Asymptotic representations of shifted quantum affine algebras from critical K-theory

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Abstract

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In this thesis we explore the geometric representation theory of shifted quantum affine algebras $\mathscr{A}^\mu$, using the critical K-theory of certain moduli spaces of infinite flags of quiver representations resembling the moduli of quasimaps to Nakajima quiver varieties. These critical K-theories become $\mathscr{A}^\mu$-modules via the so-called critical R-matrix R, which generalizes the geometric R-matrix of Maulik, Okounkov, and Smirnov. In the asymptotic limit corresponding to taking infinite instead of finite flags, singularities appear in R and are responsible for the shift in $\mathscr{A}^\mu$. The result is a geometric construction of interesting infinite-dimensional modules in the category $\mathcal{O}$ of $\mathscr{A}^\mu$, including e.g. the pre-fundamental modules previously introduced and studied algebraically by Hernandez and Jimbo. Following Nekrasov, we provide a very natural geometric definition of qq-characters for our asymptotic modules compatible with the pre-existing definition of q-characters.

When $\mathscr{A}^\mu$ is the shifted quantum toroidal $\mathfrak{gl}_1$ algebra, we construct asymptotic modules DT_μ and PT_μ whose combinatorics match those of (1-legged) vertices in Donaldson–Thomas and Pandharipande–Thomas theories. Such vertices control enumerative invariants of curves in toric 3-folds, and finding relations between (equivariant, K-theoretic) DT and PT vertices with descendent insertions is a typical example of a wall-crossing problem. We prove a certain duality between our DT_μ and PT_μ modules which, upon taking q-/qq-characters, provides one such wall-crossing relation.

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Chapter 1: Overview

The use of representation theory to shed light on problems in enumerative geometry dates back to as early as Hurwitz numbers and the Frobenius formula. In more recent years, a variety of enumerative problems have been extracted from (supersymmetric) quantum field theories, particularly gauge theories, whose symmetries often form a quantum group. There is therefore a rich interaction between these quantum groups and the geometric constructions and enumerative problems associated to these theories. This is the context for the contents of this thesis.

The following narrative in geometric representation theory forms the foundation of the results of this thesis; a review of the relevant topics is in Chapter 2. Let $\mathcal{M}_Q$ be a Nakajima quiver variety with the standard torus T acting on it. The seminal work [22] showed that the equivariant K-theory $K_{\mathsf{T}}(\mathcal{M}_Q)$ is a module for the quantum affine algebra $\mathcal{A} \coloneqq \mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$ where $\mathfrak{g}_Q$ is a Lie algebra associated to the quiver Q. Later, it was explained by [20, 25] that there are geometric correspondences called *stable envelopes* on $\mathcal{M}_Q$ which can be used to construct the R-matrix R of $\mathcal{A}$. Then, by FRT reconstruction [28], $\mathcal{A}$ is reconstructed from R as a certain collection of matrix elements which act on $K_{\mathsf{T}}(\mathcal{M}_Q)$. Hence the representation theory of $\mathcal{A}$ can be studied effectively by geometric means. For example, [22] proves several properties of q-characters (the analogue for $\mathcal{A}$-modules of characters of $\mathfrak{g}_Q$-modules) conjectured earlier by [10]. For these and other applications, it is an important fact that for appropriately chosen components $\mathcal{M}_Q(\mathbf{w}) \subset \mathcal{M}_Q$, the modules $V(u) \coloneqq K_{\mathsf{T}}(\mathcal{M}_Q(\mathbf{w}))$ form a basis for finite-dimensional modules of $\mathcal{A}$. Here u is an evaluation parameter, or, geometrically, the weight of the (constant) action of the framing torus of $\mathcal{M}_Q$.

The main construction of Chapter 3 is parallel to this narrative, and is based heavily on material from [21]. Using a generalization of stable envelopes to *critical K-theory*, we give a geometric construction of certain sub-modules (expected to be simple) of semi-infinite tensor products $V(u) \otimes V(\hbar u) \otimes V(\hbar^2 u) \cdots$ where $\hbar^{-1}$ is the weight of the symplectic form on $\mathcal{M}_Q$. The infinite nature of the tensor product creates singularities in the resulting R-matrix, and so the output of FRT reconstruction is not $\mathcal{A}$ but rather a *shifted* quantum affine algebra $\mathcal{A}^\mu := \mathscr{U}_q^\mu(\widehat{\mathfrak{g}}_Q)$. This geometric incarnation of $\mathcal{A}^\mu$ and some of its (infinite-dimensional) modules complements the existing literature where these objects are studied purely algebraically, e.g. in [8, 15] and the precursor [16].

We refer to both these semi-infinite tensor products and their sub-modules as *asymptotic modules*. Our original motivation for studying them comes from the enumerative geometry of curves, in particular Donaldson–Thomas (DT) theory and its successor Pandharipande–Thomas (PT) theory; this material is also reviewed in Chapter 2. These theories have an intimate relationship with the Fock representation $F(u) := K_{\mathsf{T}}(\mathsf{Hilb}(\mathbb{C}^2))$ of the quantum toroidal algebra $\mathscr{U}_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1)$. Namely, the enumerative invariants of relevance, called DT and PT *vertices*, can be written in the restricted "1-legged" setting as weighted sums over certain elements in the semi-infinite tensor product of $F(u)$. Note that although $\mathscr{U}_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1)$ is not a quantum affine algebra, all the previous constructions are nonetheless still applicable. Hence we construct in Chapter 4 two asymptotic modules DT_μ and PT_μ for the shifted quantum toroidal algebra $\mathscr{U}_{x,y}^n(\widehat{\widehat{\mathfrak{gl}}}_1)$. The DT_μ module is analogous to modules constructed earlier in [7, 6], but the PT_μ module is new and its existence verifies calculations done in [11]. From these modules we can directly obtain DT and PT 1-legged vertices by taking q- and even qq-characters. The more general qq-characters are a one-parameter refinement of q-characters for *finite-dimensional* $\mathcal{A}$-modules of the form $V(u)$, and were first introduced in [23] for the study of supersymmetric gauge theories. Part of the content of Chapter 4 is a geometric definition of qq-characters for our *infinite-dimensional* asymptotic modules. The qq-character of DT_μ is shown to be precisely the fully-equivariant, K-theoretic DT vertex (with an insertion).

The precise relation between DT and PT vertices, and invariants formed from them in general, is a well-known open problem. For us, these vertices and invariants are taken in the general setting of equivariant K-theory, and various limits in this setting simplify these vertices, e.g. the cohomological limit. In such limits there are some results, including the wall-crossing proof of [32] in the Calabi–Yau limit where all equivariant weights vanish, and the conjecture of [24] in the cohomological limit with the simplest kind of insertion. In Chapter 4 we prove a certain duality between our DT_μ and PT_μ modules which should categorify some DT/PT formulas distinct from the aforementioned ones. In particular, taking q-characters yields a transformation formula between DT and PT insertions in the Calabi–Yau limit, written in terms of the antipode of $\mathscr{U}^n_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1)$.

Chapter 2: Some background

We begin with a brief overview of the (extensive) modern machinery for studying enumerative invariants of various moduli of curves in a 3-fold X. Most of these subjects are of course of independent interest outside of the world of enumerative geometry, but their synthesis in the primary application of § 4.2 is ultimately enumerative.

- § 2.1 reviews equivariant K-theory and its use in the Donaldson–Thomas (DT) and Pandharipande–Thomas (PT) theories of 1-dimensional sheaves on X. Enumerative invariants of these theories are encapsulated by objects called DT and PT *equivariant vertices*, which are expected to be related in specific ways.

- § 2.2 fixes notation for Nakajima quiver varieties $\mathcal{M}_Q$ and reviews some of their properties. Then it discusses quasimaps to $\mathcal{M}_Q$, which for $\mathcal{M}_Q = \mathsf{Hilb}(\mathbb{C}^2)$ is exactly (1-legged) PT theory. For general $\mathcal{M}_Q$, quasimaps generalize PT theory and are especially amenable to techniques of geometric representation theory.

- § 2.3 reviews quantum affine algebras and, for those of "geometric type", their *geometric* construction via $\mathcal{M}_Q$. Their action on the equivariant K-theory of $\mathcal{M}_Q$ implies they are suitable algebras for describing and controlling enumerative invariants in quasimap theory, and also, in particular, DT and PT theory.

Everything is done over $\mathbb{C}$.

2.1 K-theoretic sheaf counting

The main goal of this section is Definition 2.9, of equivariant K-theoretic vertices for DT and PT theories, and the nature of the DT/PT correspondence.

2.1.1 Equivariant K-theory

Let X be a scheme with the action of a reductive group G which we will usually take to be a torus. We fix some notation for equivariant K-theory.

Definition 2.1. Let $\mathsf{Coh}_\mathsf{G}(X)$ and $\mathsf{Vect}_\mathsf{G}(X)$ be the categories of G-equivariant coherent sheaves and G-equivariant vector bundles on X. The two flavors of **G-equivariant K-theory** of X are the K-groups

$$K_\mathsf{G}(X) \coloneqq K(\mathsf{Coh}_\mathsf{G}(X))$$
$$K_\mathsf{G}^\circ(X) \coloneqq K(\mathsf{Vect}_\mathsf{G}(X)).$$

There is an inclusion

$$K_\mathsf{G}^\circ(X) \subset K_\mathsf{G}(X). \tag{2.1}$$

Both are modules for the ring

$$\Bbbk_\mathsf{G} \coloneqq K_\mathsf{G}(\mathrm{pt}) = R(\mathsf{G})$$

where $R(\mathsf{G})$ is the representation ring of G. Note that this is independent of X, so all G-equivariant K-theories will be $\Bbbk_\mathsf{G}$-modules.

Equivalently, let $\mathsf{Perf}_\mathsf{G}(X) \subset D^b\mathsf{Coh}_\mathsf{G}(X)$ be the sub-category of *perfect* complexes, i.e. those locally quasi-isomorphic to a bounded complex of vector bundles. Then

$$K_\mathsf{G}(X) = K(D^b\mathsf{Coh}_\mathsf{G}(X)), \quad K_\mathsf{G}^\circ(X) = K(\mathsf{Perf}_\mathsf{G}(X)).$$

If X is a non-singular variety, then every coherent sheaf is perfect and hence (2.1) becomes an equality.

Remark. Our notation and terminology are slightly unconventional: it is more common to refer to $K^\circ(X)$ as the (zeroth) *K-theory* of X, while $K(X)$ is sometimes called the (zeroth) *G-theory* of X and denoted $G(X)$.

Definition 2.2. In what follows, the group G will usually be a torus $\mathsf{T} = \prod_i \mathbb{C}^\times_{x_i}$, where the notation $\mathbb{C}^\times_x$ indicates that when $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication, the coordinate on $\mathbb{C}$ has weight x. (This is opposite the usual convention where the coordinate has weight x^{-1}; we do so to save on signs in § 4.2.) Then,

$$\Bbbk := \Bbbk_{\mathsf{T}} = \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$$

Let $f \mapsto \overline{f}$ be the involution $x_i \mapsto x_i^{-1}$ on $\Bbbk$. Let $|\cdot|$ be a norm on $\Bbbk$ such that $|x_i| < 1$ for all i; when we speak of a completion of $\Bbbk$, it is with respect to such a norm.

Definition 2.3. Let T be a torus and V be a T-module. View $V = V(x_1, \ldots, x_n)$ as a polynomial in the weights of T. The **plethystic exponential** of V is

$$\mathsf{S}^\bullet_u(V) := \exp\left(\sum_{k>0} \frac{u^k}{k} V(x_1^k, \ldots, x_n^k)\right) = \prod_i \frac{1}{1 - uw_i} \tag{2.2}$$

and is extended multiplicatively to (a completion of) $\Bbbk$ as $\mathsf{S}^\bullet_u(V - W) := \mathsf{S}^\bullet_u(V)/\mathsf{S}^\bullet_u(W)$. The output can be interpreted as living in $\mathrm{Frac}(\Bbbk)$. Better, when V has no constant term, i.e. T acts on V with no fixed weights, it is an element of *localized* K-theory

$$\Bbbk_{\mathrm{loc}} := K_{\mathsf{T},\mathrm{loc}}(\mathrm{pt}) := \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{1-w} \;\middle|\; 1 \neq w \in \Bbbk\right]$$

For convenience, set $\wedge^\bullet_u(f) := \mathsf{S}^\bullet_{-u}(-f)$ and use the convention that if u is omitted then $u = 1$. Then $\mathsf{S}^\bullet(V)$ and $\wedge^\bullet(V)$ are (the K-theory classes of) the symmetric and exterior algebras of V respectively. Note that both $\mathsf{S}^\bullet_u$ and $\wedge^\bullet_u$ may be extended to (a completion of) $K_{\mathsf{T}}(X)$ using the same formula (2.2).

An important tool in equivariant K-theory is **equivariant localization**, which reduces K-theoretic computations on X to its T-fixed locus X^{T}. For us, its simplest form will suffice.

Proposition 2.4 (Equivariant localization). *If X is non-singular, then for $\mathcal{F} \in K_{\mathsf{T}}(X)$,*

$$\chi(X, \mathcal{F}) = \sum_{F \subset X^{\mathsf{T}}} \chi\left(F, \frac{\mathcal{F}\big|_F}{\wedge^\bullet_{-1} \mathcal{N}_{F/X}^\vee}\right) \tag{2.3}$$

where F ranges over all components of the fixed locus X^{T}, and $\mathcal{N}_{F/X}$ is the normal bundle of F.

Unless otherwise specified, K-theoretic operations such as χ are all taken to be equivariant (with respect to whatever equivariance is present). In particular, (2.3) is valued in $\Bbbk_{\text{loc}}$. For a mild simplification, all fixed loci in what follows actually consist of isolated points, so the χ on the rhs will be unnecessary.

Definition 2.5. Suppose X is singular but can be exhibited as the zero locus $\{s = 0\} \subset M$ of a non-trivial section of a bundle $\mathcal{E}$ on a non-singular ambient space M. This is a common situation, at least locally, in moduli problems. Then $\mathcal{O}_X$ is the zeroth cohomology of the Koszul complex $\wedge^\bullet \mathcal{E}^\vee$, and to preserve the deformation invariance of invariants computed on X, one should work with the **virtual classes**

$$\mathcal{O}_X^{\text{vir}} \coloneqq \wedge^\bullet_{-1} \mathcal{E}^\vee, \quad \mathcal{T}_X^{\text{vir}} \coloneqq (\mathcal{T}_M - \mathcal{E}^\vee)\Big|_X \tag{2.4}$$

of the structure sheaf and tangent sheaf in K-theory. It is straightforward to verify the *virtual* equivariant localization formula

$$\chi(X, \mathcal{O}_X^{\text{vir}} \otimes \mathcal{F}) = \sum_{F \subset X^{\mathsf{T}}} \frac{\chi(F, \mathcal{O}_F^{\text{vir}} \otimes \mathcal{F}\big|_F)}{\wedge^\bullet_{-1} \mathcal{N}_{F/X}^{\text{vir}}}$$

by applying usual localization to $\chi(X, -) = \chi(M, \iota_*(-))$, where $\iota \colon X \to M$ is the inclusion.

Remark. Different presentations of X as a zero locus yield different virtual classes (2.4) in general.

Definition 2.6. Let

$$\widehat{\mathsf{S}}^\bullet(V) \coloneqq \mathsf{S}^\bullet(V) \otimes \det(V)^{1/2}$$

be the **symmetrized** symmetric algebra. This has the important "self-duality" property $\widehat{\mathsf{S}}^\bullet(V^*) = (-1)^{\operatorname{rk} V}\widehat{\mathsf{S}}^\bullet(V)$. Similarly, define the symmetrized virtual structure sheaf

$$\widehat{\mathcal{O}}_X^{\text{vir}} \coloneqq \mathcal{O}_X^{\text{vir}} \otimes \det(\mathcal{T}_X^{\text{vir}})^{-1/2}.$$

Symmetrized quantities may require the introduction of square roots $x_i^{\pm 1/2}$ of the equivariant weights, and we implicitly assume $\Bbbk$ contains these square roots by adjoining them as necessary.

For various reasons, it is better to study $\chi(\widehat{\mathcal{O}}_X^{\text{vir}} \otimes \mathcal{F})$ than $\chi(\mathcal{O}_X^{\text{vir}} \otimes \mathcal{F})$. This has the effect of replacing $\wedge^\bullet$ in localization formulas with the symmetrized $\widehat{\wedge}^\bullet$.

2.1.2 DT and PT

If X is a non-singular quasi-projective threefold, **Donaldson–Thomas (DT) theory** (in rank 1) is concerned with the moduli space of ideal sheaves

$$\mathsf{Ideals}(X) = \bigsqcup_{\beta \in H_*(X)} \mathsf{Ideals}(X, \beta)$$

where β denotes the homology class of the subscheme cut out by the ideal sheaf. We focus on mostly on $\mathsf{Ideals}(X, \text{points})$ and $\mathsf{Ideals}(X, \text{curves})$, i.e. when $\beta \in H_0(X)$ and $H_2(X)$ respectively. A major goal in K-theoretic enumerative geometry is to understand DT curve counts such as

$$\chi\left(\mathsf{Ideals}(X, \text{curves}), \widehat{\mathcal{O}}^{\text{vir}} \otimes \mathcal{F}\right) \tag{2.5}$$

for certain sheaves $\mathcal{F} \in K^\circ(\mathsf{Ideals}(X))$. In fact, DT theory fits within a much larger framework where one studies moduli spaces of stable (complexes of) sheaves, for varying stability parameters. For example, the stability parameter for the closely related **Pandharipande–**

Thomas (PT) theory forces the curves being counted to be *pure* 1-dimensional subschemes. PT moduli spaces in the "1-legged" setting of interest will be a special case of the *quasimap* moduli of § 2.2; in this section we will take the following more combinatorial approach to both DT and PT theories.

When X is toric, its torus $\mathsf{T} = (\mathbb{C}^\times)^3$ naturally acts on $\mathsf{Ideals}(X)$, whose T-fixed points restricted to each chart $\mathbb{C}^3 = \operatorname{Spec}\mathbb{C}[x, y, z] \subset X$ are monomial ideals

$$I \subset \mathbb{C}[x, y, z].$$

Equivalently, one can study the $\mathbb{C}[x, y, z]$-module $V \coloneqq \mathbb{C}[x, y, z]/I$. There is an analogous such $\mathbb{C}[x, y, z]$-module in PT theory. Localization in both theories relies on understanding the combinatorics of which $\mathbb{C}[x, y, z]$-modules can appear, for which there is a convenient pictorial depiction.

Definition 2.7. Given a $\mathbb{C}[x, y, z]$-module V, its associated **box configuration** $\pi \coloneqq \pi(V) \subset \mathbb{Z}^3$ has a box drawn at $(i, j, k) \in \mathbb{Z}^3$ for each non-zero weight $x^i y^j z^k \in V$. All V we will be concerned with are uniquely specified by $\pi(V)$, and so we work with V and π interchangeably. Let

$$\chi_\pi \coloneqq \sum_{(i,j,k)\in\pi} x^i y^j z^k \in \Bbbk_{\mathrm{loc}}$$

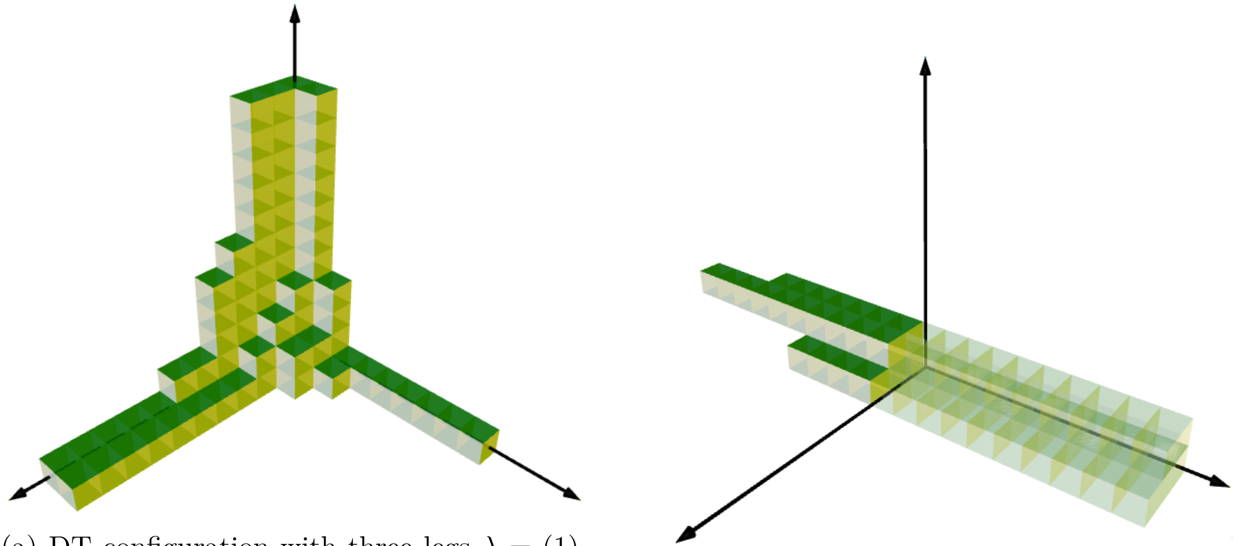
denote the T-character of V.

- Let λ be a partition. We say π contains an infinite **leg** of **profile** λ in the x **direction** if

  $$(j, k) \in \lambda \implies (i, j, k) \in \pi \ \forall i \geq 0.$$

  Legs in the y and z directions are defined similarly, permuting all coordinates cyclically. The normalized **volume** of π is

  $$|\pi| \coloneqq \sum_{(i,j,k)\in\pi} \left(1 - (\#\text{ of legs containing }(i, j, k))\right).$$



(a) DT configuration with three legs $\lambda = (1)$, $\mu = (3,1)$, $\nu = (2)$

(b) PT configuration with one leg $\lambda = (2,2,1)$

Figure 2.1: DT and PT box configurations

- A finite volume $\pi \subset (\mathbb{Z}_{\geq 0})^3$ is a **DT configuration** if $V = \mathbb{C}[x,y,z]/I$ for a monomial ideal, as discussed earlier. DT configurations are exactly 3d partitions (also known as plane partitions) with legs, as in Figure 2.1a. These generalize the classical notion of (2d) partitions. As it is straightforward to work out how weights must be arranged in monomial ideals, we will not supply a precise definition of 3d partitions.

- A finite volume $\pi \subset \mathbb{Z}$ is a **1-legged PT configuration** if it has at most one non-trivial leg and V is torsion-free in the direction of the leg. This means all additional boxes not in the leg are on the "negative" side of the leg, as in Figure 2.1b where the leg is drawn transparent for clarity. The correct combinatorics in the 2- and 3-legged cases are more complicated and irrelevant for us; see [27] for a description.

Let $\Pi_{\mathsf{DT}}(\lambda,\mu,\nu)$ and $\Pi_{\mathsf{PT}}(\lambda,\mu,\nu)$ denote the sets of DT and PT configurations, respectively, with legs λ,μ,ν along x,y,z (though we have not defined 2- or 3-legged PT configurations).

Remark. An alternate way to view PT 1-legged configurations with leg λ in the positive x direction is that their complement in the cylinder $\{(i,j,k)\}_{(j,k)\in\lambda}$ is a DT 1-legged configuration with leg λ pointing in the *negative* x direction. This will be an especially useful perspective for § 4.2.2.

Geometrically, even the simplest case $\mathsf{Ideals}(\mathbb{C}^3, \text{points}) = \mathsf{Hilb}(\mathbb{C}^3)$ is a wildly singular space with components of unknown dimension in general. It is better to study it as a *critical locus* in a smooth ambient space which we now describe.

Definition 2.8. Let $\mathbb{C}\langle x, y, z\rangle$ be the free algebra on x, y, z, viewed as the non-commutative analogue of $\mathbb{C}^3$, and let

$$M \coloneqq \mathsf{Hilb}(\mathbb{C}\langle x, y, z\rangle)$$

be the **non-commutative Hilbert scheme**. An ideal $[I] \in M$ corresponds to a tuple

$$(X, Y, Z, v), \quad X, Y, Z \in \mathrm{End}(V),\ v \in V,\ \langle X, Y, Z\rangle v = V \tag{2.6}$$

modulo $\mathrm{GL}(V)$, where $V \coloneqq \mathbb{C}\langle x, y, z\rangle/I$, the X, Y, Z are operators of multiplication by x, y, z, and v is the image of $1 \in \mathbb{C}\langle x, y, z\rangle$. Parameterizing M in this way shows it is smooth, and $\mathsf{Hilb}(\mathbb{C}^3) \subset M$ is the locus where X, Y, Z commute. In fact, this locus is a *critical locus*

$$\mathsf{Hilb}(\mathbb{C}^3) = \{d\phi = 0\} \subset M, \quad \phi \coloneqq \mathrm{tr}([X, Y]Z).$$

Critical loci are a special and very important case of the setup of Definition 2.5, which then provides virtual sheaves $\mathcal{O}^{\mathrm{vir}}$ and $\mathcal{T}^{\mathrm{vir}}$ for $\mathsf{Hilb}(\mathbb{C}^3)$. Using (2.6) or otherwise,

$$\begin{aligned} \mathcal{T}_\pi M &= \chi_\pi + \overline{\chi}_\pi \chi_\pi (x^{-1} + y^{-1} + z^{-1} - 1) \\ \mathcal{T}_\pi^{\mathrm{vir}} \mathsf{Hilb}(\mathbb{C}^3) &= \mathcal{T}_\pi M - (xyz)^{-1} \mathcal{T}_\pi^* M \\ &= \chi_\pi - \frac{\overline{\chi}_\pi}{xyz} - \overline{\chi}_\pi \chi_\pi \frac{(1-x)(1-y)(1-z)}{xyz} \end{aligned} \tag{2.7}$$

where the twist by $(xyz)^{-1}$ is so that the section $d\phi \otimes (xyz)^{-1}$ is T-equivariant.

For toric X, localization on $\mathsf{Ideals}(X, \text{curves})$ will generally involve 3d partitions π with non-trivial legs, whose localization contribution $\widehat{\mathsf{S}}^\bullet(\mathcal{T}_\pi^{\mathrm{vir}})$ may still be computed using the formulas (2.7).

Definition 2.9. Let $\mathcal{I}(\lambda,\mu,\nu) \subset \mathsf{Ideals}(\mathbb{C}^3, \text{curves})$ consist of ideal sheaves with asymptotics λ,μ,ν along x,y,z, so that its T-fixed points are exactly $\Pi_{\mathsf{DT}}(\lambda,\mu,\nu)$. The **(equivariant) DT vertex** with **insertion** $\mathcal{F} \in K^\circ_{\mathsf{T}}(\mathsf{Ideals}(\mathbb{C}^3))$ is the series

$$\begin{aligned} \mathsf{V}_{\mathsf{DT}(\lambda,\mu,\nu)}(\mathcal{F}) &\coloneqq \chi\left(\mathcal{I}(\lambda,\mu,\nu), \widehat{\mathcal{O}}^{\text{vir}} \otimes \mathcal{F} \otimes Q^{|\cdot|}\right) \\ &= \sum_{\pi \in \Pi_{\mathsf{DT}}(\lambda,\mu,\nu)} \widehat{\mathsf{S}}^\bullet\left(T^{\text{vir}}_\pi\right) \cdot \mathcal{F}\Big|_\pi \cdot Q^{|\pi|} \in \Bbbk_{\text{loc}}((Q)). \end{aligned} \tag{2.8}$$

Though we have not defined the PT moduli space, all previous discussion and formulas apply without modification to the analogous **(equivariant) 1-legged PT vertex**, which can be defined as

$$\mathsf{V}_{\mathsf{PT}(\lambda,\emptyset,\emptyset)}(\mathcal{F}) \coloneqq \sum_{\pi \in \Pi_{\mathsf{PT}}(\lambda,\emptyset,\emptyset)} \widehat{\mathsf{S}}^\bullet\left(T^{\text{vir}}_\pi\right) \cdot \mathcal{F}\Big|_\pi \cdot Q^{|\pi|} \in \Bbbk_{\text{loc}}[[Q]]. \tag{2.9}$$

The general 3-legged PT moduli has fixed loci of positive dimension, so we omit any definition of $\mathsf{V}_{\mathsf{PT}(\lambda,\mu,\nu)}(\mathcal{F})$.

Note that when π contains non-trivial legs, $\mathcal{T}^{\text{vir}}_\pi$ contains *poles*

$$\mathcal{T}^{\text{e}}_\pi(\lambda) \coloneqq \frac{T_\lambda \mathsf{Hilb}(\mathbb{C}^2)}{1-x}, \quad \mathcal{T}^{\text{e}}_\pi(\mu) \coloneqq \frac{T_\mu \mathsf{Hilb}(\mathbb{C}^2)}{1-y}, \quad \mathcal{T}^{\text{e}}_\pi(\nu) \coloneqq \frac{T_\nu \mathsf{Hilb}(\mathbb{C}^2)}{1-z} \tag{2.10}$$

corresponding to deforming the *profiles* of the legs (λ,μ,ν) themselves. It is standard to normalize away the contributions of these terms from vertices V_{DT} and V_{PT}, i.e. to use

$$\mathcal{T}^{\text{vertex}}_\pi \coloneqq \mathcal{T}^{\text{vir}}_\pi - \mathcal{T}^{\text{e}}_\pi(\lambda) - \mathcal{T}^{\text{e}}_\pi(\mu) - \mathcal{T}^{\text{e}}_\pi(\nu) \in \Bbbk$$

in place of $\mathcal{T}^{\text{vir}}_\pi$. Such a normalization is unimportant for us and we do not perform it.

Remark. Quantities such as (2.5) are built from these vertices along with other terms called *edges* which carry not only the redistributed terms (2.10) but also contributions from pairwise intersections of toric charts. Since edges have no combinatorial complexity, the main problem in DT theory is to gain good control over vertices.

The natural question of how invariants such as (2.5) compare in DT and PT theory reduces to a comparison of DT and PT vertices. When no insertions are present, the following general conjecture has only been proved in particular cases, e.g. in the **Calabi–Yau (CY) limit** $xyz = 1$ where all equivariant variables vanish and vertices V become numerical generating functions for DT and PT configurations.

Conjecture 2.10 (DT/PT vertex correspondence)**.**

$$\mathsf{V}_{\mathsf{PT}(\lambda,\mu,\nu)} = \frac{\mathsf{V}_{\mathsf{DT}(\lambda,\mu,\nu)}}{\mathsf{V}_{\mathsf{DT}(\emptyset,\emptyset,\emptyset)}}. \tag{2.11}$$

As for insertions, since arbitrary coherent sheaves on DT or PT moduli are very complicated, generally we use only specific insertions derived from $\mathsf{S}^\bullet$ or $\wedge^\bullet$ of the universal sheaf on DT or PT moduli.

Definition 2.11. Let $\mathcal{I} \to \mathbb{C}^3 \times \mathsf{Ideals}(\mathbb{C}^3)$ be the universal ideal sheaf for the fine moduli space $\mathsf{Ideals}(\mathbb{C}^3)$. Let $\pi\colon \mathbb{C}^3 \times \mathsf{Ideals}(\mathbb{C}^3) \to \mathsf{Ideals}(\mathbb{C}^3)$ be the projection and define

$$\mathcal{U}niv \coloneqq \pi_*\mathcal{I}.$$

spaces. Let $\mathcal{U}niv$ also denote the analogous construction for PT moduli spaces. In both cases, if π is a box configuration, a standard calculation shows

$$\mathcal{U}niv\Big|_\pi = 1 - (1-x)(1-y)(1-z)\chi_\pi \in \Bbbk.$$

Vertices with insertions of the form $\mathsf{S}^\bullet_u(f(\mathcal{U}niv))$, where f is any polynomial, are called **descendent vertices**.

There have been very few results or even conjectures on DT/PT correspondences for descendent vertices, e.g. see [24, Conjecture 5.3.1] for one in cohomology. In § 4.2 we provide and prove something resembling such a correspondence for the 1-legged case (though it should hold for the general 3-legged case too) in the CY limit, and speculate on how to deform it away from the CY limit.

2.2 Quasimap theory

The space $\mathsf{Hilb}(\mathbb{C}^2)$ of boundary conditions λ, μ, ν for DT and PT theories is an example of a *Nakajima quiver variety.* These are remarkable algebraic symplectic varieties with rich geometric representation theory. We review their salient properties, in preparation for § 2.3. Then we define *quasimaps* to Nakajima quiver varieties. Quasimaps are a form of curve counting which more closely resembles the venerable Gromov–Witten theory of stable maps. In the special case of $\mathsf{Hilb}(\mathbb{C}^2)$, the resulting moduli of quasimaps can be taken as the definition of PT 1-legged moduli, whose definition was not provided in § 2.1.2.

2.2.1 Nakajima quiver varieties

Definition 2.12. Let Q be a quiver. The **Nakajima quiver variety** $\mathcal{M}_Q$ is the algebraic symplectic reduction of

$$N_Q \coloneqq \bigoplus_{\text{vertices } i} T^* \operatorname{Hom}(W_i, V_i) \oplus \bigoplus_{\text{edges } i \to j} T^* \operatorname{Hom}(V_i, V_j) \tag{2.12}$$

by the group $\mathsf{G}_{\mathbf{V}} \coloneqq \prod_i \operatorname{GL}(V_i)$. More explicitly, $\mathcal{M}_Q$ is the GIT quotient

$$\mathcal{M}_Q \coloneqq \mu^{-1}(0) /\!\!/_\theta \mathsf{G}_{\mathbf{V}}$$

where $\mu\colon N_Q \to \mathfrak{g}_{\mathbf{V}}^*$ is the moment map for the $\mathsf{G}_{\mathbf{V}}$-action and $\theta\colon \mathbb{C}^\times \to \mathsf{G}_{\mathbf{V}}$ is a GIT stability parameter which we suppress from the notation. The *dimension vectors* $\mathbf{v} = (\dim V_i)_i$ and $\mathbf{w} = (\dim W_i)_i$ index irreducible components $\mathcal{M}_Q(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}_Q$, and we let $\mathcal{M}_Q(\mathbf{w}) \coloneqq \bigsqcup_{\mathbf{v}} \mathcal{M}_Q(\mathbf{v}, \mathbf{w})$.

Let $\mathbf{V} \coloneqq \bigoplus_i V_i$ and $\mathbf{W} \coloneqq \bigoplus_i W_i$, and equip M with coordinates

$$(\mathbf{a}, \mathbf{a}^*) \in T^* \operatorname{Hom}(\mathbf{W}, \mathbf{V}) \cong \operatorname{Hom}(\mathbf{W}, \mathbf{V}) \oplus \operatorname{Hom}(\mathbf{V}, \mathbf{W})$$
$$(\mathbf{b}, \mathbf{b}^*) \in T^* \operatorname{Hom}(\mathbf{V}, \mathbf{V}) \cong \operatorname{Hom}(\mathbf{V}, \mathbf{V}) \oplus \operatorname{Hom}(\mathbf{V}, \mathbf{V}).$$

The moment map in these coordinates is

$$\mu(\mathbf{a}, \mathbf{a}^*, \mathbf{b}, \mathbf{b}^*) = [\mathbf{b}, \mathbf{b}^*] + \mathbf{a}\mathbf{a}^* \in \mathfrak{g}_{\mathbf{V}} \cong \mathfrak{g}_{\mathbf{V}}^* \tag{2.13}$$

where the identification is by the standard pairing $\langle -, - \rangle \colon \mathfrak{g}_{\mathbf{V}} \times \mathfrak{g}_{\mathbf{V}} \to \mathbb{C}$. Importantly, μ is a quadratic function.

The W_i can be viewed as additional *framing* vertices in Q, one for each existing vertex. They carry the action of $\mathsf{G}_{\mathbf{W}} \coloneqq \prod_i \mathrm{GL}(W_i)$. More generally, there is an action of

$$\mathsf{G} \coloneqq \mathbb{C}^\times_\hbar \times \mathsf{G}_{\mathbf{W}} \times \cdots$$

on M which descends to $\mathcal{M}_Q$, where $\mathbb{C}^\times_\hbar$ acts by scaling the symplectic form with weight $\hbar^{-1}$ and $\cdots$ represents extra equivariance which may appear for certain $\mathcal{M}_Q$. Let $\mathsf{T}_{\mathbf{W}} \subset \mathsf{T}$ be the maximal tori of $\mathsf{G}_{\mathbf{W}} \subset \mathsf{G}$ respectively.

Example 2.13. The simplest Nakajima quiver varieties, and also the ones of primary importance to us, arise when Q is a *type* A_1 *quiver*, i.e. Q has a single node with possibly some loops.

- If there are no loops, $\mathcal{M}_Q(w) = T^* \mathsf{Gr}(w) \coloneqq \bigoplus_{k=0}^w T^* \mathsf{Gr}(k, w)$ is the Grassmannian of dimension-k subspaces in $\mathbb{C}^w$.

- If there is one loop, $\mathcal{M}_Q(w) = \mathsf{Inst}_w(\mathbb{C}^2) = \bigoplus_{n \geq 0} \mathsf{Inst}_w(\mathbb{C}^2, n)$ is the moduli of rank-w instantons on $\mathbb{C}^2$ (of charge n). Here the $\mathbb{C}^\times_\hbar$ symmetry embeds diagonally into a larger torus $\mathbb{C}^\times_x \times \mathbb{C}^\times_y$ which scales the two coordinates of $\mathbb{C}^2$ with weights x and y respectively, so that $\hbar^{-1} = xy$. When the rank is $w = 1$, the moduli of instantons is better known as the *Hilbert scheme of points* $\mathsf{Hilb}(\mathbb{C}^2)$.

Definition 2.14. In circumstances where we need to work with morphisms of quiver representations, a point $(\mathbf{a}, \mathbf{a}^*, \mathbf{b}, \mathbf{b}^*) \in \mathcal{M}_Q$ will sometimes be denoted as

$$\left(\mathbf{V} \underset{\mathbf{a}}{\overset{\mathbf{a}^*}{\rightleftarrows}} \mathbf{W}\right) \in \mathcal{M}_Q,$$

or simply $(\mathbf{V} \rightleftarrows \mathbf{W})$, with the understanding that $\mathbf{V}$ includes the data of quiver maps $\mathbf{b}, \mathbf{b}^*$. Let $\mathcal{T}aut_i$ denote the tautological bundle with fiber V_i over this point.

As a GIT quotient, $\mathcal{M}_Q = \mu^{-1}(0) /\!\!/_\theta \mathsf{G}_\mathbf{V}$ can be viewed as the θ-stable locus in the associated stack

$$\mathfrak{M}_Q \coloneqq [\mu^{-1}(0)/\mathsf{G}_\mathbf{V}].$$

We denote the θ-stable locus in a space X by X^s, since θ is usually implicit.

Remark. GIT stability for quiver representations has a useful equivalent reformulation in terms of slope stability of sub-objects, by work of King and Crawley–Bovey (see [13, Section 3.2]). The reformulated stability criterion is generally much more combinatorial.

Proposition 2.15.

$$T_{(\mathbf{V}\rightleftarrows\mathbf{W})}\mathcal{M}_Q = T_{(\mathbf{V}\rightleftarrows\mathbf{W})}N_Q - (1+\hbar)\bigoplus_i V_i^* V_i.$$

Proof. Since $X^s \subset X$ is open, $T_p X^s = T_p X$ for points $p \in X^s$. Applying this to $\mathcal{M}_Q \subset \mathfrak{M}_Q$ the desired formula, where the second term includes both the moment map equation $\mu = 0$ and the quotient by $\mathsf{G}_\mathbf{V}$. Since N_Q is an affine space, the first term is essentially just (2.12) itself. □

Definition 2.16. Let $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$ be a splitting, with associated sub-torus $\mathbb{C}_u^\times \subset \mathsf{T}_\mathbf{W}$

which acts trivially on the factors in $\mathbf{w}'$ and by weight u on the factors in $\mathbf{w}''$. Then

$$\mathcal{M}_Q(\mathbf{w})^{\mathbb{C}_u^\times} = \mathcal{M}_Q(\mathbf{w}') \times \mathcal{M}_Q(\mathbf{w}'') \subset \mathcal{M}_Q(\mathbf{w}). \tag{2.14}$$

There is a partial ordering on components $Z_\eta := \mathcal{M}_Q(\eta, \mathbf{w}') \times \mathcal{M}_Q(\mathbf{v} - \eta, \mathbf{w}'') \subset \mathcal{M}_Q(\mathbf{v}, \mathbf{w})$ given by

$$Z_\eta > Z_{\eta'} \iff \theta \cdot \eta > \theta \cdot \eta'. \tag{2.15}$$

The minimal component is called the **vacuum** component.

2.2.2 Quasimaps from $\mathbb{C}$

Let $C := \mathbb{P}^1$ with coordinate z around zero, and let $\mathbb{C}_z^\times$ act on C by scaling z (with weight also denoted z).

Definition 2.17. Let Q be a quiver. A **(stable) quasimap** $f\colon C \dashrightarrow \mathcal{M}_Q$ is a map $f\colon C \to \mathfrak{M}_Q$ which generically lands in $\mathcal{M}_Q \subset \mathfrak{M}_Q$, i.e. $f(p) \notin \mathcal{M}_Q$ for only a finite set of points in C where we say the quasimap is *singular*. Concretely, this is the data of

- a collection $\boldsymbol{\mathcal{V}} = \bigoplus_i \mathcal{V}_i$ and $\boldsymbol{\mathcal{W}} = \bigoplus_i \mathcal{W}_i$ of vector bundles on C (where the $\mathcal{W}_i$ are always trivial), and
- a section of the bundle (cf. (2.12))

$$\bigoplus_{\text{vertices } i} T^* \operatorname{Hom}(\mathcal{W}_i, \mathcal{V}_i) \oplus \bigoplus_{\text{edges } i \to j} T^* \operatorname{Hom}(\mathcal{V}_i, \mathcal{V}_j).$$

Let $\mathsf{QMaps}(C \to \mathcal{M}_Q)$ denote the moduli space of these quasimaps. We will primarily be interested in the ones non-singular at $\infty \in C$, whose moduli space we denote by

$$\mathsf{QMaps}(\mathbb{C} \to \mathcal{M}_Q) \subset \mathsf{QMaps}(C \to \mathcal{M}_Q).$$

Understanding $\mathbb{C}_z^\times$-fixed loci in $\mathsf{QMaps}(\mathbb{C} \to \mathcal{M}_Q)$ will be important for equivariant localization later. Note that a $\mathbb{C}^\times_z$-fixed quasimap can only be singular at $0 \in C$, and must be constant everywhere else.

Proposition 2.18. *Let $f \in \mathsf{QMaps}(\mathbb{C} \to \mathcal{M}_Q)$ be a $\mathbb{C}^\times_z$-fixed quasimap to*

$$f(\infty) = (\mathbf{V} \underset{\mathbf{a}}{\overset{\mathbf{a}^*}{\rightleftarrows}} \mathbf{W}) \in \mathcal{M}_Q.$$

Then f is equivalent to a $\mathbb{Z}$-graded flag $(\mathbb{V} \underset{\mathbf{a}}{\overset{\mathbf{a}^*}{\rightleftarrows}} \mathbb{W})$ of quiver representations of the form

$$\begin{array}{ccccccccccccc}
\cdots & \xhookrightarrow{z} & \mathbb{V}[-2] & \xhookrightarrow{z} & \mathbb{V}[-1] & \xhookrightarrow{z} & \mathbb{V}[0] & \xhookrightarrow{z} & \mathbb{V}[1] & \xhookrightarrow{z} & \cdots & \xhookrightarrow{z} & \mathbb{V}[\infty] = \mathbf{V} \\
 & & & & & & a \uparrow \downarrow a^* & & a \uparrow \downarrow a^* & & & & a \uparrow \downarrow a^* \\
 & & & & & & \mathbf{W} & \overset{z}{=\!=} & \mathbf{W} & \overset{z}{=\!=} & \cdots & \overset{z}{=\!=} & \mathbb{W}[\infty] = \mathbf{W}.
\end{array} \tag{2.16}$$

Proof. Since $f\big|_{C\setminus\{0\}}$ is a non-singular and *constant* map, the bundles $\boldsymbol{\mathcal{V}}\big|_{C\setminus\{0\}}, \boldsymbol{\mathcal{W}}\big|_{C\setminus\{0\}}$ are trivial and, along with their quiver maps, can be identified with the vector spaces $\mathbf{V}, \mathbf{W}$ and their (identical) quiver maps. Hence the only non-trivial data of f are the $\mathbb{C}[z]$-modules

$$\mathbb{V}_k \coloneqq \bigoplus_{n \in \mathbb{Z}} \mathbb{V}_k[n] \coloneqq H^0\left(\mathcal{V}_k\big|_{C\setminus\{\infty\}}\right),$$

where $\mathbb{V}_k[n]$ is the weight-n piece. Multiplication by z induces embeddings

$$\mathbb{V}_k[n] \hookrightarrow \mathbb{V}_k[n+1] \hookrightarrow \cdots \hookrightarrow \mathbb{V}_k[\infty] = V_k \tag{2.17}$$

compatible with quiver maps. Repeating with $\mathbb{W}_k \coloneqq H^0(\mathcal{W}_k\big|_{C\setminus\{\infty\}})$ gives the bottom row of (2.16), since $\mathbb{W}_k[n] = W_k$ for $n \geq 0$ and zero otherwise. □

Example 2.19. When $\mathcal{M}_Q = \mathsf{Hilb}(\mathbb{C}^2)$, flags (2.16) which are also fixed under the natural torus T acting on $\mathbb{C}^2$ correspond directly to 1-legged PT fixed points as in Definition 2.7. The leg goes along the z direction with profile given by the partition $f(\infty) \in \mathsf{Hilb}(\mathbb{C}^2)^{\mathsf{T}}$. In fact one can take $\mathsf{QMaps}(\mathbb{C} \to \mathsf{Hilb}(\mathbb{C}^2))$ as the *definition* of the 1-legged PT moduli space, which was left undefined in § 2.1.2.

Remark. The identification in Example 2.19 of PT theory on $\mathbb{C}^3$ with quasimaps to $\mathsf{Hilb}(\mathbb{C}^2)$ is a special case of a more general equivalence [19] of quasimaps with sheaf counting theories on threefolds, which can be formulated for ADE surface fibrations over curves.

One can define an equivariant *quasimap vertex* (with insertions) in complete analogy to Definition 2.9 for DT/PT vertices. We will not do so explicitly here.

2.3 Quantum affine algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this section, assuming some familiarity with the Drinfeld–Jimbo quantum group $\mathscr{U}_q(\mathfrak{g})$, we review the quantum affine algebra $\mathscr{U}_q(\widehat{\mathfrak{g}})$ in preparation for the *shifted* quantum affine algebras in § 3.3. Quantum affine algebras are simultaneously:

- the quantum group associated to the affine Lie algebra $\widehat{\mathfrak{g}}$;
- an affinized version of the quantum group $\mathscr{U}_q(\mathfrak{g})$.

The former perspective yields the *Jimbo presentation* of $\mathscr{U}_q(\widehat{\mathfrak{g}})$ while the latter yields the *Drinfeld presentation*. Our preferred approach to $\mathscr{U}_q(\widehat{\mathfrak{g}})$, described in § 2.3.2, is geometric rather than algebraic but is more closely aligned with Drinfeld's presentation, which we now give for completeness.

Definition 2.20. Let $C = (c_{ij})_{i,j\in I}$ be the Cartan matrix for $\mathfrak{g}$, and factor it as $C = DB$ where $D = \operatorname{diag}((d_i)_{i\in I})$ is diagonal and B is symmetric. The **quantum affine algebra**

$$\mathscr{A} \coloneqq \mathscr{U}_q(\widehat{\mathfrak{g}})$$

is generated by coefficients of the *Drinfeld currents*

$$e_i(u) \coloneqq \sum_{r\in\mathbb{Z}} e_{i,r}u^{-r}, \quad f_i(u) \coloneqq \sum_{r\in\mathbb{Z}} f_{i,r}u^{-r}, \quad \psi_i^{\pm}(u) \coloneqq \sum_{r\geq 0} \psi_{i,\pm r}^{\pm}u^{\mp r}$$

for all $i \in I$. For brevity, set $x_i^+ \coloneqq e_i$ and $x_i^- \coloneqq f_i$. Setting $q_i \coloneqq q^{d_i}$, the defining relations are

$$\psi_{i,0}^{\pm} \text{ invertible}, \quad [\psi_i^s(u), \psi_j^{s'}(v)] = 0 \;\; \forall s, s' \in \{\pm\}, \tag{2.18a}$$

$$(u - q_i^{\pm c_{ij}}v)x_i^{\pm}(u)x_j^{\pm}(v) = (q_i^{\pm c_{ij}}u - v)x_j^{\pm}(v)x_i^{\pm}(u), \tag{2.18b}$$

$$(u - q_i^{\pm c_{ij}}v)\psi_i^s(u)x_j^{\pm}(v) = (q_i^{\pm c_{ij}}u - v)x_j^{\pm}(v)\psi_i^s(u) \;\; \forall s \in \{\pm\}, \tag{2.18c}$$

$$[x_i^+(u), x_j^-(v)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\delta(u/v)\left(\psi_i^+(u) - \psi_i^-(u)\right), \tag{2.18d}$$

$$\mathrm{Sym}_{u_1,\ldots,u_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \binom{1-c_{ij}}{r}_{q_i} x_i^{\pm}(u_1)\cdots x_i^{\pm}(u_r)x_j^{\pm}(v)x_i^{\pm}(u_{r+1})\cdots x_i^{\pm}(u_{1-c_{ij}}) = 0. \tag{2.18e}$$

It is common to define an alternate set of generators $\{h_{i,m}\}_{i\in I,\, m\in\mathbb{Z}}$ in place of $\{\psi_{i,r}^{\pm}\}_{i\in I, r\in\mathbb{Z}}$, by

$$\psi_i^{\pm}(u) =: q_i^{\pm h_i} \exp\left(\pm(q_i - q_i^{-1})\sum_{m\geq 1} h_{i,\pm m}u^{\mp m}\right). \tag{2.19}$$

(Set $h_{i,0} \coloneqq h_i$). Here $\{h_i\} \subset \mathfrak{h}$ is the usual basis for the finite Cartan.

Remark. Affine Lie algebras $\widehat{\mathfrak{g}}$ have a central element C. We have set $C = 1$ in our $\mathscr{U}_q(\widehat{\mathfrak{g}})$, which is sometimes called the quantum *loop* algebra and denoted $\mathscr{U}_q(L\mathfrak{g})$ to distinguish it from the more general algebra where C is present.

A similar construction for arbitrary $\mathfrak{g}$ leads to quantum *affinized* algebras. If the original Lie algebra is itself an affine Lie algebra, its quantum affinization yields quantum *toroidal* algebras $\mathscr{U}_q(\widehat{\widehat{\mathfrak{g}}})$. The only quantum toroidal algebra of importance to us is the remarkable algebra $\mathscr{U}_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ described in § 4.2.1, which does not actually arise this way as $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$ is not an affine Lie algebra. Nonetheless, all discussion about $\mathscr{U}_q(\widehat{\mathfrak{g}})$ is equally applicable to

$\mathscr{U}_q(\widehat{\widehat{\mathfrak{gl}}}_1)$ unless indicated otherwise.

Definition 2.21. There are some sub-algebras of $\mathscr{A}$ of interest, particularly for the study of $\mathscr{A}$-modules.

- The **positive** (resp. **negative**) part $\mathscr{A}_+ \subset \mathscr{A}$ (resp. $\mathscr{A}_- \subset \mathscr{A}$) is the sub-algebra generated by all Drinfeld currents $e_i(u)$ (resp. $f_i(u)$).
- The **loop Cartan** $\mathscr{A}_0 \subset \mathscr{A}$ is the sub-algebra generated by the commuting currents $\psi_i^\pm(u)$. Note that, from (2.19), it contains elements h_i of the finite Cartan $\mathfrak{h} \subset \mathfrak{g}$ in the form $q^{h_i} = \psi_i^+(0)$.
- The **quantum affine Borel** $\mathscr{B} \coloneqq \mathscr{U}_q(\widehat{\mathfrak{b}}) \subset \mathscr{A}$ is the usual (positive) Borel in the Drinfeld–Jimbo sense. For $\mathfrak{g} = \mathfrak{sl}_2$ [1] it is known that $\mathscr{B}$ is generated by

$$\{e_{i,r}, f_{i,s}, \psi_{i,r}^+, (\psi_{i,0}^+)^{-1} \mid i \in I,\ r \geq 0,\ s > 0\}$$

  but in general this is only a sub-algebra of $\mathscr{B}$. (The $\mathfrak{g} = \widehat{\mathfrak{gl}_1}$ case is discussed in § 4.2.1.)

Recall that $\mathscr{A}$ carries a standard coproduct Δ coming from its presentation as a Drinfeld–Jimbo quantum group. In general, there is no known explicit formula for Δ on Drinfeld generators, but it satisfies some triangularity properties the most important of which is

$$\Delta(\psi_i^\pm(u)) \in \psi_i^\pm(u) \otimes \psi_i^\pm(u) + \mathscr{A}_+ \otimes \mathscr{A}_-. \tag{2.20}$$

Later in Example 2.31 we provide a proof of this triangularity for a certain class of $\mathscr{A}$.

Remark. It is possible to replace Δ with a *Drinfeld coproduct* Δ^∞ which takes a very simple form on Drinfeld generators, but we will not do so. In the geometric R-matrix language of § 2.3.2, Δ^∞ arises from the infinite-slope R-matrix while Δ arises from the *geometric* slope around zero.

2.3.1 Modules and their properties

Our primary interest in the quantum affine algebras $\mathscr{A} = \mathscr{U}_q(\widehat{\mathfrak{g}})$ stems from their action on the K-theory of Nakajima quiver varieties, as in [22] or, later, Theorem 2.29, so we begin by reviewing some facts about their modules in general.

Definition 2.22. An **ℓ-weight** is a collection

$$\Psi \coloneqq \left\{ \Psi_i(u) \coloneqq \sum_{m\geq 0} \Psi_{i,m} u^m \;\middle|\; \Psi_{i,m} \in \mathbb{C},\ \Psi_{i,0} \neq 0 \right\}_{i\in I}.$$

Given a $\mathscr{A}$-module V, its **ℓ-weight space** of ℓ-weight Ψ is

$$V_\Psi \coloneqq \left\{ v \in V \;\middle|\; (\psi^+_{i,r} - \Psi_{i,r})^p v = 0 \ \ \forall p \gg 0,\, i \in I,\, r \geq 0 \right\}.$$

We say V has a **non-trivial** ℓ-weight Ψ if $V_\Psi \neq 0$. If V is generated by a vector v_0 such that

$$\mathscr{A}^+ v_0 = 0, \quad \psi_i^{\pm}(u) v_0 = \Psi_i(u) v_0 \ \forall i \in I.$$

then it is of **highest weight** Ψ. We denote the unique simple module of highest weight Ψ by $L(\Psi)$. The analogous definition applies for simple modules of **lowest weight** Ψ, denoted $L^\vee(\Psi)$.

Definition 2.23. The **category** $\mathcal{O} = \mathcal{O}(\mathscr{A})$ consists of all $\mathscr{A}$-modules V such that:

- V is a sum of its weight spaces $V_\omega \coloneqq \{v \in V \mid q^{h_i} v = \omega(h_i) v \ \forall i \in I\}$ for $\omega \in \mathfrak{h}^*$;
- $\dim V_\omega < \infty$ for all $\omega \in \mathfrak{h}^*$;
- there is a finite set of weights $\{\omega_i\}_{i=1}^n \subset \mathfrak{h}^*$ such that if $\dim V_\omega > 0$ then $\omega \leq \omega_i$ for some i.

The *dual* **category** $\mathcal{O}^\vee$ consists of all V such that $V^* \in \mathcal{O}$. When we wish to treat $V \in \mathcal{O}$ as a $\mathscr{B}$-module via restriction we will write $V \in \mathcal{O}(\mathscr{B})$, and similarly for $\mathcal{O}^\vee(\mathscr{B})$.

Proposition 2.24. *Let $\mathfrak{r}$ be the set of all ℓ-weights Ψ whose components $\Psi_i(u)$ are all rational functions of u.*

- *$L(\Psi) \in \mathcal{O}$ iff $\Psi \in \mathfrak{r}$, and the same for $L^\vee(\Psi) \in \mathcal{O}^\vee$.*
- *If $V \in \mathcal{O}$ then all its non-trivial ℓ-weights are in $\mathfrak{r}$.*
- *If $V \in \mathcal{O}$ is finite-dimensional then all of its non-trivial ℓ-weights can be written in the form*
$$\Psi_i(u) = q_i^{\deg P_i - \deg Q_i} \frac{P_i(u)}{P_i(uq_i^2)} \frac{Q_i(uq_i^2)}{Q_i(u)}, \quad P_i, Q_i \in 1 + u\mathbb{C}[u]. \tag{2.21}$$

Definition 2.25. Let V be an $\mathscr{A}$-module and let S be the antipode of $\mathscr{A}$.

- The element $a \in \mathscr{A}$ acts on the **left dual** V^* as $S(a)^t$.
- The element $a \in \mathscr{A}$ acts on the **right dual** ${}^*V$ as $S^{-1}(a)^t$.

In general $S^2 \neq \mathrm{id}$, so left and right duals are distinct. Their names indicate which tensor factor the dual appears in for the evaluation maps

$$\mathrm{ev}\colon V^* \otimes V \to \mathbb{C}, \quad \mathrm{ev}\colon V \otimes {}^*V \to \mathbb{C}.$$

The flipped versions $V \otimes V^* \to \mathbb{C}$ and ${}^*V \otimes V \to \mathbb{C}$ are not $\mathscr{A}$-linear.

Lemma 2.26. *For $\Psi \in \mathfrak{r}$,*

$${}^*(L(\Psi)) = L^\vee(\Psi^{-1}), \quad L^\vee(\Psi)^* = L(\Psi^{-1}). \tag{2.22}$$

Proof. We prove the second statement; the first is analogous. Let $V = L^\vee(\Psi)$ have lowest weight vector v_0. Then $V^* = L(\Phi)$ is a highest weight module, whose highest weight vector v_0^* is dual to v_0. The evaluation pairing sends

$$v_0^* \otimes v_0 \mapsto 1.$$

Using the triangularity (2.20) of the coproduct, compute

$$\psi^{\pm} \cdot (v_0^* \otimes v_0) = \Psi\Phi \cdot (v_0^* \otimes v_0). \tag{2.23}$$

It follows that $\Psi\Phi = 1$, so $\Phi = \Psi^{-1}$. □

Remark. The other permutations of duals and highest/lowest weights in (2.22) yield false statements, e.g. in general $L(\Psi)^* \neq L^\vee(\Psi^{-1})$ because (2.23) may contain extra terms. However, setting $L(\Psi)^* = L^\vee(\Psi')$, it is possible (for $\mathfrak{g}$ finite type) to derive a formula for Ψ', see [9, Corollary 6.9].

2.3.2 Geometric R-matrix

Let $\mathfrak{V}$ be a set of vector spaces closed under tensor products and duals. Suppose that there is a matrix

$$R_{W(u_1),V(u_2)} = R_{W,V}(u_1/u_2) \in \mathrm{End}(W \otimes V) \otimes \Bbbk[u_1^{\pm}, u_2^{\pm}]$$

associated to any pair $W, V \in \mathfrak{V}$, such that:

- (compatibility with tensor product) for any $W, V, V' \in \mathfrak{V}$,

$$R_{W(u_1),U(u_2)\otimes V(u_3)} = R^{(12)}_{W(u_1),U(u_2)} R^{(13)}_{W(u_1),V(u_3)} \tag{2.24}$$

  and similarly for $R_{W(u_1)\otimes U(u_2),V(u_3)}$;

- (compatibility with duals) for any $W, V \in \mathfrak{V}$,

$$R_{W^*,V} = \left((R_{W,V})^{-1}\right)^{t_1}, \quad R_{W,V^*} = \left((R_{W,V})^{-1}\right)^{t_2}, \quad R_{W^*,V^*} = (R_{W,V})^{t_{12}} \tag{2.25}$$

  where t_k means transpose with respect to the k-th factor.

Proposition 2.27 (FRT reconstruction, [28]). *Let $\mathscr{A} \subset \prod_{V\in\mathfrak{V}} \operatorname{End}(V)$ be the sub-algebra generated by matrix elements*

$$\langle w_1|R_{W,V}(u)|w_2\rangle \in \operatorname{End}(V) \otimes \Bbbk[u^{\pm}]$$

taken in the "auxiliary space" W for all choices of $W \in \mathfrak{V}$. Then $\mathscr{A}$ is a Hopf algebra, with coproduct $\Delta\colon \mathscr{A} \to \mathscr{A}\widehat{\otimes}\mathscr{A}$ given by the restriction of the projection

$$\prod_{V\in\mathfrak{V}} \operatorname{End}(V) \to \prod_{V,V'\in\mathfrak{V}} \operatorname{End}(V\otimes V'). \tag{2.26}$$

All vector spaces $V \in \mathfrak{V}$ thereby become $\mathscr{A}$-modules.

Let $\mathcal{M} \coloneqq \mathcal{M}_Q(\mathbf{w})$ be a Nakajima quiver variety and $\mathsf{T} = \mathbb{C}^\times_\hbar \times \mathsf{T}_\mathbf{W} \times \cdots$ be the torus acting on it. Denote

$$V_\mathbf{w}(\mathbf{u}) \coloneqq K_\mathsf{T}(\mathcal{M}_Q(\mathbf{w}))$$

where $\mathbf{u} = (u_1, \ldots, u_r)$ are the weights of $\mathsf{T}_\mathbf{w}$. There is a remarkable collection of geometric correspondences which will yield R-matrices for $\{V_\mathbf{w}(\mathbf{u})\}_\mathbf{w}$, as follows.

Definition 2.28 ([25, Section 2.1]). Let X be an algebraic symplectic variety with the action of a torus T. Let $\hbar^{-1}$ be the weight of the symplectic form, and $\mathsf{A} \coloneqq \ker(\hbar) \subset \mathsf{T}$. Denote $\mathscr{A}$-fixed components by $F \subset X^\mathsf{A}$. Fix the data of:

- an *attracting chamber* $\mathfrak{C} \subset \mathsf{Lie}(\mathsf{A})$, which defines attracting sets

  $$\operatorname{Attr}_\mathfrak{C}(F) \coloneqq \{x \in F \mid \lim_{t\to 0} \sigma(t) \cdot x \in F \ \forall \sigma \in \mathfrak{C}\},$$

  and splits the normal bundle $\mathcal{N}_{F/X} = \mathcal{N}_+ \oplus \mathcal{N}_-$ into attracting and repelling directions respectively;

- a *polarization* $T^{1/2} \in K_\mathsf{T}(X)$, which chooses "half" the tangent bundle by satisfying $T_X = T^{1/2} + \hbar \cdot (T^{1/2})^\vee$;

- a *slope* $s \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ which should be suitably generic. When unspecified, it lies in the anti-ample alcove next to 0 (see [25, Section 2.2]), and we call it a *geometric* slope.

The **stable envelope** $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}, s} \in K_{\mathsf{T}}(X \times X^{\mathsf{A}})$ is the unique K-theory class satisfying:

- (support) $\mathrm{supp}\, \mathrm{Stab}\big|_{X \times F}$ is the *full attracting set*

$$\mathrm{Attr}^f_{\mathfrak{C}}(F) := \bigsqcup_{F' \preceq F} \mathrm{Attr}_{\mathfrak{C}}(F'),$$

  where the attracting partial order $F' \preceq F$ means $F' \subset \overline{\mathrm{Attr}_{\mathfrak{C}}(F)}$;

- (normalization) if $T^{1/2}_{\neq 0} = T^{1/2}_{+} \oplus T^{1/2}_{-} \subset T^{1/2}\big|_F$ is the splitting of the non-trivial A-weights into attracting and repelling ones,

$$\mathrm{Stab}\bigg|_{F \times F} = (-1)^{\mathrm{rk}\, T^{1/2}_{+}} \sqrt{\frac{\det N_-}{\det T^{1/2}_{\neq 0}}} \otimes \mathcal{O}_{\mathrm{Attr}}\bigg|_{F \times F}; \tag{2.27}$$

- (degree condition) for $F_1, F_2 \subset X^{\mathsf{A}}$,

$$\deg_{\mathsf{A}} \mathrm{Stab}\bigg|_{F_2 \times F_1} \otimes s\bigg|_{F_1} \subset \deg_{\mathsf{A}} \mathrm{Stab}\bigg|_{F_2 \times F_2} \otimes s\bigg|_{F_2}$$

  where $\deg_{\mathsf{A}}$ of a Laurent polynomial $f \in K_{\mathsf{A}}(\mathrm{pt})$ is the convex hull of its non-zero weights.

Convolution by Stab is an operator $K_{\mathsf{T}}(X^{\mathsf{A}}) \to K_{\mathsf{T}}(X)$. By localization, it is equivalently an operator in $\mathrm{End}(K_{\mathsf{T},\mathrm{loc}}(X^{\mathsf{A}}))$ which we also denote Stab.

Theorem 2.29 ([25]). *Split* $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$ *and let* $\mathbb{C}^\times_u \subset \mathsf{T}_{\mathbf{w}}$ *be the associated sub-torus as in Definition 2.16. Let* $\mathfrak{C}_\pm$ *be its two chambers and*

$$\mathrm{Stab}_{\mathfrak{C}} \in \mathrm{End}(V_{\mathbf{w}'}(1) \otimes V_{\mathbf{w}''}(u))$$

be the corresponding stable envelopes at a slope s. Then the collection of operators

$$R_{\mathfrak{C}_-\leftarrow\mathfrak{C}_+}(u) \coloneqq \mathrm{Stab}_{\mathfrak{C}_-}\cdot(\mathrm{Stab}_{\mathfrak{C}_+})^{-1} \in \mathrm{End}(V_{\mathbf{w}'}(1)\otimes V_{\mathbf{w}''}(u)) \quad \forall \mathbf{w}',\mathbf{w}''$$

satisfy the conditions for FRT reconstruction. The reconstructed quantum group is independent of the slope s as an algebra, but not as a Hopf algebra.

We refer to these operators as **geometric R-matrices** and write them in the basis with partial ordering $>$ as in Definition 2.16. The reconstructed quantum group for $\mathcal{M}_Q$ is the quantum affine algebra denoted $\mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$, where $q \coloneqq \sqrt{\hbar}$, which therefore acts on all $V_{\mathbf{w}}(u)$. Compatibility of stable envelopes with splittings such as (2.14) then implies that these $V_{\mathbf{w}}(u)$ are tensor products of

$$V_i(u) \coloneqq V_{\mathbf{w}_i}(u)$$

where $\mathbf{w}_i = (0,\ldots,0,1,0,\ldots,0)$ is non-zero except for at the i-th vertex.

Remark. When Q is of Lie type with associated Lie algebra $\mathfrak{g}$,

$$\mathscr{U}_q(\widehat{\mathfrak{g}}) = \mathscr{U}_q(\widehat{\mathfrak{g}}_Q)/(\text{some central elements}). \tag{2.28}$$

These central elements act by scalars which are functions only of the framing $\mathbf{W}$. They can be implicitly modded out from $\mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$ by a certain normalization of the geometric R-matrix, which has the effect of scaling the original loop Cartan generator $\psi_i^{\pm}(u)$ by an extra factor of $\mathsf{S}_u^{\bullet}(-(1-\hbar)W_i)$. (See [20, Section 6.1] for a discussion in cohomology).

If a quantum affine algebra $\mathscr{A} = \mathscr{U}_q(\widehat{\mathfrak{g}})$ (for $\mathfrak{g}$ not necessarily of finite type) can be obtained by FRT reconstruction from a geometric R-matrix, we say it is **of geometric type**. In finite type this means $\mathfrak{g}$ is ADE, i.e. simply-laced. Many properties that hold for quantum affine algebras in general are easily proved for those of geometric type by arguments directly from the R-matrix.

Proposition 2.30 (Khoroshkin–Tolstoy factorization)**.** *Let $R(u) \in \mathrm{End}(V \otimes V)$ be a geometric R-matrix. Then it has a Gauss factorization (chosen to converge around either $u = 0$ or $u = \infty$)*

$$R = R_- R_\infty R_+ \tag{2.29}$$

such that $R_\pm$ and R_∞ are in the image of $\mathscr{A}_\pm \widehat{\otimes} \mathscr{A}_\mp$ and $\mathscr{A}_0 \widehat{\otimes} \mathscr{A}_0$ respectively.

The matrix elements of $R_\pm$ and R_∞ in the Gauss factorization around $u = \infty$ generate the (positive) quantum affine Borel $\mathscr{B}$, and this can be taken as an alternate definition of $\mathscr{B}$ when $\mathscr{A}$ is of geometric type (cf. Definition 2.21).

Example 2.31. For $\mathscr{A}$ of geometric type, let $\mathfrak{W}$ be the set of its simple highest weight modules. The *vacuum* matrix elements

$$\langle w_0 | R_{W,V} | w_0 \rangle = \langle w_0 | (R_{W,V})_\infty | w_0 \rangle \in \mathrm{End}(V)$$

for highest weight vectors $w_0 \in W$, ranging over all $W \in \mathfrak{W}$, together generate the image of $\mathscr{A}_0$. As w_0 is highest weight, it follows that

$$\Delta(\langle w_0 | R | w_0 \rangle) = \langle w_0 | R^{(13)} R^{(12)} | w_0 \rangle = \langle w_0 | R^{(12)} | w_0 \rangle \otimes \langle w_0 | R^{(13)} | w_0 \rangle + \cdots \in \mathrm{End}(V \otimes V)$$

where $\cdots$ contains terms of the form $\langle w_0 | R^{(12)} | w \rangle \otimes \langle w | R^{(13)} | v_0 \rangle$ for $w \neq w_0$. This is exactly the triangularity (2.20) of the coproduct stated in $\mathrm{End}(V)$ (instead of in $\mathscr{A}$).

2.3.3 Weights and tautological bundles

Let $\mathscr{A}$ be of geometric type. Given a geometric R-matrix $R \in \mathrm{End}(W \otimes V)$, it is possible to explicitly compute the diagonal operator R_∞ in the Gauss decomposition $R = R_- R_\infty R_+$. This is important since for $\mathfrak{g}$ of finite type, vacuum matrix elements of R_∞ will correspond directly to the Drinfeld generators $\psi_i^{\pm}(u)$, i.e.

$$\psi_i^{\pm}(u) = \langle w_0 | R_\infty | w_0 \rangle \in \mathrm{End}(V)$$

where $w_0 \in W$ is highest weight. The ℓ-weights of V will therefore be certain eigenvalues of R_∞.

Remark. When $\mathfrak{g}$ is not of finite type, the $\psi_i^{\pm}(u)$ are generated by vacuum matrix elements of R_∞ but are in general *not* the vacuum matrix elements themselves. In the case of $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$, this can be seen in § 4.2.2.

Proposition 2.32. *Let $v \otimes v' \in V_{\mathbf{w}}(1) \otimes V_{\mathbf{w}'}(u)$, and N_- denote the repelling part of the normal bundle to the associated fixed point in $\mathcal{M}_Q(\mathbf{w}) \times \mathcal{M}_Q(\mathbf{w}')$. Then*

$$\langle v \otimes v' | R_\infty | v \otimes v' \rangle = \widehat{\mathsf{S}}^\bullet \left((1 - \hbar^{-1}) N_+ \right).$$

Proof. Elements of R_∞ come from the diagonal parts of the stable envelopes forming R. The desired formula follows from the normalization (2.27) on these diagonal parts, see [25, Section 2.3.6]. □

Example 2.33. From Proposition 2.15, if $\mathbf{V} \oplus \mathbf{V}'$ is the quiver data of a fixed point in $\mathcal{M}_Q(\mathbf{w}) \times \mathcal{M}_Q(\mathbf{w}')$ then

$$N_+ = \bigoplus_i W_i^* \mathbf{V}_i' + \hbar \bigoplus_i W_i' V_i^* - \bigoplus_{i,j} C_{ij} V_i V_j' \tag{2.30}$$

for some matrix $\mathbf{C}_Q = (C_{ij})_{ij}$ (namely the "equivariant Cartan matrix" of Q, see [20, Section 2.2.5]). When the stability θ has all positive components, the highest weight element v_0 corresponds to the fixed component with quiver data $\mathbf{V} = 0$, by (2.15), so that the vacuum matrix element

$$\langle v_0 | R_\infty | v_0 \rangle = \widehat{\mathsf{S}}^\bullet \left((1 - \hbar^{-1}) \mathbf{W}^* \cdot \mathcal{T}aut \right)$$

is multiplication by some plethystic expression of the tautological bundle $\mathcal{T}aut$ (whose fibers are $\mathbf{V}$).

Recall that there is a discrepancy (2.28) between $\mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$ and $\mathscr{U}_q(\widehat{\mathfrak{g}})$. Accounting for this, for $\mathfrak{g}$ of finite type the actual geometric formula for $\psi_i^\pm(u) \in \mathscr{U}_q(\widehat{\mathfrak{g}})$ is

$$\psi_i^\pm(u) =: \widehat{\mathsf{S}}^\bullet_u \left((1-\hbar^{-1})\widehat{\mathcal{T}aut_i}\right) \tag{2.31}$$

where $\widehat{\mathcal{T}aut_i} := \mathcal{T}aut_i - W_i$ is the appropriate modification of $\mathcal{T}aut_i$. (See (4.14) for the analogue for $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$.) To match this with the form (2.21) for ℓ-weights Ψ, set

$$P_i(u) = \wedge^\bullet_{-1} W_i, \quad Q_i(u) = \wedge^\bullet_{-1} \mathcal{T}aut_i \tag{2.32}$$

and $q_i = q = \hbar^{1/2}$ (since $\mathfrak{g}$ is simply-laced). As $v \in V$ varies, the Q_i will change but the P_i remain fixed.

Remark. Importantly, the $q_i^{\deg P_i - \deg Q_i}$ prefactor only arises because we use $\widehat{\mathsf{S}}^\bullet$ instead of $\mathsf{S}^\bullet$. From (2.19), this prefactor is the eigenvalue of $q_i^{h_i}$, so that

$$\lambda(h_i) = \deg P_i - \deg Q_i \tag{2.33}$$

where λ is the non-affine part of the ℓ-weight Ψ.

2.3.4 Example: $\mathfrak{g} = \mathfrak{sl}_2$

For completeness, we now give a derivation from first principles of the quantum affine algebra $\mathscr{U}_q(\widehat{\mathfrak{sl}}_2)$ acting on $K_\mathsf{T}(T^* \mathsf{Gr}(2))$. As $T^* \mathsf{Gr}(2) = \{\mathrm{pt}\} \sqcup T^*\mathbb{P}^1 \sqcup \{\mathrm{pt}\}$, the only non-trivial component is $T^*\mathbb{P}^1$. Let $\mathsf{A} := \mathrm{diag}(1, u)$ act on $T^*\mathbb{P}^1$ with weights as depicted in Figure 2.2, and let $\mathbb{C}^\times_\hbar$ scale cotangent coordinates by $\hbar^{-1}$.

Example 2.34 (Stable envelope of $T^*\mathbb{P}^1$)**.** We give the explicit calculation of the stable envelope $\mathrm{Stab}_+ := \mathrm{Stab}_{\mathfrak{C}_+, T^{1/2}, s}$ for:

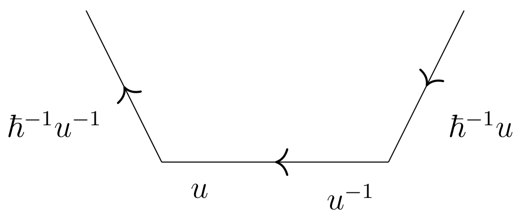


Figure 2.2: Toric diagram of $T^*\mathbb{P}^1$ with coordinates drawn at each chart. Arrows indicate the attracting direction of Example 2.34.

- the attracting chamber $\mathfrak{C}_+ \coloneqq \{u \to 0\}$;
- the polarization $T^{1/2}$ given by cotangent fibers;
- the "geometric" slope $s = -\epsilon \in \mathbb{R} = \mathrm{Pic}(T^*\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

A good way to compute $\mathrm{Stab} \in K_{\mathsf{T}}(X \times X^{\mathsf{A}})$ is to pick an "initial" $\mathcal{F} \in K_{\mathsf{T}}(X \times X^{\mathsf{A}})$ which satisfies the support and normalization conditions of Definition 2.28, but not necessarily the degree condition, and then to perform a "Gram–Schmidt" procedure to satisfy the degree condition. In this process we view stable envelopes as operators in $\mathrm{End}(K_{\mathsf{G},\mathrm{loc}}(X^{\mathsf{A}}))$ by localization.

Since it is easy to use the toric geometry to compute $\mathcal{O}_{\mathrm{Attr}(F)}\big|_{F'}$ for fixed points F, F', the elements $\mathcal{O}_{\mathrm{Attr}(F)}$ make good initial vectors (once scaled appropriately, according to the normalization (2.27)). The resulting matrix

$$\begin{pmatrix} | & | \\ \mathcal{O}_{\mathrm{Attr}(\infty)} & \mathcal{O}_{\mathrm{Attr}(0)} \\ | & | \end{pmatrix} = \begin{pmatrix} \sqrt{\hbar}(1-1/u) & 1-\hbar/u \\ 0 & 1-\hbar u \end{pmatrix}$$

is not yet a stable envelope, because the off-diagonal entry in red violates the slope condition that its u-degree is contained in $\deg_u(\sqrt{\hbar}(1-1/u)\cdot(1/u)^s) = (-\epsilon-1, -\epsilon) \subset \mathbb{R}$. Adding the appropriate multiple of the first column corrects for this, to yield

$$\mathrm{Stab}_+ = \begin{pmatrix} \sqrt{\hbar}(1-1/u) & 1-\hbar \\ 0 & 1-\hbar u \end{pmatrix}. \tag{2.34}$$

Similar work for the opposite attracting chamber $\mathfrak{C}_- \coloneqq \{u \to \infty\}$ yields

$$\mathrm{Stab}_- = \begin{pmatrix} 1 - \hbar/u & 0 \\ 1 - \hbar & \sqrt{\hbar}(1-u) \end{pmatrix}. \tag{2.35}$$

The $T^*\mathbb{P}^1$ component of the geometric R-matrix is $(\mathrm{Stab}_-)^{-1} \cdot \mathrm{Stab}_+$. From (2.34) and (2.35), the geometric R-matrix in full is therefore

$$R \coloneqq R_{V(1),V(u)} = \begin{pmatrix} 1 & & & \\ & \dfrac{\sqrt{\hbar}(1-u)}{\hbar - u} & \dfrac{u(\hbar-1)}{\hbar - u} & \\ & \dfrac{\hbar - 1}{\hbar - u} & \dfrac{\sqrt{\hbar}(1-u)}{\hbar - u} & \\ & & & 1 \end{pmatrix}. \tag{2.36}$$

This is the R-matrix for $\mathscr{U}_q(\widehat{\mathfrak{gl}}_2)$ in the representation $V(1) \otimes V(u)$ where V is the defining representation of $\mathfrak{gl}_2$.

Example 2.35 (Drinfeld generators)**.** We now compute Drinfeld generators from R-matrix elements in (2.36). The identification for $\mathscr{U}_q(\widehat{\mathfrak{gl}}_n)$ (and other types, with a bit more work) is given explicitly in [5]: Drinfeld generators are matrix elements of the terms $R_\pm, R_\infty$ in the Gauss decomposition of Proposition 2.30. For $n = 2$, this means

$$R = \begin{pmatrix} 1 & \\ e^\pm(u) & 1 \end{pmatrix} \begin{pmatrix} \psi_1^\pm(u) & \\ & \psi_2^\pm(u) \end{pmatrix} \begin{pmatrix} 1 & f^\pm(u) \\ & 1 \end{pmatrix}$$

where $e^\pm(u), f^\pm(u)$ are the expansion of $e(u), f(u)$ around $u^\mp = 0$ respectively (and the Gauss decomposition is taken to converge in the same region). More explicitly, if $\{|\downarrow\rangle, |\uparrow\rangle\}$

is the basis of V used in (2.36), then

$$e^{\pm}(u) = \langle\downarrow|R_+|\uparrow\rangle = \begin{pmatrix} 0 & \dfrac{(\hbar^{1/2}-\hbar^{-1/2})u}{1-u} \\ 0 & 0 \end{pmatrix}, \quad f^{\pm}(u) \coloneqq \langle\uparrow|R_-|\downarrow\rangle = \begin{pmatrix} 0 & 0 \\ \dfrac{\hbar^{1/2}-\hbar^{-1/2}}{1-u} & 0 \end{pmatrix}. \tag{2.37}$$

The sub-algebra $\mathscr{U}_q(\widehat{\mathfrak{sl}}_2)$ consists of only *one* loop Cartan current, given by

$$\psi^{\pm}(u) \coloneqq \frac{\psi_2^{\pm}(u)}{\psi_1^{\pm}(u)} = \frac{\langle\uparrow|R_\infty|\uparrow\rangle}{\langle\downarrow|R_\infty|\downarrow\rangle} = \begin{pmatrix} \sqrt{\hbar}\dfrac{1-u/\hbar}{1-u} & 0 \\ 0 & \dfrac{1}{\sqrt{\hbar}}\dfrac{1-u\hbar}{1-u} \end{pmatrix}. \tag{2.38}$$

Remark. It follows immediately from (2.37) that

$$\begin{aligned} e(u) &= e^+(u) - e^-(u) = (\hbar^{1/2}-\hbar^{-1/2})\delta(u)E_{\uparrow\downarrow} \\ f(u) &= f^+(u) - f^-(u) = (\hbar^{1/2}-\hbar^{-1/2})\delta(u)E_{\downarrow\uparrow} \end{aligned} \tag{2.39}$$

where $\{E_{ij}\}$ is the standard basis of $\mathrm{End}(V)$. One can now verify the Drinfeld relations (2.18) if desired. For example,

$$\psi^+(u) - \psi^-(u) = \delta(u)\,\mathrm{Res}_{u=1}\,\psi^{\pm}(u) = \left(\hbar^{1/2}-\hbar^{-1/2}\right)\delta(u)\left(E_{\uparrow\uparrow}-E_{\downarrow\downarrow}\right)$$

where $\delta(u) \coloneqq \sum_{r\in\mathbb{Z}} u^{-r}$. Using that $\delta(u/v)\delta(u) = \delta(u)\delta(v)$, the relation (2.18d) is immediate.

Example 2.36 (Coproduct)**.** From (2.26), one can compute for $\mathscr{U}_q(\widehat{\mathfrak{gl}}_2)$ that

$$\begin{aligned} \Delta(\psi_1^{\pm}(u)) &= \psi_1^{\pm}(u)\otimes\psi_1^{\pm}(u) + e(u)\psi_1^{\pm}(u)\otimes\psi_1^{\pm}(u)f(u) \\ \Delta(e(u)\psi_1^{\pm}(u)) &= \psi_1^{\pm}(u)\otimes e(u)\psi_1^{\pm}(u) + e(u)\psi_1^{\pm}(u)\otimes\left(e(u)\psi_1^{\pm}(u)f(u)+\psi_2^{\pm}(u)\right). \end{aligned}$$

Using that Δ is an algebra homomorphism, $\Delta(e(u)) = 1\otimes e(u) + e(u)\otimes\psi^{\pm}(u)$. Proceeding similarly, the coproduct for $\mathscr{U}_q(\widehat{\mathfrak{sl}}_2)$ is therefore given by

$$\begin{aligned} \Delta(e(u)) &= 1 \otimes e(u) + e(u) \otimes \psi^{\pm}(u) \\ \Delta(f(u)) &= f(u) \otimes 1 + \psi^{\pm}(u) \otimes f(u) \\ \Delta(\psi^{\pm}(u)) &= \psi^{\pm}(u) \otimes \psi^{\pm}(u) + e(u) \otimes f(u). \end{aligned} \tag{2.40}$$

Example 2.37 (Left/right duals)**.** It is clear from (2.37) and (2.38) that

$$V(u) = L\left(\widehat{\mathsf{S}}^{\bullet}_u(1-\hbar^{-1})\right) = L^{\vee}\left(\widehat{\mathsf{S}}^{\bullet}_u(1-\hbar)\right).$$

We can compute the action of $\psi^{\pm}(u)$ on the left and right duals $V(u)^*$ and ${}^*V(u)$ using (2.25) and thereby identify their highest weights. Repeating the calculation of (2.38) with $(R^{-1})^{t_2}$ and $(R^{t_2})^{-1}$ gives

$$S(\psi^{\pm}(u))^t = \begin{pmatrix} \frac{1}{\sqrt{\hbar}}\frac{1-u\hbar^2}{1-u\hbar} & 0 \\ 0 & \sqrt{\hbar}\frac{1-u}{1-u\hbar} \end{pmatrix}, \quad S^{-1}(\psi^{\pm}(u))^t = \begin{pmatrix} \frac{1}{\sqrt{\hbar}}\frac{1-u}{1-u/\hbar} & 0 \\ 0 & \sqrt{\hbar}\frac{1-u/\hbar^2}{1-u/\hbar} \end{pmatrix}$$

respectively. It follows that $V(u)^* = L(\widehat{\mathsf{S}}^{\bullet}_u(-1+\hbar))$, as expected from Lemma 2.26, and ${}^*V(u) = L(\widehat{\mathsf{S}}^{\bullet}(\hbar^{-1}-\hbar^{-2}))$.

The actions of $e(u)$ and $f(u)$ on dual modules can similarly be determined, but they will be unimportant to us.

Chapter 3: Asymptotic representations

We now construct and study certain sub-modules ${}^{>}\overrightarrow{V}(u)_v$ of semi-infinite tensor products $V(u) \otimes V(\hbar u) \otimes V(\hbar^2 u) \otimes \cdots$ of equivariant K-theories of Nakajima quiver varieties $\mathcal{M}_Q$. Viewing these modules as limiting cases of finite tensor products, we call them *asymptotic* modules. Note that the use of the term "asymptotic" here is in contrast to e.g. the asymptotic representation theory of the symmetric group, since the object which grows is the module as opposed to the algebra acting on it.

Different aspects of the module ${}^{>}\overrightarrow{V}(u)_v$ have different historical roots, which we now attempt to enumerate in tandem with the contents of each section.

- § 3.1 provides a geometric construction of ${}^{>}\overrightarrow{V}(u)_v$ as a component of the (equivariant) *critical K-theory* of a moduli space ${}^{>}\overrightarrow{\mathcal{M}}_Q$ related to the moduli of quasimaps $\mathsf{QMaps}(\mathbb{C} \to \mathcal{M}_Q)$. Critical K-theory, also sometimes called singularity K-theory, is the K-theoretic refinement of the more well-studied *critical cohomology*, i.e. cohomology of the sheaf of vanishing cycles, which has been a useful tool in DT theory almost since its inception (see [31] for an overview).

- § 3.2 explains how to construct R-matrices for ${}^{>}\overrightarrow{V}(u)_v$ from an asymptotic limit of *critical R-matrices* on critical K-theory, thereby making them modules for an FRT reconstructed quantum group. As in § 2.3.2, following the philosophy of [20], these R-matrices arise from stable envelopes but now in critical K-theory.

- § 3.3 identifies the reconstructed quantum groups acting on ${}^{>}\overrightarrow{V}(u)_v$ as *shifted* quantum affine algebras $\mathscr{U}_q^{\mu}(\widehat{\mathfrak{g}}_Q)$ and briefly reviews their definition and properties. The necessity of some modification to the original algebra $\mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$ associated to $\mathcal{M}_Q$, due to the asymptotic nature of ${}^{>}\overrightarrow{V}(u)_v$, was first recognized in [16]. In certain cases, our modules ${}^{>}\overrightarrow{V}(u)_v$ are examples of the *pre-fundamental modules* studied there.

This geometric incarnation of shifted quantum affine algebras $\mathscr{U}_q^\mu(\widehat{\mathfrak{g}}_Q)$ and some of their (infinite-dimensional) modules complements the existing literature where these objects are studied purely algebraically, e.g. [8, 15].

The material in this chapter, particularly § 3.1, owes a *great* intellectual debt to the work in progress [21] of Nakajima and Okounkov.

3.1 Geometric construction

Let $\mathcal{M}_Q$ be a Nakajima quiver variety and $V(u) \coloneqq K_{\mathsf{T}}(\mathcal{M}_Q(\mathbf{w}))$. For an appropriate sub-torus $\mathsf{T}' \subset \mathsf{T}$ of the framing torus,

$$V(u) \otimes V(\hbar u) \otimes \cdots \otimes V(\hbar^{L-1}u) = K_{\mathsf{T}'}(\mathcal{M}_Q(L \cdot \mathbf{w}));$$

see Definition 3.4 for details. Note that $V(u) \otimes V(\hbar u)$ is already not simple as a $\mathscr{U}_q(\widehat{\mathfrak{g}}_Q)$-module. The main purpose of this section is to give a geometric construction of certain (expected to be simple) sub-modules of these tensor products, which, in the *infinite leg limit* $L \to \infty$ of § 3.1.4, become the modules ${}^{>}\overrightarrow{V}(u)_v$ of interest in later sections.

The geometric construction goes as follows. We exhibit $\mathcal{M}_Q$ as a critical locus for a function ϕ_Q in § 3.1.1, and then in § 3.1.2 deform it to a function $\widetilde{\phi}_Q$ whose critical locus is a moduli space ${}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}$ of *finitary quasimaps*. The critical K-theory of the deformed $\widetilde{\phi}_Q$ is the desired sub-module, and it embeds into the critical K-theory of the original ϕ_Q, as we show in § 3.1.3.

3.1.1 Critical loci and K-theory

Let X be an algebraic variety with the action of a reductive group G. Inside $D^b\mathsf{Coh}_G(X)$ lies the full subcategory $\mathsf{Perf}_G(X)$ of *perfect* complexes. When X is singular, the quotient

$$D_{\text{sg},G}(X) \coloneqq D^b\mathsf{Coh}_G(X)/\mathsf{Perf}_G(X)$$

is non-trivial and is known as the *equivariant singularity category* of X. Its non-equivariant version was introduced and first studied by Orlov in [26] in relation to Landau–Ginzburg models, but has since become important the study of singularities in general. Since $D^b\mathsf{Coh}$ and Perf coincide on the smooth locus of X, elements of $D_{\text{sg},G}(X)$ have support only on the singular locus $X_{\text{sg}} \subset X$. As such, $D_{\text{sg},G}(X)$ is closely related to $D^b\mathsf{Coh}_G(X_{\text{sg}})$.

When X is a (global) critical locus, the notion of the singularity category yields the following useful alternative for the equivariant K-theory $K_G(X)$.

Definition 3.1. Let M be a smooth ambient variety with the action of a reductive group G. Suppose M has a map

$$\phi\colon M \to \mathbb{C},$$

called a **potential**, which is G-equivariant if G scales the target $\mathbb{C}$ with a weight κ^{-1}. For simplicity, assume ϕ has only one singular value at $0 \in \mathbb{C}$. Then the critical locus

$$X \coloneqq \text{crit}(\phi) \coloneqq \{d\phi = 0\}$$

is contained in the fiber $M_0 \coloneqq \phi^{-1}(0) \subset M$ as its singular locus. The **G-equivariant critical K-theory** of ϕ is the K-group

$$K_{\text{crit},\mathsf{G}}(\phi) \coloneqq K(D_{\text{sg},\mathsf{G}}(M_0)) = K_{\mathsf{G}}(M_0)/K^{\circ}_{\mathsf{G}}(M_0).$$

Critical loci are an important special case of Definition 2.5, since they are zero loci of very special sections. Namely, an equivalent way to describe X is as the zero locus of

$$d\phi \otimes \kappa \colon \mathcal{O}_M \to \mathcal{T}_M^* \otimes \kappa,$$

where the twist is necessary to make the section G-equivariant. Assuming non-degeneracy of ϕ, we get

$$\mathcal{O}_X^{\mathrm{vir}} = \wedge^\bullet (\kappa \mathcal{T}_M^*), \quad \mathcal{T}_X^{\mathrm{vir}} = (\mathcal{T}_M - \kappa \mathcal{T}_M^*)\Big|_X. \tag{3.1}$$

Note that $\mathcal{T}_X^{\mathrm{vir}}$ has the *self-duality* property that $(\mathcal{T}_X^{\mathrm{vir}})^* = -\kappa^{-1}\mathcal{T}_X^{\mathrm{vir}}$. This is often a strong constraint on formulas arising from virtual localization.

Proposition 3.2. *Let $X = N /\!\!/\!\!/_\theta G$ be an algebraic symplectic reduction with moment map $\mu \colon N \to \mathfrak{g}^*$. Assume the GIT stability condition θ admits no strictly semistable points, i.e. $N^{ss} = N^s$. Then*

$$X = \mathrm{crit}(\phi) \subset M \coloneqq [(N^s \times \mathfrak{g})/G] \tag{3.2}$$

for the potential $\phi \colon (v, \xi) \mapsto \langle \mu(v), \xi \rangle$.

Proof. Clearly $\partial_\xi \phi = \mu$ imposes the moment map equation $\mu = 0$. On the other hand, $\partial_v \phi = \xi \cdot v$ by the definition of the moment map, and by the stability assumption $\xi \cdot v = 0$ implies $\xi = 0$. This removes the extra factor $\mathfrak{g}$ in M. It follows that

$$\mathrm{crit}(\phi) = \left(\mu^{-1}(0)^s \times \{0\}\right)/G$$

which is exactly the definition of X. ∎

Example 3.3. Let $\mathcal{M}_Q(\mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathcal{M}_Q(\mathbf{v}, \mathbf{w})$ be a Nakajima quiver variety, with notation as in § 2.2.1. Let $M_Q \coloneqq (N_Q^s \times \mathfrak{g})/G$. Then Proposition 3.2 exhibits $\mathcal{M}_Q$ as the critical locus of

$$\phi_Q(\mathbf{a}, \mathbf{a}^*, \mathbf{b}, \mathbf{b}^*, \boldsymbol{\xi}) \coloneqq \mathrm{tr}(\mathbf{a}\mathbf{a}^*\boldsymbol{\xi}) + \mathrm{tr}([\mathbf{b}, \mathbf{b}^*]\boldsymbol{\xi}) : M_Q \to \mathbb{C} \tag{3.3}$$

(cf. (2.13)). Extend the torus T acting on $\mathcal{M}_Q$ to

$$\widetilde{\mathsf{T}} \coloneqq \mathsf{T} \times \mathbb{C}^\times_\kappa,$$

identify $\mathsf{T} = \mathsf{T} \times \{1\} \subset \widetilde{\mathsf{T}}$, and let $\boldsymbol{\xi} \in \mathfrak{g}$ be scaled with weight $\hbar\kappa^{-1}$ so that ϕ_Q has weight κ^{-1}.

Remark. There is actually a good amount of freedom in the choice of the space N^s in (3.2). For example, it can be chosen much smaller than the θ-(semi)stable locus of N by prematurely imposing some constraints of the moment map equation $\mu = 0$. Although such changes do not affect the space $\operatorname{crit}(\phi)$, they *do* affect its virtual sheaves $\mathcal{O}^{\mathrm{vir}}$ and $\mathcal{T}^{\mathrm{vir}}$.

3.1.2 Finitary quasimaps

In this section we modify the construction of Example 3.3 to obtain an infinitesimal version of *quasimaps* to $\mathcal{M}_Q$. Concretely, viewing $\mathcal{M}_Q$ as a moduli of quiver representations $\left(\mathbf{V} \rightleftarrows \mathbf{W}\right)$, we will make all quiver maps $\mathbb{C}[z]$-equivariant for an additional variable z representing the coordinate on the source curve.

Definition 3.4. Fix an integer $L > 1$ and suppose the framing is of the form

$$\mathbf{W} = \overline{\mathbf{W}} \otimes \mathbb{C}^L, \quad \mathbb{C}^L =: \bigoplus_{k=0}^{L-1} \mathbb{C}u_k$$

with dimension vector $\mathbf{w} = L \cdot \overline{\mathbf{w}}$. Let

$$\mathsf{T}' \subset \mathsf{T}, \quad \widetilde{\mathsf{T}}' \subset \widetilde{\mathsf{T}}$$

be the sub-tori where $\mathsf{T}_{\mathbf{W}}$ is replaced with $\mathsf{T}_{\overline{\mathbf{W}}}$ and the $\mathbb{C}^\times_\hbar$ factor acts with weight $\hbar^k$ on u_k. Then define

$$\boldsymbol{\Xi} \coloneqq (u_k \mapsto u_{k+1}) \in \operatorname{End}(\mathbb{C}^L) \subset \operatorname{End}(\mathbf{W})$$

where we set $u_{L+1} \coloneqq 0$. Note that Ξ is T'-equivariant of weight $\hbar$. Deform ϕ_Q to get a new potential

$$\widetilde{\phi}_Q \coloneqq \phi_Q - \mathrm{tr}(\mathbf{a}\Xi\mathbf{a}^*) \colon M_Q \to \mathbb{C}, \tag{3.4}$$

which is only T'-equivariant due to the deformation term. Let ${}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}(\overline{\mathbf{w}}) \coloneqq \mathrm{crit}(\widetilde{\phi}_Q)$.

Proposition 3.5. *Let z be a variable acting as $(\boldsymbol{\xi}, \Xi)$ on quiver representations $\left(\mathbf{V} \rightleftarrows \mathbf{W}\right)$. Then*

$${}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}(\overline{\mathbf{w}}) \subset \mathcal{M}_Q(L \cdot \overline{\mathbf{w}})$$

consists of the quiver representations which are $\mathbb{C}[z]$-equivariant.

Proof. As in the undeformed case, $\partial_\xi \widetilde{\phi}_Q = \mu$ imposes the moment map equation $\mu = 0$. We have

$$\partial_{\mathbf{a}} \widetilde{\phi}_Q = \mathbf{a}^* \xi - \Xi \mathbf{a}^*, \quad \partial_{\mathbf{a}^*} \widetilde{\phi}_Q = \xi \mathbf{a} - \mathbf{a} \Xi$$

which make $\mathbf{a}$ and $\mathbf{a}^*$ compatible with the action of z. Similarly,

$$\partial_{\mathbf{b}} \widetilde{\phi}_Q = \mathbf{b}^* \xi - \xi \mathbf{b}^*, \quad \partial_{\mathbf{b}^*} \widetilde{\phi}_Q = \xi \mathbf{b} - \mathbf{b} \xi$$

make $\mathbf{b}$ and $\mathbf{b}^*$ compatible with z. □

Remark. It may be helpful to think of $\mathbb{C}[z]$-equivariant quiver representations as usual quiver representations but where every vertex of the quiver carries an additional loop (corresponding to multiplication by z). In affine type ADE, such quivers and their representations are also known as $N = 1$ ADE quivers, from their origins in $N = 1$ supersymmetric gauge theories, and have been of interest since [4].

Denote points in ${}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}$ by $(\mathbb{V} \rightleftarrows \mathbb{W})$ to distinguish them from points $(\mathbf{V} \rightleftarrows \mathbf{W}) \in \mathcal{M}_Q$, and decompose them into z-graded pieces

$$\mathbb{V} = \bigoplus_{k=0}^{L-1} \mathbb{V}[k], \quad \mathbb{W} = \bigoplus_{k=0}^{L-1} \mathbb{W}[k].$$

By construction, $\mathbb{W}[k] = \overline{\mathbf{W}}$ for all k. Hence $\left(\mathbb{V} \overset{\mathbf{a}^*}{\underset{\mathbf{a}}{\rightleftarrows}} \mathbb{W}\right)$ is equivalently a length-L chain

$$\begin{array}{ccccccc} \mathbb{V}[0] & \xrightarrow{\xi} & \mathbb{V}[1] & \xrightarrow{\xi} & \cdots & \xrightarrow{\xi} & \mathbb{V}[L-1] \\ {\scriptstyle \mathbf{a}}\uparrow\downarrow{\scriptstyle \mathbf{a}^*} & & {\scriptstyle \mathbf{a}}\uparrow\downarrow{\scriptstyle \mathbf{a}^*} & & & & {\scriptstyle \mathbf{a}}\uparrow\downarrow{\scriptstyle \mathbf{a}^*} \\ \overline{\mathbf{W}} & \overset{\Xi}{=\!=\!=} & \overline{\mathbf{W}} & \overset{\Xi}{=\!=\!=} & \cdots & \overset{\Xi}{=\!=\!=} & \overline{\mathbf{W}} \end{array} \tag{3.5}$$

of quiver representations $(\mathbb{V}[k] \overset{\mathbf{a}^*}{\underset{\mathbf{a}}{\rightleftarrows}} \overline{\mathbf{W}})$. Different stability conditions θ impose different conditions on which $\mathbb{V}[k]$ may appear. Note that each $(\mathbb{V}[k] \overset{\mathbf{a}^*}{\underset{\mathbf{a}}{\rightleftarrows}} \overline{\mathbf{W}})$ need not belong to $\mathcal{M}_Q(\overline{\mathbf{w}})$, which may impose *more* stability conditions than $\mathcal{M}_Q(L \cdot \overline{\mathbf{w}})$ particularly on the framing maps $\mathbf{a}, \mathbf{a}^*$.

Example 3.6. Let Q be the type A_1 quiver, i.e. each of V and W correspond to a *single* vertex. This already includes the simplest cases of $\mathcal{M}_Q = T^* \mathsf{Gr}$ and $\mathcal{M}_Q = \mathsf{Hilb}(\mathbb{C}^2)$. Let $\left(\mathbb{V} \rightleftarrows \mathbb{W}\right) \in {}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}$. Then $\mathbb{V}$ is either generated or co-generated by W as a $\mathbb{C}[z]$-module, depending on whether the stability condition for $\mathcal{M}_Q$ is θ or θ^{-1}. In the former case,

$$V = \mathbb{V}[0] \supset \mathbb{V}[1] \supset \mathbb{V}[2] \supset \cdots \supset \mathbb{V}[L-1] \tag{3.6}$$

forms a *descending* flag. The quiver varieties for θ and θ^{-1} are (non-canonically) isomorphic, with a non-trivial involution on equivariant variables sending $\hbar \mapsto \hbar^{-1}$. In the latter case, using this isomorphism, (3.6) can be identified with

$$\mathbb{V}[L-1]^* \subset \mathbb{V}[L-2]^* \subset \cdots \subset \mathbb{V}[0]^* = V^*, \tag{3.7}$$

which is an *ascending* flag.

For more complicated quivers, ξ need not always be injective or surjective, and it is best to use the reformulation of stability due to King and Crawley–Bovey (see [13, Section 3.2]) to get a combinatorial condition for which $\mathbb{V}[k]$ may appear. For us, this will be unnecessary, as later we will impose an ansatz (S) which restricts us to the case of descending flags.

Proposition 3.7. *Let* $(\mathbb{V} \rightleftarrows \mathbb{W}) \in {}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}$ *and* T_N *be its tangent space on the component* $N_Q \subset M_Q$ *of its pre-image. Then*

$$T^{\mathrm{vir}}_{(\mathbb{V}\rightleftarrows\mathbb{W})} {}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)} = (T_N - T_N^*) + \left(\hbar - \hbar^{-1}\right) \bigoplus_i \mathbb{V}_i^* \mathbb{V}_i \in \Bbbk_{\mathsf{T}'}.$$

Proof. Apply (3.1) for the torus T', where $\kappa = 1$. By construction,

$$T_{(\mathbb{V}\rightleftarrows\mathbb{W})} M_Q = T_N + (\hbar - 1) \bigoplus_i \mathbb{V}_i^* \mathbb{V}_i$$

where the $\hbar \bigoplus_i \mathbb{V}_i^* \mathbb{V}_i$ comes from the operator ξ, and the $-\bigoplus_i \mathbb{V}_i^* \mathbb{V}_i$ comes from the quotient by $G_{\mathbf{V}}$ in M_Q (cf. Proposition 2.15). The desired formula follows as $T^{\mathrm{vir}}_{(\mathbb{V}\rightleftarrows\mathbb{W})} = T_{(\mathbb{V}\rightleftarrows\mathbb{W})} M_Q - T^*_{(\mathbb{V}\rightleftarrows\mathbb{W})} M_Q$. □

3.1.3 Comparison with ordinary K-theory

We now investigate the critical K-theories $K_{\mathrm{crit}}(\phi_Q)$ and $K_{\mathrm{crit}}(\widetilde{\phi}_Q)$ and their relation to the ordinary K-theory $K(\mathcal{M}_Q)$. The main result is that

$$K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}_Q) \subset K_{\mathrm{crit},\widetilde{\mathsf{T}}'}(\phi_Q) = K_{\mathsf{T}'}(\mathcal{M}_Q), \tag{3.8}$$

where the inclusion is Proposition 3.9 and the equality is an application of Proposition 3.8.

Proposition 3.8 (Dimensional reduction, [17])**.** *Let* $E \to Y$ *be a vector bundle, and suppose* $Z = \{s = 0\} \subset Y$ *is the zero locus of a section* $s \in H^0(E)$*. Let*

$$\phi\colon\ \mathrm{tot}(E^\vee) \to \mathbb{C}$$

be the function on the total space of $E^\vee$ induced by s; in coordinates,

$$\phi(y,f) \coloneqq f(s(y)), \quad y \in Y,\ f \in E_y^\vee.$$

Let $\mathbb{C}_\kappa^\times$ act by dilation on the bundle $E^\vee$, and therefore on $\mathrm{tot}(E^\vee)$. Then there is an equivalence of triangulated categories

$$D_{\mathrm{sg},\mathbb{C}_\kappa^\times}(\phi^{-1}(0)) \cong D^b\mathsf{Coh}(Z).$$

In our setting, to obtain $Z = \mathcal{M}_Q$, the bundle E is a $\mathfrak{g}^*$-bundle over $Y = N_Q /\!\!/_\theta G$ and the section $s = \mu$ is the moment map. Then ϕ is precisely ϕ_Q. Note that the extra $\mathbb{C}_\kappa^\times$-equivariance on the left hand side is part of the enlarged torus $\widetilde{\mathsf{T}}'$ and not T'.

The inclusion in (3.8) is conditional on the map

$$\iota_*\colon K_{\mathrm{crit},\widetilde{\mathsf{T}}'}(\phi_Q) \to K_{\mathsf{T}'}(M_Q) \tag{3.9}$$

being injective, where M_Q is the (smooth) ambient space. For such a map to be well-defined it is also necessary that everything in $K^\circ_{\widetilde{\mathsf{T}}'}(\phi_Q)$ arises by pullback from $K^\circ_{\mathsf{T}'}(M_Q)$. We will assume similarly for $K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}_Q)$.

Proposition 3.9 (Deformation of potential)**.** *Assuming injectivity of* (3.9),

$$K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}_Q) \subset K_{\mathrm{crit},\mathsf{T}'}(\phi_Q).$$

Proof. The $\mathbb{C}_\kappa^\times$ action can be used to contract $\widetilde{\phi}_Q^{-1}(0)$ into $\phi_Q^{-1}(0)$ in a T'-equivariant way inside the ambient space M_Q. By hypothesis, both $K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}_Q)$ and $K_{\mathrm{crit},\mathsf{T}'}(\phi_Q)$ embed into $K_{\mathsf{A}'}(M_Q)$ for a suitable torus A', and the desired inclusion follows. □

Remark. To avoid the dependency on whether (3.9) is injective or not, it is possible in many applications to manually verify that $K_{\mathsf{T}'}(\mathcal{M}_Q)$ has the sub-module $K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}_Q)$ of interest, see e.g. Example 3.21.

3.1.4 Infinite leg limit and stability

Let $(\mathbb{V} \overset{\mathbf{a}^*}{\underset{\mathbf{a}}{\rightleftarrows}} \mathbb{W}) \in {}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}(\overline{\mathbf{w}})$. We are interested in the formal $L \to \infty$ limit for stability conditions satisfying the following ansatz:

$$\begin{array}{l}\text{The GIT stability condition } \theta \text{ defining } \mathcal{M}_Q \text{ is such that } \mathbb{V} \text{ is } \textit{generated} \\ \text{as a } \mathbb{C}[z]\text{-module, i.e. } \mathbb{V} \text{ is a } \textit{descending} \text{ flag as in (3.6).}\end{array} \tag{S}$$

For the stabilities θ defining $\mathcal{M}_Q = T^* \mathsf{Gr}(w)$ and $\mathcal{M}_Q = \mathsf{Hilb}(\mathbb{C}^2)$, Example 3.6 shows that one of $\theta^{\pm 1}$ satisfies the ansatz. This is no longer true on more complicated quivers. For example, for the θ defining Hilbert schemes on ADE surfaces, neither of $\theta^{\pm 1}$ satisfies the ansatz.

Definition 3.10. Let $V(u) \coloneqq K_{\mathsf{T}'}(\mathcal{M}_Q(\overline{\mathbf{w}}))$. Define the **asymptotic module**

$$\overrightarrow{V}(u) \coloneqq V(u) \otimes V(\hbar u) \otimes V(\hbar^2 u) \otimes \cdots = \lim_{L \to \infty} K_{\mathsf{T}'}(\mathcal{M}_Q(L \cdot \overline{\mathbf{w}})) \tag{3.10}$$

which has a basis of *arbitrary* infinite sequences $(\mathbb{V}[k] \rightleftarrows \mathbb{W}[k])_{k \geq 0}$ of (T-fixed) quiver representations. Applying (3.8), ${}^{>}\overrightarrow{\mathcal{M}}_Q^{(L)}(\overline{\mathbf{w}}) \subset \mathcal{M}_Q(L \cdot \overline{\mathbf{w}})$ at each finite L induces a sub-module

$${}^{>}\overrightarrow{V}(u) \subset \overrightarrow{V}(u)$$

corresponding to the critical K-theory of the space ${}^{>}\overrightarrow{\mathcal{M}}_Q$ of infinite *descending flags* of quiver representations

$$\begin{array}{ccccccccc}
\mathbb{V}[0] & \supset & \mathbb{V}[1] & \supset & \mathbb{V}[2] & \supset & \cdots & \supset & \mathbb{V}[\infty] \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & & & \uparrow\downarrow \\
\overline{\mathbf{W}} & = & \overline{\mathbf{W}} & = & \overline{\mathbf{W}} & = & \cdots & = & \overline{\mathbf{W}}.
\end{array}$$

Note that such flags are eventually constant, and one can split

$$ {}^{>}\overrightarrow{V}(u) =: \bigoplus_{v \in V(u)} {}^{>}\overrightarrow{V}(u)_v \tag{3.11} $$

where ${}^{>}\overrightarrow{V}(u)_v$ consists of those flags where $(\mathbb{V}[\infty] \rightleftarrows \overline{\mathbf{W}}) = v$.

In what follows, the ${}^{>}\overrightarrow{V}(u)_v$ will be the primary objects of interest. They will become modules for a quantum group $\mathscr{A}^\mu$ closely related to the quantum affine algebra $\mathscr{A}$ for which $V(u)$ is a module.

Remark. The choice in ansatz (S) of a *descending* flag, instead of an *ascending* flag, is arbitrary and is made so that the results of § 4.2 more immediately resemble DT combinatorics. As in Example 3.6, the opposite choice does not change the underlying variety $\mathcal{M}_Q$ but has the effect of an involution on equivariant variables. In terms of the quantum group $\mathscr{A}^\mu$ which later will act on these modules, this is the Cartan involution.

One may wonder whether there exists a (non-GIT) stability condition for N^s in Proposition 3.2, for the ambient space M_Q, such that the infinite leg limit produces actual quasimaps as in (2.16). Certainly our choice in ansatz (S) does not do so.

3.2 Asymptotic R-matrices

In this section, we construct from an *asymptotic R-matrix* the action of a quantum group ${}^{>}\overrightarrow{\mathscr{A}}_v$ on the asymptotic module ${}^{>}\overrightarrow{V}(u)_v \subset \overrightarrow{V}(u)$ of § 3.1.4. In the case $\mathfrak{g} = \mathfrak{sl}_2$, for which all non-asymptotic objects were computed in § 2.3.4, we compute these asymptotic modules in detail in § 3.2.3 to illustrate and check the general theory.

3.2.1 General construction

Very generally, let $V(u)$ be an evaluation module for the quantum affine algebra $\mathscr{A}$ and consider the semi-infinite tensor products

$$\begin{aligned}\overrightarrow{V}(u) &\coloneqq V(u) \otimes V(\hbar u) \otimes V(\hbar^2 u) \otimes \cdots \\ \overleftarrow{V}(u) &\coloneqq \cdots \otimes V(\hbar^{-3}u) \otimes V(\hbar^{-2}u) \otimes V(\hbar^{-1}u),\end{aligned}$$

the first of which has already appeared in (3.10). Note that the order of tensor factors matters, as (certain subspaces of) these tensor products will become modules for a quantum group related to $\mathscr{A}$, which is not co-commutative. Since all constructions for $\overrightarrow{V}(u)$ vs $\overleftarrow{V}(u)$ are identical modulo the direction of the semi-infinite tensor and $\hbar \leftrightarrow \hbar^{-1}$, we write all definitions for $\overrightarrow{V}(u)$ only.

Recall from § 2.3.2 that the action of $\mathscr{A}$ on $V(u)$ is constructed by taking all matrix elements of R-matrices $R_W(u) \coloneqq R_{W,V(u)} \in \mathrm{End}(W \otimes V(u))$ over an auxiliary module W, for all possible W. Hence to construct a quantum group acting on $\overrightarrow{V}(u)$, it suffices to construct R-matrices $\overrightarrow{R}_W(u) \coloneqq R_{W,\overrightarrow{V}(u)} \in \mathrm{End}(W \otimes \overrightarrow{V}(u))$ for all auxiliary W. Formally, one would like to use the product

$$R_W(u)R_W(\hbar u)R_W(\hbar^2 u)\cdots \in \mathrm{End}(W \otimes \overrightarrow{V}(u)), \tag{3.12}$$

where the term $R_W(\hbar^k u)$ acts on the W and $V(\hbar^k u)$ factors in $W \otimes \overrightarrow{V}(u)$. However, the issue is that this product may be divergent. More precisely, if

$$\begin{aligned}\overrightarrow{V}^{(L)}(u) &\coloneqq V(u) \otimes V(\hbar u) \otimes \cdots V(\hbar^{L-1}a) \\ \overrightarrow{R}_W^{(L)}(u) &\coloneqq R_W(u)R_W(\hbar u)\cdots R_W(\hbar^{L-1}u) \in \mathrm{End}(W \otimes \overrightarrow{V}^{(L)}(u))\end{aligned}$$

denote the truncation of (3.10) and (3.12) to their first L terms, then

$$\lim_{L\to\infty} \overrightarrow{R}_W^{(L)}(u)$$

may diverge as $O(\hbar^{kL})$. The solution is to take the *leading-order term* in the $L \to \infty$ asymptotic of $\overrightarrow{R}_W(u)$.

Definition 3.11. The **asymptotic R-matrix** $\overrightarrow{R}_W(u) \in \operatorname{End}(W \otimes \overrightarrow{V}(u))$ is the leading-order term

$$\overrightarrow{R}_W(u) \coloneqq \lim_{L\to\infty} \hbar^{\cdots} \overrightarrow{R}_W^{(L)}(u).$$

Let $\overrightarrow{\mathscr{A}}$ be the quantum group associated by FRT reconstruction to these $\overrightarrow{R}_W(u)$, so that $\overrightarrow{V}(u)$ is an $\overrightarrow{\mathscr{A}}$-module.

Remark. This definition is slightly misleading, because the $L \to \infty$ asymptotic $\hbar^{\cdots}$ is in general *different* for each simple sub-module of $\overrightarrow{V}(u)$. Hence taking the overall leading-order term will lose the R-matrix data for most sub-modules, i.e. $\overrightarrow{\mathscr{A}}$ acts by zero most of the time. It is better to first fix a subspace $U \subset \overrightarrow{V}(u)$ such that $\overrightarrow{R}_W(u)$ preserves $W \otimes U$, and then take the leading-order term for the smaller

$$\overrightarrow{R}_W(u)\Big|_{W\otimes U} \in \operatorname{End}(W \otimes U),$$

as we will do in § 3.2.2.

Example 3.12. Although the diagonal matrix elements in each $R_W(\hbar^k u)$ are finite and invertible at $u = 0, \infty$, infinite products of them may not be. A typical example of this behavior is the leading-order asymptotic

$$\lim_{N\to\infty} \prod_{i=0}^{N-1} \frac{1-\hbar^{i+1}u}{1-\hbar^i u} = \lim_{N\to\infty} \frac{1-\hbar^N u}{1-u} = \frac{1}{1-u}$$

where we take $|\hbar| < 1$; note that the result has a zero at $u = \infty$. This behavior will lead to a *shift* in the resulting quantum group associated to $\overrightarrow{R}_W(u)$, as discussed in § 3.3.

3.2.2 Critical R-matrices and sub-modules

Now we specialize the discussion to quantum affine algebras $\mathscr{A} = \mathscr{U}_q(\widehat{\mathfrak{g}})$ of geometric type, and modules $V(u) = K_{\mathsf{T}'}(\mathcal{M}_Q(\overline{\mathbf{w}}))$ associated to a Nakajima quiver variety. The construction of § 3.1, culminating in Definition 3.10, yields the critical K-theories

$$ {}^{>}\overrightarrow{V}(u) \subset \overrightarrow{V}(u). \tag{3.13}$$

The following shows that the asymptotic R-matrix $\overrightarrow{R}_W(u)$ preserves the subspace $W \otimes {}^{>}\overrightarrow{V}(u) \subset W \otimes \overrightarrow{V}(u)$.

Proposition 3.13 ([21])**.** *Let* $\mathsf{A}' \coloneqq \ker(\hbar) \subset \mathsf{T}'$. *There is a well-defined stable envelope* Stab *in critical K-theory, and a commutative diagram*

$$\begin{array}{ccc} K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}\big|_{M^{\mathsf{A}'}}) & \xrightarrow{\mathrm{Stab}} & K_{\mathrm{crit},\mathsf{T}'}(\widetilde{\phi}) \\ \big\downarrow & & \big\downarrow \\ K_{\mathrm{crit},\widetilde{\mathsf{T}}'}(\phi\big|_{M^{\mathsf{A}'}}) & \xrightarrow{\mathrm{Stab}} & K_{\mathrm{crit},\widetilde{\mathsf{T}}'}(\phi). \end{array}$$

Proof sketch. There is a well-known equivalence [26] of $D_{\mathrm{sg}}(\phi^{-1}(0))$ with the category of *matrix factorizations* $\mathsf{MF}(M, \phi)$, which is a quotient of the category of 2-periodic complexes $(\mathcal{E}_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} \mathcal{E}_0)$ on the ambient space M with $d_0 d_1 = \phi$. Hence, at the level of K-theory, convolution with Stab induces a functor $\mathsf{MF}(M^{\mathsf{A}'}, \phi\big|_{M^{\mathsf{A}'}}) \to \mathsf{MF}(M, \phi)$ once we check that its support is compatible with ϕ, namely

$$\mathrm{Attr}^f \subset \{p_1^*\phi\big|_{M^{\mathsf{A}'}} - p_2^*\phi\} \subset M^{\mathsf{A}'} \times M$$

where p_1, p_2 are the two projections from $M^{\mathsf{A}'} \times M$. This is clear for the main stratum $\mathrm{Attr} \subset \mathrm{Attr}^f$ since ϕ is A'-invariant. More generally this is also clear for Attr^f since, for

Nakajima quiver varieties, ϕ is pulled back from the affinization Spec $H^0(M)$ and is therefore constant among the various strata of Attr^f. $\square$

The resulting R-matrices on critical K-theory are called **critical R-matrices**. In particular, for $K_{\mathsf{crit},\widetilde{\mathsf{T}}'}(\phi) = K_{\mathsf{T}'}(\mathcal{M}_Q)$, the critical R-matrix is just the usual geometric R-matrix. In the infinite leg limit we get the following.

Definition 3.14. Let ${}^{>}\overrightarrow{\mathscr{A}}$ be the quantum group arising from FRT reconstruction applied to

$$ {}^{>}\overrightarrow{R}_W(u) \coloneqq \overrightarrow{R}_W(u)\Big|_{W\otimes {}^{>}\overrightarrow{V}(u)}. $$

The inclusion (3.13) becomes an inclusion of ${}^{>}\overrightarrow{\mathscr{A}}$-modules.

In fact ${}^{>}\overrightarrow{V}(u)$ is *not* simple, as suggested by the decomposition (3.11). This is the infinite leg limit of the non-simplicity of tensor products $V(u)\otimes V(\hbar u)$ where evaluation parameters form a geometric sequence in $\hbar$. Hence it is better to pick a simple sub-module and apply FRT reconstruction to the R-matrix there. Note that the element

$$ \overrightarrow{v} \coloneqq v\otimes v\otimes \cdots \in {}^{>}\overrightarrow{V}(u) $$

must be a lowest weight vector for ${}^{>}\overrightarrow{\mathscr{A}}$, since raising/lowering operators can only change finitely many of the tensor factors at a time, but lowering only a finite number of tensor factors must land outside ${}^{>}\overrightarrow{V}(u)$.

Definition 3.15. Let

$$ {}^{>}\overrightarrow{V}(u)_v \coloneqq \langle \overrightarrow{v}\rangle \subset {}^{>}\overrightarrow{V}(u) $$

be the sub-module generated by $\overrightarrow{v}$ (cf. (3.11)). Let ${}^{>}\overrightarrow{\mathscr{A}}_v$ denote the FRT reconstructed quantum group associated to it.

Remark. The quantum groups ${}^{>}\overrightarrow{\mathscr{A}}_v$ and the related ${}^{>}\overrightarrow{\mathscr{A}}$ will be explicitly identified in § 3.3.2 as *shifted* versions of the quantum affine algebra $\mathscr{A}$.

In everything that follows, we assume the following conjecture. Prior to taking the infinite leg limit, i.e. for the *finite-dimensional* $K_{\mathrm{crit},\mathsf{T}'}({}^{>}\overrightarrow{\mathcal{M}}_{Q}^{(L)})$, it should follow from a computation of ℓ-weights to show that the only *dominant* ℓ-weight (in the sense of [10]) is the vacuum one.

Conjecture 3.16. ${}^{>}\overrightarrow{V}(u)_v$ is simple as a ${}^{>}\overrightarrow{\mathscr{A}}_v$-module.

Lemma 3.17. *Suppose $v \in V(u)$ has ℓ-weight $\Psi(u)$. Then*

$$ {}^{>}\overrightarrow{V}(u)_v = L^{\vee}\left(\Psi^{\infty}(u)\right) $$

for ${}^{>}\overrightarrow{\mathscr{A}}_v$, where $\Psi^{\infty}(u)$ is the leading-order asymptotic of $\prod_{k\geq 0}\Psi(\hbar^k u)$.

Proof. We have already argued that $\overrightarrow{v}$ is lowest weight, so it suffices to compute its ℓ-weight. By the triangularity of coproduct, as in (2.20) or Example 2.31, its ℓ-weight is $\Psi^{\infty}(u)$. □

In general, if $v \in V(u)$ is a highest weight vector then ${}^{>}\overrightarrow{V}(u)_v$ is the trivial ${}^{>}\overrightarrow{\mathscr{A}}$-module, and the reconstructed ${}^{>}\overrightarrow{\mathscr{A}}_v$ is uninteresting. Otherwise ${}^{>}\overrightarrow{V}(u)_v$ is infinite-dimensional in general, regardless of whether the original $V(u)$ is finite-dimensional or not.

Definition 3.18. All constructions in this section are equally valid if one begins with $\overleftarrow{V}(u)$ instead of $\overrightarrow{V}(u)$, and we denote the resulting modules

$$ {}^{>}\overleftarrow{V}(u)_v \subset {}^{>}\overleftarrow{V}(u) \subset \overleftarrow{V}(u). $$

Let ${}^{>}\overleftarrow{\mathscr{A}}_v$ be the FRT reconstructed quantum group acting on ${}^{>}\overleftarrow{V}(u)_v$.

Note that the ${}^{>}\overleftarrow{V}(u)_v$ are *highest* weight modules, and the analogue of Lemma 3.17 (with $\prod_{k<0}\Psi(\hbar^k u)$) applies to compute their highest weights.

3.2.3 Example: $\mathfrak{g} = \mathfrak{sl}_2$

Let V be the defining representation of $\mathfrak{sl}_2$, and let $V(u)$ be the associated evaluation representation of $\mathscr{A} \coloneqq \mathscr{U}_q(\widehat{\mathfrak{sl}}_2)$. Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a basis for both representations, where $|\uparrow\rangle$

is the highest weight vector, and let $\{E_{ij}\}$ be the standard basis of $\mathrm{End}(V)$. The Drinfeld generators we will use for $\mathscr{A}$ are

$$\begin{aligned} e(u) &\coloneqq (\hbar^{1/2} - \hbar^{-1/2})\delta(u)E_{\uparrow\downarrow}, \\ f(u) &\coloneqq (\hbar^{1/2} - \hbar^{-1/2})\delta(u)E_{\downarrow\uparrow}, \\ \psi^{\pm}(u) &\coloneqq \sqrt{\hbar}\frac{1-u/\hbar}{1-u}E_{\uparrow\uparrow} + \frac{1}{\sqrt{\hbar}}\frac{1-u\hbar}{1-u}E_{\downarrow\downarrow}. \end{aligned} \tag{3.14}$$

These were computed from first principles in § 2.3.4. Note that the stability ansatz (S) means we must take the *opposite* coproduct (and antipode) from (2.40).

Example 3.19 (Asymptotic modules)**.** A basis for $\overrightarrow{V}(u)$ is given by infinite strings of $\uparrow$ and $\downarrow$. Let $|\overrightarrow{\uparrow_k}\rangle$ denote the basis element where the first k tensor factors are $|\uparrow\rangle$ and the rest are $|\downarrow\rangle$. Interpret $k = \infty$ to mean that all tensor factors are $|\uparrow\rangle$. Then

$$L^\vee_\uparrow(u) \coloneqq {}^>\overrightarrow{V}(u)_\uparrow, \quad L^\vee_\downarrow(u) \coloneqq {}^>\overrightarrow{V}(u)_\downarrow$$

are the trivial module with basis $\{|\overrightarrow{\uparrow_\infty}\rangle\}$ and the infinite-dimensional module with basis $\{|\overrightarrow{\uparrow_k}\rangle\}_{k\geq 0}$ respectively. In particular,

$$|\overrightarrow{\uparrow_\infty}\rangle \in L^\vee_\uparrow(u), \quad |\overrightarrow{\uparrow_0}\rangle \in L^\vee_\downarrow(u)$$

are the lowest ℓ-weight vectors. Let $\Psi_\uparrow$ and $\Psi_\downarrow$ denote their ℓ-weights, respectively, so that by Lemma 3.17

$$\begin{aligned} L^\vee_\uparrow(u) &= L^\vee(\Psi_\uparrow), \quad \Psi_\uparrow = 1 - u/\hbar \\ L^\vee_\downarrow(u) &= L^\vee(\Psi_\downarrow), \quad \Psi_\downarrow = \frac{1}{1-u}. \end{aligned}$$

As $L^\vee_\uparrow(u)$ is a trivial module, we focus on $L^\vee_\downarrow(u)$ in what follows. For example, more explicitly,

$$\Psi_\downarrow = \lim_{L\to\infty} \hbar^{L/2} \prod_{k=0}^{L-1} \left(\frac{1}{\sqrt{\hbar}} \frac{1-u\hbar}{1-u} \right) \Big|_{u\mapsto \hbar^k u} \tag{3.15}$$

Example 3.20 (Presentation of $L^\vee_\downarrow(u)$)**.** Let $\overrightarrow{e}(u)$, $\overrightarrow{f}(u)$, and $\overrightarrow{\psi}^\pm(u)$ be the analogues of (3.14) acting on $L^\vee_\downarrow(u)$.

- By the triangularity (2.20) of coproduct,

$$\overrightarrow{\psi}^\pm(u) \propto \psi^\pm(u) \otimes \psi^\pm(u) \otimes \cdots + (\text{off-diagonal})$$

  where $\propto$ is a reminder that we are only taking leading-order terms on the rhs. It follows that

$$\langle \overrightarrow{\uparrow_k} | \overrightarrow{\psi}^\pm(u) | \overrightarrow{\uparrow_k} \rangle = \Psi_\downarrow \cdot \prod_{i=0}^{k-1} \frac{\langle \uparrow | \psi^\pm(u\hbar^k) | \uparrow \rangle}{\langle \downarrow | \psi^\pm(u\hbar^k) | \downarrow \rangle} = \hbar^k \frac{1-u/\hbar}{(1-u\hbar^{k-1})(1-u\hbar^k)}.$$

- From (2.40),

$$\overrightarrow{e}(u) \propto \psi^\pm(u) \otimes \cdots \otimes \psi^\pm(u) \otimes e(u) \otimes 1 \otimes 1 \otimes \cdots$$

  and similarly for $\overrightarrow{f}(u)$. Hence the only non-zero matrix elements of $\overrightarrow{e}(u)$ and $\overrightarrow{f}(u)$ are

$$\langle \overrightarrow{\uparrow_{k+1}} | \overrightarrow{e}(u) | \overrightarrow{\uparrow_k} \rangle, \quad \langle \overrightarrow{\uparrow_{k-1}} | \overrightarrow{f}(u) | \overrightarrow{\uparrow_k} \rangle.$$

This yields an explicit description of $L^\vee_\downarrow(u)$. Up to constants, it agrees with the *negative pre-fundamental module* studied in [16, Section 4], where it is given as a representation of the Borel sub-algebra which will be defined in § 3.3.

Example 3.21 (Lowest weight vector)**.** One may manually verify that $|\overrightarrow{\uparrow_\infty}\rangle \in \overrightarrow{V}(u)$ is a lowest weight vector. From Example 3.20, the k-th term in

$$\overrightarrow{f}(u) \, |\overrightarrow{\uparrow_\infty}\rangle$$

arises from flipping the k-th $|\uparrow\rangle$ in to a $|\downarrow\rangle$ with all other tensor factors unchanged. The coefficient of this k-th term is therefore

$$\langle\downarrow|f(u\hbar^k)|\uparrow\rangle \prod_{i=k+1}^{\infty} \langle\uparrow|\psi^{\pm}(u\hbar^i)|\uparrow\rangle \propto \delta(u\hbar^k)(1-u\hbar^k) = 0, \tag{3.16}$$

where we used that $\delta(u/v)f(u) = \delta(u/v)f(v)$ for any f. As k was arbitrary, it follows that $\overrightarrow{f}(u)\,|\overrightarrow{\uparrow_\infty}\rangle = 0$.

Remark. Calculations as in Example 3.21 first appeared in [7] for $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$. The resulting lowest weight modules are discussed in § 4.2.2.

Remark (Coproduct). Recall that coproduct Δ is defined using the projection (2.26), which in the asymptotic setting requires some care. Namely, we must ensure that the leading-order terms in $R_{W,V}$ and $R_{W,V'\otimes V''}$ have the *same* $\hbar$-asymptotic in order for the coproduct to be well-defined. For example, let

$$R_L(u) \coloneqq \left(\prod_{k\geq 0} R_{V,V}(\hbar^k u)\right)\Bigg|_{L_\downarrow^\vee(u)}$$

so that its $O(\hbar^{L/2})$ term (as $L\to\infty$) is the asymptotic R-matrix of $V(1)\otimes L_\downarrow^\vee(u)$ (cf. (3.15)). Similarly, the $O(\hbar^L)$ term of $R_L^{(12)}(u)R_L^{(13)}(v)$ is the asymptotic R-matrix for $V(1)\otimes L_\downarrow^\vee(u)\otimes L_\downarrow^\vee(v)$. As these are two *different* $\hbar$-asymptotics, it follows that

$$\Delta(E)\Big|_{E\otimes E} = 0$$

where $E \subset \operatorname{End}(L_\downarrow^\vee)$ consists of asymptotic R-matrix elements. On the other hand, $\Delta(E)$ has non-zero components in $\operatorname{End}(L_\downarrow^\vee \otimes V(u))$, since there are no additional $\hbar^{\cdots}$ terms coming from the $V(u)$ term.

The general way to formulate coproducts for asymptotic R-matrices can be found in Definition 3.27.

3.3 Shifted quantum affine algebras

In this section we define and collect some facts about shifted quantum affine algebras $\mathscr{U}_q^\mu(\widehat{\mathfrak{g}})$ and its shifted quantum affine Borel $\mathscr{U}_q^\mu(\widehat{\mathfrak{b}})$ in preparation for use in § 4.2. These shifted algebras were formally introduced in [8], but are essentially the K-theoretic analogue of the shifted *Yangians* $\mathscr{Y}^\mu(\mathfrak{g})$ first introduced in limited generality in [3, 18] and later, in full generality, popularized by [2] in their study of Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories. For us, they will naturally arise from FRT reconstruction applied to asymptotic R-matrices $\overrightarrow{R}(u) = R_- R_\infty R_+$. Recall that the usual quantum affine algebra $\mathscr{A}$ has loop Cartan generators of the form

$$\psi^\pm(u) = q^{\pm h} \exp(\cdots) \in \mathbb{C}[[u^\mp]].$$

The term "shifted" means to instead take $\psi^\pm(u) \in u^{b_\pm}\mathbb{C}[[u^\mp]]$ for some constants $b_\pm \in \mathbb{Z}$. Such a degree shift arises from singularities at $u = 0, \infty$ in the diagonal R_∞. An important precursor to these shifted algebras can be found in [16], where it was recognized that such singularities prevent the original algebra $\mathscr{A}$ from acting on the asymptotic modules of § 3.2.1.

Definition 3.22. Let $\{\alpha_i\}_{i \in I}$ be the simple positive roots of $\mathfrak{g}$. Given coweights μ_+, μ_-, set

$$b_{i,\pm} \coloneqq \alpha_i(\mu_\pm).$$

The **shifted quantum affine algebra** $\mathscr{U}_q^{\mu_+,\mu_-}(\widehat{\mathfrak{g}})$ is generated by coefficients of the Drinfeld currents

$$e_i(u) \coloneqq \sum_{r \in \mathbb{Z}} e_{i,r} u^{-r}, \quad f_i(u) \coloneqq \sum_{r \in \mathbb{Z}} f_{i,r} u^{-r}, \quad \psi_i^\pm(u) \coloneqq \sum_{r \geq -b_{i,\pm}} \psi_{i,\pm r}^\pm u^{\mp r}$$

for all $i \in I$. These currents satisfy the same defining relations (2.18) as those of $\mathscr{U}_q(\widehat{\mathfrak{g}})$ but with (2.18a) replaced by

$$\psi^{\pm}_{i,-b_{i,\pm}} \text{ invertible}, \quad [\psi_i^s(u), \psi_j^{s'}(v)] = 0 \;\; \forall s, s' \in \{\pm\}. \tag{3.17}$$

The appropriate modification of (2.19) is

$$\psi_i^{\pm}(u) =: u^{\pm b_{i,\pm}} q_i^{\pm h_i} \exp\left(\pm(q_i - q_i^{-1}) \sum_{m \geq 1} h_{i,\pm m} u^{\mp m}\right). \tag{3.18}$$

It is immediate from the homogeneity of the defining relations that $\mathscr{U}_q^{\mu_+,\mu_-} \cong \mathscr{U}_q^{0,\mu_++\mu_-}$ by scaling $f_i(u), \psi_i^{\pm}(u)$ by $u^{-b_{i,+}}$. Hence $\mathscr{U}_q^{\mu_+,\mu_-}$ depends only on $\mu \coloneqq \mu_+ + \mu_-$, and we denote

$$\mathscr{A}^{\mu} \coloneqq \mathscr{U}_q^{\mu}(\widehat{\mathfrak{g}}) \coloneqq \mathscr{U}_q^{0,\mu}(\widehat{\mathfrak{g}}).$$

Definition 3.23. The **naive shifted quantum affine Borel** $\overline{\mathscr{B}}^{\mu} \subset \mathscr{A}^{\mu}$ is the sub-algebra generated by

$$\{e_{i,r}, f_{i,s}, \psi^{+}_{i,r}, (\psi^{+}_{i,0})^{-1} \mid i \in I,\ r \geq 0,\ s > \max(0, \alpha_i(\mu))\}. \tag{3.19}$$

For μ an anti-dominant coweight, $\overline{\mathscr{B}}^{\mu} = \overline{\mathscr{B}}^{0}$. Note that $\overline{\mathscr{B}}^{0}$ does *not* necessarily agree with the actual quantum affine Borel $\mathscr{B}$, see Definition 2.21. The problem is that there is no known Drinfeld–Jimbo presentation of shifted quantum affine algebras in general. When $\mathscr{A}$ is of geometric type, the correct (non-naive) definition is Definition 3.33.

Lemma 3.24. *$\overline{\mathscr{B}}^{\mu}$ is isomorphic to the sub-algebra of $\mathscr{A}$ generated by the same elements as in* (3.19).

Proof. The bound on s was chosen so that commutation relations between the elements of (3.19) do not involve non-zero coefficients of $\psi_i^-(u)$ in any way. □

Remark. For μ dominant, the relations defining the shifted *Yangian* $\mathscr{Y}^{\mu}(\widehat{\mathfrak{g}})$ (see [2, Appendix B]) are essentially the rational degeneration of the relations in $\overline{\mathscr{B}}^{\mu}$. One may wonder to what extent the usual strengthening of this degeneration into an algebra homomorphism, as in e.g. [12], can be generalized to the shifted setting.

3.3.1 Modules and their properties

The material in this section is completely analogous to that of § 2.3.1 for *unshifted* quantum affine algebras, and so we omit some details all of which can be found in the recent preprint [15].

Definition 3.25. The **shifted category** $\mathcal{O}(\mathscr{A}^\mu)$ consists of all $\mathscr{A}^\mu$-modules V satisfying the same conditions as in Definition 2.23 for the unshifted category $\mathcal{O}$. Likewise we define the *dual* shifted category $\mathcal{O}^\vee(\mathscr{A}^\mu)$.

Note that while ℓ-weights for modules in the unshifted category $\mathcal{O}$ are rational functions of *degree zero*, the analogue for $\mathcal{O}^\mu$ are rational functions of degrees prescribed by μ. Namely if we replace the concept of rational ℓ-weights with that of **rational ℓ_μ-weights**

$$\mathfrak{r}_\mu \coloneqq \left\{ (\Psi_i(u))_{i\in I} \mid \deg \Psi_i(u) = \alpha_i(\mu)\ \forall i \in I \right\}$$

then Proposition 2.24 continues to hold for $\mathcal{O}^\mu$ verbatim, as below.

Proposition 3.26. *Let V be an $\mathscr{A}^\mu$-module.*

- *If $V \in \mathcal{O}(\mathscr{A}^\mu)$ then all its non-zero ℓ-weight spaces V_Ψ have $\Psi \in \mathfrak{r}_\mu$.*
- *$L(\Psi) \in \mathcal{O}(\mathscr{A}^\mu)$ iff $\Psi \in \mathfrak{r}_\mu$, and the same for $L^\vee(\Psi) \in \mathcal{O}^\vee(\mathscr{A}^\mu)$.*

Hence we continue to denote highest and lowest weight modules by $L(\Psi)$ and $L^\vee(\Psi)$ respectively, with no ambiguity; they live in $\mathcal{O}(\mathscr{A}^\mu)$ for μ given by the degree of Ψ. In spite of these Ψ being ℓ_μ-weights, we continue to refer to them as ℓ-weights.

Definition 3.27. Later, in Proposition 3.32, we will prove that all $\mathscr{A}^\mu$ associated to $\mathscr{A}$ of geometric type can be geometrically realized via FRT reconstruction. It follows that there is a **shifted coproduct**

$$\Delta_{\mu_1,\mu_2}\colon \mathscr{A}^\mu \to \mathscr{A}^{\mu_1}\widehat{\otimes}\mathscr{A}^{\mu_2}$$

for any coweights $\mu = \mu_1 + \mu_2$. From the perspective of R-matrix elements, the shift is necessary to compensate for *different* $\hbar$ asymptotics on the source and target, e.g. see the discussion at the end of § 3.2.3.

The unshifted $\Delta_{0,0}\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is the standard Drinfeld–Jimbo coproduct, and in particular Δ_{μ_1,μ_2} is *not* the shifted Drinfeld coproduct $\Delta^\infty_{\mu_1,\mu_2}$ which arises from R-matrices at infinite slope. Previously, Δ_{μ_1,μ_2} had only been constructed for the case $\mathfrak{g} = \mathfrak{sl}_n$, by explicit formulas in [8, Section 10]. Our shifted coproduct, obtained from FRT reconstruction, applies to $\mathscr{A}^\mu$ (of any shift) for all $\mathscr{A}$ of geometric type.

Remark. Properties of $\mathscr{A}$ derived from geometric R-matrices should carry over to $\mathscr{A}^\mu$, once shifts are taken into account. For example, the shifted coproduct has the triangularity property

$$\Delta_\mu(\psi_i^\pm) \in \psi_i^\pm \otimes \psi_i^\pm + \mathscr{A}_-^{\mu_1} \otimes \mathscr{A}_+^{\mu_2} \tag{3.20}$$

by exactly the same argument as in Example 2.31.

In § 3.3.3, it will be useful to study modules for the shifted Borel instead of the entire shifted algebra. Then the following lemma, which has not yet been relevant, becomes helpful.

Lemma 3.28 ([15, Corollary 4.10])**.** *Let μ be anti-dominant. If $V \in \mathcal{O}(\mathscr{A}^\mu)$ is simple, then it is also simple in $\mathcal{O}(\mathscr{B}^\mu)$.*

Definition 3.29. By Lemma 3.28, the notation $L(\Psi)$ is unambiguous when $\Psi \in \mathfrak{r}_\mu$ for μ anti-dominant. Otherwise we write $L_{\mathscr{B}^\mu}(\Psi)$ for the simple highest weight $\mathscr{B}^\mu$-module, to distinguish it from $L(\Psi)$, and similarly for $L^\vee_{\mathscr{B}^\mu}$.

3.3.2 From asymptotic R-matrices

Now we specialize the discussion to the quantum affine algebra $\mathscr{A} = \mathscr{U}_q(\widehat{\mathfrak{g}})$ of geometric type, i.e. reconstructed from the geometric R-matrices of a Nakajima quiver variety $\mathcal{M}_Q$.

As we will require the results of § 2.3.3, further specialize to $\mathfrak{g}$ of finite type. Away from finite type, counterexamples to Lemma 3.30 and Proposition 3.31 can be found in § 4.2.2 for $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$.

Let $V(u) = K_{\mathsf{T}'}(\mathcal{M}_Q(\overline{\mathbf{w}}))$ be a simple lowest weight $\mathscr{A}$-module. In § 3.2.2 we defined the ${}^{>}\overrightarrow{\mathscr{A}}_v$-module ${}^{>}\overrightarrow{V}(u)_v$ for any $v \in V(u)$. In this section we identify the quantum group ${}^{>}\overrightarrow{\mathscr{A}}_v$ as a shifted quantum affine algebra, namely a shift of the original $\mathscr{A}$.

Lemma 3.30. *Let $v \in V(u)$ have ℓ-weight*

$$\hbar^{(\deg P_i - \deg Q_i)/2} \frac{P_i(u)}{P_i(u\hbar)} \frac{Q_i(u\hbar)}{Q_i(u)} \tag{3.21}$$

as in (2.21). Then ${}^{>}\overrightarrow{V}(u)_v = L^\vee\left(P_i(u)/Q_i(u)\right)$.

Proof. This is a restatement in a more convenient form of Lemma 3.17, which says the lowest weight vector in ${}^{>}\overrightarrow{V}(u)_v$ has ℓ-weight

$$\lim_{L\to\infty} \hbar^{\cdots} \prod_{k=0}^{L-1} \frac{P_i(u)}{P_i(u\hbar)} \frac{Q_i(u\hbar)}{Q_i(u)}\bigg|_{u\mapsto \hbar^k u} = \frac{P_i(u)}{P_i(0)} \frac{Q_i(0)}{Q_i(u)}.$$

Recall that the P_i and Q_i are normalized to have constant term 1. □

Proposition 3.31. *Let $v \in V(u)$ have weight $\lambda \in \mathfrak{h}^*$. Then*

$${}^{>}\overrightarrow{\mathscr{A}}_v = \mathscr{A}^\mu := \mathscr{U}_q^\mu(\widehat{\mathfrak{g}})$$

for the coweight $\mu = \lambda^\vee$.

Proof. It suffices to first show that all loop Cartan generators $\psi_i^\pm(u) \in ({}^{>}\overrightarrow{\mathscr{A}}_v)_0$ are μ-shifted, and then to show that all commutation relations in ${}^{>}\overrightarrow{\mathscr{A}}_v$ are identical to those of $\mathscr{A}$. The latter is clear since elements in ${}^{>}\overrightarrow{\mathscr{A}}_v$ can be viewed as (infinite) coproducts of elements in $\mathscr{A}$, and coproduct is an algebra homomorphism.

We start with the lowest weight vector in ${}^{>}\overrightarrow{V}(u)_v$, which by Lemma 3.30 has ℓ-weight in $\mathfrak{r}_\mu$ for μ such that $\alpha_i(\mu) = \deg P_i - \deg Q_i$. On the other hand, recall that the $\hbar^{\cdots}$ prefactor in (3.21) corresponds to the eigenvalue of q^{h_i}, so that $\lambda(h_i) = \deg P_i - \deg Q_i$ as in (2.33). Hence $\mu = \lambda^\vee$.

All other vectors in ${}^{>}\overrightarrow{V}(u)_v$ arise by raising various tensor factors a finite number of times in total. A single raising operator in $V(u)$ can only change the ℓ-weight by multiplication by a term of u-degree zero, due to the form of (3.21) for weights in $V(u)$. Hence *all* ℓ-weights, not just the lowest ℓ-weight, belong to $\mathfrak{r}_\mu$. □

In general, different simple sub-modules ${}^{>}\overrightarrow{V}(u)_v \subset {}^{>}\overrightarrow{V}(u)$ admit $\mathscr{A}^\mu$ actions with *different* shifts μ. The algebra ${}^{>}\overrightarrow{\mathscr{A}}$ of Definition 3.14 which acts on the whole ${}^{>}\overrightarrow{V}(u)$ should therefore be viewed as some quotient of the algebra obtained by modifying the Drinfeld presentation of $\mathscr{A}^\mu$ so that the loop Cartan generators are the *bi-directionally infinite* and commuting $\psi_i^\pm(u) = \sum_{r\in\mathbb{Z}} \psi_{i,r}^\pm u^{\mp n}$, with no restriction on invertibility of any $\psi_{i,r}^\pm$. Then $\mathscr{A}^\mu$ is the quotient of this algebra by the relations (cf. (3.17))

$$\psi_{i,-b_{i,\pm}}^\pm \text{ invertible}, \quad \psi_{i,r}^\pm = 0 \; \forall r < -b_{i,\pm}.$$

The cohomological analogue of ${}^{>}\overrightarrow{\mathscr{A}}$ first appeared in [2, Appendix B] as the "Cartan doubled Yangian", as a way to consistently define shifted Yangians for any shift.

Proposition 3.32. *For arbitrary shift μ, the shifted algebra $\mathscr{A}^\mu$ can be geometrically realized as in Proposition 3.31.*

Proof. By Proposition 3.31, it suffices to find an $\mathscr{A}$-module $V_{\mathbf{w}}(u) = K_{\mathsf{T}'}(\mathcal{M}_Q(\mathbf{w}))$ and a vector $v \in V_{\mathbf{w}}(u)$ of weight $\mu^\vee$. Given any weight $\lambda \in \mathfrak{h}^*$, one can find a simple finite-dimensional $\mathscr{A}$-module, say of highest weight $\lambda_0 \neq \lambda$, containing it as a non-trivial weight. Then any $V_{\mathbf{w}}(u)$ of the same highest weight λ_0 must also contain λ as a non-trivial weight. □

Definition 3.33. Let $R(u) = R_- R_\infty R_+$ be the Gauss decomposition around $u = \infty$ of an R-matrix for $\mathscr{A}^\mu$. The **shifted quantum affine Borel** $\mathscr{B}^\mu := \mathscr{U}_q^\mu(\widehat{\mathfrak{b}}) \subset \mathscr{A}^\mu$ is the sub-algebra generated by coefficients of matrix elements of $R_\pm$ and R_∞ expanded around $u = \infty$.

For $\mathfrak{g} = \mathfrak{sl}_2$ this agrees with the naive shifted quantum affine Borel. For $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$, it is explicitly presented in Definition 4.16. In both cases, Lemma 3.24 therefore continues to hold.

3.3.3 Restriction, induction, and duality

Let $\Psi \in \mathfrak{r}_\mu$ be an ℓ_μ-weight. One would like to claim the modules $L(\Psi)$ and $L^\vee(\Psi^{-1})$ are dual, as in Lemma 2.26 when $\mu = 0$, but when $\mu \neq 0$ these are modules for the two *different* algebras $\mathscr{A}^\mu$ and $\mathscr{A}^{-\mu}$ respectively. In this situation, it is desirable to make them modules for a smaller, common sub-algebra so that they may be compared. We give one such procedure in this section which will be useful in § 4.2.3.

Remark. We use actual quantum affine Borel $\mathscr{B}^\mu$ in this section. When $\mathscr{A}$ is of geometric type, Definition 3.33 applies and there is no problem whenever Lemma 3.24 is applicable to $\mathscr{B}^\mu$. Otherwise one may replace $\mathscr{B}^\mu$ with the naive quantum affine Borel $\overline{\mathscr{B}}^\mu$, to which all discussion of this section is equally applicable.

Definition 3.34. Let μ be dominant, and Ψ be an ℓ_μ-weight so that

$$L(\Psi) \in \mathcal{O}(\mathscr{A}^\mu), \quad L^\vee(\Psi^{-1}) \in \mathcal{O}^\vee(\mathscr{A}^{-\mu}).$$

By restriction, consider them as elements

$$L(\Psi) \in \mathcal{O}(\mathscr{B}^\mu), \quad L^\vee(\Psi^{-1}) \in \mathcal{O}(\mathscr{B}^{-\mu}) = \mathcal{O}^\vee(\mathscr{B})$$

using that $\mathscr{B}^{-\mu} = \mathscr{B}$. By Lemma 3.24, $\mathscr{B}^\mu \subset \mathscr{B}$ is a sub-algebra. Hence define the **induced $\mathscr{B}$-module**

$$\widetilde{L}(\Psi) := \operatorname{ind}_{\mathscr{B}^\mu}^{\mathscr{B}} L(\Psi) \in \mathcal{O}(\mathscr{B}).$$

Note that by Lemma 3.28, $L^\vee(\Psi^{-1}) = L^\vee_{\mathscr{B}}(\Psi^{-1})$ as $\mathscr{B}$-modules. The same is *not* necessarily true for $L(\Psi) \in \mathcal{O}(\mathscr{B}^\mu)$.

Proposition 3.35.

1. *$\widetilde{L}(\Psi) \in \mathcal{O}(\mathscr{B})$ is highest weight of weight Ψ.*

2. *If $\chi(\widetilde{L}(\Psi)) \preceq \chi\left(L^\vee(\Psi^{-1})\right)$, then*

$$\widetilde{L}(\Psi) = L^\vee(\Psi^{-1})^* \in \mathcal{O}(\mathscr{B}).$$

In particular $L(\Psi)$ is simple as a $\mathscr{B}^\mu$-module.

Proof. The extra generators of $\mathscr{B}$ which are not present in $\mathscr{B}^\mu$ are all lowering elements, e.g. for the naive $\overline{\mathscr{B}}$ and $\overline{\mathscr{B}}^\mu$ the discrepancy is

$$\{f_{i,s} \mid i \in I,\, 0 < s \leq \alpha_i(\mu)\}.$$

As the original L_Ψ was highest weight, it follows that $\widetilde{L}_\Psi$ is still highest weight. Furthermore, induction does not modify the highest ℓ-weight.

By Lemma 2.26, $L^\vee(\Psi^{-1})^* = L(\Psi) \in \mathcal{O}(\mathscr{B})$, so it must be a quotient of $\widetilde{L}(\Psi)$. Taking characters,

$$\chi(\widetilde{L}(\Psi)) \succeq \chi\left(L^\vee(\Psi^{-1})^*\right) = \chi\left(L^\vee(\Psi^{-1})\right).$$

If in addition we have the reverse equality, then it follows that $\widetilde{L}(\Psi) = L^\vee(\Psi^{-1})^*$, as desired. ∎

Example 3.36. Let $V_i(u) := K_{\mathsf{T}'}(\mathcal{M}_Q(\mathbf{w}_i))$ where $\mathbf{w}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is non-zero except at the i-th entry (and the GIT stability condition is $\theta > 0$).

- From (2.31), $V_i(u)$ is highest weight of highest ℓ-weight $\Psi^{(i)}$, defined as

$$\Psi^{(i)}_j(u) := \delta_{i,j} \frac{1-u}{1-\hbar u}.$$

Recall that the simple modules $L(\Psi^{(i)})$ are known as *Kirillov–Reshetikhin modules.*

- Let $v \in V_i(u)$ be the highest weight vector. From Lemma 3.17, ${}^{>}\overrightarrow{V}(u)_v = L^\vee(\Psi^{(i),+})$ where

$$\Psi_j^{(i),+}(u) \coloneqq \delta_{i,j}(1-u).$$

In fact since v is highest weight, so is the resulting $\overrightarrow{v}$, and hence this is the *trivial* module. (The module $L_\uparrow(u)$ in § 3.2.3 is an example of this, but the $\mathfrak{gl}_2$ vs $\mathfrak{sl}_2$ discrepancy shifts the ℓ-weight $\Psi^{(i),+}$ a little.)

Hence $L(\Psi^{(i),+})$ is a trivial module for $\mathscr{A}^\mu$ where $\alpha_j(\mu) = \delta_{i,j}$. When the induced

$$\widetilde{L}(\Psi^{(i),+}) \in \mathcal{O}(\mathscr{B})$$

is still a simple module, it is therefore the *positive pre-fundamental* module $L(\Psi^{(i),+}) \in \mathcal{O}(\mathscr{B})$. This is the case for $\mathfrak{g} = \mathfrak{sl}_2$, for which $\mathscr{B} = \mathscr{B}^1 \otimes \mathbb{C}[f_1]$ where f_1 is the u^{-1} coefficient of $f(u)$. Therefore $\widetilde{L}(1-u) \in \mathcal{O}(\mathscr{B})$ has basis

$$v_k \coloneqq f_{1,1}^k \overrightarrow{v}.$$

One can manually verify, if desired, that none of these vectors generate sub-modules. The resulting formulas $\psi^\pm(u)v_k = \hbar^{-k}(1-u)v_k$ and $e(u)v_k \propto v_{k-1}$ agree with, up to rescaling, the presentation of the positive pre-fundamental module in [16, Section 5] (as they should).

Chapter 4: Vertices from critical K-theory

The close resemblance between the combinatorics of the $\mathscr{A}^\mu$-modules ${}^{>}\overrightarrow{V}(u)_v$ and the combinatorics of localization for (1-legged) DT/PT/quasimaps suggests that appropriate characters of the modules should recover certain *vertices* for these theories, as defined in sections 2.1.2 and 2.2.2. In fact, for a better choice of stability condition in the ansatz (S), it is possible that ${}^{>}\overrightarrow{V}(u)$ is literally the critical K-theory of the quasimap moduli $\mathsf{QMaps}(\mathbb{C} \to \mathcal{M}_Q)$ and characters are exactly quasimap vertices, though we will not pursue this direction here.

- § 4.1 defines q- and qq-characters for modules in $\mathcal{O}(\mathscr{A}^\mu)$ and $\mathcal{O}^\vee(\mathscr{A}^\mu)$ in various degrees of generality. In particular we discuss possible purely representation-theoretic formulations of qq-characters, from the R-matrix alone, and also a definition of qq-characters (originally only defined for finite-dimensional modules) for the infinite-dimensional modules ${}^{>}\overrightarrow{V}(u)_v$.
- § 4.2 applies everything to the setting $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$ to produce modules DT_μ whose q- and qq-characters are exactly 1-legged DT vertices (with leg μ). A similar construction produces modules PT_μ which capture PT 1-legged combinatorics, and we attempt to extract a DT/PT correspondence (with descendents) using a duality between DT_μ and PT_μ.

4.1 Characters

Let $\mathscr{A}^\mu$ be a shifted quantum affine algebra which we assume is of geometric type. Let $\mathcal{O} = \mathcal{O}(\mathscr{A}^\mu)$, noting that everything that follows is equally applicable to $\mathcal{O}^\vee = \mathcal{O}^\vee(\mathscr{A}^\mu)$. In this section we study successive refinements of the following classical notion of character.

Definition 4.1. Let $\mathbf{Q} \coloneqq (Q_i)_{i \in I}$ be formal variables. Given $V \in \mathcal{O}$ and $v \in V$, let $\mathrm{wt}(v) \in \mathfrak{h}^*$ denote its weight. The **character** of V is the graded trace

$$\chi(V) \coloneqq \mathrm{tr}_V \left(\mathbf{Q}^{\mathrm{wt}(-)} \right) \in \mathbb{Z}((\mathbf{Q}))$$

where Q_i records the i-th component of $\mathrm{wt}(v)$. For $V, W \in \mathcal{O}$, write

$$\chi(V) \preceq \chi(W)$$

to mean that each monomial in $\chi(V)$, with multiplicity, appears in $\chi(W)$ with at least the same multiplicity.

4.1.1 q-characters

We define q-characters for $V \in \mathcal{O}$ in a slightly more general and abstract fashion than in [10] where they originally appeared for *finite-dimensional* representations. Recall from Proposition 3.26 that all non-trivial ℓ-weights of any $V \in \mathcal{O}$ or $V \in \mathcal{O}^\vee$ have all *rational* components; let $\mathfrak{r}$ denote the set of such ℓ-weights.

Definition 4.2. Let $V \in \mathcal{O}$. Let $\mathbb{Z}^{\mathfrak{r}}$ be the ring of (possibly infinite) linear combinations of symbols $[\Psi]$ for $\Psi \in \mathfrak{r}$. The q**-character** of V is

$$\chi_q(V) \coloneqq \sum_{\Psi} \dim(V_\Psi) \cdot [\Psi] \in \mathbb{Z}^{\mathfrak{r}}.$$

This records the spectrum of the loop Cartan generators $\psi_i^{\pm}(u)$ on V.

Proposition 4.3. *$\chi_q \colon \mathcal{O} \to \mathbb{Z}^{\mathfrak{r}}$ is an injective ring homomorphism.*

Proof. That χ_q is a ring homomorphism follows from the triangularity in Example 2.31 of coproducts. That it is injective follows from the linear independence of χ_q of *simple* modules, which is immediate from Proposition 3.26. □

For reasons which will become completely clear in § 4.2, we will work with a *reduced* version of the q-character where we forget some data.

Definition 4.4. Recall that an ℓ-weight Ψ consists of a weight $\lambda(\Psi) \in \mathfrak{h}^*$ along with its actual affine part; from this, define a map $\lambda \colon \mathfrak{r} \to \mathfrak{h}^*$. Introduce formal variables $\mathbf{Q} \coloneqq (Q_i)_{i\in I}$ and let

$$\mathrm{ev}_i \colon \mathbb{Z}^{\mathfrak{r}} \to \Bbbk((\mathbf{Q})), \quad [\Psi] \mapsto \mathbf{Q}^{\lambda(\Psi)}\Psi_i,$$

which only retains the grading by weight and forgets the grading by ℓ-weight. The **reduced q-characters** are

$$\overline{\chi_q}^{(i)} \coloneqq \mathrm{ev}_i \circ \chi_q.$$

Equivalently, $\overline{\chi_q}^{(i)}$ is the graded trace $\overline{\chi_q}^{(i)} = \mathrm{tr}\left(\mathbf{Q}^{\mathrm{wt}(-)} \cdot \psi_i^{\pm}(u)\right)$.

Remark. It is important to retain the grading by weight in $\overline{\chi_q}^{(i)}$, since otherwise

$$\overline{\chi_q}\Big|_{\mathbf{Q}=1} = \mathrm{tr}(\psi_i^{\pm}(u)) = \mathrm{tr}(\psi_i^{\pm}(0))$$

is independent of the spectral parameter u which plays an important role in all asymptotic constructions. One can verify the last equality manually for $\mathfrak{g} = \mathfrak{sl}_2$ (which is enough for all $\mathfrak{g}$ of finite type) and $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$.

For $\mathscr{A}$ of geometric type, the loop Cartan generator $\psi_i^{\pm}(u)$ acts by multiplication by $\widehat{\mathsf{S}}_u^{\bullet}((1-\hbar^{-1})\widehat{\mathcal{T}aut_i})$, see (2.31). This yields the following geometric description of reduced q-characters.

Proposition 4.5. *Let* $V(u) = K_{\mathsf{T}}(\mathcal{M}_Q(\mathbf{w}))$. *Then*

$$\overline{\chi_q}^{(i)}(V(u)) = \sum_{\mathbf{v}} \mathbf{Q}^{\mathbf{v}} \chi\left(\mathcal{M}_Q(\mathbf{v},\mathbf{w}), \widehat{\mathsf{S}}_u^{\bullet}\left((1-\hbar^{-1})\widehat{\mathcal{T}aut_i}\right) \otimes \wedge_{-1}^{\bullet}(T^{\vee})\right) \in \Bbbk_{\mathsf{T},\mathrm{loc}}((\mathbf{Q})). \quad (4.1)$$

Note that, as tautological bundles are still well-defined in our construction of modules like ${}^{>}\overrightarrow{V}(u)$, the geometric formula (4.1) is equally applicable to them, with $\mathcal{M}_Q$ replaced by the asymptotic ${}^{>}\overrightarrow{\mathcal{M}}_Q$. Such a limiting procedure for q-characters was historically one of the original reasons asymptotic modules were studied; see [16].

4.1.2 qq-characters

In [23], Nekrasov introduces a deformation of the q-character via a geometric construction analogous to that of (4.1). We denote the deformation variable by κ and view it as an equivariant variable of a larger torus $\widetilde{\mathsf{T}} \supset \mathsf{T}$ (as in Example 3.3).

Definition 4.6. Let $V(u) = K_{\mathsf{T}}(\mathcal{M}_Q(\mathbf{w}))$. The **reduced** qq**-characters** of $V(u)$ are

$$\overline{\chi_{qq}}^{(i)}(V(u)) = \sum_{\mathbf{v}} \mathbf{Q}^{\mathbf{v}} \chi\left(\mathcal{M}_Q(\mathbf{v},\mathbf{w}), \widehat{\mathsf{S}}^{\bullet}_{u}\left((1-\hbar^{-1})\widehat{\mathcal{T}aut_i}\right) \otimes \wedge^{\bullet}_{-\kappa}(\mathcal{T}^{\vee}) \otimes \kappa^{-\frac{\mathrm{rk}(\mathcal{T})}{2}}\right) \in \Bbbk_{\widetilde{\mathsf{T}},\mathrm{loc}}((\mathbf{Q})) \tag{4.2}$$

where $\mathcal{T}$ is the tangent sheaf. Clearly when $\kappa = 1$ these become reduced q-characters as in (4.1). The $\kappa^{-\mathrm{rk}(\mathcal{T})/2}$ factor is not present in Nekrasov's original formula; we insert it so that the localization contribution

$$\kappa^{-\frac{\mathrm{rk}(\mathcal{T})}{2}} \frac{\wedge^{\bullet}_{-\kappa}(\mathcal{T}^{\vee})}{\wedge^{\bullet}_{-1}(\mathcal{T}^{\vee})} = \widehat{\mathsf{S}}^{\bullet}\left((1-\kappa)\mathcal{T}^{\vee}\right) \tag{4.3}$$

is symmetrized.

Remark. Unlike q-characters, qq-characters are *not* homomorphisms due to the tangent character $\mathcal{T}$. While tautological insertions like $\widehat{\mathcal{T}aut_i}$ are linear in the quiver data $\mathbf{V}$, the tangent character $\mathcal{T}$ is *quadratic* (see Proposition 2.15 for an explicit formula). On $V_1(u_1) \otimes V_2(u_2)$, each term in $\overline{\chi_{qq}}^{(i)}$ will therefore contain contributions from cross-terms in addition to contributions from $V_1(u_1)$ and $V_2(u_2)$ themselves.

With the content of § 4.1.1 on q-characters in mind, one may wonder if there is a purely representation-theoretic formulation of (4.2), as in Definition 4.4 for q-characters.

Proposition 4.7. *Let $R = R_{V,V}(u)$ be the R-matrix for $V(u)$, and R_∞ be the diagonal part* *of its Gauss decomposition. Let* $\pi\colon V(u)\otimes V(u)\to V(u)$ *be projection to the diagonal. Then*

$$\overline{\chi_{qq}}^{(i)}(V(u))\Big|_{\kappa=\hbar^{-1}} = \mathrm{tr}_{V(u)}\left(\mathbf{Q}^{\mathrm{wt}(-)}\cdot\psi_i^{\pm}(u)\cdot(\pi R_\infty\pi^{-1})\right).$$

Proof. The presence $\psi_i^{\pm}(u)$ term is clear from § 4.1.1. The remaining contributions are diagonal elements of the form $\langle v\otimes v|R_\infty|v\otimes v\rangle$. From Proposition 2.32 and (2.30), such a matrix element is exactly $\widehat{\mathsf{S}}^\bullet\left((1-\hbar^{-1})T^\vee\right)$ where T is the tangent space in $\mathcal{M}_Q$ of the fixed point associated to v, which is the specialization of (4.3) at $\kappa=\hbar^{-1}$. $\square$

Remark. It appears difficult in representation theory to deform κ away from the sub-torus $\mathbb{C}_\hbar^\times$, especially considering that the quantum groups $\mathscr{U}_q(\widehat{\mathfrak{g}})$ do not naturally have such an extra parameter κ.

In a slightly different direction, one may also wonder if qq-characters can be defined for modules in $\mathcal{O}$ or $\mathcal{O}^\vee$ in general. Our geometric construction of the modules ${}^{>}\overrightarrow{V}(u)_v$ from § 3.1 suggests that their qq-characters should be a limiting (infinite leg) case of (4.2); certainly this is already the case for q-characters and the term $\widehat{\mathsf{S}}_u^\bullet\left((1-\hbar^{-1})\widehat{\mathcal{T}aut_i}\right)$. Although the deformation parameter κ is specialized to $\kappa=1$ starting from § 3.1.2, it is possible to manually insert the κ deformation geometrically.

Definition 4.8. Recall that only the sub-torus $\mathsf{T}'\subset\widetilde{\mathsf{T}}'$ acts on ${}^{>}\overrightarrow{\mathcal{M}}_Q$ in its presentation as an equivariant critical locus. Let ${}^{>}_{\kappa}\overrightarrow{\mathcal{M}}_Q$ denote the same space as ${}^{>}\overrightarrow{\mathcal{M}}_Q$, but with κ acting on flags

$$\begin{array}{ccccccc} \mathbb{V}[0] & \xrightarrow{\boldsymbol{\xi}} & \mathbb{V}[1] & \xrightarrow{\boldsymbol{\xi}} & \mathbb{V}[1] & \xrightarrow{\boldsymbol{\xi}} & \cdots \\ \mathbf{a}\uparrow\downarrow\mathbf{a}^* & & \mathbf{a}\uparrow\downarrow\mathbf{a}^* & & \mathbf{a}\uparrow\downarrow\mathbf{a}^* & & \\ \overline{\mathbf{W}} & \overset{\Xi}{=\!=} & \overline{\mathbf{W}} & \overset{\Xi}{=\!=} & \overline{\mathbf{W}} & \overset{\Xi}{=\!=} & \cdots \end{array} \tag{4.4}$$

by scaling both $\boldsymbol{\xi}$ and Ξ with weight κ. Then the action of $(\boldsymbol{\xi},\Xi)$ has $\widetilde{\mathsf{T}}'$-weight $\kappa\hbar^{-1}$. Let ${}_\kappa\widehat{\mathcal{T}aut_i}$ denote the same modification of the tautological bundles $\widehat{\mathcal{T}aut_i}$.

The following computation supplies a very clean geometric interpretation for remaining terms in the infinite leg limit.

Lemma 4.9. *Given* $\left(\mathbf{V} \rightleftarrows \mathbf{W}\right) \in \mathcal{M}_Q$*, let* $T(\mathbf{V}, \mathbf{W})$ *be its tangent space. Similarly, given* $\left(\mathbb{V} \rightleftarrows \mathbb{W}\right) \in \underset{\kappa}{>}\overrightarrow{\mathcal{M}}_Q$*, let* $\overrightarrow{T}(\mathbb{V}, \mathbb{W})$ *be its virtual tangent space. Then*

$$\widehat{\mathsf{S}}^\bullet\left((1-\kappa)T(\mathbb{V}, \mathbb{W})^*\right) = \widehat{\mathsf{S}}^\bullet\left(\overrightarrow{T}(\mathbb{V}, \mathbb{W})^*\right). \tag{4.5}$$

Note that (4.5) is far from true if the $\widehat{\mathsf{S}}^\bullet$ are not symmetrized.

Proof. We use the notation from Definition 2.12 for Nakajima quiver varieties. Denote by $T_N(\mathbf{V}, \mathbf{W}) \coloneqq T_{(\mathbf{V} \rightleftarrows \mathbf{W})} N_Q$ the tangent space of the pre-image in the $N_Q \subset M_Q$ component. Using the formula for $T(\mathbf{V}, \mathbf{W})$ from Proposition 2.15,

$$(1-\kappa)T(\mathbb{V}, \mathbb{W})^* = (T_N(\mathbb{V}, \mathbb{W})^* - \kappa T_N(\mathbb{V}, \mathbb{W})^*) + \bigoplus_i \mathbb{V}_i^* \mathbb{V}_i \left(\kappa(1+\hbar^{-1}) - (1+\hbar^{-1})\right).$$

Similarly, the appropriate modification of Proposition 3.7 to $\kappa \neq 1$ gives

$$\overrightarrow{T}(\mathbb{V}, \mathbb{W})^* = \left(T_N(\mathbb{V}, \mathbb{W})^* - \kappa^{-1} T_N(\mathbb{V}, \mathbb{W})\right) + \bigoplus_i \mathbb{V}_i^* \mathbb{V}_i \left((\kappa\hbar^{-1} - 1) - (\hbar - \kappa^{-1})\right). \tag{4.6}$$

As $\widehat{\mathsf{S}}^\bullet(w) = -\widehat{\mathsf{S}}^\bullet(w^{-1})$ for a monomial w, the desired equality follows. ☐

Definition 4.10. Given $v \in V(u)$, let $\underset{\kappa}{>}\overrightarrow{\mathcal{M}}_{Q,v} \subset \underset{\kappa}{>}\overrightarrow{\mathcal{M}}_Q$ be the subvariety of flags whose slices are eventually v. The reduced qq-characters of ${}^{>}\overrightarrow{V}(u)_v$ are

$$\overline{\chi_{qq}}^{(i)}\left({}^{>}\overrightarrow{V}(u)_v\right) \coloneqq \chi\left(\underset{\kappa}{>}\overrightarrow{\mathcal{M}}_{Q,v},\, \hat{\mathcal{O}}^{\text{vir}} \otimes \widehat{\mathsf{S}}^\bullet_u \left((1-\kappa\hbar^{-1}) \cdot {}_\kappa\widehat{\mathcal{T}aut_i}\right)\Big|_{\kappa=1}\right). \tag{4.7}$$

The restriction of the tautological term to $\{\kappa = 1\}$ is for agreement with the original (4.2), though one cannot help but feel such a restriction is unnatural.

Since $\mathcal{T}^{\text{vir}}$ is of the form $\mathcal{F} - \kappa\mathcal{F}^\vee$, in the limit $\kappa = 1$ it follows that the localization contribution $\widehat{\mathsf{S}}^\bullet(T^{\text{vir}})$ is trivial, i.e. (4.3) vanishes, and what remains is the q-character.

4.2 Example: $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$

At last, it is time to venture beyond quantum affine algebras. In this section we study asymptotic modules for the remarkable quantum toroidal $\mathfrak{gl}_1$ algebra $\mathscr{A}$ and its shifted versions $\mathscr{A}^n$. The algebra $\mathscr{A}$ is reconstructed from the geometric R-matrix of $\mathsf{Hilb}(\mathbb{C}^2)$, and therefore governs aspects of many enumerative theories of curves on $\mathbb{C}^3$ whose boundary conditions lie in $\mathsf{Hilb}(\mathbb{C}^2)$. We review properties of $\mathscr{A}^n$ in § 4.2.1. In § 4.2.2 we realize the critical K-theory of DT 1-legged moduli as a $\mathscr{A}^n$ module for an appropriate shift. In § 4.2.2 we relate this DT module with an analogous PT module and attempt to extract DT/PT correspondences from representation theory using q- and qq-characters.

In spirit, $\mathscr{A}$ is the quantum affinization $\mathscr{U}_q(\widehat{\mathfrak{g}})$ of $\mathfrak{g} = \mathfrak{gl}_1$, even though $\widehat{\mathfrak{gl}}_1$ is not an affine Lie algebra. Since $\mathsf{Hilb}(\mathbb{C}^2)$ is acted on by a torus $(\mathbb{C}^\times)^2$ which scales the axes of $\mathbb{C}^2$ with weights x and y, one expects $\mathscr{A}$ to be a *two-parameter* deformation of $\widehat{\widehat{\mathfrak{gl}}}_1$ (as opposed to the usual single parameter q). In fact it is convenient to introduce a third variable z such that

$$xyz = 1. \tag{4.8}$$

Note that $z = \hbar$ in the notation for $\mathsf{Hilb}(\mathbb{C}^2)$ as a Nakajima quiver variety.

Definition 4.11. Let $n_+, n_- \in \mathbb{Z}$ be shifts. Let

$$\begin{aligned} g^\pm(u,v) &\coloneqq (u - x^{\pm 1}v)(u - y^{\pm 1}v)(u - z^{\pm 1}v) \\ \tau_m &\coloneqq \frac{(1-x^m)(1-y^m)(1-z^m)}{m}. \end{aligned}$$

The **shifted quantum toroidal $\mathfrak{gl}_1$ algebra** $\mathscr{U}_{x,y}^{n_+,n_-}(\widehat{\widehat{\mathfrak{gl}}}_1)$ is generated by Drinfeld currents

$$e(u) \coloneqq \sum_{r\in\mathbb{Z}} e_r u^{-r}, \quad f(u) \coloneqq \sum_{r\in\mathbb{Z}} f_r u^{-r}, \quad \psi^\pm(u) \coloneqq \sum_{r\geq -n_\pm} \psi^\pm_{\pm r} u^{\mp r}.$$

For brevity, set $x^+ \coloneqq e$ and $x^- \coloneqq f$. The defining relations are (cf. (2.18))

$$\psi^\pm_{-n_\pm} \text{ invertible with } \psi^+_{-n_+}\psi^-_{-n_-} = 1, \quad [\psi^s(u), \psi^{s'}(v)] = 0 \ \ \forall s, s' \in \{\pm\}, \tag{4.9a}$$

$$g^\pm(u,v)x^\pm(u)x^\pm(v) = g^\pm(v,u)x^\pm(v)x^\pm(u), \tag{4.9b}$$

$$g^\pm(u,v)\psi^s(u)x^\pm(v) = g^\pm(v,u)x^\pm(v)\psi^s(u) \ \ \forall s \in \{\pm\}, \tag{4.9c}$$

$$[x^+(u), x^-(v)] = \frac{1}{\tau_1}\delta(u/v)\left(\psi^+(u) - \psi^-(u)\right), \tag{4.9d}$$

$$[x^\pm_0, [x^\pm_1, x^\pm_{-1}]] = 0. \tag{4.9e}$$

As with (shifted) quantum affine algebras, it is common to define an alternate set of generators $\psi^\pm_{-n_\pm}$ and $\{h_m\}_{m\in\mathbb{Z}\setminus\{0\}}$, in place of $\{\psi^\pm_r\}_{r\in\mathbb{Z}}$, by (cf. (3.18))

$$\psi^\pm(u) \eqqcolon u^{\pm n_\pm}\psi^\pm_{\mp n_\pm} \exp\left(\sum_{m\geq 1} \tau_m h_{\pm m} u^{\mp m}\right).$$

It is immediate from the homogeneity of the defining relations that $\mathscr{U}^{n_+,n_-}_{x,y} \cong \mathscr{U}^{0,n_++n_-}_{x,y}$ by scaling $f(u), \psi^\pm(u)$ by u^{-n_+}. Hence $\mathscr{U}^{n_+,n_-}_{x,y}$ depends only on $n \coloneqq n_+ + n_-$, and we denote

$$\mathscr{A}^n \coloneqq \mathscr{U}^n_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1) \coloneqq \mathscr{U}^{0,n}_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1).$$

When $n = 0$ we write $\mathscr{A} \coloneqq \mathscr{A}^0$.

Remark. Assuming all other relations, (4.9e) is equivalent to the Serre relation

$$\mathrm{Sym}_{S_3}[x^\pm_{i_1}, [x^\pm_{i_2}, x^\pm_{i_3}]] = 0.$$

Without the Serre relation, the resulting algebra is known as the *Ding–Iohara–Miki algebra*.

4.2.1 Elliptic Hall algebra

There is a different presentation of the unshifted $\mathscr{A}$ which makes computations more explicit, where $\mathscr{A}$ is viewed as the Hall algebra of coherent sheaves on an elliptic curve. The main goal of this section is to write down a PBW basis and a shifted quantum Borel using this presentation.

Definition 4.12. Let $\mathbf{Z} \coloneqq \mathbb{Z}^2$ and $\mathbf{Z}^\times \coloneqq \mathbf{Z} \setminus \{(0,0)\}$. Set

$$\mathbf{Z}_+ \coloneqq \{(a,b) \in \mathbf{Z}^\times \mid a > 0 \text{ or } a = 0,\, b > 0\}, \quad \mathbf{Z}_- \coloneqq -\mathbf{Z}_+.$$

The **elliptic Hall algebra** $\mathscr{E} = \mathscr{E}_{x,y}$ is generated by elements

$$\{e_{\mathbf{a}} \mid \mathbf{a} \in \mathbf{Z}^\times\}, \quad \{K_{\mathbf{a}} \mid \mathbf{a} \in \mathbf{Z}\}.$$

For $\mathbf{a} = (a_1, a_2) \in \mathbf{Z}^\times$, set $\deg(\mathbf{a}) \coloneqq \gcd(a_1, a_2)$. When $\deg(\mathbf{a}) = 1$, it is convenient to define

$$\phi_{\mathbf{a}}(u) \coloneqq \sum_{k \geq 0} \phi_{k\mathbf{a}} u^{-k} \coloneqq \exp\left(\sum_{m=1}^{\infty} \tau_m e_{m\mathbf{a}} u^{-m}\right). \tag{4.10}$$

Then the defining relations of $\mathscr{E}_{x,y}$ are:

1. elements $K_{\mathbf{a}}$ are central, $K_{\mathbf{0}} \coloneqq 1$, and $K_{\mathbf{a}} K_{\mathbf{b}} \coloneqq K_{\mathbf{a}+\mathbf{b}}$;

2. if $\mathbf{a}, \mathbf{b}$ are collinear, then
$$[e_{\mathbf{a}}, e_{\mathbf{b}}] \coloneqq \delta_{\mathbf{a}+\mathbf{b}} \frac{K_{\mathbf{a}}^{-1} - K_{\mathbf{a}}}{\tau_{\deg(\mathbf{a})}}; \tag{4.11}$$

3. if $\mathbf{a}, \mathbf{b}$ are such that $\deg(\mathbf{a}) = 1$ and the triangle with vertices $(0,0), \mathbf{a}, \mathbf{a}+\mathbf{b}$ has no interior lattice points, then

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] \coloneqq -\epsilon_{\mathbf{a},\mathbf{b}} K_{\alpha(\mathbf{a},\mathbf{b})} \frac{\phi_{\mathbf{a}+\mathbf{b}}}{\tau_1}. \tag{4.12}$$

Here $\epsilon_{\mathbf{a},\mathbf{b}} \coloneqq \operatorname{sign}(\det(\mathbf{a},\mathbf{b})) \in \{\pm 1\}$ for linearly independent $\mathbf{a}, \mathbf{b}$, and

$$\alpha(\mathbf{a},\mathbf{b}) \coloneqq \frac{1}{2}\left(\epsilon_{\mathbf{a}}\mathbf{a} + \epsilon_{\mathbf{b}}\mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b})\right) \cdot \begin{cases} \epsilon_{\mathbf{a}} & \text{if } \epsilon_{\mathbf{a},\mathbf{b}} = 1 \\ \epsilon_{\mathbf{b}} & \text{if } \epsilon_{\mathbf{a},\mathbf{b}} = -1 \end{cases}$$

where $\epsilon_{\mathbf{a}} \coloneqq \pm 1$ for $\mathbf{a} \in \mathbf{Z}_{\pm}$.

There is a bi-grading on $\mathscr{E}$ given by

$$\deg e_{\mathbf{a}} \coloneqq \mathbf{a}, \quad \deg K_{\mathbf{a}} \coloneqq (0,0)$$

whose components we call the *horizontal* and *vertical* degrees respectively.

Proposition 4.13 ([29]). *Let $\overline{\mathscr{E}} \coloneqq \mathscr{E}/\langle K_{(0,1)} = 1\rangle$. Then there is an isomorphism $\mathscr{A} \cong \overline{\mathscr{E}}$ given by*

$$e_n \mapsto e_{(1,n)}, \quad f_n \mapsto e_{(-1,n)}, \quad h_{\pm m} \mapsto e_{(0,\pm r)}, \quad \psi_0^+ \mapsto K_{(1,0)}.$$

Remark. The element $K_{(0,1)} \in \mathscr{E}$ is identified with the central charge that should be in $\mathscr{U}_{x,y}(\widehat{\widehat{\mathfrak{gl}}}_1)$, which we have set to 1 in $\mathscr{A}$ (as we have also done for quantum affine algebras). The full centrally-extended algebra can be found in e.g. [6, Section 2.1]. Also, for $\mathscr{A}^{\mu}$ in general, one can remove the "extra" relation $\psi^+_{-n_+}\psi^-_{-n_-} = 1$ in (4.9a) (cf. (2.18a)), resulting in yet another (split) central extension. However, this extension is slightly unnatural in $\mathscr{E}$.

For each $\mathbf{a} \in \mathbf{Z}^{\times}$ with $\deg(\mathbf{a}) = 1$, there is a *slope sub-algebra* generated by elements $\{e_{k\mathbf{a}}, K_{k\mathbf{a}}\}_{k\in\mathbb{Z}}$. The relation (4.11) means these slope sub-algebras are all isomorphic to the quantum Heisenberg algebra $\mathscr{U}_q(\widehat{\mathfrak{gl}}_1)$. This yields a beautiful picture of $\mathscr{A}$ as a collection of $\mathscr{U}_q(\widehat{\mathfrak{gl}}_1)$, one for each rational slope, and makes manifest a certain rotational symmetry of $\mathscr{A}$ (which actually extends to a symmetry by the universal cover of $\mathrm{SL}(2,\mathbb{Z})$). In fact, one can even use (4.12) to normal-order monomials by slope, as follows.

Definition 4.14. Given $\mathbf{a} = (r\cos\theta, r\sin\theta) \in \mathbf{Z}^{\times}$, let $\arg\mathbf{a} \coloneqq \theta \in [-\pi/2, 3\pi/2)$. A monomial of the form $K_{\mathbf{b}} e_{\mathbf{a}^{(1)}} e_{\mathbf{a}^{(2)}} \cdots e_{\mathbf{a}^{(n)}} \in \mathscr{E}$ is **slope-ordered** if

$$\frac{3\pi}{2} > \arg \mathbf{a}^{(1)} \geq \arg \mathbf{a}^{(2)} \geq \cdots \geq \arg \mathbf{a}^{(n)} \geq -\frac{\pi}{2}$$

and if $\arg \mathbf{a}^{(i)} = \arg \mathbf{a}^{(i+1)}$ then $\deg(\mathbf{a}^{(i)}) \geq \deg(\mathbf{a}^{(i+1)})$.

Proposition 4.15 ([29, Proposition 5.1])**.** *$\mathscr{E}$ has a basis given by slope-ordered monomials.*

The bounds on $\arg \mathbf{a}$ are chosen so that in slope-ordered monomials, lowering operators precede raising operators, just as in the *normal-ordering* $\mathscr{A} = \mathscr{A}_+ \mathscr{A}_0 \mathscr{A}_-$, where $\mathscr{A}_\pm$ are sub-algebras generated by $\{e_{\mathbf{a}} \mid \mathbf{a} \in \mathbf{Z}_\pm\}$ and $\mathscr{A}_0$ is the loop Cartan (cf. Definition 2.21).

Definition 4.16. The **shifted quantum affine Borel** $\mathscr{B}^n \subset \mathscr{A}^n$ is the sub-algebra generated by

$$\{e_{(a,b)} \mid (a < 0,\ b > \max(0, -na)) \text{ or } (a \geq 0,\ b \geq 0)\}. \tag{4.13}$$

(Here the symbol $e_{\mathbf{a}} \in \mathscr{A}^n$ means to write $e_{\mathbf{a}} \in \mathscr{A}$ in terms of generators in the Drinfeld presentation, and to take the same expression in $\mathscr{A}^n$.) Note that $\mathscr{B} := \mathscr{B}^0 \subset \mathscr{A}$ is the usual quantum Borel, i.e. the one preserved by the standard coproduct Δ on $\mathscr{A}$.

Lemma 4.17. *$\mathscr{B}^n$ is isomorphic to the sub-algebra of $\mathscr{A}$ generated by the same elements as in* (4.13)*.*

Proof. By recursive applications of (4.12), commutators between various $e_{\mathbf{a}} \in \mathscr{B}^n$ reduce to commutators involving e_k, f_s, ψ_k^+ for $k \geq 0$ and $s > n$. Such commutators are independent of $\psi^-(z)$, as in the proof of Lemma 3.24. □

The content of § 3.3.1 on modules for shifted quantum affine algebras applies verbatim to $\mathscr{A}^n$-modules.

4.2.2 Fock, DT and PT modules

Let Q be the Jordan quiver, with one vertex and a single loop, so that $\mathcal{M}_Q(1) = \mathsf{Hilb}(\mathbb{C}^2)$. Let

$$F(u) \coloneqq K_{\mathsf{T}}(\mathsf{Hilb}(\mathbb{C}^2))$$

denote the **Fock module** of $\mathscr{A}$, with basis vectors $|\lambda\rangle$ corresponding to (structure sheaves of) T-fixed points. We use $F(u)$ to construct asymptotic modules which have bases indexed by DT and PT 1-leg configurations.

Definition 4.18. There is a well-known identification

$$F(u) \cong \mathbb{k} \otimes \mathbb{Q}[p_1, p_2, \ldots], \quad \mathbb{k} = \mathbb{Z}[u^{\pm}, x^{\pm}, y^{\pm}]$$

of $F(u)$ with the ring of symmetric polynomials over $\mathbb{k}$, where the grading given by $\deg(p_k) = k$ agrees with the grading $\mathsf{Hilb}(\mathbb{C}^2) = \bigsqcup_n \mathsf{Hilb}(\mathbb{C}^2, n)$ by the instanton charge n. As symmetric polynomials,

$$|\lambda\rangle = P_\lambda$$

where $\{P_\lambda\}$ are Macdonald (q,t) polynomials in Haiman's normalization [14, Section 6.1], with $(q,t) = (x,y)$.

Definition 4.19. Let λ be a partition, with entries denoted $\lambda_0 \geq \lambda_1 \geq \cdots \geq 0$. Write $(i,j) \in \lambda$ if $0 \leq j < \lambda_i$, i.e. there is a square at (i,j) in the Young diagram of λ. Let

$$\chi_\lambda \coloneqq \sum_{(i,j)\in\lambda} x^i y^j \in \mathbb{k}$$

be the character of λ.

Proposition 4.20 ([30])**.** *The $\mathscr{E}$-action on $F(u)$ is characterized by:*

- *(level)* $K_{(1,0)} = z^{-1/2}$ *and* $K_{(0,1)} = 1$*;*

- *(horizontal and vertical generators)* $e_{(-m,0)} = m\frac{\partial}{\partial p_m}$ *for* $m > 0$*, and*

$$e_{(0,m)}P_\lambda = \operatorname{sign}(m)\left(\frac{1}{1-x^m}\sum_{i\geq 0} x^{m\lambda_i}y^{mi}\right)P_\lambda.$$

Using (4.11) one can obtain a formula for the action of $e_{(m,0)}$ for $m > 0$. More importantly, using (4.10) and the isomorphism with $\mathscr{A}$,

$$\begin{aligned}\psi^\pm(u)\left|\lambda\right\rangle &= z^{-1/2}\phi_{(0,1)}(u)\left|\lambda\right\rangle \\ &= z^{-1/2}\mathsf{S}_u^\bullet\left((1-z)\left(1-(1-x)(1-y)\chi_\lambda\right)\right)\left|\lambda\right\rangle \\ &= \widehat{\mathsf{S}}_u^\bullet\left((1-z)\left(1-(1-x)(1-y)\chi_\lambda\right)\right)\left|\lambda\right\rangle.\end{aligned} \tag{4.14}$$

It follows that $F(u) = L^\vee(\widehat{\mathsf{S}}_u^\bullet(1-z))$.

Remark. The R-matrix $R_{F,F}$ for $F(u)$ and its Gauss factorization is given in [25, Section 8.2]. Using it, one can compute that there is a discrepancy between the vacuum matrix element

$$\langle\emptyset|(R_{F,F})_\infty|\emptyset\rangle = \widehat{\mathsf{S}}_u^\bullet\left((1-z)\left(\frac{1}{(1-x)(1-y)} - \chi_\lambda\right)\right)$$

and $\psi^\pm(u)$. This sort of discrepancy occurs whenever vertices of the quiver have self-loops.

Definition 4.21. Applying the construction of § 3.2 to $F(u)$ yields two types of asymptotic modules, which we denote

$$\begin{aligned}\mathsf{DT}_\mu(u) &\coloneqq {}^>\overrightarrow{F}(u)_\mu \\ \mathsf{PT}_\mu(u) &\coloneqq {}^>\overleftarrow{F}(z^{-1}u)_\mu.\end{aligned}$$

Denote elements of each by

$$\begin{aligned}|\overrightarrow{\lambda}\rangle &\coloneqq |\lambda^{(0)}\rangle\otimes|\lambda^{(1)}\rangle\otimes|\lambda^{(2)}\rangle\otimes\cdots\in\mathsf{DT}_\mu \\ |\overleftarrow{\lambda}\rangle &\coloneqq \cdots\otimes|\lambda^{(-3)}\rangle\otimes|\lambda^{(-2)}\rangle\otimes|\lambda^{(-1)}\rangle\in\mathsf{PT}_\mu\end{aligned}$$

where $\lambda^{(i)} \geq \lambda^{(i+1)}$ for all i, in both $\overrightarrow{\lambda}$ and $\overleftarrow{\lambda}$.

By construction, the elements of the suggestively-named DT_μ and PT_μ correspond precisely to box configurations underlying the DT and PT vertices with one non-trivial leg μ in the z direction. (To be more precise, elements in PT_μ are the *complement* of 1-legged PT configurations inside the infinite cylinder $\mu \times \mathbb{Z}$.) In a slight abuse of notation, $\overrightarrow{\lambda}$ and $\overleftarrow{\lambda}$ will also denote these box configurations. As with partitions, let

$$\chi_{\overrightarrow{\lambda}} \coloneqq \sum_{(i,j,k)\in\overrightarrow{\lambda}} x^i y^j z^k = \sum_{k\geq 0} z^k \chi_{\lambda^{(k)}}$$

denote the T-character of $\overrightarrow{\lambda}$, and similarly for $\chi_{\overleftarrow{\lambda}}$.

Proposition 4.22.

- $\mathsf{DT}_\mu \in \mathcal{O}^\vee(\mathscr{A}^{-1})$ *with*

$$\psi^\pm(u)\,|\overrightarrow{\lambda}\rangle = -\frac{1}{1-u^{-1}}\widehat{\mathsf{S}}^\bullet_u\left(-(1-x)(1-y)(1-z)\chi_{\overrightarrow{\lambda}}\right)|\overrightarrow{\lambda}\rangle + (\textit{off-diagonal}). \tag{4.15}$$

- $\mathsf{PT}_\mu \in \mathcal{O}(\mathscr{A}^1)$ *with*

$$\psi^\pm(u)\,|\overleftarrow{\lambda}\rangle = -(1-u^{-1})\widehat{\mathsf{S}}^\bullet_u\left(-(1-x)(1-y)(1-z)\chi_{\overleftarrow{\lambda}}\right)|\overleftarrow{\lambda}\rangle + (\textit{off-diagonal}). \tag{4.16}$$

Proof. Straightforward computation using (4.14), e.g. as in Lemma 3.17, keeping in mind that $|z| = |xy|^{-1} > 1$. (For PT_μ, there is an overall shift by u^{-1} due to the identification $\mathscr{A}^{1,0} \cong \mathscr{A}^{0,1} = \mathscr{A}^1$.) □

Let $\mathscr{B}^n \subset \mathscr{A}^n$ denote the shifted Borel. By restriction, we can view DT_μ and PT_μ as modules

$$\mathsf{DT}_\mu \in \mathcal{O}(\mathscr{B}), \quad \mathsf{PT}_\mu \in \mathcal{O}(\mathscr{B}^1),$$

and we will implicitly do so from here onward. By Lemma 3.28, DT_μ is simple as a $\mathscr{B}$-module. It is not clear a priori if PT_μ is simple as a $\mathscr{B}^1$-module; this will be a consequence of the proof of Proposition 4.23.

Note that DT_μ is a module for the *unshifted* Borel, and was first constructed in [7] via a similar semi-infinite tensor product construction but at infinite slope, where explicit formulas for the action of Drinfeld generators may be written. On the other hand PT_μ truly requires the *shifted* Borel $\mathscr{B}^1$. Computations of Gaiotto and Rapčak in [11] in twisted M-theory have already suggested that PT_μ carries a $\mathscr{B}^1$-module structure, but a systematic construction of PT_μ as a $\mathscr{B}^1$-module has not yet been presented in the literature.

Remark. Both [7] and [11] consider *fully* 3-*legged* DT and PT configurations. Presumably our asymptotic construction of the 1-legged DT_μ and PT_μ can be made 3-legged by starting with a generalized Fock module containing partitions with two infinite legs, though it is not immediately clear how the combinatorics of 3-legged PT configurations (which can vary in positive-dimensional families) would arise.

4.2.3 1-legged DT/PT duality

Proposition 4.22 means that the results of § 3.3.3 are applicable and yield a comparison of DT_μ and PT_μ as follows. The q- and qq-characters of these modules will be related to DT and PT 1-leg descendent vertices, and so such a comparison has immediate consequences for the DT/PT correspondence. Let

$$M(Q) \coloneqq \prod_{n>0} \frac{1}{(1-Q^n)^n} \in \mathbb{Z}[[Q]]$$

be the MacMahon function.

Proposition 4.23. *Both* $\mathsf{DT}_\mu \in \mathcal{O}(\mathscr{B})$ *and* $\mathsf{PT}_\mu \in \mathcal{O}(\mathscr{B}^1)$ *are simple, and*

$$\operatorname{ind}_{\mathscr{B}^1}^{\mathscr{B}} \mathsf{PT}_\mu = (\mathsf{DT}_\mu)^*.$$

Proof. Note that the discrepancy between $\mathscr{B}^1$ and $\mathscr{B}$ involves n generators in horizontal degree $-n$, for each $n \in \mathbb{Z}_{>0}$. Hence

$$\chi(\operatorname{ind}_{\mathscr{B}^1}^{\mathscr{B}} \mathsf{PT}_\mu) \preceq M(Q)\chi(\mathsf{PT}_\mu) = \chi(\mathsf{DT}_\mu)$$

where the equality is the numerical DT/PT correspondence. The desired results then follow from Proposition 3.35. □

Let $\widetilde{\mathsf{PT}}_\mu \coloneqq \operatorname{ind}_{\mathscr{B}^1}^{\mathscr{B}} \mathsf{PT}_\mu$. When $\mu = \emptyset$, the simple module

$$\widetilde{\mathsf{PT}}_\emptyset = L_{\mathscr{B}}\left(-1 + u^{-1}\right)$$

has been studied in [6] as the *positive fundamental module* for $\mathscr{B}$. In particular, its explicit description in [6, Proposition 4.9] exactly matches our description of it as the $\mathscr{B}$-module induced from the trivial $\mathscr{B}^1$-module.

Proposition 4.24. $\chi_q(\widetilde{\mathsf{PT}}_\mu) = M(Q)\chi_q(\mathsf{PT}_\mu)$.

Proof. For $\mathscr{B}$-modules, χ_q is computed using eigenvalues of $\psi^+(u)$. Let $gv \in \widetilde{\mathsf{PT}}_\mu$, where

$$g \coloneqq e_{\mathbf{a}^{(1)}} \cdots e_{\mathbf{a}^{(n)}} \quad e_{\mathbf{a}^{(i)}} \in \mathscr{B}/\mathscr{B}^1$$

is a slope-ordered monomial and $v \in \mathsf{PT}_\mu$ is an eigenvector of $\psi^+(u)$; ranging over all such g and v, these gv form a basis of $\widetilde{\mathsf{PT}}_\mu$. We claim that

$$\psi^+(u)gv = g\psi^+(u)v + (\text{off-diagonal}), \tag{4.17}$$

which immediately yields the desired identity of χ_q.

Coefficients of $\psi^+(u)$ are monomials in $\{h_k\}_{k>0}$. Recall from (4.12) that $\operatorname{ad}(h_k)$ changes the bi-degree of elements of $\mathscr{A}$ by exactly $(0, k)$. For example,

$$[h_k, e_m] = e_{m+k}, \quad [h_k, f_m] = -f_{m+k} \quad k \neq 0. \tag{4.18}$$

It follows that $\deg\left([\psi_i^+, g]\right) = \deg(g) + (0, i)$. Write $[\psi_i^+, g]$ as a sum of slope-ordered monomials. To get an extra on-diagonal term on the rhs of (4.17), it is therefore necessary that at least one of these monomials is of the form gh for $\deg h = (0, i)$. The appearance of such a monomial is impossible due to (4.18), where no such h appears on the rhs. □

Remark. Let $\mathsf{PT}_\mu = L(\Psi_\mu) \in \mathcal{O}(\mathscr{A}^1)$, so that Proposition 4.24 says

$$\chi_q(L_{\mathscr{B}}(\Psi_\mu)) = \chi_q(L(\Psi_\mu))M(Q).$$

Formulas of this form, for q-characters of simple $\mathscr{B}$-modules, are known for finite-dimensional representations of quantum affine algebras, see [15, Theorem 8.1]. Presumably they continue to hold for quantum affinizations, e.g. toroidal algebras, in general.

Corollary 4.25. *Let $S\colon \mathscr{B} \to \mathscr{B}^{op}$ be the antipode. Then*

$$M(Q)\overline{\chi_q}(\mathsf{PT}_\mu) = \operatorname{tr}_{\mathsf{DT}_\mu}\left(Q^{|\cdot|} \cdot S(\psi^+(u))\right).$$

Proof. By definition, $\psi^+(u)$ acts on $\widetilde{\mathsf{PT}}_\mu = (\mathsf{DT}_\mu)^*$ as $S(\psi^+(u))^t$. The transpose is irrelevant when taking trace. □

4.2.4 Characters as vertices

Finally, we relate reduced q- and qq-characters of DT_μ and PT_μ to the more traditional notions of DT and PT (1-legged) vertices with descendents, so that Corollary 4.25 bears some resemblance to a DT/PT correspondence. Such a relation should not be unexpected, since the critical locus ${}^{>}\overrightarrow{\mathcal{M}}_Q$ underlying these modules is strongly related to the (1-legged part of the) DT moduli space.

To discuss qq-characters, we must work with the *larger* torus $\widetilde{\mathsf{T}} \supset \mathsf{T}$ where there is an extra variable κ, as we did in § 4.1.2. For this section only, set

$$z \coloneqq \kappa^{-1}\hbar.$$

Then $\mathsf{T} = \{xyz = 1\} \subset \widetilde{\mathsf{T}}$, and restriction to T corresponds to the degeneration from qq- to q-characters and also makes this new z agree with the z of the previous sections.

Proposition 4.26. *Let $\mathcal{U}niv$ be the universal sheaf on DT or PT moduli spaces. Then*

$$\overline{\chi_{qq}}(\mathsf{DT}_\mu) = \mathsf{V}_{\mathsf{DT}(\emptyset,\emptyset,\mu)}\left(\widehat{\wedge}^{\bullet}_{u/z}\mathcal{U}niv\Big|_{z\mapsto 1/xy}\right) \tag{4.19}$$

$$\overline{\chi_{q}}(\mathsf{PT}_\mu) = \mathsf{V}_{\mathsf{PT}(\emptyset,\emptyset,\mu)}(\widehat{\mathsf{S}}^{\bullet}_{u/z}\mathcal{U}niv)\Big|_{z\mapsto 1/xy}. \tag{4.20}$$

Note that the restriction of (4.19), in its entirety, to the Calabi–Yau torus $\{xyz = 1\}$ gives the analogue of (4.20) for DT. On the other hand, (4.20) does not hold away from $\{xyz = 1\}$, i.e. $\overline{\chi_{qq}}(\mathsf{PT}_\mu)$ is not naturally a PT vertex.

Proof. We will match contributions in the formula (4.7) for qq-characters with corresponding terms in DT or PT theory.

Let $\pi = \overrightarrow{\lambda}$ be a DT configuration, namely a 3d partition with leg μ in the z direction. In q- and qq-characters, the tautological insertion

$$(1 - z^{-1}) \cdot {}_{\kappa}\widehat{\mathcal{T}aut}\Big|_{\overrightarrow{\lambda}} = -z^{-1}(1 - z)\sum_{k\geq 0} z^k\left(1 - (1 - x)(1 - y)\chi_{\lambda^{(k)}}\right)$$

is equal to $-z^{-1}\mathcal{U}niv\big|_{\pi}$. The overall minus sign changes $\widehat{\mathsf{S}}^{\bullet}$ into $\widehat{\wedge}^{\bullet}$. For a PT configuration π corresponding to $\overleftarrow{\lambda} \in \mathsf{PT}_\mu$,

$$(1 - z^{-1}) \cdot {}_{\kappa}\widehat{\mathcal{T}aut}\Big|_{\overleftarrow{\lambda}} = -z^{-1}(1 - z)\sum_{k<0} z^k\left(1 - (1 - x)(1 - y)\chi_{\lambda^{(k)}}\right)$$

is equal to $z^{-1}\mathcal{U}niv\Big|_{\pi}$, using that $\chi_\pi = \delta(z)\chi_\mu - \chi_{\overleftarrow{\lambda}}$. This concludes the proof of (4.20).

To finish the proof of (4.19), we must match contributions from $\mathcal{T}^{\mathrm{vir}}$ on both sides. Let $\pi = \overrightarrow{\lambda}$ be a DT configuration. From (4.6) or otherwise, the $\mathcal{T}^{\mathrm{vir}}$ contribution in the qq-character is $\mathcal{T}^{\mathrm{vir}}_{\overrightarrow{\lambda}} = \mathcal{E} - (xyz)^{-1}\mathcal{E}^*$ where

$$\mathcal{E} \coloneqq \frac{1}{1-z^{-1}}\chi_{\overrightarrow{\lambda}} + \frac{(xy)^{-1}}{1-z}\overline{\chi}_{\overrightarrow{\lambda}} + \chi_{\overrightarrow{\lambda}}\overline{\chi}_{\overrightarrow{\lambda}}\left(x^{-1}+y^{-1}+z^{-1}-1\right)$$

which by direct calculation equals the formula (2.7) for $\mathcal{T}^{\mathrm{vir}}$ of DT moduli space. For PT, the discrepancy between $\chi_{\overleftarrow{\lambda}}$ and χ_π prevents the same equality from holding. □

Returning to Corollary 4.25, whose lhs now resembles something one might see in a DT/PT correspondence, it seems difficult to give a geometric meaning to the antipode $S(\psi^+(u))$ on the rhs. The resulting DT/PT descendent transformation would be of the form

$$\mathsf{V}_{\mathsf{PT}(\emptyset,\emptyset,\mu)}(\widehat{\mathsf{S}}^{\bullet}_{u/z}\,\mathcal{U}niv)\Big|_{xyz=1} = \frac{\mathsf{V}_{\mathsf{DT}(\emptyset,\emptyset,\mu)}(\mathcal{E})}{\mathsf{V}_{\mathsf{DT}(\emptyset,\emptyset,\emptyset)}}\Big|_{xyz=1} \tag{4.21}$$

for some mysterious sheaf $\mathcal{E}$. Note that the overall form of (4.21) is strange since one expects a non-trivial insertion for $\mathsf{V}_{\mathsf{DT}(\emptyset,\emptyset,\emptyset)}$ in a true DT/PT descendent correspondence, but it is unclear how such an insertion would arise in light of Proposition 4.24. It is also unclear how to lift each ingredient of Corollary 4.25 from q-characters to qq-characters, i.e. to deform (4.21) away from $\{xyz = 1\}$.

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