TOPICS IN STOCHASTIC PORTFOLIO THEORY:
Pathwise Generation of Trading Strategies, and Portfolio Theory in Open Markets

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Abstract

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This thesis generalizes stochastic portfolio theory in two different aspects. The first part demonstrates the functional generation of portfolios in a pathwise way. This notion of functional generation of portfolios was first introduced by E.R. Fernholz, to construct a variety of portfolios solely in the terms of the individual companies' market weights. I. Karatzas and J. Ruf developed recently another approach to the functional construction of portfolios, which leads to very simple conditions for strong relative arbitrage with respect to the market. Both of these notions of functional portfolio generation are generalized in a pathwise, probability-free setting; portfolio generating functions, possibly less smooth than twice-differentiable, involve the current market weights, as well as additional bounded-variation functionals of past and present market weights.

This generalization leads to a wider class of functionally-generated portfolios than was heretofore possible to analyze, and to improved conditions for outperforming the market portfolio over suitable time-horizons.

The second part develops portfolio theory in open markets. An open market is a subset of the entire equity market, composed of a certain fixed number of top-capitalization stocks. Though the number of stocks in open market is fixed, the constituents of the market change over time as each company's rank by its market capitalization fluctuates. When one is allowed to invest also in money market, an open market resembles the entire 'closed' equity market in the sense that most of the results that are valid for the entire market, continue to hold when investment is restricted to the open market. One of these results is the equivalence of market viability (lack of arbitrage) and the existence of numéraire portfolio (portfolio which cannot be outperformed). When access to the money market is prohibited, the class of portfolios shrinks significantly in open markets. In such a case, we discuss the Capital Asset Pricing Model, how to construct functionally-generated portfolios, and the concept of universal portfolio in open market setting.

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Introduction

This thesis consists of two main chapters. Chapter 1 presents the functional generation of trading strategies in a pathwise sense. The concept of functional generation of portfolios was introduced by Fernholz [7], [9] and has been one of the essential components of stochastic portfolio theory; see [10] for an overview. Portfolios generated by appropriate functions of the individual companies' market weights have wealth dynamics which can be expressed solely in terms of these weights, and do not involve any stochastic integration. Constructing such portfolios does not require any statistical estimation of parameters, or any optimization. Completely observable quantities such as the current values of 'market weights', whose temporal evolution is modeled in terms of continuous semimartingales, are the only ingredients needed for building these portfolios. Once this structure has been discerned, the mathematics underpinning its construction involves just simple calculus. Then the goal is to construct such portfolios that outperform a reference portfolio, for example, the market, under suitable structural conditions.

A new functional generation of trading strategies, which we call 'additive', as opposed to Fernholz's 'multiplicative' generation, was recently discovered by Karatzas and Ruf [15]. This new methodology weakens the assumptions on the market: asset prices and market weights are continuous semimartingales, and trading strategies are constructed from 'regular' functions of these semimartingales. Strategies generated in this additive manner require simpler conditions for strong relative arbitrage with respect to the market, over appropriate time horizons; see also [8].

In a different, but related, development, Föllmer [11] showed almost 40 years ago that certain aspects of Itô calculus can be developed 'path by path', without any probability structure; in particular, without any semimartingale assumption. Once a given function has finite quadratic variation/covariation along a given nested sequence of partitions over a fixed time interval of finite length, change of variable formulas can be proven by Taylor expansion in a surprisingly simple way. Then Würmli [28] introduced in this setting the concept of local times and the corresponding pathwise Tanaka formula. This allows the change of variable formula to be applied to less regular functions, by involving appropriately defined pathwise local times. Such local times have been further developed recently; see [3], [6], and [22].

Most part of Chapter 1 is based on the paper [14] and generalizes both additive and multiplicative functional generation of trading strategies in several ways. First, we use pathwise Itô calculus to show how to construct trading strategies, generated additively or multiplicatively from a given function, depending on the market weights and in a manner completely devoid of probability considerations. The only analytic structure we impose is that market weights admit continuous covariations in a pathwise sense. Secondly, we allow generating functions that depend on an additional argument of finite variation. Introducing new arguments, other than the market weights, provides extra flexibility for constructing portfolios; see [23], [24], [26]. We present various types of additional such arguments, to the effect that a variety of new trading strategies can be generated from a function depending on them; these strategies yield new sufficient conditions for outperforming the market.

The pathwise Tanaka formula is also applied to the construction of portfolios from generating functions rougher than heretofore possible. The classical Itô formula applies to functions which are at least twice-differentiable; whereas the Tanaka formula only requires absolute continuity. Usage of the Tanaka formula not only broadens the class of portfolio-generating functions very considerably, but also yields a lot of identities involving pathwise local times of functions. These identities are pathwise generalizations of those in semimartingale theory. Using them as building blocks, expressions for the ranked (in descending order) functions amongst m given functions are derived, in terms of the original ones and of appropriate pathwise local times. More specifically, we represent the Föllmer-Itô integral with respect to the k-th ranked function $X_{(k)}$ among m continuous functions $X_1, \cdots, X_m$, as the sum of same integrals of the original functions, and collision local time terms generated whenever these functions collide. This representation helps us generate trading strategies depending on the ranked market weights. These results regarding pathwise local time and Tanaka formula are summarized in Section 1.2, which contains results of the paper [17].

Several sufficient conditions for strong relative arbitrage via additively and multiplicatively generated trading strategies are also provided in Chapter 1. The existing sufficient condition in [15] requires the generating function to be ‘Lyapunov’, or the corresponding ‘Gamma function’ to be nondecreasing. By contrast, the new sufficient conditions depend on the intrinsic nondecreasing structure of the generating function itself. These new conditions show that trading strategies outperforming the market can be generated from a much richer collection of functions depending on the market weights, and on an additional argument of finite variation. We provide some interesting examples of such trading strategies.

Chapter 2 focuses on the portfolio theory in open markets and is based on the paper [18]. Equity markets are conventionally thought of as being ‘closed’, in the sense that they are almost universally assumed to consist of a given, fixed number of stocks at all times. However, this assumption fails to represent most real markets, where new stocks enter and some others exit due to privatization, bankruptcy, or simply bad luck.

The number of companies in the U.S. stock market has undergone wide fluctuations. In 1975, there were around 4,800 U.S. domiciled firms listed on the NYSE, Amex, and Nasdaq. This number reached a peak of 7,500 listed firms in 1997, and then decreased by more than half to 3,600 firms 20 years later in 2017.

To mitigate the assumption of a fixed, immutable collection of companies, and to model stock markets more realistically, we study here markets that are ‘open’. These are constructed by restricting, at any given time, our investing space from the entire market to the subset composed of a certain fixed number n of top-capitalization stocks at that time. More precisely, within the entire stock market, we keep track of the price dynamics of all stocks, rank them by order of market capitalization, consider an open market consisting of the top n stocks, and only invest in stocks that belong to this open market. High-capitalization indexes, such as the S&P 500 index, where one invests only in the $n = 500$ highest-capitalization companies, and any given stock is replaced by another one when its capitalization falls, are of this type.

In Chapter 2, some results from closed markets which remain valid also in open markets, are presented. The main result of this type involves the concept of market viability, which is understood as “lack of a certain egregious form of arbitrage”; this condition prohibits financing non-trivial liabilities starting from arbitrarily small initial capital. The result shows that in an open stock market, with an access to the money market, viability is equivalent to any one of the following conditions: (i) a portfolio with the local martingale numéraire property exists; (ii) a local martingale deflator exists; (iii) the market has locally finite maximal growth; (iv) a growth-optimal portfolio exists; and (v) a portfolio with the log-optimality property exists. We provide precise definitions for these terms, and show that this equivalence can be formulated in terms of the drifts and covariations of the underlying stock prices, modeled by continuous semimartingales.

When access to the money market is forbidden, and one is only allowed to invest in a fixed number n of top-capitalization stocks, the class of eligible portfolios diminishes significantly, as portfolios must satisfy conditions of “self-financibility”. Under this extra condition, we provide a connection of the above viability theory to the Capital Asset Pricing Model (CAPM), develop a way for constructing functionally-generated portfolios, and discuss the concept of universal portfolio in open markets.

Chapter 1: Pathwise Generation of Trading Strategies

This chapter is devoted to the pathwise functional-generation of trading strategies. Section 1.1 and 1.2 present the elements of pathwise Itô-Tanaka calculus with help from the relevant notion of local time needed for our purposes. Section 1.3 defines trading strategies and regular functions, then discusses how to generate the former from the latter in ways both additive and multiplicative. Section 1.4 gives sufficient conditions for such trading strategies to generate strong arbitrage relative to the market. Section 1.5 presents the way to construct trading strategies depending on the ranked market weights. Section 1.6 provides examples of trading strategies generated from entropic functions, and corresponding sufficient conditions for strong arbitrage.

1.1 Pathwise Itô Formulae

Throughout this chapter, we consider a time-horizon $[0,T]$ of finite length $T \in (0,\infty)$. For a given subset V of a Euclidean space, we shall denote by $C([0,T],V)$ the space of continuous V-valued functions defined on $[0,T]$; whereas $CBV([0,T],V)$ stands for the space of those functions in $C([0,T],V)$ whose components are of bounded variation.

In what follows, we let $X = (X_1,\dots,X_d)'$ be a function in $C([0,T],\mathbb{R}^d)$, representing a vector of quantities defined on $[0,T]$, whose values change over time. We require the components of X to admit continuous covariations (Definition 1.1.1) in the pathwise sense with respect to a given, refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$. The sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is such that each partition is of the form $\mathbb{T}_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{N(\mathbb{T}_n)}^{(n)} = T\}$ for $n \in \mathbb{N}$, as well as $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$, and the mesh size $\|\mathbb{T}_n\| := \max_{t_j\in\mathbb{T}_n} |t_{j+1} - t_j|$ decreases to zero as $n \to \infty$.

We fix such a sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions throughout this chapter. Here and below, t_j and t_{j+1} are consecutive points in the partition $\mathbb{T}_n$, i.e., $t_j < t_{j+1}$, $\mathbb{T}_n \cap (t_j, t_{j+1}) = \emptyset$. Also, when we write

$t_j \in \mathbb{T}_n$ and $t_j \leq t$ simultaneously, we set $t_{j+1} = t$ when j is the biggest index satisfying $t_j \leq t$. With this notation, the notion of pathwise quadratic covariation of X along $(\mathbb{T}_n)_{n \in \mathbb{N}}$ is defined as follows.

Definition 1.1.1 (Pathwise quadratic covariation)**.** A function $X = (X_1, X_2, \ldots, X_d)'$ in $C([0,T], \mathbb{R}^d)$ is said to have *pathwise quadratic covariation* along a given nested sequence of partitions $(\mathbb{T}_n)_{n \in \mathbb{N}}$ of $[0,T]$, if for every $1 \leq i, k \leq d$ the limit as $n \to \infty$ of the sequence

$$\sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \leq t}} \big(X_i(t_{j+1}) - X_i(t_j)\big)\big(X_k(t_{j+1}) - X_k(t_j)\big), \qquad n \in \mathbb{N} \tag{1.1.1}$$

exists in $\mathbb{R}$ for all $t \in [0,T]$, and the resulting mapping, denoted by $t \mapsto [X_i, X_k](t)$, is real-valued and continuous. We call $[X_i, X_k]$ the *pathwise quadratic covariation* of X_i and X_k, and denote the *pathwise quadratic variation* of X_i by $[X_i] := [X_i, X_i]$.

We stress that the existence of pathwise covariations and quadratic variations for the components of X depends heavily on the choice of the nested, or "refining", sequence $(\mathbb{T}_n)_{n \in \mathbb{N}}$ of partitions. Example 5.3.2 in [2], and the arguments following it, illustrate this fact. Note also that the existence of pathwise covariations and quadratic variations is required for Itô's formula to hold in a pathwise sense. The following result is the original one-dimensional pathwise Itô formula, from [11].

Theorem 1.1.2 (Pathwise Itô formula)**.** *Fix a function $X \in C([0,T], \mathbb{R})$ admitting quadratic variation along the given nested sequence of partitions $\mathbb{T} = (\mathbb{T}_n)_{n \geq 1}$ of $[0,T]$. Then for every twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, the pathwise change of variable formula*

$$f\big(X(t)\big) - f\big(X(0)\big) = \int_0^t f'\big(X(s)\big)dX(s) + \frac{1}{2}\int_0^t f''\big(X(s)\big)d[X](s) \tag{1.1.2}$$

holds for all $t \in [0,T]$. The last integral of the right-hand side of (1.1.2) *is a Lebesgue-Stieltjes* *integral, and the first, so-called "Föllmer-Itô", integral is defined as the pointwise limit*

$$\int_0^t f'\big(X(s)\big)dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\leq t}} f'\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big). \tag{1.1.3}$$

We shall need a higher-dimensional pathwise Itô formula with an extra 'input' as additional argument. For this purpose, we consider $A = (A_1, \ldots, A_m)' \in CBV([0,T], \mathbb{R}^m)$ and $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, as well as the quantity $f(X_1(t), \ldots, X_d(t), A_1(t), \ldots, A_m(t))$, that depends on time $t \in [0,T]$. We say that such a function f is in $C^{j,k}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$, if it is j-times continuously differentiable with respect to the first d components and k-times continuously differentiable to the last m components. We also denote by $\partial_i f, \partial^2_{i,k} f$ the first- and second-order partial derivatives for the first d components of f $(1 \leq i, k \leq d)$, and by $D_\ell f$ the $(d+\ell)^{\text{th}}$ partial derivative for the last m components of f $(1 \leq \ell \leq m)$.

We present now the following version of the pathwise Itô formula involving both components X and A. The idea of proof is similar to that of Theorem 1.1.2.

Theorem 1.1.3 (Multidimensional pathwise Itô formula)**.** *Fix a function $X \in C([0,T], \mathbb{R}^d)$ having pathwise quadratic covariations along a given sequence of partitions $\mathbb{T} = (\mathbb{T}_n)_{n\geq 1}$ of $[0,T]$, and a function $A \in CBV([0,T], \mathbb{R}^m)$. Then for every function f of class $C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$, the pathwise change of variable formula*

$$\begin{aligned} f\big(X(t), A(t)\big) - f\big(X(0), A(0)\big) &= \int_0^t \sum_{i=1}^d \partial_i f\big(X(s), A(s)\big) dX_i(s) \\ &+ \sum_{\ell=1}^m \int_0^t D_\ell f\big(X(s), A(s)\big) dA_\ell(s) + \frac{1}{2} \sum_{i,k=1}^d \int_0^t \partial^2_{i,k} f\big(X(s), A(s)\big) d[X_i, X_k](s) \end{aligned} \tag{1.1.4}$$

holds for all $t \in [0,T]$. Here, the last two integrals in (1.1.4) *are of the Lebesgue-Stieltjes type,* *and the first Föllmer-Itô integral is defined as the pointwise limit*

$$\int_0^t \sum_{i=1}^d \partial_i f\big(X(s), A(s)\big)\,dX_i(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \sum_{i=1}^d \partial_i f\big(X(t_j), A(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big). \tag{1.1.5}$$

Proof. Using the telescoping sum representation, we obtain

$$\begin{aligned} f\big(X(t),A(t)\big) - f\big(X(0), A(0)\big) &= \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \Big\{f\big(X(t_{j+1}), A(t_{j+1})\big) - f\big(X(t_j), A(t_j)\big)\Big\} \\ &= \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \Big\{f\big(X(t_{j+1}), A(t_{j+1})\big) - f\big(X(t_{j+1}), A(t_j)\big)\Big\} \qquad (1.1.6) \\ &\quad + \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \Big\{f\big(X(t_{j+1}), A(t_j)\big) - f\big(X(t_j), A(t_j)\big)\Big\}. \qquad (1.1.7) \end{aligned}$$

The Taylor expansion, applied to the components of the function A in the sum (1.1.6), gives

$$\begin{aligned} &\sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \Big\{f\big(X(t_{j+1}), A(t_{j+1})\big) - f\big(X(t_{j+1}), A(t_j)\big)\Big\} \qquad (1.1.8) \\ &= \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \sum_{\ell=1}^m D_\ell f\big(X(t_{j+1}), A(t_j)\big)\big(A_\ell(t_{j+1}) - A_\ell(t_j)\big) + \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \sum_{\ell=1}^m r\big(A_\ell(t_{j+1}) - A_\ell(t_j)\big), \end{aligned}$$

where the summand in the last remainder term is bounded by

$$r\big(A_\ell(t_{j+1}) - A_\ell(t_j)\big) \le \phi\Big(\max_{t_j} \big|A_\ell(t_{j+1}) - A_\ell(t_j)\big|\Big)\Big|A_\ell(t_{j+1}) - A_\ell(t_j)\Big|$$

for some function ϕ with the property $\lim_{x\to 0}\phi(x) = 0$. Since A is continuous and of bounded variation, the last double sum of the right-hand side of (1.1.8) tends to zero as $n \to \infty$, and the sum (1.1.6) converges to the Lebesgue-Stieltjes integral $\sum_{\ell=1}^m \int_0^t D_\ell f\big(X(s), A(s)\big)dA_\ell(s)$ as $n \to \infty$. On the other hand, again by Taylor expansion applied to the components of the function X in the sum

(1.1.7), we obtain

$$\sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f\big(X(t_{j+1}), A(t_j)\big) - f\big(X(t_j), A(t_j)\big) \Big\}$$

$$= \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i=1}^{d} \partial_i f\big(X(t_j), A(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big) \tag{1.1.9}$$

$$+ \frac{1}{2} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i,k=1}^{d} \partial^2_{i,k} f\big(X(t_j), A(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big)\big(X_k(t_{j+1}) - X_k(t_j)\big) \tag{1.1.10}$$

$$+ \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i,k=1}^{d} r\big(X_{i,k}(t_{j+1}) - X_{i,k}(t_j)\big), \tag{1.1.11}$$

where the summand in the last remainder term (1.1.11) is bounded by

$$\psi\Big(\max_{t_j,i,k} \big| X_{i,k}(t_{j+1}) - X_{i,k}(t_j) \big| \Big) \big(X_i(t_{j+1}) - X_i(t_j)\big)\big(X_k(t_{j+1}) - X_k(t_j)\big),$$

for some function ψ with the property $\lim_{x \to 0} \psi(x) = 0$. Since X is continuous and has pathwise quadratic covariation in the sense of (1.1.1), the double sum (1.1.11) approaches zero as $n \to \infty$. The sum (1.1.10) converges to the Lebesgue-Stieltjes integral

$$\frac{1}{2} \sum_{i,k=1}^{d} \int_0^t \partial^2_{i,k} f\big(X(s), A(s)\big) d\langle X_i, X_k \rangle(s),$$

again by the existence of the pathwise quadratic covariation of X. As all the other terms converge, the remaining sum (1.1.9) also converges to some limit, which we call 'Föllmer-Itô integral' as in (1.1.5). □

1.2 Pathwise Local Time and Tanaka Formulae

1.2.1 *Definition of Local Time and Tanaka Formulae*

We consider in this subsection a one-dimensional function X in $C([0,T],\mathbb{R})$ and present the notion of pathwise quadratic variation of X along the partition $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of $[0,T]$, introduced by Föllmer.

Definition 1.2.1 (Pathwise quadratic variation)**.** A function $X \in C([0,T],\mathbb{R})$ is said to have finite *pathwise quadratic variation along a given sequence of partitions* $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of $[0,T]$ if, as $n \to \infty$, the mesh size

$$\|\mathbb{T}_n\| := \max_{t_j\in\mathbb{T}_n} |t_{j+1} - t_j| \tag{1.2.1}$$

tends to zero, and the sequence of measures $\mu^n := \sum_{t_j\in\mathbb{T}_n} |X(t_{j+1}) - X(t_j)|^2 \cdot \delta_{t_j}$, $n \in \mathbb{N}$ converges vaguely to a locally finite measure μ without atoms; here δ_t denotes the Dirac measure at $t \in [0,T]$. We write $Q(\mathbb{T})$ for the collection of all continuous functions having quadratic variation along $\mathbb{T}$; and denote by $[X](t) := \mu([0,t])$ the quadratic variation of X on $[0,T]$, for $t \in [0,T]$.

For a sequence of measures $(\mu^n)_{n\in\mathbb{N}}$ on $[0,T]$, vague convergence is equivalent to the pointwise convergence of their cumulative distribution functions at all continuity points of the limiting distribution function; and if this is continuous, the convergence is uniform. Thus, we are led to the following result (Lemma 1.3 of [3]). This guarantees, in particular, that the quadratic variation $[X]$ of a real-valued function X as in Definition 1.2.1 coincides with that of X in Definition 1.1.1.

Lemma 1.2.2. $X \in C([0,T],\mathbb{R})$ *belongs to the collection* $Q(\mathbb{T})$ *of Definition 1.2.1 if, and only if, there exists a continuous function* $[X]$ *such that*

$$\sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \left|X(t_{j+1}) - X(t_j)\right|^2 \xrightarrow{n\to\infty} [X](t) \tag{1.2.2}$$

for every $t \in [0,T]$*. If this property holds, the convergence in* (1.2.2) *is uniform.*

Remark 1.2.3. The assumption in Definition 1.2.1 that the mesh size in (1.2.1) goes to zero as $n \to \infty$, imposed on the sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions, is actually stronger than the assumption on the sequence of partitions usually imposed in other works involving pathwise local time. For example, in [3], [6], [22], the authors define the 'oscillation' of the function X along the partition $\mathbb{T}_n$ as

$$osc(X, \mathbb{T}_n) := \max_{t_j \in \mathbb{T}_n} \max_{r,s \in [t_j, t_{j+1}]} \big|X(s) - X(r)\big|,$$

and require $osc(X, \mathbb{T}_n) \to 0$ as $n \to \infty$, instead of the mesh size going to zero. This is because it is enough to work with Lebesgue partitions generated by X when defining the pathwise local time and deriving the pathwise Tanaka formula. Since the function X is uniformly continuous on the compact interval $[0, T]$, the decrease to zero of the mesh size does imply that the oscillation of X also shrinks to zero.

One reason for imposing here the stronger condition of Definition 1.2.1 on $(\mathbb{T}_n)_{n\in\mathbb{N}}$, is to follow our original definition of pathwise quadratic covariation/variation (cf. Definition 1.1.1). Another reason is that we are going to involve an additional (vector of) continuous functions A when generating trading strategies, and the oscillation of this A also has to shrink to zero along the sequence of partitions $(\mathbb{T}_n)_{n\in\mathbb{N}}$. In other words, by using the 'mesh' assumption instead of the 'oscillation', we can get rid of such 'dependence' of the sequence of partitions $(\mathbb{T}_n)_{n\in\mathbb{N}}$ on both X and A. □

The very first definition of pathwise local time was given in the unpublished diploma thesis [28] by Würmli. This original local time is called "$\mathbb{L}^2$-local time" of a path X along a sequence of partitions $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$. Using this notion of local time, Würmli established the expression (1.2.7) below for f in $H^2(\mathbb{R}, \mathbb{R})$, the Sobolev space of functions in $\mathbb{L}^2(\mathbb{R}, \mathbb{R})$ which are twice weakly differentiable. Since then, many versions of pathwise Tanaka formulae (generalized Itô formulae) and different definitions of local times have been introduced and studied; these vary according to the regularity of the path X, the function f, and the notion of "convergence for local time". Weaker convergence in defining a local time requires more regularity on the part of the function f. Some of these versions are stated in Section 2 of [22], and in [6], for continuous paths with quadratic variation. Similar results for rougher paths (with finite p-th variation, $p > 2$) can be found in Section 3 of [3], and in [17]. We present here the following version of local time and Tanaka's formula, which we consider most appropriate in our setting.

We have the following definition of continuous local time, with the notation

$$(\!|a, b]\!] = \begin{cases} (a, b], & a \leq b, \\ (b, a], & b \leq a, \end{cases} \tag{1.2.3}$$

used in [3], [6], [17], [22]. We adopt this notation throughout.

Definition 1.2.4 (Continuous local time)**.** We say that a function $X \in C([0, T], \mathbb{R})$ has *a continuous local time along the given nested sequence of partitions* $\mathbb{T} = (\mathbb{T}_n)_{n \in \mathbb{N}}$, if $\lim_{n \to \infty} \|\mathbb{T}_n\| = 0$, the 'discrete local times'

$$\mathbb{R} \ni x \mapsto L_t^{X, \mathbb{T}_n}(x) := \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \leq t}} \mathbb{1}_{(\!|X_{t_j}, X_{t_{j+1}}]\!]}(x) \left| X(t_{j+1}) - x \right| \in [0, \infty), \qquad n \in \mathbb{N} \tag{1.2.4}$$

converge uniformly to a continuous limit $x \mapsto L_t^{X, \mathbb{T}}(x)$ as $n \to \infty$ for every fixed $t \in [0, T]$, and the resulting mapping $(t, x) \mapsto L_t^{X, \mathbb{T}}(x)$ is jointly continuous. We call this limit *continuous local time of* X *along* $\mathbb{T}$, and write $\mathcal{L}(\mathbb{T})$ for the collection of functions X in $C([0, T], \mathbb{R})$ which admit a continuous local time along the given nested sequence of partitions $\mathbb{T} = (\mathbb{T}_n)_{n \in \mathbb{N}}$.

The existence of continuous local time for 'typical price paths' is shown in Theorem 3.5 of [22]. In order to simplify notation, we shall write $L_t^X(x)$, or simply $L_t(x)$, whenever the context is unambiguous. With this definition, we now derive the pathwise Tanaka formula.

For an absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ and any two real numbers a and b, we have $f(b) - f(a) = \int_a^b f'(x)dx$, where f' is a Radon-Nikodým derivative of f. We assume that f' is right-continuous with left-limits (RCLL) and of finite variation on compact intervals (since every function of finite variation has only countably many discontinuities, its RCLL version is also a weak derivative of f, and we work with this version). By applying the integration by parts formula with the notation (1.2.3), we obtain

$$\begin{aligned}
f(b)-f(a) &= \int_a^b f'(x)dx \\
&= \begin{cases} \int_a^b f'(x)(b-x)^0 dx = -f'(x)(b-x)\Big|_{x=a}^{b} + \int_{(a,b]}(b-x)df'(x), & \text{if } a \le b \\ -\int_b^a f'(x)(b-x)^0 dx = f'(x)(b-x)\Big|_{x=b}^{a} - \int_{(b,a]}(b-x)df'(x), & \text{if } b < a \end{cases} \\
&= \begin{cases} f'(a)(b-a) + \int_{(a,b]}(b-x)df'(x), & \text{if } a \le b \\ f'(a)(b-a) - \int_{(b,a]}(b-x)df'(x), & \text{if } b < a \end{cases} \\
&= f'(a)(b-a) + \int_{\mathbb{R}} \mathbb{1}_{(\!|a,b]\!]}(x)|b-x|df'(x). \qquad (1.2.5)
\end{aligned}$$

Thus, using the telescoping summation $f\big(X(t)\big) - f\big(X(0)\big) = \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big(f\big(X(t_{j+1})\big) - f\big(X(t_j)\big)\Big)$ for the given sequence of partitions $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of $[0,T]$, the above equality becomes

$$\begin{aligned}
f\big(X(t)\big) - f\big(X(0)\big) - \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f'\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big) &= \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \int_{\mathbb{R}} \mathbb{1}_{(\!|X_{t_j}, X_{t_{j+1}}]\!]}(x)\Big|X(t_{j+1}) - x\Big|df'(x) \\
&= \int_{\mathbb{R}} L_t^{X,\mathbb{T}_n}(x)df'(x), \qquad (1.2.6)
\end{aligned}$$

thanks to the definition (1.2.4). Since $L_t^{X,\mathbb{T}_n}(\cdot)$ converges uniformly to $L_t^{X,\mathbb{T}}(\cdot)$ for $X \in \mathcal{L}(\mathbb{T})$, we now arrive at the following result, which is Theorem 2.6 of [22].

Theorem 1.2.5 (Pathwise Tanaka formula)**.** *Let $X \in \mathcal{L}(\mathbb{T})$ and $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, with right-continuous Radon-Nikodým derivative f' of finite variation. Then, the one-dimensional Föllmer-Itô integral of* (1.1.3) *exists, and we have the generalized change of variable formula*

$$f\big(X(t)\big) - f\big(X(0)\big) = \int_0^t f'\big(X(s)\big)dX(s) + \int_{\mathbb{R}} L_t(x)df'(x), \qquad 0 \le t \le T. \qquad (1.2.7)$$

Remark 1.2.6 (Occupation density formula). For every twice continuously differentiable function f, comparing the last terms in (1.1.2) and (1.2.7), we obtain for $X \in Q(\mathbb{T}) \cap \mathcal{L}(\mathbb{T})$

$$\int_0^t f''\big(X(s)\big)d[X](s) = 2\int_{\mathbb{R}} f''(x)L_t(x)dx.$$

Thus, we also have

$$\int_0^t g\big(X(s)\big)d[X](s) = 2\int_{\mathbb{R}} g(x)L_t(x)dx, \tag{1.2.8}$$

for any continuous function g. The discrete local times $L_t^{X,\mathbb{T}_n}(\cdot)$ defined in (1.2.4), as well as their pointwise limit $L_t(\cdot)$, have a compact support $[m_t, M_t]$ for every fixed $t \in [0, T]$ where

$$m_t := \min_{0\le s\le t} X(s), \qquad M_t := \max_{0\le s\le t} X(s), \tag{1.2.9}$$

and the continuity of $L_t(\cdot)$ gives the boundedness of $L_t(\cdot)$. Furthermore, for any $a < b$, the indicator function $\mathbb{1}_{(a,b]}(\cdot)$ can be written as the pointwise limit of a sequence of bounded continuous functions. Thus, by the bounded convergence theorem, (1.2.8) holds for every function of the form $g(\cdot) = \mathbb{1}_B(\cdot)$ with $B = (a, b]$. On the other hand, the collection of all Borel sets B for which (1.2.8) holds with $g = \mathbb{1}_B$ forms a Dynkin system (closed under set subtraction and increasing countable union) and so, by the Dynkin System Theorem (Theorem 2.1.3 in [16]), the identity (1.2.8) holds for every function of the form $g = \mathbb{1}_A$, with Borel set $A \in \mathscr{B}(\mathbb{R})$:

$$\int_0^t \mathbb{1}_A\big(X(s)\big)d[X](s) = 2\int_A L_t(x)dx. \tag{1.2.10}$$

This so-called *occupation density formula* generalizes a very familiar property of semimartingale local time.

1.2.2 *Properties of Local Time*

In this subsection, we present new results regarding the pathwise local time of Definition 1.2.4. First, a pathwise version of the classical Tanaka-Meyer formula is a direct consequence of Theorem 1.2.5 with the choice of function $f(x) = (x - a)^+$.

Corollary 1.2.7 (Pathwise Tanaka-Meyer formula)**.** *For every function X in $\mathcal{L}(\mathbb{T})$, the pathwise Tanaka-Meyer formula*

$$L_t(a) = \big(X(t) - a\big)^+ - \big(X(0) - a\big)^+ - \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)dX(s) \tag{1.2.11}$$

holds for all $(t, a) \in [0, T] \times \mathbb{R}$. Here, the last integral represents the pointwise limit

$$\int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{(a,\infty)}\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big). \tag{1.2.12}$$

Remark 1.2.8 (Additional Tanaka-Meyer formulae). We obtain formulae analogous to (1.2.11) by applying Theorem 1.2.5 with the choices $g(x) = (x - a)^-$ and $h(x) = |x - a|$, respectively:

$$L_t(a) = \big(X(t) - a\big)^- - \big((X(0) - a\big)^- + \int_0^t \mathbb{1}_{(-\infty,a)}\big(X(s)\big)dX(s), \tag{1.2.13}$$

and

$$2L_t(a) = |X(t) - a| - |X(0) - a| - \int_0^t \operatorname{sign}\big(X(s) - a\big)dX(s). \tag{1.2.14}$$

The integral terms are defined as

$$\int_0^t \mathbb{1}_{(-\infty,a)}\big(X(s)\big)dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{(-\infty,a)}\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big)$$

and

$$\int_0^t \operatorname{sign}\big(X(s)-a\big)dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \operatorname{sign}\big(X(t_j)-a\big)\big(X(t_{j+1})-X(t_j)\big),$$

respectively, with the notation $\operatorname{sign}(x) := \mathbb{1}_{[0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$.

Next, we have the following result which is reminiscent of very familiar properties of semimartingale local time. Note that the continuous local time $t \mapsto L_t(a)$ of Definition 1.2.4 is, for fixed a, nondecreasing in t, thus of finite first variation in this variable, and the integrals in this lemma should be understood as Lebesgue-Stieltjes integrals with respect to this temporal variable.

Lemma 1.2.9. *For every function X in $\mathcal{L}(\mathbb{T})$ and every $a \in \mathbb{R}$, we have the following identities*

$$\int_0^t \mathbb{1}_{\{X_s=a\}} dL_s(a) = L_t(a), \qquad \int_0^t \mathbb{1}_{\{X_s<a\}} dL_s(a) = \int_0^t \mathbb{1}_{\{X_s>a\}} dL_s(a) = 0. \tag{1.2.15}$$

Proof. By the Monotone Convergence Theorem and (1.2.4), we have

$$\begin{aligned}
\int_0^t \mathbb{1}_{\{X_s<a\}} dL_s(a) &= \lim_{m\to\infty} \int_0^t \mathbb{1}_{(-\infty,\ a-\frac{1}{2^m}]}\big(X(s)\big) dL_s(a) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{(-\infty,\ a-\frac{1}{2^m}]}\big(X(t_j)\big)\big(L_{t_{j+1}}(a) - L_{t_j}(a)\big) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{(-\infty,\ a-\frac{1}{2^m}]}\big(X(t_j)\big) \mathbb{1}_{(\!\![X_{t_j}, X_{t_{j+1}}]\!\!]}(a)\big|X(t_{j+1}) - a\big| \\
&= 0;
\end{aligned}$$

that is because, for fixed m, we can take n sufficiently large so that $osc(X, \mathbb{T}_n) < 1/2^m$, which guarantees

$$\mathbb{1}_{(-\infty,\ a-\frac{1}{2^m}]}\big(X(t_j)\big) \mathbb{1}_{(\!\![X_{t_j}, X_{t_{j+1}}]\!\!]}(a) \equiv 0.$$

In a similar manner, we can show that $\int_0^t \mathbb{1}_{\{X_s>a\}} dL_s(a) = 0$ holds for any given $a \in \mathbb{R}$ and the result follows. □

Using the Tanaka-Meyer formulae (1.2.11), (1.2.13), and (1.2.14), we can establish several identities involving pathwise local times of continuous functions. Each of these identities corresponds to an identity involving local times for continuous semimartingales. First, we have the following identity.

Lemma 1.2.10. *For every nonnegative function $X \in \mathcal{L}(\mathbb{T})$, the identity*

$$L_t^X(0) = \int_0^t \mathbb{1}_{\{X_s=0\}} dX(s) \tag{1.2.16}$$

holds, where the last integral stands for the limit

$$\int_0^t \mathbb{1}_{\{X_s=0\}} dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big) = \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\ X(t_{j+1}).$$

Proof. Because of the nonnegativity of X, the pathwise Tanaka-Meyer formula of (1.2.11) for $a = 0$ becomes

$$\begin{aligned}
L_t^X(0) &= X(t) - X(0) - \int_0^t \mathbb{1}_{(0,\infty)}\big(X(s)\big) dX(s) \\
&= \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \big(X(t_{j+1}) - X(t_j)\big) - \int_0^t \mathbb{1}_{(0,\infty)}\big(X(s)\big) dX(s) \\
&= \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \Big(1 - \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\Big)\big(X(t_{j+1}) - X(t_j)\big) = \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big),
\end{aligned}$$

which is the desired result. The second equality just uses the telescoping sum representation, and the third equality follows from (1.2.12) for $a = 0$. □

Remark 1.2.11. When Z is a nonnegative semimartingale on a probability space, the equation (1.2.16) is just the well-known identity (see, for example, [21]) in semimartingale theory

$$L_t^Z(0) = \int_0^t \mathbb{1}_{\{Z_s=0\}} dZ(s).$$

Next, we offer another representation of continuous local times.

Lemma 1.2.12. *For every function $X \in \mathcal{L}(\mathbb{T})$, the equation*

$$L_t^X(0) = L_t^{X^+}(0) = \int_0^t \mathbb{1}_{\{X_s=0\}} dX^+(s) \tag{1.2.17}$$

holds, where the integral stands for the following limit

$$\int_0^t \mathbb{1}_{\{X_s=0\}} dX^+(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X^+(t_{j+1}) - X^+(t_j)\big) = \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\ X^+(t_{j+1}),$$

in a manner similar to the integral in (1.2.16).

Proof. For any $t_j \in \mathbb{T}_n$, using the equation (1.2.6) with the function $f(x) = x^+$ evaluated at $X_{t_{j+1}}$ and X_{t_j}, we obtain

$$X^+(t_{j+1}) = X^+(t_j) + \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\big(X^+(t_{j+1}) - X^+(t_j)\big) + L_{t_{j+1}}^{X,\mathbb{T}_n}(0) - L_{t_j}^{X,\mathbb{T}_n}(0)$$

as in the proof of Corollary 1.2.7. Now, if we multiply $\mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)$, and $\mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)$, respectively, on both sides, all terms on the right side except the local times terms vanish:

$$\mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\ X^+(t_{j+1}) = \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\Big(L_{t_{j+1}}^{X,\mathbb{T}_n}(0) - L_{t_j}^{X,\mathbb{T}_n}(0)\Big), \tag{1.2.18}$$

and

$$\mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)\ X^+(t_{j+1}) = \mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)\Big(L_{t_{j+1}}^{X,\mathbb{T}_n}(0) - L_{t_j}^{X,\mathbb{T}_n}(0)\Big). \tag{1.2.19}$$

Applying Lemma 1.2.10 to the nonnegative path X^+, and using (1.2.18), we obtain

$$L_t^{X^+}(0) = \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X^+(t_j)\big)\ X^+(t_{j+1}) = \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\ X^+(t_{j+1}) \tag{1.2.20}$$

$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\Big(L^{X,\mathbb{T}_n}_{t_{j+1}}(0) - L^{X,\mathbb{T}_n}_{t_j}(0)\Big)$$

$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\mathbb{1}_{\{0\}}\big(X(t_j)\big)\Big(L^{X,\mathbb{T}_n}_{t_{j+1}}(0) - L^{X,\mathbb{T}_n}_{t_j}(0)\Big)$$

$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\Big(L^{X,\mathbb{T}_n}_{t_{j+1}}(0) - L^{X,\mathbb{T}_n}_{t_j}(0)\Big) \tag{1.2.21}$$

where the second last equality follows from (1.2.15). The limit of the sum in (1.2.21) represents the Lebesgue-Stieltjes integral which is equal to $L^X_t(0)$ in (1.2.15), thus, we established $L^{X+}_t(0) = L^X_t(0)$. Similarly, starting from (1.2.19) and proceeding as before, we obtain

$$\begin{aligned}\mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)\ X^+(t_{j+1}) &= \mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)\Big(L^{X,\mathbb{T}_n}_{t_{j+1}}(0) - L^{X,\mathbb{T}_n}_{t_j}(0)\Big)\\ &= \mathbb{1}_{(-\infty,0)}\big(X(t_j)\big)\mathbb{1}_{\{0\}}\big(X(t_j)\big)\Big(L^{X,\mathbb{T}_n}_{t_{j+1}}(0) - L^{X,\mathbb{T}_n}_{t_j}(0)\Big) = 0.\end{aligned} \tag{1.2.22}$$

Thus, from (1.2.22), the right-hand side of (1.2.20) is equal to

$$\lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\ X^+(t_{j+1}) = \int_0^t \mathbb{1}_{\{X_s=0\}} dX^+(s),$$

and the claim (1.2.17) follows. □

We also have the following twin result of Lemma 1.2.12.

Lemma 1.2.13. *For every function $X \in \mathcal{L}(\mathbb{T})$, the equation*

$$L^X_t(0) = L^{X^-}_t(0) = \int_0^t \mathbb{1}_{\{X_s=0\}} dX^-(s) \tag{1.2.23}$$

holds, where the integral stands for the limit

$$\int_0^t \mathbb{1}_{\{X_s=0\}} dX^-(s) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n\\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\ X^-(t_{j+1}).$$

Proof. The continuity of $L_t^X(\cdot)$ and $L_t^{-X}(\cdot)$ gives

$$L_t^X(0) = \lim_{\epsilon\to 0} L_t^X(\epsilon) = \lim_{\epsilon\to 0} L_t^{(-X)}(-\epsilon) = L_t^{(-X)}(0),$$

where the second equality used the identity $L_t^X(\epsilon) = L_t^{(-X)}(-\epsilon)$ from Definition 1.2.4. Then, we apply Lemma 1.2.12 with the simple identity $(-X)^+ = X^-$ to obtain

$$L_t^{(-X)}(0) = L_t^{X^-}(0) = \int_0^t \mathbb{1}_{\{X_s=0\}} dX^-(s)$$

and thus the result. □

The following result provides an expression for the local time of the maximum of two given continuous functions; Lemma 1.2.12 plays an essential role in its proof.

Theorem 1.2.14. *For $X, Y \in \mathcal{L}(\mathbb{T})$, if $X \vee Y$ is also in $\mathcal{L}(\mathbb{T})$, we have*

$$L_t^{X\vee Y}(0) = \int_0^t \mathbb{1}_{\{Y_s<0\}} dL_s^X(0) + \int_0^t \mathbb{1}_{\{X_s<0\}} dL_s^Y(0) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} d(X^+ \vee Y^+)(s). \qquad (1.2.24)$$

Proof. Consider the function $Z = X \vee Y$, and use Lemma 1.2.12 to obtain the decomposition of its local time at the origin as follows:

$$\begin{aligned} L_t^Z(0) &= \int_0^t \mathbb{1}_{\{Z_s=0\}} dZ^+(s) \\ &= \int_0^t \mathbb{1}_{\{Y_s<X_s=0\}} dZ^+(s) + \int_0^t \mathbb{1}_{\{X_s<Y_s=0\}} dZ^+(s) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} dZ^+(s). \end{aligned} \qquad (1.2.25)$$

Since two paths Z^+ and X^+ coincide on the set $\{s \in [0,T] \; : \; Y_s < X_s\}$, the first term on the right-most side of (1.2.25) is

$$\int_0^t \mathbb{1}_{\{Y_s<0\}} \mathbb{1}_{\{X_s=0\}} dX^+(s) = \int_0^t \mathbb{1}_{\{Y_s<0\}} dL_s^X(0),$$

on the strength of (1.2.17) in Lemma 1.2.12. By the same token, the second term on the right-most side of (1.2.25) is equal to

$$\int_0^t \mathbb{1}_{\{X_s<0\}} dL_s^Y(0).$$

For the last term in (1.2.25), we use the fact $Z^+ = X^+ \vee Y^+$, and the result (1.2.24) follows. □

We have a very similar expansion for the local time of the mininum of two functions, instead of the maximum; the proof is completely analogous to that of Theorem 1.2.14.

Theorem 1.2.15. *For $X, Y \in \mathcal{L}(\mathbb{T})$, if $X \wedge Y$ is also in $\mathcal{L}(\mathbb{T})$, we have*

$$L_t^{X\wedge Y}(0) = \int_0^t \mathbb{1}_{\{Y_s>0\}} dL_s^X(0) + \int_0^t \mathbb{1}_{\{X_s>0\}} dL_s^Y(0) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} d(X^+ \wedge Y^+)(s). \quad (1.2.26)$$

Proof. Let $Q = X \wedge Y$ and use Lemma 1.2.12 twice, to obtain the decomposition

$$\begin{aligned} L_t^Q(0) &= \int_0^t \mathbb{1}_{\{Q_s=0\}} dQ^+(s) \\ &= \int_0^t \mathbb{1}_{\{Y_s>X_s=0\}} dQ^+(s) + \int_0^t \mathbb{1}_{\{X_s>Y_s=0\}} dQ^+(s) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} dQ^+(s) \\ &= \int_0^t \mathbb{1}_{\{Y_s>0\}} \mathbb{1}_{\{X_s=0\}} dX^+(s) + \int_0^t \mathbb{1}_{\{X_s>0\}} \mathbb{1}_{\{Y_s=0\}} dY^+(s) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} dQ^+(s) \\ &= \int_0^t \mathbb{1}_{\{Y_s>0\}} dL_s^X(0) + \int_0^t \mathbb{1}_{\{X_s>0\}} dL_s^Y(0) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} d(X^+ \wedge Y^+)(s), \end{aligned}$$

as before. Note that in the last term, we use $Q^+ = X^+ \wedge Y^+$. □

Combining Theorems 1.2.14 and 1.2.15, we have the following algebraic identity. This generalizes results of [29], [30], valid for continuous semimartingales; see also [20], [21].

Theorem 1.2.16. *For $X, Y \in \mathcal{L}(\mathbb{T})$ with $X \vee Y, X \wedge Y \in \mathcal{L}(\mathbb{T})$, we have the identity for any $t \in [0, T]$,*

$$L_t^{X\vee Y}(0) + L_t^{X\wedge Y}(0) = L_t^X(0) + L_t^Y(0). \quad (1.2.27)$$

Proof. The elementary identity $(a \vee b) + (a \wedge b) = a + b$ holds for arbitrary numbers a and b.

Therefore, the sum of the last terms of (1.2.24) and (1.2.26) can be expressed as

$$\begin{aligned}
&\int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} d(X^+ \vee Y^+)(s) + \int_0^t \mathbb{1}_{\{X_s=Y_s=0\}} d(X^+ \wedge Y^+)(s) \\
&= \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \mathbb{1}_{\{X_{t_j}=Y_{t_j}=0\}} \big(X^+(t_{j+1}) \vee Y^+(t_{j+1}) + X^+(t_{j+1}) \wedge Y^+(t_{j+1}) \big) \\
&= \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \mathbb{1}_{\{X_{t_j}=Y_{t_j}=0\}} \big(X^+(t_{j+1}) + Y^+(t_{j+1}) \big) \\
&= \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \mathbb{1}_{\{Y_{t_j}=0\}} \mathbb{1}_{\{X_{t_j}=0\}} X^+(t_{j+1}) + \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \mathbb{1}_{\{X_{t_j}=0\}} \mathbb{1}_{\{Y_{t_j}=0\}} Y^+(t_{j+1}) \\
&= \int_0^t \mathbb{1}_{\{Y_s=0\}} dL^X_s(0) + \int_0^t \mathbb{1}_{\{X_s=0\}} dL^Y_s(0),
\end{aligned}$$

where the last equality follows from Lemma 1.2.12. Then, the result follows by adding the two equations (1.2.24) and (1.2.26). □

1.2.3 *Dynamics of Ranked Functions*

For given d continuous functions $X_1, \cdots, X_d \in \mathcal{L}(\mathbb{T})$, we define

$$X_{(k)}(\cdot) := \max_{1 \le i_1 < \cdots < i_k \le d} \min\{X_{i_1}(\cdot), \cdots, X_{i_k}(\cdot)\}, \tag{1.2.28}$$

which represents the k-th rank function $X_{(k)}(\cdot)$ of $X_1, \cdots, X_d$, in descending order. More explicitly, for any $t \in [0, T]$, we have

$$\max_{1\le i \le d} X_i(t) = X_{(1)}(t) \ge X_{(2)}(t) \ge \cdots \ge X_{(d-1)}(t) \ge X_{(d)}(t) = \min_{1 \le i \le d} X_i(t), \tag{1.2.29}$$

so that these ranked functions represent the original functions arranged in descending order. In this subsection, we assume that ranked functions $X_{(k)}$, as well as their differences $X_{(k)} - X_{(\ell)}$ for $1 \le k, \ell \le d$ also belong to the space $\mathcal{L}(\mathbb{T})$. Then, we have the following extension of Theorem 1.2.16

for m continuous functions.

Theorem 1.2.17. *For every $t \in [0,T]$, we have the identity*

$$\sum_{k=1}^{d} L_t^{X_{(k)}}(0) = \sum_{i=1}^{d} L_t^{X_i}(0).$$

Proof. Using induction on Theorem 1.2.16, we can prove this identity in exactly the same manner as in the proof of Theorem 2.2 in [1]. □

Our next aim is to derive expressions of the descending ranked functions $X_{(k)}$ for $k = 1, \cdots, d$ in terms of the original functions $X_1, \cdots, X_d$ and appropriate local times, as in Theorem 2.3 of [1]. In this result of [1], expressions such as "$dX_i(t)$" appear, and represent Itô integration with respect to a semimartingale integrator. However, in our setting, such expression "$dX_i(t)$" makes sense only when a certain type of integrand, namely $f'(X_i(t))$, is given, as the Föllmer-Itô integral

$$\int_0^t f'\big(X_i(s)\big)dX_i(s)$$

is defined as the pointwise limit in (1.1.3). This integral has different representations depending on the regularity of the test-function $f : \mathbb{R} \to \mathbb{R}$ as in Theorems 1.1.2, and 1.2.5.

In particular, with $X \in \mathcal{L}(\mathbb{T})$, the identity (1.2.7) in Theorem 1.2.5 shows that this integral can be evaluated as

$$f\big(X(t)\big) - f\big(X(0)\big) - \int_{\mathbb{R}} L_t(x)df'(x)$$

for every function f satisfying the conditions of Theorem 1.2.5. Thus, in the following argument, we fix an absolutely continuous function f, and assume that the Radon-Nikodým derivative f' of this latter function is of bounded variation.

For any $t \in [0,T]$ and a sequence $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions, we fix $t_j \in \mathbb{T}_n$ such that $t_j \leq t$. Then, we define

$$S_t(k) := \{i : X_i(t) = X_{(k)}(t)\} \qquad \text{and} \qquad N_t(k) := \big|S_t(k)\big|. \tag{1.2.30}$$

Here, $N_t(k)$ is the number of functions which are at rank k at time t.

We start with an expression

$$f'\big(X_{(k)}(t_j)\big)\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big), \tag{1.2.31}$$

whose sum over all t_j's satisfying $t_j \in \mathbb{T}_n$ and $t_j \leq t$ will converge as $n \to \infty$ to the integral

$$\int_0^t f'\big(X_{(k)}(s)\big)dX_{(k)}(s). \tag{1.2.32}$$

Using the definition (1.2.30) and the fact that

$$\sum_{i=1}^{d} N_{t_j}(k)^{-1}\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} = 1, \tag{1.2.33}$$

we have

$$\begin{aligned} X_{(k)}(t_{j+1}) - X_{(k)}(t_j) &= \sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)}\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big) \\ = \sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)}\big(X_i(t_{j+1}) - X_i(t_j)\big) &+ \sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)}\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big). \end{aligned} \tag{1.2.34}$$

Then, by plugging (1.2.34) into (1.2.31), the expression (1.2.31) becomes

$$\sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)} f'\big(X_i(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big) \tag{1.2.35}$$

$$+ \sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)} f'\big(X_{(k)}(t_j)\big)\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big). \tag{1.2.36}$$

Since

$$\sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \leq t}} f'\big(X_i(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big) \longrightarrow \int_0^t f'\big(X_i(s)\big)dX_i(s) \qquad \text{as} \qquad n \to \infty,$$

for each i, the sum

$$\sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i=1}^{d} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_{t_j}(k)} f'\big(X_i(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big)$$

of (1.2.35) over t_j's also converges by virtue of (1.2.33), and we denote this limit as

$$\sum_{i=1}^{d} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(s)=X_i(s)\}}}{N_s(k)} f'\big(X_i(s)\big) dX_i(s). \tag{1.2.37}$$

For the expression (1.2.36), we decompose it into

$$\sum_{i=1}^{d} \frac{f'\big(X_{(k)}(t_j)\big)}{N_{t_j}(k)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{+} \tag{1.2.38}$$

$$- \sum_{i=1}^{d} \frac{f'\big(X_{(k)}(t_j)\big)}{N_{t_j}(k)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{-}. \tag{1.2.39}$$

Note that $f'\big(X_{(k)}(t_j)\big)$ is bounded for every $t_j \in \mathbb{T}_n$, because f' is of bounded variation and the continuous functions $X_{(k)}$'s have compact support over $[0, T]$. The sum of the expression in (1.2.38) over the t_j's, namely,

$$\sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i=1}^{d} \frac{f'\big(X_{(k)}(t_j)\big)}{N_{t_j}(k)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{+}$$

converges to

$$\sum_{i=1}^{d} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(k)}-X_i)^+}(0) \qquad \text{as} \qquad n \to \infty \tag{1.2.40}$$

as a result of Lemma 1.2.12. Similarly, the sum of (1.2.39) over the t_j's converges to

$$\sum_{i=1}^{d} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(k)}-X_i)^-}(0) \qquad \text{as} \qquad n \to \infty \tag{1.2.41}$$

by virtue of Lemma 1.2.13. By the identities

$$(X_{(k)} - X_{(h)})^+ = \begin{cases} X_{(k)} - X_{(h)}, & \text{if } h > k \\ 0, & \text{if } h \le k, \end{cases} \qquad (X_{(k)} - X_{(h)})^- = \begin{cases} X_{(h)} - X_{(k)}, & \text{if } h < k \\ 0, & \text{if } h \ge k, \end{cases} \tag{1.2.42}$$

with Theorem 1.2.17, the integrals (1.2.40) and (1.2.41) become

$$\sum_{h=k+1}^{d} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(k)}-X_{(h)})}(0) \quad \text{and} \quad \sum_{h=1}^{k-1} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(h)}-X_{(k)})}(0) \tag{1.2.43}$$

respectively. Here, the local time $L_s^{(X_{(k)}-X_{(h)})}(0)$ in (1.2.43) is called a “collision local time” of order $h-k+1$ among ranked functions $X_{(1)}, \cdots, X_{(d)}$. Thus, the sum of the expression (1.2.36) over t_j's converges as $n \to \infty$ to

$$\sum_{h=k+1}^{d} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(k)}-X_{(h)})}(0) - \sum_{h=1}^{k-1} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(h)}-X_{(k)})}(0). \tag{1.2.44}$$

To sum up, (1.2.31) is represented as the sum of (1.2.35) and (1.2.36). The sums of (1.2.31), (1.2.35), and (1.2.36) over t_j's converge to the integrals (1.2.32), (1.2.37), and (1.2.44), respectively. Therefore, we arrive at the following “integration along ranks" formula.

Theorem 1.2.18. *Let f be an absolutely continuous function which admits a Radon-Nikodým derivative f' of bounded variation. Then, the Föllmer-Itô integral of k-th rank function $X_{(k)}(\cdot)$ among given d functions $X_1, \cdots, X_d$, is expressed as*

$$\begin{aligned} \int_0^t f'\big(X_{(k)}(s)\big) dX_{(k)}(s) &= \sum_{i=1}^{d} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(s)=X_i(s)\}}}{N_s(k)} f'\big(X_i(s)\big) dX_i(s) \\ &\quad + \sum_{h=k+1}^{d} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(k)}-X_{(h)})}(0) - \sum_{h=1}^{k-1} \int_0^t \frac{f'\big(X_{(k)}(s)\big)}{N_s(k)} dL_s^{(X_{(h)}-X_{(k)})}(0) \end{aligned} \tag{1.2.45}$$

for $k = 1, \cdots, d$.

1.3 Construction of Trading Strategies

We place ourselves from now onward in the context of a frictionless equity market with a fixed number $d \geq 2$ of companies. We also consider a vector of functions $S = (S_1, \ldots, S_d)' \in C([0,T], [0,\infty)^d)$, where $S_i(t)$ represents the capitalization of the i-th company at time $t \in [0,T]$. Here we take $S_i(0) > 0$ and allow $S_i(t)$ to vanish at some time $t > 0$, for all $i = 1, \ldots, d$; but we assume also that the total capitalization $\Sigma(t) := S_1(t) + \cdots + S_d(t)$ does not vanish at any time $t \in [0,T]$. We also assume that each company has a single share of stock outstanding, thus $S_i(t)$ also represents the stock price of the i-th company at time t.

With these ingredients, we define another vector $\mu = (\mu_1, \ldots, \mu_d)'$ of continuous functions that consists of the companies' relative market weights

$$\mu_i(t) := \frac{S_i(t)}{\Sigma(t)} = \frac{S_i(t)}{S_1(t) + \cdots + S_d(t)}, \qquad t \in [0,T], \quad i = 1, \ldots, d. \tag{1.3.1}$$

We assume that the components of μ admit finite quadratic covariations $[\mu_i, \mu_j]$, $1 \leq i, j \leq d$ along a given, fixed, nested sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[0,T]$, in the manner of Definition 1.1.1.

We also consider a vector of functions $A = (A_1, \ldots, A_m)' \in CBV([0,T], \mathbb{R}^m)$, along with the vector μ of market weights. For the purposes of this section, the components of A will model the evolution of an observable, but non-tradable, quantity related to the market weights. In what follows, we consider functions of the form $G(\mu(\cdot), A(\cdot))$ which depend on the vector of market weights μ and on A. Examples of such functions A appear in Section 1.4 below. With this notation, we have the following definition of trading strategy with respect to the pair (μ, A), in the manner of [15] and [14].

Definition 1.3.1 (Trading strategies)**.** For the market weights $\mu \in C([0,T], \mathbb{R}^d)$, suppose that $\vartheta = (\vartheta_1, \ldots, \vartheta_d)'$ is a d-dimensional function, for which the Föllmer-Itô integral

$$\int_0^{\cdot} \vartheta(t) d\mu(t) \equiv \int_0^{\cdot} \sum_{i=1}^{d} \vartheta_i(t) d\mu_i(t) := \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \leq t}} \sum_{i=1}^{d} \vartheta_i(t_j) \big(\mu_i(t_{j+1}) - \mu_i(t_j)\big)$$

with respect to μ exists in $\mathbb{R}$. We write $\vartheta \in \mathscr{L}(\mu)$ to express this. We shall say that $\vartheta \in \mathscr{L}(\mu)$ is a *trading strategy with respect to* μ, if it is 'self-financed' in the sense that its value $V^{\vartheta}(\cdot) := \sum_{i=1}^{d} \vartheta_i(\cdot)\mu_i(\cdot)$ satisfies

$$V^{\vartheta}(\cdot) - V^{\vartheta}(0) = \int_0^{\cdot} \sum_{i=1}^{d} \vartheta_i(t) d\mu_i(t). \tag{1.3.2}$$

Here $\vartheta_i(t)$ stands for the dollar amount invested in the i-th asset at time t. Since $\mu_i(t)$ is the relative market weight of this asset, $\vartheta_i(t)\mu_i(t)$ is the relative dollar amount invested in asset i at time t, and $V^{\vartheta}(t)$ the total value of investment relative to the market across all assets. "Self-financing" means that there are neither infusions nor withdrawals of capital: gains are re-invested, losses have to be absorbed.

The preceding pathwise Itô formula (1.1.4) suggests that integrands $\vartheta \in \mathscr{L}(\mu)$ of the special form $\vartheta(t) = \partial f(\mu(t), A(t))$, for some function $f \in C^{2,1}(\mathbb{R}^{(d+m)}, \mathbb{R})$, play a very important role for integrators $\mu \in C([0,T], \mathbb{R}^d)$ that admit finite quadratic covariations $[\mu_i, \mu_j]$, $1 \le i, j \le d$ along an appropriate nested sequence of partitions. This gives rise to the following definition.

Definition 1.3.2 (Regular function)**.** For the pair (μ, A) of the market weights $\mu \in C([0,T], \mathbb{R}^d)$ and $A \in CBV([0,T], \mathbb{R}^m)$, we say that $G \in C(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$ is *regular*, if

1. there exists a function $\nabla G = (\nabla_1 G, \ldots, \nabla_d G)' : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ such that the vector $\vartheta = (\vartheta_1, \ldots, \vartheta_d)'$ with components

$$\vartheta_i(t) := \nabla_i G(\mu(t), A(t)), \qquad i = 1, \ldots, d, \qquad 0 \le t \le T, \tag{1.3.3}$$

   is in $\mathscr{L}(\mu)$, and

2. the continuous function

$$\Gamma^G(\cdot) := G\big(\mu(0), A(0)\big) - G\big(\mu(\cdot), A(\cdot)\big) + \int_0^{\cdot} \nabla G\big(\mu(s), A(s)\big) d\mu(s), \tag{1.3.4}$$

   which we call occasionally the *Gamma function* of G, has finite variation on compact intervals of $[0,T]$.

Example 1.3.3. (Smooth regular function) As foretold in the discussion preceding Definition 1.3.2, any function G in $C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$ is regular for the pair (μ, A). If then we set

$$\vartheta_i(t) := \nabla_i G(\mu(t), A(t)) \equiv \partial_i G(\mu(t), A(t)), \qquad i = 1, \dots, d, \qquad 0 \le t \le T, \tag{1.3.5}$$

the resulting $\vartheta = (\vartheta_1, \dots, \vartheta_d)'$ is in $\mathscr{L}(\mu)$ from Theorem 1.1.3. Furthermore, we apply the pathwise Itô formula (1.1.4) to the function G, to deduce that the continuous function of finite variation in (1.3.4) can be cast in the notation of Theorem 1.1.3 as

$$\Gamma^G(\cdot) = -\sum_{\ell=1}^{m} \int_0^{\cdot} D_\ell G\big(\mu(s), A(s)\big) dA_\ell(s) - \frac{1}{2} \sum_{i,k=1}^{d} \int_0^{\cdot} \partial^2_{i,k} G\big(\mu(s), A(s)\big) d[\mu_i, \mu_k](s). \tag{1.3.6}$$

Regular functions in Example 1.3.3 have to be sufficiently smooth (at least $C^{2,1}$), for the pathwise Itô formula to apply. However, usage of the pathwise Tanaka formula accommodates regular functions which are considerably less smooth.

Example 1.3.4 (Regularity with less smoothness). To use the pathwise Tanaka formula of Theorem 1.2.5 in place of the pathwise Itô formula of Theorem 1.1.3, we need a more specific form of regular function G than that of Example 1.3.3. We assume that μ and A have the same dimension d. Then, we set

$$X_i := \mu_i - A_i, \qquad i = 1, \dots, d, \tag{1.3.7}$$

and assume that each X_i belongs to $\mathcal{L}(\mathbb{T})$, i.e., admits a local time from Definition 1.2.4. For any absolutely continuous functions f_i with right-continuous Radon-Nikodým derivatives f_i' of finite variation for every $i = 1, \dots, d$, we define the function $G(m, a) := \sum_{i=1}^d f_i(m_i - a_i)$ for $(m, a) \in \mathbb{R}^{2d}$ and evaluate it along the pair (μ, A) as

$$G\big(\mu(t), A(t)\big) = \sum_{i=1}^{d} f_i\big(X_i(t)\big) = \sum_{i=1}^{d} f_i\big(\mu_i(t) - A_i(t)\big), \qquad 0 \le t \le T. \tag{1.3.8}$$

We claim that the function G as in (1.3.8) is regular for the pair (μ, A). To see this, we start by noting that we can only consider such generating functions G, represented as the sum of individual functions f_i for $i = 1, \dots, d$, because there is no 'multidimensional Tanaka formula' that can be applied to G directly. However, we *can* apply Theorem 1.2.5 to each component $f_i(X_i(\cdot))$ separately, and sum up to obtain

$$G\big(\mu(t), A(t)\big) = G\big(\mu(0), A(0)\big) + \sum_{i=1}^{d} \bigg\{ \int_0^t f_i'\big(X_i(s)\big) dX_i(s) + \int_{\mathbb{R}} L_t^{X_i}(x) df_i'(x) \bigg\}. \tag{1.3.9}$$

We recall from Theorem 1.2.5 that each f_i' is the RCLL derivative of f_i, and a function of bounded variation. Furthermore, the Föllmer-Itô integral in (1.3.9), defined via the recipe (1.1.3), can be decomposed as

$$\int_0^t f_i'\big(X_i(s)\big) dX_i(s) = \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)\big(\mu_i(t_{j+1}) - \mu_i(t_j)\big) \tag{1.3.10}$$

$$- \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)\big(A_i(t_{j+1}) - A_i(t_j)\big), \tag{1.3.11}$$

because the last limit (1.3.11) exists as $A \in CBV([0,T], \mathbb{R}^d)$. Thus, the limit (1.3.10) also exists, and we denote the two limits (1.3.10) and (1.3.11) as $\int_0^t f_i'(X_i(s)) d\mu_i(s)$ and $\int_0^t f_i'(X_i(s)) dA_i(s)$, respectively. Then, by setting

$$\vartheta_i(t) = \nabla_i G\big(\mu(t), A(t)\big) := f_i'\big(X_i(t)\big), \qquad i = 1, \dots, d, \tag{1.3.12}$$

for $0 \le t \le T$, we see that $\vartheta \in \mathscr{L}(\mu)$. Whereas, on account of (1.3.9) and (1.3.10), the function of (1.3.4) is seen to be of bounded variation, as it takes the form

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \vartheta_i(s) dA_i(s) - \sum_{i=1}^{d} \int_{\mathbb{R}} L_t^{(\mu_i - A_i)}(x) df_i'(x), \quad 0 \le t \le T. \tag{1.3.13}$$

From now on, we shall only consider $C^{2,1}$-smooth regular functions as in Example 1.3.3, or regular functions G of the form (1.3.8) in Example 1.3.4.

1.3.1 *Additively Generated Trading Strategies*

We would like now to introduce an additively-generated trading strategy, starting from a regular function in the pathwise sense. For any given function G which is regular for the pair (μ, A), where μ is the vector of market weights and A in $CBV([0,T], \mathbb{R}^m)$, we consider the vector ϑ with components

$$\vartheta_i(\cdot) = \nabla_i G\big(\mu(\cdot), A(\cdot)\big), \quad i = 1, \dots, d \tag{1.3.14}$$

as in (1.3.3) of the Definition 1.3.2, and the vector $\varphi = (\varphi_1, \dots, \varphi_d)'$ with

$$\varphi_i(t) := \vartheta_i(t) - Q^{\vartheta}(t) - C(0), \qquad i = 1, \dots, d, \quad 0 \leq t \leq T \tag{1.3.15}$$

as its components. Here,

$$Q^{\vartheta}(t) := V^{\vartheta}(t) - V^{\vartheta}(0) - \int_0^t \sum_{i=1}^{d} \vartheta_i(s) d\mu_i(s) \tag{1.3.16}$$

is the so-called “defect of self-financibility" at time $t \in [0, T]$ of the integrand ϑ in (1.3.14), and $V^{\varphi}(t)$ the “value” of the strategy φ as in Definition 1.3.1; whereas

$$C(0) := \sum_{i=1}^{d} \nabla_i G\big(\mu(0), A(0)\big)\mu_i(0) - G\big(\mu(0), A(0)\big) \tag{1.3.17}$$

is the so-called “defect of balance" at time $t = 0$ for the regular function G. By analogy with Proposition 2.3 of [15], the vector $\varphi = (\varphi_1, \dots, \varphi_d)'$ of (1.3.15) defines a trading strategy with respect to μ.

Definition 1.3.5 (Additive generation)**.** We say that the trading strategy φ of the form (1.3.15) is *additively generated* by $G : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, which is assumed to be regular for the pair (μ, A).

Proposition 1.3.6. *Consider the trading strategy φ, generated additively as in* (1.3.15) *by a regular function G for the pair (μ, A), where $\mu = (\mu_1, \dots, \mu_d)'$ is the vector of market weights and $A \in CBV([0,T], \mathbb{R}^m)$. This strategy has value*

$$V^{\varphi}(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t), \quad 0 \le t \le T \tag{1.3.18}$$

as in Definitions 1.3.2, and its components can be represented, for $i = 1, \dots, d$, as

$$\begin{aligned}\varphi_i(t) &= \nabla_i G\big(\mu(t), A(t)\big) + \Gamma^G(t) + G\big(\mu(t), A(t)\big) - \sum_{j=1}^d \mu_j(t) \nabla_j G\big(\mu(t), A(t)\big) \\ &= V^{\varphi}(t) + \nabla_i G\big(\mu(t), A(t)\big) - \sum_{j=1}^d \mu_j(t) \nabla_j G\big(\mu(t), A(t)\big).\end{aligned} \tag{1.3.19}$$

Proof. The proof does not involve any usage of Itô or Tanaka formula; it is exactly the same as that of Proposition 4.3 of [15], if we change $G(\mu(t))$ and $D_j G(\mu(t))$ there, into $G(\mu(t), A(t))$ and $\nabla_j G(\mu(t), A(t))$ in our present context. □

The decomposition (1.3.18) suggests that we can think of the quantity $\Gamma^G(\cdot)$ in (1.3.4), (1.3.6), or (1.3.13), as expressing the "cumulative earnings" of the strategy φ in (1.3.15) around the "baseline" $G(\mu(\cdot), A(\cdot))$.

Remark 1.3.7 (Balanced generating function). When the function G in Proposition 1.3.6 is 'balanced', i.e.,

$$G\big(\mu(t), A(t)\big) = \sum_{j=1}^d \mu_j(t)\, \nabla_j G\big(\mu(t), A(t)\big), \qquad 0 \le t \le T \tag{1.3.20}$$

holds, the additively generated trading strategy φ of (1.3.19) takes the considerably simpler form

$$\varphi_i(t) = \nabla_i G\big(\mu(t), A(t)\big) + \Gamma^G(t), \qquad i = 1, \dots, d. \tag{1.3.21}$$

Remark 1.3.8 (Additively generated portfolio). For an additively generated trading strategy φ with positive value $V^\varphi > 0$, the corresponding portfolio weights are defined as

$$\pi_i(t) := \frac{\varphi_i(t)\mu_i(t)}{V^\varphi(t)} = \frac{\varphi_i(t)\mu_i(t)}{\sum_{i=1}^d \varphi_i(t)\mu_i(t)}, \qquad i = 1, \dots, d,$$

or with the help of (1.3.18) and (1.3.19), as

$$\pi_i(t) = \mu_i(t)\left(1 + \frac{\nabla_i G\big(\mu(t), A(t)\big) - \sum_{j=1}^d \mu_j(t)\nabla_j G\big(\mu(t), A(t)\big)}{G\big(\mu(t), A(t)\big) + \Gamma^G(t)}\right). \tag{1.3.22}$$

1.3.2 *Multiplicatively Generated Trading Strategies*

Next, we introduce the notion of multiplicatively generated trading strategy. We suppose that $G : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is regular for the pair (μ, A), as in Definition 1.3.2, where μ is the vector of market weights and A in $CBV([0,T], \mathbb{R}^m)$, and that the scalar function $1/G(\mu(\cdot), A(\cdot))$ is locally bounded. This holds, for example, if G is bounded away from zero. We consider the vector $\eta = (\eta_1, \dots, \eta_d)'$ with components

$$\eta_i(\cdot) := \nabla_i G\big(\mu(\cdot), A(\cdot)\big) \exp\Big(\int_0^\cdot \frac{d\Gamma^G(t)}{G\big(\mu(t), A(t)\big)}\Big) \tag{1.3.23}$$

in the notation of (1.3.4), (1.3.14) for $i = 1, \dots, d$. The integral is well-defined, as $1/G(\mu(\cdot), A(\cdot))$ is assumed to be locally bounded. Moreover, we have $\eta \in \mathscr{L}(\mu)$, since $\vartheta = \nabla G(\mu, A) \in \mathscr{L}(\mu)$ from Definition 1.3.2, and the exponential term is a locally bounded function. We turn this η into a trading strategy $\psi = (\psi_1, \dots, \psi_d)'$ as before, by setting

$$\psi_i := \eta_i - Q^\eta - C(0), \qquad i = 1, \dots, d \tag{1.3.24}$$

in the manner of (1.3.15), and with Q^η, $C(0)$ defined as in (1.3.16) and (1.3.17).

Definition 1.3.9 (Multiplicative generation)**.** The trading strategy $\psi = (\psi_1, \dots, \psi_d)'$ of (1.3.24) is said to be *multiplicatively generated* by the regular function G for the pair (μ, A).

We have the following result, multiplicative counterpart of Proposition 1.3.6, for smooth regular function $G \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$.

Proposition 1.3.10. *Consider the trading strategy $\psi = (\psi_1, \ldots, \psi_d)'$, generated as in* (1.3.24) *by $G \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$ which is regular for the pair (μ, A) and $1/G(\mu(\cdot), A(\cdot))$ is locally bounded. The value generated by this strategy is given by*

$$V^{\psi}(t) = G\big(\mu(t), A(t)\big) \cdot \exp\Big(\int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s), A(s)\big)}\Big) > 0, \qquad 0 \le t \le T \tag{1.3.25}$$

in the notation of (1.3.4)*. This strategy ψ can be represented for $i = 1, \ldots, d$ as*

$$\psi_i(t) = V^{\psi}(t)\bigg(1 + \frac{\nabla_i G\big(\mu(t), A(t)\big) - \sum_{j=1}^d \mu_j(t)\nabla_j G\big(\mu(t), A(t)\big)}{G\big(\mu(t), A(t)\big)}\bigg). \tag{1.3.26}$$

Proof. We follow the argument in Proposition 4.8 of [15], using the pathwise Itô formula instead of the standard Itô formula for semimartingales. With the notation

$$K(t) := \exp\Big(\int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s), A(s)\big)}\Big) \tag{1.3.27}$$

in (1.3.25), the pathwise Itô formula (Theorem 1.1.3) yields

$$\begin{aligned}
G\big(\mu(t), A(t)\big)K(t) &= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \partial_i G\big(\mu(s), A(s)\big)K(s)d\mu_i(s) \\
&\quad + \int_0^t K(s)d\Gamma^G(s) + \int_0^t \sum_{i=1}^m D_i G\big(\mu(s), A(s)\big)K(s)dA_i(s) \\
&\quad + \frac{1}{2}\int_0^t \sum_{i=1}^d \sum_{j=1}^d \partial^2_{i,j} G\big(\mu(s), A(s)\big)K(s)d[\mu_i, \mu_j](s) \\
&= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \partial_i G\big(\mu(s), A(s)\big)K(s)d\mu_i(s) \\
&= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \eta_i(s)d\mu_i(s)
\end{aligned}$$

$$= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \psi_i(s) d\mu_i(s).$$

Here, the second equality uses the expression in (1.3.6), and the last equality relies on Proposition 2.3 of [15]. Since (1.3.25) holds at time zero, it follows that (1.3.25) holds at any time $t \in [0, T]$. The justification for (1.3.26) is exactly the same as that of Proposition 4.8 in [15]. □

For the trading strategy ψ, multiplicatively generated from the less smooth generating function G in Example 1.3.4 with the notation of (1.3.8), (1.3.12), we have a result similar to Proposition 1.3.10. The proof requires additional attention and computation, as there is no 'product rule' that can be applied to such functions, less regular than those we have been dealing with thus far.

Proposition 1.3.11. *The trading strategy ψ, generated multiplicatively as in* (1.3.24) *by the regular function G of the form* (1.3.8)*, has value $V^\psi(t)$ as in* (1.3.25) *and its components can be represented as in* (1.3.26).

Proof. We recall the notation (1.3.7), (1.3.8), (1.3.27) and consider the following telescoping expansion over the refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions:

$$G\big(\mu(t), A(t)\big)K(t) - G\big(\mu(0), A(0)\big)K(0) = \sum_{i=1}^d \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f_i\big(X_i(t_{j+1})\big)K(t_{j+1}) - f_i\big(X_i(t_j)\big)K(t_j) \Big\}$$

$$= \sum_{i=1}^d \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f_i\big(X_i(t_{j+1})\big)\big(K(t_{j+1}) - K(t_j)\big) \Big\} \tag{1.3.28}$$

$$+ \sum_{i=1}^d \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ \Big(f_i\big(X_i(t_{j+1})\big) - f_i\big(X_i(t_j)\big) \Big) K(t_j) \Big\}. \tag{1.3.29}$$

Then, we can further expand the last double sum (1.3.29) as

$$\sum_{i=1}^d \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ \Big(f_i\big(X_i(t_{j+1})\big) - f_i\big(X_i(t_j)\big) \Big) K(t_j) \Big\}$$

$$= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f_i'\big(X_i(t_j)\big) K(t_j) \big(X_i(t_{j+1}) - X_i(t_j)\big) + \int_{\mathbb{R}} \mathbb{1}_{(\!| X_{t_j}, X_{t_{j+1}}]\!]}(x) \big| X_{t_{j+1}} - x \big| K(t_j) df_i'(x) \Big\}$$

$$= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f_i'\big(X_i(t_j)\big) K(t_j) \big(\mu_i(t_{j+1}) - \mu_i(t_j)\big) \Big\} \tag{1.3.30}$$

$$- \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big\{ f_i'\big(X_i(t_j)\big) K(t_j) \big(A_i(t_{j+1}) - A_i(t_j)\big) \Big\} \tag{1.3.31}$$

$$+ \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j) \int_{\mathbb{R}} \big(L_{t_{j+1}}^{X_i, \mathbb{T}_n}(x) - L_{t_j}^{X_i, \mathbb{T}_n}(x) \big) df_i'(x), \tag{1.3.32}$$

where the first equation is from (1.2.5), and the last follows from (1.3.7) and (1.2.4).

Next, we show that the sum of (1.3.28), (1.3.31), and (1.3.32) vanishes as $n \to \infty$. First, since the mesh size goes to zero as $n \to \infty$, the limit of the sum (1.3.28) is a Lebesgue-Stieltjes integral

$$\sum_{i=1}^{d} \int_0^t f_i\big(X_i(s)\big) dK(s) = \int_0^t G\big(\mu(s), A(s)\big) dK(s),$$

because $f_i(X_i(\cdot))$ is bounded on the compact interval $[0, T]$ for each $i = 1, \dots, d$. From (1.3.13), the change of variable formula for Lebesgue-Stieltjes integral gives

$$\int_0^t G\big(\mu(s), A(s)\big) dK(s) = \int_0^t K(s) d\Gamma^G(s) = \int_0^t K(s) \big(d\Gamma_1^G(s) - d\Gamma_2^G(s) \big), \tag{1.3.33}$$

where

$$\Gamma_1^G(t) := \sum_{i=1}^{d} \int_0^t \vartheta_i(s) dA_i(s), \qquad \Gamma_2^G(t) := \sum_{i=1}^{d} \int_{\mathbb{R}} L_t^{X_i}(x) df_i'(x).$$

It follows that the limit of the sum in (1.3.31) is $-\sum_{i=1}^{d} \int_0^t K(s) \vartheta_i(s) dA_i(s)$. On the other hand, the last integral of (1.3.33) can be expressed as the limit of the sum

$$\int_0^t K(s) d\Gamma_2^G(s) = \lim_{n \to \infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j) \big\{ \Gamma_2^G(t_{j+1}) - \Gamma_2^G(t_j) \big\}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j) \int_{\mathbb{R}} \big(L^{X_i}_{t_{j+1}}(x) - L^{X_i}_{t_j}(x)\big) df_i'(x),$$

which coincides with the limit of the sum (1.3.32). Therefore, the limits of the sums (1.3.28), (1.3.31), and (1.3.32) are equal to zero; whereas, the remainder term on the right-hand side of (1.3.28), (1.3.29) is the sum (1.3.30), whose limit we denote as

$$\sum_{i=1}^{d} \int_0^t f_i'(X_i(s))K(s)d\mu_i(s) = \sum_{i=1}^{d} \int_0^t \eta_i(s)d\mu_i(s),$$

from (1.3.8), and (1.3.23). We obtain this way

$$G\big(\mu(t), A(t)\big)K(t) - G\big(\mu(0), A(0)\big)K(0) = \sum_{i=1}^{d} \int_0^t \eta_i(s)d\mu_i(s) = \sum_{i=1}^{d} \int_0^t \psi_i(s)d\mu_i(s),$$

where the last equality follows from $\sum_{i=1}^{d} \mu_i(\cdot) \equiv 1$ and (1.3.24). The result (1.3.25) then follows from the self-financibility of ψ and the relationship

$$V^{\psi}(0) = \sum_{i=1}^{d} \psi_i(0)\mu_i(0) = \sum_{i=1}^{d} \big(\vartheta_i(0) - C(0)\big)\mu_i(0) = G\big(\mu(0), A(0)\big)K(0).$$

The equation (1.3.26) can be justified in the same manner as Proposition 1.3.10. □

Remark 1.3.12 (Balanced generating function). When the function G of Propositions 1.3.10 and 1.3.11 is 'balanced' as in (1.3.20), the multiplicatively generated trading strategy ψ in (1.3.26), takes the far simpler form

$$\psi_i(t) = \vartheta_i(t) \cdot \exp\Big(\int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s), A(s)\big)}\Big), \qquad i = 1, \dots, d \tag{1.3.34}$$

Remark 1.3.13 (Multiplicatively generated portfolio). The portfolio weights corresponding to the multiplicatively generated trading strategy ψ, are similarly defined for $i = 1, \ldots, d$ as

$$\pi_i(t) := \frac{\psi_i(t)\mu_i(t)}{\sum_{i=1}^d \psi_i(t)\mu_i(t)} = \mu_i(t)\left(1 + \frac{\vartheta_i(t) - \sum_{j=1}^d \mu_j(t)\vartheta_j(t)}{G\big(\mu(t), A(t)\big)}\right). \tag{1.3.35}$$

Here, the last expression follows with the help of (1.3.25) and (1.3.26). For a 'balanced' function G as in (1.3.20), this last expression simplifies considerably, to

$$\pi_i(t) = \frac{\mu_i(t)\vartheta_i(t)}{G\big(\mu(t), A(t)\big)}, \qquad i = 1, \ldots, d.$$

1.4 Strong Relative Arbitrage

We consider the vector $\mu = (\mu_1, \ldots, \mu_d)'$ of market weights as in (1.3.1). For a given trading strategy φ with respect to μ, let us recall the value process $V^\varphi = \sum_{i=1}^d \varphi_i \mu_i$ from Definition 1.3.1. We recall that $V^\varphi(t)$ represents the relative value, at time t, of the strategy φ with respect to the market. For some fixed $T_* \in (0, T]$, we say that φ is *strong arbitrage relative to the market*[1] over the time-horizon $[0, T_*]$, if we have

$$V^\varphi(t) \geq 0, \quad \forall\, t \in [0, T_*], \qquad \text{along with} \qquad V^\varphi(T_*) > V^\varphi(0). \tag{1.4.1}$$

The value process of a trading strategy generated functionally, either additively or multiplicatively, admits a simple representation in terms of the generating function G and the derived function Γ^G as in (1.3.18), (1.3.25). In turn, this representation provides sufficient conditions for strong relative arbitrage with respect to the market. We find such conditions on trading strategies generated by a regular function $G(\mu(\cdot), A(\cdot))$, which depends not only on the vector of market weights μ, but also on an additional finite-variation process A related to μ.

[1]When the market weight vector μ is modeled as a vector of semimartingales in a probability space, 'strong' arbitrage requires that the strict inequality of (1.4.1) holds with probability 1, whereas 'weak' arbitrage only demands the same inequality with some positive probability.

We have not specified yet the function $A \in CBV([0,T],\mathbb{R}^m)$, so it is time to consider some plausible candidates for this function of finite variation. A first suitable one would be the d-dimensional vector

$$A = [\mu] = \big([\mu_1], [\mu_2], \ldots, [\mu_d]\big)' \tag{1.4.2}$$

of quadratic variations for the market weights. We can also think of a more general candidate; namely, the S_d^+-valued covariation process of market weights. Here, S_d^+ is the collection of symmetric, nonnegative-definite $d \times d$ matrices, and we will use double bracket [[ ]] to distinguish this d^2-dimensional vector from (1.4.2): namely,

$$A = [[\mu]], \quad (A)_{i,j} = [\mu_i, \mu_j] \qquad 1 \le i, j \le d. \tag{1.4.3}$$

Choosing A as in (1.4.3), we can match the integrators of the two integrals in (1.3.6), and the resulting expression for $\Gamma^G(\cdot)$ can then be cast as one integral.

A few more functions of finite variation for the process A are listed below:

1. The moving average $\bar{\mu}$ of μ defined by

$$\bar{\mu}_i(t) := \begin{cases} \frac{1}{\delta}\int_0^t \mu_i(s)ds + \frac{1}{\delta}\int_{t-\delta}^0 \mu_i(0)ds, & t \in [0,\delta), \\ \frac{1}{\delta}\int_{t-\delta}^t \mu_i(s)ds, & t \in [\delta, T], \end{cases} \qquad i = 1, \ldots, d.$$

2. The running maximum μ^* and the running minimum μ_* of the market weights, with the components $\mu_i^*(t) := \max_{0\le s\le t} \mu_i(s)$, $\mu_{*i}(t) := \min_{0\le s\le t} \mu_i(s)$, respectively, for $i = 1, \ldots, d$.

Since the vectors $\bar{\mu}, \mu^*$, and μ_* are d-dimensional, $m = d$ holds for these choices of A. Empirical results using the moving average $\bar{\mu}$ are given in Section 3 of [24]. The running maximum μ^* and the running minimum μ_* will appear in Section 1.6.

We first consider conditions leading to strong relative arbitrage with respect to the market with general A as the third input of generating function G. Then we present some examples of G with specific finite variation function A chosen from among the above candidates.

1.4.1 *Additively Generated Strong Relative Arbitrage*

We start with a condition leading to additively generated strong arbitrage, which is similar to Theorem 5.1 of [15].

Theorem 1.4.1. *Fix a function $G : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)$ which is regular for the pair (μ, A), and such that the function $\Gamma^G(\cdot)$ in (1.3.4) is nondecreasing. Here, μ is the vector of market weights and A is in $CBV([0, T], \mathbb{R}^m)$, as before.*

For some real number $T_* > 0$, suppose that

$$\Gamma^G(T_*) > G\big(\mu(0), A(0)\big) \tag{1.4.4}$$

holds. Then the trading strategy φ, additively generated by the regular function G as in Definition 1.3.5, is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \le t \le T$.

Proof. Since $\Gamma^G(\cdot)$ is nondecreasing, we obtain

$$V^\varphi(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) \ge \Gamma^G(0) = 0, \quad \text{for every} \quad t \in [0, T_*],$$

from (1.3.18). We also have

$$V^\varphi(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) \ge \Gamma^G(T_*) > G\big(\mu(0), A(0)\big) = V^\varphi(0)$$

for every $t \in [T_*, T]$. The last equality holds because $\Gamma^G(0) = 0$. □

Remark 1.4.2. With $A = [[\mu]]$ as in (1.4.3), the function $\Gamma^G(\cdot)$ of (1.3.6) is nondecreasing when

$$-\sum_{i,k=1}^{d} \int_0^{\cdot} \Big(D^1_{(i,k)} + \frac{1}{2}\partial^2_{i,k}\Big) G\big(\mu(s), [[\mu]](s)\big) d[\mu_i, \mu_j](s)$$

is nondecreasing. Here, $D^1_{(i,k)}$ denotes the first-order partial derivative operator with respect to the (i, k)-th entry of $[[\mu]]$. Also, we substitute from (1.4.3), (1.3.6) into (1.4.4) to obtain the more explicit form

$$-\sum_{i,k=1}^{d}\int_0^{T_*}\Big(D^1_{(i,k)}+\frac{1}{2}\partial^2_{i,k}\Big)G\big(\mu(s),[[\mu]](s)\big)\,d[\mu_i,\mu_j](s)>G\big(\mu(0),[[\mu]](0)\big)$$

of the condition (1.4.4) for strong relative arbitrage. Thus, unlike the situation of Theorem 3.7 in [15], we can have a nondecreasing Γ^G and a chance for effecting strong relative arbitrage, even without 'concavity' of G in μ. □

Concave functions such as $x \mapsto -x^2$ and $x \mapsto -x\log x$, when used to generate trading strategies, produce nondecreasing functions Γ^G in (1.3.6); this is because such functions have negative semidefinite Hessians $\partial^2 G = (\partial_{i,k}G)_{1\le i,k\le d}$, which play the role of integrand in the last integral of (1.3.6). Such concavity is known to lead to "diversity-weighted" investment strategies, as explained in Definition 3.4.1 of [7]. However, these concave functions had to be twice-differentiable to apply Itô's rule. Now, we can use concave but not differentiable functions, while still being able to generate portfolios with the help of the Tanaka formula. Typical examples are $x \mapsto -x^+ := -\max(x,0)$ and $x \mapsto -x^- := -\min(x,0)$.

Example 1.4.3 (On the "size effect"). Consider a constant $\xi \in (0,1)$ and the function

$$f(x) := \frac{1}{d} - (x-\xi)^+, \qquad x \in \mathbb{R}$$

which satisfies the conditions in Theorem 1.2.5. Here d is the dimension of the market weight vector μ. Then, for the pair (μ, A) with $A \equiv 0$, we have $X \equiv \mu$ in (1.3.7), and set $f = f_i$ for $i = 1,\dots,d$ to obtain the analogue

$$G(\mu(t)) = 1 - \sum_{i=1}^{d}\big(\mu_i(t)-\xi\big)^+ \tag{1.4.5}$$

of (1.3.8), which is nonnegative by construction. Here, ξ plays the role of threshold on the market weights: we only include in our generating function those stocks whose market weights exceed the threshold level ξ. From (1.3.12) and (1.3.13),

$$\vartheta_i(t) = -\mathbb{1}_{\{\mu_i(t)\geq\xi\}}, \quad i = 1, \ldots, d, \quad \text{and} \quad \Gamma^G(t) = \sum_{i=1}^{d} L_t^{\mu_i}(\xi), \tag{1.4.6}$$

for $0 \leq t \leq T$. Note that the function $\Gamma^G(\cdot)$ is nondecreasing, and increases whenever a market weight hits the threshold ξ. The trading strategy φ, additively generated as in (1.3.15), can be represented by Proposition 1.3.6 as

$$\varphi_i(t) = -\mathbb{1}_{\{\mu_i(t)\geq\xi\}} + \sum_{j=1}^{d} \mathbb{1}_{\{\mu_j(t)\geq\xi\}}\mu_j(t) + V^\varphi(t), \qquad i = 1, \ldots, d, \tag{1.4.7}$$

and has value

$$V^\varphi(t) = 1 - \sum_{i=1}^{d}(\mu_i(t) - \xi)^+ + \sum_{i=1}^{d} L_t^{\mu_i}(\xi), \quad 0 \leq t \leq T.$$

Since the function in (1.4.6) is nondecreasing, we can use the strong arbitrage condition in Theorem 1.4.1: Strong arbitrage relative to the market exists over every time horizon $[0, t]$ with $T_* \leq t \leq T$, satisfying

$$\Gamma^G(T_*) = \sum_{i=1}^{d} L_{T_*}^{\mu_i}(\xi) > G\big(\mu(0)\big) = 1 - \sum_{i=1}^{d} \big(\mu_i(0) - \xi\big)^+.$$

In the expression of $\varphi_i(t)$ in (1.4.7), the sum $\sum_{j=1}^{d} \mathbb{1}_{\{\mu_j(t)\geq\xi\}} + V^\varphi(t)$ is a universal term, same for all indices $i = 1, \ldots, d$. Thus, φ invests one currency unit less to this universal baseline amount for those 'big-capitalization stocks', whose market weights exceed the threshold ξ. Therefore, we can interpret the strategy φ of (1.4.7) as outperforming the market by investing more in 'small-capitalization stocks'. This is in broad agreement with results in Stochastic Portfolio Theory, to the effect that "tilting" in favor of small capitalization stocks, as opposed to their larger brethren, can lead to superior results under appropriate conditions. □

From (1.3.18), the value $V^\varphi(t)$ at time t of the additively generated trading strategy φ in (1.3.15), has two additive components, $G(\mu(t), A(t))$ and $\Gamma^G(t)$. In Theorem 1.4.1, we derived the strong arbitrage condition from the “non-decrease" of $\Gamma^G(\cdot)$, but there is no a priori reason to differentiate between $G(\mu(t), A(t))$ and $\Gamma^G(t)$. If $t \mapsto G(\mu(t), A(t))$ is nondecreasing, it is possible derive a strong arbitrage condition like Theorem 1.4.1, switching the roles of $G(\mu(t), A(t))$ and $\Gamma^G(t)$. However, it is difficult to find functions G for which $t \mapsto G(\mu(t), A(t))$ is monotone, because G must depend on the market weights $\mu(\cdot)$ and these fluctuate all the time. Thus, we have to 'extract a nondecreasing structure' from the generating function $G(\mu(\cdot), A(\cdot))$, and use this nondecreasing structure, instead of G itself, to derive a new strong arbitrage condition. This is done as follows.

Theorem 1.4.4. *Fix a function $G : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)$ which is regular for the pair (μ, A), where μ is the vector of market weights and A is in $CBV([0,T], \mathbb{R}^m)$, and such that the following conditions are satisfied:*

1. *$V^\varphi(\cdot) = G(\mu(\cdot), A(\cdot)) + \Gamma^G(\cdot) \geq 0$, with $\Gamma^G(\cdot)$ as in (1.3.4) or (1.3.6);*
2. *there exists a function $F(\mu(\cdot), A(\cdot))$ satisfying $G(\mu(t), A(t)) \geq F(\mu(t), A(t))$ for all $t \in [0,T]$, and the mapping $t \mapsto F(\mu(t), A(t))$ is nondecreasing;*
3. *$\Gamma^G(\cdot) \geq -\kappa$ holds by some constant κ.*

For some real number $T_* > 0$, suppose that

$$F\big(\mu(T_*), A(T_*)\big) > G\big(\mu(0), A(0)\big) + \kappa \tag{1.4.8}$$

holds. Then the additively generated strategy φ of Definition 1.3.5 is strong arbitrage relative to the market over every time horizon $[0,t]$ with $T_* \leq t \leq T$.

Proof. The first inequality of (1.4.1) is satisfied by the first condition above. From the last two conditions and (1.3.18), (1.4.8), we obtain also the second inequality of (1.4.1), since

$$V^\varphi(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) \geq F\big(\mu(t), A(t)\big) - \kappa$$

$$\geq F\big(\mu(T_*), A(T_*)\big) - \kappa > G\big(\mu(0), A(0)\big) = V^{\varphi}(0)$$

for every $t \in [T_*, T]$. □

In Theorem 1.4.4, the function $F(\mu(\cdot), A(\cdot))$ can be seen as the 'nondecreasing structure extracted from G'. This result states that the generating function G can lead to strong arbitrage relative to the market without necessarily being "Lyapunov", as in Theorem 5.1 of [15]. There can be strong relative arbitrage even if $\Gamma^G(\cdot)$ is nonincreasing. This is intuitively plausible already on the basis of the representation (1.3.18) when $G(\mu(\cdot), A(\cdot))$ grows faster than $\Gamma^G(\cdot)$ decays. An application of Theorem 1.4.4 will appear in Section 1.6 (see also Example 5.6 of [14]).

1.4.2 *Multiplicatively Generated Strong Relative Arbitrage*

In order to simplify the arguments, we assume in this subsection that the regular function G takes nonnegative values and satisfies $G(\mu(0), A(0)) = 1$. This can be achieved by replacing G by $G/G(\mu(0), A(0))$ if $G(\mu(0), A(0)) > 0$, or by $G + 1$ if $G(\mu(0), A(0)) = 0$.

Theorem 1.4.5. *Let us fix $G : \mathbb{R}^d \times \mathbb{R}^m \to [\alpha, \beta]$ with $0 < \alpha < 1 \leq \beta < \infty$, which is regular for the pair (μ, A) with the market weights μ and $A \in CBV([0,T], \mathbb{R}^m)$, and for which Γ^G in (1.3.4) is nondecreasing. For some real number $T_* > 0$, suppose that*

$$\Gamma^G(T_*) > \beta \log\left(\frac{1}{\alpha}\right) \tag{1.4.9}$$

holds. Then, the multiplicatively generated strategy ψ of Definition 1.3.9 is strong arbitrage relative to the market over every time-horizon $[0, t]$ with $T_* \leq t \leq T$.

Proof. First, we note that $V^{\psi}(\cdot) > 0$ from (1.3.25). Taking logarithms on both sides of (1.3.25), we obtain

$$\log V^{\psi}(t) = \log G\big(\mu(t), A(t)\big) + \int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s), A(s)\big)} \geq \log \alpha + \frac{1}{\beta}\Gamma^G(t)$$

$$\geq \log \alpha + \frac{1}{\beta}\Gamma^G(T_*) > 0 = \log G\big(\mu(0), A(0)\big) = \log V^\psi(0)$$

for all $T_* \leq t \leq T$, and the result follows. Here $G(\mu(0), A(0)) = 1$, because of the normalization on G imposed at the beginning of this subsection. □

Since the market weights μ_i, $i = 1, \dots, d$ and the continuous function A are bounded on the compact interval $[0, T]$, a regular function G as in Example 1.3.3 or Example 1.3.4, depending on the pair (μ, A), is also bounded. Thus, the boundedness condition in Theorem 1.4.5 just requires the lower bound α to be strictly positive. Also, in (1.4.9), finding tighter bounds α, β of G yields smaller T_* satisfying the arbitrage condition (1.4.9). See Remark 1.6.2 for further discussion regarding the bounds on G in the case of specific entropy functions.

Example 1.4.6 (On the "size effect", revisited). Recall the generating function G of (1.4.5) in Example 1.4.3, and add a very small constant $\epsilon > 0$ to have

$$G\big(\mu(t)\big) = (1 + \epsilon) - \sum_{i=1}^{d} \big(\mu_i(t) - \xi\big)^+,$$

with the same ϑ and the same Gamma function as in (1.4.6). The reason for inserting the constant $\epsilon > 0$ is to ensure the uniform bounds $\epsilon \leq G(\mu(\cdot)) \leq 1 + \epsilon$ regardless of the choice of $\xi \in (0, 1)$, so that $1/G$ is locally bounded. The trading strategy ψ, multiplicatively generated by this G as in Definition 1.3.9, can be represented by Theorem 1.3.10 as

$$\psi_i(t) = -K(t)\mathbb{1}_{\{\mu_i(t)\geq\xi\}} + \sum_{j=1}^{d} K(t)\mathbb{1}_{\{\mu_j(t)\geq\xi\}}\mu_j(t) + V^\varphi(t), \tag{1.4.10}$$

for $i = 1, \dots, d$, and its value is given as

$$V^\psi(t) = \Big((1 + \epsilon) - \sum_{i=1}^{d} \big(\mu_i(t) - \xi\big)^+\Big)K(t), \quad \text{with}$$
$$K(t) := \exp\Big(\int_0^t \sum_{i=1}^{d} \frac{dL_s^{\mu_i}(\xi)}{1 + \epsilon - \sum_{j=1}^{d}(\mu_j(s) - \xi)^+}\Big).$$

From Theorem 1.4.5, strong arbitrage with respect to the market exists over every time horizon $[0, t]$ with $T_* \le t \le T$, satisfying the inequality

$$\Gamma^G(T_*) = \sum_{i=1}^{d} L_{T_*}^{\mu_i}(\xi) > (1+\epsilon)\log\Big(\frac{1+\epsilon - \sum_{i=1}^{d}\big(\mu_i(0)-\xi\big)^+}{\epsilon}\Big).$$

In the manner of Example 1.4.3, the strategy ψ in (1.4.10) invests $K(t)$ unit of currency less than the 'baseline amount', $\sum_{j=1}^{d} K(t)\mathbb{1}_{\{\mu_j(t)\ge\xi\}}\mu_j(t) + V^{\varphi}(t)$, in those 'big-capitalization stocks' whose market weight exceeds the threshold ξ at time t. Because $K(\cdot)$ is nondecreasing, ψ keeps investing less and less money to those 'big-capitalization stocks' as time goes by, and the "size effect" increases gradually. □

The conditions of Theorem 1.4.5 resemble those of Theorem 1.4.1. We also have the following formulation, which is similar to Theorem 1.4.4.

Theorem 1.4.7. *Fix a regular function $G : \mathbb{R}^d \times \mathbb{R}^m \to (0, \infty)$ for the pair (μ, A), such that the following conditions hold:*

1. *there exists a function $F : \mathbb{R}^{d+m} \to (0, \infty)$ such that $G(\mu(t), A(t)) \ge F(\mu(t), A(t))$ for all $t \in [0, T]$, and the mapping $t \mapsto F(\mu(t), A(t))$ is nondecreasing;*

2. *$\Gamma^G(\cdot)$ is nonincreasing and $\Gamma^G(\cdot) \ge -\kappa$ holds by some positive constant κ.*

For some real number $T_* > 0$, suppose that

$$\log F\big(\mu(T_*), A(T_*)\big) > \frac{\kappa}{F\big(\mu(0), A(0)\big)} \tag{1.4.11}$$

holds. Then the multiplicatively generated strategy ψ of Definition 1.3.9 is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \le t \le T$.

Proof. First, note that $\Gamma^G(\cdot)$ is nonpositive, because of the condition *(ii)* and the fact $\Gamma^G(0) = 0$.

Again, from (1.3.25), we have

$$\log V^{\psi}(t) = \log G\big(\mu(t), A(t)\big) + \int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s), A(s)\big)} \geq \log F\big(\mu(t), A(t)\big) - \frac{\kappa}{\min_{0\leq s\leq t} G\big(\mu(s), A(s)\big)}$$
$$\geq \log F\big(\mu(t), A(t)\big) - \frac{\kappa}{\min_{0\leq s\leq t} F\big(\mu(s), A(s)\big)}$$
$$\geq \log F\big(\mu(T_*), A(T_*)\big) - \frac{\kappa}{F\big(\mu(0), A(0)\big)} > 0 = \log V^{\psi}(0),$$

for all $T_* \leq t \leq T$, by virtue of the conditions *1, 2* and (1.4.11). □

1.5 Rank-Based Trading Strategies

We have identified so far the stocks by their names, but it is sometimes advantageous to identify the stocks by their ranks rather than names, with regard to the distribution of capital. For this purpose, we construct trading strategies depending on the ranked market weights, by combining results from subsection 1.2.3 and section 1.3.

Recalling the notation (1.2.28) of ranked functions, we define the vector $\boldsymbol{\mu}$ of ranked market weights

$$\boldsymbol{\mu} = (\mu_{(1)}, \cdots, \mu_{(d)})'. \tag{1.5.1}$$

In what follows, we assume that the components of $\boldsymbol{\mu}$ (as well as those of μ) admit finite quadratic covariations $[\mu_{(k)}, \mu_{(\ell)}]$, $1 \leq k, \ell \leq d$ along a given sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[0, T]$ in the manner of Definition 1.1.1.

In order to generate trading strategies which depend on the ranked market weights $\boldsymbol{\mu}$, we consider the pair $(\boldsymbol{\mu}, A)$, rather than (μ, A), for some function $A \in CBV([0, T], \mathbb{R}^m)$. The generating function G then takes the form $G\big(\boldsymbol{\mu}(\cdot), A(\cdot)\big) = G\big(\mu_{(1)}(\cdot), \cdots, \mu_{(d)}(\cdot), A_1(\cdot), \cdots, A_m(\cdot)\big)$. As before, depending on the smoothness of the generating function G, we have two classes of regular functions, which now take the ranked market weights $\boldsymbol{\mu}$, instead of μ.

Example 1.5.1 (Smooth regular function, revisited). For any function $G \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$, the Itô formula of Theorem 1.1.3 gives

$$G\big(\boldsymbol{\mu}(t), A(t)\big) - G\big(\boldsymbol{\mu}(0), A(0)\big) = \int_0^t \sum_{k=1}^d \partial_k G\big(\boldsymbol{\mu}(s), A(s)\big) d\mu_{(k)}(s) \tag{1.5.2}$$
$$+ \sum_{j=1}^m \int_0^t D_j G\big(\boldsymbol{\mu}(s), A(s)\big) dA_j(s) + \frac{1}{2} \sum_{k,\ell=1}^d \int_0^t \partial^2_{k,\ell} G\big(\boldsymbol{\mu}(s), A(s)\big) d[\mu_{(k)}, \mu_{(\ell)}](s).$$

From Theorem 1.2.18, the Föllmer-Itô integral on the right-hand side of (1.5.2) is represented as

$$\int_0^t \sum_{k=1}^d \partial_k G\big(\boldsymbol{\mu}(s), A(s)\big) d\mu_{(k)}(s) = \sum_{i=1}^d \int_0^t \sum_{k=1}^d \frac{\mathbb{1}_{\{\mu_{(k)}(s)=\mu_i(s)\}}}{N_s(k)} \partial_k G\big(\boldsymbol{\mu}(s), A(s)\big) d\mu_i(s) \tag{1.5.3}$$
$$+ \sum_{k=1}^d \bigg\{ \sum_{h=k+1}^d \int_0^t \frac{\partial_k G\big(\boldsymbol{\mu}(s), A(s)\big)}{N_s(k)} dL_s^{(\mu_{(k)}-\mu_{(h)})}(0) - \sum_{h=1}^{k-1} \int_0^t \frac{\partial_k G\big(\boldsymbol{\mu}(s), A(s)\big)}{N_s(k)} dL_s^{(\mu_{(h)}-\mu_{(k)})}(0) \bigg\}.$$

By setting

$$\vartheta_i(\cdot) = \nabla_i G\big(\boldsymbol{\mu}(\cdot), A(\cdot)\big) := \sum_{k=1}^d \frac{\mathbb{1}_{\{\mu_{(k)}(\cdot)=\mu_i(\cdot)\}}}{N_\cdot(k)} \partial_k G\big(\boldsymbol{\mu}(\cdot), A(\cdot)\big), \qquad i = 1, \dots, d, \tag{1.5.4}$$

we can show from (1.5.2) that the Gamma function in (1.3.4)

$$\Gamma^G(t) = -\sum_{k=1}^d \bigg\{ \sum_{h=k+1}^d \int_0^t \frac{\partial_k G\big(\boldsymbol{\mu}(s), A(s)\big)}{N_s(k)} dL_s^{(\mu_{(k)}-\mu_{(h)})}(0) - \sum_{h=1}^{k-1} \int_0^t \frac{\partial_k G\big(\boldsymbol{\mu}(s), A(s)\big)}{N_s(k)} dL_s^{(\mu_{(h)}-\mu_{(k)})}(0) \bigg\}$$
$$- \sum_{j=1}^m \int_0^t D_j G\big(\boldsymbol{\mu}(s), A(s)\big) dA_j(s) - \frac{1}{2} \sum_{k,\ell=1}^d \int_0^t \partial^2_{k,\ell} G\big(\boldsymbol{\mu}(s), A(s)\big) d[\mu_{(k)}, \mu_{(\ell)}](s), \tag{1.5.5}$$

is of finite variation. Therefore, $G \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R})$ is regular for the pair $(\boldsymbol{\mu}, A)$.

The same procedure as in Example 1.3.4 yields the following class of regular functions depending on the pair $(\boldsymbol{\mu}, A)$.

Example 1.5.2 (Regularity with less smoothness, revisited). We assume that μ and A have the same dimension d and set

$$X_k := \mu_{(k)} - A_k, \qquad k = 1, \dots, d.$$

We then consider the following generating function

$$G\big(\boldsymbol{\mu}(t), A(t)\big) = \sum_{k=1}^{d} f_k\big(X_k(t)\big) = \sum_{k=1}^{d} f_k\big(\mu_{(k)}(t) - A_k(t)\big), \qquad 0 \le t \le T, \tag{1.5.6}$$

for any absolutely continuous functions f_k with right-continuous Radon-Nikodým derivatives f_k' of finite variation for every $k = 1, \dots, d$. From the pathwise Tanaka formula of Theorem 1.2.5, we obtain

$$\begin{aligned} G\big(\boldsymbol{\mu}(t), A(t)\big) - G\big(\boldsymbol{\mu}(0), A(0)\big) - \sum_{k=1}^{d} \int_{\mathbb{R}} L_t^{X_k}(x) df_k'(x) &= \sum_{k=1}^{d} \int_0^t f_k'\big(X_k(s)\big) dX_k(s) \\ &= \sum_{k=1}^{d} \int_0^t f_k'\big(X_k(s)\big) d\mu_{(k)}(s) - \sum_{k=1}^{d} \int_0^t f_k'\big(X_k(s)\big) dA_k(s) \end{aligned} \tag{1.5.7}$$

Here, in the last equality, we decomposed the Föllmer-Itô integral as in (1.3.10) and (1.3.11). Applying Theorem 1.2.18 to $\sum_{k=1}^{d} \int_0^t f_k'\big(X_k(s)\big) d\mu_{(k)}(s)$ on the right-hand side of (1.5.7) and setting

$$\vartheta_i(\cdot) = \nabla_i G\big(\boldsymbol{\mu}(\cdot), A(\cdot)\big) := \sum_{k=1}^{d} \frac{\mathbb{1}_{\{\mu_{(k)}(\cdot) = \mu_i(\cdot)\}}}{N_\cdot(k)} f_k'\big(X_k(\cdot)\big), \qquad i = 1, \dots, d, \tag{1.5.8}$$

yields the representation

$$\begin{aligned} \Gamma^G(t) = &- \sum_{k=1}^{d} \Bigg(\sum_{h=k+1}^{d} \int_0^t \frac{f_k'\big(X_k(s)\big)}{N_s(k)} dL_s^{(X_{(k)} - X_{(h)})}(0) - \sum_{h=1}^{k-1} \int_0^t \frac{f_k'\big(X_k(s)\big)}{N_s(k)} dL_s^{(X_{(h)} - X_{(k)})}(0) \Bigg) \\ &+ \sum_{k=1}^{d} \int_0^t f_k'\big(X_k(s)\big) dA_k(s) - \sum_{k=1}^{d} \int_{\mathbb{R}} L_t^{(\mu_{(k)} - A_k)}(x) df_k'(x), \qquad 0 \le t \le T, \end{aligned} \tag{1.5.9}$$

of the Gamma function defined in (1.3.4). This shows that G of the form (1.5.6) is also regular for the pair $(\boldsymbol{\mu}, A)$.

From ϑ defined in (1.5.4) or (1.5.8), we can generate trading strategies depending on the ranked market weights $\boldsymbol{\mu}$ both additively and multiplicatively as in subsections 1.3.1 and 1.3.2.

Example 1.5.3 (Quadratic function for the largest capitalization stock). Consider a quadratic function depending only on the largest market weight

$$G\big(\boldsymbol{\mu}(t)\big) = 1 - \mu_{(1)}^2(t). \tag{1.5.10}$$

Here, note that $A \equiv 0$ and the equations (1.5.4) and (1.5.5) give

$$\vartheta_i(t) = -\frac{2\mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)}, \qquad \Gamma^G(t) = \sum_{h=2}^{d}\int_0^t \frac{2\mu_{(1)}(s)}{N_s(1)}dL_s^{(\mu_{(1)}-\mu_{(h)})}(0) + [\mu_{(1)}, \mu_{(1)}](t),$$

for $0 \le t \le T$.

From (1.3.19), the additively generated trading strategy φ takes the form

$$\begin{aligned}\varphi_i(t) &= -\frac{2\mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)} + \big(1 - \mu_{(1)}^2(t)\big) + \Gamma^G(t) + \sum_{j=1}^{d}\mu_j(t)\frac{2\mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)=\mu_j(t)\}}}{N_t(1)} \\ &= -\frac{2\mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)} + 1 + \mu_{(1)}^2(t) + \Gamma^G(t),\end{aligned}$$

where the last equality follows from the identity $\sum_{j=1}^{d}\mathbb{1}_{\{\mu_{(1)}(t)=\mu_j(t)\}} = N_t(1)$. This trading strategy φ invests the universal baseline amount $1 + \mu_{(1)}^2(t) + \Gamma^G(t)$ to all stocks at time t, except for the largest-capitalization stocks, in which φ invests $2\mu_{(1)}(t)/N_t(1)$ currency unit less. Since $\Gamma^G(\cdot)$ is nondecreasing, Theorem 1.4.1 shows that φ is strong arbitrage relative to the market over $[0, t]$ with $t \ge T_*$ satisfying

$$\sum_{h=2}^{d}\int_0^{T_*} \frac{2\mu_{(1)}(s)}{N_s(1)}dL_s^{(\mu_{(1)}-\mu_{(h)})}(0) + [\mu_{(1)}, \mu_{(1)}](T_*) > 1 - \mu_{(1)}^2(0).$$

On the other hand, in order to generate trading strategy ψ multiplicatively, we assume that $\mu_{(1)}(t) \le r$ holds for every $t \ge 0$ for some constant $1/d < r < 1$. This guarantees that G of (1.5.10) is bounded away from zero:

$$0 < 1 - r^2 \le G\big(\boldsymbol{\mu}(t)\big) = 1 - \mu_{(1)}^2(t) < 1.$$

From (1.3.26) and (1.3.25), the multiplicatively generated trading strategy ψ and its value V^{ψ} are given as

$$\psi_i(t) = \left(- \frac{2\mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)} + 1 + \mu^2_{(1)}(t) \right) \exp\left(\int_0^t \frac{d\Gamma^G(s)}{1-\mu^2_{(1)}(s)} \right),$$

$$V^{\psi}(t) = \big(1 - \mu^2_{(1)}(t)\big) \exp\left(\int_0^t \frac{d\Gamma^G(s)}{1-\mu^2_{(1)}(s)} \right).$$

As in the case of φ above, this ψ also invests

$$\frac{2\mu_{(1)}(t)}{N_t(1)} \exp\left(\int_0^t \frac{d\Gamma^G(s)}{1-\mu^2_{(1)}(s)} \right)$$

currency unit less to the largest capitalization stock than to other stocks at time t. The inequality (1.4.9) of Theorem 1.4.5 yields the strong arbitrage condition

$$\sum_{h=2}^{d} \int_0^{T_*} \frac{2\mu_{(1)}(s)}{N_s(1)} dL_s^{(\mu_{(1)}-\mu_{(h)})}(0) + [\mu_{(1)}, \mu_{(1)}](T_*) > \log\left(\frac{1-\mu^2_{(1)}(0)}{1-r^2} \right).$$

The strategies φ and ψ are another examples of "size effect", showing that they outperform the market by investing less in the largest-capitalization stock. □

The following example, depending only on the largest-capitalization stock, is similar to Examples 1.4.3 and 1.4.6.

Example 1.5.4 (Hinge function for the largest capitalization stock). Consider a function G of the form (1.5.6)

$$G\big(\boldsymbol{\mu}(t), A(t)\big) = f_1\big(\mu_{(1)}(t)\big) := 1 - \big(\mu_{(1)}(t) - \xi\big)^+, \qquad 0 \le t \le T, \tag{1.5.11}$$

for some fixed constant $0 < \xi < 1$ which plays the role of threshold on the largest market weights. Here, note that $A \equiv 0$, as well as $f_2 = \cdots = f_d \equiv 0$ in (1.5.6). From (1.5.8) and (1.5.9), we obtain

$$\vartheta_i(t) = -\frac{\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)} \mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}}, \qquad \Gamma^G(t) = \sum_{h=2}^{d} \int_0^t \frac{\mathbb{1}_{\{\mu_{(1)}(s)\ge\xi\}}}{N_s(1)} dL_s^{(\mu_{(1)}-\mu_{(h)})}(0) + L_t^{\mu_{(1)}}(\xi),$$

for $0 \le t \le T$.

Due to (1.3.19) and the identity $\sum_{j=1}^{d} \mathbb{1}_{\{\mu_{(1)}(t)=\mu_j(t)\}} = N_t(1)$, the additively generated trading strategy φ is represented as

$$\begin{aligned}\varphi_i(t) &= -\frac{\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)}\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}} + 1 - \big(\mu_{(1)}(t)-\xi\big)^+ + \Gamma^G(t) + \mu_{(1)}(t)\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}}\\ &= -\frac{\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)}\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}} + 1 + \xi\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}} + \Gamma^G(t).\end{aligned}$$

This trading strategy φ invests the universal baseline amount $1+\xi\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}}+\Gamma^G(t)$ to all stocks at time t, but $1/N_t(1)$ currency unit less to the largest-capitalization stock whenever its market weight exceeds the threshold ξ. Theorem 1.4.1 shows that φ is strong arbitrage relative to the market over $[0,t]$ with $t \ge T_*$ satisfying

$$\sum_{h=2}^{d}\int_0^{T_*}\frac{\mathbb{1}_{\{\mu_{(1)}(s)\ge\xi\}}}{N_s(1)}dL_s^{(\mu_{(1)}-\mu_{(h)})}(0) + L_{T_*}^{\mu_{(1)}}(\xi) > 1 - \big(\mu_{(1)}(0)-\xi\big)^+.$$

On the other hand, from (1.3.26) and (1.3.25), the multiplicatively generated trading strategy ψ and its value V^ψ are given as

$$\psi_i(t) = \left(-\frac{\mathbb{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}{N_t(1)}\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}} + 1 + \xi\mathbb{1}_{\{\mu_{(1)}(t)\ge\xi\}}\right)\exp\left(\int_0^t\frac{d\Gamma^G(s)}{1-\big(\mu_{(1)}(s)-\xi\big)^+}\right),$$

$$V^\psi(t) = \Big(1-\big(\mu_{(1)}(t)-\xi\big)^+\Big)\exp\left(\int_0^t\frac{d\Gamma^G(s)}{1-\big(\mu_{(1)}(s)-\xi\big)^+}\right).$$

This trading strategy ψ also invests

$$\frac{1}{N_t(1)}\exp\left(\int_0^t\frac{d\Gamma^G(s)}{1-\big(\mu_{(1)}(s)-\xi\big)^+}\right)$$

currency unit less to the largest-capitalization stock whenever its market weight exceeds ξ at time

t. The inequality (1.4.9) of Theorem 1.4.5 gives the strong arbitrage condition

$$\sum_{h=2}^{d} \int_0^{T_*} \frac{\mathbb{1}_{\{\mu_{(1)}(s) \geq \xi\}}}{N_s(1)} dL_s^{(\mu_{(1)} - \mu_{(h)})}(0) + L_{T_*}^{\mu_{(1)}}(\xi) > \log \left(\frac{1 - \big(\mu_{(1)}(0) - \xi\big)^+}{\xi} \right).$$

1.6 Examples of Entropic Functions

In this section, we present some interesting examples of trading strategies, additively and multiplicatively generated from variants of the 'entropy function', along with the corresponding conditions for strong relative arbitrage introduced in Section 1.4.

Consider the Gibbs entropy function

$$H(x) = -\sum_{i=1}^{d} x_i \log(x_i), \qquad x \in (0,1)^d, \tag{1.6.1}$$

with values in $(0, \log d)$. Being nonnegative, twice-differentiable and concave, this function is one of the most frequently used as portfolio-generating functions in stochastic portfolio theory. See [7], [10], [15] for its usage in generating portfolios, and also [23], [24] for some variants of portfolios generated by this function.

Example 1.6.1 (Entropy function). In order to compare the trading strategy generated by the original entropy function, with those generated from variants of functions related to it, we first derive and summarize the trading strategies generated by the original entropy function. Consider the "shifted entropy"

$$G\big(\mu(t)\big) := -\sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i(t)\big) = -\log p - \sum_{i=1}^{d} \mu_i(t) \log\big(\mu_i(t)\big), \tag{1.6.2}$$

for some given real constant $p \geq 1$. This quantity coincides with the original entropy $H(\mu(t))$ in (1.6.1) when $p = 1$; the reason for inserting the additive constant will be explained in the following remark. From (1.3.6), (1.3.19), and (1.3.26), the additively generated trading strategy φ, and the multiplicatively generated trading strategy ψ from this entropy function, can be represented as

$$\varphi_i(t) = -\log\big(p\mu_i(t)\big) + \Gamma^G(t), \qquad \psi_i(t) = -\exp\left(\int_0^t \frac{d\Gamma^G(s)}{G\big(\mu(s)\big)}\right)\log\big(p\mu_i(t)\big), \tag{1.6.3}$$

for $i = 1, \ldots, d$, where

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{d[\mu_i](s)}{2\mu_i(s)}$$

is nondecreasing in t. The values of these trading strategies are given via (1.3.18) and (1.3.25). Note that φ and ψ in (1.6.3) have relatively simple forms, because G in (1.6.2) is 'almost balanced', in the sense that

$$G(\mu(\cdot)) - 1 = \sum_{j=1}^{d} \mu_j(\cdot)\partial_j G(\mu(\cdot))$$

holds; compare this with (1.3.20), and (1.6.3) with (1.3.21) and (1.3.34). Then, the condition (1.4.4) for additively generated strong arbitrage in Theorem 1.4.1 becomes

$$\sum_{i=1}^{d} \int_0^{T_*} \frac{d[\mu_i](s)}{2\mu_i(s)} > -\sum_{i=1}^{d} \mu_i(0)\log\big(p\mu_i(0)\big), \tag{1.6.4}$$

whereas the condition (1.4.9) for multiplicatively generated strong arbitrage in Theorem 1.4.5 is

$$\sum_{i=1}^{d} \int_0^{T_*} \frac{d[\mu_i](s)}{2\mu_i(s)} > \beta \log\left(\frac{-\sum_{i=1}^{d} \mu_i(0)\log\big(p\mu_i(0)\big)}{\alpha}\right).$$

Here, the constants α, β are the lower and upper bounds on G in Theorem 1.4.5. We discuss these bounds on G in the Remark 1.6.2 below. □

Remark 1.6.2. The construction of trading strategies described in the previous sections does not require any optimization or statistical estimation of parameters. However, the relative performance of trading strategies with respect to the market can be improved by introducing a parameter, or set of parameters, in the generating function G. Though the original entropy function is as in (1.6.2) with $p = 1$, we purposely inserted a constant p inside the logarithm. To achieve strong relative arbitrage faster, or to find the smallest such T_* satisfying (1.6.4), or more generally (1.4.4), it helps to be able to make smaller the 'threshold' value $G(\mu(0), A(0))$ on the right-hand side of the inequality, while keeping the 'growth rate' of $\Gamma^G(\cdot)$ fixed.

It is in this spirit, that we introduced the parameter p in (1.6.2); inserting such a constant $p > 1$ inside the logarithm makes the initial value $G(\mu(0))$ smaller by the amount $\log p$, compared to the case $p = 1$; at the same time, this does not affect $\Gamma^G(\cdot)$, as subtracting a constant $\log p$ from G does not change any derivatives of G with respect to the market weights. However, if we pick p so large that $-\sum_{i=1}^d \mu_i(t)\log\mu_i(t) < \log p$ holds at some time t, then $G(\mu(t), A(t))$ has a negative value. Theoretically, $-\sum_{i=1}^d \mu_i(t)\log\mu_i(t)$ has the minimum value of 0 only when one of the market weights, say $\mu_1(t)$, is equal to 1, and all the other weights $\mu_i(t)$ for $i = 2, \ldots, d$ vanish, which does not happen in the real world. Empirically, $-\sum_{i=1}^d \mu_i(t)\log\mu_i(t)$ is always bounded away from zero, and we can guarantee this condition theoretically by imposing a weak condition on the market weights. For example, restricting the maximum value of the market weights, say

$$\max_i \mu_i(\cdot) \leq 0.5$$

yields an additional condition on the market weights, namely; there must be an index $j \in \{1, \ldots, d\}$ such that

$$\mu_j(t) \geq \frac{0.5}{d-1}, \qquad 0 \leq t \leq T,$$

thanks to the identity $\sum_{i=1}^d \mu_i \equiv 1$. Then, the value of $-\sum_{i=1}^d \mu_i(t)\log\mu_i(t)$ should be bigger than $-\frac{0.5}{d-1}\log(\frac{0.5}{d-1})$, and bounded away from 0 at all times. Finding a suitable value of $p > 1$, while maintaining $G(\mu(\cdot))$ bounded away from 0 (and bigger than some positive constant α) should be statistically done and depends on d, the number of stocks. It is quite straightforward that $G(\mu(\cdot))$ is bounded from above by some constant β, as the function $x \mapsto -x\log x$ has the maximum value $1/e$.

Making the initial value of $G(\mu(0), A(0))$ small, while keeping the growth rate of $\Gamma^G(\cdot)$, is also beneficial for calculating the 'excess return rate' of trading strategies with respect to the market.

The excess return rate of the trading strategy φ at time $t \in (0, T]$ can be defined as

$$R^{\varphi}(t) := \frac{V^{\varphi}(t) - V^{\varphi}(0)}{V^{\varphi}(0)}, \tag{1.6.5}$$

and from (1.3.18), this can be represented as

$$R^{\varphi}(t) = \frac{G\big(\mu(t), A(t)\big) + \Gamma^{G}(t) - G\big(\mu(0), A(0)\big)}{G\big(\mu(0), A(0)\big)}$$

in the case of additively generated trading strategy. Thus, if we somehow make the value $G(\mu(0), A(0))$, the denominator of above fraction, smaller, while keeping the value of $\Gamma^{G}(t)$ in the numerator, we can obtain larger excess return rates for the trading strategy φ. In the following examples, we use this method to decrease the initial value $G(\mu(0), A(0))$ of the generating function, by inserting an appropriate constant p whenever possible. □

The following two examples use for the component A two "polar opposite" functions of finite variation, the running maximum and minimum, respectively, of the market weights:

$$\mu_i^*(t) := \max_{0 \le s \le t} \mu_i(s), \qquad \mu_{*i}(t) := \min_{0 \le s \le t} \mu_i(s). \tag{1.6.6}$$

Example 1.6.3 (Entropy function with running maximum). Consider an entropic function of the type

$$G\big(\mu(t), A(t)\big) \equiv G\big(\mu(t), \mu^*(t)\big) := -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_i^*(t), \tag{1.6.7}$$

with the notation $A \equiv \mu^* = (\mu_1^*, \dots, \mu_d^*)'$. As before, $p \ge 1$ is a constant as in Remark 1.6.2, and the initial value $G(\mu(0), \mu^*(0)) = -\log p - \sum_{i=1}^{d} \mu_i(0) \log \mu_i(0)$ is the same as in Example 1.6.1. We then obtain the derivatives

$$\partial_i G\big(\mu(t), \mu^*(t)\big) = -\log \mu_i^*(t), \qquad \partial^2_{i,j} G\big(\mu(t), \mu^*(t)\big) = 0,$$

$$D_i G\big(\mu(t), \mu^*(t)\big) = -\frac{\mu_i(t)}{\mu_i^*(t)},$$

for $1 \le i,\ j \le d$. From (1.3.6), we also have

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{\mu_i(s)}{\mu_i^*(s)} d\mu_i^*(s) = \sum_{i=1}^{d} \big(\mu_i^*(t) - \mu_i(0)\big) = \sum_{i=1}^{d} \mu_i^*(t) - 1,$$

along with the fact that the increment $d\mu_i^*(s)$ is positive only when $\mu_i(s) = \mu_i^*(s)$. As the function G of (1.6.7) is linear in $\mu_i(\cdot)$, the second-order partial derivatives with respect to μ_i of G vanish, and the nondecreasing structure of $\Gamma^G(\cdot)$ comes solely from $\mu_i^*(\cdot)$. Also from (1.3.18), and (1.3.19), the trading strategy φ generated additively from this function in (1.6.7), is expressed as

$$\varphi_i(t) = -\log\big(p\mu_i^*(t)\big) + \sum_{j=1}^{d} \mu_j^*(t) - 1, \qquad i = 1, \ldots, d;$$

and the value of this trading strategy is given as

$$V^{\varphi}(t) = -\sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i^*(t)\big) + \sum_{i=1}^{d} \mu_i^*(t) - 1.$$

The strong relative arbitrage condition (1.4.4) in Theorem 1.4.1 takes the form

$$\sum_{i=1}^{d} \mu_i^*(T_*) > 1 - \sum_{i=1}^{d} \mu_i(0) \log\big(p\mu_i(0)\big).$$

On the other hand, from (1.3.25), and (1.3.26), the trading strategy ψ generated multiplicatively by the function in (1.6.7), is given as

$$\psi_i(t) = -K(t) \log\big(p\mu_i^*(t)\big), \qquad i = 1, \ldots, d;$$

and the associated value is

$$V^{\psi}(t) = -K(t) \sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i^*(t)\big), \quad \text{with}$$

$$K(t) := \exp\left(- \int_0^t \sum_{i=1}^{d} \frac{d\mu_i^*(s)}{\sum_{j=1}^d \mu_j(s) \log\big(p\mu_j^*(s)\big)} \right).$$

The strong relative arbitrage condition (1.4.9) in Theorem 1.4.5 takes the form

$$\sum_{i=1}^{d} \mu_i^*(T_*) > 1 + \beta \log\left(\frac{-\sum_{i=1}^d \mu_i(0) \log\big(p\mu_i(0)\big)}{\alpha} \right).$$

Here α, β are again lower and upper bounds on G, and these depend on the parameter p and the condition imposed on the market weights, as described in Remark 1.6.2. Empirical results regarding this example can be found in the next section. □

The function $\Gamma^G(\cdot)$ which represents the "cumulative earnings" of the next example is nonincreasing, but surprisingly, the empirical value $V^{\varphi}(\cdot)$ and $V^{\psi}(\cdot)$ of trading strategies grow asymptotically in the long run as the value of G grows, as indicated in the empirical results of the next section. Thus, in this case, it is more appropriate to apply Theorems 1.4.4 and 1.4.7 regarding the strong arbitrage condition.

Example 1.6.4 (Entropy function with running minimum). Consider the function

$$G\big(\mu(t), A(t)\big) \equiv G\big(\mu(t), \mu_*(t)\big) := -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t), \tag{1.6.8}$$

with $A \equiv \mu_* = (\mu_{*1}, \dots, \mu_{*d})'$ in the notation of (1.6.6). As before, p is a constant and the initial value $G(\mu(0), \mu_*(0))$ is the same as in previous examples. Similarly as before, (1.3.6) gives

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{\mu_i(s)}{\mu_{*i}(s)} d\mu_{*i}(s) = \sum_{i=1}^{d} \int_0^t 1 \; d\mu_{*i}(s) = \sum_{i=1}^{d} \mu_{*i}(t) - 1, \tag{1.6.9}$$

which is a nonpositive and nonincreasing function of t.

We first consider the trading strategy φ additively generated from this function, which is expressed as

$$\varphi_i(t) = -\log\big(p\mu_{*i}(t)\big) + \sum_{j=1}^{d} \mu_{*j}(t) - 1, \qquad i = 1, \dots, d \tag{1.6.10}$$

by (1.3.19). Note that $\varphi_i(t)$ admits the lower bound

$$\varphi_i(t) = -\log p - \log \mu_{*i}(t) + \mu_{*i}(t) + \sum_{\substack{j=1 \\ j\neq i}}^{d} \mu_{*j}(t) - 1 \geq -\log p - \log \mu_i(0) + \mu_i(0) - 1, \quad (1.6.11)$$

because the function $x \mapsto -\log x + x$ is decreasing in the interval $x \in (0, 1)$ and, thus, the quantity $\varphi_i(t)$ is positive provided that $\log(p\mu_i(0)) < \mu_i(0) - 1$ holds. By (1.3.18), the value of this trading strategy is given as

$$V^{\varphi}(t) = -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) + \Big(\sum_{i=1}^{d} \mu_{*i}(t) - 1 \Big). \quad (1.6.12)$$

While $\Gamma^G(t) = \sum_{i=1}^{d} \mu_{*i}(t) - 1$, the last term on the right-hand side of (1.6.12), is nonincreasing, the second term $-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t)$ asymptotically increases as the mapping $t \mapsto -\log \mu_{*i}(t)$ is nondecreasing. Actually, as we can see in the next section, the value of this trading strategy grows in the long run. We can apply Theorem 1.4.4, rather than Theorem 1.4.1, to find a strong arbitrage condition, because $\Gamma^G(\cdot)$ in this example is not nondecreasing.

In order to apply Theorem 1.4.4, we first need to show that $V^{\varphi}(\cdot) \geq 0$ holds. From (1.6.11), we obtain

$$\begin{aligned} -\log \mu_{*i}(t) &\geq -\sum_{j=1}^{d} \mu_{*i}(t) - \log \mu_i(0) + \mu_i(0) \geq -1 - \log \mu_i(0) + \mu_i(0) \\ &\geq -1 - \log \Big(\max_{j=1,\ldots,d} \mu_j(0) \Big) + \max_{j=1,\ldots,d} \mu_j(0) \end{aligned}$$

holds for all $i = 1, \ldots, d$. The last inequality follows from the fact that the function $x \mapsto -\log x + x$ is decreasing in the interval $x \in [0, 1]$. Then, we also obtain

$$-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \geq -1 - \log \Big(\max_{j=1,\ldots,d} \mu_j(0) \Big) + \max_{j=1,\ldots,d} \mu_j(0),$$

because $-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t)$ is just the weighted average of $\{-\log \mu_{*i}(t)\}_{1\leq i\leq d}$ with weights $\mu_i(t)$

with $\sum_{i=1}^{d} \mu_i(t) = 1$. Thus, $V^{\varphi}(t)$ in (1.6.12) admits the lower bound

$$V^{\varphi}(t) \geq -\log p - 2 - \log\big(\max_j \mu_j(0)\big) + \max_j \mu_j(0)$$

for any $t \in [0, T]$, and $V^{\varphi}(\cdot) \geq 0$ is guaranteed when

$$p \leq e^{-2-\log\big(\max_j \mu_j(0)\big)+\max_j \mu_j(0)} \tag{1.6.13}$$

holds. Regarding the second condition of Theorem 1.4.4, we have

$$\begin{aligned} G\big(\mu(t), \mu_*(t)\big) &= -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \geq -\log p - \sum_{i=1}^{d} \mu_i(t) \log\big(\max_{i=1,\ldots,d} (\mu_{*i}(t))\big) \\ &= -\log p - \max_{i=1,\ldots,d} \big\{\log \mu_{*i}(t)\big\} := F\big(\mu(t), \mu_*(t)\big). \end{aligned} \tag{1.6.14}$$

Now the mapping $t \mapsto \mu_{*i}(t)$ is nonincreasing, so $F(\mu(t), \mu_*(t))$ is nondecreasing in t. Finally, the last condition of Theorem 1.4.4 follows easily from (1.6.9), as

$$\Gamma^{G}(t) \geq -1 := -\kappa. \tag{1.6.15}$$

Thus, Theorem 1.4.4 shows that the additively generated strategy φ in (1.6.10) is strong arbitrage relative to the market over every time horizon $[0, t]$ with $T_* \leq t \leq T$, satisfying the condition

$$\sum_{i=1}^{d} \mu_i(0) \log \mu_i(0) - \max_{i=1,\ldots,d} \big\{\log \mu_{*i}(T_*)\big\} > 1.$$

Next, from (1.3.26), the trading strategy ψ multiplicatively generated by the function (1.6.8) is represented as

$$\psi_i(t) = -K(t) \log\big(p\mu_{*i}(t)\big), \qquad i = 1, \ldots, d;$$

with the value

$$V^{\psi}(t) = -K(t) \sum_{i=1}^{d} \mu_i(t) \log \big(p\mu_{*i}(t)\big), \quad \text{with}$$
$$K(t) := \exp\left(- \int_0^t \sum_{i=1}^{d} \frac{d\mu_{*i}(s)}{\sum_{j=1}^{d} \mu_j(s) \log \big(p\mu_{*j}(s)\big)} \right).$$

For the strong arbitrage condition, we apply Theorem 1.4.7. Since $F(\mu(t), \mu_*(t))$, κ from (1.6.14), (1.6.15) satisfy the conditions *(i), (ii)* (with appropriate choice of p to make F positive), the strong relative arbitrage condition (1.4.11) becomes

$$\log\left(- \max_{i=1,\dots,d} \big\{ \log p\mu_{*i}(T_*) \big\} \right) > \frac{-1}{\log p + \max_{i=1,\dots,d} \big\{ \log \mu_i(0) \big\}}.$$

Remark 1.6.5. In Remark 1.6.2, we needed to find a suitable value for p satisfying an inequality, for instance, $-\sum_{i=1}^{d} \mu_i(t) \log(\mu_i(t)) \geq \log p$ for all $t \in [0, T]$ in Example 1.6.1, to ensure $G \geq 0$. This inequality usually depends on the values $\mu_i(t)$, $t \in [0, T]$ which are not observable at time 0. Thus, we need to impose some condition on the market weights, or statistically analyze historical market data to find an appropriate value for p, before we can construct the trading strategy.

However, in Example 1.6.4, due to its unique structure, we can analytically find a suitable value of p without any statistical estimation at time $t = 0$. Indeed, from (1.6.14), we have that

$$G\big(\mu(t), \mu_*(t)\big) \geq -\log p - \max_{i=1,\dots,d} \big\{ \log \mu_{*i}(t) \big\} \geq -\log p - \max_{i=1,\dots,d} \big\{ \log \mu_i(0) \big\}$$

holds; and setting

$$p = \frac{1}{\max_{i=1,\dots,d} \mu_i(0)}$$

guarantees the condition $G(\mu(t), \mu_*(t)) \geq 0$ for all $t \in [0, T]$. Note that this p depends only on quantities observable at time 0. Actually, any p satisfying (1.6.13) also guarantees the nonnegativity condition of G, because

$$G\big(\mu(\cdot), \mu_*(\cdot)\big) \geq V^{\varphi}(\cdot) = G\big(\mu(\cdot), \mu_*(\cdot)\big) + \Gamma^G(\cdot) \geq 0$$

holds due to the nonpositivity of $\Gamma^G(\cdot)$. Of course, one can perform a statistical estimation of p using past market data, to obtain a better value of p while satisfying both $G(\mu(\cdot), \mu_*(\cdot)) \geq 0$ and $V^{\varphi}(\cdot) \geq 0$. □

We next give an example which combines the last two examples with ranked market weights.

Example 1.6.6 (Ranked entropy function with running maximum and minimum). For computational simplicity, we assume that the stock market is composed of two stocks, i.e., $d = 2$ and consider a function

$$G\big(\boldsymbol{\mu}(t), A(t)\big) := -\mu_{(1)}(t) \log \mu^*_{(1)}(t) - \mu_{(2)}(t) \log \mu_{*(2)}(t). \tag{1.6.16}$$

Here, note that we set $A(t) := \big(\mu^*_{(1)}(t), \mu_{*(2)}(t)\big)$ with the notation

$$\mu^*_{(1)}(t) := \max_{0 \leq s \leq t} \max\big(\mu_1(s), \mu_2(s)\big), \qquad \mu_{*(2)}(t) := \min_{0 \leq s \leq t} \min\big(\mu_1(s), \mu_2(s)\big).$$

Along with the fact that the increments $d\mu^*_{(1)}(s)$, $d\mu_{*(2)}(s)$ are nonzero only when $\mu_{(1)}(s) = \mu^*_{(1)}(s)$ and $\mu_{(2)}(s) = \mu_{*(2)}(s)$, respectively, as well as the fact that the increment $dL_s^{\mu_{(1)}-\mu_{(2)}}(0)$ is positive only when $N_s(1) = N_s(2) = 2$, we obtain from (1.5.5)

$$\Gamma^G(t) = \frac{1}{2}\int_0^t \log\left(\frac{\mu^*_{(1)}(s)}{\mu_{*(2)}(s)}\right) dL_s^{\mu_{(1)}-\mu_{(2)}}(0) + \mu^*_{(1)}(t) + \mu_{*(2)}(t) - 1, \qquad 0 \leq t \leq T. \tag{1.6.17}$$

From (1.5.4), we have

$$\vartheta_i(\cdot) = -\frac{\mathbb{1}_{\{\mu_{(1)}(\cdot)=\mu_i(\cdot)\}}}{N_{\cdot}(1)} \log \mu^*_{(1)}(\cdot) - \frac{\mathbb{1}_{\{\mu_{(2)}(\cdot)=\mu_i(\cdot)\}}}{N_{\cdot}(2)} \log \mu_{*(2)}(\cdot), \qquad i = 1,\ 2.$$

Since the function (1.6.16) satisfies the balance condition (1.3.20), the additively generated trading strategy φ is expressed via (1.3.21) as

$$\varphi_i(t) = \vartheta_i(t) + \Gamma^G(t), \qquad i = 1,\ 2,$$

with the value

$$\begin{aligned} V^{\varphi}(t) = G\big(\boldsymbol{\mu}(t), A(t)\big) + \Gamma^G(t) = &- \mu_{(1)}(t) \log \mu^*_{(1)}(t) - \mu_{(2)}(t) \log \mu_{*(2)}(t) \\ &+ \frac{1}{2}\int_0^t \log\left(\frac{\mu^*_{(1)}(s)}{\mu_{*(2)}(s)}\right) dL_s^{\mu_{(1)}-\mu_{(2)}}(0) + \mu^*_{(1)}(t) + \mu_{*(2)}(t) - 1. \end{aligned}$$

As the Gamma function (1.6.17) contains both a nondecreasing term $\mu^*_{(1)}(t)$ and a nonincreasing term $\mu_{*(2)}(t)$ in t, we directly tackle the condition (1.4.1) in order to find the strong arbitrage for φ, rather than apply either Theorem 1.4.1 or 1.4.4. Since the inequality

$$V^{\varphi}(t) \geq \frac{1}{2}\int_0^t \log\left(\frac{\mu^*_{(1)}(s)}{\mu_{*(2)}(s)}\right) dL_s^{\mu_{(1)}-\mu_{(2)}}(0) + \mu^*_{(1)}(t) - 1$$

holds for every $t \geq 0$, the trading strategy φ is strong arbitrage relative to the market over every time-horizon $[0, t]$ with $T_* \leq t \leq T$ satisfying

$$\frac{1}{2}\int_0^{T_*} \log\left(\frac{\mu^*_{(1)}(s)}{\mu_{*(2)}(s)}\right) dL_s^{\mu_{(1)}-\mu_{(2)}}(0) + \mu^*_{(1)}(T_*) \geq 1 + V^{\varphi}(0) = 1 - \sum_{k=1}^{2} \mu_{(k)}(0) \log \mu_{(k)}(0).$$

Chapter 2: Portfolio Theory in Open Markets

This chapter discusses stochastic portfolio theory in open markets. Section 2.1 defines open markets, investment strategies, and portfolios, as well as other notions which will be needed throughout this chapter. Section 2.2 develops arbitrage theory in open markets, along with the concepts of market viability and numéraire. We provide definitions and properties for these concepts, then state and prove the main result of this chapter. Section 2.3 explores stock portfolios, the Capital Asset Pricing Model, functional generation of portfolios, as well as the so-called "universal portfolio", in open markets. Section 2.4 provides some concluding remarks. We emphasize here that throughout this chapter, we are abandoning 'pathwise setting' of Chapter 1 such that stock prices are modeled as semimartingales in a fixed probability space.

2.1 Portfolios in Open Markets

Let us suppose that the "whole equity market universe" is composed of N stocks and that we are only interested in investing in the top n largest capitalization stocks, for some fixed $1 \leq n < N$. For example, when our investing universe is the entire U.S. stock market, by setting $n = 500$ we are investing in those large companies which form the S&P 500 index. In order to invest in these top n stocks, we must keep track of the rank of each stock's capitalization at all times, and put together a portfolio composed of the n stocks with the largest capitalizations.

Throughout this chapter, we fix two positive integers n and N satisfying $1 \leq n < N$ as above. We suppose that trading is continuous in time, with no transaction costs or taxes, and that shares are infinitely divisible. As in the previous chapter, we also assume that each stock has a single share outstanding, and the price of a stock is equal to its capitalization; thus, we use the terms 'price of stock' and 'capitalization of stock' interchangeably. We also assume that stock prices are discounted by the money market, and adopt the convention that the money market pays and charges zero interest.

2.1.1 *Stock Prices and Their Ranks*

We place ourselves on a fixed, filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})$, and present the following definition of price process in the market described above.

Definition 2.1.1 (Price process)**.** For an N-dimensional vector $S \equiv (S_1, \cdots, S_n, \cdots, S_N)$ of continuous semimartingales on the probability space $(\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})$, we call S a *vector of price processes* if each component is strictly positive, i.e., the inequalities $S_i(t) > 0$ hold for all $i \in \{1, \cdots, N\}$ at any time $t \geq 0$. The component processes of S represent the stock prices, or the capitalizations, of N companies.

As in (1.2.29), for any $t \geq 0$, we rank the components of the vector process $S = (S_1, \cdots, S_N)$ in descending order

$$\max_{i=1,\cdots,N} S_i(t) = S_{(1)}(t) \geq S_{(2)}(t) \geq \cdots \geq S_{(N-1)}(t) \geq S_{(N)}(t) = \min_{i=1,\cdots,N} S_i(t); \tag{2.1.1}$$

but here, we use the lexicographic rule for breaking ties that always assigns a higher rank (i.e., a smaller (k)) to the smallest index i.

Definition 2.1.2 (Price process by rank)**.** For the vector S of price processes in Definition 2.1.1, we call the N-dimensional vector process

$$\boldsymbol{S}(t) \equiv \big(S_{(1)}(t), \cdots, S_{(N)}(t)\big), \quad t \geq 0 \tag{2.1.2}$$

where each component is defined via (2.1.1), the *vector of price processes by rank*. In particular, the k-th component $\boldsymbol{S}_k(t) = S_{(k)}(t)$ of the vector $\boldsymbol{S}(t)$ represents the price of the k-th ranked stock among all N companies at time t.

Each component of the vector process $\boldsymbol{S}(\cdot)$ is also a continuous semimartingale, from the results in [1]. Along with the notation (2.1.1), we define a process $\{1, \cdots, N\} \times [0, \infty) \ni (i, t) \mapsto u_i(t) \in \{1, \cdots, N\}$, such that each $u_i(\cdot)$ is predictable and satisfies

$$S_i(t) = S_{\left(u_i(t)\right)}(t), \quad \forall\, t \geq 0, \tag{2.1.3}$$

for every $i = 1, \cdots, N$. In other words, $u_i(t)$ is the rank of the i-th stock $S_i(t)$ at time t, for any given index $i = 1, \cdots, N$. Note that for every fixed $t \geq 0$, the function $u_\cdot(t) : \{1, \cdots, N\} \to \{1, \cdots, N\}$ is a bijection, because we break ties using the lexicographic rule when defining (2.1.1).

2.1.2 *Cumulative Return Processes*

In this subsection, we present the notion of cumulative returns of the market. We first define the stochastic logarithm $\mathcal{L}(Y)$ of a positive continuous semimartingale Y with $Y(0) = 1$ by

$$\mathcal{L}(Y) := \int_0^\cdot \frac{dY(t)}{Y(t)} \tag{2.1.4}$$

and consider the vector $R \equiv (R_1, \cdots, R_N)$, whose every component is the stochastic logarithm of the corresponding normalized component of S in Definition 2.1.1:

$$R_i := \mathcal{L}\left(\frac{S_i}{S_i(0)}\right), \qquad i = 1, \cdots, N. \tag{2.1.5}$$

Each component process R_i is again a semimartingale, and represents the *cumulative returns of the i-th stock*, since its dynamic is represented as

$$dR_i(t) = \frac{dS_i(t)}{S_i(t)}, \quad t \geq 0, \quad \text{and} \quad R_i(0) = 0 \quad \text{for} \quad i = 1, \cdots, N. \tag{2.1.6}$$

We posit the semimartingale decomposition

$$R_i = A_i + M_i, \quad i = 1, \cdots, N, \tag{2.1.7}$$

for each component of the vector $R = (R_1, \cdots, R_N)$. Here, the component A_i of the vector process $A \equiv (A_1, \cdots, A_N)$ with $A_i(0) = 0$ is adapted, continuous and of finite variation on compact time intervals; whereas each component M_i of the vector process $M \equiv (M_1, \cdots, M_N)$ is a continuous local martingale with $M_i(0) = 0$, for $i = 1, \cdots, N$. We think of the finite variation processes A_i as the 'drift components', and of the local martinagles M_i as the 'noise components', of R.

We define next the continuous, nondecreasing scalar process

$$O := \sum_{i=1}^{N} \left(\int_0^{\cdot} |dA_i(t)| + d[M_i, M_i](t) \right), \tag{2.1.8}$$

where $\int_0^T |dA_i(t)|$ denotes the total variation of A_i on the interval $[0, T]$ for $T \geq 0$ and $[M_i, M_j]$ represents the covariation process of the continuous semimartingales M_i and M_j for $1 \leq i, j \leq N$. We note that $[R_i, R_j] = [M_i, M_j]$ holds from (2.1.7). This scalar process O plays the role of an "operational clock" for the vector R. All processes A_i and $[M_i, M_j]$ for $1 \leq i, j \leq N$ are absolutely continuous with respect to this clock, and thus, by the Radon-Nikodým Theorem, there exist two predictable processes

$$\alpha \equiv (\alpha_i)_{1 \leq i \leq N} \quad \text{and} \quad c \equiv (c_{i,j})_{1 \leq i,j \leq N}, \tag{2.1.9}$$

vector-valued and matrix-valued, respectively, such that

$$A = \int_0^{\cdot} \alpha(t) dO(t), \quad \text{and} \quad C \equiv [M, M] = \int_0^{\cdot} c(t) dO(t). \tag{2.1.10}$$

Here and in what follows, we write $C \equiv (C_{i,j})_{1 \leq i,j \leq N}$ for the nonnegative-definite, matrix-valued process of covariations

$$C_{i,j} := [M_i, M_j] = [R_i, R_j], \quad \text{for} \quad 1 \leq i, j \leq N. \tag{2.1.11}$$

The component α_i in (2.1.9) represents the *local rate of return* of the i-th stock in the market; whereas the entry $c_{i,j}$ stands for the *local covariation rate* of the i-th and j-th stocks. We call the collection of local rates α, c in (2.1.9) the *local characteristics* of the market, and these rates are measured with respect to the operational clock O in (2.1.8).

For a continuous vector-valued semimartingale $Y = (Y_1, \cdots, Y_N)$, we denote by $\mathcal{I}(Y)$ the class of predictable vector processes $\pi = (\pi_1, \cdots, \pi_N)$ which are integrable with respect to the vector Y. In particular, for the collection $\mathcal{I}(R)$ of the vector R in (2.1.5) and (2.1.7), we have a very convenient characterization: A predictable vector process $\pi = (\pi_1, \cdots, \pi_N)$ belongs to $\mathcal{I}(R)$, if and only if

$$\int_0^T \Big(|\pi'(t)\alpha(t)| + \pi'(t)c(t)\pi(t)\Big) dO(t) < \infty, \quad \text{for any} \quad T \geq 0. \tag{2.1.12}$$

We denote then by

$$\int_0^\cdot \sum_{i=1}^N \pi_i(t) dR_i(t) \equiv \int_0^\cdot \pi'(t) dR(t) = \int_0^\cdot \pi'(t) dA(t) + \int_0^\cdot \pi'(t) dM(t),$$

the stochastic integral of $\pi \in \mathcal{I}(R)$, with respect to the vector semimartingale R.

2.1.3 *Investment Strategies and Portfolios*

Along with the N-dimensional vector S of Definition 2.1.1, representing the stock prices of the market, we introduce the following notions.

Definition 2.1.3 (Investment strategy, wealth process, and numéraire)**.** We call an N-dimensional vector of predictable process $\vartheta \equiv (\vartheta_1, \cdots, \vartheta_N)$ *investment strategy*, if it is integrable with respect to the price vector S, i.e., $\vartheta \in \mathcal{I}(S)$. For any nonnegative real number x, we call

$$X(\cdot; x, \vartheta) := x + \int_0^\cdot \vartheta'(t) dS(t) \equiv x + \int_0^\cdot \sum_{i=1}^N \vartheta_i(t) dS_i(t) \tag{2.1.13}$$

the *wealth process* generated by ϑ with initial capital x. We call the wealth process *numéraire*, if

$X(\cdot; 1, \vartheta) > 0$ holds for the normalized initial capital $x = 1$. The collection of all numéraires is denoted by $\mathcal{X}$.

The i-th component $\vartheta_i(t)$ represents the units of investment (or number of shares) held in the i-th stock at time t, and plays the role of integrand with integrator $dS_i(t)$ in the stochastic integral of (2.1.13). The requirement $X(0) = x = 1$ in defining numéraires is a simple normalization, because $X(\cdot; cx, c\vartheta) = cX(\cdot; x, \vartheta)$ holds for any positive real number c.

Since we consider investment only in the top n stocks, we need a similar definition of investment strategy for this particular case.

Definition 2.1.4 (Investment strategy among the top n stocks)**.** We call an investment strategy $\vartheta \in \mathcal{I}(S)$ an *investment strategy among the top n stocks*, if the "sensoring" equalities

$$\vartheta_i(t)\mathbf{1}_{\{u_i(t)>n\}} = 0, \quad \text{for} \quad i = 1, \cdots, N, \quad t \geq 0, \tag{2.1.14}$$

hold with the notation (2.1.3).

The wealth process and the numéraire associated with this investment strategy ϑ among the top n stocks, are defined in the same manner as in Definition 2.1.3. We denote by $\mathcal{T}(n)$ the collection of N-dimensional predictable processes ϑ satisfying the condition (2.1.14), and by $\mathcal{I}(S) \cap \mathcal{T}(n)$ the collection of investment strategies among the top n stocks.

The collection of all numéraires generated by investment strategies $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ among the top n stocks, is denoted by $\mathcal{X}^n$.

The condition (2.1.14) prohibits the strategy ϑ from investing in stock i at time $t \geq 0$, if this stock fails to rank at that time among the top n stocks in terms of capitalization. We present another definition, that of a portfolio rule, which plays the role of integrand with respect to the integrator $dR_i(t)$ of (2.1.5).

Definition 2.1.5 (Portfolio)**.** We call an N-dimensional predictable, vector-valued process $\pi \equiv (\pi_1, \cdots, \pi_N) \in \mathcal{I}(R)$ a *portfolio*, if it is integrable with respect to the cumulative return vector R of

(2.1.5). We call a portfolio $\pi \in \mathcal{I}(R)$ a *portfolio among the top n stocks*, if the equalities

$$\pi_i(t)\mathbf{1}_{\{u_i(t)>n\}} = 0, \quad \text{for} \quad i = 1, \cdots, N, \quad t \geq 0, \tag{2.1.15}$$

hold with the notation (2.1.3). We denote the collection of portfolios among the top n stocks by $\mathcal{I}(R) \cap \mathcal{T}(n)$.

Since the function $u_\cdot(t) : \{1, \cdots, N\} \to \{1, \cdots, N\}$ is bijective for every $t \geq 0$, the collection $\left\{\mathbf{1}_{\{u_i(t)=k\}}\right\}_{k=1,\cdots,N}$ constitutes a partition of unity for any given $i = 1, \cdots, N$, $t \geq 0$, and the conditions (2.1.14), (2.1.15) can also be formulated respectively as

$$\vartheta_i(t) = \sum_{k=1}^{n} \vartheta_i(t)\mathbf{1}_{\{u_i(t)=k\}} = \vartheta_i(t)\mathbf{1}_{\{u_i(t)\leq n\}}, \quad \text{for} \quad i = 1, \cdots, N, \quad t \geq 0, \tag{2.1.16}$$

$$\pi_i(t) = \sum_{k=1}^{n} \pi_i(t)\mathbf{1}_{\{u_i(t)=k\}} = \pi_i(t)\mathbf{1}_{\{u_i(t)\leq n\}}, \quad \text{for} \quad i = 1, \cdots, N, \quad t \geq 0. \tag{2.1.17}$$

We present next the connection between investment strategies ϑ and portfolios π. For any scalar continuous semimartingale Z with $Z(0) = 0$, we denote the stochastic exponential of Z by

$$\mathcal{E}(Z) := \exp\left(Z - \frac{1}{2}[Z, Z]\right). \tag{2.1.18}$$

It can be shown that this is also the unique process satisfying the linear stochastic integral equation

$$\mathcal{E}(Z) = 1 + \int_0^{\cdot} \mathcal{E}(Z)(t)dZ(t). \tag{2.1.19}$$

It is straightforward to check that the stochastic logarithm operator $\mathcal{L}(\cdot)$ in (2.1.4), is the inverse of the stochastic exponential operator $\mathcal{E}(\cdot)$ in (2.1.18).

We introduce now the *cumulative returns process of a portfolio* π as in Definition 2.1.5, via the vector stochastic integral

$$R_\pi := \int_0^\cdot \pi'(t)dR(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t)dR_i(t), \tag{2.1.20}$$

and consider its stochastic exponential

$$X_\pi := \mathcal{E}(R_\pi) = \mathcal{E}\Big(\int_0^\cdot \sum_{i=1}^N \pi_i(t)dR_i(t)\Big). \tag{2.1.21}$$

In particular, we note that X_π is positive. Then, from (2.1.21), (2.1.20), and (2.1.6), we obtain the dynamics

$$\frac{dX_\pi(t)}{X_\pi(t)} = dR_\pi(t) = \sum_{i=1}^N \pi_i(t)dR_i(t) = \sum_{i=1}^N \pi_i(t)\frac{dS_i(t)}{S_i(t)}, \quad X_\pi(0) = 1. \tag{2.1.22}$$

By setting

$$\vartheta_i := \frac{X_\pi \pi_i}{S_i} \quad \text{for} \quad i = 1, \cdots, N, \tag{2.1.23}$$

we arrive at the equation (2.1.13) with $X(\cdot; 1, \vartheta)$ replaced by $X_\pi(\cdot)$. Thus, from the portfolio $\pi \in \mathcal{I}(R)$, we can obtain the corresponding investment strategy ϑ and its numéraire $X(\cdot; 1, \vartheta)$, via the recipe (2.1.23). Here, we denote the numéraire generated by the portfolio π by $X(\cdot; 1, \vartheta) := X_\pi$, as in (2.1.21).

Conversely, for a given investment strategy ϑ generating a positive wealth process, i.e., the numéraire $X(\cdot; 1, \vartheta)$, we define a predictable, vector-valued process $\pi \equiv (\pi_1, \cdots, \pi_N)$ as

$$\pi_i := \frac{S_i \vartheta_i}{X(\cdot; 1, \vartheta)} \quad \text{for} \quad i = 1, \cdots, N. \tag{2.1.24}$$

It can be easily checked that π is indeed a portfolio, i.e., R-integrable and (2.1.13) can be written as

$$X(\cdot; 1, \vartheta) = 1 + \int_0^\cdot X(t; 1, \vartheta) \sum_{i=1}^N \pi_i(t)dR_i(t),$$

with the help of (2.1.6). This last equation gives the dynamics in (2.1.22) with $X_\pi(\cdot) \equiv X(\cdot; 1, \vartheta)$.

Thus, whether we start from an investment strategy ϑ (generating a numéraire) or from a portfolio π, the counterpart can always be obtained via (2.1.24) or (2.1.23), respectively, and we will denote the corresponding numéraire $X(\cdot;, 1, \vartheta)$ in (2.1.13) by X_π as in (2.1.21).

In the relationship (2.1.24), the product $S_i(t)\vartheta_i(t)$ represents the amount of wealth invested in i-th stock at time t, thus $\pi_i(t)$ can be interpreted as the proportion of current wealth invested in i-th stock at time t. The remaining proportion of wealth

$$\pi_0 := 1 - \sum_{i=1}^{N} \pi_i \tag{2.1.25}$$

is then considered to be placed in the money market.

We present now a few more concepts regarding portfolios. For any two portfolios π, ρ in $\mathcal{I}(R)$, we consider the covariation process between the cumulative returns R_π, R_ρ in (2.1.20), namely

$$C_{\pi\rho} := [R_\pi, R_\rho] = \int_0^{\cdot} c_{\pi\rho}(t)dO(t), \quad \text{with} \quad c_{\pi\rho} := \pi' c \rho = \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i c_{i,j} \rho_j. \tag{2.1.26}$$

Here, we recall the definitions of the matrix-valued processes c and C in (2.1.9), (2.1.10) and note the notational consistency with (2.1.26). In particular, when the portfolio is given as the unit vector e^i of $\mathbb{R}^N$ for some $i = 1, \cdots, N$, we use the subscript 'i' instead of 'e^i' to write $C_{i\rho} \equiv C_{e^i\rho}$ and $c_{i\rho} \equiv c_{e^i\rho}$ in order to ease notation. This convention is consistent with the actual equalities $C_{i,j} = C_{e^ie^j}$ and $c_{i,j} = c_{e^ie^j}$ for $1 \le i, j \le N$.

By recalling the wealth process X_π generated by the portfolio π as in (2.1.21), (2.1.22), we can express the logarithm of X_π as

$$\log X_\pi = R_\pi - \frac{1}{2}C_{\pi\pi} = \int_0^{\cdot} \pi'(t)dA(t) - \frac{1}{2}C_{\pi\pi} + \int_0^{\cdot} \pi'(t)dM(t). \tag{2.1.27}$$

We call the finite-variation part of $\log X_\pi$ the *cumulative growth of the portfolio* π, and denote it by

$$\Gamma_\pi := A_\pi - \frac{1}{2}C_{\pi\pi}, \quad \text{where} \quad A_\pi := \int_0^\cdot \pi'(t)dA(t). \tag{2.1.28}$$

In a similar manner, the local martingale part of the decomposition in (2.1.27) is denoted by

$$M_\pi := \int_0^\cdot \pi'(t)dM(t). \tag{2.1.29}$$

In particular, the cumulative return R_π in (2.1.21) is the stochastic logarithm $\mathcal{L}(X_\pi)$ of X_π, and has 'drift' component A_π as in (2.1.28), from (2.1.20) and (2.1.7); whereas the natural logarithm $\log X_\pi$ in (2.1.27) of X_π has 'drift' term Γ_π.

We further define the predictable processes

$$\alpha_\pi := \pi'\alpha, \qquad \gamma_\pi := \pi'\alpha - \frac{1}{2}\pi' c\pi = \alpha_\pi - \frac{1}{2}c_{\pi\pi}, \tag{2.1.30}$$

and call α_π the *rate of return*, and γ_π the *growth rate*, of the portfolio π. The 'drift parts' A_π and Γ_π, of $\mathcal{L}(X_\pi)$ and $\log X_\pi$, respectively, are then represented as the integrals of these rates with respect to the 'operational clock' in (2.1.8):

$$A_\pi = \int_0^\cdot \alpha_\pi(t)dO(t), \qquad \Gamma_\pi = \int_0^\cdot \gamma_\pi(t)dO(t). \tag{2.1.31}$$

2.1.4 *Portfolios among the Top n Stocks*

In this subsection we provide definitions, similar to those introduced in the previous subsections, for portfolios that invest only among the top n stocks.

For $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ and $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$, representing a strategy that invests only among the top n stocks and a portfolio among the top n stocks, respectively, the equations (2.1.20)-(2.1.25) can be used in the same manner. In particular, the bidirectional connections (2.1.23) and (2.1.24) between $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ and $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ still hold, because of the similarity in the conditions (2.1.14) and (2.1.15).

We define next a new N-dimensional vector $\widetilde{R} \equiv (\widetilde{R}_1, \cdots, \widetilde{R}_N)$ by

$$\widetilde{R}_i(t) := \int_0^t \mathbf{1}_{\{u_i(s) \le n\}} dR_i(s), \quad \text{for} \quad i = 1, \cdots, N, \quad t \ge 0. \tag{2.1.32}$$

Each component $\widetilde{R}_i(t)$ represents the cumulative return of the i-th stock, accumulated over $[0, t]$ but only at times when the stock ranks among the top n by capitalization. Then, for $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$, (2.1.20) can be also cast as

$$R_\pi = \int_0^\cdot \sum_{i=1}^N \pi_i(t) dR_i(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t) \mathbf{1}_{\{u_i(t) \le n\}} dR_i(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t) d\widetilde{R}_i(t), \tag{2.1.33}$$

where the second equality follows from (2.1.17). We then have the semimartingale decomposition

$$\widetilde{R}_i = \widetilde{A}_i + \widetilde{M}_i, \qquad i = 1, \cdots, N, \tag{2.1.34}$$

where

$$\widetilde{A}_i(t) := \int_0^t \mathbf{1}_{\{u_i(s) \le n\}} dA_i(s), \qquad \widetilde{M}_i(t) := \int_0^t \mathbf{1}_{\{u_i(s) \le n\}} dM_i(s), \qquad i = 1, \cdots, N, \tag{2.1.35}$$

from (2.1.7). In the decomposition $R_\pi = A_\pi + M_\pi$, with A_π as in (2.1.28) and M_π as in (2.1.29), we note that A_π and M_π can be expressed in terms of the components of $\widetilde{A}$ and $\widetilde{M}$, respectively, as

$$A_\pi = \int_0^\cdot \sum_{i=1}^N \pi_i(t) dA_i(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t) d\widetilde{A}_i(t), \quad M_\pi = \int_0^\cdot \sum_{i=1}^N \pi_i(t) dM_i(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t) d\widetilde{M}_i(t) \tag{2.1.36}$$

by analogy with (2.1.33). Also in a manner similar to (2.1.11), we define

$$\widetilde{C}_{i,j} := [\widetilde{M}_i, \widetilde{M}_j] = [\widetilde{R}_i, \widetilde{R}_j], \quad \text{for} \quad 1 \le i, j \le N. \tag{2.1.37}$$

Note the relationship

$$d\widetilde{C}_{i,j}(t) = d[\widetilde{R}_i, \widetilde{R}_j](t) = \mathbf{1}_{\{u_i(t)\le n\}}\mathbf{1}_{\{u_j(t)\le n\}} d[R_i, R_j](t) = \mathbf{1}_{\{u_i(t)\le n\}}\mathbf{1}_{\{u_j(t)\le n\}} dC_{i,j}(t) \tag{2.1.38}$$

between $\widetilde{C}$ and C. We further define a vector-valued process $\widetilde{\alpha} \equiv (\widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_N)$ and a matrix-valued process $\widetilde{c} \equiv (\widetilde{c}_{i,j})_{1\le i,j\le N}$ as

$$\widetilde{\alpha}_i(t) := \mathbf{1}_{\{u_i(t)\le n\}}\alpha_i(t), \qquad i = 1, \cdots, N, \tag{2.1.39}$$

$$\widetilde{c}_{i,j}(t) := \mathbf{1}_{\{u_i(t)\le n\}}\mathbf{1}_{\{u_j(t)\le n\}} c_{i,j}(t), \qquad 1 \le i, j \le N; \tag{2.1.40}$$

then it is straightforward to obtain the relationships

$$\widetilde{A} = \int_0^{\cdot} \widetilde{\alpha}(t) dO(t), \quad \text{and} \quad \widetilde{C} \equiv [\widetilde{M}, \widetilde{M}] = \int_0^{\cdot} \widetilde{c}(t) dO(t) \tag{2.1.41}$$

in accordance with (2.1.10), where the vector-valued and matrix-valued processes $\widetilde{A} \equiv (\widetilde{A}_1, \cdots, \widetilde{A}_N)$ and $\widetilde{C} \equiv (\widetilde{C}_{i,j})_{1\le i,j\le N}$, respectively, are as in (2.1.35), (2.1.37).

The definition of $C_{\pi\rho}$ in (2.1.26) can be also invoked when $\pi, \rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$, but we have also with the help of (2.1.33) and (2.1.37) the alternative representation

$$C_{\pi\rho} = [R_\pi, R_\rho] = \left[\int_0^{\cdot} \sum_{i=1}^N \pi_i(t) d\widetilde{R}_i(t), \int_0^{\cdot} \sum_{j=1}^N \rho_j(t) d\widetilde{R}_j(t) \right] = \int_0^{\cdot} \sum_{i=1}^N \sum_{j=1}^N \pi_i(t)\rho_j(t) d\widetilde{C}_{i,j}(t). \tag{2.1.42}$$

In particular, consider the portfolio π among the top n stocks, defined as

$$\pi(\cdot) := \mathbf{1}_{\{u_i(\cdot)\le n\}} e^i \tag{2.1.43}$$

for a fixed $i = 1, \cdots, N$. This portfolio π invests all wealth in the i-th stock, when this stock ranks among the top n; otherwise, it puts all wealth in the money market. From (2.1.33), the identity

$$R_\pi = \widetilde{R}_i \tag{2.1.44}$$

holds, and we shall use the subscript '$\widetilde{i}$' instead of 'π' to write $X_\pi \equiv X_{\widetilde{i}}$ and

$$C_{\widetilde{i}\rho} \equiv C_{\pi\rho} = \int_0^{\cdot} \sum_{j=1}^{N} \rho_j(t) d\widetilde{C}_{i,j}(t), \qquad \text{as well as} \qquad c_{\widetilde{i}\rho} \equiv c_{\pi\rho} = \sum_{j=1}^{N} \rho_j(t)\widetilde{c}_{i,j}(t), \tag{2.1.45}$$

in order to ease notation for the specific π in (2.1.43). This convention is consistent with the equalities $C_{\widetilde{i},\widetilde{j}} = [\widetilde{R}_i, \widetilde{R}_j] = \widetilde{C}_{i,j}$ for $1 \le i, j \le N$.

It is useful to write succinctly the above relationships in this subsection, between symbols with tilde and corresponding symbols without tilde, in matrix notation. We do this by introducing the predictable matrix-valued process $D \equiv (D_{i,j})_{1 \le i,j \le N}$ with entries

$$D_{i,j}(t) := \begin{cases} \mathbf{1}_{\{u_i(t) \le n\}} & i = j, \\ 0 & i \ne j, \end{cases} \tag{2.1.46}$$

for each $t \ge 0$. Here, we note that $D(t)$ is a diagonal, idempotent matrix, whose (i,i)-th entry is 1 if the i-th stock belongs to the top n stocks at time $t \ge 0$, otherwise it is zero. Because at least $N - n$ diagonal entries of $D(t)$ are zero, $D(\cdot)$ is always singular. Then, any N-dimensional predictable process ν in $\mathcal{T}(n)$ as in Definition 2.1.4, satisfies $D\nu = \nu$; in particular,

$$D\vartheta = \vartheta, \qquad D\pi = \pi, \tag{2.1.47}$$

hold for all $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ and $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ from the conditions (2.1.16) and (2.1.17). Also, the identities (2.1.32), (2.1.35), (2.1.38), (2.1.39), and (2.1.40) can be reformulated as

$$d\widetilde{R}(t) = D(t)dR(t), \quad d\widetilde{A}(t) = D(t)dA(t), \quad d\widetilde{M}(t) = D(t)dM(t), \tag{2.1.48}$$

$$d\widetilde{C}(t) = D(t)dC(t)D(t),$$

as well as

$$\widetilde{\alpha} = D\alpha, \quad \widetilde{c} = DcD. \tag{2.1.49}$$

Moreover, we have another expression of the type (2.1.30) for $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$: using the property (2.1.47), we write

$$\alpha_\pi = \pi'\alpha = \pi' D\alpha = \pi'\widetilde{\alpha}, \qquad \gamma_\pi = \pi'\alpha - \frac{1}{2}\pi' c\pi = \pi'\widetilde{\alpha} - \frac{1}{2}\pi' DcD\pi = \pi'\widetilde{\alpha} - \frac{1}{2}\pi'\widetilde{c}\pi. \tag{2.1.50}$$

We present now the following results regarding the integrability condition with respect to R (or $\widetilde{R}$), which will be used in the next section.

Lemma 2.1.6 (Null portfolio)**.** *For an N-dimensional predictable process $\eta \in \mathcal{T}(n)$, suppose that $\eta'\widetilde{\alpha} = 0$ and $\widetilde{c}\eta = 0$ hold in the $(\mathbb{P} \otimes O)$-a.e. sense.*

Then η is a portfolio, i.e., $\eta \in \mathcal{I}(R) \cap \mathcal{T}(n)$, and the identity $R_\eta = \int_0^\cdot \eta'(t)d\widetilde{R}(t) \equiv 0$ holds. In this case, we call η a null portfolio.

Proof. As $\eta \in \mathcal{T}(n)$, we have $D\eta = \eta$, or $\eta' = \eta' D$. Recalling (2.1.48), (2.1.34), and (2.1.41), we have

$$\begin{aligned}\int_0^\cdot \eta'(t)dR(t) &= \int_0^\cdot \eta'(t)D(t)dR(t) = \int_0^\cdot \eta'(t)d\widetilde{R}(t) \\ &= \int_0^\cdot \eta'(t)d\widetilde{A}(t) + \int_0^\cdot \eta'(t)d\widetilde{M}(t) = \int_0^\cdot \eta'(t)\widetilde{\alpha}(t)dO(t) + \int_0^\cdot \eta'(t)d\widetilde{M}(t).\end{aligned} \tag{2.1.51}$$

The first integral on the right-hand side of (2.1.51) vanishes, thanks to the assumption $\eta'\widetilde{\alpha} = 0$. The second integral $\int_0^\cdot \eta'(t)d\widetilde{M}(t)$ is a continuous local martingale, and has quadratic variation $\int_0^\cdot \eta'(t)\widetilde{c}(t)\eta(t)dO(t)$ from (2.1.41). This quadratic variation also vanishes on account of the assumption $\widetilde{c}\eta = 0$, and the result follows. □

Lemma 2.1.7 (Integrability condition with respect to R)**.** *An N-dimensional predictable vector process $\pi \in \mathcal{T}(n)$ belongs to $\mathcal{I}(R)$, if and only if*

$$\int_0^T \Big(|\pi'(t)\widetilde{\alpha}(t)| + \pi'(t)\widetilde{c}(t)\pi(t) \Big) dO(t) < \infty, \quad \textit{for any} \quad T \geq 0. \tag{2.1.52}$$

Proof. From the assumption $\pi \in \mathcal{T}(n)$, we have $D\pi = \pi$, and $\pi' D = \pi'$. The condition (2.1.12) can be rewritten with the help of (2.1.49) as

$$\int_0^T \Big(|\pi'(t)\alpha(t)| + \pi'(t)c(t)\pi(t) \Big) dO(t) = \int_0^T \Big(|\pi'(t)D(t)\alpha(t)| + \pi'(t)D(t)c(t)D(t)\pi(t) \Big) dO(t)$$
$$= \int_0^T \Big(|\pi'(t)\widetilde{\alpha}(t)| + \pi'(t)\widetilde{c}(t)\pi(t) \Big) dO(t) < \infty.$$

□

2.2 Numéraires and Market Viability

This section presents the fundamental result in the arbitrage theory of an equity market, in open market context. Before we state and prove the result, we explain in succession several necessary concepts.

2.2.1 *Auxiliary Market*

Consider a portfolio $\rho \in \mathcal{I}(R)$ which generates the numéraire X_ρ as in Definition 2.1.5 and (2.1.21), and fix ρ throughout this subsection. We regard this portfolio ρ as a 'baseline', in the sense that want to compare the relative performance of any other portfolio $\pi \in \mathcal{I}(R)$ with respect to ρ, by understanding the relative wealth process

$$X_\pi^\rho := \frac{X_\pi}{X_\rho}. \tag{2.2.1}$$

As the wealth X_π is denominated relative to X_ρ in (2.2.1), we consider an *auxiliary market*, in which all the components of the price vector S in Definition 2.1.1 are denominated in units of X_ρ:

$$S_i^\rho := \frac{S_i}{X_\rho}, \quad i = 1, \cdots, N. \tag{2.2.2}$$

Here, we also consider the money market $S_0 \equiv 1$, with $S_0^\rho := 1/X_\rho$, as we assume that the money market pays and charges zero interest in the introductory part of Section 2.1. Since S_0^ρ is no longer trivial, we will consider the $(N + 1)$-dimensional vector $S^\rho \equiv (S_0^\rho, S_1^\rho, \cdots S_N^\rho)$ as the price process vector in this auxiliary market.

Recalling the notation (2.1.20) and (2.1.26), we define two $(N + 1)$-dimensional vectors of semimartingales $R^\rho \equiv (R_0^\rho, \cdots, R_N^\rho)$, and $\widetilde{R}^\rho \equiv (\widetilde{R}_0^\rho, \cdots, \widetilde{R}_N^\rho)$ with components

$$R_0^\rho := C_{\rho\rho} - R_\rho, \qquad \text{and} \qquad R_i^\rho := R_0^\rho + (R_i - C_{i\rho}), \qquad \text{for} \quad i = 1, \cdots, N. \tag{2.2.3}$$

$$\widetilde{R}_0^\rho := R_0^\rho = C_{\rho\rho} - R_\rho, \qquad \text{and} \qquad \widetilde{R}_i^\rho := \widetilde{R}_0^\rho + (\widetilde{R}_i - C_{\widetilde{i}\rho}), \qquad \text{for} \quad i = 1, \cdots, N. \tag{2.2.4}$$

Proposition 1.29 of [13] shows that the vector R^ρ plays the role of *cumulative return* in the auxiliary market, as the relative wealth process X_π^ρ of (2.2.1) admits the representation

$$X_\pi^\rho = \mathcal{E}(R_\pi^\rho), \qquad \text{where} \qquad R_\pi^\rho := R_{\pi-\rho} - C_{\pi-\rho,\rho} = \int_0^\cdot \sum_{i=0}^N \pi_i(t) dR_i^\rho(t). \tag{2.2.5}$$

Here, we also recall the 'money market proportion' π_0 of a portfolio π in (2.1.25). We present the additional representation in particular for portfolios $\rho, \pi \in \mathcal{I}(R)$ among top n stocks.

Proposition 2.2.1. *For any two portfolios $\rho, \pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks, the process R_π^ρ in* (2.2.5) *admits the additional representation*

$$R_\pi^\rho = \int_0^\cdot \sum_{i=0}^N \pi_i(t) d\widetilde{R}_i^\rho(t). \tag{2.2.6}$$

Proof. Since $\widetilde{R}_0^\rho = R_0^\rho$ in (2.2.3) and (2.2.4), it is enough to show

$$\int_0^\cdot \sum_{i=1}^N \pi_i(t) d\big(R_i - C_{i\rho}\big)(t) = \int_0^\cdot \sum_{i=1}^N \pi_i(t) d\big(\widetilde{R}_i - C_{\widetilde{i\rho}}\big)(t).$$

Thanks to the condition (2.1.17) and the definition (2.1.32), this can be easily checked:

$$\begin{aligned}\int_0^\cdot \sum_{i=1}^N \pi_i(t) d\big(R_i - C_{i\rho}\big)(t) &= \int_0^\cdot \sum_{i=1}^N \pi_i(t) \mathbf{1}_{\{u_i(t)\le n\}} d\big(R_i - C_{i\rho}\big)(t) \\ &= \int_0^\cdot \sum_{i=1}^N \pi_i(t) d\big(\widetilde{R}_i - C_{\widetilde{i\rho}}\big)(t)\end{aligned}$$

where, in the last equality, we used the string of identities

$$\mathbf{1}_{\{u_i(t)\le n\}} dC_{i\rho}(t) = \mathbf{1}_{\{u_i(t)\le n\}} d[R_i, R_\rho](t) = d[\widetilde{R}_i, R_\rho](t) = dC_{\widetilde{i\rho}}. \tag{2.2.7}$$

□

In the special case $\pi \equiv e^i$, that is, when the portfolio π invests all wealth in the i-th stock at all times, the relative wealth process X_π^ρ and its stochastic logarithm R_π^ρ in (2.2.1), (2.2.5) become

$$X_\pi^\rho = \frac{S_i}{X_\rho} = S_i^\rho, \qquad R_\pi^\rho = R_i^\rho,$$

and Proposition 2.2.1 yields

$$S_i^\rho = \mathcal{E}(R_i^\rho) \tag{2.2.8}$$

for any given $i = 1, \cdots, N$. Therefore, the component R_i^ρ of (2.2.3) is the stochastic logarithm of the i-th component of the price vector S^ρ in the auxiliary market, and the vector R^ρ plays the role of cumulative returns in the auxiliary market.

By analogy with (2.1.22), we also have

$$\frac{dX_\pi^\rho(t)}{X_\pi^\rho(t)} = dR_\pi^\rho(t) = \sum_{i=0}^{N} \pi_i(t) dR_i^\rho(t) = \sum_{i=0}^{N} \pi_i(t) \frac{dS_i^\rho(t)}{S_i^\rho(t)}, \quad X_\pi^\rho(0) = 1, \tag{2.2.9}$$

for ρ, π in $\mathcal{I}(R)$, from (2.2.5), (2.2.8). It is very important that the summation in (2.2.9) should include the index $i = 0$, as indeed it does, in contrast to the summation in (2.1.22).

2.2.2 *Supermartingale Numéraire and Local Martingale Numéraire*

We introduce now the notions of supermartingale numéraire and local martingale numéraire.

Definition 2.2.2 (Supermartingale numéraire and local martingale numéraire)**.** A given portfolio $\rho \in \mathcal{I}(R)$ is called *supermatingale numéraire portfolio (local martingale numéraire portfolio) in the whole market*, if the relative wealth process $X_\pi^\rho = X_\pi / X_\rho$ of (2.2.1) is a supermartingale (local martingale) for every portfolio $\pi \in \mathcal{I}(R)$ in the market. In this case, the wealth process X_ρ is called a *supermartingale numéraire (local martingale numéraire, respectively) in the whole market*.

Similarly, a given portfolio $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks is called *supermatingale numéraire portfolio (local martingale numéraire portfolio) among the top n stocks*, if the relative wealth process X_π^ρ is a supermartingale (local martingale) for every portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks. In this case, the wealth process X_ρ is called *supermartingale numéraire (local martingale numéraire, respectively) among the top n stocks*.

By Fatou's lemma, every nonnegative local martingale is a supermartingale; thus, every local martingale numéraire is in particular a supermatingale numéraire. We also have the following uniqueness result for supermartingale (local martingale) numéraires (respectively, among the top n stocks).

Lemma 2.2.3. *There is a unique (modulo null portfolios) supermartingale (local martingale) numéraire portfolio in the entire market (respectively, among the top n stocks).*

Proof. Suppose that there are two local martingale (or two supermartingale) numéraire portfolios ρ and ν with the same initial wealth $X_\rho(0) = X_\nu(0)$. Then, the relative wealth process X_ρ/X_ν and its reciprocal X_ν/X_ρ are positive supermartingales. From the Doob-Meyer decomposition of semimartingales, it is easy to show that a continuous, positive supermartingale Y is almost everywhere constant, if its reciprocal is also a supermartingale. Thus, $X_\rho \equiv X_\nu$ almost everywhere, and the two portfolios ρ and ν generate the same wealth process. □

It can be shown that the supermartingale numéraire is actually the local martingale numéraire, thus the two numéraires are equivalent, in the whole market where no constraint is imposed on portfolios. This is Proposition 2.4 of [13], which we repeat here.

Proposition 2.2.4. *For a portfolio $\rho \in \mathcal{I}(R)$, the following statements are equivalent:*

(1) *ρ is a supermartingale numéraire portfolio in the whole market.*

(2) *ρ is a local martingale numéraire portfolio in the whole market.*

(3) *The equality $A_i = C_{i\rho}$ holds for all $i = 1, \cdots, N$.*

The statement (3) gives a very simple structural condition, derived from the cumulative return process of the market, which characterizes this equivalence. It is no surprise that the result also holds for portfolios among the top n stocks; but in this case, the cumulative return process vector R in (2.1.5) should be replaced by $\widetilde{R}$ of (2.1.32).

Proposition 2.2.5. *For a portfolio $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$, the following statements are equivalent:*

$\widetilde{(1)}$ *ρ is a supermartingale numéraire portfolio among the top n stocks.*

$\widetilde{(2)}$ *ρ is a local martingale numéraire portfolio among the top n stocks.*

$\widetilde{(3)}$ *The equality $\widetilde{A}_i = C_{\widetilde{i}\rho}$ holds for all $i = 1, \cdots, N$.*

Proof. We first assume statement $\widetilde{(3)}$, which is equivalent to the requirement that $\widetilde{R}_i - C_{\widetilde{i}\rho} = \widetilde{M}_i$ is a local martingale for all $i = 1, \cdots, N$ from (2.1.34). Recalling the notation of (2.2.4), (2.1.26) with the identities (2.1.17), (2.1.33) and (2.2.7), we obtain that the process

$$\begin{aligned}
\widetilde{R}_0^\rho = C_{\rho\rho} - R_\rho &= \int_0^\cdot \sum_{i=1}^N \rho_i(t) dC_{i\rho}(t) - \int_0^\cdot \sum_{i=1}^N \rho_i(t) d\widetilde{R}_i(t) \\
&= \int_0^\cdot \sum_{i=1}^N \rho_i(t) \mathbf{1}_{\{u_i(t)\le n\}} dC_{i\rho}(t) - \int_0^\cdot \sum_{i=1}^N \rho_i(t) d\widetilde{R}_i(t) \\
&= \int_0^\cdot \sum_{i=1}^N \rho_i(t) dC_{\widetilde{i}\rho}(t) - \int_0^\cdot \sum_{i=1}^N \rho_i(t) d\widetilde{R}_i(t) = -\int_0^\cdot \sum_{i=1}^N \rho_i(t) d\widetilde{M}_i(t)
\end{aligned} \tag{2.2.10}$$

is then also a local martingale. This in turn implies that all the components $\widetilde{R}_i^\rho = \widetilde{R}_0^\rho + (\widetilde{R}_i - C_{\widetilde{i}\rho})$ for $i = 1, \cdots, N$ in (2.2.4) are local martingales as well. Moreover, from Proposition 2.2.1, the processes R_π^ρ and X_π^ρ are also local martingales for every portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks, so the implication $\widetilde{(3)} \Rightarrow \widetilde{(2)}$ has been proved.

Since statement $\widetilde{(2)}$ trivially implies statement $\widetilde{(1)}$, it remains to establish the implication $\widetilde{(1)} \Rightarrow \widetilde{(3)}$. Assuming statement $\widetilde{(1)}$, we first fix any i in $\{1, \cdots, N\}$, consider a specific portfolio π among the top n stocks defined as in (2.1.43), and recall the notation $X_\pi \equiv X_{\widetilde{i}}$ as well as $R_\pi \equiv \widetilde{R}_i$. Then, the processes

$$X^\rho_{\rho+\widetilde{i}} = \frac{X_{\rho+\widetilde{i}}}{X_\rho}, \qquad X^\rho_{\rho-\widetilde{i}} = \frac{X_{\rho-\widetilde{i}}}{X_\rho},$$

are supermartingales. In view of Proposition 2.2.1 along with (2.1.44), all processes

$$\mathcal{L}(X^\rho_{\rho+\widetilde{i}}) = R^\rho_{\rho+\widetilde{i}} = \widetilde{R}_i - C_{\widetilde{i}\rho}, \qquad \mathcal{L}(X^\rho_{\rho-\widetilde{i}}) = R^\rho_{\rho-\widetilde{i}} = -(\widetilde{R}_i - C_{\widetilde{i}\rho}),$$

are local supermartingales, implying that $\widetilde{R}_i - C_{\widetilde{i}\rho}$ is a local martingale. Since $i \in \{1, \cdots, N\}$ can be chosen arbitrarily, we arrive at statement $\widetilde{(3)}$. □

Remark 2.2.6 (Representation of wealth relative to the supermartingale numéraire). When $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is a supermartingale numéraire portfolio among the top n stocks, statement $\widetilde{(3)}$ of

Proposition 2.2.5 implies that $\widetilde{R}_i - C_{\widetilde{i\rho}} = \widetilde{M}_i$ is a local martingale for all $i = 1, \cdots, N$.

Then, Proposition 2.2.1 with the notation (2.2.4) yields the following representation of the relative wealth process X^{ρ}_{π} for any portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks, namely,

$$\begin{aligned} X^{\rho}_{\pi} &= \mathcal{E}\left(\int_0^{\cdot} \sum_{i=0}^{N} \pi_i(t) d\widetilde{R}^{\rho}_i(t)\right) = \mathcal{E}\left(\widetilde{R}^{\rho}_0 + \int_0^{\cdot} \sum_{i=1}^{N} \pi_i(t) d\widetilde{M}_i(t)\right) \\ &= \mathcal{E}\left(\int_0^{\cdot} \sum_{i=1}^{N} \big(\pi_i(t) - \rho_i(t)\big) d\widetilde{M}_i(t)\right) = 1 + \int_0^{\cdot} X^{\rho}_{\pi}(t) \sum_{i=1}^{N} \big(\pi_i(t) - \rho_i(t)\big) d\widetilde{M}_i(t), \end{aligned}$$

where the second-to-last equality is from (2.2.10). Thus, the relative wealth process X^{ρ}_{π} is a stochastic integral with respect to the local martingale vector $\widetilde{M}$, defined in (2.1.35).

Remark 2.2.7 (A reformulation of statement $\widetilde{(3)}$). The statement $\widetilde{(3)}$ of Proposition 2.2.5 can be reformulated using the 'rate processes' $\widetilde{\alpha}$, $\widetilde{c}$ of (2.1.41), namely,

$$\int_0^{\cdot} \widetilde{\alpha}_i(t) dO(t) = \widetilde{A}_i = C_{\widetilde{i\rho}} = \int_0^{\cdot} \sum_{j=1}^{N} \rho_j(t) d\widetilde{C}_{i,j}(t) = \int_0^{\cdot} \sum_{j=1}^{N} \rho_j(t) \widetilde{c}_{i,j}(t) dO(t),$$

with the help of (2.1.45). Thus, we have the following statement $\widetilde{(3)}$ of Proposition 2.2.5, namely

$$\widetilde{(3)}' \qquad \widetilde{\alpha} = \widetilde{c}\rho, \qquad (\mathbb{P} \otimes O) - \text{a.e.} \tag{2.2.11}$$

in matrix notation. In the same manner, the statement (3) of Proposition 2.2.4 also has the equivalent formulation:

$$(3)' \qquad \alpha = c\rho, \qquad (\mathbb{P} \otimes O) - \text{a.e.} \tag{2.2.12}$$

2.2.3 *Structural Conditions*

In this subsection, we present yet another equivalent requirement for statement $\widetilde{(3)}$ of Proposition 2.2.5. This new formulation does not involve ρ, the supermartingale numéraire portfolio among the top n stocks, and is in the form of what we call 'structural conditions'. First, we note that $\widetilde{c}$ of (2.1.49) is a singular symmetric matrix, thus not invertible, from the fact that D is singular. Before proceeding to the next result, we need the following definition of 'pseudo-inverse' for the matrix-valued process $\widetilde{c}$ of (2.1.40):

$$\widetilde{c}^{\dagger} := \lim_{m\to\infty} \left(\left(\widetilde{c} + \frac{1}{m} I\right)^{-2} \widetilde{c} \right), \tag{2.2.13}$$

where I is the identity operator on $\mathbb{R}^N$. This process $\widetilde{c}^{\dagger}$ will play the role of 'pseudo-inverse' for $\widetilde{c}$, because it is easily checked that

1. $\widetilde{c}^{\dagger}$ is the inverse of $\widetilde{c}$ when restricted on **range**($\widetilde{c}$),
2. $\widetilde{c}\widetilde{c}^{\dagger}$ coincides with the projection operator of $\mathbb{R}^N$ onto **range**($\widetilde{c}$),
3. $\widetilde{c}^{\dagger}$ is predictable, since matrix inversion is a continuous operation when restricted to strictly positive-definite matrices.

We are now ready to present the structural conditions.

Proposition 2.2.8. *The existence of the supermartingale numéraire portfolio among the top n stocks, is equivalent to the conjunction of the conditions:*

$$(i) \quad \widetilde{\alpha} \in \boldsymbol{range}(\widetilde{c}), \qquad (\mathbb{P} \otimes O) - a.e., \tag{2.2.14}$$

$$(ii) \quad \int_0^T \widetilde{\alpha}'(t)\widetilde{c}^{\dagger}(t)\widetilde{\alpha}(t)dO(t) < \infty, \quad \textit{for any} \quad T \geq 0. \tag{2.2.15}$$

Proof. First, we assume that the supermartingale numéraire portfolio ρ among the top n stocks exists; then, from statement *(3')* of (2.2.11), the identity $\widetilde{\alpha} = \widetilde{c}\rho$ holds. The condition *(i)* follows immediately, and we obtain $\widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha} = \widetilde{\alpha}$ from the property (b) above. This also implies that the set $\{\widetilde{\alpha} \in \textbf{range}(\widetilde{c})\}$ is predictable. We set the predictable process

$$\nu := \widetilde{c}^{\dagger}\widetilde{\alpha}, \tag{2.2.16}$$

which is **range**$(\widetilde{c})$-valued in the $(\mathbb{P}\otimes O)$-a.e. sense, and satisfies $\widetilde{\alpha} = \widetilde{c}\,\nu$. Then, every supermartingale numéraire portfolio among the top n stocks should be of the form

$$\rho = \nu + \eta = \widetilde{c}^{\,\dagger}\widetilde{\alpha} + \eta, \tag{2.2.17}$$

for a suitable predictable process η which is in **ker**$(\widetilde{c})$, the kernel of $\widetilde{c}$, $(\mathbb{P}\otimes O)$-a.e. We have $\widetilde{c}\eta = 0$ and $\eta'\widetilde{\alpha} = 0$, thus η is a null portfolio in the sense of Lemma 2.1.6.

On the other hand, the assumption that the supermartingale numéraire portfolio among the top n stocks exists, implies that some N-dimensional process of the form $\rho = \widetilde{c}^{\,\dagger}\widetilde{\alpha}+\eta$ of the form (2.2.17) should be a portfolio, i.e., R-integrable. The integrability condition (2.1.52) in Lemma 2.1.7 with the observation

$$\rho'\widetilde{c}\rho = \rho'\widetilde{c}\,(\widetilde{c}^{\,\dagger}\widetilde{\alpha} + \eta) = \rho'\widetilde{\alpha} = \widetilde{\alpha}'\rho = \widetilde{\alpha}'\widetilde{c}^{\,\dagger}\widetilde{\alpha},$$

gives the condition (ii).

We next assume the conjunction of conditions *(i), (ii)* and find the supermartingale numéraire portfolio among the top n stocks. We define the two predictable processes

$$\nu := \widetilde{c}^{\,\dagger}\widetilde{\alpha}, \qquad \text{and} \qquad \rho := D\nu = D\widetilde{c}^{\,\dagger}\widetilde{\alpha}, \tag{2.2.18}$$

and claim that ρ is the supermartingale numéraire portfolio among the top n stocks. Thanks to the condition *(i)*, we obtain the identity $\widetilde{c}\nu = \widetilde{c}\,\widetilde{c}^{\,\dagger}\widetilde{\alpha} = \widetilde{\alpha}$, $(\mathbb{P}\otimes O)$-a.e. Then, we note the series of identities

$$\nu'\widetilde{c}\nu = \nu'\widetilde{\alpha} = \widetilde{\alpha}'\nu = \widetilde{\alpha}'\widetilde{c}^{\,\dagger}\widetilde{\alpha}, \qquad (\mathbb{P}\otimes O) - \text{a.e.},$$

as well as

$$\rho'\widetilde{c}\rho = \nu' D\widetilde{c}D\nu = \nu'\widetilde{c}\nu = \widetilde{\alpha}'\widetilde{c}^{\,\dagger}\widetilde{\alpha}, \qquad \rho'\widetilde{\alpha} = \nu' D\widetilde{\alpha} = \nu'\widetilde{\alpha} = \widetilde{\alpha}'\widetilde{c}^{\,\dagger}\widetilde{\alpha}, \qquad (\mathbb{P}\otimes O) - \text{a.e.} \tag{2.2.19}$$

Here, we used the identities $D\widetilde{\alpha} = \widetilde{\alpha}$, and $D\widetilde{c}D = \widetilde{c}$ which can be obtained from (2.1.49).

Combining equations of (2.2.19) with the condition *(ii)* yields the integrability condition (2.1.52) for $\rho \equiv \pi$ in Lemma 2.1.7, i.e., $\rho \in \mathcal{I}(R)$. Also, from the construction (2.2.18), we have $D\rho = DD\nu = D\nu = \rho$, thus $\rho \in \mathcal{T}(n)$. Therefore, we have shown that ρ is a portfolio among the top n stocks, i.e., $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$.

Furthermore, we deduce

$$\widetilde{c}\rho = \widetilde{c}D\nu = \widetilde{c}\nu = \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha} = \widetilde{\alpha}, \qquad (\mathbb{P} \otimes O) - \text{a.e.}, \tag{2.2.20}$$

where the second equation uses the identity $\widetilde{c}D = \widetilde{c}$, a consequence of (2.1.49) and of the fact that D is idempotent. Thus, we have obtained the condition (2.2.11), which is equivalent to statement $\widetilde{(3)}$ of Proposition 2.2.5, and ρ is indeed the supermartingale numéraire portfolio among the top n stocks. □

The conjunction of the two conditions in Proposition 2.2.8 can be formulated as one equivalent condition, as follows. We first recall the 'growth rate' γ_π of the portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ among the top n stocks in (2.1.50). We denote $\mathbb{R}^N \cap \mathcal{T}(n)$ the collection of elements in $\mathbb{R}^N$ such that at most n components are nonzero; then $\pi(t)$ takes values in $\mathbb{R}^N \cap \mathcal{T}(n)$ for each $t \geq 0$, by the property (2.1.15). Let us define the $[0, \infty]$-valued process

$$\widetilde{g} := \sup_{p \in \mathbb{R}^N} \Big(p'\widetilde{\alpha} - \frac{1}{2} p'\widetilde{c}p \Big) = \sup_{p \in \mathbb{R}^N \cap \mathcal{T}(n)} \Big(p'\widetilde{\alpha} - \frac{1}{2} p'\widetilde{c}p \Big). \tag{2.2.21}$$

The last equality follows because of the identities

$$p'\widetilde{\alpha} - \frac{1}{2}p'\widetilde{c}p = p'D\widetilde{\alpha} - \frac{1}{2}p'D\widetilde{c}Dp = \widetilde{p}'\widetilde{\alpha} - \frac{1}{2}\widetilde{p}'\widetilde{c}\widetilde{p},$$

valid for any $p \in \mathbb{R}^N$, where $\widetilde{p} := Dp \in \mathbb{R}^N \cap \mathcal{T}(n)$, by recalling the properties $\widetilde{\alpha} = D\widetilde{\alpha}$ and $\widetilde{c} = D\widetilde{c}D$ which can be deduced from (2.1.49).

This process $\widetilde{g}$ in (2.2.21) can be interpreted as the *maximal growth rate* achievable for all portfolios among the top n stocks. Note that $\widetilde{g}$ is predictable, because the supremum can be restricted over a countable, dense subset of $\mathbb{R}^N$. We then easily rewrite the process $\widetilde{g}$ in the form

$$\widetilde{g} = \frac{1}{2}\big(\widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha}\big)\mathbf{1}_{\{\widetilde{\alpha}\in\mathbf{range}(\widetilde{c})\}} + \infty\mathbf{1}_{\{\widetilde{\alpha}\notin\mathbf{range}(\widetilde{c})\}}, \tag{2.2.22}$$

and the supremum of (2.2.21) is attained if and only if $\widetilde{g} < \infty$, at $p \equiv \rho := D\widetilde{c}^{\dagger}\widetilde{\alpha}$ as in (2.2.18) and (2.2.19). Then, the conjunction of conditions *(i) + (ii)* in Proposition 2.2.8 becomes simply

$$\widetilde{G}(T) < \infty, \qquad \text{for all} \quad T \geq 0, \tag{2.2.23}$$

where $\widetilde{G}$ is an adapted nondecreasing process

$$\widetilde{G} := \int_0^{\cdot} \widetilde{g}(t)dO(t). \tag{2.2.24}$$

We call this $\widetilde{G}$ the *aggregate maximal growth from portfolios among the top n stocks*; and say that the market consisting of the top n stocks has *locally finite growth*, if the process $\widetilde{G}$ satisfies the condition (2.2.23). We formalize this argument into the next proposition.

Proposition 2.2.9. *The requirement of* (2.2.23) *of locally finite growth among the top n stocks, is equivalent to the conjunction of the two conditions (i) + (ii) of Proposition 2.2.8, thus sufficient and necessary for a supermartinagle numéraire portfolio among the top n stocks to exist. In this case, we have*

$$\widetilde{G} = \Gamma_{\rho},$$

where ρ is a supermartingale numéraire portfolio among the top n stocks.

We present the following results which will be used later.

Lemma 2.2.10. *Suppose the market has locally finite growth among the top n stocks, i.e., that* (2.2.23) *holds, and let ρ be the supermartingale numéraire portfolio among the top n stocks. Re-*

calling (2.1.50)*,* (2.1.31)*,* (2.1.26)*, and* (2.1.36)*, we have*

$$\widetilde{G} = \Gamma_\rho = \frac{1}{2} C_{\rho\rho},$$

as well as the representation

$$\frac{1}{X_\rho} = \mathcal{E}(-M_\rho). \tag{2.2.25}$$

Proof. As with (2.2.18) in the proof of Proposition 2.2.8, the supermartingale numéraire portfolio ρ among the top n stocks is of the form $D\widetilde{c}^\dagger\widetilde{\alpha}$. With (2.2.19), the claim $\widetilde{G} = \Gamma_\rho$ is easily obtained. Furthermore, again by (2.2.19) with (2.1.50), we have

$$\gamma_\rho = \rho'\widetilde{\alpha} - \frac{1}{2}\rho'\widetilde{c}\rho = \frac{1}{2}\rho'\widetilde{c}\rho = \frac{1}{2}c_{\rho\rho} = \widetilde{g} \tag{2.2.26}$$

thus $\Gamma_\rho = \frac{1}{2}C_{\rho\rho}$, as well as $A_\rho = C_{\rho\rho}$. We then write (2.2.5), (2.2.6) with $\pi \equiv (0, \cdots, 0) \in \mathcal{I}(R) \cap \mathcal{T}(n)$:

$$\frac{1}{X_\rho} = X_\pi^\rho = \mathcal{E}(\widetilde{R}_0^\rho) = \mathcal{E}(C_{\rho\rho} - R_\rho) = \mathcal{E}(C_{\rho\rho} - A_\rho - M_\rho) = \mathcal{E}(-M_\rho).$$

□

Lemma 2.2.11. *Let ρ be the supermartingale numéraire portfolio among the top n stocks. For any investment strategy $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ among the top n stocks, and for any initial capital $x \geq 0$, let us recall the wealth process $X \equiv X(\cdot; x, \vartheta)$ generated by ϑ and x in the manner of* (2.1.13)*. Then there exists a process $\eta = (\eta_1, \cdots, \eta_N) \in \mathcal{I}(\widetilde{M}) \cap \mathcal{T}(n)$, such that*

$$\frac{X}{X_\rho} = x + \int_0^\cdot \sum_{i=1}^N \eta_i(t) d\widetilde{M}_i(t). \tag{2.2.27}$$

Conversely, for any $x \geq 0$ and $\eta \in \mathcal{I}(M) \cap \mathcal{T}(n)$, there exists a process $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ such that (2.2.27) *holds.*

Proof. From (2.2.25) and (2.1.36), we have $d\big(1/X_\rho(t)\big) = \big(1/X_\rho(t)\big)\sum_{i=1}^N \big(-\rho_i(t)\big)d\widetilde{M}_i(t)$, as well as the dynamics

$$dX(t) = \sum_{i=1}^N \vartheta_i(t)dS_i(t) = \sum_{i=1}^N \vartheta_i(t)S_i(t)d\widetilde{R}_i(t) = \sum_{i=1}^N \vartheta_i(t)S_i(t)\big(d\widetilde{A}_i(t) + d\widetilde{M}_i(t)\big),$$

from (2.1.34). Combining two equations via Itô's formula, we obtain

$$\begin{aligned} d\big(X(t)/X_\rho(t)\big) = &\sum_{i=1}^N \frac{\vartheta_i(t)S_i(t)}{X_\rho(t)}\big(d\widetilde{A}_i(t) + d\widetilde{M}_i(t)\big) + \frac{X(t)}{X_\rho(t)}\sum_{i=1}^N \big(-\rho_i(t)\big)d\widetilde{M}_i(t) \\ &+ \sum_{i=1}^N\sum_{j=1}^N \frac{\vartheta_i(t)S_i(t)}{X_\rho(t)}\big(-\rho_j(t)\big)d[\widetilde{M}_i, \widetilde{M}_j](t). \end{aligned}$$

Here, the finite variation terms vanish because of the relationship $d\widetilde{A}_i(t) = \sum_{j=1}^N \rho_j(t)d[\widetilde{M}_i, \widetilde{M}_j](t)$ for $i = 1, \cdots, N$, which is valid on the strength of condition $\widetilde{(3)}$ in Proposition 2.2.5. Thus, by setting

$$\eta_i(t) := \frac{\vartheta_i(t)S_i(t) - X(t)\rho_i(t)}{X_\rho(t)}, \quad i = 1, \cdots, N,$$

it is straightforward to check $\eta \in \mathcal{T}(n)$, and the result follows. The converse can be easily shown by reversing the above procedure. □

2.2.4 *Local Martingale Deflator and Market Viability*

We present a few more concepts and state the main result of this section.

Definition 2.2.12 (Local martingale deflator)**.** We call an adapted, right-continuous and left-limited (RCLL) process Y, a *local martingale deflator among the top n stocks*, if it satisfies $Y(0) = 1$, $Y > 0$, and the process YX is a local martingale for every $X \in \mathcal{X}^n$ of Definition 2.1.4. We denote by $\mathcal{Y}^n$ the collection of all local martingale deflators among the top n stocks.

Since $X \equiv 1 \in \mathcal{X}^n$, every deflator $Y \in \mathcal{Y}^n$ is in particular local martingale.

Definition 2.2.13 (cumulative withdrawal stream)**.** We denote by $\mathcal{K}$ the collection of all nondecreasing, adapted and right-continuous processes K with $K(0) = 0$. Any element K of $\mathcal{K}$ is called *cumulative withdrawal process*, and $K(t)$ represents for the cumulative capital withdrawn up to time $t \geq 0$; actual withdrawals in each infinitesimal interval $(t, t + dt]$ are represented as $dK(t)$. We say that $K \in \mathcal{K}$ is nonzero, if $\mathbb{P}\big(K(\infty) > 0\big) > 0$.

For $x \geq 0$, $\vartheta \in \mathcal{I}(S)$ or $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$, the wealth process $X(\cdot; x, \vartheta)$ defined in (2.1.13) is said to *finance* a given cumulative withdrawal process $K \in \mathcal{K}$, if $X \geq K$ holds. In this case, we say the process K is *financeable from the initial capital* $x \geq 0$ *with the investment strategy* ϑ.

We denote by $\mathcal{K}(x)$, $\mathcal{K}^n(x)$ the subset of $\mathcal{K}$ consisting of cumulative capital withdrawal processes financeable from initial capital x; namely:

$$\mathcal{K}(x) := \{K \in \mathcal{K} \mid \exists\, \vartheta \in \mathcal{I}(S) \text{ such that } X(\cdot; x, \vartheta) \geq K\},$$
$$\mathcal{K}^n(x) := \{K \in \mathcal{K} \mid \exists\, \vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n) \text{ such that } X(\cdot; x, \vartheta) \geq K\},$$

We introduce also the collection of cumulative withdrawal processes in $\mathcal{K}$ which can be financed starting from any positive initial capital:

$$\mathcal{K}(0+) := \bigcap_{x>0} \mathcal{K}(x) \subset \mathcal{K}, \qquad \mathcal{K}^n(0+) := \bigcap_{x>0} \mathcal{K}^n(x) \subset \mathcal{K}.$$

Definition 2.2.14 (Superhedging capital)**.** For any cumulative withdrawal process $K \in \mathcal{K}$, we call the quantities

$$x(K) := \inf\{x \geq 0 \mid K \in \mathcal{K}(x)\} = \inf\{x \geq 0 \mid \exists\, \vartheta \in \mathcal{I}(S) \text{ such that } X(\cdot; x, \vartheta) \geq K\},$$
$$x^n(K) := \inf\{x \geq 0 \mid K \in \mathcal{K}^n(x)\} = \inf\{x \geq 0 \mid \exists\, \vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n) \text{ such that } X(\cdot; x, \vartheta) \geq K\} \tag{2.2.28}$$

the *superhedging capital* associated with the withdrawal stream K in the entire market, and in the market consisting of the top n stocks, respectively. We follow here the standard convention that the infimum of an empty set is equal to infinity.

Lemma 2.2.15. *Suppose that $\mathcal{Y}^n$ is nonempty. For a fixed cumulative withdrawal process $K \in \mathcal{K}$, we assume that it is financeable from the initial capital $x \geq 0$ with investment strategy $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$, i.e.,*

$$X \equiv X(\cdot; x, \vartheta) = x + \int_0^{\cdot} \sum_{i=1}^{N} \vartheta_i(t) dS_i(t) \geq K.$$

Then, the process

$$Y(X - K) + \int_0^{\cdot} Y(t-) dK(t)$$

is a nonnegative local martingale, thus also a supermartingale, for every local martingale deflator $Y \in \mathcal{Y}^n$ among the top n stocks. In particular, $Y(X - K)$ is nonnegative supermartingale, for every $Y \in \mathcal{Y}^n$. Furthermore, for the quantity $x^n(K)$ of (2.2.28) *we have the inequality*

$$x^n(K) \geq \sup_{Y \in \mathcal{Y}^n} \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\infty} Y(t-) dK(t)\Big]. \tag{2.2.29}$$

Proof. For every $Y \in \mathcal{Y}^n$, integration by parts gives

$$Y(X - K) = YX - \int_0^{\cdot} Y(t-) dK(t) - \int_0^{\cdot} K(t-) dY(t),$$

thus

$$Y(X - K) + \int_0^{\cdot} Y(t-) dK(t) = YX - \int_0^{\cdot} K(t-) dY(t). \tag{2.2.30}$$

Both terms on the right-hand side of (2.2.30) are local martingales, and the terms on the left hand side of (2.2.30) are nonnegative; thus the first claim follows. Also, the process $\int_0^{\cdot} Y(t-) dK(t)$ is nondecreasing, therefore $Y(X - K)$ is nonnegative supermartingale. We denote the left-hand side of (2.2.30) by $Q := Y(X - K) + \int_0^{\cdot} Y(t-) dK(t)$, then we obtain

$$Q(0) = x \geq \mathbb{E}^{\mathbb{P}}\Big[Q(\infty)\Big] \geq \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\infty} Y(t-) dK(t)\Big].$$

By taking the supremum over $Y \in \mathcal{Y}^n$ and then the infimum over the initial capital $x \geq 0$, the last claim follows. $\square$

Definition 2.2.16 (Viability)**.** We say that the entire market is *viable* if, whenever $x(K) = 0$ holds for some cumulative withdrawal process $K \in \mathcal{K}$, we have $K \equiv 0$.

In the same manner, we say the market consisting of the top n stocks is *viable*, if whenever $x^n(K) = 0$ holds for some cumulative withdrawal process $K \in \mathcal{K}$, we have $K \equiv 0$.

The viability of the market consisting of the top n stocks, is actually equivalent to the identity

$$\mathcal{K}^n(0+) = \{0\};$$

whereas the failure of such viability implies the strict inclusion $\mathcal{K}^n(0+) \supset \{0\}$. When the viability of the market consisting of the top n stocks fails, there exists a nonzero cumulative withdrawal process $K \in \mathcal{K}$ which is financeable from any initial capital $x > 0$, no matter how minuscule; or equivalently, there exists an investment strategy $\vartheta_m \in \mathcal{I}(R) \cap \mathcal{T}(n)$ for each $m \in \mathbb{N}$, such that

$$X(\cdot; \frac{1}{m}, \vartheta_m) \geq K.$$

We further present the following lemma; it can be proven in the same manner as Exercise 2.22 of [13].

Lemma 2.2.17. *The market consisting of the top n stocks fails to be viable if, and only if, there exist a real number $T \geq 0$ and a nonnegative $\mathcal{F}(T)$-measurable random variable h with $\mathbb{P}[h > 0] > 0$ such that for every $m \in \mathbb{N}$, there exists an $X^m \in \mathcal{X}^n$ with $X^m(T) \geq mh$.*

The following result presents another equivalent characterization of viability for the market consisting of the top n stocks.

Proposition 2.2.18 (Boundedness in probability)**.** *The market consisting of the top n stocks is viable if, and only if,*

$$\lim_{m\to\infty} \sup_{X\in\mathcal{X}^n} \mathbb{P}[X(T) > m] = 0, \qquad \forall\, T \geq 0. \tag{2.2.31}$$

Proof. We first assume that the market consisting of the top n stocks is not viable. Then, from Lemma 2.2.17, there exist a real number $T \geq 0$, a nonnegative $\mathcal{F}(T)$-measurable random variable h with $\mathbb{P}[h > 0] > 0$, and a sequence $(X^m)_{m\in\mathbb{N}}$ of wealth processes $X^m \in \mathcal{X}^n$ satisfying $X^m(T) \geq mh$. Pick $\epsilon > 0$ sufficiently small, so that $\mathbb{P}[h > \epsilon] > \epsilon$ holds. We then have

$$\liminf_{m\to\infty} \mathbb{P}[X^m(T) > \epsilon m] \geq \liminf_{m\to\infty} \mathbb{P}[X^m(T) > mh, h > \epsilon] \geq \epsilon,$$

thus the condition (2.2.31) is violated.

Conversely, we assume that for some $T \geq 0$, there exist $\epsilon > 0$ and a sequence $(X^m)_{m\in\mathbb{N}} \subset \mathcal{X}^n$ such that $\mathbb{P}[X^m(T) > m2^m] > \epsilon$ hold for all $m \in \mathbb{N}$. Consider the set

$$H := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{X^k(T) > k2^k\right\} \in \mathcal{F}(T),$$

and note that $\mathbb{P}(H) \geq \epsilon$. For every $m \in \mathbb{N}$, the inclusion

$$H \subseteq \bigcup_{k=m+1}^{\infty} \{X^k(T) > k2^k\}$$

holds, so there exists a sufficiently large number $K_m > m$ such that the set

$$H_m := H \cap \Big(\bigcup_{k=m+1}^{K_m} \{X^k(T) > k2^k\}\Big) \in \mathcal{F}(T)$$

satisfies $\mathbb{P}[H \setminus H_m] \leq \frac{\mathbb{P}[H]}{2^{m+1}}$. Then, the countable intersection

$$E := \bigcap_{m=1}^{\infty} H_m \in \mathcal{F}(T)$$

is a subset of H, and we have

$$\mathbb{P}[H \setminus E] = \mathbb{P}\Big[\bigcup_{m=1}^{\infty}(H \setminus H_m)\Big] \leq \sum_{m=1}^{\infty} \frac{\mathbb{P}[H]}{2^{m+1}} = \frac{\mathbb{P}[H]}{2},$$

thus, $\mathbb{P}[E] \geq \frac{\mathbb{P}[H]}{2}$ and $\mathbb{P}[E] \geq \frac{\epsilon}{2} > 0$. Let us define a sequence of numéraires $(\Xi^m)_{m\in\mathbb{N}}$

$$\Xi^m := \sum_{k=m+1}^{K_m} 2^{-(k-m)} X^k, \qquad \text{for each } m \in \mathbb{N},$$

and it is straightforward that $\Xi^m \in \mathcal{X}^n$, as all $X^k \in \mathcal{X}^n$ for $k \in \mathbb{N}$. Furthermore, for every $m \in \mathbb{N}$, we have $E \subseteq H_m \subseteq \{\Xi^m(T) > m\}$, from which $\Xi^m(T) \geq m\mathbf{1}_E$ follows. Set $h := \mathbf{1}_E \in \mathcal{F}(T)$, then

$$\mathbb{P}[h > 0] = \mathbb{P}[E] \geq \frac{\epsilon}{2} > 0.$$

Lemma 2.2.17 yields that the market consisting of the top n stocks is not viable. □

We are now ready to state and prove the main result of this section.

Theorem 2.2.19. *The following statements are equivalent:*

(1) The market consisting of the top n stocks is viable.

(2) There exists a local martingale deflator among the top n stocks, i.e., $\mathcal{Y}^n \neq \emptyset$.

(3) The supermartingale numéraire among the top n stocks exists.

(4) The market consisting of the top n stocks has locally finite growth; namely, the condition (2.2.23) *of the aggregate maximal growth process* $\widetilde{G}$ *among the top n stocks of* (2.2.24) *holds.*

Proof. The implication *(4) ⇒ (3)* follows from Proposition 2.2.9. The implication *(3) ⇒ (2)* also follows easily, because the supermartingale numéraire among the top n stocks is a local martingale numéraire among the top n stocks from Proposition 2.2.5, and the reciprocal of the local martingale numéraire among the top n stocks is a local martingale deflator among the top n stocks.

In order to prove *(2)* $\Rightarrow$ *(1)*, let $Y \in \mathcal{Y}^n$ be a local martingale deflator and pick a cumulative withdrawal process $K \in \mathcal{K}$ such that $x^n(K) = 0$. From (2.2.29) of Lemma 2.2.15, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\infty Y(t-)dK(t)\right] = 0.$$

Since Y is strictly positive and K is nondecreasing with $K(0) = 0$, it follows that $K(\infty) = 0$ holds $\mathbb{P}$-a.e., which is equivalent to $K \equiv 0$. The market consisting of the top n stocks is then viable.

The remaining part is to show the implication *(1)* $\Rightarrow$ *(4)*, which is quite technical. Suppose that the market fails to have locally finite growth among the top n stocks, i.e., one of the structural conditions (2.2.14), (2.2.15) is violated. Thus, we need to consider two cases:

1. the set $\{\widetilde{\alpha} \notin \textbf{range}(\widetilde{c})\}$ fails to be $(\mathbb{P} \otimes O)$-null,

2. the set $\{\widetilde{\alpha} \notin \textbf{range}(\widetilde{c})\}$ is $(\mathbb{P} \otimes O)$-null, but $\mathbb{P}[\widetilde{G}(T) = \infty] > 0$ holds for some $T > 0$.

We shall show that the market is not viable in each of the cases *(A)* and *(B)* below.

$*$ **Case (A)**. Recalling the notation (2.2.13) with its properties (a)-(c), we first note that the predictable process

$$\varphi := \frac{1}{||\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha}||^2}(\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha})\mathbf{1}_{\{\widetilde{\alpha}\notin\textbf{range}(\widetilde{c})\}}, \tag{2.2.32}$$

is well-defined, because $\widetilde{\alpha} \notin \textbf{range}(\widetilde{c})$ holds if and only if $\widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha} \neq \widetilde{\alpha}$. Note that $D\varphi = \varphi$, thus $\varphi \in \mathcal{T}(n)$, thanks to the properties $D\widetilde{\alpha} = \widetilde{\alpha}$, $D\widetilde{c} = \widetilde{c}$ from (2.1.49). Since the process $\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha}$ is orthogonal to $\textbf{range}(\widetilde{c})$, we have $\widetilde{c}\varphi = 0$. Furthermore, we have $\varphi'\widetilde{\alpha} = \mathbf{1}_{\{\widetilde{\alpha}\notin\textbf{range}(\widetilde{c})\}}$, because

$$(\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha})'\widetilde{\alpha} = ||\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha}||^2 + (\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha})'(\widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha}) = ||\widetilde{\alpha} - \widetilde{c}\widetilde{c}^{\dagger}\widetilde{\alpha}||^2.$$

Thus, from Lemma 2.1.7, φ is a portfolio among the top n stocks, i.e., $\varphi \in \mathcal{I}(R) \cap \mathcal{T}(n)$. Also, the local martingale vanishes: $\int_0^{\cdot} \varphi'(t)d\widetilde{M}(t) \equiv 0$, because its quadratic variation process vanishes

$$\Big[\int_0^{\cdot} \varphi'(t)d\widetilde{M}(t)\Big] = \int_0^{\cdot} \varphi'(t)\widetilde{c}(t)\varphi(t)dO(t) \equiv 0. \tag{2.2.33}$$

Thus,

$$\int_0^{\cdot} \varphi'(t)d\widetilde{R}(t) = \int_0^{\cdot} \varphi'(t)d\widetilde{A}(t) = \int_0^{\cdot} \varphi'(t)\widetilde{\alpha}(t)dO(t) = \int_0^{\cdot} \mathbf{1}_{\{\widetilde{\alpha}\notin\mathbf{range}(\widetilde{c})\}}(t)dO(t) =: K.$$

We define the vector process $\vartheta \equiv (\vartheta_1, \cdots, \vartheta_N)$ with components given by $\vartheta_i = \varphi_i/S_i$ for $i = 1, \cdots, N$. It is then easy to check that $m\vartheta$ is an investment strategy among the top n stocks, i.e., $m\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$, for any $m \in \mathbb{N}$, and

$$X(\cdot; 0, m\vartheta) = \int_0^{\cdot} m\vartheta'(t)dS(t) = m\int_0^{\cdot} \varphi'(t)dR(t) = m\int_0^{\cdot} \varphi'(t)d\widetilde{R}(t) = mK.$$

In other words, for any $m \in \mathbb{N}$, the wealth process generated by the investment strategy $m\vartheta$ among the top n stocks has vanishing local martingale part, and is equal to the non-trivial, nondecreasing part mK of finite variation. This process mK can be arbitrarily scaled by the multiplicative constant $m \in \mathbb{N}$, and thus $x^n(K) = 0$, by recalling (2.2.28). We conclude that the market consisting of the top n stocks is not viable.

$*$ **Case (B)**. We assume that the set $\{\widetilde{\alpha} \notin \mathbf{range}(\widetilde{c})\}$ is $(\mathbb{P}\otimes O)$-null, but $\mathbb{P}[\widetilde{G}(T) = \infty] > 0$ holds for some $T > 0$. In this case, the aggregate maximal growth process $\widetilde{G}$ of (2.2.24) becomes

$$\widetilde{G} = \frac{1}{2}\int_0^{\cdot} \widetilde{\alpha}'(t)\widetilde{c}^{\dagger}(t)\widetilde{\alpha}(t)dO(t). \tag{2.2.34}$$

We consider first the portfolio $\rho := D\widetilde{c}^{\dagger}\widetilde{\alpha} \in \mathcal{T}(n)$ as in (2.2.18), and also set $\rho^m := \rho\mathbf{1}_{\{||\rho||\leq m\}} \in \mathcal{I}(R) \cap \mathcal{T}(n)$. The log-wealth process of (2.1.27) can be represented, with the help of (2.1.36) and

(2.2.19), as

$$\log X_{\rho^m} = \frac{1}{2}\int_0^{\cdot} \mathbf{1}_{\{||\rho(t)||\leq m\}}\rho'(t)\widetilde{c}(t)\rho(t)dO(t) + \int_0^{\cdot} \mathbf{1}_{\{||\rho(t)||\leq m\}}\rho'(t)d\widetilde{M}(t). \tag{2.2.35}$$

Note that the first integral on the right-hand side of (2.2.35), namely

$$2G^m := \int_0^{\cdot} \mathbf{1}_{\{||\rho(t)||\leq m\}}\rho'(t)\widetilde{c}(t)\rho(t)dO(t),$$

is the quadratic variation of the local martingale $\int_0^{\cdot} \mathbf{1}_{\{||\rho(t)||\leq m\}}\rho'(t)d\widetilde{M}(t)$, which is the second integral on the right-hand side of (2.2.35). The Dambis-Dubins-Schwarz representation (cf. Theorem 3.4.6 and Problem 3.4.7 of [16]), with the scaling property of Brownian motion, implies that there exists a Brownian motion W^m, on a possibly enlarged filtered probability space, such that

$$\log X_{\rho^m} = G^m + \sqrt{2}W^m(G^m), \tag{2.2.36}$$

for every $m \in \mathbb{N}$. The sequence $\{G^m(T)\}_{m\in\mathbb{N}}$ is nondecreasing and converges to

$$\frac{1}{2}\int_0^{\cdot} \rho'(t)\widetilde{c}(t)\rho(t)dO(t) = \frac{1}{2}\int_0^{\cdot} \widetilde{\alpha}'(t)\widetilde{c}^{\dagger}(t)\widetilde{\alpha}(t)dO(t) = \widetilde{G}(T),$$

as in (2.2.34), again with the help of (2.2.19). The strong law of large numbers for Brownian motion gives

$$\lim_{m\to\infty}\mathbb{P}\left[\frac{W^m\big(G^m(T)\big)}{G^m(T)} \leq -\frac{1}{2\sqrt{2}}, \quad \widetilde{G}(T) = \infty\right] = 0.$$

From the representation (2.2.36), we obtain

$$\lim_{m\to\infty}\mathbb{P}\left[\frac{\log X_{\rho^m}(T)}{G^m(T)} \leq \frac{1}{2}, \quad \widetilde{G}(T) = \infty\right] = 0.$$

Therefore, in case *(B)*, the collection of random variables $\{X_{\rho^m}(T) \mid m \in \mathbb{N}\} \subseteq \{X(T) \mid X \in \mathcal{X}^n\}$ fails to be bounded in probability, and Proposition 2.2.18 concludes that the market consisting of the top n stocks is not viable. □

2.2.5 *Growth Optimality and Relative Log-optimality*

The results in the previous subsection characterize the supermartingale numéraire portfolio among the top n stocks, via the 'structural condition', in terms of $\widetilde{\alpha}$ and $\widetilde{c}$. More specifically, in the argument leading to Proposition 2.2.8 and in the proof of Lemma 2.2.10, the *maximal growth rate among the top n stocks* $\widetilde{g}$ of (2.2.21) is attained when the portfolio is the supermartingale numéraire portfolio among the top n stocks, as in (2.2.26). In this subsection, we reformulate this property and show that the supermartingale numéraire portfolio is 'optimal' in some sense among portfolios of top n stocks.

Definition 2.2.20 (Relative growth and growth optimality)**.** We define the *relative growth* of a given portfolio $\pi \in \mathcal{I}(R)$ with respect to another portfolio $\rho \in \mathcal{I}(R)$ as

$$\Gamma_\pi^\rho := \Gamma_\pi - \Gamma_\rho, \tag{2.2.37}$$

namely, the difference between the finite variation process of the log-relative wealth process $\log X_\pi^\rho = \log(X_\pi / X_\rho)$ from (2.1.28), (2.2.1).

We call a portfolio $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ *growth-optimal among the top n stocks*, if for every portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ the process $\Gamma_\pi^\rho = \Gamma_\pi - \Gamma_\rho$ is non-increasing.

Proposition 2.2.21. *A portfolio is growth-optimal among the top n stocks, if and only if it is a supermartingale numéraire portfolio among the top n stocks.*

Proof. (i) Let us first assume that $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is the supermartingale numéraire portfolio among the top n stocks. From Proposition 2.2.8 and (2.2.11), we know that $\widetilde{\alpha} \in$ **range**$(\widetilde{c})$ and $\widetilde{\alpha} = \widetilde{c}\rho$ hold $(\mathbb{P} \otimes O) - a.e.$ Recalling (2.1.50), (2.2.21) and the fact that the supremum of g

is attained by the supermartingale numéraire portfolio among the top n stocks, the comparison $\gamma_\rho = \widetilde{g} \geq \gamma_\pi$ holds $(\mathbb{P} \otimes O) - a.e.$ for every $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$. Thus, ρ is growth-optimal.

(ii) Next, we assume that $\nu \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is a growth-optimal portfolio among the top n stocks. We pick a portfolio $\varphi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ satisfying $\widetilde{c}\varphi = 0$ and $\varphi'\widetilde{\alpha} = 1$ on the set $\{\widetilde{\alpha} \notin \textbf{range}(\widetilde{c})\}$ (for example, as in (2.2.32) in the proof of Theorem 2.2.19). We then have $\gamma_{\nu+\varphi} = \gamma_\nu + 1$ on $\{\widetilde{\alpha} \notin \textbf{range}(\widetilde{c})\}$ from (2.1.50), violating the growth-optimality of ν. This implies that the latter set is $(\mathbb{P} \otimes O)$-null. In particular, $\widetilde{g} < \infty$ in the $(\mathbb{P} \otimes O) - a.e.$ sense, from (2.2.22).

On the other hand, we let $\rho := D\widetilde{c}^{\dagger}\widetilde{\alpha} \in \mathcal{T}(n)$ and define $\rho^m := \rho \mathbf{1}_{\{||\rho|| \leq m\}} \in \mathcal{I}(R) \cap \mathcal{T}(n)$ for $m \in \mathbb{N}$. The equation (2.2.26) yields $\gamma_\nu \geq \gamma_{\rho^m} = \widetilde{g}\mathbf{1}_{\{||\rho|| \leq m\}}$, and thus $\gamma_\nu \geq \widetilde{g}$ holds $(\mathbb{P} \otimes O) - a.e.$ by taking the limit $m \to \infty$. We conclude that ν is also a supermartingale numéraire portfolio among the top n stocks. $\square$

The supermartingale numéraire portfolio among the top n stocks is 'optimal' also in another sense, as follows.

Definition 2.2.22. A portfolio $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is called *relatively log-optimal among the top n stocks*, if for all portfolios $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ and for all stopping times τ of $\mathcal{F}$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\log X^{\rho}_{\pi}(\tau)\big)^+\Big] < \infty, \qquad \text{and} \qquad \mathbb{E}^{\mathbb{P}}\Big[\log X^{\rho}_{\pi}(\tau)\Big] \leq 0. \tag{2.2.38}$$

Proposition 2.2.23. *A portfolio is relatively log-optimal among the top n stocks, if and only if it is a supermartingale numéraire portfolio among the top n stocks.*

Proof. (i) We first suppose that $\rho \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is the supermartingale numéraire portfolio among the top n stocks. Then, we obtain

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\log X^{\rho}_{\pi}(\tau)\big)^+\Big] = \int_0^\infty \mathbb{P}(X^{\rho}_{\pi}(\tau) > e^t)dt \leq \int_0^\infty \mathbb{P}(X^{\rho}_{\pi}(\tau) > t)dt \leq \mathbb{E}^{\mathbb{P}}[X^{\rho}_{\pi}(\tau)] \leq 1,$$

where the last inequality is from the Optional Sampling Theorem. By applying Jensen's inequality to this last inequality, the second condition of (2.2.38) also holds, and we conclude that ρ is relatively log-optimal among the top n stocks.

(ii) For the converse implication, we assume that $\nu \in \mathcal{I}(R) \cap \mathcal{T}(n)$ is relatively log-optimal among the top n stocks. As in the proof of Proposition 2.2.21, we pick a portfolio $\varphi \in \mathcal{I}(R) \cap \mathcal{T}(n)$ as in (2.2.32), satisfying $\widetilde{c}\varphi = 0$ and $\varphi'\widetilde{\alpha} = 1$ on the set $\{\widetilde{\alpha} \notin \mathbf{range}(\widetilde{c})\}$. By recalling (2.1.27), (2.1.28), (2.1.31), (2.1.36), and (2.1.50), straightforward computations show

$$\begin{aligned}\log X^{\nu}_{\nu+\varphi} = \log X_{\nu+\varphi} - \log X_{\nu} &= \int_0^{\cdot} \big(\gamma_{\nu+\varphi}(t) - \gamma_{\nu}(t)\big) dO(t) + \int_0^{\cdot} \varphi'(t) d\widetilde{M}(t) \\ &= \int_0^{\cdot} \mathbf{1}_{\{\widetilde{\alpha} \notin \mathbf{range}(\widetilde{c})\}}(t) dO(t).\end{aligned} \tag{2.2.39}$$

Here, the last integral on the right-hand side of (2.2.39) vanishes, because of the equation (2.2.33) above. The relative log-optimality of ν implies that the set $\{\widetilde{\alpha} \notin \mathbf{range}(\widetilde{c})\}$ is $(\mathbb{P} \otimes O)$-null. We then consider a process $\rho := D\widetilde{c}^{\dagger}\widetilde{\alpha} \in \mathcal{T}(n)$ of (2.2.18), as in the proof of Proposition 2.2.21. Note that $\widetilde{\alpha} = \widetilde{c}\rho$ holds $(\mathbb{P} \otimes O) - a.e.$ from (2.2.20), or equivalently, $\widetilde{A}_i = C_{\widetilde{i}\rho}$ hold for $i = 1, \cdots, N$, from Remark 2.2.7. This last requirement implies that $A_\pi = C_{\pi\rho}$, thus $R_\pi - C_{\pi\rho}$ is local martingale for every $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n)$. We further define

$$\nu^m := \nu \mathbf{1}_{\{\widetilde{\alpha} = \widetilde{c}\nu\}} + \nu \mathbf{1}_{\{\widetilde{\alpha} \neq \widetilde{c}\nu\}} \mathbf{1}_{\{||\rho|| > m\}} + \rho \mathbf{1}_{\{\widetilde{\alpha} \neq \widetilde{c}\nu\}} \mathbf{1}_{\{||\rho|| \leq m\}}, \qquad \text{for} \quad m \in \mathbb{N},$$

and it is easy to check that $\nu^m \in \mathcal{I}(R) \cap \mathcal{T}(n)$ for all $m \in \mathbb{N}$.

We now claim that the ratio X_ν / X_{ν^m} for every $m \in \mathbb{N}$ is a local martingale. Proposition 2.2.1 implies that it is sufficient to show $R^{\nu^m}_{\nu} = R_\pi - C_{\pi\nu^m} =: Q$ is a local martingale, where we set $\pi := \nu - \nu^m \in \mathcal{I}(R) \cap \mathcal{T}(n)$. On the set $\zeta := \{\widetilde{\alpha} \neq \widetilde{c}\nu, ||\rho|| \leq m\}$, we have $\nu^m = \rho$, thus Q is local martingale. On the complement set ζ^c, we have $\pi = \nu - \nu^m = 0$, thus $Q = 0$. In other words, we showed that

$$Q = \int_0^{\cdot} \mathbf{1}_{\zeta}(t) dQ(t) = \int_0^{\cdot} \mathbf{1}_{\zeta}(t) d(R_\pi - C_{\pi\rho})(t)$$

is local martingale, verifying our claim that X_ν/X_{ν^m} is a local martingale for every $m \in \mathbb{N}$. As the ratio is positive, X_ν/X_{ν^m} is also a supermartingale.

If we assume that $\mathbb{P}[X_\nu(T) \neq X_{\nu^m}(T)] > 0$ were true for some $T > 0$, we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\log \frac{X^\nu(T)}{X^{\nu^m}(T)}\right] < \log \mathbb{E}^{\mathbb{P}}\left[\frac{X^\nu(T)}{X^{\nu^m}(T)}\right] \leq 0,$$

contradicting the relative log-optimality of ν. Thus, we conclude that $X_\nu = X_{\nu^m}$, from the continuity of X_ν/X_{ν^m}, and $\nu - \nu^m$ is a null portfolio in the sense of Lemma 2.1.6. We then have $\widetilde{c}\nu^m = \widetilde{c}\nu = \widetilde{c}\rho = \widetilde{\alpha}$, $(\mathbb{P} \otimes O) - a.e$, on the set $\zeta = \{\widetilde{\alpha} \neq \widetilde{c}\nu, ||\rho|| \leq m\}$ defined above, which implies that ζ is $(\mathbb{P} \otimes O)$-null. Since this property is true for every $m \in \mathbb{N}$, the identity $\widetilde{\alpha} = \widetilde{c}\nu$ is valid $(\mathbb{P} \otimes O) - a.e$, thus ν is the supermartingale numéraire portfolio among the top n stocks. □

In part (ii) of the proofs of both Proposition 2.2.21 and Proposition 2.2.23, we did not assume the existence of supermartingale numéraire portfolio among the top n stocks. Thus, the existence of growth-optimal or relatively log-optimal portfolio among the top n stocks is equivalent to the existence of the supermartingale numéraire portfolio among the top n stocks, and we can add the following two statements to the list of equivalences in Theorem 2.2.19:

(5) A growth-optimal portfolio among the top n stocks exists.

(6) A relatively log-optimal portfolio among the top n stocks exists.

2.2.6 *The Optional Decomposition*

Suppose that we are given a nonnegative, adapted process with RCLL paths and $X(0) = x \geq 0$. In this subsection, we characterize the condition when X belongs to $\mathcal{X}^n$ of Definition 2.1.4, i.e., when X is the wealth process generated by an investment strategy that invests in the top n stocks of the market, and study how can we construct this strategy from X. The following Theorem 2.2.25, which we call the Optional Decomposition Theorem, gives, along with its Corollary 2.2.26, the answer to this question.

We first present the following result, originally from Theorem 1 of [25]. See also Propositions 2.3 and 3.2 of [19]. We recall for this purpose the semimartingale vector $\widetilde{M}$ defined in (2.1.35) and write $\mathcal{M}_{loc}^{\perp}(\widetilde{M})$ the collection of scalar local martingales L with RCLL paths, satisfying $L(0) = 0$ and the orthogonality $[L, \widetilde{M}_i] = 0$ for all $i = 1, \cdots, N$.

Lemma 2.2.24. *If the supermartingale numéraire portfolio ρ among the top n stocks exists, then the collections $\mathcal{Y}^n$ of local martingale deflators among the top n stocks, defined in Definition 2.2.12, admits the representation:*

$$\mathcal{Y}^n = \left\{ \frac{1}{X_\rho} \mathcal{E}(L) : L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M}) \quad with \quad \Delta L > -1 \right\}. \tag{2.2.40}$$

In order to simplify the proof of the Optional Decomposition Theorem, we shall work under the following assumption. The general case of the Theorem can be proven as in the Subsection 3.1.3 of [13].

Assumption (A) : All local martingales on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})$ have continuous paths.

Theorem 2.2.25 (Optional Decomposition)**.** *Suppose that the market consisting of the top n stocks is viable. For a nonnegative, adapted process X with RCLL paths satisfying $X(0) = x \geq 0$, the following statements are equivalent:*

(1) The process YX is a supermartingale, for every $Y \in \mathcal{Y}^n$.

(2) There exist an investment strategy $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ among the top n stocks, and a cumulative withdrawal process $K \in \mathcal{K}$, such that

$$X = x + \int_0^{\cdot} \sum_{i=1}^{N} \vartheta_i(t) dS_i(t) - K. \tag{2.2.41}$$

Proof. We first show the implication *(2)* $\Longrightarrow$ *(1)*. For any $Y \in \mathcal{Y}^n$, we write $Y = \mathcal{E}(L)/X_\rho$ for some $L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M})$ with $\Delta L > -1$ from Lemma 2.2.24, where we denote by ρ the supermartingale numéraire portfolio among the top n stocks. Then, we have from Lemma 2.2.11,

$$YX + YK = \frac{x + \int_0^{\cdot} \sum_{i=1}^{N} \vartheta_i(t) dS_i(t)}{X_\rho} \mathcal{E}(L) = \Big(x + \int_0^{\cdot} \sum_{i=1}^{N} \eta_i(t) d\widetilde{M}_i(t)\Big)\mathcal{E}(L),$$

for some process $\eta \in \mathcal{I}(\widetilde{M}) \cap \mathcal{T}(n)$. The last expression is a product of two nonnegative, orthogonal local martingales, thus it is a nonnegative local martingale. The claim that YX is a supermartingale follows.

We now show the implication *(1)* $\Longrightarrow$ *(2)* which is more involved, under the above Assumption (A). We assume that *(1)* holds and recall the collections $\mathcal{Y}^n$ and $\mathcal{M}_{loc}^{\perp}(\widetilde{M})$ of (2.2.40). All processes in $\mathcal{M}_{loc}^{\perp}(\widetilde{M})$ have continuous paths under the Assumption (A). From Lemma 2.2.24, $(X/X_\rho)\mathcal{E}(L)$ is a supermartingale for every $L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M})$, and in particular, X/X_ρ is a supermartingale itself. The Doob-Meyer and Kunita-Watanabe decompositions give

$$\frac{X}{X_\rho} = x + M_\eta + L - B, \quad \text{where} \quad M_\eta := \int_0^{\cdot} \sum_{i=1}^{N} \eta_i(t) d\widetilde{M}_i(t).$$

Here, $\eta \equiv (\eta_1, \cdots, \eta_N) \in \mathcal{I}(\widetilde{M})$, $L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M})$ and B is an adapted, nondecreasing and right-continuous process with $B(0) = 0$, i.e., B is a cumulative withdrawal process in $\mathcal{K}$. Recalling the diagonal matrix D of (2.1.46) with its property $Dd\widetilde{M}(t) = d\widetilde{M}(t)$, we further define $\widetilde{\eta} := D\eta \in \mathcal{I}(\widetilde{M}) \cap \mathcal{T}(n)$, and we have

$$M_\eta = \int_0^{\cdot} \eta'(t) d\widetilde{M}(t) = \int_0^{\cdot} (D\eta)'(t) d\widetilde{M}(t) = M_{\widetilde{\eta}}.$$

Consequently, we obtain

$$\frac{X}{X_\rho} = x + M_{\widetilde{\eta}} + L - B, \quad \text{with} \quad \widetilde{\eta} \in \mathcal{I}(\widetilde{M}) \cap \mathcal{T}(n), \quad L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M}). \tag{2.2.42}$$

We next show that $L \equiv 0$ in (2.2.42). Again from Lemma 2.2.24, $(1/X_\rho)\mathcal{E}(mL)$ is a local martingale, thus $(X/X_\rho)\mathcal{E}(mL)$ is a supermartingale for every $m \in \mathbb{N}$. Since $[\mathcal{E}(mL), \widetilde{M}_i] = 0$ for $i = 1, \cdots, N$, we have $[\mathcal{E}(mL), M_{\widetilde{\eta}}] = 0$ and consequently, $\mathcal{E}(mL)M_{\widetilde{\eta}}$ is a local martingale as a product of two orthogonal local martingales. Thus, from (2.2.42), the process

$$\mathcal{E}(mL)(L - B) = \mathcal{E}(mL)\frac{X}{X_\rho} - \mathcal{E}(mL)(x + M_{\widetilde{\eta}})$$

is a local supermartingale for every $m \in \mathbb{N}$. On the other hand, the integration by parts gives

$$\mathcal{E}(mL)(L-B) = \int_0^\cdot (L-B)(t-)d\mathcal{E}(mL)(t) + \int_0^\cdot \mathcal{E}(mL)(t)dL(t) + \int_0^\cdot \mathcal{E}(mL)(t)d\big([mL, L] - B\big)(t).$$

Then, the last integrator $m[L, L] - B$ should be a local supermartingale for every $m \in \mathbb{N}$, which implies $[L, L] \equiv 0$, thus $L \equiv 0$.

As a result, the equation (2.2.42) becomes

$$\frac{X}{X_\rho} = x + M_{\widetilde{\eta}} - B,$$

and we apply the product rule to obtain the decomposition of $X = X_\rho(X/X_\rho)$:

$$X = x + \int_0^\cdot X(t-)\rho'(t)dR(t) + \int_0^\cdot X_\rho(t)d(M_{\widetilde{\eta}} - B)(t) + \int_0^\cdot X_\rho(t)dC_{\widetilde{\eta}\rho}(t),$$

in conjunction with (2.1.22) and (2.1.42). Moreover, the condition $(\widetilde{3})$ of Proposition 2.2.5 implies $C_{\widetilde{\eta}\rho} = A_{\widetilde{\eta}} = R_{\widetilde{\eta}} - M_{\widetilde{\eta}}$, and we deduce

$$X = x + \int_0^\cdot \big(X(t-)\rho'(t) - X_\rho(t)\widetilde{\eta}'(t)\big)dR(t) - \int_0^\cdot X_\rho(t)dB(t).$$

Therefore, if we define

$$\vartheta_i(t) := \frac{X(t-)\rho'(t) - X_\rho(t)\widetilde{\eta}'(t)}{S_i(t)}, \quad i = 1, \cdots, N, \qquad K := \int_0^\cdot X_\rho(t)dB(t),$$

then it is easy to check that $\vartheta \equiv (\vartheta_1, \cdots, \vartheta_N) \in \mathcal{I}(S) \cap \mathcal{T}(n)$ and $K \in \mathcal{K}$. □

Corollary 2.2.26. *Suppose that the market consisting of the top n stocks is viable. For a nonnegative, adapted process X with RCLL paths satisfying $X(0) = x \geq 0$, the following statements are then equivalent:*

(1) The process YX is a local martingale, for every $Y \in \mathcal{Y}^n$.

(2) There exists an investment strategy $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ among the top n stocks, such that

$$X = x + \int_0^{\cdot} \sum_{i=1}^{N} \vartheta_i(t) dS_i(t). \tag{2.2.43}$$

Proof. We first assume *(1)*; then YX is a supermartingale for every $Y \in \mathcal{Y}^n$. From Theorem 2.2.25, we have a decomposition (2.2.41) for some $\vartheta \in \mathcal{I}(S) \cap \mathcal{T}(n)$ and $K \in \mathcal{K}$. In particular, if we take $Y = 1/X_\rho$, the reciprocal of the local martingale numéraire, we obtain

$$YK = \frac{X(\cdot; x, \vartheta)}{X_\rho} - YX,$$

with the notation in (2.1.13). Since the terms on the right-hand side are local martingales, YK is a local martingale, and so is

$$YK - \int_0^{\cdot} K(t-) dY(t) = \int_0^{\cdot} Y(t) dK(t).$$

However, the last integral is nondecreasing and is a supermartingale (as a non-negative local martingale), and therefore identically equal to zero. Thus, $K \equiv 0$ as Y is positive, and the statement *(2)* follows.

In order to show the reverse implication, we assume *(2)*, then X/X_ρ is a local martingale where ρ is the local martingale numéraire portfolio among the top n stocks, as before. From Lemma 2.2.11, X/X_ρ can be cast as a stochastic integral with respect to the local martingale vector

$\widetilde{M}$. Furthermore, from Lemma 2.2.24, every $Y \in \mathcal{Y}^n$ is of the form $Y = (1/X_\rho)\mathcal{E}(L)$ for some local martingale L satisfying $[L, \widetilde{M}_i] = 0$ for $i = 1, \cdots, N$. Therefore, the product $YX = (X/X_\rho)\mathcal{E}(L)$ of these two orthogonal local martingales is again a local martingale. □

2.2.7 *Entire Market Versus Top n Market*

We present first the following result, which can be easily proven from the equivalence between the existence of supermartingale numéraire portfolio and the market viability.

Theorem 2.2.27. *The existence of a supermartingale numéraire portfolio in the whole market, implies the existence of supermartingale numéraire portfolio among the top n stocks.*

Proof. From Theorem 2.34 of [13], the existence of a supermartingale numéraire portfolio in the whole market, is equivalent to the viability of the whole market. The viability of the whole market implies the viability of the market consisting of the top n stocks, thanks to the inequality $0 \leq x(K) \leq x^n(K)$ in Definition 2.2.14. We conclude that there exists a supermartingale numéraire portfolio among the top n stocks, from Theorem 2.2.19. □

Theorem 2.2.27 shows that the viability of the entire market, composed of N stocks, implies the viability of the 'top n market'. Thus, if the entire market is viable, there exist both a supermartingale numéraire portfolio for the whole market, and a supermartingale numéraire portfolio among the top n stocks, and the former dominates the latter in the sense of growth-optimality. In the following proposition, we study this dominance by expressing the asymptotic behavior of log-relative wealth process between these two portfolios in terms of the 'local characteristics' of the market. We first need the following definitions which are similar to those in (2.2.21)-(2.2.24).

We call a $[0, \infty]$-valued, predictable process

$$g := \sup_{p \in \mathbb{R}^N} \left(p'\alpha - \frac{1}{2} p' c p \right) \tag{2.2.44}$$

maximal growth rate achievable in the whole market. This process can be rewritten in the form

$$g = \frac{1}{2}\big(\alpha' c^{\dagger}\alpha\big)\mathbf{1}_{\{\alpha\in\textbf{range}(c)\}} + \infty\mathbf{1}_{\{\alpha\notin\textbf{range}(c)\}}, \tag{2.2.45}$$

and the supremum of (2.2.44) is attained if and only if $g < \infty$, at $p \equiv \rho := c^{\dagger}\alpha$, i.e., when ρ is the supermartingale numéraire portfolio of whole market. Here, $c^{\dagger}$ is the 'pseudo-inverse' of c, defined as in (2.2.13). Then, the viability of the whole market can be shown to be equivalent to the condition

$$G(T) := \int_0^T g(t)dO(t) < \infty, \qquad \text{for all} \quad T \geq 0. \tag{2.2.46}$$

Here, the adapted nondecreasing process G is called as *aggregate maximal growth of whole market*.

When the whole market is viable, the growth rates $\widetilde{g}$ of (2.2.22) and g of (2.2.45) have simpler forms

$$\widetilde{g} = \frac{1}{2}\widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha} = \gamma_{\widetilde{\rho}}, \qquad g = \frac{1}{2}\alpha' c^{\dagger}\alpha = \gamma_{\rho}, \tag{2.2.47}$$

respectively, as from (2.2.26), with $\widetilde{\rho} := D\widetilde{c}^{\dagger}\widetilde{\alpha}$ the supermartingale numéraire portfolio among the top n stocks, and $\rho := c^{\dagger}\alpha$ the supermartingale numéraire portfolio for the whole market. We denote the *difference of aggregate maximal growth between the whole market and the top n market* by

$$\mathcal{G} := G - \widetilde{G} = \int_0^{\cdot} \big(g(t) - \widetilde{g}(t)\big)dO(t) = \int_0^{\cdot} \big(\gamma_{\rho}(t) - \gamma_{\widetilde{\rho}}(t)\big)dO(t) = \Gamma_{\rho} - \Gamma_{\widetilde{\rho}}. \tag{2.2.48}$$

Since ρ is also a growth-optimal portfolio (as a supermartingale numéraire portfolio) in the whole market, the relative growth $\Gamma^{\rho}_{\widetilde{\rho}} = \Gamma_{\widetilde{\rho}} - \Gamma_{\rho}$ of Definition 2.2.20 is non-increasing, from which we conclude that $\mathcal{G}$ is nondecreasing and nonnegative.

Proposition 2.2.28. *Suppose that the whole market is viable and let ρ and $\widetilde{\rho}$ be the supermartingale numéraire portfolio for the whole market and the supermartingale numéraire portfolio among the top n stocks, respectively. Then, the asymptotic growth rate of the log-relative wealth process*

$\log X^{\rho}_{\widetilde{\rho}}$ *is the same as* $-\mathcal{G}$ *of* (2.2.48)*, namely:*

$$\lim_{T\to\infty} \frac{1}{\mathcal{G}(T)} \log\left(\frac{X_{\widetilde{\rho}}(T)}{X_{\rho}(T)}\right) = -1 \quad \textit{holds} \quad \mathbb{P}-\textit{a.e. on the set} \quad \{\lim_{T\to\infty} \mathcal{G}(T) = \infty\}. \tag{2.2.49}$$

Proof. We recall the notations (2.1.27)-(2.1.31) and write for $T \geq 0$,

$$\log X^{\rho}_{\widetilde{\rho}}(T) = \log X_{\widetilde{\rho}}(T) - \log X_{\rho}(T) = \int_0^T \big(\gamma_{\widetilde{\rho}}(t) - \gamma_{\rho}(t)\big) dO(t) + \int_0^T \big(\widetilde{\rho}(t) - \rho(t)\big)' dM(t). \tag{2.2.50}$$

The first integral on the right-hand side is just $-\mathcal{G}(T)$ of (2.2.48) and it can be rewritten as

$$-\mathcal{G}(T) = \int_0^T \big(\gamma_{\widetilde{\rho}}(t) - \gamma_{\rho}(t)\big) dO(t) = \frac{1}{2}\int_0^T \big(\widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha} - \alpha' c^{\dagger}\alpha\big)(t) dO(t) \tag{2.2.51}$$

from (2.2.47). On the other hand, from $\rho = c^{\dagger}\alpha$ and $\widetilde{\rho} = D\widetilde{c}^{\dagger}\widetilde{\alpha}$, we obtain series of equations as in (2.2.19):

$$(\widetilde{\rho} - \rho)' c(\widetilde{\rho} - \rho) = \widetilde{\rho}' c\widetilde{\rho} + \rho' c\rho - \widetilde{\rho}' c\rho - \rho' c\widetilde{\rho} = \widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha} + \alpha' c^{\dagger}\alpha - 2\widetilde{\rho}' c\rho,$$

as well as

$$\widetilde{\rho}' c\rho = \widetilde{\rho}' cc^{\dagger}\alpha = \widetilde{\rho}'\alpha = \widetilde{\alpha}'(\widetilde{c}^{\dagger})' D\alpha = \widetilde{\alpha}'(\widetilde{c}^{\dagger})'\widetilde{\alpha}.$$

Combining these equations, we have

$$(\widetilde{\rho} - \rho)' c(\widetilde{\rho} - \rho) = \alpha' c^{\dagger}\alpha - \widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha}.$$

Thus, the quadratic variation of the last integral on the right hand-side of (2.2.50) is written as

$$\begin{aligned}\left[\int_0^T \big(\widetilde{\rho}(t) - \rho(t)\big)' dM(t)\right] &= \int_0^T (\widetilde{\rho} - \rho)' c(\widetilde{\rho} - \rho)(t) dO(t) \\ &= \int_0^T (\alpha' c^{\dagger}\alpha - \widetilde{\alpha}'\widetilde{c}^{\dagger}\widetilde{\alpha})(t) dO(t) = 2\mathcal{G}(T).\end{aligned}$$

The Dambis-Dubins-Schwarz representation (Theorem 3.4.6, Problem 3.4.7 of [16]) with the scaling property of Brownian motion implies that there exists a Brownian motion W, on a possibly enlarged filtered probability space, such that

$$\log X^{\rho}_{\widetilde{\rho}}(T) = -\mathcal{G}(T) + \sqrt{2}W\big(\mathcal{G}(T)\big). \tag{2.2.52}$$

The strong law of large numbers for Brownian motion gives the result (2.2.49). □

The expression (2.2.51) shows that the asymptotic growth rate of the log-relative wealth process $\log X^{\rho}_{\widetilde{\rho}}$ is expressed in terms of 'local characteristics' of the market: $\alpha, \widetilde{\alpha}, c$, and $\widetilde{c}$.

2.3 Stock Portfolios in Open Markets

The open market described in the previous section consists of the top n stocks in terms of capitalization, and of the money market. The existence of this money market gives us a flexibility to construct portfolios among top n stocks. To be more specific, for any given portfolio $\pi \in \mathcal{I}(R)$, pre-multiplying it the diagonal matrix D of (2.1.46) transforms it into a new portfolio $D\pi$ among top n stocks. The proportion of assets, which is supposed to be invested in 'bottom' $N - n$ stocks by π, is now assigned to the money market by $D\pi$. In the absence of the money market, building portfolios among top n stocks is more subtle, and this section focuses on these subtleties.

2.3.1 *Stock Portfolios and the Market Portfolio*

An important subclass of portfolios in Definition 2.1.5 is the collection of portfolios π satisfying $\sum_{i=1}^{N} \pi_i \equiv 1$, or $\pi_0 \equiv 0$ in (2.1.25). Such a portfolio never invests in the money market; and this condition can be formulated as $\pi \in \Delta^{N-1}$, where we denote

$$\Delta^{N-1} := \Big\{(x_1, \cdots, x_N) \in \mathbb{R}^N \;\Big|\; \sum_{i=1}^{N} x_i = 1\Big\}.$$

Definition 2.3.1 (Stock Portfolio)**.** We call a portfolio $\pi \in \mathcal{I}(R)$ *stock portfolio*, if it takes values in Δ^{N-1}, i.e., satisfies $\sum_{i=1}^{N} \pi_i \equiv 1$. We denote the collection of stock portfolios by $\mathcal{I}(R) \cap \Delta^{N-1}$.

We call a stock portfolio π *stock portfolio among the top n stocks*, if in addition it belongs to $\mathcal{T}(n)$, i.e., satisfies the condition (2.1.15), or equivalently, (2.1.17). We denote the collection of stock portfolios among the top n stocks by $\mathcal{I}(R) \cap \Delta^{N-1} \cap \mathcal{T}(n)$.

Remark 2.3.2 (Self-financibility of stock portfolios). For any stock portfolio π, we sum over (2.1.24) for all indices $i = 1, \cdots, N$ to obtain

$$1 \equiv \sum_{i=1}^{N} \pi_i(\cdot) = \frac{\sum_{i=1}^{N} S_i(\cdot)\vartheta_i(\cdot)}{X(\cdot; 1, \vartheta)},$$

and from (2.1.13),

$$X(\cdot; 1, \vartheta) = 1 + \int_0^{\cdot} \sum_{i=1}^{N} \vartheta_i(t) dS_i(t) = \sum_{i=1}^{N} \vartheta_i(\cdot) S_i(\cdot).$$

This last equation shows the self-financing property of the stock portfolios; recall the equation (1.3.2) of Definition 1.3.1.

Before we present the most important example of stock portfolios, we introduce the notation

$$\Sigma := S_1 + \cdots + S_N, \tag{2.3.1}$$

representing the total capitalization of whole equity market.

Example 2.3.3 (Market portfolio). Suppose that an investment strategy ϑ is given as $\vartheta \equiv \mathbf{1}/\Sigma(0) \equiv (1, 1, \cdots, 1)/\Sigma(0)$ with initial wealth $x = 1$. Then, its wealth process is just the total capitalization normalized by its initial value:

$$X(\cdot; 1, \vartheta) = \frac{\Sigma(\cdot)}{\Sigma(0)}. \tag{2.3.2}$$

Whereas, from (2.1.24), the corresponding portfolio $\pi \equiv \mu \equiv (\mu_1, \cdots, \mu_N)$ can be expressed as

$$\mu_i(\cdot) = \frac{S_i(\cdot)}{\Sigma(\cdot)} = \frac{S_i(\cdot)}{S_1(\cdot) + \cdots + S_N(\cdot)}, \quad \text{for} \quad i = 1, \cdots, N. \tag{2.3.3}$$

We call this special stock portfolio μ the *market portfolio*, and its component processes in (2.3.3) *market weights* as in (1.3.1); it is considered as the most important stock portfolio, as its wealth process gives the evolution of total market capitalization.

In an analogous manner, we define the *top n market portfolio*, which we denote by $\widetilde{\mu} \equiv (\widetilde{\mu}_1, \cdots, \widetilde{\mu}_N)$, with components

$$\widetilde{\mu}_i(\cdot) := \frac{\widetilde{S}_i(\cdot)}{\widetilde{\Sigma}(\cdot)} = \frac{\widetilde{S}_i(\cdot)}{\widetilde{S}_1(\cdot) + \cdots + \widetilde{S}_N(\cdot)}, \quad \text{for} \quad i = 1, \cdots, N, \tag{2.3.4}$$

where

$$\widetilde{\Sigma} := \sum_{i=1}^{N} \widetilde{S}_i = S_{(1)} + \cdots + S_{(n)}, \qquad \text{and} \qquad \widetilde{S}_i(\cdot) := \mathbf{1}_{\{u_i(\cdot) \leq n\}} S_i(\cdot), \qquad \text{for} \quad i = 1, \cdots, N. \tag{2.3.5}$$

The denominator $\widetilde{\Sigma}$ of (2.3.4) represents the sum of the capitalizations of the top n stocks; thus, $\widetilde{\mu}_i(t)$ is the proportion of the capitalization of stock i, if this stock belongs to the top n, to the total capitalization of the top n stocks at time t. In other words, $\widetilde{\mu}_i$ can be interpreted as the 'market weight' of i-th stock in the restricted market composed of the top n stocks by capitalization. It is easy to check that $\widetilde{\mu}$ is a stock portfolio among the top n stocks, i.e., $\widetilde{\mu} \in \mathcal{I}(R) \cap \Delta^{N-1} \cap \mathcal{T}(n)$.

2.3.2 *Capital Asset Pricing Model*

The Capital Asset Pricing Model posits that individual stocks cannot systematically outperform the market. In our open market setting, this requirement can be cast as saying that each individual stock, whenever it belongs to the top n stocks, cannot outperform the top n market. In this subsection, we briefly discuss this model for the top n market. Recalling the top n stock portfolio $\widetilde{\mu}$ defined in (2.3.4), we have the next definition.

Definition 2.3.4 (CAPM)**.** We say that the top n market is in the realm of the *Capital Asset Pricing Model (CAPM)*, if

$$\widetilde{R}_i = \int_0^{\cdot} \beta_i(t) dR_{\widetilde{\mu}}(t) + N_i, \qquad i = 1, \cdots, N, \tag{2.3.6}$$

hold for appropriate processes $\beta_i \in \mathcal{I}(R_{\widetilde{\mu}})$, $i = 1, \cdots, N$, and for continuous local martingales N_i with $N_i(0) = 0$ which are orthogonal to $R_{\widetilde{\mu}}$ for all $i = 1, \cdots, N$:

$$[N_i, R_{\widetilde{\mu}}] \equiv 0.$$

The following proposition characterizes this property, in terms of the local characteristics of the top market introduced in Section 2.1.4.

Proposition 2.3.5 (Characterization of CAPM)**.** *The top n market is in the realm of the CAPM if, and only if, the following two conditions hold.*

1. *There exists a scalar "leverage" predictable process b such that*

$$\sum_{i=1}^{N} \int_0^T |b(t)| \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}} > 0\}} |dC_{\widetilde{i}\widetilde{\mu}}(t)| < \infty, \qquad \textit{for} \quad T \geq 0, \tag{2.3.7}$$

and the equalities hold $(\mathbb{P} \otimes O)$*-a.e.:*

$$\widetilde{\alpha}_i = b c_{\widetilde{i}\widetilde{\mu}}, \quad \textit{on} \quad \{c_{\widetilde{\mu}\widetilde{\mu}} > 0\} \quad \textit{for} \quad i = 1, \cdots, N. \tag{2.3.8}$$

2. *On the set* $\{c_{\widetilde{\mu}\widetilde{\mu}} = 0\}$*, we have* $(\mathbb{P} \otimes O)$*-a.e.:*

$$\alpha_{\widetilde{\mu}} = 0 \quad \Longleftrightarrow \quad \widetilde{\alpha}_i = 0, \qquad i = 1, \cdots, N. \tag{2.3.9}$$

When these conditions are satisfied, the process b of (2.3.7) *and the processes* $\beta_i \in \mathcal{I}(R_{\widetilde{\mu}})$ *of* (2.3.6) *can be chosen, respectively, as*

$$b = \frac{\alpha_{\widetilde{\mu}}}{c_{\widetilde{\mu}\widetilde{\mu}}} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}} > 0\}}, \tag{2.3.10}$$

$$\beta_i = \frac{c_{\widetilde{i}\widetilde{\mu}}}{c_{\widetilde{\mu}\widetilde{\mu}}} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}} > 0\}} + \frac{\widetilde{\alpha}_i}{\alpha_{\widetilde{\mu}}} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}} = 0,\ \alpha_{\widetilde{\mu}} \neq 0\}}, \qquad i = 1, \cdots, N. \tag{2.3.11}$$

Proof. Let us assume first that the top n market is in the realm of the CAPM. Recalling the notation (2.1.45), we have

$$C_{\widetilde{i\mu}} = [\widetilde{R}_i, R_{\widetilde{\mu}}] = \int_0^{\cdot} \beta_i(t)d[R_{\widetilde{\mu}}, R_{\widetilde{\mu}}](t) + [N_i, R_{\widetilde{\mu}}] = \int_0^{\cdot} \beta_i(t)dC_{\widetilde{\mu\mu}}(t),$$

which implies that $c_{\widetilde{i\mu}} = \beta_i c_{\widetilde{\mu\mu}}$ also hold $(\mathbb{P} \otimes O)$-a.e., for $i = 1, \cdots, N$. On $\{c_{\widetilde{\mu\mu}} > 0\}$, it follows that $\beta_i = c_{\widetilde{i\mu}}/c_{\widetilde{\mu\mu}}$ for $i = 1, \cdots, N$. Moreover, since $\widetilde{R}_i - \int_0^{\cdot} \beta_i(t)dR_{\widetilde{\mu}}(t)$ is a local martingale, we obtain $\widetilde{A}_i = \int_0^{\cdot} \beta_i(t)dA_{\widetilde{\mu}}(t)$, and also $\widetilde{\alpha}_i = \beta_i \alpha_{\widetilde{\mu}}$ holds $(\mathbb{P} \otimes O)$-a.e. for $i = 1, \cdots, N$. As a consequence, the identities of (2.3.8)

$$\widetilde{\alpha}_i = \frac{\alpha_{\widetilde{\mu}}}{c_{\widetilde{\mu\mu}}} c_{\widetilde{i\mu}} = bc_{\widetilde{i\mu}}, \qquad \text{hold for} \quad i = 1, \cdots, N, \quad (\mathbb{P} \otimes O) - \text{a.e. on} \quad \{c_{\widetilde{\mu\mu}} > 0\},$$

with b given as in (2.3.10). Also, the $(\mathbb{P} \otimes O)$-a.e. identities $\widetilde{\alpha}_i = \beta_i \alpha_{\widetilde{\mu}}$, combined with $\alpha_{\widetilde{\mu}} = \widetilde{\mu}'\alpha$, lead to the condition (B). Finally, $b = \widetilde{\alpha}_i / c_{\widetilde{i\mu}}$ on $\{c_{\widetilde{\mu\mu}} > 0,\ c_{\widetilde{i\mu}} \neq 0\}$ implies that $|b|\mathbf{1}_{\{c_{\widetilde{\mu\mu}}>0\}}|c_{\widetilde{i\mu}}| \leq |\widetilde{\alpha}_i|$ hold for $i = 1, \cdots, N$, and thus the condition (2.3.7):

$$\sum_{i=1}^{N} \int_0^T |b(t)|\mathbf{1}_{\{c_{\widetilde{\mu\mu}}>0\}}|dC_{\widetilde{i\mu}}(t)| \leq \sum_{i=1}^{N} \int_0^T |d\widetilde{A}_i(t)| < \infty, \qquad \text{for all} \quad T \geq 0.$$

Conversely, suppose that the conditions (A) and (B) are valid. For $i = 1, \cdots, N$, defining β_i via (2.3.11), we have

$$\int_0^T |\beta_i(t)||dA_{\widetilde{\mu}}(t)| \leq \int_0^T |b(t)|\mathbf{1}_{\{c_{\widetilde{\mu\mu}}>0\}}|dC_{\widetilde{i\mu}}(t)| + \int_0^T |d\widetilde{A}_i(t)| < \infty,$$

as well as

$$\int_0^T |\beta_i(t)|^2 dC_{\widetilde{\mu\mu}}(t) = \int_0^T \frac{|c_{\widetilde{i\mu}}(t)|^2}{c_{\widetilde{\mu\mu}}(t)} \mathbf{1}_{\{c_{\widetilde{\mu\mu}}>0\}} dO(t) \leq \int_0^T c_{\widetilde{i,i}}(t)dO(t) = \widetilde{C}_{i,i}(T) < \infty.$$

These inequalities imply that $\beta_i \in \mathcal{I}(R_{\widetilde{\mu}})$ for $i = 1, \cdots, N$. Furthermore, recalling the semimartingale decomposition (2.1.34), we observe that

$$\begin{aligned}\int_0^{\cdot} \beta_i(t) dR_{\widetilde{\mu}}(t) &= \int_0^{\cdot} \beta_i(t)\widetilde{\mu}'(t) d\widetilde{A}(t) + \int_0^{\cdot} \beta_i(t)\widetilde{\mu}'(t) d\widetilde{M}(t) \\ &= \int_0^{\cdot} \beta_i(t)\mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)>0\}} b(t) dC_{\widetilde{\mu}\widetilde{\mu}}(t) + \int_0^{\cdot} \beta_i(t)\mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)=0\}} \widetilde{\mu}'(t) d\widetilde{A}(t) \\ &\quad + \int_0^{\cdot} \beta_i(t)\widetilde{\mu}'(t) d\widetilde{M}(t),\end{aligned} \tag{2.3.12}$$

from (2.3.10). The first two integrals on the right hand side of (2.3.12) can be expressed as

$$\int_0^{\cdot} \beta_i(t)\mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)>0\}} b(t) dC_{\widetilde{\mu}\widetilde{\mu}}(t) = \int_0^{\cdot} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)>0\}} b(t) dC_{\widetilde{i}\widetilde{\mu}}(t) = \int_0^{\cdot} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)>0\}} d\widetilde{A}_i(t),$$

and

$$\int_0^{\cdot} \beta_i(t)\mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)=0\}} \widetilde{\mu}'(t) d\widetilde{A}(t) = \int_0^{\cdot} \mathbf{1}_{\{c_{\widetilde{\mu}\widetilde{\mu}}(t)=0\}} d\widetilde{A}_i(t),$$

for $i = 1, \cdots, N$, on account of (2.3.11). Thus, we obtain

$$\int_0^{\cdot} \beta_i(t) dR_{\widetilde{\mu}}(t) = \widetilde{A}_i + \int_0^{\cdot} \beta_i(t)\widetilde{\mu}'(t) d\widetilde{M}(t) = \widetilde{R}_i - \int_0^{\cdot} \big(e^i - \beta_i(t)\widetilde{\mu}(t)\big)' d\widetilde{M}(t) =: \widetilde{R}_i - N_i,$$

which is (2.3.6), where we define $N_i = \int_0^{\cdot} \big(e^i - \beta_i(t)\widetilde{\mu}(t)\big)' d\widetilde{M}(t)$ for $i = 1, \cdots, N$. We observe that the identities $(e^i - \beta_i\widetilde{\mu})'\widetilde{c}\widetilde{\mu} = c_{\widetilde{i}\widetilde{\mu}} - \beta_i c_{\widetilde{\mu}\widetilde{\mu}} = 0$ hold on the set $\{c_{\widetilde{\mu}\widetilde{\mu}} > 0\}$ from the definition (2.3.11), as well as on the set $\{c_{\widetilde{\mu}\widetilde{\mu}} = 0\}$ since $c_{\widetilde{i}\widetilde{\mu}} = 0$ holds there. Finally, we obtain

$$[N_i, R_{\widetilde{\mu}}] = \int_0^{\cdot} \big(e^i - \beta_i(t)\widetilde{\mu}(t)\big)' \widetilde{c}(t)\widetilde{\mu}(t) dO(t) \equiv 0, \qquad i = 1, \cdots, N,$$

which shows that the top n market is in the realm of the CAPM. □

2.3.3 *Functionally Generated Portfolios*

Functionally generated portfolios were discussed (in a pathwise sense) in Chapter 1. In this subsection, we adopt the original method of (multiplicative) functional generation of portfolios, which was introduced in [9], to construct stock portfolios in the top n open market. Given a function $G : \Delta_+^{N-1} \to (0, \infty)$ of class C^2 with the notation

$$\Delta_+^{N-1} := \Delta^{N-1} \cap \mathbb{R}_+^N = \Big\{(x_1, \cdots, x_N) \in \mathbb{R}^N \,\Big|\, x_i \geq 0 \text{ for } i = 1, \cdots, N, \ \sum_{i=1}^{N} x_i = 1\Big\}, \tag{2.3.13}$$

we can generate a portfolio π^G from G, depending on the vector of market weights μ. The formula (11.2) of [10], colloquially known as the 'master formula', gives a simple way to compare the relative wealth process of π^G with respect to the 'market', namely, the market portfolio μ (see, Chapter III of [10] for an overview). In what follows, we impose extra conditions on so-generated portfolios to make them invest only in the top n stocks, and derive a new master formula to compare the wealth of these portfolios, relative to $\widetilde{\mu}$, the market portfolio among the top n stocks. The reason of using Fernholz's original approach, rather than newer methods presented in Chapter 1, will be explained later.

Recalling the rank notation in Definition 2.1.2, we define the random permutation process $p_k(t)$ of $\{1, \cdots, N\}$ such that for $k = 1, \cdots, N$,

$$S_{p_k(t)}(t) = S_{(k)}(t), \tag{2.3.14}$$

$$p_k(t) < p_{k+1}(t) \quad \text{if} \quad S_{(k)}(t) = S_{(k+1)}(t).$$

$p_k(t)$ represents the index name of the stock at rank k at time t, breaking ties with the lexicographic rule, so it is the inverse permutation of $u_i(t)$, introduced in (2.1.3): $u_i(t) = k \iff p_k(t) = i$, for all $t \geq 0$.

For any continuous semimartingale Y, we denote the local time accumulated at the origin by $Y(\cdot)$ up to time $t \geq 0$ by $L^Y(t)$;

$$L^Y(t) := \frac{1}{2}\bigg(|Y(t)| - |Y(0)| - \int_0^t \operatorname{sign}\big(Y(s)\big)\,dY(s)\bigg), \quad \text{where} \quad \operatorname{sign}(x) = 2 \times \mathbf{1}_{(0,\infty)}(x) - 1.$$

Then, $L^{S_{(k)}-S_{(\ell)}}(t)$ can be interpreted as the 'collision local time' accumulated up to time t, whenever the k-th and ℓ-th ranked processes of S collide. In order to simplify the local time terms throughout this subsection, we introduce the following definition which prohibits the accumulation of local times of 'triple collisions' between the stock prices.

Definition 2.3.6. The components of the price vector $S = (S_1, \cdots, S_N)$ in Definition 2.1.1 are called *pathwise mutually nondegenerate*, if

(i) the set $\{t : S_i(t) = S_j(t)\}$ has Lebesgue measure zero, $\mathbb{P}$-a.e., for all $i \neq j$;

(ii) $L^{S_{(k)}-S_{(\ell)}}(t) \equiv 0$ holds $\mathbb{P}$-a.e., for $|k - \ell| \geq 2$.

Proposition 2.3.7. *Suppose that the components of the price vector S are pathwise mutually nondegenerate. Then, with the notation* (2.3.5)*, the wealth process $X_{\widetilde{\mu}}$ of $\widetilde{\mu}$ admits the representation*

$$X_{\widetilde{\mu}}(\cdot) = \frac{\widetilde{\Sigma}(\cdot)}{\widetilde{\Sigma}(0)} \exp\Big(-\frac{1}{2}\int_0^{\cdot} \frac{1}{\widetilde{\Sigma}(t)} dL^{S_{(n)}-S_{(n+1)}}(t)\Big). \tag{2.3.15}$$

Proof. From Proposition 2.2.1 and the fact that $\widetilde{\mu}$ is a stock portfolio, we have

$$X_{\widetilde{\mu}}(\cdot) = \mathcal{E}\Big(\int_0^{\cdot} \sum_{i=1}^{N} \widetilde{\mu}_i(t) dR_i(t)\Big) = \mathcal{E}\Big(\int_0^{\cdot} \sum_{i=1}^{N} \frac{\widetilde{S}_i(t)}{\widetilde{\Sigma}(t) S_i(t)} dS_i(t)\Big) = \mathcal{E}\Big(\int_0^{\cdot} \sum_{i=1}^{N} \sum_{k=1}^{n} \frac{\mathbf{1}_{\{u_i(t)=k\}}}{\widetilde{\Sigma}(t)} dS_i(t)\Big).$$

On the other hand, from Proposition 4.1.11 of [7], we have

$$\sum_{i=1}^{N} \mathbf{1}_{\{u_i(t)=k\}} dS_i(t) = dS_{(k)}(t) - \frac{1}{2} dL^{S_{(k)}-S_{(k+1)}}(t) + \frac{1}{2} dL^{S_{(k-1)}-S_{(k)}}(t),$$

for $k = 1, \cdots, N$ and $t \geq 0$, with the conventions $L^{S_{(0)}-S_{(1)}} \equiv 0$ and $L^{S_{(N)}-S_{(N+1)}} \equiv 0$. Thus, we obtain

$$X_{\widetilde{\mu}}(\cdot) = \mathcal{E}\Big(\int_0^{\cdot} \sum_{k=1}^{n} \frac{dS_{(k)}(t)}{\widetilde{\Sigma}(t)} + \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n} \frac{dL^{S_{(k-1)}-S_{(k)}}(t)}{\widetilde{\Sigma}(t)} - \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n} \frac{dL^{S_{(k)}-S_{(k+1)}}(t)}{\widetilde{\Sigma}(t)}\Big)$$

$$= \mathcal{E}\Big(\int_0^{\cdot} \frac{d\widetilde{\Sigma}(t)}{\widetilde{\Sigma}(t)} - \frac{1}{2}\int_0^{\cdot} \frac{dL^{S_{(n)}-S_{(n+1)}}(t)}{\widetilde{\Sigma}(t)} \Big) = \frac{\widetilde{\Sigma}(\cdot)}{\widetilde{\Sigma}(0)} \exp\Big(-\frac{1}{2}\int_0^{\cdot} \frac{dL^{S_{(n)}-S_{(n+1)}}(t)}{\widetilde{\Sigma}(t)} \Big).$$

□

The exponential term of (2.3.15) captures the 'leakage', the effect caused by stocks which cross over from the top n league to the bottom. Due to this effect, we need to keep track of the collision local time $L^{S_{(n)}-S_{(n+1)}}$ in order to compute $X_{\widetilde{\mu}}$.

We next present Fernholz's original method of constructing rank-dependent portfolios from generating functions. We write $\mu_{(k)}$ to represent the k-th ranked market weight among $\mu_1, \cdots, \mu_N$ for $k = 1, \cdots, N$, and introduce the vector $\boldsymbol{\mu} = (\mu_{(1)}, \cdots, \mu_{(N)})$ with components $\mu_{(k)} = S_{(k)}/\Sigma$, $k = 1, \cdots, N$, as in (2.1.1), (2.3.1). The following result is based on Theorem 4.2.1 of [7].

Theorem 2.3.8 (Functionally generated portfolios based on ranked market weights)**.** *Suppose that the price vector S is pathwise mutually nondegenerate. Let $p_k(\cdot)$, $k = 1, \cdots, N$ be the random permutation process defined by* (2.3.14) *and let $\boldsymbol{G}$ be a function defined on a neighborhood U of Δ_+^{N-1}. Suppose that there exists a positive C^2 function G such that for $(x_1, \cdots, x_N) \in U$,*

$$\boldsymbol{G}(x_1, \cdots, x_N) = G(x_{(1)}, \cdots, x_{(N)}). \tag{2.3.16}$$

Then $\boldsymbol{G}$ generates the stock portfolio $\pi^{\boldsymbol{G}}$ such that for $k = 1, \cdots, N$,

$$\pi^{\boldsymbol{G}}_{p_k(t)}(t) = \left(\frac{D_k G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{\ell=1}^{N} \mu_{(\ell)} \frac{D_\ell G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} \right) \mu_{(k)}(t), \qquad \textit{for} \quad t \geq 0. \tag{2.3.17}$$

The log-relative wealth process of $\pi^{\boldsymbol{G}}$ with respect to the market portfolio μ, can be expressed via the 'master formula' :

$$\log\left(\frac{X_{\pi^{\boldsymbol{G}}}}{X_\mu}\right) = \log\left(\frac{G(\boldsymbol{\mu})}{G\big(\boldsymbol{\mu}(0)\big)}\right) - \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{N-1} \left(\frac{\pi^{\boldsymbol{G}}_{p_k(t)}(t)}{\mu_{(k)}(t)} - \frac{\pi^{\boldsymbol{G}}_{p_{k+1}(t)}(t)}{\mu_{(k+1)}(t)} \right) dL^{\mu_{(k)}-\mu_{(k+1)}}(t)$$
$$- \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{D^2_{k,\ell} G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} d[\mu_{(k)}, \mu_{(\ell)}](t). \tag{2.3.18}$$

The portfolio $\pi^{\mathbf{G}}$ generated via the recipe (2.3.17) is easily checked to be a stock portfolio, i.e., $\pi^{\mathbf{G}} \in \mathcal{I}(R) \cap \Delta^{N-1}$; however, it is not generally a portfolio among the top n stocks, because $\pi^{\mathbf{G}}_{p_k(t)}(t)$ may have a nonzero value for $k > n$ at some time $t \geq 0$. In order to make it a portfolio among the top n stocks, we need to impose two conditions on the function G in Theorem 2.3.8:

(A) G is 'balanced', i.e., satisfies the identity

$$G(x_1, \cdots, x_N) = \sum_{j=1}^{N} x_j D_j G(x_1, \cdots, x_N), \qquad \text{for} \quad \text{any} \quad x \in U, \tag{2.3.19}$$

(B) $G(x)$ depends only on the first n components of x.

If the condition (A) is satisfied, then the portfolio $\pi^{\mathbf{G}}$ of (2.3.17) has a simpler representation as

$$\pi^{\mathbf{G}}_{p_k(t)}(t) = \frac{D_k G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} \mu_{(k)}(t), \qquad \text{for} \quad t \geq 0. \tag{2.3.20}$$

Moreover, if the condition (B) holds as well, then $D_k G\big(\boldsymbol{\mu}\big) = 0$ for $k > n$, thus $\pi^{\mathbf{G}}_{p_k(t)}(t) = 0$ for $k > n$. This means that the portfolio $\pi^{\mathbf{G}}$ does not invest in the $i = p_k(t)$-th stock at time t, if the rank k of this i-th stock is bigger than n at time t.

Definition 2.3.9 (Admissible generating function in open market)**.** We call a function G in Theorem 2.3.8 an *admissible generating function of market consisting of the top n stocks*, if it satisfies conditions (A) and (B) above.

Corollary 2.3.10. *If G in Theorem 2.3.8 is an admissible generating function of market consisting of top n stocks, then* **G** *generates the stock portfolio among the top n stocks* $\pi^{\mathbf{G}} \in \mathcal{I}(R) \cap \mathcal{T}(n) \cap \Delta^{N-1}$*, given as* (2.3.20) *for* $k = 1, \cdots, N$*. In this case, we have the master formula*

$$\log\left(\frac{X_{\pi^{\mathbf{G}}}}{X_{\mu}}\right) = \log\left(\frac{G(\boldsymbol{\mu})}{G\big(\boldsymbol{\mu}(0)\big)}\right) - \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n} \left(\frac{D_k G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} - \frac{D_{k+1} G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)}\right) dL^{\mu_{(k)}-\mu_{(k+1)}}(t)$$
$$- \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{D^2_{k,\ell} G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} d[\mu_{(k)}, \mu_{(\ell)}](t). \tag{2.3.21}$$

Example 2.3.11 (Balanced functions). By solving the partial differential equation of (2.3.19), a balanced function G can be shown to be homogeneous of degree 1, i.e, the identity

$$G(ax) = aG(x) \tag{2.3.22}$$

holds for any $x \in U$ and $a > 0$. From this simple characterization of balanced functions, we illustrate three types of balanced functions here:

(i) $G(x) = \frac{1}{c_1 + \cdots + c_N} \sum_{i=1}^N c_i x_i$,

(ii) $G(x) = \big(\prod_{i=1}^N x_i \big)^{1/N}$,

(iii) $G(x) = \big(\sum_{i=1}^N x_i^p \big)^{\frac{1}{p}}$.

These functions are closely related to 'three Pythagorean means'; (i) and (ii) are just the weighted-arithmetic and geometric means of the components of x, and (iii) becomes the harmonic mean when $p = -1$. A plethora of examples of these types can be found in the literature. The "capitalization-weighted portfolios" of large and small stocks from Example 6.2, Example 6.3 of [15], or from Example 4.3.2 of [7] are special cases of (i). The "equal-weighted portfolio", which holds equal weights across all assets, in Section 4.3 of [15], is generated by (ii). The portfolio generated by (iii) for $0 < p < 1$ is called "diversity-weighted portfolio", and is discussed in detail in Example 3.4.4 and Section 6.2 of [7]. Diversity-weighted portfolios with negative parameter $p < 0$ in (iii) are the main subject of [27].

We can slightly generalize and make these functions satisfy conditions (A) and (B) as well:

(i)' $G(x) = \sum_{i=1}^n c_i x_i$,

(ii)' $G(x) = \prod_{i=1}^n x_i^{c_i}, \quad \text{with} \quad \sum_{i=1}^n c_i = 1$,

(iii)' $G(x) = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}$,

for some constants c_i's and p.

The following example further devolops Example 4.3.2 of [7], and shows that the top n market portfolio $\widetilde{\mu}$, defined in (2.3.4), can be generated functionally.

Example 2.3.12 (Top n market portfolio). Consider the function

$$\mathbf{G}(x) = G(x_{(1)}, \cdots, x_{(n)}) = \sum_{k=1}^{n} x_{(k)}$$

satisfying the conditions (A) and (B) above. Corollary 2.3.10 implies that $\mathbf{G}$ generates the portfolio

$$\pi^{\mathbf{G}}_{p_k(\cdot)}(\cdot) = \frac{\mu_{(k)}(\cdot)}{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)}\mathbf{1}_{\{k\le n\}} = \frac{S_{(k)}(\cdot)}{S_{(1)}(\cdot) + \cdots + S_{(n)}(\cdot)}\mathbf{1}_{\{k\le n\}}.$$

This coincides with the top n market portfolio $\widetilde{\mu}$, because

$$\frac{S_{(k)}(\cdot)\mathbf{1}_{\{k\le n\}}}{S_{(1)}(\cdot) + \cdots + S_{(n)}(\cdot)} = \frac{S_{p_k(\cdot)}(\cdot)\mathbf{1}_{\{k\le n\}}}{\widetilde{\Sigma}(\cdot)} = \widetilde{\mu}_{p_k(\cdot)}(\cdot),$$

holds for $k = 1, \cdots, N$, from (2.3.4). The master formula (2.3.21) is then

$$\log\left(\frac{X_{\widetilde{\mu}}}{X_{\mu}}\right) = \log\left(\frac{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)}{\mu_{(1)}(0) + \cdots + \mu_{(n)}(0)}\right) - \frac{1}{2}\int_0^{\cdot} \frac{dL^{\mu_{(n)}-\mu_{(n+1)}}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(n)}(t)}. \tag{2.3.23}$$

In Corollary 2.3.10, the portfolio $\pi^{\mathbf{G}}$ is indeed a stock portfolio among the top n stocks; but the master formula (2.3.21) compares its performance with the market portfolio μ, which is not a portfolio among the top n stocks. In the open market setting, since we only consider portfolios among the top n stocks, it is more appropriate to compare a portfolio's performance with respect to $\widetilde{\mu}$, rather than μ. This can be done by combining (2.3.21) and (2.3.23).

Corollary 2.3.13 (Master formula in top n market)**.** *For a functionally generated portfolio $\pi^{\mathbf{G}}$ as in Corollary 2.3.10, the master formula, which compares the log-relative wealth of $\pi^{\mathbf{G}}$ to that generated by $\widetilde{\mu}$, the top n market, is given as*

$$\log\left(\frac{X_{\pi^{\mathbf{G}}}}{X_{\widetilde{\mu}}}\right) = \log\left(\frac{G(\boldsymbol{\mu})}{G\big(\boldsymbol{\mu}(0)\big)}\right) - \log\left(\frac{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)}{\mu_{(1)}(0) + \cdots + \mu_{(n)}(0)}\right) \tag{2.3.24}$$

$$- \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n} \left(\frac{D_k G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} - \frac{D_{k+1} G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} \right) dL^{\mu_{(k)}-\mu_{(k+1)}}(t)$$

$$+ \frac{1}{2}\int_0^{\cdot} \frac{dL^{\mu_{(n)}-\mu_{(n+1)}}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(n)}(t)} - \frac{1}{2}\int_0^{\cdot} \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{D^2_{k,\ell} G\big(\boldsymbol{\mu}(t)\big)}{G\big(\boldsymbol{\mu}(t)\big)} d[\mu_{(k)}, \mu_{(\ell)}](t).$$

We call this formula of (2.3.24), the 'master formula for the top n market' to distinguish it from the formula of (2.3.18), which we call the 'master formula in the entire market'.

Example 2.3.14 (Diversity-weighted portfolio). Consider a function

$$\mathbf{G}(x) = G(x_{(1)}, \cdots, x_{(N)}) = \left(\sum_{k=1}^{n} x_{(k)}^p \right)^{1/p}$$

with a fixed constant $p \in (0, 1)$. Corollary 2.3.10 implies that $\mathbf{G}$ generates the "diversity-weighted portfolio"

$$\pi^{\mathbf{G}}_{p_k(\cdot)}(\cdot) = \frac{\mu^p_{(k)}(\cdot)}{\mu^p_{(1)}(\cdot) + \cdots + \mu^p_{(n)}(\cdot)} \mathbf{1}_{\{k \le n\}}, \qquad k = 1, \cdots, N.$$

The master formula in the top n market in (2.3.24) is then given as

$$\begin{aligned}
\log\left(\frac{X_{\pi^{\mathbf{G}}}}{X_{\widetilde{\mu}}}\right) &= \frac{1}{p}\log\left(\frac{\mu^p_{(1)}(\cdot) + \cdots + \mu^p_{(n)}(\cdot)}{\mu^p_{(1)}(0) + \cdots + \mu^p_{(n)}(0)}\right) - \log\left(\frac{\mu_{(1)}(\cdot) + \cdots + \mu_{(n)}(\cdot)}{\mu_{(1)}(0) + \cdots + \mu_{(n)}(0)}\right) \\
&- \frac{1}{2}\int_0^{\cdot} \frac{\mu^{p-1}_{(n)}(t)}{\mu^p_{(1)}(t) + \cdots + \mu^p_{(n)}(t)} dL^{\mu_{(n)}-\mu_{(n+1)}}(t) + \frac{1}{2}\int_0^{\cdot} \frac{dL^{\mu_{(n)}-\mu_{(n+1)}}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(n)}(t)} \qquad (2.3.25) \\
&- \frac{1-p}{2}\int_0^{\cdot} \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{\mu^{p-1}_{(k)}(t)\mu^{p-1}_{(\ell)}(t)}{\big(\mu^p_{(1)}(t) + \cdots + \mu^p_{(n)}(t)\big)^2} d[\mu_{(k)}, \mu_{(\ell)}](t) \\
&+ \frac{1-p}{2}\int_0^{\cdot} \sum_{k=1}^{n} \frac{\mu^{p-2}_{(k)}(t)}{\mu^p_{(1)}(t) + \cdots + \mu^p_{(n)}(t)} d[\mu_{(k)}, \mu_{(k)}](t).
\end{aligned}$$

Here, in the first integral of (2.3.25), we use the fact that the local time process $L^{\mu_{(k)}-\mu_{(k+1)}}(\cdot)$ is flat off the set $\{s \ge 0 : \mu_{(k)}(s) = \mu_{(k+1)}(s)\}$ for $k = 1, \cdots, n-1$.

We now conclude this subsection with some remarks. We used Theorem 2.3.8 as the starting point of the subsection and this is based on Fernholz's multiplicative generation of portfolios;

compare (2.3.17) with (1.3.35). In contrast, the additive generation discussed in subsection 1.3.1 is not appropriate in open markets, because additively generated strategies (and corresponding portfolios) allocate ‘cumulative earnings’, represented by Γ^G, uniformly across all stocks in the market. The existence of Γ^G term in (1.3.19) or (1.3.21) makes these strategies impossible to avoid investing in those stocks with $\vartheta_i \equiv \nabla_i G \equiv 0$; whereas the multiplicatively generated strategy ψ in (1.3.34) invests nothing in the i-th stock if the generating function is balanced and $\vartheta_i \equiv \nabla_i G \equiv 0$. The conditions (A) and (B) above were imposed exactly to achieve this.

Another reason of adopting Theorem 2.3.8 is to use the master formula (2.3.18). This compares explicitly the log values of two different wealth processes $X_{\pi^{\mathbf{G}}}$ and X_μ. In chapter 1, as our goal was to construct strategies (or portfolios) which outperform the market portfolio μ, we computed the value V^ϑ of strategy ϑ *relative to* μ. However, in open markets, we need to compare the performance of portfolios with respect to $\widetilde{\mu}$ and this was done by successive usage of two master formulae (2.3.21) and (2.3.23).

2.3.4 *Universal Portfolio*

We explore in this subsection a universal portfolio in open markets. This portfolio was first introduced by Cover [4] in discrete time, and its extension to continuous time was developed in [12]. More recent work under the setting of Stochastic Portfolio Theory can be found in [5].

Recalling the notation Δ_+^{N-1} from (2.3.13), we need first the following notation

$$\Delta_+^{N-1,n} := \Big\{ x \in \mathbb{R}^N \,\Big|\, x_k \geq 0 \;\text{ for }\; k = 1, \cdots, N, \;\; \sum_{k=1}^{n} x_k = 1, \;\; x_{n+1} = \cdots = x_N = 0 \Big\} \qquad (2.3.26)$$

throughout this subsection. Since we are only allowed to invest in the top n stocks in an open market, the notion of Cover’s ‘constant rebalanced portfolio’ needs to be amended, as follows.

Definition 2.3.15 (constant rebalanced portfolio by rank)**.** If a stock portfolio $\pi \in \mathcal{I}(R) \cap \mathcal{T}(n) \cap$

Δ^{N-1} among the top n stocks satisfies

$$\pi_{p_k(t)}(t) = \xi_k \quad \text{for} \quad t \geq 0, \quad k = 1, \cdots, N \tag{2.3.27}$$

with some $\xi = (\xi_1, \cdots, \xi_N) \in \Delta_+^{N-1,n}$, we call π a *constant rebalanced portfolio among the top n stocks by rank*. This portfolio re-balances at all times to maintain a constant proportion ξ_k of current wealth invested in the k-th ranked stock, for $k \leq n$. We denote the collection of constant rebalanced portfolios among the top n stocks by $\mathcal{CR}^n$.

Proposition 2.3.16. *Every constant rebalanced portfolio among the top n stocks by rank is functionally generated.*

Proof. For a fixed $\xi \in \Delta_+^{N-1,n}$, consider a function

$$\mathbf{G}(x) = G(x_{(1)}, \cdots, x_{(N)}) = \prod_{k=1}^{n} x_{(k)}^{\xi_k}. \tag{2.3.28}$$

It is easy to check that G is an admissible generating function of market consisting of the top n stocks, and it generates the portfolio via the recipe (2.3.20):

$$\pi^{\mathbf{G}}_{p_k(t)}(t) = \xi_k, \quad \text{for} \quad t \geq 0, \quad k = 1, \cdots, N.$$

Since ξ is chosen arbitrarily from $\Delta_+^{N-1,n}$, the claim follows. □

Thanks to Proposition 2.3.16, for every $\xi \in \Delta_+^{N-1,n}$ there exists a corresponding portfolio $\pi \in \mathcal{CR}^n$ as in Definition 2.3.15, and we write $X_\xi(t)$ to represent the wealth process of π at time t in the manner of (2.1.21), namely,

$$X_\xi(t) \equiv X_\pi(t) = \mathcal{E}\left(\int_0^t \sum_{i=1}^{N} \pi_i(s) dR_i(s)\right) = \mathcal{E}\left(\sum_{k=1}^{n} \xi_k \int_0^t \sum_{i=1}^{N} \mathbb{1}_{\{u_i(s)=k\}} dR_i(s)\right) \qquad \text{for} \quad t \geq 0. \tag{2.3.29}$$

For $T > 0$ fixed, we define

$$X^*(T) := \sup_{\pi \in \mathcal{CR}^n} X_\pi(T) = \sup_{\xi \in \Delta_+^{N-1,n}} X_\xi(T). \tag{2.3.30}$$

This $X^*(T)$ represents the maximal wealth at time T, achievable over all constant rebalanced portfolios among the top n stocks by rank. We show in the following that a $\mathcal{F}(T)$-measurable random vector of weights $\pi^*(T) \equiv \xi^*$ exists, which attains the supremum in (2.3.30), namely, that $X^*(T) = X_{\pi^*(T)}(T) = X_{\xi^*}(T)$ holds.

Lemma 2.3.17. *For a fixed $T > 0$, the mapping $\Delta_+^{N-1,n} \ni \xi \mapsto X_\xi(T) \in \mathbb{R}$ is continuous.*

Proof. For $\xi, \zeta \in \Delta_+^{N-1,n}$, we have

$$\begin{aligned}
\log X_\xi(T) - \log X_\zeta(T) &= \log\left(\frac{X_\xi(T)}{X_{\widetilde{\mu}}(T)}\right) - \log\left(\frac{X_\zeta(T)}{X_{\widetilde{\mu}}(T)}\right) \\
&= \sum_{k=1}^{n} \log\left(\frac{\mu_{(k)}(T)}{\mu_{(k)}(0)}\right)^{(\xi_k - \zeta_k)} \\
&\quad - \frac{1}{2}\int_0^T \sum_{k=1}^{n-1}\left(\frac{\xi_k - \zeta_k}{\mu_{(k)}(t)} - \frac{\xi_{k+1} - \zeta_{k+1}}{\mu_{(k+1)}(t)}\right) dL^{\mu_{(k)}-\mu_{(k+1)}}(t) - \frac{1}{2}\int_0^T \frac{\xi_n - \zeta_n}{\mu_{(n)}(t)} dL^{\mu_{(n)}-\mu_{(n+1)}}(t) \\
&\quad - \frac{1}{2}\int_0^T \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{\xi_k\xi_\ell - \zeta_k\zeta_\ell}{\mu_{(k)}(t)\mu_{(\ell)}(t)} d[\mu_{(k)}, \mu_{(\ell)}](t) + \frac{1}{2}\int_0^T \sum_{k=1}^{n} \frac{\xi_k\zeta_k}{\mu_{(k)}^2(t)} d[\mu_{(k)}, \mu_{(k)}](t).
\end{aligned} \tag{2.3.31}$$

In the last equality, we used the master formula (2.3.24) twice, and applied it to the functions of the form (2.3.28) for ξ and ζ, respectively. Since the functions $\mu_{(k)}(\cdot)$, $L^{\mu_{(k)}-\mu_{(k+1)}}(\cdot)$, $[\mu_{(k)}, \mu_{(\ell)}](\cdot)$ for $1 \leq k, \ell \leq n$ on the right-hand side (2.3.31) are all continuous, they are bounded on the compact interval $[0, T]$. Thus, we obtain the estimate $\left|\log X_\xi(T) - \log X_\zeta(T)\right| \leq ||\xi - \zeta||K_T$ for some positive constant K_T, which depends on $\min_{0 \leq t \leq T} \mu_{(k)}(t)$, $L^{\mu_{(k)}-\mu_{(k+1)}}(T)$, $[\mu_{(k)}, \mu_{(k)}](T)$ for $k = 1, \cdots, n$, and this proves the continuity. □

Definition 2.3.18 (Best retrospectively chosen vector of weights)**.** The continuity shown in Lemma 2.3.17 shows that there exists a vector $\xi^* \equiv \pi^*(T) \in \Delta_+^{N-1,n}$ which attains the supremum in (2.3.30) for a fixed $T \in (0, \infty)$. We call this $\mathcal{F}(T)$-measurable, $\Delta_+^{N-1,n}$-valued random variable $\pi^*(T)$ the *best retrospectively chosen vector of weights among the top n stocks for the given* $T \in (0, \infty)$.

Even though $\pi^*(T)$ was meant to outperform all constant rebalanced portfolios among the top n stocks by rank at $T > 0$, constructing it requires the knowledge of stock prices over the entire interval $[0, T]$, that is, ahead of time. Cover [4] introduced a remarkable way to construct a portfolio, called "universal portfolio", depending only on past stock prices, whose long-run performance is almost as good as that of the best retrospectively chosen vector of weights. Cover's idea of building the universal portfolio, was to determine its weights by averaging the performances of all constant portfolio weights, at any time $t \geq 0$.

Definition 2.3.19 (Universal portfolio)**.** With the notation $\Delta_+^{N-1,n}$ of (2.3.26), the portfolio $\hat{\pi}$, defined as

$$\hat{\pi}_{p_k(t)}(t) := \frac{\int_{\Delta_+^{N-1,n}} \xi_k X_\xi(t) d\xi}{\int_{\Delta_+^{N-1,n}} X_\xi(t) d\xi} \qquad \text{for} \quad t \geq 0, \quad k = 1, \cdots, N, \tag{2.3.32}$$

is called *universal portfolio among the top n stocks*.

From the notation $\Delta_+^{N-1,n}$, we have $\hat{\pi}_{p_k(t)}(t) = 0$ for all $t \geq 0$ for $k > n$; i.e., $\hat{\pi}$ invests only in the top n stocks, thus it belongs to $\mathcal{I}(R) \cap \mathcal{T}(n) \cap \Delta^{N-1}$, the collections of stock portfolios among the top n stocks. We next compute the wealth of the universal portfolio.

Proposition 2.3.20. *The wealth process $X_{\hat{\pi}}$ is given as*

$$X_{\hat{\pi}}(t) = \frac{\int_{\Delta_+^{N-1,n}} X_\xi(t) d\xi}{\int_{\Delta_+^{N-1,n}} d\xi}, \qquad \textit{for} \quad t \geq 0. \tag{2.3.33}$$

Proof. Let $Z(t)$ denote the right-hand side of (2.3.33). We have

$$\begin{aligned}
\frac{dZ(t)}{Z(t)} = \frac{\int_{\Delta_+^{N-1,n}} dX_\xi(t) d\xi}{\int_{\Delta_+^{N-1,n}} X_\xi(t) d\xi} &= \frac{\int_{\Delta_+^{N-1,n}} X_\xi(t) \sum_{i=1}^N \sum_{k=1}^n \xi_k \mathbb{1}_{\{u_i(t)=k\}} dR_i(t) d\xi}{\int_{\Delta_+^{N-1,n}} X_\xi(t) d\xi} \\
&= \sum_{i=1}^N \sum_{k=1}^n \hat{\pi}_{p_k(t)}(t) \mathbb{1}_{\{u_i(t)=k\}} dR_i(t) = \sum_{i=1}^N \hat{\pi}_i(t) dR_i(t) = \frac{dX_{\hat{\pi}}(t)}{X_{\hat{\pi}}(t)}.
\end{aligned}$$

Here, the second, third and last equalities are from (2.3.29), (2.3.32), and (2.1.21), respectively. Since $X_{\hat{\pi}}(0) = Z(0) = 1$, the result follows. $\square$

We are now ready to compare the long-run performance of the universal portfolio with the best retrospectively chosen vector of weights.

Theorem 2.3.21. *Suppose that the portfolio μ, defined in* (2.3.3)*, satisfies*

$$\mu_{(1)}(t) \geq \cdots \geq \mu_{(n)}(t) \geq \delta, \quad \textit{for all} \quad t \geq 0 \quad \textit{for some} \quad \delta > 0,$$
$$\limsup_{T\to\infty} \frac{1}{T}[\mu_{(k)}, \mu_{(k)}](T) < \infty, \qquad \limsup_{T\to\infty} \frac{1}{T} L^{\mu_{(k)}-\mu_{(k+1)}}(T) < \infty, \quad \textit{for} \quad k = 1, \cdots, n. \tag{2.3.34}$$

Then, the best retrospectively chosen vector of weights and the universal portfolio have the same asymptotic growth rate; that is,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_{\pi^*(T)}(T) - \log X_{\hat{\pi}}(T)\Big) = 0, \tag{2.3.35}$$

where $\pi^*(T)$ and $\hat{\pi}$ are as in Definitions 2.3.18 and 2.3.19, respectively.

Proof. Since $X_{\pi^*(T)}(T) \geq X_{\xi}(T)$ holds for every $\xi \in \Delta_+^{N-1,n}$ for every $T \geq 0$, the inequality "$\geq$" of (2.3.35) is obvious from (2.3.33).

We now show the reverse inequality. Let $\xi^* \in \Delta_+^{N-1,n}$ be the corresponding vector of weights $\pi^*(T)$ as in Definition 2.3.18. For any $\xi \in \Delta_+^{N-1,n}$ satisfying $||\xi^* - \xi|| \leq \eta$ for some $\eta > 0$, we have the estimate

$$\begin{aligned}\frac{1}{T}\Big(\log X_{\xi}(T) - \log X_{\xi^*}(T)\Big) &\geq -\frac{\eta}{T}\Big(\frac{a_n}{\delta} \max_{1\leq k\leq n} L^{\mu_{(k)}-\mu_{(k+1)}}(T) + \frac{b_n}{\delta^2} \max_{1\leq k\leq n} [\mu_{(k)}, \mu_{(k)}](T)\Big)\\ &=: -\frac{\eta}{T}K_T,\end{aligned}$$

in the same manner as in the proof of Lemma 2.3.17, for some positive constants a_n and b_n depending on n. Due to the condition (2.3.34), we can take η sufficiently small such that $\frac{\eta}{T}K_T \leq \epsilon$

holds for every $T \geq 1$, for any given $\epsilon > 0$. To summarize, for any given $\epsilon > 0$, there exists $\eta > 0$ such that

$$\frac{1}{T}\big(\log X_{\xi}(T) - \log X_{\xi^*}(T)\big) \geq -\epsilon \tag{2.3.36}$$

holds for every $\xi \in B(\xi^*, \eta)$ and for every $T \geq 1$. Here, $B(\xi^*, \eta)$ is the intersection of $\Delta_+^{N-1,n}$ and $||\cdot||$-ball in $\mathbb{R}^N$ centered at ξ^* with radius η. We denote $V_{B(\xi^*,\eta)}$ and $V_{\Delta_+^{N-1,n}}$ the volume of $B(\xi^*, \eta)$ and the volume of the subset $\Delta_+^{N-1,n}$ of $\mathbb{R}^N$, respectively.

From (2.3.33) and Jensen's inequality, we have

$$\begin{aligned}
\left(\frac{X_{\hat{\pi}}(T)}{X_{\pi^*(T)}(T)}\right)^{\frac{1}{T}} = \left(\frac{\int_{\Delta_+^{N-1,n}} X_{\xi}(T)d\xi}{X_{\xi^*}(T)\, V_{\Delta_+^{N-1,n}}}\right)^{\frac{1}{T}} &\geq \left(\frac{\int_{B(\xi^*,\eta)} X_{\xi}(T)d\xi}{X_{\xi^*}(T)\, V_{\Delta_+^{N-1,n}}}\right)^{\frac{1}{T}} \\
&\geq \frac{\big(V_{B(\xi^*,\eta)}\big)^{\frac{1}{T}-1} \int_{B(\xi^*,\eta)} X_{\xi}(T)^{\frac{1}{T}} d\xi}{\big(X_{\xi^*}(T)\big)^{\frac{1}{T}} \, \big(V_{\Delta_+^{N-1,n}}\big)^{\frac{1}{T}}} \\
&= \frac{\big(V_{B(\xi^*,\eta)}\big)^{\frac{1}{T}-1}}{\big(V_{\Delta_+^{N-1,n}}\big)^{\frac{1}{T}}} \int_{B(\xi^*,\eta)} \left(\frac{X_{\xi}(T)}{X_{\xi^*}(T)}\right)^{\frac{1}{T}} d\xi \geq \left(\frac{V_{B(\xi^*,\eta)}}{V_{\Delta_+^{N-1,n}}}\right)^{\frac{1}{T}} e^{-\epsilon},
\end{aligned}$$

where the last inequality is from (2.3.36). Taking logarithms, then letting $T \to \infty$ for any given $\epsilon > 0$, the desired inequality follows. □

2.4 Conclusion

Most of the results in Section 2.2, including the main Theorem 2.2.19, a foundational result of equity market structure and of the study of arbitrage in open markets, can be formulated quite simply in terms of the local characteristics $\widetilde{\alpha}$ and $\widetilde{c}$ of the open market, defined in (2.1.39), (2.1.40). In particular, the supermartingale numéraire portfolio ρ in the top n open market, if it exists, should satisfy the equation $\widetilde{\alpha} = \widetilde{c}\rho$ of (2.2.11). From this equation, we were able to conclude that the supermartingale numéraire portfolio ρ in the open market takes the form of $\rho = D\widetilde{c}^{\dagger}\widetilde{\alpha}$. Here, multiplying by the diagonal matrix D of (2.1.46) makes the portfolio invest only in the top n stocks, while maintaining its supermartingale numéraire property.

However, as foretold in the introductory part of Section 2.3, we cannot use this technique to deal with stock portfolios; multiplying by D a stock portfolio in order to make it invest only in the top n stocks, destroys its self-financing property. For example, a unit vector $\pi := e^1 = (1, 0, \cdots, 0)$ is a stock portfolio which invests all capital into the first stock, but $D\pi$ is not a stock portfolio as it invests all wealth into the money market whenever the first stock fails to belong to the top n market. Thus, for stock portfolios in open markets, a different approach is offered. Fernholz's functional generation of stock portfolios with ranked market weights, under the extra conditions (A) and (B) of Definition 2.3.9, provides a systematic way to construct stock portfolios that invest only in the top n open market. This approach also yields the 'master formula' in Corollary (2.3.13), which allows comparing these portfolios with the top n market portfolio, $\widetilde{\mu}$. As an application of this formula, we could prove that Cover's result on the universal portfolio is also valid in open markets.

Nonetheless, there are a lot of limitations when considering stock portfolios in open markets. First, the balance condition (2.3.19) significantly restricts the class of generating functions in open markets. Moreover, the local time terms which appear on the right-hand side of the master formula (2.3.24), make it very difficult to find stock portfolios in open markets which outperform $\widetilde{\mu}$. These difficulties are an inevitable price to pay for dealing with stock portfolios in open markets.

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