Moduli of Surfaces and Applications to Curves

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Abstract

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This thesis has two parts. In the first part, we construct a moduli scheme $F[n]$ that parametrizes tuples $(S_1, S_2, \ldots, S_{n+1}, p_1, p_2, \ldots, p_n)$ where S_1 is a fixed smooth surface over Spec R and S_{i+1} is the blowup of S_i at a point p_i, $\forall 1 \leqslant i \leqslant n$. We show this moduli scheme is smooth and projective. We prove that $F[n]$ has smooth divisors $D_{i,j}^{(n)}$, $\forall 1 \leqslant i < j \leqslant n$, which correspond to tuples that map $p_j \mapsto p_i$ under the projection morphism $S_j \to S_i$. When $R = k$ is an algebraically closed field, we demonstrate that the Chow ring $\mathbf{A}^*(F[n])$ is generated by these divisors over $\mathbf{A}^*(S_1^n)$. We end by giving a precise description of $\mathbf{A}^*(F[n])$ when S_1 is a complex rational surface.

In the second part of this thesis, we focus on finding a characterization of the smooth surfaces S on which a smooth very general curve of genus g embeds as an ample divisor. Our results can be summarized as follows: if the Kodaira dimension of S is $\kappa(S) = -\infty$ and S is not rational, then S is birational to $C \times \mathbb{P}^1$. If $\kappa(S)$ is 0 or 1, then such an embedding does not exist if the genus of C satisfies $g \geqslant 22$. If $\kappa(S) = 2$ and the irregularity of S satisfies $q(S) = g$, then S is birational to the symmetric square $\mathrm{Sym}^2(C)$.

We analyze the conditions that need to be satisfied when S is a rational surface. The case in which S is of general type and $q(S) = 0$ remains mainly open; however, we provide a partial answer to our question if S is a complete intersection.

Contents

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Introduction

Origin of the thesis

The idea for this thesis started from Daniel Litt. He found Proposition 6.1 in Harris's and Mumford's paper "On the Kodaira Dimension of the Moduli Space of Curves" (see [20]), which states the following: if a surface S contains a very general curve of genus $g \geqslant 22$ that moves in a non-trivial linear system, then S is birational to $C \times \mathbb{P}^1$. Litt thought to analyze a similar question in a different setting: can we characterize the smooth surfaces on which a smooth very general curve C embeds as an ample divisor such that $\dim |C| = 0$? As a starting point, de Jong and Litt gave me the idea that $\mathrm{Pic}^0(C)$ should be a simple abelian variety.

To attack this question, we divided the problem into multiple cases, one for each minimal model of the surface S. This is the content of Chapters 6-12. Our result can be summarized as follows: if $\kappa(S) = -\infty$ and S is not rational, then S is birational to $C \times \mathbb{P}^1$. If the Kodaira dimension of S is 0 or 1, then such an embedding does not exists if the genus of C satisfies $g \geqslant 21$. If $\kappa(S) = 2$ and $q(S) = g$, then S is birational to the symmetric square $\mathrm{Sym}^2(C)$.

There are a few cases left open. If S is a rational surface, we analyze the conditions that need to be satisfied in this situation (see Prop. 9.5 and 9.10). For the case in which S is of general type and $q(S) = 0$, we prove the following partial result: if

$S \hookrightarrow \mathbb{P}^r$ is a complete intersection and the composed morphism $C \hookrightarrow S \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture, then C is not ample on S if its genus is higher than 16.

While working on the proofs above, we encountered the following scenario: say we have a series of morphisms $S_{n+1} \to \cdots \to S_1$, where S_1 is a smooth minimal surface and each map $S_{i+1} \to S_i$ is the blowup of some point $p_i \in S_i$. Say, for example, S_1 is a K3 surface, so it has 19 moduli (see Thm. 10.2). Then S_{n+1} has $19 + 2n$ moduli, since we add 2 moduli each time we blow up a new point. This line of thought made us consider the following scenario: say we fix a smooth surface S_1 over Spec R. Consider all the ordered sequences of morphisms $S_{n+1} \to \cdots \to S_1$, where $S_{i+1} \to S_i$ is the blowup of S_i at a point p_i. Can we thoroughly construct a moduli space that parametrizes these objects? The answer is yes.

In Chapters 1- 5, we construct a moduli scheme that parametrizes these sequences of blowups $S_{n+1} \to \cdots \to S_1$. We prove that the moduli functor is represented by a smooth projective scheme of dimension $2n$ over Spec R, as expected. We find smooth divisors $D_{i,j}^{(n)}$ that correspond to tuples $(S_i, p_i)_{i=1}^n$ where $p_j \mapsto p_i$ under the projection map $S_j \to S_i$. When $R = k$ is an algebraically closed field, we prove that these divisors generate the Chow ring $\mathbf{A}^*(F[n])$ over $\mathbf{A}^*(S_1^n)$. We end by giving a precise description of this Chow ring in the case where S_1 is a complex rational surface.

One of the resons why I found this problem interesting is that the moduli scheme has a very beautiful and natural construction, it behaves "as expected". All the intuitive guesses we had about these moduli spaces and their properties while developing our theory turned out to be correct.

Second of all, this construction is a new step forward in the study of moduli of surfaces. At the moment, one can find in the literature various constructions of moduli spaces parametrizing certain minimal surfaces. This moduli scheme stands out because it parametrizes non-minimal surfaces.

Lastly, looking at our results regarding the study of very general curves on smooth surfaces, here is what we can say based the open cases: if a curve C of high genus g is embedded on a smooth surface S as an ample divisor, then S is either a surface of general type of irregularity 0 (this is a vast class of surfaces), or it has a very specific blowup of $\mathbb{P}^2$. It is interesting to note that S might be a very general class of surfaces, or a class given by very strict conditions.

When it comes to future work, we see some clear ways to expand on the thesis below. First, one can extend the moduli construction over a non-affine base, i.e. replace Spec R by an arbitrary scheme. Another step is to work on the open cases. Ideally, with more work one could conclude that the surface S cannot be rational. One can also work on proving S cannot be a complete intersection (removing the requirement involving the Maximal Rank Conjecture).

Main statements

Let S_1 be a fixed smooth projective surface over a commutative ring R. Our goal is to study and parametrize surfaces obtained through a series of n ordered blowups of the base surface S_1. More specifically, we focus our attention on tuples:

$$(S_1, S_2, \ldots, S_{n+1}, p_1, p_2, \ldots, p_n),$$

where $p_i \in S_i$ and S_{i+1} is the blowup of S_i at p_i, for all $1 \leqslant i \leqslant n$.

To parametrize these tuples, we define the functor $\mathcal{F}[n] = \mathcal{F}[S_1, n]$, whose objects are as follows: for any R-scheme B, an element in $\mathcal{F}[n](B)$ is a tower of morphisms:

$$
\begin{tikzcd}
\Sigma_{B,n+1} \arrow[r, "\pi_{B,n+1}"] & \Sigma_{B,n} \arrow[r, "\pi_n"] & \Sigma_{B,n-1} \arrow[r, "\pi_{n-1}"] & \dots \arrow[r] & \Sigma_{B,2} \arrow[r, "\pi_2"] & \Sigma_{B,1} = B \times S_1 \arrow[d, "\pi_1"] \\
 & & & & & B, \arrow[u, bend left, "p_1"] \arrow[ul, bend left, "p_2"] \arrow[uulll, phantom] \arrow[ulll, bend left, "p_{n-1}"] \arrow[ullll, bend left, "p_n"]
\end{tikzcd}
$$

such that the following conditions are satisfied:

(1) $\pi_1 = \mathrm{pr}_1$ is the projection onto the first factor;

(2) for each $1 \leqslant i \leqslant n$, the morphism $p_i : B \to \Sigma_{B,i}$ is a section of the composed map $\Sigma_{B,i} \to B$;

(3) for each $1 \leqslant i \leqslant n$, the morphism $\pi_{i+1} : \Sigma_{B,i+1} \to \Sigma_{B,i}$ is the blowup of $\Sigma_{B,i}$ along the locus $p_i(B)$.

Theorem. *The functor $\mathcal{F}[n]$ has a fine moduli scheme $F[n]$. The space $F[n]$, together with all the schemes in its universal family, are smooth and projective over Spec R. Moreover, the top scheme in the universal family of $F[n]$ is the moduli space of the functor $\mathcal{F}[n+1]$, and it can be identified with $Bl_{\triangle}(F[n] \times_{F[n-1]} F[n])$.*

We think about this moduli scheme $F[n]$ inductively. First, it is intuitively easy to see that $F[0] = \mathrm{Spec}\, R$, $F[1] = S_1$, and $F[2] = \mathrm{Bl}_{\triangle}(S_1 \times S_1)$. Moreover, the universal family over $F[0]$ is given by the structure morphism $S_1 \to \mathrm{Spec}\, R$, and the universal family over $F[1]$ is $\mathrm{Bl}_{\triangle}(S_1 \times S_1) \to S_1 \times S_1 \xrightarrow{\mathrm{pr}_1} S_1$. Using these base cases as guidance, we form the following hypothesis, which proves to be correct: the top scheme in the universal family over $F[n]$ is $\mathrm{Bl}_{\triangle}(F[n] \times_{F[n-1]} F[n])$, and it is actually the moduli scheme corresponding to the "next" functor $\mathcal{F}[n+1]$. We prove this claim by constructing a direct 1-to-1 correspondence for every R-scheme B:

$$\mathrm{Mor}(B, \mathrm{Bl}_\triangle(F[n] \times_{F[n-1]} F[n])) \cong \mathcal{F}[n+1](B).$$

Additionally, since we build these schemes inductively, it is easy to prove at the end of the construction that they are indeed smooth and projective.

The moduli scheme $F[n]$ comes equipped with divisors $D_{i,j}^{(n)}, \forall 1 \leqslant i < j \leqslant n$, which can be described as follows: given an R-scheme B, a B-point of such a divisor $D_{i,j}^{(n)}$ corresponds to a sequence of blow-ups $(\Sigma_{B,i}, p_i)_{i=1}^n$ for which p_j maps to p_i under the projection map $\Sigma_{B,j} \to \Sigma_{B,i}$. These divisors arise naturally from the construction of the moduli space $F[n]$: the projection map $F[n] \to S_1^n$ decomposes as a sequence of blowups, and these divisors inside $F[n]$ are exactly the inverse images of the exceptional divisors corresponding to the blowups. In Proposition 3.5, we show that the divisors $D_{i,j}^{(n)}$ are smooth.

Now assume $R = k$ is an algebraically closed field. Using a theorem of Keel (see Thm. 4.1), we prove that the Chow ring of the moduli space $F[n]$ is generated by the classes of the divisors $\{D_{i,j}^{(n)}\}_{1 \leqslant i < j \leqslant n}$ over the Chow ring $\mathbf{A}^*(S_1^n)$. Additionally, we prove two key relations among these classes in the Chow ring $\mathbf{A}^*(F[n])$: let d be any divisor class of S_1 and $d_i^* \in \mathbf{A}^*(F[n])$ be the image of d under composed morphism $F[n] \to S_1^n \xrightarrow{\mathrm{pr}_i} S_1$, $\forall 1 \leqslant i \leqslant n$. Then:

(i) $D_{i,j}^{(n)} d_i^* = D_{i,j}^{(n)} d_j^*, \forall 1 \leqslant i < j \leqslant n$;

(ii) $D_{i,j}^{(n)} D_{j,k}^{(n)} = D_{i,k}^{(n)} D_{j,k}^{(n)}, \forall 1 \leqslant i < j < k \leqslant n$.

Intuitively, relation (i) above holds because the projection map $F[n] \to S_1^n$ maps $D_{i,j}^{(n)}$ onto the diagonal $\triangle_{i,j} \subset S_1^n$. Relation (ii) can be explained as follows: for any $i < j < k$, the left hand side parametrizes sequences of blowups $(\Sigma_{B,i}, p_i)_{i=1}^n$ where p_k

maps to p_j and p_j maps to p_i. The right hand side parametrizes sequences of blowups $(\Sigma_{B,i}, p_i)_{i=1}^n$ where p_k maps to p_j and p_k maps to p_i. Hence the two sides agree.

In the case where the base surface S_1 is rational and the base field is the complex numbers $\mathbb{C}$, another lemma of Keel (see Prop. 5.2) helps us conclude that the canonical map $\mathbf{A}^*(F[n]) \xrightarrow{\text{cl}} \mathbf{H}^{2*}(F[n])$ is an isomorphism. With this in mind, we set out to compute the Betti numbers of $F[n]$. Using the spreading out method, we define the surface S_1 over a finite field $\mathbb{F}_q$ and construct the moduli space $F[n] \otimes \mathbb{F}_q$ over this new surface. We compute the polynomial $R_n(q)$ that gives the number of $\mathbb{F}_q$-rational points of $F[n] \otimes \mathbb{F}_q$. Together with the Grothendieck-Lefschetz Trace formula and the identities between the Betti numbers in étale and singular cohomology, we conclude that $R_n(q)$ coincides with the Poincare polynomial of $F[n]$, so we immediately recover the Betti numbers we wanted (see Lemma 5.11). Lastly, we use these Betti numbers to conclude that the relations (i) and (ii) above are sufficient to give a full description of $\mathbf{A}^*(F[n])$.

For example, in Corollary 9.11 we present the Chow ring of the moduli space $\mathbf{A}^*(F[\mathbb{P}^2, n])$ as follows:

$$\mathbf{A}^*(F[\mathbb{P}^2, n]) \cong \frac{\mathbb{Z}[H_i^*, D_{j,k}^{(n)}]_{1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant n}}{\langle D_{j,k}^{(n)}(D_{i,j}^{(n)} - D_{i,k}^{(n)}), D_{j,k}^{(n)}(H_j^* - H_k^*), H_i^{*3}, P_{i,j}(-D_{i,j}^{(n)}) \rangle},$$

where, $\forall 1 \leqslant i \leqslant n$, H_i^* is the image of the hyperplane class $H \in \mathbf{A}^*(\mathbb{P}^2)$ under the composition $F[\mathbb{P}^2, n] \to (\mathbb{P}^2)^n \xrightarrow{\text{pr}_i} \mathbb{P}^2$, and $P_{i,j}(t)$ are certain quadratic polynomials (see Thm. 4.1).

Let $b_{n,j}$ be the j-th Betti number of the moduli space $F[\mathbb{P}^2, n]$. From the discussion above, we get the following recursive relation:

$$b_{n+1,j} = b_{n,j} + (n+1)b_{n,j-1} + b_{n,j-2}.$$

Here is how we can interpret the relation above: we have the forgetful map $\Pi_{n+1} : F[n+1] \to F[n]$ which induces a map on the Chow rings $\Pi_{n+1}^* : \mathbf{A}^*(F[n]) \to \mathbf{A}^*(F[n+1])$. Given this map, we can think of the identity above as follows: compared to the moduli space $F[n]$, the space $F[n+1]$ has $n+1$ extra divisors: $H_{n+1}^*, D_{1,n+1}^{(n+1)}, \ldots D_{n,n+1}^{(n+1)}$. A generator in $\mathbf{A}^j(F[n+1])$ is either a class inherited from $\mathbf{A}^j(F[n])$ under the map Π_{n+1}^* (this accounts for $b_{n,j}$ generators), or it is a product between a generator class coming from $\mathbf{A}^{j-1}(F[n])$ and one of the $n+1$ new divisor classes (this accounts for $(n+1)b_{n,j-1}$ generators), or it is a product between $(H_{n+1}^*)^2$ and a generator class coming from $\mathbf{A}^{j-2}(F[n])$ (this accounts for $b_{n,j-2}$ generators).

Now we switch gears entirely and go back to the original problem suggested by Daniel Litt. Recall that we want to characterize smooth surfaces S on which a smooth very general curve C of genus g embeds as an ample divisor such that $\dim |C| = 0$. Our general strategy will be to break down the problem into multiple cases, one for each minimal model of S.

We start this second half of the thesis by recalling some basic facts about curves and surfaces. We recall the Enriques-Kodaira classification of complex surfaces, which we will use extensively. Moreover, we recall the basic results of Brill-Noether theory and the Maximal Rank Conjecture.

We continue with some preliminary results. Since C is a very general curve, $\mathrm{Pic}^0(C)$ is a simple abelian variety (see Thm. 8.1). Additionally, since C is ample on S, the pullback map $\iota^* : \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}^0(C)$ is injective. Putting these two together, we conclude that either $\mathrm{Pic}^0(S) = 0$ or the map ι^* is an isomorphism.

Now, we can assume that every (-1)-curve E on S satisfies $C \cdot E \geqslant 2$ (if $C \cdot E = 1$, we can contract E and preserve both the smoothness and ampleness of C). Let S_0 be a minimal model of S and $S = S_n \to \cdots \to S_0$ the corresponding sequence of blow-downs. We prove the following inequality, where K_S, K_{S_0} are the canonical divisors of S and S_0, respectively:

$$C \cdot K_{S_0} \geqslant C_0 \cdot K_{S_0} + 2n.$$

In addition, if $\mathrm{Pic}^0(S) = 0$, then the number of moduli of S is at least $3g-3$ (see Lemma 8.7).

We analyze first the scenario in which a smooth very general curve C of genus g is embedded as an ample divisor on a smooth surface of Kodaira dimension $-\infty$. If S is not rational, then it is birational to $C \times \mathbb{P}^1$ (see Prop. 11.3). The case in which S is rational remains mainly open. However, we analyze the conditions that need to be satisfied in this situation (see Prop. 9.5 and 9.10). While these conditions are not sufficient to conclude such an embedding $C \hookrightarrow S$ exists, we show that there exist cases where this conditions could potentially be satisfied. We make this precise in Chapter 9.

Next, we analyze the case in which a smooth very general curve C of genus g is embedded as an ample divisor on a smooth surface of Kodaira dimension 0. We know that the minimal model of S, denoted by S_0, is either an abelian surface, a K3 surface, an Enriques surface, or a bielliptic surface. We show that S_0 is an abelian surface if and only if the genus of C is $g = 2$ and S is the Jacobian of C. If S_0 is bielliptic, such an embedding is not possible. If S_0 is a K3 or Enriques surface, then $g \leqslant 18$. To conclude the latter, we combine the following facts: S_0 has a fixed (bounded) number of moduli, $C \cdot K_S$ is bounded below by $2n$, and S has at least $3g-3$ moduli.

When $\kappa(S) = 1$, we show that S is an elliptic fibration over $\mathbb{P}^1$ (see Lemma 11.3). We use the moduli spaces constructed by Miranda and Seiler which parametrize these elliptic fibrations (see Thm. 11.7 and 11.9), and proceed with the same analysis as above to conclude that such an embedding does not exist if $g \geqslant 21$.

Lastly, we analyze the case in which S is of general type. If $q(S) = g$, then S is birational to the symmetric product $\mathrm{Sym}^2(C)$. This result is due to Mendes Lopes and Pardini (see [28]). The case where $q(S) = 0$ remains mainly open. However, we prove the following partial result: if $S \hookrightarrow \mathbb{P}^r$ is a complete intersection such that the composed map $C \hookrightarrow S \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture, then $g \leqslant 15$. For this part, we mainly use the results of Brill-Noether theory (see Thm. 7.11).

Structure of the thesis

We provide here an outline of each chapter.

In Chapter 1, we define thoroughly the moduli functor whose objects are the tuples $(S_1, \ldots, S_{n+1}, p_1, \ldots, p_n)$, where S_{i+1} is the blowup of S_i at p_i, $\forall 1 \leqslant i \leqslant n$, and S_1 is a fixed smooth projective surface over $\operatorname{Spec} R$. We check that our definition has all the required properties of a contravariant functor. We end with a few remarks on the natural maps that this functor possesses.

In Chapter 2, we prove that the aforementioned functor has a fine moduli scheme, which we denote by $F[n]$. We demonstrate that $F[n]$ and all the schemes in its universal family are smooth and projective over $\operatorname{Spec} R$.

In Chapter 3, we define the divisors $D_{i,j}^{(n)}$ of the moduli scheme $F[n]$, where $1 \leqslant i < j \leqslant n$. We prove that each $D_{i,j}^{(n)}$ parametrizes tuples $(S_1, \ldots, S_{n+1}, p_1, \ldots, p_n)$ for which $p_j \mapsto p_i$ under the projection map $S_j \to S_i$, then show they are smooth.

In Chapter 4, we let $R = k$ be an algebraically closed field. We prove that the Chow ring of the moduli space $F[n]$ is generated by the classes of the divisors $\{D_{i,j}^{(n)}\}_{1 \leqslant i < j \leqslant n}$ over the Chow ring $\mathbf{A}^*(S_1^n)$. We end the chapter by proving two key relationships between these generators.

In Chapter 5, we give a precise description of the Chow ring $\mathbf{A}^*(F[n])$ in the case where the base surface S_1 is rational over Spec $\mathbb{C}$.

In Chapter 6, we shift our focus to the study of smooth very general smooth curves on smooth surfaces. We explain our question and its origins, then state the main result.

In Chapter 7, we state all the basic facts about curves and surfaces which we need later on. Among these facts is a quick introduction to Brill-Noether theory and the statement of the Maximal Rank Conjecture.

In Chapter 8, we start building towards a proof of our main result by presenting some preliminary results. If C is a very general curve that embeds as an ample divisor on a smooth surface S, then $\mathrm{Pic}^0(C)$ is a simple abelian variety and $\mathrm{Pic}^0(S)$ is either 0 or isomorphic to $\mathrm{Pic}^0(C)$. We give a lower bound for the intersection number $C \cdot K_S$, where K_S is the canonical divisor of S, and finish by showing that S must have at least $3g-3$ moduli.

In Chapter 9, we analyze the case where the surface S has Kodaira dimension $-\infty$. If S is not rational, then it is birational to $C \times \mathbb{P}^1$. The case in which S is rational remains mostly open. However, we analyze the conditions that need to be satisfied in this situation.

In Chapter 10, we analyze the case in which the surface S has Kodaira dimension 0. We show that C cannot be embedded on S as an ample divisor if its genus is higher than 18.

In Chapter 11, we analyze the case in which the surface S has Kodaira dimension 1. We show that C cannot be embedded on S as an ample divisor if its genus is higher than 20.

In Chapter 12, we analyze the case where the surface S has Kodaira dimension 2. If $q(S) = g$, then S has to be birational to the symmetric product $\mathrm{Sym}^2(C)$. The case $q(S) = 0$ remains mostly open. However, we show that if $S \hookrightarrow \mathbb{P}^r$ is a complete intersection and the composed morphism $C \hookrightarrow S \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture, then C cannot be embedded on S as an ample divisor if its genus is higher than 15.

Chapter 1

Definition of the moduli problem

Let S_1 be a fixed smooth projective surface over a ring R. Our goal is to study and parametrize surfaces obtained through a series of n ordered blowups of the base surface S_1. More specifically, we focus our attention on tuples:

$$(S_1, S_2, \ldots, S_{n+1}, p_1, p_2, \ldots, p_n),$$

where $p_i \in S_i$ and S_{i+1} is the blowup of S_i at p_i, for all $1 \leqslant i \leqslant n$.

We define formally the functor $\mathcal{F}[S_1, n]$ as follows:

Definition 1.1. *Let S_1 be a smooth projective surface over a ring R and $n \geqslant 0$ an integer. Consider the contravariant functor:*

$$\mathcal{F}[n] = \mathcal{F}[S_1, n] : Sch(R) \to Sets$$

defined as follows:

- *For any R-scheme B, an object in $\mathcal{F}[S_1, n](B)$ is a tower of morphisms:*

$$
\begin{tikzcd}
\Sigma_{B,n+1} \arrow[r, "\pi_{B,n+1}"] & \Sigma_{B,n} \arrow[r, "\pi_n"] & \Sigma_{B,n-1} \arrow[r, "\pi_{n-1}"] & \dots \arrow[r] & \Sigma_{B,2} \arrow[r, "\pi_2"] & \Sigma_{B,1} = B \times S_1 \arrow[d, shift left, "\pi_1"] \\
 & & & & & B, \arrow[u, bend left, "p_1"] \arrow[ul, bend left, "p_2"] \arrow[ulll, bend left, "p_{n-1}"] \arrow[ullll, bend left, "p_n"]
\end{tikzcd}
$$

such that the following conditions are satisfied:

(1) $\pi_1 = pr_1$ *is the projection onto the first factor;*

(2) *for each* $1 \leqslant i \leqslant n$, *the morphism* $p_i : B \to \Sigma_{B,i}$ *is a section of the composed map* $\Sigma_{B,i} \to B$;

(3) *for each* $1 \leqslant i \leqslant n$, *the morphism* $\pi_{i+1} : \Sigma_{B,i+1} \to \Sigma_{B,i}$ *is the blowup of* $\Sigma_{B,i}$ *along the locus* $p_i(B)$.

A shortened notation for such a family is $(\Sigma_{B,i}, \pi_i, p_i)_{i=1}^n$. *Notice that* Σ_{n+1} *is not included in this notation; however, this scheme is uniquely defined by the data given, so the notation is consistent. For brevity throughout the paper, we will also use the following notation for the same family:* $\Sigma_{B,\leqslant n+1} \to B$.

- *For every* R*-scheme* B, $\mathcal{F}[n](B)$ *is the set of all families over* B, *up to isomorphism. An isomorphism between two families is defined as:*

$$\Theta = (\theta_i)_{i=1}^n : (\Sigma_{B,i}, \pi_i, p_i)_{i=1}^n \xrightarrow{\sim} (\Sigma'_{B,i}, \pi'_i, p'_i)_{i=1}^n,$$

where, $\forall 1 \leqslant i \leqslant n$, $\theta_i : \Sigma_i \to \Sigma'_i$ *are isomorphisms that commute with the maps of the two families.*

- *Let* B_1, B_2 *be* R*-schemes, and let* $f : B_1 \to B_2$ *be an* R*-morphism between the two schemes. There exists a natural contravariant map* $\mathcal{F}(f) : \mathcal{F}[n](B_2) \to \mathcal{F}[n](B_1)$

given as follows: for every family over B_2, we obtain a family over B_1 by pulling back the schemes $\Sigma_{B_2,i}$ and the sections p_i along f, as in the figure below. To conclude that the tower of morphisms over B_1 is indeed a valid object in $\mathcal{F}[n](B_1)$, we need to know that $\Sigma_{B_1,i+1}$ is the blowup of $\Sigma_{B_1,i}$ along the locus $p_i^*(B_1)$, for all $1 \leqslant i \leqslant n$. This is an immediate application of Lemma 2.1 from Chapter 2.

$$
\begin{tikzcd}
\Sigma_{B_1,n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{B_2,n+1} \arrow[d] \\
\Sigma_{B_1,n} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{B_2,n} \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] \\
\Sigma_{B_1,1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{B_2,1} \arrow[d] \\
B_1 \arrow[r, "f"] \arrow[u, bend left, "p_1^*"] \arrow[uuu, bend left=60, "p_n^*"] & B_2. \arrow[u, bend right, "p_1"'] \arrow[uuu, bend right=60, "p_n"']
\end{tikzcd}
$$

• *The identity map on an R-scheme $id\colon B \to B$ corresponds to the identity map on sets $\mathcal{F}(id) = id\colon \mathcal{F}[n](B) \to \mathcal{F}[n](B)$.*

• *Let B_1, B_2, B_3 be R-schemes. Let $f : B_1 \to B_2$ and $g : B_2 \to B_3$ be R-morphisms. Then $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$. This follows from the uniqueness of pullbacks (up to isomorphism): if the middle family is the pullback of the family on the right along the map g, and the family on the left is the pullback of the middle family along the map f, then the family on the left is the pullback of family on the right along the map $g \circ f$:*

$$
\begin{tikzcd}
\Sigma_{B_1,\leqslant n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{B_2,\leqslant n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{B_3,\leqslant n+1} \arrow[d] \\
B_1 \arrow[r, "f"] & B_2 \arrow[r, "g"] & B_3
\end{tikzcd}
$$

Remark 1.2. *Let B be a R-scheme, and let $\Sigma_{B,\leqslant n+1} \to B$ be a family in $\mathcal{F}[n](B)$. For any point $x \in B$, the fiber over x in this family is a sequence $S_{n+1} \to \cdots \to S_1$,* *where S_{i+1} is the blowup of S_i at some point p_i, $\forall 1 \leqslant i \leqslant n$. Therefore, this fiber corresponds to a tuple $(S_1, \dots, S_{n+1}, p_1, \dots, p_n)$ as the ones introduced in the beginning of the chapter.*

Remark 1.3. *For every integer $n \geqslant 0$, there exists a natural forgetful map:*

$$\mathcal{F}[n+1] \to \mathcal{F}[n]$$

which sends families $\Sigma_{B,\leqslant n+2} \to B$ in $\mathcal{F}[n+1](B)$ to families $\Sigma_{B,\leqslant n+1} \to B$ in $\mathcal{F}[n](B)$, for all R-schemes B.

Remark 1.4. *For every integer $n > 0$, there exists a natural transformation of functors:*

$$\mathcal{F}[n] \to S_1^n$$

which, for any R-scheme B, maps a family $(\Sigma_{B,i}, \pi_i, p_i)_{i=1}^n$ over B to $(\overline{p_1}, \overline{p_2}, \dots, \overline{p_n})$, where $\overline{p_i} \in S_1(B)$ is the composition map $B \xrightarrow{p_i} \Sigma_{B,i} \to \dots \to \Sigma_{B,1} = B \times S_1 \xrightarrow{pr_2} S_1$.

Chapter 2

Construction of the moduli scheme

In this section we prove that the functor $\mathcal{F}[n]$ has a fine moduli scheme, which we denote by $F[n]$. We show that $F[n]$ and all the schemes in its universal family are smooth and projective over Spec R.

Before we start working towards our main result, note that we reserve the following notation for the universal family over $F[S_1, n]$:

$$\begin{tikzcd}
\Sigma_{n+1,n+1} \arrow[r, "\pi_{n+1,n+1}"] & \Sigma_{n+1,n} \arrow[r, "\pi_{n+1,n}"] & \dots \arrow[r] & \Sigma_{n+1,2} \arrow[r, "\pi_{n+1,2}"] & \Sigma_{n+1,1} = F[n] \times S_1 \arrow[d, "\pi_{n+1,1}"] \\
& & & & F[n], \arrow[u, bend left, "\sigma_{n+1,1}"] \arrow[ul, bend left, "\sigma_{n+1,2}"] \arrow[ulll, bend left, "\sigma_{n+1,n}"]
\end{tikzcd}$$

Example. *We start by constructing $F[0]$, $F[1]$ (which represent the functors $\mathcal{F}[0]$ and $\mathcal{F}[1]$, respectively), and their universal families. It is easy to see that $F[0] =$ Spec R, since every scheme $B \in Sch(R)$ comes equipped with a structure morphism to Spec R. Moreover, we notice that the universal family over $F[0]$ is, by definition, $\Sigma_{1,1} =$ Spec $R \times S_1 = S_1$:*

$$\begin{tikzcd} \Sigma_{B,1} \arrow[r] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{1,1} \arrow[d, "pr_1"] \\ B \arrow[r] & F[0] \end{tikzcd} \quad \cong \quad \begin{tikzcd} B \times S_1 \arrow[r] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & S_1 \arrow[d, "pr_1"] \\ B \arrow[r] & Spec\ R \end{tikzcd}$$

Next, we want to show that $F[1] \cong S_1$. Intuitively, every object in this moduli space is a triple (S_1, S_2, p_1) such that $S_2 = Bl_{p_1}S_1$, so this triple is uniquely identified by the point $p_1 \in S_1$. More concretely, we need to show that $\mathcal{F}[1](B) \cong Mor(B, S_1)$, i.e. that each family over B corresponds uniquely to a morphism $B \to S_1$. The equivalence goes as follows: say we start with a family over B, like in the figure below. The corresponding morphism $B \to S_1$ is $f = pr_2 \circ p_1$. Conversely, say we start with a morphism $f : B \to S_1$. This map gives a section $p_1 = id \times f : B \to \Sigma_{B,1} = S_1 \times B$, and $\Sigma_{B,2}$ is the blowup of $\Sigma_{B,1}$ along this section.

$$\begin{tikzcd} & \Sigma_{B,2} \arrow[d, "\pi_2"'] \\ S_1 & \Sigma_{B,1} = B \times S_1 \arrow[l, "pr_2"'] \arrow[d, "\pi_1"'] \\ & B \arrow[ul, "f = pr_2 \circ p_1"] \arrow[u, bend right, "p_1 = id \times f"'] \end{tikzcd}$$

The top scheme in the universal family over $F[1]$ is $\Sigma_{2,2} = Bl_{\triangle}(S_1 \times S_1)$. This follows immediately from the figure above, considering the special case where $B = F[1] \cong S_1$ and $f = id : S_1 \to S_1$:

$$\begin{tikzcd} \Sigma_{2,2} \arrow[d, "\pi_{2,2}"] \\ \Sigma_{2,1} \arrow[d, "\pi_{2,1}"] \\ F[1] \arrow[u, bend left, "p_{2,1}"] \end{tikzcd} \quad \cong \quad \begin{tikzcd} Bl_{\triangle}(S_1 \times S_1) \arrow[d, "bl_{\triangle}"] \\ S_1 \times S_1 \arrow[d, "pr_1"] \\ S_1 \arrow[u, bend left, "\triangle"] \end{tikzcd}$$

Before we give the general construction of the moduli scheme corresponding to the functor $\mathcal{F}[n]$, we state and prove the following lemmas, which will be the key ingredients in the construction of the moduli scheme.

Lemma 2.1. *Let $A, B, \Sigma_{B,1}$ be schemes over R. Let $\pi : \Sigma_{B,1} \to B$ be a smooth morphism, let $\sigma : B \to \Sigma_{B,1}$ be a section of π, and let $\Sigma_{B,2}$ be the blowup of $\Sigma_{B,1}$ along the locus $\sigma(B)$. Given $f : A \to B$ an R-morphism, let $\Sigma_{A,1}$ and σ^* be the pullbacks along the map f of $\Sigma_{B,1}$ and σ, respectively. Then the following statements hold:*

(i) The composed morphism $\Sigma_{B,2} \to B$ is smooth.

(ii) The blowup of $\Sigma_{A,1}$ along the locus $\sigma^*(A)$, denoted by $\Sigma_{A,2}$, is the pullback of $\Sigma_{B,2}$ along the map $\Sigma_{A,1} \to \Sigma_{B,1}$.

$$\begin{tikzcd}
\Sigma_{A,2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{B,2} \arrow[d] \\
\Sigma_{A,1} \arrow[r] \arrow[d, "\pi^*"] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{B,1} \arrow[d, "\pi"'] \\
A \arrow[r, "f"] \arrow[u, bend left, "\sigma^*"] & B \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

Proof. (i) We show that the morphism $\Sigma_{B,2} \to B$ is smooth by proving that it is flat, locally of finite presentation, and has smooth fibers (see [37], Tag 02K5). Affine locally, on the level of rings, we are given a smooth morphism $\pi : T \to T'$, and $\sigma : T' \twoheadrightarrow T$ a section of π. Let $I = \ker(\sigma) \subset T'$. The blowup of Spec T' along I is defined to be:

$$\mathrm{Bl}_I(\mathrm{Spec}\ T') = \mathrm{Proj}(T' \oplus I \oplus I^2 \oplus \dots).$$

The map σ is a section of the smooth morphism π, which means that I is a regular ideal. Since I is regular, then $\mathrm{Bl}_I(\mathrm{Spec}\ T')$ is of finite presentation over Spec T (see [37], Tag 0BIQ).

To show that $\Sigma_{B,2} \to B$ is flat, it is enough to prove that $(T' \oplus I \oplus I^2 \oplus \dots)$ is flat over T. We know that $T \to T'$ is smooth, hence T' is flat over T. We show inductively that I^n is flat over T. By the following short exact sequence, it is enough to prove that T'/I^n is T-flat:

$$0 \to I^n \to T' \to T'/I^n \to 0.$$

When $n = 1$, $T'/I \cong T$, so the claim is true. For the inductive step, consider another short exact sequence:

$$0 \to I^{n-1}/I^n \to T'/I^n \to T'/I^{n-1} \to 0.$$

Since I is a regular ideal, then I^{n-1}/I^n is a locally free finite T-module, hence it is flat. By the induction hypothesis, T'/I^{n-1} is flat over T. Putting these two facts together, we conclude that T'/I^n is flat over T, completing the induction step.

To finish the proof of statement (i), we are left to show that the morphism $\Sigma_{B,2} \to B$ has smooth fibers. By part (ii) below, the fiber over every point $x \in B$ is as follows, where $k(x)$ is the residue field of x:

$$\begin{tikzcd}
V' = \mathrm{Bl}_x V \arrow[d] \\
V \arrow[d] \\
\mathrm{Spec}\ k(x). \arrow[u, bend left, "x"]
\end{tikzcd}$$

The scheme V is smooth over $k(x)$, hence $V' = \mathrm{Bl}_x V$ is also smooth over $k(x)$. With this, we conclude that the morphism $\Sigma_{B,2} \to B$ is smooth.

(ii) This claim is true because blowups commute with base change. To make this precise, we work affine locally, on the level of rings, where we have the following figure:

$$\begin{tikzcd} S' = S \otimes_T T' \arrow[d, bend right, "\sigma_S"'] & & T' \arrow[ll] \arrow[d, bend left, "\sigma"] \\ S \arrow[u, "\pi_S"] & & T \arrow[ll] \arrow[u, "\pi"] \end{tikzcd}$$

Let $\pi_S : S \to S'$ be the pullback of π and σ_S be the pullback of σ. Let $I_S = \ker(\sigma_S) \subset S'$. The blowup of Spec S' along I_S is defined to be:

$$\mathrm{Bl}_{I_S}(\mathrm{Spec}\ S') = \mathrm{Proj}(S' \oplus I_S \oplus I_S^2 \oplus \dots).$$

On the level of rings, our claim boils down to showing that:

$$(S' \oplus I_S \oplus I_S^2 \oplus \dots) = (S \oplus I \oplus I^2 \oplus \dots) \otimes_{T'} S',$$

which translates to showing that:

$$I_S^n = I^n \otimes_{T'} S' = I^n \otimes_T S.$$

The statement above follows from the following figure. Note that T'/I^n is flat over T, so tensoring the top short exact sequence with S preserves the exactness of the resulting sequence:

$$\begin{tikzcd} 0 \arrow[r] & I^n \otimes_T S \arrow[r] \arrow[d] & T' \otimes_T S \arrow[r] \arrow[d, "\cong"] & T'/I^n \otimes_T S \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & I_S^n \arrow[r] & S' \arrow[r] & S'/I_S^n \arrow[r] & 0 \end{tikzcd}$$

$\square$

Lemma 2.2. *Let A, B, C be R-schemes. Let $A \xrightarrow{h} C$ be an R-morphism that factors as $A \xrightarrow{f} B \xrightarrow{g} C$. Given any R-morphism $V_C \xrightarrow{v_C} C$, let $V_B \xrightarrow{v_B} B$ be its pullback along g, and $V_A \xrightarrow{v_A} A$ be its pullback along h. There exists a unique map $f' : V_A \to V_B$ which makes the top triangle commutative and the left square cartesian.*

$$\begin{tikzcd}
V_A \arrow[rr, "h'"] \arrow[dr, dashed, "\exists! f'"] \arrow[dd, "v_A"'] & & V_C \arrow[dd, "v_C"] \\
& V_B \arrow[ur, "g'"'] \arrow[dd, "v_B"' near start] & \\
A \arrow[rr, "h" near end] \arrow[dr, "f"'] & & C \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. First, we are given the maps $V_A \xrightarrow{h'} V_C$ and $V_A \xrightarrow{f \circ v_A} B$ that form a commutative square with the maps $V_C \xrightarrow{v_C} C$ and $B \xrightarrow{g} C$. Since the right square is cartesian, there exists a unique map $f' : V_A \to V_B$ making all squares and all triangles commutative. We are left to show that the maps f' makes the left square cartesian.

Consider a scheme M together with morphisms $m_A : M \to A$, $m_{V_B} : M \to V_B$ that make a commutative diagram with $A \xrightarrow{f} B$, $V_B \xrightarrow{v_B} B$. We need to show there exists a unique map $m_{V_A} : M \to V_A$ such that $m_A = v_A \circ m_{V_A}$ and $m_{V_C} = f' \circ m_{V_A}$.

$$\begin{tikzcd}
M \arrow[drrr, bend left, "m_{V_C}"] \arrow[ddrr, bend left, "m_{V_B}"] \arrow[dr, "m_{V_A}"] \arrow[dddr, bend right, "m_A"'] & & & \\
& V_A \arrow[rr, "h'"] \arrow[dr, "f'"] \arrow[dd, "v_A"'] & & V_C \arrow[dd, "v_C"] \\
& & V_B \arrow[ur, "g'"'] \arrow[dd, "v_B"' near start] & \\
& A \arrow[rr, "h" near end] \arrow[dr, "f"'] & & C \\
& & B \arrow[ur, "g"'] &
\end{tikzcd}$$

Let $m_{V_C} = g' \circ m_{V_B}$. We claim that the maps $M \xrightarrow{m_{V_C}} V_C \xrightarrow{v_C} C$ and $M \xrightarrow{m_A} A \xrightarrow{h} C$

coincide. This is true because:

$$\begin{aligned} v_C \circ m_{V_C} &= v_C \circ g' \circ m_{V_B} \\ &= g \circ v_B \circ m_{V_B} \\ &= g \circ f \circ m_A \\ &= h \circ m_A. \end{aligned}$$

Now, given that the big square is cartesian, there exists a unique map $m_{V_A} : M \to V_A$ such that $m_{V_C} = h' \circ m_{V_A}$ and $m_A = v_A \circ m_{V_A}$. However, we need to show more that this, specifically that $m_{V_B} = f' \circ m_{V_A}$. Assume by contradiction this is not the case; then we have two different maps $M \to V_B$ that commute with the cartesian square on the right, which is impossible. Hence, we found a map $m_{V_A} : M \to V_A$ such that $m_B = v_A \circ m_{V_A}$ and $m_{V_B} = f' \circ m_{V_A}$. Assume by contradiction this map is not unique; then we would have two maps that commuted with the big cartesian square, which is again impossible. Hence the conclusion holds and the left square is cartesian. □

Theorem 2.3. *The functor $\mathcal{F}[n]$ has a fine moduli scheme $F[n]$. Moreover, the following is true:*

(a) For every $n \geqslant 0$, the moduli space $F[n]$, together with all the schemes $\Sigma_{n+1,1}$, $\ldots$, $\Sigma_{n+1,n+1}$ in its universal family, are smooth and projective over Spec R;

(b) For every $n \geqslant 1$, the top scheme $\Sigma_{n+1,n+1}$ in the universal family of $F[n]$ can be identified as:

$$\Sigma_{n+1,n+1} \cong Bl_{\triangle}(F[n] \times_{F[n-1]} F[n]),$$

where the cartesian product is induced by the forgetful map $F[n] \to F[n-1]$;

(c) *For every $n \geqslant 0$, the top scheme $\Sigma_{n+1,n+1}$ in the universal family of $F[n]$ is the moduli scheme representing the functor $\mathcal{F}[n+1]$. Under this identification, the map*

$$\Pi_{n+1} = \pi_{n+1,1} \circ \cdots \circ \pi_{n+1,n+1} : \Sigma_{n+1,n+1} \cong F[n+1] \to F[n]$$

corresponds to the forgetful functor $\mathcal{F}[n+1] \to \mathcal{F}[n]$.

Proof. We will prove these statements inductively over n. In Example 2 we constructed the moduli schemes $F[0] \cong \operatorname{Spec} R$ with its universal family $\Sigma_{1,1} \cong S_1$, and $F[1] \cong S_1$. The map $\Pi_1 : \Sigma_{1,1} \to F[0]$ is the structure morphism $\Pi_1 : S_1 \to \operatorname{Spec} R$, so it trivially corresponds to the forgetful functor $\mathcal{F}[1] \to \mathcal{F}[0]$. Hence the statements (a) and (c) are true for $n = 0$.

For the inductive step, assume the moduli space $F[k]$ exists, for all $k < n$. Assume $F[k], \Sigma_{k+1,1}, \ldots, \Sigma_{k+1,k+1}$ are all smooth projective schemes. We prove the following:

(i) Let $W = \Sigma_{n,n}$ be the top scheme in the universal family over $F[n-1]$. We start by constructing a family over W in $\mathcal{F}[n](W)$. We then show that W is the fine moduli scheme corresponding to the functor $\mathcal{F}[n]$, and that the family we defined over W is the universal family;

(ii) We show that $F[n], \Sigma_{n+1,1}, \ldots, \Sigma_{n+1,n+1}$ are smooth and projective, and that we can identify $\Sigma_{n+1,n+1}$ with $\mathrm{Bl}_{\triangle}(F[n] \times_{F[n-1]} F[n])$;

(iii) We prove that the map $\Pi_n : \Sigma_{n,n} \cong F[n] \to F[n-1]$ corresponds to the forgetful functor $\mathcal{F}[n] \to \mathcal{F}[n-1]$.

The first step is to construct the family over W in $\mathcal{F}[n](W)$. This family is obtained as follows: let $\Pi_n : W = \Sigma_{n,n} \to F[n-1]$ be the composed morphism. Let $\Sigma_{W,\leqslant n} \to W$ be the pullback of the family $\Sigma_{n,\leqslant n} \to F[n-1]$ along the map Π_n (see figure below). In particular, notice that $\Sigma_{W,n} \cong W \times_{F[n-1]} W$. To construct the top scheme $\Sigma_{W,n+1}$, let $p_{W,n} := \triangle \colon W \to \Sigma_{W,n} = W \times_{F[n-1]} W$ be the diagonal embedding and $\Sigma_{W,n+1} = \mathrm{Bl}_{\triangle}(W \times_{F[n-1]} W)$:

$$\begin{tikzcd}
\Sigma_{W,n+1} = \mathrm{Bl}_{\triangle}(W \times_{F[n-1]} W) \arrow[d] & \\
\Sigma_{W,n} = W \times_{F[n-1]} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Sigma_{n,n} = W \arrow[d] \arrow[dd, bend left, "\Pi_n"] \\
\Sigma_{W,\leqslant n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Sigma_{n,\leqslant n-1} \arrow[d] \\
W \arrow[r, "\Pi_n"] \arrow[uu, bend left, "p_{W,n}=\triangle"] & F[n-1].
\end{tikzcd}$$

First, notice that the tower of morphisms $\Sigma_{W,\leqslant n+1} \to W$ is indeed a family in $\mathcal{F}[n](W)$, as a result of Lemma 2.1. We claim that W is the fine moduli space for $\mathcal{F}[n]$, and that the family constructed above is the universal family over $F[n]$. We prove our statement by using this family over W to build the correspondence $\mathcal{F}[n](B) \cong \mathrm{Mor}(B, W)$, for any R-scheme B.

We start by constructing a functor map:

$$C_1 : \mathcal{F}[n] \to \mathrm{Mor}(-, W).$$

Let B be an R-scheme and $\Sigma_{B,\leqslant n+1} \to B$ a family in $\mathcal{F}[n](B)$. We need to associate a morphism $B \to W$ to this family. The truncated family $\Sigma_{B,\leqslant n} \to B$ is an element of $\mathcal{F}[n-1](B)$, so it corresponds uniquely to a morphism $f_{n-1} : B \to F[n-1]$ that gives the figure below. Now, recall that the family $\Sigma_{B,\leqslant n+1} \to B$ comes equipped with a section $p_n : B \to \Sigma_n$, so we obtain the desired map $f_W : B \to W$ by composing $B \xrightarrow{p_n} \Sigma_{B,n} \xrightarrow{\mathrm{pr}_2} W$:

$$\begin{tikzcd}
\Sigma_{B,n} \arrow[r, "\mathrm{pr}_2"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n,n} = W \arrow[d] \\
\Sigma_{B,\leqslant n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n,\leqslant n-1} \arrow[d] \\
B \arrow[r, "f_{n-1}"] \arrow[uu, bend left, "p_n"] & F[n-1].
\end{tikzcd}$$

Second, we construct a functor map:

$$C_2 : \mathrm{Mor}(-, W) \to \mathcal{F}[n].$$

Let B be an R-scheme and $f : B \to W$ an R-morphism. We want to obtain a corresponding family in $\mathcal{F}[n](B)$. To do so, we pull back the family $\Sigma_{W,\leqslant n+1} \to W$ along f. The resulting tower of morphisms is indeed a family in $\mathcal{F}[n](B)$, as a result of Lemma 2.1:

$$\begin{tikzcd}
\Sigma_{B,\leqslant n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{W,\leqslant n+1} \arrow[d] \\
B \arrow[r] \arrow[u, bend left, "p^*_{W,\bullet}"] & W. \arrow[u, bend right, "p_{W,\bullet}"']
\end{tikzcd}$$

Now we want to show that the maps C_1 and C_2 are inverses of each other. Say we start with a morphism $f : B \to W$ and we construct a family over B by pulling back $\Sigma_{W,\leqslant n+1} \to W$ along f. Then we obtain the following figure:

$$\begin{tikzcd}
\Sigma_{B,n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{W,n+1} \arrow[d] & \\
\Sigma_{B,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{W,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n,n} \cong W \arrow[d] \\
\Sigma_{B,\leqslant n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{W,\leqslant n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n,\leqslant n-1} \arrow[d] \\
B \arrow[r, "f"] \arrow[uu, bend left, "p_n"] & W \arrow[r] & F[n-1].
\end{tikzcd}$$

Notice the following maps are equivalent, which shows that $C_1 \circ C_2 = \mathrm{id}$:

$$
\begin{aligned}
[B \xrightarrow{f_W} W] &= [B \xrightarrow{p_n} \Sigma_{B,n} \to \Sigma_{W,n} \to \Sigma_{n,n}] \\
&= [B \xrightarrow{f} W \xrightarrow{p_{W,n}} \Sigma_{W,n} \to \Sigma_{n,n}] \\
&= [B \xrightarrow{f} W \xrightarrow{\triangle} W \times_{F[n-1]} W \xrightarrow{\mathrm{pr}_2} W] \\
&= [B \xrightarrow{f} W].
\end{aligned}
$$

Lastly, say we start with a family over B in $\mathcal{F}[n](B)$, denoted by $\Sigma_{B,\leqslant n+1} \to B$. As before, we obtain the corresponding morphism $f_W : B \to W$, which is the composed morphism $B \xrightarrow{p_n} \Sigma_{B,n} \xrightarrow{\mathrm{pr}_2} \Sigma_{n,n} \cong W$. We want to show that if we pull back $\Sigma_{W,\leqslant n+1} \to W$ along f_W, we recover the family we started with:

$$
\begin{tikzcd}
\Sigma_{B,\leqslant n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{W,\leqslant n+1} \arrow[d] \\
B \arrow[r, "f_W"] \arrow[u, bend left, "p_{\leqslant n}"] & W. \arrow[u, bend right, "p_{W,\leqslant n}"']
\end{tikzcd}
$$

Since the truncated family $\Sigma_{B,\leqslant n} \to B$ is an object in $\mathcal{F}[n-1](B)$, it corresponds uniquely to a morphism $f : B \to F[n-1]$ which gives this figure:

$$
\begin{tikzcd}
\Sigma_{B,n+1} \arrow[d] & \\
\Sigma_{B,n} \arrow[r, "\mathrm{pr}_2"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{n,n} \arrow[d] \\
\Sigma_{B,\leqslant n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \Sigma_{n,\leqslant n-1} \arrow[d] \\
B \arrow[r, "f"] \arrow[uu, bend left, "p_n"] & F[n-1].
\end{tikzcd}
$$

We first claim that the map $B \xrightarrow{f} F[n-1]$ factors as $B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]$. This is true because of the following equivalence of maps:

$$
\begin{aligned}
[B \xrightarrow{f} F[n-1]] &= [B \xrightarrow{\mathrm{id}} B \xrightarrow{f} F[n-1]] \\
&= [B \xrightarrow{p_n} \Sigma_{B,n} \to B \xrightarrow{f} F[n-1]] \\
&= [B \xrightarrow{p_n} \Sigma_{B,n} \xrightarrow{\mathrm{pr}_2} \Sigma_{n,n} \xrightarrow{\Pi_n} F[n-1]] \\
&= [B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]].
\end{aligned}
$$

Since the map $B \xrightarrow{f} F[n-1]$ factors as $B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]$, we can use Lemma 2.2 repeatedly, from the bottom up, to recover the maps $\Sigma_i \to \Sigma_{W,i}$, for all $i \leqslant n$, which make every rectangle in the figure below cartesian, and every triangle commutative:

$$
\begin{tikzcd}
\Sigma_{B,n+1} \arrow[dd] & & \\
& \Sigma_{W,n+1} \arrow[dd] & \\
\Sigma_{B,\leqslant n} \arrow[rr] \arrow[dr] \arrow[dd] & & \Sigma_{n,\leqslant n} \arrow[dd] \\
& \Sigma_{W,\leqslant n} \arrow[ur] \arrow[dd] & \\
B \arrow[rr, "f"] \arrow[dr, "f_W"'] & & F[n-1] \\
& W \arrow[ur, "\Pi_n"'] &
\end{tikzcd}
$$

To finish proving that $C_2 \circ C_1 = \mathrm{id}$, we need to show that the section $p_n : B \to \Sigma_{B,n}$ is the pullback along f_W of the diagonal embedding $p_{W,n} = \triangle \colon W \to \Sigma_{W,n} = W \times_{F[n-1]} W$. This follows from the figure below: by the definition of f_W, it is the composition $B \xrightarrow{p_n} \Sigma_{B,n} \to \Sigma_{W,n} = W \times_{F[n-1]} W \to W$, so all maps in the figure below commute as expected, and p_n is indeed the pullback of $p_{W,n} = \triangle$. By Lemma 2.1, we obtain that $\Sigma_{B,n+1}$ is the pullback of $\Sigma_{W,n+1}$, and the proof of (i) is complete.

$$
\begin{tikzcd}
B \arrow[dr, "p_n"] \arrow[drr, bend left, "f_W \times f_W"] \arrow[drrr, bend left, "f_W"] \arrow[ddr, bend right, "\mathrm{id}"'] & & & \\
& \Sigma_{B,n} \arrow[r] \arrow[d] & W \times_{F[n-1]} W \arrow[r] \arrow[d] & W \arrow[d] \\
& B \arrow[r, "f_W"] & W \arrow[r] \arrow[u, bend left, "\triangle"] & F[n-1]
\end{tikzcd}
$$

To finish the proof of part (ii), we show inductively that $F[n]$ and all the schemes in its universal family are smooth projective schemes over Spec R. For the base case, recall that $F[0] \cong \operatorname{Spec} R$ and $\Sigma_{1,1} \cong S_1$ have this property. Inductively, assume that $F[n-1]$ and all the schemes in its universal family are smooth and projective over Spec R. From the arguments above, the top scheme $\Sigma_{n,n}$ over $F[n-1]$ can be identified with $F[n]$, so we know that $F[n]$ is also a smooth and projective. The first scheme in the universal family over $F[n]$ is, by definition, $\Sigma_{n+1,1} \cong F[n] \times S_1$, which also satisfies these properties. Now, by construction, all the maps $\Sigma_{n+1,i} \to F[n]$ are smooth (this is an application of Lemma 2.1), which means the corresponding sections $\sigma_{n+1,i} : F[n] \to \Sigma_{n+1,i}$ are regular embeddings. Going up in the tower of morphisms, we can conclude step by step that $\Sigma_{n+1,2}, \ldots, \Sigma_{n+1,n+1}$ are smooth and projective, since each of them is obtained by blowing up a smooth projective scheme along a smooth projective subscheme. This concludes claim (ii).

Lastly, we need to show that $\Pi_n : \Sigma_{n,n} \cong F[n] \to F[n-1]$ corresponds to the forgetful functor $\mathcal{F}[n] \to \mathcal{F}[n-1]$. For any R-scheme B and any morphism $f : B \to F[n]$, let $\Sigma_{B,\leqslant n+1} \to B$ be its corresponding family in $\mathcal{F}[n](B)$. We need to show that $\Pi_n \circ f : B \to F[n-1]$ corresponds to the truncated family $\Sigma_{B,\leqslant n} \to B$ in $\mathcal{F}[n-1](B)$. This follows immediately from the figure below, and the proof of (iii) is complete:

$$\begin{tikzcd}
\Sigma_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n+1,n+1} \arrow[d] & \\
\Sigma_{\leqslant n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n+1,\leqslant n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n,\leqslant n} \arrow[d] \\
B \arrow[r, "f"] & F[n] \arrow[r, "\Pi_n"] & F[n-1].
\end{tikzcd}$$

□

Remark 2.4. *As a corollary of the construction outlined in the proof of Theorem 2.3, we obtain the following ascending ladder. Note that each square is cartesian, by construction:*

$$\begin{tikzcd}
\dots \arrow[r] & \Sigma_{5,5} \arrow[d, "\pi_{5,5}"'] & & & & \\
\dots \arrow[r] & \Sigma_{5,4} \arrow[r] \arrow[d, "\pi_{5,4}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{4,4} \arrow[d, "\pi_{4,4}"'] & & & \\
\dots \arrow[r] & \Sigma_{5,3} \arrow[r] \arrow[d, "\pi_{5,3}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{4,3} \arrow[r] \arrow[d, "\pi_{4,3}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{3,3} \arrow[d, "\pi_{3,3}"'] & & \\
\dots \arrow[r] & \Sigma_{5,2} \arrow[r] \arrow[d, "\pi_{5,2}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{4,2} \arrow[r] \arrow[d, "\pi_{4,2}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{3,2} \arrow[r] \arrow[d, "\pi_{3,2}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{2,2} \arrow[d, "\pi_{2,2}"'] & \\
\dots \arrow[r] & \Sigma_{5,1} \arrow[r] \arrow[d, "\pi_{5,1}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{4,1} \arrow[r] \arrow[d, "\pi_{4,1}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{3,1} \arrow[r] \arrow[d, "\pi_{3,1}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{2,1} \arrow[r] \arrow[d, "\pi_{2,1}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{1,1} \arrow[d, "\pi_{1,1}"'] \\
\dots \arrow[r] & F[4] \arrow[r, "\Pi_4"'] & F[3] \arrow[r, "\Pi_3"'] & F[2] \arrow[r, "\Pi_2"'] & F[1] \arrow[r, "\Pi_1"'] & F[0].
\end{tikzcd}$$

In particular, since $\Sigma_{n,n} \cong F[n]$, for all $n \geqslant 1$, we obtain the following identification, $\forall 1 \leqslant i \leqslant n+1$:

$$\Sigma_{n+1,i} \cong F[n] \times_{F[i-1]} F[i]. \tag{2.1}$$

In light of Equation 2.1, we look back to the universal family over $F[n]$ and give another description of the projection maps $\pi_{n+1,*}$ and the sections $\sigma_{n+1,*}$. Before we do so, we need to establish some notation:

Notation 2.5. *Let B be an R-scheme. A point in $F[n](B)$ corresponding to a family $(\Sigma_{B,i}, \pi_i, p_i)_{i=1}^n$ will simply be denoted as $(p_1, \ldots, p_n)$. Similarly, a B-point of $\Sigma_{n+1,i} \cong F[n] \times_{F[i-1]} F[i]$ will simply be denoted by $(p_1, \ldots, p_n; p_i')$, with the understanding that $(p_1, \ldots, p_n)$ is the corresponding point in $F[n](B)$ and $(p_1, \ldots, p_{i-1}, p_i')$ is the corresponding point in $F[i]$.*

Proposition 2.6. *Let B be a R-scheme and $(\Sigma_{n+1,i}, \pi_{n+1,i}, \sigma_{n+1,i})_{i=1}^n$ be the universal family over the moduli scheme $F[n]$. Using Notation 2.5 above, the morphisms $\pi_{n+1,*}$ and $\sigma_{n+1,*}$ map B-points as follows:*

(a)
$$\sigma_{n+1,i} : F[n](B) \to \Sigma_{n+1,i}(B)$$
$$(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n; p_i);$$

(b)
$$\pi_{n+1,i} : \Sigma_{n+1,i}(B) \to \Sigma_{n+1,i-1}(B)$$
$$(p_1, \ldots, p_n; p_i') \mapsto (p_1, \ldots, p_n; \overline{p_i'}),$$

where $\overline{p_i'} : B \to \Sigma_{B,i-1}$ is the image of p_i' under the projection map $\Sigma_{B,i} \to \Sigma_{B,i-1}$.

Proof. (a) Let $f : F[n] \to F[i]$ be the natural projection map, which maps B-points $(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_i)$. We obtain the following diagram:

$$\begin{tikzcd}
F[n] \arrow[dr, "\sigma_{n+1,i}"] \arrow[drr, bend left, "\triangle\circ f"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \Sigma_{n+1,i} = F[n] \times_{F[i-1]} F[i] \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_1"] & \Sigma_{i+1,i} = F[i] \times_{F[i-1]} F[i] \arrow[d, shift left, "\mathrm{pr}_1"] \\
& F[n] \arrow[r, "f"] & F[i], \arrow[u, shift left, "\triangle"]
\end{tikzcd}$$

where the section $\sigma_{n+1,i} : F[n] \to \Sigma_{n+1,i}$ is defined to be the unique morphism making the two triangles of the diagram commute. It is clear from the figure that $\sigma_{n+1,i}$ maps a B-point of $F[n]$, denoted by $(p_1, \ldots, p_n)$, to $(p_1, \ldots, p_n; p_i) \in \Sigma_{n+1,i}(B)$.

(b) We start by showing that, for all n, the morphism

$$\pi_{n+1,n+1} : \Sigma_{n+1,n+1} \to \Sigma_{n+1,n}$$

maps B-points as follows:

$$(p_1, \ldots, p_n, p_{n+1}) \mapsto (p_1, \ldots, p_n; \overline{p_{n+1}}),$$

where $\overline{p_{n+1}}$ is the image of p_{n+1} under the projection map $\Sigma_{B,n+1} \to \Sigma_{B,n}$.

Since $\Sigma_{n+1,n} = F[n] \times_{F[n-1]} F[n]$, it is enough to show that:

$$\begin{aligned}
\mathrm{pr}_1 \circ \pi_{n+1,n+1} &: (p_1, \ldots, p_{n+1}) \to (p_1, \ldots, p_{n-1}, p_n), \\
\mathrm{pr}_2 \circ \pi_{n+1,n+1} &: (p_1, \ldots, p_{n+1}) \to (p_1, \ldots, p_{n-1}, \overline{p_{n+1}}).
\end{aligned}$$

We already know that $\mathrm{pr}_1 \circ \pi_{n+1,n+1} = \Pi_{n+1} : \Sigma_{n+1,n+1} \to F[n]$ corresponds to the forgetful functor $\mathcal{F}[n+1] \to \mathcal{F}[n]$, as a result of Theorem 2.3.

To show the second identity, let $f_{n+1} : B \to F[n+1]$ be a B-point of $F[n+1]$, and let $f_n : B \to F[n]$ be the 'truncated' point $B \xrightarrow{f_{n+1}} F[n+1] \xrightarrow{\Pi_{n+1}} F[n]$. The maps f_{n+1} and f_n produce the following diagram, where p_{n+1} is the pullback of f_{n+1}, and $p_1, \ldots, p_n$ are the pullbacks of $\sigma_{n+1,1}, \ldots, \sigma_{n+1,n}$, respectively:

$$
\begin{tikzcd}
B \arrow[dr, "p_{n+1}"] \arrow[ddr, bend left, "p_n"] \arrow[ddddr, bend left, "p_1"] \arrow[dddddr, bend right, "\mathrm{id}"'] \arrow[drr, bend left, "f_{n+1}"] & & & \\
& \Sigma_{B,n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n+1,n+1} \cong F[n+1] \arrow[d, "\pi_{n+1,n+1}"] & \\
& \Sigma_{B,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n+1,n} \arrow[r, "\mathrm{pr}_2"] \arrow[d] & \Sigma_{n,n} \cong F[n] \arrow[d] \\
& \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
& \Sigma_{B,1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{n+1,1} \arrow[r] \arrow[d] & \Sigma_{n,1} \arrow[d] \\
& B \arrow[r, "f_n"] & F[n] \arrow[r] \arrow[u, bend right, "\sigma_{n+1,1}"'] \arrow[uuu, bend right, "\sigma_{n+1,n}"'] & F[n-1].
\end{tikzcd}
$$

Notice the following maps are equivalent:

$$
\begin{aligned}
&[B \xrightarrow{f_{n+1}} \Sigma_{n+1,n+1} \xrightarrow{\pi_{n+1,n+1}} \Sigma_{n+1,n} \xrightarrow{\mathrm{pr}_2} \Sigma_{n,n}] = \\
&\qquad = [B \xrightarrow{p_{n+1}} \Sigma_{B,n+1} \to \Sigma_{n+1,n+1} \to \Sigma_{n+1,n} \to \Sigma_{n,n}] \\
&\qquad = [B \xrightarrow{p_{n+1}} \Sigma_{B,n+1} \to \Sigma_{n} \to \Sigma_{n+1,n} \to \Sigma_{n,n}] \\
&\qquad = [B \xrightarrow{\overline{p_{n+1}}} \Sigma_{B,n} \to \Sigma_{n+1,n} \to \Sigma_{n,n}].
\end{aligned}
$$

By the equivalence above, it follows that $\mathrm{pr}_2 \circ \pi_{n+1,n+1}$ maps a B-point $(p_1, \ldots, p_{n+1})$ in $\Sigma_{n+1,n+1}$ to $(p_1, \ldots, \overline{p_{n+1}})$ in $\Sigma_{n,n}$, and this concludes the proof of our initial statement.

The behavior of the more general map $\pi_{n+1,i}$ becomes clear from the following cartesian square:

$$
\begin{tikzcd}
\Sigma_{n+1,i} \cong F[n] \times_{F[i-1]} F[i] \arrow[r] \arrow[d, "\pi_{n+1,i}"] & \Sigma_{i,i} \cong F[i] \arrow[d, "\pi_{i,i}"] \\
\Sigma_{n,i-1} \cong F[n] \times_{F[i-2]} F[i-1] \arrow[r] & \Sigma_{i,i-1} \cong F[i-1] \times_{F[i-2]} F[i-1]
\end{tikzcd}
$$

We know that the top horizontal map $\Sigma_{n,i} \to \Sigma_{i,i}$ is the projection onto the second factor; the morphism $\Sigma_{n,i-1} \to \Sigma_{i,i-1}$ acts as the forgetful map $F[n] \to F[i-1]$ onto the first factor, and as the identity on the second factor. We know the behavior of $\pi_{i,i}$ from the previous paragraph, hence we conclude that $\pi_{n+1,i}$ maps B-points $(p_1, \ldots, p_n; p'_i) \mapsto (p_1, \ldots, p_n; \overline{p_i})$, where $\overline{p'_i}$ is the projection of p'_i under the map $\Sigma'_{B,i} \to \Sigma_{B,i-1}$.

$\square$

Notation 2.7. *As a result of Proposition 2.6, we change notation and denote the section $\sigma_{n+1,i}$ as $\triangle_{i,n+1}$. We will use this notation throughout the rest of the paper.*

Chapter 3

Divisors of the moduli scheme

Definition 3.1. *The moduli scheme $F[n]$ comes equipped with divisors $D_{i,j}^{(n)}, \forall 1 \leqslant i < j \leqslant n$, which arise naturally from the construction outlined in the previous section. As we have seen already, we can construct $F[n] = \Sigma_{n,n}$ as a series of blowups of the base variety $\Sigma_{n,1} = F[n-1] \times S_1$. We start by defining the divisors $D_{1,n}^{(n)}, \ldots, D_{n-1,n}^{(n)}$. To do so, recall that, for all $1 \leqslant i \leqslant n-1$, $\Sigma_{n,i+1}$ is obtained by blowing up of the previous variety $\Sigma_{n,i}$ along the locus $\triangle_{i,n}(F[n-1])$. We define $D_{i,n}^{(n)}$ on $\Sigma_{n,i+1}$ to be the exceptional divisor of this blowup:*

$$D_{i,n}^{(n)} \subset Bl_{\triangle_{i,n}} \Sigma_{n,i} = \Sigma_{n,i+1} \longrightarrow \begin{tikzcd} \Sigma_{n,i} \arrow[r] & F[n-1] \arrow[l, bend left, "\triangle_{i,n}"] \end{tikzcd}.$$

By abuse of notation, the divisor $D_{i,n}^{(n)}$ on $F[n] = \Sigma_{n,n}$ is the strict transform of this exceptional divisor coming from $\Sigma_{n,i+1}$ in the tower of blowups $\Sigma_{n,n} \to \Sigma_{n,n-1} \to \cdots \to \Sigma_{n,i+1}$. We define the other divisors inductively as follows: assume we have defined $D_{i,j}^{(n-1)}$ on $F[n-1]$, where $1 \leqslant i < j \leqslant n-1$. Then $D_{i,j}^{(n)} = D_{i,j}^{(n-1)} \times S_1$ is a divisor on $\Sigma_{n,1} = F[n-1] \times S_1$. By abuse of notation, $D_{i,j}^{(n)} \subset F[n] = \Sigma_{n,n}$ is defined as the strict transform of this divisor in the tower of blowups $\Sigma_{n,n} \to \Sigma_{n,n-1} \to \cdots \to \Sigma_{n,1}$.

Proposition 3.2. *For every $n \geqslant 1$, the natural projection map $F[n] \to S_1^n$ is a composition of blowups. Under this map, every divisor $D_{i,j}^{(n)}$ is mapped surjectively onto the diagonal $\triangle_{i,j}$.*

Proof. In the previous section, we constructed a tower of blowups:

$$F[n] = \Sigma_{n,n} \xrightarrow{\pi_{n,n}} \Sigma_{n,n-1} \xrightarrow{\pi_{n,n-1}} \cdots \to \Sigma_{n,2} \xrightarrow{\pi_{n,2}} \Sigma_{n,1} = F[n-1] \times S_1.$$

Inductively, using that $F[0] \cong \operatorname{Spec} R$, we obtain a morphism $F[n] \to S_1^n$ that decomposes as a series of blowups. Given the behavior of the maps $\pi_{n,i}$, outlined in Proposition 2.6, this map coincides with the natural projection map.

Next, we show inductively on n that this projection morphism $F[n] \to S_1^n$ maps surjectively any divisor $D_{i,j}^{(n)}$ onto the diagonal $\triangle_{i,j}$. Recall that $F[2] = \mathrm{Bl}_{\triangle_{1,2}}(S_1 \times S_1)$ and $D_{1,2}^{(2)}$ is the exceptional divisor of this blowup, so indeed the projection morphism $F[2] \to S_1 \times S_1$ maps $D_{1,2}^{(2)} \to \triangle_{1,2}$. In the induction step, consider a divisor $D_{i,j}^{(n)}$ of the moduli space $F[n]$. If $j < n$, then, by construction, the morphism $\Sigma_{n,n} \to \Sigma_{n,1} = F[n-1] \times S_1$ maps $D_{i,j}^{(n)} \mapsto D_{i,j}^{(n-1)} \times S_1$, so the conclusion follows. If $j = n$, the divisor $D_{i,n}^{(n)}$ on $F[n] = \Sigma_{n,n}$ is the strict transform of the exceptional divisor:

$$\begin{tikzcd} D_{i,n}^{(n)} \subset \mathrm{Bl}_{\triangle_{i,n}}\Sigma_{n,i} = \Sigma_{n,i+1} \arrow[r] & \Sigma_{n,i} \arrow[r] & F[n-1]. \arrow[l, bend left, "\triangle_{i,n}"] \end{tikzcd}$$

Given the behavior of the maps $\pi_{n,i}$, outlined in Proposition 2.6, the projection morphism $F[n] = \Sigma_{n,n} \to \Sigma_{n,i} \to S_1^n$ maps $D_{i,n}^{(n)} \subset \Sigma_{n,n}$ onto the blowup locus $\triangle_{i,n}\ (F[n-1]) \subset \Sigma_{n,i}$, which is then mapped onto the diagonal $\triangle_{i,n} \subset S_1^n$, and the proof is complete. $\square$

Proposition 3.3. *(a) Let $1 \leqslant i < j < n$, $1 \leqslant k \leqslant n$. The divisor $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ is the inverse image of $D_{i,j}^{(n-1)} \subset F[n-1]$ under the projection map $\Sigma_{n,k} \to F[n-1]$.*

(b) Let $1 \leqslant i < k \leqslant n$. The divisor $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ is the inverse image of the exceptional divisor $D_{i,n}^{(n)} \subset \Sigma_{n,i+1}$ under the projection map $\Sigma_{n,k} \to \Sigma_{n,i+1}$.

Proof. (a) When $k = 1$, $\Sigma_{n,1} = F[n-1] \times S_1$. Recall that the divisor $D_{i,j}^{(n)} \subset \Sigma_{n,1}$ is defined to be $D_{i,j}^{(n-1)} \times S_1$, therefore it is indeed the inverse image of $D_{i,j}^{(n-1)}$ under the map

$$\pi_{n,1} = \mathrm{pr}_1 : \Sigma_{n,1} \to F[n-1].$$

When $k > 1$, recall we have the following figure:

$$\begin{tikzcd} \Sigma_{n,k+1} = \mathrm{Bl}_{\triangle_{k,n}} \Sigma_{n,k} \arrow[r, "\pi_{n,k+1}"] & \Sigma_{n,k} \arrow[r] & F[n-1]. \arrow[l, bend left, "\triangle_{k,n}"] \end{tikzcd}$$

The divisor $D_{i,j}^{(n)} \subset \Sigma_{n,k+1}$ was defined to be the strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. We claim that the strict transform coincides with the total transform. By construction, the moduli space $F[n-1]$ is irreducible and the map $\triangle_{k,n}$ is a regular embedding, hence the exceptional divisor $D_{k,n}^{(n)}$ of the blowup $\Sigma_{n,k+1} \to \Sigma_{n,k}$ is also irreducible. In other words, the only way statement (a) could fail is if the exceptional divisor $D_{k,n}^{(n)} \subset \Sigma_{n,k+1}$ is completely contained in the inverse image $\pi_{n,k+1}^{-1}(D_{i,j}^{(n)})$, which could happen only if the blowup locus $\triangle_{k,n}(F[n-1]) \subset \Sigma_{n,k}$ was completely contained in $D_{i,j}^{(n)}$. However, the projection $\Sigma_{n,k} \to F[n-1]$ maps $D_{i,j}^{(n)} \twoheadrightarrow D_{i,j}^{(n-1)}$ and $\triangle_{k,n}(F[n-1]) \twoheadrightarrow F[n-1]$, so the induction step is complete and claim (a) is correct.

(b) When $k > i+1$, we have a similar picture as before:

$$\begin{tikzcd} \Sigma_{n,k+1} = \mathrm{Bl}_{\triangle_{k,n}} \Sigma_{n,k} \arrow[r, "\pi_{n,k+1}"] & \Sigma_{n,k} \arrow[r] & F[n-1]. \arrow[l, bend left, "\triangle_{k,n}"] \end{tikzcd}$$

The divisor $D_{i,n}^{(n)} \subset \Sigma_{n,k+1}$ is defined to be the strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. Using the same argument as in (a), it is enough to prove that the blowup locus $\triangle_{k,n}(F[n-1])$ is not fully contained in $D_{i,n}^{(n)}$. We prove this inductively for $k = i+1, \ldots, n$.

Consider the base case $k = i+1$. A B-point of $\triangle_{i+1,n}(F[n-1]) \subset \Sigma_{n,i+1}$ can be summarized as $(p_1, \ldots, p_{n-1}; p_{i+1})$, and a B-point of $D_{i,n}^{(n)} \subset \Sigma_{n,i+1}$ can be summarized as $(p_1, \ldots, p_{n-1}; p'_{i+1})$, where $p'_{i+1} \in \Sigma_{i+1}$ maps to p_i under the projection map $\Sigma_{B,i+1} \to \Sigma_{B,i}$. Given these descriptions, clearly $\triangle_{i+1,n}(F[n-1]) \not\subset D_{i,n}^{(n)}$. For the induction step, recall that a B-point of $\triangle_{k,n}(F[n-1])$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k)$. By induction, we know that $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ is the inverse image of the exceptional divisor of the blowup $\Sigma_{n,i+1} \to \Sigma_{n,i}$, hence a B-point of $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p'_k)$, where $p'_k \in \Sigma_{B,k}$ maps to p_i under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$. Thus, any point in the blowup locus $\triangle_{k,n}(F[n-1])$ of the form $(p_1, \ldots, p_{n-1}; p_k)$ where p_k does not map to p_i under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$ is not a point of $D_{i,n}^{(n)}$, so the induction step is finished and the proof is complete. $\square$

Proposition 3.4. *Let $1 \leqslant i < j \leqslant n$ and $D_{i,j}^{(n)}$ be a divisor of $F[n]$ as defined above. A B-point of $F[n]$, denoted by a tuple $(p_1, p_2, \ldots, p_n)$ consistent with Notation 2.5, is a point of $D_{i,j}^{(n)}$ if and only if the projection $\Sigma_{B,j} \to \Sigma_{B,i}$ maps the point p_j to the point p_i.*

Proof. Let $D_{i,j}^{(n)}$ be a divisor of $F[n]$, as constructed above. If $j < n$, then by Proposition 3.3, $D_{i,j}^{(n)}$ is the inverse image of $D_{i,j}^{(n-1)}$ under the forgetful map $F[n] \to F[n-1]$. If the claim holds true for $D_{i,j}^{(n-1)} \subset F[n-1]$, then it also holds true for $D_{i,j}^{(n)} \subset F[n]$. In other words, we can assume without loss of generality that $j = n$.

Let $j = n$; by Proposition 3.3, the divisor $D_{i,n}^{(n)} \subset F[n] = \Sigma_{n,n}$ is the inverse image of the blowup locus $\triangle_{i,n}(F[n-1]) \subset \Sigma_{n,i}$ under the projection map $\Sigma_{n,n} \to \Sigma_{n,i}$:

$$
\begin{tikzcd}
D_{i,n}^{(n)} = \mathrm{pr}^{-1}(\triangle_{i,n}\,(F[n-1]) \subset \Sigma_{n,n} \arrow[r, "\mathrm{pr}"] & \Sigma_{n,i} \arrow[r] & F[n-1]. \arrow[l, bend left, "\triangle_{i,n}"]
\end{tikzcd}
$$

Recall that, by Proposition 2.6, the projection morphism $\Sigma_{n,n} \to \Sigma_{n,i}$ maps points as follows:

$$(p_1, \dots, p_{n-1}, p_n) \mapsto (p_1, \dots, p_{n-1}; \overline{p_n}),$$

where $\overline{p_n} \in \Sigma_{B,i}$ is the projection of p_n under the map $\Sigma_{B,n} \to \Sigma_{B,i}$. On the other hand, the section $\triangle_{i,n}\colon F[n-1] \to \Sigma_{n,i}$ maps points as follows:

$$(p_1, \dots, p_{n-1}) \mapsto (p_1, \dots, p_{n-1}; p_i).$$

In conclusion, a B-point $F[n]$ is a point of $D_{i,n}^{(n)}$ if and only if $p_n \to p_i$ under the map $\Sigma_{B,n} \to \Sigma_{B,i}$.

$\square$

Proposition 3.5. *Let $1 \leqslant i < j \leqslant n$. The divisor $D_{i,j}^{(n)} \subset F[n]$ is smooth over Spec R.*

Proof. We will prove this statement inductively over n. When $n = 2$, we have only one divisor of this form, namely $D_{1,2}^{(2)}$, which is the exceptional divisor of the blowup of $S_1 \times S_1$ along the diagonal, so it is clearly smooth. Inductively, we need to analyze two cases: either $j < n$ or $j = n$.

Assume $j < n$. We will show inductively over k that $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ is smooth. For the base case $k = 1$, recall that the divisor $D_{i,j}^{(n)} \subset \Sigma_{n,1} = F[n-1] \times S_1$ is defined to be $D_{i,j}^{(n-1)} \times S_1$. By the induction hypothesis, $D_{i,j}^{(n-1)} \subset F[n-1]$ is smooth, so the base case is true. Inductively, assume $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ is smooth. Recall that $\Sigma_{n,k+1} = \mathrm{Bl}_{\triangle_{k,n}} \Sigma_{n,k}$, and $D^{(n)}_{i,j} \subset \Sigma_{n,k+1}$ is defined to be strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. To show smoothness is preserved, it is enough to prove that the intersection of the divisor with the blowup locus $D^{(n)}_{i,j} \cap \triangle_{k,n} (F[n-1])$ inside $\Sigma_{n,k}$ is itself smooth. We prove the following, which, by the induction hypothesis, concludes this step of the proof:

$$D^{(n)}_{i,j} \cap \triangle_{k,n} (F[n-1]) \cong D^{(n-1)}_{i,j}.$$

As a result of Proposition 2.6 and Proposition 3.3, a B-point of $D^{(n)}_{i,j} \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p'_k)$, where $p_j \to p_i$ under the projection map $\Sigma_{B,j} \to \Sigma_{B,i}$. By Proposition 2.6, a B-point of $\triangle_{k,n} (F[n-1]) \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k)$. In conclusion, a B-point of $D^{(n)}_{i,j} \cap \triangle_{k,n} (F[n-1])$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k)$, where $p_j \to p_i$ under the projection map $\Sigma_{B,j} \to \Sigma_{B,i}$. We see a clear correspondence between the B-points of $D^{(n)}_{i,j} \cap \triangle_{k,n} (F[n-1])$ and those of $D^{(n-1)}_{i,j}$, for any $B \in Sch(R)$. This means that the functors of points of the two schemes are isomorphic. By the Yoneda Lemma, the two schemes in question are isomorphic and the induction step is complete.

Now assume that $j = n$. We will show inductively that $D^{(n)}_{i,n} \subset \Sigma_{n,k}$ is smooth, for all $i + 1 \leqslant k \leqslant n$. For the base case $k = i + 1$, recall that the divisor $D^{(n)}_{i,n} \subset \Sigma_{n,i+1}$ is the exceptional divisor of the blowup of $\Sigma_{n,i}$ along the locus $\triangle_{i,n} (F[n-1])$. By Theorem 2.3, $\Sigma_{n,i}$ is smooth and the blowup locus $\triangle_{i,n} (F[n-1])$ is regularly embedded, hence the exceptional divisor $D^{(n)}_{i,n} \subset \Sigma_{n,i+1}$ is smooth. To prove the induction step is enough to show, as before, that the intersection $D^{(n)}_{i,j} \cap \triangle_{k,n} (F[n-1]) \subset \Sigma_{n,k}$ with the blowup locus is itself smooth. We prove the following, which, by the induction hypothesis, concludes this step of the proof:

$$D_{i,n}^{(n)} \cap \triangle_{k,n}(F[n-1]) \cong D_{i,k}^{(n-1)}.$$

As a result of Proposition 2.6 and Proposition 3.3, a B-point of $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p'_k)$, where $p'_k \to p_i$ under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$. By Proposition 2.6, a B-point of $\triangle_{k,n}(F[n-1]) \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k)$. In conclusion, a B-point of $D_{i,n}^{(n)} \cap \triangle_{k,n}(F[n-1])$ can be identified as $(p_1, \ldots, p_{n-1}; p_k)$, where $p_k \to p_i$ under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$. There is a clear correspondence between the B-points of $D_{i,n}^{(n)} \cap \triangle_{k,n}(F[n-1])$ and those of $D_{i,k}^{(n-1)}$, for any $B \in Sch(R)$, which means that the functors of points of the two schemes are isomorphic. By the Yoneda Lemma, the two schemes in question are isomorphic and the induction step is complete. $\square$

Chapter 4

The Chow ring of the moduli scheme

In this chapter we assume that the underlying ring R is an algebraically closed field k. In this setup, the moduli space $F[n]$ and all the schemes in its universal family are smooth projective varieties.

We prove in the main result of this chapter that the Chow ring of the moduli space $F[n]$ is generated by the classes of the divisors $\{D_{i,j}^{(n)}\}_{1\leqslant i<j\leqslant n}$ over the Chow ring $\mathbf{A}^*(S_1^n)$. We conclude this chapter by proving certain key relations among these classes in the Chow ring $\mathbf{A}^*(F[n])$. In the next chapter, we prove these relations are sufficient to give a precise description of the Chow ring $\mathbf{A}^*(F[n])$ in the special case where S_1 is a rational surface and the base field is the complex numbers $\mathbb{C}$.

We start with a theorem by Keel, which is the key in our proof:

Theorem 4.1. *Let Y be a variety and let $i : X \hookrightarrow Y$ be a regularly embedded subvariety. Let $\tilde{Y}$ be the blowup of Y along X. Suppose the map of bivariate rings $i^* : \mathbf{A}^*(Y) \to \mathbf{A}^*(X)$ is surjective. Then:*

$$\mathbf{A}^*(\tilde{Y}) \cong \frac{\mathbf{A}^*(Y)[T]}{(P(T), T \cdot ker(i^*))},$$

where $P(T) \in \mathbf{A}^*(Y)[T]$ *is any polynomial whose constant term is* $[X]$ *and whose restriction to* $\mathbf{A}^*(X)$ *is the Chern polynomial of the normal bundle* $N = N_X Y$*, i.e.:*

$$i^* P(T) = T^d + T^{d-1} c_1(N) + \cdots + c_{d-1}(N) T + c_d(N),$$

where $d = codim(X, Y)$*. This isomorphism is induced by*

$$\pi^* : \mathbf{A}^*(Y)[T] \to \mathbf{A}^*(\tilde{Y})$$

and by sending $-T$ *to the class of the exceptional divisor.*

Proof. See [23], Appendix, Theorem 1. □

Remark 4.2. *The moduli space* $F[n]$ *is a smooth projective variety, for any* $n \geqslant 1$*, hence its bivariate ring* $\mathbf{A}^*(F[n])$ *is isomorphic to its Chow ring* $\mathbf{CH}^*(F[n])$ *(see [16], Ch. 17).*

Corollary 4.3. *Let* $1 \leqslant i < n$*. Let* $\Sigma_{n,i+1}$ *and* $\Sigma_{n,i}$ *be two of the varieties in the universal family over the moduli space* $F[n-1]$*. The Chow ring of* $\Sigma_{n,i+1}$ *has the following description:*

$$\mathbf{A}^*(\Sigma_{n,i+1}) \cong \frac{\mathbf{A}^*(\Sigma_{n,i})[D_{i,n}^{(n)}]}{\langle P_{i,n}(-D_{i,n}^{(n)}), D_{i,n}^{(n)} \cdot \ker(\triangle_{i,n}^*) \rangle},$$

where $P_{i,n}$ *is a quadratic polynomial with coefficients in* $\mathbf{A}^*(\Sigma_{n,i})$*.*

Proof. In the universal family over the moduli space $F[n-1]$, $\Sigma_{n,i+1}$ is the blowup of $\Sigma_{n,i}$ along the locus $\triangle_{i,n}\ (F[n-1])$, and the corresponding exceptional divisor is

$D_{i,n}^{(n)}$:

$$D_{i,n}^{(n)} \subset \mathrm{Bl}_{\triangle_{i,n}}\Sigma_{n,k} = \Sigma_{n,i+1} \longrightarrow \begin{tikzcd} \Sigma_{n,i} \arrow[r] & F[n-1] \arrow[l, bend left, "\triangle_{i,n}"] \end{tikzcd}.$$

By Theorem 2.3, $\Sigma_{n,i}$ is a variety and $\triangle_{i,n}\colon F[n-1] \hookrightarrow \Sigma_{n,i}$ is a regularly embedded subvariety. Moreover, since $\triangle_{i,n}$ is a section of the map $\Sigma_{n,i} \to F[n-1]$, the corresponding map on Chow rings $\triangle_{i,n}^*\colon \mathbf{A}^*(\Sigma_{n,i}) \to \mathbf{A}^*(F[n-1])$ is surjective. The conclusion follows immediately as an application of Theorem 4.1, $\square$

Remark 4.4. *By Proposition 3.3, the induced map on Chow rings:* $\pi_{n,i+1}^* : \mathbf{A}^*(\Sigma_{n,i}) \to \mathbf{A}^*(\Sigma_{n,i+1})$ *sends the class of any divisor* $D_{j,k}^{(n)}$ *to the class of the divisor with the same name inside* $\mathbf{A}^*(\Sigma_{n,i})$. *This means that the notation remains consistent when we state the next results.*

To generalize Corollary 4.5, we first need to give an alternative way of defining the divisors $D_{i,j}^{(n)} \subset F[n]$. Recall that the natural projection map $F[n] \to S_1^n$ decomposes as a series of blowups; more specifically, we encounter the following situation:

$$F[n] = \Sigma_{n,n} \to \ldots \to \Sigma_{j,i+1} \times S_1^{n-j} \to \begin{tikzcd} \Sigma_{j,i} \times S_1^{n-j} \arrow[r] & \ldots \arrow[r] & F[j] \times S_1^{n-j} \arrow[ll, bend left, "\triangle_{i,j} \times \mathrm{id}"] \end{tikzcd} \to \cdots \to S_1^n.$$

The scheme $\Sigma_{j,i+1} \times (S_1)^{n-j}$ is the blowup of $\Sigma_{j,i} \times (S_1)^{n-j}$ along the locus $\triangle_{i,j} \times \mathrm{id}$. We can define $D_{i,j}^{(n)} \subset \Sigma_{j,i+1} \times (S_1)^{n-j}$ to be the exceptional divisor of this blowup. By abuse of notation, we define $D_{i,j}^{(n)} \subset F[n] = \Sigma_{n,n}$ to be the strict transform of this exceptional divisor in the tower of blowups. Following an identical argument as the one in Proposition 3.3, more is true: $D_{i,j}^{(n)} \subset F[n]$ is actually the full inverse image of this divisor in the tower of blowups. Therefore, by Theorem 4.1, we conclude that:

$$\mathbf{A}^*(\Sigma_{j,i+1} \times S_1^{n-j}) \cong \frac{\mathbf{A}^*(\Sigma_{j,i} \times S_1^{n-j})[D_{i,j}^{(n)}]}{\langle P_{i,j}(-D_{i,j}^{(n)}), D_{i,j}^{(n)} \cdot \ker(\triangle_{i,j}^*)\rangle},$$

where $P_{i,j}$ is a quadratic polynomial with coefficients in $\mathbf{A}^*(\Sigma_{j,i} \times (S_1)^{n-j})$.

Applying this procedure step by step from S_1^n all the way up to $\Sigma_{n,n} = F[n]$, we immediately obtain the Chow ring of the moduli space $F[n]$ for any $n \geqslant 1$:

Theorem 4.5. *With notation as in Corollary 4.3, the Chow ring of the moduli space $F[S_1, n]$ is as follows:*

$$\mathbf{A}^*(F[n]) \cong \frac{\mathbf{A}^*(S_1^n)[D_{i,j}^{(n)}]_{1 \leqslant i < j \leqslant n}}{\langle D_{i,j}^{(n)} \cdot ker(\triangle_{i,j}^*), P_{i,j}(-D_{i,j}^{(n)})_{1 \leqslant i < j \leqslant n} \rangle}.$$

Proposition 4.6. *Let d be any divisor class of S_1 and $d_i^* \in \mathbf{A}^*(F[n])$ the image of d under the composed morphism $F[n] \to S_1^n \xrightarrow{pr_i} S_1$. The following relations hold in the Chow ring $\mathbf{A}^*(F[n])$:*

(i) $D_{i,j}^{(n)} d_i^* = D_{i,j}^{(n)} d_j^*, \forall 1 \leqslant i < j \leqslant n$;

(ii) $D_{i,j}^{(n)} D_{j,k}^{(n)} = D_{i,k}^{(n)} D_{j,k}^{(n)}, \forall 1 \leqslant i < j < k \leqslant n$.

Proof. (i) The first identity follows immediately from the fact that the natural projection $F[n] \to S_1^n$ maps $D_{i,j}^{(n)}$ surjectively onto the diagonal $\triangle_{ij}$, for all $1 \leqslant i < j \leqslant n$.

(ii) The second identity requires some work. First, we show that it suffices to prove this relation in the case where $k = n$. If $i < j < k < n$, we can assume inductively that a similar relation holds in the Chow ring of $F[S_1, n-1]$:

$$D_{i,j}^{(n-1)} D_{j,k}^{(n-1)} = D_{i,k}^{(n-1)} D_{j,k}^{(n-1)}.$$

Now, recall that inside the variety $\Sigma_{n,1} = F[n-1] \times S_1$, the divisors $D_{i,j}^{(n)}, D_{i,k}^{(n)}, D_{j,k}^{(n)}$ are defined to be $D_{i,j}^{(n-1)} \times S_1, D_{i,k}^{(n-1)} \times S_1, D_{j,k}^{(n-1)} \times S_1$, respectively. Hence, it is clear that relation (ii) holds in the Chow ring $\mathbf{A}^*(\Sigma_{n,1})$. Furthermore, by Remark 4.4, the projection morphism $\Sigma_{n,n} \to \Sigma_{n,1}$ induces a map on Chow rings $\mathbf{A}^*(\Sigma_{n,1}) \to \mathbf{A}^*(\Sigma_{n,n})$

that sends the classes of the divisors $D_{i,j}^{(n)}, D_{i,k}^{(n)}, D_{j,k}^{(n)}$ to the classes of the divisors with the same name inside $\Sigma_{n,1}$, which means relation (ii) also holds in the Chow ring $\mathbf{A}^*(F[S_1, n])$.

We are left to show that the relation (ii) holds true when $k = n$. Recall that $D_{j,n}^{(n)}$ is obtained as the exceptional divisor of $\Sigma_{n,j}$ blown up along $\triangle_{j,n}(F[n-1])$:

$$\begin{tikzcd} D_{j,n}^{(n)} \subset \mathrm{Bl}_{\triangle_{j,n}}\Sigma_{n,j} = \Sigma_{n,j+1} \arrow[r] & \Sigma_{n,j} \arrow[r] & F[n-1]. \arrow[l, bend left, "\triangle_{j,n}"] \end{tikzcd}$$

By Remark 4.4, it suffices to show that relation (ii) holds in the Chow ring of $\Sigma_{n,j+1}$. As a result of Theorem 4.1, the Chow ring of $\Sigma_{n,j+1}$ is as follows:

$$\mathbf{A}^*(\Sigma_{n,j+1}) \cong \frac{\mathbf{A}^*(\Sigma_{n,j})[D_{j,n}^{(n)}]}{\langle P_{j,n}(-D_{j,n}^{(n)}), D_{j,n}^{(n)} \cdot \ker(\triangle_{j,n}^*)\rangle},$$

so we need to show that:

$$D_{i,j}^{(n)} - D_{i,n}^{(n)} \in \ker(\triangle_{j,n}^*\colon \mathbf{A}^*(\Sigma_{n,j}) \to \mathbf{A}^*(F[n-1])).$$

To prove the relation above, recall that $\Sigma_{n,j} \cong F[n-1] \times_{F[j-1]} F[j]$. By Proposition 3.3, under this isomorphism, $D_{i,j}^{(n)} \cong D_{i,j}^{(n-1)} \times_{F[j-1]} F[j]$. Next, we claim that $D_{i,n}^{(n)} \cong F[n-1] \times_{F[j-1]} D_{i,j}^{(j)}$. To see that this isomorphism holds inside $\Sigma_{n,j}$, we first need to check it on $\Sigma_{n,i+1}$. For this, consider the following diagram:

$$\begin{tikzcd} D_{i,n}^{(n)} \subset \Sigma_{n,i+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{j,i+1} \supset D_{i,j}^{(j)} \arrow[d] \\ \Sigma_{n,i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma_{j,i} \arrow[d] \\ F[n-1] \arrow[r] \arrow[u, bend left, "\triangle_{i,n}"] & F[j-1]. \arrow[u, bend right, "\triangle_{i,j}"'] \end{tikzcd}$$

The bottom square is cartesian and the section $\triangle_{i,n}$ is the pullback of $\triangle_{i,j}$ along the map $F[n-1] \to F[j-1]$, therefore the exceptional divisor $D^{(n)}_{i,n} \subset \Sigma_{n,i+1}$ is the pullback of the exceptional divisor $D^{(j)}_{i,j} \subset \Sigma_{j,i+1}$, so $D^{(n)}_{i,n} \cong F[n-1] \times_{F[j-1]} D^{(j)}_{i,j}$ inside $\Sigma_{n,i+1}$. Clearly, this relation lifts to $\Sigma_{n,j}$.

In conclusion, inside $\Sigma_{n,j} \cong F[n-1] \times_{F[j-1]} F[j]$, we have $D^{(n)}_{i,j} \cong D^{(n-1)}_{i,j} \times_{F[j-1]} F[j]$, $D^{(n)}_{i,n} \cong F[n-1] \times_{F[j-1]} D^{(j)}_{i,j}$. Additionally, we also know that $\triangle_{j,n} \colon F[n-1] \to \Sigma_{n,j} = F[n-1] \times_{F[j-1]} F[j]$ acts like a 'truncated' diagonal embedding. Putting all this information together, it becomes clear that:

$$D^{(n)}_{i,j} - D^{(n)}_{i,n} \in \ker(\triangle^*_{j,n} \colon \mathbf{A}^*(\Sigma_{n,j}) \to \mathbf{A}^*(F[n-1])),$$

and the proof is complete. $\square$

Chapter 5

The Chow ring of the moduli scheme for rational surfaces

As a special case of the theory developed above, we give a precise description of the Chow ring $\mathbf{A}^*(F[n])$ when the base surface S_1 is a smooth projective rational surface over the complex numbers (Spec R = Spec $\mathbb{C}$). The result relies on a few key ideas. First, the canonical map $\mathbf{A}^*(F[n]) \xrightarrow{\text{cl}} \mathbf{H}^{2*}(F[n])$ is an isomorphism. Second, for some prime p and any $q = p^l$, where $l \gg 0$, we can define the moduli space $F[n] \otimes \mathbb{F}_q$ over the finite field $\mathbb{F}_q$. The number of $\mathbb{F}_q$-points on $F[n] \otimes \mathbb{F}_q$ is given by a polynomial $R_n(q)$ that coincides with the Poincare polynomial of $F[n]$. We use this fact to derive precise formulas for the Betti numbers of $F[n]$; using these formulas, we show that the relations in Proposition 4.6 are enough to give a complete description of $\mathbf{A}^*(F[n])$.

Definition 5.1. *A scheme X of characteristic zero is called an HI (for Homology Isomorphism) scheme if the canonical map from the Chow groups of X to the homology groups:*

$$\mathbf{A}_*(X) \xrightarrow{cl} \mathbf{H}_{2*}(X)$$

is an isomorphism.

Proposition 5.2. *Let Y be a variety and $i : X \hookrightarrow Y$ be a regularly embedded subvariety. Let $\tilde{Y}$ be the blowup of Y along X. If X and Y are HI schemes, then so is $\tilde{Y}$.*

Proof. See [23], Appendix, Theorem 2. $\square$

Proposition 5.3. *Let S_1 be a smooth projective variety over an algebraically closed field k of characteristic zero. If S_1^n is an HI scheme, for all $n \geqslant 0$, then so is $F[n]$.*

Proof. We prove inductively over n something stronger: $\forall n, j \geqslant 0$, the variety $F[n] \times S_1^j$ is an HI scheme.

The base cases are easy. If $n = 0$, then $F[0] \cong \operatorname{Spec} k$, so $F[0] \times S_1^j \cong S_1^j$, which is an HI scheme by hypothesis. If $n = 1$, then $F[1] \cong S_1$, so $F[1] \times S_1^j \cong S_1^{j+1}$, which is again an HI scheme by hypothesis.

For the induction step, assume $F[m] \times S_1^j$ is an HI scheme, for all $m \leqslant n$ and $j \geqslant 0$. We want to show that $F[n+1] \times S_1^j$ is an HI scheme, for all $j \geqslant 0$. Recall that we can obtain $F[n+1]$ from $F[n] \times S_1$ as a series of blowups:

$$\begin{tikzcd}
F[n+1] = \Sigma_{n+1,n+1} \arrow[r] & \Sigma_{n+1,n} \arrow[r] & \dots \arrow[r] & \Sigma_{n+1,2} \arrow[r] & \Sigma_{n+1,1} = F[n] \times S_1 \arrow[d] \\
& & & & F[n]. \arrow[u, bend left, "\triangle_{1,n+1}"] \arrow[ul, bend left, "\triangle_{2,n+1}"] \arrow[ulll, bend left, "\triangle_{n,n+1}"]
\end{tikzcd}$$

By the induction hypothesis, we know that $F[n]$ and $F[n] \times S_1$ are HI varieties. By Theorem 2.3, the blowup locus $\triangle_{i,n+1}\colon F[n] \hookrightarrow \Sigma_{n+1,i}$ is a regular embedding, for all $1 \leqslant i \leqslant n$. Applying Proposition 5.2 repeatedly, we conclude step by step that $\Sigma_{n+1,2}, \Sigma_{n+1,3}, \dots, \Sigma_{n+1,n+1} \cong F[n+1]$ are all HI varieties. More generally, the fact that $F[n+1] \times S_1^j$ is an HI variety, for any $j \geqslant 0$, follows immediately if we look instead at the following tower of morphisms:

$$\begin{tikzcd}
\Sigma_{n+1,n+1} \times S_1^j \arrow[r] & \Sigma_{n+1,n} \times S_1^j \arrow[r] & \dots \arrow[r] & \Sigma_{n+1,2} \times S_1^j \arrow[r] & \Sigma_{n+1,1} \times S_1^j = F[n] \times S_1 \times S_1^j \arrow[d] \\
 & & & & F[n] \times S_1^j . \arrow[u, bend left] \arrow[ul, bend left] \arrow[ulll, bend left]
\end{tikzcd}$$

□

Proposition 5.4. *Let $k \geqslant 1$ and $S_1, \dots, S_k$ be complex smooth projective rational surfaces. Then $\prod_{i=1}^{k} S_i$ is an HI variety.*

Proof. We prove this statement inductively over k. As the base case, we show that a complex smooth projective rational surface S is an HI variety. By the Enriques-Kodaira classification of complex surfaces (see Thm. 7.9), there exist smooth projective surfaces $S_{n-1}, \dots, S_1, S_0$, and morphisms:

$$S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0,$$

such that each $S_{i+i} \to S_i$ is the contractions of a (-1)-curve and S_0 is a minimal rational surface (either $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_a$, for $a = 0$ or $a \geqslant 2$).

Both $\mathbb{P}^2$ and $\mathbb{F}_a$ have algebraic cell decompositions, which means they are HI varieties. Knowing that S_0 is an HI surface, we can apply Proposition 5.2 repeatedly, obtaining step by step that $S_1, S_2, \dots, S_n = S$ are HI varieties, since each of them is obtained by blowing up a smooth HI surface at a smooth point.

Inductively, assume the statement is true for all $i < k$. Let $S_1, \dots, S_k$ be complex smooth projective rational surfaces. We need to show that the product $\prod_{i=1}^{k} S_i$ is also an HI variety. As before, for each surface S_i we have a sequence of morphisms:

$$S_i = S_{i,n_i} \to S_{i,n_i-1} \to \cdots \to S_{i,1} \to S_{i,0},$$

such that each $S_{i,j+i} \to S_{i,j}$ is the contractions of a (-1)-curve, and $S_{i,0}$ is a minimal rational surface (either $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_a$). Putting these morphism together, we obtain the following sequence:

$$\prod_{i=1}^{k} S_{i,n_i} \to \cdots \to \prod_{i=1}^{k-1} S_{i,n_i} \times S_{k,0} \to \cdots \to \prod_{i=1}^{k-2} S_{i,n_i} \times S_{k-1,0} \times S_{k,0} \to \cdots \to \prod_{i=1}^{k} S_{i,0}.$$

Since each surface $S_{i,0}$ admits a cell decomposition, so does $\prod_{i=1}^{k} S_{i,0}$, which means that this product is an HI variety. The last morphism in the sequence:

$$S_{1,1} \times \prod_{i=2}^{k} S_{i,0} \to S_{1,0} \times \prod_{i=2}^{k} S_{i,0}$$

is the blowup of $\prod_{i=1}^{k} S_{i,0}$ along the locus $\{\text{pt}\} \times \prod_{i=2}^{k} S_{i,0}$, where $\text{pt} \in S_{i,0}$ is some smooth point. The base scheme is an HI variety, as noted earlier. By the induction hypothesis, the blowup locus is an HI variety. Thus, as a consequence of Proposition 5.2, the blown up variety $S_{1,1} \times \cdots \times S_{k,0}$ is also an HI scheme. It is easy to see that we can apply the same proposition in a similar fashion multiple times (from right to left) to conclude that every variety in the main sequence above is HI. The induction step is complete, and the claim is true. □

Remark 5.5. *In the proof above, we used that the Hirzebruch surface $\mathbb{F}_a$ has an algebraic cell decomposition. One way to see this is to note that $\mathbb{F}_a$ is a toric variety, and all toric varieties admit a cell decomposition.*

Corollary 5.6. *Let S_1 be a complex smooth projective rational surface and $F[S_1, n] =$*

$F[n]$ its associated moduli variety. There exists a canonical isomorphism:

$$\mathbf{A}^*(F[n]) \xrightarrow{\cong} \mathbf{H}^{2*}(F[n]).$$

Proof. This is an immediate result of Proposition 5.3 and Proposition 5.4. $\square$

Setup 5.7. *Let S be a complex surface. There exists $R \subset \mathbb{C}$ a finitely generated $\mathbb{Z}$-algebra such that S is defined over R, i.e. there exists a surface S_R over Spec R such that the following square is cartesian:*

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_R \arrow[d] \\ \mathrm{Spec}\ \mathbb{C} \arrow[r] & \mathrm{Spec}\ R. \end{tikzcd}$$

Now, for any $m \subset R$ maximal ideal, the field $\kappa(m) = R/m$ is finite, so there exists some prime p and $q = p^l$, where $l \gg 0$, such that $\kappa(m) \subseteq \mathbb{F}_q$. We obtain the following figure:

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_R \arrow[d] & S_{\kappa(m)} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{\mathbb{F}_q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{\mathbb{F}_{\overline{q}}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\ \mathrm{Spec}\ \mathbb{C} \arrow[r] & \mathrm{Spec}\ R & \mathrm{Spec}\ \kappa(m) \arrow[l] & \mathrm{Spec}\ \mathbb{F}_q \arrow[l] & \mathrm{Spec}\ \mathbb{F}_{\overline{q}}. \arrow[l] \end{tikzcd}$$

For the rest of this section, when we say a complex surface S can be defined as a surface $S_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$, it means we do a procedure as above.

Proposition 5.8. *Let S be a complex smooth projective rational surface. There exists a prime integer p and $q = p^l$, for some $l \gg 0$, such that S can be defined as a smooth projective rational surface $S_{\mathbb{F}_q}$ over Spec $\mathbb{F}_q$. For this choice of p and q, there exists a quadratic polynomial $r(t)$ with the property that, for any $a \geqslant 1$ and $q' = q^a$, the number of $\mathbb{F}_{q'}$-points on $S_{\mathbb{F}_{q'}}$ equals $r(q')$.*

Proof. Let S be a complex smooth projective rational surface. By the Enriques-Kodaira classification of surfaces, there exist complex smooth projective surfaces $S_{n-1}, \ldots, S_1, S_0$, and a sequence of morphisms:

$$S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0 \to \operatorname{Spec} \mathbb{C},$$

such that each $S_{i+i} \to S_i$ is the contractions of a (-1)-curve and S_0 is a minimal rational surface (either $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_a$, for $a = 0$ or $a \geqslant 2$).

As in the Setup 5.7 above, we can find $R \subset \mathbb{C}$ a finitely generated $\mathbb{Z}$-algebra such that all the surfaces S_i are defined over Spec R. Moreover, we can pick R in such a way that, if S_0 is $\mathbb{P}^2$ or $\mathbb{F}_a$, then the surface $S_{0,R}$ is either $\mathbb{P}^2_R$ or $\mathbb{F}_{a,R}$, respectively:

$$\begin{tikzcd}
S_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_{n,R} \arrow[d] & S_{n,\kappa(m)} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{n,\mathbb{F}_q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{n,\mathbb{F}_{q'}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[d] & \vdots \arrow[l] \arrow[d] & \vdots \arrow[l] \arrow[d] & \vdots \arrow[l] \arrow[d] \\
S_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_{1,R} \arrow[d] & S_{1,\kappa(m)} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{1,\mathbb{F}_q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{1,\mathbb{F}_{q'}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\
S_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_{0,R} \arrow[d] & S_{0,\kappa(m)} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{0,\mathbb{F}_q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] & S_{0,\mathbb{F}_{q'}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} R & \operatorname{Spec} \kappa(m) \arrow[l] & \operatorname{Spec} \mathbb{F}_q \arrow[l] & \operatorname{Spec} \mathbb{F}_{q'}. \arrow[l]
\end{tikzcd}$$

If $S_{0,R}$ is $\mathbb{P}^2_R$, then the number of $\mathbb{F}_{q'}$-points on $S_{0,\mathbb{F}_{q'}}$ is $(q')^2 + q' + 1$. If $S_{0,R}$ is $\mathbb{F}_{a,R}$, then the number of $\mathbb{F}_{q'}$-points on $S_{0,\mathbb{F}_{q'}}$ is $(q')^2 + 2q' + 1$. In every blowup $S_{i+1} \to S_i$, we replace one smooth point of S_i with a copy of $\mathbb{P}^1$, so the number of $\mathbb{F}_{q'}$-points on $S_{\mathbb{F}_{q'}}$ is given by a polynomial $r(q')$ that satisfies:

$$r(q') = \begin{cases} (q')^2 + (n+1)q' + 1, & \text{if } S_0 = \mathbb{P}^2, \\ (q')^2 + (n+2)q' + 1, & \text{if } S_0 = \mathbb{F}_a. \end{cases}$$

□

Proposition 5.9. *Let S_1 be a complex smooth projective rational surface. Let p be a prime integer and $q = p^l$, as in Proposition 5.8. Let $r(t)$ be the quadratic polynomial corresponding to S_1 from Proposition 5.8. For any $a \geqslant 1$ and $q' = q^a$, the number of $\mathbb{F}_{q'}$-points on the moduli space $F[n]_{\mathbb{F}_{q'}} = F[S_{1,\mathbb{F}_{q'}}, n]$ is given by a polynomial $R_n(q)$ which has the following formula:*

$$R_n(q') = \prod_{i=0}^{n-1}(r(q') + iq').$$

Proof. We prove the statement inductively, using the fact that the number of $\mathbb{F}_{q'}$-points of $S_{1,\mathbb{F}_{q'}}$ blown up at n points equals $r(q') + nq'$.

When $n = 1$, $F[1]_{\mathbb{F}_{q'}} \cong S_{1,\mathbb{F}_{q'}}$ has exactly $r(q')$ points over $\mathbb{F}_{q'}$. Assume the statement of the theorem is true for all $k \leqslant n$. We want to show that

$$R_{n+1}(q') = R_n(q')(r(q') + nq').$$

Recall that we have a forgetful map $F[n+1]_{\mathbb{F}_{q'}} \to F[n]_{\mathbb{F}_{q'}}$. Using Notation 2.5, every point in $F[n]_{\mathbb{F}_{q'}}$ is of the form $x = (p_1, \ldots, p_n)$, and its fiber under the forgetful map is isomorphic to $S_{n+1,\mathbb{F}_{q'}}$. The surface $S_{n+1,\mathbb{F}_{q'}}$ is obtained by blowing up $S_{1,\mathbb{F}_{q'}}$ at n consecutive points, hence it has $r(q') + nq'$ rational points. Since every fiber of the forgetful map has the same number of $\mathbb{F}_{q'}$-points, namely $r(q') + nq'$, the equation above is true and the inductive step is complete. □

Definition 5.10. *Let X be a smooth, irreducible complex algebraic variety. The Poincaré polynomial of X is:*

$$P_X(q) = \sum_{i=1}^{2\dim X} b_i q^i,$$

where b_i is the rank of the i^{th} singular homology group $H^i(X,\mathbb{Z})$.

Lemma 5.11. *Let S_1 be a complex smooth projective rational surface. Let p be a prime integer and $q = p^l$, as in Proposition 5.8. The Poincaré polynomial of $F[n] = F[S_1, n]$, denoted by $P_n(q)$, coincides with the polynomial $R_n(q)$ which gives the number of $\mathbb{F}_q$-points on the moduli space $F[n]_{\mathbb{F}_q}$.*

Proof. Let S_1 be a smooth projective rational surface over Spec $\mathbb{C}$. Let $X = F[n] = F[S_1, n]$ be the moduli space corresponding to S_1 over Spec $\mathbb{C}$. We can regard S_1 as a rational surface $S_{1,\mathbb{F}_q}$ over $\mathbb{F}_q$, as in Proposition 5.8 above. Let $X_{\mathbb{F}_q}$ be the moduli space associated to $S_{1,\mathbb{F}_q}$. Since X is smooth and projective, the Betti numbers corresponding to the l-adic cohomology (where $l \neq 0 \mod p$) are independent of l, and they coincide with the Betti numbers corresponding to the ordinary (integral) cohomology of the topological space X (see [30]):

$$b_i = \text{rk } H^i(X,\mathbb{Z}) = \text{rk } H^i(X,\mathbb{Q}) = \text{rk } H^i_{\acute{e}t}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l).$$

One the other hand, we have the Grothendieck-Lefschetz Trace Formula (see [30], Thm. 13.4, p. 292), which states the following:

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2n} (-1)^i \text{tr}(\text{Frob}_q | H^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)).$$

Since X is proper, $H^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) = H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. As a consequence of the Weil conjectures (see [8]), we have:

$$\text{tr}(\text{Frob}_q | H^i(X \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_l)) = z_{i,1} + \cdots + z_{i,b_i},$$

where $z_{i,1}, \ldots, z_{i,b_i}$ are the eigenvalues of the Frobenius map. These eigenvalues satisfy

$|z_{i,j}| = q^{i/2}$, for all j. Now, if we replace the field $\mathbb{F}_q$ by $\mathbb{F}_{q'}$, where $q' = q^a$, then:

$$\operatorname{tr}(\mathrm{Frob}_{q^a}|H^i(X_{\overline{\mathbb{F}}_{q^a}}, \mathbb{Q}_l)) = z_{i,1}^a + \cdots + z_{i,b_i}^a.$$

Thus, we have that for all $a \geqslant 1$, $R_n(q^a) = \sum_{i=1}^{2n}(-1)^i \sum_{j=0}^{b_j} z_{i,j}^a$, where $|z_{i,j}| = q^{i/2}, \forall i, j$. It follows immediately that $H^{2i+1}(X_{\overline{\mathbb{F}}_{q^a}}, \mathbb{Q}_l) = 0$, for all i and $l \neq 0 \mod p$, and $z_{2i,j} = q^i$, for all i, j. With this, we conclude our statement:

$$R_n(q) = P_n(q) = \sum_{i=1}^{n} b_{2i} q^i.$$

$\square$

Let $r(q) = q^2 + kq + 1$ be the Poincaré polynomial of S_1. This means that $\mathbf{A}^*(S_1, \mathbb{Z})$ is generated in degree 1 by k classes $d_1, \ldots, d_k$, and by one class in degree 2.

Theorem 5.12. *Let S_1 be a complex smooth projective rational surface. Let $\Pi : F[n] \to S_1^n$ be the natural projection map and $pr_i : S_1^n \to S_1$ the projection onto the i-th copy, $\forall 1 \leqslant i \leqslant n$. Let $pr_i^* \circ \Pi^* : \mathbf{A}^*(S_1) \to \mathbf{A}(F[n])$ be the induced map on Chow rings and $d_{i,1}, \ldots, d_{i,k}$ be the images of the classes $d_1, \ldots, d_k$, respectively. The Chow ring of the moduli space $\mathbf{A}^*(F[n])$ is:*

$$\mathbf{A}^*(F[n]) \cong \frac{(\mathbf{A}^*(S_1))^{\otimes n}[D_{i,j}^{(n)}]_{1 \leqslant i < j \leqslant n}}{\langle D_{j,k}^{(n)}(D_{i,j}^{(n)} - D_{i,k}^{(n)}), D_{j,k}^{(n)}(d_{i,j} - d_{i,k}), P_{i,j}(-D_{i,j}^{(n)})\rangle}.$$

Proof. Let $R = \frac{(\mathbf{A}^*(S_1))^{\otimes n}[D_{i,j}^{(n)}]_{1 \leqslant i < j \leqslant n}}{\langle D_{j,k}^{(n)}(D_{i,j}^{(n)} - D_{i,k}^{(n)}), D_{j,k}^{(n)}(d_{i,j} - d_{i,k}), P_{i,j}(-D_{i,j}^{(n)})\rangle}$. We claim the following composition of morphisms is an isomorphism, after tensoring by $\mathbb{Q}$:

$$R \twoheadrightarrow \mathbf{A}^*(F[n], \mathbb{Z}) \xrightarrow{\cong} \mathbf{H}^*(F[n], \mathbb{Z}) \twoheadrightarrow \mathbf{H}^*(F[n], \mathbb{Q}).$$

By Theorem 4.5, we know that $\mathbf{A}^*(F[n])$ is generated over $\mathbf{A}^*(S_1^n)$ by the classes of the divisors $\{D_{i,j}^{(n)}\}_{1 \leqslant i < j \leqslant n}$. Moreover, by Proposition 4.6, we know that the following relations hold in the Chow ring $\mathbf{A}^*(F[n])$:

$$\begin{aligned} D_{j,k}^{(n)}(d_{i,j} - d_{i,k}) &= 0 \\ D_{j,k}^{(n)}(D_{i,j}^{(n)} - D_{i,k}^{(n)}) &= 0 \\ P_{i,j}(-D_{i,j}^{(n)}) &= 0. \end{aligned} \tag{5.1}$$

We show the relations above are sufficient by looking at the Betti numbers of the moduli space $F[n]$. By definition, the j-th Betti number of $F[n]$ gives us the number of codimension j linearly independent generators of $\mathbf{A}^*(F[n])$ as a $\mathbb{Z}$-module. Recall the formula from Proposition 5.9 describing the relation between the Poincaré polynomial of $F[n+1]$ and that of $F[n]$:

$$P_{n+1}(q) = (q^2 + (n+k)q + 1)P_n(q).$$

Let $b_{n,j}$ be the j-th Betti number of the moduli space $F[n]$. The relation above translates to an equality between the Betti numbers as follows:

$$b_{n+1,j} = b_{n,j} + (n+k)b_{n,j-1} + b_{n,j-2}.$$

We give the following interpretation to the relation above: recall we have the forgetful map $\Pi_{n+1} : F[n+1] \to F[n]$, which induces a map on the Chow rings $\Pi_{n+1}^* : \mathbf{A}^*(F[n]) \to \mathbf{A}^*(F[n+1])$. Given this map, we can think of the identity above as follows: compared to the moduli space $F[n]$, the space $F[n+1]$ has $n+k$ extra divisors: $d_{n+1,1}, \ldots, d_{n+1,k}, D_{1,n+1}^{(n+1)}, \ldots D_{n,n+1}^{(n+1)}$. A generator in $\mathbf{A}^j(F[n+1])$ is either a class inherited from $\mathbf{A}^j(F[n])$ under the map Π^*_{n+1} (this accounts for $b_{n,j}$ generators), or it is a product between a generator class coming from $\mathbf{A}^{j-1}(F[n])$ and one of the $n+k$ new divisor classes (this accounts for $(n+k)b_{n,j-1}$ generators), or it is a product between a generator class coming from $\mathbf{A}^{j-2}(F[n])$ and the one generator class coming from $\mathbf{A}^2(S_1)$ under the projection map $\mathrm{pr}_{n+1} : S_1^{n+1} \to S_1$ (this accounts for $b_{n,j-2}$ generators). It is easy to see that these are the only generators, since the divisors $D^{(n)}_{i,j}$ satisfy the identities in 5.1. $\square$

Corollary 5.13. *When $S_1 = \mathbb{P}^2_{\mathbb{C}}$, the Chow ring of the moduli space $\mathbf{A}^*(F[\mathbb{P}^2, n])$ is:*

$$\mathbf{A}^*(F[\mathbb{P}^2, n]) \cong \frac{\mathbb{Z}[H_i^*, D^{(n)}_{j,k}]_{1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant n}}{\langle D^{(n)}_{j,k}(D^{(n)}_{i,j} - D^{(n)}_{i,k}), D^{(n)}_{j,k}(H_j^* - H_k^*), H_i^{*3}, P_{i,j}(-D^{(n)}_{i,j})\rangle},$$

where, $\forall 1 \leqslant i \leqslant n$, H_i^* is the image of the hyperplane class $H \in \mathbf{A}^*(\mathbb{P}^2)$ under the composition $F[\mathbb{P}^2, n] \to (\mathbb{P}^2)^n \xrightarrow{pr_i} \mathbb{P}^2$.

Chapter 6

A Question about Very General Curves

We now switch gears entirely and go back to the original problem suggested by Daniel Litt. We refer to the Introduction for more information. Recall that we want to characterize smooth surfaces S on which a very general curve C of genus g embeds as an ample divisor.

Notation. *Throughout the rest of the thesis, a curve (resp. surface) is a complex projective variety of dimension 1 (resp. 2), reduced and irreducible.*

An early result related to this question is the following Proposition from the paper "On the Kodaira Dimension of the Moduli Space of Curves" by Harris and Mumford (see [20]):

Proposition 6.1. *Assume for some g that the Kodaira dimension of $\mathcal{M}_g$ is at least 0. Then if C is a very general curve of genus g (i.e. the corresponding point $[C] \in \mathcal{M}_g$ lies in no subvariety defined over $\mathbb{Q}$), and S is an algebraic surface containing C on which C moves in a non-trivial linear system, then S is birational to $C \times \mathbb{P}^1$.*

Proof. See [20]. $\square$

The study of the Kodaira dimension of the moduli space of genus g curves $\mathcal{M}_g$ has a long history. We recall the main results here, noting that not much is known when the genus satisfies $17 \leqslant g \leqslant 21$:

- for $g \leqslant 16$, $\mathcal{M}_g$ is uniruled, therefore $\kappa(\mathcal{M}_g) = -\infty$ (see [4], [5], [6], [15], [33], [36], [35], [38]);
- for $g \geqslant 22$, $\mathcal{M}_g$ is of general type, so the Kodaira dimension satisfies $\kappa(\mathcal{M}_g) = 3g - 3$ (see [11], [19], [20]).

Corollary. *Let S be a surface. Assume S contains a very general curve C of genus $g \geqslant 22$ such that $\dim |C| \geqslant 1$. Then the surface S is birational to $C \times \mathbb{P}^1$.*

We thought to analyze a similar question in a different setting:

Question. *Let C be a very general smooth curve of genus g which embeds on a smooth surface S as an ample divisor such that $\dim |C| = 0$. What can we say about the surface S? Is S always birational to $C \times \mathbb{P}^1$?*

On this question, we will show the following:

Theorem 6.2. *Let C be a very general smooth curve of genus g which embeds on a smooth surface S as an ample divisor such that $\dim |C| = 0$. The following statements hold:*

(i) If the Kodaira dimension of S satisfies $\kappa(S) = -\infty$ and S is not rational, then S is birational to $C \times \mathbb{P}^1$;

(ii) If the Kodaira dimension of S is 0 or 1, then such an embedding does not exist if the genus of C is $g \geqslant 21$;

(iii) If S is of general type and its regularity is $q(S) = g$, then S is birational to the symmetric square $Sym^2(C)$.

There are a few cases left open. If S is a rational surface, we analyze of the conditions that need to be satisfied (see Prop. 9.5 and 9.10). For the case in which S is of general type and $q(S) = 0$, we prove the following partial result: if $S \hookrightarrow \mathbb{P}^r$ is a complete intersection and the composed morphism $C \hookrightarrow S \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture, then C is not ample on S if its genus is higher than 15.

Chapter 7

Curves and Surfaces

In this chapter we recall some well-known statements related to the study of smooth curves and surfaces. This section has three parts. First, we recall some basic results about curves on surfaces. Second, we outline the Enriques-Kodaira classification of surfaces. Lastly, we sketch the main results of Brill-Noether theory, finishing with the statement of the Maximal Rank Conjecture.

Here are a few basic facts about curves and surfaces:

Proposition 7.1 (Projection formula). *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then there exists a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$.*

Proof. See [21], Exercise II.5.1(d). □

Theorem 7.2 (Torelli). *Let C, C' be complete smooth curves over an algebraically closed field k. Assume there exists an isomorphism $(J(C), \lambda_C) \cong (J(C'), \lambda_{C'})$ between the canonically polarized Jacobian varieties of C and C'. Then C and C' are isomorphic.*

Proof. See, for example, [29]. □

Proposition 7.3 (Genus formula). *Let S be a smooth surface and K_S its canonical divisor. Let C be a smooth genus g curve on S. Then $C(K_S + C) = 2g - 2$.*

Proof. See [21], Proposition V.1.5. □

Proposition 7.4 (Nakai-Moishezon). *Let S be a smooth surface and D a divisor on S. Then D is ample if and only if $D^2 > 0$ and $D \cdot B > 0$, for all irreducible curves B on S.*

Proof. See [21], Proposition V.1.10. □

We now turn our focus to the Enriques-Kodaira classification of surfaces:

Definition 7.5. *Let S be a surface. A curve E on S is called a (-1)-curve if it is smooth, rational, and $E^2 = -1$. The surface S is called minimal if it doesn't contain any (-1)-curves.*

Theorem 7.6 (Castelnuovo Contractibility Criterion). *Let S be a smooth surface. Let E be a (-1)-curve on S. There exists a smooth surface S' and a morphism $\pi : S \to S'$ such that π contracts E to a point p and (S', π) is isomorphic to the blowup of S at p.*

Proof. See [2], Theorem II.17. □

Proposition 7.7. *Let S be a smooth surface. Let E be a (-1)-curve on S and $\pi : S \to \overline{S}$ the contraction of E. Then $K_S = \pi^* K_{\overline{S}} + E$, where K_S and $K_{\overline{S}}$ are the canonical divisors of S and $\overline{S}$, respectively, and $\pi^* : Pic(\overline{S}) \to Pic(S)$ is the pullback map on the Picard groups.*

Proof. See [21], Proposition V.3.3. □

Every time we contract a (-1)-curve E on the surface S, the rank of the Neron-Severi group $\mathrm{NS}(S)$ decreases by 1. This means that we can obtain a minimal surface starting from S in a finite number of steps. Moreover, the following stronger results holds:

Theorem 7.8. *Let S be a surface. Then S is birational to a minimal surface. Moreover, if S is non-ruled, this minimal surface is unique.*

Proof. See [2], Theorem V.19. □

Before we invoke the well-known Enriques-Kodaira classification of smooth minimal surfaces, we recall the definition of the *Kodaira dimension* of a smooth surface S:

$$\kappa(S) = \min\Big(c \in \mathbb{N} \,|\, \Big(\frac{h^0(S, nK_S)}{n^c}\Big)_{n \geqslant 1} \text{ is bounded from above}\Big).$$

By convention, when the plurigenera $h^0(S, nK_S)$ vanish for all $n \gg 0$, one sets $\kappa(S) = -\infty$.

We are now ready to state the Enriques-Kodaira classification of complex smooth minimal surfaces, focusing on specific properties of these surfaces which we will use later. For more details, see [1]:

Theorem 7.9 (Enriques-Kodaira Classification)**.** *Let S be a smooth minimal surface. The following holds:*

(a) if $\kappa(S) = -\infty$, then S is one of the following:

(i) $S \cong \mathbb{P}^2$;

(ii) S is a ruled surface onto a smooth curve B, such that all fibers are isomorphic to $\mathbb{P}^1$. If the genus of B is 0, then S is a Hirzebruch surface $\mathbb{F}_a$, with $a = 0$ or $a \geqslant 2$;

(b) if $\kappa(S) = 0$, then S is one of the following:

(i) Abelian surface: these satisfy $q(S) = 2$, $p_g(S) = 1$, $K_S \sim 0$;

(ii) K3 surface: these satisfy $q(S) = 0$, $p_g(S) = 1$, $K_S \sim 0$;

(iii) Enriques surface: these satisfy $q(S) = p_g(S) = 0$, $K_S \nsim 0$, $2K_S \sim 0$;

(iv) Bielliptic suface: these satisfy $q(S) = 1$, $p_g(S) = 1$, $nK_S \sim 0$ for $n \in \{2, 3, 4, 6\}$;

(c) if $\kappa(S) = 1$, then S admits an elliptic fibration onto a smooth curve B;

(d) if $\kappa(S) = 2$, then S is called a surface of general type.

In this last part of this chapter, we state the main statements of Brill-Noether theory. For more informations, see [18]:

Definition 7.10. *Let C be a smooth curve. A map $f : C \to \mathbb{P}^r$ is called nondegenerate if the image of C does not lie in a hyperplane.*

Theorem 7.11. *Let C be a general smooth genus g curve. There exist a nondegenerate map $f : C \to \mathbb{P}^r$ of degree d if and only if the Brill-Noether number $\rho(g, d, r)$ is nonnegative:*

$$\rho(g, d, r) = (r + 1)d - rg - r(r + 1) \geqslant 0.$$

Proof. See [18]. □

Theorem 7.12. *Let C be a general smooth genus g curve. Let $G^2_d(C)$ be the (smooth) projective variety parametrizing all linear systems of degree d and dimension 2 on C. A general point of $G^2_d(C)$ corresponds to a morphism $f : C \to \mathbb{P}^2$ which maps C birationally onto a plane curve with only nodal singularities.*

Proof. See [18]. □

As seen in [26], for fixed d, g, r satisfying $\rho(d, g, r) \geqslant 0$, there exists a unique component of the Kontsevich space of stable maps $\overline{M}_g(\mathbb{P}^r, d)$ that dominates the moduli space of curves $\overline{M}_g$, whose general member is non-degenerate and whose relative dimension over $\overline{M}_g$ is $\rho(d, g, r) + \dim \operatorname{Aut}(\mathbb{P}^r)$. A stable map corresponding to points in this component is called a Brill-Noether curve.

Theorem 7.13 (Maximal Rank Conjecture)**.** *If $r \geqslant 3$ and $C \subset \mathbb{P}^r$ is a general Brill-Noether curve of degree d, then the dimension of the space of polynomials of degree k which vanish on C is given by:*

$$\begin{cases} \binom{r+k}{k} - (kd + 1 - g) & , \textit{if } kd + 1 - g \leqslant \binom{r+k}{k} \textit{ and } k \geqslant 2; \\ 0 & , \textit{otherwise.} \end{cases}$$

Proof. See [26]. □

Chapter 8

Preliminary results

In this chapter we prove some preliminary facts that will be very useful in the proof of Theorem 6.2. If C is a very general curve of genus g, then $\mathrm{Pic}^0(C)$ is a simple abelian variety. Additionally, if C embeds on a smooth surface S as an ample divisor, then either $\mathrm{Pic}^0(S) = 0$ or $\mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(C)$. We finish the chapter by giving a lower bound on the intersection number $C \cdot K_S$, where K_S is the canonical divisor of S. Lastly, we show that when $\mathrm{Pic}^0(S) = 0$ and $\dim |C| = 0$, S must have at least $3g - 3$ moduli.

To start, we focus our attention on the Picard group of the curve C and of the surface S, more specifically on the component of the identity of the Picard group $\mathrm{Pic}^0(-)$. An abelian variety X is called *simple* if 0 and X are the only abelian subvarieties of X. We use the following theorems of Koizumi and Mumford to derive a very useful corollary:

Theorem 8.1. *Let C be a very general smooth curve. Then $Pic^0(C)$ is a simple abelian variety.*

Proof. The proof of this statement can be found in [25]. $\square$

Theorem 8.2. *Let S be a smooth surface. Let $\iota : C \hookrightarrow S$ be an ample divisor on S. Then the pullback morphism $\iota^* : Pic^0(S) \hookrightarrow Pic^0(C)$ is injective.*

Proof. See [32, p. 99]. □

Proposition 8.3. *Let S be a smooth surface. Let $C \hookrightarrow S$ be a very general smooth curve that embeds as an ample divisor on S. Then one of the following conditions must hold:*

(i) $Pic^0(S) \cong Pic^0(C)$;

(ii) $Pic^0(S) = 0$.

Proof. By Theorem 8.2, the inclusion $\iota : C \hookrightarrow S$ induces an injective map $\iota^* : \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}^0(C)$, so $\mathrm{Im}(\iota^*)$ is an abelian subvariety of $\mathrm{Pic}^0(C)$. By Theorem 8.1, $\mathrm{Pic}^0(C)$ is simple, so $\mathrm{Im}(\iota^*)$ is either $\mathrm{Pic}^0(C)$ or 0. This means that ι^* is either an isomorphism, which corresponds to case (i), or $\mathrm{Pic}^0(S) = 0$, which corresponds to case (ii). □

Out strategy in later chapters will be to prove Theorem 6.2 in several steps, analyzing every possible minimal model of S. Before we get to that part, we first prove some lemmas that will be very helpful for some of the cases:

Lemma 8.4. *Let S be a smooth surface, C be a smooth ample curve on S, and E be a (-1)-curve on S. Let $\pi : S \to \overline{S}$ be the contraction of E and $\overline{C} = Im(C)$. Then $\overline{C}$ is an ample divisor. Moreover, if $C \cdot E = 1$, then $\overline{C}$ is also smooth.*

Proof. We start by showing that $\overline{C}$ is ample using the Nakai-Moishezon Criterion for Ampleness. Let $\pi^* : \mathrm{Pic}(\overline{S}) \to \mathrm{Pic}(S)$ be the pullback map on the Picard groups. If $C \cdot E = r$, then $\pi^*\overline{C} = C + rE$, and:

$$\overline{C}^2 = (\pi^*\overline{C})^2 = (C + rE)^2 = C^2 + 2rC \cdot E + r^2E^2 = C^2 + r^2 > 0. \tag{8.1}$$

Now, let $\overline{D} \subset \overline{S}$ to be any irreducible curve on $\overline{S}$. Say $\pi^*\overline{D} = D + nE$, for some integer $n \geqslant 0$, where D is the strict transform of $\overline{D}$ on the surface S. We obtain:

$$\overline{C} \cdot \overline{E} = \pi^*\overline{C} \cdot \pi^*\overline{D} = (C + rE)(D + nE) = C \cdot D + rn > 0. \quad (8.2)$$

As a consequence of inequalities 8.1 and 8.2, we conclude that $\overline{C} \subset \overline{S}$ is an ample divisor.

For the second part, assume $C \cdot E = 1$. Since C and E don't have any common irreducible components, then

$$C \cdot E = \sum_{P \in C \cap E} (C.E)_P,$$

where we define the intersection multiplicity $(C.E)_P$ of C and E at P to be the length of $\mathcal{O}_{P,X}/(f, g)$, where f, g are local equations for C, E at P. Since $C \cdot E = 1$, there exists a unique point $P \in C \cap E$, with intersection multiplicity 1. This means that contracting E does not affect the smoothness of C, i.e. $\overline{C}$ is still smooth. $\square$

By Lemma 8.4, we can contract all (-1)-curves E on S that satisfy $C \cdot E = 1$ and still preserve the smoothness of C. Therefore, we can assume without loss of generality that every (-1)-curve E on S satisfies $C \cdot E \geqslant 2$. We are now ready to state and prove the lower bound for the intersection number $C \cdot K_S$. Here is our setup:

Setup 8.5. *Let S be a smooth surface and S_0 a smooth minimal model of S. There exists an integer $n \geqslant 0$ and a sequence of morphisms $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$ with the property that each map $S_i \to S_{i-1}$ is the contraction of a (-1)-curve of S_i. Let C be a smooth ample divisor on S such that $C \cdot E \geqslant 2$, for all (-1) curves $E \subset S$. Let C_0 be the image of C on S_0.*

Lemma 8.6. *Let S, S_0, n, C, C_0 be as in the Setup 8.5 above. Let K_S, K_{S_0} be the canonical divisors of S and S_0, respectively. The following inequality holds:*

$$C \cdot K_S \geqslant C_0 \cdot K_{S_0} + 2n. \tag{8.3}$$

Proof. We prove this statement inductively. Let $E \subset S$ be a (-1)-curve on S and $\pi : S \to \overline{S}$ its contraction. Let $\overline{C}$ be the image of C under π. By Lemma 8.4, we know that $\overline{C}$ is an ample divisor. Using Proposition 7.7, we obtain:

$$C \cdot K_S = C \cdot (\pi^* K_{\overline{S}} + E) = C \cdot \pi^* K_{\overline{S}} + C \cdot E = \pi_* C \cdot K_{\overline{S}} + C \cdot E = \overline{C} \cdot K_{\overline{S}} + C \cdot E \geqslant \overline{C} \cdot K_{\overline{S}} + 2.$$

To conclude the statement, we need one more fact. Let E' be another (-1)-curve on S, and let $\overline{E'}$ be its image in $\overline{S}$. We claim that if $C \cdot E' \geqslant 2$, then $\overline{C} \cdot \overline{E'} \geqslant 2$. Assume $\pi^* \overline{E'} = E' + tE$, for some $t \geqslant 0$. We obtain the following inequality, which completes the proof:

$$\begin{aligned} \overline{C} \cdot \overline{E'} &= \pi^* \overline{C} \cdot \pi^* \overline{E'} = (C + rE)(E' + tE) \\ &= C \cdot E' + rE \cdot E' + tC \cdot E + rtE^2 \\ &= C \cdot E' + 2rt - rt \geqslant C \cdot E' \geqslant 2. \end{aligned}$$

□

To finish the chapter, we give a lower bound for the number of moduli of S when $\mathrm{Pic}^0(S) = 0$:

Lemma 8.7. *Let S be a smooth surface with irregularity $q(S) = 0$, and C a very general curve of genus g on S satisfying* $\dim |C| = 0$. *Then the number of moduli of S is at least $3g - 3$.*

Proof. We can find a smooth projective family $\mathcal{C} \subset \mathcal{S}$ over a base scheme B of finite type over $\overline{\mathbb{Q}}$ such that $C \subset S$ is a $\mathbb{C}$-valued point of B. Moreover, for every $t \in B$, the fiber $C_t \subset S_t$ satisfies $\mathrm{Pic}^0(S_t) = \dim |C_t| = 0$. This family gives the following figure:

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{C} \arrow[r, "{[C\subset S]}"] \arrow[d, equal] & B \arrow[r] \arrow[d] & M_S, \\ \mathrm{Spec}\ \mathbb{C} \arrow[r, "{[C]}"] & M_g & \end{tikzcd}$$

where M_g is the coarse moduli space of genus g curves and M_S is a "moduli space of surfaces". We will make this more precise in the remark below.

Now, since $C \subset S$ is a $\mathbb{C}$-valued point of B and C is a very general curve, then the image of the map $\mathrm{Spec}\ \mathbb{C} \to M_g$ is the generic point, which means $B \to M_g$ is dominant. Additionally, we observe that the map $B \to M_S$ must have 0-dimensional fibers. To see this, assume by contradiction that there exists a 1-dimensional fiber. Thus there exists a surface S' that admits a 1-dimensional family of curves on it. Since $\mathrm{Pic}^0(S') = 0$, then these curves must move in the same linear system, but that contradicts $\dim |C_t| = 0$, for any C_t in the family, so we are done.

In conclusion, the moduli space M_S has dimension at least $3g-3$, as claimed. □

Remark 8.8. *We want to make the definition of the "moduli of surfaces" M_S above more precise. In our applications below, the surface S is obtained by blowing up a smooth minimal surface S_0 at n consecutive points, and the minimal surface is always a K3 surface, an Enriques surface, or an elliptic fibration over $\mathbb{P}^1$. As we will see in the next chapters, all these minimal surfaces are parametrized by coarse moduli spaces. Our claim is that S admits a moduli space, as well.*

To see this, recall that in the previous chapters, we constructed the moduli space $F[n]$ parametrizing n-fold blowups of a smooth surface S_0 over an affine base Spec R (see Thm. 2.3). We can generalize this construction such that the affine base is *replaced by the moduli space of K3 surfaces, or the moduli space of Enriques surfaces, and so on. At least for the purpose of our dimension count, this conclusion follows easily.*

Chapter 9

Surfaces of Kodaira dimension $-\infty$

In this chapter we analyze the scenario in which a very general curve C of genus g is embedded as an ample divisor on a smooth surface of Kodaira dimension $-\infty$ such that $\dim |C| = 0$. We show that either S is birational to $C \times \mathbb{P}^1$, or its minimal model is $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_a$, where $a \neq 1$. The latter case remains mainly open. However, we analyze the conditions that need to be satisfied in this situation. While these conditions are not sufficient to conclude that such an embedding $C \hookrightarrow S$ exists, we show that there exist cases where this conditions could potentially be satisfied.

To start, let S be a smooth surface satisfying $\kappa(S) = -\infty$. Let S_0 be a smooth minimal model of S. By the Enriques-Kodaira classification of minimal surfaces, we know that S_0 is either $\mathbb{P}^2$ or a ruled surface, i.e. it admits a surjective morphism onto a smooth curve B such that every fiber is isomorphic to $\mathbb{P}^1$. We show first that, if S is not rational, then it is birational to $C \times \mathbb{P}^1$.

Lemma 9.1. *Let $\pi : S_0 \to B$ be a ruled minimal surface over a smooth curve B. Let $\pi^* : Pic^0(B) \to Pic^0(S_0)$ be the pullback map on the Picard groups. Then π^* is injective.*

Proof. First, notice that $\mathcal{O}_B = \pi_* \mathcal{O}_{S_0}$. This is true because every fiber is connected, so the result follows by base change. Let $\mathcal{L} \in \mathrm{Pic}^0(B)$ such that $\pi^*\mathcal{L} = \mathcal{O}_{S_0}$. We apply the projection formula and conclude that π^* is injective:

$$\mathcal{L} = \mathcal{O}_B \otimes_{\mathcal{O}_B} \mathcal{L} \cong \pi_*\mathcal{O}_{S_0} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L} \cong \pi_*(\mathcal{O}_{S_0} \otimes_{\mathcal{O}_{S_0}} \pi^*\mathcal{L}) \cong \pi_*\mathcal{O}_{S_0} = \mathcal{O}_B.$$

□

Lemma 9.2. *Let S be a smooth surface and $C \hookrightarrow S$ a very general smooth curve embedded on S as an ample divisor. If S has Kodaira dimension $\kappa(S) = -\infty$ and it is not rational, then it is birational to $C \times \mathbb{P}^1$.*

Proof. From the Enriques-Kodaira classification of surfaces (see Thm. 7.9), we know that the minimal model of S is a ruled surface $S_0 \to B$, where $B \neq \mathbb{P}^1$. As a consequence of Theorem 8.2 and Lemma 9.1, we have the following sequence of morphisms: $\mathrm{Pic}^0(B) \hookrightarrow \mathrm{Pic}^0(S_0) \xrightarrow{\cong} \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}^0(C)$. On the other hand, Lemma 8.1 gives us that $\mathrm{Pic}^0(C)$ is simple, therefore either $\mathrm{Pic}^0(B) = 0$ or the composed map $\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(C)$ is an isomorphism. Since $B \neq \mathbb{P}^1$, then we are in the second case. By the Torelli theorem (see Thm. 7.2), we conclude that the morphism $C \to B$ is an isomorphisms, hence S is birational to $C \times \mathbb{P}^1$. □

We now begin our analysis of the case in which S is a rational surface, i.e. its minimal model S_0 is either $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_e$, where $e \neq 1$. Here is a partial result to our question, which follows from our research below:

Lemma 9.3. *Let S be a smooth surface and $C \hookrightarrow S$ a very general smooth curve embedded on S as an ample divisor. If S is rational and the genus of C satisfies $g \geqslant 22$, then the image of C on the smooth minimal model of S is not a nodal curve.*

Proof. This is a consequence of Lemmas 9.11 and 9.6 below. □

Assume we are in the case in which $S_0 = \mathbb{P}^2$. To start, notice that a very general smooth curve C of genus g can be embedded on a smooth surface whose minimal model is $\mathbb{P}^2$. To see this, recall that any curve can be mapped birationally to $\mathbb{P}^2$ such that its image has only nodal singularities (see Thm. 7.12). Once we blow up these nodal singularities, the curve C will be embedded on the resulting blown up surface. The question we try to tackle next is whether C can ever be an ample divisor on S. This question seems difficult to answer in the most general of circumstances. In this paper, we decided to focus on a special case, as follows:

Setup 9.4. *Assume that $C \hookrightarrow S$ is a very general curve embedded on S as an ample divisor such that $\dim |C| = 0$, where S is obtained by blowing up $\mathbb{P}^2$ at n distinct points $p_1, \ldots, p_n$. Let $E_1, \ldots, E_n$ be the corresponding exceptional divisors. The Picard group of S is isomorphic to $\mathbb{Z}^{\oplus n+1}$, and it is generated by the classes $H, E_1, \ldots, E_n$, which satisfy $H^2 = 1$, $E_i^2 = -1$, $H \cdot E_i = E_i \cdot E_j = 0$. Let $C \sim dH - \sum_{i=1}^n m_i E_i$ be the divisor class of C.*

Proposition 9.5. *Let S and C be as in the Setup 9.4 above. Then the following conditions need to be satisfied:*

(a) $d > 0$ *and* $m_i \geqslant 2$*, for all* $i = 1, \ldots, n$*;*

(b) $d(d-3) - \sum_{i=1}^n m_i(m_i - 1) = 2g - 2$*;*

(c) $C^2 = d^2 - \sum_{i=1}^n m_i^2 > 0$*;*

(d) $3d - \sum_{i=1}^n m_i > 2 - 2g$*;*

(e) $3d - \sum_{i=1}^n m_i \leqslant 1 - g$*;*

(f) $d \geqslant \frac{2g+6}{3}$*;*

(g) $d - m_i \geqslant \frac{g+2}{2}$*;*

(h) $2n \geqslant 3g + 5$*;*

(i) $(C + K_S)^2 = (d-3)^2 - \sum_{i=1}^{n}(m_i - 1)^2 \geqslant g - 2$;

(j) $\max\{m_i\}_{i=1}^{n} \geqslant \frac{g}{9}$.

Proof. (a) Since C is ample, then $d > 0$. As a result of Lemma 8.4, we can assume without loss of generality that $C \cdot E_i \geqslant 2$, for all $i = 1, \ldots, n$.

(b) This is the genus formula combined with the fact that $K_S = -3H + \sum_{i=1}^{n} E_i$.

(c) Since C is ample, then $C^2 > 0$.

(d) This is true because $C \cdot K_S = 2g - 2 - C^2 < 2g - 2$.

(e) Consider the space of homogeneous polynomials of degree d in three variables which vanish to order at least m_i at the point p_i, for all $i = 1, \ldots, n$. The naive dimension count for this space is $\binom{d+2}{2} - \sum_{i=1}^{n}\binom{m_i+1}{2}$. The actual dimension is always at least the expected one. Since we are assuming that $\dim |C| = 0$, this translates to the following:

$$\binom{d+2}{2} - \sum_{i=1}^{n}\binom{m_i+1}{2} \leqslant 1.$$

Combining the equation above with the genus formula, we obtain the desired inequality.

(f) This is an immediate application of Brill-Noether theory (see Theorem 7.11).

(g) Fix a point p_i among the n singular points of $C \to \mathbb{P}^2$. When we project from the point p_i onto a line, we obtain a composed map $C \to \mathbb{P}^2 \to \mathbb{P}^1$ of degree

$d - m_i$. The inequality now follows immediately from Brill-Noether theory (see Thm. 7.11).

(h) This follows from a moduli count. From Lemma 8.7, we know that S needs to have at least $3g-3$ moduli. On the other hand, since S is the blowup of $\mathbb{P}^2$ at n distinct points, the number of moduli of S is $2n - \dim(\mathrm{Aut}(\mathbb{P}^2)) = 2n-8$, and the conclusion follows.

(i) Consider the following short exact sequence:

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + C) \to \mathcal{O}_C(K_S) \to 0,$$

which gives the following long exact sequence in cohomology:

$$0 \to H^0(S, \mathcal{O}_S(K_S)) \to H^0(S, \mathcal{O}_S(K_S + C)) \to H^0(C, \mathcal{O}_C(K_S)) \to \dots$$

Since $h^0(S, \mathcal{O}_S(K_S)) = 0$ and $h^0(C, \mathcal{O}_C(K_S)) = g$, then $h^0(S, \mathcal{O}_S(K_S+C)) = g$. Now, pick two very general global sections in $\Gamma(\mathcal{O}_C(K_S))$ that have no common zeros and let $\sigma, \sigma' \in \Gamma(\mathcal{O}_S(K_S + C))$ be very general lifts of these sections. Let $D = V(\sigma) \subset S$. The following short exact sequence:

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(K_S + C) \to \underbrace{\mathcal{O}_D(K_S + C|_D)}_{\mathcal{L}} \to 0$$

gives a long exact sequence on cohomology:

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(K_S + C)) \to H^0(D, \mathcal{L}) \to \dots$$

Since $h^0(S, \mathcal{O}_S) = 1$ and $h^0(S, \mathcal{L}) = g$, we obtain:

$$h^0(D,\mathcal{L}) \geqslant g-1. \tag{9.1}$$

We note here that $\mathcal{L}$ is a line bundle on D of degree $(K_S+C)^2$. Now, recall that we picked another general section $\sigma' \in H^0(S, \mathcal{O}_S(K_S+C))$. Let σ'' be its image in $H^0(D,\mathcal{L})$ and $D' = V(\sigma') \subset S$. By our initial assumptions about σ and σ', we know that σ'' vanishes at a finite number of points. Now consider the following short exact sequence:

$$0 \to \mathcal{O}_D \xrightarrow{\sigma''} \mathcal{L} \to \mathcal{L}|_{D\cap D'} \to 0,$$

which gives the following long exact sequence on cohomology:

$$0 \to H^0(D,\mathcal{O}_D) \to H^0(D,\mathcal{L}) \to H^0(D\cap D', \mathcal{L}|_{D\cap D'}) \to \dots$$

We claim that $H^0(D,\mathcal{O}_D) = \mathbb{C}$. Together with the fact that $H^0(D\cap D', \mathcal{L}|_{D\cap D'})$ is a vector space of dimension $(K_S+C)^2$, we conclude our statement:

$$1 + (K_S+C)^2 \geqslant H^0(D,\mathcal{L}) \geqslant g-1.$$

Hence, we are left to show that $H^0(D,\mathcal{O}_D) = \mathbb{C}$. By the following short exact sequence, it suffices to show that $h^1(S, \mathcal{O}_S(-K_S-C)) = 0$:

$$0 \to \mathcal{O}_S(-K_S-C) \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

If we show that K_S+C is big and nef on the surface S, then the condition above follows from a strong version of Kodaira Vanishing Theorem (see [27], Theorem

4.3.1). The divisor $K_S + C$ is nef because the base locus of its linear system has dimension 0. Given that it is nef, it suffices to show that $(K_S + C)^2 > 0$ to conclude that it is also big (see [27], Remark 4.3.2).

Now, we know that $(K_S + C)^2 = D \cdot D' \geqslant 0$, since D and D' don't have any components in common. Thus, we need to rule out the case when $(K_S + C)^2 = D \cdot D' = 0$, i.e. D and D' do not meet. If this is the case, then $K_S + C$ is base point free and $h^0(D, \mathcal{O}_D) \geqslant g - 1$. Since D is a reduced CM 1-dimensional scheme over the complex numbers, this means that D has $g - 1 > 1$ connected components. Thus, we get a base point free linear system whose general member is disconnected, which implies that the image of our linear system is a curve (see [22], Theorem 7.1):

$$|K_S + C| : S \to \mathbb{P}^{g-1}.$$

On the other hand, this curve has to be the canonical image of C because $(K_S + C)|_C = K_C$. Hence, we get morphisms $C \to S \to C$ whose composition is the identity. Denote $S \to C' \to C$ the Stein factorization of the second morphism. Then C' is a curve, but that means that $C = C'$, i.e. the fibers $S \to C$ are connected, which gives us the contradiction we wanted.

(j) Let $\overline{m_i} = \frac{1}{n}\sum_{i=1}^{n} m_i$ and $\overline{m_i^2} = \frac{1}{n}\sum_{i=1}^{n} m_i^2$. The genus formula gives us the following:

$$2g - 2 = d^2 - 3d - \sum_{i=1}^{n} m_i^2 + \sum_{i=1}^{n} m_i = d^3 - 3d - n\overline{m_i^2} + n\overline{m_i} \Rightarrow$$

$$\Rightarrow n = \frac{1}{\overline{m_i^2} - \overline{m_i}}(d^2 - 3d - 2g + 2).$$

On the other hand, C is ample on S, which means $C^2 > 0$:

$$d^2 - n\overline{m_i^2} = d^2 - \frac{\overline{m_i^2}}{\overline{m_i^2} - \overline{m_i}}(d^2 - 3d - 2g + 2) > 0 \Rightarrow$$

$$\Rightarrow -\overline{m_i}d^2 + 3\overline{m_i^2}d + \overline{m_i^2}(2g-2) > 0.$$

Let $\alpha = \frac{\overline{m_i^2}}{\overline{m_i}}$. By Brill-Noether theory (see Thm. 7.11), we know that $d \geqslant \frac{2g+6}{3}$, so the conclusion follows:

$$d^2 - 3\alpha d - \alpha(2g-2) < 0 \Rightarrow (\frac{2g+6}{3})(\frac{2g+6}{3} - 3\alpha) \leqslant d(d-3\alpha) < \alpha(2g-2)$$
$$\Rightarrow \frac{g}{9} < \frac{(g+3)^2}{9(g+1)} < \alpha \leqslant \max\{m_i\}_{i=1}^n.$$

□

Now that we have established some preliminary conditions that need to be satisfied, we start by ruling out one of the easiest cases, where all the multiplicities are the same: $m_1 = \cdots = m_n = m$. We show that $C \hookrightarrow S$ cannot be ample:

Lemma 9.6. *Let S be a rational surface obtained by blowing up $\mathbb{P}^2$ at n distinct points $p_1, \ldots, p_n$, as in the Setup 9.4 above. Assume $C \hookrightarrow S$ is a very general curve of genus $g \geqslant 22$ such that $C \sim dH - \sum_{i=1}^n mE_i$ is its divisor class on S. Then C is not an ample divisor of S.*

Proof. Assume by contradiction that C is ample. From the proof of Proposition 9.5 above, we know that

$$m > \frac{d^2}{3d + 2g - 2}.$$

On the other hand, from Proposition 9.5(f) we know that $3d \geqslant 2g + 6$. Combining these two facts, we obtain:

$$m > \frac{d^2}{3d + 2g - 2} > \frac{d^2}{6d - 6} > \frac{d}{6}.$$

Lastly, since C is assumed to be ample, it should satisfy $C^2 = d^2 - nm^2 > 0$, but that produces a contradiction:

$$d^2 > nm^2 > \frac{nd^2}{36} \Rightarrow \frac{3g + 5}{2} < n < 36 \Rightarrow g < 22.$$

$\square$

Now, Brill-Noether theory gives us a bit more (see Theorem 7.12). A general point of the variety $G^2_d(C)$ corresponds to a map $C \to \mathbb{P}^2$ that maps C birationally onto a plane curve with only nodes as singularities. Thus, we quickly deduce the following corollary:

Corollary 9.7. *Let C be a very general curve of genus $g \geqslant 22$. Assume C is embedded on a rational surface S as in Setup 9.4 above. Let d be the degree of the composed map $C \to S \to \mathbb{P}^2$. Then this map corresponds to a non-general point of $G^2_d(C)$.*

Remark 9.8. *Using a computer program, we were able to find numbers $n, d, m_1, \ldots, m_n$ for every $g \gg 0$ such that all the conditions in Proposition 9.5 are satisfied. From our simulations, it seems like we can always find solutions that have the following format: 8 points have the same (high) multiplicity m, and $n2 = n - 8$ nodal points, i.e. of multiplicity 2.*

Here are a few examples:

$g = 50$	$d = 241$	$n2 = 70$	$m = 85$
$g = 50$	$d = 285$	$n2 = 70$	$m = 102$

$$\begin{array}{llll} g = 100 & d = 651 & n2 = 145 & m = 230 \\ g = 100 & d = 733 & n2 = 158 & m = 259 \\ g = 150 & d = 948 & n2 = 221 & m = 335 \\ g = 150 & d = 1030 & n2 = 228 & m = 364 \\ g = 150 & d = 1112 & n2 = 231 & m = 393 \\ g = 200 & d = 1245 & n2 = 306 & m = 440 \\ g = 200 & d = 1327 & n2 = 307 & m = 469. \end{array} \tag{9.2}$$

Further discussion. Ideally, we want to find a way to use that C is ample to its full potential. We already know that the points $p_1, \ldots, p_n$ are not in general position. However, let's assume every subset of 9 points are in general position. Then there exist infinitely many (-1)-curves on S. More specifically, for every set of indices $\{i_1, \ldots, i_9\} \subset \{1, \ldots, n\}$, there are infinitely many (-1)-curves E whose divisor class is of the form $eH - \sum_{r=1}^{9} n_{i_r} E_{i_r}$. Since every such curve has genus 0 and self-intersection -1, then the following system of equations has an infinite number of solutions:

$$\begin{cases} e^2 - \sum_{r=1}^{9} n_{i_r}^2 = -1 \\ e^2 - 3e - \sum_{r=1}^{9} n_{i_r}(n_{i_r} - 1) = -2 \end{cases}$$

Conversely, given a tuple $(e, n_{i_1}, \ldots, n_{i_9})$ that satisfies the equations above, the corresponding divisor $E = eH - \sum_{r=1}^{9} n_{i_r} E_{i_r}$ need not be a (-1)-curve (see [10], §1). However, the "expected dimension" of $h^0(S, \mathcal{O}_S(E))$ is $\binom{e+2}{2} - \sum_{r=1}^{9} \binom{n_{i_r}+1}{2} = 1$, which means that E is effective and hence $C \cdot E > 0$. Here are a few example of effective divisors with these properties: $H - E_{i_1} - E_{i_2}$, $2H - E_{i_1} - \cdots - E_{i_5}$, $3H - 2E_{i_1} - E_{i_2} - \cdots - E_{i_7}$. These divisors impose the following (further) restrictions on C:

$$d - m_{i_1} - m_{i_2} > 0$$

$$2d - m_{i_1} - m_{i_2} - m_{i_3} - m_{i_4} - m_{i_5} > 0$$

$$3d - 2m_{i_1} - m_{i_2} - m_{i_3} - m_{i_4} - m_{i_5} - m_{i_6} - m_{i_7} > 0.$$

We note that all the divisors C of the form found in (9.2) satisfy the inequalities above. However, it would be interesting to know if other divisors E like the ones above could impose further restrictions on $d, m_1, \ldots, m_n$, leading to a contradiction together with the other conditions.

We now switch to do a similar analysis of the case in which S_0 is the Hirzebruch surface $\mathbb{F}_e$, for some $e = 0$ or $e \geqslant 2$. As before, we will focus on a special case, as follows:

Setup 9.9. *Let $C \hookrightarrow S$ be a very general curve embedded on S as an ample divisor such that* $\dim |C| = 0$. *Assume S is obtained by blowing up the Hirzebruch surface $S_0 = \mathbb{F}_e$ at n distinct points $p_1, \ldots, p_n$, and $E_1, \ldots, E_n$ are the corresponding exceptional divisors. The Picard group of S is generated by the classes $\sigma, f, E_1, \ldots, E_n$, which satisfy $\sigma^2 = -e$, $f^2 = 0$, $\sigma \cdot f = 1$, $E_i^2 = -1$, $\sigma \cdot E_i = f \cdot E_i = E_i \cdot E_j = 0$. Let $C \sim a\sigma + bf - \sum_{i=1}^n m_i E_i$ be the divisor class of C.*

Proposition 9.10. *Let S and C be as in Setup 9.9 above. Then the following conditions need to be satisfied:*

(a) $m_i \geqslant 2$, for all $i = 1, \ldots, n$;

(b) $a > 0$ and $b > ae$;

(c) $-a^2e + 2ab - a(3e+2) - 2b - \sum_{i=1}^n m_i(m_i - 1) = 2g - 2$*;*

(d) $C^2 = -a^2e + 2ab - \sum_{i=1}^n m_i^2 > 0$*;*

(e) $a \geqslant \frac{g+2}{2}$*;*

(f) $2n \geqslant 3g + e + 2$*.*

Proof. (a) We explained in Lemma 8.4 why we can assume without loss of generality that $C \cdot E_i \geqslant 2$, for all $i = 1, \ldots, n$.

(b) Let $C_0 \sim a\sigma + bf$ be the image of C on $S_0 = \mathbb{F}_e$. Since C_0 is ample on $\mathbb{F}_e$, it must satisfy $a > 0$ and $b > ae$ (see [21], Corollary V.2.18);

(c) This is the genus formula combined with the fact that $K_S = -2\sigma - (2+e)f + \sum_{i=1}^n E_i$ (see [21], Corollary 2.11);

(d) Since C is ample, then $C^2 > 0$.

(e) This is an immediate application of Brill-Noether theory (see Thm. 7.11).

(f) This follows from a moduli count. From Lemma 8.7, we know that S needs to have at least $3g-3$ moduli. On the other hand, since S is the blowup of $\mathbb{F}_e$ at n distinct points, the number of moduli of S is $2n - \dim(\mathrm{Aut}(\mathbb{F}_e)) = 2n - (e+5)$, and the conclusion follows.

□

Now that we have established some preliminary conditions that need to be satisfied, we start by ruling out one of the easiest cases, in which all the multiplicities are the same: $m_1 = \cdots = m_n = m$. We show that $C \hookrightarrow S$ cannot be ample if $g \geqslant 22$:

Lemma 9.11. *Let S be a rational surface obtained by blowing up the Hirzebruch surface $S_0 = \mathbb{F}_e$ at n distinct points $p_1, \ldots, p_n$, as in the Setup 9.9 above. Assume*

$C \hookrightarrow S$ is a very general curve of genus $g \geqslant 22$ such that $C \sim a\sigma + bf - \sum_{i=1}^{n} mE_i$ is its divisor class in S. Then C is not an ample divisor of S.

Proof. Assume by contradiction that C is ample. From Proposition 9.10(c) above, we have:

$$-a^2e + 2ab - a(3e+2) - 2b - nm(m-1) = 2g - 2,$$

which means that:

$$n = \frac{1}{m(m-1)}(-a^2e + 2ab - a(3e+2) - 2b - 2g + 2).$$

On the other hand, from Proposition 9.10(d), we have:

$$\begin{aligned} C^2 &= -a^2e + 2ab - nm^2 > 0 \\ \Rightarrow -a^2e + 2ab - \frac{m}{m-1}(-a^2e + 2ab - a(3e+2) - 2b - 2g + 2) &> 0 \\ \Rightarrow^2 e - 2ab + m(3e+2g) &> 0, \end{aligned}$$

which translates to:

$$m > \frac{2ab - a^2e}{3e + 2g}. \tag{9.3}$$

Now, combining Equation 9.3 with $C^2 > 0$, we obtain:

$$n\Big(\frac{2ab - a^2e}{3e + 2g}\Big)^2 < nm^2 < 2ab - a^2e. \tag{9.4}$$

Combining Inequality 9.4 above with the fact that $b > ae$ (see Prop. 9.10(b)), we obtain:

$$nab < (3e + 2g)^2. \tag{9.5}$$

If $e > 0$, then we combine the results (b), (e), (f) of Proposition 9.10 with Equation 9.5 to get the following inequality, which leads to a contradiction, since $g \geqslant 22$:

$$\frac{3g+e+2}{2} \cdot \frac{g+2}{2} \cdot \frac{(g+2)e}{2} < (3e+2g)^2 \Rightarrow 3g^3e + g^2e^2 < 12^2e^2 + 8^2g^2.$$

Lastly, if $e = 0$, then $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In this situation we have two projections onto $\mathbb{P}^1$, which means we have the added condition $b \geqslant \frac{g+2}{2}$, by Brill-Noether theory (see Thm. 7.11). Since $g \geqslant 22$, Proposition 9.10(f) implies that $n \geqslant 36$. Using the same inequalities as before, we get a contradiction, and the proof is complete:

$$36\Big(\frac{g+2}{2}\Big)^2 < nab < (2g)^2.$$

$\square$

Chapter 10

Surfaces of Kodaira dimension 0

In this chapter we analyze the case in which a very general curve C of genus g is embedded as an ample divisor on a smooth surface S of Kodaira dimension 0 such that $\dim |C| = 0$. We know that the minimal model of S is either an abelian surface, a K3 surface, an Enriques surface, or a bielliptic surface. We show that if $g \geqslant 19$, such an embedding does not exist.

10.1 Abelian Surfaces

Theorem 10.1. *Let $C \hookrightarrow S$ be a very general curve embedded on a smooth surface whose minimal model is an abelian surface. Then C has genus 2 and S is the Jacobian of C.*

Proof. Let S be a smooth surface whose minimal model S_0 is an abelian surface. Let $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$ be the corresponding sequence of blow-downs. By the Enriques-Kodaira classification (see Thm. 7.9), we know that S_0 satisfies:

$$p_g(S_0) = 1, q(S_0) = 2, K_{S_0} \sim 0.$$

By Theorem 8.3, we know that $q(S)$ must be either 0 or g. Since the irregularity of a surface is a birational invariant, we must have $q(S) = q(S_0) = g = 2$. Thus, we are in the situation where the genus of the curve C is 2. Now, by the moduli count, we get:

$$2 = 2g - 2 = C \cdot K_S + C^2 > C \cdot K_S \geqslant C_0 \cdot K_{S_0} + 2n = 2n, \tag{10.1}$$

which means that $n = 0$, i.e C is embedded on a simple abelian surface. We have a natural candidate for such a surface, which is the Jacobian of C.

Given that S is an abelian surface, the inclusion $C \hookrightarrow S$ factors through the Jacobian of C: $C \hookrightarrow J \xrightarrow{h} S$. Since S is simple, the map h must be surjective. Assume by contradiction that h is not an isomorphism, i.e. there exists $x \in J$ such that $h(x) = 0$. We know that the canonical map $C \to J$ is a closed immersion and the image of C is an ample divisor on J. In particular, this means that the curves $x + C$ and C intersect inside J, which means there exist $y, z \in C$ such that $x + y = z$. We conclude:

$$h(z) = h(x + y) = h(x) + h(y) = h(y),$$

but this contradicts the map $C \hookrightarrow S$ being an embedding, so we are done.

□

10.2 K3 Surfaces

Lemma 10.2. *The moduli space of polarized K3 surfaces of a given degree* $2d$ *is a quasi-projective variety of dimension 19.*

Proof. See [1]. □

Theorem 10.3. *There exists no smooth surface* S *whose minimal model is a K3* *surface, that contains a smooth ample very general curve C of genus $\geqslant 19$ satisfying* $\dim |C| = 0$.

Proof. Assume by contradiction we can find such an embedding $C \hookrightarrow S$, where C is a very general curve of genus $g \geqslant 19$. Let S_0 be the minimal K3 surface associated to S and $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$ be the corresponding sequence of blow-downs. By the Enriques-Kodaira classification (see Thm. 7.9), we know that S_0 satisfies:

$$p_g(S_0) = 1, q(S_0) = 0, K_{S_0} \sim 0.$$

The curve C is ample on S, hence $C^2 > 0$. Using this fact together with the genus formula and Lemma 8.6, we obtain:

$$2g - 2 = C \cdot K_S + C^2 > C \cdot K_S \geqslant C_0 \cdot K_{S_0} + 2n = 2n. \tag{10.2}$$

Now we count moduli: by Lemma 10.2 stated above, the number of moduli of S_0 is 19. It follows that the number of moduli of S is $19 + 2n$, since we add 2 moduli for each point we blow up from S_0 all the way up to S_n. On the other hand, Lemma 8.7 states that the number of moduli of S needs to be at least $3g - 3$. Putting this information and Equation 10.2 together, we obtain a contradiction:

$$3g - 3 \leqslant 19 + 2n \leqslant 19 + (2g - 4) = 2g + 15 \Rightarrow g \leqslant 18.$$

□

10.3 Enriques Surfaces

Lemma 10.4. *The moduli space of Enriques surfaces is an irreducible smooth Artin stack of dimension 10.*

Proof. See [9]. $\square$

Theorem 10.5. *There exists no smooth surface S whose minimal model is an Enriques surface, that contains a smooth ample very general curve C of genus $\geqslant 10$ satisfying* $\dim |C| = 0$.

Proof. Assume by contradiction we can find such an embedding $C \hookrightarrow S$, where C is a very general curve of genus $g \geqslant 10$. Let S_0 be the minimal Enriques surface associated to S and $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$ be the corresponding sequence of blow-downs. By the Enriques-Kodaira classification (see Thm. 7.9), we know that S_0 satisfies:

$$p_g(S_0) = q(S_0) = 0, K_{S_0} \nsim 0, 2K_{S_0} \sim 0.$$

The curve C is ample on S, hence $C^2 > 0$. Using this fact together with the genus formula and Lemma 8.6, we obtain:

$$2g - 2 = C \cdot K_S + C^2 > C \cdot K_S \geqslant C_0 \cdot K_{S_0} + 2n = 2n. \tag{10.3}$$

Now we count moduli: by Theorem 10.2 stated above, the number of moduli of S_0 is 10. It follows that the number of moduli of S is $10 + 2n$, since we add 2 moduli for each point we blow up from S_0 all the way up to S_n. On the other hand, Lemma 8.7 states that the number of moduli of S needs to be at least $3g - 3$. Putting this information and Equation 10.3 together, we obtain a contradiction:

$$3g - 3 \leqslant 10 + 2n \leqslant 10 + (2g - 4) = 2g + 6 \Rightarrow g \leqslant 9.$$ $\square$

10.4 Bielliptic Surfaces

Theorem 10.6. *There exists no smooth surface S whose minimal model is a bielliptic surface, that contains a smooth ample very general curve C.*

Proof. Assume by contradiction we can find such an embedding $C \hookrightarrow S$. Let S_0 be the minimal bielliptic surface associated to S and $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$ be the corresponding sequence of blow-downs. By the Enriques-Kodaira classification (see Thm. 7.9), we know that S_0 satisfies:

$$p_g(S_0) = 0, q(S_0) = 1, nK_{S_0} \sim 0, n \in \{2, 3, 4, 6\}.$$

By Theorem 8.3, we know that $q(S)$ must be either 0 or g. Since the irregularity of a surface is a birational invariant, we must have that $q(S) = q(S_0) = g = 1$. Thus, we are in the situation where the genus of the curve C is 1. Using this fact together with the genus formula and Lemma 8.6, we obtain the following contradiction:

$$0 = 2g - 2 = C \cdot K_S + C^2 > C \cdot K_S \geqslant C_0 \cdot K_{S_0} + 2n = 2n.$$

□

Chapter 11

Surfaces of Kodaira dimension 1

In this chapter we analyze the scenario in which a very general curve C of genus g is embedded as an ample divisor on a smooth surface S of Kodaira dimension 1 such that $\dim |C| = 0$. We show that if $g \geqslant 21$, such an embedding does not exist.

Setup 11.1. *Let S be a smooth surface of Kodaira dimension 1 and S_0 its unique smooth minimal model. By the Kodaira-Enriques classification of surfaces (see Theorem. 7.9), we know that S_0 is an elliptic surface, i.e. there exists a surjective morphism $S_0 \to B$ onto a smooth curve, such that all but finitely many fibers are smooth irreducible curves of genus 1. As before, let C be our ample very general curve on S satisfying* $\dim |C| = 0$*, and let C_0 be the image of C in S_0. We will be working with the following morphisms:*

$$C \hookrightarrow S \twoheadrightarrow S_0 \twoheadrightarrow B. \tag{11.1}$$

Lemma 11.2. *Let $\pi : S_0 \to B$ be an elliptic surface over B and $\pi^* : Pic^0(B) \to Pic^0(S_0)$ the pullback morphism on the Picard groups. Then π^* is injective.*

Proof. First, notice that $\mathcal{O}_B = \pi_* \mathcal{O}_{S_0}$. This is true because every fiber is connected, so the result follows by base change. Consider $\mathcal{L} \in \mathrm{Pic}^0(B)$ such that $\pi^*\mathcal{L} = \mathcal{O}_{S_0}$. We apply the projection formula and conclude that π^* is injective:

$$\mathcal{L} = \mathcal{O}_B \otimes_{\mathcal{O}_B} \mathcal{L} \cong \pi_*\mathcal{O}_{S_0} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L} \cong \pi_*(\mathcal{O}_{S_0} \otimes_{\mathcal{O}_{S_0}} \pi^*\mathcal{L}) \cong \pi_*\mathcal{O}_{S_0} = \mathcal{O}_B.$$

□

Lemma 11.3. *Let C, S, S_0, B as in the Setup 11.1 above. Then $B \cong \mathbb{P}^1$.*

Proof. As a consequence of Theorem 8.2 and Lemma 11.2, we have the following sequence of morphisms: $\mathrm{Pic}^0(B) \hookrightarrow \mathrm{Pic}^0(S_0) \xrightarrow{\cong} \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}^0(C)$. By Lemma 8.1, since C is a very general curve, we have that $\mathrm{Pic}^0(C)$ is simple, therefore either $\mathrm{Pic}^0(B) = 0$ or the composed map $\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(C)$ is an isomorphism.

If the composed map $\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(C)$ is an isomorphism, then by the Torelli theorem for curves (see Thm. 7.2), the composed map $C \to B$ is an isomorphism. However, if this is the case, we get a minimal elliptic surface $S_0 \to C$ with a section σ. As a consequence of Theorem 11.5 below, we obtain $C^2 = -\sigma^2 \leqslant 0$, which contradicts the ampleness of C. In conclusion, we must have that $\mathrm{Pic}^0(B) = 0$, i.e. $B \cong \mathbb{P}^1$.

□

As a consequence of the Lemma 11.3 above, S_0 is an elliptic fibration over $\mathbb{P}^1$. Now we need to analyze two scenarios: either the fibration admits a section, or it doesn't. The simpler case is when the fibration has a section, so we will analyze that case first. The second case builds upon the first one, as we will see momentarily.

Proposition 11.4. *Let $\pi : S_0 \to \mathbb{P}^1$ be a minimal elliptic surface. Then the irregularity of S is zero.*

Proof. See [31], Corollary 2.4. □

Now, assume $S_0 \to \mathbb{P}^1$ is an elliptic fibration that admits a section σ. In this case, we have some nice properties, summarized by Miranda (see [31]):

Theorem 11.5. *Let $\pi : S_0 \to \mathbb{P}^1$ be a minimal elliptic surface over $\mathbb{P}^1$ with section σ. Then $R^1\pi_*\mathcal{O}_{S_0} \cong \mathcal{O}_{\mathbb{P}^1}(-N)$ for some $N \geqslant 0$. The surface S_0 is a product if and only if $N = 0$. Moreover, if $N > 0$, then:*

(i) $\sigma^2 = -N$

(ii) $K_{S_0} \sim (N-2)F$*, where F is the class of the fiber of π.*

Proof. See [31], Corollary 2.4. □

Remark 11.6. *Observe that, in our setup, we are in the case where $N > 0$. To see this, assume by contradiction that $N = 0$ and $S_0 = E \times \mathbb{P}^1$, where E is an elliptic curve. Recall we have the following sequence of morphisms: $Pic^0(S_0) \xrightarrow{\cong} Pic^0(S) \hookrightarrow Pic^0(C)$. Since $Pic^0(S_0) \cong Pic^0(E \times \mathbb{P}^1) \cong Pic^0(E)$, this contradicts $Pic^0(C)$ being a simple abelian surface of dimension $g \geqslant 2$.*

In his paper (see [31]), Miranda constructed a coarse moduli space parametrizing Weierstrass fibrations over $\mathbb{P}^1$, which are in a 1-to-1 correspondence with minimal elliptic surfaces over $\mathbb{P}^1$ that admit a section σ. We refer the reader to the original paper for more information. Here is the main result we need:

Theorem 11.7. *Let N be a positive integer. There exists a coarse moduli space that parametrizes minimal elliptic surfaces $\pi : S \to \mathbb{P}^1$ that admit a section and satisfy $N = -\deg R^1_\pi \mathcal{O}_S$. The dimension of this moduli space is $10N - 2$.*

Proof. See [31]. □

Generalizing Miranda's work, Seiler constructed the moduli space of polarized minimal elliptic surfaces (see [34]). To deal with the missing section $\sigma : \mathbb{P}^1 \to S$, he worked instead with the associated Jacobian of such a fibration, which is obtained as follows:

Definition 11.8. *Let $f : S \to B$ be an elliptic surface over an algebraically closed field k. Let $K = k(B)$ be the function field of B. The general fiber S_K of f is a curve of genus 1 over K, therefore its Jacobian J_K is an elliptic curve over K. This Jacobian extends to an elliptic surface $j : J \to B$, which is called the Jacobian fibration associated to f. This fibration comes equipped with a section, namely the closure of the zero divisor on S_K.*

Here is Seiler's main result:

Theorem 11.9. *Let N be a positive integer. There exists a coarse moduli space that parametrizes minimal elliptic surfaces $\pi : S \to \mathbb{P}^1$ that have k multiple fibers and satisfy $N = -\deg R^1_p\mathcal{O}_J$, where $p : J \to \mathbb{P}^1$ is the associated Jacobian fibration. The dimension of this moduli space is* $10N + k - 2$.

Now that we have the moduli count, we are almost ready to prove the main statement. Before we do so, we need two more lemmas from Seiler:

Theorem 11.10. *Let $f : S \to C$ be an elliptic fibration with multiple fibers m_iD_i, where $1 \leqslant i \leqslant k$. Let $L = (R^1f_*\mathcal{O}_S)^\vee$. Then:*

$$\omega_S = f^*(L \otimes \omega_C) \otimes \mathcal{O}_S(\sum_i (m_i - 1)D_i).$$

Proof. See [34], Theorem 1.1. □

Lemma 11.11. *Let $f : S \to C$ be an elliptic surface and $j : J \to C$ be the corresponding Jacobian fibration. Then $R^1f_*\mathcal{O}_S \cong R^1j_*\mathcal{O}_J$.*

Proof. See [34], Lemma 1.3. □

Now that we stated all the necessary prerequisites, we turn to the main theorem of this section:

Theorem 11.12. *There exists no smooth surface S of Kodaira dimension 1 on which one can embed a smooth very general curve C of genus $g \geqslant 21$ as an ample divisor satisfying* $\dim |C| = 0$.

Proof. Assume by contradiction that such a surface S exists. Let S_0 be its unique smooth minimal model and $S = S_n \to \cdots \to S_0$ the corresponding sequence of blow-downs. From Lemma 11.3, we know that S_0 is an elliptic surface over $\mathbb{P}^1$ with (possible) multiple fibers $m_1 D_1, \ldots, m_k D_k$, where $k \geqslant 0$.

Let $p : J \to \mathbb{P}^1$ be the Jacobian fibration associated to S_0 and $N = -\deg R^1_p \mathcal{O}_J$. By Remark 11.6, we know that $N > 0$, so the moduli space parametrizing such surfaces S_0 has dimension $10N+k-2$. As before, this means that S has $10N+k-2+2n$ moduli. On the other hand, since C is a very general curve on S and $\mathrm{Pic}^0(S) = 0$, Lemma 8.7 gives us that S must have at least $3g-3$ moduli, so we conclude our first inequality:

$$3g - 3 \leqslant 10N + k - 2 + 2n. \tag{11.2}$$

On the other hand, let d be the degree of the composed map $C \hookrightarrow S \to S_0 \to \mathbb{P}^1$. Since C is a very general curve, Brill-Noether theory (see Thm. 7.11) gives the following bound on d:

$$\rho(g, 1, d) = 2d - g - 2 \geqslant 0 \iff d \geqslant \frac{g+2}{2}. \tag{11.3}$$

Now, as a consequence of Theorem 11.5 and Lemma 11.11, we obtain $C_0 \cdot K_{S_0} = d(N-2)+\sum_{i=1}^{k}(m_i-1)C_0 \cdot D_i$. Combining this fact with the genus formula, Lemma 8.6, and Equation 11.3, we conclude:

$$
\begin{aligned}
2g-2 = C^2 + C\cdot K_S > C\cdot K_S \geqslant C_0\cdot K_{S_0} + 2n \\
&= (N-2)d + \sum_{i=1}^{k}(m_i-1)C_0\cdot D_i + 2n \\
&\geqslant (N-2)d + k + 2n \geqslant \frac{(N-2)(g+2)}{2} + k + 2n,
\end{aligned}
$$

from which we derive that:

$$
2n + k < 2g - 2 - \frac{(N-2)(g+2)}{2}. \tag{11.4}
$$

Finally, we can put equations 11.2 and 11.4 together to obtain:

$$
3g-3 \leqslant 10N - 2 + k + 2n \leqslant 10N - 2 + 2g - 2 - \frac{(N-2)(g+2)}{2}, \tag{11.5}
$$

which is a contradiction when $g \geqslant 21$. In conclusion, there exists no smooth surface S of Kodaira dimension 1 on which one can embed a smooth very general curve C of genus $g \geqslant 21$ as an ample divisor. $\square$

Chapter 12

Surfaces of Kodaira dimension 2

In this chapter we analyze the scenario in which a very general curve C of genus g is embedded as an ample divisor on a surface of general type. From Proposition 8.3, we know that the irregularity $q(S)$ of the surface must be either 0 or g. If $q(S) = g$, then S is birational to the symmetric product $\mathrm{Sym}^2(C)$. The case where $q(S) = 0$ remains mainly open. However, we prove the following statement: if $S \hookrightarrow \mathbb{P}^r$ is a complete intersection such that the composed map $C \hookrightarrow S \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture, then $g \leqslant 15$.

12.1 Surfaces of general type with $q(S) = g$

In the case where S is of general type and satisfies $q(S) = g$, we have a complete characterization due to Mendes Lopes and Pardini (see [28]):

Theorem 12.1. *Let C be a very general genus g curve embedded on a smooth surface S of general type with irregularity $q(S) = g$. Then S is birational to $Sym^2(C)$.*

Proof. We follow the proof of Theorem 1.1 in [28]. Let S_0 be the (unique) smooth minimal model of S. We first claim that S_0 is not a product of curves. To prove this, assume by contradiction that $S_0 \cong C_1 \times C_2$, where $g(C_1) + g(C_2) = g$ and $g(C_1), g(C_2) \geqslant 2$. By Theorem 8.2, we obtain the following sequence of morphisms:

$$\mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2) = \mathrm{Pic}^0(S_0) \xrightarrow{\cong} \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}^0(C),$$

but this contradicts the simpleness of $\mathrm{Pic}^0(C)$ (see Thereom 8.1). Therefore, we can assume that S_0 is not a product of curves.

Let $d = C^2 > 0$. There exists a d-dimensional system of curves $\mathcal{C}$ on S which are numerically equivalent to C (see [28], Proposition 4.3). Moreover, all smooth elements of $\mathcal{C}$ are isomorphic to C. Now, if $d > 1$, then S is not of general type (see [3], §0). Therefore, $C^2 = 1$ and the conclusion follows by [3], Proposition 0.18. □

Remark 12.2. *We remark here that the situation above can happen. Fix $x \in C$ and consider the composed morphism $f_x : C \hookrightarrow C \times C \to Sym^2(C)$ that maps $y \mapsto x + y$. Let C_x be the image of C in $Sym^2(C)$. First of all, it is easy to see that $C_x^2 = 1$. Moreover, C_x is ample on $Sym^2(C)$ because its inverse image in $C \times C$ is ample (it is the union of $C \times \{x\}$ and $\{x\} \times C$).*

12.2 Surfaces of general type with $q(S) = 0$

While this case remains mostly unsolved, we would like to discuss a simple scenario in which $q(S) = 0$, i.e. when S is a complete intersection. Our result is the following:

Theorem 12.3. *Let C be a very general smooth curve. Let S be a complete intersection in $\mathbb{P}^r$ of multidegree $(d_1, \ldots, d_{r-2})$, where $2 \leqslant d_1 \leqslant \cdots \leqslant d_{r-2}$. Assume C is embedded in S as an ample divisor such that the composed map $C \hookrightarrow \mathbb{P}^r$ satisfies the Maximal Rank Conjecture. Then the genus of C satisfies $g \leqslant 15$.*

Proof. Assume by contradiction that we can embed a very general curve C of genus $g \geqslant 16$ on a complete intersection S, where S has multidegree $(d_1, \ldots, d_{r-2})$ and $2 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{r-2}$, such that $C \to \mathbb{P}^r$ is a general degree d Brill-Noether curve.

Since S is a complete intersection, then $\omega_S = \mathcal{O}_S(\sum_{i=1}^{r-2} d_i - r - 1)$, which means:

$$C \cdot K_S = d(\sum_{i=1}^{r-2} d_i - r - 1). \tag{12.1}$$

On the other hand, $C \to \mathbb{P}^r$ is a non-degenerate map of degree d, so by Brill-Noether theory (see Thm. 7.11), we obtain:

$$d \geqslant \frac{r}{r+1} g + r. \tag{12.2}$$

Now, we claim the following inequality holds:

$$\sum_{i=1}^{r-2} d_i - r - 1 \leqslant 2. \tag{12.3}$$

To prove this, assume by contradiction that $\sum_{i=1}^{r-2} d_i - r - 1 \geqslant 3$. This means that $C \cdot K_S \geqslant 3d$. We combine the genus formula, the ampleness of C, and Equations 12.1 and 12.2 to obtain a contradiction:

$$2g - 2 = C^2 + C \cdot K_S > C \cdot K_S \geqslant 3d \geqslant 3(\frac{r}{r+1} g + r).$$

Since Inequality 12.3 fails for $r \geqslant 8$, we are left to analyze the cases where $3 \leqslant r \leqslant 7$. The composed map $C \to \mathbb{P}^r$ satisfies the Maximal Rank Conjecture (see Thm. 12.4), which means that in order to have a degree k polynomial vanishing on $C \hookrightarrow \mathbb{P}^r$, the following inequality needs to be satisfied:

$$\binom{k+r}{k} > kd + 1 - g. \tag{12.4}$$

We combine Inequalities 12.4 and 12.2 with the fact that $g \geqslant 16$ to obtain:

$$\binom{k+r}{k} > k\frac{r^2 + 17r}{r+1} - 15. \tag{12.5}$$

If $r = 3$, then S is a complete intersection of type (d_1) on $\mathbb{P}^3$. Equation 12.5 gives us the following bound:

$$\binom{3+k}{k} > 15k - 15 \Rightarrow k \geqslant 6,$$

which means that the degree d_1 of S is at least 6. On the other hand, Inequality 12.3 implies $d_1 \leqslant 6$. Hence d_1 must be exactly 6. Combining Inequalities 12.4 and 12.2, we obtain:

$$\binom{9}{6} \geqslant \frac{7}{2}g + 19 \Rightarrow g \leqslant 18.$$

Therefore, we only need to analyze the cases where $d_1 = 6$ and $g = 16, 17, 18$. Inequality 12.2 gives us the following bounds on d:

$$g = 16 \Rightarrow d \geqslant 15$$

$$g = 17 \Rightarrow d \geqslant 16$$

$$g = 18 \Rightarrow d \geqslant 17,$$

but all these bounds contradict the ampleness of C:

$$0 < C^2 = 2g - 2 - C \cdot K_S = 2g - 2 - 2d \leqslant 0.$$

If $r = 4$, then S is a complete intersection of type (d_1, d_2) on $\mathbb{P}^4$. Equation 12.5 gives us the following bound:

$$\binom{4+k}{k} > \frac{84}{5}k - 15 \Rightarrow k \geqslant 4,$$

which means that the degrees d_1, d_2 of S are at least 4, but that contradicts Inequality 12.3.

If $r = 5$, then S is a complete intersection of type (d_1, d_2, d_3) and $C \hookrightarrow S \hookrightarrow \mathbb{P}^5$. Equation 12.5 gives us the following bound:

$$\binom{5+k}{k} > \frac{55}{3}k - 15 \Rightarrow k \geqslant 3,$$

which means the degrees d_1, d_2, d_3 of S are at least 3, but that contradicts Inequality 12.3.

If $r = 6$, then Inequality 12.3 implies that $\sum_{i=1}^{4} d_i < 10$. This means that either $d_1 = d_2 = d_3 = d_4 = 2$ or $d_1 = d_2 = d_3 = 2$ and $d_4 = 3$. If we are in the first case, the space of polynomials of degree 2 that vanish on C must be at least 4, which by the Maximal Rank Conjecture (see Thm. 12.4), translates to:

$$\binom{6+2}{2} - (2d + 1 - g) \geqslant 4 \Rightarrow 23 + g \geqslant 2d.$$

Combining the above with Equation 12.2, we obtain that $g \leqslant 15$, which is a contradiction. If we are in the second case, the space of polynomials of degree 2 that vanish on C must be at least 3, which by the Maximal Rank Conjecture (see Thm. 12.4), translates to:

$$\binom{6+2}{2} - (2d+1-g) \geqslant 3 \Rightarrow 24 + g \geqslant 2d.$$

Combining the above with equation 12.2, we obtain that $g \leqslant 16$. In the special case when $g = 16$, we apply Inequality 12.2 again and get the following contradiction:

$$30 = 2g - 2 > K_S \cdot C = 2d \geqslant 2(\frac{6}{7} \cdot 16 + 6) \geqslant 39.$$

If $r = 7$, then Inequality 12.3 implies that $\sum_{i=1}^{5} d_i < 11$. This means that $d_1 = \cdots = d_5 = 2$, so the space of polynomials of degree 2 that vanish on C must be at least 5, which by the Maximal Rank Conjecture (see 12.4), translates to:

$$\binom{7+2}{2} - (2d+1-g) \geqslant 5 \Rightarrow 30 + g \geqslant 2d.$$

Combining the above with equation 12.2, we obtain that $g \leqslant 21$. On the other hand, we apply Inequality 12.2 again and get the following contradiction, finishing the proof:

$$2g - 2 > C \cdot K_S = 2d \geqslant 2(\frac{7}{8}g + 7) \Rightarrow g > 64.$$

□

Remark 12.4. *We note that Verra ([38]) proved the unirationality of the moduli space of genus g curves $\mathcal{M}_g$ for $11 \leqslant g \leqslant 14$ by embedding a general smooth curve C on a complete intersection, hence this situation does happen.*

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