2 \in I \cup E_1$ and $r_2 R r_1$. Then $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing $f : [0,1] \rightarrow [0,1]$.*

Proof. Consider the proof in Lemma 1.7, but that when $r_2 \in E_1$, we simply let δ_1, p_1 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}^{+}_{r_1}$. Before, we let $\mathbb{P}$ here be a full-dimensional convex subset of $\mathbb{P}^{+r_2}_{r_1}$ when $r_1 \notin c\left(\{r_1, r_2\}\right)$, but now such subset need not exist as $r_2 \notin I$. To compensate for this, since $\delta_2, p_2 \in \mathbb{P}^{+}_{r_2}$ implies that δ_2, p_2 FOSD r_2 due to Lemma 1.8, we replace the argument "$\mathbb{E}_{r_2} u_{r_1}\left(x\right) < \mathbb{E}_{p_1} u_{r_1}\left(x\right)$" with "$\mathbb{E}_{r_2} u_{r_1}\left(x\right) < \mathbb{E}_{p_2} u_{r_1}\left(x\right)$". Everything else goes through as in the proof in Lemma 1.7, giving us the desired result. □

For $r \in E_2$, given Lemma 1.8, any increasing utility function $u_r : X \rightarrow [0,1]$ will accomplish $c\left(A\right) = \arg\max_{p \in A} \mathbb{E}_p u_r\left(x\right)$ for all $A \in \mathbb{A}^{r}_{R^{\uparrow}(r)}$. With this freedom, we construct u_r in the following way. Consider for an increasing utility function u_p, the object $\rho^p = \left(\rho^p_2, ..., \rho^p_{|X|-1}\right) \in (0,1)^{|X|-2}$ where $\rho^p_i := \frac{u_p(x_i) - u_p(x_{i-1})}{u_p(x_{i+1}) - u_p(x_{i-1})}$ (that is, ρ^p_i satisfies $u_p\left(x_i\right) = \rho^p_i u_p\left(x_{i+1}\right) + \left(1 - \rho^p_i\right) u_p\left(x_{i-1}\right)$). There is a one-to-one relationship between u_p and ρ^{ρ}. It is an algebraic exercise to show that $u_p = f \circ u_q$ for some concave and increasing $f : [0,1] \rightarrow [0,1]$ if and only if $\rho^p_i \geq \rho^q_i$ for all $i \in \{2, ..., |X| - 1\}$. Take $r \in E_2$ and define $\rho^r := \left(\inf_{p \in K}\left(\rho^p_2\right), ..., \inf_{p \in K}\left(\rho^p_{|X|-1}\right)\right)$, where $K_r := \left(I \cup E_2\right) \cap \{p : rRp\} \subseteq \Delta\left(X\right)$, and subsequently construct u_r using ρ^r. It is easy to show that R being risk-consistent implies K_r is nonempty for all $r \in E_2 \backslash \{\delta_b, \delta_w\}$, and so u_r is defined other than when $r \in \{\delta_b, \delta_w\}$.

For the non-generic case where for some $j \in \{b,w\}$ we have $\delta_j \in E_2$ and K_{δ_j} is not defined, this implies $pR\delta_j$ for all $p \in \Delta(X)\setminus\{\delta_b,\delta_w\}$. Then, we define $\rho_i^{\delta_j} = \frac{1}{2}(1) + \frac{1}{2}\sup_{p\in\Delta X\setminus\{\delta_b,\delta_w\}}\rho_i^p$ for all i and construct u_{δ_j} correspondingly. Utility functions indexed by such a δ_j and that by any $p \in \Delta X\setminus\left\{\delta_j\right\}$ now satisfy $\rho_i^{\delta_j} \geq \rho_i^p$, with equality when p also is a δ_j falling into this special case (there are at most two of them, δ_b and δ_w).

We now show that for $r_1, r_2 \in \Delta(X)$ where r_2Rr_1, we have $\rho^{r_1} \geq \rho^{r_2}$. This is already shown for any $r_1, r_2 \in I \cup E_1$ by Corollary 1.2. It also is shown for the special cases in the previous paragraph. Henceforth we restrict attention to the remaining cases. Say $r_1 \in E_2$, $r_2 \in I \cup E_1$, but $\rho_i^{r_1} < \rho_i^{r_2}$ for some i. Then $\inf_{p\in K_{r_1}}\left(\rho_i^p\right) < \rho_i^{r_2}$, so $\rho_i^p < \rho_i^{r_2}$ for some $p \in K_{r_1}$. However, $p \in K_{r_1}$ implies r_2Rp since R is transitive; since $p \in I \cup E_2$, this contradicts Corollary 1.2. Say $r_1 \in I \cup E_1$, $r_2 \in E_2$, but $\rho_i^{r_1} < \rho_i^{r_2}$ for some i. Then $\rho_i^{r_1} < \inf_{p\in K_{r_2}}\left(\rho_i^p\right)$, so $\rho_i^{r_1} < \rho_i^p$ for all $p \in K_{r_2}$. But $r_1 \in K_{r_2}$, a contradiction. Finally, for $r_1, r_2 \in E_2$ and r_2Rr_1, either $K_{r_1} = K_{r_2}$ or $K_{r_1} \subsetneq K_{r_2}$. If the earlier, it is immediately that $\rho^{r_1} = \rho^{r_2}$. If the later, then $\rho_i^{r_1} = \inf_{p\in K_{r_1}}\left(\rho_i^p\right) \leq \inf_{p\in K_{r_2}}\left(\rho_i^p\right) = \rho_i^{r_2}$ for all i, as desired.

Thus, we have now shown that for any $r_1, r_2 \in \Delta(X)$, $\rho^{r_1} \geq \rho^{r_2}$ whenever r_2Rr_1, or equivalently $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing $f : [0,1] \to [0,1]$.

1.9.5 Proof of Proposition 1.4, Part 1

Suppose $c(\{p,q\}) = \{p,q\}$. Without loss of generality, either $\arg\max_R\{p,q,p^\alpha q\} = p$ or $\arg\max_R\{p,q,p^\alpha q\} = p^\alpha q$. If the former, then $\arg\max_R\{p,q\} = \arg\max_R\{p,p^\alpha q\} = p$, hence the utility functions used in the two choice problems $c(\{p,q\})$ and $c(\{p,p^\alpha q\})$ are both u_p. It is immediately that, since $p^\alpha q$ is a mixture of p and q, we have $c(\{p,p^\alpha q\}) = \{p,p^\alpha q\}$, and by Transitivity we have $c(\{q,p^\alpha q\}) = \{q,p^\alpha q\}$.[40] If the latter, then $\arg\max_R\{p^\alpha q,q\} =$

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[40] Since $\sum_{x\in X} p(x)\,u_p(x) = \sum_{x\in X} q(x)\,u_p(x) = \sum_{x\in X}\left[\alpha p(x) + (1-\alpha)\,q(x)\right]u_p(x)$.

$\arg\max_R \{p, p^\alpha q\} = p^\alpha q$. Suppose for contradiction $p \notin c(\{p, p^\alpha q\})$, then $p^\alpha q \notin c(p^\alpha q, q)$ since $p^\alpha q$ is a mixture of p and q.[41] But by Transitivity we would have $c(\{p, q\}) = \{q\}$, a contradiction. We have hence proved the second property of Betweenness in Definition 1.6. The first property is immediate using this second property, Continuity, and FOSD (the latter two are axioms/implications of AREU).

1.9.6 Proof of Proposition 1.4, Parts 2-4

We first prove point 3. Using Proposition 1.4 Part 1, we know that indifference curves are linear and do not intersect. Take an arbitrary indifference curve and consider two points p, q on it that lie in the interior of the triangle. Let p' and q' be mean-preserving contractions of p and q such that the line connecting p', q' is parallel to the line connecting p, q. Since p', q' are mean-preserving contractions, $\min(\{p, q\}, R)\ R \min(\{p', q'\}, R)$, and so AREU posits that $c(\{p', q'\})$ is explained by a more concave utility function than the one used for $c(\{p, q\})$, corresponding to a weakly steeper indifference curve. Figure **??** provides an illustration. Point 4 is proven analogously. The consequence of these unidirectional fanning, along with continuity, rules out the possibility of c being both strictly risk averse and strictly risk loving in this triangle, i.e., point 2 of the proposition.

1.9.7 Proof of Proposition 1.5

The proof for utility representation is three-fold. First, we show that with Axiom 1.6 and 1.7, for each time $r \in T$, the set of all choice problems such that the earliest payment arrives at

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[41] Since $\sum_{x \in X} p(x)\, u_{p^\alpha q}(x) < \sum_{x \in X} [\alpha p(x) + (1-\alpha) q(x)]\, u_{p^\alpha q}(x)$ implies $\sum_{x \in X} [\alpha p(x) + (1-\alpha) q(x)]\, u_{p^\alpha q}(x) < \sum_{x \in X} q(x)\, u_{p^\alpha q}(x)$.

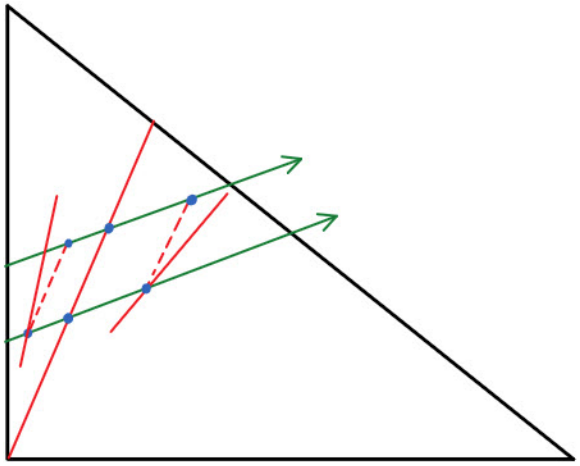

Figure 1.9.1: AREU with Transitivity
Indifference curves fan out when AREU is combined with Transitivity and risk aversion Proposition 1.4). Arrows correspond to direction of mean-preserving spread.

this time can be explained by a nonempty set of Discounted Utility specifications, where an element of this set is $(\tilde{u}, \delta)$, a utility function and a discount factor. Second, we show that at least one utility function u can be supported for all $r \in T$, and set as δ_r as the corresponding discount factor supporting u for r; this is the more involved part of the proof and uses Axiom 1.8. Lastly with Axiom 1.8 again, we show the desired relationship between δ_r and $\delta_{r'}$ for any two r, r'.

By Lemma 1.1 and Lemma 1.2, for any $r \in T$ where $r < \bar{t}$, c satisfies WARP and Stationarity over $S := \{A \in \mathcal{A} : \Psi(A) = (\cdot, r)\}$ (S is the collection of choice sets such that the soonest available payment arrives at time r). Take $\epsilon > 0$ such that $r + \epsilon < \bar{t}$. For each (x, r)

1.9.8 Proof of Proposition 1.7

Fix c. First we show that with Axiom 1.10 and Axiom 1.9, for each equity ratio $r \leq 1$, the set of all choice problems where the greatest equity is r can be explained by the maximization of $x + v_r(y)$ for some unique $v_r : \mathbb{R}_+ \to \mathbb{R}$. For each alternative $(x, y) \in X$, the revealed preference relation generated from $c : \mathcal{S} \to \mathcal{A}$, where $\mathcal{S} = \left\{A \in \mathcal{A} : r(A) = e_{(x,y)} \text{ and } (x,y) \in A\right\}$, satisfies acyclicity and does not violate quasi-linearity. Combined with Continuity, acyclicity gives us a set of utility functions $u_{(x,y)}(\cdot)$ where for all $A \in \mathcal{S}$, $c(A)$ is the set of maximizers of $u_{(x,y)}(\cdot)$ in A. With non-violation of quasi-linearity, any admissible $u_{(x,y)}(\cdot)$ must be a strictly increasing transformation of $x + v_{(x,y)}(y)$ for some $v_{(x,y)} : \mathbb{R}_+ \to \mathbb{R}$. Otherwise, since for any pair of income distributions $\{(x', y'), (x'', y'')\}$ there are infinitely many shifted copies $\{(x'+a, y'), (x''+a, y'')\}$ such that $e_{(x'+a,y')}, e_{(x''+a,y'')} \geq e_{(x,y)}$ and $(x,y) \notin c\{(x,y), (x+a,y), (x'+a,y')\}$, a violation of quasi-linearity must occur.

Fix an r, we now show that $v_{(x,y)}$ must coincide for all (x,y) where $e_{(x,y)} = r$. Consider the set of choice problems $\mathcal{S}=\{A \in \mathcal{A} : r(A) = r\}$. Note that c satisfies WARP and Quasi-linearity on $\mathcal{S}$. To see this, take any two choice problems A_1, A_2 in $\mathcal{S}$. For each $i = 1, 2$, there must be an alternative $(x_i, y_i) \in A_i$ such that $e_{(x_i,y_i)} = r$ and $e_{(x',y')} \geq r$ for all other (x', y') in A_i. Consider an income distribution (x^*, y^*) such that $x^* \leq \min\{x_1, x_2\}$ and $y^* \leq \min\{y_1, y_2\}$ and $e_{(x^*,y^*)} = r$. Due to $(x_i, y_i) \in \Psi(A_i \cup \{(x^*, y^*)\})$, Axiom 1.10 (Fairness Dependence) and Monotonicity (so that (x^*, y^*) is not chosen), $c(A_i) = c(A_i \cup \{(x^*, y^*)\})$. But $(x^*, y^*) \in \Psi(A_1 \cup A_2 \cup \{(x^*, y^*)\})$, so by Axiom 1.10 again $c(A_1 \cup \{(x^*, y^*)\})$ and $c(A_2 \cup \{(x^*, y^*)\})$, which as established are just $c(A_1)$ and $c(A_2)$, cannot generate a violation of WARP or quasi-linearity. Consequently, $v_{(x,y)}$ must coincide for all (x,y) such that $e_{(x,y)} = r$.

Finally we show that for all $r > r'$, $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$ (reminder: higher r implies greater attainable equity). Suppose not, our goal is to substantiate a contradiction of Axiom 1.11 in the choice correspondence. Fix any $y, y' \in \mathbb{R}_+$ such that $y > y'$. Define $\tilde{v}_r = v_r(y) - v_r(y')$ and $\tilde{v}_{r'} = v_{r'}(y) - v_{r'}(y')$. We want to show $\tilde{v}_r \geq \tilde{v}_{r'}$. Suppose for contraction this is not true, let z be any value such that $\tilde{v}_r < z < \tilde{v}_{r'}$. Find a number b such that $\max\{(z-b)/y, (z-b)/y'\} < r'$ and $(z-b) \geq 0$, which is clearly possible for fixed r', y, and

y' since $r' > 0$ and b can be arbitrarily close to z from below. Define $x := z - b$, $x' := 2z - b$, let (x_0, y_0) be some income distribution such that $e_{(x_0,y_0)} = r'$ and $x_0 < x$, $y_0 < y$, and similarly let (x_1, y_1) be some income distribution such that $e_{(x_1,y_1)} = r$ and $x_1 < x$, $y_1 < y$ (these are always possible). Consider the set $A := \{(x, y), (x', y'), (x_0, y_0)\}$. $c(A)$ comes from maximizing the utility function $\hat{x} + v_{r'}(\hat{y})$, and (x_0, y_0) will never be chosen since it is strictly less than (x, y) in each component. Likewise, $c(A \cup \{(x_1, y_1)\})$ comes from maximizing the utility function $\hat{x} + v_r(\hat{y})$ and both (x_0, y_0), (x_1, y_1) will not be chosen. We essentially introduced reference points that won't be chosen, forcing the choice to be between (x, y) and (x', y'). Now note that the way z was obtained gives us $\tilde{v}_r + z < 2z < \tilde{v}_{r'} + z$, and so $\tilde{v}_r + z - b < 2z - b < \tilde{v}_{r'} + z - b$. The first and second inequality are equivalent to $v_r(y) + x < v_r(y') + x'$ and $v_{r'}(y) + x > v_{r'}(y') + x'$ respectively. Finally, the latter gives us $c(A) = \{(x, y)\}$ (where $(x_0, y_0) \in A$) and the former gives us $c(A \cup \{(x_1, y_1)\}) = \{(x', y')\}$; since $A \subset A \cup \{(x_1, y_1)\}$, this is a contradiction of Axiom 1.11. This establishes $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$, for all $r > r'$.

Remark. Some proofs are not included, and can be requested from me (rc@xzlim.com).

CHAPTER 2: SET-DEPENDENT RISK AVERSION: EVIDENCE

Joint with Silvio Ravaioli

2.1 Introduction

Correct measures of risk attitudes have important economics implications. Whereas it is well-known that risk aversion depends on wealth, little is known about the effect of available alternatives on risk aversion—even when the level of wealth is fixed. For instance, when an agent chooses a riskier option R over a safer option S, but reverses her decision in a different choice set. While wealth effect is manifested in the curvature of a (Bernoulli) utility function, changes in risk aversion due to available alternatives—which may lead to violations of the Weak Axiom of Revealed Preferences—calls for a representation that is richer than the maximization of a single utility function.

In a laboratory setting, we provide a direct test for changes in risk aversion when a third option, which is either extremely risky or extremely safe, is added to a choice set. The experiment is conducted using a within-subject design. Basically, the third option expands the choice set in way that increases the amount of risk an agent can take/avoid. Under standard theory, risk aversion should be unaffected by this type of modification—risk aversion is set-independent. We find that, in violation of standard theory, subjects displayed increased risk aversion when a safe option was added to a choice set. On the contrary, there is no detectable change in risk aversion when an extremely risky option was added instead. This finding is in line with the Avoidable Risk Expected Utility model by Lim (2019), which proposes higher risk aversion when risk is avoidable than when it is not.

Throughout the experiment, the lotteries we use take the form "50% chance of x (low prize) and 50% chance of y (high prize)." Lotteries differ in the spread of their prizes; an example of a relatively low risk lottery is 50% chance of \$9 and 50% chance of \$6, and a riskier lottery is 50% chance of \$15 and 50% chance of \$3. It is deemed riskier because $\$15 > \$9$ and $\$3 < \$6$. Section 2.2 describes our hypothesis, the experimental design, and how we came up with the specifications of the lotteries involved. By using lotteries of this form, we separate ourselves from the (more typical) experimental settings where a lottery is a single probability of a single prize (for example, 30% chance of \$4 and \$0 otherwise). Using 50/50 lotteries allow us to study changes in risk aversion independent of probability weighting and provides a simpler (and possibly more realistic) representation.

Our experiment is primarily motivated by models of set-dependent risk aversion. Bleichrodt & Schmidt (2002) proposes a model that restricts attention to binary comparisons, in which a decision maker uses a different utility function when a sure prize is present than when it is not. When a sure prize is present, she uses a utility function that is more concave.

Similar in spirit, Lim (2019) introduces a model of reference dependent risk aversion. In their model, the decision maker uses the safest alternative in a choice problem as reference point, and a safer reference leads to a more concave utility function. In particular, Lim (2019)'s model applies to any finite choice sets, providing the theoretical foundation of a direct test via the use of a third option.

The intuition is as follows; when the amount of risk a decision maker *can* avoid is increased—i.e., when the safest available lottery has become even safer—the decision maker becomes more risk averse. Our experiment provides a direct test of this intuition, and the result concurs.

Our findings are in line with other evidence of changes in risk aversion. The Allais (1953)'s paradox is one example: In two binary choice problems where one contains a sure prize and the other doesn't, subjects overwhelmingly choose the sure prize even though it is in violation of expected utility's predictions. This behavior is consistent with the interpretation that risk aversion is higher when a sure prize is present than when it is not.[1] Wakker & Deneffe (1996) introduced a novel method of eliciting risk aversion without involving a sure prize. They find that, in general, subjects display less risk aversion relatively to elicitation using certainty equivalent/probability equivalent (both of which involve a sure prize). Herne (1999) used lotteries of the form "probability p on prize x" to study set effect on choices. A lottery with a higher probability of a smaller prize is deemed safer. Their experiment showed that for two choice sets $\{(p_1,x_1),(p_2,x_2),(p_3,x_3)\}$ and $\{(p_2,x_2),(p_3,x_3),(p_4,x_4)\}$ such that $p_i < p_{i+1}$ for $i = 1,2,3$, subjects' choices display greater risk aversion for the latter choice set.

Although many studies document behavior that are in line with our findings, a notable exception is Kroll & Vogt (2012). In their study, the addition of a riskier lottery decreases risk aversion, a finding that is not present in our experiment. However, since they rely on the use of certainty equivalence to measure risk aversion, they did not test for changes in risk aversion when, instead, a safer lottery was added (since a sure prize—the certainty equivalent—is always present).

The rest of the paper is organized as follows. In Section 2.2 we describe our methodology, experimental design, and the dataset. In Section 2.3 we provide analysis of our findings, and Section 2.4 concludes.

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[1] Lim (2019) shows that as long as the support of the prize space has size three, which is the case for the majority of Allais-type experiments, a violation of the Independence axiom in the standard direction (where the safe prize is chosen whenever available) implies a more concave utility function for the choice set that contains a safe prize.

2.2 Experimental Design

The key feature of the Avoidable Risk Expected Utility model (of Lim (2019)) is that the introduction of safer alternatives increases risk aversion. To test this, we designed a lab experiment involving simple choices in a controlled environment. Each trial of the task involves choosing a lottery from a binary or trinary choice. Trinary choice sets are different from binary ones only by the addition of a third option, which can either be safer or riskier than the options already available. After testing whether the predicted effect occurs in the dataset (H1), we need to verify the asymmetric prediction (H2) and whether the result is affected by the characteristics of the safer option (H3).

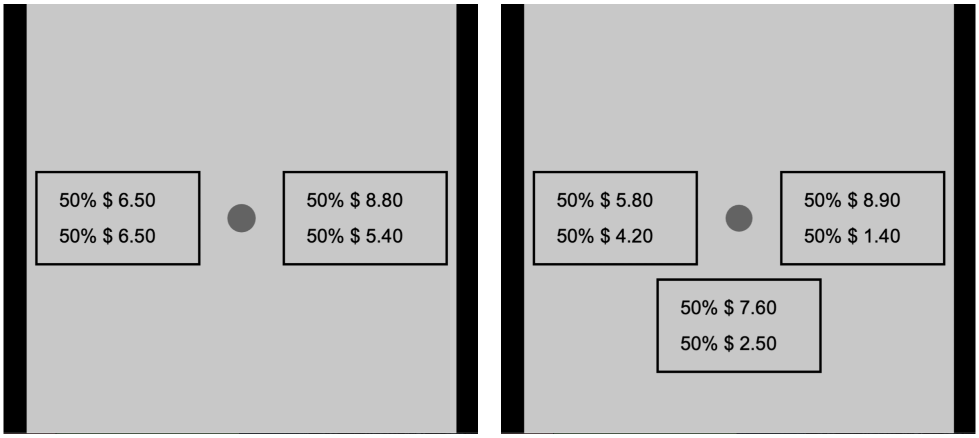
Hypothesis 1: Avoidable Risk Effect. The addition of a safer option makes the agents more risk averse.

Hypothesis 2: No Compromise Effect. The addition of a riskier option does *not* make the agent more risk seeking.

Hypothesis 3: Robustness of the Avoidable Risk Effect. The avoidable risk effect is robust with respect to the characteristics of the safer option: certainty (safer option being degenerate) and magnitude (expected value of the safer option compared to the other options).

2.2.1 Task

In each round, participants choose one lottery between the two or three options available. Each lottery is expressed as a pair of outcomes (dollar amounts), each with probability 50% of being realized. The lotteries appear on the screen in four possible locations (top, right, bottom, left) equally distant from the fixation point placed at the center of the screen, as shown in Figure 2.2.1). Each round starts with only the fixation point on the screen. After 500 ms, two (three) empty boxes appear at random locations. In this way we are providing information about the number and position of the screen of the lotteries, without revealing any further detail about the values. After 500 ms, the values also appear on the screen. No action is allowed for the first 3 seconds, past which the subject can select one option by using the keyboard (arrow keys to select, spacebar to confirm) and without time limit. Upon confirmation, no feedback is provided and a new round starts.



(a) Binary choice (b) Trinary choice

The figures illustrate samples of binary and trinary choice sets. Lotteries appeared simultaneously in randomized locations on the screen. Subjects used arrow keys to select an option and spacebar to confirm. A minimum observation time of 3 seconds was enforced. Each choice set contains two (left) or three lotteries (right).

Figure 2.2.1: Screenshot of the Experiment

The whole task includes 120 rounds, with the possibility to take short breaks every 30 rounds. Each round is distinct (i.e., every choice set is faced exactly once). All participants observed the same list of choice sets in a random order. We also randomize at the subject level the location of the lotteries on the screen and the position of the two outcomes (high and low prize).

2.2.2 Dataset

The experiment includes dataset of 120 choice sets, divided into 90 "test rounds" and 30 "control rounds."

The test rounds are grouped in 15 *conditions*. Each condition is characterized by two parameters: a baseline dollar amount $d \in [\$3, \$12]$ and a CRRA coefficient $\alpha \in [0.3, 0.7]$.[2] A procedural algorithm takes a pair (d, α) and generates 2 *target options*—S (safer) and R (riskier)—and 2 *third options*—SS (safest) and RR (riskiest). SS, S, R, RR are ordered by increasing spread of prizes, i.e., the lower prize of SS is greater than the lower prize of S, the higher prize of SS is less than the higher prize of S, and so on. Therefore, no lottery first-order stochastically dominates another.

Figure 2.2.2 illustrates the relationship between these lotteries. The target options, S and R, lie on the same indifference curve under CRRA utility and risk aversion coefficient α. The third options, SS and RR, are designed to be less appealing and lie below the indifference curve connecting S and R. Therefore, choosing SS or RR would reflect extreme risk averse / risk seeking preferences.

Additional to SS, we used other types of safe lotteries, $SS2$, $SS3$, and $SS4$ to test whether absolute certainty and attractiveness affect our finding. Lotteries SS and $SS3$ are degenerate lotteries, with 100% probability of receiving a fixed dollar amount. They differ in that $SS3$ receives a smaller penalization with respect to the benchmark d, making it less unappealing compared to SS. Lotteries $SS2$ and $SS4$ are non-degenerate lotteries. They are obtained by

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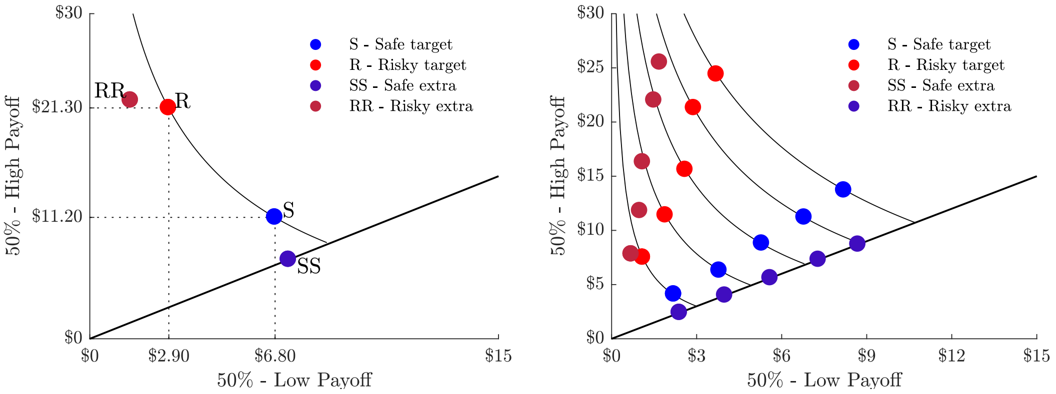
[2] The range of parameters for α was calibrated based on estimated risk preferences in other lab experiments (Harrison & Rutström (2008)), as well as a pilot study we conducted in the Summer 2019. This range allows to implement the within-subject analysis of the data despite the large heterogeneity in risk preferences across participants.

adding jitters to SS and $SS3$: In both cases, starting from the degenerate lottery, we create a new pair of outcomes such that low payoff is 10 cents less and the high payoff is 10 cents higher. In this way we can test if the main effect is robust to small variation in the magnitude as well as the certainty features of the safer option introduced in the choice set.



(a) Lotteries in one condition (fixed (d, α)). (b) Lotteries used in different conditions

(a) Starting from the indifference curve characterized by the pair of parameters d (baseline dollar amount) and α (relative risk coefficient), an algorithm generates the target lotteries S (safe) and R (risky) on the curves, and the third options SS (safer) and RR (riskier) below the curve. (b) d and α were varied to obtain different conditions.

Figure 2.2.2: Trials Generating Process

The control rounds are designed to detect heuristics (e.g., maximizing the downside, minimizing the upside), and are not part of the main analyses. Like the target rounds, control rounds include binary and trinary choice sets, and within each round there are no first-order stochastically dominated options. These choice sets contain options with large positive (resp. negative) risk premia, for which we expect choices in favor of the riskier (resp. safer) options. Systematically failing to do so would suggest extreme risk preferences (a possible confounding factor for our test) or decision processes that follow simple heuristics, such as maximizing the lowest (resp. highest) payoffs or minimizing (resp. maximizing) variance.

2.2.3 Procedure

The experiment was run in CELSS (Columbia Experimental Laboratory of Social Sciences, Columbia University, New York, USA) between August and September 2019. It was coded in MATLAB (Release 2018b) using Psychotoolbox 3 (Psychophysics Toolbox Version 3). 55 (paid) volunteers were recruited using the platform ORSEE (Online Recruitment System for Economic Experiments) and were naive to the main purpose of the study. All subjects provided written, informed consent. The experiment took on average 45 minutes, including instructions, payment, and an Allais task.[3]

Upon completion, subjects received payment in cash that depended on the choices they had made. One trial was randomly selected for implementation, for which the chosen lottery was played and the outcome was paid. Each subject also received a $10 show-up fee. The average payment was $16.70.

Instructions were provided both (i) on the computer screen as slides that can be browsed by each subject at the desired pace and (ii) as a paper printout that is available to the subject throughout the experiment. The two versions of the instructions contained the same information verbatim. At the beginning of the experiment, subjects were informed of the payment structure, the no-deception policy of the laboratory, and that their decisions would not affect the questions they would face in other rounds. The average laboratory session contains five subjects.

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[3] The results of the Allais task is omitted in the main body, it can be found in Section 2.6, Figure 2.6.3 and Figure 2.6.4.

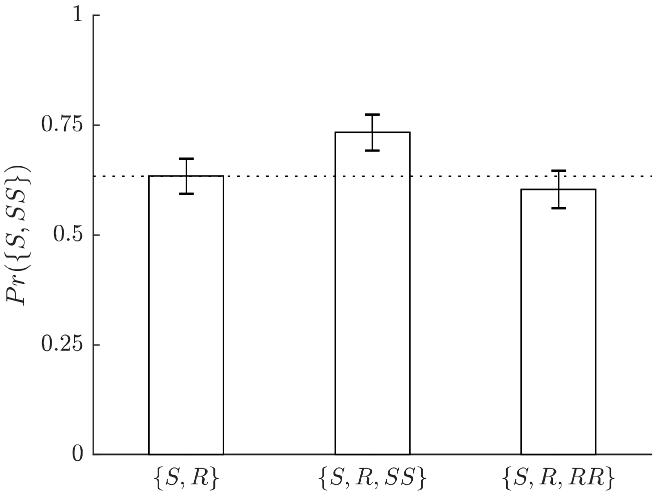
2.3 Results

We analyze the frequency of choosing a safer option (either S or SS) in the binary and trinary choice. In the binary choice trials, participants chose the safe lottery S 63.4% of the times. The frequency of choosing a safe options in trinary choices trials with SS is significantly higher (**H1**, 73.3%, p<10-6, Wilcoxon signed-rank test within subject), whereas it is not significantly different when RR was added (**H2**, 60.4%, p=0.149, Wilcoxon signed-rank test within subject). The frequencies and confidence intervals are shown in Figure 2.3.1. In other words, subjects chose safer options more often when a very safe third option SS was present, but they did not choose riskier options more often when a very risky third option RR was added. Section 2.4 shows that this is consistent with increased risk aversion when SS was added and unchanged risk aversion when RR was added.

The result is robust when we look at the distribution of choices across participants: 69% of the participants become more risk averse when SS is introduced (22% unchanged, 9% less risk averse), and only 26% are more risk averse when RR was added (27% unchanged, 47% less). The pair of scatter plots displayed in Figure 2.3.2 show the distribution of safe and risky choices across participants and how they vary after the introduction of the third lottery.

2.3.1 Variation of Safe Options

Moreover, to investigate the effect of absolute certainty, we created a set of options labeled $SS2$. Each lottery $SS2$ is obtained from SS by adding 10 cents to the high payoff and removing 10 cents from the low payoff. When $SS2$ were added (instead of SS), behavior in aggregate are similar to those when SS was added. This is shown in Figure 2.3.3 and Table 2.1 (**H3**, 74.2% safe choices, p<10-5 Wilcoxon signed-rank test, 62% participants more



The figure reports, from left to right, the (empirical) proportions of *safe group* choices (choosing S or SS) for the benchmark $\{S,R\}$, when SS was added, and when RR was added. Confidence intervals are calculated at 95% using subject-level cluster robust standard errors.

Figure 2.3.1: Frequency of Safe(r) Choices

risk averse, only 4% less).

We also used two other types of safe options labeled $SS3$ (degenerate) and $SS4$ (non-degenerate). They are analogous to SS (degenerate) and $SS2$ (non-degenerate), but with a smaller penalization in expected value (about 20% of the expected value instead of 50%).[4] This allows us to test whether the effect on risk aversion depends on the attractiveness of the added safe options. We find that behavior in aggregate are similar between the $SS, SS2$ and their more attractive counterparts, $SS3, SS4$, as shown in Figure 2.3.3 and Table 2.1.

Figure 2.6.1 reports the panel data on the direction of risk aversion switches across $SS-$

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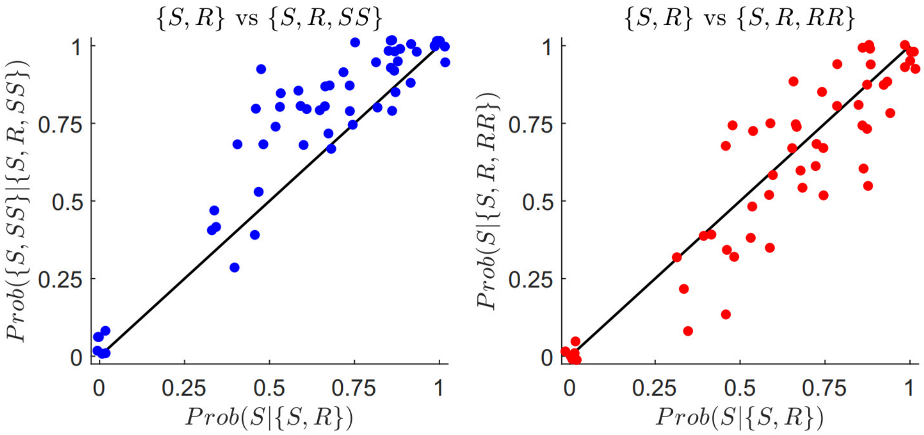
[4] SS was penalized so that subjects' choices concentrate at the target options, S and R. In order to reduce similarity between trials, we use a procedural algorithm that penalizes SS by a random percentage between 45% and 55%. The risky options RR are generated in a similar way, by reducing the lowest value of R by 40-60% and increasing the higher value of R by a smaller fraction (30-60% of the subtracted value).



The figure reports the (empirical) proportion of *safe group* choices (choosing S or SS, as opposed to R or RR) in binary choice sets (x-axis) and after a third option was added (y-axis) for each participant (N=55). On the left, the third option is SS. On the right, the third option is RR.

Figure 2.3.2: Frequency of Safe(r) Choices at the Subject Level

$SS4$.

2.3.2 WARP Violations

Finally, WARP violations were overwhelmingly in the direction of switching from R to S when $SS-SS4$ was added, reflecting increased risk aversion, but equally likely in either direction when RR was added, reflecting inconclusive change in risk aversion.[5]This is reported in Table 2.2.

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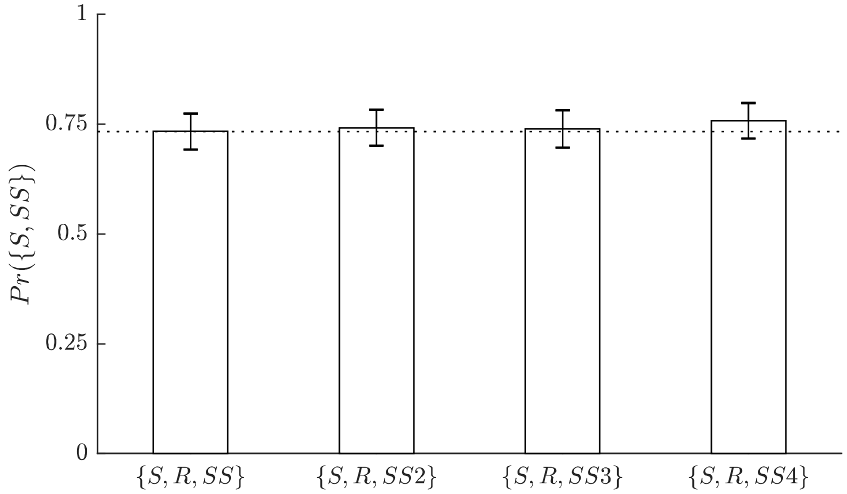
[5] WARP stands for the Weak Axioms of Revealed Preferences. It is violated when a subject chooses a from $\{a,b,c\}$ but b from $\{a,b\}$.



The figure reports the (empirical) proportion of *safe group* choices (choosing S, SS, $SS2$, $SS3$, or $SS4$) when, from left to right, SS, $SS2$, $SS3$, and $SS4$ was added to the binary choice set. Confidence interval is calculated at 95% using subject-level cluster robust standard errors.

Figure 2.3.3: Absolute Certainty and Attractiveness

2.4 Changes in Risk Aversion

Our analysis deems "chooses S over R" as more risk averse than "chooses R over S". In addition to being intuitive, this approach is backed by a formal definition of *more risk averse*, which we now introduce.

Suppose we restrict attention to the class of utility functions defined by *constant relative risk aversion* (CRRA), commonly used across economics and finance research for their intuitive properties. Then, variation in the *Arrow-Pratt risk aversion coefficients* agree with our notion of changes in risk aversion. The same is true if, instead, we restrict attention to the *constant absolute risk aversion* (CARA) class.

Let p and q be two 50/50 lotteries where the high prize of lottery p is h_p and its low prize is l_p. Then, restricting attention to CRRA utility functions, there exists an Arrow-

Table 2.1: Safe and Risky Choices Across Choice Sets

	$\{S,R\}$	$\{S,R,SS\}$	$\{S,R,SS2\}$	$\{S,R,SS3\}$	$\{S,R,SS4\}$	$\{S,R,RR\}$
S	63.39%	62.42%	62.91%	57.21%	56.00%	60.36%
R	36.61%	26.67%	25.82%	26.06%	24.24%	32.48%
Z		10.91%	11.27%	16.73%	19.76%	7.15%
Safe group	63.39%	73.33%	74.18%	73.94%	75.76%	60.36%
Risky group	36.61%	26.67%	25.82%	26.06%	24.24%	39.64%
Wilcoxon SRT		<10^-6	<10^-5	<10^-5	<10^-6	0.1489

Note: This table reports proportion of each choice (row) from binary and trinary choice sets (column). Choice probabilities are expressed for each option as well as for groups. The p-values for the Wilcoxon signed-rank test are calculated within subject with respect to the null hypothesis $\Pr(\text{safe group}|\{S,R,Z\}) = \Pr(\text{safe group}|\{S,R\})$, where $Z = SS, SS2, SS3, SS4, RR$, safe group contains $S, SS-SS4$, and risky group contains R and RR.

Pratt coefficient $\bar{\alpha}$ such that $EU_{\alpha}(q) \geq EU_{\alpha}(p)$ if and only if $\alpha \geq \bar{\alpha}$. The existence of $\bar{\alpha}$ is guaranteed as long as one lottery does not first-order stochastically dominate the other. In other words, we can formalize the statement "an agent who chooses q over p is *more risk averse* than an agent who chooses p over q" using CRRA utilities: this behavior implies a higher risk aversion coefficient. [6]

We use this fact to test our hypotheses, and then conclude that risk aversion has increased from choice set A to choice set B if subject's choices imply a greater risk aversion coefficient in B than in A. Specifically, each choice from a choice set is consistent with a (deterministic) range of CRRA coefficients. We compare the implied ranges to conclude changes in risk aversion. Formally, where $U_{\alpha}(L)$ is the expected utility of L under a CRRA utility function with coefficient α:

Fact. *Let SS, S, R, RR be four 50/50 lotteries sorted by increasing spread of low and high*

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[6] A similar argument can also be made when we restrict attention to CARA utilities instead, although $\bar{\alpha}$ will be different. This is because of the "single coefficient" nature of CARA and CRRA, that a single observation is sufficient to pin down the range of α that explains an underlying behavior. The same comparative statics cannot be done with DARA, etc.

Table 2.2: Types of WARP Violations

$Z =$	SS	$SS2$	$SS3$	$SS4$	RR
WARP (Z)	11%	11%	17%	20%	7%
WARP (not Z)	68%	71%	65%	62%	70%
WARP Violation	20%	18%	18%	18%	23%
Conditional on WARP violations:					
$R \rightarrow S$	71%	74%	69%	75%	50%
$S \rightarrow R$	29%	26%	31%	25%	50%

Note: This table reports the frequency of WARP violations between $\{S,R\}$ and $\{S,R,Z\}$, where $Z = SS, SS2, SS3, SS4, RR$. The first row reports WARP-compliance as a result of choosing Z in $\{S,R,Z\}$. The second row reports WARP-compliance as a result of choosing S in both $\{S,R\}$ and $\{S,R,Z\}$ or R in both $\{S,R\}$ and $\{S,R,Z\}$. All WARP violations are either switches from R in $\{S,R\}$ to S in $\{S,R,Z\}$, which are reported in row $R \rightarrow S$, or S in $\{S,R\}$ to R in $\{S,R,Z\}$, which are reported in row $S \rightarrow R$. $R \rightarrow S$ reflects increased risk aversion when Z was added, and $R \rightarrow S$ reflects decreased risk aversion when Z was added.

prizes. There exists a unique CRRA coefficient $\bar{\alpha}$ such that

1. $U_\alpha(S) > U_\alpha(R)$ *if and only if* $\alpha > \bar{\alpha}$,
2. $\min\{U_\alpha(SS), U_\alpha(S)\} > U_\alpha(R)$ *if and only if* $\alpha > \bar{\alpha}$,
3. $\min\{U_\alpha(R), U_\alpha(RR)\} > U_\alpha(S)$ *if and only if* $\alpha < \bar{\alpha}$.

Now we use this fact to categorize changes in risk attitude: (1) If a subject chooses R from $\{S,R\}$ and SS or S from $\{S,R,SS\}$, her behavior is inconsistent with a persistent CRRA coefficient in the direction of becoming more risk averse in the presence of SS. (2) If a subject chooses S in $\{S,R\}$ and R in $\{S,R,SS\}$, then she has become more risk seeking in the presence of SS. (3) Any other choice combinations from $\{S,R\}$ and $\{S,R,SS\}$ is consistent with a single CRRA coefficient.

Although our method of classifying changes in risk aversion is formalized with CRRA, it is also consistent with a broader intuition. When a subject switches from choosing R, a higher spread lottery, to S, a lower spread lottery, it has natural interpretation of increased risk aversion. Similarly, the opposite behavior corresponds to decreased risk aversion. Moreover, choosing R in $\{S, R\}$ but SS in $\{S, R, SS\}$ means, despite having chosen a high spread lottery (R) over a low spread one (S), she now prefers the lowest spread lottery above all else. Again, this suggests that her risk aversion is greater in $\{S, R, SS\}$ than in $\{S, R\}$.

To simplify notation, it suffices to categorize choices into the *safe group* when S or SS is chosen, and the *risky group* when R or RR is chosen. For $Z = SS, RR$, when we observe a risky group choice from $\{S, R\}$ but a safe group choice from $\{S, R, Z\}$, we say that the addition of Z increases risk aversion; likewise, a safe group choice from $\{S, R\}$ but a risky group choice from $\{S, R, Z\}$ suggests that the addition of Z decreases risk aversion. Any other combination of choices between $\{S, R\}$ and $\{S, R, Z\}$ is consistent with a single CRRA coefficient.

This method streamlines our analysis. In Figure 2.3.1, the y-axis reports the proportion of safe group choices; therefore, increased in safe group choices from $\{S, R\}$ to $\{S, R, SS\}$ reflects increased risk aversion. In Figure 2.3.2, safe group choices are plotted on both axes. If a subject lies above the diagonal line, it meant she made more safe group choices from $\{S, R, SS\}$ than from $\{S, R\}$, which implies increased risk aversion when SS was added.

2.5 Model Fitting

In this section, we perform two model fitting exercises with our dataset. In the first part, we consider expected utility with a risk aversion coefficient that may vary across conditions. In the second part, we show the relationship between Lim (2019)'s Avoidable Risk Expected Utility (AREU) and Kahneman & Tversky (1979)'s Prospect Theory. In the third part, we investigate the compatibility of our dataset with models of random utility, and argues that they are in general incompatible.

2.5.1 Expected Utility

Consider a stochastic expected utility (EU) model (Blavatskyy (2007)) in which the (true) utility of a 50/50 lottery p is given by the pair of equally likely outcomes (x_1, x_2) and a CRRA coefficient α,

$$U_\alpha(p) = \mathbb{E}_p u(x) = \sum_{i=1}^{2} \frac{1}{2} \frac{(x_i)^{1-\alpha} - 1}{1-\alpha},$$

Higher values of α reflect greater risk aversion. We consider stochasticity in choice by introducing noise, $\hat{U}_\alpha(p) = U_\alpha(p) + \epsilon$, where ϵ is a logit error (T1EV).[7] Equivalently, given the true utility and an accuracy parameter λ, the probability of choosing lottery p_i from choice set $A = \{p_1, p_2, ..., p_J\}$ is characterized by

$$\Pr(p_i, A) = \frac{e^{\lambda U_\alpha(p_i)}}{\sum_j e^{\lambda U_\alpha(p_j)}},$$

where higher values of λ results in fewer deviations from maximizing $U_\alpha(p)$.

To fit the EU model, we allow α to differ across binary choice sets, when SS was added, when RR was added, etc. This allows for the same participant to display different risk preferences when third options are added to the choice set.[8] Consistently with the predictions of AREU, estimated parameters for risk aversion differed across choice sets systematically.

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[7] T1EV stands for Type 1 Extreme Value.

[8] For “All data”, we used all trials, including control trials. To resolve the lack of fit from subjects that either always choose the safest alternatives or always choose the riskiest alternatives (which result in risk aversions above or below a certain threshold to be indistinguishable by the fitting exercise, we imposed a constraint that prevents any estimator to predict one choice over another with an accuracy greater than 95%.

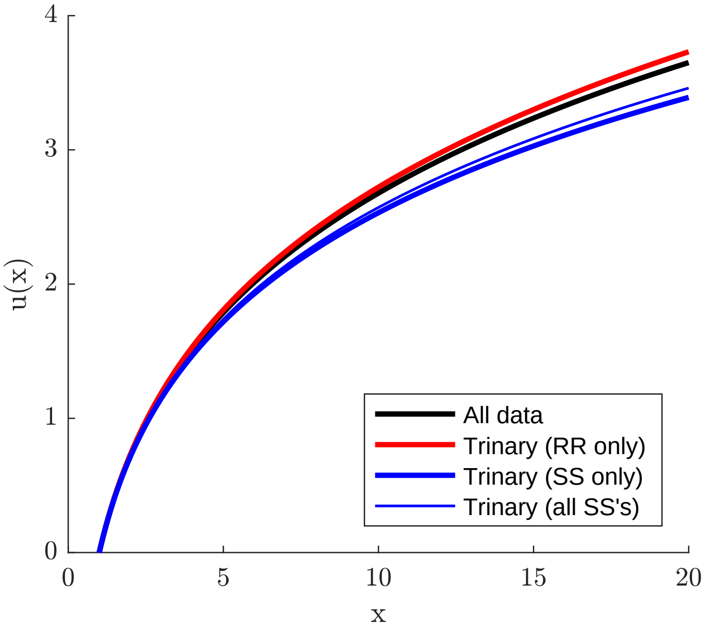

Figure 2.5.1: Model Fitting: Expected Utility Under Different Risk Aversion Coefficients

The estimated risk aversion coefficient α is 0.87 when all choice sets are pooled together. For $\{S, R, SS\}$ (and variations of SS described in Subsection 2.3.1), we observe increased risk aversion: the estimated parameters for α are 0.92 for SS trials and 0.91 when we pool together all $SS - SS4$. On the contrary, the estimated α for RR trials is very close to the benchmark, at 0.86 (versus 0.87). These findings are in line with AREU, which suggests increased risk aversion when the safest alternative become safer (but not when the riskiest alternative changes).

2.5.2 Prospect Theory

Prospect Theory (PT) is another benchmark model for risk preferences. In PT, lotteries are evaluated with respect to a reference point. The reference point determines gains and losses, which is evaluated differently. This is used to capture loss aversion, a prominent concept used to explain various anomalies such as the endowment effect and the equity premium puzzle.

A well-known issue with PT is the identification of the reference point. It turns out that the intuition behind AREU can be used as a compass for the determination of (PT's) reference point, where we adopt the max-min criterion.

In our exercise, the reference point is defined as the "highest minimum outcome" from the available lotteries. This is the outcome that can be guaranteed if the subject simply chooses the safest alternative. In our setting, adding a safer option raises the reference point whilst adding a riskier option does not affect it.[9] Using the conventional functional form of PT,

$$v_\alpha(x, x_{RP}) = \begin{cases} (x - x_{RP})^\alpha & \text{if } x \geq x_{RP} \\ -\beta(x_{RP} - x)^\alpha & \text{if } x < x_{RP} \end{cases},$$

we estimate the three parameters: α (risk aversion), β (loss aversion), and λ (accuracy in the choice stage, as in the EU model) with the entire dataset. The fitted model has the typical S-shaped gain-loss utilities with risk aversion ($\alpha = 0.80 < 1$) and loss aversion ($\beta = 1.85 > 1$). The high degree of loss aversion—where the reference point is the max-min outcome of a choice set—is consistent with the observed shift towards safer options when very safe options are available. On the contrary, there is no significant variation in choice preferences when riskier lotteries are made available.

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[9] If the new lottery is riskier (larger spread of values, and not stochastically dominant), its lower outcome cannot be the highest one, and the reference point is unchanged. This means that risk preferences should also be unchanged by adding a risky lottery. Instead, if the new lottery is safer (smaller spread of values), its lowest outcome becomes the new reference point, higher than the previous one. Moving the reference point has two effects: on the concavity of the curve above and below the reference point, and on the range of values that are perceived as losses, and therefore discounted further in the evaluation process.

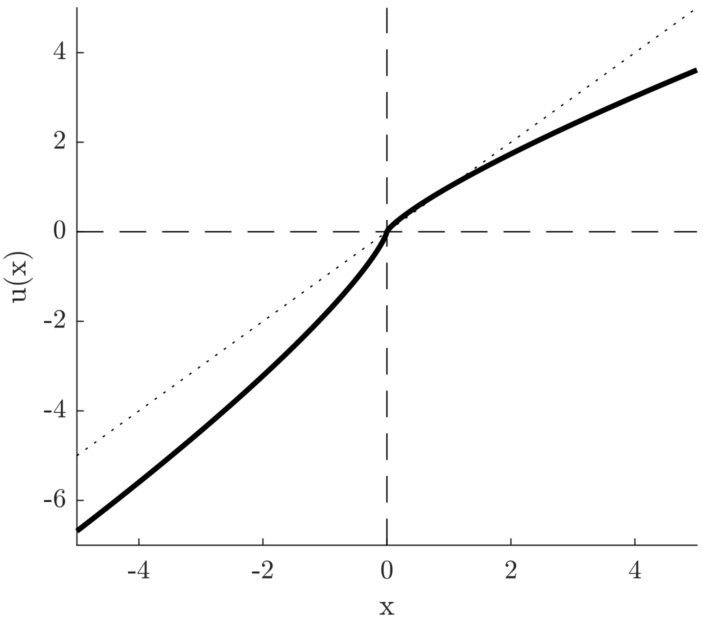

Figure 2.5.2: Model Fitting: Prospect Theory

2.5.3 Random Utility Models

We also investigate the compatibility of our results with random utility models (RUM). In RUM, each alternative p has a utility value $U(p)$ that is subjected to mean-zero noise ϵ_p. Facing a choice set A, the agent chooses p from A to maximize $U(p) + \epsilon_p$, resulting in stochastic choice

$$\Pr(p, A) = \Pr\left[U(p) + \epsilon_p \geq U(q) + \epsilon_q \forall q \in A\right].$$

Under the most general setup, a RUM is captured by a set of probabilities, one for each alternative in each choice set: $\Pr(p, A)$ where $p = SS, S, R, RR$ and $A = \{S, R\}, \{S, R, SS\}, \{S, R, RR\}$. Moreover, $\Pr(p, A) \geq \Pr(p, B)$ if $A \subset B$.

In our dataset, panel data allows us to rule out RUMs, which dictates that the probability of each choice from $\{S, R, SS\}$ is independent of whether a subject has chosen S from $\{S, R\}$

or R from $\{S,R\}$. Figure 2.6.1 shows that this is clearly not the case: For those who chose S from $\{S,R\}$, the vast majority, 76%, chose S from $\{S,R,SS\}$, and only 9% chose R from $\{S,R,SS\}$; on the contrary, for those who have chosen R from $\{S,R\}$, only 40% chose S from $\{S,R,SS\}$ and as many as 57% chose R from $\{S,R,SS\}$.

However, our dataset can be reconciled with RUM if we consider multiple populations of representative agents. For example, suppose we have two population of agents, and each population has their own RUM parameters, then our dataset can be obtained with the correct calibrations due to over-fitting (with 7 parameters for 5 equations).[10]

Another property of RUM is stochastic monotonicity,

$$\Pr(p,A) \geq \Pr(p,B) \text{ if } A \subset B,$$

which simply says that an alternative p is not chosen more often when more options are available. Our dataset does not display stochastic monotonicity violation with the aggregate data, where all trials are clustered together. However, this is partly due to fact that in most trials, subjects choose S in $\{S,R\}$ overwhelming (in aggregate, this happens 63% of the time). This makes it harder for the frequency of S to go even higher.

Indeed, as illustrated in Figure 2.5.3, for trials where S is chosen from $\{S,R\}$ less than half of the time, stochastic monotocity is violated in that S is chosen more often in $\{S,R,SS\}$ than in $\{S,R\}$. On the contrary, stochastic monotonicity holds for R throughout. That is, for all trials—even the ones where R is chosen from $\{S,R\}$ less than half of the time—R is chosen even less often in $\{S,R,SS\}$ than in $\{S,R\}$, again suggesting that introducing SS increases

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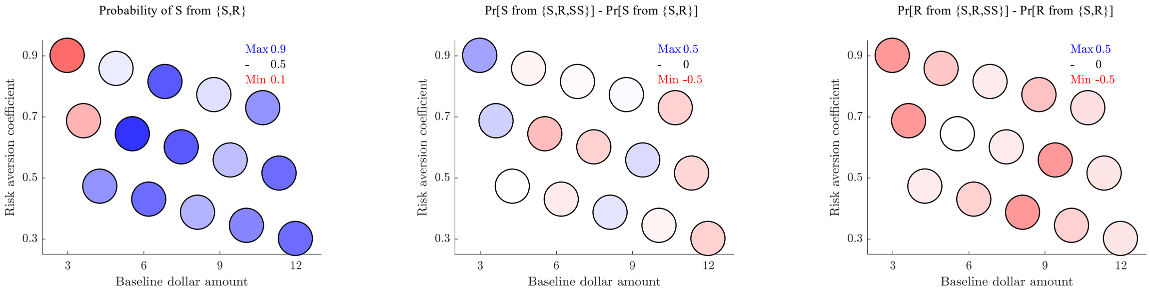
[10] The 7 parameters are $\Pr(S,\{S,R\})$, $\Pr(S,\{S,R,SS\})$ and $\Pr(R,\{S,R,SS\})$ for each of the two types of agents and the weight of each type. The 5 equations are the equations for, on the dataset, $\Pr(S,\{S,R\})$, $\Pr(S,\{S,R,SS\}|S,\{S,R\})$, $\Pr(R,\{S,R,SS\}|S,\{S,R\})$, $\Pr(S,\{S,R,SS\}|S,\{S,R\})$ and $\Pr(R,\{S,R,SS\}|R,\{S,R\})$.

risk aversion.

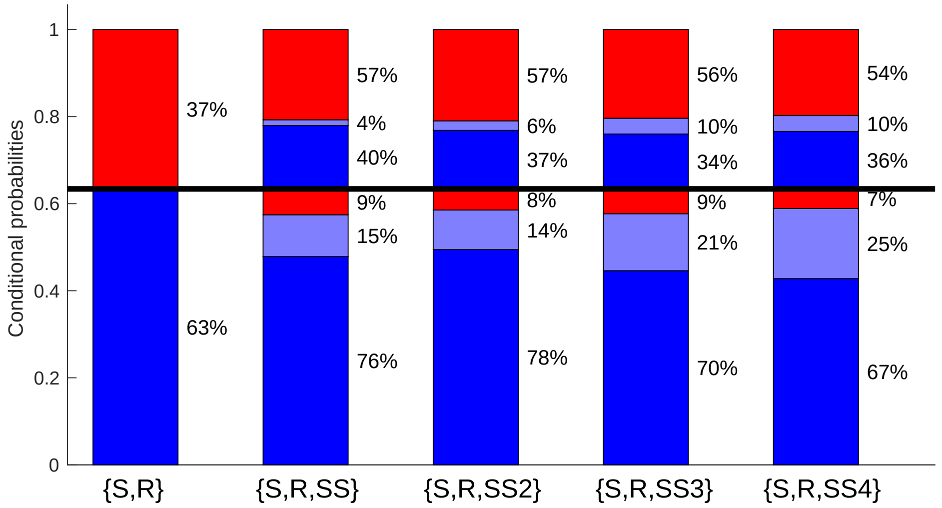


(a) (b) (c)

Each circle is a *condition*, it characterizes a set of $\{S,R,SS\}$ as described in Section 2.2. Figure (a) reports the frequency of S from $\{S,R\}$ from each condition. Figure (b) reports the increase in frequency of S from $\{S,R\}$ to $\{S,R,SS\}$. Figure (c) reports the increase in frequency of R from $\{S,R\}$ to $\{S,R,SS\}$.

Figure 2.5.3: Heatmaps of Choices by Condition

2.6 Appendix



Unconditional probabilities [after excluding decoy]

Other	0	0.11	0.11	0.17	0.2
Risky R	0.37 [0.37]	0.27 [0.3]	0.26 [0.29]	0.26 [0.31]	0.24 [0.3]
Safe S	0.63 [0.63]	0.62 [0.7]	0.63 [0.71]	0.57 [0.69]	0.56 [0.7]

This figure reports panel data on changes in risk aversion. The leftest bar reports the probability that S was chosen (from $\{S,R\}$) in Blue, and R in Red. The second bar reports that, conditional on choosing R from $\{S,R\}$, 57% chose R from $\{S,R,SS\}$, 4% chose SS, and 40% chose S. Similarly, conditional on choosing S from $\{S,R\}$, 9% chose R from $\{S,R,SS\}$, 15% chose SS, and 76% chose S. The third, fourth, and rightest bars report the same information but for $\{S,R,SS2\}$, $\{S,R,SS3\}$, and $\{S,R,SS4\}$.

Figure 2.6.1: Panel Data on Changes in Risk Aversion

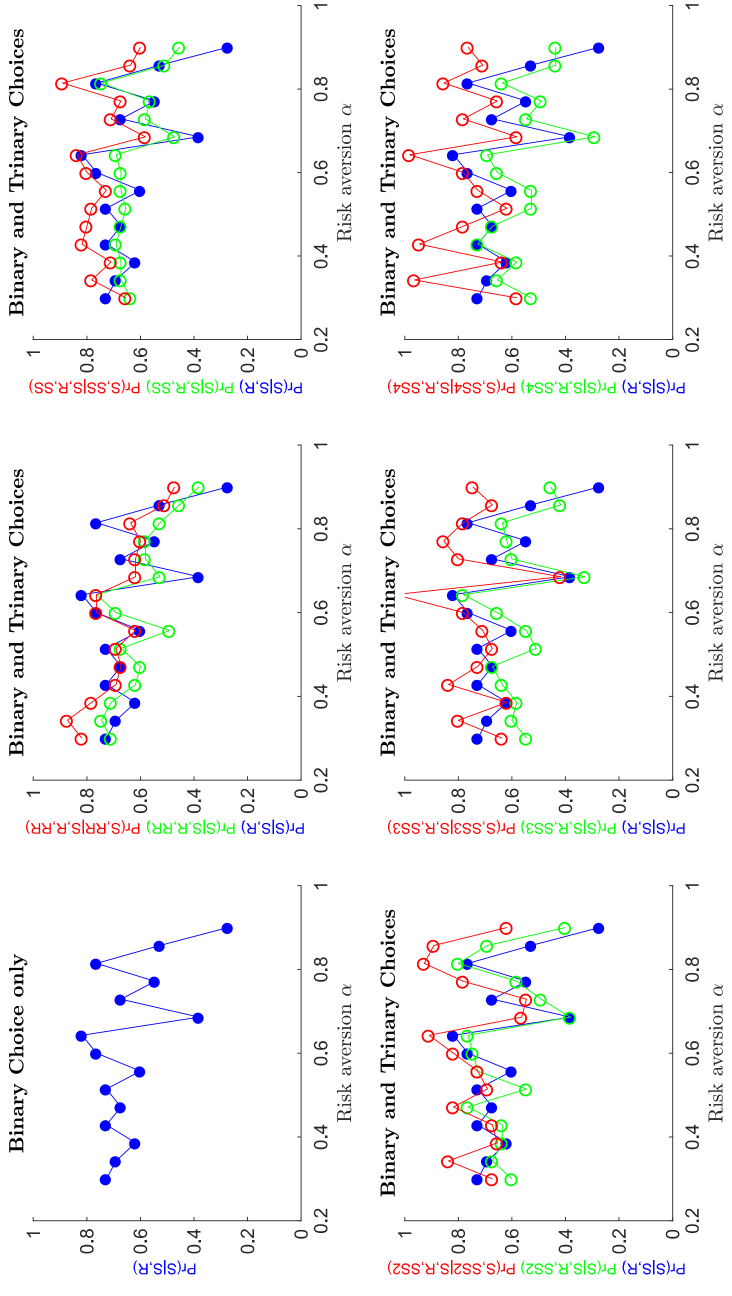
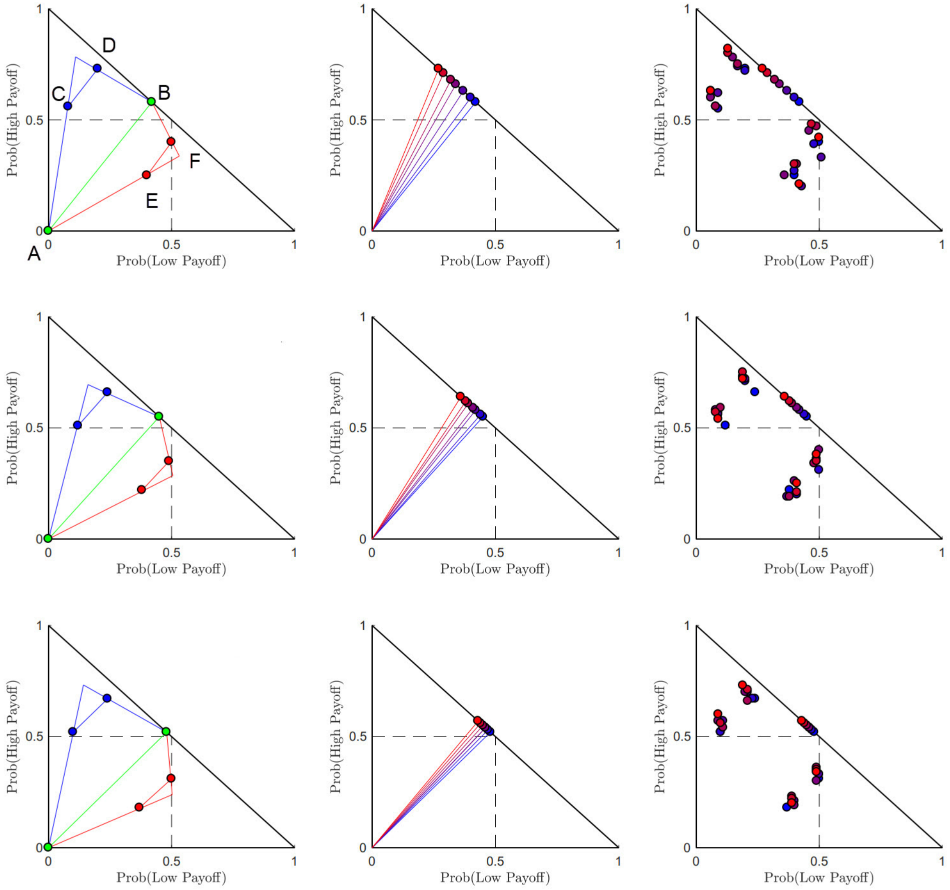
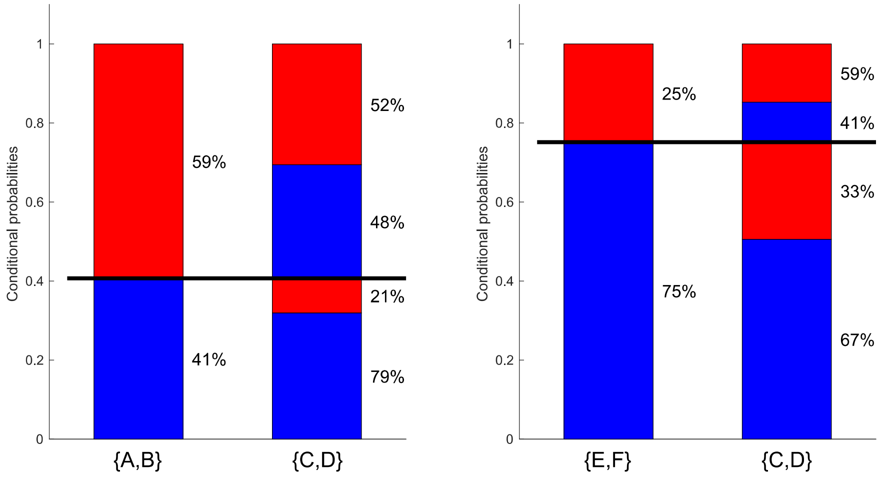


Figure 2.6.2: Choice Frequencies Across Trials Generated with Different CRRA Coefficients



Subjects in our experiment took an Allais Task at the end of the experiment. In this task, they faced multiple choice sets containing lotteries of the same color on the leftest figures. The Green choice set contains degenerate lottery, and no other choice sets do. Blue and Red choice sets are related to the Green choice set by common ratios.

Figure 2.6.3: Trials Generating Process for Allais Task



These figures report the outcome of our Allais Task. C,D correspond to the Green lotteries in Figure 2.6.3, where C is the degenerate lottery. If a subject chooses B from $\{A,B\}$ but C from $\{C,D\}$, the subject committed the Allais paradox, which is a violation of expected utility maximization. In the bars, Blue corresponds to A,C,E and Red corresponds to B,D,F. The left figure, for example, reports that 59 of the choices from $\{A,B\}$ are B, and conditional on choosing B from $\{A,B\}$, 52% chose D from $\{C,D\}$ and 48% chose C.

Figure 2.6.4: Data from Allais Task

CHAPTER 3: CHOICE AND ATTENTION OVER TIME

3.1 Introduction

Past choices may affect future choices, especially when past choices lead to the consideration of alternatives that are otherwise ignored. For example, you may be unaware of this wonderful island called Penang in Malaysia, a great vacation spot, even though it is actually in your choice set. However, if you were attending the Asian Meeting of the Econometric Society and had to choose a place in the region for vacation, Penang may then and forever be an option that you are aware of. Building on this intuition, we provide a model that studies attention across time.

The framework of Limited Attention by Masatlioglu et al. (2012), which does not take choice over time into account, has inspired a strand of literature that attributed some violation of rationality in observed choices to an agent's failure to consider all options, perhaps due to limited attention. In these models, an agent has a "rational" preference ranking over alternatives, but only pays attention to a subset of alternatives, which varies from one choice set to another, leading to irrational choice.

Yet, with choices treated to be independent from one another, models of this kind left an open-ended question about an economist's ability to separately pin down utility functions and consideration sets. This poses a challenge to properly measure welfare—when an agent chooses a over b, was it because she preferred a, or that she did not consider b?

The present work provides a natural solution by analyzing *sequences* of choices. Through a richer setting, for which datasets are increasingly available, we exploit the intuition that the agent is aware of and hence considers the alternatives that she chose in the past. For instance, if an agent was using an iPad, but has now converted to using Surface Go, we conclude that she prefers a Surface Go to an iPad.

Under this model, preference rankings are uniquely pinned down (Proposition 3.1) except for an alternative that is never chosen, which is not the case when sequences of choices are not considered.[1]

3.2 Axioms

3.2.1 Preliminaries

Let X be a countable set of alternatives, and let $\mathcal{A}$ be the set of all finite subsets X that has at least size two. Our primitive (dataset) is a choice function that maps each infinite sequence of choice sets to an infinite sequence of choices, $c : \mathcal{A}^{\mathrm{N}} \to X^{\mathrm{N}}$, where for every sequence of choice sets $(A_n) \in \mathcal{A}^{\mathrm{N}}$ and any natural number k, the corresponding choice is an element of the corresponding choice set: $c\left((A_n)\right)_k \in A_k$.

In this paper, we make the simplifying assumption that the agent acts as if she is not building a bundle, and hence not strategically making choices in anticipation of the type of choice sets she would face in the future. Formally, for all $(A_n), (B_n) \in \mathcal{A}^{\mathrm{N}}$ such that $A_k = B_k$ for all $k \leq K \in \mathbb{N}$, $c\left((A_n)\right)_k = c\left((B_n)\right)_k$ for all $k \leq K$. In other words, if two sequences of choice sets are identical up to a certain point, the corresponding choices up to that point are identical as well.

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[1] By an alternative that is never chosen, we mean an alternative that is not a choice from any choice set in any sequence. There is at most one such alternative, and there is no way to know whether this alternative is worse than other alternatives or simply never paid attention to. This is formalized in Subsection 3.3.2.

With this assumption, we can equivalently characterize c using a set of cross-sectional choice functions: Let $\mathbb{A}$ be the set of all finite sequences where each element is a member of $\mathcal{A}$. Define by $\tilde{c} : \mathbb{A} \times \mathcal{A} \to X$ the cross-sectional choice function that maps a choice set $A \in \mathcal{A}$ to a choice $x \in A$ given a finite sequence of past choice sets $(A_1, ..., A_K) \in \mathbb{A}$. The set of cross-sectional choice functions, varied by elements of $\mathbb{A}$, is fully and uniquely defined using our primitive c, and in fact fully characterizes c. For a fixed finite sequence of past choice sets $(A_1, ..., A_K)$, the choice function $\tilde{c}((A_1, ..., A_K)) : \times\mathcal{A} \to X$ can be viewed as a standard choice function.

3.2.2 Stability

The primary goal of our model is to capture how past choices may affect future choices. Hence, for a *fixed* choice set that the agent faces today, the model allows for *different* choices depending on what past choice sets she faced. Analogously, given a *fixed* sequence of past choice sets, her choices today (from different choice sets) may violate the standard rationality assumption as well. This latter statement corresponds to the lack of restrictions on the cross-sectional choice functions.

Instead, we introduce a notion of rationality *within* each sequence of choices:

Axiom 3.1 (Stability)**.** *For any* $(A_n) \in \mathcal{A}^N$ *and* $k_1 < k_2 < k_3$,

$$c((A_n))_{k_1} = x,\ c((A_n))_{k_2} = y \text{ with } x \in A_{k_2} \text{ and } y \in A_{k_3}$$

implies $c((A_n))_{k_3} \neq x$.

If an agent were fully rational, her behavior across time (i.e. within a sequence of choice sets) should be in full compliance with WARP. However, such a setting would deprive us of any opportunity in separating utility from attention—WARP may be due to standard utility maximization or conveniently paying attention to the right things.[2] On the other hand, if an agent's behavior is completely unrestricted, we are unable to identify any form of utility maximization.

Axiom 3.1 posits that an agent does not make flip-flopping choices. That is, although the agent can "switch" her choice between x and y, she won't be going back and fourth between them.

To illustrate, suppose an agent first chose x in the presence of y, and then chose y in the presence of x. The latter WARP-violating choice may be due to y being in the consideration set now but not before. The axiom permits this behavior, but posits that from here on the choice between x and y stabilizes.

Imagine now that y was not in the earlier choice set, so choosing y (over x) in the latter choice set does not violate WARP. Nonetheless, the same argument holds in that both x and y were chosen in the past and should both be considered going forward, and so future choices between x and y should be consistent.

3.2.3 Past Dependence

Next, we introduce an axiom that allows past choices to affect future choices.

Axiom 3.2 (Past Dependence)**.** *For any* $(A_n) \in \mathcal{A}^N$, $B \in \mathcal{A}$, *and* $K \in \mathbb{N}$,

$$\tilde{c}\left((A_1, \ldots, A_K)\right)(B) \in \tilde{c}\left((A_1, \ldots, A_{K-1})\right)(B) \cup \tilde{c}\left((A_1, \ldots, A_{K-1})\right)(A_K).$$

---

[2] WARP stands for the Weak Axioms of Revealed Preferences. There are many (roughly) equivalent definitions. Here, since we consider choice functions (as opposed to correspondence), we use "if x is chosen in y's presence, then y is never chosen in x's presence".

This postulate says the following: After having faced choice sets $A_1, ..., A_K$, what an agent chooses from B is either (i) what she would have chosen if she had not faced the most recent choice set, that is, $\tilde{c}\left((A_1, ..., A_{K-1})\right)(B)$ or (ii) what she chose from the most recent choice set, that is, $\tilde{c}\left((A_1, ..., A_{K-1})\right)(A_K)$.

If past choices *do not* affect future choices, then $\tilde{c}\left((A_1, ..., A_K)\right)(B) = \tilde{c}\left((A_1, ..., A_{K-1})\right)(B)$; that is, the agent's choice from B does not depend on whether or not she had faced A_k. Axiom 3.2 allows for past choices to affect behavior, but it only permits a very specific type of departure: the new choice is exactly the most recent choice. It seeks to captures the difference between today (the onset of the present choice) and yesterday (prior to the most recent choice), where one difference in the agent's attention structure is that the most recent choice must have gained her attention, and may now be chosen.

While the axiom specializes the effect of past choices on future choices into a framework of attention, it also restricts it to a *verifiable* setting. For instance, another way of modeling attention is to say that every alternative that receives attention in A_K—whether or not it is chosen—also receives the agent's attention when she faces B. However, since the primitive is a dataset of choices as opposed to a dataset of attention, there is no way for an analyst to know which alternatives receive attention without further assumptions, with the sole exception of the choice itself.

3.2.4 Default Attention

Everything starts somewhere. Even without past choices, certain alternatives in a choice set receive attention (at the very least, the one that was chosen). We capture this notion of *default attention* using the following axiom. First, for a fixed c, we define the set of choices without history by the cross-sectional choice function $\tilde{c}(\emptyset) : \mathcal{A} \to X$, where $() \in \mathbb{A}$ is the empty sequence. Consider the following postulate.

Axiom 3.3 (Default Attention)**.** *If* $c\left(\left(A_n\right)\right)_j = x$ *with* $y \in A_j$ *and either*

$$c\left(\left(A_n\right)\right)_i = y \text{ where } i < j \tag{3.2.1}$$

or

$$\tilde{c}\left(\emptyset\right)\left(A_j\right) = y, \tag{3.2.2}$$

then $\tilde{c}\left(\emptyset\right)\left(B_k\right) = x$ *implies* $c\left(\left(B_n\right)\right)_k \neq y$ *for all* $\left(B_n\right) \in \mathcal{A}^N$ *and* $k \in \mathbb{N}$.

The postulate that this axiom characterizes is two-fold.

First, it captures the intuition that if, in the choice set B_k, the agent pays attention to alternative x even when there is no history, then she will always pay attention to x when she faces B_k.

Building on this, it posits that if x is chosen when y receives attention—either because y was chosen in the past (Equation 3.2.1) or y is always considered in the underlying choice set (Equation 3.2.2) —then y should not be chosen from any choice set in which z is always considered.

The fact that certain alternatives acquire attention *by default* is an essential feature of our attention model. It can be thought of as the set of most salient alternative insofar as to attract attention. Without this property, we cannot rule out the possibility that an agent only considers alternatives that were previously chosen, which denies us the opportunity of identifying a consistent preference.

3.3 Model

We are ready for the representation theorem!

3.3.1 Attention Across Time

Definition 3.1. c admits an Attention Across Time (AAT) representation if there exist a *utility function* $u : X \longrightarrow \mathbb{R}$ and a *default attention* set $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}, \Gamma(A) \subseteq A$, such that

$$\tilde{c}\left(\left(A_1, \ldots, A_k\right)\right)(B) = \underset{x \in \tilde{\Gamma}\left(\left(A_1, \ldots, A_k\right)\right)(B)}{\arg\max} u(x)$$

where

$$\tilde{\Gamma}\left(\left(A_1, \ldots, A_k\right)\right)(B) := \Gamma(B) \cup \left[\hat{c}\left(\left(A_1, \ldots, A_k\right)\right) \cap B\right]$$

and

$$\hat{c}\left(\left(A_1, \ldots, A_k\right)\right) := \tilde{c}(\emptyset)\left(A_1\right) \cup \left\{\tilde{c}\left(\left(A_1, \ldots A_{l-1}\right)\right)\left(A_l\right) : l = 2, \ldots, k\right\}.$$

Theorem 3.1. *c satisfies Axioms 3.1-3.3 if and only if it admits an Attention Across Time (AAT) representation.*

AAT prescribes the following choice procedure: When an agent faces choice set B, she considers all of the alternatives that she would always consider by default as well as all of the options that she chose before. Hence, the set of alternatives among which she maximizes utility is defined by a *history-dependent attention* set $\tilde{\Gamma}\left(\left(A_1, \ldots, A_k\right)\right)(B)$, which is the union of the default attention set $\Gamma(B)$ and (history-dependent) past choices $\hat{c}\left(\left(A_1, \ldots, A_k\right)\right)$.

Under this model, the economist knows definitively that an agent prefers a to b if she chose b in the history and chooses a over b now, since both a and b are within the attention set of the latter choice problem. In fact, one way to elicit such preference is to first introduce a choice set under which the agent would choose b, and then ask the agent to choose from a choice set from which she would normally choose a. This is in line with the general technique of *intervention* used in economics research. In our setting, if the underlying issue is that the choice of a over b is subjected to the possibility that b wasn't considered at all, then one resolution is to first draw the agent's attention to b.

3.3.2 Uniqueness

In fact, for any fixed c, utility functions are ordinally unique for all but at most one never-chosen alternative. We formalize this statement:

For a given c, define by $\hat{X} := \{x \in X : c((A_n))_i = x \text{ for some } (A_n) \text{ and } i\}$ the set of alternatives that are *sometimes chosen*. It can be shown that there is at most one alternative that is *never chosen*, that is, $|X \backslash \hat{X}| \leq 1$.[3]

Proposition 3.1. *Suppose c admits an AAT representation with specifications (u_1, Γ_1) and (u_2, Γ_2). For any $x, y \in \hat{X}$, $u_1(x) > u_1(y)$ if and only if $u_2(x) > u_2(y)$.*

In other words, by analyzing choice sequences, our model proposes a solution that resolves not some but all issues of non-uniqueness in an attention framework. On the contrary, if we were to analyze past and future choices in isolation of one another, we may be unable to conclude whether certain choices are the result of preference or lack of attention, which poses a fundamental economics problem that concerns welfare, incentives, and beyond.

In fact, almost all preferences can be identified in as little as a single sequence of choice sets, and this sequence is independent of c. To formalize this statement, recall that if x is chosen over y after y was previously chosen, this allows us to conclude that x is better than y, since y received attention when x was chosen. Formally:

---

[3] Say $z \in X \backslash \hat{X}$, so $\tilde{c}(\emptyset)(\{z, x\}) = x$ for all $x \neq z$, which means $x \in \hat{X}$ for all $x \neq z$.

Definition 3.2. Define $x \succ_{(A_n)} y$ if for some $i < j$, $c\left((A_n)\right)_i = y$ and $c\left((A_n)\right)_j = x$ with $y \in A_j$.

Proposition 3.2. *Suppose X is finite. There exists (A_n) such that for any choice function c that admits an ATT representation, there exists a subset of alternatives $\bar{X} \subseteq X$ of size $|\bar{X}| \geq |X| - 1$ such that $\left(\succ_{(A_n)}, \bar{X}\right)$ is complete and transitive.*

This captures the simple intuition that if we ask the right questions in the right order, we can essentially force an agent to pay attention to the universe of available alternatives. Consequently, this allows us to identify preferences.

3.3.3 Features of AAT

The fact that utility functions can be pinned down is closely linked to a feature, also a testable prediction, of the model:

Proposition 3.3. *Suppose c admits an AAT representation. If for some $(A_n) \in \mathcal{A}^{\mathbf{N}}$ and $i < j$,*

$$c\left((A_n)\right)_i = x,\ c\left((A_n)\right)_j = y,\ \text{and } x \in A_j,$$

*then for all $(B_n) \in \mathcal{A}^{\mathbf{N}}$ and $k < l$, it is **not** the case that*

$$c\left((B_n)\right)_k = y,\ c\left((B_n)\right)_l = x,\ \text{and } y \in B_l.$$

This results states that, between any two sequences of choices, if in one of the sequences the agent first chose x and then chose y in the presence of x, then it is never the case that she would first choose y but then x in the presence of y in the other sequence. This provides a notion in which the choices between x and y, albeit involving switches, only involves switches in one direction. Upon that switch, we learn the agent's preference between x and y.

Proposition 3.3 applies to the case of $(A_n) = (B_n)$, a special case that implies Axiom 3.1.

Immediate corollaries are available from Proposition 3.3 to outline simple and intuitive testable predictions; we provide two:

Corollary 3.1. *Suppose c admits an AAT representation.*

1. *If $x, y \in \hat{c}((A_1, \dots, A_k))$ and $\{x, y\} \subseteq B \cap D$, then $\tilde{c}((A_1, \dots, A_k))(B) = x$ implies $\tilde{c}((A_1, \dots, A_k))(D) \neq y$.*
2. *If $\{x, y\} \subseteq \hat{c}((A_1, \dots, A_k)) \cap \hat{c}\left(\left(A'_1, \dots, A'_l\right)\right)$, then $\tilde{c}((A_1, \dots, A_k))(B) = x$ implies $\tilde{c}\left(\left(A'_1, \dots, A'_l\right)\right)(B) \neq y$.*

First, if both x and y were chosen in the past, they should be considered in any future choice sets. Hence when they are both present, whether it is in choice set B or choice set D, the choice between x and y is consistent in the sense that the agent will not choose x over y in one choice set but chooses y over x in another (although she is allowed to choose something else). In other words, choices between chosen options will satisfy WARP. Analogously, for a fixed present choice set B, the choice between x and y does not depend on what past choice sets were faced as long as both x and y were chosen in the past.

3.4 AAT and Choice with Limited Attention

This paper introduces a model that complements, instead of competes with, other attention-based models in which choices are assumed to take place independently. We study such complementarity with Masatlioglu et al. (2012)'s Choice with Limited Attention (CLA) next.

3.4.1 Choice with Limited Attention

Recall that $\tilde{c}(\emptyset) : \mathcal{A} \to X$ is the choice function *without history*. Our model, AAT, puts no restrictions on $\tilde{c}(\emptyset) : \mathcal{A} \to X$.[4] On the other hand, Masatlioglu et al. (2012) introduces a model also based on attention but does not consider sequences of choices; instead, it puts restrictions on $\tilde{c}(\emptyset) : \mathcal{A} \to X$.

We now reconcile two, demonstrate that they are compatible with each other, and argue that this finding is intuitive.

3.4.2 AAT with CLA

Definition 3.3 (Masatlioglu et al. (2012))**.** A regular choice function $\bar{c} : \mathcal{A} \to X$ is a choice with limited attention (CLA) if there exist $\bar{u} : X \to \mathbb{R}$ and $\bar{\Gamma} : \mathcal{A} \to \mathcal{A}$ such that

$$\bar{c}(A) = \underset{x \in \bar{\Gamma}(x)}{\arg\max} \bar{u}(x)$$

and $\bar{\Gamma}$ is an *attention filter*, that is, $y \notin \bar{\Gamma}(A)$ implies $\bar{\Gamma}(A \setminus \{y\}) = \bar{\Gamma}(A)$.

In Masatlioglu et al. (2012), an option x is inferred to be preferred to another option y if, in some choice set where x is chosen in the presence of y, dropping y results in x no longer chosen. This refines the standard definition that simply requires x to be chosen in the presence of y, since we do not know whether y is considered or not. The underlying intuition is that, if y were not initially considered, then dropping y should not result in a change in the consideration set, and hence x should remain chosen. If instead x is no longer chosen, it must be that y was considered, and hence x is inferred to be revealed preferred to y.

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[4] That is, if $f : \mathcal{A} \to X$ is a choice function, there exists c such that c admits an AAT representation where $\tilde{c}(\emptyset)(A) = f(A)$ for all $A \in \mathcal{A}$.

We introduce an axiom that substantiates this inference with actual choices. That is, if x is inferred to be preferred to y a la Masatlioglu et al. (2012), then this preference is *manifested* in future choices.

Recall that $\hat{X} := \{x \in X : c\left(\left(A_n\right)\right)_i = x \text{ for some } \left(A_n\right) \text{ and } i\}$ is the set of alternatives that are *sometimes chosen*, and its complement $X \backslash \hat{X}$ is the set of lotteries that are *never chosen*.

Axiom 3.4. *Suppose $\tilde{c}\left(\emptyset\right)\left(T\right) = x$ and $\tilde{c}\left(\emptyset\right)\left(T \backslash \{y\}\right) \neq x$. If $c\left(\left(A_n\right)\right)_i = x$, then $c\left(\left(A_n\right)\right)_j \neq y$ for all $j > i$ such that $x \in A_j$.*

This postulate reflects the following intuition: If x is revealed preferred to y a la Masatlioglu et al. (2012), and this method of revealed preference is true, then we should *never* observe a sequence of choices where x was chosen in the past but y is chosen over x in the future. It turns out that this is the sufficient and necessary for the default attention set Γ in ATT to be an attention filter.

Theorem 3.2. *c satisfies Axioms 3.1-3.4 if and only if it admits an Attention Across Time (AAT) representation where Γ is an attention filter.*

Although *attention* is a shared foundation for AAT and CLA, the two models infer attention, and hence revealed preference, in different ways. In CLA, we infer that y receives attention because dropping it from a choice set changes the choice; in AAT, we infer that y receives attention because it is chosen in the past. Theorem 3.2 reconciles these two methods of inferring attention.

This reconciliation allows us to study how attention at a given time is related to attention across time. For example, it allows us to use a future choices to provide direct evidence that certain alternatives received attention in the past—even if they were not chosen. In Masatlioglu et al. (2012)'s CLA, revealed preference relies on the assumption that dropping alternatives that are not considered would not result in a change in the attention set. When we take choice sequences into account, we might wonder if the fact that x is revealed preferred to y in the sense of CLA means we will observe evidence of x chosen over y after y was chosen in the past. It turns out that the answer is yes:

Proposition 3.4. *Suppose c admits an ATT representation where Γ is an attention filter. If $\tilde{c}(\emptyset)(T) = x$ and $\tilde{c}(\emptyset)(T \setminus \{y\}) \neq x$ and $y \in \hat{X}$, then there exists a sequence of choice sets (A_n) such that $c((A_n))_i = y$ and $c((A_n))_j = x$ where $y \in A_j$ and $i < j$.*

3.4.3 Local CLA

In fact, if c admits an AAT representation where Γ is an attention filter, then every cross-sectional choice function is a CLA. Furthermore, this allows us to recover CLA's axiom, WARP(LA), showing again that a dataset of choice sequences can manifest the underlying intuition of CLA.

Definition 3.4 (Masatlioglu et al. (2012))**.** A cross-sectional choice function $\tilde{c} : \mathcal{A} \to X$ satisfies WARP with Limited Attention (WARP(LA)) if for any $S \in \mathcal{A}$, there exists $x^* \in S$ such that, for any T including x^*,

$$\text{if } \tilde{c}(T) \in S \text{ and } \tilde{c}(T) \neq \tilde{c}(T \setminus \{x^*\}), \text{ then } \tilde{c}(T) = x^*.$$

Proposition 3.5. *Suppose c admits an AAT representation where Γ is an attention filter, then*

1. *$\tilde{c}(\emptyset) : \mathcal{A} \to X$ is a CLA and satisfies WARP(LA).*
2. *Moreover, for any history $(A_1, \ldots, A_n) \in \mathbb{A}$,*
    a) *$\tilde{\Gamma}((A_1, \ldots, A_n)) : \mathcal{A} \to \mathcal{A}$ is also an attention filter,*

b) $\tilde{c}((A_1,\ldots,A_n)) : \mathcal{A} \to X$ *is also a CLA and satisfies WARP(LA).*

Part 1 of Proposition 3.5 connects a particular type of attention set, the *attention filter* of Masatlioglu et al. (2012), to our model. Under AAT, if Γ is an *attention filter*, then choices are in line with CLA. This straightforward observation demonstrates that AAT can be thought of as an enriching framework for other attention-based models in which history is not considered. Since any history-less choice functions $\tilde{c}(\emptyset) : \mathcal{A} \to X$ can be admitted, AAT is applicable to wide range of models that are used to explain choices without taking past choices into account.

Instead, AAT places restrictions on future behavior based on past choices. In the case of CLA, this leads to Part 2 of the proposition. Given any history of past choices sets $(A_1,\ldots,A_n)$, the present (regular) choice function is also a CLA. In other words, if an agent takes history into account when making choices, and the default attention set happens to be an attention filter, then her behavior would be in compliance with CLA at every step of the time.

Moreover, since we obtained this form of local CLA using a separate behavioral postulate that is based on choices over time, Axiom 3.4, this can be viewed as an external justification for WARP(LA), and captures how WARP(LA) would be manifested in future choices.[5]

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[5] For interested readers, it can be shown that Axiom 3.1-Axiom 3.3 and WARP(LA) (for $\tilde{c}(\emptyset) : \mathcal{A} \to X$) is *not* sufficient for an AAT representation where Γ is an attention filter. We provide one counterexample: Let $X = \{x,y,z\}$. Take c represented by AAT with specifications $u(z) > u(x) > u(y)$ and $\Gamma(\{x,y,z\}) = \{x\}$, $\Gamma(\{x,y\}) = \Gamma(\{y,z\}) = \{y\}$, $\Gamma(\{x,z\}) = \{z\}$. Since $\tilde{c}(\emptyset)(\{x,y,z\}) = x$ and $\tilde{c}(\emptyset)(\{x,y\}) \neq x$, a CLA representation (which is implied by an AAT representation where Γ is an attention filter) requires $u(x) > u(z)$. Separately, since $\tilde{c}(\emptyset)(\{x,y,z\}) = x$ and $\tilde{c}(\{x,y,z\})(\{x,z\}) = z$, $u(z) > u(x)$ is necessary in any AAT representation. So c does not admit an AAT representation where Γ is an attention filter, although $\tilde{c}(\emptyset) : \mathcal{A} \to X$ satisfies WARP(LA) since it is a CLA with specifications $u(x) > u(z) > u(y)$ and $\Gamma(\{x,y,z\}) = \{x,y,z\}$, $\Gamma(\{x,y\}) = \Gamma(\{y,z\}) = \{y\}$, $\Gamma(\{x,z\}) = \{z\}$.

3.5 Extension to Choice Correspondences

Indifferences can be accommodated in ATT, where we assume that when facing a future choice problem, the decision maker considers, from every past choice set A, both the choice and all other alternatives that were indifferent to it.

3.5.1 Preliminaries

We begin by making the following changes to the preliminaries (X remains as a countable set of alternatives): Let $C : \mathcal{A}^{\mathrm{N}} \to \mathcal{A}^{\mathrm{N}}$, $C\left((A_n)\right)_i \subseteq A_i$ be a choice correspondence, therefore more than one choice can be chosen from any choice set in any sequence of choice sets. Like before, we assume *future independence*: for all $(A_n), (B_n) \in \mathcal{A}^{\mathrm{N}}$ such that $A_k = B_k$ for all $k \leq K \in \mathbb{N}$, we have $C\left((A_n)\right)_k = C\left((B_n)\right)_k$ for all $k \leq K$. In other words, if two sequences of choice sets are identical up to a certain point, the corresponding choices up to that point are identical as well. Finally, based on C we also define the cross-sectional choice correspondence $\tilde{C} : \mathbb{A} \times \mathcal{A} \to \mathcal{A}$, which maps a choice set $A \in \mathcal{A}$ to a choice $x \in A$ given a finite sequence of past choice sets $(A_1, \ldots, A_K) \in \mathbb{A}$.

3.5.2 Axioms

We modify Axioms 3.5-3.7 to accommodate indifferences. Additionally, we introduce a new axiom, Axiom 3.8, which captures the basic intuition that if x is indifferent to a, and y is indifferent to b, then having observed a switch from y to x means we won't observe a switch from b to a.

The first two modifications are straightforward adaptations from Axiom 3.1 and Axiom 3.2 to choice correspondences:

Axiom 3.5 (Stability*)**.** *For any* $(A_n) \in \mathcal{A}^N$ *and* $k_1 < k_2 < k_3$,

$$x \in C\left((A_n)\right)_{k_1}, y \in C\left((A_n)\right)_{k_2} \text{ with } x \in A_{k_2} \backslash C\left((A_n)\right)_{k_2} \text{ and } y \in A_{k_3}$$

implies $x \notin C\left((A_n)\right)_{k_3}$.

Axiom 3.6 (Past Dependence*)**.** *For any* $(A_n) \in \mathcal{A}^N$, $B \in \mathcal{A}$, *and* $K \in \mathbb{N}$,

$$\tilde{C}\left((A_1, \ldots, A_K)\right)(B) \subseteq \tilde{C}\left((A_1, \ldots, A_{K-1})\right)(B) \cup \tilde{C}\left((A_1, \ldots, A_{K-1})\right)(A_K).$$

Next we modify Axiom 3.3, but in addition to simply rewriting it for choice correspondences, we add a property that extends the intuition of default attention to alternatives that are indifferent:

Definition 3.5. Define xWy if for some $(A_n) \in \mathcal{A}^{\mathbf{N}}$ and $i \in \mathbb{N}$, $\{x, y\} \subseteq C\left((A_n)\right)_i$.

Axiom 3.7 (Default Attention*)**.** *If* $x \in C\left((A_n)\right)_j$, $y \in A_j \backslash C\left((A_n)\right)_j$ *and at least one of the following holds:*

1. $y \in C\left((A_n)\right)_i$ *where* $i < j$,
2. $y \in \tilde{C}(\emptyset)\left(A_j\right)$.

Then for all x^*Wx *and* y^*Wy, $x^* \in \tilde{C}(\emptyset)(B_k) \cup C\left((B_n)\right)_k$ *implies* $y^* \notin C\left((B_n)\right)_k$ *for all* $(B_n) \in \mathcal{A}^N$ *and* $k \in \mathbb{N}$.

Finally we add an axiom that captures the crucial restriction that reflects a consistent preference ranking—if x is indifferent to a, and y is indifferent to b, then having observed a switch from y to x means we won't observe a switch from b to a.

Definition 3.6. Define aSx if for some $a \in C\left(\left(A_n\right)\right)_i$, $x \in c\left(\left(A_n\right)\right)_j$, and $a \in A_j \backslash c\left(\left(A_n\right)\right)_j$ for some $i < j$.

Axiom 3.8. *Suppose xWa and yWb. If ySx, then not aSb.*

We are ready for the representation theorem.

Definition 3.7. C admits an Attention Across Time (AAT) representation if there exist a *utility function* $u : X \longrightarrow \mathbb{R}$ and a *default attention* set $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$, $\Gamma(A) \subseteq A$, such that

$$\tilde{C}\left(\left(A_1, \ldots, A_k\right)\right)(B) = \underset{x \in \tilde{\Gamma}\left(\left(A_1, \ldots, A_k\right)\right)(B)}{\arg\max} \; u(x)$$

where

$$\tilde{\Gamma}\left(\left(A_1, \ldots, A_k\right)\right)(B) := \Gamma(B) \cup \left[\hat{C}\left(\left(A_1, \ldots, A_k\right)\right) \cap B\right]$$

and

$$\hat{C}\left(\left(A_1, \ldots, A_k\right)\right) := \tilde{C}(\emptyset)\left(A_1\right) \cup \left\{\tilde{C}\left(\left(A_1, \ldots A_{l-1}\right)\right)\left(A_l\right) : l = 2, \ldots, k\right\}.$$

Theorem 3.3. *C satisfies Axioms 3.5-3.8 if and only if it admits an Attention Across Time (AAT) representation.*

3.6 Related Literature

The basic intuition that switches reveal preferences is not new. Caplin & Dean (2011) uses this idea to study search and satisficing. Built on a dataset where agents contemplate on how to choose from a single choice set, where search is observed through temporary choices, their models of alternative-based search (ABS) and reservation-based search (RBS) intersect with AAT only in the trivial case where there are no switches to begin with.

This technical distinction is wholly accounted for by the fact that the models are built on two different primitives as they seek to study two fundamentally different behavior. In ABS and RBS, the agent faces just one choice set, but her decision evolves as she searches through the choice set. A "switch" in this setting occurs when an option was temporary chosen during this choice process but is not the final choice, as it was replaced by an option that is *revealed better*. In our model, AAT, a switch never occurs when the underlying choice set does not change. Instead, it is precisely because the underlying choice set has changed that other options have gained the agent's attention.

This conceptual difference is also reflected by the technology used to characterize the respective behaviors. The key axiom in Caplin & Dean (2011) is acyclicity of switches (with an additional requirement for RBS to capture satisficing). While acyclicity of switches is necessary for AAT, it is not sufficient: The evolution of attention is fully prescribed in AAT, which is unlike ABS and RBS's search processes. Consequently, preference rankings are generically pinned down under AAT but not the in other two models.[6]

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[6] Proposition 3.1 shows that the preference ranking between any two alternatives is pinned down, other than for (at most) one alternative that is *never chosen* in any choice set of any sequence.

3.7 Proofs

3.7.1 Notations

First we introduce simplifying notations.

1. We use $\boldsymbol{A}$ to denote a sequence of choice sets (of any length).
2. Denote by $[\boldsymbol{A}]_{t=k}^{l}$ the subsequence of $\boldsymbol{A}$ including only elements in positions k to l (if $k = l$, this is a sequence of length 1).
3. We use upper-case characters (without bold) to represent choice sets.
    a) When we say $A \in \boldsymbol{A}$, it means a choice set in the sequence of choice sets $\boldsymbol{A}$.
4. ABC represents the sequence of choice sets that starts with A, followed by B, and ends with C.
5. We use lower-case characters to represent alternatives.
    a) When we say Ax, we mean x chosen from the set A which in this case is it's own sequence.
6. We may also say $A^{\ni y}x$, which additionally indicates that y was in A.
    a) So $AxByC^{\ni x}z$ means a sequence of choice sets ABC from which x, y, z are chosen respectively, and x was also in C.
7. Sometimes, we say $\exists ABy$, which means there exists a sequence that starts with A, followed by B, from which y is chosen.
    a) Likewise, $\boldsymbol{A}By$ means a sequence of choice sets $\boldsymbol{A}$, followed by B, from which y is chosen.

b) Finally we may write $\boldsymbol{A}xzBy$ to characterize a sequence of choice sets $\boldsymbol{A}$, from which x is chosen from some $A \in \boldsymbol{A}$ and z is chosen from some $C \in \boldsymbol{A}$ (in no particular order), followed by B from which y is chosen.

3.7.2 Implications of Axioms

Condition 3.1. If $\boldsymbol{A}Bx$, either Bx or $\boldsymbol{A}x$.

Lemma 3.1. *If c satisfies Axiom 3.2, then c satisfies Condition 3.1.*

Proof. Take $\boldsymbol{A}Bx$, and suppose not Bx. Let K be the length of $\boldsymbol{A}$. By Axiom 3.2, either $[\boldsymbol{A}]_{t=1}^{K-1}\,[\boldsymbol{A}]_{t=K}^{K}\,x$ or $[\boldsymbol{A}]_{t=1}^{K-1}\,Bx$ (or both). If it is the former, we are done since $\boldsymbol{A}x$. Suppose it is the latter, then by Axiom 3.2 again we have either $[\boldsymbol{A}]_{t=1}^{K-2}\,[\boldsymbol{A}]_{t=K-1}^{K-1}\,x$ or $[\boldsymbol{A}]_{t=K}^{K-2}\,Bx$ (or both). Again, if it is the former we are done, otherwise we keep moving backward until we find q such that $[\boldsymbol{A}]_{t=1}^{q}\,[\boldsymbol{A}]_{t=q+1}^{q+1}\,x$. This process must end when $q = 0$, at which point it must be that $[\boldsymbol{A}]_{t=1}^{1}\,x$, so $\boldsymbol{A}x$. □

Condition 3.2. If $\boldsymbol{A}x$, then Bx for some B.

Lemma 3.2. *If c satisfies Axiom 3.2, then c satisfies Condition 3.2.*

Proof. This is due to Condition 3.1. Say $\boldsymbol{A}x$, and in particular x is chosen from the K-th element, or equivalently $[\boldsymbol{A}]_{t=1}^{K-1}\,[\boldsymbol{A}]_{t=K}^{K}\,x$. By Condition 3.1, either $[\boldsymbol{A}]_{t=K}^{K}\,x$ or $[\boldsymbol{A}]_{t=1}^{K-1}\,x$. If the former, let $B = [\boldsymbol{A}]_{t=K}^{K}$ and we are done. If the latter, by Condition 3.1 again, either $[\boldsymbol{A}]_{t=K-1}^{K-1}\,x$ or $[\boldsymbol{A}]_{t=1}^{K-2}\,x$. If the former, let $B = [\boldsymbol{A}]_{t=K-1}^{K-1}$ and we are done. Otherwise we keep going backward until we find q such that $[\boldsymbol{A}]_{t=q}^{q}\,x$. This process must end when $q = 0$, at which point it must be that $[\boldsymbol{A}]_{t=1}^{1}\,x$, so Bx where $B = [\boldsymbol{A}]_{t=1}^{1}$. □

Condition 3.3. If $\boldsymbol{A}xyBx$, then not $\boldsymbol{C}xyBy$.

Lemma 3.3. *If c satisfies Axiom 3.1 and Axiom 3.3, then c satisfies Condition 3.3.*

Proof. Suppose for contradiction $\boldsymbol{A}xyBx$ and $\boldsymbol{C}xyBy$, then in particular $x, y \in B$. Say without loss of generality $\{x, y\}x$, that is, x is chosen from $\{x, y\}$ without history. Now suppose $\boldsymbol{A}xyBx\,\{x, y\}\,z_1$ and $\boldsymbol{C}xyBy\,\{x, y\}\,z_2$. There are two cases:

1. Say $z_1 = z_2$. If $z_1 = z_2 = y$, then $\boldsymbol{A}xyBx\,\{x, y\}\,z_1$ violates Axiom 3.1. In instead $z_1 = z_2 = x$, then $\boldsymbol{C}xyBy\,\{x, y\}\,z_2$ violates Axiom 3.1.
2. Say $z_1 \neq z_2$, and say $z_1 = y$. Then since $\{x, y\}x$ and $\boldsymbol{A}xyBx\,\{x, y\}\,y$, Axiom 3.3 implies $\nexists \boldsymbol{D}y\,\{x, y\}\,x$, but $\boldsymbol{C}xyBy\,\{x, y\}\,x$ is a contradiction. If instead $z_1 = x$, then since $\{x, y\}x$ and $\boldsymbol{C}xyBy\,\{x, y\}\,y$, Axiom 3.3 implies $\nexists \boldsymbol{D}y\,\{x, y\}\,x$, but $\boldsymbol{A}xyBx\,\{x, y\}\,x$ is a contradiction.

□

Condition 3.4. Let $\{c(\boldsymbol{A})\}$ denote the (unordered) set of choices made from the sequence of choice sets $\boldsymbol{A}$. If $\{c(\boldsymbol{A})\} = \{c(\boldsymbol{B})\}$, then $\tilde{c}(\boldsymbol{A})(A) = \tilde{c}(\boldsymbol{B})(A)$ for all A.

Condition 3.4 states that the choice in A depends on the history only insofar as it depends on the set of realized choices in the history, and not specifically on past choice sets, nor the order of said choices. That is, if two history $\boldsymbol{A}$ and $\boldsymbol{B}$ resulted in identical sets of choices $\{c(\boldsymbol{A})\} = \{c(\boldsymbol{B})\}$, then the choice from A would be the same ($\tilde{c}(\boldsymbol{A})(A) = \tilde{c}(\boldsymbol{B})(A)$).

Lemma 3.4. *If c satisfies Condition 3.1, Condition 3.3, and Axiom 3.3, then c satisfies Condition 3.4.*

Proof. Say Az. By Condition 3.1, $\tilde{c}(\boldsymbol{A})(A), \tilde{c}(\boldsymbol{B})(A) \in \{c(\boldsymbol{A})\} \cup \{z\}$. Let $\tilde{c}(\boldsymbol{A})(A) = c_1$ and $\tilde{c}(\boldsymbol{B})(A) = c_2$, and suppose for contradiction $c_1 \neq c_2$. Say $c_1, c_2 \in \tilde{c}(\boldsymbol{A})$, then Condition 3.3 Part 1 is violated. Instead, suppose without loss of generality $c_1 \notin \tilde{c}(\boldsymbol{A})$, so $c_1 = z$ and

$c_2 \in \tilde{c}(\boldsymbol{A})$ by Condition 3.1. Since Az and $\boldsymbol{B}Ac_2$ where $c_2 \neq z$, Axiom 3.3 implies $\nexists \boldsymbol{D}c_2Az$, but since $c_2 \in \tilde{c}(\boldsymbol{A})$ and $\tilde{c}(\boldsymbol{A})(A) = c_1 = z$, $\boldsymbol{A}c_2Ac_1$ is a contradiction. □

Condition 3.5. If $\boldsymbol{A}yB^{\ni y}x$, then not $\boldsymbol{C}xD^{\ni x}y$.

Lemma 3.5. *If c satisfies Axiom 3.1 and Condition 3.3, then c satisfies Condition 3.5.*

Proof. Suppose for contradiction $\boldsymbol{A}yB^{\ni y}x$ and $\boldsymbol{C}xD^{\ni x}y$. Define $\boldsymbol{E} = \boldsymbol{A}B$ and $\boldsymbol{F} = \boldsymbol{C}D$. So $\boldsymbol{E}xy$ and $\boldsymbol{F}xy$. Consider $\boldsymbol{E}xy\{x,y\}z_1$ and $\boldsymbol{F}xy\{x,y\}z_2$. Clearly, $z_1, z_2 \in \{x,y\}$. Hence by Condition 3.3, $z_1 = z_2$. Suppose without loss of generality that $z_1 = z_2 = x$, and hence $\boldsymbol{C}xD^{\ni x}y\{x,y\}^{\ni y}x$. This is an event of a back-and-fourth switching, which violates Axiom 3.1. □

Corollary 3.2. *Say c satisfies Condition 3.5.*

1. *If $\boldsymbol{A}xyB^{\ni y}x$, then not $\boldsymbol{A}xyD^{\ni x}y$.*

2. *If $\boldsymbol{A}xyB^{\ni y}x$, then not $\boldsymbol{C}xyB^{\ni y}x$.*

Proof. This follows immediately from Condition 3.5 □

Condition 3.6. If $AyBy$, then not $BzAz$.

Lemma 3.6. *If c satisfies Axiom 3.3, then c satisfies Condition 3.6.*

Proof. This is a direct implication of Axiom 3.3: Suppose for contradiction $AyBy$ and $BzAz$. So $y, z \in A, B$. Also, Ay and Bz. $BzAz$ combined with Ay is a direct violation of Axiom 3.3 under $AyBy$. □

The intuition is straightforward: Under $AyBy$ and Bz, we learn that y is revealed better than z since z would have been chosen in B without history but the choice of y from A prior to facing B induced a choice of y in B. Hence, there must not exist $\boldsymbol{A}A^{\ni y}z$ where A is a choice set in which y always receives attention.

Condition 3.7. If $\{x,y\}\,x$ and $A\,\{x,y\}\,y$ for some A, then $\{x,y\}\,xAy\,\{x,y\}\,x$, and hence there exists $\boldsymbol{B}xCy^{\ni x}$ for some $\boldsymbol{B}, C$.

Lemma 3.7. *If c satisfies Condition 3.1 and Axiom 3.3, then c satisfies Condition 3.7.*

Proof. Suppose $\{x,y\}\,x$ and $A\,\{x,y\}\,y$ for some A. By Condition 3.1, Ay. By Condition 3.1 again, either $\{x,y\}\,xAy$ or $\{x,y\}\,xAx$. The latter would violate Condition 3.6 (which by Lemma 3.6 is implied by Axiom 3.3), since $\{x,y\}\,xAx$ and $Ay\,\{x,y\}\,y$. Hence, $\{x,y\}\,xAy$. Consider $\{x,y\}\,xAy\,\{x,y\}\,z$. Clearly, $z \in \{x,y\}$. Since $\{x,y\}\,x$ and $A\,\{x,y\}\,y$ for some A, Axiom 3.3 implies that $\nexists \boldsymbol{C}y\,\{x,y\}\,x$. So $z = y$. To conclude, $\{x,y\}\,xAy\,\{x,y\}\,x$, and we constructed an example of $\boldsymbol{B}xCy^{\ni x}$ where $\boldsymbol{B} = \{x,y\}\,A$ and $C = \{x,y\}$. □

3.7.3 Proof of Theorem 3.1

Fix c. As is common, necessity is straightforward, and we will focus on showing sufficiency. The game plan goes as follows. Stage 1, we construct $\succ$. Stage 2, we show that the constructed $\succ$ has some desirable properties. Stage 3, we construct Γ and show that $(\succ, \Gamma)$ explains choices. Let's begin!

Stage 1, Construction of $\succ$ First we partition alternatives $x \in X$ into two parts: The group where an alternative is sometimes chosen, $\hat{X} := \{x \in X : x = c\left((A_n)\right)_i \text{ for some } (A_n) \text{ and } i\}$, and the group where an alternatives is never chosen, $X \backslash \hat{X}$. By Condition 3.2, $\hat{X}$ can be equivalently characterized by $\{x \in X : x = \tilde{c}\,(\emptyset)\,(A) \text{ for some } A \in \mathcal{A}\}$; that is $x \in \hat{X}$ if and only if $\exists A \in \mathcal{A}$ such that Ax. Moreover, $|X \backslash \hat{X}| \leq 1$: say $z \in X \backslash \hat{X}$, so $\{z,x\}\,x$ for all $x \neq z$, which means $x \in \hat{X}$ for all $x \neq z$.

Fix any two alternatives $x, y \in \hat{X}$, and suppose without loss of generality that $\{x,y\}\,x$. Fix any A such that Ay.

1. If $Ay\,\{x,y\}\,x$, we set $x \succ_S y$. (We use subscript S to denote an observed *switch*.)
2. If $Ay\,\{x,y\}\,y$, we set $y \succ_D x$. (We use subscript D to denote, in the absence of a switch, we know y is better cause x is paid attention to *by default* in $\{x,y\}$.)

Note that the conclusion of whether $x \succ_S y$ or $y \succ_D x$ does not depend on A as long as Ay. Suppose not, so for some A we have $Ay\,\{x,y\}\,x$ and for some A' we have $A'y\,\{x,y\}\,y$, this violates Condition 3.4.

Moreover, note that for every pair $x, y \in \hat{X}$, either $x \succ_S y$ or $y \succ_D x$ (and never both), meaning the joint relation $\succ_S \cup \succ_D$ is complete and antisymmetric (not $x \succ y$ and $y \succ x$) on $\hat{X}$.

For every pair x, y such that $x \in \hat{X}$ and $y \in X \backslash \hat{X}$, set $x \succ_P y$.

Finally, for any $x \in X$, set $x \succ_R x$.

Stage 2, Properties of $\succ$ We argued that $\succ_S \cup \succ_D$, a subset of the Cartesian product $\hat{X} \times \hat{X}$, is complete and antisymmetric. Note that $\succ_P$, a subset of $\hat{X} \times X \backslash \hat{X}$, is complete by construction and clearly antisymmetric. Hence the joint relation $\succ := \succ_S \cup \succ_D \cup \succ_P \cup \succ_R$, a subset of $X \times X$, is complete, antisymmetric, and reflexive.

We will soon argue that $\succ$ is transitive; but first we make the following observations.

Lemma 3.8. *Suppose c satisfies Condition 3.7 and Condition 3.5. If either $x \succ_S y$ or $x \succ_D y$, then there exists $\mathbf{A}yB^{\ni y}x$. Moreover, if $x \succ_i y$, $y \succ_j z$, $z \succ_k x$ where $i, j, k \in \{S, D\}$, then $\nexists \mathbf{C}xyz$.*

Proof. First we prove the first statement. Say $x \succ_S y$, then by the construction of $\succ_S$ there exists $\mathbf{A}yB^{\ni y}x$. Say $x \succ_D y$, then by the construction of $\succ_D$ the sufficient condition of Condition 3.7 is satisfied, hence we have $\mathbf{A}yB^{\ni y}x$. Next we prove the second statement. Say

$x \succ_i y$, $y \succ_j z$, $z \succ_k x$ where $i, j, k \in \{S, D\}$ and suppose for contradiction there exists $\boldsymbol{C}xyz$. Now consider $\boldsymbol{C}xyz\,\{x,y,z\}\,w$. Clearly, $w \in \{x,y,z\}$. Suppose without loss of generality that $w = x$, then $\boldsymbol{C}xyz\,\{x,y,z\}\,x$ and $\boldsymbol{A}x\boldsymbol{B}^{\ni x}z$ (due to $z \succ_S x$ or $z \succ_D x$ and the first statement) violate Condition 3.5. $\square$

Claim. $\succ$ is transitive.

Proof. Take any $x, y, z \in X$.

Suppose some of x, y, z are in $X \backslash \hat{X}$. Since $|X \backslash \hat{X}| \leq 1$, there is exactly one of x, y, z are in $X \backslash \hat{X}$, say without loss of generality $x, y \in \hat{X}$ and $z \in X \backslash \hat{X}$. Then either $x \succ y$ or $y \succ x$. Say without loss of generality $x \succ y$, then since $x \succ y, z$, no transitivity violation is possible.

Now suppose all of x, y, z are in $\hat{X}$. Suppose for contradiction that transitivity is violated, that is, $x \succ y$, $y \succ z$, and $z \succ x$. Since $x, y, z \in \hat{X}$, each of these $\succ$'s is either $\succ_S$ or $\succ_D$. In the following cases of transitivity violations, we show the existence of $\boldsymbol{C}xyz$ for cases 1-3, which violates Axioms 1-3 as outlined by Lemma 3.8.

1. Suppose none of the $\succ$'s are $\succ_D$, that is, $x \succ_S y$, $y \succ_S z$, and $z \succ_S x$. By the construction of $\succ_S$: $\{x,y\}\,x$, $\{y,z\}\,y$, $\{x,z\}\,z$. By Condition 3.1, $\{x,y\}\,x\,\{y,z\}\,y$. By Axiom 3.3 and the construction of $z \succ_S x$, $\{x,y\}\,x\,\{x,z\}\,z$. So by Axiom 3.2, $\{x,y\}\,x\,\{y,z\}\,y\{x,z\}\,z$. So there exists $\boldsymbol{C}xyz$, and by Lemma 3.8 we have contradiction.

2. Suppose exactly one of the $\succ$'s is $\succ_D$. Without loss of generality, say $x \succ_S y$, $y \succ_S z$, and $z \succ_D x$. By the constructions of $\succ_S$ and $\succ_D$: $\{x,y\}\,x$, $\{y,z\}\,y$, $\{x,z\}\,x$. Since $z \succ_D x$, $Az\,\{x,z\}\,z$ for some A. By Condition 3.1, $Az\,\{x,z\}\,z\,\{x,y\}\,x$. By Axiom 3.3 and the construction of $z \succ_S x$, $Az\,\{x,z\}\,z\,\{y,z\}\,y$. So by Axiom 3.2, $Az\,\{x,z\}\,z\,\{x,y\}\,x\,\{y,z\}\,y$. So there exists $\boldsymbol{C}xyz$, and by Lemma 3.8 we have contradiction.

3. Suppose exactly two of the $\succ$'s are $\succ_D$. Without loss of generality, say $x \succ_D y$, $y \succ_S z$, and $z \succ_D x$. By the construction of $z \succ_D x$ and Condition 3.7, $\exists \boldsymbol{B} x C z^{\ni x}$. By the construction of $y \succ_S z$, $Az\{y,z\}y$ for some A and $\{y,z\}y$. So by Axiom 3.3, $\boldsymbol{B} x C z^{\ni x}\{y,z\}y$. So there exists $\boldsymbol{C} xyz$, and by Lemma 3.8 we have contradiction.

4. Finally, suppose all three $\succ' s$ are $\succ_D$. Suppose without loss of generality that $\{x,y,z\}x$. From $z \succ_D x$ and Condition 3.7, $\{x,z\}xAz\{x,y\}z$ for some A. Now consider $\{x,z\}xAz\{x,y\}z\{x,y,z\}w$. If $w = y$, this violates Condition 3.1. If $w = x$, this violates Axiom 3.1. Hence

$$\{x,z\}xAz\{x,y\}z\{x,y,z\}z. \tag{3.7.1}$$

From $y \succ_D z$ and Condition 3.7, $\{y,z\}zBy\{y,z\}y$ for some B. Now consider $\{y,z\}zBy\{y,z\}y\{x,y,z\}w$. Since $\{x,y,z\}x$ and Equation 3.7.1, $w = x$ would violate Axiom 3.3. If $w = z$, this violates Axiom 3.1. Hence

$$\{y,z\}zBy\{y,z\}y\{x,y,z\}y. \tag{3.7.2}$$

From $x \succ_D y$ and Condition 3.7, $\{x,y\}yCx\{x,y\}x$ for some C. Now consider $\{x,y\}yCx\{x,y\}x\{x,y,z\}w$. If $w = z$, this violates Condition 3.1. If $w = y$, this violates 3.1. Hence

$$\{x,y\}yCx\{x,y\}x\{x,y,z\}x,$$

but this along with 3.7.2 and $\{x,y,z\}x$ violates 3.3, a contradiction.

$\square$

Stage 3, Model Explains Choices Now that we established completeness and transitivity of $\succ$ on X, and since X is countable, let $u : X \longrightarrow \mathbb{R}$ be real-valued function such that $u(x) > u(y)$ if and only if $x \succ y$. Moreover, define by $\Gamma(A) := \{\tilde{c}(\emptyset)(A)\}$, for all $A \in \mathcal{A}$, the default attention set $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$.

Throughout, we use c_{model} to label the choice function *given by the model*, and from it $\tilde{c}_{\text{model}}$ the cross-sectional choice functions.

Lemma 3.9. *The constructed* (u, Γ) *explains* c *when restricting attention to the first choice sets (i.e., without history). That is, for all* $A \in \mathcal{A}$*,*

$$\tilde{c}(\emptyset)(A) = \arg\max_{x \in \tilde{\Gamma}(\emptyset)(A)} u(x).$$

Proof. This is a direct consequence of how Γ was constructed, $\Gamma(A) := \{\tilde{c}(\emptyset)(A)\}$ and the fact that $\tilde{\Gamma}(\emptyset)(A) = \Gamma(A)$. □

We now show that (u, Γ) explains the entire c.

Take any sequence of choice sets (A_n), and suppose for contradiction that, for some i,

$$c((A_n))_i \neq \arg\max_{x \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_i)} u(x).$$

Let $\boldsymbol{i}$ be the set of all such i's; they point to the set of all choice sets in (A_n) from which the actual choice is not the same as the model prediction. Denote the minimum element of $\boldsymbol{i}$ by $i^* := \min \boldsymbol{i}$, which is well-defined. $i^* \neq 1$ due to Lemma 3.9.

Consider $i^* \geq 2$. Denote by

$$c^R = c((A_n))_{i^*}$$

the *realized choice* and

$$c^P = c_{\text{model}}\left(\left(A_n\right)\right)_{i^*} = \operatorname*{arg\,max}_{x \in \tilde{\Gamma}\left(\left(A_1,\ldots,A_{i^*-1}\right)\right)\left(A_{i^*}\right)} u\left(x\right)$$

the *model prediction*.

Claim 3.1. $c^R, c^P \in \hat{X}$.

Proof. It is by definition that $c^R \in \hat{X}$. To establish $c^P \in \hat{X}$, we use the construction of Γ: Since $c^P \in \tilde{\Gamma}\left(\left(A_1,\ldots,A_{i^*-1}\right)\right)\left(A_{i^*}\right)$, denote by A_j the earliest choice set in $\left(A_n\right)$ from which c^P is chosen. Since c^P is chosen from A_{i^*}, A_j exists and is well-defined. Hence

$$c^P \in \tilde{\Gamma}\left(\left(A_1,\ldots,A_{j-1}\right)\right)\left(A_j\right)$$

and

$$c^P \notin \hat{c}_{\text{model}}\left(\left(A_1,\ldots,A_{j-1}\right)\right) := \left\{c_{\text{model}}\left(\left(A_n\right)\right)_k : k \leq j-1\right\}.$$

By the definition of the history-dependent attention set,

$$\tilde{\Gamma}\left(\left(A_1,\ldots,A_{j-1}\right)\right)\left(A_j\right) = \left(\Gamma\left(A_j\right) \cup \hat{c}_{\text{model}}\left(\left(A_1,\ldots,A_{j-1}\right)\right)\right) \cap A_j,$$

it must be that $c^P \in \Gamma\left(A_j\right)$, that is, c^P is in the default attention set of A_j. Moreover, given the way Γ was constructed (including only the alternatives that are chosen without history), the model predicts $\tilde{c}_{\text{model}}\left(\emptyset\right)\left(A_j\right) = c^P$. Finally by Lemma 3.9, $\tilde{c}\left(\emptyset\right)\left(A_j\right) = \tilde{c}_{\text{model}}\left(\emptyset\right)\left(A_j\right) = c^P$, and hence $c^P \in \hat{X}$. $\square$

Now that we know $c^R, c^P \in \hat{X}$, we leverage the way $u\left(c^R\right)$ and $u\left(c^P\right)$ were defined to complete the proof.

Since c^P is the model's prediction, we know that $c^P \in \tilde{\Gamma}\left(\left(A_1, \ldots, A_{i^*-1}\right)\right)\left(A_{i^*}\right)$.

Suppose first that c^R is also in $\tilde{\Gamma}\left(\left(A_1, \ldots, A_{i^*-1}\right)\right)\left(A_{i^*}\right)$. Since the model predicts c^P to be chosen over c^R even though c^R was paid attention to, we have $u\left(c^P\right) > u\left(c^R\right)$. Given $c^P, c^R \in \hat{X}$, this means either $c^P \succ_S c^R$ or $c^P \succ_D c^R$, both imply there exists

$$\boldsymbol{A} c^R \boldsymbol{B}^{\ni c^R} c^P \tag{3.7.3}$$

by Lemma 3.8.

Meanwhile, since $c^P \in \tilde{\Gamma}\left(\left(A_1, \ldots, A_{i^*-1}\right)\right)\left(A_{i^*}\right)$, either

$$c^P \in \hat{c}_{\text{model}}\left(\left(A_1, \ldots, A_{i^*-i}\right)\right) := \left\{c_{\text{model}}\left(\left(A_n\right)\right)_k : k \leq i^* - 1\right\} \tag{3.7.4}$$

or

$$c^P = \tilde{c}\left(\emptyset\right)\left(A_{i^*}\right) \tag{3.7.5}$$

or both.

Under Equation 3.7.4, since A_{i^*} is by definition the earliest choice set in $\left(A_n\right)$ from which the model prediction and the actual choice disagree, this implies $c\left(\left(A_n\right)\right)_k = c_{\text{model}}\left(\left(A_n\right)\right)_k = c^P$ for some $k < i^*$. Along with $c^R = c\left(\left(A_n\right)\right)_{i^*}$ where $c^P \in A_{i^*}$, this gives $\left(A_1, \ldots, A_{i^*-1}\right) c^P A_{i^*}^{\ni c^P} c^R$, and so we have for some $\boldsymbol{B}$,

$$\boldsymbol{B} c^P A_{i^*}^{\ni c^P} c^R. \tag{3.7.6}$$

If instead we have Equation 3.7.5, then by Lemma 3.9 and $c^R = c\left(\left(A_n\right)\right)_{i^*}$ we have for some

$\boldsymbol{B}$,

$$A_{i^*} c^P \text{ and } \boldsymbol{B} A_{i^*} c^R. \tag{3.7.7}$$

These cases each results in a contradiction:

	Equation 3.7.6	Equation 3.7.7
$\boldsymbol{A} c^R B^{\ni c^R} c^P$ (Equation 3.7.3)	Violates Condition 3.5	Violates Axiom 3.3

Suppose now that c^R is not in $\tilde{\Gamma}((A_1, ..., A_{i^*-1}))(A_{i^*})$. That is,

$$c^R \notin \Gamma(A_{i^*}),$$

which by Lemma 3.9 means

$$c^R \neq \tilde{c}(\emptyset)(A_{i^*}), \tag{3.7.8}$$

and also

$$c^R \notin \hat{c}_{\text{model}}((A_1, ..., A_{i^*-i})) := \{c_{\text{model}}((A_n))_k : k \leq i^* - 1\},$$

which, by the fact that i^* is the earliest choice set in (A_n) from which the model and actual choices disagree, implies

$$c^R \neq c((A_n))_l \text{ for all } l < i^*. \tag{3.7.9}$$

But then $c^R = c((A_n))_{i^*}$, Equation 3.7.8, and Equation 3.7.9 result in a direct contradiction of Condition 3.1, a consequence of Axiom 3.2 (Lemma 3.1).

What we have done thus far is to show that if the model predictions and and actual choices mismatch for the first time in a sequence, that necessarily leads to a contradiction. But this means no mismatches can ever happen. Therefore,

$$c((A_n))_k = c_{\text{model}}((A_n))_k = \underset{x \in \tilde{\Gamma}((A_1, ..., A_{k-1}))(A_k)}{\arg\max} u(x)$$

for all $(A_n) \in \mathcal{A}^{\mathbf{N}}$ and $k \in \mathbb{N}$.

3.7.4 Proof of Proposition 3.1

If c admits an AAT representation, it satisfies Axiom 3.2. By Lemma 3.1 and Lemma 3.2, it also satisfies Condition 3.1 and Condition 3.2 respectively. Suppose for contradiction c admits AAT representations with specifications (u_1, Γ_1) and (u_2, Γ_2) but $u_1(x) > u_1(y)$ and $u_2(x) < u_2(y)$ for some $x, y \in \hat{X}$. Since $x, y \in \hat{X}$, there exist $A, B \in \mathcal{A}$ such that Ax, By by Condition 3.2. Now consider $AxBz$. By Condition 3.1, $z \in \{x, y\}$. Under AAT, $\{x, y\} \subseteq \tilde{\Gamma}((A))(B)$, so by (u_1, Γ_1) we have $z = x$ but by (u_2, Γ_2) we have $z = y$, hence they cannot both represent c.

3.7.5 Proof of Proposition 3.2

Consider any sequence that begins with all possible binary choice problems $(A_1, \ldots, A_K)$ (in any order), and then repeats itself, with no restriction on what happens next. Since X is finite, the set of all binary choice problems is finite, so this sequence is possible. Note that all but at most one alternative will been chosen in the first half of the sequence (if z was not chosen, then then all other alternatives have been chosen, so z is the only alternative that is not chosen), denote this set by $\bar{X}$. Then, during the repetition of $(A_1, \ldots, A_K)$, either $x \succ_{(A_n)} y$ or $y \succ_{(A_n)} x$ for all $x, y \in \bar{X}$. Moreover, since c admits an ATT representation, due to Proposition 3.1 and $\bar{X} \subseteq \hat{X}$, we have $x \succ_{(A_n)} y$ only if $u(x) > u(y)$. Hence $\succ_{(A_n)}$ is also transitive.

3.7.6 Proof of Proposition 3.3

This is proven in Lemma 3.5. Or simply notice that

$$c\left(\left(A_n\right)\right)_i = x,\ c\left(\left(A_n\right)\right)_j = y, \text{ and } x \in A_j$$

implies $\tilde{\Gamma}\left(\left(A_1, \ldots, A_{j-1}\right)\right)\left(A_j\right) \supset \{x, y\}$, and so $c\left(\left(A_n\right)\right)_j = y$ implies $u(y) > u(x)$. But the same analysis can be done on

$$c\left(\left(B_n\right)\right)_k = y,\ c\left(\left(B_n\right)\right)_l = x, \text{ and } y \in B_l$$

to imply $u(x) > u(y)$, a contradiction.

3.7.7 Proof of Theorem 3.2

We start with sufficiency.

The key is to show that revealed preference in CLA is also revealed preference in AAT. The rest is completed with a proof technique similar to that for CLA in Masatlioglu et al. (2012), which allows for an arbitrary completion of preference, and other intrinsic compatibility between AAT and CLA.

Recall that $\hat{X} := \{x \in X : c\left(\left(A_n\right)\right)_i = x \text{ for some } \left(A_n\right) \text{ and } i\}$

Stage 1, AAT Since c satisfies Axiom 3.1-Axiom 3.3, by Theorem 3.1 it admits an AAT representation. We use the specification as constructed by the proof of Theorem 3.1, that is, $\Gamma(A) = c(A)$ for all A and for all $x \in \hat{X}$ and $y \in X \backslash \hat{X}$ we have $u(x) > u(y)$.

Stage 2, xPy implies $u(x) > u(y)$

Lemma 3.10. *Suppose c admits an AAT representation. If $x, y \in \hat{X}$,then there exists a sequence of two choice sets, (A_1, A_2), such that $\{x, y\} \subseteq \hat{c}((A_1, A_2))$.*

Proof. Say c admits an AAT representation with u and Γ. Suppose without loss $u(x) > u(y)$.

1. Suppose $\tilde{c}(\emptyset)(\{x, y\}) = x$, and hence $x \in \Gamma(\{x, y\})$. Since $y \in \hat{X}$, by Lemma 3.2 there exists $A \in \mathcal{A}$ such that $\tilde{c}(\emptyset)(A) = y$, and hence $y \in \Gamma(A)$. By AAT representation, $\{x, y\} \subseteq \tilde{\Gamma}(A)(\{x, y\})$, then since $u(x) > u(y)$, we have $\tilde{c}(A)(\{x, y\}) = x$. So $\{x, y\} \subseteq \hat{c}((A, \{x, y\}))$.

2. Suppose $\tilde{c}(\emptyset)(\{x, y\}) = y$, and hence $y \in \Gamma(\{x, y\})$. Since $x \in \hat{X}$, by Lemma 3.2 there exists $A \in \mathcal{A}$ such that $\tilde{c}(\emptyset)(A) = x$, and hence $x \in \Gamma(A)$. By AAT representation, $\{x, y\} \subseteq \tilde{\Gamma}(A)(\{x, y\})$, then since $u(x) > u(y)$, we have $\tilde{c}(A)(\{x, y\}) = x$. So $\{x, y\} \subseteq \hat{c}((A, \{x, y\}))$.

□

Definition 3.8. Define xPy if for some $T \in \mathcal{A}$, we have $\tilde{c}(\emptyset)(T) = x$ and $\tilde{c}(\emptyset)(T \setminus \{y\}) \neq x$.

Lemma 3.11. *Suppose c admits an AAT representation. Consider any $x, y \in \hat{X}$. Either $\exists (A_n)$ such that $c((A_n))_i = y$ and $c((A_n))_j = x$ where $y \in A_j$ and $i < j$ or $\exists (A_n)$ such that $c((A_n))_i = x$ and $c((A_n))_j = y$ where $x \in A_j$ and $i < j$ (and not both).*

Proof. Since $x, y \in \hat{X}$, Lemma 3.10 guarantees the existence of (A_1, A_2) such that $\{x, y\} \subseteq \hat{c}((A_1, A_2))$. Suppose without loss of generality $\tilde{c}((A_1, A_2))(\{x, y\}) = x$, then $(A_1, A_2, \{x, y\})$ is a sequence where y was chosen in the past and x is chosen over y in the future. Furthermore, since c admits an ATT representation, we must have $u(x) > u(y)$, which excludes the possibility of any sequence where x was chosen in the past and y is chosen over x in the future. □

Lemma 3.12. *If c (which is assumed to satisfy Axiom 3.1-Axiom 3.3) further satisfies Axiom 3.4, then xPy implies $u(x) > u(y)$.*

Proof. Suppose xPy. ☐

1. Suppose $y \in \hat{X}$. Since xPy, $x \in \hat{X}$. Then by Axiom 3.4 and Lemma 3.11, $\exists (A_n)$ such that $c((A_n))_i = y$ and $c((A_n))_j = x$ where $y \in A_j$ and $i < j$, so $u(x) > u(y)$ since (u, Γ) explains choices.
2. Suppose $y \notin \hat{X}$, then $u(x) > u(y)$ is by construction.

Stage 3, Constructing the attention filter Now we show that we can reconstruct our default attention Γ into an attention filter Γ^* while continue to explain c.

First we restate a result provided in Masatlioglu et al. (2012) (with notational revisions):

Lemma 3.13 (Masatlioglu et al. (2012))**.** *Fix a choice function c. If P is acyclic, for any arbitrary completion from P to $\succ$ that is transitive and an attention filter constructed by*

$$\Gamma^*(A) := \{x \in A : \tilde{c}(\emptyset)(A) \succ x\} \cup \{\tilde{c}(\emptyset)(A)\},$$

we have

$$\tilde{c}(\emptyset)(A) = \{x \in \Gamma^*(A) : x \succ y \text{ for all } y \in \Gamma^*(A)\}.$$

In Masatlioglu et al. (2012), an axiom (WARP(LA)) is used to guarantee the acyclicity of P. In our case, Lemma 3.12 guarantees that P is a subset of $\succ_u$ (defined by $x \succ_u y$ if $u(x) > u(y)$), which is complete and transitive, so P is acyclic. Moreover, $\succ_u$ is itself a transitive completion of P.

Stage 4, Shows that u, Γ^* explains choice We complete our proof by showing that our choice function c admits an AAT representation with u, Γ^*, where

$$\Gamma^*(A) := \{x \in A : u(\tilde{c}(\emptyset)(A)) > u(x)\} \cup \{\tilde{c}(\emptyset)(A)\}.$$

Lemma 3.13 already states that Γ^* constructed this way is an attention filter, and Γ^* along with u explains $\tilde{c}(\emptyset) : \mathcal{A} \to X$. It remains to show that choices *with history* is also explained.

The intuition is straightforward: We added alternatives to Γ to form Γ^*, but all those that are added are worse than what should be chosen, and hence does not affect the choice. Formally:

First note that $\Gamma \subseteq \Gamma^*$ (since $\Gamma(A) = \tilde{c}(\emptyset)(A)$). Moreover, by construction of Γ^*, for all $(A_1, .., A_n)$ and A,

$$\tilde{\Gamma}^*(A_1, .., A_n)(A) \setminus \tilde{\Gamma}(A_1, .., A_n)(A) \subseteq \{x : u(\tilde{c}(\emptyset)(A)) > u(x)\}.$$

Suppose

$$\tilde{c}((A_1, ..., A_n))(A) = y,$$

then it must be that

$$u(y) \geq \max\left\{u(x) : x \in \tilde{\Gamma}(A_1, .., A_n)(A)\right\} \geq \max\left\{u(x) : x \in \tilde{\Gamma}(\emptyset)(A)\right\} = u(\tilde{c}(\emptyset)(A)).$$

by the AAT representation (the second weak inequality is due to $\tilde{\Gamma}(A_1, .., A_n)(A) \supseteq \tilde{\Gamma}(\emptyset)(A)$. So $u(y) \geq u(\tilde{c}(\emptyset)(A))$, and hence

$$u(y) \geq u(\tilde{c}(\emptyset)(A)) \geq \max\left\{u(x) : x \in \tilde{\Gamma}^*(A_1, .., A_n)(A) \setminus \tilde{\Gamma}(A_1, .., A_n)\right\}.$$

Since $y \in \tilde{\Gamma}^*(A_1,..,A_n)(A)$ and $u(y) \geq u(x)$ for all $x \in \tilde{\Gamma}^*(A_1,..,A_n)(A)$, the model prediction with u and Γ^* conforms with choice (for any $(A_1,..,A_n)$ and A). And we are done!

Necessity Since c that admits an AAT representation, it satisfies Axiom 3.1-Axiom 3.3.

Suppose c admits an AAT representation where Γ is an *attention filter*. If $\tilde{c}(\emptyset)(T) = x$ and $\tilde{c}(\emptyset)(T \setminus \{y\}) \neq x$, it must be that $y \in \Gamma(T)$ since Γ is an attention filter and $\Gamma(T) \neq \Gamma(T \setminus \{y\})$. Since $\tilde{c}(\emptyset)(T) = x$, it must be that $u(x) > u(y)$. So for all (A_n), if $c((A_n))_i = x$, then $c((A_n))_j \neq y$ for all $j > i$ such that $x \in A_j$. Hence Axiom 3.4.

3.7.8 Proof of Proposition 3.4

This is a direct implication of Lemma 3.11 and the fact that $u(x) > u(y)$, which is given by $\tilde{c}(\emptyset)(T) = x$ and $\tilde{c}(\emptyset)(T \setminus \{y\}) \neq x$.

3.7.9 Proof of Proposition 3.5

Part 1 is immediate, where equivalence between CLA and WARP(LA) is proven in Masatlioglu et al. (2012)'s Theorem 3.

For Part 2, we just need to show that $\tilde{\Gamma}((A_1,...,A_K)) : \mathcal{A} \to \mathcal{A}$ is an attention filter for all $(A_1,...,A_K) \in \mathbb{A}$. Recall

$$\begin{aligned}\tilde{\Gamma}((A_1,...,A_K))(A) &= [\Gamma(A) \cup \hat{c}((A_1,...,A_K))] \cap A \\ &= \Gamma(A) \cup [\hat{c}((A_1,...,A_K)) \cap A]\end{aligned}$$

Since Γ is an attention filter, if for some for some $C \subseteq X$, $\Gamma^*(A) = \Gamma(A) \cup C$ for all A, then

$\Gamma^*(A)$ is also an attention filter: If $y \notin \Gamma^*(A)$, then $y \notin \Gamma(A)$, so $\Gamma(A\backslash\{y\}) = \Gamma(A)$, and so

$$\begin{aligned}\Gamma^*(A\backslash\{y\}) &= \Gamma(A\backslash\{y\}) \cup C \\ &= \Gamma(A) \cup C \\ &= \Gamma^*(A).\end{aligned}$$

Finally, letting $C := \hat{c}((A_1, ..., A_K))$ completes the proof for 2(a), which implies 2(b).

3.7.10 Proof of Theorem 3.3

Stage 1, Equivalent Classes First, we show that under the axioms, if x and y are sometimes chosen simultaneously, and y and z are sometimes chosen simultaneously, then x and z are sometimes chosen simultaneously:

Lemma 3.14. *Suppose C satisfies Axiom 3.7. For all xWy, if $x \in C((A_n))_i$ and $y \in C((A_n))_j \cup \tilde{C}(\emptyset)(A_i)$ for some $j < i$, then $y \notin A_i \backslash C((A_n))_i$.*

Proof. This is by direct application of Axiom 3.7. Suppose xWy. Suppose for contradiction $x \in C((A_n))_i$, $y \in B_i$, and $y \in C((A_n))_j \cup \tilde{C}(\emptyset)(A_i)$ for some $j < i$, but $y \notin C((A_n))_i$, then by Axiom 3.7, $x \in C((B_n))_k$ implies $y \notin C((B_n))_k$ for all $(B_n) \in \mathcal{A}^{\mathbb{N}}$ and $k \in \mathbb{N}$, but this contradicts xWy. □

Lemma 3.15. *If xWy and yWz, then xWz.*

Proof. First we show that there exists a sequence of choice sets $(A_1, ..., A_K)$ such that x, y, z were chosen at some point, $\{x, y, z\} \subseteq \hat{C}((A_1, ..., A_K)) := \{C((A_n))_i : i \leq K\}$.

Since yWz, there exists a sequence of choice sets (B_n) such that $\{y, z\} \subseteq C((B_n))_i$ for some $i \in \mathbb{N}$.

1. Suppose $x \in \tilde{C}(\emptyset)(\{x,y\})$, then either $x \in \tilde{C}((B_1,...,B_i))(\{x,y\})$ or $y \in \tilde{C}((B_1,...,B_i))(\{x,y\})$, and the latter implies $x \in \tilde{C}((B_1,...,B_i))(\{x,y\})$ due to $x \in \tilde{C}(\emptyset)(\{x,y\})$ and Lemma 3.14. Hence $\{x,y,z\} \subseteq \hat{C}((B_1,...,B_i,\{x,y\}))$.

2. Suppose $x \notin \tilde{C}(\emptyset)(\{x,y\})$, so $\tilde{C}(\emptyset)(\{x,y\}) = \{y\}$. Since xWy, x is *sometimes chosen* ($x \in \hat{X}$), so through an argument analogous to Lemma 3.2, Axiom 3.6 implies $x \in \tilde{C}(\emptyset)(D)$ for some $D \in \mathcal{A}$. Due to Lemma 3.14 and $\tilde{C}(\emptyset)(\{x,y\}) = \{y\}$, $\tilde{C}((D))(\{x,y\}) = \{x,y\}$. Now consider $\tilde{C}((D,\{x,y\}))(B_i)$ where B_i is defined above as the choice set from which both y and z were chosen in some sequence. If $\tilde{C}((D,\{x,y\}))(B_i) \cap \{x,y,z\} \neq \emptyset$, then due to $\{x,y\} \subseteq \hat{C}((D,\{x,y\}))$, xWy and yWz, once or twice application(s) of Lemma 3.14 guarantees $z \in \tilde{C}((D,\{x,y\}))(B_i)$, and we are done with $\{x,y,z\} \subseteq \hat{C}((D,\{x,y\},B_i))$. Suppose for contradiction $\tilde{C}((D,\{x,y\}))(B_i) \cap \{x,y,z\} = \emptyset$, let $a \in \tilde{C}((D,\{x,y\}))(B_i)$:
   a) Suppose $a \in \tilde{C}(\emptyset)(D)$, then aWx, and so $x \notin \tilde{C}((D,\{x,y\}))(B_i)$ is a contradiction of $x \in \hat{C}((D,\{x,y\}))$ and Lemma 3.14.
   b) Suppose $a \notin \tilde{C}(\emptyset)(D)$, then through an argument analogous to Lemma 3.1, Axiom 3.6 implies $a \in \tilde{C}(\emptyset)(B_i)$.
      i. If $a \in C((B_n))_i$, then aWy, and so $y \notin \tilde{C}((D,\{x,y\}))(B_i)$ is a contradiction of $y \in \hat{C}((D,\{x,y\}))$ and Lemma 3.14.
      ii. If $a \notin C((B_n))_i$ (so $a \in B_i \backslash C((B_n))_i$), then $y \in C((B_n))_i$, $a \in \tilde{C}(\emptyset)(B_i) \backslash C((B_n))_i$, $y \in \hat{C}((D,\{x,y\}))$ but $a \in \tilde{C}((D,\{x,y\}))(B_i)$ contradicts Lemma 3.14.

We have now established that there exists a sequence of choice sets $(A_1,...,A_K)$ such that $\{x,y,z\} \subseteq \hat{C}((A_1,...,A_K))$. Now consider $\tilde{C}((A_1,...,A_K))(\{x,y,z\})$. Since xWy

and $\{x,y\} \subset \hat{C}((A_1,...,A_K))$, Lemma 3.14 requires $\{x,y\} \subset \tilde{C}((A_1,...,A_K))(\{x,y,z\})$ or $\{x,y\} \cap \tilde{C}((A_1,...,A_K))(\{x,y,z\}) = \emptyset$. Similarly, since yWz and $\{y,z\} \subset \hat{C}((A_1,...,A_K))$, Lemma 3.14 requires $\{y,z\} \subset \tilde{C}((A_1,...,A_K))(\{x,y,z\})$ or $\{y,z\} \cap \tilde{C}((A_1,...,A_K))(\{x,y,z\}) = \emptyset$. Therefore $\tilde{C}((A_1,...,A_K))(\{x,y,z\}) = \{x,y,z\}$, and hence xWz.

□

Using Lemma 3.15, given C, we can group alternatives into equivalent classes, denoted by $[x]_\sim$ where x is an (arbitrary) representative alternative and $y \in [x]_\sim$ if (and only if) yWx.

Stage 2, Representation Without Indifferences Suppose there are at least two non-intersecting indifferent classes (otherwise, the representation is simply $\Gamma(A) = \tilde{C}(\emptyset)(A)$ and $u(x) = u(y)$ for all $x, y \in X$).

Definition 3.9. We call X^* an *Indifference-Removed Set* if $\cup_{x \in X^*} [x]_\sim = X$ and $[x]_\sim \cap [y]_\sim = \emptyset$ for all $x, y \in X^*$ and $x \neq y$.

That is, we pick an alternative x, remove all alternatives that are sometimes simultaneously chosen with x, remove them from X, and repeat until we are done. Consider the choice correspondence restricted to X^* and the set of all finite choice sets of at least size two that X^* generates: $\mathcal{A}^*$. This choice correspondence, $C^* : \mathcal{A}^{*\mathbf{N}} \to \mathcal{A}^{*\mathbf{N}}$, is essentially a choice function, since we artificially removed all indifferences. Therefore, it admits an ATT representation using Theorem 3.1.

Stage 3, Representation for C Pick any *Indifference-Removed Set* $X^* \subseteq X$. Since C satisfies Axioms 3.5-3.8 and $C^* : \mathcal{A}^{*\mathbf{N}} \to \mathcal{A}^{*\mathbf{N}}$ is essentially a choice function, it admits an ATT representation with (u^*, Γ^*), where $u^* : X^* \to \mathbb{R}$ and $\Gamma^* : \mathcal{A}^* \to \mathcal{A}^*$.

Now we construct $u : X \longrightarrow \mathbb{R}$ from u^* in the following way. For all $x \in \hat{X}^*$ (the set of chosen things in X^* under C^*),

$$u(x) := u^*(x).$$

For $\bar{x} \in X^* \backslash \hat{X}^*$, we ask whether $\bar{x} \in \hat{X}$ or not. If not, then we set

$$u(x) := u^*(x).$$

If instead $\bar{x} \in \hat{X}$ (but $\bar{x} \in X^* \backslash \hat{X}^*$), this issue arises because there exists some set in $\mathcal{A}$—but no set in $\mathcal{A}^*$—from which $\bar{x}$ is chosen without history (recall that by an argument analogous to Lemma 3.2, if $\bar{x} \in \hat{X}$, $\bar{x} \in \tilde{C}(\emptyset)(A)$ for some $\mathcal{A}$). This means that the ATT representation (u^*, Γ^*) did not correctly capture $\bar{x}$'s preference ranking in C. We now solve this issue.

Since $\bar{x} \in X^* \backslash \hat{X}^*$, $\tilde{C}(\emptyset)(\{\bar{x}, z\}) = \{z\}$ for all $z \in X^* \backslash \{\bar{x}\}$. Since $\bar{x} \in \hat{X}$, there exists $A \in \mathcal{A}$ such that $\bar{x} \in \tilde{C}(\emptyset)(A)$. Therefore we can partition $X^* \backslash \{\bar{x}\}$ into two parts: $\bar{x}^+$ and $\bar{x}^-$:

$$\bar{x}^+ := \left\{z \in X^* \backslash \{\bar{x}\} : \tilde{C}(A)(\{\bar{x}, z\}) = \{z\}\right\},$$
$$\bar{x}^- := \left\{z \in X^* \backslash \{\bar{x}\} : \tilde{C}(A)(\{\bar{x}, z\}) = \{\bar{x}\}\right\},$$

the things that, *in anticipation*, are better than and worse than $\bar{x}$ respectively. Note that $\tilde{C}(A)(\{\bar{x}, z\}) = \{\bar{x}, z\}$ is impossible by construction of X^*.

Since $|X^* \backslash \hat{X}^*| \leq 1$ (if $\tilde{C}(\emptyset)(\{\bar{x}, z\}) = \{z\}$ for all $z \in X^* \backslash \{\bar{x}\}$, then $z \in \hat{X}^*$ for all $z \in X^* \backslash \{\bar{x}\}$), $\bar{x}$'s preference ranking is well-defined, and we want to define $u(\bar{x})$ to reflect that. One way to do that is:

$$u(\bar{x}) := \sup_{z \in \bar{x}^-} u(z) + \epsilon$$

for some $\epsilon > 0$ and add 2ϵ to every $u(z)$ such that $z \in \bar{x}^+$,

$$u_{\text{new}}(z) := u_{\text{old}}(z) + 2\epsilon \text{ for all } z \in \bar{x}^+,$$

(The reason we do this is to go around the possible issue of $\sup_{z\in\bar{x}^-} u(z) = \inf_{z\in\bar{x}^+} u(z)$.)

Consequently, for all $x, y \in X^* \cap \hat{X}$,

$$u(x) > u(y) \text{ if and only if } ySx. \tag{3.7.10}$$

Finally, for all $x \in X^*$,

$$u(y) := u^*(x) \text{ for all } y \in [x]_{\sim}. \tag{3.7.11}$$

Note, consequently, that for all $x, y \in \hat{X}$, there exists $x^*, y^* \in X^* \cap \hat{X}$ such that x^*Wx and y^*Wy such that

$$u(x) > u(y) \text{ if and only if } y^*Sx^*, \tag{3.7.12}$$

as a consequence of Equation 3.7.10 and Equation 3.7.11. Moreover, for $y \in X\backslash\hat{X}$ (if it exists),

$$u(x) > u(y) \text{ for all other } x \in X, \tag{3.7.13}$$

since this means $y \in X^*\backslash\hat{X}^*$ and therefore $u(y) < u(x)$ for all $x \in X^*$ by construction of u^* for C^*.

For Γ, we simply construct it by

$$\Gamma(A) := \tilde{C}(\emptyset)(A).$$

Stage 4, Model Explains Choices This proof is similar to that for Theorem 3.1, with two main differences. First, we need to deal with timely predictions of indifferences, which is primarily done using Lemma 3.15. Second, we will demonstrate contradictions of Axiom 3.7 and Axiom 3.8 when x is predicted (but not chosen) but y is chosen (but not predicted), where either x or y (or both) is not in X^*.

Lemma 3.16. *The constructed* (u, Γ) *explains* C *when restricting attention to the first choice sets (i.e., without history). That is, for all* $A \in \mathcal{A}$*,*

$$\tilde{C}(\emptyset)(A) = \arg\max_{x \in \tilde{\Gamma}(\emptyset)(A)} u(x).$$

Proof. This is a direct consequence of how Γ was constructed, $\Gamma(A) := \left\{\tilde{C}(\emptyset)(A)\right\}$ and the fact that $\tilde{\Gamma}(\emptyset)(A) = \Gamma(A)$. Moreover for all $x, y \in \tilde{C}(\emptyset)(A)$, it is by definition xWy and therefore $u(x) = u(y)$. Hence $\tilde{C}(\emptyset)(A) = \arg\max_{x \in \tilde{\Gamma}(\emptyset)(A)} u(x)$. □

We now show that (u, Γ) explains the entire C.

Take any sequence of choice sets (A_n), and suppose for contradiction that, for some i,

$$C((A_n))_i \neq \arg\max_{x \in \tilde{\Gamma}((A_1, \ldots, A_{i-1}))(A_i)} u(x)$$

Let $\boldsymbol{i}$ be the set of all such i's; they point to the set of all choice sets in (A_n) from which the actual choice(s) is not the same as the model prediction(s). Denote the minimum element of $\boldsymbol{i}$ by $i^* := \min \boldsymbol{i}$, which is well-defined. $i^* \neq 1$ due to Lemma 3.9.

Consider $i^* \geq 2$. Denote by

$$C^R = C((A_n))_{i^*}$$

the *realized choice* and

$$C^P = C_{\text{model}}\left((A_n)\right)_{i^*} = \underset{x \in \tilde{\Gamma}((A_1, \ldots, A_{i^*-1}))(A_{i^*})}{\arg\max} u(x)$$

the *model prediction*.

If $C^R \neq C^P$, there are three cases: (i) $C^R \cap C^P \neq \emptyset$ where $C^R \backslash C^P \neq \emptyset$, (ii) $C^R \cap C^P \neq \emptyset$ where $C^R \backslash C^P \neq \emptyset$, and (iii) $C^R \cap C^P = \emptyset$. We now show that each admits a contradiction. Note too that since C^P is a well-defined maximization problem, $C^P \neq \emptyset$. Clearly, $C^R \neq \emptyset$ by definition.

Claim 3.2. $C^R \cap C^P \neq \emptyset$ where $C^R \backslash C^P \neq \emptyset$ is impossible.

Proof. Suppose for contradiction there exists $y \in C^R \backslash C^P$ and $x \in C^R \cap C^P$. Since y and x are both in C^R, yWx, therefore $u(y) < u(x)$ is not possible due to construction of u (Equation 3.7.11). Therefore if $y \notin C^P$, it must be that $y \notin \tilde{\Gamma}((A_1, \ldots, A_{i^*-1}))(A_{i^*})$, which means (by definition of $\tilde{\Gamma}$) $y \notin \Gamma(A_{i^*})$ and $y \notin \hat{C}_{\text{model}}((A_1, \ldots, A_{i^*-1}))$. By Lemma 3.16, $y \notin \Gamma(A_{i^*})$ implies

$$y \notin \tilde{C}(\emptyset)(A_{i^*}).$$

And by the fact that A_{i^*} is the first choice set in (A_n) from which the model and actual choice disagrees, $\hat{C}_{\text{model}}((A_1, \ldots, A_{i^*-1}))$ implies

$$y \notin \hat{C}((A_1, \ldots, A_{i^*-1})).$$

Finally, since y is neither the default choice of A_{i^*} nor chosen in the past, $y \in C^R$ contradicts Axiom 3.6 by an argument analogous to Lemma 3.1. □

Claim 3.3. $C^R \cap C^P \neq \emptyset$ where $C^R \backslash C^P \neq \emptyset$ is impossible.

Proof. Suppose for contradiction there exists $y \in C^P \backslash C^R$ and $x \in C^R \cap C^P$. Therefore $x, y \in \tilde{\Gamma}((A_1, ..., A_{i^*-1}))(A_{i^*})$ and $u(x) = u(y)$. By $u(x) = u(y)$, we note that xWy (due to construction of u, Equation 3.7.11). By $y \in \tilde{\Gamma}((A_1, ..., A_{i^*-1}))(A_{i^*})$, either $y \in \Gamma(A_{i^*})$ or $y \in \hat{C}_{\text{model}}((A_1, ..., A_{i^*-1}))$ (or both). If $y \in \Gamma(A_{i^*})$, by Lemma 3.16 we have

$$y \in \tilde{C}(\emptyset)(A_{i^*}).$$

If instead $y \in \hat{C}_{\text{model}}((A_1, ..., A_{i^*-1}))$, then since A_{i^*} is the first choice set in (A_n) from which the model and actual choice disagrees,

$$y \in \hat{C}((A_1, ..., A_{i^*-1})).$$

Therefore y is either the default choice of A_{i^*} or chosen in the past, and combined with the fact that yWx and $x \in C^R$, we have a contradiction of Axiom 3.7 by Lemma 3.14. □

Claim 3.4. $C^R \cap C^P = \emptyset$ is impossible.

Proof. Suppose for contradiction $C^R \cap C^P = \emptyset$. Take $y \in C^R$ and $x \in C^P$. Suppose $y \in \tilde{\Gamma}((A_1, ..., A_{i^*-1}))(A_{i^*})$, so the fact that $y \notin C^P$ and $x \in C^P$ means $u(x) > u(y)$. If $y \notin \hat{X}$, an immediate contradiction to $y \in C^R$ is established. If $x \notin \hat{X}$, then by Equation 3.7.13 it is impossible that $u(x) > u(y)$. Therefore $x, y \in \hat{X}$. By construction of $u : X \longrightarrow \mathbb{R}$, and specifically by Equation 3.7.12, there exists x^*Wx and y^*Wy (it is possible that $x^* = x$ or $y^* = y$ or both) such that

$$y^* S x^*. \tag{3.7.14}$$

Now since $x \in C^P$, it must be that either $x \in \Gamma\left(A_{i^*}\right)$ and therefore by Lemma 3.16,

$$x \in \tilde{C}\left(\emptyset\right)\left(A_{i^*}\right), \tag{3.7.15}$$

or $x \in \hat{C}_{\text{model}}\left(\left(A_1, \ldots, A_{i^*-1}\right)\right)$ and therefore by the fact that A_{i^*} is the first disagreement between the model and choice in $\left(A_n\right)$,

$$x \in \hat{C}\left(\left(A_1, \ldots, A_{i^*-1}\right)\right). \tag{3.7.16}$$

Both these cases result in contradictions:

	Equation 3.7.15	Equation 3.7.16
y^*Sx^*(Equation 3.7.3)	Violates Axiom 3.7	Violates Axiom 3.8

□

Ruling all all three cases means we are left with $C^R = C^P$, therefore

$$C\left(\left(A_n\right)\right)_i = \operatorname*{arg\,max}_{x\in\tilde{\Gamma}((A_1,\ldots,A_{i-1}))(A_i)} u\left(x\right)$$

for all $\left(A_n\right)$ and i.

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