Time Evolution of the Kardar-Parisi-Zhang Equation

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Abstract

The use of the non-linear SPDEs are inevitable in both physics and applied mathematics since many of the physical phenomena in nature can be effectively modeled in random and non-linear way. The Kardar-Parisi-Zhang (KPZ) equation is well-known for its applications in describing various statistical mechanical models including randomly growing surfaces, directed polymers and interacting particle systems. We consider the upper and lower tail probabilities for the centered (by time/24) and scaled (according to KPZ time^{1/3} scaling) one-point distribution of the Cole-Hopf solution of the KPZ equation. We provide the first tight bounds on the lower tail probability of the one point distribution of the KPZ equation with narrow wedge initial data. Our bounds hold for all sufficiently large times $T$ and demonstrates a crossover between super-exponential decay with exponent $5/2$ (and leading pre-factor $4/15\pi T^{1/3}$) for tail depth greater than $T^{2/3}$ (deep tail), and exponent 3 (with leading pre-factor at least $1/12$) for tail depth less than $T^{2/3}$ (shallow tail). We also consider the case when the initial data is drawn from a very general class. For the lower tail, we prove an upper bound which demonstrates a crossover from super-exponential decay with exponent 3 in the shallow tail to an exponent $5/2$ in the deep tail. For the upper tail, we prove super-exponential decay bounds with exponent $3/2$ at all depths in the tail. We study the correlation of fluctuations of the narrow wedge solution to the KPZ equation at two different times. We show that when the times are close to each other, the correlation approaches one at a power-law rate with exponent $2/3$, while when the two times are remote from each other, the correlation tends to zero at a power-law rate with exponent $-1/3$. 
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The Kardar-Parisi-Zhang (KPZ) equation proposed in 1986 [KPZ86], has been applied to many fields such as the random growth of interfaces, transport in one-dimension (1D) and Burgers turbulence, directed polymers, chemical reaction fonts, bacterial growth, slow combustion, coffee stains, conductance fluctuations in Anderson localization, polar active fluids, Bose Einstein superfluids, quantum entanglement growth. The (1+1) dimensional KPZ describes the stochastic growth of an interface height function \( \mathcal{H}(T, X) \) for \( T > 0 \) and \( X \in \mathbb{R} \),

\[
\partial_t \mathcal{H}(T, X) = \frac{1}{2} \partial^2_X \mathcal{H}(T, X) + \frac{1}{2} (\partial_X \mathcal{H}(T, X))^2 + \xi(T, X)
\]

starting from the initial condition \( \mathcal{H}(T = 0, X) \). Here, \( \xi \) is a Gaussian process (often referred as the space-time white noise) taking values in the space of distributions. In particular, \( \xi \) induce a continuous linear map \( \phi \mapsto \xi(\phi) \) from the compactly supported smooth test functions in \( (0, \infty) \times \mathbb{R} \) into the space of square integrable random variable on some fixed probability space \( (\Omega, \mathbb{P}) \) and this map is characterized through the covariance function \( \mathbb{E}[\xi(\phi)\xi(\psi)] = \langle \phi, \psi \rangle \) where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \)-scalar product and \( \mathbb{E} \) is the expectation w.r.t. the probability measure \( \mathbb{P} \). In the physically relevant Cole-Hopf solution considered in [BG97b], it is specified that \( \mathcal{H}(T, X) = \log \mathcal{Z}(T, X) \) where \( \mathcal{Z} \) is the unique solution of the stochastic heat equation (SHE)

\[
\partial_T \mathcal{Z}(T, X) = \frac{1}{2} \partial^2_X \mathcal{Z}(T, X) + \mathcal{Z}(T, X)\xi(T, X).
\]

The mathematical analysis of the KPZ equation has offered many outstanding challenges: notably, for developing a robust solution theory; for showing that the equation accurately approximates the evolution of physical growth processes; and for studying properties of the solution’s probability distribution and various asymptotics. Indeed, the Cole-Hopf solution has been shown in [BG97b] to coincide with limits of certain discrete growth processes; while the development of solution theory has
been the object of intense recent activity, including the theory of regularity structures [Hai13]; energy solutions [GJ14], [GJ14]; paracontrolled distributions [GIP15, GP17]; and the renormalization group [KM17]. This thesis is mainly focused on the last two challenges mentioned at the beginning of this paragraph. More precisely, we study the one point tail probabilities of the KPZ equation and its time correlation. In Chapter 3 (based on [CGH19]), we investigate the correlation function of the solution of the KPZ equation at two different times.

One of the key inputs to the time correlation project is an estimate on the tail probabilities of the KPZ equation. The discussion on the tail probabilities will be based on [CG18a, CG18b] and are contained in Chapter 1 and 2. Our analysis is based on an exact identity (see Proposition 0.1.3 of Chapter 1) between the KPZ equation and the Airy point process (which arises at the edge of the spectrum of the random Hermitian matrices) and the Gibbs property of the KPZ line ensemble (see Section 2.2 of Chapter 2).

The rest of the chapter is organized as follows. Section 0.1 presents an informal description of two of the main tail probability results of [CG18a, CG18b]. Section 0.2 informally discusses the time correlations results of [CGH19]. Both of these two sections are accompanied by a brief discussions (see Section 0.2.1 and 0.2.1) on the proof ideas. Our works bear connections with different fields, namely, the integrable PDEs and Riemann-Hilbert techniques, free-Fermion models, large deviation theory, half-space KPZ, tail probabilities and time correlations in other integrable probabilistic systems. In Section 0.3 we will give an overview of those connections and extensions of our results. In Section 0.4 we discuss future directions and the ongoing works which are churned out of the main results of this thesis. We end this chapter with a discussion on the other research topics that I pursued during my PhD.

0.1 Tail probabilities of the KPZ equation

In a seminal work, [ACQ11] found the exact one point distribution of the fundamental solution of the KPZ equation, namely, the Cole-Hopf solution $\mathcal{H}^{nw}$ where the initial data for $Z^{nw}$ is the delta initial measure, i.e., $Z^{nw}(0, 0) = 1$ and $Z^{nw}(0, X) = 0$ for $X \neq 0$. The superscript $nw$ is used as an abbreviation of the word ‘narrow wedge’. The fundamental solution $\mathcal{H}^{nw}$ is often referred as the

\footnote{We denote the correlation function between any two random variables $X$ and $Y$ by $\text{Corr}(X, Y)$ which is given by the ratio of covariance $\text{Cov}(X, Y)$ and the product of the square root of variances of $X$ and $Y$.}
narrow wedge solution in recognition of the fact that the initial data $\log Z_{nw}(0, X)$ can be obtained by narrowing a wedge centered at 0. In a joint work with Prof. Ivan Corwin [CG18a, CG18b], we investigated the upper and lower tail probability of $H_{nw}$ uptosome centering and scaling

$$h_T(\alpha, X) = \frac{H_{nw}(\alpha T, T^{2/3}X) + \frac{\alpha T}{2T}}{T^{1/3}} + \frac{X^2}{2\alpha}.$$ (0.1.1)

In [CG18a, Theorem 1.1], we provide the first tight bounds on the lower tail probability of the one point distribution of $h_T(X)$. Our bounds hold for all sufficiently large times $T$ and demonstrates a crossover between super-exponential decay with exponent $\frac{5}{2}$ and exponent 3. In what follows, we present an informal statement of Theorem 1.1.1 of Chapter 1 (see also Proposition 3.2.4 of Chapter 3).

**Theorem 0.1.1 (Informal Statement of Theorem 1.1.1).** There exists $s_0 > 0$ such that for all $T > 0$ and $s > s_0$,

$$\mathbb{P}(h_T(2, X) \leq -s) = \Theta\left(\exp\left(-T^{\frac{1}{3}}\frac{4s^2}{15\pi}\right)\right) + \Theta\left(\exp\left(-\frac{s^3}{12}\right)\right).$$ (0.1.2)

Here, $y = \Theta(x)$ means that $y$ is upper and lower bounded by some constant multiples of $x$ where the associated constants does not depend on the value of $x$.

In contrast, the (super-exponential) decay exponent of the upper tail probability is always $\frac{3}{2}$ (see [CG18b, Theorem 1.10]) irrespective of the tail depth. In what follows, we state an informal version of Theorem 2.1.9 of Chapter 2.

**Theorem 0.1.2 (Informal statement of Theorem 2.1.9).** There exists $s_0 > 0$ such that for all $T > 1$ and $s > s_0$,

$$\mathbb{P}(h_T(2, X) \geq s) = \Theta\left(\exp\left(-\frac{4s^3}{3}\right)\right).$$ (0.1.3)

### 0.1.1 Proof Ideas

In this section, we give an overview of the set of tools that we have used for proving Theorem 0.1.1 and 0.1.2. One of our main tools for showing Theorem 0.1.1 is the following identity which connects the Laplace transform of $Z_{nw}$ to the Airy point process. See Section 1.1.1 for discussion on its origin and related identities.
Proposition 0.1.3 (Theorem 2.2 of [BG16]). Let $Z^{\text{nw}}(T, X)$ be the unique solution to the SHE with $Z^{\text{nw}}(0, X) = \delta_{X=0}$. Denote the ordered points of the Airy point process (Section 1.3) by $a_1 > a_2 > \ldots$. Then, for any $T, u > 0$, we have\footnote{A similar result holds for any $X$ up to multiplying $Z^{\text{nw}}$ by a Gaussian factor – see [ACQ11, Proposition 1.4].}

\[
E_{\text{SHE}} \left[ \exp \left( -u Z^{\text{nw}}(2T, 0) \exp \left( \frac{T^2}{12} \right) \right) \right] = E_{\text{Airy}} \left[ \prod_{k=1}^{\infty} \frac{1}{1 + u \exp \left( \frac{T^3}{12} a_k \right)} \right] \quad (0.1.4)
\]

Setting $u = \exp \left( \frac{T^3}{12} s \right)$ and rewriting the above result in terms of $\Upsilon_T$ from (1.1.2), we find

\[
E_{\text{SHE}} \left[ \exp \left( -\exp \left( h_T(2,0) + T^3 \Upsilon_T \right) \right) \right] = E_{\text{Airy}} \left[ \prod_{k=1}^{\infty} \frac{1}{1 + \exp \left( \frac{T^3}{12} (s + a_k) \right)} \right] \quad (0.1.5)
\]

Let $G$ be a Gumbel random variable. Then, the function $\exp \left( -\exp(x) \right)$ is equal $\mathbb{P}(G < -x)$. Armed with this, one can now see that the left hand side of (0.1.5) is equal to $\mathbb{P}(h_T(2,0) + T^3 G < -s)$ where $\Upsilon_T$ and $G$ are independent. Due to the rapid decay\footnote{The lower tail probability of a Gumbel distribution has double exponential decay.} of the lower tail probability of a Gumbel distribution, when $s$ is large and $T$ is greater than some $T_0 > 0$, $\mathbb{P}(h_T + T^{-\frac{5}{3}} G < -s)$ is approximately $\mathbb{P}(h_T(2,0) < -s)$ which is exactly the tail we are looking to control. Now consider the right-hand side of (0.1.5). If $s + a_k \gg 0$ then the corresponding term in the product will be exponentially small, whereas if $s + a_k \ll 0$ then the term will be very close to 1. Thus, the tail decay on the left-hand side is linked with the number of exponentially small terms (and their exponential factors) on the right-hand side.

Typically, the Airy point process is close to the zeros of the Airy function (Proposition 1.3.5), and hence $a_k \sim -\left( \frac{3\pi}{2} k \right)^{\frac{2}{3}}$ (Proposition 1.3.6). Plugging in this estimate readily yields decay like $\exp \left( -\frac{4}{15\pi} T^\frac{1}{3} s^{\frac{5}{2}} \right)$. The Airy points may, however, differ from these typical locations. For instance, $a_1$ (which is GUE Tracy-Widom distributed) may dip below $-s$ in which case the product in the expectation on the right-hand side of (0.1.5) becomes very close to 1. The probability of such a drastic dip behaves like $\exp \left( -\frac{1}{12} s^3 \right)$. Of course, there are many other scenarios in which the Airy points deviate from their typical locations in less drastic ways, and the contributions of those to the overall expectation need to be controlled and ultimately contribute to the other terms in our bounds in Theorem 1.1.1. We give a brief overview of this in Section 1.1.2. Proposition 1.2.1 (which follows directly from Proposition 1.3.2) contains precise statements of the bounds that we prove on the behavior of the right-
hand side of (0.1.5).

The main tool for showing the bounds on the upper tail probability is an explicit integral formulas
for moments of the SHE with delta initial data. This is stated as follows.

**Theorem 0.1.4 (Theorem 1.1 of [Gho18])**. Let \( \mathcal{H}_{\text{nw}}(T, X) \) be the fundamental solution of the KPZ
equation. Then, for any \( k \in \mathbb{N} \)

\[
\mathbb{E}[\exp(k\mathcal{H}_{\text{nw}}(T, X))] = \frac{1}{(2\pi i)^k} \int_{C_1} \cdots \int_{C_k} \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - z_B - 1} e^{\frac{T}{2} \sum_{j=1}^k z_j^2 + X \sum_{j=1}^k z_j} \prod_{j=1}^k dz_j
\]

(0.1.6)

where \( C_j \) is the line \( \alpha_j + i\mathbb{R} \) such that \( \alpha_1 > \alpha_2 + 2 > \ldots > \alpha_k + (k - 1) \)

By analyzing the integral formula (0.1.6) and recalling (0.1.1), we show in Lemma 2.4.3 that
\( \mathbb{E}[\exp(kT^{1/3}h_T(2, X))] \) is upper and lower bounded by some constant multiples of \( \exp(Tk^3/12) \) for
all \( T > 1 \). Based on this bound, the upper tail probability of \( h_T(2, X) \) turns out to be the exponential
of negative of the Legendre-Fenchel dual of \( k^3/12 \). In Chapter 2, we clarify this correspondence to
show (2.1.9) through the use of Markov’s inequality and Paley-Zygmund’s inequality.

The tail probabilities of the narrow wedge solution to the KPZ (as in Theorem 0.1.1 and 0.1.2)
is used in Chapter 2 (see Theorem 2.1.2 and 2.1.4) to obtain similar bounds for the solution under
general initial data. One of the main tools for proving such extension is the following convolution
formula which writes one point distribution of the KPZ with general initial data as a convolution of
the exponentials of the narrow wedge solution and the initial data.

For general initial data \( \mathcal{H}_0(\cdot) := \mathcal{H}(0, \cdot) \) and for a fixed pair \( T > 0 \) and \( X \in \mathbb{R} \), the Cole-Hopf
solution \( \mathcal{H}(T, X) \) of the KPZ equation satisfies

\[
\mathcal{H}(2T, X) \overset{d}{=} \log \left( \int_{-\infty}^{\infty} e^{\mathcal{H}_{\text{nw}}(2T,Y)+\mathcal{H}_0(X-Y)}dY \right)
\]

(0.1.7)

\[
\overset{d}{=} -\frac{T}{12} + \log \left( \int_{-\infty}^{\infty} e^{T\frac{1}{2}h_T(2T,Y)+\frac{1}{2}Y}+\mathcal{H}_0(X-Y)dY \right).
\]

The convolution formula when combined with the one-point tail probabilities of Theorem 0.1.1
and 0.1.2 and the Gibbs property of the KPZ line ensemble (see Section 2.2) yields the tail bounds
of Theorem 2.1.2 and 2.1.4 This is indeed an oversimplification of the intricacies we overcome to
complete the proof. For more detailed discussion, we refer to Section 2.1.1 of Chapter 2.
0.2 Time correlation of the KPZ equation

Recently, the study of the multi-time distributions of the models in the KPZ universality class became an active area of research. We refer to \cite{Joh19, BL17, FS16, FO19, BG19} for the results in various models and further references. In a recent work \cite{CGH19}, we found tight upper and lower bounds on the two time the correlation of the narrow wedge solution of the KPZ equation. We consider two scenarios: (a) when two time points are far apart and (2) when they are closer to each other. Recall that the correlation between two random variables $X$ and $Y$ is defined to be

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}}, \quad \text{where} \quad \text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y].$$

Theorem 1.1 of \cite{CGH19} gives bounds on the correlation between $h_T(1, 0)$ and $h_T(\alpha, 0)$ for $\alpha > 2$. In this case of two remote times, correlations decay as $\alpha^{-1/3}$ in the limit of high $\alpha$. In what follows, we provide an informal statement of the first main result of Chapter 3.

**Theorem 0.2.1** (Informal Statement of Theorem 3.1.1). For all $T > 1$ and $\alpha > 2$,

$$\text{Corr}(h_T(1, 0), h_T(\alpha, 0)) = \Theta(\alpha^{-1/3}). \quad (0.2.1)$$

**Theorem 0.2.2** (Informal Statement of Theorem 3.1.2). For all $T > 1$ and $\beta < \frac{1}{2}$ satisfying $\beta T > 1$,

$$1 - \text{Corr}(h_T(1, 0), h_T(1 + \beta, 0)) = \Theta(\beta^{2/3}). \quad (0.2.2)$$

We observe a crossover of the exponents (from $\frac{1}{3}$ to $\frac{2}{3}$) of decay of correlation in going from (a) to (b). This crossover is in corroboration with the experimental results obtained by Takeuchi and Sano \cite{TS10} (later supported by the work of Nardis, Le Doussal and Takeuchi \cite{DNLDT17}).
0.2.1 Proof Ideas

Theorem 0.2.1 and Theorem 0.2.2 investigate the correlation between the narrow wedge solution at two different time points. The main challenge of computing such correlation can be exemplified by the fact that there is no description of the two time distribution of the KPZ. However, by the linearity and the time reversal property of the stochastic heat equation, we derive a composition law\(^4\) see Proposition 3.2.2 which writes the height function at a later time \(T_1 (T_2 > 0)\) as a convolution of the narrow wedge height profile at \(T_2\) and an independent narrow wedge solution evolved for \(T_1 - T_2\) amount of time. Combining the composition law with the tail bounds of Theorem 0.1.1 and 0.1.2 and the Gibbs property, we bound the tails of \(h_{T_1}(1 + \beta, 0) - h_{T_2}(1, 0)\) in Theorem 3.1.5 for \(\beta > 0\) such that \(\beta T > 1\). Armed with those tail probability bound, we control the two time correlation by bounding the covariance and the variances. For a detailed discussion on the proofs, we refer to Section 3.1.2 of Chapter 3.

0.3 Connections and extensions

We discuss various applications and extensions of our results and methods. Section 0.3.1 describes the relationship between our analysis and an inverse-scattering problem generalizing the Painlevé II equation. Section 0.3.2 explains how our results relate to the lower-tail decay for positive temperature free-Fermions. Section 0.3.3 discusses extending our analysis to study the KPZ equation large deviation rate function, as well as relates our work to recent physics literature. Sections 0.3.4, 0.3.5 and 0.3.6 touch upon extensions of our methods and results to (respectively) the KPZ equation upper-tail decay, general initial data, half-space geometry, and certain discretizations of the KPZ equation like ASEP or the stochastic six vertex model. We discuss how the time correlation results (Theorem 0.2.1 and 0.2.2) fit into the broader endeavor to contain the time correlation structure of the other integrable systems in Section 0.3.7.

0.3.1 An integro-differential generalization of Painlevé II

Using the explicit form of the Airy kernel and the fact (1.1.13) that expectations of multiplicative functions of determinant point processes can be written as Fredholm determinants we can rewrite the

\(^4\)In the same spirit as (0.1.7)
equality in Proposition 0.1.3 (actually 0.1.5) as

\[ \mathbb{E}_{\text{SHE}} \left[ \exp \left( - \exp \left( \left( T \frac{1}{2} \left( h_T(2,0) + s \right) \right) \right) \right) \right] = \det (I - K)_{L^2(s,\infty)} =: Q(s) \quad (0.3.1) \]

where \( K \) is the Airy kernel deformed by a Fermi-factor:

\[ K(x, x') = \int_{-\infty}^{\infty} dr \sigma(r) \text{Ai}(x + r) \text{Ai}(x' + r), \quad \text{with} \quad \sigma(r) = \frac{1}{1 + e^{-T \frac{3}{5} r}}. \quad (0.3.2) \]

It was proved in [ACQ11] Section 5.2 (following [TW02]) that for any choice of \( \sigma(r) \) (which is smooth except at a finite number of points at which it has bounded jumps, and which approaches 0 at \(-\infty\) and 1 at \(+\infty\) exponentially fast) and the resulting \( Q(s) \) satisfies

\[ Q(s) = \exp \left( - \int_{s}^{\infty} dx (x - s) \int_{-\infty}^{\infty} dr \sigma'(r) q_r^2(x) \right), \]

where \( q_r(s) \) solves the following integro-differential generalization of Painlevé II

\[ \frac{d^2}{ds^2} q_r(s) = \left( s + r + 2 \int_{-\infty}^{\infty} dr' \sigma'(r') q_r^2(s) \right) q_r(s), \quad \text{with} \quad q_r(s) \sim \text{Ai}(r + s) \quad \text{as} \quad s \to +\infty. \quad (0.3.3) \]

If \( \sigma(r) = 1_{r \geq 0} \) then the above equation recovers the Hastings-McLeod solution to Painlevé II. The derivation of the above result in [ACQ11] Section 5.2 came from an attempt to directly study the lower tail for the KPZ equation. Due to the complexity of this equation, [ACQ11] was unable to even show that the lower tail decays to zero and resorted to a more indirect route via the results of [Mue91]. [SMP17] managed to extract asymptotics from this equation via a WKB approximation along with a self-consistency ansatz for the form of the solution to a Schrödinger equation in which the potential depends upon the solution. It would be valuable to make this approach rigorous, and below we mention possible ways to start. Before doing so, let us note that we may reverse the direction of inference and try to use our methods for studying the KPZ tail (via the Airy point process) to deduce results for the solution to (0.3.3).

The connection problem for (0.3.3) asks how the Airy behavior as \( s \to \infty \) propagates through as \( s \to -\infty \). This problem also falls under the realm of inverse scattering on the line [D179] [BDT88] for the Airy operator. For the Hastings-McLeod solution of the Painlevé II equation, this problem
has been resolved to a great level of detail using the steepest descent method for an associated $2 \times 2$ Riemann-Hilbert problem \cite{DZ93, DZ95, DIK08, BBD08, BB17, Fok+06}.

For a general choice of $\sigma(r)$, the kernel $K$ may be rewritten as

$$K(x, x') = \int_{-\infty}^{\infty} dr \sigma'(r) \frac{\text{Ai}(x + r)\text{Ai}'(x' + r) - \text{Ai}'(x + r)\text{Ai}(x' + r)}{x - x'}$$

and hence takes the form of an integrable integral operator. As shown in \cite{Its90}, the associated $Q(s)$ can be written in terms of an operator valued \cite{R} Riemann-Hilbert problem. The analysis of such problems is considerably more involved than in the finite dimensional (namely $2 \times 2$) matrix setting (cf. \cite{IK14, IK16} for some recent advances in this direction).

The approach developed in this present paper may offer an alternative to studying the operator valued Riemann-Hilbert problem. In our analysis there is nothing particularly special about the choice of $\sigma(r)$ (which translates into the choice of multiplicative functional). For another $\sigma(r)$ we could just as well similarly derive asymptotics for $Q(s)$. Turning this into a solution to the connection problem in (0.3.3) may still be a challenge. Should this work, the study of the operator valued Riemann-Hilbert problem would be reduced to the study of the $2 \times 2$ matrix problem associated with the Hastings-McLeod and Ablowitz-Segur solutions. We do not pursue this idea further in the present text and leave it for further investigation.

### 0.3.2 Positive temperature free-Fermions

Positive temperature free-Fermions and the related Moshe–Neuberger–Shapiro \cite{MNS94} matrix model\footnote{This is a one-parameter ($b \geq 0$) unitarily invariant measure on $n \times n$ Hermitian matrices $H$ with density (relative to the Lebesgue measure on algebraically independent entries of $H$) given proportional to $e^{-(2b+1)\text{Tr}(H^2)}\int_{U(n)} dU e^{2\text{Tr}(UHU^\dagger H)}$, where the integral is with respect to the Haar measure on the unitarity group $U(n)$. When $b = 0$ the measure reduces to the GUE.} have recently been studied in \cite{DLDS15, LW17} (and earlier in \cite{Joh07} in grand-canonical form). These ensembles are indexed by an inverse temperature $\beta$. When $\beta \to \infty$ this recovers the Gaussian Unitary Ensemble. \cite{Joh07, DLDS15, LW17} consider taking the number of Fermions (or matrix dimension) $N \to \infty$. When $\beta$ is fixed, the distribution of the rightmost Fermion converges to the GUE Tracy-Widom distribution (see \cite{LW17} Theorem 2(a)); when $\beta$ tends to 0 sufficiently
fast relative to $N$ going to infinity, the rightmost Fermion converges to a Gumbel distribution; and when $\beta$ tends to 0 and $N$ tend to infinity in a critical manner, there is a crossover between the GUE Tracy-Widom and Gumbel distribution. The limit of the correlation kernel for Fermion point process at the edge converges under this critical scaling to the Fermi-factor deformation of the Airy kernel given in (0.3.2). Hence, the Fredholm determinant (0.3.1) gives the probability that the right-most limiting Fermion is to the left of $s$, and Proposition 1.2.1 provides the lower tail probability decay of that distribution.

0.3.3 Large deviation rate function

Theorem 1.1.1 shows that there is a crossover between two types of tail decay which occurs when $s$ is of order $T^{2/3}$. This can be understood in terms of large deviations. For $z \leq 0$ let

$$\Phi_-(z) = \lim_{T \to \infty} T^{-2} \log \left( \mathbb{P} \left( H_{nw}(2T, 0) + \frac{T}{12} \leq zT \right) \right).$$

The existence of the above limit is not a priori clear\footnote{[BGS17] has an approach to proving the existence of such rate functions for first and directed last passage percolation. Whether this approach lifts to positive temperature models like KPZ remains to be seen.}. In terms of $\Phi_-$, Theorem 1.1.1 suggests that $\Phi_-(z) \approx \frac{1}{12}(-z)^3$ for $z$ near 0 and $\Phi_-(z) \approx \frac{4}{15\pi}(-z)^{5/2}$ for $z$ near $-\infty$.

In the physics literature, the crossover between the exponents $\frac{5}{2}$ to 3 seems to have been first predicted via weak noise theory by [KK07] in the context of directed polymers, and quite recently by [MKV16] in the context of the KPZ equation. Weak noise theory (WNT), sometimes also called ‘optimal fluctuation theory’ studies the large deviations of the noise necessary to produce a given space-time trajectory of the KPZ equation (or more general systems). It is a valid method only under ‘weak coupling’ or when there is an exceedingly small parameter in front of the noise term. In many instances, this approach is only valid for short times (when the noise is, through rescaling, effectively weak). However, for the KPZ equation it seems that it remains valid for longer times, if one probes deep enough into the tail. WNT has a long and rich history within physics dating back to the 1960s in condensed matter physics [HL66; ZL66; Lif68] and was introduced into the study of the noisy Burgers equation by Fogedby in the late 90s [Fog98]. It also goes under names such as the ‘instanton method’ in turbulence, ‘macroscopic fluctuation theory’ in lattice gases [Ber+15], and ‘WKB method’ in reaction-diffusion systems (see [MS17] for a more extensive history).
mathematics, the WNT for diffusions goes under the name Freidlin-Wentzell theory. For field valued / infinite dimensional diffusion processes [BDM08] and for certain non-linear stochastic PDEs [HW15; CD16], it has recently received some rigorous treatment. WNT alone does not provide the $\frac{5}{2}$ exponent or associated prefactor. Once the large deviations for the sample path (e.g. evolution of the KPZ equation) is determined, one still needs to solve a Hamiltonian variational problem to figure out the most likely trajectory among all those which achieve a given one-point large deviation. In the physics literature, [KK07; KK09; MKV16] worked through this calculation for KPZ with flat initial data and predicted the $\frac{5}{2}$ exponent along with a prefactor of $\frac{8}{15\pi}$. [KMS16] worked with parabolic initial data (which interpolates between flat and narrow wedge) and predicted that the prefactor becomes $\frac{4}{15\pi}$ in the narrow wedge limit. These short-time predictions have been confirmed through exact formulas in [Le +16; KLD].

Weak noise theory does not provide any explanation to the crossover of the exponents of $\Phi_-(z)$ from $\frac{5}{2}$ to $3$ as $z$ goes from $0$ to $-\infty$. Recently, this crossover has been studied via analysis of the integro-differential equation discussed in Section 0.3.1 [LDMS16] performed a rough (non-rigorous) analysis of the equation and predicted the existence of a LDP with speed $T^2$ and cubic behavior for small $z$. However, their analysis missed the behavior of $\Phi_-(z)$ for $z \ll 0$ and hence did not predict that the $\frac{5}{2}$ exponent remains for long time. Via non-rigorous WKB approximation analysis, [SMP17] predicted not only that the $\frac{5}{2}$ to $3$ crossover should hold for all times sufficiently large, but also predicted a formula for the large deviation rate function $\Phi_-(z)$ from (0.3.4). The [SMP17] prediction

$$\Phi_-(z) = \frac{4}{15\pi^6}(1 - \pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^7} z - \frac{1}{2\pi^2} z^2$$  \hspace{1cm} (0.3.5)$$

indeed recovers the desired small and large $z$ asymptotics. [Har+18] has performed simulations which numerically confirm the $\frac{5}{2}$ exponent for short and moderate values of time. The cubic exponent is harder to access numerically.

We now explain how our present work could be extended to prove a formula for $\Phi_-(z)$. The core challenge is that there was no proved large deviation theory for the empirical density of the Airy point process (such as done for the GUE point process in [BG97a] – see also [LS17] and references therein). Since there are infinitely many points in the Airy point process, one cannot naïvely apply the Coulomb-gas / electrostatics approach to formulate a large deviation principle. Indeed, we have
derived what should be the rate function in [Cor+18], though only in a physics way. We hope to give a rigorous proof in future.

In light of (0.1.4) and the argument used to prove Theorem 1.1.1, $\Phi^{-}(z)$ should be given by

$$\Phi^{-}(z) = \lim_{T \to \infty} \frac{1}{T^2} \log \mathbb{E} \left[ \exp \left( - \sum_{i=1}^{\infty} \varphi_{T, -zT^{2/3}}(a_i) \right) \right],$$

where the $a_i$ are the Airy point process, and $\varphi_{t,s}(a) := \log \left(1 + \exp \left(T^{1/3}(a + s)\right)\right)$. For large $T$, $\varphi_{T, -zT^{2/3}}(T^{2/3}a) \approx T(a - z)_+$ (where $(\cdot)_+ := \max(\cdot, 0)$). Letting $\mu_T(\cdot) = T^{-1} \sum_{i \geq 1} \delta_{a_iT^{-2/3}(\cdot)}$ denote the scaled empirical Airy point process measure,

$$\Phi^{-}(z) = \lim_{T \to \infty} \frac{1}{T^2} \log \mathbb{E} \left[ \exp \left( - T^2 \int_{\mathbb{R}} d\mu_T(a)(a - z)_+ \right) \right].$$

Now, assume that for a suitable class of functions $\mu$, the empirical measure $\mu_T$ satisfies $\mathbb{P}(\mu_T \approx \mu) \approx \exp \left( - T^2 I(\mu) \right)$ for a rate functional $I$. Then, we would expect that

$$\Phi^{-}(z) = \min_{\mu} \left( \int_{\mathbb{R}} d\mu(a)(a - z)_+ + I(\mu) \right) \quad (0.3.6)$$

where the minimum is over the class of functions upon which $I$ is finite.

Assuming (0.3.6), we can derive upper bounds on $\Phi^{-}$. For instance $I(\mu)$ should be minimized and equal to 0 for the limiting density\(^8\) of the Airy point process $\mu_+(a) = \pi^{-1/2} a^{1/2} 1_{a \leq 0}$. Plugging this choice into (0.3.6) and evaluating the integral gives $\Phi^{-}(z) \leq \frac{4}{10\pi} (-z)^{3/2}$. On the other hand, consider the limiting density of the Airy point process conditioned on $a_1 \leq zT^{2/3}$ (after the scaling discussed above). Since that density will be supported strictly on $(-\infty, z]$, the integral in (0.3.6) will be zero. For that density, $I(\mu) = \frac{(-z)^3}{12}$, as can be determined by the known large deviations for $a_1$ in Proposition 1.4.1 Thus we find that $\Phi^{-}(z) \leq \frac{(-z)^3}{12}$.

A month after initially posting [CG18a], we (along with I. Corwin, P. Le Doussal, A. Krajcenbrink and L.-C.-Tsai in [Cor+18]) derived (based on a non-rigorous limit of the GUE LDP from [BG97a] (which is recently been made rigorous in [Zho19]), see also [DM06, DM08]) an electrostatic formula\(^9\)

\[^8\] This can be calculated, for instance, by taking the trace of the Airy kernel.
\[^9\] The conjectured formula is that $I(\mu)$ is finite only when $\int da (\mu(a) - \mu_+(a)) = 0$ and otherwise

$$I(\mu) = - \int \log |a_1 - a_2| \prod_{i=1}^{2} \left[ \mu_i(a_i) - \mu_+(a_i) \right] + \frac{4}{3} \int_0^{\infty} da |a|^{3/2} \mu_+(a).$$

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for $I(\mu)$. Assuming the validity of this, we were able to solve the variational problem and confirm the formula for $\Phi_-(z)$ in (0.3.5). Recently, [Zho19] gave a rigorous upper and lower bound to the large deviation rate function of the empirical density of the Airy point process. Proving the Airy large deviation principle remains a challenge for future work.

L.-C.-Tsai [Tsa18] found a proof of the formula for $\Phi_-(z)$ in (0.3.5) based on the stochastic Airy operator representation for the point process. Essentially, the large deviation problem is transferred onto a large deviation problem for the driving noise for that operator which, being Gaussian, is readily studied via standard theory. This is obviously an over-simplification and there are some real subtleties which presently restrict the class of test functions for which this approach can be applied. Presently the result of [Tsa18] does not control the finite $T$ tail behavior, though it is believable that the method could be extended to recover bounds such as proved here in Theorem [1.1.1]. In certain applications, it is important to have uniform tail bounds for all times.

Recently, [CC19] found another proof of the large deviation rate function $\Phi_-(z)$ by expressing $\det(I - K)_{L^2(s,\infty)}$ of (0.3.1) in terms of a $2 \times 2$ matrix valued Riemann-Hilbert problem (RHP) and solving the corresponding RHP by the method of nonlinear steepest descent. Their method also recover the tail behavior at finite $T$. However, it is not yet clear if this approach would also recover the tail bounds as time $T$ goes to $\infty$.

0.3.4 Upper tail and general initial data

Unlike for the lower tail, the upper tail probability $\mathbb{P}(\Upsilon_T > s)$ can be studied via Fredholm determinants [CQ13, Proposition 10]. In Chapter 2 we have provided some analysis of this upper tail behavior. The upper tail probability of the SHE had been studied before in a number of places. For instance, see [CD15, CJK13, KKK17] in regards to its connection to the moments and the intermittency property [GM90, GKM07] of the SHE. Again, there is a question of whether results are suitable to taking $T$ large. The only such result is [CQ13, Corollary 14] which shows that for some constants $c_1, c_2, c_1', c_2'$, and $s, T \geq 1$, $\mathbb{P}(\Upsilon_T > s) \leq c_1 \exp(-c_1' s T^{1/3}) + c_2 \exp(-c_2' s^{2/3})$. When $s \ll T^{3/2}$ the second bound is active and one sees the expected $3/2$ power-law in the exponent. However, as $s \gg T^{3/2}$, the leading term above become $c_1 \exp(-c_1' s T^{1/3})$ and only demonstrates exponential decay. Our result (Theorem 2.1.9) shows that $c_1 \exp(-c_1' s T^{1/3})$ is not a tight upper bound for $\mathbb{P}(\Upsilon_T > s)$ in
this regime of \( s \). In fact, the \( 3/2 \) power-law is shown to be valid for all \( s \) even as \( T \) grows (with upper and lower bounds of this sort).

Some works have focused on the large \( s \) but fixed \( T \) upper tail, e.g. \([CJK13]\) showed that 
\[
\log \mathbb{P}( \log Z(T, X) > s ) \asymp -s^{3/2} \quad \text{as } s \to \infty \quad \text{where } Z(0, X) \equiv 1.
\]
These results are not suitable for taking \( T \) and \( s \) large together. Our results (Theorems 2.1.4, 2.1.9, and 2.1.12) provide the first upper and lower bound for the upper tail probability which are well-adapted to taking \( T \) large. In particular, we showed that for a wide range of initial data the exponent of the upper tail decay is always \( 3/2 \) (a result which was not proved before for any specific initial data). However, the constants in the exponent for our bounds on the upper tail probability are not optimal.

It is natural to speculate on the values of these optimal coefficients for the upper tail. Remarkably, for the upper tail (the \( 3/2 \) exponent region) it seem that for all deterministic initial data, the upper tail coefficient remains the same\(^{10}\). However, for Brownian initial data, the coefficient changes by a factor of 2. \([CH16]\) introduced a method (based on the KPZ line ensemble Gibbs property) to extend tail probabilities from narrow wedge initial data to general initial data. In \([CH16, Theorem 13]\), the inputs came from \([ACQ11]\) and \([MF14]\) and were far from optimal. In Chapter 2, we employed the bounds from Theorem 2.1.9 (which contains the bound for the narrow wedge solution) to extend both upper and lower bound (2.1.9) to general initial data.

The upper tail large deviation rate should be \( T \) (instead of \( T^2 \) for the lower tail) and it is predicted in \([LDMS16, SMP17]\) that the rate function is \( 4/3 s^{3/2} \). see e.g. \([LDMS16]\) for a non-rigorous predictions about this. Recently, \([DT19]\) proved the upper tail large deviation of the narrow wedge solution of the the KPZ equation. For a discrete analog (the log-gamma polymer) and a semi-discrete analog (the O’Connell-Yor polymer) such an upper tail bound is proved in \([GS13]\) and \([Jan15]\) respectively.

0.3.5 Half-space KPZ

The \((1 + 1)\)-dimensional SHE \( Z^{hs}(T, X) \) in the half space \( \mathbb{R}_+ \) with delta initial data at the origin is uniquely defined (see \([CS16]\)) by the SPDE in (0.0.1) and the Robin boundary condition (parameterized by \( A \in \mathbb{R} \)) which is formally given as 
\[
\partial_X Z^{hs}(T, X)|_{X=0} = AZ^{hs}(T, 0).
\]
for all \( T \geq 0 \).

The above half space SHE/KPZ equation has been recently studied in \([CS16, Par17]\) where it arises as the scaling limit of a corresponding ASEP. In the spirit of Proposition 0.1.3, \([Bar+17]\) (see \([Par17]\))

\(^{10}\)For instance, for flat and narrow wedge initial data, the upper tail seems to have the same 4/3 coefficient.
Corollary 1.3]) computed a Laplace transform formula for the half-space SHE in terms of the (Pfaffian) GOE point process. Using this, [Tsa18] proved the half-space KPZ large deviation rate function. Recently, [Kim19] proved tight bounds on the lower tail of the half-space KPZ equation by extending our methods. The half-space problem has also received attention in the physics literature, see [KLD18a; MV18].

0.3.6 Other integrable probabilistic systems

Integrable probabilistic systems in the KPZ universality class [Cor12] fall into two classes – determinantal (i.e., free Fermion) or non-determinantal. For determinantal models like the longest increasing subsequence, polynuclear growth model, directed last passage percolation with geometric (or, exponential) weights and the totally asymmetric simple exclusion process (TASEP) works, such as [BDJ99; Bai+01], have obtained optimal lower tail estimates via analysis of $2 \times 2$ Riemann-Hilbert problems related to Painlevé equations. Coulomb gas methods or loop equation also provide means to extract lower tail estimates in these contexts – see for instance [DM06; BN12; CP13; CP15]. So far, our present work on the KPZ equation provides the only lower tail bounds for non-determinantal models.

Besides studying one-point lower tail decay and large deviations, there is much interest in understanding the large deviations of the entire space-time trajectory. For TASEP a recent attempt at this has been made in [OT19]. The rate is still $N^2$, though the rate function is only bounded above and below in [OT19]. The stochastic six vertex model [BCG16] is a discrete time analogs of (T)ASEP. There has been significant efforts (summarized, for instance, in [Res10]) to study large deviations and surface tensions for the six vertex model. Until recently, the only rigorous results (i.e., large deviations for limit shapes) are for determinantal models such as uniform Aztec diamond or rhombus tilings (see, for example, [KOS06; KO07]). [Agg18] has now proved an arctic circle for the square ice model. This is, however, only the boundary of the limit shape and the method there does not seem able to capture the internal shape.

Using the methods considered in this paper, we should be able to access tail / large deviation type results for a few other non-determinantal models. The starting point for our result is the identity in Proposition 0.1.3 (of Chapter I) which matches the SHE Laplace transform with a multiplicative $\text{For TASEP, the lower tail corresponds to the upper tail for the current of particles to pass the origin.}$
function of the Airy point process. Similar formulas exist for the asymmetric simple exclusion process (ASEP) [BO16, Theorem 1.1], stochastic six vertex model [Bor16 Corollary 4.4], q-TASEP [OP17 Proposition 6.1]. The methods of this paper should extend to these other models though will likely involve some new analysis (such as of q-Laplace transforms and the associated variants of Painlevé which arise for these different models).

0.3.7 Broader context of time correlation

The main results, Theorems 3.1.1 and 3.1.2 fit into a broader effort in the last few years to understand the temporal correlation structure for the KPZ equation and other models in its eponymous universality class. The experimental observations of [TS12] about temporal correlations brought this question into focus. Soon after, due to the applicability of the replica trick, there were a few informative non-rigorous investigations in the physics community of the two-time distribution for the KPZ equation. Using certain combinatorial approximations, [Dot13; Dot15; Dot16] derived a formula for the two-time distribution. Later work of [NLD17] argued against the validity of this approximation and the resulting formula, and derived a formula for the two-time distribution when one of the fluctuations is taken deep in its tail (which simplifies some combinatorics in the Bethe ansatz used there). That work also provided an argument based on this approximate formula for the type of decay of correlation bounds proved here (along with predictions for the limiting values of the constants in those bounds). Further details are contained in [NLD18] and an independent calculation leading to the same conclusion is provided in [LD17]. In [NLDT17], time correlations are further investigated using numerical simulations.

Ours are the first rigorous results regarding the temporal correlations of the KPZ equation or any other positive temperature models. As we will review momentarily, there has been considerable recent mathematical activity in the analysis of temporal correlations of zero-temperature models in the KPZ universality class which enjoy determinantal structure. While these results have no direct implications for the KPZ equation, they do inform our expectation for its behaviour. On the other hand, with the exception of slow decorrelation, the methods used in the works that we now mention do not seem to generalize to positive temperature.

Slow decorrelation, shown to hold true for many KPZ class models in [Fer08] and [CFP12], implies that, for any $\eta < 1$ and any pair of times $T_1 = T$ and $T_2 = T + T^\eta$, as $T \nearrow \infty$ the fluctuations
(up to centring and scaling) converge to the same limiting random variable. This choice corresponds to very adjacent times in our setup – i.e., taking $\beta \to 0$ as an inverse power of $T$ in Theorem 3.1.2.

Studying exponential last passage percolation as well as limiting Airy process variational formulas for the two-time distribution, [FS16] presented two non-rigorous approaches to studying remote and adjacent correlation decay. Such results are proved in [FOI9] and [BG19] for exponential and Poissonian last passage percolation for narrow wedge initial condition. Recently, [BGZ19] obtained temporal correlation of the directed last passage percolation in $\mathbb{Z}^2$ with exponential passage time. Neither of the latter works rely on explicit formulas for the two-time distribution, but rather on more probabilistic characterizations. The work of [FOI9] is also able to address more general initial data than just narrow wedge.

There are, in fact, some proven explicit formulas for the two-time distribution of certain zero-temperature determinantal KPZ class models. The first formula was derived in [Joh17] and concerns Brownian last passage percolation – and, through a limit transition, the KPZ fixed point. The formula is complicated enough that extracting remote and adjacent correlation seems arduous. Recently, [Joh18] has derived a new and much more manageable formula for geometric last passage percolation which has permitted the study, in [Joh19], of limits of the two-time distribution for adjacent and remote times. So far, explicit formulas have not been used to prove correlation results. In [BL17], multi-time formulas for periodic last passage percolation (or TASEP) have been derived in the time scale on which the system relaxes to equilibrium. However, these formulas are again rather complicated and it is unclear how tractable they may render the zero-temperature counterparts to the questions that we consider. Quite recently, [JR19] and [Liu19] proved multi-time formulas and asymptotics in the non-periodic case. As of yet, there are no multi-time exact formulas known for the KPZ equation or any other positive temperature KPZ class model.

Excepting [FO19], which works more generally, the articles that we have mentioned treat specific types of initial data. Recently, there have been significant advances [MQR16], [Ham19], [DOV18] regarding the KPZ fixed point with general initial data. In fact, all these works contain Hölder continuity results for the fixed point; see also [Pim18] and [HS18] for related results. Our spatial and temporal fluctuation results (and consequential modulus of continuity estimates) agree in the limit of high $T$ with the Hölder continuity of the KPZ fixed point.

We close this discussion by recalling our initial motivation to study this problem. As explained in
[DD07] (a work providing many helpful references on the subject), aging is a phenomenon in glassy materials in which “older systems relax in a slower manner than younger ones”. The nature of this relaxation can be studied via the two-time correlation; aging corresponds to correlation crossing over from one to zero as the times move from being adjacent to remote. In this sense, our main theorems prove that the KPZ equation does, indeed, age.

0.4 Future directions & ongoing works

Our works open the doors to a hosts of new directions and provide necessary tools to some of the ongoing works. We discuss below few of such new directions and works.

(a) The large deviation theory of the KPZ equation is an important direction. The upper and the lower tail large deviation rate functions of the narrow wedge solution of the KPZ are obtained in [DT19] and [Tsai18; CC19] respectively. An interesting direction would be to investigate large deviation under general initial data. In an ongoing work [GL], we seek to find out the upper tail large deviation rate function of the KPZ under general initial condition. Our work will show that the rate function \( \Psi_+ (z) \) is \( \frac{4}{3} z^{-\frac{4}{3}} \) (same as in the narrow wedge case) under any compactly supported initial data. However, the rate function changes by a factor of 2 when the initial data is Brownian. One of the future direction is to investigate the lower tail large deviation under general initial data.

(b) Theorem 2.1.2 of Chapter 2 falls short of an optimal lower bound to the lower tail probability of the KPZ equation under general initial data\(^{12}\). To prove this, one of the key estimates is the tail bounds of the Radon-Nikodym derivative of the KPZ line ensemble (see Section 2.2 of Chapter 2) with respect to a set of independent Brownian bridges. Similar results are obtained for the Airy-line ensemble in [Ham16] and we hope to investigate this problem for the KPZ line ensemble in a future.

(c) Meerson and his co-authors ([MKV16; KMS16; JKM16]) investigated the short time and finite time asymptotics of the tails of the KPZ equation under various initial conditions. Their argument relies on the use of the (non-rigorous) weak noise theory. It is not yet clear how this

\(^{12}\)It is worth noting that we have near optimal lower bound to the lower tail probability under the narrow wedge initial data in [CG18a].
techniques is related to ours. In an ongoing work [DG], we are giving uniform in time (on a neighborhood around 0) bounds to the upper and lower tail probabilities of the narrow wedge solution of the KPZ equation. One of our future goal is to develop a rigorous version of the weak noise theory and apply that to find the tail estimates of a large class of nonlinear SPDEs.

(d) KPZ equation is the cornerstone of the KPZ universality class. The models in the KPZ universality class has a non-degenerate limit when the scaling exponents of the fluctuation, the space and the time follows the ratio 1:2:3. One of our main tools for proving Theorem 1.1.1, 2.1.9 is a Laplace transform formula of $Z^{nw}$ of [BG16, Theorem 1]. Similar formulas are also present for other integrable models in the KPZ universality class like asymmetric simple exclusion processes, stochastic six vertex model, etc (see [BO16]). In an ongoing work [Bai+], we are using the result of [BO16] to find the tail probabilities of the stochastic six vertex model. In future, we like to investigate the tail probabilities of other integrable models.

(e) In [CGH19], we only found the two time correlation of the KPZ equation under narrow wedge initial condition. On the other hand, there are conjectures [FS16, TS10] about the nature of the two time correlations under other initial data. One of our future goal is to characterize the two time correlation of the KPZ equation under general initial condition.

0.5 Other research topics

In my graduate research, I have used probability theory to explore various other research directions such as quantum field theory, optimal transport, number theory, vertex models, interacting particle systems, random Schrodinger operators and parameters estimation in Ising model. In this section, we will briefly touch upon my contributions in those other research topics. Since these are not closely related to the main focus of this thesis, we try to keep our discussion short and nontechnical.

**Conformal Bootstrap:** Quantum field theory is one of the most important theoretical tool to understand subatomic world. Conformal field theory (CFT) is a special type of quantum field theory which is invariant under the angle-preserving transformations. A cornerstone in the field of CFT is the ‘conformal bootstrap’ formulas which connects the correlation functions of the CFT with the building blocks of the CFT namely, the ’conformal blocks’. In an ongoing joint work with Guillaume Remi, Xin Sun and Yi Sun, we are giving a rigorous proof of the conformal bootstrap formulas for the two
dimensional Liouville CFT. In a recent work [Gho+20], we have used tools from the probability theory to give an explicit description of the ‘conformal blocks’ which have so far remained mysterious. Our work shows a probabilistic AGT (Alday-Gaiotto-Tachikawa) correspondence for the two dimensional Liouville CFT with 1-point on torus, instigate a new direction of research in the mathematical theory of CFT and shed some light in rigorously proving some of the fundamental problems in nonperturbative quantum field theory such as conformal bootstrap.

**Optimal Transport:** In every walk of life, the task of transportation is always challenging. The quest to build a mathematical theory of optimal transport dated back to 300 years and the discoveries of the last three decades put this topic at the forefront of the modern mathematics. In a recent joint work [GS19] with Bodhisattva Sen, we have applied optimal transport of distributions to study one of the fundamental problems in statistics namely, what is a consistent notion of multidimensional ordering (or, ranks) of data? One of the important innovations of our work is that the our proposed multidimensional ranks are direct generalization of the one dimensional ranks. Our work solved an open problem of showing the uniform convergence of the estimated multidimensional rank and quantile functions under minimal possible assumptions.

**Persistence of random polynomials:** The study of the random polynomials has important applications in various fields including number theory, quantum chaos and statistical mechanics. One of the open problems in the random polynomials is to show the universality of the exponent of log density of the set of real polynomials with a given degree and prescribed number of real roots. This conjecture was partly proved twenty years ago and left an indelible impact in the theory of elliptic curves. In an ongoing joint work with Sumit Mukherjee, we are aiming to prove this conjecture in full generality.

**Six vertex model:** Six vertex (6V) model is one of the important lattice models in equilibrium statistical mechanics. We consider 6V model in $\mathbb{Z}^2$ or, in a subdomain $\Sigma \subset \mathbb{Z}^2$. In 6V model, there are six possible configurations at each site of $\Sigma$. Figure 1 shows those configurations and the associated weights for stochastic, symmetric and asymmetric six vertex model.

The weight of a lattice configuration in $\Sigma$ is equal to the product of the weights of the configurations of the vertices in $\Sigma$. This induces a measure on a set of non-crossing, up-right (solid) lines on $\Sigma$. The height function of a vertex $v \in \Sigma$ is defined to be the number of non-crossing solid lines moved across the time-space zone between $(0,0)$ and $v$. In a recent joint work with Ivan Corwin, Hao Shen and Li-Cheng Tsai [Cor+18], we investigated the height function fluctuation of the stochastic and
symmetric 6V model. We showed that the height function of the stochastic 6V model under the weak noise scaling converges (see [Cor+18, Theorem 1.1]) to the (Cole-Hopf) solution of the KPZ equation. On the other other hand, the gradient field of the height function of the stochastic Gibbs states of the symmetric 6V model converges (see [Cor+18, Theorem 1.6]) to the solution of the stochastic Burger’s equation. To show those convergence, we introduce the Markov duality method which is one of the main contributions of our work. It opens the door to various other questions which we wish to investigate in future.

**Second class particle in exclusion processes:** The study of interacting particle systems (IPS) began in the early days of particle physics and still now, it is an active area of research. The applications of IPS spans in the different areas of science and the mathematical theory of IPS is enriched to a great extent by studying those practical problems. The asymmetric simple exclusion processes (ASEP) is one of the fundamental models of IPS which has been investigated extensively in the last four decades. The study of the exclusion processes with hierarchical divisions among particles is one of the interesting directions due to its immense practical applications and the associated mathematical challenges.

In a recent joint work with Axel Saenz and Ethan Zell [GSZ19], we considered ASEP on one dimensional integer lattice $\mathbb{Z}$ with two class of particles. The ASEP is an interacting particle system on $\mathbb{Z}$ with each of the vertices of $\mathbb{Z}$ occupied by at most one particle. Every particle carries an exponential clock of rate 1, and all clocks are independent. The evolution of the particles is Markovian and may be described as follows: when the clock rings, the particle decides to move to the right (or left) by one with probability $p \in (0, 1)$ (or $1 - p$). The particle moves to the corresponding target site if it was unoccupied, and nothing happens if the target site is occupied (i.e. the jump is suppressed). In a two-species ASEP, every particle is labeled as either first class or second class, depending on its jump hierarchy. For instance, a first class particle can interchange its position with a second class particle which was sitting on a neighboring target site; on the other hand, a second class particle is not allowed
to interchange its position with any other first or second class particle. We found the limiting law of the interface between two class of particles when there are infinitely many first class particles and finitely many second class particles and initially, the particles are occupying all the sites on the left half side of $\mathbb{Z}$. The limiting law of the interface (see [GSZ19, Theorem 1]) is found to be same as law of a minimum of $n$ uniform random samples where $n$ is equal to the number of second class particles in the system.

In a joint work with Patrik Ferrari and Peter Nejjar [FNG17], we found the shock fluctuation of the second class particle in totally asymmetric simple exclusion processes (when $p = 1$). We consider the TASEP with non-random initial condition and density $\lambda$ on $\mathbb{Z}^-$ and $\rho$ on $\mathbb{Z}^+$, and a second class particle initially at the origin. For $\lambda < \rho$, there is a shock and the second class particle moves with speed $1 - \lambda - \rho$. For large time $t$, we show that the position of the second class particle fluctuates on the $t^{1/3}$ scale and the limiting law is given in terms of the difference of two independent GOE Tracy-Widom\(^{13}\) random variables.

**Stochastic PDE limit of the Dynamic ASEP:** The study of the exactly solvable IPS plays a central role in quantum field theory. Recently, Alexei Borodin [Bor17] introduced the dynamic asymmetric simple exclusion process which is a generalization of ASEP and constructed by the representation theory of the quantum group $E_{\tau, \eta}(sl_2)$. In a recent joint work with Ivan Corwin and Konstantin Matetski [CGM19], we have shown that the dynamic ASEP under very weakly asymmetric scaling converges to the solution of the Ornstein-Uhlenbeck partial differential equation with a linear drift. In future, we like to investigate the KPZ type limit of the dynamic ASEP.

**Spectral rigidity of random Schrödinger operators:** In recent years, the study of spatial conditioning of the point processes became a reliable tool to investigate the structure of the Gibbs states in various statistical mechanical models where the particles are interacting with each other and are exposed to some external potential. In continuum, these systems are well represented by the spectrum of the random Schrödinger operators

$$\mathcal{H}_I := -\frac{1}{2} \Delta + V + \xi$$

where $\Delta$ is a Laplacian, $V$ is the binding potential acting on the functions in the $L^2$-space of the do-

\(^{13}\)It is the limiting distribution of the largest eigenvalue of a Gaussian orthogonal ensemble (GOE).
main $I$ and $\xi$ is a Gaussian process. The interaction between particles are modelled by using different noises. Among various notions of spatial conditioning, the study of the number rigidity (introduced by [Gho17]) of the point processes with strong interaction became a new trend. In a recent work with Pierre Yves Gaudreau Lamarre and Yuchen Liao [GGL19], we built a unified framework to study the number rigidity of the spectrum of the random Schrödinger operator for different classes of noises. Our framework is based on the semi-group theory of $\hat{H}_I$ and eloquently nourished from the theory of large deviation of the self-intersection time of Brownian bridges. However, one major limitation of this technique is that it falls short of proving the rigidity for the eigenvalues of the stochastic Airy-$\beta$ operators for any $\beta > 0$.

**Parameters estimation of Ising models:** Study of Ising models is a growing area which has received significant attention in Statistics and Machine Learning in recent years. Suppose $\beta > 0, B \neq 0$ are unknown parameters and $A_n$ is a sequence of $n \times n$ symmetric matrix with nonnegative entries with $0$ on the diagonal. For $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$, define a p.m.f. $P_{\beta,B}(\cdot)$ by setting

$$P_{n,\beta,B}(X = x) = \frac{1}{Z_n(\beta, B)} e^{\beta x'A_n x + B \sum_{i=1}^n x_i}.$$ 

This is the Ising model with coupling matrix $A_n$, and inverse temperature parameter $\beta$ and magnetization parameter $B$. In a joint work with Sumit Mukherjee [GM18], we have studied joint estimation of the inverse temperature and magnetization parameters $(\beta, B)$ of an Ising model. We give a general bound on the rate of convergence of the bi-variate pseudolikelihood estimator of the parameters. Based on this, we provided a sufficient condition in terms of the coupling matrix for the existence of $\sqrt{n}$-consistent bivariate estimator of $(\beta, B)$. 

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Chapter 1: Lower tail of the KPZ equation

1.1 Introduction

The $(1 + 1)$d stochastic heat equation (SHE) with multiplicative space time white noise $\xi$ is defined in (0.0.1) where $T \geq 0$ and $X \in \mathbb{R}$. The SHE is ubiquitous, modeling the density of particles diffusing in space-time random environments (with random killing / branching [Mol; Kho14] or random drifts [BC17; CG17]). Via the Feynman-Kac formula, it is the partition function for the continuum directed polymer model [ACQ11; Com17; HHF85]. Taking logarithms formally leads to the Kardar-Parisi-Zhang (KPZ) equation which is a paradigm for random interface growth [KPZ86] and a testing ground for the study of non-linear stochastic PDEs [Hai13; GJ14; GIP15; HQ15; GP17]. The KPZ equation’s spatial derivative formally solves the stochastic Burgers equation – a continuum model for turbulence [FNS77; BCJL94], interacting particle systems and driven lattice gases [BKS85].

The Cole-Hopf solution to the KPZ equation with narrow wedge initial data is given by

$$H_{nw}(T, X) := \log Z_{nw}(T, X), \quad \text{with} \quad Z_{nw}(0, X) = \delta_{X=0}. \quad (1.1.1)$$

The well-definedness of $\log Z_{nw}$ for all $T > 0$ and $X \in \mathbb{R}$ relies upon the almost-sure strict positivity of $Z_{nw}$ proved in [Mue91] to hold for a wide class of initial data (including the delta function). This is the physically relevant notion of solution and has been shown to arise quite generally from various regularization or discretization schemes for the equation and noise [BC95; BG97b; CT15; CS16; Hai13; HS15; HQ15; GIP15; GP17; GJ14]. The Cole-Hopf solution also coincides with the solutions constructed from regularity structures [Hai13], paracontrolled distributions [GIP15] and energy solution methods [GP17].

1. Due to the nonlinearity of the KPZ equation and the roughness of the white noise, it is challenging to construct a solution theory for it directly. Smoothing the noise in space, [BC95] showed that the logarithm of the smoothed noise SHE solves the KPZ equation with the same smoothed noise, up to an Itô correction whose size diverges as the smoothing disappears. More recently, the techniques of regularity structures [Hai13]; [Hai14], energy solution [GJ14; GP18], paracontrolled distributions [GIP15; GP17], and renormalization group [Kup16] have been used to construct the solution theory of the KPZ equation directly. These solutions all agree with the Cole-Hopf solution.

2. The solution theory for this stochastic PDE is classical [Wal86; Cor12; Qua12], based on Itô stochastic integrals or martingale problems.
This paper establishes tight bounds on the lower tail probability that $Z^{nw}(T, X)$ is close to zero, or equivalently that $H^{nw}(T, X)$ is very negative. The first result in this direction was the aforementioned almost-sure positivity of $Z^{nw}$ established in [Mue91] via large deviation bounds and a comparison principle. Using Malliavin calculus, [MN08] proved a quantitative upper bound on the decay of the lower tail probability. Working with the SHE on an interval with Dirichlet boundary conditions and constant initial data, they show that for any $\delta > 0$ there are constants $c_1, c_2 > 0$ so that $\mathbb{P}(H^{nw}(T, X) \leq -s) \leq c_1 \exp \left(-c_2 s^{3/2-\delta}\right)$. Using Talagrand’s concentration of measure methods, [MF14] improved the exponent. In particular, [MF14] considered the full-line SHE with $Z^{nw}(0, X) = \delta$ initial data (this is the setting we address in this paper) and proved a similar bound to [MN08] but with the $3/2 - \delta$ exponent replaced by the Gaussian exponent 2. Quite recently, using Malliavin calculus [HL18] extended these sort of results to noises with more general covariance structure. There is some work in progress [Kho18] which seeks to use stochastic analytic methods to prove a lower bound with exponent $3/2$ on this tail probability. As we prove here, the exponents accessed in earlier work are not optimal and, moreover, these previous results are (in a sense we now describe) not well adapted to study the long (or intermediate) time solution tail.

When time increases, the KPZ equation shows an overall decay at linear rate $-T/24$ with fluctuations which grows like $T^{1/3}$. [ACQ11] proved (see also [SS10] for a less rigorous treatment done in parallel, and [CLDR10; Dot10] for physics results) that when $Z^{nw}(0, X) = \delta_{X=0}$,

$$\lim_{T \to \infty} \mathbb{P}(\Upsilon_T \leq s) = F_{\text{GUE}}(s), \quad \text{where} \quad \Upsilon_T := \frac{H^{nw}(2T, 0) + T^{1/2}}{T^{1/4}}. \quad (1.1.2)$$

The $T^{1/3}$ scaling is a characteristic of models in the KPZ universality class, as is the limiting GUE Tracy-Widom distribution $F_{\text{GUE}}(s)$ [Cor12]. We consider $H^{nw}$ at time $2T$ to simplify some factors of 2 in formulas. Reinterpreting the tail bounds of [MN08; MF14] in terms of the lower tail of $\Upsilon_T$, one sees that their effectiveness degrades as $T$ grows (i.e. they do not reflect the centering or scaling associated with the long-time fluctuations).

While the distributional limit in (1.1.2) does not control the tails of $\Upsilon_T$ for finite $T$, it does suggest

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3To avoid confusion, let us distinguish our present investigation from earlier work of [E+97; EVE99; EVE00] which studied the stochastic Burgers equation (the spatial derivative of KPZ) but with a noise which is smooth in space and white in time. In that case, which has no direct relationship to our work, the tail of the local slope has $-\frac{7}{2}$ power-law (not exponential) decay. A proxy for the question we consider here, [EA95] studied the tail behavior of the invicid Burgers equation with white-noise initial data, showing cubic exponential decay. That result, however, also has no direct bearing on our present work.
a natural conjecture. For $s$ large, $F_{\text{GUE}}(-s) \approx e^{-\frac{1}{12} s^3}$ (see Proposition 1.4.1 herein, or [TW94, BBD08, RRV11]). Thus one might expect a similar lower tail bound for $\Upsilon_T$, at least for large enough $T$. As we prove in Theorem 1.1.1 this is only half true. In fact, there are two types of decay regimes for the lower tail $P(\Upsilon_T < -s)$: For $T^{2/3} \gg s \gg 0$ a cubic exponent controls the tail decay whereas for $s \gg T^{2/3}$ the tail exponent becomes $\frac{5}{2}$ and the leading constant in the exponential is $\frac{4}{15\pi} T^{1/3}$ instead of $\frac{1}{12}$ in the first regime (in fact, in Theorem 1.1.1 we are only able to lower bound the prefactor to the cubic exponent by $\frac{1}{12}$).

The existence of these two regimes has been discussed extensively in the physics literature for many years. The KPZ equation is believed to be the unique heteroclinic orbit between the Edwards-Wilkinson (or weak coupling) and KPZ (or strong coupling) fixed points [MQR16]. The cubic exponent is (as explained above) representative of the KPZ fixed point behavior. On the other hand, the Edwards-Wilkinson fixed point is described by a Gaussian process, suggesting naively a tail exponent of 2, not $\frac{5}{2}$. The $\frac{5}{2}$ exponent was first demonstrated in [KK07, KK08, KK09] (see also more recent works of [MKV16, KMS16] and the discussion in the footnote of Section 0.3.3) by studying the short time deep tail. Since the Gaussian Edwards-Wilkinson fixed point arises as a limit of the short time shallow tail, there is no contradiction.

Studying the short time, deep tail is equivalent to putting a small constant in front of the noise. In that case, it is possible to reformulate the tail behavior in terms of a large deviation problem for the underlying space-time white noise and then to optimize over all possible instances of the noise which realize the desired one-point deviation. Though this approach only applies to short time, quite interestingly the behavior it predicts seems to remain valid for all times, provided one goes deep enough into the tail.

This ‘weak noise theory’ or ‘optimal fluctuation theory’ generalizes Freidlin-Wentzell theory for stochastic differential equations. In the mathematics literature, it is only recently that this sort of approach for nonlinear SPDEs has begun to be has begun to be put on a rigorous mathematical footing. Namely, [HW15, CD16] take the first step of this approach (computing the rate function for a given space-time trajectory) for certain SPDEs (not presently including KPZ) in a periodic setting. Besides adapting this to the KPZ equation on the line, there is significant work needed to extract close-form one-point tail behavior by optimizing over all possible space-time trajectories.

While the cubic shallow tail exponent should be quite universal (i.e., for all KPZ class models),
the $\frac{5}{2}$ exponent’s universality is much less clear. For instance, for some discrete KPZ class models like last passage percolation with bounded entries, the very deep tail will be controlled by the tail behavior of the underlying noise and is unlikely to conform with this exponent. On the other hand, the $\frac{5}{2}$ exponent seems to show up when studying the total current of certain periodic space KPZ class particle systems (i.e., in terms of height functions, this corresponds with the spatial average height function). This behavior was first demonstrated in the physics literature for TASEP in [DL98] and then extended to ASEP in [DA99] and tested numerically for other models in [LK99]. Studying the tail behavior for other integrable models (as discussed in Section 0.3.6) and extending the weak noise theory to other systems should help to shed further light on the generality of this $\frac{5}{2}$ exponent.

Between the cubic and $\frac{5}{2}$ exponent regimes, there is a crossover in the tail of $\Upsilon_T$ at depth of order $T^{2/3}$, or for $H_{nw}^{\text{inv}}(2T,0)$ at a depth of order $T$. As $T$ goes to infinity, this crossover corresponds with the large deviation rate function (generally denoted by $\Phi_-(z)$) for the KPZ equation which has speed $T^2$ (see Section 0.3.3). Recently there has been significant work focused on determining this rate function. [SMP17] computed $\Phi_-(z)$ (see (0.3.5) for the formula) and showed that it crosses over between cubic and $\frac{5}{2}$ power-law behavior as $|z|$ moves from near zero to near infinity. That calculation involves a non-rigorous WKB approximation analysis of an integro-differential generalization of the Painlevé II equation which [ACQ11] related to the distribution of the KPZ equation. Making this WKB approximation rigorous seems to require solving an inverse scattering problem – see Sections 0.3.1 further discussion on this.

In Section 0.3.3 we outline another approach to computing $\Phi_-(z)$ via large deviations for the Airy point process. A month after posting the first version of this paper, along with P. Le Doussal, A. Kajenbrink and L.-C.-Tsai, we filled in the details of this outline. In particular, we conjectured an electrostatic formula for this rate function of the empirical measure for the Airy point process and solved the resulting variation problem. This approach (which has not yet been made rigorous) yields the same formula for $\Phi_-(z)$ as in [SMP17].

The first rigorous proof of the rate function $\Phi_-(z)$ came half a year after the first version of this paper posted and is due to L.-C.-Tsai [Tsa18]. Instead of proving a general Airy point process large deviation principle, he considers just the special test function (namely $a \mapsto (a - z)_+$ discussed in Section 0.3.3) which arises in this application and uses the stochastic Airy operator representation for the point process to reduce the computation of $\Phi_-(z)$ into a large deviation result for the underlying
Brownian motion driving that operator. This approach is delicate and relies on a special property of the test function which does not hold in general. Besides the three approaches mentioned so far (integro-differential equation, electrostatic Airy rate function, and stochastic Airy operator), there is another non-rigorous approach introduced recently in [KLD18b] which relies on computing cumulants of certain linear statistics of the Airy point process. All four approaches are discussed in the recent survey paper [KLD19].

Before stating our main result, it should be emphasized that the tail result which we prove, and the above mentioned large deviation rate function results are complementary in their nature. Our result demonstrates that for all time there are precisely two regimes of tail decay (and gives their behavior), and the large deviation results identify the long-time limit of the crossover mechanism between these two regimes.

We now state the main result of this chapter.

**Theorem 1.1.1.** Let \( \Upsilon_T \) denote the centered and scaled KPZ solution with narrow wedge initial data as in (1.1.1). Fix \( \epsilon, \delta \in (0, \frac{1}{3}) \) and \( T_0 > 0 \). Then, there exist \( S = S(\epsilon, \delta, T_0), C = C(T_0) > 0, K_1 = K_1(\epsilon, \delta, T_0) > 0 \) and \( K_2 = K_2(T_0) > 0 \) such that for all \( s \geq S \) and \( T \geq T_0 \),

\[
\mathbb{P}(\Upsilon_T \leq -s) \leq e^{-\frac{4(1-C\epsilon)}{15\pi}T^{\frac{2}{3}}s^{\frac{1}{2}}} + e^{-K_1s^{3-\delta}}e^{T^{1/3}s} + e^{-\frac{(1-C\epsilon)}{12}s^{3}}, \tag{1.1.3}
\]

and

\[
\mathbb{P}(\Upsilon_T \leq -s) \geq e^{-\frac{4(1+C\epsilon)}{15\pi}T^{\frac{2}{3}}s^{\frac{1}{2}}} + e^{-K_2s^{3}}. \tag{1.1.4}
\]

We prove this in Section 1.2. Note that the right side of (1.1.3) is a sum of three terms. The first dominates the other two when \( s \gg T^{\frac{2}{3}} \). In the region \( T^{\frac{2}{3}} \gg s \gg 0 \), the second and third terms dominate, and when \( T \to \infty \), the third dominates the second and recovers the \( \frac{1}{12}s^{3} \) tail behavior of the GUE Tracy-Widom distribution. There is a similar interplay between the two terms in (1.1.4), though in this lower bound we do not recover the \( \frac{1}{12} \) constant as \( T \to \infty \).

In [CG18b](see Chapter 2), we have extended the upper bound (1.1.3) to general initial data. We do not yet have a matching lower bound, and expect that the coefficients in the exponents will depend on the initial data. That work relies on Theorem 1.1.1 as an input, and also uses the Brownian Gibbs property for the KPZ line ensemble [CH16].

\(^{4}\)We expect this is just a limitation of our result and would follow from a finer analysis.
We now briefly explain the three steps in our proof, though to simplify the exposition we will leave off the $\epsilon$ and $\delta$'s which are present in the statement and proof.

**Step 1:** The first step in our proof is to reduce the study of the lower tail asymptotics for the KPZ equation to the large parameter ($u$ in (0.1.4)) asymptotics of the SHE Laplace transform. This is the content of the proof of Theorem 1.1 and the desired SHE Laplace transform asymptotics are then recorded as Proposition 1.2.1. The fundamental identity which allows us to prove these asymptotics in Proposition 1.2.1 is the one-point formula [ACQ11; SS10; CLDR10; Dot10]. Recently, [BG16] reformulated that result as an identity between the Laplace transform of the SHE and the expectation of a specific multiplicative functional of the Airy point process (see Proposition 1.2). Armed with this, we reduce Proposition 3.1 to Proposition 4.2 which studies Airy point process asymptotics and whose proof is the main technical feat of this paper.

**Step 2:** The proof of Proposition 1.3.2 relies upon three results (Theorems 1.1.3, 1.1.4 and 1.1.5) about large deviations of the number of Airy points in large intervals and their rigidity around typical locations. Theorems 1.1.3 and 1.1.4 respectively probe the lower and upper large deviation tails for the fluctuations of the number of Airy points in a large interval $[-s, \infty)$. The mean number of points grows (Proposition 1.1.2) like $\frac{2}{3\pi} s^{\frac{3}{2}}$ and these theorems probe the probability of finding a different constant than $\frac{2}{3\pi}$. On the lower tail, Theorem 1.1.3 shows that the exponential decay power law has exponent 3, while Theorem 1.1.4 shows that the corresponding upper tail exponent is $\frac{3}{2}$. To our knowledge, such large deviation result are new for the Airy point process (cf. Sections 1.1.2 and 0.3.3 for further discussion). Theorem 1.1.5 controls the maximum (over the entire Airy point process) deviation of points outside bands around their typical locations. We do not expect this result is nearly as tight Theorems 1.1.3 and 1.1.4 but it suffices for our purposes. Using these three theorems we can establish control to the probabilities of various scenarios for the Airy point process and hence establish precise upper and lower bounds on the expectation value needed to prove Proposition 1.3.2.

**Step 3:** The proofs of Theorems 1.1.3, 1.1.4 and 1.1.5 are each rather different. The first two rely on the determinantal structure of the Airy point process (Section 1.3.1), while the third uses its relation to the stochastic Airy operator (Section 1.3.3). The proof of Theorem 1.1.3 is technically the most challenging. Via Markov’s inequality, it reduces to a bound on the cumulant generating function for the number of Airy points in the interval $[-s, \infty)$, when the parameter $\nu$ of the generating function

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5The variance grows like $\log(s)$ and the fluctuations satisfy a central limit theorem in this scale [Sos00].
is of order $s^{\frac{3}{2}}$ (see Section 1.1.3). Theorem 1.1.6 relates (via standard determinantal methods) this generating function $F(x; v)$ to the Ablowitz-Segur solution to the Painlevé II equation, and then proves the needed decay bound on the generating function using a delicate analysis of an asymptotic formula (given in recent work in [Bot17] in terms of oscillatory Jacobi elliptic functions) for this solution to Painlevé II. The proof of Theorem 1.1.4 is considerably simpler. It uses the fact that the number of Airy points in an interval equals (in law) the sum of independent Bernoulli random variables (with parameters related to the eigenvalues of the Airy kernel projected onto the interval). The theorem follows by combining Bennett’s concentration inequality on such sums, along with estimates on mean and variance given in Proposition 1.1.2. Theorem 1.1.5 uses the identity in law (Proposition 1.3.4) between the Airy point process and the spectrum of the stochastic Airy operator. The typical locations of the Airy points are given by the zeros of the Airy function, and the estimate on uniform deviations from bands around those typical locations can be reduced (through operator manipulations such used in [RRVT11; Vir14]) to an exponential tail estimate (proved in Lemma 1.3.7) on of the maximum oscillation of Brownian motion.

The rest of this introduction records the main results (summarized above) which go into our proof of Theorem 1.1.1. Section 1.1.1 provides the key identity relating the Laplace transform of the SHE and the expectation of a multiplicative functional of the Airy point process. Section 1.1.2 records the Airy point process large deviation and rigidity estimates that we rely upon. Section 1.1.3 records the precise asymptotics of the Ablowitz-Segur solution of the Painlevé II equation needed in the proof of Theorem 1.1.3.

1.1.1 Laplace transform formula

The starting point for our study is the exact formula characterizing the one-point distribution of the SHE with delta initial data. This was simultaneously and independently computed in [ACQ11; SS10; CLDR10; Dot10] (rigorous proof provided in [ACQ11]). That formula can, by straightforward manipulations, be reformulated (Proposition 0.1.3) in terms the expectation of a multiplicative functional of the Airy point process (Section 1.3). This was done in [BGI16 Theorem 2.2], and the resulting formula offers a major benefit since it enables one to bring to bare on the KPZ equation the vast range of tools and understanding developed for the Airy point process. In fact, prior to our present work,

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6The Brownian motion is the driving noise for the stochastic Airy operator.
it was not clear how to prove directly that the formula in [ACQ11; SS10; CLDR10; Dot10] defines a probability distribution. Armed with Proposition 0.1.3 such a result is immediate, and the lower tail decay becomes more tractable.

Proposition 0.1.3 is a special limit case of a general matching between stochastic vertex models and Macdonald measures in [Bor16, Corollary 4.4]. In special cases, the Macdonald measures reduce to determinantal Schur measures and hence are analyzable in the spirit of this paper (see [BO16; BBW16; Bar+17] or Sections 0.3.5 and 0.3.6 for further discussion).

1.1.2 Rigidity bounds for Airy point process

The Airy point process $\alpha_1 > \alpha_2 > \ldots$ (see Section 1.3) is a determinantal point process on the real line introduced by Tracy and Widom [TW94] as the scaling limit of the edge of the spectrum of the Gaussian unitary ensemble (GUE). [TW94] found that $F(s) := P(\alpha_1 < s)$ can be written in terms of the Hastings-McLeod (HM) solution of Painlevé II:

$$F(s) = \exp \left( - \int_0^\infty (y-x)u_{HM}^2(y)dy \right)$$

where $u_{HM}(y)$ is the solution of the Painlevé II equation (introduced in [Pai00; Pai02] – see also the review [Fok+06]) with specific boundary behavior as $x \to \infty$:

$$u_{HM}'' = xu_{HM} + 2u_{HM}^3, \quad ('') = \frac{d}{dx},$$

$u_{HM}(x) = \frac{x^{-\frac{1}{2}}}{2\sqrt{\pi}} e^{-\frac{x}{2}} (1 + o(1))$, as $x \to \infty$.

This solution was introduced in [HM80] wherein they determined an asymptotic formula for $u_{HM}(x)$ as $x \to -\infty$ (this is called solving the connection problem). Plugging this into (1.1.5), allowed [TW94] to demonstrate that $F(-s)$ decays like $\exp(-\frac{3}{12}s^3)$. Later, using the nonlinear steepest descent technique pioneered by [DZ93; DZ95; DIK08; BB08] determined smaller order terms in the asymptotic expansion of $F(-s)$. Similar results have been established for other determinantal point processes related to KPZ class models, e.g. [BDJ99, Bai+07].

In order to make rigorous the heuristic described in the last section we need to establish sufficiently

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3The hard part is to prove that the lower tail probability decays to 0.
precise control over the deviations of a large number of Airy points from their typical locations. Controlling deviations of eigenvalues from their typical locations is a central theme in some random matrix universality works (see, for example [ESY09, EYY12] and subsequent works). We require very precise upper and lower bounds on large deviations which do not seem to be present in the existing literature. In fact, we must ultimately rely upon the Ablowitz-Segur solution of Painlevé II to establish suitably precise bounds.

Our rigidity bound are established in terms of counting Airy points in intervals. Define

$$
\chi^{Ai}: \mathcal{B} (\mathbb{R}) \to \mathbb{Z}_{\geq 0}, \quad \chi^{Ai}(B) := \# \{ k : a_k \in B \}, \quad \forall B \in \mathcal{B} (\mathbb{R})
$$

where \(\mathcal{B} (\mathbb{R})\) denotes the Borel \(\sigma\)-algebra of \(\mathbb{R}\). The cumulants of \(\chi^{Ai}(B)\) have been studied in [Sos00] when the Borel set \(B\) is a semi-infinite interval of the form \([-s, \infty)\) or a finite interval of the form \([-ks, -(k-1)s)\). Following [Sos00] Theorem 1, we can record the following formulas for the expectation and the variance of the random variable \(\chi^{Ai}(B)\).

**Proposition 1.1.2.** Define intervals \(\mathcal{B}_k(s) := [-ks, -(k-1)s)\) for \(k \in \mathbb{Z}_{\geq 1}\) and \(\mathcal{B}_1(s) := [-s, \infty)\). For any \(s > 0\),

$$
\mathbb{E}_{\text{Airy}} \left[ \chi^{Ai}(\mathcal{B}_1(s)) \right] = \frac{2}{3\pi} s^{\frac{3}{2}} + \mathcal{O}(s),
$$

$$
\text{Var}_{\text{Airy}} \left( \chi^{Ai}(\mathcal{B}_1(s)) \right) = \frac{3}{4\pi^2} \log(s) + \mathcal{E}(s),
$$

where \(\sup_{s \geq 0} |\mathcal{O}(s)| < \infty\), and \(\sup_{s \geq 0} |\mathcal{E}(s)| < \infty\). For any \(k \in \mathbb{Z}_{\geq 1}\), there exists \(s_0 = s_0(k) > 0\) and \(C = C(k) > 0\) such that for all \(s > s_0\),

$$
\text{Var}_{\text{Airy}} \left( \chi^{Ai}(\mathcal{B}_k(s)) \right) \leq C \log(s).
$$

These estimates can be used to prove a central limit theory for linear statistics (including the

\[\text{As pointed out by [Haa07] (see the footnote on page 16), there is a mistake in the calculation leading up to [Sos00] Theorem 1] which erroneously produced the constant } \frac{1}{12 \pi^2} \text{ instead of the correct constant } \frac{1}{4 \pi^2} \text{ for the variance of } \chi^{Ai}(\mathcal{B}_1(s)) \text{ recorded in Proposition 1.1.2. In fact, for our purposes, the exact value of this constant is unnecessary. For the variance of } \chi^{Ai}(\mathcal{B}_1(s)) \text{ with } k \in \mathbb{Z}_{\geq 1} \text{ the constant should also be modified from that in [Sos00] Theorem 1]. Since [Haa07] does not provide the fixed value in this case (and to avoid redoing the calculation) we can easily argue that there must be some constant } c(k) > 0 \text{ such that } \text{Var}_{\text{Airy}} \left( \chi^{Ai}(\mathcal{B}_1(s)) \right) \leq c(k) \log(s). \text{ To see this, define } X = \chi^{Ai}(\mathcal{B}_1(ks)), X = \chi^{Ai}(\mathcal{B}_1((k-1)s)), \text{ and } Z = \chi^{Ai}(\mathcal{B}_1(s)). \text{ Since } X = Y + Z, \text{Var}_{\text{Airy}}(X) = \text{Var}_{\text{Airy}}(Y) + \text{Var}_{\text{Airy}}(Z) + \text{Cov}_{\text{Airy}}(Y, Z). \text{ Using the bounds for the variance of } X \text{ and } Y, \text{ and the inequality } |\text{Cov}(Y, Z)| \leq \sqrt{\text{Var}(Y)\text{Var}(Z)}, \text{ we arrive at the desired type of bound for } \text{Var}(Z). \text{ (We thank Thomas Claeys for pointing us to the work of [Haa07].)}\]
number of particles in large intervals) for the Airy point process (see, e.g. [Sos00]). By studying higher order cumulants, [DE13, Theorem 5.2] derives a moderate deviation result for $\chi^{Ai}(B_n)$ where $B_n := [-n, n]$. However, their result does not probe far enough into the tails of the distribution (it is still effectively Gaussian) to be of use in our desired application.

The theorems which we now state effectively show that the deviations of $\chi^{Ai}([-s, \infty))$ have the same exponential order (up to some small correct terms) tail behavior as the deviations of $a_1$. In other terms, the probability of having far too few or far too many points in a large interval is similar to the probability of having the first point far to the left or to right.

**Theorem 1.1.3.** For any $\delta \in (0, \frac{2}{5})$, there exist $s_0 = s_0(\delta) > 0$ and $K = K(\delta) > 0$ such that for all $s \geq s_0$ and $c > 0$

$$P\left(\chi^{Ai}([-s, \infty)) - E[\chi^{Ai}([-s, \infty))] \leq -cs^{\frac{3}{2}}\right) \leq \exp\left(-cs^{\frac{3}{2}} - (1 - Ks^{-\frac{2}{11}})\right). \quad (1.1.7)$$

**Theorem 1.1.4.** Recall $\mathcal{B}_k(s)$ from Proposition 1.1.2. Fix any $k \in \mathbb{Z} \geq 1$, $c > 0$ and $\epsilon \in (0, 1)$. Then, there exists $s = s_0(k, \epsilon)$ such that for all $s \geq s_0$,

$$P\left(\chi^{Ai}(\mathcal{B}_k(s)) - E[\chi^{Ai}(\mathcal{B}_k(s))] \geq cs^{\frac{3}{2}}\right) \leq \exp\left(-cs^{\frac{3}{2}}\left(\log(cs^{\frac{3}{2}}) - (1 + \epsilon) \log(\log(s))\right)\right).$$

Theorems 1.1.3 and 1.1.4 are respectively proved in Sections 1.3.5 and 1.3.6. The proof of Theorem 1.1.3 is based on a connection between the cumulant generating function of $\chi^{Ai}([-s, \infty))$ and the Ablowitz-Segur solution of Painlevé II (Section 1.1.3). The proof of Theorem 1.1.4 is simpler, relying on estimates in Proposition 1.1.2 along with Bennett's concentration inequality.

In addition to controlling the number of Airy points in large intervals, we require some uniform bound on the distance between the points and their typical locations. Let $\lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of the Airy operator (see Section 1.3.3). As shown in Proposition 1.3.6, $\lambda_n \approx \left(\frac{3\pi}{2}n\right)^{\frac{2}{3}}$. The following result follows directly from combining Proposition 1.3.5 with $\beta = 2$, and Proposition 1.3.4. Proposition 1.3.5 is a similar bound for the Airy$_\beta$ point process, and its proof relies on studying the spectrum of the stochastic Airy operator. To control the deviations of that random operator's spectrum, we prove a result (Lemma 1.3.7) which precisely controls the oscillations of Brownian motion. We do not claim that the next rigidity result is optimal and it may be possible to prove similar
(or better) results about uniform rigidity of Airy points via other methods, e.g. \[\text{[BEY14, Theorem 3.1]}\].

**Theorem 1.1.5.** For $\epsilon \in (0, 1)$, let $C_{\epsilon}^{Ai}$ be the smallest real number such that for all $k \geq 1$

$$
(1 - \epsilon)\lambda_k - C_{\epsilon}^{Ai} \leq -a_k \leq (1 + \epsilon)\lambda_k + C_{\epsilon}^{Ai}
$$

(1.1.8)

Then, for all $\epsilon, \delta \in (0, 1)$ there exist $s_0 = s_0(\epsilon, \delta)$, and $\kappa = \kappa(\epsilon, \delta)$ such that for $s \geq s_0$,

$$
\mathbb{P}(C_{\epsilon}^{Ai} \geq s) \leq \kappa \exp\left(-\kappa s^{1-\delta}\right).
$$

(1.1.9)

### 1.1.3 Asymptotics of Ablowitz-Segur solution of Painlevé II

The proof of Theorem 1.1.3 relies on Markov’s inequality which shows that for any $v > 0$,

$$
\mathbb{P}\left(\chi^{Ai}([-s, \infty)) - \mathbb{E}\chi^{Ai}([-s, \infty)) \right) \leq -cs^\frac{3}{2} \leq e^{-cvs^\frac{3}{2} + v\mathbb{E}\chi^{Ai}([-s, \infty))} \mathbb{F}(-s; v)
$$

(1.1.10)

where

$$
\mathbb{F}(x; v) := \mathbb{E}\left[\exp\left(-v\chi^{Ai}([x, \infty))\right)\right].
$$

In (1.1.10) we choose $v = s^{\frac{3}{2}-\delta}$. In order to extract asymptotics of $\mathbb{F}(x; v)$ (see Theorem 1.1.6), we rely on a connection to the Ablowitz-Segur (AS) solution to the Painlevé II equation.

The Ablowitz-Segur (AS) solution $u_{AS}(\cdot; \gamma)$ of the Painlevé II equation is an one parameter family of solutions to (1.1.6) characterized by the following boundary condition

$$
u_{AS}(x; \gamma) = \sqrt{\gamma} \frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{x}{2}} (1 + o(1)) \text{ as } x \to \infty.
$$

(Here \(o(1)\) means any function which goes to 0 as \(x \to \infty\).) For fixed $\gamma \in (0, 1)$, \[\text{[AS76, AS77]}\] solved the connection problem (behavior as $x \to -\infty$). The case $\gamma = 1$ is the Hastings-McLeod solution analyzed in \[\text{[HM80]}\], and the case when $\gamma > 1$ was subsequently studied in \[\text{[Kap92]}\].
Theorem 1.1.6. For $K^{Ai}$ the Airy point process correlation kernel (Section 1.3) and $\gamma = 1 - e^{-v}$,

$$F(x; v) = \det \left( I - \gamma K^{Ai} \right)_{L^2([x, \infty))} = \exp \left( - \int_{x}^{\infty} (y - x) u_{AS}^2(y; \gamma) dy \right).$$  \hspace{1cm} (1.1.11)

Fix any $\delta \in (0, \frac{2}{3})$ and set $v = s^\frac{3}{2} - \delta$. Then, as $s$ goes to $\infty$,

$$\log F(-s; v) \leq -\frac{2}{3\pi} vs^{\frac{3}{2}} + O(s^{3 - \frac{14}{11}}).$$ \hspace{1cm} (1.1.12)

The first part of this result, (1.1.11), contains two equalities. The first follows from general theory relating multiplicative functions of determinantal point processes to Fredholm determinants (see [AGZ10, Section 3.4] for background on Fredholm determinants): For a determinantal point process $X$ with state space $\mathcal{X}$ and correlation kernel $K^X$, and a function $\phi : \mathcal{X} \rightarrow \mathbb{C}$,

$$E \left[ \prod_{x \in X} \phi(x) \right] = \det \left( 1 - (1 - \phi) K^X \right)_{L^2(X)}.$$ \hspace{1cm} (1.1.13)

This identity requires $(1 - \phi) K^X$ to be trace-class (see [Bor11] for more details). The second equality in (1.1.11) relies on the integrable structure of the Airy kernel [TW94, Section 1.C].

Proving the second part of the theorem, namely (1.1.12), requires a close analysis of the AS solution to Painlevé II, as is provided in Section 1.5.

The AS solution has received some attention recently in [BCP09, BB17] due to the fact that $\gamma K^{Ai}$ represents the kernel for a thinned version of the Airy point process – each particle is removed with probability $1 - \gamma$. This thinning represents one way to achieve a crossover between the GUE Tracy-Widom distribution and more classical extreme value statistics. The study of positive temperature free-Fermions in Section 0.3.2 represents another such mechanism.

[AS76, AS77] solved the connection problem for the AS solution for $\gamma \in (0, 1)$ fixed. For our application, $\gamma$ (or equivalently $v$) fixed would only yield an exponent of $s^{\frac{3}{2} - \delta}$ in Theorem 1.1.3 (not the desired $s^{3 - \delta}$). Recently, utilizing Riemann-Hilbert steepest descent, [Bot17] computed the asymptotic form of the AS solution $u_{AS}(x; \gamma)$ as $x \rightarrow -\infty$ for a more general range of $\gamma$. The formulas are written in terms of Jacobi elliptic theta functions and take different forms depending on the values of $\gamma$. In particular, setting $\tau := - \frac{1}{(-\tau)^{\frac{1}{2}}} \log(1 - \gamma)$, [Bot17] computes asymptotic formulas in three different
ranges of parameters: (a) \( \tau \in (0, (-x)^{-\delta}] \); (b) \( \tau \in (0, \frac{2}{3}\sqrt{2} - \eta] \); (c) \( \tau \in (\frac{3}{2}\sqrt{2} - 8\log(-x)^{\frac{3}{2}}(\tau - x)^{2}, \infty) \).

Here \( \delta, \eta > 0 \) are arbitrary small numbers and \( 8 \in (-\infty, \frac{7}{4}] \). For \( \tau \in (0, \frac{2}{3}\sqrt{2}) \) the resulting asymptotic form of \( u_{AS}(x; \gamma) \) as \( x \to -\infty \) is pseudoperiodic, thus making it rather challenging to compute the integral in the exponential in (1.1.11) (as necessary to recover asymptotics for \( F(x; v) \)). As \( \tau \) approaches 0 and \( \frac{2}{3}\sqrt{2} \) the oscillations die out, though due to different mechanisms in each case.

[Bot17; Bot16] managed to translate his asymptotic result for \( u_{AS} \) into a corresponding result for \( F \) only in the (c) region\(^7\) For region (a), [Bot17] demonstrated a simplified form of \( u_{AS}(x; \gamma(x)) \) for \( \tau \in (0, (-x)^{-\delta}) \) for any fixed \( \delta > 0 \). However, this simplified form still retains its oscillatory nature which is one of the difficulties in getting a full expansion for \( F(-s; 1 - e^{-s^{3/2-n}}) \). Recently, [BB17] showed that for any \( 0 < \epsilon < \frac{1}{4} \), there exist constants \( s_0 = s_0(\epsilon) \) and \( c_j' = c_j'(\epsilon) \) for \( j = 1, 2 \) so that for \( s \geq s_0 \) and \( 0 \leq v = -\log(1 - \gamma) < s^{\frac{1}{2} - \epsilon} \),

\[
\log F(-s; v) = -\frac{2v}{3\pi} s^{3} + \frac{v^2}{4\pi^2} \log(8s^{3}) + \log \left(G\left(\frac{1 + iv}{2\pi}\right)G\left(1 - \frac{iv}{2\pi}\right)\right) + r(s, v). \tag{1.1.14}
\]

Here \( G(x) \) is the Barnes G-function and \( |r(s, v)| \leq c_1' \frac{v^3}{s^2} + c_2' 2^v \) for all \( s \geq s_0, 0 \leq v \leq s^{\frac{1}{2} - \epsilon} \).

Since (1.1.14) gives the full expansion of \( \log F(-s; s^{3-\delta}) \) only when \( \delta > \frac{2}{3} \), plugging it into the right side of (1.1.10) only yields a leading term (in the upper bound of the lower tail probability of \( \chi^{Ai}([-s, \infty]) \)) like \( \exp(-cs^{2-}) \). However, Theorem 1.1.3 asks that the upper bound is like \( \exp(-cs^{3-}) \). In Section 1.5 we demonstrate how we can work with \( \delta \) close to 0. Presently we cannot justify a full expansion of \( F(-s; v) \) in Theorem 1.1.6 like that of (1.1.14). However, the weaker result in Theorem 1.1.6 suffices for our present needs.

Outline

Rest of this paper is organized as follows. Section 0.3 includes a brief discussion of how our results and methods connect to other problems and may be extended in other directions. Section 1.2 reduces the proof of our main result (Theorem 1.1.1) to a result (Proposition 1.2.1) for a cumulant generating function. Proposition 1.2.1 is subsequently proved in Section 1.3.2 by reducing it to a result (Proposition 1.3.2) about the Airy point process. The rest of Section 1.3 develops and proves various properties about the Airy point process, including the key rigidity estimates stated in the introduction\(^7\) [Bot17] achieved this for \( \tau > \frac{2}{3}\sqrt{2} \) based on the lack of oscillations in \( u_{AS} \) for such \( \tau \), and [Bot16] provided an extension to the full region (c) (and slightly beyond).
as Theorems 1.1.3, 1.1.4 and 1.1.5. Proposition 1.3.2 is proved in Section 1.4. Finally, Section 1.5 contains a discussion on asymptotics of the Ablowitz-Segur solution to Painlevé II and a proof of Theorem 1.1.6 stated earlier in the introduction.

1.2 Proof of the main result

Recall $\Upsilon_T$ from (1.1.2). The random variable $\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))$ is equal to the conditional probability $P(G \leq -T^{1/3}(\Upsilon_T + s)|\Upsilon_T)$ where $G$ is a Gumbel random variable independent of $\Upsilon_T$. Thus, the expected value of $\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))$ is equal to the probability $P(\Upsilon_T + T^{-1/3}G \leq -s)$ which is approximately equal to $1(\Upsilon_T \leq -s)$ for large enough $s$ and for all $T$ greater than some $T_0 > 0$. Motivated by this heuristic, we prove Theorem 1.1.1 by estimating the Laplace transform formula $E[\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))]$. We first state in Proposition 1.2.1 matching upper and lower bounds on the Laplace transform formula. Then, using Proposition 1.2.1, we finish the proof of Theorem 1.1.1 in Section 1.2.1.

Proposition 1.2.1. Fix $\epsilon, \delta \in (0, \frac{1}{3})$ and $T_0 > 0$. Then, there exist $s_0 = s_0(\epsilon, \delta, T_0)$, $C = C(T_0) > 0$, $K_1 = K_1(\epsilon, \delta, T_0) > 0$ and $K_2 = K_2(T_0) > 0$ such that for all $s \geq s_0$, one has

$$E\left[ \exp \left( - \exp \left( T^{\frac{1}{3}}(\Upsilon_T + s) \right) \right) \right] \leq e^{-4(1-C\epsilon)/15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}} + e^{-K_1 s^{3-\delta} - \epsilon T^{1/3}s} + e^{-\frac{(1-C\epsilon)}{12} s^3} \quad (1.2.1)$$

and

$$E\left[ \exp \left( - \exp \left( T^{\frac{1}{3}}(\Upsilon_T + s) \right) \right) \right] \geq e^{-4(1+C\epsilon)/15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}} + e^{-K_2 s^{3}}. \quad (1.2.2)$$

We postpone the proof of Proposition 1.2.1 to Section 1.3.2.

1.2.1 Proof of Theorem 1.1.1

We show that (1.2.1) (resp. (1.2.2)) implies (1.1.3) (resp. (1.1.4)) of Theorem 1.1.1. Let us first show that (1.2.1)$\Rightarrow$(1.1.3). Observe that using Markov’s inequality

$$P(\Upsilon_T \leq -s) = P\left( \exp \left( - \exp \left( T^{\frac{1}{3}}(\Upsilon_T + s) \right) \right) \geq e^{-1} \right) \leq eE\left[ \exp \left( - \exp \left( T^{\frac{1}{3}}(\Upsilon_T + s) \right) \right) \right].$$

(1.2.1) bounds the right-hand side and yields (1.1.3).
Now we show that \((1.2.2) \Rightarrow (1.1.4)\). Take \(\bar{s} := (1 - \epsilon)^{-1} s\). Observe that

\[
\mathfrak{R} := \mathbb{E} \left[ \exp \left( - \exp \left( T^{\frac{1}{3}} (\Upsilon_T + \bar{s}) \right) \right) \right] \\
\leq \mathbb{E} \left[ 1 \{ \Upsilon_T \leq -s \} + 1 \{ \Upsilon_T > -s \} \exp \left( - \exp (\epsilon \bar{s} T^{\frac{1}{3}}) \right) \right].
\]

where \(1 \{ A \} \) is an indicator function. The above inequality implies that

\[
P(\Upsilon_T \leq -s) \geq \mathfrak{R} - \exp \left( - \exp (\epsilon \bar{s} T^{\frac{1}{3}}) \right), \tag{1.2.3}
\]

It follows from \((1.2.2)\) that

\[
\mathfrak{R} \geq \exp \left( - \left( 1 + C \epsilon + C' \epsilon \right) \frac{4}{15 \pi} T^{\frac{1}{3}} s^{\frac{2}{3}} \right) + \exp (-K_2 s^3) \tag{1.2.4}
\]

for all \(s \geq S = S(\epsilon, \delta)\). Here, the \(C' \epsilon\) terms appears because \(s^{\frac{2}{3}} \leq \bar{s}^{\frac{2}{3}} (1 + C' \epsilon)\) for some \(C' > 0\).

Now, we notice that there exists \(S' = S'(\epsilon, T_0)\) such that for all \(s \geq S'\),

\[
\exp (\epsilon \bar{s} T^{\frac{1}{3}}) \geq T^{\frac{1}{3}} \frac{4 s^{\frac{2}{3}}}{15 \pi} - \log \epsilon, \quad \text{and} \quad \exp \left( - \exp (\epsilon \bar{s} T^{\frac{1}{3}}) \right) \leq \epsilon \exp \left( - \frac{4}{15 \pi} T^{\frac{1}{3}} s^{\frac{2}{3}} \right). \tag{1.2.5}
\]

Plugging the lower bound \((1.2.4)\) on \(\mathfrak{R}\) and the upper bound \((1.2.5)\) on \(\exp(-\exp(\epsilon \bar{s} T^{\frac{1}{3}}))\) into the right-hand side of \((1.2.3)\) yields, for all \(s \geq \max\{S, S'\}\),

\[
P(\Upsilon_T \leq -s) \geq (1 - \epsilon) \exp \left( - \left( 1 + (C + C') \epsilon \right) \frac{4}{15 \pi} T^{\frac{1}{3}} s^{\frac{2}{3}} \right) + \exp (-K_2 s^3).
\]

The multiplicative factor \((1 - \epsilon)\) can be absorbed into the exponential factor \((1 + (C + C') \epsilon))\) on the right-hand side above; and rewriting it as \((1 + C \epsilon)\) for a slightly modified constant \(C\) yields the right side of \((1.1.4)\), thus completing the proof of Theorem 1.1.1. \(\blacksquare\)

### 1.3 Airy point process

To prove Proposition \(1.2.1\) we use Proposition \(0.1.3\) which connects the SHE and the Airy point process. In this section we recall or prove various properties about the Airy point process. Section \(1.3.1\) reviews its determinantal structure. Section \(1.3.2\) contains a proof of Proposition \(1.2.1\). Section
1.3.3 relates the Airy point process to the stochastic Airy operator and derives properties about the typical point locations and deviations from there. Section 1.3.4 contains a heuristic explanation for certain terms in our tail bound. Finally, Sections 1.3.5, 1.3.6, and 1.3.7 provide proofs of, respectively, Theorem 1.1.3, Theorem 1.1.4 and Proposition 1.3.5.

1.3.1 Determinantal point process definition

The Airy point process (written here as \( \chi_{\text{Ai}} \) or \( a_1 > a_2 > \cdots \)) is a simple determinantal point process [AGZ10, Section 4.2]. Let us briefly review these terms. Denote the Borel \( \sigma \)-algebra of the real line \( \mathbb{R} \) by \( B(\mathbb{R}) \) and let \( \mu \) be a sigma finite measure over \( \mathbb{R} \). A point process is a probability distribution on locally finite configurations of the real points, or in other words, a non-negative integer-valued random measure \( \chi \) on the measure space \( (\mathbb{R}, B(\mathbb{R}), \mu) \). A point process \( \chi \) is called simple if \( \mu(\{\exists x : \chi(x) \neq 0\}) = 0 \). For any \( k \geq 1 \), the \( k \)-point correlation function of \( \chi \) with respect to the measure \( \mu \) is the locally integrable function \( \rho_k : \mathbb{R}^k \to [0, \infty) \) such that for any mutually disjoint \( B_1, \ldots, B_k \in B(\mathbb{R}) \),

\[
\mathbb{E}_\nu\left[ \prod_{i=1}^k \chi(B_i) \right] = \int_{B_1 \times \ldots \times B_k} \rho_k(x_1, \ldots, x_k)d\mu(x_1)\ldots d\mu(x_k).
\]

A simple point process \( \chi \) is determinantal if there exists \( K^\chi : \mathbb{R}^2 \to \mathbb{C} \) such that for all \( k \geq 1 \),

\[
\rho_k(x_1, \ldots, x_k) = \det \left[ K^\chi(x_i, x_j) \right]_{1 \leq i,j \leq k}.
\]

We refer to \( K^\chi \) as the correlation kernel of \( \chi \).

The Airy point process correlation kernel \( K^\chi_{\text{Ai}} \) relative to Lebesgue measure \( \mu \) on \( \mathbb{R} \) is

\[
K^\chi_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x-y} = \int_0^\infty \text{Ai}(x+r)\text{Ai}(y+r)dr. \quad (1.3.1)
\]

We will write \( \chi_{\text{Ai}} \) to denote the Airy point process (random) measure. We may also write \( \sum_{i=1}^\infty \delta_{a_i} \) for random points \( a_1 > a_2 > \cdots \). We will use both of these notations.

An integral operator \( R : L^2(M) \to L^2(M) \) with kernel \( K : \mathbb{R}^2 \to \mathbb{C} \) written as

\[
(Rf)(x) = \int K(x, y)f(y)d\mu(y), \quad \text{for } f \in L^2(M)
\]

Recall the Airy function \( \text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos(tx + t^3/3)dt \).

This follows from a calculation like in Proposition 1.1.2 which shows that almost-surely there are infinitely many particles in \( \chi_{\text{Ai}} \) but only finitely many to the right of any given point.
is locally admissible if for any compact set \( D \subset \mathbb{R} \), the operator \( \hat{\mathcal{R}}_D = 1_D \mathcal{R} 1_D \), having kernel
\[ K_D(x, y) = 1_D(x)K(x, y)1_D(y), \]
has the following representation:

\[
(\hat{\mathcal{R}}_D f)(x) = \sum_{k=1}^{n} \lambda_k \phi_k(x) \langle \phi_k, f \rangle_{L^2(M)}, \quad K_D(x, y) = \sum_{k=1}^{n} \lambda_k \phi_k(x) \phi_k(y)
\]

(1.3.2)

where \( n \) may be finite or infinite, \( \{ \phi_k \}_{k \in \mathbb{N}} \subset L^2(M) \) are orthonormal eigenfunctions and the eigenvalues \( (\lambda^D_k)_{k=1}^{n} \) of \( K_D \) are positive and satisfy \( \sum_{k=1}^{n} \lambda^D_k < \infty \). We call \( \mathcal{R} \) good if for all compact \( D \) and all \( 1 \leq k \leq n \), \( \lambda^D_k \in (0, 1) \). For a determinantal point process with locally admissible and good correlation kernel, for any compact set \( D \subset \mathbb{R} \), \( \chi(D) \) equals in distribution the sum of \( n \) (same \( n \) as in (1.3.2)) independent Bernoulli random variables with the respective probabilities of equality to 1 given by the \( \lambda^D_1, \ldots, \lambda^D_n \) – see [AGZ10, Section 4.2].

**Lemma 1.3.1.** The kernel (1.3.1) of the Airy point process \( K^{Ai} \) is locally admissible and good.

We use this result in proving Theorem [1.1.4](see [AGZ10, Proposition 4.2.30] for a proof).

### 1.3.2 Proof of Proposition [1.2.1]

As above, let \( a_1 > a_2 > \ldots \) denote the Airy point process. Denote

\[
I_s(x) := \frac{1}{1 + \exp \left( T^{3/4} (s + x) \right)} \quad \text{and} \quad J_s(x) := \log \left( 1 + \exp \left( T^{3/4} (s + x) \right) \right)
\]

(1.3.3)

so that for any \( x \in \mathbb{R} \), we have \( I_s(x) = \exp(-J_s(x)) \). Proposition 0.1.3 connects \( \mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \) with the Laplace transform of the SHE. We now state upper and lower bounds on this expectation and then subsequently complete the proof of Proposition 1.2.1.

**Proposition 1.3.2.** Fix any \( \epsilon, \delta \in (0, \frac{1}{3}) \) and \( T_0 > 0 \). Then, there exist \( s_0 = s_0(\epsilon, \delta, T_0) \), an absolute constant \( C > 0 \), \( K_1 = K_1(\epsilon, \delta, T_0) > 0 \) and \( K_2 = K_2(T_0) > 0 \) such that for all \( s \geq s_0 \) and \( T \geq T_0 \),

\[
\mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \leq e^{-\frac{4(1-C\epsilon)}{15\pi} T^{1/3} s^{5/2}} + e^{-K_1 s^{3-\delta} - e T^{1/3} s} + e^{-\frac{(1-C\epsilon)}{12} s^3}
\]

(1.3.4)

and

\[
\mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \geq e^{-\frac{4(1+C\epsilon)}{15\pi} T^{1/3} s^{5/2}} + e^{-K_2 s^3}.
\]

(1.3.5)

**Proof of Proposition 1.2.1** Using (0.1.5), (1.2.1)-(1.2.2) follows from (2.3.14)-(1.3.5).
1.3.3 Stochastic Airy operator

As observed in [ES07] and proved in [RRV11], the Airy point process equals in distribution the negated spectrum of the *stochastic Airy operator*. This yields a way to compute the typical locations of the points and establish a uniform bound (Proposition 1.3.5) on the deviations from those locations. This bound is used in the proof of (1.3.5) of Proposition 1.3.2. It is not, however, tight enough to suffice for all of our needs, hence our need for Theorems 1.1.3 and 1.1.4.

**Definition 1.3.3** (Stochastic Airy operator). Let $D = D(R^+)$ be the space of the generalized functions, i.e., the continuous dual of the space $C^\infty_0$ of all smooth compactly supported test functions endowed with the topology of compact convergence. For any function $f$, we denote its $k$-th derivative by the symbol $f^{(k)}$ and define its action on any test function $\phi \in C^\infty_0$ by

$$ \langle \phi, f^{(k)}(x) \rangle := (-1)^k \int f(x)\phi^{(k)}(x)dx. $$

Define the space of functions $H^1_{loc} = H^1_{loc}(R)$, where for any $f \in H^1_{loc}$ and any compact set $I \subset R$, we have $f^{(1)}1_I \in L^2(R)$. The $\beta > 0$ stochastic Airy operator $\mathcal{H}^{nw}_\beta$ is a linear map $\mathcal{H}^{nw}_\beta : H^1_{loc} \to D$ with $\mathcal{H}^{nw}_\beta f = -f^{(2)} + x f + \frac{2}{\sqrt{\beta}} f B'$.

Here, $B$ is a standard Brownian motion and $B'$ is its derivative which belongs to the space $D$. The non-random part of $\mathcal{H}^{nw}_\beta$ is the *Airy operator* $\mathcal{A} = -\partial^2_x + x$. Define the Hilbert space

$$ L^* := \{ f : f(0) = 0, \| f \|_* < \infty \} \quad \text{where} \quad \| f \|_* = \int_0^\infty ((f')^2 + (1 + x)f^2)dx. $$

A pair $(f, \Lambda) \in L^* \times R$ is an eigenfunction/value pair for $\mathcal{H}^{nw}_\beta$ if $\mathcal{H}^{nw}_\beta f = \Lambda f$ (likewise for $\mathcal{A}$).

**Proposition 1.3.4** ([RRV11]). Let $a = (a_1 > a_2 > \ldots)$ denote the Airy point process and $\Lambda = (\Lambda_1 < \Lambda_2 < \ldots)$ denote the eigenvalues of $\mathcal{H}^{nw}_2$. Then, $a$ and $-\Lambda$ are equal in distribution.

Results obtained in [RRV11][Vir14] show that the spectrum of $\mathcal{H}^{nw}_\beta$ lies within a uniform random band around the spectrum of the Airy operator $\mathcal{A}$. The following is a strengthening of such a result

---

To see that $fB' \in D$, observe that $\int_0^y fB'dx = -\int_0^y B'f'dx + f(y)B_y - f(0)B_0$ by integration by parts. One can now see that the latter is continuous function. Thus, its derivative $fB'$ belongs to the space $D$. 

---
wherein the tail decay of the band width (here $C_\epsilon$) is controlled. This result is proven for generic $\beta$, although, we need it later only for $\beta = 2$.

**Proposition 1.3.5.** Denote the eigenvalues of the Airy operator $\mathcal{A}$ by $(\lambda_1 < \lambda_2 < \ldots)$ and the eigenvalues of $\mathcal{H}_{\beta}^{\text{nw}}$ by $(\Lambda_1^\beta, \Lambda_2^\beta, \ldots)$. For any $\epsilon \in (0,1)$, we define the random variable $C_\epsilon$ as the minimal real number such that for all $k \geq 1$,

$$(1 - \epsilon)\lambda_k - C_\epsilon \leq \Lambda_k^\beta \leq (1 + \epsilon)\lambda_k + C_\epsilon$$

Then, for all $\epsilon, \delta \in (0,1)$ there exist $s_0 = s_0(\epsilon, \delta)$, and $\kappa = \kappa(\epsilon, \delta)$ such that for $s \geq s_0$,

$$\mathbb{P}(C_\epsilon \geq s \sqrt{\beta}) \leq \exp\left(-\kappa s^{\frac{1}{\delta}}\right). \quad (1.3.6)$$

Notice that (1.3.6) demonstrates a concentration inequality for the supremum of the deviations of the eigenvalues of $\mathcal{H}_{\beta}^{\text{nw}}$ around their typical locations. We defer the proof of this proposition until Section 1.3.7.

Finally, we state a result on the position of the eigenvalues of the Airy operator $\mathcal{A}$. Classical works [MT59, Tit58] have addressed this question for more general operators $-\partial_x^2 + V(x)$ for $V(x)$ satisfying certain regularity conditions. For the Airy operator, $\lambda_k$ coincides with the $k$-th zero of the Airy function.

**Proposition 1.3.6 ([MT59]).** Denote the eigenvalues of the Airy operator $\mathcal{A}$ by $\lambda = (\lambda_1 < \lambda_2 < \ldots)$. Then for any $n \geq 1$, $\lambda_n$ satisfies

$$\frac{1}{\pi} \int_{0}^{\lambda_n} \sqrt{(\lambda_n - x)} \, dx = n - \frac{1}{4} + \mathcal{R}(n), \quad \text{or} \quad \lambda_n = \left(\frac{3\pi}{2}(n - \frac{1}{4} + \mathcal{R}(n))\right)^{\frac{2}{3}}. \quad (1.3.7)$$

where $|\mathcal{R}(n)| \leq K/n$ for some large constant $K$.

1.3.4 Heuristics for Proposition 1.3.2

There are two main contributions to $\mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} \mathcal{I}_s(a_k) \right]$ – typical and atypical values of $a$. Owning to Proposition 1.3.5 the typical values of $a$ are close to the negatives of the Airy operator eigenvalues, whose locations are estimated in Proposition 1.3.6.
The asymptotic formula in (1.3.7) leads (as we now show) to the \( \exp \left( - \frac{4}{15\pi} T^{1/3} s^{5/3} \right) \) term in (2.3.14) and (1.3.5)\(^{11}\). Replacing \( a_k \) by \(-\lambda_k\) yields

\[
\log \left( \prod_{k=1}^\infty I_s(a_k) \right) \approx \sum_{k=1}^\infty J_s(-\lambda_k) = - \sum_{k=1}^\infty \log \left( 1 + \exp \left( T^{1/3} (s - \lambda_k) \right) \right).
\]

When \( s \gg \lambda_k \) and \( T \) is bounded away from 0, \( \log \left( 1 + \exp(T^{1/3} (s - \lambda_k)) \right) \approx T^{1/3} (s - \lambda_k) \). By Proposition 1.3.6 \( \lambda_k \approx (3\pi k/2)^{\frac{2}{3}} \), hence

\[
\sum_{\{k: \lambda_k < s\}} \log \left( 1 + \exp \left( T^{1/3} (s - \lambda_k) \right) \right) \approx T^{1/3} \sum_{k < \frac{s^{2/3}}{\pi}} \left( s - \left( \frac{3\pi k}{2} \right)^{\frac{2}{3}} \right)
\]

\[
\approx T^{1/3} \left( \frac{2}{3\pi} s^{\frac{2}{3}} - \frac{3}{5} \cdot \left( \frac{3\pi}{2} \right)^{\frac{2}{3}} \cdot \left( \frac{2}{3\pi} s^{\frac{2}{3}} \right)^{\frac{2}{3}} \right) = \frac{4}{15\pi} T^{1/3} s^{5/3}.
\]

To obtain the last approximation we replace the sum \( \sum_{k < s} k^{\frac{2}{3}} \) by the integral \( \int_0^x z^{\frac{2}{3}} dz \) which is equal to \( \frac{3}{2} \cdot x^{\frac{5}{3}} \). Thus (1.3.8) accounts for the first term in (2.3.14) and (1.3.5).

To complete the above heuristic we must show that the sum of \( J_s(-\lambda_k) \) over all \( \lambda_k > s \) can be ignored. For all \( \lambda_k > s \), one has \( 0 \leq J_s(-\lambda_k) \leq \exp \left( T^{1/3} (s - \lambda_k) \right) \). Using this,

\[
0 \leq \sum_{\{k: \lambda_k > s\}} J_s(-\lambda_k) \leq \sum_{k \geq \frac{2}{3\pi} s^{\frac{2}{3}}} \exp \left( T^{1/3} \left( s - \left( \frac{3\pi k}{2} \right)^{\frac{2}{3}} \right) \right) \leq \int_{\frac{2}{3\pi} s^{\frac{2}{3}}}^\infty \exp \left( T^{1/3} \left( s - \left( \frac{3\pi z}{2} \right)^{\frac{2}{3}} \right) \right) dz.
\]

The final integrand is less than 1 inside \([\frac{2}{3\pi} s^{\frac{2}{3}}, \infty]\) and thanks to the inequality (Lemma 1.4.6)

\[
s - \left( \frac{3\pi z}{2} \right)^{\frac{2}{3}} \leq - \left( \frac{3\pi (z - \frac{2}{3\pi} s^{\frac{2}{3}})}{2} \right)^{\frac{1}{3}} \text{ for all } z \geq \left( \frac{2}{3\pi} \right) s^{\frac{2}{3}} + \sqrt{\frac{2}{3\pi} s^{\frac{3}{3}}}.
\]

we obtain the following bound

\[
\int_{\frac{2}{3\pi} s^{\frac{2}{3}}}^\infty \exp \left( T^{1/3} \left( s - \left( \frac{3\pi z}{2} \right)^{\frac{2}{3}} \right) \right) dz \leq \sqrt{\frac{2}{3\pi} s^{\frac{3}{3}}} + \int_0^\infty \exp \left( - T^{1/3} \left( \frac{3\pi z}{2} \right)^{1/3} \right) dz.
\]

The final integral evaluates to a constant times \( (T/2)^{\frac{1}{3}} \int_0^\infty z^{\frac{2}{3}} \exp(-z) dz = (T/2)^{\frac{1}{3}} \Gamma(3) \). Thus, when \( T \) is bounded away from 0, the contribution of the eigenvalues which are greater than \( s \) is of the

\(^{11}\)The \( c \) error factor comes from various approximation errors and the fact that the replacement is only true with high probability.
order $O(s^{\frac{3}{2}})$ which is certainly less than $s^{\frac{3}{2}}$ for enough large $s$.

The other terms in the bounds (2.3.14) and (1.3.5) come from the atypical deviations of the Airy points from their typical locations. For instance, if $a_1$ is very negative, this will clearly effect the validity of the above heuristic. The proof of Proposition [1.3.2] boils down to controlling these atypical deviations and measuring their effect on the multiplicative functional in question.

Before we prove Proposition [1.3.2], we give proofs of Theorems [1.1.3] [1.1.4] and Proposition [1.3.5] which provide important control over the atypical deviations of the Airy point process.

1.3.5 Proof of Theorem [1.1.3]

Let us denote $A := \left\{ \chi_{\text{Ai}}(\left[ -s, \infty \right)) - E[\chi_{\text{Ai}}(\left[ -s, \infty \right))] \leq -cs^{\frac{3}{2}} \right\}$. Using Markov’s inequality, we find that for any $\lambda > 0$

$$P(A) \leq \exp \left( -\lambda cs^{\frac{3}{2}} + \lambda E[\chi_{\text{Ai}}(\left[ -s, \infty \right))] \right) E\left[ \exp \left( -\lambda \chi_{\text{Ai}}(\left[ -s, \infty \right])) \right) \right].$$

Set $\lambda = s^{\frac{3}{2}} - \delta$. Owing to Proposition [1.1.2] and Theorem [1.1.6]

$$E[\chi_{\text{Ai}}(\left[ -s, \infty \right))] = \frac{2}{3\pi} s^{\frac{3}{2}} + D(s),$$

$$E\left[ \exp \left( -\lambda \chi_{\text{Ai}}(\left[ -s, \infty \right])) \right) \right] = F(-s; \lambda) \leq \exp \left( -\frac{2\lambda}{3\pi} s^{\frac{3}{2}} + K s^{3-\frac{13\delta}{11}} \right)$$

where $K = K(\delta)$ is a large constant and $s$ is large enough. Thus

$$P(A) \leq \exp \left( -cs^{3-\delta} + K s^{3-\frac{13\delta}{11}} + D(s) \right).$$

Recalling that $|D(s)|$ is uniformly bounded for all $s > 0$, we find the desired bound.

1.3.6 Proof of Theorem [1.1.4]

Fix any $k \in \mathbb{Z}_{\geq 0}$. By Lemma [1.3.1] the kernel of the Airy point process is locally admissible and good. Thus (as discussed before Lemma [1.3.1]) for any compact set $D$, $\chi_{\text{Ai}}(D) \overset{d}{=} \sum_{i=1}^{\infty} X_i$ where the $X_i$’s are independent Bernoulli random variables satisfying $P(X_i = 1) = 1 - P(X_i = 0) = \lambda_i^D$. Here $\lambda_i^D$’s are the eigenvalues of the operator $\mathbb{1}_D K_{\text{Ai}} \mathbb{1}_D$. Choose a sequence of compact set $D_n$ increasing
to the interval $\mathcal{B}_k$. By Bennett’s concentration inequality \cite{Ben62},

$$
\mathbb{P}
\left(
\chi_{\text{Ai}}(D_n) - \mathbb{E}[\chi_{\text{Ai}}(D_n)] \geq cs^3
\right)
\leq
\exp\left(-\sigma_n^2 h\left(\frac{cs^3}{\sigma_n^2}\right)\right)
$$

(1.3.9)

where $h(u) := (1 + u) \log(1 + u) - u$. By the dominated convergence theorem, as $n \to \infty$, $\mu_n := \mathbb{E}[\chi_{\text{Ai}}(D_n)] \to \mathbb{E}[\chi_{\text{Ai}}(\mathcal{B}_k)]$ and $\sigma_n^2 := \text{Var}(\chi_{\text{Ai}}(D_n)) \to \text{Var}(\chi_{\text{Ai}}(\mathcal{B}_k))$. By Proposition 1.1.2 for $s$ large enough,

$$
\text{Var}(\chi_{\text{Ai}}(\mathcal{B}_k)) \leq C \log s
$$

for some constant $C > 0$. Therefore, for any given $\epsilon > 0$, there exist $S_0 = S_0(\epsilon)$ and $N_0 = N_0(\epsilon)$ such that for all $s \geq S_0$ and $n \geq N_0$,

$$
\sigma_n^2 \leq C(1 + \epsilon) \log s.
$$

(1.3.10)

Since $h(u) \geq u(\log u - 1)$, we find $\sigma_n^2 h(cs^3/\sigma_n^2) \geq cs^3 \left(\log(cs^3) - \log \sigma_n^2 - 1\right)$. Plugging the upper bound (1.3.10) on $\sigma_n^2$ into this inequality and exponentiating yields

$$
\exp\left(-\sigma_n^2 h(cs^3/\sigma_n^2)\right) \leq \exp\left(-cs^3 \left(\log(cs^3) - (1 + \epsilon) \log \log s\right)\right)
$$

(1.3.11)

for all $n \geq N_0$ and $s$ sufficiently large. Now, Fatou’s lemma shows

$$
\mathbb{P}\left(\chi_{\text{Ai}}(\mathcal{B}_k) - \mathbb{E}[\chi_{\text{Ai}}(\mathcal{B}_k)] \geq cs^3\right)
\leq
\liminf_{n \to \infty} \mathbb{P}\left(\chi_{\text{Ai}}(D_n) - \mathbb{E}[\chi_{\text{Ai}}(D_n)] \geq cs^3\right).
$$

(1.3.12)

Owing to (1.3.9) and (1.3.11), we find that

$$
\text{r.h.s of (1.3.12)} \leq \limsup_{n \to \infty} \exp\left(-\sigma_n^2 h(cs^3/\sigma_n^2)\right) \leq \exp\left(-cs^3 \left(\log(cs^3) - (1 + \epsilon) \log \log s\right)\right).
$$

1.3.7 Proof of Proposition 1.3.5

We start with a lemma about the tails of the distribution of Brownian motion oscillations.
Lemma 1.3.7. Let $B_x$ be a Brownian motion on $[0, \infty)$ and define

$$Z := \sup_{x>0} \sup_{y \in [0,1)} \frac{|B_{x+y} - B_x|}{6 \sqrt{\log(3 + x)}}. \quad (1.3.13)$$

Then, letting $\tilde{B}_x = \int_{x}^{x+1} B_y \, dy$ and $\tilde{B}'_x = \frac{d}{dx} \tilde{B}_x \big( = B_{x+1} - B_x \big)$, we have that (1) $\max\{|\tilde{B}'_x|, |\tilde{B}_x - B_x|\} \leq 6Z \sqrt{\log(3 + x)}$, and (2) there exist $K_1, K_2, s_0 > 0$ such that for all $s > s_0$

$$\mathbb{P}(Z \geq s) \leq K_1 e^{-K_2 s^2}. \quad (1.3.14)$$

Proof. The proof of (1) follows from the following inequalities:

$$|\tilde{B}'_x| = |B_{x+1} - B_x| \leq 6 \sqrt{\log(3 + x)} \sup_{y \in [0,1)} \frac{|B_{x+y} - B_x|}{6 \sqrt{\log(3 + x)}} \leq 6Z \sqrt{\log(3 + x)},$$

$$|\tilde{B}_x - B_x| \leq \int_0^1 |B_{x+y} - B_x| \, dy \leq \sup_{y \in [0,1)} |B_{x+y} - B_x| \leq 6Z \sqrt{\log(3 + x)}.$$

Turning to (2), for any $y \in [0, 1)$,

$$|B_{x+y} - B_x| \leq |B_{x+y} - B_{[x]}| + |B_{[x]} - B_{[x]}| + |B_{[x]} - B_{[x]}| \leq 2 \sup_{y \in [0,1]} |B_{[x]+y} - B_{[x]}| + 2 \sup_{y \in [0,1]} |B_{[x]+y} - B_{[x]}|.$$

Therefore

$$\sup_{y \in [0,1)} \frac{|B_{x+y} - B_x|}{\sqrt{\log(3 + x)}} \leq 2 \sup_{y \in [0,1]} \frac{|B_{[x]+y} - B_{[x]}|}{\sqrt{\log(3 + x)}} + 2 \sup_{y \in [0,1]} \frac{|B_{[x]+y} - B_{[x]}|}{\sqrt{\log(3 + x)}}. \quad (1.3.15)$$

To study $Z$ we must take the sup over all positive real $x$ of the above bound. However, at the cost of replacing $3 + x$ by $2 + x$ in the denominator, using (1.3.15) we can bound $Z \leq 4W$ where

$$W := \max_{n \in \mathbb{Z}_{\geq 1}} \frac{W_n}{6 \sqrt{\log(2 + n)}}, \quad \text{where} \quad W_n := \sum_{y \in [0,1)} |B_{n+y} - B_n|.$$

The $\{W_n\}_{n \in \mathbb{Z}_{\geq 1}}$ are iid, and an application of the reflection principle shows that

$$\mathbb{P}(W_n \geq a) \leq 2 \mathbb{P}(|B_{n+1} - B_n| \geq a/2) \leq \frac{2}{a} e^{-a^2/8}. \quad (1.3.16)$$
The union bound shows that
\[ \mathbb{P}(Z \geq s) \leq \mathbb{P}(4W \geq s) = \mathbb{P}\left( \bigcup_{n=0}^{\infty} \frac{W_n}{6\sqrt{\log(2+n)}} \geq \frac{s}{4} \right) \leq \sum_{n=0}^{\infty} \mathbb{P}\left( W_n \geq \frac{3}{2} s \sqrt{\log(2+n)} \right). \]

Combining this with (2.3.20) yields the desired decay bound as long as \( s \) is large enough. ■

**Proof of Proposition 1.3.5** We will make use the following convention: For any two operator \( A, B : H_{\text{loc}}^{1} \to D \), we write \( A \leq B \) if for all \( f \in L^* \), \( \langle f, Af \rangle \preceq \langle f, Bf \rangle \preceq \). If \( A \leq B \), then \( \lambda_k^A \leq \lambda_k^B \) where \( \lambda_k^A \) and \( \lambda_k^B \) are \( k \)-th lowest eigenvalues of the operators \( A \) and \( B \) respectively.

In our proof we bound \( H_{\text{nw}}^\beta \) above/below by the Airy operator plus/minus an error with well-controlled tails. This requires establishing a random operator bound on \( B' \). Decomposing Brownian motion \( B_x = \bar{B}_x + (B_x - \bar{B}_x) \) (\( \bar{B}_x \) is defined in Lemma 1.3.7) we find that for \( f \in C_0^\infty \),
\[ \langle f, B'f \rangle = \int_0^{\infty} f^2 B'_x dx + \int_0^{\infty} f(x)f'(x)(\bar{B}_x - B_x)dx. \] (1.3.17)

**Claim:** Fix \( \epsilon, \delta \in (0, 1) \). Let \( K_1 = K_1(\delta) \geq 1 \) (resp. \( K_2 = K_2(\delta) \geq 1 \)) be a constant such that \( \sqrt{\log(3+x)} \leq x^\delta \) (resp. \( \log(3+x) \leq x^\delta \)) for all \( x \geq K_1 \) (resp. \( x \geq K_2 \)). Define
\[ U_\epsilon := \max \left\{ Z\left( \left( \frac{Z}{\epsilon} \right)^{\frac{\delta}{(1-\delta)}} + K_1^\delta \right), Z^2\left( \left( \frac{Z}{\epsilon} \right)^{\frac{2\delta}{(1-\delta)}} + K_2^\delta \right) \right\}. \]

Then
\[ -10\epsilon A - 6(1 + \frac{1}{2} \epsilon^{-1})U_\epsilon \leq B' \leq 10\epsilon A + 6(1 + \frac{1}{2} \epsilon^{-1})U_\epsilon \] (1.3.18)

**Proof of Claim:** Recall that \( \bar{B}_x' = B_{x+1} - B_x \). From Lemma 1.3.7 \( |\bar{B}_x'| \leq 6Z \sqrt{\log(3+x)} \) (see (1.3.13) for \( Z \)). Thus, we will start by establishing the following bound, valid for all \( x \geq 0 \):
\[ Z\sqrt{\log(3+x)} \leq \max \left\{ Z\left( (Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta \right) + \epsilon x, Z\sqrt{(Z/\epsilon)^{\frac{2\delta}{(1-\delta)}} + K_2^\delta + \epsilon^2 x} \right\}. \] (1.3.19)

We explain the derivation of the first bound by \( Z\left( (Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta \right) + \epsilon x \), as the second bound follows.
a similar type of argument. Let \( z_0 := \max\{ (Z/\epsilon)^{\frac{1}{1-\delta}}, K_1 \} \). For \( x < z_0 \),

\[
Z \sqrt{\log(3 + x)} \leq Z \sqrt{\log(3 + z_0)} \leq Z \cdot z_0^{\delta} \leq Z \left( (Z/\epsilon)^{\frac{\delta}{1-\delta}} + K_1^\delta \right) \leq Z \left( (Z/\epsilon)^{\frac{\delta}{1-\delta}} + K_1^\delta \right) + \epsilon x.
\]

The second inequality uses \( \sqrt{\log(3 + z_0)} \leq z_0^{\delta} \), the third uses \( \max\{ a, b \} \leq a + b \) for \( a, b \geq 0 \). For \( x \geq z_0 \),

\[
Z \sqrt{\log(3 + x)} \leq Z(1 + \sqrt{\log(3 + x)}) \leq Z + \epsilon x^{1-\delta} \sqrt{\log(3 + x)} \leq Z + \epsilon x \leq Z \left( (Z/\epsilon)^{\frac{\delta}{1-\delta}} + K_1^\delta \right) + \epsilon x.
\]

The second inequality uses \( Z \leq \epsilon x \) (as \( (Z/\epsilon)^{\frac{1}{1-\delta}} \leq x \)), the third uses \( \sqrt{\log(3 + x)} \leq x^{\delta} \) (since \( x \geq K_1 \)), the fourth uses \( (Z/\epsilon)^{\frac{\delta}{1-\delta}} + K_1^\delta \geq 1 \).

Combining (1.3.19) with the definition of \( U_\epsilon \), we see that for all \( x \geq 0 \),

\[
Z \sqrt{\log(3 + x)} \leq \max \left\{ U_\epsilon + \epsilon x, \sqrt{U_\epsilon + \epsilon^2 x} \right\}.
\]

This, along with Lemma 1.3.7 establishes that for all \( x \geq 0 \),

\[
|\vec{B}'_x| \leq 6Z \sqrt{\log(3 + x)} \leq 6(U_\epsilon + \epsilon x), \quad |\vec{B}_x - B_x| \leq 6Z \sqrt{\log(3 + x)} \leq 6\sqrt{U_\epsilon + \epsilon^2 x}. \tag{1.3.20}
\]

Using the formula for \( \prec f, B' f \succ \) in (1.3.17), along with the inequality \( |f'(x)f(x)(\vec{B}_x - B_x)| \leq 3\epsilon (f'(x))^2 + (12\epsilon)^{-1} f(x)^2 |\vec{B}_x - B_x|^2 \) (which follows by applying \( ab \leq \frac{1}{2} (a^2 + b^2) \)) we have that

\[
|\prec f, B' f \succ | \leq \int_0^\infty f^2(x)(\epsilon + |\vec{B}_x|)dx + 3\epsilon \int_0^\infty (f'(x))^2 + (12\epsilon)^{-1} \int_0^\infty (f(x))^2 |\vec{B}_x - B_x|^2 dx.
\]

Plugging the bounds from (1.3.20) into the above expression yields

\[
|\prec f, B' f \succ | \leq 6U_\epsilon \|f\|^2 + 7\epsilon \langle f, A f \rangle + 3\epsilon \int_0^\infty (f'(x))^2 dx + 3\epsilon^{-1} \int_0^\infty f^2(U_\epsilon + \epsilon^2 x) dx
\]

\[
\leq 6 \left( 1 + (2\epsilon)^{-1} \right) U_\epsilon \|f\|^2 + 10\epsilon \langle f, A f \rangle.
\]

which implies (1.3.18) as claimed.

Now, we turn to complete the proof of Proposition 1.3.5. In (1.3.18), we have shown that for any
\[ f \in L^*, \]
\[ -\frac{20}{\sqrt{\beta}} \prec f, Af \succ -\frac{12}{\sqrt{\beta}} \mathcal{U}_e \| f \|^2 \leq - f, A f \succ \frac{2}{\sqrt{\beta}} B' f \succ \leq \frac{20}{\sqrt{\beta}} \prec f, A f \succ + \frac{12}{\sqrt{\beta}} \mathcal{U}_e \| f \|^2 \]

Adding \(- f, Af \succ\) on each sides of the above inequalities and combining those with the definition that \( \mathcal{H}_{\beta}^{nw} = A + \frac{2}{\sqrt{\beta}} B' \) yields
\[ A \left( 1 - \frac{20}{\sqrt{\beta}} \epsilon \right) - \frac{12}{\sqrt{\beta}} \left( 1 + \frac{1}{2 \epsilon} \right) \mathcal{U}_e \leq \mathcal{H}_{\beta}^{nw} \leq A \left( 1 + \frac{20}{\sqrt{\beta}} \epsilon \right) + \frac{12}{\sqrt{\beta}} \left( 1 + \frac{1}{2 \epsilon} \right) \mathcal{U}_e \]

which shows
\[ \left( 1 - \frac{20}{\sqrt{\beta}} \epsilon \right) \lambda_k - \frac{12}{\sqrt{\beta}} \left( 1 + \frac{1}{2 \epsilon} \right) \mathcal{U}_e \leq \Lambda_{\beta} \leq \left( 1 + \frac{20}{\sqrt{\beta}} \epsilon \right) \lambda_k - \frac{12}{\sqrt{\beta}} \left( 1 + \frac{1}{2 \epsilon} \right) \mathcal{U}_e \]

for all \( k \in \mathbb{N} \). Replacing \( \epsilon \mapsto \frac{\sqrt{\beta}}{20} \epsilon \) and using the tail bound (1.3.14) on \( \mathcal{U}_e \) yields Proposition 1.3.5. ■

1.4 Proof of Proposition 1.3.2

We prove the upper bound (2.3.14) in Section 1.4.1 and the lower bound (1.3.5) in Section 1.4.2. Before giving these proofs, we recall the behavior of the tail of \( a_1 \) (the GUE Tracy-Widom distribution). There have been numerous works [BN12, DV13, BBD08, RRV11] to find the exact tails of \( a_1 \) and the below proposition follows from these (e.g. [RRV11, Theorem 1.3]).

**Proposition 1.4.1.** Let \( a_1 \) denote the top particle in the Airy point process (which follows the Tracy-Widom GUE distribution). Then \( o(1) \) goes to zero as \( s \) goes to infinity
\[ \mathbb{P}(a_1 < -s) = \exp \left( -\frac{1}{12} (s^3 + o(1)) \right). \tag{1.4.1} \]

1.4.1 Proof of the upper bound (2.3.14)

Recall \( I_s(\cdot) \) and \( J_s(\cdot) \) from (1.3.3), related by \( I_s(\cdot) = \exp(J_s(\cdot)) \). Thus, in order to obtain an upper bound on \( \mathbb{E}[\prod_{k=1}^{\infty} I_s(a_k)] \), we derive a lower bound on \( \sum_{k=1}^{\infty} J_s(a_k) \) by comparing the Airy point process with the corresponding eigenvalues \( \lambda_k \) of the Airy operator (Section 1.3.3). Let us denote \( D_k := (-\lambda_k - a_k)_+ = \max\{ -\lambda_k - a_k, 0 \} \).
Lemma 1.4.2. Fix some $\epsilon \in (0, 1/3)$. Denote $\theta_0 = [2s^{3/2}/3\pi]$. There exists $S_0 = S_0(\epsilon) > 0$ and a constant $R > 0$ such that for all $s \geq S_0$,

$$
\sum_{k=1}^{\infty} J_s(\alpha_k) \geq T^{1/2} \left( \frac{4s^{3/2}}{15\pi} (1 - 8\epsilon) - \sum_{k=1}^{\theta_0} D_k - R \right) .
$$

(1.4.2)

Proof. Using monotonicity of $J_s(\cdot)$ and the inequality (1.1.8), we obtain the following

$$
\sum_{k=1}^{\infty} J_s(\alpha_k) = \sum_{k=1}^{\infty} J_s \left( -\lambda_k - (-\lambda_k - \alpha_k)_+ + (-\lambda_k - \alpha_k)_- \right) \geq \sum_{k=1}^{\infty} J_s(-\lambda_k - D_k). 
$$

(1.4.3)

We divide the sum on the right side of (1.4.3) into three ranges: $[1, \theta_1], (\theta_1, \theta_2)$ and $[\theta_2, \infty)$ where $\theta_1$ and $\theta_2$ are defined as (recall $R(n)$ from Proposition 1.3.6)

$$
K := \sup_{n \geq 1} \{ |nR(n)| \}, \quad \theta_1 := \lfloor 4K \rfloor, \quad \theta_2 := \left\lfloor \frac{2s^{3/2}}{3\pi} + \frac{1}{2} \right\rfloor.
$$

Note that as $\theta_1$ does not depend on $s$, but $\theta_2$ does, we choose $s$ large enough so $\theta_1 < \theta_2$.

Claim:

$$
\sum_{k=1}^{\theta_1} J_s(-\lambda_k - D_k) \geq T^{1/2} \left( \theta_1 \left( s - \frac{3\pi(4K + 1)}{2} \right)^{\frac{3}{2}} - \sum_{k=1}^{\theta_1} D_k \right).
$$

(1.4.4)

PROOF OF CLAIM: Since $\log(1 + \exp(a)) \geq a$ for any $a \in \mathbb{R}$, $J_s(\cdot) \geq T^{1/2}(s + \cdot)$. Using this and monotonicity of $\lambda_k$ in $k$, we find that

$$
\sum_{k=1}^{\theta_1} J_s(-\lambda_k - D_k) \geq \sum_{k=1}^{\theta_1} T^{1/2}(s - \lambda_k - D_k) \geq T^{1/2} \left( \theta_1 (s - \lambda_1) - \sum_{k=1}^{\theta_1} D_k \right).
$$

From (1.3.7), $\lambda_1 \leq (3\pi(1 - 1/4 + K/\theta_1)/2)^{\frac{3}{2}} \leq (3\pi(4K + 1)/2)^{\frac{3}{2}}$; hence (1.4.4) follows immediately.

Claim:

$$
\sum_{k=\theta_1+1}^{\theta_2-1} J_s(-\lambda_k - D_k) \geq T^{1/2} \left( \frac{4s^{3/2}}{15\pi} (1 - 3\epsilon) - (\theta_1 + 1)s - \sum_{k=\theta_1+1}^{\theta_2-1} D_k \right).
$$

(1.4.5)

PROOF OF CLAIM: We assume that $s \geq (3\pi \epsilon^{-1}/4)^{\frac{3}{2}}(1 + \epsilon)$. Observe that

$$
\sum_{k=\theta_1+1}^{\theta_2-1} J_s(-\lambda_k - D_k) \geq T^{1/2} \sum_{k=\theta_1+1}^{\theta_2-1} \left( s - \frac{3\pi K}{2} \right)^{\frac{3}{2}} - \sum_{k=\theta_1+1}^{\theta_2-1} D_k.
$$

(1.4.6)
This uses \( \log(1 + \exp(a)) \geq a \) for all \( a \in \mathbb{R} \) and \( \lambda_k \leq (3\pi k/2)^{3/5} \) for all \( k > \theta_1 \). Now we bound

\[
\sum_{k=\theta_1+1}^{\theta_2-1} \left( s - \left( \frac{3\pi k}{2} \right)^{3/5} \right) \geq \sum_{k=\theta_1+1}^{\theta_2-1} \left( s - \left( \frac{3\pi k}{2} \right)^{3/5} \right) \geq \int_{\theta_1+1}^{\theta_2-1} \left( s - \left( \frac{3\pi z}{2} \right)^{3/5} \right) dz
\]

\[
\geq \int_0^{\theta_2-1} \left( s - \left( \frac{3\pi z}{2} \right)^{3/5} \right) dz - (\theta_1 + 1)s = (\theta_2 - 1)s - \frac{3}{5} \cdot \left( \frac{3}{5} \right)^{3/5} (\theta_2 - 1)^{3/5} - (\theta_1 + 1)s
\]

Noting that \( (1 + \epsilon) \frac{2s^2}{3\pi} \leq \theta_2 - 1 \leq \frac{2s^2}{3\pi} + 1 \) we may bound the above expression such that combining with (1.4.6) we arrive at the claimed inequality (1.4.5).

Plugging into (1.4.3) the bounds (1.4.4), (1.4.5), and \( \sum_{k=\theta_2}^{\infty} J_s(-\lambda_k - D_k) \geq 0 \) yields

\[
\sum_{k=1}^{\infty} J_s(a_k) \geq \frac{T_1^{\frac{1}{2}}}{2\pi} \left( \frac{4^{\frac{5}{2}}}{15\pi} (1 - 3\epsilon) - s - \sum_{k=1}^{\theta_2-1} D_k - \theta_1 \left( \frac{3\pi (K + 1)}{2} \right)^{3/2} \right) \tag{1.4.7}
\]

To finally arrive at the desired inequality in (1.4.2), we use two more bounds. Since we may assume \( s \leq \frac{4s^2}{3\pi} \) for all \( s \geq S_0 := (3\pi \epsilon^{-1/4})^{2/5} (1 + \epsilon) \), we can replace \( -s \) by \( -\frac{4s^2}{3\pi} \) in the right side of (1.4.7). Finally, for all \( \epsilon < 1 \), \( \theta_1 (3\pi (K + 1)/2)^{3/2} \) can be bounded above by a large constant \( R \) (independent of \( s \) and \( \epsilon \)). Incorporating these bounds into (1.4.7) yields (1.4.2). \( \square \)

**Proof of (2.3.14) in Proposition (3.2)** Multiplying (1.4.2) by \(-1\) and exponentiating yields

\[
\prod_{k=1}^{\infty} I_s(a_k) \leq \exp \left( - T^{\frac{1}{2}} \left( \frac{4s^5}{15\pi} (1 - 8\epsilon) - \sum_{k=1}^{\theta_0} D_k - R \right) \right).
\]

Recalling \( \theta_0 = \lceil 2s^2 / 3\pi \rceil \) and defining \( S_{\theta_0} := \sum_{k=1}^{\theta_0} D_k \) we have that

\[
1 \{ S_{\theta_0} < \epsilon s \theta_0 \} \prod_{k=1}^{\infty} I_s(a_k) \leq \exp \left( - T^{\frac{1}{2}} \frac{4s^5}{15\pi} (1 - 11\epsilon) \right).
\]

If \( S_{\theta_0} \geq \epsilon s \theta_0 \), then there exists at least one \( k \in [1, \theta_0] \cap \mathbb{Z} \) such that \( D_k \) is greater than \( \epsilon s \). Thus,

\[
\{ S_{\theta_0} \geq \epsilon s \theta_0 \} \subset \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \}.
\]

Summarizing the discussion above, we have that

\[
E \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] = \mathbb{E} \left[ 1 \{ S_{\theta_0} < \epsilon s \theta_0 \} \prod_{k=1}^{\infty} I_s(a_k) \right] + \mathbb{E} \left[ 1 \{ S_{\theta_0} \geq \epsilon s \theta_0 \} \prod_{k=1}^{\infty} I_s(a_k) \right] \tag{1.4.8}
\]

\[
\leq \exp \left( - T^{\frac{1}{2}} \frac{4s^5}{15\pi} (1 - 11\epsilon) \right) + \mathbb{E} \left[ \prod_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right] \prod_{k=1}^{\infty} I_s(a_k).
\]

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We may bound indicator functions
\[ 1\left\{ \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right\} \leq 1\left\{ \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \cap \{ a_1 \geq -(1-\epsilon)s \} \right\} + 1\{ a_1 \leq -(1-\epsilon)s \}. \]

Since \( I_s(a_k) \leq 1 \) for all \( k \in \mathbb{Z}_{>0} \), when \( a_1 \geq -(1-\epsilon)s \),
\[ \prod_{k=1}^{\infty} I_s(a_k) \leq \frac{1}{1 + \exp\left(T \frac{1}{\epsilon s}(s+a_1)\right)} \leq \exp\left(-\epsilon sT^{\frac{1}{\pi}}\right). \]

Combining these observations and taking expectations implies
\[ \mathbb{E}\left[ 1\left\{ \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right\} \prod_{k=1}^{\infty} I_s(a_k) \right] \leq \exp\left(-\epsilon sT^{\frac{1}{\pi}}\right) \mathbb{P}\left( \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right) + \mathbb{P}(a_1 \leq -(1-\epsilon)s). \] (1.4.9)

By Proposition 1.4.1 there exists \( C > 0 \) such that for \( s \) large enough \( \mathbb{P}(a_1 \leq -(1-\epsilon)s) \leq \exp\left(-\frac{s^3}{12}(1-C\epsilon)\right) \).

Combining (1.4.8), (1.4.9) and (1.4.10) in Lemma 1.4.3 we find (2.3.14). □

**Lemma 1.4.3.** Fix \( \epsilon, \delta \in (0, 1/3) \). There exist \( S_0 = S_0(\eta, \delta) > 0 \) and \( K_1 = K_1(\eta, \delta) > 0 \) such that the following holds for all \( s \geq S_0 \). Divide the interval \([-s, 0]\) into \([2\epsilon^{-1}]\) segments \( Q_i := [-j\epsilon s/2, -(j-1)\epsilon s/2) \) for \( j = 1, \ldots, [2\epsilon^{-1}] \). Denote the right and left end points of \( Q_j \) by \( q_j \) and \( p_j \). Define \( k_j := \inf\{ k : -\lambda_k \geq q_j \} \) (\( \lambda_1 < \lambda_2 < \ldots \) are the Airy operator eigenvalues). Then (recalling \( \theta_0 = \lfloor 2s^2/3\pi \rfloor \)),
\[ \mathbb{P}\left( a_{k_j} \leq p_j \right) \leq \exp(-K_1 s^{3-\delta}) \quad \forall j \in \{1, \ldots, [2\eta^{-1}]\} \] (1.4.10)
\[ \mathbb{P}\left( \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right) \leq \exp(-K_1 s^{3-\delta}). \]

**Proof.** We prove the first line of (1.4.10). For \( 1 \leq j \leq [2\epsilon^{-1}] \), when \( a_{k_j} \leq p_j = -2^{-1}(j\epsilon s) \),
\[ \chi^{Ai}([-2^{-1}(j\epsilon s), \infty)) \leq k_j \leq \#\{ k : -\lambda_k \geq -2^{-1}(j-1)\epsilon s \}. \] (1.4.11)

Owing to Propositions 1.1.2 and 1.3.6 we have
\[ \#\{ k : -\lambda_k \leq -x \} =: \frac{2}{3\pi}x^3 + C_1(x), \quad \text{and} \quad \mathbb{E}\left[ \chi^{Ai}([-x, \infty)) \right] =: \frac{2}{3\pi}x^3 + C_2(x) \] (1.4.12)
where \( \sup_{x \geq 0} \{ |C_1(x)|, |C_2(x)| \} < \infty \). Combining (1.4.11) and (1.4.12) shows that when \( a_{kj} \leq p_j \),

\[
\chi^{AI}([-2^{-1}(j\epsilon s), \infty)) - \mathbb{E} [\chi^{AI}([-2^{-1}(j\epsilon s), \infty))] \\
\leq \#\{ k : -\lambda_k \geq -2^{-1}(j-1)\epsilon s \} - \#\{ k : -\lambda_k \geq -2^{-1}j\epsilon s \} + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s) \\
\leq \frac{(\epsilon s)^{\frac{3}{2}}}{3\sqrt[3]{2\pi}} ((j-1)^{\frac{3}{2}} - j^{\frac{3}{2}}) + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s) \\
\leq -M \sqrt{j}(\epsilon s)^{\frac{3}{2}} + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s).
\]

for some \( M > 0 \). Therefore,

\[
\mathbb{P}(a_{kj} \leq p_j) \leq \mathbb{P}\left( \chi^{AI}(\lfloor p_j, \infty \rfloor) - \mathbb{E}[\chi^{AI}(\lfloor p_j, \infty \rfloor)] \leq -M \sqrt{j}(\epsilon s)^{\frac{3}{2}} + 2 \sup_{x \geq 0} \{ |C_1(x)|, |C_2(x)| \} \right).
\]

For large enough \( s \), \(-M \sqrt{j}(\epsilon s)^{\frac{3}{2}} + 2 \sup_{x \geq 0} \{ |C_1(x)|, |C_2(x)| \} \leq -\frac{M}{2} \sqrt{j}(\epsilon s)^{\frac{3}{2}} \) for all \( j \in \{1, \ldots, [2\epsilon^{-1}] \} \).

The first line of (1.4.10) follows by applying (1.1.7) of Theorem 1.1.3 which shows that there exist \( S_0(\epsilon, \delta) \) and \( K_1 = K_1(\epsilon, \delta) \) such that for all \( s \geq S_0 \),

\[
\mathbb{P}\left( \chi^{AI}([-j\epsilon s, \infty)) - \mathbb{E}[\chi^{AI}([-j\epsilon s, \infty))] \leq -\frac{M}{2} \sqrt{j}(\epsilon s)^{\frac{3}{2}} \right) \leq \exp\left(-K_1s^{3-\delta}\right).
\]

Turning to the second line of (1.4.10), we assume (as allowed by (1.3.7)) that \( s \) is large enough so \( \lambda_{\theta_0} < s \). We claim then that

\[
\bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \subset \bigcup_{j=1}^{[2\epsilon^{-1}]} \{ a_{kj} \leq p_j \}. \quad (1.4.13)
\]

To see this, consider any integer \( 1 \leq k \leq \theta_0 \) and assume that \( D_k \geq \epsilon s \). Let \( j \) be such that \(-\lambda_k \in Q_{j-1}\). Since \( Q_{j-1} \) is to the right of \( Q_j = [p_j, q_j] \), it follows that \( a_k \leq -\lambda_k - \epsilon s \). Moreover, \( a_{kj} \leq a_k \) because \(-\lambda_{kj} < -\lambda_k \). Combining these yields

\[
a_{kj} < a_k \leq -\lambda_k - \epsilon s = (\lambda_{kj} - \lambda_k) - \lambda_k - \epsilon s \leq -\lambda_{kj} - \frac{\epsilon s}{2},
\]

where the last inequality uses \( 0 \leq (\lambda_{kj} - \lambda_k) \leq \frac{\epsilon s}{2} \) (as \( \lambda_{kj}, \lambda_k \in Q_{j-1}\)). Hence, the distance between \( a_{kj} \) and \( \lambda_{kj} \) is greater than or equal to \( \epsilon s/2 \). This shows \( a_{kj} \leq p_j \), and hence (1.4.13).
The first line of (1.4.10) along with (1.4.13) implies that
\[ P \left( \bigcup_{k=1}^{\theta_0} \{ D_k \geq \epsilon s \} \right) \leq \sum_{i=1}^{\theta_0} P( a_{k_i} \leq p_i ) \leq \left[ 2\epsilon^{-1} \right] \exp \left( -K_1 s^{3-\delta} \right). \]

As long as \( s \) is sufficiently large, the \( \left[ 2\epsilon^{-1} \right] \) prefactor can be absorbed into the exponent at the cost of slightly modifying \( K_1 \).

### 1.4.2 Proof of the lower bound (1.3.5)

In order to obtain a lower bound on \( E \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \), we derive an upper bound on \( \sum_{k=1}^{\infty} J_s(a_k) \).

**Lemma 1.4.4.** There exists \( B > 0 \) and \( S_0 \) such that for all \( \epsilon \in (0, 1/3) \) and all \( s \geq S_0 \),
\[ \sum_{k=1}^{\infty} J_s(a_k) \leq L_{T,\epsilon}(s + C^{A_i}) \quad (1.4.14) \]

where
\[ L_{T,\epsilon}(x) := T^{\frac{1}{2}} \left( 4\frac{x^2}{15\pi} (1 + 3\epsilon) + 2x + B \right) + \frac{x^2}{3(1-\epsilon)^2} + \sqrt{\frac{2}{\pi}} \frac{x^4}{(1-\epsilon)^3} + \frac{4}{T\pi(1-\epsilon)^3}. \]

**Proof.** Using the monotonicity of \( J_s(\cdot) \) and the inequality (1.1.8), we obtain
\[ \sum_{k=1}^{\infty} J_s(a_k) \leq \sum_{k=1}^{\infty} J_s(- (1-\epsilon)\lambda_k + C^{A_i}) = (\bar{I}) + (\bar{II}) + (\bar{III}), \quad (1.4.15) \]

where \( (\bar{I}), (\bar{II}) \) and \( (\bar{III}) \) equal the sum of \( J_s(- (1-\epsilon)\lambda_k + C^{A_i}) \) over all integers \( k \) in the intervals \( [1, \theta'_1], (\theta'_1, \theta'_2) \) and \( [\theta'_2, \infty) \) respectively, and (similar to Section 1.4.1) \( \theta'_1 \) and \( \theta'_2 \) are
\[ \theta'_1 := \theta_1 := 4 \sup_{n \in \mathbb{Z}_{>0}} n |\mathcal{R}(n)|, \quad \theta'_2 := \left[ \frac{2(s + C^{A_i})^{\frac{3}{2}}}{3\pi (1-\epsilon)^{\frac{3}{2}}} + \frac{1}{2} \right]. \]

For any integer \( 1 \leq k \leq \theta'_1 \), \( J_s(- (1-\epsilon)\lambda_k + C^{A_i}) \leq J_s(- (1-\epsilon)\lambda_1 + C^{A_i}). \) Using this upper bound and the inequality \( \log(1 + \exp(a)) \leq a + \pi/2 \) for \( a > 0 \), we obtain
\[ (\bar{I}) \leq \theta'_1 J_s(- (1-\epsilon)\lambda_1 + C^{A_i}) \leq \theta'_1 T^{\frac{1}{2}} (s - (1-\epsilon)\lambda_1 + C^{A_i}) + \frac{\pi \theta'_2}{2}. \quad (1.4.16) \]
Claim:

\[ (\Pi) \leq T^{1/3} \left( \frac{4(s+C_{\epsilon}^{Ai})^2}{15\pi} (1 + 3\epsilon) + (2 - \theta'_1)(s + C_{\epsilon}^{Ai}) - \frac{3(3\pi)^{2/3}(\theta'_1)^{5/3}}{52^{4/3}} \right) + \frac{\pi(\theta'_1 - \theta'_1)^3}{2} \]  \hfill (1.4.17)

**Proof of Claim:** For integer \( k \in (\theta'_1, \infty) \), it follows from the definition of \( \theta'_1 \) that

\[ \lambda_k \geq \left( \frac{3\pi(k - \frac{1}{4}) - |R(k)|}{2} \right)^{\frac{2}{3}} \geq \left( \frac{3\pi(k - \frac{1}{2})}{2} \right)^{\frac{2}{3}}. \]  \hfill (1.4.18)

This and monotonicity of \( J_s(\cdot) \) implies that

\[ J_s\left((1 - \epsilon)\lambda_k + C_{\epsilon}^{Ai}\right) \leq J_s\left((1 - \epsilon)\left(\frac{3\pi(k - \frac{1}{2})}{2}\right)^{\frac{2}{3}} + C_{\epsilon}\right). \]

Leveraging this and using the inequality \( J_s(a) \leq a + \pi/2 \) for any \( a > 0 \), we obtain

\[ (\Pi) \leq \sum_{k=\theta'_1 + 1}^{\theta'_2 - 1} \left( T^{1/3} f_s(k) + \frac{\pi}{2} \right), \quad \text{where} \quad f_s(z) := s + C_{\epsilon}^{Ai} - (1 - \epsilon)\left(\frac{3\pi(z - \frac{1}{2})}{2}\right)^{\frac{2}{3}}. \]  \hfill (1.4.19)

Bounding the sum in (1.4.19) by the corresponding integral we find

\[ (\Pi) \leq T^{1/3} \int_{\theta'_1}^{\theta'_2} f_s(z)dz + \frac{\pi(\theta'_2 - \theta'_1)}{2}. \]  \hfill (1.4.20)

To bound \( \int_{\theta'_1}^{\theta'_2} f_s(z)dz \), we observe that

\[ \int_{\frac{1}{2}}^{\theta'_2} f_s(z)dz \leq (s + C_{\epsilon}^{Ai}) \left( \frac{2(s + C_{\epsilon}^{Ai})^{3/2}}{3\pi(1 - \epsilon)^{3/2}} + \frac{3}{2} \right) - (1 - \epsilon) \frac{3}{5} \cdot \left( \frac{3\pi}{2} \right)^{\frac{2}{3}} \cdot \left( \frac{2(s + C_{\epsilon}^{Ai})^{3/2}}{3\pi(1 - \epsilon)^{3/2}} \right)^{\frac{2}{3}} \]

\[ = \frac{4(s + C_{\epsilon}^{Ai})^{3/2}}{15\pi(1 - \epsilon)^{3/2}} + \frac{3}{2}(s + C_{\epsilon}^{Ai}) \leq \frac{4(s + C_{\epsilon}^{Ai})^{3/2}}{15\pi} - (1 + 3\epsilon) + \frac{3}{2}(s + C_{\epsilon}^{Ai}), \]

\[ \int_{\frac{1}{2}}^{\theta'_1} f(z)dz \geq (s + C_{\epsilon}^{Ai}) \left( \theta'_1 - \frac{1}{2} \right) - \int_{\frac{1}{2}}^{\theta'_1} \left( \frac{3\pi(z - 1)}{2} \right)^{\frac{2}{3}} dz \]

\[ = (s + C_{\epsilon}^{Ai}) \left( \theta'_1 - \frac{1}{2} \right) - \frac{3}{5} \cdot \left( \frac{3\pi}{2} \right)^{\frac{2}{3}} \cdot \left( \theta'_1 - \frac{1}{2} \right)^{\frac{2}{3}}. \]

Combining these bounds with (1.4.20) yields the upper bound on (\( \Pi \)) in (1.4.17).

Claim:
\begin{equation}
(\tilde{\text{III}}) \leq \sqrt{\frac{2}{\pi}} \left( s + C_\epsilon^{\text{A}1} \right)^{\frac{3}{4}} + \frac{4}{T\pi(1-\epsilon)^{\frac{3}{2}}}. \tag{1.4.21}
\end{equation}

**Proof of Claim:** Using the inequality \( \log(1 + z) \leq z \) for all \( z \geq 0 \), we find

\[
J_s\left(- (1-\epsilon) \lambda_k + C_\epsilon^{\text{A}1}\right) \leq \exp \left( T^{\frac{1}{3}} (s - (1-\epsilon) \lambda_k + C_\epsilon^{\text{A}1}) \right). \tag{1.4.22}
\]

Plugging the lower bound on \( \lambda_k \) from (1.4.18) into (1.4.22), we find (recalling \( f_s(z) \) from (1.4.19))

\[
(\tilde{\text{III}}) \leq \sum_{k=\theta_2'}^\infty \exp \left( T^{\frac{1}{3}} f_s(k) \right). \tag{1.4.23}
\]

Noting that \( f_s(k) < f_s(\theta_2') < 0 \) for all \( k > \theta_2' \), we find that for all \( k > \theta_2' + \sqrt{3\theta_2'} \),

\[
f_s(k) < (1-\epsilon) \left( \frac{3\pi(\theta_2' - \frac{1}{2})}{2} \right)^{\frac{1}{3}} - (1-\epsilon) \left( \frac{3\pi(k - \frac{1}{2})}{2} \right)^{\frac{1}{3}} \leq -(1-\epsilon) \left( \frac{3\pi(k - \theta_2')}{2} \right)^{\frac{1}{3}}.
\]

The first inequality uses \( f_s(\theta_2') < 0 \) and the second follows from Lemma 1.4.6 (we assume \( s \) is large enough so \( \theta_2' - \frac{1}{2} > 27 \)). Utilizing this estimate yields

\[
(\tilde{\text{III}}) \leq \sum_{k=\theta_2'+\sqrt{3\theta_2'}}^{\theta_2'+\sqrt{3\theta_2'}} \exp \left( T^{\frac{1}{3}} f_s(k) \right) + \sum_{k=\theta_2'+\sqrt{3\theta_2'}}^\infty \exp \left( T^{\frac{1}{3}} f_s(k) \right)
\]

\[
\leq \sqrt{3\theta_2'} + \sum_{k=\theta_2'+\sqrt{3\theta_2'}}^\infty \exp \left( -(1-\epsilon) T^{\frac{1}{3}} \left( \frac{3\pi(k - \theta_2')}{2} \right)^{\frac{1}{3}} \right)
\]

\[
\leq \sqrt{3\theta_2'} + \int_0^\infty \exp \left( -(1-\epsilon) T^{\frac{1}{3}} \left( \frac{3\pi z}{2} \right)^{\frac{1}{3}} \right) dz
\]

\[
= \sqrt{3\theta_2'} + \frac{4}{T\pi(1-\epsilon)^{\frac{3}{2}}} \leq \sqrt{\frac{2}{\pi}} \left( s + C_\epsilon^{\text{A}1} \right)^{\frac{3}{4}} + \frac{4}{T\pi(1-\epsilon)^{\frac{3}{2}}}. \tag{1.4.24}
\]

The first inequality follows from (1.4.23); the second follows from the bound

\[
\exp \left( T^{\frac{1}{3}} f_s(k) \right) \leq \begin{cases} 
1, & k \in [\theta_2', \theta_2' + \sqrt{3\theta_2'}], \\
\exp \left( -(1-\epsilon) T^{\frac{1}{3}} \left( \frac{3\pi(k - \theta_2')}{2} \right)^{\frac{1}{3}} \right), & k \in [\theta_2' + \sqrt{3\theta_2'}, \infty);
\end{cases}
\]

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the third uses that the sum is bounded by the integral; and the last uses that for $s$ large enough, 
$$\sqrt{3\theta_2} \leq \sqrt{\frac{2}{\pi} \frac{(s+C^A_{\epsilon})^\frac{2}{3}}{(1-\epsilon)^\frac{2}{3}}}.$$ This completes the proof of (1.4.21).

Plugging the upper bounds of (I), (II) and (III) obtained in (1.4.16), (1.4.17) and (1.4.21) respectively into (1.4.15), we arrive at (1.4.14).

\[\text{Proof of (1.3.5).} \]

\textbf{Claim:} Fix any $\epsilon, \delta \in (0, 1/3)$ and $T_0 > 0$. Then, there exists $\kappa = \kappa(\epsilon, \delta, T_0) > 0$ and $S_0 = S_0(\epsilon, \delta, T_0) > 0$ such that for all $s \geq S_0$ and $T > T_0$

$$E_{\text{Airy}} \left[ 1(a_1 \geq -s) \prod_{k=1}^{\infty} I(a_k) \right] \geq \left( 1 - 2 \exp(-\kappa s^{1-2\delta}) \right) \exp \left( -T \frac{4s^{\frac{2}{3}}}{15\pi} (1 + 9\epsilon) \right). \quad (1.4.25)$$

\textbf{Proof of Claim:} Negating both sides of (1.4.14) and exponentiating yields $\prod_{k=1}^{\infty} I(a_k) \geq \exp \left( -L_{T,\epsilon}(s + C^A_{\epsilon}) \right)$. Along with the monotonicity of $L_{T,\epsilon}(\cdot)$, this implies

$$E_{\text{Airy}} \left[ 1(a_1 \geq -s) \prod_{k=1}^{\infty} I(a_k) \right] \geq \mathbb{P}(a_1 \geq -s, C^A_{\epsilon} < s^{1-\delta}) \exp \left( -L_{T,\epsilon}(s + s^{1-\delta}) \right). \quad (1.4.26)$$

Taking $s$ large enough we have the bounds

$$T \frac{\frac{2}{\pi} \frac{(s + s^{1-\delta})^{\frac{2}{3}}}{15\pi \epsilon}}{3(1-\epsilon)^{\frac{2}{3}}} \leq T \frac{\frac{2}{\pi} \frac{(s + s^{1-\delta})^{\frac{2}{3}}}{15\pi \epsilon}}{(1-\epsilon)^{\frac{2}{3}}}, \quad \sqrt{\frac{\frac{2}{\pi} \frac{(s + s^{1-\delta})^{\frac{2}{3}}}{15\pi \epsilon}}{(1-\epsilon)^{\frac{2}{3}}}} \leq T \frac{\frac{4s^{\frac{2}{3}}}{15\pi \epsilon}}{(1-\epsilon)^{\frac{2}{3}}}, \quad \frac{4}{T \pi(1-\epsilon)^{\frac{2}{3}}} \leq T \frac{\frac{4s^{\frac{2}{3}}}{15\pi \epsilon}}{(1-\epsilon)^{\frac{2}{3}}}.$$

Using these bounds we find that

$$L_{T,\epsilon}(s + s^{1-\delta}) \leq T \frac{\frac{4s^{\frac{2}{3}}}{15\pi \epsilon}}{(1-\epsilon)^{\frac{2}{3}}}.$$

Thanks to (1.1.9) of Theorem 1.1.8 there exists $\kappa = \kappa(\epsilon, \delta)$ and $S_0 = S_0(\epsilon, \delta)$ such that for all $s \geq S_0$, $\mathbb{P}(C^A_{\epsilon} < s^{1-\delta}) > 1 - \exp(-\kappa s^{1-2\delta})$. Moreover, using (1.4.1), we find that for large enough $s$, $\mathbb{P}(a_1 \leq -s) \leq \kappa \exp \left( -\kappa s^{1-2\delta} \right)$. This implies that for large enough $s$,

$$\mathbb{P}(a_1 \geq -s, C^A_{\epsilon} < s^{1-\delta}) \geq \mathbb{P}(a_1 \geq -s) + \mathbb{P}(C^A_{\epsilon} < s^{1-\delta}) - 1 \geq 1 - 2 \exp \left( -\kappa s^{1-2\delta} \right).$$

Plugging this and (1.4.27) into (1.4.26) yields (1.4.25).
**Claim:** Fix $\epsilon \in (0, 1/3)$ and $T_0 > 0$. Then, there exists $K = K(\epsilon, T_0) > 0$ and $S_0 = S_0(\epsilon, T_0) > 0$ such that for all $s \geq S_0$,

\[
\mathbb{E}_{\text{Airy}} \left[ \mathbb{I} (a_1 < -s) \prod_{k=1}^{\infty} I (a_k) \right] \geq \exp \left( -K s^3 \right).
\] (1.4.28)

**Proof of Claim:** We begin with a brief description of our proof technique. Let us denote $\theta'_0 := \lceil s^1 + \delta \rceil$. We consider a finite of sequence of intervals

\[ I_1 := [-s^2, -s), I_2 := [-2s^2, -s^2), \ldots, I_{\theta'_0} := [-\theta'_0 s^2, -(\theta'_0 - 1)s^2). \]

The length of each of the interval is $s^2$ and there are $\theta'_0$ intervals. For any integer $\ell \in (1, \theta'_0] \cap \mathbb{Z}$ (resp. $\ell = 1$), note that $\sum_{a_k \in I_\ell} J_s (a_k)$ is less than or equal to $\sum_{a_k \in I_\ell} J_s (-\ell - 1) s^2$ (resp. $\sum_{a_k \in I_1} J_s (-s)$) with equality when all the $a_k$ in the interval $I_\ell$ coincide with the right end point $-(\ell - 1) s^2$ (resp. $-s$). We show that with high probability the number of Airy points inside the interval $I_\ell$ cannot differ considerably from its expected value. Based on this, we argue that the probability of an abundant accumulation of the Airy points inside any of the intervals $I_1, \ldots, I_{\theta'_0}$ is small in comparison to $\mathbb{P}(a_1 \leq -s)$. Moreover, the contributions of those Airy points which fall into any of those intervals are bounded from above by the result of moving the points to the right endpoint of the interval. Finally, using the upper tail estimate of $C^A_\epsilon$ (see (1.1.9) of Theorem 1.1.5), we show that the $a_k$’s which fall in the region $(-\infty, -\theta'_0 s^2)$ hardly contribute to the product $\sum_{k=1}^{\infty} I_s (a_k)$.

Now, we provide the details of the above sketch. First, we find an upper bound on $\sum_{a_k \in \tilde{I}} J_s (a_k)$ where $\tilde{I} := \bigcup_{\ell=1}^{\theta'_0} I_\ell$. Recall that the number of $a_k$’s in a Borel set $D$ is given by $\chi^{Ai} (D)$. By replacing all the $a_k$’s inside the interval $I_\ell$ by the right endpoint of the interval, we obtain

\[
\sum_{a_k \in I_\ell} J_s (a_k) \leq \begin{cases} 
\chi^{Ai} (I_\ell) \log \left( 1 + \exp \left( T_1 \frac{4}{3} (s - (\ell - 1)s^2) \right) \right) & \text{when } \ell > 1, \\
\chi^{Ai} (I_1) \log(2) & \text{when } \ell = 1.
\end{cases}
\]

Next, using Theorem 1.1.4, we observe that for large enough $s$, $\chi^{Ai} (I_\ell)$ is bounded above by $\mathbb{E}[\chi^{Ai} (I_\ell)] + \epsilon s^3$ with probability greater than $1 - K_1 \exp(-K_2 s^3 \log s)$. Owing to Proposition 1.1.2 there exists
a constant $M$ such that for large enough $s$,
\[
\mathbb{E}[\chi^A(I_s)] = \frac{2}{3\pi} (\ell^2 - (\ell - 1)^2) s^3 + \mathcal{D}(\ell s^2) - \mathcal{D}((\ell - 1)s^2) \leq \frac{M\sqrt{\ell}s^3}{\pi}.
\]

Consequently, with probability exceeding $1 - \theta'_0 K_1 \exp \left( - K_2 s^3 \log s \right)$
\[
\sum_{a_k \in \tilde{I}} J_s(a_k) \leq \left( \frac{Ms^3}{\pi} + \epsilon s^3 \right) \left( \log 2 + \sum_{\ell=2}^{\theta'_0} \sqrt{\ell} \log \left( 1 + \exp \left( \frac{1}{2} (s - (l - 1)s^2) \right) \right) \right).
\]

Since $\log(1 + x) \leq x$ for all $x > 0$ and $(\ell - 1)s^2 - s \geq (l - 1)s^2(1 - \epsilon)$ for all $s \geq \epsilon^{-1}$, we conclude there exists a constant $C$ such that for large enough $s$, with probability exceeding $1 - \theta'_0 K_1 \exp \left( - K_2 s^3 \log s \right)$
\[
\sum_{a_k \in \tilde{I}} J_s(a_k) \leq s^3 \left( \frac{M}{\pi} + \epsilon \right) \left( \log 2 + \sum_{\ell=2}^{\theta'_0} \sqrt{\ell} \exp \left( - (\ell - 1)(1 - \epsilon)T_1^3 s^2 \right) \right) \leq C s^3. \tag{1.4.29}
\]

We now turn to bound the remaining sum $\sum_{a_k < -\theta'_0 s^2} J_s(a_k)$. For this, we consider the following decomposition
\[
\sum_{k: a_k < -\theta'_0 s^2} J_s(a_k) = (A) + (B),
\]
\[
(A) := \sum_{k: a_k < -\theta'_0 s^2, \lambda_k \leq \theta'_0 s^2} J_s(a_k), \quad (B) := \sum_{k: a_k < -\theta'_0 s^2, \lambda_k > \theta'_0 s^2} J_s(a_k).
\]

Proposition 1.3.6 shows that $\# \{ \lambda_k \leq \theta'_0 s^2 \} \leq C s^{\frac{9}{2} + \frac{3\delta}{2}}$ for large enough $s$ and some constant $C > 0$.

This, along with the bound $\log(1 + a) \leq a$ for all $a > 0$ implies
\[
J_s(a_k) \leq \exp \left( T_1^s (s - \theta'_0 s^2) \right) \leq \exp \left( - (1 - \epsilon)T_1^s s^3 \right)
\]
when $a_k \leq -\theta'_0 s^2$ and $s \geq \epsilon^{-\frac{1}{2}}$. Thus, for large enough $s$,
\[
(A) \leq C s^{\frac{9}{2} + \frac{3\delta}{2}} \exp \left( - (1 - \epsilon)T_1^s s^3 \right) \leq s^3. \tag{1.4.30}
\]

Now, we turn to bound $(B)$. Recall the inequality $J_s(a_k) \leq J_s(- (1 - \epsilon)\lambda_k + C_\epsilon^A)$ which we
obtain by using monotonicity of $J_s$ and the inequality (1.1.8). We will now employ Theorem 1.1.5 but to avoid confusion in notion let us temporarily rename the variables $s$ and $\delta$ in the statement of Theorem 1.1.5 by $\tilde{s}$ and $\tilde{\delta}$. Then, taking $\tilde{s} = s^{3+\frac{\delta}{2}}$ and $\tilde{\delta} = \frac{\delta}{2(3+\delta/2)}$, the corollary implies there exist $\kappa = \kappa(\epsilon, \delta) > 0$ and $S_0 = S_0(\epsilon, \delta) > 0$ such that for all $s \geq S_0$, $\mathbb{P}(C_{\epsilon_1} < s^{3+\frac{\delta}{2}}) \geq 1 - \exp\left(-\kappa s^{3+\frac{\delta}{4}}\right)$. Since $\theta'_0s^{2} \approx s^{3+\delta}$, we have $s + s^{3+\delta} \leq (1 - \epsilon)\theta'_0s^{2}$ for large enough $s$. Consequently, for large enough $s$

$$\mathbb{P}\left(\sum_{\lambda_k > \theta'_0s^2} J_s((1 - \epsilon)(\theta'_0s^2 - \lambda_k) - s) \geq \sum_{\lambda_k > \theta'_0s^2} J_s((1 - \epsilon)(\theta'_0s^2 - \lambda_k) - s) \right) \geq 1 - \exp\left(-\kappa s^{3+\frac{\delta}{4}}\right). \quad (1.4.31)$$

Plugging the inequality (1.4.35) in Lemma 1.4.5 into (1.4.31) and using (1.4.30) along with the fact that $(\theta'_0s^2)^{\frac{\delta}{2}} \leq Cs^3$ for some constant $C$, we find that for large enough $s$

$$\mathbb{P}\left(\left(\bigcup_{\lambda_k > \theta'_0s^2} J_s((1 - \epsilon)(\theta'_0s^2 - \lambda_k) - s) \right) \geq 1 - \exp\left(-\kappa s^{3+\frac{\delta}{4}}\right).$$

Combining this with the probability bound computed on the event in (1.4.29) implies that there exists a constant $C = C(\epsilon, \delta, T_0) > 0$ such that for $s$ large enough

$$\mathbb{P}(A) \geq 1 - \theta'_0K_1 \exp(-K_2s^3 \log s) - \exp(-\kappa s^{3+\frac{\delta}{4}}), \quad (1.4.32)$$

where $A := \left\{ \sum_{k=1}^{\infty} J_s(a_k) \leq C s^3 \right\}$. Negating both sides above, exponentiating, then multiplying by $\mathbb{1}(a_1 \leq -s)$ and taking expectation, we obtain

$$\mathbb{E}_{\text{Airy}} \left[ \mathbb{1}(a_1 \leq -s) \prod_{k=1}^{\infty} J_s(a_k) \right] \geq \mathbb{P}(\{a_1 \leq -s\} \cap A) \exp(-Cs^3). \quad (1.4.33)$$

It thus remains to estimate

$$\mathbb{P}(\{a_1 \leq -s\} \cap A) \geq \mathbb{P}(a_1 \leq -s) + \mathbb{P}(A) - 1 \quad (1.4.34)$$

$$\geq \exp(-s^3) - \theta'_0K_1 \exp(-K_2s^3 \log s) - \exp(-\kappa s^{3+\frac{\delta}{4}}).$$

The first inequality uses $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for any events $A$ and $B$. The second uses the lower bound on $\mathbb{P}(a_1 \leq -s)$ in (1.4.1) and the lower bound in (1.4.32). Combining (1.4.34) with
Now we may complete the proof of (1.3.5) by combining (1.4.25) and (1.4.28) with
\[
\mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \right] = \mathbb{E}\left[ \mathbf{1}(a_1 \geq -s) \prod_{k=1}^{\infty} I_s(a_k) \right] + \mathbb{E}\left[ \mathbf{1}(a_1 < -s) \prod_{k=1}^{\infty} I_s(a_k) \right].
\]

\[\blacksquare\]

**Lemma 1.4.5.** As above, set \( \theta'_0 = \lceil s^{1+\delta} \rceil \). Then, for all \( s \) such that \( \theta'_0 s^2 > 27 \),
\[
\sum_{\lambda_k > \theta'_0 s^2} J_s((1 - \epsilon)(\theta'_0 s^2 - \lambda_k) - s) \leq \sqrt{\frac{2}{\pi}}(\theta'_0 s^2)^{\frac{3}{2}} \log 2 + \frac{4}{T \pi (1 - \epsilon)^3}. \tag{1.4.35}
\]

**Proof.** For \( s \) large enough, (1.3.7) implies that
\[
\left\{ k : \lambda_k > \theta'_0 s^2 \right\} \subseteq \left\{ k : k \geq \frac{2}{3 \pi} (\theta'_0 s^2)^{\frac{3}{2}} \right\}.
\]

This implies that following (first) inequality
\[
\sum_{\lambda_k > \theta'_0 s^2} J_s((1 - \epsilon)(\theta'_0 s^2 - \lambda_k) - s) \leq \sum_{k \geq \frac{2}{3 \pi} (\theta'_0 s^2)^{\frac{3}{2}}} J_s((1 - \epsilon)(\theta'_0 s^2 - \lambda_k) - s)
\]
\[
\leq \sqrt{\frac{2}{\pi}}(\theta'_0 s^2)^{\frac{3}{2}} \log 2 + \sum_{k' = \sqrt{\frac{2}{\pi}}(\theta'_0 s^2)^{\frac{3}{2}}}^{\infty} \exp \left( - (1 - \epsilon) T \frac{1}{3} \left( \frac{3 \pi (k' - \frac{1}{2})}{2} \right)^{1/3} \right). \tag{1.4.36}
\]

To show the second inequality, let \( \theta''_0 := \frac{2}{3 \pi} (\theta'_0 s^2)^{\frac{3}{2}} \) and \( \theta''''_0 := \frac{2}{3 \pi} (\theta'_0 s^2)^{\frac{3}{2}} + \sqrt{\frac{2}{\pi}}(\theta'_0 s^2)^{\frac{3}{2}}. \) Using \( J_s(x) \leq \log 2 \) and \( \log(1 + x) \leq x \) for \( x \leq 0 \), along with Lemma 1.4.6 (similarly to (1.4.24)),
\[
J_s((1 - \epsilon)(\theta'_0 s^2 - \lambda_k) - s) \leq \begin{cases} 
\log 2 & \text{k} \in [\theta''''_0, \theta''_0] \cap \mathbb{Z} \\
\exp \left( - (1 - \epsilon) T \frac{1}{3} \left( \frac{3 \pi (k'''' - \frac{1}{2})}{2} \right)^{1/3} \right), & \text{k} \in (\theta''''_0, \infty) \cap \mathbb{Z}.
\end{cases}
\]

Using this bound and substituting \( k' = k - \theta''_0 \), we obtain
\[
\sum_{k \geq \frac{2}{3 \pi} (\theta'_0 s^2)^{\frac{3}{2}}} J_s((1 - \epsilon)(\theta'_0 s^2 - \lambda_k) - s) \leq (\theta''''_0 - \theta''_0) \log 2 + \sum_{k' > \theta''''_0 - \theta''_0} \exp \left( - (1 - \epsilon) T \frac{1}{3} \left( \frac{3 \pi (k' - \frac{1}{2})}{2} \right)^{1/3} \right),
\]
which implies the second inequality in (1.4.36). Bounding the sum by a corresponding integral and evaluating yields the bound in (1.4.35).

Lemma 1.4.6. Fix $a > 27$. Then, we have $(a + x)^{\frac{2}{3}} \geq a^{\frac{2}{3}} + x^{\frac{1}{3}}$ for all $x \geq \sqrt{3a}$.

Proof. Observe that for all $x \geq \sqrt{3a}$ and $a > 27$, one can write $x < x^2$, and using $3a^{\frac{2}{3}} \leq a$, one has $3a^{\frac{2}{3}}x^{\frac{1}{3}} \leq ax$ and $3a^{\frac{2}{3}}x^{\frac{2}{3}} \leq ax^{\frac{2}{3}} \leq ax$. Combining these inequalities yields

$$(a + x)^2 = a^2 + x^2 + 2ax \geq a^2 + x + 3a^{\frac{4}{3}}x^{\frac{1}{3}} + 3a^{\frac{2}{3}}x^{\frac{2}{3}} = (a^{\frac{2}{3}} + x^{\frac{1}{3}})^3.$$  

1.5 Ablowitz-Segur solution of Painlevé II

Recall (Section 1.1.3) the Ablowitz-Segur solution $u_{AS}(x; \gamma)$ of Painlevé II. We restate [Bot17, Theorem 1.10] which provides the asymptotic form of $u_{AS}(x; \gamma)$ as $x \to -\infty$. Lemmas 1.5.3 and 1.5.4 result from the analysis of this form. We will combine those two lemmas in Section 1.5.2 to yield a proof of Theorem 1.1.6.

1.5.1 Asymptotic form for Ablowitz-Segur solution of Painlevé II

In order to restate [Bot17, Theorem 1.10] we introduce a few special functions. For a real variable $\aleph \in (0, \infty)$, define $\kappa = \kappa(\aleph) \in (0, 1)$ implicitly as follows

$$\aleph = \frac{2}{3} \sqrt{\frac{2}{1 + \kappa^2} \left( E(\kappa') - \frac{2\kappa^2}{1 + \kappa^2} K(\kappa') \right)},$$

where $\kappa' := \sqrt{1 - \kappa^2}$, and $K$ and $E$ are standard complete elliptic integrals

$$K(\kappa) := \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - \kappa^2\xi^2)}}, \quad \text{and} \quad E(\kappa) := \int_0^1 \sqrt{\frac{1 - \xi^2\kappa^2}{1 - \xi^2}} d\xi.$$

It follows from [Bot17, Proposition 3.2] that $\kappa$ is uniquely defined for all $\aleph \in (0, \frac{2}{3}\sqrt{2})$.

Further, define (using $\kappa = \kappa(\aleph)$)

$$V = V(\aleph) := -\frac{2}{3\pi} \sqrt{\frac{2}{1 + \kappa^2} \left( E(\kappa) - \frac{1 - \kappa^2}{1 + \kappa^2} K(\kappa) \right)}, \quad \text{and} \quad \tau = \tau(\aleph) := 2\frac{K(\kappa)}{K(\kappa')}.$$
Define the Jacobi theta and elliptic functions (with \( q = e^{i\pi \tau} \) and \( z \in \mathbb{C} \))

\[
\theta_2(z, q) = 2 \sum_{m=0}^{\infty} q^{(m+1)^2} \cos((2m+1)\pi z), \quad \theta_3(z, q) = 1 + 2 \sum_{m=1}^{\infty} q^{m^2} \cos(2\pi mz),
\]

\[
\text{cd} \left( 2zK \left( \frac{1-\kappa}{1+\kappa}, \frac{1-\kappa}{1+\kappa} \right) \right) = \frac{\theta_3(0, q) \theta_2(z, q)}{\theta_2(0, q) \theta_3(z, q)}, \quad z \in \mathbb{C} \setminus \bigcup_{m,n \in \mathbb{Z}} \left\{ \frac{1}{2} + \frac{\tau}{2} + m + \tau n \right\}.
\]

The below asymptotic formula follows immediately\(^{14}\) from \([\text{Bot17}]\) Theorem 1.10. In order to state it, let us define a bit more notation which will also be used in the subsequent lemmas.

Fix, throughout what follows, some \( \delta \in (0, \frac{2}{3}) \). For \( s > 0 \), define \( \gamma = \gamma(s) := 1 - \exp(-s^{\frac{3}{2}} - \delta) \), \( v = v(s) := -\log(1 - \gamma(s)) = s^{\frac{3}{2}} - \delta \) and an interval \( \Psi_s := \left[ -s , -\frac{3^{2/3}}{2} s^{1-2/3} \right) \). For \( x \in (-\infty, 0) \) and \( s > 0 \) define \( \mathcal{N} = \mathcal{N}(x, s) := \frac{v(s)}{(-x)^{3/2}} \). For \( x \in \Psi_s \), it follows that \( \mathcal{N}(x, s) \in (0, \frac{2}{3} \sqrt{2}) \) and hence \( \kappa = \kappa(\mathcal{N}(x, s)) \) is a function\(^{15}\) of \( s > 0 \) and \( x \in \Psi_s \). In fact, in Proposition 1.5.1 and Lemmas 1.5.3 and 1.5.4 we will generally deal with functions of \( s \) and \( x \), though often suppress the explicit dependence (mainly, to keep equations from growing to lengthy).

**Proposition 1.5.1.** For any fixed\(^{16}\) \( \zeta \in (0, \frac{\sqrt{2}}{3}) \) there exist \( s_0 = s_0(\zeta) > 0 \), \( c_0 = c_0(\zeta) > 0 \) such that for all \( s \geq s_0 \), (using \( \kappa = \kappa(\mathcal{N}(x, s)) \), \( V = V(\mathcal{N}(x, s)) \), \( \gamma = \gamma(s) \), \( v = v(s) \) as above, and \( J_1 = J_1(x, s) \))

\[
\left| J_1 \right| \leq c_0(-x)^{-1/6}, \quad \text{for all } x \in \Psi_s \text{ which satisfy } (-x)^{\frac{3}{2}} \left( \frac{2\sqrt{2}}{3} - \zeta \right) \geq v. \quad (1.5.2)
\]

Continuing with the notation introduced above, the following result controls the small \( \mathcal{N} \) behavior of \( \kappa(\mathcal{N}) \) and \( V(\mathcal{N}) \). The bound \(^{14}\) follows immediately from equation (3.5) in \([\text{Bot17}]\) Proposition 1.5.3 and 1.5.4 we will generally deal with functions of \( s \) and \( x \), though often suppress the explicit dependence (mainly, to keep equations from growing to lengthy).

\(^{14}\)Since our notation is slightly different than that of \([\text{Bot17}]\), let us match it here. The parameter \( \varepsilon \) in \([\text{Bot17}]\) is equal to 1 in our case. \([\text{Bot17}]\) has parameters \( s \) and \( \delta \) which do not correspond to our notation. Let us denote them as \( s \) and \( \delta \). Then in terms of our notation, \( s = \gamma \) and \( \delta = \zeta \). In contrast to \([\text{Bot17}]\), we treat \( v \) as being parameterized by an underlying \( s \), whereas he treated \( v \) as a free variable in its own right. Finally, since we only utilize equation (1.26) from \([\text{Bot17}]\), we do not need to make use of his constants \( v_1, f_1, \) or \( c_1 \). His \( v_1 \) translates into our \( s_1 \) constant, and his \( c_0 \) is the same as ours.

\(^{15}\)This follows from \([\text{Bot17}]\) Proposition 3.2.1 as noted above.

\(^{16}\)The result, in fact, holds for \( \zeta \in (0, \frac{\sqrt{2}}{3}) \). The more restrictive range \( (0, \frac{\sqrt{2}}{3}) \) in \([\text{Bot17}]\) Theorem 1.10] is only needed for equation (1.27) therein, not (1.26). However, since we only need this result for small \( \zeta \) we do not provide the explanation for this wider range’s validity.

\(^{17}\)The condition assumed on \( x \) in (1.5.2) is equivalent to \( \mathcal{N} \leq \frac{2\sqrt{2}}{3} - \zeta \).
3.2] and the bound (1.5.4) follows immediately from equation (3.9) in [Bot17 Corollary 3.3].

**Proposition 1.5.2.** Define $\Omega_1(N)$ and $\Omega_2(N)$ via

$$
\kappa(N) = 1 - 2 \sqrt{\frac{N}{\pi}} \frac{2N}{\pi} - \frac{29}{8} \left( \frac{N}{\pi} \right)^{3/2} + \Omega_1(N),
$$

(1.5.3)

$$
V(N) = -\frac{2}{3\pi} - \frac{N}{2\pi^2} \log(N) + \frac{N}{2\pi^2} (1 + \log(16\pi)) + \Omega_2(N).
$$

(1.5.4)

Then, there exists $N_0 \in (0, \frac{2}{3}\sqrt{2})$ and $C = C(N_0)$ such that for all $N \leq N_0$, $|\Omega_1(N)| \leq CN^2$ and $|\Omega_2(N)| \leq CN^2$.

Combining Propositions 1.5.1 and 1.5.2 we may simplify the asymptotic formula of $u_{AS}(x; \gamma)$.

**Lemma 1.5.3.** Recall the notation from Propositions 1.5.1 (namely, $N = N(x, s)$, $\gamma = \gamma(N(x, s))$, $V = V(N(x, s))$, $\gamma = \gamma(s)$, $v = v(s)$, and $\Psi_s$). Define $\phi = \phi(x, s)$ via the relation

$$
\pi(-x)^{\frac{3}{2}} V = -\frac{2}{3} (-x)^{\frac{3}{2}} + \frac{v}{2\pi} \log(8(-x)^{\frac{3}{2}}) + \phi.
$$

(1.5.5)

Fix any $\eta_0 \in (0, \frac{2}{3})$. Then, there exists $s_0 = s_0(\eta_0) > 0$, $C = C(\eta_0) > 0$ and $C' = C'(\eta_0) > 0$ such that for all pairs $(x, s)$ which satisfy $s \geq s_0$, $x \in \Psi_s$ and $N = (-x)^{-\eta}$ for some $\eta \in (\eta_0, \frac{2}{3})$, we have that (with the notation $J_2 = J_2(x, s)$ and $J_3 = J_3(x, s)$),

$$
u_{AS}(x; \gamma) = (-x)^{-\frac{1}{4}} \sqrt{\frac{\eta}{\pi}} \cos \left( \pi(-x)^{\frac{3}{2}} V \right) + J_2,
$$

(1.5.6)

$$
\phi = \frac{v}{2\pi} \left( 1 - \log(v/2\pi) + J_3 \right)
$$

(1.5.7)

where $|J_2| \leq C(-x)^{\frac{1}{2} - \frac{3\eta}{2}}$ and $|J_3| \leq C'(-x)^{-\eta}$.

**Proof.** It follows from [SN17 (22.11.4)] that

$$
\text{cd}(z, \kappa) = \frac{2\pi}{K(\kappa) K'(\kappa)} \sum_{n=0}^{\infty} (-1)^n \frac{\tilde{q}^{n+\frac{1}{2}}}{1 - \tilde{q}^{2n+1}} \cos \left( (2n + 1) \zeta \right)
$$

(1.5.8)

where $\zeta := \frac{\pi z}{2K(\kappa)}$ and $\tilde{q} := \exp(-\pi K(\kappa')/K(\kappa))$.

We claim that there exists $0 \leq \kappa_0 < 1$ and $C_1 = C_1(\kappa_0) > 0$ such that for all $\kappa \leq \kappa_0$,

$$
\cos \left( \pi z/2K(\kappa) \right) - C_1 \kappa^2 \leq \text{cd}(z, \kappa) \leq \cos \left( \pi z/2K(\kappa) \right) + C_1 \kappa^2.
$$

(1.5.9)
Owing to [Fet69] and [SN17, (19.5.5), (19.5.8)], there exist \(0 \leq \kappa_0 < 1\) and \(0 < C_2 = C_2(\kappa_0) < C_3 = C_3(\kappa_0)\) such that for all \(\kappa \leq \kappa_0\)

\[
C_2 \kappa^4 + \frac{\kappa^2}{16} \leq \bar{q} \leq \frac{\kappa^2}{16} + C_3 \kappa^4, \quad \frac{\pi}{2} + C_2 \kappa^2 \leq K(\kappa) \leq \frac{\pi}{2} + C_3 \kappa^2. \tag{1.5.10}
\]

When \(\kappa \leq \kappa_0\), substituting (1.5.10) into (1.5.8) yields

\[
|\text{cd}(z, k) - \cos\left(\frac{\pi z}{2K(\kappa)}\right)| \leq \sum_{n=1}^{\infty} \frac{(C_3 \kappa)^{2n}}{1 - \frac{\kappa^2}{16} - C_2 \kappa^4} + C_2 \kappa^2.
\]

For small enough \(\kappa\), the r.h.s. above is bounded by \(C_1 \kappa^2\) (for \(C_1 = C_1(\kappa_0)\)) which proves (1.5.9).

Return to the proof of the lemma, define \(\tilde{\kappa} := \frac{1 - \kappa}{1 + \kappa}\) (recall \(\kappa = \kappa(\kappa)\) as above). If we further define \(\tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_1(\kappa)\) by the relation \(\tilde{\kappa} = \sqrt{\frac{2}{\kappa} + \mathcal{H}_1}\), then (1.5.3) implies that there exists a constant \(C_4 > 0\) such that for small enough \(\kappa > 0\), \(|\mathcal{H}_1| \leq C_4 \kappa\). Thus, as \(\kappa\) goes to zero, so too does \(\tilde{\kappa}\).

Now, recall that we have assumed that \(\kappa\) satisfies \(\kappa = (x)\eta\) for some \(\eta \in (\eta_0, \frac{2}{3})\). This implies that \(\kappa \leq (x)\eta_0\) and hence, as \(-x\) goes to \(\infty\), \(\kappa\) and \(\tilde{\kappa}\) both go to zero. Combining this deduction with (1.5.9), we conclude that there exists \(x_1 = x_1(\eta_0) > 0\) and \(C_5 = C_5(\eta_0) > 0\) such that (with the notation \(\tilde{\mathcal{H}}_2 = \tilde{\mathcal{H}}_2(x, s)\) defined by the below relation),

\[
\text{cd}\left(2(-x)^{3/2} V K(\tilde{\kappa}), \tilde{\kappa}\right) = \cos\left(\pi (-x)^{\frac{3}{2}} V\right) + \tilde{\mathcal{H}}_2 \tag{1.5.11}
\]

where \(|\tilde{\mathcal{H}}_2| \leq C_5 (-x)^{-\eta}\) for all \((-x) \geq x_1\).

Using (1.5.11) along with the expansion for \(\kappa\) provided by (1.5.3), we see that there exists a \(x_2 = x_2(\eta_0) > 0\) and \(C_6 = C_6(\eta_0)\) such that (with the notation \(\tilde{\mathcal{H}}_3 = \tilde{\mathcal{H}}_3(x, s)\)),

\[
\sqrt{-\frac{x}{2} \frac{1 - \kappa}{\sqrt{1 + \kappa^2}}} \text{cd}\left(2(-x)^{3/2} V K(\tilde{\kappa}), \tilde{\kappa}\right) = \sqrt{\frac{v}{\pi (-x)^{\frac{3}{2}}}} \cos\left(\pi (-x)^{\frac{3}{2}} V\right) + \tilde{\mathcal{H}}_3 \tag{1.5.12}
\]

where \(|\tilde{\mathcal{H}}_3| \leq C_6 (-x)^{\frac{1}{2} - \frac{3\eta}{2}}\) for all \((-x) \geq x_2\).

We may now apply (1.5.1) and combine that with the deduction above in (1.5.12). The first result requires that \(s \geq s_0, x \in \Psi_s\) and \((-x)^{\frac{3}{2}} \left(\frac{2\sqrt{3}}{3} - \zeta\right) \geq v\), and the second requires that \((-x) \geq x_2\). This second condition can be ensured by possibly increasing the value of \(s_0\). In applying (1.5.1), we may use the inequality \(|J_1| \leq c_0(-x)^{-\frac{1}{10}} \leq c_0(-x)^{\frac{1}{2} - \frac{3\eta}{2}}\) (thanks to (1.5.2) and \(\frac{1}{2} - \frac{3\eta}{2} > -\frac{1}{10}\).
Combining this bound with the bound on \(J_3\) in (1.5.12), we see that \(J_2 := J_1 + J_3\) satisfies the desired bound \(|J_2| \leq C(−x)^{\frac{3}{2}−\frac{3\eta}{2}}\) for some constant \(C\). This proves the bound on the error \(J_2\) in (1.5.6).

Now, we turn to prove (1.5.7). Owing to (1.5.4), there exist \(s_0' = s_0'(\eta_0) > 0\) and \(C_\eta = C_\eta(\eta_0) > 0\) such that for all \(s \geq s_0'\) and \(x \in \Psi_s\) satisfying \(\mathcal{N}(x, s) = (−x)^{-\eta}\) for some \(\eta \in (\eta_0, \frac{2}{3})\), one has

\[
\pi(−x)^{\frac{3}{2}}V = -\frac{2}{3}(−x)^{\frac{3}{2}} + \frac{v}{2\pi} \log(8(−x)^{\frac{3}{2}}) - \frac{v}{\pi} \log(v/2\pi) + \frac{v}{2\pi} + vJ_3
\]

(1.5.13)

where \(J_3 = J_3(\mathcal{N}) := \Omega_2(\mathcal{N})/v\). By substituting (1.5.13) into (1.5.5), we arrive at the desired error bound on \(J_3\) in (1.5.7).

The next lemma highlights the critical oscillatory cancelation which enables us to prove Theorem 1.1.6 (done in Section 1.5.2). Let us introduce a shorthand notation (the first equality is the definition and the second follows from (1.5.5))

\[
\psi(x, s) := 2\pi(−x)^{\frac{3}{2}}V(\mathcal{N}(x, s)) = -\frac{4}{3}(−x)^{\frac{3}{2}} + \frac{v}{\pi} \log(8(−x)^{\frac{3}{2}}) + 2\phi(x, s).
\]

**Lemma 1.5.4.** Recall that we have fixed \(\delta \in (0, \frac{2}{3})\) throughout this section. For \(\theta \in (0, \delta)\), there exist \(s_0' = s_0'(\delta, \theta) > 0\) and \(C = C(\delta, \theta) > 0\) such that for all \(s \geq s_0\)

\[
\left| \int_{−s}^{s^{1−\frac{2\theta}{3}}} \frac{(x + s)}{(-x)^{\frac{3}{2}}} \cos(\psi(x, s)) \, dx \right| \leq Cs^{\frac{3}{2}}s^{-(\delta−\theta)}.
\]

(1.5.14)

**Proof.** We will apply Lemma 1.5.3 to provide an asymptotic expansion for \(\psi(x, s)\) (via \(V(\mathcal{N}(x, s))\)). To do this, fix \(\eta_0 = \delta−\theta\), which is in \((0, \frac{2}{3})\) since \(\delta \in (0, \frac{2}{3})\) and \(\theta \in (0, \delta)\). We must verify two conditions to apply the lemma: For large enough \(s\), and for \(x\) in the domain of integration \([-s, −s^{1−\frac{2\theta}{3}}]\) (1) \(x \in \Psi_s = [-s, −s^{2/3}s^{1-\frac{2\eta}{3}}]\), and (2) \(\mathcal{N}(x, s) = \frac{\nu(s)}{(-x)^{\frac{3}{2}}} = \frac{s^{\frac{3}{2}-\delta}}{(-x)^{\frac{3}{2}}}\) equals \((−x)^{-\eta}\) for some \(\eta \in (\eta_0, \frac{2}{3})\). Condition (1) is immediate. Condition (2) follows by considering the two endpoints \(x = −s\) and \(x = −s^{1−\frac{2\theta}{3}}\). In the first case, we find that \(\eta = \delta\) and in the second case, \(\eta = (\delta−\theta)/(1−\frac{2\theta}{3})\), which is bounded below by \(\delta−\theta\). Clearly, for intermediate \(x\), \(\eta\) ranges between these two extremes which are contained in the interval \([\delta−\theta, \frac{2}{3}]\) as desired. Thus, conditions (1) and (2) are both confirmed. By
applying Lemma 1.5.3, there exists \( s_0 = s_0(\delta, \theta) > 0 \) and \( C' = C'(\delta, \theta) > 0 \) such that for all \( s \geq s_0 \)

\[
|J_3(x, s)| = \left| \frac{2\pi}{v} \phi(x, s) - 1 + \log(v/2\pi) \right| \leq C'(-x)^{-(\delta - \theta)}
\]  

(1.5.15)

for any \( x \in [-s, -s^{1-\frac{2}{3}\theta}] \) (here we use \( J_3 \) as in (1.5.7) and use that \( \eta_0 = \delta - \theta \)).

Given this control over the expansion for \( \phi \) (and hence \( \psi \)) we now turn to estimating the integral in (1.5.14). In order to capture the scale of oscillations and hence bound their net effect, it will be necessary for us to divide the domain of integration \([-s, -s^{1-\frac{2}{3}\theta}]\) into a disjoint (except for endpoints) union of consecutive closed intervals \( I_1, I_2, \ldots, I_K \). Here \( K \) and the intervals are chosen by the following prescription. Denote \( I_j = [a_j, b_j] \) and let \( b_1 = -s^{1-\frac{2}{3}\theta} \). Inductively in \( j \geq 1 \) let

\[
a_j = b_j - \pi(-b_j)^{-\frac{1}{2}}, \quad \text{and} \quad b_{j+1} = a_j.
\]

Let \( K \) denote the minimal integer \( k \) such that \( b_k < -s \). Finally, reassign \( b_K = -s \). This produces the desired intervals. Note that each interval has length of order between \( s^{-\frac{1}{2}} \) and \( s^{\frac{1}{3}\theta - \frac{1}{2}} \), and the total number of intervals \( K \) is of order \( s^{\frac{3}{2}} \). These intervals are chosen so as to contain roughly one period of oscillation. This enables us to control the sum of oscillatory effects.

For any \( 1 \leq j < K \), we may parameterize the interval \( I_j \) via the function \( I_j(t) = b - \pi(-b)^{-\frac{1}{2}}t \), as \( t \) ranges over \([0, 1]\). Let us fix some \( j \) and for the moment drop the subscripts on \( a_j, b_j \) and \( I_j(t) \).

We claim the following bounds. There exist \( s_1 = s_1(\delta, \theta) > 0 \) and \( C = C(\delta, \theta) > 0 \) such that for all \( s \geq s_1 \) (note, the error terms \( J_4, J_5, J_6, J_7, J_8, J_9 \) below are functions of \( t, b \) and \( s \))

\[
\frac{4}{3} \left( -I(t) \right)^{\frac{3}{2}} = \frac{4}{3}(-b)^{\frac{3}{2}} + 2\pi t + J_4 \quad \text{where} \quad |J_4| = C(-b)^{-\frac{3}{2}},
\]

\[
\phi(I(t), s) = \phi(b, s) + J_5 \quad \text{where} \quad |J_5| = C(-b)^{-(\delta - \theta)},
\]

\[
\frac{v}{\pi} \log \left( 8(-b)^{\frac{3}{2}} \right) = \frac{v}{\pi} \log \left( \frac{1}{2} (b)^{-\frac{3}{2}} \right) + J_6 \quad \text{where} \quad |J_6| \leq C(-b)^{-\frac{3}{2}-(\delta - \theta)/(1-\frac{2}{3}\theta)}.
\]

The first bound above follows from Taylor’s expansion of \((1 + x)^{\frac{3}{2}}\); the second bound follows directly from (1.5.15); and the third bound needs a bit more argument. Combining Taylor’s expansion of \((1 + x)^{\frac{3}{2}}\) with that of \( \log(1 + x) \) yields the first two terms and an error of order \( v(-b)^{-3} \). Since \( b \in [-s, -s^{1-\frac{2}{3}\theta}] \), it follows that we may upper-bound \( s \leq (-b)^{(1-\frac{2}{3}\theta)^{-1}} \), and hence \( v \leq
\(-b\left(\frac{3}{2} - \delta\right)(1 - \frac{2}{3} \theta)^{-1}\). This enables us to reexpress the order of \(v(-b)^{-3}\) entirely in terms of \(b\) as claimed.

Observe that of \(J_4, J_5\) and \(J_6\), the largest error term is \(J_5\). This is because \(\delta - \theta \leq \frac{2}{5}\) whereas all other exponents have negative powers exceeding \(\frac{3}{2}\). Thus, combining the three bounds with the definition of \(\psi\), we find that for all \(s \geq s_1\)

\[
\psi(I(t), s) = \psi(b, s) + 2\pi t + \frac{3vt}{2}(-b)^{-\frac{3}{2}} + J_7 \quad \text{where} \quad |J_7| \leq 3C(-b)^{-(\delta - \theta)}.
\]  

(1.5.16)

We may now use (1.5.16) to bound the integral on the l.h.s. of (1.5.14) over the interval \(I\) (that is, \(I_j\) for any \(1 \leq j < K\)). As we show below, there exists \(s_2 = s_2(\delta, \theta) > 0\) and \(C = C(\delta, \theta)\) such that for all \(s \geq s_2\) and all \(I = I_j\) for \(1 \leq j < K\),

\[
\left| \int_{I} \frac{(x + s)}{(-x)^{\frac{3}{2}}} \cos\left(\psi(x, s)\right) dx \right| \leq C|b + s| \left(\frac{v|b|^{-\frac{3}{2}}}{\xi(b, s)} + |b|^{-2(\delta - \theta)}\right) + C|b|^{-\frac{1}{2} - \frac{7}{11}}
\]  

(1.5.17)

where \(\xi(b, s) = 2\pi + \frac{3}{2}v|b|^{-\frac{3}{2}}\).

To show this, observe first that by parameterizing the interval \(I\) via \(I(t)\) for \(t \in [0, 1]\) we have

\[
\int_{I} \frac{(x + s)}{(-x)^{\frac{3}{2}}} \cos\left(\psi(x, s)\right) dx = \pi \left(\frac{b + s}{(-b)^{\frac{3}{2}}} + J_8\right) \cos\left(\psi(I(t), s)\right) dt
\]  

(1.5.18)

where the error term \(J_8 = J_8(t, b, s)\) comes from Taylor’s expansion and can be bounded uniformly in \(t\) and for all intervals \(I\) by

\[
|J_8| \leq C(-b)^{(1 - \frac{2}{3} \theta)^{-1} - 2} \leq C(-b)^{-\frac{7}{11}}
\]

for some constant \(C > 0\). The second bound comes by taking the worst value of \(\theta \in (0, \frac{2}{5})\).

Using (1.5.16), \(\cos\left(\psi(I(t), s)\right) = \cos\left(\psi_t(b, s) + J_7\right)\). Expanding the sum in the cosine yields

\[
\cos\left(\psi(I(t), s)\right) = \cos\left(\psi_t(b, s)\right) \cos(J_7) - \sin\left(\psi_t(b, s)\right) \sin(J_7).
\]  

(1.5.19)
The bound on $J_7$ in (1.5.16) implies that for some $C = C(\delta, \theta) > 0$,

$$\max \left\{ |\cos(J_7) - 1|, |\sin(J_7)| \right\} \leq C(-b)^{-2(\delta - \theta)}.$$

Substituting this into (1.5.19) yields

$$\cos \left( \psi(t, s) \right) = \cos \left( \psi(b, s) \right) + J_9 \quad \text{where} \quad |J_9| \leq C(-b)^{-2(\delta - \theta)}.$$

We may finally substitute this back into (1.5.18) and evaluate the main contribution to the integral as well as the error terms. This yields (recall $\xi(b, s)$ defined below (1.5.17) and note that the value of the constant $C$ may change between lines)

$$\text{l.h.s. of (1.5.17)} \leq \pi \left| b + s \right| \left( \int_0^1 \cos \left( \psi(t, s) \right) dt \right) + C\left| b \right|^{-2(\delta - \theta)} + C\left| b \right|^{-\frac{1}{2} - \frac{7}{14}}$$

$$= \pi \left| b + s \right| \left( \left| \sin(\psi(b, s)) \right| - \sin(\psi(b, s) + \frac{3}{2}v|b|^{-\frac{3}{2}}) \right) \xi(b, s) + C\left| b \right|^{-2(\delta - \theta)} + C\left| b \right|^{-\frac{1}{2} - \frac{7}{14}}$$

$$\leq \pi \left| b + s \right| \left( \frac{3}{2}v|b|^{-\frac{3}{2}} \xi(b, s) \right) + C\left| b \right|^{-2(\delta - \theta)} + C\left| b \right|^{-\frac{1}{2} - \frac{7}{14}}$$

where, in the third line we have used the inequality $\left| \sin(x) - \sin(x + y) \right| \leq |y|$ which holds for all $y$. Redefining $C$ to absorb all needed constants yields the r.h.s. of (1.5.17) as desired.

Now, we turn to the final step of the proof where we sum the contributions over all the intervals $I_1, \ldots, I_K$. Summing (1.5.17) over $1 \leq j < K$ yields

$$\left| \sum_{j=1}^{K-1} \int_{I_j} \frac{x + s}{(-x)^{\frac{3}{2}}} \cos \left( \psi(x, s) \right) dx \right| \leq \sum_{j=1}^{K-1} \left( \frac{C|b_j + s|}{|b_j|} \left( \frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j, s)} + |b_j|^{-2(\delta - \theta)} \right) + C|b_j|^{-\frac{1}{2} - \frac{7}{14}} \right).$$

(1.5.20)

We may use the bound $\sum_{j=1}^{K-1} \frac{\pi(b_j + s)}{(-b_j)} \leq 2 \int_{s}^{s + \frac{3}{3}} \left( -x \right)^{-\frac{3}{2}} \frac{1}{\pi} (x + s) dx \leq \frac{8}{9}s^{\frac{3}{2}}$ to see that (the constant
\[ C \text{ may change from the l.h.s to r.h.s below) } \]

\[
\sum_{j=1}^{K-1} \frac{C|b_j + s|}{|b_j|} \left( \frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j, s)} + |b_j|^{-2(\delta - \theta)} \right) \leq C s^{\frac{3}{2}} \max_{1 \leq j \leq K-1} \left( \frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j, s)} + |b_j|^{-2(\delta - \theta)} \right) (1.5.21)
\]

\[
\leq C s^{\frac{3}{2}} s^{-(\delta - \theta)}.
\]

The second inequality comes from noting that \( v = v(s) = s^{\frac{3}{2} - \delta} \), the maximal value of \( |b_j| \) to a negative power is realized when \( j = 1 \), in which case \( |b_1| = s^{1-\frac{3}{2}\theta} \), and \( -\delta - \theta \geq -2(\delta - \theta)(1 - \frac{3}{2}\theta) \) for \( 0 \leq \theta < \frac{2}{5} \). As for the other term in the summation on the r.h.s. of (1.5.20), we may bound it via an integral as

\[
\sum_{j=1}^{K-1} C|b_j|^{-\frac{3}{2} - \frac{7}{11}} \leq C \int_{-s}^{s} \left( -x \right)^{-\frac{3}{2} - \frac{7}{11}} dx \leq C s^{1-\frac{7}{11}}.
\]

Combining this bound with (1.5.21) shows that

\[ l.h.s. (1.5.20) \leq C s^{\frac{3}{2}} s^{-(\delta - \theta)}. \]

Here we have used that \( s^{1-\frac{7}{11}} = s^{\frac{3}{2}} s^{-\frac{3}{2} - \frac{7}{11}} \) and that clearly \( s^{-\frac{3}{2} - \frac{7}{11}} \leq s^{-(\delta - \theta)} \). Finally, we must deal with the summand when \( j = K \). However, this is clearly bounded by a constant. Thus, we arrive at the desired bound (1.5.14) and complete the proof of the lemma.

\[ \square \]

1.5.2 Proof of Theorem 1.1.6

Recall from (1.1.11) that (for \( v = v(s) = s^{\frac{3}{2} - \delta} \) with \( \delta \in (0, \frac{2}{5}) \) fixed)

\[ \log F(-s; v) = -\int_{-s}^{\infty} (x + s)u^{2}_{AS}(x; \gamma) dx, \quad \text{with } \gamma = 1 - e^{-v}. \]

We seek to prove that

\[ \int_{-s}^{\infty} (x + s)u^{2}_{AS}(x; \gamma) dx \geq \frac{2}{3\pi} vs^{\frac{3}{2}} + O(s^{3-\frac{125}{11}}). \]

Most of the contribution to the l.h.s. integral comes from \( x \) near \( -s \). With this in mind, for
\( \theta \in (0, \delta) \), divide the integral into two parts

\[
(a) := \int_{-s^{-1/3}}^{\infty} (x + s) u_{\text{AS}}^2(x; \gamma) \, dx \quad \text{and,} \quad (b) := \int_{-s}^{-s^{-1/3}} (x + s) u_{\text{AS}}^2(x; \gamma) \, dx.
\]

We use the obvious lower-bound \( (a) \geq 0 \). For \( (b) \) we use the asymptotic expansion for \( u_{\text{AS}} \) given in Lemma 1.5.3. The assumptions of this lemma were previously verified for this range of \( x \) in the beginning of the proof of Lemma 1.5.4, so we do not repeat it here. We may now use the expansion provided in (1.5.6) for \( u_{\text{AS}} \) to show that (using \( \eta_0 = \delta - \theta \) in the lemma), there exists \( s_0 = s_0(\delta, \theta) > 0 \) such that for all \( s \geq s_0 \)

\[
(b) := \int_{-s}^{-s^{-1/3}} (x + s) \left( (-x)^{-1/4} \sqrt{\frac{v}{\pi}} \cos \left( \pi (-x)^{3/2} V \right) + J_2 \right)^2 \, dx,
\]

where \( |J_2| \leq C(-x)^{\frac{1}{2} - \frac{3}{2}(\delta - \theta)} \). Squaring the expansion term and using the identity that \( 2 \cos(y)^2 = 1 + \cos(2y) \) we can write

\[
(b) := \frac{v}{2\pi} \int_{-s}^{-s^{-1/3}} \frac{x + s}{(-x)^{1/2}} \, dx + \frac{v}{2\pi} \int_{-s}^{-s^{-1/3}} \frac{x + s}{(-x)^{1/2}} \cos \left( 2\pi (-x)^{3/2} V \right) \, dx \tag{b_1}
\]

\[
+ \int_{-s}^{-s^{-1/3}} (x + s) \left( 2(-x)^{-1/4} \sqrt{\frac{v}{\pi}} \cos \left( \pi (-x)^{3/2} V \right) J_2 + J_2^2 \right) \, dx. \tag{b_2}
\]

By extending the domain of integration to \([-s, -1]\) we may lower bound

\[
(b_1) = \frac{v}{2\pi} \int_{-s}^{-1} \frac{x + s}{(-x)^{1/2}} \, dx - \frac{v}{2\pi} \int_{-s^{-1/3}}^{-1} \frac{x + s}{(-x)^{1/2}} \, dx \geq \frac{2}{3\pi} vs^\frac{3}{2} + J_{10}
\]

where the term \( J_{10} \) comes from bounding the second integral along with an order \( vs \) residual from evaluating the first integral. It can be bounded (for some constant \( C > 0 \)) by

\[
|J_{10}| \leq Cs^{3-\delta-\frac{1}{4}\theta}.
\]

The term \( \frac{2}{3\pi} vs^\frac{3}{2} \) will constitute the main contribution.

The oscillatory integral in \( (b_2) \) is lower-bounded by applying Lemma 1.5.4 which shows that
\[ |(b_2)| \geq Cus^{\frac{3}{2}}s^{-(\delta-\theta)} \text{ for some constant } C > 0. \]

We may bound

\[ |(b_3)| \leq \int_{-s}^{-s^2} (x+s) \left( 2(-x)^{-\frac{3}{4}} \sqrt{\frac{u}{\pi}} |J_2| + |J_2|^2 \right) dx \leq s^{3} C \left( s^{\frac{3}{2}(\delta-\theta)} - \frac{1}{2} \delta + s^{\frac{3}{2}(\delta-\theta)} \right), \]

where the second inequality follows from our bound on \(|J_2|\) along with extending the integral from \(-s\) to 0 and evaluating.

Combining the above bounds we find that

\[ (b) \geq \frac{2}{3\pi} us^{\frac{3}{2}} + J_{11} \]

where there is a constant \(C > 0\) such that

\[ |J_{11}| \leq Cs^{3-\delta} \left( s^{-\frac{2}{3}\theta} + s^{-\delta+\theta} + s^{-\delta+\frac{2}{3}\theta} + s^{-2\delta+3\theta} \right) \]

We are given \(\delta \in (0, \frac{3}{5})\) but free to choose \(\theta \in (0, \delta)\) so as to minimize \(|J_{11}|\). Choosing \(\theta = \frac{6}{11} \delta\) results in the best (i.e., lowest) upper bound of \(|J_{11}| \leq s^{3-\frac{12}{11}\delta}\). This is small compared to \(\frac{2}{3\pi} us^{\frac{3}{2}}\) (which is of order \(s^{3-\delta}\)), and hence our proof is completed. \(\blacksquare\)
Chapter 2: Tails under general initial data

2.1 Introduction

In this chapter we consider the following question: How does the initial data for an SPDE affect the statistics of the solution at a later time? Namely, we consider the Kardar-Parisi-Zhang (KPZ) equation (or equivalently, the stochastic heat equation (SHE)) and probe the lower and upper tails of the centered (by time $/24$) and scaled (by time$^{1/3}$) one-point distribution for the solution at finite and long times. Our main results (Theorems 2.1.2 and 2.1.4) show that within a very large class of initial data, the tail behavior for the KPZ equation does not change in terms of the super-exponential decay rates and at most changes in terms of the coefficient in the exponential. These results are the first tail bounds for general initial data which capture the correct decay exponents and which respect the long-time scaling behavior of the solution. In what follows, we consider very general initial data as now describe.

Definition 2.1.1. Fix $\nu \in (0, 1)$ and $C, \theta, \kappa, M > 0$. A measurable function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies $Hyp(C, \nu, \theta, \kappa, M)$ if:

1. $f(y) \leq C + \frac{\nu}{2^{2/3}} y^2, \quad \forall y \in \mathbb{R}, \quad \text{(2.1.1)}$

2. there exists a subinterval $\mathcal{I} \subset [-M, M]$ with $|\mathcal{I}| = \theta$ such that

$$f(y) \geq -\kappa, \quad \forall y \in \mathcal{I}. \quad \text{(2.1.2)}$$

For a measurable function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, and $T > 0$ consider the solution to the KPZ equation with initial data $\mathcal{H}_0$ chosen such that

$$T^{-\frac{1}{2}} \mathcal{H}_0((2T)^{\frac{2}{3}} y) = f(y). \quad \text{(2.1.3)}$$
We consider the KPZ equation with this initial data and run until time $T$. Namely, let

$$h_T^f(y) := \frac{\mathcal{H}(2T, (2T)^{\frac{2}{3}} y) + \frac{T}{12} - \frac{2}{3} \log(2T)}{T^{\frac{1}{3}}}.$$  (2.1.4)

Figure 2.1: Schematic plot of the density (top) and log density (bottom) of $h_T^f(0)$. Letting $s$ denote the horizontal axis variable, there are four regions which display different behaviors. Region I (deep lower tail, when $s \ll -T^{2/3}$): the log density has power law decay with exponent $5/2$. Region II (shallow lower tail, when $-T^{2/3} \ll s \ll 0$): the log density has power law decay with exponent $3$. Region III (center, when $s \approx 0$): the density depends on initial data as predicted by the KPZ fixed point. Region IV (upper tail, when $s \gg 0$): the log density has power law decay with exponent $3/2$. The universality of the power law exponents (in regions I, II and IV) for general initial data constitutes the main contribution of this paper.

Our first main result (Theorem 2.1.2) provides an upper bound on the lower tail that holds uniformly over $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$, and $T > 1$. The proof of this and our other main results are deferred to the later sections of the paper.

**Theorem 2.1.2.** Fix any $\epsilon, \delta \in (0, \frac{1}{3})$, $C, M, \theta > 0$, $\nu \in (0, 1)$, and $T_0 > 0$. There exist $s_0 = s_0(\epsilon, \delta, C, M, \theta, \nu, T_0)$ and $K = K(\epsilon, \delta, T_0) > 0$ such that for all $s \geq s_0$, $T \geq T_0$, and $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$ (recall $h_T^f(y)$ is defined in (2.1.3) and (2.1.4)),

$$\mathbb{P}\left(h_T^f(0) \leq -s\right) \leq e^{-T^{1/3} \frac{4(1-\nu)^{\frac{1}{6}}}{12 \pi}} + e^{-Ks^{3-\delta} - \epsilon s T^{1/3}} + e^{-\frac{(1-\nu)^{\frac{3}{2}}}{12}}.$$  (2.1.5)

**Remark 2.1.3.** There are three regions of the lower tail (see I, II, and III in Figure 2.1). In each region (and for $T$ large) a different one of the three terms on the r.h.s. of (2.1.5) becomes active. For

$^{1}$Notice that the initial data and time horizon are both dependent on $T$. This allows for a much wider class of initial data which are adapted to the KPZ fixed point scaling.
instance, for region I when $s \gg T^{2/3}$, the largest term in our bound is the first term in the r.h.s. of (2.1.5). Likewise, the middle term in the r.h.s. of (2.1.5) is active in region II and the last term in region III. We presently lack a matching lower bound for the lower tail probability. This is known for only the narrow wedge (see Theorem 1.1.1). See Section 0.3.3 for some discussion regarding physics literature related to this tail. Let us also note that one can get similar bound as in (2.1.5) on $P(h_f^T(y) \leq -s)$ when $y \neq 0$. This is explained in Section 2.1.1. Finally, observe that two important choices of initial data — narrow wedge and Brownian motion — do not fit into this class\(^2\). The narrow wedge result is in fact a building block for the proof of this result, while Brownian follows as a fairly easy corollary (see Section 2.1.2).

Our second main result pertains to the upper tail and shows upper and lower bounds which hold uniformly over $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$, and $T > \pi$.

**Theorem 2.1.4.** Fix any $\nu \in (0, 1)$ and $C, \theta, \kappa, M > 0$. For any $T_0 > 0$, there exist $s_0 = s_0(C, \nu, \theta, \kappa, M, T_0) > 0$, $c_1 = c_1(T_0) > c_2 = c_2(T_0) > 0$ such that for all $s \geq s_0$, $T > T_0$ and $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$,

$$e^{-c_1 s^{3/2}} \leq P(h_T^f(0) \geq s) \leq e^{-c_2 s^{3/2}}. \quad (2.1.6)$$

We may further specify values of $c_1$ and $c_2$ for which (2.1.6) holds, provided we assume $T_0 > \pi$. In that case, for any $\epsilon, \mu \in (0, 1/2)$, there exists $s_0 = s_0(\epsilon, \mu, C, \nu, \theta, \kappa, M, T_0) > 0$ such that for all $s \geq s'_0$, $T \geq T_0$, and $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$, (2.1.6) holds with the following choices for $c_1 > c_2$:

1. If $s_0 \leq s < \frac{1}{8^3} \left(1 - \frac{2\mu}{3}\right)^{-1} T^{\frac{2}{3}}$ then we may take $c_1 = \frac{8}{3} (1 + \mu)(1 + \epsilon)$ and $c_2 = \frac{\sqrt{2}}{3} (1 - \mu)(1 - \epsilon)$.

2. If $s \geq \max\{s_0, \frac{9}{16} \epsilon^{-2} (1 - \frac{2\mu}{3})^{-1} T^{\frac{2}{3}}\}$ then we may take $c_1 = 8 \sqrt{3} (1 + \mu)(1 + \epsilon)$ and $c_2 = \frac{\sqrt{2}}{3} (1 - \mu)(1 - \epsilon)$.

3. If $\max\{s_0, \frac{1}{8^3} \left(1 - \frac{2\mu}{3}\right)^{-1} T^{\frac{2}{3}}\} \leq s \leq \max\{s_0, \frac{9}{16} \epsilon^{-2} (1 - \frac{2\mu}{3})^{-1} T^{\frac{2}{3}}\}$ then we may take $c_1 = 2^{9/2} \epsilon^{-3} (1 + \mu)$ and $c_2 = \frac{\sqrt{2}}{3} (1 - \mu) \epsilon$.

**Remark 2.1.5.** In Theorems 2.1.9 and 2.1.12 we prove similar results for narrow wedge and Brownian initial data. The upper and lower bounds on the constants $c_1$ and $c_2$ are not optimal. In fact,
it is not clear to us how the initial data translates to the optimal value of \( c_1 \) or \( c_2 \). There, however, some predictions in the physics literature – see Section 0.3.4. The condition \( T_0 > \pi \) assumed in the second part of Theorem 2.1.4 could be replaced by an arbitrary lower bound, though the resulting conditions on \( s, c_1 \) and \( c_2 \) would need to change accordingly. This value \( \pi \) turns out to work well in the computations leading to this result; in particular see (2.4.19).

2.1.1 Proof sketch

The fundamental solution to the SHE \( \mathcal{Z}^{\text{nw}}(T, X) \) corresponds to delta initial data \( \mathcal{Z}_0(X) = \delta_{X=0} \). For any positive \( T \), this results in a strictly positive solution, hence the corresponding KPZ equation solution is well-defined for \( T > 0 \) and this initial data is termed narrow wedge since in short time \( \mathcal{Z}(T, X) \) is well-approximated by the Gaussian heat-kernel whose logarithm is a very thin parabola \( \frac{X^2}{2T} \). The Cole-Hopf transform of \( \mathcal{Z}^{\text{nw}}(T, X) \) is denoted in (1.1.1) of Chapter 1 by \( \mathcal{H}^{\text{nw}}(T, X) := \log \mathcal{Z}^{\text{nw}}(T, X) \).

The proof of our main results relies upon a combination of three ingredients: (1) lower tail bounds for the narrow wedge initial data proved in Chapter [1] (2) Gibbsian line ensemble techniques applied to the KPZ line ensemble [CH16], and (3) explicit integral formulas for moments of the SHE with delta initial data. Now, we give an overview of our proofs. A more involved discussion of the KPZ line ensemble is contained in Section 2.2.

To prove Theorem 2.1.2 one of our main tools is the upper and lower bound for the lower tail of the one point distribution of the narrow wedge solution of the KPZ equation given in Theorem 1.1.1. However, to use this result, we need a connection between the solution of the KPZ equation under general initial conditions and the narrow wedge solution. This connection is made through the identity (0.1.7) (which follows from the Feynman-Kac formula) which represents the one point distribution of the KPZ equation started from \( \mathcal{H}_0 \) as a convolution between the spatial process \( \Upsilon_T(\cdot) \) and the initial data \( \mathcal{H}_0(\cdot) \).

**Proposition 2.1.6** (Lemma 1.18 of [CH16]). For \( \mathcal{H}_0 \) as in (2.1.3), we have

\[
\mathcal{H}(2T, (2T)^{\frac{3}{2}} X) + \frac{T}{3} - \frac{2}{3} \log(2T) \quad \overset{d}{=} \quad \frac{1}{T^{\frac{3}{2}}} \log \left( \int_{-\infty}^{\infty} e^{T^{\frac{3}{2}} \left( \Upsilon_T(Y) + f(X-Y) \right)} dY \right). \tag{2.1.7}
\]

To employ this identity, we need tail bounds for the entire spatial process \( \Upsilon_T(\cdot) \). Presently, exact
formulas amenable to rigorous asymptotics are only available for one-point tail probabilities, and not multi-point. However, by using the Gibbs property for the KPZ line ensemble (introduced in [CH16] and recalled here in Section 2.2) we will be able to extend this one-point tail control to the entire spatial process. Working with the Gibbs property is a central technical aspect of our present work and forms the backbone of the proof of Theorem 2.1.2.

Besides the KPZ line ensemble, another helpful property of the narrow wedge KPZ solution is the stationarity of the spatial process $\Upsilon_T(\cdot)$ after a parabolic shift.

**Proposition 2.1.7** (Proposition 1.4 of [ACQ11]). The one point distribution of $\Upsilon_T(y) + \frac{y^2}{2T^{\frac{1}{3}}}$ does not depend on the value of $y$.

The proof of Theorem 2.1.4 shares a similar philosophy with that of Theorem 2.1.2. We first prove an upper (as Theorem 2.1.9) and a lower bound for the upper tail probability of $\Upsilon_T(0)$. The proof of Theorem 2.1.9 employs a combination of the one-point Laplace transform formula (see Proposition 2.4.6) and moment formulas (see the proof of Lemma 2.4.5) for $Z_{nw}$.

The rest of the proof of Theorem 2.1.4 is based on the Gibbs property of the KPZ line ensemble and the FKG inequality of the KPZ equation. The FKG inequality of the KPZ equation is, for example (as shown in [CQ13, Proposition 1]) a consequence of the positive associativity of its discrete analogue, the asymmetric simple exclusion process (ASEP).

**Proposition 2.1.8** (Proposition 1 of [CQ13]). Let $H$ be the Cole-Hopf solution to KPZ started from initial data $H_0$. Fix $k \in \mathbb{Z}_{>0}$. For any $T_1, \ldots, T_k \geq 0$, $X_1, \ldots, X_k \in \mathbb{R}$ and $s_1, \ldots, s_k \in \mathbb{R}$,

$$
\mathbb{P}\left( \bigcap_{\ell=1}^{k} \{ H(T_\ell, X_\ell) \leq s_\ell \} \right) \geq \prod_{\ell=1}^{k} \mathbb{P}(H(T_\ell, X_\ell) \leq s_\ell).
$$

A simply corollary of this result is that for $T_1, T_2 \in \mathbb{R}_{>0}$, $X_1, X_2 \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$,

$$
\mathbb{P}(H(T_1, X_1) > s_1, H(T_2, X_2) > s_2) \geq \mathbb{P}(H(T_1, X_1) > s_1) \mathbb{P}(H(T_2, X_2) > s_2). \quad (2.1.8)
$$

2.1.2 Narrow wedge and Brownian initial data results

Neither narrow wedge nor two-sided Brownian initial data belongs to the class of functions in Definition 2.1.1. The analogue of Theorem 2.1.2 for the narrow wedge is stated and proved in Chapter 1.
(see Theorem 1.1.1). We record here the analogues of Theorem 2.1.4 for the narrow wedge case and Theorems 2.1.2 and 2.1.4 for the Brownian case.

Our general initial data results rely upon upper and lower bounds on the upper tail probability of $\Upsilon_T(\cdot)$ which are, in fact, new (see Section 0.3.4 for a discussion of previous work).

**Theorem 2.1.9.** For any $T_0 > 0$, there exist $s_0 = s_0(T_0) > 0$ and $c_1 = c_1(T_0) > c_2 = c_2(T_0) > 0$ such that for all $s \geq s_0$ and $T > T_0$

$$e^{-c_1 s^{3/2}} \leq \Pr(\Upsilon_T(0) \geq s) \leq e^{-c_2 s^{3/2}}. \quad (2.1.9)$$

We may further specify values of $c_1$ and $c_2$ for which (2.1.9) holds, provided we assume $T_0 > \pi$.

In that case, for any $\epsilon \in (0, \frac{1}{2})$, there exists $s_0 = s_0(\epsilon, T_0) > 0$ such that for all $s \geq s_0$ and $T \geq T_0$, (2.1.9) holds with the following choices for $c_1 > c_2$:

1. If $s_0 \leq s < \frac{1}{8} \epsilon^2 T_1 \frac{3}{2}$ then we may take $c_1 = \frac{4}{3}(1 + \epsilon)$ and $c_2 = \frac{4}{3}(1 - \epsilon)$.

2. If $s \geq \max\{s_0, \frac{9}{16} \epsilon^2 T_1 \frac{3}{2}\}$ then we may take $c_1 = 4 \sqrt{3}(1 + \epsilon)$ and $c_2 = \frac{4}{3}(1 - \epsilon)$. Furthermore, for $c_1 = \frac{4}{3}(1 + \epsilon)$ there exists a sequence $\{s_n\}_{n \geq 1}$ with $s_n \to \infty$ as $n \to \infty$ such that $\Pr(\Upsilon_T > s_n) > e^{-c_1 s_n^{3/2}}$ for all $n$.

3. If $\max\{s_0, \frac{1}{8} \epsilon^2 T_1 \frac{3}{2}\} \leq s \leq \max\{s_0, \frac{9}{16} \epsilon^2 T_1 \frac{3}{2}\}$ then we may take $c_1 = 2^{7/2} \epsilon^{-3}$ and $c_2 = \frac{4}{3} \epsilon$.

**Remark 2.1.10.** Part (i) of Theorem 2.1.9 shows that $\Pr(\Upsilon_T(0) > s)$ is close to $\exp(-4 s \frac{3}{2}/3)$ when $s \ll T_1 \frac{3}{2}$. This is in agreement with the fact that the tail probabilities of $\Upsilon_T(0)$ should be close to the tails of the Tracy-Widom GUE distribution as $T$ increases to $\infty$. Part (ii) of Theorem 2.1.9 shows that the upper bound to $\Pr(\Upsilon_T(0) > s)$ is close $\exp(-4 s \frac{3}{2}/3)$ when $s \gg T_1 \frac{3}{2}$. We also have some lower bound which is not tight. However, part (ii) further tells that the lower bound for $\Pr(\Upsilon_T(0) > s)$ cannot differ much from $\exp(-4 s \frac{3}{2}/3)$ for all large $s$. In the regime $s = O(T_1 \frac{3}{2})$, we do not have tight upper and lower bounds in (2.1.9), although, the decay exponent of $\Pr(\Upsilon_T(0) > s)$ will still be equal to $3/2$.

Our next two results are about the tail probabilities for the KPZ equation with two sided Brownian motion initial data; as this initial data falls outside our class, some additional arguments are necessary. Define $\mathcal{H}^{Br}_0 : \mathbb{R} \to \mathbb{R}$ as $\mathcal{H}^{Br}_0(x) := B(x)$ where $B$ is a two sided standard Brownian motion with
\[ B(0) = 0. \] Denote the Cole-Hopf solution of the KPZ equation started from this initial data \( \mathcal{H}^{Br}_0 \) by \( \mathcal{H}^{Br}(\cdot, \cdot) \) and define

\[
h^{Br}_T(y) := \frac{\mathcal{H}^{Br}(2T, (2T)^{\frac{2}{3}} y) + \frac{T}{12} - \frac{2}{3} \log(2T)}{T^{\frac{1}{3}}} \quad \forall T > 0. \tag{2.1.10}
\]

We first state our result on the lower tail of \( h^{Br}_T(0) \).

**Theorem 2.1.11.** Fix \( \epsilon, \delta \in (0, \frac{1}{3}) \) and \( T_0 > 0 \). There exist \( s_0 = s_0(\epsilon, \delta, T_0) \) and \( K = K(\epsilon, \delta, T_0) > 0 \) such that for all \( s \geq s_0 \) and \( T \geq T_0 \),

\[
\mathbb{P}(h^{Br}_T(0) \leq -s) \leq e^{-T^{-1/3}4(1-\epsilon)\frac{s^{5/2}}{15\pi}} + e^{-Ks^{3-\delta} - csT^{1/3}} + e^{-\frac{(1-\epsilon)s^3}{12}}. \tag{2.1.11}
\]

Our last result of this section is about the upper tail probability of \( h^{Br}_T(0) \).

**Theorem 2.1.12.** Fix \( \epsilon, \mu \in (0, \frac{1}{2}) \) and \( T_0 > 0 \). Then, there exists \( s_0 = s_0(\epsilon, \mu, T_0) \) such that for all \( s \geq s_0 \) and \( T \geq T_0 \),

\[
e^{-c_1s^{3/2}} \leq \mathbb{P}(h^{Br}_T(0) > s) \leq e^{-c_2s^{3/2}} + e^{-\frac{1}{5\sqrt{3}}(\mu s)^{3/2}}
\]

where \( c_1 > c_2 \) depend on the values of \( \epsilon, \mu \) and \( T_0 \) as described in Theorem 2.1.4.

In Theorem 2.1.12 the second term of the upper bound (on the right-hand side of the equation) comes from the fact that Brownian motion is random, and the first term arises in an analogous way as it does for deterministic initial data in Theorem 2.1.4.

As proved in [Bor+15, Theorem 2.17], \( h^{Br}_T(0) \) converges in law to the Baik-Rains distribution (see [BR00; Fs0; IS04; PS04; BFP10]). The following corollary strengthens the notion of that convergence and implies that the moments of \( h^{Br}_T(0) \) converge to the moments of the limiting Baik-Rains distribution. This answers a question posed to us by Jean-Dominique Deutschel (namely, that the variance converges).

**Corollary 2.1.13.** Let \( X \) be a Baik-Rains distributed random variable (see [Bor+15, Definition 2.16]). Then, \( \mathbb{E}[e^{t|X|}] < \infty \) and for all \( t \in \mathbb{R} \),

\[
\mathbb{E}[e^{t|h^{Br}_T(0)|}] \to \mathbb{E}[e^{t|X|}], \quad \text{as } T \to \infty. \tag{2.1.12}
\]
Proof. Theorems 2.1.11 and 2.1.12 show that $e^{t|h|_B^2(0)}$ is uniformly integrable. The dominated convergence theorem, along with [Bor+15, Theorem 2.17] yields (2.1.12) and $E[e^{t|X|}] < \infty$. ■

2.2 KPZ line ensemble

This section reviews (following the work of [CH16]) the KPZ line ensemble and its Gibbs property. We use this construction in order to transfer one-point information (namely, tail bounds) into spatially uniform information for $\Upsilon_T(y)$ (see (1.1.2)). It is through this mechanism that we can escape the bonds of exact formulas and generalize the conclusions of Chapter 1 to general initial data.

Definition 2.2.1. Fix intervals $\Sigma \subset \mathbb{N}$ and $\Lambda \subset \mathbb{R}$. Let $X$ be the set of all continuous functions $f : \Sigma \times \Lambda \mapsto \mathbb{R}$ endowed with the topology of uniform convergence on the compact subsets of $\Sigma \times \Lambda$. Denote the sigma field generated by the Borel subsets of $X$ by $\mathcal{C}$.

A $\Sigma \times \Lambda$-indexed line ensemble $L$ is a random variable in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that it takes values in $X$ and is measurable with respect to $(\mathcal{B}, \mathcal{C})$. In simple words, $L$ is a collection of $\Sigma$-indexed random continuous curves, each mapping $\Lambda$ to $\mathbb{R}$.

Fix two integers $k_1 \leq k_2$, $a < b$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. A $\{k_1, \ldots, k_2\} \times (a, b)$-indexed line ensemble is called a free Brownian bridge line ensemble with the entrance data $\vec{x}$ and the exit data $\vec{y}$ if its law, denoted here as $\mathbb{P}_{k_1, k_2, (a, b), \vec{x}, \vec{y}}$, is that of $k_2 - k_1 + 1$ independent Brownian bridges starting at time $a$ at points $\vec{x}$ and ending at time $b$ at points $\vec{y}$. We use the notation $\mathbb{E}_{k_1, k_2, (a, b), \vec{x}, \vec{y}}$ for the associated expectation operator.

Consider a continuous function $H : [0, \infty) \rightarrow \mathbb{R}$, which we call a Hamiltonian. Given $H$ and two measurable functions $f : [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$ and $g : [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$, we define a $\{k_1, \ldots, k_2\} \times (a, b)$-indexed line ensemble with the entrance data $\vec{x}$, the exit data $\vec{y}$, boundary data $(f, g)$ and $H$ to be the law of $\mathbb{P}_{k_1, k_2, (a, b), \vec{x}, \vec{y}, f, g}^H$ on curves $L_{k_1}, \ldots, L_{k_2} : [0, \infty) \rightarrow \mathbb{R}$ which is given in terms of the following Radon-Nikodym derivative

$$
\frac{d\mathbb{P}_{k_1, k_2, (a, b), \vec{x}, \vec{y}, f, g}^H}{d\mathbb{P}_{k_1, k_2, (a, b), \vec{x}, \vec{y}}^\text{free}}(L_{k_1}, \ldots, L_{k_2}) = \frac{W_{k_1, k_2, (a, b), \vec{x}, \vec{y}, f, g}^H(L_{k_1}, \ldots, L_{k_2})}{Z_{k_1, k_2, (a, b), \vec{x}, \vec{y}, f, g}^H}
$$

$$
W_{k_1, k_2, (a, b), \vec{x}, \vec{y}, f, g}^H(L_{k_1}, \ldots, L_{k_2}) = \exp \left\{ - \sum_{k=k_1-1}^{k_2} \int_a^b H(L_{k+1}(u) - L_k(u)) du \right\}
$$
with the convention $L_{k_1-1} = f$ and $L_{k_2+1} = g$. Here, the normalizing constant is given by

$$Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g} = \mathbb{P}_{\text{free}}^{k_1,k_2,(a,b)} \left[ W_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g} (L_{k_1}, \ldots, L_{k_2}) \right]$$

where the curves $(L_{k_1}, \ldots, L_{k_2})$ are distributed via $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$. Throughout this paper we will restrict our attention to one parameter family of Hamiltonians indexed by $T \geq 0$:

$$H_T(x) := e^{T^{1/3}x}.$$

A $\Sigma \times \Lambda$-indexed line ensemble $\mathcal{L}$ satisfies the $\mathbf{H}$-Brownian Gibbs property if for any subset $K = \{k_1, k_1 + 1, \ldots, k_2\} \subset \Sigma$ and $(a, b) \subset \Lambda$, one has the following distributional invariance

$$\text{Law} \left( \mathcal{L} \big|_{K \times (a,b)} \right) \text{ conditional on } \mathcal{L} \big|_{\Sigma \times \Lambda \setminus K \times (a,b)} = \mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$$

where $\vec{x} = (a_{k_1}, \ldots, a_{k_2})$, $\vec{y} = (b_{k_1}, \ldots, b_{k_2})$ and $f = L_{k_1-1}|_{(a,b)}$, $g = L_{k_2+1}$ with $f = -\infty$ if $k_1 - 1 \notin \Sigma$ and $g = +\infty$ if $k_2 + 1 \notin \Sigma$. This is a spatial Markov property — the ensemble in a given region has marginal distribution only dependent on the boundary-values of said region.

Denote the sigma field generated by the curves with indices outside $K \times (a, b)$ by $\mathcal{F}_{\text{ext}}(K \times (a, b))$. The random variable $(a, b)$ is a $K$-stopping domain if $\{a \leq b \geq b\} \in \mathcal{F}_{\text{ext}}(K \times (a, b))$. Let $C^K(a, b)$ be the set of continuous functions $(f_{k_1}, \ldots, f_{k_2})$ where $f_i : (a, b) \to \mathbb{R}$ and define

$$C^K := \left\{ (a, b, f_{k_1}, \ldots, f_{k_2}) : a < b \text{ and } (f_{k_1}, \ldots, f_{k_2}) \in C^K(a, b) \right\}.$$

Denote the set of all Borel measurable functions from $C^K$ to $\mathbb{R}$ by $\mathcal{B}(C^K)$. Then, a $K$-stopping domain $(a, b)$ is said to satisfy the strong $\mathbf{H}$-Brownian Gibbs property if for all $F \in \mathcal{B}(C^K)$, following holds $\mathbb{P}$-almost surely,

$$\mathbb{E} \left[ F(a, b, \mathcal{L} \big|_{K \times (a,b)}) \big| \mathcal{F}_{\text{ext}}(K \times (a, b)) \right] = \mathbb{E}_{\mathbf{H}}^{k_1,k_2,(\ell,r),\vec{x},\vec{y},f,g} \left[ F(\ell, r, L_{k_1}, \ldots, L_{k_2}) \right]$$

(2.2.1) where $\ell = a$, $r = b$, $\vec{x} = \{L_i(a)\}_{i=k_1}^{k_2}$, $\vec{y} = \{L_i(b)\}_{i=k_1}^{k_2}$, $f(\cdot) = L_{k_1-1}(\cdot)$ (or $+\infty$ if $k_1 - 1 \notin \Sigma$) and $g(\cdot) = L_{k_2+1}(\cdot)$ (or $-\infty$ if $k_2 + 1 \notin \Sigma$). On the l.h.s. of (2.2.1), $\mathcal{L} \big|_{K \times (a,b)}$ is the restriction of the $\mathbb{P}$-distributed curves and on the r.h.s. $L_{k_1}, \ldots, L_{k_2}$ is $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(\ell,r),\vec{x},\vec{y},f,g}$-distributed.
Remark 2.2.2. When \( k_1 = k_2 = 1 \) and \( (f, g) = (\pm \infty, -\infty) \) the measure \( P_{(k_1, k_2)}(a, b) \) is same as the measure of a free Brownian bridge started from \( \vec{x} \) and ended at \( \vec{y} \).

The following lemma demonstrates a sufficient condition under which the strong \( H \)-Brownian Gibbs property holds.

**Lemma 2.2.3** (Lemma 2.5 of [CH16]). Any line ensemble which enjoys the \( H \)-Brownian Gibbs property also enjoys the strong \( H \)-Brownian Gibbs property.

Line ensembles with the \( H \)-Brownian Gibbs property benefit from certain stochastic monotonicities.

**Definition 2.2.4** (Domination of measure). Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be two \((\Sigma \times \Lambda)\)-indexed line ensembles with respective laws \( P_1 \) and \( P_2 \). We say that \( P_1 \) dominates \( P_2 \) if there exists a coupling of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) such that \( L^j_1(x) \geq L^j_2(x) \) for all \( j \in \Sigma \) and \( x \in \Lambda \).

The next proposition relates the narrow wedge KPZ equation to the KPZ line ensemble.

**Proposition 2.2.5** (Theorem 2.15 of [CH16]). Fix any \( T > 0 \). Then there exists an \( \mathbb{N} \times \mathbb{R} \)-indexed line ensemble \( \mathcal{H}_T = \{ \mathcal{H}^n_T(x) \}_{n \in \mathbb{N}, x \in \mathbb{R}} \) satisfying the following properties:

1. The lowest indexed curve \( \mathcal{H}^1_T(X) \) is equal in distribution (as a process in \( X \)) the Cole-Hopf solution \( \mathcal{H}^{nw}(T, X) \) of KPZ started from the narrow wedge initial data.
2. \( \mathcal{H}_T \) satisfies the \( H_1 \)-Brownian Gibbs property (see Definition 2.2.1).
3. Define the scaled KPZ line ensemble \( \{ \Upsilon_T^{(n)}(x) \}_{n \in \mathbb{N}, x \in \mathbb{R}} \) as follows

\[
\Upsilon_T^{(n)}(x) := \frac{\mathcal{H}^n_T((2T)^{\frac{3}{2}}x) + T}{T^{\frac{1}{2}}}
\]

Then, \( \{ 2^{-\frac{1}{4}} \Upsilon_T^{(n)}(x) \}_{n \in \mathbb{N}, x \in \mathbb{R}} \) satisfies the \( H_{2T} \)-Brownian Gibbs property.

The following proposition is a monotonicity result which shows that two line ensembles with the same index set can be coupled in such a way that if the boundary conditions of one ensemble dominates the other, then likewise do the curves.

\[\text{Note, we do not require the full strength of the result proved in Theorem 2.15 of [CH16]. That result also proves uniform over } T \text{ of the local Brownian nature of the top curve } \Upsilon_T^{(n)}(x) \text{ as } x \text{ varies.}\]

\[\text{This pesky } 2^{-\frac{1}{4}} \text{ compensates for the fact that it is missing in the denominator of } \Upsilon_T^{(n)}(x).\]
**Proposition 2.2.6** (Lemmas 2.6 and 2.7 of [CH16]). Fix an interval $K = \{k_1, \ldots, k_2\} \subset \Sigma$ for some fixed positive integers $k_1 < k_2$, $(a, b) \subset \Lambda$ for $a < b$ and two pairs of vectors $\vec{x}_1, \vec{x}_2$ and $\vec{y}_1, \vec{y}_2$ in $\mathbb{R}^{k_2-k_1+1}$. Consider any two pairs of measurable functions $f, \bar{f} : (a, b) \to \mathbb{R} \cup \{-\infty\}$ and $g, \bar{g} : (a, b) \to \mathbb{R} \cup \{-\infty\}$ such that $\bar{f}(s) \leq f(s)$, $\bar{g}(s) \leq g(s)$ for all $s \in (a, b)$ and $x^{(k)}_2 \leq x^{(k)}_1$, $y^{(k)}_2 \leq y^{(k)}_1$ for all $k \in K$. Let $\mathcal{Q} = \{\mathcal{Q}^{(n)}(x)\}_{n \in K, x \in (a, b)}$ and $\mathcal{Q} = \{\mathcal{Q}^{(n)}(x)\}_{n \in K, x \in (a, b)}$ be two $K \times (a, b)$-indexed line ensembles in the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{\mathbb{P}})$ respectively such that $\mathbb{P}$ equals to $\mathbb{P}^{k_1, k_2, (a, b), \vec{x}_1, \vec{x}_2, f, g}_{\mathcal{H}_2}$ and $\hat{\mathbb{P}}$ equals to $\hat{\mathbb{P}}^{k_1, k_2, (a, b), \vec{x}_1, \vec{x}_2, \vec{y}_2, \bar{f}, \bar{g}}_{\mathcal{H}_2}$. If $H : [0, \infty) \to \mathbb{R}$ is convex, then, there is a coupling (i.e., a common probability space upon which both measures are supported) between $\mathbb{P}$ and $\hat{\mathbb{P}}$ such that $\mathcal{Q}^{(j)}(s) \leq \mathcal{Q}^{(j)}(s)$ for all $n \in K$.

Let us provide the basic idea behind how we use Lemma 2.2.6. Note that by $\mathcal{H}$-Brownian Gibbs property the lowest indexed curve $2^{-\frac{1}{4}} Y_T^{(1)}(\cdot)$ of the $\mathbb{N}$-indexed KPZ line ensemble $\{2^{-\frac{1}{4}} Y_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$, when restricted to the interval $(a, b)$, has the conditional measure $\mathbb{P}^{1, 1, (a, b), 2-\frac{1}{4} Y_T^{(1)}(a), 2-\frac{1}{4} Y_T^{(1)}(b), +\infty, 2-\frac{1}{4} Y_T^{(2)}}_{\mathcal{H}_2}$. On the other hand, replacing $2^{-\frac{1}{4}} Y_T^{(2)}$ by $-\infty$, $\mathbb{P}^{1, 1, (a, b), 2-\frac{1}{4} Y_T^{(1)}(a), 2-\frac{1}{4} Y_T^{(1)}(b), +\infty, -\infty}_{\mathcal{H}_2}$ is the probability measure of a Brownian bridge on the interval $(a, b)$ with the entrance and exit data $2^{-\frac{1}{4}} Y_T^{(1)}(a)$ and $2^{-\frac{1}{4}} Y_T^{(1)}(b)$ respectively. Lemma 2.2.6 constructs a coupling between these two measures on the curve $2^{-\frac{1}{4}} Y_T^{(1)}(\cdot)_{(a, b)}$ such that

$$
\mathbb{P}^{1, 1, (a, b), 2-\frac{1}{4} Y_T^{(1)}(a), 2-\frac{1}{4} Y_T^{(1)}(b), +\infty, 2-\frac{1}{4} Y_T^{(2)}}_{\mathcal{H}_2} (A) \leq \mathbb{P}^{1, 1, (a, b), 2-\frac{1}{4} Y_T^{(1)}(a), 2-\frac{1}{4} Y_T^{(1)}(b), +\infty, -\infty}_{\mathcal{H}_2} (A)
$$

(2.2.2)

for any event $A$ whose chance increases under the pointwise decrease of $Y_T^{(1)}$.

In most of our applications of this idea, it is easy to find upper bounds on the r.h.s. of (2.2.2) using Brownian bridge calculations. Via (2.2.2), those bounds transfers to the spatial process $Y_T^{(1)}(\cdot)$. Since, by Proposition 2.2.5 this curve is equal in law to $\Upsilon_T(\cdot)$ (the scaled and centered narrow wedge KPZ equation solution), these bounds in conjunction with the convolution formula of Proposition 2.1.6 embodies the core of our techniques to generalize the tail bounds from narrow wedge to general initial data. The following lemma is used in controlling the probabilities which arise on r.h.s. of (2.2.2).

**Lemma 2.2.7.** Let $B(\cdot)$ be a Brownian bridge on $[0, L]$ with $B(0) = x$ and $B(L) = y$. Then,

$$
\mathbb{P}\left( \inf_{t \in [0, L]} B(t) \leq \min\{x, y\} - s \right) \leq e^{-\frac{2s^2}{L}}.
$$

(2.2.3)

\footnote{If increase is replaced by decrease, then, the inequality (2.2.2) is reversed.}

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**Proof.** Due to symmetry, we may assume $\min\{x, y\} = y$. Note that $\tau = \min\{t \in [0, 1] : B(t) \leq y\}$ is a stopping time for the natural filtration of $B(\cdot)$. Thanks to the resampling invariance property of the Brownian bridge measure, $\{B(t)\}_{t \in [\tau, L]}$ conditioned on the sample paths outside the interval $(\tau, L)$ is again distributed as a Brownian bridge with $B(\tau) = B(L) = y$. Now, applying [KS91, (3.40)] (see also Lemma 2.11 of [CH16]), we get

$$
\mathbb{P}\left( \inf_{t \in [\tau, L]} B(t) \leq \min\{x, y\} - s \middle| \mathcal{F}([0, \tau]) \right) = e^{-\frac{s^2}{2}}. \tag{2.2.4}
$$

Here, $\mathcal{F}([0, \tau])$ denotes the natural filtration of $\{B\}_{t \in [0, L]}$ stopped at time $\tau$. Taking expectation of (2.2.4) with respect to the $\sigma$-algebra $\mathcal{F}_\tau$ and noting $e^{-\frac{s^2}{2}} \leq e^{-\frac{s^2}{L}}$ yields (2.2.3).

It is worth noting that Proposition 4.3.5.3 of [MMFM] contains an exact formulas for the left hand side of (2.2.3). The next result (which follows from [GT11, (3.14)]) is used in Theorem 2.1.12.

**Lemma 2.2.8.** Let $B(\cdot)$ be a two-sided standard Brownian motion with $B(0) = 0$. Then, for any given $\xi \in (0, 1)$, there exists $s_0 = s_0(\xi)$ such that for all $c > 0$ and $s \geq s_0$,

$$
\mathbb{P}\left( B(t) \geq s + ct^2 \text{ for some } t \in \mathbb{R} \right) \leq \frac{1}{\sqrt{3}} e^{-\frac{(1-\xi)\sqrt{s}}{\sqrt{3}c}}. \tag{2.3.1}
$$

### 2.3 Lower tail under general initial data

In this section, we prove Theorems 2.1.2 and 2.1.11. Starting with the tail bounds of Theorem 1.1.1, we first bound the lower tail probabilities of the narrow wedge solution at a countable set of points of $\mathbb{R}$ (see Lemma 2.3.1). Combining this with the Brownian Gibbs property of the narrow wedge solution and the growth conditions of initial data (given in Definition 2.1.1), we prove the lower tail bound of Theorem 2.1.2 in Section 2.3.1 via the convolution formula of Proposition 2.1.6. By controlling the fluctuations of a two sided Brownian motion in small intervals, we prove the lower tail bound of Theorem 2.1.11 (see Section 2.3.2) in a similar way.
2.3.1 Proof of Theorem 2.1.2

Recall that the initial data $\mathcal{H}_0$ is defined from $f$ via (2.1.2). Also recall the definition of $\Upsilon_T(\cdot)$ from (1.1.2). Fix the sequence $\{\zeta_n\}_{n \in \mathbb{Z}}$ where $\zeta_n := \frac{n}{\sqrt{4\alpha T}}$. Let us define the following events

\[
\begin{align*}
A^T := & \left\{ \int_{-\infty}^{\infty} e^{T^{1/3} (\Upsilon_T(y) + f(-y))} dy \leq e^{-T^{1/3} s} \right\}, \\
E_n := & \left\{ \Upsilon_T(\zeta_n) \leq - \frac{(1 + 2 - \nu)\zeta_n^2}{2^{2/3}} - (1 - \epsilon)s \right\}, \\
F_n := & \left\{ \Upsilon_T(y) \leq - \frac{(1 + \nu)y^2}{2^{2/3}} - \left(1 - \frac{\epsilon}{2}\right)s \text{ for some } y \in (\zeta_n, \zeta_{n+1}) \right\}.
\end{align*}
\]

Here, we suppress the dependence on the various variables. By (2.1.7) of Proposition 2.1.6 \( P(h_T^f(0) \leq -s) = P(A^T) \) which we need to bound. To begin to bound this, note that

\[
P(A^T) \leq P \left( \bigcup_{n \in \mathbb{Z}} E_n \right) + P \left( A^T \cap \left( \bigcup_{n \in \mathbb{Z}} E_n \right)^c \right) \leq \sum_{n \in \mathbb{Z}} P(E_n) + P \left( A^T \cap \left( \bigcup_{n \in \mathbb{Z}} E_n \right)^c \right). \tag{2.3.1}
\]

We focus on bounding separately the two terms on the right side of (2.3.1).

**Lemma 2.3.1.** There exist $s_0 = s_0(\epsilon, \delta, C, \nu, T_0)$ and $K_* = K_*(\epsilon, \delta, T_0) > 0$ such that for all $T \geq T_0$ and $s \geq s_0$,

\[
\sum_{n = -\infty}^{\infty} P(E_n) \leq e^{-T^{1/3} \left( 4(1 - \epsilon)^{5/2} \frac{s^2}{15\pi} \right)} + e^{-K_* s^{3-\delta} - \epsilon s T^{1/3}} + e^{-\frac{(1 - \epsilon)s^3}{12}}. \tag{2.3.2}
\]

**Proof.** Recall that the one point distribution of $\Upsilon_T(y) + \frac{y^2}{2^{2/3}}$ is independent of $y$ (see Proposition 2.1.7). Setting $s_n := (1 - \epsilon)s + \frac{\nu n^2}{2^{2/3}}$ and invoking Propositions 2.1.7 and Theorem 1.1.1, we write

\[
P(E_n) = P(\Upsilon_T(0) \leq -s_n) \leq e^{-T^{1/3} (1 - \epsilon)^{\frac{5}{2}} \frac{n^2}{15\pi} \frac{s^2}{15\pi}} + e^{-K_* s^{3-\delta} - \epsilon s T^{1/3}} + e^{-\frac{(1 - \epsilon)s^3}{12}}. \tag{2.3.3}
\]

Applying the reverse Minkowski inequality, we get $s_n^\alpha \geq \left((1 - \epsilon)s\right)^\alpha + (\nu n^2/2^{2/3} s^2)^\alpha$ for all $\alpha \geq 1$.

Plugging this into (2.3.3) and summing over all $n \in \mathbb{Z}$, we get

\[
\sum_{n \in \mathbb{Z}} P(E_n) \leq e^{-T^{1/3} (1 - \epsilon)^{\frac{5}{2}} \frac{n^2}{15\pi}} \sum_{n \in \mathbb{Z}} e^{-K_* s^{3-\delta} - \epsilon s T^{1/3}} + e^{-\frac{(1 - \epsilon)s^3}{12}} \sum_{n \in \mathbb{Z}} e^{-K_* n^6 \frac{s^3}{2^{2/3} 12}} \\
+ e^{-K_* s^{3-\delta} - \epsilon s T^{1/3}} \sum_{n \in \mathbb{Z}} e^{-K_* \frac{n^3 (3 - \delta)}{2^{2/3} (3 - \delta)}} \cdot \epsilon \frac{\nu n^2 s^2}{2^{2/3}} T^{1/3}. \tag{2.3.4}
\]
for three positive constants $K_1$, $K_2$ and $K_3$. By a direct computation, we observe

$$\sum_{n \in \mathbb{Z}} e^{-T^{1/3} K_1 s^{-5} |n|^5} \leq K'_1 T^{-7/4} s^5, \quad \sum_{n \in \mathbb{Z}} e^{-K_2 s^{6/3}} \leq K'_2 s^6, \quad (2.3.5)$$

$$\sum_{n \in \mathbb{Z}} e^{-K_3 s^{2(3-\delta)} s^2 T^{1/4}} \leq K'_3 \left(s^{3(2-\delta)} + s^2 T^{-1/4}\right). \quad (2.3.6)$$

Combining (2.3.5) and (2.3.6) with (2.3.4) yields (2.3.2). \[\square\]

Now it suffices to control the second term on the right side of (2.3.1). We start by showing:

**Lemma 2.3.2.** Under the assumption that $f$ belongs to the class $\text{Hyp}(C, \nu, \theta, \kappa, M)$, there exists $s_1 = s_1(C, \nu, \theta, \kappa, M)$ such that for all $s \geq s_1$,

$$\bigcap_{n \in \mathbb{Z}} \{E_n^c \cap F_{n+1}^c\} \subset (A^f)^c. \quad (2.3.7)$$

**Proof.** Assume the events on the l.h.s. of (2.3.7) occur. Appealing to (2.1.2), we observe

$$\int_{-\infty}^{\infty} e^{-T^{1/3} (\Upsilon_T(y)+f(y))} dy \geq \int_{I} e^{-T^{1/3} \left(\frac{(1+\nu/2)}{2} y^2 + (1-\frac{1}{2})s-\kappa\right)} dy \geq \theta e^{-T^{1/3} \left(\frac{1+\nu/2}{2} M^2 + \kappa - \frac{\epsilon^2}{2}\right)} e^{-T^{1/4} s}. $$

Clearly, there exists $s_1 = s_1(C, \nu, \theta, \kappa, M)$ such that the right side above is bounded below by $e^{-T^{1/4} s}$ for all $s \geq s_1$. This shows the claimed containment of the events in (2.3.7). \[\square\]

Owing to (2.3.7) and then, Bonferroni’s union bound,

$$\mathbb{P}\left(A^f \cap \left(\bigcup_{n \in \mathbb{Z}} E_n^c\right)^c\right) = \mathbb{P}\left(A^f \cap \left(\bigcap_{n \in \mathbb{Z}} E_n^c\right) \cap \left(\bigcup_{n \in \mathbb{Z}} F_n\right)\right) \leq \sum_{n \in \mathbb{Z}} \mathbb{P}\left(E_n^c \cap E_{n+1}^c \cap F_n\right). \quad (2.3.8)$$

We obtain an upper bound of the r.h.s. of (2.3.8) in the following lemma.

**Lemma 2.3.3.** There exists $s_2 = s_2(\epsilon) > 0$ such that for all $s \geq s_2$

$$\sum_{n \in \mathbb{Z}} \mathbb{P}\left(E_n^c \cap E_{n+1}^c \cap F_n\right) \leq e^{-s^{3+\delta}}. \quad (2.3.9)$$

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Combining (2.3.8) with (2.3.9) of Lemma 2.3.3 yields

$$\mathbb{P}\left(A^c \cap \left( \bigcup_{n \in \mathbb{Z}} E_n \right)^c \right) \leq e^{-s^{3+\delta}}$$  \hspace{1cm} (2.3.10)

for some $\delta > 0$. Plugging the bounds (2.3.2) and (2.3.10) into the r.h.s. of (2.3.1) yields (2.1.5). To complete the proof of Theorem 2.1.2 it only remains to prove Lemma 2.3.3 which we show below.

**Proof of Lemma 2.3.3** We aim to bound $\mathbb{P}(E_n^c \cap E_{n+1}^c \cap F_n)$. By Proposition 2.2.5 $Y_T$ equals in law the curve $Y_T^{(1)}$ of the scaled KPZ line ensemble $\{2^{-\frac{1}{2}} Y_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$. Hence, without loss of generality, we replace $Y_T$ by $Y_T^{(1)}$ in the definitions of $E_n$ and $F_n$ for the rest of this proof. By the $H_{2T}$-Brownian Gibbs property of $\{2^{-\frac{1}{2}} Y_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$,

$$\mathbb{P}(E_n^c \cap E_{n+1}^c \cap F_n) = \mathbb{E}\left[1 (E_n^c \cap E_{n+1}^c) \cdot \mathbb{E}\left[1 (F_n) \mathbf{F}_{\text{ext}}(\{1\}, (\zeta_n, \zeta_{n+1}))\right]\right]$$

$$= \mathbb{E}\left[1 (E_n^c \cap E_{n+1}^c) \cdot \mathbb{P}_{\mathbf{H}_{2T}}^{1.1, (\zeta_n, \zeta_{n+1}), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_n), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_{n+1}), +\infty, 2^{-\frac{1}{3}} Y_T^{(2)}(F_n)}\right].$$

Recall $\mathbf{F}_{\text{ext}}(\{1\}, (\zeta_n, \zeta_{n+1}))$ is the $\sigma$-algebra generated by $\{Y_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$ outside the set $\{Y_T^{(1)}(x) : x \in (\zeta_n, \zeta_{n+1})\}$. Via Proposition 2.2.6 there exists a monotone coupling between the probability measure $\mathbb{P}_{\mathbf{H}_{2T}} := \mathbb{P}_{\mathbf{H}_{2T}}^{1.1, (\zeta_n, \zeta_{n+1}), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_n), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_{n+1}), +\infty, 2^{-\frac{1}{3}} Y_T^{(2)}}$ and another measure $\mathbb{P}_{\mathbf{H}_{2T}} := \mathbb{P}_{\mathbf{H}_{2T}}^{1.1, (\zeta_n, \zeta_{n+1}), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_n), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_{n+1})} \mathbf{P}_{\text{free}}^{1.1, (\zeta_n, \zeta_{n+1}), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_n), 2^{-\frac{1}{3}} Y_T^{(1)}(\zeta_{n+1})}$ such that

$$\mathbb{P}_{\mathbf{H}_{2T}}(F_n) \leq \mathbb{P}_{\mathbf{H}_{2T}}(F_n).$$  \hspace{1cm} (2.3.11)

The r.h.s. of (2.3.11) is a probability with respect a Brownian bridge measure. For the rest of the proof, we use shorthand notation $\theta_n := (1 - \epsilon)s + 2^{-\frac{1}{3}} (1 + 2^{-1} \nu) \zeta_n^2$ for $n \in \mathbb{Z}$. The probability of the event $F_n$ increases under the pointwise decrease of the end points of $Y_T^{(1)}$. Using $\{E_n^c \cap E_{n+1}^c\} = \{Y_T^{(1)}(\zeta_n) \geq -\theta_n\} \cap \{Y_T^{(1)}(\zeta_{n+1}) \geq -\theta_{n+1}\}$ and Proposition 2.2.5,

$$1 (E_n^c \cap E_{n+1}^c) \times \mathbb{P}_{\mathbf{H}_{2T}}(F_n) \leq \mathbb{P}_{\text{free}}^{1.1, (\zeta_n, \zeta_{n+1}), -2^{-\frac{1}{3}} \theta_n, -2^{-\frac{1}{3}} \theta_{n+1}}(F_n).$$  \hspace{1cm} (2.3.12)
Combining (2.3.11) and (2.3.12) yields

\[
1 (\mathcal{E}_n^c \cap \mathcal{E}_{n+1}^c) \times \mathbb{P}_{H_2T}^{1.1, (\zeta_n, \zeta_{n+1}), 2^{-\frac{1}{3}} \frac{V^{(1)}(\zeta_n)}{V^{(1)}(\zeta_{n+1})}, +\infty, 2^{-\frac{1}{3}} \frac{V^{(2)}}{V^{(1)}}} (F_n)
\]

\[
\leq \mathbb{P} \left( \min_{x \in [\zeta_n, \zeta_{n+1}]} B(t) \leq 2^{-\frac{1}{3}} \{ \theta_n \land \theta_{n+1} \} - \frac{\epsilon s}{24} - \frac{\nu \zeta_n^2}{4} \right) \tag{2.3.13}
\]

where \( B(\cdot) \) is a Brownian bridge such that \( B(\zeta_n) = -2^{-\frac{1}{3}} \theta_n \) and \( B(\zeta_{n+1}) = -2^{-\frac{1}{3}} \theta_{n+1} \). Applying Lemma 2.2.7 yields r.h.s. of (2.3.13) \( \leq e^{-2^{1/3} s^{1/3} \left( \frac{16}{27} + \frac{\nu \zeta_n^2}{4} \right)^2} \). Combining this upper bound with (2.3.13) and taking the expectations, we arrive at

\[
\mathbb{P} (\mathcal{E}_n^c \cap \mathcal{E}_{n+1}^c \cap F_n) \leq e^{-2^{1/3} s^{1/3} \left( \frac{16}{27} + \frac{\nu n^2}{4} (1 + \delta) \right)^2}. \tag{2.3.14}
\]

Summing both side of (2.3.14) over \( n \in \mathbb{Z} \), we obtain (2.3.9).

\[\square\]

2.3.2 Proof of Theorem 2.1.11

This proof is similar to that of Theorem 2.1.2. We use the same notations \( \zeta_n, E_n \) and \( F_n \) introduced in the beginning of the proof of Theorem 2.1.2 and additionally define

\[
\mathcal{A}^B = \left\{ \int_{-\infty}^{\infty} e^{T^{1/3} (Y_T(y) + B(-y))} dy \leq e^{-T^{1/3} s} \right\}
\]

where \( B \) is a two sided Brownian motion with diffusion coefficient \( 2^{1/3} \) and \( B(0) = 0 \). In particular, \( B(y) \overset{d}{=} \tilde{B}(2^{2/3} y) \) where \( \tilde{B}(\cdot) \) is standard two sided Brownian motion. Owing to (2.1.7), \( \mathbb{P}(h^B(0) \leq -s) = \mathbb{P}(B^B) \) which we need to bound. As in (2.3.1), we write

\[
\mathbb{P}(\mathcal{A}^B) \leq \sum_{n \in \mathbb{Z}} \mathbb{P}(E_n) + \mathbb{P}(\mathcal{A}^B \cap \left( \bigcap_{n \in \mathbb{Z}} E_n \right)^c). \tag{2.3.15}
\]

We can use (2.3.2) of Lemma 2.3.1 to bound \( \sum_n \mathbb{P}(E_n) \). While the conclusion of Lemma 2.3.2 does not hold in the present case, we will show that it does hold with high probability.

Lemma 2.3.4. There exist \( s_1 = s_1(\epsilon, \delta) \), \( c_1 = c_1(\epsilon) \), \( c_2 = c_2(\epsilon) > 0 \) such that for all \( s \geq s_1 \),

\[
\mathbb{P} \left( \bigcap_{n \in \mathbb{Z}} \left( E_n^c \cap F_n^c \right) \cap \mathcal{A}^B \right) \leq c_1 e^{-c_2 s^{3/2} \delta}. \tag{2.3.16}
\]
Combining (2.3.9) of Lemma 2.3.3 and (2.3.16) of Lemma 2.3.4 yields
\[ \mathbb{P}\left( A^B \cap \left( \bigcup_{n \in \mathbb{Z}} E_n \right)^c \right) \leq c_2 e^{-c_1 s^{3+\delta}}. \] (2.3.17)

Applying (2.3.17) and (2.3.2) to (2.3.15), we obtain (2.1.11). To complete the proof of Theorem 2.1.11, we now need to prove Lemma 2.3.4 which is given as follows.

**Proof of Lemma 2.3.4**  Observe first that
\[ \bigcap_{n \in \mathbb{Z}} \{ E_n^c \cap F_n^c \} \cap \mathcal{A}^{Br} \subseteq \left\{ \int_{-\infty}^{\infty} e^{-T^{1/3} \left( \frac{(1+\nu)y^2}{2/3} - \frac{1}{2} B(y) \right)} dy \leq 1 \right\}. \] (2.3.18)

Note that if \( B(y) \geq -\frac{\epsilon}{2} s \) for all \( y \in [-1/s^{1+\delta}, 1/s^{1+\delta}] \), then, \( \frac{(1+\nu)y^2}{2/3} - \frac{1}{2} B(y) \leq -\frac{\epsilon}{8} s \) for all \( y \in [-1/s^{1+\delta}, 1/s^{1+\delta}] \) which implies
\[ \int_{-\infty}^{\infty} e^{-T^{1/3} \left( \frac{(1+\nu)y^2}{2/3} - \frac{1}{2} B(y) \right)} dy \geq \frac{2}{s^{1+\delta}} e^{\frac{\epsilon}{8} s T^{1/3}} > 1 \]
when \( s \) is large. Hence, there exists \( s_1 = s_1(\epsilon, \delta) \) such that for all \( s \geq s_1 \), one has
\[ \left\{ \int_{-\infty}^{\infty} e^{-T^{1/3} \left( \frac{(1+\nu)y^2}{2/3} - \frac{1}{2} B(y) \right)} dy \leq 1 \right\} \subseteq \left\{ \min_{y \in [-1/s^{1+\delta}, 1/s^{1+\delta}]} B(y) < -\frac{\epsilon}{4} s \right\}.

Thanks to this containment, we get
\[ \mathbb{P}\left( \bigcap_{n \in \mathbb{Z}} \{ E_n^c \cap F_n^c \} \cap \mathcal{A}^{Br} \right) \leq \mathbb{P}\left( \min_{y \in [-1/s^{1+\delta}, 1/s^{1+\delta}]} B(y) < -\frac{\epsilon}{4} s \right). \] (2.3.19)

We bound the r.h.s. of (2.3.19), via the reflection principle as
\[ \mathbb{P}\left( \min_{y \in [-1/s^{1+\delta}, 1/s^{1+\delta}]} B(y) \leq -\frac{\epsilon}{4} s \right) \leq \mathbb{P}\left( 2|X_1| + 2|X_2| \geq \frac{\epsilon}{4} s \right) \] (2.3.20)
where \( X_1, X_2 \) are independent Gaussians with variance \( 2^{1/3} s^{-(1+\delta)} \). By tail estimates, it follows that the r.h.s. of (2.3.20) is bounded above by \( c_1 e^{-c_2 s^{3+\delta}} \) for some constants \( c_1, c_2 > 0 \) which only depend on \( \epsilon \). Plugging this into (2.3.19) and combining with (2.3.18), we find (2.3.16).
2.4 Upper Tail under narrow wedge initial data

The aim of this section is to prove Theorem 2.1.9. To achieve this, we first state a few auxiliary results which combine together to prove Theorem 2.1.9. These auxiliary results are proved in the end of Section 2.4. Recall the definition of $\Upsilon_T$ from (1.1.2). Our first result of this section (Proposition 2.4.1) gives an upper and lower bound for the probability $P(\Upsilon_T(0) \geq s)$. These bounds are close to optimal when $s \gg T^{2/3}$. When $s = O(T^{2/3})$ or $s \ll T^{2/3}$, those bounds are not optimal (see Remark 2.4.2). In those cases, we obtain better bounds using Proposition 2.4.3.

**Proposition 2.4.1.** Fix some $\zeta \leq \epsilon \in (0, 1)$ and $T_0 > 0$. There exists $s_0 = s_0(\epsilon, \zeta, T_0)$ such that for all $s \geq s_0$ and $T \geq T_0$,

\[
P(\Upsilon_T(0) > s) \leq e^{-T^{1/3}\zeta s} + e^{-\frac{4}{3}(1-\epsilon)s^{3/2}},
\]

(2.4.1)

\[1 - \exp\left(-e^{-\zeta s T^{1/3}}\right)P(\Upsilon_T(0) \leq s) \geq e^{-T^{1/3}(1+\zeta)s} + e^{-\frac{4}{3}(1+\epsilon)s^{3/2}}.
\]

(2.4.2)

**Remark 2.4.2.** Proposition 2.4.1 implies that for $s \ll T^{2/3}$

\[
\exp\left(-\frac{4}{3}(1+\epsilon)s^{3/2}\right) \leq P(\Upsilon_T(0) > s) \leq \exp\left(-\frac{4}{3}(1-\epsilon)s^{3/2}\right).
\]

To see this, we first note that

\[
\text{r.h.s. of (2.4.1)} \leq \exp\left(-\frac{4}{3}(1-\epsilon)s^{3/2}\right), \quad \text{when } s \ll T^{2/3}.
\]

Using the approximation $1 - \exp\left(-e^{-\zeta s T^{1/3}}\right) \approx \exp(-\zeta s T^{1/3})$, we see that (2.4.2) implies

\[
P(\Upsilon_T(0) > s) \geq \exp\left(e^{-\zeta s T^{1/3}}\right)\left(e^{-(1+\zeta)s T^{1/3}} - e^{-\zeta s T^{1/3}} + e^{-\frac{4}{3}(1+\epsilon)s^{3/2}}\right).
\]

(2.4.3)

The r.h.s. of (2.4.3) is bounded below by $\exp\left(-\frac{4}{3}(1+\epsilon)s^{3/2}\right)$ when $s \ll T^{2/3}$. Note, when $s \gg T^{2/3}$, the dominating term of the r.h.s. of (2.4.1) is $\exp(-\zeta s T^{1/3})$ which we show in our next theorem is the not correct order of decay of $P(\Upsilon_T(0) > s)$.
**Proposition 2.4.3.** Fix $\epsilon \in (0, 1)$. Then, for all pairs $(s, T)$ satisfying $s \geq \frac{9}{16} e^{-2T \frac{3}{2}}$ and $T > \pi$,

\[
\mathbb{P}(\Upsilon_T(0) > s) \leq e^{-\frac{4(1-\epsilon)}{3}s^{3/2}} \quad (2.4.4)
\]

\[
\mathbb{P}(\Upsilon_T(0) > s) \geq e^{-4\sqrt{3}(1+3\epsilon)s^{3/2}} \quad (2.4.5)
\]

Furthermore, for all $s \in \left[\frac{1}{8}e^{2T^{\frac{3}{2}}}, \frac{9}{16} e^{-2T^{\frac{3}{2}}}\right],$

\[
\mathbb{P}(\Upsilon_T(0) > s) \geq \frac{1}{2} e^{-27/2e^{-3s^{3/2}}} \quad (2.4.6)
\]

Moreover, for any $0 < T_0 \leq \pi$ and $\epsilon \in (0, 3/5)$, there exist $c_1 = c_1(T_0) > c_2 = c_2(T_0) > 0$ such that for all $T \in [T_0, \pi]$ and $s \geq \frac{9}{16} e^{-2T^{\frac{3}{2}}} + 24T_0^{-\frac{1}{3}}(1-\epsilon)^{-1}\log(T_0/\pi)$,

\[
e^{-c_1s^{3/2}} \leq \mathbb{P}(\Upsilon_T(0) > s) \leq e^{-c_2s^{3/2}} \quad (2.4.7)
\]

**Proposition 2.4.4.** Fix $\epsilon \in (0, 1)$, $T > \pi$ and $c > \frac{4}{9}(1 + \frac{1}{3}\epsilon)$. Then, there exists $\{s_n\}_n$ such that $s_n \to \infty$ as $n \to \infty$ and $\mathbb{P}(\Upsilon_T(0) > s_n) \geq e^{-cs_n^{3/2}}$ for all $n \in \mathbb{N}$.

**2.4.1 Proof of Theorem 2.1.9**

We first show (2.1.9) when $T_0 \in (0, \pi)$. Fix $\epsilon \in (0, \frac{3}{4})$ and define $s_0 = \frac{9}{16} e^{-2T_0^{\frac{3}{2}}} + 3(1-\epsilon)^{-1}T_0^{-\frac{1}{3}}|\log T_0|$. Then, for all $T \in [T_0, \pi]$ and $s \geq s_0$, (2.1.9) follows from (2.4.7).

Now, we show (2.1.9) for $T_0 > \pi$. Fix $\zeta = \epsilon \in (0, \frac{1}{2})$. Proposition 2.4.1 says that there exists $s_0 = s_0(\epsilon, T_0)$ such that (2.4.4) and (2.4.5) holds for all $s \geq s_0$ and $T > T_0$.

(i) For all $s \in (0, \frac{1}{8}e^{2T_0^{\frac{3}{2}}})$, we note

\[
\frac{4}{3}(1+\epsilon)s^{\frac{3}{2}} \leq 2s^{\frac{3}{2}} \leq \frac{1}{\sqrt{2}}\epsilon sT^{\frac{1}{2}} \quad (2.4.8)
\]

where the first and second inequalities follow from $\epsilon \leq \frac{1}{2}$ and $s \leq \frac{1}{8}e^{2T_0^{\frac{3}{2}}}$ respectively. Furthermore, there exists $s'_0 = s'_0(\epsilon, T_0)$ such that for all $s \geq s'_0$, one has

\[
\exp\left(-\frac{1}{\sqrt{2}}\epsilon sT^{\frac{1}{2}}\right) \geq 2\exp\left(-\epsilon sT^{\frac{1}{2}}\right). \quad (2.4.9)
\]
Combining (2.4.8) and (2.4.9) yields
\[ \exp\left(-\frac{4}{3}(1 + \epsilon)s^\frac{3}{2}\right) \geq 2 \exp(-\epsilon s T^\frac{1}{3}), \quad \forall s \in (s'_0, \frac{1}{8}\epsilon^2 T^\frac{2}{3}). \] (2.4.10)

Plugging this into the r.h.s. of (2.4.1) yields
\[ P(\Upsilon_T(0) > s) \leq 2 \exp \left( -\frac{4}{3}(1 - \epsilon)s^\frac{3}{2} \right) \] (2.4.11)

for all \( s \in (\max\{s_0, s'_0\}, \frac{1}{8}\epsilon^2 T^\frac{2}{3}) \) where \( s_0 = s_0(\epsilon, T_0) \) comes with Proposition 2.4.1. Moreover, applying (2.4.10) in (2.4.3), we observe
\[ P(\Upsilon_T(0) > s) \leq \frac{1}{2} \exp \left( -\frac{4}{3}(1 + \epsilon)s^3/2 \right). \] (2.4.12)

Combining (2.4.11) and (2.4.12), we obtain (2.1.9) with \( c_1 \leq \frac{4}{3}(1 + \epsilon) \) and \( c_2 \geq \frac{4}{3}(1 - \epsilon) \) for all \( s \in (s'_0, \frac{1}{8}\epsilon^2 T^\frac{2}{3}) \) for some \( s''_0 = s''_0(\epsilon, T_0) \).

(ii) When \( s \geq \frac{n}{16}\epsilon^{-2} T^\frac{2}{3} \), we first apply Proposition 2.4.3. Using (2.4.4) and (2.4.5), yields (2.1.9) with \( c_1 \leq 4\sqrt{3}(1 + \epsilon) \) and \( c_2 \geq \frac{4}{3}(1 - \epsilon) \). The second part of the claim follows from Proposition 2.4.4.

(iii) For all \( s \in (\frac{1}{8}\epsilon^2 T^\frac{2}{3}, \frac{n}{16}\epsilon^{-2} T^\frac{2}{3}) \), appealing to (2.4.6) of Lemma 2.4.3, we get \( c_1 \leq 2\sqrt{2}\epsilon^{-3} \). Furthermore, one has the following bound on the r.h.s. of (2.4.1)
\[ \exp\left(-\epsilon s T^\frac{1}{3}\right) + \exp\left(-\frac{4}{3}(1 - \epsilon)s^\frac{3}{2}\right) \leq 2 \exp\left(-\min\{\epsilon s T^\frac{1}{3}, \frac{4}{3}(1 - \epsilon)s^\frac{3}{2}\}\right). \] (2.4.13)

For all \( \epsilon \leq \frac{1}{2} \) and \( s \in (\frac{1}{8}\epsilon^2 T^\frac{2}{3}, \frac{n}{16}\epsilon^{-2} T^\frac{2}{3}) \), the r.h.s. of (2.4.13) is bounded above by \( \exp(-\frac{4}{3}\epsilon s^\frac{3}{2}) \).

Plugging this bound into (2.4.5), we get
\[ P(\Upsilon_T(0) > s) \leq 2e^{-\frac{4}{3}\epsilon s^3/2}, \quad \forall s \in \left( \max\{s_0, \frac{1}{8}\epsilon^2 T^\frac{2}{3}\}, \max\{s_0, \frac{9}{16}\epsilon^{-2} T^\frac{2}{3}\} \right). \]

Therefore, (2.1.9) holds when \( s \) lies in the interval \( (\max\{s_0, \frac{1}{8}\epsilon^2 T^\frac{2}{3}\}, \max\{s_0, \frac{9}{16}\epsilon^{-2} T^\frac{2}{3}\}) \) with \( c_1 \leq 2\sqrt{2}\epsilon^{-3} \) and \( c_2 \geq \frac{4}{3}\epsilon \). This completes the proof of Theorem 2.1.9.
2.4.2 Proof of Proposition 2.4.3

To prove Proposition 2.4.3 we need the following lemma. Let

$$\psi_T(k) = \begin{cases} \frac{k! T_k^3}{2\sqrt{T} k^{3/2}} & \text{when } T \geq \pi \\ \frac{\pi(k-1/2)k! T_k^3}{2T^{k/2}k^{3/2}} & \text{when } T < \pi. \end{cases}$$

**Lemma 2.4.5.** Fix $k \in \mathbb{N}$ and $T_0 \in \mathbb{R}_+$. Then, we have

$$C \psi_T(k) \leq \mathbb{E} \left[ \exp \left( k T_T^4 \left(2.4.15\right) \right) \right] \leq 69 \psi_T(k) \tag{2.4.14}$$

where $C = C(k, T_0) > 0$ is bounded below by 1 for all $T > T_0 > \pi$ and by $T_0^{(k-1)/2} \pi^{-k/2}$ for all $T \in [T_0, \pi]$.

**Proof.** Recall that $Z(2T, 0) = \exp(T_T^4 \left(2.4.15\right) - \frac{T}{T_T^4})$. The moments of $Z(2T, 0)$ are given by\(^6\)

$$\mathbb{E} \left[ \exp \left( k T_T^4 \left(2.4.15\right) \right) \right] = \sum_{\lambda \vdash k} \frac{1}{m_1! m_2! \ldots} \int_{-\infty}^{\infty} \frac{dw_1}{2\pi i} \ldots \int_{-\infty}^{\infty} \frac{dw_{\ell(\lambda)}}{2\pi i} \det \left[ \frac{1}{w_j + \lambda_j - w_i} \right]_{i,j = 1}^{\ell(\lambda)} \times \exp \left[ T \sum_{j=1}^{\ell(\lambda)} \left( \frac{\lambda_j^3}{12} + \lambda_j \left( w_j + \frac{\lambda_j}{2} - 1 \right)^2 \right) \right]. \tag{2.4.15}$$

Here, $\lambda \vdash k$ denotes that $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ partitions $k$, $\ell(\lambda) = \# \{ i : \lambda_i > 0 \}$ and $m_j = \# \{ i : \lambda_i = j \}$. By Cauchy’s determinant formula,

$$\det \left[ \frac{1}{w_i + \lambda_i - w_j} \right] = \prod_{i=1}^{\ell(\lambda)} \prod_{i<j} \frac{\ell(\lambda)}{\lambda_i} \frac{(w_i - w_j + \lambda_i - \lambda_j)(w_j - w_i)}{(w_i + \lambda_j - w_j)(w_j + \lambda_i - w_i)}. \tag{2.4.16}$$

Applying (2.4.16) to (2.4.15) followed by substituting $iz_j = T_T^4 \left( w_j + \frac{\lambda_j}{2} - \frac{1}{2} \right)$ in (2.4.15) and deforming the contours to the real axis (note that no pole will be crossed) implies that

$$\text{r.h.s. of (2.4.15)} = \sum_{\lambda \vdash k} \prod_{i=1}^{\ell(\lambda)} \frac{T_T^4 \lambda_i^3}{m_1! m_2! \ldots} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{i=1}^{\ell(\lambda)} \frac{dz_i}{2\pi i} e^{-T_T^4 \lambda_i z_i^2} \prod_{i<j} \frac{T_T^4 (\lambda_i + \lambda_j)^2}{4} + (z_i - z_j)^2$$

\(^6\)These formulas were formally derived in [BC14] with a proof given as [Gho18, Theorem 2.1].
Taking \( \lambda = (k) \) (i.e., \( \lambda_1 = k \) and \( \lambda_i = 0 \) for all \( i \geq 2 \)), evaluating the single integral and noting that all the terms on the r.h.s. above are positive yields the lower bound in (2.4.14) when \( T_0 > \pi \). In the case when \( T_0 < \pi \), the term corresponding to \( \lambda = (k) \) is bounded below by \( T_0^{(k_0-1)/2} \pi^{k/2} \psi_T(k) \) for all \( T \in [T_0, \pi] \). This yields the lower bound in (2.4.14) when \( T_0 < \pi \).

For the upper bound, we first show that if \( \lambda \) is a partition of \( k \) not equal to \( (k,1) \) then

\[
\frac{k^3}{12} - \sum_{j=1}^{\ell(\lambda)} \frac{\lambda_j^3}{12} \geq \frac{k^2 - k}{4} \tag{2.4.17}
\]

with equality only when \( \lambda = (k - 1, 1) \). We prove this by induction. It is straightforward to check that (2.4.17) holds when \( k = 1, 2 \). Assume (2.4.17) holds when \( k = k_0 - 1 \). Now we show it for \( k = k_0 \).

Let us assume that \( \lambda \) is a partition of \( k_0 \) and write

\[
\frac{k_0^3}{12} - \sum_{j=1}^{\ell(\lambda)} \frac{\lambda_j^3}{12} = \frac{(k_0 - 1)^3 + 1}{12} + \frac{(k_0 - 1)^3 + 1}{12} - \sum_{j=1}^{\ell(\lambda)} \frac{\lambda_j^3}{12}.
\]

The right hand side of the above display is equal to \( \frac{k_0^3}{12} - \frac{(k_0 - 1)^3 + 1}{12} = \frac{k_0^2 - k_0}{4} \) when \( \lambda = (k_0 - 1, 1) \). It suffices to show

\[
\frac{(k_0 - 1)^3 + 1}{12} - \sum_{j=1}^{\ell(\lambda)} \frac{\lambda_j^3}{12} \geq 0 \tag{2.4.18}
\]

when \( \lambda \neq (k_0), (k_0 - 1, 1) \). In the case when \( \lambda \ell(\lambda) = 1 \), the above inequality follows by our assumption since \( (\lambda_1, \ldots, \lambda_{\ell(\lambda)-1}) \) is a partition of \( k_0 - 1 \). For \( \lambda \ell(\lambda) > 1 \), we write

\[
\frac{(k_0 - 1)^3 + 1}{12} - \sum_{j=1}^{\ell(\lambda)} \frac{\lambda_j^3}{12} = \frac{(k_0 - 1)^3}{12} - \sum_{j=1}^{\ell(\lambda)-1} \frac{\lambda_j^3}{12} - \frac{(\lambda_{\ell(\lambda)} - 1)^3}{12} - \frac{(\lambda_{\ell(\lambda)} - 1)(\lambda_{\ell(\lambda)} - 2)}{4}.
\]

Note that \( (\lambda_1, \ldots, \lambda_{\ell(\lambda)} - 1) \) is a partition of \( k_0 - 1 \). Since \( \lambda_{\ell(\lambda)} < k_0 \) and (2.4.17) holds for \( k = k_0 - 1 \), the right hand side of the above display is greater than 0. This shows (2.4.18) and hence, proves (2.4.17).

We return to the proof of the upper bound in (2.4.14). Observe that by bounding the cross-product
over $i < j$ by 1 and using Gaussian integrals, we may bound

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{\ell(\lambda)} \frac{dz_i e^{-T^\frac{1}{2} \lambda_i z_i^2}}{T^\frac{1}{2} \lambda_i} \prod_{i < j}^{\ell(\lambda)} \frac{T \frac{3}{4} (\lambda_i - \lambda_j)^2}{4} + (z_i - z_j)^2 \leq \prod_{i=1}^{\ell(\lambda)} \frac{\sqrt{2\pi}}{\sqrt{2T \lambda_i^\frac{3}{2}}}.
\]

(2.4.19)

When $T > \pi$, the r.h.s. of (2.4.19) is bounded above by $(\pi/T)^{k/2}$. Owing to this, (2.4.17), and $m_1!m_2! \cdots \leq k!$, we get

\[
E\left[ \exp\left(kT^\frac{1}{2} \Upsilon_T(0)\right)\right] \leq \left(1 + k^\frac{3}{2} e^{-k^2/2} \frac{\#\{\lambda : \lambda \vdash k\}}{k!}\right) \times \begin{cases} \frac{k!}{2\sqrt{\pi}Tk^\frac{3}{2}} & T \geq \pi \\ \frac{\pi^{(k-1)/2}k!e^{k^3/4}}{2Tk^{k/2}k^\frac{3}{2}} & T < \pi. \end{cases}
\]

(2.4.20)

Applying Siegel’s bound (see [Apo76, pp. 316-318], [Kno70, pp. 88-90]) on the number partition of any integer $k \geq 1$, we find that

\[
k^\frac{3}{2} e^{-k^2/2} \frac{\#\{\lambda : \lambda \vdash k\}}{k!} \leq k^\frac{3}{2} e^{-k^2/2 + \pi \sqrt{2k^{3/2}}} \leq 68 \quad \forall k \in \mathbb{N}.
\]

(2.4.21)

Combining (2.4.21) with (2.4.20) completes the proof of the upper bound in (2.4.14).

\[\blacksquare\]

**Proof of (2.4.4).** Combining Markov’s inequality and the second inequality of (2.4.14), we get

\[
P(\Upsilon_T(0) \geq s) \leq 69 \exp\left(-\max_{k \in \mathbb{N}} \left[k s T^\frac{1}{2} - \log \psi_T(k)\right]\right).
\]

(2.4.22)

By Stirling’s formula $\psi_T(k) = \exp\left(\frac{Tk^3(1+O(k^{-3/2}))}{12}\right)$. Set $k_0 = \lfloor 2s^\frac{1}{2} T^{-\frac{1}{3}} \rfloor$. When $s \geq \frac{9}{16} e^{-2T^\frac{1}{3}}$,

\[
k_0 s T^\frac{1}{2} - \log \psi_T(k_0) \geq k_0 s T^\frac{1}{2} - \frac{Tk_0^3(1 + O(e^{\frac{3}{2}}))}{12} \geq \frac{4(1 - \epsilon)s^{\frac{3}{2}}}{3}.
\]

(2.4.23)

The first inequality of (2.4.23) follows by noting that $k_0 \geq c\epsilon^{-1}$ for some positive constant $c$. We get the second inequality of (2.4.23) by noticing that $\lfloor 2s^\frac{1}{2} T^{-\frac{1}{3}} \rfloor \geq 2s^\frac{1}{2} T^{-\frac{1}{3}} - 1 \geq 2s^\frac{1}{2} T^{-\frac{1}{3}} (1 - \frac{2\epsilon}{3})$. Finally, (2.4.4) follows by plugging (2.4.23) into the r.h.s. of (2.4.22).

\[\blacksquare\]
Fixing now $k_0 = \lceil 2 \cdot (3(1 + 5\epsilon/6)s)^{-1/3} \rceil$, we observe that

$$
\exp \left( k_0 s T^{1/3} \right) \leq \frac{1}{2} \frac{k_0!}{2 \sqrt{\pi T k_0^3}} \exp \left( \frac{k_0^3 T}{12} \right). \tag{2.4.24}
$$

To prove this inequality first note that

$$
k_0 s T^{1/3} \leq \left( 2 \cdot (3(1 + 5\epsilon/6)s)^{1/3} T^{-1/3} + 1 \right) s T^{1/3} \leq 2\sqrt{3} \left( 1 + \frac{5\epsilon}{12} + \frac{2\epsilon}{3\sqrt{3}} \right)^{3/2}. \tag{2.4.25}
$$

where the first inequality follows from $\lceil k \rceil \leq k + 1$ and the second inequality is obtained using $s \geq \frac{9}{16} \epsilon^{-2} T^{2/3}$. Moreover, using $k! \geq k^{3/2}$ which holds for all $k \in \mathbb{Z}_{\geq 3}$, we see

$$
\text{r.h.s. of (2.4.24)} \geq \frac{1}{4\sqrt{\pi T}} \exp \left( 2\sqrt{3} \left( 1 + \frac{5\epsilon}{4} s^{3/2} \right) \right). \tag{2.4.26}
$$

Now, (2.4.24) follows from (2.4.25) and (2.4.26) by noting that $\frac{5}{4} \geq \frac{5}{12} + \frac{2}{3\sqrt{3}}$ and $T \leq \frac{64}{27} (\epsilon^2 s)^{3/2}$. Combining the first inequality of (2.4.14) with (2.4.24) yields

$$
\mathbb{P}(\Upsilon_T(0) > s) \geq \mathbb{P}(E), \quad \text{with} \quad E = \left\{ \exp \left( k_0 T^{1/3} \Upsilon_T(0) \right) > \frac{1}{2} \mathbb{E}[\exp(k_0 T^{1/3} \Upsilon_T(0))] \right\}. \tag{2.4.27}
$$

Claim 2.4.1. Fix $p, q > 1$ such that $p^{-1} + q^{-1} = 1$. Then,

$$
\mathbb{P}(E) \geq 2^{-q} \mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \right]^q \mathbb{E} \left[ \exp(pk_0 T^{1/3} \Upsilon_T(0)) \right]^{q/p}. \tag{2.4.28}
$$

Proof. Let us write

$$
\mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \right] = \mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \mathbb{1}(\mathcal{E}^c) \right] + \mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \mathbb{1}(E) \right]. \tag{2.4.29}
$$

The first term on the r.h.s. of (2.4.29) is bounded above by $\frac{1}{2} \mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \right]$. To bound the second term, we use Hölder’s inequality

$$
\mathbb{E} \left[ \exp(k_0 T^{1/3} \Upsilon_T(0)) \mathbb{1}(E) \right] \leq \left[ \mathbb{E} \left[ \exp(pk_0 T^{1/3} \Upsilon_T(0)) \right] \right]^{p/2} \mathbb{P}(E)^{1/2}. \tag{2.4.30}
$$

where $p^{-1} + q^{-1} = 1$. Plugging the upper bound of (2.4.30) into the r.h.s. of (2.4.29) and simplifying
yields (2.4.28) and proves the claim. ■

Returning to the proof of (2.4.5), thanks to (2.4.14), we find that

\[ \text{r.h.s. of } (2.4.28) \geq \exp\left( - \frac{g(p^2 - 1)k_0^3T(1 + O(\epsilon^{3/2}))}{12} \right). \]

From \( p^{-1} + q^{-1} = 1 \), it follows that \( q(p^2 - 1) = p(p + 1) \). Taking \( p = 1 + \epsilon/6 \) and recalling that 
\[ k_0 = \lceil (2 \cdot (3(1 + 5\epsilon/6)s)^{\frac{1}{2}})T^{-\frac{1}{3}} \rceil, \]
we get l.h.s. of (2.4.28) \( \geq 2^{-q} \exp\left( - 4\sqrt{3}(1 + 3\epsilon/2)s^\frac{3}{2} \right) \). Since \( q = 6\epsilon^{-1} + 1 \), we find that the r.h.s. of the above inequality is bounded below by \( \exp(-4\sqrt{3}(1 + 3\epsilon)s^\frac{3}{2}) \) for all \( s \geq \frac{9}{16} \epsilon^{-2}T \) and \( T \geq T_0 \geq \pi \). This completes the proof. ■

*Proof of (2.4.6).* Fix \( k_0 = \lceil (2 \cdot (3(1 + 5\epsilon/6)s)^{\frac{1}{2}})T^{-\frac{1}{3}} \rceil \). Our aim is to obtain a lower bound for the r.h.s. of (2.4.27). Applying (2.4.28) with \( p = q = 2 \) yields

\[ \mathbb{P}(\Upsilon_T(0) > s) \geq \frac{1}{2} \exp\left( - \frac{7k_0^3T}{12} \right). \] (2.4.31)

For \( k_0 \geq 2 \), we have \( k_0 \leq 2(k_0 - 1) \) which implies \( k_0 \leq 4 \cdot (3(1 + \epsilon))^{\frac{1}{2}}T^{-\frac{1}{3}} \) and hence \( \mathbb{P}(\Upsilon_T(0) > s) \geq \frac{1}{2} \exp(-2^6s^\frac{3}{2}) \). When \( k_0 = 1 \), r.h.s. (2.4.31) \( \geq \frac{1}{2} \exp(-2^\frac{7}{2} \epsilon^{-3} s^\frac{3}{2} \) for all \( s \geq \frac{1}{8} \epsilon T^\frac{2}{3} \). ■

*Proof of (2.4.7).* We first prove the second inequality of (2.4.7). Fix \( T \in [T_0, \pi] \). Applying Markov’s inequality yields

\[ \mathbb{P}(\Upsilon_T(0) \geq s) \leq 69 \exp\left( - \max_{k \in \mathbb{N}} \left[ ksT^\frac{1}{3} - \log \psi_T(k) \right] \right). \] (2.4.32)

Owing to Stirling’s formula, we get \( \psi_T(k) = \exp(Tk^3(1 + O(k^{-3/2})) - \frac{k}{2} \log T_0) \). Set \( k_0 = \lfloor 2s^\frac{1}{2}T^{-\frac{1}{3}} \rfloor \) and when \( s \geq \frac{9}{16} \epsilon^{-3}T^2 + 24T_0^{-\frac{1}{3}}(1 - \epsilon)^{-1} \log(T_0/\pi) \), we have

\[ k_0sT^\frac{1}{3} - \log \psi_T(k_0) \geq k_0sT^\frac{1}{3} - \frac{T_0^3(1 + O(\epsilon^{\frac{3}{2}}))}{12} - \frac{k}{2} \log T_0 \geq \frac{4(1 - \epsilon)}{3} s^\frac{3}{2} + \frac{k_0}{2} \log T(2.4.33) \]

for some constant \( c = c(\epsilon, T_0) > 0 \). The first inequality of (2.4.33) follows since \( k_0 \geq c\epsilon^{-1} \) for some positive constant \( c > 0 \) and the second inequality follows since \( \lfloor 2s^\frac{1}{2}T^{-\frac{1}{3}} \rfloor \geq 2s^\frac{1}{2}T^{-\frac{1}{3}}(1 - \frac{2\epsilon}{3}) \). Now,
we claim that the r.h.s. of (2.4.33) is bounded below by \((1 - \epsilon)s^{3/2}\). To see this, we write

\[
\frac{k_0}{2} \log T_0 \geq \min \{ s^{1/2}T^{-1/2} \log T_0, 0 \} \geq -\frac{1}{24} s^{3/2} (1 - \epsilon)(T_0/T)^{1/3} \geq -\frac{1}{24} (1 - \epsilon)s^{3/2}
\]

where the first inequality follows since \(k_0 \leq 2s^{1/2}T^{-1/2}\), the second inequality holds since \(s \geq 24T_0^{-3/2}(1 - \epsilon)^{-1}|\log (T_0/\pi)|\) and the last inequality is obtained by noting that \(T_0 \leq T\). Substituting the inequalities in the above display in the r.h.s. of (2.4.33) proves the claim. As a consequence, for all \(T \in [T_0, \pi]\),

\[
\max_{k \in \mathbb{N}} \left[ ksT^{1/2} - \log \psi_T(k) \right] \geq k_0 s T^{1/2} - \log \psi_T(k_0) \geq (1 - \epsilon)s^{3/2}.
\]

Applying the inequality in the above display in the r.h.s. of (2.4.32) yields the second inequality of (2.4.7).

Now, we turn to show the first inequality of (2.4.7). Fix \(k_0 = \lfloor 4s^{1/2}T^{-1/2} \rfloor\). We claim that for all \(T \in [T_0, \pi]\)

\[
\exp \left( k_0 s T^{1/2} \right) \leq \frac{1}{2} \left( T_0/T \right)^{k_0 - 1} \frac{k_0!}{2\sqrt{\pi} T k_0^{3/2}} \exp \left( \frac{k_0^3 T}{12} \right). \tag{2.4.34}
\]

To prove (2.4.34) we note

\[
k_0 s T^{1/2} \leq \left( 4 s^{1/2} T^{-1/2} + 1 \right) s T^{1/2} \leq 4 \left( 1 + \frac{\epsilon}{3} \right)s^{3/2} \tag{2.4.35}
\]

where the first inequality follows since \(\lfloor k \rfloor \leq k + 1\) and the second inequality is obtained using \(s \geq \frac{9}{16} \epsilon^{-2} T^{2/3}\). Since we know \(T_0 \leq T \leq \pi\) and \(k_0^3 T = \left( \lfloor 4s^{1/2}T^{-1/2} \rfloor \right)^3 T \geq 64s^{3/2}\),

\[
\text{r.h.s. of (2.4.34)} \geq \left( \frac{T_0}{\pi} \right)^{k_0 - 1} \frac{k_0!}{4\pi k_0^{3/2}} \exp \left( \frac{64}{12} s^{3/2} \right) = \left( \frac{T_0}{\pi} \right)^{k_0 - 1} \frac{k_0!}{4\pi k_0^{3/2}} \exp \left( (5 + 3^{-1})s^{3/2} \right). \tag{2.4.36}
\]

By using the fact that \(s \geq \frac{9}{16} \epsilon^{-2} T^{2/3} + 24T_0^{-1/3}(1 - \epsilon)|\log (T_0/\pi)|\) and \(\epsilon < 3/5\), we get

\[
k_0 = \lfloor 4s^{1/2}T^{-1/2} \rfloor \geq 4s^{1/2} T^{-1/2} > 3 \epsilon^{-1} > 5,
\]

\[
\frac{1}{3}s^{3/2} > 2s^{1/2} T_0^{-1/3} |\log (T_0/\pi)| \geq \frac{k_0 - 1}{2} |\log (T_0/\pi)|.
\]

Now, (2.4.34) follows from (2.4.35), (2.4.36) and the inequalities of the above display by noting that \(4(1 + \epsilon/3) \leq 5\), \(k_0 \geq 6\) and \((T_0/\pi)^{(k_0 - 1)/2} \exp (3^{-1}s^{3/2}) \geq 1\).
For any $T \in [T_0, \pi]$, combining the first inequality of (2.4.14) with (2.4.34) yields

$$\mathbb{P}(\forall T(0) > s) \geq \mathbb{P}(\tilde{E}), \quad \text{where } \tilde{E} = \left\{ \exp \left(k_0 T^{\frac{1}{2}} \forall T(0) \right) > \frac{1}{2} \mathbb{E}\left[ \exp \left(k_0 T^{\frac{1}{2}} \forall T(0) \right) \right] \right\}.$$ 

Applying (2.4.28) with $p = q = 2$ shows

$$\mathbb{P}(\forall T(0) > s) \geq \frac{1}{2} \exp \left(-\frac{7k_0^3 T}{12} \right) \geq \exp \left(-cs^\frac{3}{2} \right) \quad (2.4.37)$$

for some absolute constant $c > 0$. The last inequality of the above display follows since $k_0 = [4s^{\frac{1}{2}} T^{\frac{1}{2}}]$. Note that (2.4.37) implies the first inequality of (2.4.7). This completes the proof. ■

### 2.4.3 Proof of Proposition 2.4.4

We prove this by contradiction. Assume there exists $M > 0$ such that $\mathbb{P}(\forall T(0) > s) \leq e^{-cs^\frac{3}{2}}$ for all $s \geq M$. Dividing the expectation integral into $(-\infty, 0]$, $[0, M]$ and $(M, \infty)$, we have

$$\mathbb{E}\left[ \exp(k \forall T(0) T^{\frac{1}{2}}) \right] \leq 1 + M kT^{\frac{1}{2}} e^{kMT^{\frac{1}{2}}} + \int_M^\infty kT^{\frac{1}{2}} e^{ksT^{\frac{1}{2}} - cs^\frac{3}{2}} ds \quad (2.4.38)$$

Observing that

$$\arg\max_{s \geq 0} \left\{ ksT^{\frac{1}{2}} - cs^\frac{3}{2} \right\} = \frac{4k^2 T^2}{9e^2}, \quad (2.4.39)$$

we may choose $k$ to be a sufficiently large integer such that the r.h.s. of (2.4.39) exceeds $M$. Then, approximating the integral of (2.4.38) by $C'kT^{\frac{1}{2}} \exp(\max_{s \geq 0} \left\{ ksT^{\frac{1}{2}} - cs^\frac{3}{2} \right\})$ for some absolute constant $C' = C'(k)$ and plugging in the value of the maximizer from (2.4.39), we find

$$\mathbb{E}\left[ \exp(k \forall T(0) T^{\frac{1}{2}}) \right] \leq (M + 1) kT^{\frac{1}{2}} + C'kT^{\frac{1}{2}} e^{\frac{4k^3 T}{27e^2}} \quad (2.4.40)$$

Applying $c > \frac{3}{4}(1 + \frac{1}{3} \epsilon)$ into (2.4.40) shows that the r.h.s. of (2.4.40) is less than $e^{(1-\epsilon)k^3 T^{\frac{1}{2}}}$ which contradicts (2.4.14). Hence, the claim follows.
Our proof of Proposition 2.4.1 relies on a Laplace transform formula for \( Z_{nw}^T(0) \) which was proved in [BG16] and follows from the exact formula for the probability distribution of \( \Upsilon_T(0) \) of [ACQ11]. It connects \( Z_{nw}^T(0) \) with the Airy point process \( a_1 > a_2 > \ldots \). The latter is a well studied determinantal point process in random matrix theory (see, e.g., [AGZ10, Section 4.2]).

For convenience, we introduce following shorthand notations:

\[
I_s(x) := \frac{1}{1 + \exp(T \frac{1}{3} (x - s))}, \quad J_s(x) := \log \left( 1 + \exp(T \frac{1}{3} (x - s)) \right).
\]

It is worth noting that \( I_s(x) = \exp(-J_s(x)) \).

**Proposition 2.4.6** (Theorem 1 of [BG16]). For all \( s \in \mathbb{R} \),

\[
\mathbb{E}_{\text{KPZ}} \left[ \exp \left( - \exp \left( T \frac{1}{3} (\Upsilon_T(0) - s) \right) \right) \right] = \mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right]. \tag{2.4.41}
\]

We start our proof of Proposition 2.4.1 with upper and lower bounds on the r.h.s. of (2.4.41).

**Proposition 2.4.7.** Fix some \( \zeta \leq \epsilon \in (0, 1) \) and \( T_0 > 0 \). Continuing with the notation of Proposition 2.4.6 there exists \( s_0 = s_0(\epsilon, \zeta, T_0) \) such that for all \( s \geq s_0 \),

\[
1 - \mathbb{E} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \leq e^{-\zeta s T^{1/3}} + e^{-\frac{4}{3} (1-\epsilon) s^{3/2}}, \tag{2.4.42}
\]

\[
1 - \mathbb{E} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \geq e^{-(1+\zeta)s T^{1/3}} + e^{-\frac{4}{3} (1+\epsilon) s^{3/2}}. \tag{2.4.43}
\]

We defer the proof of Proposition 2.4.7 to Section 2.4.4.

**Proof of Proposition 2.4.1** Define \( \bar{s} := (1+\zeta)s \) and \( \theta(s) := \exp \left( - \exp \left( T \frac{1}{3} (\Upsilon_T(0) - s) \right) \right) \). Thanks to (2.4.41), we have \( \mathbb{E}_{\text{KPZ}}[\theta(s)] = \mathbb{E}_{\text{Airy}}[\prod_{k=1}^{\infty} I_s(a_k)] \). Note that

\[
\theta(s) \leq 1(\Upsilon_T(0) \leq \bar{s}) + 1(\Upsilon_T(0) > \bar{s}) \exp(-\zeta s T^{1/3})).
\]
Rearranging, taking expectations and applying (2.4.41), we arrive at

\[ P(\Upsilon_T(0) > \bar{s}) \leq \left( 1 - \exp(-\exp(\zeta s T^{1/3})) \right)^{-1} \left( 1 - \mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \right). \] (2.4.44)

By taking \( s \) sufficiently large and \( T \geq T_0 \), we may assume that \( 1 - \exp(-\exp(\zeta s T^{1/3})) \geq \frac{1}{2} \). Plugging this bound and (2.4.42) into the r.h.s. of (2.4.44) yields

\[ P(\Upsilon_T(0) > \bar{s}) \leq e^{-\zeta s T^{1/3}} + e^{-\frac{4}{3} (1-\epsilon) s^{3/2}} \]

for all \( s \geq s_0 \) where \( s_0 \) depends on \( \epsilon, \zeta \) and \( T_0 \). This proves (2.4.1).

We turn now to prove (2.4.2). Using Markov’s inequality,

\[ P(\Upsilon_T(0) \leq s) = P\left( \theta(\bar{s}) \geq \exp\left( -e^{-\zeta s T^{1/3}} \right) \right) \leq \exp\left( -e^{-\zeta s T^{1/3}} \right) \cdot \mathbb{E}[\theta(\bar{s})]. \]

Rearranging yields \( 1 - \exp\left( -e^{-\zeta s T^{1/3}} \right) P(\Upsilon_T(0) \leq s) \geq 1 - \mathbb{E}[\theta(\bar{s})] \). Finally, applying (2.4.41) and (2.4.43) to the r.h.s. of this result, we get (2.4.2).

**Proof of Proposition 2.4.7**

**Proof of (2.4.42).** We start by noticing the following trivial lower bound

\[ \mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \geq \mathbb{E}_{\text{Airy}} \left[ \prod_{k=1}^{\infty} I_s(a_k) \mathbbm{1}(A) \right] \] (2.4.45)

where \( A = \{ a_1 \leq (1 - \zeta)s \} \). Setting \( k_0 := \left\lfloor \frac{2}{3\pi} s^{\frac{3}{4} + 2\epsilon} \right\rfloor \) we observe that

\[ \prod_{k=1}^{k_0} I_s(a_k) \mathbbm{1}(A) = \exp\left( -\sum_{k=1}^{k_0} J_s(a_k) \mathbbm{1}(A) \right) \geq \exp\left( -\frac{2}{3\pi} s^{\frac{3}{4} + 2\epsilon} e^{-T^{1/3}s\zeta} \right). \] (2.4.46)

where inequality is obtained via \( J_s(a_k) \leq e^{-T^{1/3}s\zeta} \) which follows on the event \( A \). Our next task is to bound \( \prod_{k > k_0} I_s(a_k) \) from below. To achieve this, we recall the result of [CG18a, Proposition 4.5] which shows that for any \( \epsilon, \delta \in (0, 1) \) we can augment the probability space on which the Airy point

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process is defined so that there exists a random variable $C^{\text{Ai}}_\varepsilon$ satisfying

$$(1 + \varepsilon)\lambda_k - C^{\text{Ai}}_\varepsilon \leq a_k \leq (1 - \varepsilon)\lambda_k + C^{\text{Ai}}_\varepsilon \quad \text{for all } k \geq 1 \quad \text{and} \quad \mathbb{P}(C^{\text{Ai}}_\varepsilon \geq s) \leq e^{-s^{1-\delta}}$$

for all $s \geq s_0$ where $s_0 = s_0(\varepsilon, \delta)$ is a constant. Here, $\lambda_k$ is the $k$-th zero of the Airy function (see [CG18a Proposition 4.6]) and we fix some $\delta \in (0, \varepsilon)$. Now, we write

$$\prod_{k > k_0} I_s(a_k) \geq \prod_{k > k_0} I_s(a_k) \mathbb{1}(C^{\text{Ai}}_\varepsilon \leq \phi(s)) \geq \exp \left( - \sum_{k > k_0} J_s((1 - \varepsilon)\lambda_k + \phi(s)) \right).$$

Appealing to the tail probability of $C^{\text{Ai}}_\varepsilon$, we have $\mathbb{P}(C^{\text{Ai}}_\varepsilon \leq \phi(s)) \geq 1 - e^{-s^{1-\frac{\delta}{2} + \frac{\delta}{3}}}$. We now claim that for some constant $C > 0$,

$$\sum_{k > k_0} J_s((1 - \varepsilon)\lambda_k + \phi(s)) \leq C \frac{T^{1/3}}{3} \exp(-sT^{1/3}).$$

To prove this note that for all $k \geq k_0$,

$$\lambda_k \leq -\left(\frac{3\pi k}{2}\right)^{\frac{2}{3}} \quad \text{and} \quad (1 - \varepsilon)\left(\frac{3\pi k}{2}\right)^{\frac{4}{3}} - \phi(s) \geq (1 - \varepsilon)\left(\frac{3\pi}{2}(k - k_0)\right)^{\frac{2}{3}}.$$

The first inequality of (2.4.49) is an outcome of [CG18a Proposition 4.6] and the second inequality follows from [CG18a Lemma 5.6]. Applying (2.4.49), we get

$$J_s((1 - \varepsilon)\lambda_k + \phi(s)) \leq e^{T^{1/3}\left(-s - (1 - \varepsilon)(3\pi k/2)^{2/3} + \phi(s)\right)} \leq e^{T^{1/3}\left(-s - (1 - \varepsilon)(k - k_0)^{2/3}\right)}.$$ (2.4.50)

Summing over $k > k_0$ in (2.4.50), approximating the sum by the corresponding integral, and evaluating yields (2.4.48).

Now, we turn to complete the proof of (2.4.42). Plugging (2.4.48) into the r.h.s. of (2.4.47) yields

$$\prod_{k > k_0} I_s(a_k) \mathbb{1}(C^{\text{Ai}}_\varepsilon \leq \phi(s)) \geq \exp\left(-\frac{C}{T^{1/3}} \exp(-sT^{1/3})\right).$$ (2.4.51)
Combining (2.4.46) and (2.4.51) yields

\[
\text{l.h.s. of (2.4.45)} \geq \exp \left( -\frac{2}{3\pi} s^\frac{9}{4} + 2\epsilon e^{-\zeta s T^\frac{1}{2}} - \frac{C}{T^\frac{1}{4}} e^{-s T^\frac{1}{2}} \right) \mathbb{P}(C_{\epsilon}^A \leq \phi(s), A). \tag{2.4.52}
\]

To finish the proof, we observe that

\[
\mathbb{P}(C_{\epsilon}^A \leq \phi(s), A) \geq 1 - \mathbb{P}(C_{\epsilon}^A \geq \phi(s)) - \mathbb{P}(A^c) \geq 1 - e^{-s^\frac{3}{2} + \frac{4}{3} \epsilon} - e^{-\frac{4}{3}(1-\epsilon)s^\frac{3}{2}} \tag{2.4.53}
\]

for all \( s \geq s_0 \). The second inequality above used \( \mathbb{P}(A^c) = \mathbb{P}(a_1 \geq (1 - \zeta)s) \leq \exp\left( -\frac{4}{3}(1 - \epsilon)s^{\frac{3}{2}} \right) \) which holds when \( s \) is sufficiently large (see [RRV11, Theorem 1.3]). Plugging (2.4.53) into the r.h.s. of (2.4.52) and rearranging yields \( e^{-(1-\epsilon)\zeta s T^\frac{1}{2}} \leq \exp\left( -\frac{2}{3\pi} s^\frac{9}{4} + 2\epsilon e^{-\zeta s T^\frac{1}{2}} - \frac{C}{T^\frac{1}{4}} e^{-s T^\frac{1}{2}} \right) \) \leq e^{-(1+\epsilon)\zeta s T^\frac{1}{2}} \) for sufficiently large \( s \). Hence (2.4.42) follows. \( \blacksquare \)

**Proof of (2.4.43).** Here, we need to get an upper bound on \( \mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \). We start by splitting \( \mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \) into two different parts (again set \( A = \{a_1 \leq (1 + \zeta)s\} \)):

\[
\mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \leq \mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \mathbb{1}(A) \right] + \mathbb{P}(A^c) \cdot \exp(-\zeta s T^\frac{1}{2}). \tag{2.4.54}
\]

Let us define \( \chi_{Ai}(s) := \#\{a_i \geq s\} \) and, for \( c \in (0, \frac{2}{3\pi}) \) fixed, define

\[
B := \left\{ \chi_{Ai}(-\zeta s) - \mathbb{E}\left[ \chi_{Ai}(-\zeta s) \right] \geq -c(\zeta s)^{\frac{3}{2}} \right\}
\]

We split the first term on the r.h.s. of (2.4.54) as follows

\[
\mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \mathbb{1}(A) \right] \leq \mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \mathbb{1}(B \cap A) \right] + \mathbb{E}\left[ \mathbb{1}(B^c \cap A) \right]. \tag{2.4.55}
\]

On the event \( B \), we may bound

\[
\prod_{k=1}^{\infty} I_s(a_k) \mathbb{1}(B) \leq \exp\left( -\left( \frac{2}{3\pi} - c \right)(\zeta s)^{\frac{3}{2}} e^{-(1+\zeta)s T^\frac{1}{2}} \right)
\]
so that
\[
\mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \mathbb{1}(B \cap A) \right] \leq \exp\left( -\left( \frac{2}{3\pi} - c \right)(\zeta s)^\frac{3}{2} e^{-(1+\zeta)sT^{\frac{1}{3}}} \right) \cdot \mathbb{P}(A). \tag{2.4.56}
\]

For large \( s \), the r.h.s. of (2.4.56) is bounded above by \( \exp\left( -e^{-(1+\zeta)sT^{\frac{1}{3}}} \right) \mathbb{P}(A) \). Thanks to Theorem 1.4 of [CG18a], we know that for any \( \delta > 0 \), there exists \( s_\delta \) such that \( \mathbb{P}(B^c) \leq e^{-c(\zeta s)^{3-\delta}} \) for all \( s \geq s_\delta \). Now, we plug these bounds into (2.4.55) which provides an upper bound to the first term on the r.h.s. of (2.4.54). As a result, we find
\[
1 - \mathbb{E}\left[ \prod_{k=1}^{\infty} I_s(a_k) \right] \geq 1 - e^{-e^{-c(1+\zeta)sT^{\frac{1}{3}}} - e^{-c(\zeta s)^{3-\delta}} + \mathbb{P}(A^c)(e^{-e^{-(1+\zeta)sT^{\frac{1}{3}}} - e^{-\zeta sT^{\frac{1}{3}}})}. \tag{2.4.57}
\]

Finally, we note that \( \mathbb{P}(A^c) \geq \exp\left( -\frac{4}{3}(1 + \epsilon)s^2 \right) \) (again thanks to [RRV11, Theorem 1.3]). Thus, the r.h.s. of (2.4.57) is lower bounded by \( \frac{1}{2} e^{-(1+\zeta)sT^{1/3}} + e^{-\frac{4}{3}(1+\epsilon)s^3/2} \) for sufficiently large \( s \). This completes the proof of (2.4.43) and hence also of Proposition 2.4.7. \( \blacksquare \)

### 2.5 Upper tail under general initial data

This section contains the proofs of Theorems 2.1.4 and 2.1.12.

#### 2.5.1 Proof of Theorem 2.1.4

Theorem 2.1.4 will follow directly from the next two propositions which leverage narrow wedge upper tail decay results to give general initial data results. The cost of this generalization is in terms of both the coefficients in the exponent and the ranges on which the inequalities are shown to hold. Recall \( h_T^f \) and \( \Upsilon_T \) from (2.1.4) and (1.1.2) respectively.

The following proposition has two parts which correspond to \( T \) being greater or, less than equal to \( \pi \). The main goal of this proposition is to provide a recipe to deduce upper bounds on \( \mathbb{P}(h_T^f(0) > s) \) by employing the upper bounds on \( \mathbb{P}(\Upsilon_T(0) > s) \). We have noticed in Theorem 2.1.9 that the latter bounds vary as \( s \) lies in different intervals and furthermore, those intervals vary with \( T \). This motivates us to choose a generic set of intervals of \( s \) based on a given \( T \) and assume upper bounds on \( \mathbb{P}(\Upsilon_T(0) > s) \) in those intervals. In what follows, we show how those translate to the upper bounds on \( \mathbb{P}(h_T^f(0) > s) \).
Proposition 2.5.1. Fix $\epsilon, \mu \in (0, \frac{1}{2})$, $\nu \in (0, 1)$, $C, \theta, \kappa, M > 0$ and assume that $f$ belongs to $\text{Hyp}(C, \nu, \theta, \kappa, M)$ (see Definition 2.1.1).

1. Fix $T_0 > \pi$. Suppose there exists $s_0 = s_0(\epsilon, T_0)$ and for any $T \geq T_0$ there exist $s_1 = s_1(\epsilon, T)$ and $s_2 = s_2(\epsilon, T)$ with $s_1 \leq s_2$ such for any $s \in [s_0, \infty)$,

$$\mathbb{P} (Y_T(0) > s) \leq \begin{cases} e^{-\frac{4}{3}(1-\epsilon)s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ e^{-\frac{4}{3}\epsilon s^{3/2}} & \text{if } s \in (s_1, s_2]. \end{cases} \quad (2.5.1)$$

Let

$$s_0 := \frac{s_0}{1 - \frac{2\mu}{3}}, \quad s_1 := \frac{\epsilon s_1}{1 - \frac{2\mu}{3}}, \quad s_2 := \frac{s_2}{1 - \frac{2\mu}{3}}. \quad (2.5.2)$$

Then, there exists $s_0' = s_0'(\epsilon, \mu, C, \nu, \theta, \kappa, M, T_0)$ such for any $T > T_0$ and any $s \in [\max\{s_0', s_0\}, \infty)$, we have

$$\mathbb{P} (h_T^f(0) > s) \leq \begin{cases} e^{-\frac{4\nu^2}{3}(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ e^{-\frac{4\nu^2}{3}\epsilon(1-\mu)s^{3/2}} & \text{if } s \in (s_1, s_2]. \end{cases} \quad (2.5.3)$$

2. Fix $T_0 \in (0, \pi)$. Then, there exists $s_0' = s_0'(C, \nu, \theta, \kappa, M, T_0)$ satisfying the following: if there exist $s_0 = s_0(T_0) > 0$ and $c = c(T_0) > 0$ such that $\mathbb{P} (Y_T(0) > s) \leq e^{-cs^{3/2}}$ for all $s \in [s_0, \infty)$ and $T \in [T_0, \pi]$, then,

$$\mathbb{P} (h_T^f(0) > s) \leq e^{-\frac{1}{2\nu^2}cs^{3/2}}, \quad \forall s \in [\max\{s_0', s_0\}, \infty), T \in (T_0, \pi]. \quad (2.5.4)$$

The next proposition provides a lower bound on $\mathbb{P} (h_T^f(0) > s)$ in terms of the upper tail probability of the narrow wedge solution.

Proposition 2.5.2. Fix $\mu \in (0, \frac{1}{2})$, $n \in \mathbb{Z}_{\geq 3}$, $\nu \in (0, 1)$, $C, \theta, \kappa, M > 0$ and $T_0 > \pi$ and assume that $f \in \text{Hyp}(C, \nu, \theta, \kappa, M)$. Then, there exist $s_0 = s_0(\mu, n, T_0, C, \nu, \theta, \kappa, M)$ and $K = K(\mu) > 0$ such that for all $s \geq s_0$ and $T \geq T_0$,

$$\mathbb{P} (h_T^f(0) > s) \geq \left( \mathbb{P} (Y_T(0) > (1 + \frac{2\mu}{3})s) \right)^2 - e^{-Ks^n}. \quad (2.5.5)$$
We prove Propositions 2.5.1 and 2.5.2 in Sections 2.5.1 and 2.5.1 respectively. In what follows, we complete the proof of Theorem 2.1.4 assuming Propositions 2.5.1 and 2.5.2.

**Proof of Theorem 2.1.4** By Theorem 2.1.9 for any $\epsilon \in (0, \frac{1}{2})$ and $T_0 > \pi$, there exists $s_0 = s_0(\epsilon, T_0)$ such that for all $T > T_0$ and $s \in [s_0, \infty)$

$$
\mathbb{P}(\Upsilon_T > s) \leq \begin{cases} 
    e^{-\frac{3}{4}(1-\epsilon)s^{3/2}} & \text{if } s \in [s_0, \frac{1}{8}e^2T] \cup \left(\frac{9}{16}e^{-2}T, \infty\right), \\
    e^{-\frac{3}{4}e s^{3/2}} & \text{if } s \in \left(\frac{1}{8}e^2T, \frac{9}{16}e^{-2}T\right].
\end{cases} \tag{2.5.6}
$$

For any $\epsilon \in (0, \frac{1}{2})$ and $T > T_0$, (2.5.6) shows that the hypothesis of part (1) of Proposition 2.5.1 is satisfied with $s_1 = \frac{1}{8}e^2T$ and $s_2 = \frac{9}{16}e^{-2}T$. Proposition 2.5.1 yields $s'_0 = s'_0(\epsilon, \mu, T_0, C, \nu, \theta, \kappa, M)$ such that for all $T \geq T_0$ and $s \in \left[\max\{s'_0, s_0/(1 - \frac{2\mu}{3})\}, \infty\right)$

$$
\mathbb{P}\left(h_T^f(0) > s\right) \leq \begin{cases} 
    e^{-\frac{3}{4}e(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in \left[\frac{s_0}{1 - \frac{2\mu}{3}}, \frac{e^3T}{8(1 - \frac{2\mu}{3})}\right] \cup \left(\frac{9e^{-2}T}{16(1 - \frac{2\mu}{3})}, \infty\right), \\
    e^{-\frac{3}{4}e(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in \left(\frac{e^3T}{8(1 - \frac{2\mu}{3})}, \frac{9e^{-2}T}{16(1 - \frac{2\mu}{3})}\right].
\end{cases} \tag{2.5.7}
$$

This shows the upper bound on $\mathbb{P}\left(h_T^f(0) > s\right)$ when $T_0 > \pi$. For any $T_0 \in (0, \pi)$, the upper bound on $\mathbb{P}(h_T^f(0) > s)$ follows from (2.5.4) for all $T \in [T_0, \pi]$.

Now, we turn to show the lower bound. Let us fix $n = 3$. Owing to Proposition 2.5.2 and the lower bound on the probability $\mathbb{P}(\Upsilon_T(0) \geq s)$ in (2.1.9) of Theorem 2.1.9, we observe that the second term $e^{-Ks^3}$ of the r.h.s. of (2.5.5) is less than the half of the first term when $s$ is large enough. Hence, there exist $s'_0 = s'_0(\epsilon, \mu, C, \nu, \theta, \kappa, M, T_0)$ such that for all $T \geq T_0 > \pi$ and $s \in \left[\max\{s'_0, s_0/(1 - \frac{2\mu}{3})\}, \infty\right)$

$$
\mathbb{P}\left(h_T^f(0) > s\right) \geq \begin{cases} 
    \frac{1}{2}e^{-\frac{3}{4}e(1+\epsilon)(1+\mu)s^{3/2}} & \text{if } s \in \left[\frac{s_0}{1 + \frac{2\mu}{3}}, \frac{e^3T}{8(1 + \frac{2\mu}{3})}\right], \\
    \frac{1}{2}e^{-\frac{3}{2}e^{-3}(1+\mu)s^{3/2}} & \text{if } s \in \left(\frac{e^3T}{8(1 + \frac{2\mu}{3})}, \frac{9e^{-2}T}{16(1 + \frac{2\mu}{3})}\right], \\
    \frac{1}{2}e^{-8\sqrt{3}(1+\epsilon)(1+\mu)s^{3/2}} & \text{if } s \in \left(\frac{9e^{-2}T}{16(1 + \frac{2\mu}{3})}, \infty\right).
\end{cases} \tag{2.5.8}
$$

The sets of three intervals of (2.5.7) and (2.5.8) are not same. Note that

$$
\left[\frac{s_0}{1 - \frac{2\mu}{3}}, \frac{e^3T}{8(1 - \frac{2\mu}{3})}\right] \subset \left[\frac{s_0}{1 + \frac{2\mu}{3}}, \frac{e^3T}{8(1 + \frac{2\mu}{3})}\right], \text{ and } \left(\frac{9e^{-2}T}{16(1 - \frac{2\mu}{3})}, \infty\right) \subset \left(\frac{9e^{-2}T}{16(1 + \frac{2\mu}{3})}, \infty\right).
$$

The first inequality uses $\epsilon \leq (1 - \frac{2\mu}{3})(1 + \frac{2\mu}{3})^{-1}$ for any $\epsilon, \mu \in (0, \frac{1}{2})$ and the second inequality uses $\mu > 0$.  

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By these containments and (2.5.7)-(2.5.8), for all $s \in \left[ \max \left\{ s_0, s_0/(1 - \frac{2\mu}{3}), s_0/(1 + \frac{2\mu}{3}) \right\}, \infty \right)$ and $T \geq T_0 > \pi$, we have $\exp(-c_1 s^{\frac{3}{2}}) \leq P(h_T^f(0) > s) \leq \exp(-c_2 s^{\frac{3}{2}})$ where

$$
\begin{align*}
\frac{\sqrt{2}}{3}(1 - \mu)(1 - \epsilon) \quad &\leq c_2 < c_1 \leq \begin{cases} 
\frac{8}{3}(1 + \epsilon)(1 + \mu) & \text{if } s \in \left[ s_0, \frac{e^{3T}}{1 - \frac{2\mu}{3}} \right], \\
2^\frac{2}{3}e^{-3}(1 + \mu) & \text{if } s \in \left( \frac{e^{3T}}{8(1 - \frac{2\mu}{3})}, \frac{9e^{-2T}}{16(1 - \frac{2\mu}{3})} \right], \\
8\sqrt{3}(1 + \epsilon)(1 + \mu) & \text{if } s \in \left( \frac{9e^{-2T}}{16(1 - \frac{2\mu}{3})}, \infty \right).
\end{cases}
\end{align*}
$$

The lower bound $P(h_T^f(0) > s) \geq e^{-2c_1 s^{\frac{3}{2}}}$ for all $T \in [T_0, \pi]$ when $T_0 \in (0, \pi)$ follows by combining the first inequality of (2.1.6) with (2.5.5) (with $n = 3$). This completes the proof. ■

**Proof of Proposition 2.5.1**

Recall $h_T^f$ and $\Upsilon_T$ from (2.1.4) and (1.1.2). By Theorem 1.1.1, $P(h_T^f(0) \geq s) = P(\tilde{A}^f)$ where

$$
\tilde{A}^f := \left\{ \int_{-\infty}^{\infty} e^{T^\frac{2}{3}(\Upsilon_T(y)+f(-y))}dy \geq e^{T^\frac{2}{3}s} \right\}.
$$

Let $\zeta_n := \frac{n}{s^{\frac{3}{2}}(1 - \mu)}$, $n \in \mathbb{Z}$ and fix $\tau \in (0, 1)$ such that $\nu + \tau < 1$. We define the following events:

$$
\begin{align*}
\tilde{E}_n &:= \left\{ \Upsilon_T(\zeta_n) \geq -\frac{1 - 2^{-1}\tau}{22/3} - \zeta_n^2 + (1 - \frac{2\mu}{3})s \right\}, \\
\tilde{F}_n &:= \left\{ \Upsilon_T(y) \geq -\frac{1 - \tau}{22/3} y^2 + (1 - \frac{\mu}{3})s \text{ for some } y \in [\zeta_n, \zeta_{n+1}] \right\}.
\end{align*}
$$

In the same way as in (2.3.1), we write

$$
P(\tilde{A}^f) \leq \sum_{n \in \mathbb{Z}} P(\tilde{E}_n) + P(\tilde{A}^f \cap \bigcup_{n \in \mathbb{Z}} \tilde{E}_n^c). \quad (2.5.9)
$$

From now on, we will fix some $T > T_0 > \pi$ and assume that there exist $s_0 = s_0(\epsilon, T_0)$, $s_1 = s_1(\epsilon, T)$ and $s_2 = s_2(\epsilon, T)$ with $s_1 \leq s_2$ such that for all $s \in [s_0, \infty)$ (2.5.1) is satisfied. In the next result, we demonstrate some upper bound on the first term on the r.h.s. of (2.5.9).
Lemma 2.5.3. There exist $\bar{s} = \bar{s}(\epsilon, T_0)$ and $\Theta = \Theta(\epsilon, T_0)$ such that for all $s \in [\max\{\bar{s}, s_0\}, \infty)$,

$$\sum_{n \in \mathbb{Z}} \mathbb{P}(\tilde{E}_n) \leq \begin{cases} \Theta e^{-\frac{2}{3}(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ \Theta e^{-\frac{2}{3}s^{3/2}} & \text{if } s \in (s_1, s_2], \end{cases}$$

(2.5.10)

where $s_0$, $s_1$ and $s_2$ are defined in (2.5.2).

Proof. We first prove (2.5.10) when $s \in [s_0, s_1]$. If $[s_0, s_1]$ is an empty interval, then, nothing to prove. Otherwise, fix any $s \in [s_0, s_1]$. Let us denote

$$S_1 := [0, (1-\epsilon)s_1], \quad S_2 := ((1-\epsilon)s_1, s_2 - s_0], \quad S_3 := (s_2 - s_0, \infty).$$

Claim 2.5.1.

$$\mathbb{P}(\tilde{E}_n) \leq \begin{cases} \exp \left( -\frac{4}{3}(1-\epsilon) \left( (1 - \frac{2\mu}{3})s + \frac{\tau \zeta^2}{25/3} \right)^{\frac{3}{2}} \right) & \text{when } \frac{\tau \zeta^2}{25/3} \in S_1 \cup S_3, \\ \exp \left( -\frac{4}{3}(1-\epsilon) \left( (1 - \frac{2\mu}{3})s + \frac{\tau \zeta^2}{25/3} \right)^{\frac{3}{2}} \right) & \text{when } \frac{\tau \zeta^2}{25/3} \in S_2, \end{cases}$$

(2.5.11)

Proof. Note that $s_0 \leq (1 - \frac{2\mu}{3})s \leq \epsilon s_1$. This implies $(1 - \frac{2\mu}{3})s + 2^{-5/3}\tau \zeta_n^2$ is bounded above by $\epsilon s_1 + (1-\epsilon)s_1 = s_1$ whenever $2^{-5/3}\tau \zeta_n^2 \leq (1-\epsilon)s_1$ whereas it is bounded below by $s_0 + s_2 - s_0 = s_2$ if $2^{-5/3}\tau \zeta_n^2 > s_2 - s_0$. Owing to this and (2.5.1), we have

$$\mathbb{P}(\tilde{E}_n) \leq \exp \left( -\frac{4}{3}(1-\epsilon) \left( (1 - \frac{2\mu}{3})s + \frac{\tau \zeta^2}{25/3} \right)^{\frac{3}{2}} \right) \quad \text{when } \frac{\tau \zeta^2}{25/3} \in S_1 \cup S_3.$$ (2.5.12)

Furthermore, $(1 - \frac{2\mu}{3})s + 2^{-5/3}\tau \zeta_n^2$ is greater than $s_0$ when $s \geq s_0$. Thanks to $\epsilon < \frac{1}{2}$, one can now see the following from (2.5.1):

$$\mathbb{P}(\tilde{E}_n) \leq \exp \left( -\frac{4}{3} \epsilon \left( (1 - \frac{2\mu}{3})s + \frac{\tau \zeta^2}{25/3} \right)^{\frac{3}{2}} \right) \quad \text{when } \frac{\tau \zeta^2}{25/3} \in S_2.$$ (2.5.13)

Combining (2.5.12) and (2.5.13), we get (2.5.11). □

Let $n_0 = n_0(s, \delta, \tau) < n'_0 = n'_0(s, \delta, \tau) \in \mathbb{N}$ be such that $2^{-5/3}\tau \zeta_n^2 \in S_2$ for all integer $n$ in
\[ [n_0, n'_0] \cup [-n'_0, -n_0]. \] Using the reverse Minkowski’s inequality,
\[
\frac{\tau \zeta^2 n}{25/3} \geq \frac{\tau \zeta^2 n_0}{25/3} + \frac{\tau \zeta^2 |n| - n_0}{25/3}, \quad \forall n \in \{[n_0, n'_0] \cup [-n'_0, -n_0]\} \cap \mathbb{Z}.
\] (2.5.14)

Owing to \( s_1 \geq \epsilon^{-1}(1 - \frac{2\mu}{3}) s \), we get
\[
\frac{\tau \zeta^2 n_0}{25/3} \geq (1 - \epsilon) s \geq \epsilon^{-1}(1 - \frac{2\mu}{3})(1 - \epsilon) s.
\] (2.5.15)

Combining (2.5.14) with (2.5.15) and invoking the reverse Minkowski’s inequality yields
\[
\left( (1 - \frac{2\mu}{3}) s + \frac{\tau \zeta^2 n_0}{25/3} \right)^{\frac{3}{2}} \geq \left( \epsilon^{-1}(1 - \frac{2\mu}{3})(1 - \epsilon) s \right)^{\frac{3}{2}} + \frac{\tau^{3/2} \zeta^3}{25/2}, \quad \text{when } \frac{\tau \zeta^2 n}{25/3} \in S_2.
\] (2.5.17)

Plugging this into (2.5.11), summing in a similar way as in the proof of Lemma 2.3.1 and noticing
\[
\epsilon \left( \epsilon^{-1}(1 - \frac{2\mu}{3})(1 - \epsilon) \right)^{\frac{3}{2}} > \epsilon^{-\frac{1}{2}}(1 - \epsilon)^{\frac{1}{2}}(1 - \mu)(1 - \epsilon) > (1 - \mu)(1 - \epsilon),
\]
we arrive at
\[
\sum_{n: 2^{-5/3} \tau \zeta^2 \in S_2} \mathbb{P}(\tilde{E}_n) \leq C_1 \exp \left( -\frac{4}{3}(1 - \epsilon)(1 - \mu)s\right) \] (2.5.16)

for some \( C_1 = C_1(\epsilon, T_0) \) when \( s \) is large enough. From the reverse Minkowski’s inequality,
\[
\left( (1 - \frac{2\mu}{3}) s + \frac{\tau \zeta^2 n_0}{25/3} \right)^{\frac{3}{2}} \geq (1 - \frac{2\mu}{3})^{\frac{3}{2}} s^{\frac{3}{2}} + \frac{\tau^{3/2} \zeta^3}{25/2}.
\] (2.5.17)

Applying (2.5.17) to the r.h.s. of (2.5.11) for all \( n \) such that \( 2^{-5/3} \tau \zeta^2 n_0 \in S_1 \cup S_3 \) and summing in a similar way as in the proof of Lemma 2.3.1 yields
\[
\sum_{n: 2^{-5/3} \tau \zeta^2 \in S_1 \cup S_3} \mathbb{P}(\tilde{E}_n) \leq C_2 \exp \left( -\frac{4}{3}(1 - \epsilon)(1 - \frac{2\mu}{3})^{3/2} s^{\frac{3}{2}} \right) \] (2.5.18)

for some \( C_2 = C_2(\epsilon, T_0) \). Adding (2.5.16) and (2.5.18) and noticing that \( (1 - \frac{2\mu}{3})^{\frac{3}{2}} \geq (1 - \mu) \), we obtain (2.5.10) if \( s \in [s_0, s_1] \cap [\bar{s}, \infty) \) where \( \bar{s} \) depends on \( \epsilon \) and \( T_0 \).

Now, we turn to the case when \( s \in \{(s_1, s_2] \cup (s_2, \infty)\} \cap [s_0, \infty) \). Owing to (2.5.1), for all \( n \in \mathbb{Z} \) for some...
and \( s \in [s_0, \infty) \),

\[
\mathbb{P}(\overline{E}_n) \leq \begin{cases} 
\exp \left( \frac{4}{3} \epsilon \left( \frac{1 - 2\mu}{3} s + \frac{\tau \zeta^2}{2^3/3} \right)^{\frac{3}{2}} \right) & \text{if } s \in (s_1, s_2], \\
\exp \left( \frac{4}{3} (1 - \epsilon) \left( \frac{1 - 2\mu}{3} s + \frac{\tau \zeta^2}{2^3/3} \right)^{\frac{3}{2}} \right) & \text{if } s \in (s_2, \infty).
\end{cases}
\] (2.5.19)

Applying (2.5.17) and summing the r.h.s. of (2.5.19) in the same way as (2.5.18), we find (2.5.10). ■

Now, we show an analogue of Lemma 2.3.2.

**Lemma 2.5.4.** There exists \( s' = s'(\epsilon, T_0, C, \nu, \theta, \kappa, M) \) such that for all \( s \geq s' \),

\[
\left( \bigcup_{n \in \mathbb{Z}} \overline{E}_n \right)^c \cap \left( \bigcup_{n \in \mathbb{Z}} \overline{F}_n \right)^c \subseteq (\overline{A}^l)^c.
\] (2.5.20)

**Proof.** Assume the event of the l.h.s. of (2.5.20) occurs. By (2.1.1) of Definition 2.1.1 and \( \tau + \nu < 1 \),

\[
\int_{-\infty}^{\infty} e^{T^{1/3}(\tau T(y) + f(-y))} \, dy \leq \int_{-\infty}^{\infty} e^{T^{1/3}(C - \frac{1-\tau}{2^3/3} y + (1-\frac{\nu}{3}) s + \frac{\nu}{2^3/3} y^2)} \, dy \leq \frac{K}{T^{1/6}} e^{(1-\frac{\nu}{3}) s T^{1/3}}.
\]

for some \( K = K(C, T, \tau, \nu) > 0 \). There exists \( s' = s'(\mu, T_0, C, \nu, \theta, \kappa, M) \) such that the right hand side of the above inequality is bounded above by \( \exp(s T^{1/3}) \), thus confirming (2.5.20). ■

Applying (2.5.20) and Bonferroni’s union bound, (see (2.3.8) for a similar inequality)

\[
\mathbb{P} \left( \overline{A}^l \cap \left( \bigcup_{n \in \mathbb{Z}} \overline{E}_n \right)^c \right) \leq \sum_{n \in \mathbb{Z}} \mathbb{P} \left( \overline{E}_{n-1}^c \cap \overline{E}_{n+1}^c \cap \overline{F}_n \right).
\] (2.5.21)

**Lemma 2.5.5.** There exists \( s'' = s''(\epsilon, \mu, T_0) \) and \( \Theta = \Theta(\epsilon, T_0) \) such that for all \( s \in [\max\{s'', s_0\}, \infty) \),

\[
\sum_{n \in \mathbb{Z}} \mathbb{P} \left( \overline{E}_{n-1}^c \cap \overline{E}_{n+1}^c \cap \overline{F}_n \right) \leq \begin{cases} 
\Theta e^{-\frac{\sqrt{3}}{4} \epsilon (1-\epsilon) (1-\mu) s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\
\Theta e^{-\frac{\sqrt{3}}{4} \epsilon (1-\mu) s^{3/2}} & \text{if } s \in (s_1, s_2).
\end{cases}
\] (2.5.22)

See (2.5.2) for the definitions of \( s_0, s_1 \) and \( s_2 \).

**Proof.** We need to bound \( \mathbb{P} \left( \overline{E}_{n-1}^c \cap \overline{E}_{n+1}^c \cap \overline{F}_n \right) \) for all \( n \in \mathbb{Z} \). Define

\[
\tilde{\mathcal{E}}_n := \left\{ \mathcal{T}^\epsilon(\zeta_n) \geq \frac{1 + 2^{-1} \tau}{2^{3/3}} \zeta^2_n - s^2 \right\}, \quad \text{for } n \in \mathbb{Z}.
\]

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\[ \mathbb{P}(E_{n-1}^c \cap \tilde{E}_{n+1}^c \cap \widetilde{F}_n) \leq \mathbb{P}((E_{n-1}^c \cap \tilde{E}_{n-1}) \cap (E_{n+1}^c \cap \tilde{E}_{n+1}) \cap \widetilde{F}_n) + \mathbb{P}(\tilde{E}_{n-1}^c) + \mathbb{P}(\tilde{E}_{n+1}^c). \]

We will bound each term on the r.h.s. above. Theorem 1.1.1 provides \( s'' := s''(\epsilon, T_0), \) \( K = K(\epsilon, T_0) > 0 \) and the following upper bound\(^6\) for \( s \geq s'' \) and \( T \geq T_0 \):

\[ \mathbb{P}(\tilde{E}_{n}^c) \leq \exp \left( -T^{\frac{3}{2}} \frac{4}{15\pi} (1 - \epsilon) \left( s^{\frac{2}{3}} + \frac{\tau \zeta_n^2}{2^{5/3}} \right) \right) + \exp \left( -K \left( s^{\frac{2}{3}} + \frac{\tau \zeta_n^2}{2^{5/3}} \right)^{3 - \epsilon} \right). \]

Summing over all \( n \in \mathbb{Z} \) (in the same way as in Lemma 2.3.1) yields

\[ \sum_{n \in \mathbb{Z}} (\mathbb{P}(\tilde{E}_{n-1}^c) + \mathbb{P}(\tilde{E}_{n+1}^c)) \leq e^{-T^{1/3} \frac{4}{15\pi} (1 - \epsilon) s^{5/3}} + e^{-K s^{2 - 2\epsilon/3}}. \quad (2.5.23) \]

**Claim 2.5.2.** There exists \( s'' = s''(\epsilon, \mu, T_0) \), such that for all \( s \geq s'' \), \( T \geq T_0 \) and \( n \in \mathbb{Z} \):

\[ \mathbb{P}((E_{n-1}^c \cap \tilde{E}_{n-1}) \cap (E_{n+1}^c \cap \tilde{E}_{n+1}) \cap \widetilde{F}_n) \leq 2\mathbb{P}(\Upsilon_T(0) \geq 2^{\frac{11}{3}} \zeta_n^2 + \frac{1}{2} (1 - 2^{\frac{2\mu}{3}}) s). \quad (2.5.24) \]

**Proof.** We parallel the proof of [CH14 Proposition 4.4] (see also [CH16 Lemma 4.1]). illustrates the main objects in this proof and the argument (whose details we now provide).

By Proposition 2.2.5, the curve \( 2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\cdot) \) from the KPZ line ensemble \( \{2^{-\frac{1}{3}} \Upsilon_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}} \) has the same distribution as \( 2^{-\frac{1}{3}} \Upsilon_T(\cdot) \). For the rest of this proof, we replace \( \Upsilon_T \) by \( \Upsilon_T^{(1)} \) in the definitions of \( \{E_n\}, \{\tilde{F}_n\} \) and \( \{\tilde{E}_n\} \). We define the following three curves:

\[ U(y) := -\frac{(1 - \tau)}{2^{2/3}} y^2 + \left( 1 - \frac{\mu}{3} \right) s, \quad L(y) := -\frac{(1 + 2^{-1} \tau)}{2^{2/3}} y^2 - s^{\frac{2}{3}}, \quad M(y) := -\frac{(1 - \tau)}{2^{2/3}} y^2 + (1 - \frac{\mu}{3}) s. \]

If \( E_{n-1}^c \cap \tilde{E}_{n-1} \) and \( E_{n+1}^c \cap \tilde{E}_{n-1} \) occurs, then, \( \Upsilon_T^{(1)}(\cdot) \) stays in between the curves \( M(\cdot) \) and \( L(\cdot) \) at the points \( \zeta_{n-1} \) and \( \zeta_{n+1} \) respectively. If \( \tilde{F}_n \) occurs, then, \( \Upsilon_T^{(1)}(\cdot) \) touches the curve \( U(\cdot) \) at some point in the interval \([\zeta_n, \zeta_{n+1}]\). Therefore, on the event \((E_{n-1}^c \cap \tilde{E}_{n-1}) \cap (E_{n+1}^c \cap \tilde{E}_{n+1}) \cap F_n, \Upsilon_T^{(1)}(\cdot) \) hits \( U(\cdot) \) somewhere in the interval \([\zeta_n, \zeta_{n+1}]\) whereas it stays in between \( M(\cdot) \) and \( L(\cdot) \) at the points

\(^6\)Taking \( \epsilon = \delta \) in Theorem 1.1.1 the r.h.s. of 1.1.3 \( \leq \exp(-T^{\frac{3}{2}} \frac{4(1 - \epsilon)s^{5/2}}{15\pi}) + \exp(-K s^{3 - \epsilon}). \)
Figure 2.2: Illustration from the proof of (3.4.16). The three parabolas are $U(\cdot)$, $M(\cdot)$ and $L(\cdot)$. The solid black curve is $\Upsilon_T^{(1)}(\cdot)$ when $\hat{E}_{n-1} \cap \hat{E}_{n-1} \cap \hat{E}_{n+1} \cap \hat{F}_n$ occurs. Note that $\Upsilon_T^{(1)}(\cdot)$ stays in between $M(\cdot)$ and $L(\cdot)$ at $\zeta_{n-1}$ and $\zeta_{n+1}$. The rightmost point in $(\zeta_n, \zeta_{n+1})$ where $\Upsilon_T^{(1)}(\cdot)$ hits $U(\cdot)$ is labeled $\sigma_n$. The event that the black curve stays above the square at $\zeta_n$ is $\mathcal{B}_n$ and $\mathbb{P}_{H^{2T}}(\mathcal{B}_n)$ (see (2.5.25) for $\mathbb{P}_{H^{2T}}$) is the probability of $\mathcal{B}_n$ conditioned on the sigma algebra $\mathcal{F}_{\text{ext}}(\{1\} \times (\zeta_n-1, \sigma_n))$. On the other hand, $\widetilde{\mathbb{P}}_{H^{2T}}(\mathcal{B}_n)$ (see (3.4.17) for $\widetilde{\mathbb{P}}_{H^{2T}}$) is the probability of $\mathcal{B}_n$ under the free Brownian bridge (scaled by $2^{\frac{3}{2}}$) measure on the interval $(\zeta_n-1, \sigma_n)$ with same starting and end point as $\Upsilon_T^{(1)}(\cdot)$. The dashed black curve is such a free Brownian bridge coupled to $\Upsilon_T^{(1)}(\cdot)$ so that $B(y) \leq \Upsilon_T^{(1)}(y)$ for all $y \in (\zeta_n-1, \sigma_n)$. Owing to this coupling, $\mathbb{P}_{H^{2T}}(\mathcal{B}_n) \geq \widetilde{\mathbb{P}}_{H^{2T}}(\mathcal{B}_n)$. The probability of $B(\sigma_n)$ staying above the bullet point is $\frac{1}{2}$ which implies that $\mathbb{P}_{H^{2T}}(\mathcal{B}_n) \geq \frac{1}{2}$. Consequently, we can bound the probability of $(\hat{E}_{n-1} \cap \hat{E}_{n-1}) \cap (\hat{E}_{n+1} \cap \hat{E}_{n+1}) \cap \hat{F}_n$ by $2\mathbb{P}(\mathcal{B}_n)$ (see (3.4.19)). The expected value of $\mathbb{P}(\mathcal{B}_n)$ can be bounded above by the upper tail probability of $\Upsilon_T^{(1)}(\zeta_n)+\frac{\zeta_n^2}{2(2T)}$ (see (3.4.24)). The upper bound in (3.4.16) follows then by invoking Proposition 2.1.7.

Let us define $\sigma_n := \sup \left\{ y \in (\zeta_n, \zeta_{n+1}) : \Upsilon_T^{(1)}(y) \geq U(y) \right\}$.

Recall that $\zeta_{n-1} < \zeta_n < \zeta_{n+1}$. Consider the following crossing event

$$\mathcal{B}_n := \left\{ \Upsilon_T^{(1)}(\zeta_n) \geq \frac{\sigma_n - \zeta_n}{\sigma_n - \zeta_{n-1}} L(\zeta_{n-1}) + \frac{\zeta_n - \zeta_{n-1}}{\sigma_n - \zeta_{n-1}} U(\sigma_n) \right\}.$$  

We will use the following abbreviation for the probability measures

$$\mathbb{P}_{H^{2T}} := \mathbb{P}_{H^{2T}}^{1,1,,(\zeta_{n-1}, \sigma_n),2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\zeta_{n-1}),2^{-\frac{1}{3}} \Upsilon_T^{(2)}(\sigma_n),+\infty,2^{-\frac{1}{3}} \Upsilon_T^{(2)}},$$  

$$\widetilde{\mathbb{P}}_{H^{2T}} := \mathbb{P}_{H^{2T}}^{1,1,,(\zeta_{n-1}, \sigma_n),2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\zeta_{n-1}),2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\sigma_n),+\infty,-\infty}.$$  

Since, $(\zeta_{n-1}, \sigma_n)$ is a $\{1\}$-stopping domain (see Definition 2.2.1) for the KPZ line ensemble, the
Now, we bound the r.h.s. of (3.4.19). Note the following holds:

\[
\mathbb{E} \left[ 1 \left( (\tilde{E}_{n-1}^c \cap \tilde{E}_{n-1}) \cap (\tilde{E}_{n+1}^c \cap \tilde{E}_{n+1}) \cap \tilde{F}_n \right) \cdot 1 \left( \tilde{\mathcal{B}}_n \right) \right] \quad (2.5.27)
\]

\[
= 1 \left( (\tilde{E}_{n-1}^c \cap \tilde{E}_{n-1}) \cap (\tilde{E}_{n+1}^c \cap \tilde{E}_{n+1}) \cap \tilde{F}_n \right) \cdot \mathbb{P}_{\mathbf{H}_{2T}}(\tilde{\mathcal{B}}_n).
\]

By Proposition 2.2.6, there exists a monotone coupling\(^9\) between the probability measures \(\mathbb{P}_{\mathbf{H}_{2T}}\) and \(\mathbb{P}_{\mathbf{H}_{2T}}\). Using this and the fact that the probability of \(\mathcal{B}_n\) increases under pointwise increase of its sample paths, we have \(\mathbb{P}_{\mathbf{H}_{2T}}(\tilde{\mathcal{B}}_n) \geq \mathbb{P}_{\mathbf{H}_{2T}}(\tilde{\mathcal{B}}_n)\). Since \(\mathbb{P}_{\mathbf{H}_{2T}}\) is the law of a Brownian bridge on the interval \((\zeta_{n-1}, \sigma_n)\) with end points \(2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\zeta_{n-1})\) and \(2^{-\frac{1}{3}} \Upsilon_T^{(1)}(\sigma_n)\), the probability that it stays above the line joining the two end points at a given intermediate point is \(\frac{1}{2}\). Therefore \(\mathbb{P}_{\mathbf{H}_{2T}}(\tilde{\mathcal{B}}_n) \geq \frac{1}{2}\). Plugging this into (3.4.18) and taking expectation yields

\[
\mathbb{P} \left( (\tilde{E}_{n-1}^c \cap \tilde{E}_{n-1}) \cap (\tilde{E}_{n+1}^c \cap \tilde{E}_{n+1}) \cap \tilde{F}_n \right) \leq 2\mathbb{E} \left[ 1 \left( (\tilde{E}_{n-1}^c \cap \tilde{E}_{n-1}) \cap (\tilde{E}_{n+1}^c \cap \tilde{E}_{n+1}) \cap \tilde{F}_n \right) \cdot 1 \left( \tilde{\mathcal{B}}_n \right) \right].
\]

(2.5.28)

Now, we bound the r.h.s. of (3.4.19). Note the following hold\(^10\) for all \(n \in \mathbb{Z}\):

\[
\frac{(\sigma_n - \zeta_n) \zeta_n^2}{\sigma_n - \zeta_{n-1}} = (\sigma_n - \zeta_n)(\zeta_n - \zeta_{n-1}) \leq \frac{1}{s^{2+2\delta}},
\]

(2.5.29)

\[
-\frac{1}{s} (\sigma_n - \zeta_n) \zeta_n^2 + (\zeta_n - \zeta_{n-1}) \sigma_n^2
\]

\[
= \frac{1}{2} \zeta_n^2 + \frac{1}{2} \sigma_n^2 - \frac{1}{2} (\sigma_n - \zeta_n)(\zeta_n - \zeta_{n-1}) + \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}{2} \sigma_n = \frac{3}{2} \zeta_n - \frac{1}{2} \sigma_n - \frac{1}
below by $-2^{-\frac{2}{3}}(1 - 8^{-1}\tau) + \frac{1}{2}(1 - \frac{2\mu}{3}) s$ when $s$ is large enough. Hence, we have

$$\text{r.h.s. of (3.4.19)} \leq 2\mathbb{P}(\tilde{B}_n) \leq 2\mathbb{P}\left(\Upsilon^{(1)}_T(\zeta_n) \geq -\frac{(1 - 8^{-1}\tau)}{2^{2/3}}\zeta_n^2 + \frac{1}{2}(1 - \frac{2\mu}{3}) s\right). \quad (2.5.32)$$

Now, the claim follows from (3.4.19) and (3.4.24) by recalling that $\Upsilon^{(1)}_T(\zeta_n^2) + \frac{\zeta_n^2}{2^{2/3}} = \Upsilon_T(0)$. ■

Using (3.4.16) and a similar analysis as in Lemma 2.5.3, there exist $s'' = s''(\epsilon, \mu, T_0)$ and $C' = C'(\epsilon, T_0)$ such that for all $s \in [\max\{s'', s_0\}, \infty)$,

$$\sum_{n \in \mathbb{Z}} \mathbb{P}\left((\tilde{E}_{n-1}^{c} \cap \tilde{E}_{n-1}^c) \cap (\tilde{E}_{n+1}^{c} \cap \tilde{E}_{n+1}^c) \cap \tilde{F}_n\right) \leq \begin{cases} C' e^{-\frac{\sqrt{2}}{3}(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ C' e^{-\frac{\sqrt{2}}{3}\epsilon(1-\mu)s^{3/2}} & \text{if } s \in (s_1, s_2]. \end{cases} \quad (2.5.33)$$

Combining this with (3.4.15), we arrive at (2.5.22).

**Final step of the proof of Proposition 2.5.1:** Define $s'_0 := \max\{\bar{s}, s', s''\}$ where $\bar{s}, s', s''$ are taken from Lemmas 2.5.3, 2.5.4 and 2.5.5 respectively.

1. Owing to (2.5.21) and (2.5.22), when $T_0 > \pi$, there exists $\Theta = \Theta(\epsilon, T_0)$ such that for all $s \in [\max\{s'', s_0\}, \infty)$

$$\mathbb{P}\left(\tilde{A}^f \cap \left(\bigcup_{n \in \mathbb{Z}} \tilde{E}_n\right)^c\right) \leq \begin{cases} \Theta e^{-\frac{\sqrt{2}}{3}(1-\epsilon)(1-\mu)s^{3/2}} & \text{when } s \in [s_0, s_1] \cup (s_2, \infty), \\ \Theta e^{-\frac{\sqrt{2}}{3}\epsilon(1-\mu)s^{3/2}} & \text{when } s \in (s_1, s_2]. \end{cases} \quad (2.5.33)$$

Plugging (2.5.33) and (2.5.10) of Lemma 2.5.3 into the r.h.s. of (2.5.9) yields (2.5.3).

2. When $T_0 \in (0, \pi)$, the proof of (2.5.3) follows in the same way as in the proof of (2.5.3) by assuming $\mathbb{P}(\Upsilon_T(0) > s) \leq e^{-cs^{3/2}}$ for all $s \geq s_0$ and $T \in [T_0, \pi]$. 

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Proof of Proposition 2.5.2

Let $I$ be a subinterval of $[-M, M]$ with $|I| = \theta$ such that $f(y) \geq -\kappa$ for all $y \in I$. Assume $s$ is large enough such that $s^{-n+2} \leq \theta$. Let $\chi_1 \leq \chi_2 \in I$ be such that $\chi_2 - \chi_1 = s^{-n+2}$. Define

$$
W_i := \left\{ \Upsilon_T(-\chi_i) \geq -\frac{\chi_i^2}{2^{2/3}} + \left(1 + \frac{2\mu}{3}\right)s \right\} \text{ for } i = 1, 2,
$$

$$
W_{\text{int}} := \left\{ \Upsilon_T(y) \geq -\frac{y^2}{2^{2/3}} + \left(1 + \frac{\mu}{3}\right)s \text{ for all } y \in (-\chi_2, -\chi_1) \right\}.
$$

We claim that there exists $s' = s'(\mu, \theta, \kappa, T_0)$ such that for all $s \geq s'$ and $T \geq T_0$

$$
\mathbb{P}(W_1 \cap W_2 \cap W_{\text{int}}) \leq \mathbb{P}(h_T^f(0) \geq s). \tag{2.5.34}
$$

To show this, assume that the event $W_1 \cap W_2 \cap W_{\text{int}}$ occurs. Then

$$
\int_{-\infty}^{\infty} e^{T^{1/3}(\Upsilon_T(y) + f(-y))} dy \geq \int_{-I} e^{T^{1/3}(\Upsilon_T(y) + f(-y))} dy \geq 2\theta e^{T^{1/3}(1+\mu/3)s - \kappa} \geq e^{T^{1/3}s}
$$

where the last inequality holds when $s$ exceeds some $s'(\mu, \theta, \kappa, T_0)$. This shows that

$$
\mathbb{P}(W_1 \cap W_2 \cap W_{\text{int}}) \leq \mathbb{P}\left(\int_{-\infty}^{\infty} e^{T^{1/3}(\Upsilon_T(y) + f(y))} dy \geq e^{T^{1/3}s}\right) = \mathbb{P}(h_T^f(0) \geq s).
$$

To finish the proof of (2.5.5) we combine (2.5.34) with (2.5.35) below and take $s_0 = \max\{s', s''\}$.

Claim 2.5.3. There exist $s'' = s''(\mu, n, T_0)$, $K = K(\mu) > 0$ such that for all $s \geq s''$ and $T \geq T_0$,

$$
\mathbb{P}(W_1 \cap W_2 \cap W_{\text{int}}) \geq \mathbb{P}(\Upsilon_T(0) > \left(1 + \frac{2\mu}{3}\right)s)^2 - e^{-Ks^n}. \tag{2.5.35}
$$

Proof. We start by writing $\mathbb{P}(W_1 \cap W_2 \cap W_{\text{int}}) = \mathbb{P}(W_1 \cap W_2) - \mathbb{P}(W_1 \cap W_2 \cap W_{\text{int}}^c)$. Using the FKG inequality from (2.1.8),

$$
\mathbb{P}(W_1 \cap W_2) \geq \mathbb{P}(W_1)\mathbb{P}(W_2) \geq \left(\mathbb{P}(\Upsilon_T(0) > \left(1 + \frac{2\mu}{3}\right)s)\right)^2 \tag{2.5.36}
$$

where the last inequality follows from Proposition 2.1.7. Note that (2.5.36) provides a lower bound.

\footnote{We use below $-I$ as a shorthand notation for $\{x : -x \in I\}$}
for the first term on the r.h.s. of (2.5.35). To complete the proof, we need to demonstrate an upper bound on $P(W_1 \cap W_2 \cap W_c \cap \text{int})$ of the form $e^{-Ks^n}$. To achieve this we go to the KPZ line ensemble and use its Brownian Gibbs property. We may replace $\Upsilon_T$ by $\Upsilon_T^{(1)}$ in all definitions without changing the value of $P(W_1 \cap W_2 \cap W_c \cap \text{int})$ (see Proposition 2.2.5). Let us define

$$P_{H_2T} := P_{H_2T}^{1,1,(-\chi_2,-\chi_1),2^{-\frac{1}{2}}\Upsilon_T^{(1)}(-\chi_2),2^{-\frac{3}{4}}\Upsilon_T^{(1)}(-\chi_1),+\infty,2^{-\frac{3}{4}}\Upsilon_T^{(2)}},$$

$$\tilde{P}_{H_2T} := P_{H_2T}^{1,1,(-\chi_2,-\chi_1),2^{-\frac{1}{4}}\Upsilon_T^{(1)}(-\chi_2),2^{-\frac{3}{4}}\Upsilon_T^{(1)}(-\chi_1),+\infty,-\infty}.$$

Using the $H_{2T}$-Brownian Gibbs property of the KPZ line ensemble $\{2^{-\frac{1}{3}}\Upsilon_T^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$,

$$P(W_1 \cap W_2 \cap W_c \cap \text{int}) = \mathbb{E}[1(W_1 \cap W_2) \cdot P_{H_2T}(W_c \cap \text{int})]. \quad (2.5.37)$$

Via Proposition 2.2.6 there exists a monotone coupling between $P_{H_2T}$ and $\tilde{P}_{H_2T}$ so that

$$P_{H_2T}(W_c \cap \text{int}) \leq \tilde{P}_{H_2T}(W_c \cap \text{int}). \quad (2.5.38)$$

Recall that $\tilde{P}_{H_2T}$ is the measure of a Brownian bridge on $(-\chi_2,-\chi_1)$ with starting and end points at $2^{-\frac{1}{2}}\Upsilon_T^{(1)}(-\chi_2)$ and $2^{-\frac{1}{4}}\Upsilon_T^{(1)}(-\chi_1)$. Applying (2.5.38) into the r.h.s. of (2.5.37) implies

$$1(W_1 \cap W_2) \cdot \tilde{P}_{H_2T}(W_c \cap \text{int}) \leq P_{\text{free}}^{1,1,(-\chi_2,-\chi_1),-\frac{\chi_2^2}{2}+2^{-\frac{3}{4}}(1+\frac{2\mu}{3})s,-\frac{\chi_1^2}{2}+2^{-\frac{3}{4}}(1+\frac{2\mu}{3})s}(W_c \cap \text{int}).$$

Therefore (using Lemma 2.2.7 for the second inequality) there exists $K = K(\mu)$ such that

$$\text{l.h.s. of (2.5.37)} \leq P_{\text{free}}^{1,1,(-\chi_2,-\chi_1),-\frac{\chi_2^2}{2}+2^{-\frac{3}{4}}(1+\frac{2\mu}{3})s,-\frac{\chi_1^2}{2}+2^{-\frac{3}{4}}(1+\frac{2\mu}{3})s}(W_c \cap \text{int}) \leq e^{-Ks^n}. \ ■$$

2.5.2 Proof of Theorem 2.1.12

Theorem 2.1.12 follows by combining all three parts of Theorem 2.1.9 with the following results which are in the same spirit of Proposition 2.5.1 and 2.5.2 respectively.

Recall $\Upsilon_T$ and $h_T^{Br}$ from (1.1.2) and (2.1.10) respectively.
Proposition 2.5.6. Fix $\epsilon, \mu \in (0, \frac{1}{2})$.

1. Fix $T_0 > \pi$. Suppose there exists $s_0 = s_0(\epsilon, T_0)$ and for any $T \geq T_0$, there exist $s_2 = s_2(\epsilon, T)$ and $s_3 = s_3(\epsilon, T)$ with $s_1 \leq s_2 \leq s_3$ such that for any $s \in [s_0, \infty)$,

$$
P(\Upsilon_T(0) > s) \leq \begin{cases} 
e^{-\frac{4}{3}(1-\epsilon)s^2} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ e^{-\frac{4}{3}cs^2} & \text{if } s \in (s_1, s_2]. \end{cases} \quad (2.5.39)
$$

Then, there exists $s'_0 = s'_0(\epsilon, \mu, T_0)$ such that for any $T > T_0$ and $s \in [\max\{s'_0, s_0\}, \infty)$, we have (recall $s_0, s_1$ and $s_2$ from (2.5.2))

$$
P(h^\text{Br}_T(0) > s) \leq \begin{cases} e^{-\frac{\sqrt{2}}{3}(1-\epsilon)(1-\mu)s^{3/2}} + e^{-\frac{1}{9\sqrt{3}}(\mu s)^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\ e^{-\frac{\sqrt{2}}{3}(1-\mu)s^{3/2}} + e^{-\frac{1}{9\sqrt{3}}(\mu s)^{3/2}} & \text{if } s \in (s_1, s_2]. \end{cases} \quad (2.5.40)
$$

2. For any $T_0 \in (0, \pi)$, there exists $s'_0 = s'_0(T_0) > 0$ satisfying the following: if there exists $s_0 = s_0(T_0) > 0$ such that $P(\Upsilon_T(0) > s) \leq e^{-cs^{3/2}}$ for all $s \geq s_0$ and $T \in [T_0, \pi]$, then,

$$
P(h^\text{f}_T(0) > s) \leq e^{-cs^{3/2}}, \quad \forall s \in [\max\{s'_0, s_0\}, \infty), T \in [T_0, \pi]. \quad (2.5.41)
$$

Proposition 2.5.7. Fix $\mu \in (0, \frac{1}{2})$, $n \in \mathbb{Z}_{\geq 3}$ and $T_0 > \pi$. Then, there exist $s_0 = s_0(\mu, n, T_0), K = K(\mu, n) > 0$ such that for all $s \geq s_0$ and $T \geq T_0$,

$$
P(h^\text{Br}_T(0) > s) \geq \left( P(\Upsilon_T(0) > (1 + \frac{2\mu}{3})s) \right)^2 - e^{-Ks^n}. \quad (2.5.42)
$$

We prove these propositions using similar arguments as in Section 2.5.1 and Propositions 2.5.6 and 2.5.7 are proved in Sections 2.5.2 and Section 2.5.2, respectively.

Proof of Theorem 2.1.12. This theorem is proved in the same way as Theorem 2.1.4 by combining Proposition 2.5.6 and Proposition 2.5.7. We do not duplicate the details. ■
Proof of Proposition 2.5.6

To prove this proposition, we use similar arguments as in Section 2.5.1. Let \( \tau \in (0, \frac{1}{2}) \) be fixed (later we choose its value). Recall the events \( \tilde{E}_n \) and \( \tilde{F}_n \) from Section 2.5.1 and define

\[
\tilde{A}^{B_t} := \left\{ \int_{-\infty}^{\infty} e^{T^{1/3}(\tau T(y) + B(-y))} dy > e^{sT^{1/3}} \right\}
\]

where \( B \) is a two sided Brownian motion with diffusion coefficient \( 2^{1/3} \) and \( B(0) = 0 \). Appealing to Proposition 2.1.6 we see that \( P(\mu_t^{B_t}(0) > s) = P(\tilde{A}^{B_t}) \). Now, we write

\[
P(\tilde{A}^{B_t}) \leq \sum_{n \in \mathbb{Z}} P(\tilde{E}_n) + P\left( \tilde{A}^{B_t} \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right)^c \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{F}_n \right)^c \right) + P\left( \tilde{A}^{B_t} \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right)^c \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{F}_n \right)^c \right).
\]

(2.5.43)

Using Lemma 2.5.3 (see (2.5.10)) and Lemma 2.5.5 (see (2.5.22)) we can bound the first two terms on the right side hand side of (2.5.43). However, unlike in Theorem 2.5.1, the last term in (2.5.43) is not zero. We now provide an upper bound to this term.

Claim 2.5.4. There exists \( s' = s'(\tau, \mu) \) such that for all \( s \geq s' \),

\[
P\left( \tilde{A}^{B_t} \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right)^c \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{F}_n \right)^c \right) \leq \exp\left(-\sqrt{(1-2\tau)\frac{y^2}{2^{2/3}} + B(-y)}\left(2\mu s T^{1/3} + \frac{1}{2} \log((2\pi)^{-1}\tau(2T)^{1/3})\right)\right)^{\frac{3}{2}}.
\]

(2.5.44)

Proof. Note that

\[
\left\{ \tilde{A}^{B_t} \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right)^c \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{F}_n \right)^c \right\} \subseteq \left\{ \int_{-\infty}^{\infty} e^{T^{1/3}(\tau T(y) + B(-y))} dy \geq e^{sT^{1/3}} \right\}.
\]

(2.5.45)

We claim that

\[
\text{r.h.s. of (2.5.45)} \subseteq \left\{ \max_{y \in \mathbb{R}} \left\{ -\frac{(1-2\tau)y^2 + B(-y)}{2^{2/3}} \right\} \geq \frac{1}{3}\mu s + \frac{1}{2} \log((2\pi)^{-1}\tau(2T)^{1/3}) \right\}.
\]

(2.5.46)

To see this by contradiction, assume the complement of the r.h.s. of (2.5.46). This implies that

\[
\int_{-\infty}^{\infty} e^{T^{1/3}(\tau T(y) + B(-y))} dy < \sqrt{(2\pi)^{-1}\tau(2T)^{1/3}e^{3^{-1}\mu s T^{1/3}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2T^{1/3}2^{2/3}}} dy = e^{3^{-1}\mu s T^{1/3}}.
\]
Therefore, (2.5.46) holds. Applying Proposition 2.2.8 (with \( \xi = \frac{1}{2} \)), we see that
\[
\mathbb{P}\left( \text{r.h.s. of (2.5.46)} \right) \leq \frac{1}{\sqrt{3}} \exp \left( -\frac{\sqrt{(1-2\tau)}}{3\sqrt{6}} \left( \frac{2\mu s}{3} + \log((2\pi)^{-1}(2T)^{1/3}) \right)^{3/2} \right),
\]
when \( s \) is large enough. Combining this with (2.5.45), we arrive at (2.5.44) showing the claim.

Now, we turn to complete the proof of Proposition 2.5.6. Choosing \( \tau = \frac{1}{8} \), we notice
\[
\text{r.h.s. of (2.5.44)} \leq \exp \left( -\frac{1}{9\sqrt{3}} \left( \mu s - \frac{3\log(16\pi)}{2} \right)^{3/2} \right), \quad \forall T > \pi.
\]

For the rest of this proof, we will fix some \( T \geq T_0 \) and assume that there exist \( s_0 = s_0(\epsilon, T_0) \), \( s_1 = s_1(\epsilon, T) \) and \( s_2 = s_2(\epsilon, T) \) with \( s_1 \leq s_2 \) such that (2.5.39) is satisfied for all \( s \in [s_0, \infty) \). Owing to (2.5.10) of Lemma 2.5.3 and (2.5.22) of Lemma 2.5.5, there exist \( \Theta = \Theta(\epsilon, T_0) \) and \( \tilde{s} = \tilde{s}(\epsilon, \mu, T_0) \) such that for all \( s \in \max\{\tilde{s}, s_0\}, \infty \),
\[
\mathbb{P}\left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right) + \mathbb{P}\left( \mathcal{A}^{Br} \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{E}_n \right) \cap \left( \bigcup_{n \in \mathbb{Z}} \tilde{F}_n \right) \right) \leq \begin{cases} 
\Theta e^{-\frac{\sqrt{2}}{3}(1-\epsilon)(1-\mu)s^{3/2}} & \text{if } s \in [s_0, s_1] \cup (s_2, \infty), \\
\Theta e^{-\frac{\sqrt{2}}{3}(1-\mu)s^{3/2}} & \text{if } s \in (s_1, s_2].
\end{cases}
\]

Combining this with (2.5.44) and plugging into (2.5.43), we get (2.5.40) for all \( T > T_0 \geq \pi \). In the case when \( T_0 \in (0, \pi) \), we obtain (2.5.41) in a similar way as in the proof of (2.5.4) of Proposition 2.5.1 by combining the inequality of the above display with \( \mathbb{P}(Y_T(0)) \leq e^{-c_\alpha s^{3/2}} \) for all \( s \geq s_0 \) and \( T \in [T_0, \pi] \).

**Proof of Proposition 2.5.7**

We use similar argument as in Proposition 2.5.2. The main difference from the proof of Proposition 2.5.2 is that we do not expect (2.5.34) to hold because the initial data is now a two sided Brownian motion, hence, (2.1.2) of Definition 2.1.1 is not satisfied. However, it holds with high probability which follows from the following simple consequence of the reflection principle for \( B \) (a two-sided Brownian motion with diffusion coefficient \( 2^\frac{3}{2} \) and \( B(0) = 0 \))
\[
\mathbb{P}(\mathcal{M}_s) \leq e^{-\frac{\mu^2}{2\pi} s^n}, \quad \text{where } \mathcal{M}_s = \left\{ \min_{y \in [-s^{-n+2}, \infty]} B(t) \leq -\frac{\mu}{\sqrt{s}} \right\}, \quad (2.5.47)
\]
To complete the proof, let us define:

\[ \tilde{W}_\pm := \left\{ \Upsilon_T(\pm s^{-n+2}) \geq -\frac{1}{2^{2/3}s^{2(n-2)}} + (1 + \frac{2\mu}{3}) \right\}, \]

\[ \tilde{W}_{\text{int}} := \left\{ \Upsilon_T(y) \geq -\frac{y^2}{2^{2/3}} + (1 + \frac{\mu}{3}) s, \quad \forall y \in [-s^{-n+2}, s^{-n+2}] \right\}. \]

We claim that there exists \( s' = s'(\mu, n, T_0) \) such that for all \( s \geq s' \) and \( T \geq T_0 \),

\[ P(\tilde{W}_T^B(0) > s) \geq P(\tilde{W}_+ \cap \tilde{W}_- \cap \tilde{W}_{\text{int}}) - e^{-\frac{2}{3n}s^n}. \] (2.5.48)

To see this, assume \( \tilde{W}_+ \cap \tilde{W}_- \cap \tilde{W}_{\text{int}} \cap \mathcal{M}_s \) occurs. Then, for \( s \) large enough,

\[ \int_{-\infty}^{\infty} e^{T^{1/3}(\Upsilon_T(y)+B(-y))} dy \geq \int_{-s-n+2}^{-s-n+2} e^{T^{1/3}\left(-\frac{1}{2^{2/3}s^{2(n-2)}} + (1 + \frac{\mu}{3}) s\right)} dy > e^{sT^{1/3}}. \] (2.5.49)

By Proposition 2.1.6 the event \{l.h.s. of (2.5.49) \geq r.h.s. of (2.5.49)\} equals \( \{h_T^B(0) > s\} \). Therefore (using (2.5.47) for the second inequality) we arrive at the claimed (2.5.48) via

\[ P(\{h_T^B(0) > s\}) \geq P(\tilde{W}_+ \cap \tilde{W}_- \cap \tilde{W}_{\text{int}} \cap \mathcal{M}_s) \geq P(\tilde{W}_+ \cap \tilde{W}_- \cap \tilde{W}_{\text{int}}) - e^{-\frac{2}{3n}s^n}. \]

To finish the proof of Proposition 2.5.7 we use a similar argument as used to prove (2.5.35). For any \( n \in \mathbb{Z}_{\geq 3} \), there exists \( s'' = s''(\mu, n, T_0) \) such that for all \( s \geq s'' \) and \( T \geq T_0 \),

\[ P(\tilde{W}_+ \cap \tilde{W}_- \cap \tilde{W}_{\text{int}}) \geq \left( P(\Upsilon_T(0) > (1 + \frac{2\mu}{3}) s) \right)^2 - e^{-Ks^n}. \]

Combining this with (2.5.48) and taking \( s_0 = \max\{s', s''\} \), we arrive at (2.5.42) for all \( s \geq s_0 \).
Chapter 3: KPZ equation correlation in time

3.1 Introduction

The purpose of this chapter is to verify for the first time the KPZ equation’s $3/3$ temporal exponent in the case of narrow wedge initial data. We will prove that the correlation of centered and scaled fluctuations at a pair of distinct moments in the time scale $T$ transitions between one as the times approach each other, and zero as they separate – and we will quantify this transition with precise power-law exponents for the speed.

First we specify notation that represents a $3 : 2 : 1$ scaled version of $H^{nw}(T, X)$ in a manner suitable for the presentation and proof of our main results. The parameter $t > 0$ specifies a time scale; the value of $\alpha > 0$ specifies time judged on this scale; and $X \in \mathbb{R}$ specifies location judged on the suitable spatial scale $T^{2/3}$. Indeed, for $T, \alpha > 0$ and $X \in \mathbb{R}$, we set

$$\Upsilon_T(\alpha, X) := T^{-1/3} \left( H^{nw}(\alpha T, T^{2/3} X) + \alpha T/24 \right).$$

The use of a pair $(\alpha, T)$ of temporal parameters may seem to introduce a touch of redundancy, but various scalings are in our view simplified by the use of these parameters; and we hope that conceptual clarity is offered by the division of roles between $T$, whose value fixes a time scale, and $\alpha$, which varies on the given scale.

3.1.1 Main results

Our first main result gives bounds on the correlation between $\Upsilon_T(1, 0)$ and $\Upsilon_T(\alpha, 0)$ for $\alpha > 2$. In this case of two remote times, correlations decay as $\alpha^{-1/3}$ in the limit of high $\alpha$. Recall that the correlation between two random variables $X$ and $Y$ is defined to be

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}}, \quad \text{where} \quad \text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y].$$
Theorem 3.1.1 (Remote correlations). There exist $T_0, c_1, c_2 > 0$ such that, for all $T > T_0$ and $\alpha > 2$, \[ c_1^{\alpha^{-1/3}} \leq \text{Corr}(\Upsilon_T(1,0), \Upsilon_T(\alpha,0)) \leq c_2^{\alpha^{-1/3}}. \] (3.1.2)

The second main result bounds the correlation between $\Upsilon_T(1,0)$ and $\Upsilon_T((\beta + 1),0)$ when $\beta$ is smaller than $1/2$. As $\beta$ approaches zero, the correlation between $\Upsilon_T(1,0)$ and $\Upsilon_T(\beta + 1,0)$ approaches one with a discrepancy of order $\beta^{2/3}$.

Theorem 3.1.2 (Adjacent correlations). For any $T_0 > 0$, there exist $c_1 = c_1(T_0) > 0$ and $c_2 = c_2(T_0) > 0$ such that, for all $T > T_0$ and $\beta \in (0,1/2)$ satisfying $\beta T > T_0$, \[ c_1^{\beta^{2/3}} \leq 1 - \text{Corr}(\Upsilon_T(1,0), \Upsilon_T(1 + \beta,0)) \leq c_2^{\beta^{2/3}}. \] (3.1.3)

These theorems invite two interesting questions. Notice that, in Theorem 3.1.1, the minimal time $t_0$ must be large enough for the bounds to hold. First, we may ask: what happens to the two bounds for small $t$? In fact, in our proof of this theorem, we show a stronger upper bound: for any $t_0 > 0$, there exists $c_2(t_0) > 0$ so that, for all $t > t_0$ and $\alpha > 2$, $\text{Corr}(\Upsilon_T(1,0), \Upsilon_T(\alpha,0)) \leq c_2^{\alpha^{-1/3}}$. Our proof does not, however, produce a matching lower bound. It is unclear to us if a new phenomenon occurs at short time that renders such a bound invalid or if instead our technique of proof is unsuitable for obtaining a bound of this form.

The second question also touches on short-time behaviour. In Theorem 3.1.2 we require $\beta T > T_0$ for some arbitrary yet fixed $T_0$. For $T$ fixed, this limits us to values of $\beta > T_0/T$. (As $T \to \infty$, this lower bound tends to zero.) What happens for $\beta$ smaller than $T_0/T$? The source of the restriction $\beta T > T_0$ comes from the application of the one-point tail behaviour from Propositions 3.2.4 and 3.2.5. These tail bounds are valid provided that time exceeds $T_0$. When the time gap $\beta T$ is less than $T_0$, we need short-time tail bounds which are presently not available in the literature.

We further mention some other natural themes related to our first two main results. Theorems 3.1.1 and 3.1.2 both probe the two-time correlation when the spatial coordinate is zero. In this context, two space-time points lie along a space-time line of velocity zero emanating from the origin. In general, space-time lines of fixed velocity emanating from the origin are called *characteristics*; and, by trivial affine shifts – see Proposition 2.1.7 – our results imply the same correlation behaviour.
along such characteristics. On the other hand, we have not addressed what happens when, say, 
\( \text{Corr}(\Upsilon_T(1,0), \Upsilon_T(\alpha,X)) \) is considered for \( X \neq 0 \). This is a problem that may plausibly be addressed by the methods of the present article. That said, if the aim is to prove results concerning more general initial data than the narrow wedge for the KPZ equation, our methods will not be useful without significant further input. Indeed, as the proof sketch offered in Section 3.1.2 will explain, we rely upon a special property of narrow wedge initial data, namely its connection to the KPZ line ensemble and this ensemble’s Gibbs property; and this essential aspect is lacking when general initial data is considered.

We state two further theorems concerning local fluctuations of the KPZ equation in space and time.

**Theorem 3.1.3.** For any \( T_0 > 0 \) there exist \( s_0 = s_0(T_0) > 0 \) and \( c = c(T_0) > 0 \) such that for all \( X \in \mathbb{R} \), \( T \geq T_0 \), \( s \geq s_0 \), and \( \epsilon \in (0,1) \),

\[
\mathbb{P}\left( \sup_{z \in [X,X+\epsilon]} \left| \Upsilon_T(1,z) + \frac{z^2}{2} - \Upsilon_T(1,X) - \frac{X^2}{2} \right| \geq s \epsilon^{1/2} \right) \leq \exp\left( -cs^{3/2} \right). \tag{3.1.4}
\]

This theorem gauges fluctuation of the parabolically shifted process \( \Upsilon_T(1,X) + \frac{X^2}{2} \) since, as we will recall in Proposition 2.1.7, this shift renders the process stationary in the \( x \) variable. Before turning to our theorem concerning fluctuations in time, we state a corollary of the preceding result, which gives an estimate on the spatial modulus of continuity for \( \Upsilon_T \). The corollary holds uniformly in the limit of high \( T \), so that, when the result is allied with the one-point convergence that will be the subject of Proposition ??, we learn that the spatial process is tight and that any subsequential limit shares the Hölder continuity known for the finite \( T \) processes.

**Corollary 3.1.4.** For \( t_0 > 0 \) and any interval \( [a,b] \subset \mathbb{R} \), define

\[
C := \sup_{X_1 \neq X_2 \in [a,b]} |X_1 - X_2|^{-\frac{1}{2}} \left( \log \frac{|b-a|}{|X_1 - X_2|} \right)^{-2/3} \left| \Upsilon_T(1,X_1) + \frac{X_1^2}{2} - \Upsilon_T(1,X_2) - \frac{X_2^2}{2} \right|. \tag{1.5}
\]

Then there exist \( s_0 = s_0(t_0,|b-a|) > 0 \) and \( c = c(t_0,|b-a|) > 0 \) such that, for \( s \geq s_0 \) and \( t > t_0 \),

\[
\mathbb{P}(C > s) \leq \exp\left( -cs^{3/2} \right).
\]

Our last result concerns the upper and lower tails for the difference in fluctuations at two times.
These tail bounds will be used in the proofs of Theorems 3.1.1 and 3.1.2.

**Theorem 3.1.5.** For any $T_0 > 0$, there exist $s_0 = s_0(T_0) > 0$, $c_1 = c_1(T_0) > 0$, $c_2 = c_2(T_0) > 0$ and $c_3 = c_3(T_0) > 0$ such that, for $s > s_0$, $T > T_0$ and $\beta \in (0, 1/2)$ for which $T \beta > T_0$,

$$
\exp \left( - c_1 s^{3/2} \right) \leq \mathbb{P} \left( \Upsilon_T(1 + \beta, 0) - \Upsilon_T(1, 0) \geq \beta^{1/3} s \right) \leq \exp \left( - c_2 s^{3/4} \right),
$$

$$
\mathbb{P} \left( \Upsilon_T(1 + \beta, 0) - \Upsilon_T(1, 0) \leq -\beta^{1/3} s \right) \leq \exp \left( - c_3 s^{3/2} \right). \tag{3.1.6}
$$

Just as in Theorem 3.1.2, we suspect different behaviour will arise in the short-time limit as $\beta \to 0$ for $t$ fixed. Because of this lack of a short-time result, we do not present a temporal modulus of continuity result arising from this theorem in the style of the spatial Corollary 3.1.4. Notice, also, that we do not record a lower bound on the probability in (3.1.6); we have not pursued this since we presently have no application for it. Finally, although we focus here on $x = 0$ at both times, it should also be possible to address two different spatial locations.

### 3.1.2 Idea of proof

Before describing our methods, we explain why existing approaches to studying the KPZ equation do not yield our main results. The KPZ equation has attracted attention of specialists in stochastic PDE via regularity structures, paracontrolled distributions or the renormalization group, for example – and of probabilists using integrable methods involving, for instance, Macdonald processes and the Bethe ansatz. It is natural enough to ask why our results do not follow from techniques in either of these areas. Stochastic PDE methods are well suited to the study of local problems but they have little to say about the distribution of KPZ equation under the characteristic $3 : 2 : 1$ scaling. For example, that the one-point distribution has fluctuations of order $T^{1/3}$ is inaccessible via these techniques. Integrable probability has been able to identify both the $t^{1/3}$ scaling as well as the limiting one-point fluctuations. However, as of yet, there has been no rigorous progress in deriving explicit formulas for the joint distribution of the KPZ equation at several space-time points. Even if such formulas existed, it might not be easy to extract our results from them. For instance, for zero temperature models (as discussed in Section 0.3.7), explicit multi-point formulas exist, but they have yet to prove valuable for extracting correlation decay results in the style of Theorems 3.1.1 and 3.1.2.

How do we proceed? Our study is principally probabilistic, and a vital aspect is the use of Gibbsian
line ensembles – objects that are born integrably but whose lives are in large part lived probabilistically. Indeed, the analysis of a Gibbsian line ensemble is one of three pivotal tools concerning the KPZ equation that will enable our proofs – tools whose use is here briefly described, and in more detail in Section [3.2].

The first tool, the *composition law* in Proposition [3.2.2] realizes the two-time distribution in terms of an exponentiated convolution of two independent narrow wedge KPZ equation solutions.

The second tool is the pertinent Gibbsian line ensemble. This is the *KPZ line ensemble*, which is recalled in Proposition [2.2.5]. The narrow wedge solution to the KPZ equation for any fixed time is embedded as the lowest indexed curve of an infinite ensemble of curves which jointly enjoy a *Brownian Gibbs property*. This property says that, fixing an index and an interval, the law of that indexed curve on that interval only depends on the boundary data (the curve indexed one above and one below on that interval, and the starting and ending points) and is comparable to the law of Brownian bridge with this endpoint pair, the comparison made succinctly via a Radon-Nikodym reweighting depending on said boundary data. Moreover, the Brownian Gibbs property implies *stochastic monotonicity* (Lemma [2.2.6]) which shows that shifting the boundary data in a given direction likewise shifts the law of the curve conditioned on that data.

The third tool consists of *tail bounds* (Propositions [3.2.4 and 3.2.5]) for the distribution of $\Upsilon_T(1, 0)$.

How do these tools combine to produce our main results? To compare $\Upsilon_T(1, 0)$ with $\Upsilon_T(\alpha, 0)$, we first use the composition law to realize $\Upsilon_T(\alpha, 0)$ in terms of the process $\Upsilon_T(1, \cdot)$ and an independent and scaled (based on the value of $\alpha$) narrow wedge KPZ equation solution which we denote later by $\Upsilon_{\alpha T \uparrow T}(\cdot)$. The composition law is a *softening* of a variational problem in which one would instead maximize the sum of the two narrow wedge solutions over their spatial argument. Our composition law only matches this variational problem in the $T \nearrow \infty$ limit, but the limiting case offers a convenient venue for a brief description of our approach.

Indeed, for the limiting maximization problem – with $T$ formally infinite – controlling the difference or correlation between $\Upsilon_T(1, 0)$ with $\Upsilon_T(\alpha, 0)$ boils down to three main steps. First we must show that the maximization likely occurs for values of $X$ near zero. Second we must show that the value of the functions in consideration at the maximizing location are close to their values at zero. And finally, we must show that one of the two random variables $\Upsilon_T(1, 0)$ or $\Upsilon_{\alpha T \uparrow T}(0)$ is small compared to the other one. Which process is small depends on whether we are working with $\alpha$ large – the
case of remote correlations – in which case $\Upsilon_T(1,0)$ should be correspondingly small; or if we are considering $\alpha = 1 + \beta$ for $\beta$ small – the case of adjacent correlations – in which case it is $\Upsilon_{\alpha T^2}T(0)$ that should be small.

These steps can be realized by using the Gibbsian line ensemble tool along with the tail bounds. Using the tail bounds we may control (with exponentially small tail probability) the boundary data for the lowest labeled curve of the KPZ line ensemble – i.e., the narrow wedge solution at fixed time. Allowing this with stochastic monotonicity enables us to transform our problems into questions involving the fluctuations of Brownian bridges, which can be treated quite classically. This general argument motif of using the Gibbs property to transfer one-point information into spatial regularity originated in [CH14] where the Airy line ensemble was introduced and studied, and has been developed in various directions in many subsequent works on Gibbsian line ensembles such as [CH16; CD18; Ham19; Ham17a; Ham17b; Ham16; HS18; BGHT; DVT; DOV18; CIW19a; CIW19b]. An example in which we closely follow a proof in the literature is Proposition 3.4.3, which mimics [Ham17b Proposition 3.5]. In most other cases, while we follow the general motif, we are forced to develop new variations since our present work is the first instance of applying Gibbsian line ensemble methods to study the temporal regularity of a KPZ class model.

Our proofs must operate when $T$ is finite, rather than in the limiting case of $T \uparrow \infty$. We have mentioned that the finite-$T$ KPZ equation composition law involves not a variational problem, but its softened convolution formula. We thus need to vary the proposed approach, arguing that the principal contribution to the integrals in the concerned composition law occurs near the integrand’s maximizer. A second complication arises from our seeking in Theorems 3.1.1 and 3.1.2 to control correlations, while the tools we have indicated merely control tail bounds on fluctuations. The needed transition is achieved in the proofs of these theorems via some rather general arguments contained in Appendix 3.8.

While the variational problem version of the composition law is a helpful, albeit only heuristic, way of thinking in the context of the KPZ equation, it is precisely the composition law for the KPZ fixed point where the narrow wedge solutions are replaced by suitably scaled Airy processes. In fact, we first developed our arguments in this simpler context. However, several recent works described next in Section 0.3.7 probe temporal correlation for the KPZ fixed point and other zero temperature models such as exponential, geometric and Brownian last passage percolation. While the methods
used in these investigations do not seem amenable (at least with present tools) to application to positive temperature models like the KPZ equation, the Gibbsian line ensemble technique is readily lifted to this level, so that we may employ it here. In Section 3.3 we describe the KPZ fixed point analogue of our work in greater detail.

Our Gibbsian line ensemble approach to studying temporal correlations has further potential. On the zero temperature side, it should be applicable to all of the just mentioned last passage percolation models. On the positive temperature side, there is a growing body of models which can be embedded into a Gibbsian line ensemble including the semi-discrete polymer [O’C12]; log-gamma / strict-weak polymers [COSZ]; and stochastic six vertex model / ASEP [BBW16; CD18]. In each of these positive temperature cases (besides the KPZ equation), there are some challenges to implementing the methods from the present work. For example, suitably strong tail bounds are yet to be demonstrated for the polymer models; and the composition law is considerably more complicated for the stochastic six vertex model and ASEP. That these models embed into Gibbsian line ensemble is a facet of their underlying integrability. Indeed, the study of Gibbsian line ensembles and their marriage of integrable and non-integrable probabilistic ideas has been quite fruitful recently.

3.1.3 Outline

Section 3.2 reviews several important and known properties of the KPZ equation, including its composition law, its relation to the KPZ line ensemble, and its one-point tail bounds. Since our analysis of the two-time distribution involves a cousin of a classical variational problem at zero temperature, and since zero temperature counterparts to our study have recently been undertaken, it is profitable – though not necessary for understanding our proofs – to view our results through the lens of zero temperature; and in Section 3.3 we do so. Our main technical contribution begins in Section 3.4 where we demonstrate how to extend one-point tail bounds to bounds on spatial fluctuation tails. While Propositions 3.4.1 and 3.4.2 contain global spatial fluctuation results which measure the size of spatial fluctuations on all of space, Propositions 3.4.3 and 3.4.4 contain local fluctuation results. This section also contains the proof of the local spatial fluctuation Theorem 3.1.3 which follows quite readily by combining Propositions 3.4.1, 3.4.2 and 3.4.3. The remote correlation decay Theorem 3.1.1, adjacent correlation Theorem 3.1.2 and spatial modulus of continuity Corollary 3.1.4 are successively proved in Sections 3.5, 3.6 and 3.7. Section 3.8, the appendix, contains several general probabilistic results.
which we develop to relate tail probabilities (to which we generally have access) to covariances and
 correlations.

3.2 Tools

In this section, we recall significant results that we will need. In Section 2.2, the KPZ line en-
semble is introduced (in Proposition 2.2.5). This ensemble enjoys two key properties: (1) its lowest
indexed curve coincides in law with the fixed time narrow wedge Cole-Hopf solution to the KPZ equa-
tion; and (2) it enjoys a certain Brownian Gibbs property. The general notion of a line ensemble and the
Brownian Gibbs property are recorded in Definition 2.2.1, while Lemma 2.2.6 records certain mono-
tonicity results associated with this Gibbs property. Section 3.2.1 contains results about the solution to
the KPZ equation such as its stationarity (Proposition 2.1.7); positive association (Propositions 2.1.8
and 3.2.1); composition law (Proposition 3.2.2); and tail bounds (Propositions 3.2.4 and 3.2.5).

Before commencing, let us record a few pieces of notation which will be used throughout. We
write \( \mathbb{N} = \{1, 2, \ldots\} \). We will often discuss probabilities without specifying the probability space.
When we use symbols for an event in such a probability space, we will often use the styles \( \mathcal{A} \) or \( \mathcal{A} \) instead of the standard \( \mathcal{A} \). For two events \( \mathcal{A} \) and \( \mathcal{B} \) we will sometimes write \( \mathbb{P}(\mathcal{A}, \mathcal{B}) \) instead of
\( \mathbb{P}(\mathcal{A} \cap \mathcal{B}) \). In the statements and proofs of many of our results, we will use the hopefully obvious
notation \( c = c(\cdot) > 0 \) to represent a positive constant \( c \) that depends on the variable in place of \( \cdot \).
In some of our proofs, we will allow constants such as this to vary line to line and within lines to
simplify the exposition and avoid introducing too many constants. Finally, we will sometimes use the
shorthand

\[
\Upsilon_T(X) := \Upsilon_T(1, x)
\]

when the time parameter is fixed and our interest is in the spatial process.

3.2.1 Input results for the KPZ equation

The first result of this section is a variant of FKG inequality which asserts that conditioning the
KPZ equation solution at time \( t \) on a larger (or smaller) value increases (or decreases) the conditional
expectation at a later time \( \alpha t \).

**Proposition 3.2.1** (Monotonicity under conditioning). For \( T > 0, \alpha > 1, X_1, X_2, r \in \mathbb{R} \) and
\( u > v \in \mathbb{R}, \)

\[
\begin{align*}
\mathbb{P}(\Upsilon_T(1, X_1) > v) & \mathbb{P}(\Upsilon_T(\alpha, X_2) > r, \Upsilon_T(1, X_1) > u) \\
\geq & \mathbb{P}(\Upsilon_T(1, X_1) > u) \mathbb{P}(\Upsilon_T(\alpha, X_2) > r, \Upsilon_T(1, X_1) > v), \\
\mathbb{P}(\Upsilon_T(1, X_1) \leq u) & \mathbb{P}(\Upsilon_T(\alpha, X_2) > r, \Upsilon_T(1, X_1) \leq v) \\
\geq & \mathbb{P}(\Upsilon_T(1, X_1) \leq v) \mathbb{P}(\Upsilon_T(\alpha, X_2) > r, \Upsilon_T(1, X_1) \leq u).
\end{align*}
\]

**Proof.** The two bounds are proved similarly hence we only treat the first. Consider the SHE with Dirac delta initial data (the proof works for general initial data too). We claim that, for \( S < T, \)

\[
\begin{align*}
\mathbb{P}(Z(S, X_1) > e^v) & \mathbb{P}(Z(T, X_2) > e^r, Z(S, X_1) > e^u) \\
\geq & \mathbb{P}(Z(S, X_1) > e^u) \mathbb{P}(Z(T, X_2) > e^r, Z(S, X_1) > e^v).
\end{align*}
\]

The proposition’s first bound follows from this claim by taking logarithms, centring and scaling.

The proof of the claim is almost the same as that of [CQ13 Proposition 1]. It relies on (1) the results of [ACQ11], which approximate \( Z(T, X) \) by the microscopic Cole-Hopf (or Gärtner) transform \( Z^\epsilon(T, X) \) of ASEP under weak asymmetry scaling; and (2) the FKG inequality for ASEP, a bound due to this model’s graphical construction (see the proof of [CQ13 Proposition 1] or [Lig05; Lig99] for details). The FKG inequality implies that the desired identity claimed for \( Z \) holds for \( Z^\epsilon \); by convergence, this bound holds in the limit as well.

The next result is the composition law for the KPZ equation. By its use, aspects of the two-time distribution will be inferred from the fixed-time narrow wedge KPZ solution. The mainstays of the result’s proof are the random semi-group property and the time-reversal symmetry enjoyed by the SHE. Our presentation of the derivation is brief in view of its similarities to the proof of [CH16 Lemma 1.18]. It is reasonable to wonder whether the composition law can be applied for study more than two disjoint times. For such a purpose, there is a composition law but it cannot be formulated purely in terms of the narrow wedge solution: it would be necessary to understand the joint distribution of the KPZ equation started from various shifted narrow wedges. In the \( T \uparrow \infty \) limit, this data is expected to be measurable with respect to the space-time Airy sheet.

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For $T > 0$, define a $T$-indexed composition map $I_T(f, g)$ between two functions $f(\cdot)$ and $g(\cdot)$:

$$I_T(f, g) := t^{-1/3} \log \int_{-\infty}^{\infty} e^{T^{1/3} \left( f(T^{-2/3}y) + g(-T^{-2/3}y) \right)} \, dy.$$  

(3.2.1)

**Proposition 3.2.2** (Composition law). *For any fixed $t > 0$ and $\alpha > 1$, there exists a spatial process $\Upsilon_{\alpha T^\uparrow T}(\cdot)$ supported on the same probability space as the KPZ equation solution such that:

1. $\Upsilon_{\alpha T^\uparrow T}(\cdot)$ is distributed according to the law of the process $\Upsilon_T(\alpha - 1, \cdot)$;

2. $\Upsilon_{\alpha T^\uparrow T}(\cdot)$ is independent of $\Upsilon_T(\cdot) := \Upsilon_T(1, \cdot)$; and

3. $\Upsilon_T(\alpha, 0) = I_T(\Upsilon_T, \Upsilon_{\alpha T^\uparrow T})$.

**Remark 3.2.3.** The notation $\Upsilon_{\alpha T^\uparrow T}(\cdot)$ will be, in Sections 3.5 and 3.6, complemented by similar notation $\Upsilon_0^\uparrow T(\cdot)$ in place of $\Upsilon_T(1, \cdot)$ and $\Upsilon_0^\uparrow \alpha T(\cdot)$ in place of $\Upsilon_T(\alpha, \cdot)$. The idea prompting this usage is that the solution from time 0 to $\alpha T$ can be constructed by combining the solution from time zero to time $T$ with the solution from $\alpha T$ to $T$—in a sense which will be clear in the proof that we now give.

**Proof of Proposition 3.2.2**. For $S < T$ and $x, y \in \mathbb{R}$, let $Z_{S,x}^{nw}(T, y)$ be the solution at time $T$ and position $y$ of the SHE started at time $S$ with Dirac delta initial data at position $x$. As these four parameters vary, we assume that all solutions are coupled on a probability space upon which their common space-time white noise is defined. It is due to this and the linearity of the SHE that

$$Z_{S,x}^{nw}(T, y) := Z_{0}^{nw}(T, y) = \int_{\mathbb{R}} Z_{0}^{nw}(S, x) Z_{S,x}^{nw}(T, y) \, dx.$$  

Since white noise is independent on non-overlapping time intervals, $Z_{0}^{nw}(S, x)$ and $Z_{S,x}^{nw}(T, y)$ are independent. The final property we use is that, for $S < T$ and $y \in \mathbb{R}$ fixed, $Z_{S,x}^{nw}(T, y)$ is equal in law and as a process in $x$ to $Z_{S,y}^{nw}(T, x)$—the change between the two expressions is in the interchange of $x$ and $y$. Writing these results in terms of $\Upsilon_T$ yields the proposition.

Our final inputs are one-point tail bounds for the KPZ equation, recently proved in [CG18a; CG18b] and discussed in Chapter 1 and 2.
**Proposition 3.2.4** (Uniform lower-tail bound). For any $T_0 > 0$, there exist $s_0 = s_0(T_0) > 0$ and $c = c(T_0) > 0$ such that, for $T > T_0$, $s > s_0$ and $X \in \mathbb{R}$,

$$
P\left( \Upsilon_T(X) + \frac{X^2}{2} \leq -s \right) \leq \exp\left(- cs^{5/2}\right).
$$

**Proof.** By Proposition 2.1.7 we may take $X = 0$. The result then follows from Theorem 2.1.2 because $\Upsilon_T$ in the quoted result is the same as $\Upsilon_T(0)$ up to a constant change of scale. ■

**Proposition 3.2.5** (Uniform upper-tail bound). For any $t_0 > 0$, there exist $s_0 = s_0(T_0) > 0$ and $c_1 = c_1(T_0) > c_2 = c_2(T_0) > 0$ such that, for $T \geq T_0$, $s > s_0$ and $X \in \mathbb{R}$,

$$
\exp\left(- c_1 s^{3/2}\right) \leq P\left( \Upsilon_T(X) + \frac{X^2}{2} \geq -s \right) \leq \exp\left(- c_2 s^{3/2}\right).
$$

**Proof.** By Proposition 2.1.7 we may take $X = 0$. The result follows from Theorem 2.1.9 ■

### 3.3 Analogous results for the KPZ fixed point

It is believed that $\Upsilon_T(\alpha, X)$ converges in the limit of high $t$ and as a time-space process to the narrow-wedge initial data solution of the **KPZ fixed point**. This important universal object is a Markov process on random functions whose existence was conjectured in [CQR15]; it has recently been constructed in [MQR16] for any fixed initial data via its transition probabilities, and in [DOV18] simultaneously for all initial data via the **Airy sheet**. A special case of the putative universality of the KPZ fixed point is the conjecture — made, for example, in [ACQ11, Conjecture 1.5] — that the process $x \mapsto 2^{1/3}(\Upsilon_T(X) + \frac{X^2}{2})$ converges to $x \mapsto A(X)$, where $A(\cdot)$ is the Airy$_2$ process introduced in [PS02]. A similar though stronger assertion has been expressed in [CH16, Conjecture 2.17] for the KPZ line ensemble: namely that, after adding in the parabolic shift $\frac{X^2}{2}$ and scaling by $2^{1/3}$, this ensemble converges in the limit of high $t$ to the Airy line ensemble constructed in [CH14].

In this section, we review our results and methods through the lens offered by zero temperature – that is, we discuss the counterpart problems and solutions in the limit where the KPZ time parameter $t$ becomes high. Positive temperature structures have zero temperature counterparts with simple and vivid interpretations, and there has been much recent effort to understand counterpart problems in the limiting $t \not\to \infty$ case. Thus it is that, while the upcoming discussion is logically needless for
comprehension of this paper’s results and proofs, we hope that this summary may aid the reader’s perspective on our results, their derivations and their relation to recent advances.

We start by noting the $t \nearrow \infty$ counterpart to the composition law Proposition 3.2.2. For functions $f$ and $g$, the high $t$ limit of $I_t(f, g)$ defined in (3.2.1) is the variational problem

$$I_t(f, g) := t^{-1/3} \log \left( t^{2/3} \int_{-\infty}^{\infty} e^{t^{1/3} \left( f(y) + g(-y) \right)} dy \right) \xrightarrow{T, t \nearrow \infty} \sup_{y \in \mathbb{R}} \left\{ f(y) + g(-y) \right\} =: I_\infty(f, g).$$

This Laplace method type of transition from the logarithm of the integral of an exponential in $I_T$ to the supremum in $I_\infty$ is the hallmark of passing from positive to zero temperature.

The form of the limiting composition law permits counterparts to our principal results to be formatted in terms of two independent Airy$_2$ processes, $A$ and $\tilde{A}$. Define $B(X) := 2^{-1/3} A(X) - X^2_2$ and $\tilde{B}(X) := 2^{-1/3} \tilde{A}(X) - \frac{X^2_2}{2}$. Set $B_1 = B(0)$ and, for $\beta > 0$, define

$$B_{1+\beta} = \sup \left\{ B(X) + \beta^{1/3} \tilde{B}(\beta X^{-2/3}) : X \in \mathbb{R} \right\}.$$

The scaling here applied to the term $\tilde{B}$ results in a process in $x$ suitable for the description of scaled last passage percolation values over a scaled duration equal to $\beta$; the resulting term may be viewed as a time $\beta$ version of the Airy$_2$ process.

The pair $(B_1, B_{1+\beta})$ is counterpart to $(\Upsilon_T(1, 0), \Upsilon_T(1 + \beta, 0))$. In law, this pair has the joint distribution of the narrow-wedge initial data KPZ fixed point at the space-time point pair $(0, 1)$ and $(0, 1+\beta)$. This distributional equality holds for given $\beta$, and no such assertion is being made regarding processes in $\beta$.

Besides the composition law, the other key tools described in Section 3.2 have direct analogues for the KPZ fixed point. The KPZ line ensemble is replaced by the Airy line ensemble [CH14]. After a parabolic shift, the latter enjoys the version of the Brownian Gibbs property given formally by $H(x) = \infty 1_{x \geq 0}$, in which intersection of adjacent curves is forbidden. Stochastic monotonicity is unaffected by the limit $t \nearrow \infty$. The KPZ fixed point also satisfies the same stationarity properties and FKG inequalities, while the tail bounds in Propositions 3.2.4 and 3.2.5 are replaced by such bounds for the GUE Tracy-Widom distribution [TW94].

Combining these alterations with the noted transition from $I_t$ to $I_\infty$ composition laws, we now state zero-temperature counterparts to our main theorems. We do not gives proofs of the statements.
in this article, although our arguments offer templates for such proofs. Indeed, the zero-temperature context is an alternative and, in certain regards, simpler mode for interpreting statements and proofs – the presentation of the following statements is intended to aid the reader who wishes to view this article through the zero-temperature prism.

First, two assertions concerning exponents for remote and adjacent two-time correlation. The recent works [FO19] and [BG19] offer corresponding theorems for certain last passage percolation models.

**Theorem 3.1.1 analogue:** There exist \( c_1, c_2 > 0 \) such that, for \( \alpha > 2 \),

\[
    c_1 \alpha^{-1/3} \leq \text{Corr}(B_1, B_\alpha) \leq c_2 \alpha^{-1/3}.
\]

**Theorem 3.1.2 analogue:** There exist \( c_1, c_2 > 0 \) such that, for \( \beta \in (0, \frac{1}{2}) \),

\[
    c_1 \beta^{2/3} \leq 1 - \text{Corr}(B_1, B_{1+\beta}) \leq c_2 \beta^{2/3}.
\]

The next spatial regularity result has been proved for scaled Brownian last passage percolation – see [Ham17b, Theorem 1.1].

**Theorem 3.1.3 analogue:** There exist \( s_0 > 0 \) and \( c > 0 \) such that, for \( X \in \mathbb{R}, s \geq s_0 \) and \( \epsilon \in (0, 1] \),

\[
    \mathbb{P}\left( \sup_{z \in [X, X+\epsilon]} |B(z) - B(X)| \geq s \epsilon^{1/2} \right) \leq \exp\left(-cs^{3/2}\right).
\]

The next modulus of continuity inference follows – see [Ham17b, Theorem 1.3] for such a result which holds uniformly over choices of \( B \) arising from a large class of initial data, in place of the narrow wedge considered here.

**Corollary 3.1.4 analogue:** For any interval \([a, b] \subset \mathbb{R}\), define

\[
    C := \sup_{X_1 \neq X_2 \in [a, b]} |X_1 - X_2|^{-1/2} \left( \log \frac{|b - a|}{|X_1 - X_2|} \right)^{-2/3} |B(X_1) - B(X_2)|.
\]

Then there exist \( s_0 = s_0(|b - a|) > 0 \) and \( c = c(|b - a|) > 0 \) such that, for \( s \geq s_0 \),

\[
    \mathbb{P}(C > s) \leq \exp\left(-cs^{3/2}\right).
\]
The spatial-temporal modulus of continuity for the KPZ fixed point is also probed in [DOV18, Proposition 1.6], a result which implies the next stated tail on the law of fluctuation between nearby times at a given location at zero temperature. A similar result is obtained for Poissonian last passage percolation in [HS18].

**Theorem 3.1.5 analogue:** There exist \( s_0 > 0 \) and \( c_1, c_2, c_3 > 0 \) such that, for \( s > s_0 \) and \( \beta \in (0, 1/2) \),

\[
\exp \left( -c_1 s^{3/2} \right) \leq \mathbb{P} \left( B_{1+\beta} - B_1 \geq \beta^{1/3} s \right) \leq \exp \left( -c_2 s^{3/4} \right),
\]

\[
\mathbb{P} \left( B_{1+\beta} - B_1 \leq -\beta^{1/3} s \right) \leq \exp \left( -c_3 s^{3/2} \right).
\]

### 3.4 Spatial process tail bounds

In this section, we prove bounds on the tails of various functionals of the spatial process \( \Upsilon_T(\cdot) \), such as its infimum, supremum and increment on a fixed interval. Four propositions will be stated and proved, the first two concerning global properties of this process, the latter two addressing local ones; in the latter vein, we will also prove the local spatial regularity Theorem 3.1.3 here. The proofs in this section rely upon: (1) the KPZ line ensemble Brownian Gibbs property, Proposition 2.2.5; (2) the monotone coupling Lemma 2.2.6; (3) Brownian bridge calculations; and (4) the one-point tail bounds Propositions 3.2.4 and 3.2.5. Note that we do not use the composition law in this section.

In the proofs of the first two propositions, the constant \( c = c(t_0, \nu) > 0 \) may change value from line to line and even between consecutive inequalities. Moreover, a bound involving \( c \) and \( s \) implicitly asserts that there exist \( s_0 = s_0(t_0, \nu) \) and \( c(t_0, \nu) \) such that, for all \( s \geq s_0 \) and \( t \geq t_0 \), the recorded bound holds. The explicit form is used in the propositions’ statements, and the abbreviating device is employed in their proofs. Recall also that \( \neg E \) denotes the complement of the event \( E \).

**Proposition 3.4.1.** For any \( T_0 > 0 \) and \( \nu \in (0, 1) \), there exist \( s_0 = s_0(T_0, \nu) > 0 \) and \( c = c(T_0, \nu) > 0 \) such that, for \( T \geq T_0 \) and \( s > s_0 \),

\[
\mathbb{P}(A) \leq \exp \left( -cs^{3/2} \right) \quad \text{where} \quad A := \left\{ \inf_{x \in \mathbb{R}} \left( \Upsilon_T(x) + \frac{(1+\nu)x^2}{2} \right) \leq -s \right\}. \quad (3.4.1)
\]
Proof. For \( n \in \mathbb{Z} \), define \( \zeta_n := n/s \) and
\[
E_n := \left\{ \Upsilon_T(\zeta_n) \leq -\frac{(1 + \nu/2)\zeta_n^2}{2} - (1 - \epsilon)s \right\},
\]
\[
F_n := \left\{ \Upsilon_T(y) \leq -\frac{(1 + \nu)y^2}{2} - (1 - \epsilon/2)s, \quad \forall y \in [\zeta_n, \zeta_{n+1}] \right\}.
\]

By the union bound,
\[
P(A) \leq \sum_{n \in \mathbb{Z}} P(E_n) + \sum_{n \in \mathbb{Z}} P(A \cap \left( \bigcap_{m \in \mathbb{Z}} \neg E_n \right) \cap F_n) + P\left( A \cap \left\{ \bigcap_{n \in \mathbb{Z}} \neg E_n \right\} \cap \left\{ \bigcap_{n \in \mathbb{Z}} \neg F_n \right\} \right).
\]

We bound each of the three right-hand summands. Note that \( A \cap \neg F_n = \emptyset \) for all \( n \in \mathbb{Z} \). Hence,
\[
P\left( A \cap \left( \bigcap_{n \in \mathbb{Z}} \neg E_n \right) \cap \left( \bigcap_{n \in \mathbb{Z}} \neg F_n \right) \right) = 0 .
\]

The first summand in (3.4.2) is bounded by
\[
\sum_{n \in \mathbb{Z}} P(E_n) \leq \sum_{n \in \mathbb{Z}} \exp\left( -c(\nu(\frac{\nu}{2})^2 + s)^{5/2} \right) \leq \sum_{n \in \mathbb{Z}} \exp\left( -c(\frac{\nu}{2})^{5/2} - cs^{5/2} \right).
\]

The first inequality here is due to the bound on \( P(E_n) \) that results from Proposition [3.2.4], while the second is obtained by applying the reverse Minkowski inequality – this being the bound that, for any \( a, b > 0 \), \( (a + b)^{5/2} \geq a^{5/2} + b^{5/2} \). The right-hand sum in (3.4.4) is now bounded above by a suitable multiple of the corresponding integral:
\[
\sum_{n \in \mathbb{Z}} \exp\left( -c(\frac{\nu}{2})^{5/2} / 2 \right) \leq s^{5} \int_{-\infty}^\infty \exp(-c|x|^5)dx \leq cs^5 .
\]

The right-hand side of (3.4.4) is thus seen to be at most \( cs^5 e^{-cs^{5/2}} \), which is in turn bounded above by \( e^{-cs^{5/2}} \), where in this instance, we distinguish between constants in an attempt at avoiding confusion.

We thus find that
\[
\sum_{n \in \mathbb{Z}} P(E_n) \leq \exp(-cs^{5/2}) .
\]
The second right-hand term in (3.4.2) remains to be addressed. Indeed, since $A \cap \{ \bigcap_{n \in \mathbb{Z}} -E_n \} \cap F_n$ is contained in $-E_n \cap -E_{n+1} \cap F_n$ for all $n \in \mathbb{Z}$, the next bound suffices in light of (3.4.3) and (3.4.6) to prove (3.4.1) and hence the proposition:

$$\sum_{n \in \mathbb{Z}} \mathbb{P}(-E_n \cap -E_{n+1} \cap F_n) \leq \exp(-c n^3).$$

(3.4.7)

To bound $\mathbb{P}(-E_n \cap -E_{n+1} \cap F_n)$ for all $n \in \mathbb{Z}$, we will make use of Proposition 2.2.5, which shows that the lowest indexed curve $\Upsilon_T^1(\cdot)$ of the KPZ line ensemble has the same distribution as $\Upsilon_T(\cdot)$. Owing to this, we may replace $\Upsilon_T(\cdot)$ in the definitions of $E_n$ and $F_n$ by $\Upsilon_T^1(\cdot)$. We will work with these modified definitions for the rest of the proof (which is to say, the derivation (3.4.7)); we will thus be able to use the $H_T$-Brownian Gibbs property associated with the KPZ line ensemble.

Let $\mathcal{F}_n = \mathcal{F}_{\text{ext}}(\{1\} \times (\zeta_n, \zeta_{n+1}))$ be the $\sigma$-algebra generated by $\{ \Upsilon_T^{(n)}(x) \}_{n \in \mathbb{N}, x \in \mathbb{R}}$ outside $\{ \Upsilon_T^{(1)}(X) : X \in (\zeta_n, \zeta_{n+1}) \}$. By the strong $H_T$-Brownian Gibbs property (2.2.1) for $\{ \Upsilon_T^{(n)} \}_{n \in \mathbb{N}, X \in \mathbb{R}}$,

$$\mathbb{P}(-E_n \cap -E_{n+1} \cap F_n) = \mathbb{E} \left[ 1_{-E_n \cap -E_{n+1}} \cdot \mathbb{E}[F_n | \mathcal{F}_n] \right] = \mathbb{E} \left[ 1_{-E_n \cap -E_{n+1}} \cdot \mathbb{P}_{H_T}(F_n) \right]$$

(3.4.8)

where $\mathbb{P}_{H_T} := \mathbb{P}^{1,1,(\zeta_n, \zeta_{n+1}), \Upsilon_T^{(1)}(\zeta_n), \Upsilon_T^{(1)}(\zeta_{n+1}), +\infty, -\infty}_{H_T}$. By Lemma 2.2.6, there exists a monotone coupling between $\mathbb{P}_{H_T}$ and $\mathbb{P}_{H_T} := \mathbb{P}^{1,1,(\zeta_n, \zeta_{n+1}), \Upsilon_T^{(1)}(\zeta_n), \Upsilon_T^{(1)}(\zeta_{n+1}), +\infty, -\infty}_{\text{free}} = \mathbb{P}^{(\zeta_n, \zeta_{n+1}), \Upsilon_T^{(1)}(\zeta_n), \Upsilon_T^{(1)}(\zeta_{n+1})}_{\text{free}}$, such that

$$\mathbb{P}_{H_T}(F_n) \leq \mathbb{P}_{H_T}(F_n).$$

(3.4.9)

Recall that $\mathbb{P}_{H_T}$ is the law of a Brownian bridge. For $n \in \mathbb{Z}$, define $\theta_n := (1 - e)s + \frac{(1 + 2^{-1} \nu)}{2} \zeta_n^2$. Then $1_{-E_n \cap -E_{n+1}} = 1_{\Upsilon_T^{(1)}(\zeta_n) > -\theta} \cdot 1_{\Upsilon_T^{(1)}(\zeta_{n+1}) > -\theta_{n+1}}$. Under Brownian bridge law on the interval $(\zeta_n, \zeta_{n+1})$, the probability of $F_n$ increases with pointwise decrease of the sample paths at the endpoints. Thus,

$$1_{-E_n \cap -E_{n+1}} \cdot \mathbb{P}_{H_T}(F_n) \leq \mathbb{P}^{(\zeta_n, \zeta_{n+1}), -\theta_n, -\theta_{n+1}}(F_n).$$

(3.4.10)
For a Brownian bridge $B(\cdot)$ with $B(\zeta_n) = -\theta_n$ and $B(\zeta_{n+1}) = -\theta_{n+1},$

$$\mathbb{P}^{\text{free}}(\zeta_n,\zeta_{n+1}, -\theta_n, -\theta_{n+1} | \mathcal{F}_n) \leq \mathbb{P} \left( \min_{T \in [\zeta_n,\zeta_{n+1}]} B(T) \leq -\{\theta_n \lor \theta_{n+1}\} - \frac{\epsilon}{2} s - \frac{\nu}{4} \epsilon^2 \zeta_n^2 \right).$$

Combining this with (3.4.10), (3.4.9) and (3.4.8) yields

$$\mathbb{P}(\neg E_n \cap \neg E_{n+1} \cap \mathcal{F}_n) \leq \mathbb{P} \left( \min_{T \in [\zeta_n,\zeta_{n+1}]} B(T) \leq -\{\theta_n \lor \theta_{n+1}\} - \epsilon^2 s - \frac{\nu}{4} \zeta_n^2 \right).$$

Employing an elementary Brownian bridge estimate [CG18b, Lemma 2.5] shows that

$$\mathbb{P}(\neg E_n \cap \neg E_{n+1} \cap \mathcal{F}_n) \leq \exp \left( -s (\frac{\nu}{4} \zeta_n^2 + \frac{\epsilon^2 s^2}{2}) \right) \leq \exp \left( -\frac{st^2}{16} \zeta_n^4 - \frac{\epsilon^2 s^3}{2} \right),$$

where the second inequality is due to $(a + b)^2 \geq a^2 + b^2$ for any $a, b > 0.$ Summing both sides of the above inequality over $n \in \mathbb{Z}$ and bounding the right-hand sum in the manner of (3.4.5), we arrive at (3.4.7) and thus complete the proof of Proposition 3.4.1. ■

**Proposition 3.4.2.** For any $t_0 > 0$ and $\nu \in (0, 1),$ there exist $s_0 = s_0(T_0, \nu) > 0$ and $c_1 = c_1(T_0, \nu) > c_2 = c_2(T_0, \nu) > 0$ such that, for $T \geq T_0$ and $s > s_0,$

$$\exp \left( -c_1 s^{3/2} \right) \leq \mathbb{P}(A) \leq \exp \left( -c_2 s^{3/2} \right) \quad \text{where} \quad A = \left\{ \sup_{x \in \mathbb{R}} \left( \mathcal{Y}_T(x) + \frac{(1-\nu)x^2}{2} \right) \geq s \right\}.$$  

(3.4.11)

**Proof.** The first inequality of (3.4.11) follows from Proposition 3.2.5 since $\left\{ \mathcal{Y}_T(0) \geq s \right\} \subseteq \left\{ \sup_{x \in \mathbb{R}} \mathcal{Y}_T(x) \geq s \right\}.$

Turning to prove the second inequality of (3.4.11), for $n \in \mathbb{Z},$ set $\zeta_n = n/s$ and

$$E_n := \left\{ \mathcal{Y}_T(\zeta_n) \geq -\frac{(1-\nu/2)x^2}{2} \zeta_n^2 + \frac{s}{2} \right\}$$

$$F_n := \left\{ \mathcal{Y}_T(x) \geq -\frac{(1-\nu)}{2} y^2 + s, \text{ for some } y \in (\zeta_n, \zeta_{n+1}) \right\}.$$ 

(These events are similar to those in the derivation of Proposition 3.4.1. Since the present events concern the upper tail, and their earlier cousins the lower tail, we replace calligraphic by sans-serif...
font to denote them.) Seeking the second inequality of (3.4.11), note that

$$\mathbb{P}(A) \leq \sum_{n \in \mathbb{Z}} \mathbb{P}(E_n) + \sum_{n \in \mathbb{Z}} \mathbb{P}(-E_n \cap -E_{n+1} \cap F_n) + \mathbb{P}\left(A \cap \left\{ \bigcap_{n \in \mathbb{Z}} -E_n \right\} \cap \left\{ \bigcap_{n \in \mathbb{Z}} -F_n \right\} \right).$$ (3.4.12)

Note that $A \cap -F_n = \emptyset$, so that the last term on the right-hand side of (3.4.12) equals zero.

For the first term on the right-hand side of (3.4.12), the second inequality in Proposition 3.4.2 shows that

$$\sum_{n \in \mathbb{Z}} \mathbb{P}(E_n) \leq \sum_{n \in \mathbb{Z}} \exp \left( -c \left( \frac{1-\nu/2}{2} \zeta_n^2 + s \right)^{3/2} \right) \leq \exp \left( -cs^{3/2} \right).$$ (3.4.13)

The second inequality here follows from the argument used in (3.4.4) and (3.4.5).

Thus, the proof of Proposition 3.4.2 will be finished if we can show that

$$\sum_{n \in \mathbb{Z}} \mathbb{P}(-E_n \cap -E_{n+1} \cap F_n) \leq \exp \left( -cs^{3/2} \right).$$ (3.4.14)

For $\overline{E}_n := \left\{ \Upsilon_T(\zeta_n) \geq -\left( \frac{1+\nu/2}{2} \right) \zeta_n^2 - s^{2/3} \right\}$, we have that

$$\mathbb{P}(-E_{n-1} \cap -E_{n+1} \cap F_n) \leq \mathbb{P}(B_n) + \mathbb{P}(-\overline{E}_{n-1}) + \mathbb{P}(-\overline{E}_{n+1}),$$

where we have defined the event $B_n = -E_{n-1} \cap \overline{E}_{n-1} \cap -E_{n+1} \cap \overline{E}_{n+1} \cap \overline{F}_n$. We will bound each term on the last displayed right-hand side. Proposition 3.2.4 implies that

$$\mathbb{P}(-\overline{E}_n) \leq \exp \left( -c(s^{2/3} + \frac{\nu}{2} \zeta_n^2)^{5/2} \right).$$

Summing over $n \in \mathbb{Z}$ in the same way as in (3.4.13) yields

$$\sum_{n \in \mathbb{Z}} (\mathbb{P}(-\overline{E}_{n-1}) + \mathbb{P}(-\overline{E}_{n+1})) \leq \exp \left( -cs^{5/2} \right).$$ (3.4.15)

It remains to show that $\sum_{n \in \mathbb{Z}} \mathbb{P}(B_n) \leq \exp \left( -cs^{3/2} \right)$. This readily follows once we show

$$\mathbb{P}(B_n) \leq 2\mathbb{P}\left( \Upsilon_T(0) \geq \frac{\nu}{8} \zeta_n^2 + \frac{1}{8}s \right).$$ (3.4.16)
Indeed, the latter right-hand side can be bounded above by appealing to the right-hand inequality in Proposition 3.2.5; the bound on the sum of \( \mathbb{P}(B_n) \) follows then by the logic that governs (3.4.13).

Assuming (3.4.16), we will first complete the proof of (3.4.14). Applying (3.4.16) and a similar analysis as in (3.4.13) shows that there exist \( C' = C' (\nu, t_0) \) such that for all \( s \geq s_0' \),

\[
\sum_{n \in \mathbb{Z}} \mathbb{P}(\{ (\sim E_{n-1}) \cap \tilde{E}_{n-1} \} \cap (\sim E_{n+1}) \cap \tilde{E}_{n+1} \cap F_n) \leq \exp \left( - C' s^{3/2} \right).
\]

Combining this with (3.4.15), we arrive at (3.4.14).

The remainder of this proof is devoted to showing (3.4.16). We will rely upon the equality in distribution between the narrow wedge solution \( \Upsilon_T (\cdot) \) and the lowest labeled curve \( \Upsilon_T^{(1)} (\cdot) \) of the KPZ line ensemble that is offered by Proposition 2.2.5. Consequently, in the definitions of the events \( E_n, F_n, \tilde{E}_n \) and \( B_n \), we may substitute \( \Upsilon_T \) with \( \Upsilon_T^{(1)} \). Our proof of (3.4.16) parallels the proof of [CH14, Proposition 4.4]; see also [CH16, Lemma 4.1].

Define three curves (and consult Figure 2.2 for an illustration of the main objects in this proof):

\[
U(y) := - \frac{(1 - \nu)}{2} y^2 + s, \quad L(y) := - \frac{(1 + \nu/2)}{2} y^2 - s^{2/3}, \quad M(y) := - \frac{(1 - \nu/2)}{2} y^2 + \frac{s}{2}.
\]

If \( \sim E_{n-1} \cap \tilde{E}_{n-1} \) and \( \sim E_{n-1} \cap \tilde{E}_{n-1} \) occurs, then \( \Upsilon_T^{(1)} (\cdot) \) stays in between the curves \( M \) and \( L \) at the points \( \zeta_{n-1} \) and \( \zeta_{n+1} \) respectively. If \( F_n \) occurs, then \( \Upsilon_T^{(1)} \) touches the curve \( U \) at some point in the interval \( [\zeta_n, \zeta_{n+1}] \). Therefore, on the event \( B_n \), the curve \( \Upsilon_T^{(1)} \) hits \( U \) somewhere in the interval \( (\zeta_n, \zeta_{n+1}) \), whereas it stays in between \( M \) and \( L \) at the points \( \zeta_{n-1} \) and \( \zeta_{n+1} \). The solid black curve in Figure 2.2 is an instance of \( \Upsilon_T (\cdot) \) on the event \( B_n \).

Let us define \( \sigma_n := \inf \left\{ y \in (\zeta_n, \zeta_{n+1}) : \Upsilon_T^{(1)}(y) \geq U(y) \right\} \). Consider the crossing event

\[
C_n := \left\{ \Upsilon_T^{(1)} (\zeta_n) \geq \frac{\sigma_n - \zeta_n}{\sigma_n - \zeta_{n-1}} L(\zeta_{n-1}) + \frac{\zeta_n - \zeta_{n-1}}{\zeta_n - \zeta_{n-1}} U(\sigma_n) \right\},
\]

which in Figure 2.2 is the event that the solid black curve stays above the solid bullet at time \( \zeta_n \). We adopt the shorthand

\[
\mathbb{P}^{1,1}_{H_T} := \mathbb{P}^{1,1}(\zeta_n, \zeta_{n-1}, \Upsilon_T^{(1)} (\zeta_n), \Upsilon_T^{(1)} (\zeta_{n-1}), \Upsilon_T^{(1)} (\sigma_n), + \infty, \Upsilon_T^{(2)}), \quad \mathbb{P}^{\text{free}}_{H_T} := \mathbb{P}^{1}(\zeta_n, \zeta_{n-1}, \Upsilon_T^{(1)} (\zeta_n), \Upsilon_T^{(1)} (\sigma_n)) \quad \text{(3.4.17)}
\]

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Recalling Definition 2.2.1 note that, since $(\zeta_{n-1}, \sigma_n)$ is a $\{1\}$-stopping domain for the KPZ line ensemble, the strong $\mathbf{H}_T$-Brownian Gibbs property (2.2.1) implies that

$$\mathbb{E}[1_{B_n} 1_{C_n} | \mathcal{F}_{ext} \left( \{1\} \times (\zeta_{n-1}, \sigma_n) \right)] = 1_{B_n} \mathbb{P}_{\mathbf{H}_t}(C_n). \quad (3.4.18)$$

By Lemma 2.2.6 there exists a monotone coupling between the laws $\mathbb{P}_{\mathbf{H}_T}$ and $\mathbb{P}_{\bar{\mathbf{H}}_T}$; in Figure 2.2 the curve $B$ is supposed to represent a sample from $\mathbb{P}_{\bar{\mathbf{H}}_T}$ coupled to $\gamma^{(1)}_T$ distributed according to $\mathbb{P}_{\mathbf{H}_T}$. Using this and that the probability of $C_n$ increases under pointwise increase of its sample paths, we find that $\mathbb{P}_{\mathbf{H}_T}(C_n) \geq \mathbb{P}_{\bar{\mathbf{H}}_T}(C_n)$. Since $\mathbb{P}_{\mathbf{H}_T}$ is the law of a Brownian bridge, there is probability one-half that it stays above the line joining the two endpoints at a given intermediate time such as $\zeta_n$. Since that linear interpolation value at time $\zeta_n$ (the empty circle in Figure 2.2) sits below the value considered in $C_n$ (the solid bullet in Figure 2.2), we see that $\mathbb{P}_{\mathbf{H}_T}(C_n) \geq 1/2$. Substituting this into (3.4.18) and taking expectation yields

$$\mathbb{P}(B_n) \leq 2 \mathbb{E}[1_{B_n} 1_{C_n}]. \quad (3.4.19)$$

To bound the right-hand side of (3.4.19), observe that

$$\frac{\sigma_n - \zeta_n}{\sigma_n - \zeta_{n-1}} L(\zeta_{n-1}) + \frac{\zeta_n - \zeta_{n-1}}{\sigma_n - \zeta_{n-1}} U(\sigma_n) \geq -\frac{(1 - \nu/8)}{2} \zeta_n^2 - \frac{(4 + 34\nu)}{2s^2} + \frac{1}{2} \left( \frac{s}{2} - s^{2/3} \right). \quad (3.4.20)$$

In order to demonstrate (3.4.20), we use the bounds:

$$\frac{(\sigma_n - \zeta_n) \zeta_n^2}{\sigma_n - \zeta_{n-1}} + \frac{(\zeta_n - \zeta_{n-1}) \sigma_n^2}{\sigma_n - \zeta_{n-1}} - \zeta_n^2 \geq (\sigma_n - \zeta_n)(\zeta_n - \zeta_{n-1}) \leq \frac{1}{s^2}, \quad (3.4.21)$$

$$\frac{-\frac{1}{2}(\sigma_n - \zeta_n) \zeta_n^2}{\sigma_n - \zeta_{n-1}} + \frac{\zeta_n^2}{2} \geq -\frac{1}{2} (\sigma_n - \zeta_n)(\zeta_n - \zeta_{n-1}) + \frac{3}{2} \frac{\zeta_n - \zeta_{n-1}}{2 \sigma_n - \zeta_{n-1}} \sigma_n^2 \geq \frac{3}{4} \zeta_n^2 - 2 \frac{|\zeta_n|}{s} \geq \frac{1}{8} \zeta_n^2 - \frac{32}{s^2}. \quad (3.4.22)$$

The first inequality of (3.4.23) uses that $(\zeta_n - \zeta_{n-1})/(\sigma_n - \zeta_{n-1}) \geq \frac{1}{2}$ and $\sigma_n^2 \geq \zeta_n^2 - 2|\zeta_n|s^{-(1+\delta)}$; the second inequality of that line follows from $8^{-1} \zeta_n^2 - 2|\zeta_n|s^{-(1+\delta)} \geq 0$ for all $|n| \geq 16$.

Owing to (3.4.20), when $C_n$ occurs, $\gamma^{(1)}_T(\zeta_n)$ will be greater than the right-hand side of (3.4.20). The latter quantity is bounded below by $-\frac{2^{-2/3}(1 - \nu/8)}{s} + s/8$ when $s$ is large enough. Hence,
omitting the indicator \(1_{B_n}\) from the right-hand side of (3.4.19), we learn that

\[
P(B_n) \leq 2 \mathbb{P}(C_n) \leq 2 \mathbb{P}\left(\mathcal{Y}_T^{(1)}(\zeta_n) \geq -\frac{(1 - \nu/8)}{2} \zeta_n^2 + \frac{s}{8}\right).
\] (3.4.24)

The claim (3.4.16) now follows by recalling from Proposition 2.1.7 that \(\mathcal{Y}_T^{(1)}(\zeta_n^2) + \frac{s^2}{2} = \mathcal{Y}_T^{(1)}(0)\). ■

Whereas Propositions 3.4.1 and 3.4.2 deal with the lower and upper tail of the infimum and supremum of the entire spatial process \(\mathcal{Y}_T\), Proposition 3.4.3 addresses the tail behaviour of small spatial increments of \(\mathcal{Y}_T\). This proposition asserts that conditioned on a good – or typical – event (3.4.25), the tails of the increments are roughly the same as for that of Brownian motion; the result’s proof is a brief but powerful application of the Brownian Gibbs technique which runs in parallel to the derivation of its zero-temperature cousin [Ham17b, Proposition 3.5]. The good event with which Proposition 3.4.3 deals has lighter than Gaussian tails, so that, without conditioning, the power law in the exponential becomes 3/2 instead of 2. The result that thus arises was recorded earlier as Theorem 3.1.3, and is proved a little later in this section, along with another consequence of Proposition 3.4.3 – namely, Proposition 3.4.4 concerning tails of spatial increments.

**Proposition 3.4.3.** For any \(X \in \mathbb{R}\), \(\epsilon \in (0, 1]\) and \(s \geq 0\), define

\[
G_{\epsilon,s}(x) = \bigcap_{y \in \{X + \epsilon - 2, X, X + \epsilon, X + 2\}} \{ -s/4 \leq \mathcal{Y}_T(y) + \frac{y^2}{2} \leq s/4 \}. 
\] (3.4.25)

There exist constants \(c_1, c_2 > 0\) such that, for \(\epsilon \in (0, 1]\), \(t > 0\), \(s \geq 4\) and \(X \in \mathbb{R}\),

\[
\mathbb{P}\left( \sup_{z \in [X, X+\epsilon]} \left| (\mathcal{Y}_T(z) + \frac{z^2}{2}) - (\mathcal{Y}_T(X) + \frac{X^2}{2}) \right| \geq s \epsilon^{1/2}, G_{\epsilon,s}(X) \right) \leq c_1 \exp(-c_2 s^2).
\]

**Proof.** By the stationarity in \(X\) of \(\mathcal{Y}_T(X) + \frac{X^2}{2}\), we may suppose that \(X = 0\). By the equality in distribution of \(\mathcal{Y}_T(\cdot)\) with \(\mathcal{Y}_T^{(1)}(\cdot)\) stated in Proposition 2.2.5, we may substitute \(\mathcal{Y}_T^{(1)}\) for \(\mathcal{Y}_T\) in the statement and proof of the desired result.
For a stochastic process \( X \) whose domain of definition includes \([0, \epsilon]\), define the events

\[
\text{Fall}_{\epsilon, s}[X] = \left\{ \inf_{z \in [0, \epsilon]} \left\{ X(z) + \frac{z^2}{2} \right\} \leq X(0) - s \epsilon^{1/2} \right\},
\]

\[
\text{Rise}_{\epsilon, s}[X] = \left\{ \sup_{z \in [0, \epsilon]} \left\{ X(z) + \frac{z^2}{2} \right\} \geq X(0) + s \epsilon^{1/2} \right\}.
\]

We will take \( X(X) = \Upsilon_T^{(1)}(X) \). To obtain Proposition 3.4.3 for \( X = 0 \), it is enough to verify two bounds:

\[
P\left( \text{Fall}_{\epsilon, s}[\Upsilon_T^{(1)}], G_{\epsilon, s}(0) \right) \leq c_1 \exp(-c_2 s^2), \quad P\left( \text{Rise}_{\epsilon, s}[\Upsilon_T^{(1)}], G_{\epsilon, s}(0) \right) \leq c_1 \exp(-c_2 s^2).
\]

We start by proving the first bound, concerning \( \text{Fall} \). By using the Brownian Gibbs property and stochastic monotonicity for the KPZ line ensemble, we bound above the probability of a large fall by the corresponding probability for a suitable Brownian bridge. Indeed, removing conditioning on the second indexed curve of the line ensemble may only cause a statistical increase in the fall suffered by the first curve. But after the removal, the first curve is distributed as a Brownian bridge.

Recalling the notation from Definition 2.2.1, we will argue that

\[
P\left( \text{Fall}_{\epsilon, s}[\Upsilon_T^{(1)}], G_{\epsilon, s}(0) \right) \leq P\left( \text{Fall}_{\epsilon, s}[\Upsilon_T^{(1)}], G_{\epsilon, s}(0) \right) \leq E\left[ 1_{\Upsilon_T^{(1)} \leq s/4} \cdot 1_{\Upsilon_T^{(2)} \geq s/4} \cdot P_{\text{free}}^{(0, 2)}[\text{Fall}_{\epsilon, s}[\Upsilon_T^{(1)}]] \right].
\]

The first line follows by dropping two of the conditions in \( G \), while the second line equality uses the Brownian Gibbs property for the KPZ line ensemble and conditional expectations with respect to the \( \sigma \)-field external to \( \Upsilon_T^{(1)} \) on the interval \([0, 2]\). The third line inequality uses the monotone coupling of Lemma 2.2.6 to remove the second curve in the conditioning. The curve \( \Upsilon_T^{(1)} \) will drop without the support offered by the second curve, so that the probability of the event \( \text{Fall} \) may only increase in
response to this removal. The fourth line inequality is due to variation over those values of \( \Upsilon_T^{(1)}(0) \) and \( \Upsilon_T^{(1)}(2) \) that satisfy the pair of conditions in the indicator functions of the preceding line. The final equality follows by coupling all the Brownian bridges together via affine shifts – clearly the largest fall occurs when the boundaries are maximally displaced in the suitable direction.

We may rewrite the final term in (3.4.26) in slightly simpler notation as

\[
P_{\text{free}}^{(0,2),s/4} \left( \text{Fall}_{\epsilon,s}[\Upsilon_T^{(1)}] \right) = P \left( \inf_{z \in [0,\epsilon]} \left\{ \tilde{B}(z) + \frac{z^2}{2} \right\} \leq -s\epsilon^{1/2} \right),
\]

where \( \tilde{B} : [0, 2] \to \mathbb{R} \) is a Brownian bridge with \( \tilde{B}(0) = 0 \) and \( \tilde{B}(2) = -2 - \frac{s}{2} \); here, we shifted the whole system down by \( s/4 \) to relocate the starting value to zero. Removing the parabola from the infimum only increases the probability. Now set \( B(z) = \tilde{B}(z) - \frac{z^2}{2}(-2 - \frac{s}{2}) \), so that \( B \) is a Brownian bridge with \( B(0) = B(2) = 0 \). Taking into account the maximal effect of this linear shift shows that

\[
P \left( \inf_{z \in [0,\epsilon]} \left( \tilde{B}(z) + \frac{z^2}{2} \right) \leq -s\epsilon^{1/2} \right) \leq P \left( \inf_{z \in [0,\epsilon]} B(z) \leq -s\epsilon^{1/2} + \frac{\epsilon}{2}(-2 - \frac{s}{2}) \right) \leq P \left( \inf_{z \in [0,\epsilon]} B(z) \leq -\frac{s}{2}\epsilon^{1/2} \right).
\]

The latter inequality is due to \( \frac{\epsilon}{2}(2 + \frac{s}{2}) \leq s\epsilon^{1/2}/2 \), a bound that holds for \( s \geq 4 \) – assuming \( \epsilon \in (0, 1] \), as we do. The right-hand probability can be estimated via the reflection principle, which yields the desired bound of the form \( c_1 \exp(-c_2s^2) \) for suitable constants \( c_1, c_2 > 0 \).

The bound for \( \text{Rise} \) is much the same – the rapid rise of a function \( f : \mathbb{R} \to \mathbb{R} \) is a rapid fall if the domain of \( f \) is viewed in the opposing direction. Indeed, the reasoning in (3.4.26) yields that

\[
P \left( \text{Rise}_{\epsilon,s}[\Upsilon_T^{(1)}], G_{\epsilon,s}(0) \right) \leq P \left( \text{Rise}_{\epsilon,s}[\Upsilon_T^{(1)}], \Upsilon_T^{(1)}(\epsilon) + \frac{\epsilon^2}{2} \leq -s/4, \Upsilon_T^{(1)}(-2 + \epsilon) + \frac{(-2+\epsilon)^2}{2} \geq -s/4 \right)
\]

\[
\leq P_{\text{free}}^{(-2+\epsilon),s/4} \left( \text{Rise}_{\epsilon,s}[\Upsilon_T^{(1)}] \right).
\]

The \( \text{Rise} \) bound has been reduced to a matter of Brownian bridge increments, treated in the manner of the counterpart argument for the \( \text{Fall} \) bound.

\textbf{Proof of Theorem 3.1.3} Applying Propositions 3.4.1 and 3.4.2 there exist \( s_0 = s_0(t_0) \) and \( c = c(t_0) \) such that, for \( x \in \mathbb{R}, t \geq t_0 \) and \( s \geq s_0 \), \( P(-G_{\epsilon,s}(x)) \leq \exp \left( -cs^3/2 \right) \). This inference and Proposition 3.4.3 yield (3.1.4); thus is the theorem proved.
We will need one more result, concerning tails of increments.

**Proposition 3.4.4.** For any \( t_0 > 0 \) and \( \nu > 0 \), there exist \( s = s(t_0, \nu) \) and \( c = c(t_0, \nu) \) such that, for \( t \geq t_0, s \geq s_0 \) and \( \theta > 1 \),

\[
\mathbb{P}\left( \sup_{x \in [0, \theta^{1/3}]} \{ \Upsilon_T(\theta^{-2/3}x) - \Upsilon_T(0) - \theta^{-1/3} \frac{x^2}{2} \} \geq \theta^{-1/3} s \right) \leq \exp(-cs^{3/4}), \tag{3.4.27}
\]

\[
\mathbb{P}\left( \inf_{x \in [0, \theta^{1/3}]} \{ \Upsilon_T(\theta^{-2/3}x) - \Upsilon_T(0) + \theta^{-1/3} \frac{x^2}{2} \} \leq -\theta^{-1/3} s \right) \leq \exp(-cs^{3/4}). \tag{3.4.28}
\]

**Proof.** We prove only (3.4.27), since a very similar argument yields (3.4.28). Let \( E \) denote the event on the left-hand side of (3.4.27). Suppose first that \( \theta < \frac{2}{3} \). Then

\[
\mathbb{P}(E) \leq \mathbb{P}\left( \sup_{x \in [0, 2]} \{ \Upsilon_T(x) - \Upsilon_T(0) \} \geq \frac{s}{2} \right) \leq \mathbb{P}\left( \sup_{x \in \mathbb{R}} \Upsilon_T(x) \geq \frac{s}{4} \right) + \mathbb{P}\left( \Upsilon_T(0) \leq -\frac{s}{4} \right). \tag{3.4.29}
\]

The first bound is due to the supremum increasing in response to the omission of the parabola and extension of the range of \( x \) to \([0, 2]\). The second bound uses \( \sup_{x \in [0, 2]} \Upsilon_T(X) \leq \sup_{x \in \mathbb{R}} \Upsilon_T(X) \) and

\[
\left\{ \sup_{x \in [0, 2]} \{ \Upsilon_T(x) - \Upsilon_T(0) \} \geq s/2 \right\} \subset \left\{ \sup_{x \in \mathbb{R}} \Upsilon_T(x) \geq s/4 \right\} \cup \left\{ \Upsilon_T(0) \leq -s/4 \right\}.
\]

Propositions 3.2.4 and 3.4.2 provide bounds on the right-hand probabilities in (3.4.29) of the form \( \exp(-cs^{3/2}) \), for some \( c = c(t_0) > 0 \) when \( s > s_0(t_0) \) is large enough. This proves (3.4.27) for \( \theta < \frac{2}{3} \).

Now suppose that \( \theta \geq \frac{2}{3} \). We may partition \([1, \theta^{1/3}] = \bigsqcup_{\ell=1}^{3K_0+1} \mathcal{I}_\ell \) where \( K_0 := \left\lfloor \frac{1}{3} \log_2 \theta \right\rfloor \) and \( \mathcal{I}_\ell := [2^{(\ell-1)/3}, 2^{\ell/3}] \) for \( 1 \leq \ell \leq 3K_0 \) and \( \mathcal{I}_{3K_0+1} := [2^{K_0}, \theta^{1/3}] \). For \( 1 \leq \ell \leq 3K_0 + 1 \), define

\[
Q_\ell := \left\{ \sup_{X \in \mathcal{I}_\ell} \{ \Upsilon_T(\theta^{-2/3}X) - \Upsilon_T(0) \} \geq \theta^{-1/3} \left( s + \nu^2 \frac{2^{(\ell-1)/3}}{2} \right) \right\}.
\]

We seek to apply Proposition 3.4.3. For an event \( G_\ell \) that is arbitrary – but which will shortly be
specified—we have that, for \( \epsilon = 2^{\ell/3} \theta^{-2/3} \) and \( s_\ell = 2^{(\ell-1)/3}\sqrt{2\nu s} \),

\[
\mathbb{P}(Q_\ell \cap G_\ell) \leq \mathbb{P}\left( \sup_{X \in [0,2^{\ell/3}]} \{ Y_T(\theta^{-2/3}X) - Y_T(0) \} \geq \theta^{-1/3} 2^{(\ell-1)/3}\sqrt{2\nu s} \right) \cap G_\ell \\
= \mathbb{P}\left( \left\{ \sup_{X \in [0,\epsilon]} \{ Y_T(X) - Y_T(0) \} \geq s_\ell^{1/2} \right\} \cap G_\ell \right).
\]

Here, the inequality results from the arithmetic-geometric mean inequality in the guise \( s + \nu 2^{(\ell-1)/3} \geq 2^{(\ell-1)/3}\sqrt{2\nu s} \); the equality is due to a change of variables.

By the definition of \( Q_\ell \), \( E \subset \bigcup_{\ell=1}^{3K_0+1} Q_\ell \). Choosing \( G_\ell = G_{\ell,s_\ell} \) from (3.4.25), by the union bound,

\[
\mathbb{P}(E) \leq \sum_{\ell=1}^{3K_0+1} \left( \mathbb{P}(Q_\ell \cap G_\ell) + \mathbb{P}(\neg G_\ell) \right).
\]

Propositions 3.2.4 and 3.2.5 furnish \( s_0 = s_0(t_0, \nu) \) and \( c = c(t_0, \nu) \) such that, for \( \ell \geq 1 \) and \( s > s_0 \),

\[
\mathbb{P}(\neg G_\ell) \leq \exp \left( -cs_\ell^{3/2} \right) = \exp \left( -c2^{(\ell-1)/2}(2\nu s)^{3/4} \right).
\]

The sum over \( \ell \) of such a bound produces an upper bound of the same form \( \exp(-cs^{3/4}) \), as desired.

Turning to \( \mathbb{P}(Q_\ell \cap G_\ell) \), we may apply Proposition 3.4.3 with \( \epsilon = \epsilon_\ell \) and \( s = s_\ell \). The result is precisely the same sort of bound as on \( \mathbb{P}(\neg G_\ell) \) above; hence, by summing over \( \ell \), we again recover the sought upper bound, of the form \( \exp(-cs^{3/4}) \). This completes the proof of (3.4.27).

\[\blacksquare\]

### 3.5 Remote correlation: Proof of Theorem 3.1.1

The composition law from Proposition 3.2.2 will figure prominently in this argument since it permits us to describe the two-time distribution in terms of spatial processes that we understand well. Recall that this result shows that, for \( \alpha > 1 \) given, we may write \( \Upsilon_T(\alpha,0) \) as the composition of independent spatial processes \( \Upsilon_T(1,\cdot) \) and \( \Upsilon_{\alpha T} \cdot \). The latter is distributed as \( \Upsilon_T(\alpha - 1,\cdot) \). We will use a suggestive shorthand, in the spirit of \( \Upsilon_{\alpha T} \cdot \):

\[
\Upsilon_{0\Uparrow T}(\cdot) := \Upsilon_T(1,\cdot), \quad \Upsilon_{0\Uparrow \alpha T}(\cdot) := \Upsilon_T(\alpha,\cdot);
\] (3.5.1)
and, when we write $\Upsilon_{0\uparrow T}$, we mean $\Upsilon_{0\uparrow T}(0)$; and likewise for $\Upsilon_{0\uparrow \alpha T}$ and $\Upsilon_{\alpha T\downarrow T}$. The shorthand is intended to suggest that the value of $\Upsilon_{0\uparrow \alpha T}(0)$ from time zero to $\alpha T$ is obtained via the integral operation $I_T$ by composing the function $\Upsilon_{0\uparrow T}(\cdot)$ with $\Upsilon_{\alpha T\downarrow T}(\cdot)$, the latter under the opposite direction of time. The shorthand will cause confusion if $\Upsilon_{0\uparrow T}$ or $\Upsilon_{0\uparrow \alpha T}$ are regarded as functions of $T$ or $\alpha$. These two parameters are, however, given: the variable ‘·’ in (3.5.1) is spatial.

This section’s goal is to prove the bounds (3.1.2) in Theorem 3.1.1 which in our new notation are:

$$c_1 \alpha^{-1/3} \leq \text{Corr}\left(\Upsilon_{0\uparrow T}, \Upsilon_{0\uparrow \alpha T}\right) \leq c_2 \alpha^{-1/3}.$$  

(3.5.2)

The derivation of Theorem 3.1.1 depends on two principal results, Propositions 3.5.1 and 3.5.2. We state these; use them to prove the theorem; and then prove them in turn.

Define $A_{\text{high}} = \{\Upsilon_{0\uparrow \alpha T} - \Upsilon_{\alpha T\downarrow T} - \Upsilon_{0\uparrow T} \geq s\}$ and $A_{\text{low}} = \{\Upsilon_{0\uparrow \alpha T} - \Upsilon_{\alpha T\downarrow T} - \Upsilon_{0\uparrow T} \leq -t^{-1/3}s\}$.

**Proposition 3.5.1.** For any $T_0 > 0$ and $\alpha_0 > 1$, there exist $s_0 = s_0(T_0, \alpha_0) > 0$ and $c = c(T_0, \alpha_0) > 0$ such that, for $s \geq s_0$, $T > T_0$ and $\alpha > \alpha_0$,

$$\mathbb{P}(A_{\text{high}}) \leq \exp(-cs^{3/4}).$$  

(3.5.3)

**Proposition 3.5.2.** For any $T_0 > 0$, there exist $s_0 = s_0(T_0) > 0$ and $c = c(T_0) > 0$ such that, for $s \geq s_0$, $T > T_0$ and $\alpha > 2$,

$$\mathbb{P}(A_{\text{low}}) \leq \exp \left( -cs^{3/2} \right).$$  

(3.5.4)

Further, for any $T_0 > 0$, $\alpha_0 > 1$ and $\delta > 0$, there exist $s_0 = s_0(T_0, \alpha_0, \delta) > 0$ and $c = c(T_0, \alpha_0, \delta) > 0$ such that, for $s \geq s_0$, $T > T_0$ and $\alpha > \alpha_0$,

$$\mathbb{P} \left( A_{\text{low}} \mid \Upsilon_{0\uparrow T} \geq \mathbb{E}[\Upsilon_{0\uparrow T}] + \delta \right) \leq \exp \left( -cs^{3/2} \right).$$  

(3.5.5)

The two results address the rarity of the events that $\Upsilon_{0\uparrow \alpha T}$ significantly exceeds, or falls below, $\Upsilon_{0\uparrow T} + \Upsilon_{\alpha T\downarrow T}$. Note that, in the latter case – via the definition of $A_{\text{low}}$ – deviation is measured on scale $t^{-1/3}$. To interpret the two results and the latter choice of scaling, it may be helpful to consider the
composition law and the supremum variational problem obtained in the high $t$ limit which was a focus of attention in Section 3.3. In the high $t$ case, a counterpart to Proposition 3.5.1 would examine the tail of the difference between the variational problem’s solution and the value of the pair sum associated to the choice of location zero at the intermediate time. In this $t \nearrow \infty$ limit, Proposition 3.5.2 would become trivial, because the difference in question can never be negative. Back in our finite $T$ world, the softening of the variational problem leads to a degree of violation to this strict ordering. The $T^{-1/3}$ tail that we study probes the extent of this violation.

**Proof of Theorem 3.1.1** We first show the following stronger version of the upper bound on the correlation in (3.5.2): for any $T_0 > 0$, there exist $c_2 = c_2(0) > 0$ such that, for $T > T_0$, and $\alpha > 2$, $|\text{Corr}(\Upsilon_{0|\alpha T}, \Upsilon_{0|T})| \leq c_2 \alpha^{-1/3}$. Recall that, by definition,

$$\text{Corr}(\Upsilon_{0|T}, \Upsilon_{0|\alpha T}) = -\frac{\text{Cov}(\Upsilon_{0|\alpha T}, \Upsilon_{0|T})}{\sqrt{\text{Var}(\Upsilon_{0|\alpha T}) \sqrt{\text{Var}(\Upsilon_{0|T})}}}.$$  \hspace{1cm} (3.5.6)

Under the coupling provided by Proposition 3.2.2, $\Upsilon_{\alpha T \downarrow T}$ and $\Upsilon_{0|T}$ are independent. Hence,

$$|\text{Cov}(\Upsilon_{0|\alpha T}, \Upsilon_{0|T})| = \text{Cov}(\Upsilon_{0|\alpha T} - \Upsilon_{\alpha T \downarrow T} - \Upsilon_{0|T}, \Upsilon_{0|T}) + \text{Var}(\Upsilon_{0|T}).$$  \hspace{1cm} (3.5.7)

Applying the Cauchy-Schwarz inequality to the first term yields

$$|\text{Cov}(\Upsilon_{0|\alpha T} - \Upsilon_{\alpha T \downarrow T} - \Upsilon_{0|T}, \Upsilon_{0|T})| \leq \sqrt{\text{Var}(\Upsilon_{0|\alpha T} - \Upsilon_{\alpha T \downarrow T} - \Upsilon_{0|T}) \text{Var}(\Upsilon_{0|T})}.$$  \hspace{1cm} (3.5.8)

Substituting (3.5.7) into and applying (3.5.8) to (3.5.6) then leads to

$$|\text{Corr}(\Upsilon_{0|\alpha T}, \Upsilon_{0|T})| \leq \frac{\sqrt{\text{Var}(\Upsilon_{0|\alpha T} - \Upsilon_{\alpha T \downarrow T} - \Upsilon_{0|T})}}{\sqrt{\text{Var}(\Upsilon_{0|\alpha T})}} + \frac{\text{Var}(\Upsilon_{0|T})}{\sqrt{\text{Var}(\Upsilon_{0|\alpha T})}}.$$  \hspace{1cm} (3.5.9)

The tail bounds on $\Upsilon_T(0)$ (or on $\Upsilon_{0|T}$) from Propositions 3.2.4 and 3.2.5 imply via Lemma 3.8.4 that there exist $c = c(T_0) > 0$ and $C = C(T_0) > 0$ such that

$$c \leq \text{Var}(\Upsilon_{0|T}) \leq C, \quad \text{and} \quad c\alpha^{2/3} \leq \text{Var}(\Upsilon_{0|\alpha T}) \leq C\alpha^{2/3}.$$  \hspace{1cm} (3.5.10)

Here, the second bound uses $\Upsilon_{0 \to \alpha T} \overset{(d)}{=} \Upsilon_{\alpha T}(1,0)\alpha^{1/3}$. Similarly, Propositions 3.5.1 and 3.5.2
imply that

\[ \text{Var}(\Upsilon_{\alpha T} - \Upsilon_{\alpha T|\Upsilon_{\alpha T}} - \Upsilon_{\alpha T|\Upsilon_{\alpha T}}) \leq C. \tag{3.5.11} \]

Substituting (3.5.10) and (3.5.11) into the right-hand side of (3.5.9) yields

\[ |\text{Corr}(\Upsilon_{\alpha T}, \Upsilon_{\alpha T})| \leq c_2 \alpha^{-1/3} \]

for a constant \( c_2 = c_2(T_0) > 0 \). This is the strengthened upper bound that we have sought.

Now we turn to prove the lower bound in (3.5.2). Assume for now that \( T > 1 \), though we will eventually need to impose that \( T > T_0 \) for \( T_0 \) sufficiently large. The lower bound will arise from an appeal to Corollary 3.8.2 with \( X := \Upsilon_{\alpha T} \) and \( Y := \Upsilon_{\alpha T|\Upsilon_{\alpha T}} \). The other two parameters in the corollary, \( C_1 \) and \( C_2 \), will be specified after the next calculation, which will inform our choice of their value. Observe that, for \( y := E[\Upsilon_{\alpha T} + \delta] \) with \( \delta > 0 \),

\[ E[\Upsilon_{\alpha T} \mid \Upsilon_{\alpha T} \geq y] \geq E[\Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y} + y + E[\Upsilon_{\alpha T} - \Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y}]. \]

This bound follows from \( E[\Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y}] = E[\Upsilon_{\alpha T}] \) – a consequence of the independence of \( \Upsilon_{\alpha T} \) and \( \Upsilon_{\alpha T | \Upsilon_{\alpha T}} \) – and the trivial \( E[\Upsilon_{\alpha T} \mid \Upsilon_{\alpha T} \geq y] \geq y \). Next note that

\[ E[\Upsilon_{\alpha T} - \Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y}] \geq E \left[ \min\{0, \Upsilon_{\alpha T} - \Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y}\} \right] \]

\[ = -t^{-1/3} \int_0^\infty \mathbb{P}\left(\Upsilon_{\alpha T} - \Upsilon_{\alpha T | \Upsilon_{\alpha T} \geq y} \leq -T^{-1/3}s \mid \Upsilon_{\alpha T} \geq y\right)ds \geq -T^{-1/3}c(\delta) \]

for some \( c(\delta) > 0 \). The last inequality is due to an application of (3.5.5) in Proposition 3.5.2.

We now apply Corollary 3.8.2 with \( X := \Upsilon_{\alpha T}, Y := \Upsilon_{\alpha T}, C_1 := y = E[\Upsilon_{\alpha T}] + \delta \) and \( C_2 := \delta - T^{-\frac{1}{3}}c(\delta) \). Notice that \( C_1 \) and \( C_2 \) both depend on the parameter \( \delta > 0 \), which is as yet unspecified. By (3.8.16),

\[ \text{Cov}(\Upsilon_{\alpha T}, \Upsilon_{\alpha T}) \geq \left(\delta - T^{-1/3}c(\delta)\right) \cdot \mathbb{P}(\Upsilon_{\alpha T} \geq y) \cdot \left(\mathbb{E}[\Upsilon_{\alpha T} \mid \Upsilon_{\alpha T} \geq y] - \mathbb{E}[\Upsilon_{\alpha T} \mid \Upsilon_{\alpha T} < y]\right), \tag{3.5.12} \]
where we recall that $y = \mathbb{E}[\Upsilon_{0\uparrow T}] + \delta$. Observe that

$$
\mathbb{E}[\Upsilon_{0\uparrow T} | \Upsilon_{0\uparrow T} \geq y] - \mathbb{E}[\Upsilon_{0\uparrow T} | \Upsilon_{0\uparrow T} < y] = \frac{\mathbb{E}[\Upsilon_{0\uparrow T} \mid \Upsilon_{0\uparrow T} \geq y] - \mathbb{E}[\Upsilon_{0\uparrow T}]}{\mathbb{P}(\Upsilon_{0\uparrow T} < y)} \\
\geq \frac{\mathbb{P}(\Upsilon_{0\uparrow T} < \mathbb{E}[\Upsilon_{0\uparrow T}] + \delta)}{\delta} \geq \delta.
$$

Substituting this into (3.5.12), we arrive at

$$
\text{Cov}(\Upsilon_{0\uparrow \alpha T}, \Upsilon_{0\uparrow T}) \geq (\delta - t^{-1/3}c(\delta)) \cdot \mathbb{P}(\Upsilon_{0\uparrow T} \geq y) \cdot \delta.
$$

Fix any $\delta > 0$. Observe that for $T > T_0 := (\frac{2c(\delta)}{\delta})^3, \delta - T^{-1/3}c(\delta) \geq \delta/2$. By the upper-bound in Proposition 3.2.5 for $T \geq T_0$, we may bound $\mathbb{E}[\Upsilon_{0\uparrow T}] < C$ and hence $y < C + \delta$, for $C = C(T_0) > 0$. Using the lower-bound in Proposition 3.2.5 we further infer that $\mathbb{P}(\Upsilon_{0\uparrow T} \geq y) > C'$ for $C' = C'(T_0, \delta) > 0$. This shows that $\text{Cov}(\Upsilon_{0\uparrow \alpha T}, \Upsilon_{0\uparrow T}) \geq (\delta - t^{-1/3}c(\delta)) \cdot C' \cdot \delta$. This right-hand side is a strictly positive constant which holds uniformly over $T > T_0$. Substituting this and the upper bounds of (3.5.10) into the right-hand side of (3.5.6) produces the desired lower bound in (3.5.2) on $\text{Corr}(\Upsilon_{0\uparrow \alpha T}, \Upsilon_{0\uparrow T})$.

### 3.5.1 Proof of Proposition 3.5.1

For this proof and Proposition 3.5.2's, we will return to writing $\Upsilon_{0\uparrow T}(0)$ in place of $\Upsilon_{0\uparrow T}$, because we will utilize $\Upsilon_{0\uparrow T}(x)$ for various values of $x$. Define

$$
\mathcal{E} := \{ \sup_{|x| < T^{2/3}} \{ \Upsilon_{0\uparrow T}(T^{-2/3}x) + \frac{x^2}{4T^{4/3}} - \Upsilon_{0\uparrow T}(0) \geq \frac{s}{2} \} \}, \\
\mathcal{G} := \{ \sup_{|x| \geq T^{2/3}} \{ \Upsilon_{0\uparrow T}(T^{-2/3}x) + \frac{x^2}{4T^{4/3}} \geq \frac{s}{4} \} \cup \{ \Upsilon_{0\uparrow T}(0) \leq -\frac{s}{4} \} \}, \\
\mathcal{B} := \{ \int_{-\infty}^{\infty} e^{T^{1/3}(\Upsilon_{0\uparrow \alpha T}(T^{-2/3}x) - \frac{x^2}{4T^{4/3}})} dx \geq e^{T^{1/3}(\Upsilon_{0\uparrow \alpha T}(0) + \frac{s}{2})} \}.
$$

The derivation has three steps. **Step I** shows that $\mathcal{A}_{\text{high}} \cap -\mathcal{E} \cap -\mathcal{G} \subset \mathcal{B}$. The desired bound on $\mathbb{P}(\mathcal{A}_{\text{high}})$ will thus result from bounds on $\mathbb{P}(\mathcal{B})$ and $\mathbb{P}(\mathcal{E} \cup \mathcal{G})$ which are provided in **Steps II and III**.
Step I: To show that $A_{\text{high}} \cap -E \cap -G \subset B$, we will argue that, on the event $-E \cap -G$,

$$
\int_{|x|<T^{2/3}} e^{T^{1/3} \left( Y_{\alpha T \downarrow T}(T^{-2/3} x) + Y_{\alpha T}(-T^{-2/3} x) \right)} dx \\
\leq e^{T^{1/3}(Y_{\alpha T}(0)) + \frac{s}{2}} \int_{|x|<T^{2/3}} e^{T^{1/3} \left( Y_{\alpha T \downarrow T}(T^{-2/3} x) - \frac{x^2}{4T^{4/3}} \right)} dx \\
\int_{|x|\geq T^{2/3}} e^{T^{1/3} \left( Y_{\alpha T \downarrow T}(T^{-2/3} x) + Y_{\alpha T}(-T^{-2/3} x) \right)} dx \\
\leq e^{T^{1/3}(Y_{\alpha T}(0)) + \frac{s}{2}} \int_{|x|\geq T^{2/3}} e^{T^{1/3} \left( Y_{\alpha T \downarrow T}(T^{-2/3} x) - \frac{x^2}{4T^{4/3}} \right)} dx.
$$

Indeed, the first bound holds because, on the event $-E$, $\sup_{|x| \leq T^{2/3}} Y_{\alpha T}(T^{-2/3} x) \leq Y_{\alpha T}(0) - \frac{x^2}{4T^{4/3}} + s/2$; while the second bound is due to the validity on the event $-G$, and for $|x| > T^{2/3}$, of the bound $Y_{\alpha T}(t^{-2/3} x) \leq s/4 - \frac{x^2}{4T^{4/3}} \leq Y_{\alpha T}(0) + s/2 - \frac{x^2}{4T^{4/3}}$.

Summing the displayed bounds and using the composition law Proposition 3.2.2 yields

$$
e^{T^{1/3} Y_{\alpha T}(0)} \leq e^{T^{1/3} \left( Y_{\alpha T}(0) + s/2 \right)} \int_{-\infty}^{\infty} e^{T^{1/3} \left( Y_{\alpha T \downarrow T}(T^{-2/3} x) - \frac{x^2}{4T^{4/3}} \right)} dx. \quad (3.5.13)
$$

On the event $A_{\text{high}}$, the left-hand side of (3.5.13) is at least $e^{T^{1/3}(Y_{\alpha T \downarrow T}(0) + Y_{\alpha T}(0) + s)}$. Applying this, and cancelling $e^{T^{1/3}(Y_{\alpha T}(0) + s/2)}$, we arrive at the inequality which defines the event $B$; hence, we conclude that $A_{\text{high}} \cap -E \cap -G \subset B$, as desired.

Step II: Here, we prove that, for $t_0 > 0$ and $\alpha_0 > 1$, there exist $s_0 = s_0(t_0, \alpha_0)$ and $c = c(t_0, \alpha_0)$ such that, for $t > t_0$, $\alpha > \alpha_0$ and $s > s_0$,

$$
\mathbb{P}(B) \leq \exp(-cs^{3/4}). \quad (3.5.14)
$$

Set $x_0 = ((\alpha - 1)t^2)^{1/3}$ and introduce two events

$$
\tilde{E} := \left\{ \sup_{|x| < x_0} \left\{ Y_{\alpha T \downarrow T}(T^{-2/3} x) - Y_{\alpha T \downarrow T}(0) - \frac{x^2}{8T^{4/3}} \right\} \geq s/4 \right\},
$$

$$
\tilde{G} := \left\{ \sup_{|x| > x_0} \left\{ Y_{\alpha T \downarrow T}(T^{-2/3} x) \right\} \geq s/8 + \frac{k_0^2}{8T^{4/3}} \right\} \cup \left\{ Y_{\alpha T \downarrow T}(0) \leq -s/8 - \frac{k_0^2}{8T^{4/3}} \right\}.
$$

We first show that there exists $s_0 = s_0(t_0)$ such that, for $s \geq s_0$, $B \subset \tilde{E} \cup \tilde{G}$. We show the
contrapositive: \( \tilde{E} \cap \tilde{G} \subset \neg B \) when \( s \) is large enough. On the event \( \neg \tilde{E} \cap \neg \tilde{G} \), we have that
\[
\int_{|x|<x_0} e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(T^{-2/3}x)-\frac{x^2}{4T^{4/3}}\right)} dx \leq e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(0)+s/4\right)} \int_{|x|<x_0} e^{-\frac{x^2}{4T}} dx , \tag{3.5.15}
\]
\[
\int_{|x|\geq x_0} e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(T^{-2/3}x)-\frac{x^2}{4T^{4/3}}\right)} dx \leq e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(0)+s/4\right)} \int_{|x|\geq x_0} e^{-\frac{(|x|+x_0)^2}{4T}} dx . \tag{3.5.16}
\]

Indeed, the first bound is due to the event \( \neg \tilde{E} \) entailing that \( Y_{\alpha T \downarrow T}(x) \leq Y_{\alpha T \downarrow T}(0) + \frac{s}{4} + \frac{x^2}{8T^{4/3}} \) for all \( x \in [-x_0, x_0] \). The second bound follows by combining the inequality \( \exp(-(|x|^2-x_0^2)/4T) \leq \exp(-(|x|-x_0)^2/4T) \) for all \( |x| \geq x_0 \) with the fact that, on the event \( \neg \tilde{G} \),
\[
\sup_{|x|\geq x_0} T^{1/3} Y_{\alpha T \downarrow T}(T^{-2/3}x) \leq T^{1/3}\left(\frac{s}{8} + \frac{k_0^2}{8T^{4/3}}\right) \leq T^{1/3}\left(Y_{\alpha T \downarrow T}(0) + \frac{s}{4} + \frac{k_0^2}{4T^{4/3}}\right) .
\]

Summing (3.5.15) and (3.5.16), and using \( \int_{-\infty}^{\infty} \left(e^{-\frac{x^2}{4T}} + e^{-\frac{(|x|+x_0)^2}{4T}}\right) dx \leq cT^{1/2} \) for some \( c > 0 \), yields
\[
\int_{-\infty}^{\infty} e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(T^{-2/3}x)-\frac{x^2}{4T^{4/3}}\right)} dx \leq cT^{1/2} e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(0)+s/2\right)} \leq e^{t^{1/3}\left(Y_{\alpha T \downarrow T}(0)+s/2\right)} ,
\]
where the second inequality holds provided that \( s > s_0(T_0) \) for some suitably large \( s_0(t_0) > 0 \). This shows that, for \( s \geq s_0(t_0) \), \( B \subset \tilde{E} \cup \tilde{G} \), as claimed.

Returning to the proof of (3.5.14), by the above proved claim, \( B \subset \tilde{E} \cup \tilde{G} \) for \( s \geq s_0(t_0) \); therefore,
\[
\mathbb{P}(B) \leq \mathbb{P}(\tilde{E}) + \mathbb{P}(\tilde{G}) . \tag{3.5.17}
\]

We first bound \( \mathbb{P}(\tilde{E}) \). Owing to the definition of \( Y \) in (3.1.1) and \( Y_{\alpha T \downarrow T} \) in Proposition 3.2.2, we have
\[
\{ (\alpha - 1)^{-1/3} Y_{\alpha T \downarrow T}(x) : x \in \mathbb{R} \} \equiv \{ Y_{(\alpha - 1)T}((\alpha - 1)^{-2/3}x) : x \in \mathbb{R} \} . \tag{3.5.18}
\]

Using this and the change of variables \( y = T^{-2/3}x \) for all \( x \in [-x_0, x_0] \) implies that, for \( \theta := \alpha - 1 \),
\[
\mathbb{P}(\tilde{E}) = \mathbb{P}\left( \sup_{|y|<\theta^{1/3}} \{ Y_{\theta T}((\theta^{-2/3}y) - Y_{\theta T}(0) - \theta^{-1/3}y^2/8 \} \geq \theta^{-1/3} s/4 \right) .
\]

Applying Proposition 3.4.4 with this \( \theta, \nu = 1/4 \), \( s \) replaced by \( s/4 \), and \( T \) replaced by \( \theta T \) (which is
still bounded below by some \( T_0' > 0 \) since \( \alpha \geq \alpha_0 > 1 \), we see that there exist \( s_0 = s_0(T_0, \alpha_0) \) and \( c = c(T_0, \alpha_0) \) such that, for \( T \geq T_0, s \geq s_0 \) and \( \alpha > \alpha_0, \mathbb{P}(\bar{E}) \leq \exp(-cs^{3/4}) \).

We now bound \( \mathbb{P}(\bar{G}) \). By the arithmetic-geometric mean inequality, \((\alpha - 1)^{-1/3} \left( \frac{s}{8} + \frac{k_0^2}{8T^{2/3}} \right) \geq s^{1/2} / 4\). This, in conjunction with (3.5.18) and the union bound, shows that

\[
\mathbb{P}(\bar{G}) \leq \mathbb{P} \left( \sup_{x \in \mathbb{R}} \Upsilon_{\alpha-1}(x) \geq \frac{s^{1/2}}{4} \right) + \mathbb{P} \left( \Upsilon_{\alpha-1}(0) \leq -\frac{s^{1/2}}{4} \right).
\]

Applying Propositions 3.4.2 and 3.2.4, there exist \( s_0 = s_0(T_0, \alpha_0) \) and \( c = c(T_0, \alpha_0) \) such that, for \( T > T_0, s \geq s_0' \) and \( \alpha > \alpha_0, \mathbb{P}(\bar{G}) \leq \exp( -cs^{3/4} ) \). Substituting the upper bounds on \( \mathbb{P}(\bar{E}) \) and \( \mathbb{P}(\bar{G}) \) into (3.5.17) and summing, we arrive at (3.5.14).

**Step III:** By Step I, \( A_{\text{high}} \cap \neg E \cap \neg G \subset B \); hence,

\[
\mathbb{P}(A_{\text{high}}) \leq \mathbb{P}(B) + \mathbb{P}(E) + \mathbb{P}(G).
\]

We may bound \( \mathbb{P}(E) \leq \exp(-cs^{3/2}) \), using Theorem 3.1.3; and we may obtain the same bound for \( \mathbb{P}(G) \), using Propositions 3.4.2 and 3.2.4. Here, we have assumed that \( s > s_0(t_0) \) and \( c = c(t_0) \). Combining these bounds with the Step II bound (3.5.14), we arrive at (3.5.3), and thus complete the proof of Proposition 3.5.1.

3.5.2 Proof of Proposition 3.5.2

We separately address the two claims (3.5.4) and (3.5.5).

**Proof of (3.5.4).** Define events

\[
\mathcal{W}_1 := \left\{ \inf_{x \in [0,1]} \left( \Upsilon_{\alpha T \downarrow T}(T^{-2/3}x) - \Upsilon_{\alpha T \downarrow T}(0) \right) \leq -T^{-1/3}s/2 \right\},
\]

\[
\mathcal{W}_2 := \left\{ \inf_{x \in [0,1]} \left( \Upsilon_{0 \uparrow T}(T^{-2/3}x) - \Upsilon_{0 \uparrow T}(0) \right) \leq -T^{-1/3}s/2 \right\}.
\]

On the event \( \neg \mathcal{W}_1 \cap \neg \mathcal{W}_2 \),

\[
\Upsilon_{\alpha T \downarrow T}(t^{-2/3}x) \geq \Upsilon_{\alpha T \downarrow T}(0) - T^{-1/3}s/2, \quad \Upsilon_{0 \uparrow T}(T^{-2/3}x) \geq \Upsilon_{0 \uparrow T}(0) - T^{-1/3}s/2
\]
for all \( x \in [0, 1] \). It follows from the composition law Proposition \ref{prop:composition1} and these inequalities that

\[
e^{T_1/3 \Upsilon_{0} T(0)} \geq \int_{[0,1]} e^{T_1/3 (\Upsilon_{\alpha T} T(T^{-2/3} x) + \Upsilon_{0} T(T^{-2/3} x))} \, dx \geq e^{T_1/3 \Upsilon_{0} T(0) + T_1/3 \Upsilon_{0} T(0) - s}.
\]

This shows that \( \neg \mathcal{W}_1 \cap \neg \mathcal{W}_2 \subset \neg \mathcal{A}_{\text{low}} \). Hence,

\[
P(\mathcal{A}_{\text{low}}) \leq P(\mathcal{W}_1) + P(\mathcal{W}_2). \tag{3.5.19}
\]

To bound \( P(\mathcal{W}_1) \) and \( P(\mathcal{W}_2) \), we set \( \epsilon_1 := ((\alpha - 1) T)^{-2/3} \) and \( \epsilon_2 := T^{-2/3} \), and rewrite via (3.5.18)

\[
P(\mathcal{W}_1) = P\left( \inf_{x \in [0, \epsilon_1]} \left( \Upsilon_{(\alpha-1)T}(x) - \Upsilon_{(\alpha-1)T}(0) \right) \leq -\frac{s}{2} \epsilon_1^{1/2} \right), \tag{3.5.20}
\]

\[
P(\mathcal{W}_2) = P\left( \inf_{x \in [0, \epsilon_2]} \left( \Upsilon_{T}(x) - \Upsilon_{T}(0) \right) \leq -\frac{s}{2} \epsilon_2^{1/2} \right). \tag{3.5.21}
\]

We will bound these probabilities via Theorem \ref{thm:shift_independence} but two aspects of this application require mention. First, \( \epsilon_1 \) and \( \epsilon_2 \) may exceed the upper bound of one assumed on \( \epsilon \) in the theorem. These quantities are, however, bounded above by a constant depending on \( T_0 \) and \( \alpha_0 \); hence, at the price of degrading the values of \( s_0 \) and \( c \) in the theorem, variants of the result concerning shifts may be satisfactorily applied. Second, the theorem involves a parabolic shift, whereas the above expressions do not. The parabolic shift can be absorbed by changing the value of \( s \). By applying Theorem \ref{thm:shift_independence} in this manner, the right-hand sides of (3.5.20) and (3.5.21) may be bounded above by \( \exp(-cs^3/2) \) for some \( c = c(T_0, \alpha_0) > 0 \), provided that \( s > s_0 \) for some \( s_0 = s_0(T_0, \alpha_0) \). Substituting this upper bound into (3.5.19) completes the proof of (3.5.4).

\textbf{Proof of (3.5.5).} In the proof of (3.5.4), we showed that \( \mathcal{A}_{\text{low}} \subset \mathcal{W}_1 \cup \mathcal{W}_2 \). Since \( \Upsilon_{0+T} \) and \( \Upsilon_{\alpha T+T} \) are independent, we may bound

\[
P(\mathcal{A}_{\text{low}} | \Upsilon_{0+T}(0) \geq y) \leq P(\mathcal{W}_1) + P(\mathcal{W}_2 | \Upsilon_{0+T}(0) \geq y) \leq \exp(-cs^3/2) \left( 1 + P(\Upsilon_{0+T}(0) \geq y)^{-1} \right),
\]

where we set \( y := \mathbb{E}[\Upsilon_{0+T}(0)] + \delta \) and where we have used the bounds on \( P(\mathcal{W}_1) \) and \( P(\mathcal{W}_2) \) established in the proof of (3.5.4). By the upper-bound in Proposition \ref{prop:exponential_bound}, we find that, for \( T \geq T_0 \),
E and hence \( y < C + \delta \) for \( C = C(T_0) > 0 \). Thus, by the lower-bound in Proposition 3.2.5,
\[
P(\Upsilon^{T_0} \uparrow T(0) \geq y) > C' \quad \text{for} \quad C' = C'(T_0, \delta) > 0.
\]

Thus we may bound above the term \( 1 + P(\Upsilon^{T_0} \uparrow T(0) \geq y) - 1 \) by a constant. By choosing suitable values of \( c = c(T_0, \alpha_0) \) and \( s_0 = s_0(T_0, \alpha_0, \delta) \), we may absorb this constant and thus show that, for \( T > T_0, s > s_0 \) and \( \alpha > \alpha_0 \),
\[
P(\Upsilon^{T_0} \uparrow T(0) \geq y) \leq \exp(-cs_0^2/2),
\]
as desired to demonstrate (3.5.5).

3.6 Adjacent Correlation: Proof of Theorem 3.1.2

We are now concerned with the correlation between \( \Upsilon_T(1+\beta, 0) \) and \( \Upsilon_T(1, 0) \). As in Section 3.5, we will rely on the composition law Proposition 3.2.2 to realize \( \Upsilon_T(1+\beta, 0) \) in terms of \( \Upsilon_T(1, \cdot) \) and \( \Upsilon_T(1+\beta, T) \downarrow T(\cdot) \).

In this section, we will use a variation on the shorthand from Section 3.5:
\[
\Upsilon_0 \uparrow T(\cdot) := \Upsilon_T(1, \cdot), \quad \Upsilon_0 \uparrow (1+\beta) T(\cdot) := \Upsilon_T(1+\beta, \cdot);
\]
and by \( \Upsilon_0 \uparrow T \) is meant \( \Upsilon_0 \uparrow T(0) \) – and likewise for \( \Upsilon_0 \uparrow (1+\beta) T \).

Theorem 3.1.5 and the next stated Proposition 3.6.1 will yield these two bounds. We state the new proposition; prove Theorem 3.1.2; and, in two ensuing subsections, prove Theorem 3.1.5 and Proposition 3.6.1.

Proposition 3.6.1. For any \( T_0 > 0 \), there exist \( s_0 = s_0(T_0) \) and \( c = c(T_0, \beta) \) such that, for \( s > s_0 \), \( T > T_0 \) and \( \beta \in (0, \frac{1}{2}) \) satisfying \( \beta T > T_0 \),
\[
P(\Upsilon_0 \uparrow (1+\beta) T - \Upsilon_0 \uparrow T - \text{Cov}(\Upsilon_0 \uparrow (1+\beta) T, \Upsilon_0 \uparrow T)) \leq c \beta^{2/3}s_0^{2/3} \exp(-cs_0^2/2).
\]

Proof of Theorem 3.1.2. Aiming to apply Lemma 3.8.3, suppress \( T \) and \( \beta \) in notation that sets \( X := \text{P}(\Upsilon_0 \uparrow (1+\beta) T - \Upsilon_0 \uparrow T - \text{Cov}(\Upsilon_0 \uparrow (1+\beta) T, \Upsilon_0 \uparrow T)) \leq c \beta^{2/3}s_0^{2/3} \exp(-cs_0^2/2).\)
We may argue that there exist \( c_1 = c_1(T_0) < c_2 = c_2(T_0) \) such that, for all \( T > T_0 / \beta \),

\[
c_1 \leq \text{Var}(Y) \leq c_2, \quad c_1 \beta^{2/3} \leq \text{Var}(X) \leq c_2 \beta^{2/3}.
\]  

(3.6.3)

Indeed, the bounds on \( \text{Var}(Y) \) follow by combining the tail bounds from Propositions \( 3.2.4 \) and \( 3.2.5 \) with the later presented tool, Lemma \( 3.8.4 \) that translates tail bounds into bounds on variance. For the bounds on \( \text{Var}(X) \), it is Theorem \( 3.1.5 \) that instead furnishes the needed tail bounds.

In view of these bounds and the Cauchy-Schwarz inequality in the guise \( \Psi^2 \leq \chi \), there exists \( \beta_0 \in (0, \frac{1}{2}) \) such that, for \( \beta < \beta_0 \) and \( T > T_0 / \beta \), \( \max \{ \chi, \Psi + \frac{1}{2} \chi \} < 1 \). This verifies the hypothesis of Lemma \( 3.8.3 \) and thus demonstrates the existence of constants \( C_1 < C_2 \) for which \( \beta < \beta_0 \) and \( T > T_0 / \beta \) imply that

\[
1 - \Theta / 2 + C_1 \chi^{3/2} \leq \text{Corr}(Z, Y) \leq 1 - \Theta / 2 + C_2 \chi^{3/2}.
\]  

(3.6.4)

We claim that

\[
c_1 \beta^{2/3} \leq \Theta \leq \chi \leq c_2 \beta^{2/3},
\]  

(3.6.5)

where here we assume that \( \beta < \beta_0 \) and \( 0 < c_1 = c_1(T_0, \beta_0) < c_2 = c_2(T_0, \beta_0) \). This claim alongside (3.6.4) proves (3.6.1) for \( \beta < \beta_0 \). The result is extended to \( \beta \in [\beta_0, 1/2) \) by possibly changing the constants and recognizing that the correlations are bounded above by one; thus the proof of Theorem \( 3.1.2 \) is complete subject to verifying the three bounds in the claim (3.6.5).

Consider the first claimed bound \( c_1 \beta^{2/3} \leq \Theta \) in (3.6.5). Observe that \( \Theta = \frac{\mathbb{E}[(X - \Psi Y)^2]}{\text{Var}(Y)} \). We may bound \( \text{Var}(Y) \) above by a constant using (3.6.3). Thus it remains to bound \( \mathbb{E}[(X - \Psi Y)^2] \geq c \beta^{2/3} \) for some \( c = c(T_0, \beta_0) \). To do this, we will appeal to the second part of Lemma \( 3.8.4 \) There are two hypotheses that we must verify for that application. The first is that \( \mathbb{E}[X - \Psi Y] \leq C \beta^{1/3} \) for some \( C = C(T_0, \beta_0) > 0 \), and the second is the upper-tail lower bound on \( X - \Psi Y \) already
given in Proposition 3.6.1. Predicated upon showing these bounds, the second part of Lemma 3.8.4 immediately implies \( \mathbb{E}[(X - \Psi Y)^2] \geq c\beta^{2/3} \) as desired. To show that \( |\mathbb{E}[X - \Psi Y]| \leq C\beta^{1/3} \) we demonstrate that \( \mathbb{E}[X] \) converges to 0, \( \Psi \leq C\beta^{1/3} \) and \( \mathbb{E}[Y] < C \) for some \( C = C(T_0, \beta_0) > 0 \). That \( \mathbb{E}[X] \to 0 \) is shown by combining the fact that \( \mathbb{E}[\Upsilon_0 T] \) and \( \mathbb{E}[\Upsilon_0 (1+\beta) T] \) converge to a common limit (due to the weak convergence result of Proposition 2.1.7) with the tail bounds affording in Propositions 3.2.4 and 3.2.5. The Cauchy-Schwarz inequality bounds \( \Psi \leq \sqrt{\text{Var}(X) / \text{Var}(Y)} \) and the inference \( \Psi \leq C\beta^{1/3} \) made from applying the bounds (3.6.3); these bounds in (3.6.3) also show that \( \mathbb{E}[Y] < C \).

The second claimed bound \( \Theta \leq \chi \) in (3.6.5) follows since \( \Theta = \chi - \Psi^2 \), and the third bound \( \chi \leq c_2 \beta^{2/3} \) is due to (3.6.3). Thus claim (3.6.5) is verified and the proof of Theorem 3.1.2 completed. ■

3.6.1 Proof of Theorem 3.1.5

There are three inequalities to prove, and the proof is divided into three numbered stages accordingly. The notation \( \Upsilon_{0\uparrow T} \) in the theorem’s statement becomes \( \Upsilon_{0\uparrow T}(0) \) in the proof, because we have cause to consider \( \Upsilon_{0\uparrow T}(x) \) for general values of \( x \).

Stage 1. Proof of \( \mathbb{P}(\Upsilon_{0\uparrow (1+\beta)} T - \Upsilon_{0\uparrow T} \geq \beta^{1/3} s) \leq \exp\left(-c_2 s^{3/4}\right) \). Seeking to bound \( \mathbb{P}(\mathcal{A}_{\text{high}}) \), define

\[
\mathcal{A}_{\text{high}} := \left\{ \Upsilon_{0\uparrow (1+\beta)} T(0) - \Upsilon_{0\uparrow T}(0) \geq \beta^{1/3} s \right\},
\]

\[
\mathcal{E} := \left\{ \sup_{x \in \mathbb{R}} \left\{ \Upsilon_{(1+\beta) T \downarrow T}(T^{-2/3} x) + \frac{x^2}{4\beta T^{1/3}} \right\} \geq \beta^{1/3} \frac{s}{2} \right\}.
\]

By the union bound, \( \mathbb{P}(\mathcal{A}_{\text{high}}) \leq \mathbb{P}(\mathcal{E}) + \mathbb{P}(\mathcal{A}_{\text{high}} \cap \neg \mathcal{E}) \); it thus suffices to bound the probabilities of the two right-hand events by expressions of the form \( \exp\left(-c s^{3/4}\right) \). Bounding \( \mathbb{P}(\mathcal{E}) \) is easier: by the definition of \( \Upsilon \),

\[
\left\{ \Upsilon_{(1+\beta) T \downarrow T}(\beta^{2/3} x) : x \in \mathbb{R} \right\} \overset{d}{=} \left\{ \beta^{1/3} \Upsilon_{\beta T}(x) : x \in \mathbb{R} \right\}.
\]

Via the change of variables \( y = (\beta T)^{-2/3} x \) and this distributional identity,

\[
\mathbb{P}(\mathcal{E}) = \mathbb{P}\left( \sup_{y \in \mathbb{R}} \left\{ \Upsilon_{\beta T}(y) + \frac{x^2}{4} \right\} \geq \frac{s}{2} \right).
\]
By Proposition 3.4.2 there exist \( s_0 = s_0(T_0) > 0 \) and \( c_0 = c_0(T_0) \) such that, for \( s > s_0, T > T_0 \) and
\( \beta \in (0, \frac{1}{2}) \) satisfying \( \beta T > T_0, \mathbb{P}(\mathcal{E}) \leq \exp(-cs^{3/2}) \) – thus obtaining the desired bound on \( \mathbb{P}(\mathcal{E}) \).

We turn to demonstrating that \( \mathbb{P}(\mathcal{A}_{\text{high}} \cap \neg \mathcal{E}) \leq \exp\left(-cs^{3/4}\right) \). Set \( y_0 := (\beta T)^{1/3} \) and
\[
\mathcal{D} := \left\{ \int_{-\infty}^{\infty} \exp\left(T^{1/3} \Upsilon_{0\uparrow T}(T^{-2/3}x) - \frac{x^2}{4\beta T}\right)dx \geq \exp\left(T^{1/3} \Upsilon_{0\uparrow T}(0) + (\beta T)^{1/3}s\right) \right\},
\]
\[
\mathcal{Q} := \left\{ \sup_{|x| \leq y_0} \left\{ \Upsilon_{0\uparrow T}(T^{-2/3}x) - \Upsilon_{0\uparrow T}(0) - \frac{x^2}{8\beta T^{4/3}} \right\} \geq \beta^{1/3}s\right\},
\]
\[
\mathcal{G} := \left\{ \sup_{|x| \geq y_0} \Upsilon_{0\uparrow T}(T^{-2/3}x) \geq \frac{y_0^2}{16\beta T^{4/3}} + \frac{\beta^{1/3}s}{8} \right\} \cup \left\{ \Upsilon_{0\uparrow T}(0) \leq -\frac{y_0^2}{16\beta T^{4/3}} - \frac{\beta^{1/3}s}{8} \right\}.
\]

To obtain the presently sought bound, we show in three steps that \( \mathcal{A}_{\text{high}} \cap \neg \mathcal{E} \subset \mathcal{D} \); that \( \mathcal{D} \subset \mathcal{Q} \cup \mathcal{G} \); and that \( \mathbb{P}(\mathcal{Q} \cup \mathcal{G}) \leq \exp(-cs^{3/4}) \).

**Step I:** We seek to show that \( \mathcal{A}_{\text{high}} \cap \neg \mathcal{E} \subset \mathcal{D} \). When \( \neg \mathcal{E} \) occurs, we have that, for \( x \in \mathbb{R} \),
\[
t^{1/3} \Upsilon_{(1+\beta)T \uparrow T}(T^{-2/3}x) \leq -\frac{x^2}{4\beta T} + (\beta T)^{1/3}s.
\]
Applying the composition law Proposition 3.2.2 to write \( \Upsilon_{0\uparrow T}(0) \) in terms of \( \Upsilon_{0\uparrow T}(\cdot) \) and \( \Upsilon_{(1+\beta)T \uparrow T}(\cdot) \),
\[
\exp\left(t^{1/3} \Upsilon_{0\uparrow T}(0)\right) \leq \exp\left((\beta T)^{1/3}s\right) \int_{-\infty}^{\infty} \exp\left(T^{1/3} \Upsilon_{0\uparrow T}(T^{-2/3}x) - \frac{x^2}{4\beta T}\right)dx.
\]
On the event \( \mathcal{A}_{\text{high}} \), this left-hand side is bounded below by \( \exp\left(T^{1/3} \Upsilon_{0\uparrow T}(0) + (\beta T)^{1/3}s\right) \). Since the bound specifying \( \mathcal{D} \) has been verified, the desired containment has been shown.

**Step II:** We seek to show that there exists \( s_0 > 0 \) such that for all \( s \geq s_0, \mathcal{D} \subset \mathcal{Q} \cup \mathcal{G} \). We will show the contrapositive \( \neg \mathcal{Q} \cap \neg \mathcal{G} \subset \neg \mathcal{D} \). Note first that
\[
\int_{-\infty}^{\infty} \exp\left(t^{1/3} \Upsilon_{0\uparrow T}(T^{-2/3}x) - \frac{x^2}{4\beta T}\right)dx
\leq \exp\left(T^{1/3} \sup_{|y| \leq y_0} \left\{ \Upsilon_{0\uparrow T}(T^{-2/3}y) - \frac{y^2}{8\beta T^{4/3}} \right\}\right) \int_{|y| \leq y_0} \exp\left(-\frac{x^2}{8\beta T}\right)dx
+ \exp\left(t^{1/3} \sup_{|y| \geq y_0} \Upsilon_{0\uparrow T}(T^{-2/3}y)\right) \int_{|y| \geq y_0} \exp\left(-\frac{x^2}{8\beta T}\right)dx.
\] (3.6.7)
Here, the first right-hand term is present because, when \( |x| \leq y_0, \Upsilon_{0\uparrow T}(T^{-2/3}x) - \frac{x^2}{4\beta T} \) is at most
and the upper bound on the lower tail of the law of $\Upsilon$ such that, for $s \geq y_0$ \{ $\Upsilon_{0,T}(T^{-2/3}y)$ \} $- \frac{x^2}{4\beta T}$ when $|x| \geq y_0$.

On the event $\neg Q \cap \neg G$, we have the bounds

$$\exp \left( t^{1/3} \sup_{|x| < y_0} \left( \Upsilon_{0,T}(T^{-2/3}x) - \frac{x^2}{8\beta T^{4/3}} \right) \right) \leq \exp \left( T^{1/3} \Upsilon_{0,T}(0) + (\beta T)^{1/3} \frac{s}{4} \right),$$

$$\exp \left( T^{1/3} \sup_{|x| \geq y_0} \Upsilon_{0,T}(T^{-2/3}x) \right) \leq \exp \left( T^{1/3} \Upsilon_{0,T}(0) + \frac{y_0^2}{8\beta T} + (\beta T)^{1/3} s \right).$$

Substitute these into (3.6.7) and note that $\int_{|x| \leq y_0} \exp \left( - \frac{x^2}{8\beta T} \right) dy$ and $\int_{|x| \geq y_0} \exp \left( - \frac{x^2 - y_0^2}{8\beta T} \right) dx$ are bounded above by $c(\beta T)^{1/2}$ for some $c > 0$. What we learn is that, for some constant $c > 0$,

$$\int_{-\infty}^{\infty} \exp \left( T^{1/3} \Upsilon_{0,T}(T^{-2/3}x) - \frac{x^2}{4\beta T} \right) dx \leq c(\beta T)^{1/2} \exp \left( T^{1/3} \Upsilon_{0,T}(0) + (\beta T)^{1/3} \frac{s}{4} \right)(3.6.8)$$

If $s > 4 \log \left( c(\beta T)^{1/2} \right) (\beta T)^{-1/3}$ then $c(\beta T)^{1/2} \leq \exp \left( (\beta T)^{1/3} \frac{s}{4} \right)$. Since $r^{-1} \log r$ is bounded for $r > 0$, we find that there exists $s_0 > 0$ such that, for $s > s_0$, the right-hand side of (3.6.8) is at most $\exp \left( T^{1/3} \Upsilon_{0,T}(0) + (\beta T)^{1/3} \frac{s}{4} \right)$. Since this bound specifies the event $\neg D$, the conclusion of this second step has been obtained.

**Step III:** We seek to show that there exist $s_0 = s_0(T_0) > 0$ and $c = c(T_0) > 0$ such that, for $s \geq s_0$, $\mathbb{P}(Q \cup G) \leq \exp(-cs^{3/4})$. Recall that the spatial process $\Upsilon_{0,T}(\cdot)$ has same law as $\Upsilon_T(\cdot)$. Apply the change of variables $x = (\beta T)^{2/3} y$ in the definition of $Q$, and set $\theta := \beta^{-1}$, to obtain

$$Q = \left\{ \sup_{|y| < \theta^{1/3}} \left\{ \Upsilon_T(\theta^{-2/3} y) - \Upsilon_T(0) - \theta^{-1/3} \frac{y^2}{8} \right\} \geq \theta^{-1/3} \frac{s}{4} \right\}.$$

Since $\beta < 1$ by assumption, $\theta > 1$. Owing to this, we can apply Proposition 3.4.4 with $\nu = 1/4$. This result controls the supremum only over positive $\theta$. However, using the reflection invariance of $\Upsilon_T$ given in Proposition 2.1.7 we may likewise control the supremum over negative $\theta$. Allowing the resulting inferences with the union bound, we conclude that there exist $s_0 = s_0(T_0)$ and $c = c(T_0)$ such that, for $s \geq s_0$, $\mathbb{P}(Q) \leq \exp(-cs^{3/4})$.

Using the upper bound on the upper tail for the law of $\sup_{x \in \mathbb{R}} \Upsilon_{0,T}(x)$ from Proposition 3.4.1 and the upper bound on the lower tail of the law of $\Upsilon_{0,T}(0)$ from Proposition 3.2.4 shows that there
exist \( s_0 = s_0(T_0) > 0, c = c(T_0) > 0 \) and \( c' = c'(T_0) > 0 \) so that

\[
P(G) \leq \exp \left( -c \left( \frac{1}{16\beta^{1/3}} + \frac{\beta^{1/3} s^2}{8} \right) \right) \leq \exp(-c's^{3/4}),
\]

where the latter inequality uses \( \frac{1}{16\beta^{1/3}} + \frac{\beta^{1/3} s}{8} \geq \sqrt{\frac{3}{8}} \) (via the arithmetic-geometric mean inequality).

Combining this bound on \( P(G) \) with the above bound on \( P(Q) \) yields the conclusion of Step III. Thus do we obtain the sought bound \( P(A_{\text{high}} \cap -E) \leq \exp \left( -cs^{3/4} \right) \) completing the derivation of the first of three inequalities asserted by Theorem 3.1.5, namely \( P(Y_{0\uparrow(1+\beta)T} - Y_{0\uparrow T} \geq \beta^{1/3}s) \leq \exp \left( -c_2s^{3/4} \right). \)

**Stage 2. Proof of** \( \exp \left( -c_1s^{3/2} \right) \leq P(Y_{0\uparrow(1+\beta)T} - Y_{0\uparrow T} \geq \beta^{1/3}s) \). **Fix** \( \theta := (\beta T)^{2/3}s^{-1} \) and define events

\[
C := \bigcap_{\epsilon \in (-1,1)} \{ Y_{(1+\beta)T \uparrow T}(\epsilon T^{-2/3}y) \geq 2\beta^{1/3}s \}, \quad D := \left\{ \inf_{y \leq \theta} Y_{(1+\beta)T \uparrow T}(T^{-2/3}y) \geq \beta^{1/3}\frac{\sqrt{s}}{4} \right\},
\]

\[
E := \left\{ \inf_{y \leq \theta} Y_{0\uparrow T}(T^{-2/3}y) - Y_{0\uparrow T}(0) \geq -\beta^{1/3}s^2 \right\}, \quad W := C \cap D \cap E.
\]

The proof has two steps. First, we show that, for \( s \) large enough, \( W \subset A_{\text{high}} \); second, we find a lower bound of the form \( P(W) \geq \exp(-cs^{3/2}). \)

**Step I:** We seek to show that \( W \subset A_{\text{high}} \) for \( s > s_0 \) for \( s_0 = s_0(T_0) > 0 \). On the event \( W \), we have

\[
\frac{1}{3}Y_{(1+\beta)T \uparrow T}(T^{-2/3}y) + T^{1/3}y_{0\uparrow T}(T^{-2/3}y) \geq T^{1/3}Y_{0\uparrow T}(0) + (\beta T)^{1/3}\frac{5s}{4}, \quad \forall y \in [-\theta, \theta].
\]

Substituting this inequality into the composition law Proposition 3.2.2 yields

\[
\exp \left( T^{1/3}Y_{0\uparrow T}(0) \right) \geq \int_{-\theta}^{\theta} \exp \left( T^{1/3}Y_{0\uparrow T}(0) + (\beta t)^{1/3}\frac{5s}{4} \right) dx = 2 \exp \left( T^{1/3}Y_{0\uparrow T}(0) + (\beta T)^{1/3}\frac{5s}{4} - \log s + \frac{2}{3} \log(\beta T) \right).
\]

The quantity in the preceding line is greater than \( \exp(T^{1/3}Y_{0\uparrow T}(0) + (\beta T)^{1/3}s) \) provided that \( s > s_0 \) for some \( s_0(T_0) > 0 \), because \( \beta T > T_0 \) by assumption. The event \( A_{\text{high}} \) is specified by the resulting
Here, this shows that \( \mathcal{W} \subset \mathcal{A}_{\text{high}} \).

**Step II:** We seek to prove that \( \mathbb{P}(\mathcal{W}) \geq \exp \left( -cs^{3/2} \right) \) where \( c = c(T_0) > 0 \) and \( s > s_0 \) for \( s_0 = s_0(T_0) > 0 \). Since \( \Upsilon_{(1+\beta)T\downarrow T} \) and \( \Upsilon_{0\uparrow T} \) are independent,

\[
\mathbb{P}(\mathcal{W}) = \mathbb{P}(\mathcal{C} \cap \mathcal{D}) \mathbb{P}(\mathcal{E}) .
\]

By setting \( \epsilon = \beta^{2/3}s^{-1} \), we may write \( \mathcal{E} = \{ \inf_{|y| \leq \epsilon} \Upsilon_{0\uparrow T}(T^{-2/3}y) - \Upsilon_{0\uparrow T}(0) \geq -\epsilon^{1/2}cs^{3/2} \} \). Thus, we may apply Theorem 3.1.3 to find that there exist \( s_0 = s_0(T_0) > 0 \) and \( c = c(T_0) > 0 \) such that \( \mathbb{P}(\mathcal{E}) \geq 1 - \exp \left( -cs^{9/4} \right) \), provided that \( s > s_0 \) and \( \beta T \geq T_0 \). We see then that the lower bound on \( \mathbb{P}(\mathcal{W}) \) will follow once we derive \( \mathbb{P}(\mathcal{C} \cap \mathcal{D}) \geq \exp \left( -cs^{3/2} \right) \). Combining the distributional equality (3.6.6), the stationarity in Proposition 2.1.7 and the lower bound in Proposition 3.2.5, we find that \( \mathbb{P}(\mathcal{C}) \geq \exp \left( -cs^{3/2} \right) \). By decomposing \( \mathbb{P}(\mathcal{C} \cap \mathcal{D}) = \mathbb{P}(\mathcal{C}) - \mathbb{P}(\mathcal{C} \cap -\mathcal{D}) \) we see that the present step II will be completed by obtaining \( \mathbb{P}(\mathcal{C} \cap -\mathcal{D}) \leq \exp \left( -cs^{2+\eta} \right) \) for any fixed \( \eta > 0 \). We will use the KPZ line ensemble to derive such a bound with the choice \( \eta = 1 \).

Using (3.6.6), Proposition 2.2.5 and changing variables, we express \( \mathbb{P}(\mathcal{C} \cap -\mathcal{D}) \) as a probability concerning the lowest indexed curve \( \Upsilon^{(1)}_{\beta T}(\cdot) \) of the KPZ\(\beta T \) line ensemble, and bound the resulting probability:

\[
\mathbb{P}(\mathcal{C} \cap -\mathcal{D}) = \mathbb{P}
\bigg( \bigcap_{\epsilon \in (-1,1)} \left\{ \Upsilon^{(1)}_{\beta T}(\epsilon s^{-1}) \geq 2s \right\} \bigcap \left\{ \inf_{|y| \leq s^{-1}} \Upsilon^{(1)}_{\beta T}(y) \leq \frac{7s}{4} \right\} \bigg)
\leq \mathbb{P}_{B(\pm s^{-1})=2s} \left( \inf_{|y| \leq s^{-1}} B(y) \geq \frac{7s}{4} \right) \leq \exp \left( -c's^{5/2} \right). \tag{3.6.9}
\]

Here, \( B \) is a Brownian bridge on \([-s^{-1}, s^{-1}]\) with starting and ending values \( 2s \), and \( c' > 0 \) is a constant. The first upper bound in (3.6.9) is due to an appeal to the Brownian Gibbs property in view of the coupling in Lemma 2.2.6. That is, we write the concerned probability as the expectation of the indicator of the first event \( \bigcap_{\epsilon \in (-1,1)} \left\{ \Upsilon^{(1)}_{\beta T}(\epsilon s^{-1}) \geq 2s \right\} \) multiplied by the conditional expectation (with respect to the sigma field generator by everything outside of the first curve on the interval \([-s^{-1}, s^{-1}]\)) of the indicator for the second event. By Lemma 2.2.6 and the nature of the second event, this conditional expectation only increases when the second curve drops to \( -\infty \) and the starting and ending points drop to \( 2s \). Doing this, and then bounding the indicator for the first event by 1, yields the first bound in (3.6.9). The second bound in (3.6.9) is due to the reflection principle.
As we have noted, the bound (3.6.9) yields Step II, namely the bound \( P(W) \geq \exp(-c_3 s^{3/2}) \).

Steps I and II completed, the second of the three stages of the proof of Theorem 3.1.5 is also finished.

**Stage 3. Proof of** \( P(\Upsilon_{0\uparrow(1+\beta)T} - \Upsilon_{0\uparrow T} \leq -\beta^{1/3}s) \leq \exp(-c_3 s^{3/2}) \). In order to bound \( P(A_{\text{low}}) \),

set

\[
A_{\text{low}} := \left\{ \Upsilon_{0\uparrow(1+\beta)T} - \Upsilon_{0\uparrow T}(0) \leq -\beta^{1/3}s \right\},
\]

\[
\mathcal{E} := \left\{ \inf_{x \in \mathbb{R}} \left( \Upsilon_{(1+\beta)T\downarrow T}(T^{-2/3}x) + \frac{(1+\nu)x^2}{2\beta T^{4/3}} \right) \leq -\beta^{1/3}s \frac{s}{4} \right\},
\]

where \( \nu \in (0,1) \) is arbitrary and fixed. The union bound shows that \( P(A_{\text{low}}) \leq P(\mathcal{E}) + P(A_{\text{low}} \cap \neg \mathcal{E}) \),

so that our task is to bound the two right-hand terms. Regarding the first (and easier) of the two, recall the distributional identity in (3.6.6) and change variables \( y = (\beta t)^{-2/3}x \) to find that

\[
P(\mathcal{E}) = P\left( \inf_{y \in \mathbb{R}} \left( \Upsilon_{\beta T}(y) + \frac{(1+\nu)x^2}{2} \right) \leq -s \frac{s}{4} \right).
\]

By Proposition 3.4.1 there exist \( s_0 = s_0(T_0) > 0 \) and \( c = c(T_0) > 0 \) such that, for \( s > s_0 \) and \( T > T_0 \frac{1}{\beta} \),

\[
P(\mathcal{E}) \leq \exp(-cs^{5/2}),
\]

which is the first of the two bounds that we seek. The second is an upper bound on \( P(A_{\text{low}} \cap \neg \mathcal{E}) \).

Fixing \( y_0 := (\beta T^2)^{1/3} \), define events

\[
\mathcal{D} := \left\{ \int_{-\infty}^{\infty} \exp\left( T^{1/3} \Upsilon_{0\uparrow T}(T^{-2/3}x) - (1+\nu)x^2 \frac{x^2}{2\beta T} \right) dx \leq \exp\left( T^{1/3} \Upsilon_{0\uparrow T}(0) - (\beta T)^{1/3}s \frac{3s}{4} \right) \right\},
\]

\[
\mathcal{Q} := \left\{ \inf_{|x| < y_0} \left( \Upsilon_{0\uparrow T}(T^{-2/3}x) - \Upsilon_{0\uparrow T}(0) + \frac{(1+\nu)x^2}{2\beta T^{4/3}} \right) \leq -\beta^{1/3}s \frac{s}{2} \right\},
\]

\[
\mathcal{G} := \left\{ \inf_{|y| \geq y_0} \Upsilon_{0\uparrow T}(T^{-2/3}y) \leq -\frac{(1+\nu)y_0^2}{2\beta T^{4/3}} - \beta^{1/3}s \frac{3s}{4} \right\} \cup \left\{ \Upsilon_{0\uparrow T}(0) \geq \frac{(1+\nu)y_0^2}{2\beta T^{4/3}} + \beta^{1/3}s \frac{3s}{4} \right\}.
\]

These events are naturally similar to \( \mathcal{D}, \mathcal{Q}, \) and \( \mathcal{G} \) from the preceding proof of the upper bound on \( P(A_{\text{high}}) \), but the inequalities are each of opposite type. The same three steps govern the proof: in
Step I, we show that $A_{\text{low}} \cap \neg \tilde{E} \subset \tilde{D}$; in Step II, that $\tilde{D} \subset \tilde{Q} \cup \tilde{F}$; and in Step III, we find an upper bound on $\mathbb{P}(\tilde{Q} \cup \tilde{F})$. The logic of the argument in each step is unchanged from its counterpart’s in the derivation of an upper bound on $\mathbb{P}(A_{\text{low}})$, and we do not record the arguments again. ■

3.6.2 Proof of Proposition 3.6.1

The derivation runs in parallel with Stage 2 of Theorem 3.1.5’s proof. Propositions 3.2.4 and 3.2.5, Theorem 3.1.5 and Lemma 3.8.4 imply that there exist $0 < C_2 = C_2(T_0) \leq C_1 = C_1(T_0)$ for which

$$C_2 \leq \text{Var}(\Upsilon_{0\uparrow T}) \leq C_1, \quad C_2 \beta^{2/3} \leq \text{Var}(\Upsilon_{0\uparrow (1+\beta)T} - \Upsilon_{0\uparrow T}) \leq C_1 \beta^{2/3}. \quad (3.6.10)$$

By (3.6.10) and the Cauchy-Schwarz inequality, there exists $C' = C'(t_0) > 1$ such that

$$\rho := \beta^{-1/3} \frac{\text{Cov}(\Upsilon_{0\uparrow (1+\beta)T} - \Upsilon_{0\uparrow T}, \Upsilon_{0\uparrow T}(0))}{\text{Var}(\Upsilon_{0\uparrow T}(0))} \leq C'.$$

Fix $\delta \in (0, 1)$ and denote $\theta := (\beta T)^{2/3}s^{-1}$. Define $\tilde{W} = \tilde{C} \cap \tilde{D} \cap \tilde{E} \cap \tilde{F}$, where

$$\tilde{C} := \bigcap_{\epsilon \in \{-1, 1\}} \left\{ \Upsilon_{(1+T)\beta \epsilon t}(T-2/3) \geq 3 \beta^{1/3}s \right\}, \quad \tilde{D} := \left\{ \inf_{|y| \leq \theta} \Upsilon_{(1+\beta)T\downarrow T}(t^{-2/3}y) \geq (2 + \delta) \beta^{1/3}s \right\},$$

$$\tilde{E} := \left\{ \inf_{|y| \leq \theta} \Upsilon_{0\uparrow T}(T-2/3y) - \Upsilon_{0\uparrow T}(0) \geq -\beta^{1/3}s/2 \right\}, \quad \tilde{F} := \left\{ \Upsilon_{0\uparrow T}(0) \leq \beta^{1/3}s/2 \rho \right\}.$$

On the event $\tilde{W}$, it follows that, for $|y| \leq \theta$,

$$T^{1/3} \Upsilon_{(1+\beta)T\downarrow T}(T-2/3y) + T^{1/3} \Upsilon_{0\uparrow T}(T-2/3y) \geq (\beta T)^{1/3}(\frac{3}{2} + \delta)s + T^{1/3} \Upsilon_{0\uparrow T}(0) \geq \rho \beta^{1/3}s + (1 + \rho \beta^{1/3})T^{1/3} \Upsilon_{0\uparrow T}(0) \quad (3.6.11)$$

where the first bound is due to $\Upsilon_{(1+\beta)T\downarrow T}(T-2/3y)$ and $\Upsilon_{0\uparrow T}(T-2/3y)$ being respectively greater than $\beta^{1/3}(2 + \delta)s$ and $\Upsilon_{0\uparrow T}(0) - \beta^{1/3}s/2$; while the second depends on $\Upsilon_{0\uparrow T}(0) \leq \beta^{1/3}s/2 \rho$. 

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Substituting (3.6.11) into the composition law Proposition 3.2.2 yields
\[
\exp\left(t^{1/3}\Upsilon_{0\uparrow(1+\beta)t}\right) \geq \int_{-\theta}^{\theta} \exp\left((\beta T)^{1/3}(1 + \delta)s + (1 + \rho\beta^{1/3})T^{1/3}\Upsilon_{0\uparrow t}(0)\right)dx \\
\geq 2 \exp\left((1 + \delta)(\beta T)^{1/3}s + (1 + \rho\beta^{1/3})T^{1/3}\Upsilon_{0\uparrow T}(0) + \frac{3}{2}\log(\beta T) - \log s\right).
\]

Since we have assumed that \(\beta T \geq T_0\), there exists \(s_0 = s_0(T_0) > 0\) such that the quantity in the preceding line is bounded below by \(\exp\left((\beta T)^{1/3}s + (1 + \rho\beta^{1/3})T^{1/3}\Upsilon_{0\uparrow T}(0)\right)\) for \(s > s_0\). Thus, we see that the occurrence of \(\tilde{\Omega}\) entails that \(\{\Upsilon_{0\uparrow(1+\beta)t} - \Upsilon_{0\uparrow T} - \rho\Upsilon_{0\uparrow T}(0) \geq (\beta T)^{1/3}s\}\), which is nothing other than the event in (3.6.2) whose probability we seek to bound below. The proof of Proposition 3.6.2 has thus been reduced to the derivation of a suitable lower bound on \(\mathbb{P}(\tilde{\Omega})\).

Owing to the independence between \(\Upsilon_{(1+\beta)t\downarrow T}\) and \(\Upsilon_{0\uparrow T}\), \(\mathbb{P}(\tilde{\Omega}) = \mathbb{P}(\tilde{C} \cap \tilde{D})\mathbb{P}(\tilde{E} \cap \tilde{F})\).

First let us bound \(\mathbb{P}(\tilde{C} \cap \tilde{D}) \geq \exp\left(-c s^{3/2}\right)\), where \(c = c(T_0) > 0\) and \(s > s_0\) for some \(s_0(T_0) > 0\). Combining the distributional equality (3.6.6), the stationarity in Proposition 2.1.7 and the lower bound in Proposition 3.2.5 we find that \(\mathbb{P}(\tilde{C}) \geq \exp\left(-c s^{3/2}\right)\). By decomposing \(\mathbb{P}(\tilde{C} \cap \tilde{D}) = \mathbb{P}(\tilde{C}) - \mathbb{P}(\tilde{C} \cap \tilde{D})\) we see that our bound on \(\mathbb{P}(\tilde{C} \cap \tilde{D})\) will be established if we can prove an upper bound \(\mathbb{P}(\tilde{C} \cap \tilde{D}) \leq \exp\left(-c s^{3/2+\eta}\right)\) for any \(\eta > 0\) fixed. This can be established for \(\eta = 1\) in precisely the manner of (3.6.9).

To obtain a lower bound on \(\mathbb{P}(\tilde{\Omega})\) — and hence to finish the proof of Proposition 3.6.1 — it suffices to show that \(\mathbb{P}(\tilde{E} \cap \tilde{F}) \geq \exp\left(-c s^{3/2}\right)\). Observe that \(\mathbb{P}(\tilde{E} \cap \tilde{F}) \geq \mathbb{P}(\tilde{F}) - \mathbb{P}(\tilde{\Omega})\). From the tail bounds in Propositions 3.2.4 and 3.2.5 there exists some constant \(C > 0\) such that \(\mathbb{P}(\tilde{F}) > C\). The event \(\tilde{E}\) may be written in the form \(\{\inf_{|y| \leq \varepsilon} \Upsilon_{0\uparrow T}(T^{-2/3}y) - \Upsilon_{0\uparrow T}(0) \geq -e^{1/2 - 3/2}\}\) by setting \(\varepsilon = \beta^{2/3}s^{-1}\); an application of Theorem 3.1.3 then implies that \(s_0 = s_0(T_0) > 0\) and \(c = c(T_0) > 0\) exist so that \(\mathbb{P}(\tilde{E}) \geq 1 - \exp\left(-c s^{9/4}\right)\), provided that \(s > s_0\) and \(\beta T \geq T_0\). Thus we find that, for \(s\) large enough, \(\mathbb{P}(\tilde{E} \cap \tilde{F}) \geq C - \exp\left(-c s^{9/4}\right) \geq \exp\left(-c s^{3/2}\right)\); and thus is the proof of Proposition 3.6.1 concluded.

3.7 Spatial modulus of continuity

Proof of Theorem 3.1.4 The cases of small and large \(|X_1 - X_2|\) will separately occupy our attention. The shorthand \(\Upsilon_t(X) := \Upsilon_T(X) + \frac{X^2}{2}\) will be employed to prevent repetitious parabolic shifting.
Define
\[
C_\ll := \sup_{X_1 \neq X_2 \in [a,b], |X_1 - X_2| < 1} \left\{ |X_1 - X_2|^{-1/2} \left( \log \frac{|b - a|}{X_1 - X_2} \right)^{-2/3} |\tilde{\Upsilon}_t(X_1) - \tilde{\Upsilon}_t(X_2)| \right\},
\]
\[
C_\gg := \sup_{X_1 \neq X_2 \in [a,b], |X_1 - X_2| > 1/2} \left\{ |X_1 - X_2|^{-1/2} \left( \log \frac{|b - a|}{X_1 - X_2} \right)^{-2/3} |\tilde{\Upsilon}_t(X_1) - \tilde{\Upsilon}_t(X_2)| \right\};
\]
and note that \( C \) from (3.1.5) is given by \( C = \max(C_\ll, C_\gg) \). Upper-tail bounds for \( C_\ll \) and \( C_\gg \) will thus extend to \( C \). Some notation will aid in the derivation of such bounds. Let \( n_0 = \inf \{ n \geq 1 : |b - a| < 2^{n-1} \} \). For \( n \geq n_0 \), dyadically partition \([a, b] \) as
\[
[a, b] = \bigcup_{k=1}^{2^n} \mathcal{J}_k^{(n)} \quad \text{where} \quad \mathcal{J}_k^{(n)} := [\alpha_k^{(n-1)}, \alpha_k^{(n)}], \quad \alpha_k^{(n)} := a + \frac{k}{2^n} (b - a), \quad \text{for} \quad k = 0, \ldots, 2^n.
\]

First we bound \( C_\gg \). Unless \(|b - a| > 1/2\), there is nothing to prove. Supposing then the latter bound, consider the dyadic partition with \( n = n_0 \). Each interval in the partition has length less than \( 1/2 \); thus, \( x_1 \) and \( x_2 \) lie in distinct intervals. Label by \( k_1 \) and \( k_2 \) the respective indices \( k \) of the right endpoint \( \alpha_k^{(n_0)} \) of the interval containing \( X_1 \) and \( X_2 \). By the triangle inequality,
\[
|\tilde{\Upsilon}_T(X_1) - \tilde{\Upsilon}_T(X_2)|
\leq 4 \max \left( |\tilde{\Upsilon}_T(X_1) - \tilde{\Upsilon}_T(\alpha_k^{(n_0)})|, |\tilde{\Upsilon}_T(\alpha_k^{(n_0)})|, |\tilde{\Upsilon}_T(\alpha_{k_2}^{(n_0)})|, |\tilde{\Upsilon}_T(X_1) - \tilde{\Upsilon}_T(\alpha_{k_2}^{(n_0)})| \right).
\]
Introducing \( B_k^{(n)} := \sup_{X \in \mathcal{J}_k^{(n)}} |\tilde{\Upsilon}_T(X) - \tilde{\Upsilon}_T(\alpha_k^{(n)})| \) and \( C_k^{(n)} := |\tilde{\Upsilon}_T(\alpha_k^{(n)})| \), we learn that
\[
C_\gg \leq (1/2)^{-1/2} \left( \log \frac{|b - a|}{1/2} \right)^{-2/3} \max_{k \in \{0, \ldots, 2^n\}} \left\{ B_k^{(n)}, C_k^{(n)} \right\}.
\]
In total, the maximum is taken over \( 2^{n_0} + 1 \) possible values of \( k \). By Theorem 3.1.3 we see that
\[
\mathbb{P}(B_k^{(n_0)} \geq s) \leq \exp \left( -cs^{3/2} \right) \text{ for some } c = c(T_0) > 0, \text{ provided that } s > s_0 \text{ for } s_0 = s_0(T_0).
\]
Proposition 3.2.5 provides a similar bound on the upper tails of the \( C_k^{(n_0)} \). A union bound transmits this inference to an upper-tail bound on \( C_\gg \), with dependences of the form \( c = c(T_0, |b - a|) \) and \( s_0 = s_0(T_0, |b - a|) \). The dependence on \(|b - a|\) arises by the absorption into \( c \) of the term \((1/2)^{-1/2} \left( \log \frac{|b - a|}{1/2} \right)^{-2/3}\) and of the term \( 2(2^{n_0} + 1) \) arising from the union bound.
As we turn to bounding above $C_\ll$, we may impose that $|X_1 - X_2| < 1$. Let $n \geq n_0$ be the smallest integer such that $|X_1 - X_2| \geq |b - a|2^{-n-1}$. Either $X_1$ and $X_2$ lie in a given interval in the dyadic partition of level $n$, or they lie in consecutive intervals. When $X_1, X_2 \in J_k^{(n)}$ for some $k \in \{0, \ldots, 2^n\}$, recall of the event $B_k^{(n)}$ and use of the triangle inequality yield

$$|X_1 - X_2|^{-1/2} \left( \log \frac{|b - a|}{|X_1 - X_2|} \right)^{-2/3} |\bar{\Upsilon}_t(X_1) - \bar{\Upsilon}_t(X_2)| \leq (|b - a|2^{-n-1})^{-1/2} \left( \log(2^{n+1}) \right)^{-2/3} 2B_k^{(n)}.$$  

When $X_1 \in J_k^{(n)}$ and $X_2 \in J_{k+1}^{(n)}$, we may set $\tilde{B}_k^{(n)} := \sup_{X \in J_k^{(n)}} |\bar{\Upsilon}_t(X) - \bar{\Upsilon}_t(v_k^{(n)})|$ to obtain

$$|X_1 - X_2|^{-1/2} \left( \log \frac{|b - a|}{|X_1 - X_2|} \right)^{-2/3} |\bar{\Upsilon}_t(X_1) - \bar{\Upsilon}_t(X_2)| \leq (|b - a|2^{-n-1})^{-1/2} \left( \log(2^{n+1}) \right)^{-2/3} 2 \max(B_k^{(n)}, \tilde{B}_k^{(n)}).$$

From these bounds, we arrive at

$$C_\ll \leq \sup_{n \geq n_0} \sup_{k \in \{0, \ldots, 2^n\}} \left( |b - a|2^{-n-1} \right)^{-1/2} \left( \log(2^{n+1}) \right)^{-2/3} 2 \max(B_k^{(n)}, \tilde{B}_k^{(n)}).$$

In contrast to the analysis of $C_\gg$, infinitely many terms must now be considered. Indeed, we have

$$\{C_\ll \geq s\} = \bigcup_{n=n_0}^{\infty} \bigcup_{k=0}^{2^n} \left\{ \max(B_k^{(n)}, \tilde{B}_k^{(n)}) \geq \left( |b - a|2^{-n-1} \right)^{1/2} \left( \log(2^{n+1}) \right)^{2/3} 2^{-1}s \right\}.$$

Apply the union bound and the tail bound in Theorem 3.1.3 with $\epsilon = 2^{-n}|b - a|$ – which parameter is at most one since $n \geq n_0$. We thus find that there exist $s_0 = s_0(t_0, |b - a|) > 0$ and $c = c(t_0, |b - a|) > 0$ such that, for $t > t_0$ and $s > s_0$,

$$\mathbb{P}(C_\ll \geq s) \leq \sum_{n=n_0}^{\infty} \sum_{k=0}^{2^n} \exp\left(-ncs^{3/2}\right) \leq \sum_{n=n_0}^{\infty} \exp\left(-ncs^{3/2}\right) \leq \exp\left(-cs^{3/2}\right);$$

here, the values of $c$ and $s_0$ change between each inequality in order to absorb the higher indexed terms in the two sums. Indeed, in the second bound, the sum over $k$ contributes a factor $2^n + 1$ which is absorbed by decreasing the constant $c$ provided that $s_0$ is high enough. The third bound arises by computing the geometric sum, expressing $n_0$ in terms of $|b - a|$, and absorbing the resulting constant.
into $c$.

These bounds $\mathbb{P}(C_\ll \geq s)$ and $\mathbb{P}(C_\gg \geq s)$ obtained, the proof is Theorem 3.1.4 is complete. ■

3.8 Appendix: tail probabilities and covariances

Several tools are stated and proved here. The conditions in (3.8.1) and (3.8.2) can be expressed in terms of conditional probabilities but since they may not exist, we formulate them as below.

**Lemma 3.8.1.** Let $X$ and $Y$ be two real-valued integrable random variables and suppose that for all $r \in \mathbb{R}, u > v \in \mathbb{R},$

\[
\mathbb{P}(Y > v)\mathbb{P}(X > r, Y > u) \geq \mathbb{P}(Y > u)\mathbb{P}(X > r, Y > v), \tag{3.8.1}
\]

\[
\mathbb{P}(Y \leq u)\mathbb{P}(X > r, Y \leq v) \geq \mathbb{P}(Y \leq v)\mathbb{P}(X > r, Y \leq u). \tag{3.8.2}
\]

1. Then, $\text{Cov}(X, Y) \geq 0.$

2. Moreover, for any $a \in \mathbb{R},$

\[
\text{Cov}(X, Y) \geq \mathbb{P}(Y \geq a) \cdot \left(\mathbb{E}[X|Y \geq a] - \mathbb{E}[X]\right) \cdot \left(\mathbb{E}[Y|Y > a] - \mathbb{E}[Y|Y \leq a]\right). \tag{3.8.3}
\]

**Proof.** (1): Since (3.8.1) holds after replacing $X$ by $X - \mathbb{E}[X]$ and $Y$ by $Y - \mathbb{E}[Y]$, we may assume that $\mathbb{E}[X] = \mathbb{E}[Y] = 0.$ Denote $X_+ := \max\{X, 0\}$ and $X_- := \max\{-X, 0\}$, so that $X = X_+ - X_-;$ and likewise for $Y.$ Thus, $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]; \mathbb{E}[Y] = \mathbb{E}[Y_+] - \mathbb{E}[Y_-];$ and

\[
\text{Cov}(X, Y) = \mathbb{E}[X_+Y_+] - \mathbb{E}[X_+Y_-] - \mathbb{E}[X_-Y_+] + \mathbb{E}[X_-Y_-].
\]

To prove that $\text{Cov}(X, Y) \geq 0,$ it suffices to show that

\[
\mathbb{E}[X_+Y_+] \geq \mathbb{E}[X_+]\mathbb{E}[Y_+], \quad \mathbb{E}[X_+Y_-] \leq \mathbb{E}[X_+]\mathbb{E}[Y_-], \tag{3.8.4}
\]

\[
\mathbb{E}[X_-Y_+] \leq \mathbb{E}[X_-]\mathbb{E}[Y_+], \quad \mathbb{E}[X_-Y_-] \geq \mathbb{E}[X_-]\mathbb{E}[Y_-].
\]

We prove (3.8.4): the bounds in the following line are derived similarly. Taking $v \to -\infty$ in (3.8.1)
yields

\[ \Pr(X > r, Y > u) \geq \Pr(X > r)\Pr(Y > u), \quad \forall r \in \mathbb{R}, u \in \mathbb{R}. \tag{3.8.5} \]

Subtracting (3.8.5) from \( \Pr(X > r) \) yields

\[ \Pr(X > r, Y \leq u) \leq \Pr(X > r)\Pr(Y \leq u), \quad \forall r \in \mathbb{R}, u \in \mathbb{R}. \tag{3.8.6} \]

Integrating (3.8.5) with respect to \((r, u)\) over \((0, \infty) \times (0, \infty)\), we see that

\[ \int_0^{\infty} \int_0^{\infty} \Pr(X > r, Y > u) drdu = \mathbb{E}[X_+ Y_+] \geq \int_0^{\infty} \int_0^{\infty} \Pr(X > r)\Pr(Y > u) drdu = \mathbb{E}[X_+]\mathbb{E}[Y_+]. \tag{3.8.7} \]

Integrating (3.8.6) with respect to \((r, u)\) over \((0, \infty) \times (-\infty, 0]\) yields

\[ \int_0^{\infty} \int_{-\infty}^{0} \Pr(X > r, Y \leq u) drdu = \mathbb{E}[X_+ Y_+] \leq \int_0^{\infty} \int_{-\infty}^{0} \Pr(X > r)\Pr(Y \leq u) drdu = \mathbb{E}[X_+]\mathbb{E}[Y_-]. \tag{3.8.8} \]

The bounds in line (3.8.4) follow from (3.8.7) and (3.8.8).

(2): Fix \( a \in \mathbb{R} \). If \( \Pr(Y \geq a) \in \{0, 1\} \), the right-hand side of (3.8.3) equals zero; in which case, (3.8.3) follows from the just derived part (1). Suppose then that \( \Pr(Y \geq a) \in (0, 1) \). Let the pair \((X', Y')\) be an independent copy of \((X, Y)\). One may write

\[ \mathbb{E}[X(Y - a)] = \frac{\mathbb{E}[1_{Y' > a}X(Y - a)_+]}{\Pr(Y > a)} - \frac{\mathbb{E}[1_{Y' \leq a}X(Y - a)_-]}{\Pr(Y \leq a)}. \tag{3.8.9} \]

We will next argue that

\[ \mathbb{E}[1_{Y' > a}X(Y - a)_+] \geq \mathbb{E}[X'1_{Y' > a}]\mathbb{E}[(Y - a)_+], \tag{3.8.10} \]

\[ \mathbb{E}[1_{Y' \leq a}X(Y - a)_-] \leq \mathbb{E}[X'1_{Y' \leq a}]\mathbb{E}[(Y - a)_-]. \tag{3.8.11} \]

In fact, we will merely show how (3.8.10) follows from (3.8.1); (3.8.11) follows in a similar way from
By (3.8.1), we see that, for \( r \in \mathbb{R} \) and \( u > 0 \),
\[
\mathbb{P}(Y' > a, X > r, Y > a + u) \geq \mathbb{P}(Y' > a, X' > r)\mathbb{P}(Y > a + u).
\]

Integrating this inequality with respect to \((r, u)\) over \([0, \infty)^2\), we obtain
\[
\mathbb{E}[\mathbf{1}_{Y' > a} X_+ (Y - a)_+] \geq \mathbb{E}[\mathbf{1}_{Y' > a} X'_+] \mathbb{E}[(Y - a)_+]. \tag{3.8.12}
\]

Subtracting (3.8.1) from \( \mathbb{P}(Y > u)\mathbb{P}(Y > v) \) with \( u = a \) yields
\[
\mathbb{P}(Y' > a, X \leq r, Y > u) \leq \mathbb{P}(Y' > a, X' \leq r)\mathbb{P}(Y > u);
\]
which, after integrating with respect to \((r, u)\) over \((-\infty, 0) \times [a, \infty)\), gives
\[
\mathbb{E}[\mathbf{1}_{Y' > a} X_-(Y - a)_+] \leq \mathbb{E}[\mathbf{1}_{Y' > a} X'_+] \mathbb{E}[(Y - a)_+]. \tag{3.8.13}
\]

Subtracting (3.8.13) from (3.8.12) yields (3.8.10).

Pursuing the proof of (3.8.3), we substitute (3.8.10) and (3.8.11) into the right-hand side of (3.8.9) to learn that
\[
\mathbb{E}[X(Y - a)] \geq \mathbb{E}[X \mathbf{1}_{Y > a}] \left( \mathbb{E}[Y|Y > a] - a \right) + \mathbb{E}[X \mathbf{1}_{Y \leq a}] \left( \mathbb{E}[Y|Y \leq a] - a \right).
\]

Subtracting \( \mathbb{E}[X]\mathbb{E}[Y - a] \) from this inequality, and simplifying, yields
\[
\text{Cov}(X, Y) \geq \left( \mathbb{E}[X \mathbf{1}_{Y > a}] - \mathbb{E}[X] \mathbb{P}(Y > a) \right) \left( \mathbb{E}[Y|Y > a] - a \right)
+ \left( \mathbb{E}[X \mathbf{1}_{Y \leq a}] - \mathbb{E}[X] \mathbb{P}(Y \leq a) \right) \left( \mathbb{E}[Y|Y \leq a] - a \right). \tag{3.8.14}
\]

Now, we obtain (3.8.3): it follows from (3.8.14) by noting that
\[
\mathbb{E}[X \mathbf{1}_{Y \leq a}] - \mathbb{E}[X] \mathbb{P}(Y \leq a) = - \left( \mathbb{E}[X \mathbf{1}_{Y > a}] - \mathbb{E}[X] \mathbb{P}(Y > a) \right)
\]
and rewriting \( \mathbb{E}[X \mathbf{1}_{Y > a}] = \mathbb{P}(Y > a)\mathbb{E}[X|Y > a] \).
Corollary 3.8.2. Let $X$ and $Y$ be integrable random variables that satisfy (3.8.1) and (3.8.2). Suppose that there exist $C_1, C_2 > 0$ such that

$$
E[X | Y \geq C_1] \geq E[X] + C_2.
$$

(3.8.15)

Then

$$
\text{Cov}(X, Y) \geq C_2 \cdot P(Y \geq C_1) \left( E[Y | Y \geq C_1] - E[Y | Y \leq C_1] \right).
$$

(3.8.16)

Proof. We obtain (3.8.16) from (3.8.3) by taking $a = C_1$ and applying (3.8.15).

Lemma 3.8.3. Let $X$, $Y$ and $Z$ be non-degenerate real-valued random variables with finite second moments such that $Z = X + Y$. Define

$$
\chi := \frac{\text{Var}(X)}{\text{Var}(Y)}, \quad \Psi := \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}, \quad \Theta := \frac{\text{Var}(X)\text{Var}(Y) - (\text{Cov}(X, Y))^2}{(\text{Var}(Y))^2}.
$$

Suppose that $\max\{\chi, |2\Psi + \chi|\} < 1$. There exist two constants $C_2 > C_1 > 0$ such that

$$
1 - \Theta/2 + C_1\chi^{3/2} \leq \text{Corr}(Z, Y) \leq 1 - \Theta/2 + C_2\chi^{3/2}.
$$

(3.8.17)

Proof. Since $\text{Var}(Z) = (1 + 2\Psi + \chi)\text{Var}(Y)$, we may rewrite

$$
\text{Corr}(Z, Y) = \frac{\text{Cov}(Z, Y)}{\sqrt{\text{Var}(Z)}\sqrt{\text{Var}(Y)}} = \frac{1 + \Psi}{\sqrt{1 + 2\Psi + \chi}}.
$$

Taylor expanding $(1 + 2\Psi + \chi)^{-1/2}$ with respect to $\Xi := \Psi + \frac{1}{2}\chi$ yields

$$
\text{Corr}(Z, Y) = (1 + \Psi) \left( 1 - \Xi + \frac{3}{2}\Xi^2 + \mathcal{O} \right)
$$

where $|\mathcal{O}| \leq C|\Xi|^3$

for some constant $C > 0$. Simplifying the product and substituting $\Xi = \Psi + \frac{1}{2}\chi$, we find that

$$
\text{Corr}(Z, Y) = 1 - \frac{1}{2}\chi + \frac{1}{2}\Psi^2 + \frac{1}{2}\Psi\chi + \frac{3}{8}\chi^2 + \frac{3}{2}\Psi(\Psi + \frac{1}{2}\chi)^2 + \mathcal{O}',
$$

where $|\mathcal{O}'| \leq C'|\Xi|^3$

(3.8.18)
for some constant $C' > 0$. By the Cauchy-Schwarz inequality, $\Psi^2 \leq \chi$; note also that $\chi \in [0, 1]$.

Thus, there exist $C_2 > C_1 > 0$ such that

$$C_1 \chi^{3/2} \leq \frac{1}{2} \Psi \chi + \frac{3}{8} \chi^2 + \frac{3}{2} \Psi (\Psi + \frac{1}{2} \chi)^2 \leq C_2 \chi^{3/2}. \quad (3.8.19)$$

Substituting (3.8.19) into (3.8.18) and noting that $\Theta = \chi - \Psi^2$ proves (3.8.17).

\[\blacksquare\]

**Lemma 3.8.4.** Fix $\theta > 0$. Let $X$ be a real-valued random variable.

1. Suppose there exist $s_0$, $\alpha > 0$ and $c > 0$ such that, for $s \geq s_0$,

$$\mathbb{P}(X \leq -s\theta) \leq e^{-cs^{\alpha}} \quad \text{and} \quad \mathbb{P}(X \geq s\theta) \leq e^{-cs^{\alpha}}. \quad (3.8.20)$$

Then there exists $C = C(s_0, c, \alpha) > 0$ such that $\text{Var}(X) \leq C\theta^2$.

2. Suppose that $|\mathbb{E}[X]| \leq C_1 \theta$ for some $C_1 > 0$, and that there exist $s_0$, $\alpha > 0$ and $c > 0$ such that, for $s \geq s_0$,

$$\mathbb{P}(X \geq s\theta) \geq e^{-cs^{\alpha}}. \quad (3.8.21)$$

Then there exists $C = C(C_1, s_0, c, \alpha) > 0$ such that $\text{Var}(X) \geq C\theta^2$.

**Proof.** To prove (1), observe that

$$\text{Var}(X) \leq \mathbb{E}[X^2] = \theta^2 \int_0^\infty 2s \left( \mathbb{P}(X > s\theta) + \mathbb{P}(X < -s\theta) \right) ds.$$

Substituting (3.8.20) into the right-hand side and integrating, we obtain the sought bound on $\text{Var}(X)$.

To prove (2), suppose first that $\mathbb{E}[X] \leq 0$. Then

$$\text{Var}(X) \geq \mathbb{E}[(X+)^2] = \theta^2 \int_0^\infty 2s \mathbb{P}(X > s\theta) d\theta.$$

Substituting (3.8.21) into this right-hand side, and integrating, results in the desired bound on $\text{Var}(X)$.
Suppose instead that $\mathbb{E}[X] > 0$. Since $\mathbb{E}[X] \leq C_1 \theta$, we have

$$\text{Var}(X) \geq \mathbb{E}[(X - C_1 \theta)^2] \geq \theta^2 \int_0^\infty 2s \mathbb{P}(X > (C_1 + s)\theta) \, ds.$$ 

Similarly, we now substitute (3.8.21) into this right-hand side, and integrate, to obtain the bound on $\text{Var}(X)$ that we seek. 

\[ \blacksquare \]
References


