Brentanian continua and their boundaries

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Abstract

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This dissertation focuses on how a specific conceptual thread of the history of mathematics unfolded throughout the centuries from its original account in Ancient Greece to its demise in the Modern era due to new mathematical developments and, finally, to its revival in the work of Brentano. In particular, we shall discuss how the notion of continuity and the connected notion of continua and boundaries developed through the ages until Brentano's revival of the original Aristotelian account against the by then established mathematical ortodoxy. Thus, this monograph hopes to fill in a gap in the present state of Brentanian scholarship as well as to present a thorough account of this specific historical thread.

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Introduction

Imagine we have a light that shines with a blue colour. At one point, though, say at noon today, this colour immediately changes and the light starts to shine a red light. Then, one could ask: what is the *last* instant in which the light's colour was blue? And, analogously, one could also ask: what is the $first$ instant that the light started to shine red? An obvious answer to both questions seems to be precisely noon. However, if we take a closer look at these answers, we will realize, first that they cannot be both true together, if we assume that there is only one instant happening at noon; for if this instant is *both* the last instant in which the light was blue and the first instant that the light was red, we will necessarily have to say that the light was both fully blue and fully red at this instant, which seems to be a contradiction.

Maybe, then, one might reply, we should say that noon is the $first$ instant the light was red and we can take an instant that is one millisecond before noon and claim that that instant is the last instant in which the light was blue. A first obvious objection to this, however, would be: but what in the world allows one to pick the later colour as the one actually present at noon and not the earlier one? Indeed, this attempted solution is guilty of some apparent inherent air of arbitrariness; one cannot present a principled reason to choose red instead of blue as the colour of the light at noon. However, there is a further problem hiding in this simple alleged solution. A natural assumption about time is that between any two instants there is at least another instant. Now, if this is true, then, looking back at our example, we must conclude that there is at least another instant between that last instant in which the light was blue and noon; and, hence, we can ask: what was the colour of the light in *that* instant. If we answer "blue", then we would contradict the claim that the instant one millisecond before noon was the *last* instant in which the light was blue; and, if we answer "red", then we would contradict the claim that the instant at noon was indeed the $first$ instant in which the light was red. Thus, we must conclude that in this particular instant between noon and a millisecond before noon, the light was neither blue not red. But we assumed that the change was immediate, that is, that no instant passed between the light being blue and then red. We definitely seem to have sunk deeply into a murky logical swamp.

Indeed, this kind of problems have a very old history, dating back at least to Ancient Greece, but they are nonetheless also relevant today, with discussions as to how to solve them coming from all directions of research. In particular, we can mention the 20th century attempts, pioneered by Whitehead, de Laguna, Broad and other authors, at constructing a "point-free" version of topology. However, the main point to keep in mind here is that all these discussions have as common ground the discussion of the topological notion of *continuity*.

This notion is, however, not an intrinsically clear one. In fact, different circumstances call for slightly different ideas regarding how to cash out this notion of continuity. Nowadays, in mathematics, the most common usage of this notion of continuity is perhaps in connection to functions. In this sense, a function — more specifically, let us think here of a simple function from real numbers to real numbers — is said to be continuous if one can draw its graph without taking the pen off the paper, i.e., if the line that we recognize as its graph does not contain any "jumps". The usual formal characterization of this property is the classical ϵ-δ property, which shall be studied in a little more depth further on in this dissertation, but which can be intuitively described here as the property according to which a "small" change of the argument of the function only produces a "small" change in its value at this argument — or, in other words, a change to a "nearby" argument point moves the value of the function also to a "nearby" value. We can see in this sense how this notion is also connected to the notion of "closeness". Nonetheless, this transition into a rigorous definition does away with the original intuition of gaplessness in the sense that we are now always considering a "close" argument point, but one that is indeed some distance away from the original point and that cannot be considered the "next" point following the original argument in the sense that there is a whole interval of points strictly between the two points we are considering.

The main goal of this dissertation, thus, is to unravel a specific thread in the history of mathematics. More specifically, we shall discuss the particular development of this notion of continuity, that started in Ancient Greece and eventually evolved into a key notion of the modern mathematical theory of point-set topology.

However, we shall look at this historical thread from the perspective of Brentano's ideas concerning these matters as they were presented in his late texts from the first decades of the 20th century. In doing so, our goal is to fill out a gap in Brentano's scholarship, as well as to offer a fairly self-contained presentation of this particular historical thread — giving special emphasis on details that share connections with Brentano's ideas on the matter. It is not only my personal opinion that problems — and, ultimately, their solutions — are much better understood in historical context, but also, Brentano himself was a fairly historically oriented thinker and, although he presented many radically new ideas, he always relied on the historical tradition of every problem he dealt with.

Thus, the first chapter of the dissertation deals with the beginning of the history of this concept of continuity, as it was presented by Aristotle and further developed both by later Ancient and by Medieval thinkers. Special attention will be given to a particular interpretation of these topics by a specific Medieval tradition that became known to contemporary writers as quasi-Aristotelianism. This is done since this tradition contains ideas that are very much in tune with the later Brentanian account of the subject, so that the study of this earlier tradition shall provide us not only with the necessary historical underpinning of Brentano's position, but also with an early presentation of the logical structure of the problems of the Aristotelian position, and of the structure of a solution that will eventually turn out to be analogous to the later Brentanian solution.

In the second chapter, we shall look at how the notion of continua developed from the original Aristotelian background assumptions — in particular, from the claim that these continua cannot be composed out of indivisibles — to the modern mathematical, or point-set theoretical, conception that relies heavily on indivisibles. It is our thesis that this change was carried out in the wake of new and very powerful integration methods that were developed around the beginning of the Early Modern period. Accordingly, we shall briefly study these methods and point out their shared characteristics which contribute to yield a logical defense of the thesis that continua are indeed composed out of indivisibles.

The third chapter, then, is concerned with Brentano's own ideas on this topic, from his criticisms of the mathematical constructions of continua in his time, to his positive ideas regarding their nature and properties. These ideas compose a fairly detailed account of the nature of continua which, I believe, amounts to a very interesting and complete description of what one might call, as will be explained later in this monograph, "the natural topology of the 3-dimensional objects of outer experience". This topology will radically differ from the orthodox mathematical topological accounts in that there will not be open sets, since a basic thesis of this account is that every bounded portion of space has its own boundary. This, in turn, will yield a very down to earth theory of contact and of change that bypasses all the problems that creep into the orthodox set-theoretical approach to defining these concepts.

It is my view that this Brentanian conception of continua relates to new, extensive and exciting work — by authors like Gotts, Gooday and Cohn, and Lando and Scott — that is being carried out today both from a philosophical and from a formal-mathematical perspective. This monograph aims at providing a different perspective to these questions and paves the way for the introduction of Brentanian concepts and ideas into this current debate.

CHAPTER 1

Aristotle and the origins of the notion of continuity

Like many other crucial notions that are nowadays commonplace in science and mathematics, our first recorded logical assessment of the notion of continuity dates from Ancient Greece. In particular, we can date the first known inquiries into this notion and the problems inherent to it to two disciples of Parmenides of Elea: Melissus of Samos and Zeno of Elea. The latter, indeed, became especially famous for his arguments that relied heavily on the infinite divisibility of space and time to derive so-called "paradoxes of motion" and, thus, vindicate his teacher's doctrine that motion was not real. For instance, we can mention the classical arrow paradox which aims to conclude that an arrow shot could never reach its intended target. For, say the target is set 10m from where the arrow was shot; then, in order to get to the target, the arrow would have to travel at least until the midway point, which is 5m away. The same argument, however, can be used again having now the midway point as the new location where the arrow begins its motion and we must conclude that in order to get to the target from this new position the arrow needs to go through the new midway point that is 2.5m away. Moreover, from every new mid point we can carry out the argument, so that by applying it an ever increasing number of times we arrive at the result that the arrow would have to travel an infinite number of distances each being half the size of the former, but all of them being nonetheless strictly positive. This, Zeno argued, would imply that this traversed distance must be infinite and, therefore, that a complete motion until the target would be indeed impossible!

Now, Aristotle, surely having as one of his goals the refusal of such paradoxes and the idea that there must be a way to dispel them by means of a proper understanding of the terms used, was the first to really present a thorough and strictly speaking logical account of this notion of "continuity".

1. Aristotle's theory of continuity

Aristotle's account of continuity can be separated into two distinct parts. On the one hand, we have his "offical" abstract account in the *Physics* and the *Categories* in which he considers continuity as a property of some things or quantities in opposition to others. On the other hand, however, we have the actual usage that Aristotle makes of the notion of continuity — specifically with respect to time — in the context of motion and transformations.

In this chapter we shall first concentrate on the former aspect and, once we have a good grasp of it, we shall discuss the enormously controversial application to motion and its development until the Medieval period.

1.1. The definition. Aristotle presents us in Book V of the *Physics* a definition of a notion of continuity, together with the related more basic notions of "being next-in-succession" and of "contiguity". The reason for this, as he recognizes in the beginning of Book III, is that

> motion is supposed to belong to the class of things which are continuous; and the infinite presents itself first in the continuous—that is how it comes about that the account of the infinite is often used in definitions of the continuous; for what is infinitely divisible is continuous. (*Physics*, 200b16-20)

Hence, as far as our main topic is concerned, Aristotle here not only claims that motion has a particular connection to the notion of a continuum — a fact that shall lead to some tensions, which shall be discussed later —, thus explaining the connection that will lead him to further discuss this latter notion, but also already stipulates a fundamental relation between continua and the notion of infinity, by recognizing a first specific property of continua, *viz.* the property of being "infinitely divisible".[1] This notion is to be contrasted with what Aristotle calls "an indivisible".[2] These can be interpreted either as extended atoms that cannot be divided because of some fundamental impossibility in its own nature, or, as we shall do henceforth in this monograph, as things that fail to be able to be divided because they lack extension in some spatial way.

This seems to be a fairly minor point, but, in fact, these two notions of an "indivisible" have extremely different logical implications for one's conception of them. Indeed, having some true extension, the first kind of indivisibles — that we might call something like "the thick conception of atoms" — can easily be though of as composing an extended continuum; something like the Archimedean axiom would guarantee that, no matter how small the extension of these thick atoms, if we juxtapose enough of them, we will arrive at a continuum that is as large as we want. Therefore, although these thick atoms were fairly common place in the mind of Ancient atomists, it seems that

[1] In fact, this property of continua is mentioned by Aristotle already in book I of the *Physics*: "the continuous is divisible *ad infinitum*" [εὶς ἄπειρον γὰρ διαιρετὸν τὸ σινεχές] (185b10-185b11).
[2] Aristotle uses the word ἀδιαιρετον, which is exactly the Greek counterpart of "indivisible".

Aristotle's notion of "an indivisible" was indeed the truly unextended version, as we shall justify more thoroughly further on in order to consider Aristotle's claim that continua cannot be made up of indivisibles.

But first, let us consider what one can call "Aristotle's technical definition of continua", which is presented further down the text of the *Physics*. This passage is important because it lays down the structure of an account of the notion of continuity that, not only was completely dominant until at least the Late Middle Ages, but also played a significant role in characterizing Brentano's views on the matter, as will become clear in a later chapter of this monograph. Thus, we shall quote the full passage here and then go on to present our reading of it.

> A thing is in succession when it is after the beginning in position or in form or in some other respect in which it is definitely so regarded, and when further there is nothing of the same kind as itself between it and that to which it is in succession [...].
>
> A thing that is in succession and touches is contiguous. The continuous [σινεχές] is a subdivision of the contiguous: things are called continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. And in whatever way that which holds them together is one, so too will the whole be one, e.g. by a rivet or glue or contact or organic union.
>
> It is obvious that of these terms 'in succession' is primary; for that which touches is necessarily in succession, but not everything that is in succession touches: and so succession is a property of things prior in definition, e.g. numbers, while contact is not. And if there is continuity there is necessarily contact, but if there is contact, that alone does not imply continuity; for the extremities of things may be together without necessarily being one; but they cannot be one without necessarily being together [ἅμα]. So natural union is last in coming to be; for the extremities must necessarily come into contact if they are to be naturally united; but things that are in contact are not all naturally united, while where there is no

> contact clearly there is no natural union either. Hence, if as some say points and units have an independent existence of their own, it is impossible for the two to be identical; for points can touch while units can only be in succession. Moreover, there can always be something between points (for all lines are intermediate between points), whereas it is not necessary that there should be anything between units; for there is nothing between the numbers one and two. (*Physics*, Book V, 226b34-227a34)[3]

At a first glance, we readily see that these definitions hint at something like a distinction we might make today between discrete and non-discrete entities, for the former are the things that can be next-in-succession, in the sense that nothing of the same character is between the previous thing and what is next-in-succession to it. The interesting thing to note, however, is that, although unfortunately[4] Aristotle was still very far from a further subdivision that modern mathematicians make inside the non-discrete class between the "truly continuous" and the "merely dense",[5] he did nonetheless provide us with another interesting distinction to be made in this domain of the non-discrete — *viz.* the distinction between "the merely contiguous" and the "truly continuous" things.

Now, in order to carry out a proper clarification of this Aristotelian characterization, we need to understand the corresponding notion of "limit" or "boundary", which shall be done in the next section.

[3] Another interesting passage is the following passage from the *Categories*:

> To quantity let us turn next. This is either discrete or continuous. [...] Of quantities that are discrete we may here instance number and speech, of quantities that are continuous line, superficies and solid, to which time and place may be added. [...] Thus is number discrete, not continuous. The same may be said about speech, if by speech the spoken word is intended. Being measured in long and short syllables, speech is an evident quantity, whose parts possess no common boundary. No common limit exists, where those parts—that is, syllables—join. Each, indeed, is distinct from the rest.
>
> A line is, however, continuous. Here we discover that limit of which we have just now been speaking. This limit or term is a point. So it is with a plane or a solid. Their parts also have such a limit— line in the case of the former, a line or a plane in the latter. Again, time and space are continuous. Time is a whole and continuous; the present, past, future are linked. Space is also this kind of a quantity. For seeing the parts of a solid themselves occupy so much space and these parts have a limit in common, it follows the parts of space also, which those parts themselves occupy, have exactly the same common limit or term as the parts of the solid. As is time, so is space, then, continuous: the parts meet at one common boundary. (*Categories*, VI, 4b22-5a14)

[4] And, of course, very much understandably.

[5] The latter being that which has the property that between any two things of this kind there is always another of the same kind in between, while the former requires a much more complicated property, which shall be discussed in a later chapter.

1.2. Boundaries. We should pay attention to the fact that, in order to carry out the aforementioned distinction between "being next-in-succession", "being contiguous" and "being continuous", Aristotle introduces the notion of "limit" [ἔσχατον]. In particular, he is assuming that things that can be contiguous or continuous, i.e., what we would now call the domain of the non-discrete, all have these limits and that these limits can touch or even fuse into a single limit, or, as Aristotle puts it, they can "become one and the same". In order to standardize our terminology, we shall use the fairly synonymous term "boundary" for ἔσχατον, which is usually translated as "limit" or "extremity". I would like to stress here that the translation into "limit" is by no means a bad translation. Indeed, it is usually the preferred word to carry out this translations in most scholarly contexts. However, the ancient Greek word ἔσχατον is closely related in meaning to the word ὅρος, which is, in its turn the root word for both ὅριον, which means literally "boundary", and for ὁρισμός, which was the corresponding Ancient Greek word for the latin *definitio* — which gave rise to the English "definition", in the sense of something that limits the meaning of a thing. Our choice of terminology here, *viz.* the translation of ἔσχατον as boundary, is simply to unify our discussion with the much later terminology found in Brentano and used in the later chapters of this dissertation. Moreover, we should note that Aristotle talks about this notion of "limit" in a way that is parallel to the use of the term "boundary" in modern mathematics, as we can see in the following passage:

> We call a limit the extremity of each thing, i.e. the first thing outside of which no part [of the thing] is to be found, and the first thing inside of which every part [of the thing] is to be found. (*Metaphysics*, Δ, 1022a4–5)

Now, to understand this notion of "boundary", it is important to go back to the notion of "an indivisible". We have mentioned that this latter term in Ancient Greece could mean different things — *viz.* extended atoms or unextended spatial figures. Now, because of the path that we shall take in the further chapters of this monograph, we shall didactically forget about extended atoms[6] and define this concept of "an indivisible" to encompass any geometrical figure that is unextended in

[6] We shall have the opportunity in the next chapter to discuss an analogous late Medieval or early Modern distinction between indivisibles and the so-called "infinitesimals", which are the counterparts of these extended atoms of Antiquity.

at least one of the three spatial dimensions.[7] Thus, a plane figure, a line, a point, these are all examples of such indivisibles.

Then, we should note as well that, although there is no clear explicit claim in Aristotle to identify these boundaries with indivisibles, it seems to be a very natural move to carry out this identification of the various possible boundaries of geometrical figures with one or another type of these indivisibles. For instance, it is extremely natural to consider two points as the boundary of a line segment, a circle as the boundary of a two-dimensional disc, a sphere as the boundary of a 3-dimensional massive ball etc.

Moreover, another example that is particularly important for Aristotle is the "now", which is understood by him to be the boundary between past and future in the continuum of time.[8,9] However, this characterization of the "now" is by no means restricted to it, but extends to all types of boundaries. Thus, in the *Metaphysics*, Aristotle says that

> [c]learly it is the same with points, lines and planes, for the same account holds, since all alike are boundaries or divisions. (*Metaphysics*, 1002b8-11)

1.3. Boundaries are not substances. An important part of Aristotle's account of continua is, for the purposes of this monograph, the claim that boundaries are not substances. This is the case because this Aristotelian claim is closely related, as we shall see later, with Brentano's idea that a continuum cannot be thought of as composed out of indivisibles. But first, let us analyse how these notions are interconnected in Aristotle's account.

[7] Here we should mention a text that has been credited to Aristotle, but whose authorship has been severely contested. It is the text whose Latin name is *De lineis insecabilibus* and that contains essentially a discussion of whether lines (and, more generally even, whether any geometrical figure that is unextended in some dimension but extended in another) can indeed be though of as indivisibles. Most of the text is, indeed, dedicated to pointing out that lines, being extended in one direction, can be divided in that direction, so that the ascription of the predicate "indivisible" to lines constitutes a contradiction. This line of reasoning is extremely interesting in its rigor, since we must recognize that, from a purely logical standpoint, truly indivisible figures must be indeed points. However, I believe that the failure to use such a rigorous approach is not an insurmountable problem for Aristotle's official account, since pretty much all of the properties necessary for his theory are indeed present in this notion of an indivisible as something that is unextended, not in general, but in at least one possible direction.

[8] "the now [...] is indivisible and is inherent in all time. For the now is an extremity of the past (no part of the future being on this side of it), and again of the future (no part of the past being on that side of it): it is, we maintain, a limit of both. And if it is proved that it is of this character and one and the same, it will at once be evident also that it is indivisible." (*Physics*, 233b32-234a4)

[9] Sorabji (1983) says about an instant that it

> is not a very short period, but rather the beginning or end (the boundary) of a period. It therefore has no size, for it is not a very short line, but rather the boundary of a line. (p. 8)

First, we should note that, in Aristotle's philosophical framework, everything that exists is essentially split into two classes of "beings": the class of substances and the class of things that belong to those substances. Moreover, the main distinguishing property between these classes is that, whereas substances can have independent existence, i.e., they do not require the existence of anything else for their own existence, the things that belong to them can only exist on top of some specific underlying substance that is to be thought, therefore, as having an ontologically prior being or existence. This has consequences regarding how each of these different metaphysical entities can come into being and cease to be. Indeed, Aristotle says in the beginning of the last passage mentioned that

> besides what has been said, there are also paradoxes about coming into existence and ceasing to exist. It is thought that in the case of a substance, if it now exists without having existed previously, or later fails to exist after previously existing, it must be in process of coming into existence or ceasing to exist. But with regards to points, lines and surfaces, when they exist at one time without existing at another, they cannot be in the process of coming into existence or ceasing to exist. For as soon as bodies have been put together, one boundary does not exist, but has ceased to exist, and when they have been divided, the boundaries exist which did not exist before (for the point, being indivisible, was not divided into two). And if the boundaries are in process of coming into existence or ceasing to exist, from what are they coming into existence?
>
> It is similar with the now in time; for this too cannot be in the process of coming into existence or ceasing to exist, and yet it is thought to be ever different, which shows that it is not a substance. Clearly it is the same with points, lines and planes, for the same account holds, since all alike are boundaries or divisions. (*Metaphysics*, 1002a28-b11)

This passage portrays very clearly how, for Aristotle, boundaries are *sui generis* objects, that, for instance, can come to be without ever being in the processes of becoming and, conversely, can cease to exist without ever being in the process of ceasing to exist. Indeed, in the *Physics* he restricts the general metaphysical claim that nothing can exist (not exist) without being previously in the process of coming into existence (ceasing to exist) explicitly to *continuous* or *divisible* things:

> Hence it is apparent that what has come into existence must previously have been in process of coming into existence [...] in the case of things which are divisible and continuous. (237b10)

Furthermore, this point is what evolved into the discussion that became known as "the problem of the ceasing instant". In particular, Kretzmann says that

> [i]n the light of the notion of instantaneous transition, just introduced, such ceasings and beginnings take place at instants — instants that serve as limits of the temporal interval during which the one or the other of the opposed conditions obtains. But since instants cannot be immediate to each other, two intervals immediate to each other must be limited relative to each other by one and the same instant. (Aristotelian instants are mere cuts in the potentially infinitely divisible temporal continuum, and one cut cannot be immediately adjacent to another. Thus, any two instants define a temporal interval, no matter how short; and any interval, no matter how short, can be divided by a middle instant into two intervals each of which is only half as long as the original.) Thus, the ceasing of one contradictory condition and the beginning of the other must occur at one and the same instant, the instant of transition. (Kretzmann (1982), p. 273)

Indeed, these elements provide, according to Sorabji (1983), the logical foundation on top of which

> Aristotle would solve the paradox of the ceasing instant. Moreover, I think that the solution is not only brilliantly ingenious, but also entirely effective. It renders unnecessary the alternative solutions of the following nine hundred years. (p. 12)

Thus, in order to go more deeply into the details of these logical elements, we shall discuss a little bit of this paradox and its Aristotelian solution in the next section. But, before that, let us complete our exposition of the basic Aristotelian position, by analysing another fact that will play a major role in Brentano's application of this metaphysical position.

1.4. Boundaries are indivisibles and, thus, cannot compose continua. Note that all of these boundaries we have been considering are unextended in at least one direction and, thus, are all indivisibles. Hence, we shall here make our first exegetical assumption and suppose that the boundaries that Aristotle talks about are indeed these indivisibles. There is an interesting passage in book B of the *Metaphysics* that attests to this interpretation. In it, Aristotle says:

> if it is a magnitude, it is corporeal; for the corporeal has being in every dimension, while the other objects of mathematics, e.g. a plane or a line, added in one way will increase what they are added to, but in another way will not do so, and a point or a unit does so in no way. (1001b10-11)

The connection seems to be between having extension in some spatial dimension and being able to increase something's size "in one way or another". More specifically, we claim that these "ways" in which something might add to something else are precisely what we would recognize now as spatial dimensions. And right after that, Aristotle claims that adding such indivisibles would surely increase the "number", but not the "size", for how can

> a magnitude proceed from one such indivisible or from many? It is like saying that the line is made out of points. (*Ibid.*, 1001b17-19)

What seems to be claimed here is that, although surely boundaries seem to play a vital role in the Aristotelian characterization of continua,[10] they are *not* assumed to compose these continua as their building blocks. That is clear, for instance, in the following passage, that opens the sixth book of the *Physics*:

> Now if the terms 'continuous', 'in contact', and 'in succession' are understood as defined above—things being continuous if their extremities are one, in contact if their extremities are together, and in succession if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible. For the extremities of two points can neither be one (since of an indivisible there can be no extremity as distinct from some other part) nor together (since that which has no parts can have no extremity, the extremity and the thing of which it is the extremity being distinct).

[10] Indeed, we find Aristotle characterizing the notion of a "body" by means of this notion of "boundary":

> If 'bounded by a surface' is the definition of body there cannot be an infinite body either intelligible or sensible. (*Physics*, 204b5)

> Moreover, if that which is continuous is composed of points, these points must be either continuous or in contact with one another: and the same reasoning applies in the case of all indivisibles. Now for the reason given above they cannot be continuous; and one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous; for that which is continuous has distinct parts, and these parts into which it is divisible are different in this way, i.e. spatially separate.
>
> Nor, again, can a point be in succession to a point or a now to a now in such a way that length can be composed of points or time of nows; for things are in succession if there is nothing of their own kind intermediate between them, whereas intermediate between points there is always a line and between nows a period of time.
>
> Again, they could be divided into indivisibles, since each is divisible into the parts of which it is composed. But, as we saw, no continuous thing is divisible into things without parts. Nor can there be anything of any other kind between; for it would be either indivisible or divisible, and if it is divisible, divisible either into indivisibles or into divisibles that are always divisible, in which case it is continuous.
>
> Moreover, it is plain that everything continuous is divisible into divisibles that are always divisible; for if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact. (*Physics*, 231a18-231b17)

This passage has become fairly famous and it essentially deduces contradictions from the assumption that points can be either next-in-succession, in contact or continuous with each other. The main result from this argument, however, is that continua cannot be mere aggregates of indivisibles — and this fact is indeed brought back by Aristotle in the last paragraph to lend its weight to the original characterization of continua as infinitely divisible. However, this position had already been explicitly made in the *Physics*, albeit not in its present fully abstract form. For instance, let us consider the following passage:

> the 'now' is not a part: a part is a measure of the whole, which must be made up of parts. Time, on the other hand, is not held to be made up of 'nows'. (218a6-8) [11]

This passage seems to raise the reader's attention to the "indivisible" character of what Aristotle calls "the now" [νῦν]. This is done, however, by means of another notion, that of "measurability". Usually, in Ancient Greek texts one sees this word used in relation to the mathematical distinction between rational and irrational magnitudes, i.e., between magnitudes that share a common unit of measurement and magnitudes that do not.[12] Here, however, we have a slightly different, albeit related meaning being used. Indeed, here the question is not whether two magnitudes share a common unit and, therefore, are rational; the question here is whether some alleged part of a magnitude (*viz.* an instant as a part of a time span) can be superimposed on the magnitude a natural number of times so that at the end of the superimposition, the whole original magnitude is covered.[13] This seems to be the way the notion of measurement acts in this context and Aristotle seems to be equating the notion of "a part of a continuous magnitude" to the notion of "being able to be superimposed on the original magnitude a natural number of times so that at the end of the superimposition, the whole original magnitude is covered". In this respect, then, he concludes that the instant, or the "now", is *not* such a part. This conclusion, however, bears a strong indication that Aristotle is indeed thinking about the "now" essentially as what in our terminology has been called "an indivisible" or "a boundary".[14]

[11] Another similar passage is:

> Necessarily, too, the now—the now so-called not derivatively but in its own right and primarily—is indivisible and is inherent in all time. For the now is an extremity of the past (no part of the future being on this side of it), and again of the future (no part of the past being on that side of it): it is, we maintain, a limit of both. And if it is proved that it is of this character and one and the same, it will at once be evident also that it is indivisible. (233b32-234a4)

[12] E.g., we can think about a pair of lines with 2m and 3m and a pair of lines with 2m and $\sqrt{2}$m. In the first case, there is another magnitude, say, a line with 1m, that can be superimposed onto the original lines a natural number of times (2 and 3 times, respectively). However, in the second, there is no such magnitude.

[13] A more formal property bearing a striking resemblance to this formulation is the Archimedean property, according to which (as applyied, say, to real numbers), given two real numbers $p > q$, it is always possible to find a natural number n, such that $nq > p$.

[14] Kretzmann (1976b) calls attention to the fact that many of the portions of a magnitude that would unquestionably be called "parts" of the magnitude also do not measure the magnitude, as by our reading of this notion. E.g., we can think about a portion of $\sqrt{2}$m of a line segment 2m long. Indeed, this criticism is reasonable in that it calls attention to a fault in the formal definition which is not easily corrected, since in our example there is not even a shared part

Therefore, after all this I believe we can be confident in our interpretation of boundaries as indivisibles and in our reading of Aristotle's position as being that boundaries are not parts of continua and, therefore, that the latter cannot be composed of the former. Indeed, here is Aquinas' conclusion regarding this topic in his *Commentary to Aristotle's Metaphysics*:

> And the truth of the matter is that mathematical entities of this kind are not substances of things [and thus cannot compose things], but are accidents which accrue to substances. But this mistake about continuous quantities is due to the fact that no distinction is made between the sort of body which belongs to the genus of substance and the sort which belongs to the genus of quantity. For body belongs to the genus of substance according as it is composed of matter and form; and dimensions are a natural consequence of these in corporeal matter. But dimensions themselves belong to the genus of quantity, and are not substances but accidents whose subject is a body composed of matter and form. (pp. 189-190)

And with these things in mind, we can now conclude that, according to Aristotle, the notion of "a continuous thing" is characterized both by its infinite (potential) divisibility[15] and by the fact that it possesses a fundamental unity that is characterized by the property that any pair of parts which exhaust the continuum must share at least a boundary. Note, however, how both properties eventually boil down to the assumption of these indivisible boundaries that are (potentially, perhaps) everywhere in the continuum, since everywhere in it is a possible place of division and since everywhere in it can be conceived of as being a place where two parts of the continuum are actually fused together — a condition that is also determinant for the Aristotelian conception.[16]

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of both segments that can be superimposed in both a (maybe different) natural number of times so as to exhaust them — i.e., the obvious answer to define a part of a magnitude in terms of the notion of being smaller and rational with the original magnitude. However, it seems that all this talk about measuring is only the setup that Aristotle could find in which to discuss the "indivisibleness" of instants, which we explained in terms of it not being extended in some dimension. Thus, although Kretzmann's criticisms are not wrong, I believe they fail to hit the overall logical kernel of Aristotle's point here, being directed more towards the contingent historical setup that it actually took in the passage. Furthermore, Aristotle's setup could be made precise by substituting this notion of "measurement" by the notion of "covering", which is what we did here.

[15] Cf. *De generatione et corruptione*, I, 2, 317a4-13.

[16] I'd like to mention here a contemporary interpretation of Aristotle that is less historical and more logico-formal, but, nonetheless, agrees almost entirely with our characterization here:

> To summarize, it seems that, for Aristotle, line segments have actual endpoints — but such endpoints are decidedly not part of the line segment. They are just its boundaries. Each endpoint is metaphysically tied to the segment it bounds, and cannot be considered in isolation from the segment, in the same sense as the smile of the Cheshire cat cannot be considered apart from the cat. The interior points on a line segment exist only potentially. There is a potential

Thus, we see how these two properties of continua are, for Aristotle, merely flip-sides of the same theoretical coin, which is the assumption of this close relation between continua and boundaries.[17] However, what is also interesting for our purposes in this monograph is to note a sort of converse of this claim that any two parts of a continuum must share a common boundary — *viz.* the fact that a pair of things that merely touch at their boundaries and, thus, are not fused together into a single continuum, must each have its own boundary which is somehow collocated with the boundary of its corresponding contiguous counterpart. Indeed, this independently follows from the fact that a "thing" or a "body", as we saw, must have a boundary, together with the fact that two things, if they are to *fail* to merge into a single continuous entity, their touching boundaries cannot become the same. In this respect, we have the following passage from Aristotle:

> In the act of dividing the continuous distance into two halves one point is treated as two, since we make it a beginning and an end; and this same result is produced by the act of counting halves as well as by the act of dividing into halves. But if divisions are made in this way, neither the distance nor the motion will be continuous; for motion if it is to be continuous must relate to what is continuous; and though what is continuous contains an infinite number of halves, they are not actual but potential halves. If he makes the halves actual, he will get not a continuous but an intermittent motion. (*Physics*, VIII, 263b1-263b6)

1.5. The discussion regarding the scope of book B of *Metaphysics*. Before we consider the more specific question of how these Aristotelian ideas regarding substances and their boundaries have been used in the discussions of motion and the instant of transition, we shall briefly talk about two points. First, we shall study in this section how much Aristotle was indeed concerned with the problems we are trying to find answers to in his work. Then, in the next section, we shall discuss the scope and meaning of his concept of "place".

> infinity of such interior points, but not an actual infinity of them. (Hellmann and Shapiro (2018), p. 4)

[17] I believe this overall account is in harmony with Sorabji's view on Aristotle. For instance, cf. Sorabji (1976), pp. 69-70:

> It means that time will be infinitely divisible, and there will be no such thing as a time-atom, that is, an indivisible period with an indivisible duration. An instant will be not a time-atom, nor any kind of period, but rather the boundary of a period, itself having no duration.

So, to start, let us note that, in the beginning of *Metaphysics* book B, Aristotle claims to be wrestling with two questions:

> whether numbers and lines and figures are a kind of substance or not, and if they are substances whether they are separate from sensible things or present in them (996a12–15)

Nonetheless, notwithstanding this clear statement of the problem, we find different opinions in the literature regarding its true scope. More precisely, we have Madigan (1999, p.29) and Mueller (2009, pp. 190-191, 204, 205) stating that Aristotle is truly only discussing the first of these questions, perhaps assuming "dialectically", as something accepted by the thinkers whose views he is discussing, that the objects in question here — *viz.* numbers, lines and figures — are undoubtedly locally present in substances, as the latter's boundaries or limits.

However, we must disagree with them and, to do so, we shall follow here a different opinion found in the literature — more specifically in Katz (2018), who believes that Aristotle does indeed address the second question, understanding the term "separate" not as local separability, which fails to hold in the case of the boundaries considered, but as what she terms the "ontological status relative to sensible bodies" (p. 24). In this sense, then, the question regarding separability is not whether points, lines and figures can exist as locally distinct entities from bodies — which indeed, as Madigan and Mueller rightly claim, they cannot —, but whether they are ontologically prior to bodies or not.

And, in this respect, it seems that Aristotle does conclude that they are also *not* separable. But, now, this claim is not merely that they never seem to occur in places which are not occupied by bodies, but that they, indeed, can *never* do so! In other words, surfaces, lines and points are *not* "more substances" than bodies.

Now, if we look into Aristotle's argument in this part of book B, his first approach is to mention "more recent thinkers", who claim that

> a body is surely less of a substance than a surface, and a surface less than a line, and a line less than a unit and a point. For a body is bounded by these; and they are thought to be capable of existing without body, but a body cannot exist without these. This is why the more recent and those who were held to be wiser thought numbers were the first principles. (1002a4-12)

However, after he does this, he goes on to criticise this position, claiming that

> if this is admitted, that lines and points are substance more than bodies, but we do not see to what sort of bodies these could belong (for they cannot be in perceptible bodies), there can be no substance. Further, these are all evidently divisions of body, one in breadth, another in depth, another in length. [...] Therefore, if on the one hand body is in the highest degree substance, and on the other hand these things are so more than body, but these are not even instances of substance, it baffles us to say what being is and what the substance of things is. (1002a15-27)

Thus, it seems that Aristotle is denying the status of "substance" to indivisibles; or, in Katz's words, that, for him,

> geometrical objects are groups of properties of certain sensible objects. ((2019), p. 467)

This is, incidentally, exactly the interpretation we found in Aquinas' *Commentary*, which is however couched in a language that is perhaps more Aristotelian than Aristotle's, in the sense that Aquinas, in the aforementioned passage, fully identifies indivisibles with accidents. This is, however, nothing more than what Katz is claiming when she says that these indivisibles — which are surely a part of the full collection of geometrical objects — are "properties" of objects.

Moreover, I agree that these positive claims are the correct interpretation of Aristotle's negative claim to the effect that indivisibles are *not* more substances than bodies. This is essentially the most important elements to extract from this discussion if our goal is to understand the much later ideas of Brentano. However, I highly recommend Katz's papers for a more in depth discussion.

1.6. The notion of "place". Before we move into an exposition of how these general ideas proposed by Aristotle were applied in the case of motion, we shall have a look at a few points regarding Aristotle's notion of "place" as it is presented in the *Physics*. The reason for doing this is that, as we shall see in chapter 3 when we consider Brentano's take on these ideas, several authors, starting from perhaps a more commonsense-based notion of "place", fail to truly grasp the main points of Brentano's very Aristotelian account, and even try to argue against this account, failing to notice that in doing so they are incurring in a *petitio principii*, since they are arguing from a certain notion of "place" against a view that does *not* share this notion, but thinks about place in a wholly different way.

Now, in *Physics* IV, Aristotle wrestles with the notion of "place". In his own words,

> we must raise the whole problem about place—not only as to what it is, but even whether there is such a thing. (209a29-30)

In particular, it is fairly clear that Aristotle wants to make sure that place is not thought of, as Furley (1982) puts it,

> as a sort of ghostly duplicate of the body itself. (p. 22)

In other words, he wants to go against a very easily considered position (even today), according to which every body has a certain spatial portion which it is said to occupy and which is usually called "the object's place". Indeed,

> when we come to a point we cannot make a distinction between it and its place. Hence if the place of a point is not different from the point, no more will that of any of the others be different, and place will not be something different from each of them. (*Physics*, 209a9-13)

Aristotle seems to be concerned with making sure that one does not fall into the trap of thinking about the place of an indivisible over and above the indivisible itself.[18] However, we find a more positive account of what it would be to think about this notion of a "place" in his final conclusion, according to which

> the place of a thing is the innermost motionless boundary of what contains it. (*Physics*, 212a20-212a21)

The book Morison (2002) is fully devoted to presenting and discussing this Aristotelian notion of a "place", and it does so very thoroughly. For instance, chapter 3 presents a detailed discussion of Aristotle's treatment of Zeno's infinite regress: if everything that exists is somewhere, and if places exist, then it would seem that places must be somewhere as well; and, then, places of places too, *ad infinitum*. As an example of Aristotle's love for terminological distinctions, his solution is to claim that there are many ways in which things might be somewhere, depending on the ontological nature

[18] As we mentioned, the main reason for considering this topic here is that, not only it does bear on the more general topic of this dissertation, but that we shall have the opportunity in a later chapter to criticise a criticism of Brentano that hinges precisely on this mistake that Aristotle is urging one not to make.

of the thing in question. For instance, a property is somewhere in virtue of being in a substance which, in its turn, is somewhere in the sense that it is in some place. Thus, although Aristotle does not explicitly tell us what it is for a place to be somewhere, one can extract from his overall approach that places are somewhere in a sense that is related to the part/whole relation: places are parts of their spatial environment.

Moreover, Morison also contributes to an explanation of the notion of "that which contains something", which is required for Aristotle's definition to make sense. In particular, one may readily see that this notion might be extremely ambiguous. For instance, consider a fish in a fishbowl and ask: what is the thing that contains the fish? Is it the water? Or the bowl? Perhaps the air in the room? Or maybe the room itself? Morison rightly concludes that, when Aristotle is talking about *the* thing that surrounds something he is indeed considering the limiting case in this "Russian doll" situation, so that what he means is something like the whole universe, which is, for Aristotle, finite and, thus, can be seen as *the* "common place of all things".[19]

Finally, Morison explains that if we were to take this latter quote too seriously and consider merely the universe as the place of something, then we would have to conclude that everything would be in the same place. Thus, if we want, as Aristotle did, to consider places of things as unique to those things and different from other thing's places, we must consider a more complicated notion which is the one picked out by Aristotle by the expression "inner boundary of the thing's surroundings" and which is, as Morison puts it, the intersection of all of the thing's surroundings — a limit in the opposite sense in which we saw the universe to be the limit of these Russian doll type structure that is composed by the various possible surroundings of something.

Thus, since I do not see anything to correct or complement on his account, I simply refer the reader to this book by Morison for reference. However, I would like to call one's attention to how this Aristotelian account, not only, as we saw, forbids one to think about the place of an indivisible over and above the indivisible itself, but also it identifies something's place with a certain boundary, *viz.* the so-called "innermost motionless boundary" of the thing that surrounds the object whose place we are concerned with. This is, indeed, a more general characterization that has as a consequence the aforementioned identification of points with their places; in particular, it is one that ties the notions of boundary and place very closely together.

[19] Cf. *Physics*, 209a32.

2. The connection with motion

Much of the discussion regarding continua and their boundaries did not, however, take place in Antiquity in the abstract framework we have been considering. Much more common were narrower discussions that had in mind the particular cases of motion or change, which, by their nature, gave rise to questions regarding the specific continuum of time and its "pointy" indivisibles that are usually called "instants". In this respect, we can mention the following passage of Aristotle that sets the stage for a few of the later inquiries into this topic:

> The 'now' is the link of time, as has been said (for it connects past and future time), and it is a limit of time (for it is the beginning of the one and the end of the other). But this is not obvious as it is with the point, which is fixed. It divides potentially, and in so far as it is dividing the 'now' is always different, but in so far as it connects it is always the same, as it is with mathematical lines. For the intellect it is not always one and the same point, since it is other and other when one divides the line; but in so far as it is one, it is the same in every respect.
>
> So the 'now' also is in one way a potential dividing of time, in another the termination of both parts, and their unity. And the dividing and the uniting are the same thing and in the same reference, but in essence they are not the same. (*Physics*, 222a10-20)

Moreover, coming back to the tight connection between continua and infinite divisibility, Aristotle says of time that it must be a continuum. Indeed, he says:

> By continuous I mean that which is divisible into divisibles that are always divisible: and if we take this as the definition of continuous, it follows necessarily that time is continuous. (232b23-25)

Now, the important point for our purposes here is that, when we think about change, we must always think about it against a background time that somehow "measures" this change. Then, under the assumption that this background time is indeed continuous, we run into problems relating to instants when something is beginning to be something or ceasing to be something. According to Kretzmann (1976b),

> [w]e may describe [...] the Aristotlelian problem of beginning and ceasing very broadly as the problem of assigning temporal limits to a process of change measured against a continuum. (p. 102)

More precisely, these notions of "beginning" and "ceasing" must be correlated with some of the particular indivisibles where one can divide this background time continuum; and, more specifically, with the alleged particular instants where something was somehow for the very first time and where something was somehow for the very last time.

However, although in the 19th century mathematicians discovered that one must distinguish between dense sets and truly continuous ones,[20] at the time of Aristotle, this was not even close to being common knowledge and all kinds of thinkers used the notion of density, i.e., the fact that between any two points there is always another point, to characterize continua in general.[21] Thus, if we ascribe to some given motion or change a first moment in which something was so-and-so, we must, according to Aristotle, conclude that there was no last instant in which this something was *not* so-and-so, for if there were such last instant, since it must be different from the first moment we specified, it is guaranteed to exist points between them; and, since they will all be prior to the first moment of something's being so-and-so, they must be moments in which this something was not so-and-so, so that the assumed last moment of something's not being so-and-so was unjustified. And, conversely, if we ascribe to some given motion or change a last moment in which something was so-and-so, then we must, according to Aristotle, conclude, by an analogous reasoning, that there was no first instant in which this something was *not* so-and-so.

Aristotle's solution, then, was to — perhaps quite arbitrarily — assume that whereas there could always be a first moment of something, there can never be a last moment of anything. More specifically, we have the following passage:

> It is also plain that unless we hold that the point of time that divides earlier from later always belongs only to the later so far as the thing is concerned, we shall be involved in the consequence that the same thing at the same moment is and

[20] More on this discussion in later chapters.

[21] Indeed, this is merely a restatement of Aristotle's characterization of continua as infinitely divisible. For every point is a possible division position; and if, say, a line segment between two points is to be divisible, there must exist (at least potentially) a point between the points that mark the segment's extremities.

> is not, and that a thing is not at the moment when it has become. (*Physics*, 263b9-12)

We shall denote this Aristotelian position as the "first but not last" thesis. It is explicitly recognized by Strobach (1998), as we can see in the following passage:

> This already presents us with quite a good reason for regarding the problematic limiting instant as an instant when the target state has in fact been reached with the limiting instant assigned to the later state (being black): if instants are densely ordered and there is a limiting instant (which it is reasonable to assume) and at every instant something has either to be the case or not, and if there is no last instant at which something is not the case, then there is a first instant at which this *is* the case. (p. 57)

However, just after recognizing this fact, he goes on to say that

> Hence Aristotle's solution does not just consist of the simple instruction: "Any problematic instant should be assigned to the later state!". There would be no motivation for doing this. In 263b10, Aristotle assigns the limiting instant to the later state because he is in this passage occupied with the end of a process and not with its *beginning*. Analogously, at the beginning of a process, the limiting instant would have to be assigned to the initial state (there is no first intermediate state reached, and thus no first instant of having left the initial state). The word ἀεί in 263b12 refers only to ends of processes. There is no difficulty in applying the solution concerning being white to being at the target place of a motion. Aristotle's position may then be summed up as follows:
>
> > There are, in such changes no first position a mobile could occupy after the terminus *a quo* and no last position it could occupy before the terminus *ad quem*. [...T]he actuality of the condition in which an object rests has intrinsic limits.
>
> Thus, the option Aristotle uses for describing the moment of change when, at the beginning and at the end of a continuous process, non-comparative properties are involved, is the either/or option. (*Ibid.*)

In this sense, Strobach is holding a position similar to Sorabji's, according to which there is no general solution as to whether the moment of transition belongs to the earlier or to the later state. Indeed, according to Sorabji,

> [o]ne of the difficulties about answering [whether the moment of transition belongs to the earlier or to the later state] was that if we said that one of these instants existed [i.e., in our terminology, if the moment of transition belonged to one of these states], but not the other, we seemed to be being arbitrary. It would be a sufficient solution, if we could show that it would not be arbitrary to prefer one instant to the other. For this purpose, we need only show that there is a reason for preferring one to the other; we need not show that it is mandatory to do so. (Sorabji (1976), p. 71)

However, I do not believe that this type of solution is in fact what Aristotle had in mind. A reason for that would be, indeed, the following passage:

> the primary time that has reference to the end of the change is something really existent; for a change may be completed, and there is such a thing as an end of change, which we have in fact shown to be indivisible because it is a limit. But that which has reference to the beginning is not existent at all; for there is no such thing as a beginning of change, nor any primary time at which it was changing. (236a10-13)

In this passage, Aristotle expressly rejects Strobach's point that we can have a first term if we are considering "the beginning of a process". Furthermore, when discussing Sorabji's account of the moment of transition in the specific cases of generation (γένεσις) and corruption (φθορά), Strobach says the following:

> It seems to me that according to the terminology of Physics VI γένεσις and φθορά are exactly analogous to discontinuous changes in objects as described in 2.3.2.1. The only difference is that they are the discontinuous changes from non-existence to existence or vice-versa. Let us call this use γένεσις^1 and φθορά^1. I think that in the other passages Aristotle uses the key terms as referring to the processes on which γένεσις^1 and φθορά^1 depend. So in the terminology of the other passages the γένεσις of an object is a process which takes place before the object exists. At any

> instant which falls within this process the object does not exist yet. The limiting instant at the end of the process is the first instant of its existence. Perishing, φθορά, however, is a process in which the object decays while still existing. Only the limiting instant at the end of the process is the first instant of its non-existence. Thus, coming-to-be and perishing in this sense are processes but they are not processes between existence and nonexistence. Rather, existence is the end-term of the process of coming-to-be, and non-existence is the end-term of the process of perishing. We might call this use γένεσις^2 and φθορά^2. As a result, then, γένεσις^1 and φθορά^1 are special kinds of discontinuous change (namely between existence and nonexistence), while γένεσις^2 and φθορά^2 are the processes on which γένεσις^1 and φθορά^1 depend. According to this view, what was said about the limiting instants of κίνησις applies to γένεσις^2 and φθορά^2, simply because γένεσις^2 and φθορά^2 are kinds of κίνησις. So I suggest that there are two different uses of γένεσις and φθορά in Aristotle but that both the uses of γένεσις and φθορά have their definite place in Aristotle's treatment of the moment of change. Under this interpretation, objects would usually have a first instant of existence, but no last instant of existence (while there is a first instant of non-existence). ((1998), p. 59)

Indeed, Strobach believes that this follows from a generalization of the argument presented by Aristotle in *Physics*, 235b17-28, which, according to him, has the form:

> [i] ... that which has changed must be in that to which it has changed ...
>
> [ii] For,
>
> [a] since it has left that from which it has changed
>
> [b] and must be somewhere
>
> [c] it must be either in that to which it has changed or in something else.
>
> [iii] If, then, that which has changed to B is in something other than B, say C, it must again be changing from C to B;
>
> [iv] for B was not assumed to be contiguous, and change is continuous.
>
> [v] Thus we have the result that the thing that has changed, at the moment when it has changed, is changing to that to which it has changed,

[vi] which is impossible:

[vii] that which has changed, therefore, must be in that to which it has changed.

[viii] So it is evident likewise that that which has come to be, at the moment when it has come to be, will be ... (*Ibid.*, p. 60)

This argument, as Strobach correctly concludes, aims at establishing the fact that at the limiting instant of a change process we have present the state which is to be achieved by the change. Also correctly, he claims that this argument can be used almost *verbatim* to conclude the analogous result for the limiting instant in the beginning of this change process. Indeed, we can find his overall view in the following quote:

> Aristotle offers a precisely motivated *description* of the moment of change; it is a mixed description, using different systematic options for different cases. We have seen that *for comparative properties he uses the neither/nor-option, while for non-comparative properties he uses the either/or option*, i.e. a sort of either/or-option which assigns the limiting instant at the beginning of a process to the initial state and the limiting instant at the end of a process to the target state.
>
> Aristotle's *classification* of the moment of change entails the denial of the existence of an instantaneous event at the limiting instant. This offers the chance to avoid a banishment of change from time, even though instants are densely ordered and exhaustively distributed to the obtaining or not obtaining of a non-comparative state: since there is no event of changing anyway, no event of changing needs to be banished from time. If there is no event of changing, then Plato's premiss that an event of changing can only take place when of two opposite states one does not obtain any more and the other does not obtain yet, is of no interest.
>
> Although Aristotle denies an instantaneous event at the limiting instant, the limiting instant has nevertheless a very special role. Aristotle makes this clear for the limiting instant at the end of a process, by ascribing to it the complicated and interesting property of being an ἐν ᾧ πρώτῳ μεταβέβληκεν. He does, however, neglect the limiting instant at the beginning of a process and he does not provide

a really thorough discussion of why there is no discontinuous change. (*Ibid.*, pp. 80-81)

In this passage, Strobach seems to be claiming that Aristotle's "first but not last" thesis applies only to "endings of processes". In the beginning of a process, on the other hand, something like a "last but not first" thesis would hold.

I believe that that this interpretation is not quite correct. Indeed, for transformations that require processes that are extended in time, Aristotle believes that the transformation takes place part by part, so that three moments have to be acknowledged: the time before the process, in which the substance was completely A; the time after the process, in which the substance was completely not-A; and the process itself, in which the substance was neither totally A or not-A, but partially both. Then, according to Aristotle, we should think about this situation as providing us with no last moment in which the substance was either completely A or partially A, but with first moments in which the substance was partially A and completely not-A.

Now, although I believe that Strobach's interpretation is not quite accurate to Aristotle's intentions, it shares, nonetheless, much in common with other interpretations of Aristotle and with further developments that took place in the latter history of the Aristotelian tradition, which shall be considered in the next sections.

Indeed, the whole reason why we are taking a look at these multiple attempts at a logical reconstruction of Aristotle's position by contemporary philosophers is to realize that, although Aristotle's account is very interesting and indeed fairly rigorous, there is nonetheless a gap in his original exposition on this matter. Moreover, it is in the context of filling this gap that we shall encounter later on a particularly interesting theory that, although couched in historically localized terminology and conceptual underpinning, is, nonetheless, if not a historical, surely a logico-formal ancestral of the much later Brentanian approach to this problem.

3. Sorabji's piecewise solution

Let us, then, have a look at one more modern interpretation of Aristotle's ideas on the topic of the instant of transition, in which this mentioned gap will be fairly clear. According to Sorabji (1976),

> [o]ne of the difficulties about answering was that if we said that one of these instants existed, but not the other, we seemed to be being arbitrary. It would be a sufficient solution, if we could show that it would not be arbitrary to prefer one instant to the other. For this purpose, we need only show that there is a reason for preferring one to the other; we need not show that it is mandatory to do so. (Sorabji (1976), p. 71)

In other words, Sorabji realizes that to escape the accusation of being arbitrary as to whether the moment of transition belongs to the former or later state, he needs only provide, for each of the main classes of transition, a reason for the moment of transition belonging to either one or the other state. And, thus, he goes on to list and analyse the various types of transition accepted by Aristotle, so as to locate in each type the condition to which the moment of transition belongs.

The first case he undertakes is that of a continuous motion. More specifically, he considers the case of some particle starting at rest, then moving and finally coming back to rest. In this context, he says that

> there is an asymmetry between the series of positions away from the position of rest and the position of rest itself. There can be no first position away from the starting point, or last position away from the finishing point in a continuous motion, or in any other continuous change. Hence there can be no first instant of being away from the starting point or last instant of being away from the finishing point. No such considerations apply to being at the position of rest. This already supplies us with a solution to our paradox, in some of its applications. For if someone were to ask. "when is the last instant of being at the position of rest, and when the first of being away from it?", we could safely reply that the latter instant does not exist. But we can go further. The asymmetry between the position of rest and the positions away from it can provide us with the excuse we want for treating rest differently from motion. It would be perfectly reasonable to mark the asymmetry by saying that just as there is no first or last instant of being away from the position of rest, so equally there is no first or last instant of motion. It would be reasonable, but not mandatory. Reasonableness is all we need in order to escape the charge of arbitrariness. (p. 72)

However, he also acknowledges the possibility of thinking about this case in a different way, as is clear from the following passage:

> at the instant of reversing direction, a ball's centre of mass is (as in all cases of coming to a halt) at a different position from that occupied at preceding instants, and (differently from ordinary cases of coming to a halt) it is also at a different position from that occupied at succeeding instants. This difference of position admittedly favours our regarding the instant of reversing direction as one of motion. But I think the consideration is outweighed by the absence of a particular direction and of a positive velocity. (p. 73)

Then, he goes on to analyse cases in which either the initial or the final state are not continuous transformations. A typical example is the one about visibility, which was introduced by Aristotle himself. About this case, Sorabji says that

> [i]f we are watching a receding aeroplane, or looking for an approaching one, we cannot normally tell at the time what will prove to be the last instant of visibility as it recedes, or the last instant of invisibility as it approaches. If we want to register this instant as it arrives, we shall normally have to wait until the new state is upon us, before we can do so, and it may then reasonably be held that we are not registering the end of the old state, but, at best, the beginning of the new. This means that, in many contexts, we have a good reason for not talking of the last instant of the old state, but (if it has one) of the first instant of the new. This solution seems to have appealed to Peter of Spain, for certain kinds of case [*sic*]. [...] Aristotle himself may have another consideration relevant to the particular example of visibility. For he classifies seeing as an energeia, and on one interpretation, an energeia has no first instant. This is how J. L. Ackrill interprets Aristotle's idea (e.g., Sens. 446b2) that "he is seeing" entails "he has seen". Ackrill treats the perfect tense "he has seen", like "he has been seeing", as implying an earlier period of seeing ("Aristotle's distinction between Energeia and Kinesis", in New Essays on Plato and Aristotle, ed. R. Bambrough, London 1965, esp. pp. 126-7). This interpretation has been disputed, but if it is correct, it implies that there will not be a first instant of seeing, and therefore not a first

> instant of seeing the approaching object in the problematic example of De Sensu 449a21-31. (p. 76)

Thus, in the case of visibility, we have more of a classic “first but not last” type of solution.

These examples, however, are enough to show us the gist of Sorabji (1976)’s solution: that the question as to whether the instant of change belongs to the former or latter state strongly depends on the particular transition in question, for in each case we shall be able to find a sufficient reason to assign the moment of transition to one state and not the other. This sufficient reason that presents itself in each particular case would, then, be enough to deter any criticism that this choice was arbitrary.

Another interesting thing to note from this way of understanding Aristotle’s ideas on the moment of transition — and, in particular, as it is presented in the last quote — is that we see mentioned the name of Peter of Spain as accepting one of the presented solutions. This is interesting because, indeed, the Medieval philosopher Peter of Spain ended up constructing a intricate and extensive classification of cases and the corresponding ascription of the moment of change in each one of them. Thus, although he would not agree perhaps with all the details of Sorabji (1976)’s cases, he does indeed agree with the overall picture of the moment of transition as belonging to earlier or later states depending on the particular transition in question. So, let us bridge this gap between Aristotle’s original exposition and the later Medieval appropriation of his ideas in the following section.

4. Antiquity after Aristotle

After Aristotle, the discussion around the notion of continua and boundaries continued and many philosophers did attempt to present new and different solutions to the problems that creep into Aristotle’s account.

For instance, some philosophers began to seriously entertain the possibility of indivisible time leaps, which are the temporal counterparts to an atomic conception of matter as composed out of elementary indivisible building blocks that have nonetheless some finite, albeit very small, size. These were called atoms, so that their time counterparts are now known as “time atoms”.

The original Aristotelian set up continued, however, to have a strong foothold on the ongoing theories that were being developed to make sense of continua. For instance, according to Gould (1971), there are

> four fragments in which something is reported about Chrysippus' view of the infinite. The first (II 482), as Arnin has constituted it, combines two seemingly incompatible reports. We are told by Stobaeus that, for Chrysippus, bodies surfaces, lines, places, the void and time are divisible *ad infinitum*. Diogenes, however, reports that Chrysippus did not speak of divisibility *ad infinitum* but rather said that a body is infinitely divisible. The reason for Chrysippus preferring the latter way of talking, as Diogenes says, is because the process of division goes on without cessation; the expression *ad infinitum* wrongly suggests that there is an infinite into which the body is divisible. Though one cannot speak with definiteness, Chrysippus seems to be taking Aristotle's stand with respect to the infinite divisibility of spatial magnitudes: namely, that to say that a body is infinitely divisible is to say that it is potentially infinite; that is, one can go on dividing without end. (p. 116)

Another Philosopher that undertook to expand and correct Aristotle's position regarding continua was Alexander of Aphrodisias, who attempted to further clarify Aristotle's original definition of contiguous and continuous by means of redefining the meaning of "in" from the original Aristotelian characterization. In his own words,

> [t]hings are said to be *in the same*, if there is nothing between them. [...] Things that fit on to each other [ἐφαρμόζοντα] and make no volume [ὄνκος] and have nothing between them are "in the same". For the limit of both becomes *one* in the case of things that touch, since they coincide, on account of there being no interval in that region. Whereas in the case of things that are continuous, even the *one* is destroyed; for things are continuous when in actuality there is no boundary [πέρας] in between. (Alexander, *apud* Simplicium, *Commentary on the Physics*, 570.1-7)

It seems that Alexander, although expressly contradicting — or at the very least, diverging from — Aristotle's original definition of contiguous and continuous, is nonetheless true to Aristotle's intention. For Aristotle himself claims that inner points of a continuum are only potentially, so that in a sense, when two objects fuse into a single continuous unity, their common boundary ceases to be, in the sense of having no actuality. This seems to be what Alexander is calling our attention to; and, thus, he seems to be merely clarifying, and not truly modifying, Aristotle's original definition.

4.1. Neo-Platonism. Aristotle's main characterization of continua kept on dominating the Ancient philosophical framework well into the late Ancient period, especially through the neo-Platonic school. Here we shall take a look into an interesting case study in this respect, which is Simplicius commentary on book VI of the *Physics*.

After presenting a layout of the book, he presents what Furley (1982) calls "a famous contrast" between the early atomic theory of Leucippus and Democritus and the later theory of Epicurus.[22] The contentious issue would be the partlessness of atoms. According to the original atomic theory, atoms are both impassive (ἀπαθε̃) and partless (ἀμερε̃); however, the later atomist Epicurus claims that, while they are impassive, they are *not* partless. According to Furley (1982), this change in the characterization of atoms springs from Aristotle's critical arguments towards the claim that partless bodies can compose larger objects. In Furley's words,

> [p]artless units can make up neither a magnitude composed of parts that are continuous with each other nor one composed of parts in contact. Thus a continuous line cannot be composed of points, since the parts of a continuous magnitude, according to Aristotle's definition, have extremities that are *one*, and points, being partless, have no extremity different themselves. Similarly, they cannot have extremities that are together and so cannot compose a larger magnitude by contact. (p. 27)

This is an interesting point of view; and, although Furley himself recognizes that "most historians" reject this interpretation, I believe he is indeed correct in his assessment.

This, however, shows us indeed how overwhelming Aristotle's account of continua was in Antiquity, to the point that rival accounts had to conform themselves to the very compelling arguments presented by Aristotle in the time between these atomist theories.

Another position in late Antiquity that reproduces Aristotle's ideas is Damascius' conviction that it was a mistake, which eventually led philosophers into the paradoxes of time, to assume that

[22]Cf. Simplicius (1997), 925.13-22.

instants (which are an example of the more abstract notion of points) have actuality. According to him, they are simply boundaries wihtout reality (Phys. 799,30-5) (SP 86 2-8) — a position that is very much in tune with Aristotle's ideas on continua and their boundaries.

A further example of the foothold of Aristotle's ideas in the neo-Platonic school is Simplicius' claim that Aristotle's argument

> called for the indivisibility of now and its not existing in actuality, which is demonstrated in the sixth book of the work in his discussion of motion.

This was the last known Greek contribution on this topic in Antiquity, since in 529 Justinian put an end to the teachings in the school in Athens. However, it shows us that by this time Aristotle's account of continua was indeed the mainstream account, even though it had by them its terminology slightly rephrased.

Notwithstanding this late hegemony, however, the Aristotelian account faded a little out of the mainstream as Late Antiquity turned into the Early Medieval period. Indeed, according to Sorabji (1983),

> [i]t was not only infinitely divisible leaps, but also the Aristotlelian paradoxes, which continued to live after the end of antiquity. But solutions did not improve in medieval Europe, if we may take as typical de treatment of Peter of Spain (died *c.* A.D. 1277) to the ceasing instant. According to a recent report [Kretzmann (1976), p. 105], he held that an instant exists, begins to exist and ceases to exist simultaneously. (p. 63)

5. The Middle Ages

The beginning of the Middle Ages was marked by an intellectual appropriation of neo-Platonic ideas into a Christian theological worldview. This much is certain. However, this was also accompanied by a lack of interest in Aristotelian ideas regarding continua. As we saw in Sorabji's last quote, what survived the end of Antiquity was more a discussion regarding the possibility of leaps and regarding the paradoxes of motion than a thorough appropriation of Aristotle's abstract account of continua and their boundaries.

However, as Aristotle's works found their way into the newly established universities of Europe, his ideas on this topic quickly became again fairly canonical. More precisely, much like most other topics touched by the Stagirite philosopher, the discussion regarding continua and indivisibles in the Late Middle Ages was heavily constrained by the account we have been analysing in this chapter.

Of course, as in other areas of inquiry, Late Medieval thinkers expanded upon Aristotle's original account, but most of those thinkers carried out all kinds of logical and semantic manoeuvres to position their accounts as close to the original Aristotelian account as possible.

Thus, for example, we have as a prime example Ockham's discussion in his *De corpore Christi* (1930, pp. 36.22-38.5 and 40.18-42.11). According to Stump (1982),

> There is something at least misleading [...] in characterizing Ockham's discussion of points here as a discussion about the *existence* of points. Ockham himself does not put the issue in those terms here. As he explains it, what is at stake in his treatment of points is not so much whether there is such things as points as what the nature of points is. [...]
>
> Ockham understands a point primarily as a limit of or a cut in a continuum. There are such cuts or limits, just as there are absences from loved ones; but a cut or a limit, like an absence, must not be thought of as an independently existing thing. This continues to be Ockham's general strategy also with lines and surfaces — he takes each to be a kind of limit which is not an entity in its own right — until he reaches his general conclusion about quantity, namely, that a substance or quality is not quantified by the addition to it of a separate, independently existing entity which is quantity but instead is quantified simply by itself, by a certain condition and arrangements of its parts. (p. 217)

Indeed, given this characterization — which I believe is correct — what we find in Ockham's discussion is nothing more than a thorough logical evolution of the ideas present in Aristotle's characterization of these limits, whose existence already depended, for him, on the continua they bounded.

Another example of this general trend that also has to do with the theme of this dissertation is the extremely complex aforementioned taxonomy introduced by Peter of Spain to specify the correct location of the moment of transition in the myriad of types of transition that can occur in nature. As we saw above, Aristotle's account of continua and boundaries presents us with a decision as to which of the two states of a transition should hold at the very moment of transition. And Peter of

Spain took it one step further in that he worked out an incredibly complex taxonomy of transitions and studied in each case where the moment of transition should belong to.

There was, however, a specially interesting (minor) trend in late Medieval thought that is especially relevant the topic of this dissertation and that relates closely with Brentano's ideas on this issue. It is, hence, to this trend that we shall turn our attention in the next section.

5.1. The instant of transition and quasi-Aristotelianism. Imagine we have a plane that is divided by lines into successive strips such that the first strip has size $\frac{1}{2}$ the size of the plane, the second $\frac{1}{4}$ the size of the plane and so on, so that the n-th strip has size $\frac{1}{2^n}$ the size of the plane. Then, we make a distinction between the odd and even numbered strips and imagine a point starting at the extremity of the plane that is farthest from the first strip. Then, as this points moves clearly it will eventually enter an odd numbered strip. But, we might ask, what is the first moment in which such an event happens?

As the point starts moving into the plane, it will immediately enter either an odd numbered strip or an even numbered one, there is no other choice. If the first option is the case, then we can conclude that the answer to our question is: immediately! If, on the other hand, we have the second case, we must also conclude the same answer. The reason is that, given an even numbered strip in our plane, there is always an odd numbered strip that closer to the starting position of the point than the given even number strip. Hence, the point must pass before through that odd numbered strip before reaching the even numbered one.

Now, what we have with such an argument is that the point will reach an odd numbered strip immediately as it starts moving. Nonetheless, there is nothing preventing us from switching odd for even and vice-versa in the above argument, thus also showing that the point will reach immediately an even numbered strip. But no odd numbered strip is an even numbered one — we seemed to have reached a contradiction!

This paradox is a version of the paradox considered by Richard Kilvington in his *Sophisma* 16.[23] It is a prime example of the kind of questions that were being discussed in the Late Middle Ages in connection with the Aristotelian ideas concerning continua and, in particular the temporal continuum.

[23] A translation of which can be found in Kretzmann (1982), Appendix F.

This is not the place for a complete discussion of the myriad of paradoxes and their multiple solutions by Scholastic philosophers, but what is of interest for us here is a particular general account on the moment of transition that provided the Aristotelian account with tools to attack these paradoxes in a way that is surely a logical ancestor of Brentano's approach in the 20th century, in the sense that it hinges on a particular approach to the metaphysics of boundaries that is, from a logico-formal perspective wholly analogous to Brentano's. However, a different question is whether these ideas were indeed a truly historical ancestor to Brentano's. This is much harder to say, since we cannot be sure what exactly in the Aristotelian tradition Breentano actually had access to. However, there is such a logical agreement between this tradition and Bretano's ideas that one must surely wonder whether this is where Brentano got his ideas about boundaries from.

We are referring to the tradition that arose in the 14th century with the work of some philosophers — such as Henry of Ghent, Hugh of Newcastle, John Baconthorpe and Landulf Caraccioli[24] — who began to take seriously a certain possible attack on what was by then the "orthodox" Aristotelian notion of continua. The problem they raised attention to has to do specifically with the application of Aristotle's general theory of continua to the particular case of change, or, more specifically, motion — which is indeed a particular case of the former broad notion of change in general, as it is, according to Aristotle, simply a change in location. In Kretzmann's words, the problem was the following:

> If the ceasing of condition not-ϕ and the beginning of condition ϕ occur at the same instant, then it seems right to say that that ceasing and that beginning are precisely simultaneous. And since condition not-ϕ obtains precisely until it ceases and condition ϕ obtains precisely as soon as it begins, that instant of transition may appear to present a violation of the *non simul* principle. In short, the elements of Aristotle's analysis of change may appear to entail this absurdity: at the instant of transition x is both not-ϕ and ϕ in the same respect.

That is, a certain change of some substance is a process in which it either has at some time t_0 some condition[25] which it fails to have at some later time t_1 or vice-versa. Now, if both the moment in

[24] On this topic, the papers Knuuttila and Lehtinen (1979), Kretzmann (1982) and Spade (1982) are particularly interesting and thorough. Also, cf. the discussion between Sorabji (1976) and Kretzmann (1976b).

[25] Now a days, perhaps the word "property" might be more appropriate here, but we shall stick to the historical vocabulary.

which the substance stops having the condition and the one in which it starts not having it coincide and, since something has some condition up to, and including, the last moment of the condition's presence and does not have it from the moment it stops having it onwards, then it seems that we must conclude that this moment of transition is a moment in which the substance both has and does not have the condition in question — something that would blatantly contradict Aristotle's principle of contradictories.[26]

The problem, however, is a real problem, according to Kretzmann, only for the class of conditions that he calls "successive conditions" and characterize as those conditions that can be realized only over a temporal interval. Indeed, he tells us that

> [t]he most important of them [i.e. genuine problems for the Aristotelian position] may be the difficulty (perhaps the impossibility) of providing a satisfactory Aristotelian account of the instant of transition between such conditions as rest and motion, conditions that can be realized only over a temporal interval. Medieval philosophers, building on Aristotle, called such conditions "successive", distinguishing them from "permanent" conditions, those that are fully realized at an instant. (Kretzmann (1982), pp. 274-275)

It must be noted, however, that this restriction of the problem to successive conditions does not make the slightest sense to me. In particular, it seems that the problem would relate essentially to the *permanent* conditions, since these are the ones that can be realized on singular instants and, therefore, the question whether one or the other holds at the moment of transition will yield either a contradiction — i.e., the answer that *both* contradictory conditions hold at the instant —

[26]Indeed, these are Baconthorpe's words with which he presents the problem:

> contradictory termini that are the termini of a single change occur at the same instant — I mean the *ultimum* of the not-being of the form to be generated and the *primum* of its being. (*Commentary on the "Sentences"*, L. III, d. 3, q. 2, art. 3)

Another characterization of the problem can be found in Spade (1982):

> In the fourteenth century, a certain group of authors thought they saw a problem with the Aristotelian analysis of the instant of transition; it appeared to them that at the instant when a thing changes from being ϕ to being not-ϕ (or vice-versa), it must be *both* ϕ and not-ϕ. In order to preserve the Law of Contradictories, therefore, these authors interpreted the word "together" (*simul*) in the common formulation of that law as referring not to temporal simultaneity, but to simultaneity by nature. Hence, contradictories may be true at the same instant of time, as in the case of instataneous transition, but they cannot be true at the same "instant of nature". (p. 298)

or something that goes against the excluded middle principle[27] — i.e., the answer that *neither* contradictory conditions holds at the instant. For successive conditions, one could much more easily accept something much closer to Aristotle's "first but not last" solution, taking the *terminus ad quo* to be the earlier *open* interval and the *terminus ad quem* to be the later *closed* interval that contains the boundary. On the other hand, a condition that requires some temporal interval to exist can obviously not be present at a single instant and this fact would be enough to conclude that these conditions would not yield problems; for how can two contradictory conditions be present at a single instant if neither of them can be realized at a single instant? Also, this would also not go against something like the excluded middle principle. Indeed, the fact that neither of the contradictory conditions are realized at the single instant in question might arguably not be a denial of the excluded middle principle, since it could arguably apply only to such conditions that could actually obtain at a single instant. Otherwise, *every* instant, which by definition must lack both a successive condition and its contradictory would indeed be a counter example for the excluded middle principle.

Now, independently of whether the problem is related to successive or permanent conditions,[28] the so-called quasi-Aristotelian Medieval philosophers introduced in connection to it what Kretzmann (1982) calls "the divided instant" (p. 276), which was originally characterized in the following passage of Baconhtorpe:

> The termini of a change are separated from each other only as much as the duration of the change that mediates between the termini, but an instantaneous change does not endure except for an instant alone; therefore its termini are separated not in accordance with with the parts of a duration, but solely in accordance with the order of nature. (*Commentary on the "Sentences"*, L. III, d. 3, q. 2, art. 3)

According to him, the usual interpretation of his contemporaries makes it so that

> [...] something false is imposed on the Philosopher. For the Philosopher there does not save the contradiction between being and not-being in that way [...]; instead, the Philosopher saves the contradiction in this way, that the instant is divided into a beginning and an end in such a way that the instant's first sign,

[27] Which is not considered by Aristotle, but that from our vantage point cannot be omitted

[28] Kretzmann (1982) recognizes that his quasi-Aristotelian Medieval thinkers did relate the problem of the instant of transition with permanent conditions (p. 275).

> which corresponds to the terminus a quo of an instantaneous change, measures the *ultimum* of the not-being, and its last sign measures the *primum* of the being [...] (*Commentary on the sentences*, L. III, d. 3, q. 2, art. 3)

The gist of the idea here is that, if one can think about the instant of transition as being composed of two distinct instants, then one is free to locate each contradictory condition in one of the composing instants and thus to reject the conclusion that these contradictory conditions must be both present in the same instant, which is the conclusion that seems to be the main problem for the quasi-Aristotelian thinkers in the original Aristotelian framework.[29] The whole talk of "instants of nature" seems to be an artifice found by these philosophers to introduce such a division in the instant of change, which is what does the main logical labor in this discussion. It is not the fact that these new instants are "of nature" that allows the quasi-Aristotelians to do away with the original problem, but the simple fact that the assumption of these moments of nature allows one to think about the original instant of change as somehow being *divided*. In this sense, then, Kretzmann's characterization is fairly on target, for it calls one's attention exactly for this divided nature of the instant of transition in the quasi-Aristotelian tradition.

That this idea is to be found originally in Aristotle was surely claimed by the so-called pseudo-Aristotelian proponents. However, it is not so far-fetched to believe that the Stagirite did have, perhaps not such a fully detailed account, but an idea that the point of transition did somehow pertain both to the former and later condition. Indeed, he says that

> [i]t is also plain that unless we hold that the point of time that divides earlier from later always belongs only to the later so far as the thing is concerned, we shall be involved in the consequence that the same thing at the same moment is and is not, and that a thing is not at the moment when it has become. It is true that the point is common to both times, the earlier as well as the later, and that, while numerically one and the same, it is not so in definition, being the end of the

[29]Here, we have a passage that sums up these ideas:

> For present purposes, then, Quasi-Aistotelianism may be described as a theory of change that developed in response to the standard medieval Aristotelian account of the instant of transition, an account that struck the Quasi-Aristotelians as entailing the simultaneous occurrence of contradictory conditions. They sought to avoid that outcome by accepting the contradiction in the temporal order and separating the contradictories in the order of nature, accordingly dividing the instant of transition into two instants or "signs" of nature, basing their solution primarily on *Physics* VIII 8. (Kretzmann (1982), pp. 280-281)

> one and the beginning of the other; but so far as the thing is concerned it always belongs to the later affection. (263b9-263b12)[30]

Moreover, the interesting point to be made here, in the context of the broader discussion of the monograph, is that this divided nature of the moment of transition, as it is conceived in the quasi-Aristotelian tradition, is perhaps what will allow the Aristotelian-inspired Brentano in the 20th century to talk about a multiplicity of indivisible boundaries being collocated in order to elucidate cases in which different continua meet. In particular, we must note the protagonism that the problem regarding the moment of transition plays in both contexts and, although this moment of transition is, for Brentano, a much more abstract notion — not being necessarily restricted to any kind of physical motion or change —, which encompasses boundary points in which, e.g., differently coloured continua touch, we must attest to the fact that the recognition of a multiplicity of collocated boundary points plays the same logical role in the Brentanian discussion and eventual solution to the problem as the divided instant in the quasi-Aristotelian solution to the paradoxes of change. Thus, this Brentanian solution too, notwithstanding its being more abstract and less couched in obscure scholastic terms such as "instants of nature", will have, in very broad terms,[31] the distinction between different indivisibles that are present at the same "point" or "instant" as its essential logical structure.

5.2. Criticisms of the pseudo-Aristotelian position. Now, although there seems to be at least a logical connection — if not a truly historical one, since it is very likely that the Aristotelian Brentano was indeed familiar with the writings of the Medeival philosophers mentioned in this discussion —, Kretzmann (1982) is, nonetheless, in favor of

> dismissing both the problem regarding the instant of transition and the Quasi-Aristotelian solution [...]. (p. 274)

For him, it is clear that

> from an Aristotelian point of view the [quasi-Aristotelian] attempt is misguided and the result unacceptable. (*ibid.*)

[30]Knuuttila and Lehtinen (1979) interpret this passage as meaning that

> Aristotle's 'solution' to the difficulty is to say that the change is instantaneous and the instant belongs to the posterior time with respect to the changing object

[31]More details in a later chapter.

This is because he believes that Aristotle has indeed long before already presented a solution to this conundrum. In particular, he believes that this solution is precisely the one we have called "first but not last position" and the one which Baconthorpe recognized as "something false" being imposed on Aristotle.

As we saw, this doctrine is extracted from a passage in *Physics*, VIII. Also, we saw how the latter part of the passage introduces what we have called the "first but not last" solution, which is far from being a clear and decisive solution and, thus, any reference to it for a solution of the problem of the instant of transition will be as undecisive and problematic as Aristotle's position itself. However, Kretzmann (1982) not only believes the problem has been successfully dealt with by Aristotle himself in this passage, but also that there are further more fundamental criticisms to be made to the quasi-Aristotelian unquestioned acceptance of the Scotist doctrine of the so-called "moments of nature". For this, he cites a fairly complicated passage of Ockham that seems to derive the impossibility of moments of nature. This passage, indeed, involves further systematic assumptions that could — and should — be called into question in a thorough evaluation of the true strength of the criticism. However, as was hinted at when talking about the relationship of this so-called "quasi-Aristotelian" doctrine with the later Brentanian account of continua, I believe that the main point here is not to think about the quasi-Aristotelian view regarding the problem of the instant of transition in terms of the moments of nature as some kind of "true and complete" solution to the problem, but merely as a first — perhaps the first in many centuries — recognition of the problem and an early attempt of solving it by means of this logical approach of "dividing" the problematic instant.

This position, however, as we mentioned above, should be held in higher regard since it seems to be the intellectual precursor to Brentano's much more abstract account that does away with Scot's notion of an "instant of nature" and retains, as we shall see in a later chapter, just what is truly important from a logical perspective, and amounts to an interesting position that is worth investigating further.

CHAPTER 2

The manifold-theoretic conception of continua

In this chapter, we shall be concerned with the history of how, in the course of the centuries following the Middle Ages, Aristotle's conception of continua as infinitely divisible and, therefore, as being unable to be composed out of ultimately simple or indivisible parts was transformed, from an undeniable canon of philosophical thought, into a theory that is completely foreign to the mathematical notions that took root and grew into the backbone of modern mathematical orthodoxy. We will see how this history unfolded in many bursts, with a few back-and-forths, but at the end of the day one must say that it ultimately culminated in the birth of set theory as the underlying theory behind all of modern mathematics and, particularly of a new mathematical discipline that became known as general or point-set topology.

Nowadays, it thoroughly accepted that any mathematical theory, from abstract algebraic group theory to differential geometry or functional analysis, can be and, indeed, is perhaps supposed to be — if one is to consider it as being actually thoroughly formalized so as to achieve the modern level of logical rigor — recast inside formal set theory, say, as a collection of sentences that follow logically from the ZFC axioms. Even category theory, which is often thought to be a new — and better, since more in line with many mathematical customs and intuitions — logical foundation for modern mathematics, is hardly ever introduced without recourse to set theoretical notions.[1] However, this omnipresence of set theoretical concepts and methods is a late 19th century creation, whose consolidation occurred well into the first half of the 20th century with the introduction of the formal axiomatic systems we know today as ZFC or BGvN and with its use in the development of many areas of mathematics, but mainly of point-set topology.

Now, this moment of consolidation of these new mathematical methods is precisely when Brentano is writing about his ideas on continua. Hence, it is only reasonable to assume — and,

[1] I believe that it is very symptomatic of this the fact that all the category theoretical formulation I know start by postulating that a category is a *set* of objects together with a *set* of "arrows" or "mappings" between these objects.

indeed, fairly clear from the few mathematicians and mathematical notions he mentions in his essays — that, in doing so, he is replying to the ideas that have been around in the mathematical discussions regarding continua and that his thoughts must be understood in the context of the different mathematical methods for defining the continuum of real numbers which were just 30 or 40 years old at the time he was struggling with his own ideas on the subject. Philosophical ideas are, as any human creations, not born *ex nihilo*, but have precursor ideas, gestation periods, fertile environments in which to sprout and grow, and enemy ideas to combat and overthrow. Brentano's ideas on continua are not different, and so are what we would like to call the "manifold-theoretic conceptions"[2] against which these ideas stood. The former are surely related to Aristotelian and Medieval conceptions, but the latter are not as well — merely — the outcome of individual geniality and vision. Even if one forgets all the other mathematical developments in course in the 19th century that surely — as we shall see — have had a role to play in the full development of this manifold-theoretic conception, and think only about the single man who might be credited with the creation of modern set-theory, *viz.* Cantor, it is already clear that his ideas did not come from nothingness itself, but were inserted in a debate that during his life already spanned at least two millennia, as we can clearly ascertain from the following passage of his (1883b):

> The concept of the 'continuum' has not only played an important role everywhere in the development of the sciences but has also evoked the greatest differences of opinion and even vehement quarrels. This lies perhaps in the fact that, because the exact and complete definition of the concept has not been bequeathed to the dissentients, the underlying idea has taken on different meanings; but it must also be (and this seems to me the most probable) that the idea of the continuum had not been thought out by the Greeks (who may have been the first to conceive it) with the clarity and completeness which would have been required to exclude the possibility of different opinions among their posterity. Thus we see that Leucippus, Democritus, and Aristotle consider the continuum as a composite which

[2] We use the word "manifold" (in German *Mannigfaltigkeit*) to make clear a the slight distinction that exists between set-theory as a mathematical theory that surely overgrew its origins and that has many features independent of the historical situation we are considering and the background ideas that were behind the historical developments of this mathematical theory at the turn of the 19th to the 20th century. Indeed, Cantor himself first used this word to refer to the objects whose formal theory he would eventually become known for creating, only later calling them "sets" (*Menge*), and the nowadays ubiquity of the word "set" seems to have come from (besides the obvious simplicity of the English word, which is surely appealing to mathematicians) from the important developments of the French analysts, who referred to Cantor's manifolds, following the latter's own lead, as "*ensembles*".

consists *ex partibus sine fine divisilibus*, but Epicurus and Lucretius construct it out of their atoms considered as finite things. Out of this a great quarrel arose among the philosophers, of whom some followed Aristotle, others Epicurus; still others, in order to remain aloof from this quarrel, declared with Thomas Aquinas that the continuum consisted neither of infinitely many nor of a finite number of parts, but of *absolutely no* parts. This last opinion seems to me to contain less an explanation of the facts than a tacit confession that one has not got to the bottom of the matter and prefers to get genteely out of its way. Here we see the *medieval-scholastic origin* of a point of view which we still find represented today, in which the continuum is thought to be an unanalysable concept, or, as others express themselves, a pure *a priori* intuition which is scarcely susceptible to a determination through concepts. Every arithmetical attempt at determination of this *mysterium* is looked on as a forbidden encroachment and repulsed with due vigour. Timid natures thereby get the impression that with the 'continuum' it is not a matter of a *mathematically logical concept* but rather of *religious dogma*. ((1932), pp. 190-191, translation from Ewald (1996), p. 903)[3]

1. Medieval dissidents and the new mathematical methods of the 17th century

Just as with so many other notions originally investigated by Aristotle, the notion of a continuum posited by the Greek philosopher remained canonical for at least a dozen centuries, during which the vast majority of scholars took his teachings as essentially true, albeit sometimes cryptic and incomplete. They struggled to understand the Aristotelian theses concerning this notion of continua, as they did for most of Aristotle's philosophical ideas, by proposing new — sometimes insightful and sometimes merely pedantic — distinctions to understand cases for which Aristotle's own considerations did not apply; and also by presenting corrections to Aristotle's arguments or even wholly new arguments to justify his original doctrines. As we saw in the last chapter, very few scholastic philosophers went against the Philosopher's teachings and, even when they did, it was

[3]Bell (2006) rightly finds strange the mention of the famous atomists Leucippus and Democritus along with Aristotle at the divisionist side of the quarrel and explains this by introducing a distinction between 'material' and 'theoretical' divibilism — a distinction that will appear below as we consider Medieval ideas on continua — and understanding Cantor's claim to be that these Greek philosophers, albeit great champions of *material* atomism or indivibilism, were nonetheless prone to accept that mathematical continua were indeed infinitely divisible — a view that is supported by Heath (1981).

done with a high degree of — perhaps not always truly sincere — caution and reluctance, not to mention the logical manoeuvres undertook to consolidate their affiliation to the great Philosopher. Thus, it is no great surprise that throughout these centuries very few philosophers, most notably Henry of Barclay and Nicholas of Autrecourt, took the path of going against the Aristotelian ortodoxy of regarding continua as not composed of indivisibles. Others, as Nicole Oresme and Nicolaus of Cusa, claim that any real continuum, albeit infinitely divisible with respect to some ideal mental operation, is in practice only finitely divisible, reaching eventually a moment in which further division would necessarily imply substantial destruction. Thus, we have:

> Divisible is used in two ways: one way it means the real separation of the parts of anything, and the other way it means division conceptually in the mind. It is not to be thought that every magnitude or continuum is divisible in the first sense, for it is naturally impossible to divide the heavens as one divides a wooden log, separating one part from another. In dividing a log or a stone or another material or destructible object, one can reach a part so small that further division would destroy its substance. But any continuum or magnitude is continually divisible conceptually in the human mind, just as astrologers divide the heavens into degrees, the degrees into minutes, the minutes into seconds, the seconds into thirds, fourths, and then fifths. The imagination can proceed thus endlessly. (Oresme (1968), pp. 45)

> Under mental consideration that which is continuous becomes divided into the ever divisible, and the multitude of parts progresses to infinity. But by actual division we arrive at an actually indivisible part which I call an atom. For an atom is a quantity, which on account of its smallness is actually indivisible. (Nicolaus of Cusa, *apud* Stones (1928), p. 447)

However, these independent thinkers of the late scholastic period could hardly scratch the surface of the well established Aristotelian doctrine of the infinite divisibility of continua. It is mainly with the mathematical developments of the 17th century that the existence of indivisibles would acquire more respectability, so that philosophically oriented mathematicians and mathematically oriented philosophers could begin to think of extended continua as being composed of an actual infinity of indivisibles.

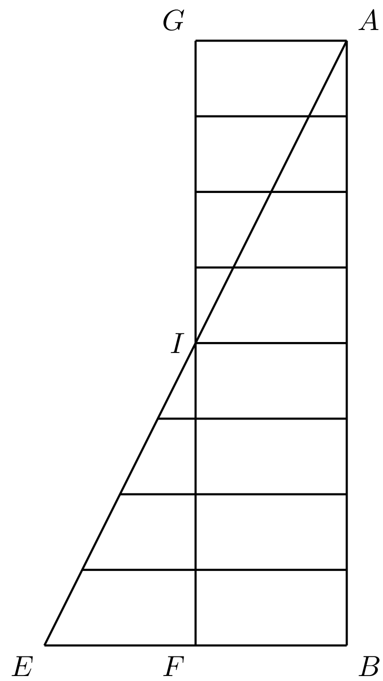
Thus, we shall begin our reconstruction of the path through which the idea of indivisibles ultimately composing continua came to the forefront of the mathematical culture in the 17th century with the method introduced by Kepler in his *Nova stereometria* for the analysis of the volume of revolution solids. Kepler is widely regarded as the first mathematician to use freely the notion of both infinitesimals and indivisibles in carrying out calculations of areas and volumes. Indeed, somewhat like the connection between atoms and indivisibles in the last chapter, these two notions often go hand in hand in the history of the developments that led to the eventual discovery of the calculus at the end of the 17th century. However, it is important for our reconstructions to keep these two notions apart. In particular, it is important to understand that, while indivisibles were though of as *heterogenea* or actual lower-dimensional portions of some mathematical continuum — *viz.* lines in a figure or surfaces in a solid —, infinitesimals were thought as *homogenea* or as parts of a continuum that, although had the same number of dimensions as the original continuum, had nevertheless a very small extension in at least one of these dimensions — as in the case of thinking about a circle as a collection of very thin triangles or, for a more modern example used for the construction of the Riemann integral, as in the case of thinking about the area below a curve as the collection of all very thin rectangles bounded above by the curve.

The example we gave above of the circle as the collection of all very thin triangles is indeed Kepler's. In fact, to compute the area of the given circle, he though of this circle as being composed of many infinitesimally small isosceles triangles whose bases were infinitesimal sections of the circumference and whose heights were the radius of the circle. Thus, the area of the circle could be estimated as being half the circumference of the circle, i.e., $\frac{1}{2}2\pi r = \pi r$, times its radius r — which is in fact the right result.

It is, however, his method for calculating the volumes of solids of revolution that is most important for our theme here because this is the method that actually introduced into the modern mathematical scene the notion of indivisibles. These solids of revolution, which are essentially the ones obtained by rotating some conic section about an axis — as, e.g., when one considers a sphere as arising from the rotation of a semi-circle about its diameter or, to use some of Kepler examples, when one considers "apples" or "lemons", which are obtained by the rotation about the same diameter of arcs that are, respectively, longer and shorter than the semi-circle —, were fairly well studied in antiquity, in particular with the work of Pappus. Notwithstanding, Kepler introduces a very fruitful technique for the calculation of their volumes by adding the areas of all cylindrical surfaces generated by rotating all the possible chords one could draw uniting two points of the original conic section. Thus, the image one gets from this method is one in which these revolution solids are thought of as being somehow ultimately composed of all these cylindrical surfaces.[4]

Another interesting example of how such intuitions came to bear on mathematical methods of integration is Galileo's proof of the mean speed theorem in his *Two new sciences* (pp. 173-174).[5] In this theorem, Galileo shows that the time it takes a uniformly accelerated body starting from rest to traverse some distance d is the same as the time taken by some other body traversing the same distance d with a velocity which is the mean between the starting and ending velocities of the first body. The actual proof involves turning things around a bit and showing that if the two motions happen in the same time interval, then the distances covered by the two motions are equal, which amounts to the same result. Indeed, let AB be the time it takes for the first body to traverse d and let EB be proportional to the highest speed the body has during the movement — viz., its final velocity. Then, by drawing the line EA, we have that all the lines parallel to EB represent the instantaneous speed which the object has at some instant t, which is the distance between each parallel line and the line GA. Now, if we let F be the midpoint between E and B, we have that the area of the rectangle $ABFG$ is the same as the area of the triangle ABE, since GF bisects AE at I. What is happening is that the area of the smaller triangle AIG is the same as that of the triangle IFE, as can be seen by understanding that for each point T between A and B, the length of the section of the line parallel to GA meeting AB at a point T that lies in the triangle AIG is identical to the length of the corresponding line segment inside IFE of the line that meets AB at T. But since the lengths of the parallel lines — in the case of the uniformly accelerated motion, the length of the sections inside the triangle ABE, and in the case of the constant velocity, the length of the sections inside the rectangle $ABFG$ — are proportional to each instantaneous speed of the body, the areas in question are proportional to the distance d, so that this distance is indeed the same for both motions.

[4] Cf. Kepler, *Opera omnia* IV, pp. 584-5.
[5] Cf. Boyer (1949), pp. 113 ff. and Baron (1969), pp. 120 ff.



For us, the interesting part of this proof is the one in which the areas of the two smaller triangles AIG and IFE are compared, for, to carry out this comparison, Galileo thinks of these respective areas as being somehow composed out of the lengths of the infinite number of line sections parallel to EB that lie inside the respective triangle. Thus, by showing that for each such length in AIG there is a corresponding length in IFE and *vice-versa*, the equality of areas is established.

This proof bears some similarities with Oresme's proof of the same proposition. However, although we have seen that Orestes does indeed take a step towards the recognition of continua as being composed of an actual infinity of indivisibles, by distinguishing between real and mental or conceptual division, Baron notes that

> [t]here seems no reason to suppose that Oresme regarded a line as the sum of its points or a surface as the sum of its lines. This is a static idea and the whole emphasis with Oresme is on the description of lines, surfaces and solids by motion. It was not necessary for him to postulate an infinity of lines drawn upon a surface; he argues, in fact, that if an ordinate be erected at any point on the base line proportional to the intensity of the quality then the total area under the summit

> line (or line of intensity) will correspond with the quantity of the quality. (Baron (1969), p. 86)

What is important to note here, though, is that even if Oresme's philosophical positions make him merely an intermediate figure in the history of acceptance of a manifold-theoretic view of continua, he undeniably did play a key sociological role in this history, in that his mathematical ideas were the bedrock on top of which this ontological conception of continua could be justified. Indeed, Baron concludes by saying that

> [i]f Oresme himself did not find it necessary to draw lines of intensity everywhere on the base line others who followed him may well have done so; for most of the sixteenth-century editions of his work and that of the Merton Calculators are illustrated with innumerable diagrams exhibiting an endless variety of geometric *forms* covered with arrays of parallel ordinates. (*Ibid.*, p. 87)

By "parallel ordinates" she has in mind the type of construction we used, following Galileo, for the proof of the mean speed theorem. These, according to her, were commonly accepted in the 16th century as constituting the areas of polygons, as the addition of diagrams containing them to the works of Oresme and the Medieval calculators would show. Undoubtedly, though, by the 17th century this idea of higher-dimensional geometric figures as being composed out of an infinity of lower-dimensional ones is elevated by the work of Galileo's student Cavalieri into a very efficient method for integrating areas of plane figures and of solids, which eventually became known as "the method of indivisibles". As we shall see below, this turned out to be a very famous method and its rigorousness was very much a discussion theme in the 17th century until its results could be reinterpreted in terms of the calculus of Newton and Leibniz.

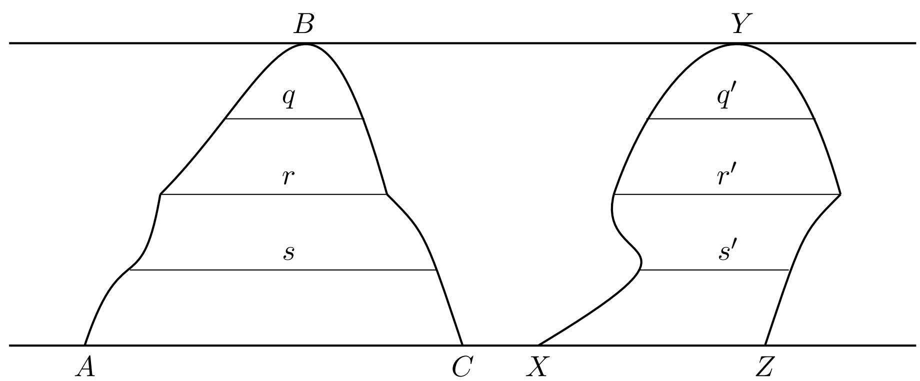
Essentially, the method of indivisibles of Cavalieri consists of the following mathematical result that became known as Cavalieri's theorem:[6]

> If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and, if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the

[6] Cf. Edwards Jr. (1979), pp. 105 ff., Boyer (1949), pp. 117 ff. and Bell (2006), pp. 69-70.

included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another. (*Geometria Indivisibilibus Continuorum Nova quadam ratione promota*, p. 484; Evans (1917), p. 448)

The idea behind this theorem is essentially a generalization of the method we used when we gave the reconstruction of Galileo's proof of the mean speed theorem for equating the areas of the two small triangles. Now, however, we are not restricting our attention to two triangles, but are considering *any* two-dimensional figure (or three-dimensional solid) constructed in the space between two parallel lines (or two parallel planes), as the figures ABC and XYZ in the following figure:



Then, what one does is to think about these figures as being composed out of the numerous parallel line segments — some representatives of which are denoted q, q', r, r', s, s' in the figure — so that, if one can establish a correspondence between the lengths of each pair q, q', r, r', s, s' etc., then the theorem guarantees, by an intricate but intelligent argument involving the superposition of the two figures and a method for comparing the non-overlapping parts, that the areas of the figures must also be equal.

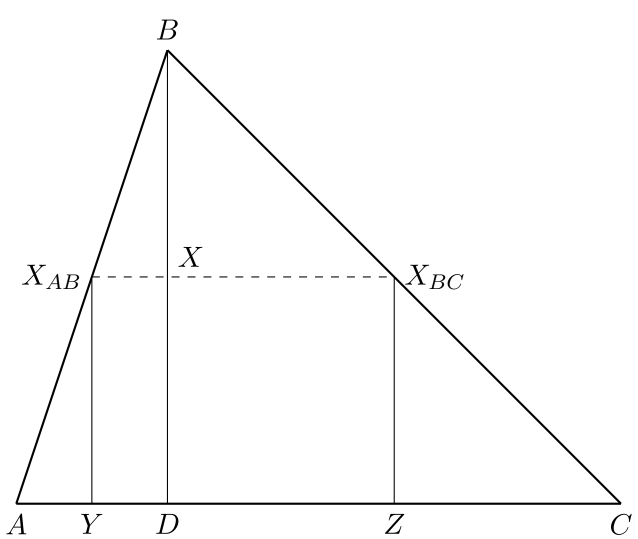
The main point to keep in mind here is the notion of "any straight line being drawn equidistant from the parallels". These lines are examples of what Cavalieri called the "indivisibles of a figure"[7]. He defines them in the following passage:

> If through opposite tangents to a given plane figure two parallel and indefinitely produced planes are drawn either perpendicular or inclined the given figure, and if one of the parallel planes is moved toward the other, still remaining parallel to it,

[7] *Geometria*, p. 114: "*indivisibilia. s. omnes lineas figurae*"

> until it coincides with it; then the single lines which during the motion form the intersections between the moving plane and the given figure, collected together, are called all the lines of the figure taken with one of them as *regula*; this when the planes are perpendicular to the given figure. When, however, the planes are inclined to the figure the lines are called all the lines of the same given figure with respect to an oblique passage (*obliqui transitus*),[8] the *regula* being likewise one of them. (*Geometria*, p. 99, translation from Andersen (1985), pp. 300-301)

This distinction between *recti* and *obliqui transita* is related to what he calls the distribution of the respective line segments in a figure. In particular, it plays a role in an alleged counterexample to Cavalieri's theorem, which is given by Cavalieri himself in *Ex. geom. sex*, pp. 238-9. Consider a triangle ABC with $AB \neq BC$. Then, draw a line BD perpendicular to the base AC. For each point X on this line, one can consider the projection of this point on the AB and BC lines, which we shall call, respectively, X_{AB} and X_{BC}. And for each such pair of projections, the respective line segments starting at each projection point and meeting the base AC at a right angle are of equal length, so that one would have to conclude from Cavalieri's method of indivisibles that the triangles ABD and DBC have the same area, which is false, since we are assuming that $AB \neq BC$.



[8]As opposed to the case in which the plane is perpendicular to the figure, which he called *recti transitus*.

It is easy to see that this construction does not fall into the assumptions of Cavalieri's theorem since the triangles ABD and DBC are not drawn between two parallel lines so that the indivisibles considered are parallel to these original lines. However, harder to see is why one requires such strong assumptions in the theorem in the first place, i.e., why this construction is not analogous to the construction in Cavalieri's theorem in the sense that it establishes a correlation between pairs of line segments of the same length that together exhaust the two triangles ABD and DBC.

Cavalieri's answer was that in this kind of constructions the line segments obtained from the points X_{AB} did *not* have the same *distribution* as those arising from the points X_{BC}. He never gave a clear definition of what this notion of distribution was to mean, but it seams that this idea would justify the stronger conditions of his theorem as necessary for the equality of the distribution of the respective line segments involved in the analysis of area by the method of indivisibles. Thus, Baron says that,

> [a]lthough this idea of the distribution of indivisibles was never very clearly developed by Cavalieri he was careful only to compare figures in which the distribution of indivisibles was, in fact, uniform. The summation of lines and planes in the Cavalierian sense is therefore formally equivalent to the addition of rectangular elements of equal thickness,
>
> $$\frac{\sum l_1 b}{\sum l_2 b} = \frac{\sum l_1}{\sum l_2},$$
>
> where $b \to 0$. Many of Cavalieri's contemporaries were quick to see this and to make the required adjustments. Wallis and Roberval, in particular, considered the use of lines and planes entirely justifiable since both derived from infinitely narrow rectangles and sections of uniform thickness. (Baron (1969), pp. 134-135)

The formal equivalence noted at the end of this passage has a somehow double meaning here. On the one hand, it shows that the assumption involved in this method regarding the composition of higher dimensional continua from lower-dimensional indivisibles is not *necessary*, in the sense that one can instead consider these continua as being composed of other continua of the same dimensionality, whose thicknesses are a constant value b, and then consider the case in which $b \to 0$. As a matter of fact, this is indeed the first notion of integration to achieve a canonical status due to its rigorous formulation by Riemann. However, the aforementioned equivalence also shows that the method of indivisibles, being equivalent to this "limit method", which was made rigorous in the work of the nineteen century analysts, is a *possible* way of understanding the composition of mathematical continua and, thus, it surely provides some strength to those who would like to go against the classical Aristotelian view of continua and towards a recognition of the manifold-theoretic constitution of these continua out of lower-dimensional indivisibles. And, indeed, given the widespread acceptance of Cavalieri's mathematical work, one can surely believe that its reliance of this manifold-theoretic conception of continua might have found warm acceptance wherever the mathematical method itself had been recognized as sound and useful.

Cavalieri, though, quite politically claimed that his method could be interpreted in both conceptions of the continua, *viz.* both if one thought it was composed of atomic indivisibles or if one held the Aristotelian position of infinite divisibility. However, from his very attempt of presenting an interpretation of the method in both conceptual schemes it seem clear how the method itself is much more in line with the first way of understanding the composition of continua. Indeed, Andersen (1985) says that

> [h]e did not state exactly how the space occupied by "all the lines" should be understood if continuous divisibility was assumed, but he argued for the existence of the ratio between two collections of lines even in this case. (p. 306)

I believe it is doubtless how this method of indivisibles, if it is to be taken at face value, involves a thorough acceptance of the thesis according to which, at least with respect to abstract geometrical two and three-dimensional objects, continuous entities have their dimensions in virtue of their being composed from a myriad of lower-dimensional entities — *viz.* plane figures from line segments and solid bodies from plane sections. Further evidence is given by Andersen (1985) in the following passage:

> Although Cavalieri did not make it clear what he thought of the composition of the continuum, one may wonder whether he did not incline more to the one possibility than to the other. The following remark from Cavalieri's letter to Galileo dated June 28, 1639, "I have not dared to say that the continuum was composed of these [the indivisibles] ... Had I dared ..." (Galilei Opere, vol. 18, p. 67) could give the impression that most likely Cavalieri conceived of the continuum as composed of indivisibles. Lur'e seems to have maintained that this was indeed Cavalieri's opinion not expressed too explicitly because he feared an opposition from the

> Catholic church holding the Aristotelian view (cf. Lombardo-Radice 1966, p. 206) (p. 307)

It is an undeniable historical fact that the period we are considering was a quite dangerous moment for those who openly went against the Catholic doctrines of the Church, which were deep-rooted in medieval Aristotelianism.[9] However, she concludes more conservatively that

> I find it likely that Cavalieri's apparent ambivalence should be ascribed to the circumstance that he was not genuinely interested in the philosophical aspects of the composition of the continuum. The function of "all the lines" was first of all, as Cavalieri himself stated in the introduction to *Exercitationes* (p. 3), to be an instrument for quadratures; and his mathematical treatment of them was independent of any conception of the continuum. (This point of view is also expressed in Lombardo-Radice 1966, e.g. p. 206, and in Cellini 1966_1 p. 9.) (*Ibid.*)

Indeed, although the method initially encountered stern criticism — mainly from Guldin[10]—, it found relatively widespread acceptance in the mathematical community of the 17th century. As an example of this acceptance of Cavalieri's method of indivisibles, we can mention its use in the mathematical work of Torricelli. According to Andersen (1985),

> During the first years after the publication of *Geometria* (1635) Torricelli took a rather sceptical [*sic*] attitude toward Cavalieri's method (cf. Lombardo Radice 1966, pp. 21-22). But about 1641 he changed his mind and found that it opened a "royal road" to quadratures (Torricelli *Opere*, vol. 1, part 1, p. 140), and he gave examples of the use of the method in his *Opera geometrica*, published in 1644. This book was well received by European mathematicians and became influential in spreading knowledge of the method of indivisibles; for several mathematicians it remained the only origin of this knowledge. (pp. 355-356)

Torricelli opens his *Quadratura Parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, with the following:

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[9] One can mention here figures like Giordano Bruno, Copernicus and Galileo himself who either were killed by the inquisition or whose lives were threatened by the it based solely on doctrinal reasons.

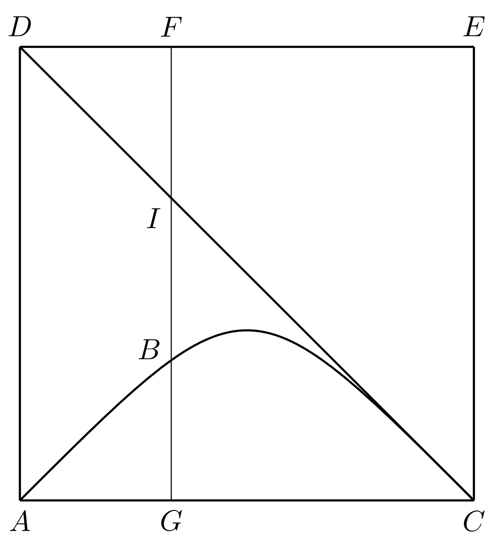
[10] Cf. *Centrobaryca*, pp. 340-342.

> Until now, the matter about the measurement of the parabola has been related in the manner of the ancients. It remains that we should approach the same measurement of the parabola with a certain new but marvelous system — namely, by the aid of the Geometry of Indivisibles and with diverse methods in this manner. (*Opera geometrica*, p. 55)

Then, before proceeding to the actual mathematical arguments, he concludes the introduction by saying that

> this [the method of indivisibles] is truly the Royal Road in the mathematical thorn hedges, that Cavalieri, creator of these wonderful inventions, first among everyone opened up and made public for the common good. (*Ibid.*, p. 56)

From these quotes, it seems clear how much Torricelli valued the new method of Cavalieri; and the proofs themselves have clear instances in which certain figures and solids are identified with the collection of all lower-dimensional indivisibles of some kind.[11] In order to see exactly in which respect this is so, it is interesting to look at one of the actual proofs presented by him to show that a certain figure which is bound by a parabola and a horizontal line has an area which is $\frac{4}{3}$ of the are of a triangle with same base and with the other vertex at the point where the diameter of the parabola intersects with it. The proof refers to the following figure



[11]E.g., a line is "[...] una lunghezza, cioè una estensione di punti continuati", Torricelli, *Opere*, II, p. 247.

and it goes like this:

> Let ABC be a parabola with tangent CD and let AD be parallel to the diameter. Let the parallelogram AE be drawn, and let a circle with diameter AD be conceived which is the base of a cone having vertex at point C and likewise is the base of some cylinder $ACED$ of the same height with the aforementioned cone.
>
> Now let some line FG be drawn parallel to AD, and let a plane parallel to the circle on AD be conceived to pass through the line itself. FG will then be to IB as the line DA is to IB — that is, as the square on DC is to the square on CI (because of the parabola). Or as the square on DA is to the square on IG (because of similar triangles) — that is, as the circle on DA is to the circle on IG — namely, as the circle on FG is to the same circle on IG. And it is this always. All the first magnitudes are equal to the line DA and therefore equal among themselves. Also, all the thirds are equal to the circle on DA, and on account of this equal among themselves. Therefore, by Lemma 18, all the firsts together — namely the parallelogram AE — will be to all the seconds together — namely, to the trilineum $ABCD$ — as all the thirds together — namely, the cylinder AE — are to all the fourths together together — that is, to the cone ACD. Therefore, the parallelogram AE is three times the trilineum $ABCD$. With the parallelogram AE cut in half, the triangle ACD will be $\frac{3}{2}$ of the trilineum $ABCD$. By conversion of the ratio, the triangle ACD will be three times the parabola itself. On account of this, from the explanation of Proposition 9, the parabola will be $\frac{4}{3}$ of its own inscribed triangle. (*Opera geometrica*, II, pp. 56-57)

The interesting thing to note in the quoted proof is how the parallelogram AE is identified with the collection of all lines from which FG is an instance, how the trilineum $ABCD$ is identified with the collection of lines from which IB is an instance, how the cylinder $ACED$ is identified with the collection of circles from which the one with diameter AD is an instance and how the ACD is identified with the collection of all the circles from which the one with diameter IG is an instance.[12] These identifications are clear instances of the general approach of considering continua

[12] Cf. Bascelli (2015) pp. 116-117.

to be constituted by a collection of lower-dimensional heterogenea and the simplicity with which these identifications are made seems to attest to how commonplace they became in the context of geometrical discussions of integration type problems in the 17th century.

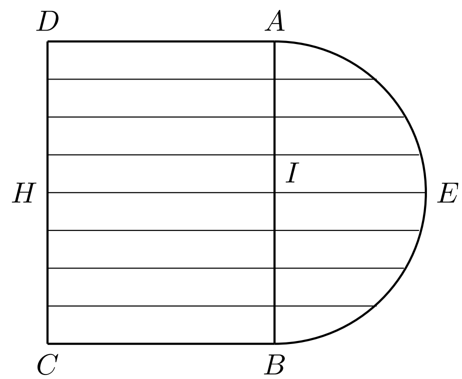
Now, whereas Cavalieri had a very technical and ontologically neutral or purely mathematical approach to the method of indivisibles, what we see in Torricelli is a full blown acceptance of these heterogeneous indivisibles, which were used in his work in a much more naive and widespread manner.[13] He even proudly claimed, perhaps unaware of Kepler's work, that his method

> proceeds with curved indivisibles, an example of which no one has yet given. Since Cavalieri himself has not provided any element of this subject in his *Geometry*, we have to assume that he should corroborate with certain examples this way of reasoning. (*Opera geometrica*, p. 174. The translation here is our translation from the French quote in De Gandt (1987))

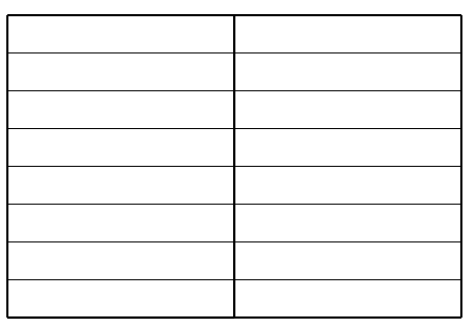
Then, he used these curved indivisibles — in particular, cylinders — to prove a formula for the volume inside a hyperboloid obtained by rotating a hyperbola about one of the axis, which shows his intent of thinking about every geometric object in terms of these lower-dimensional indivisibles.

Another episode of the history of the calculus that has some interest to our overview of mathematical developments which could have influenced the thorough acceptance in the late 19th century of the idea that continua were composed out of indivisibles is an integration method presented by Grégoire de Saint-Vincent in his *Opus geometricum*, which was published in 1647, although it seems to have been written probably between 1622-1629.[14] The method was called *ductus plani ad planum*. Essentially, this method provides a way of comparing the volumes of solids that are constructed according to a common construction method as *ductii* of certain plane figures on a ground line.

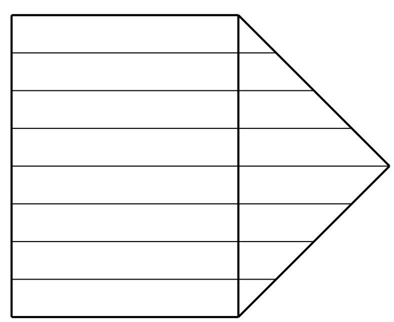
[13]Cf. De Gandt (1987), pp. 161-163.
[14]Cf. Baron (1969), p. 135.



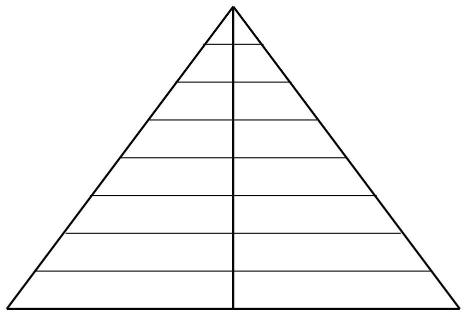
Given two plane figures, e.g., the rectangle $ABCD$ and the semi-circle AEB in the previous picture, the *ductus* of these figures is the solid one builds by considering all the lines between DA and CB parallel to HE and then constructing, for each of these lines, a rectangle with base HI and height IE, where I is the point where HE meets the ground line AB. Thus, we have that a parallelepiped is the *ductus* of two rectangles, a prism is the *ductus* of a rectangle and a triangle and a square base pyramid is the *ductus* of two right angled triangles with equal sized bases, as illustrated in the following pictures:



The *ductus* decomposition of a parallelepiped.



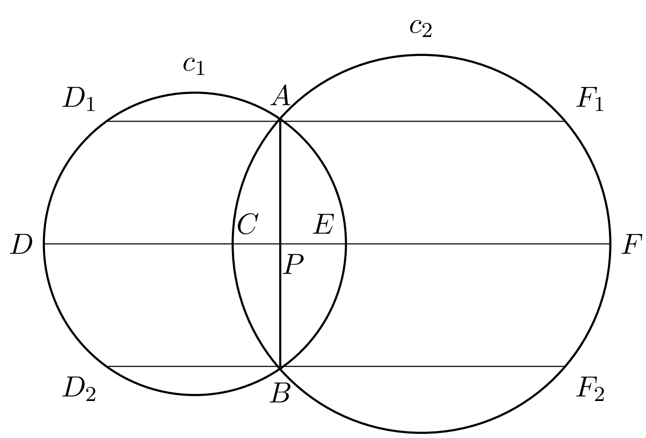
The *ductus* decomposition of a prism.



The *ductus* decomposition of a rectangular based pyramid.

This method is interesting to our present discussion because it seems to presuppose a close relationship between the solids in question and the rectangles from which the solid is constructed. Indeed, these rectangles can be thought of as that out of which the solid is *composed*, since one can infer — this is the very lesson one draws from the method — a relationship between the volumes of any two *ductii* whose composing rectangles stand in the relevant relation. This is indeed a very general result, but let us consider, for the sake of clarification, a particular example[15] of how this method can be used.

Let c_1 and c_2 be circles which intercept each other at a common chord AB — as in the following figure.



Then, for any line perpendicular to AB which intercepts it at P, c_1 at C and D, and c_2 at E

[15]Cf. Baron (1969), pp. 140-141.

and F, we have the following identities:

$$DP \cdot PC = AP \cdot PB = EP \cdot PF.$$

From these identities, one can conclude, therefore, that the volumes of the *ductus* of AD_1D_2B in ACB is the same as the volume of the *ductus* of AF_1F_2B in AEB, since the identities guarantee the equality of the areas of any pair of corresponding rectangles, one from the construction of the first *ductus* and the other from the construction of the second *ductus*.

Hence, one readily sees how this method also follows Kepler's, Cavalieri's and Torricelli's integration methods in assuming that the continua whose area or, in this case, volume is to be evaluated according to the integration method are composed of a myriad of lower-dimensional *indivisibles*, such as lines or rectangles. It, thus, provides more evidence to the presence of this manifold-theoretic conception of continua in the 17th century.

The last 17th century figure we shall consider in relation to this method of indivisibles is the interesting case of Roberval, who, albeit claiming to having discovered the method around the time Cavalieri developed his own version of it and, having used it to solve many open problems without making it public so that people would be impressed by his prowess,[16] conceived of it in a way that is much closer to the limit conception of integration which was eventually made clear with the Riemann integral in the 19th century. This is because he seems to have at least half clearly understood the composition of a certain geometrical continuum, not by means of "true indivisibles", but by means of *homogenea*, which were assumed to be very small in the dimension which the Cavalierian "true indivisibles" were to be considered unextended. The exact extent to which Roberval himself thinks of his method as being essentially different from Cavalieri's is hard to assess precisely. Jullien (2015) seems to think that Roberval had a clear conception of this difference and he says that

> Cavalieri's proposition, which involves continuum [sic] being composed via an aggregate of indivisibles of a smaller dimension, is not part of Roberval's approach. Roberval was perfectly aware of his modification of the doctrine. (p. 180)

To justify such a position, one can — as Jullien does[17] — mention the following passage from Roberval's letter to Torricelli:

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[16]In this respect, it is worthwhile to cite a beautiful passage of one of Roberval's letters to Torricelli, in which he admits the silliness of this attitude and gives precedence to Cavalieri for being the first to make this method public. This is the passage, as cited in Jullien (2015):

> Let us now discuss indivisibles, since I believe them to be of some importance. Whether or not the illustrious Cavalieri did indeed invent them before we did, I cannot be sure. However, I do know this: five years before he unveiled his finding, the doctrine of indivisibles helped me to solve some difficult problems. But do not worry: I will not claim that the invention of this sublime doctrine is my own rather than his. I cannot, and I would not even if I could. It is he who revealed it first, and thus, it is his. Let him therefore claim possession of it and make full use of it; let him be known as its inventor. May God ensure that in future, I do not allow myself to become a ridiculous intermediary in such an affair, as I have done in the past; all the more since I have not even revealed the doctrine to my own friends, my youthful pride having prevented me from deciding to make the discovery public. For I hoped that, in the meantime, I would easily become renowned for my own doctrine by solving difficult problems which I publicly submitted, every day, with the help of precisely the same technique. And I have certainly not been disappointed. Indeed, once I had fully developed the theory, inspired as I was by an intense enthusiasm, and once I had extended its field of application to points, lines, surfaces, angles, solids and finally to numbers too, I had no difficulty in achieving such results as to delight my friends and frustrate my rivals. My successes were thus rather too much like those of a child; I concealed a doctrine which would itself have been worthy of the following line of poetry:
>
> *Nec ferre videt sua gaudia ventos.*
>
> Having discovered a gold mine, I put on display a few of the gold nuggets that I had collected from it, in order to be taken for a rich and happy person; meanwhile, another man showed to everyone the same gold mine which he had also discovered, and, to unanimous applause, brought it into the public domain. Hence, I would almost certainly be ridiculed were I now to claim to have also discovered it.

[17]Cf. Julien (2015), p. 180

> There is, however, a small difference between Cavalieri's method and ours. Our method considers the indivisibles of any surface in terms of an infinity of lines, and the indivisibles of any volume in terms of an infinity of surfaces. [...]
>
> We consider a line to be composed as if it were made up of infinite lines, or of an infinite number of lines, a surface of surfaces, a solid of solids, an angle of angles, an indefinite number of indefinite units, and even better, a plane by planes (*plano* $-$ *planum* in latin) made up of plane by plane, and so forth; each one of these categories has its own properties.

Honestly, this passage certainly seems more confusing than illuminating, since in it Roberval seems to recognize a small difference between his and Cavalieri's methods, then to claim that his method conceives of continua as being composed of lower-dimensional indivisibles and finally concludes that continua must composed out of *homogenea* — together with other magnitudes such as angles and "indefinite numbers", which are not even part of our discussion regarding continua.

Jullien (2015) seems to explain away this and analogous inconsistencies in Roberval's terminology as merely a simplification, so that, in speaking of indivisibles as if they were true Cavalierian lower-dimensional *heterogenea* whose infinite multitude actually composed a certain higher-dimensional continua, Roberval would always have in the back of his head their "true" interpretation as very thin *homogenea*, a finite number of which would exhaust the dimensions of the continua in question. Indeed, Jullien says that

> if indivisibles are indeed homogeneous, the algorithms that are available require discrete quantities to be added together. It is therefore necessary — for the sake of calculation — to identify, fictitiously, the small lines with points. This method is not entirely unique to Roberval. A short time before him, Simon Stevin made use of some fairly similar notions. (Jullien (2015), p. 185)

We have already seen this when considering Cavalieri's method in the quote from Baron (1969) and we noted then that this equivalence between seeing a certain continuum as composed of lower-dimensional indivisibles and seeing it as composed of many very thin *homogenea* has the two-fold consequence of, on the one hand, retroactively giving some respectability to the anti-Aristotelian conception of continua as composed of indivisibles, since this conception would yield a mathematical method which is essentially equivalent to the undeniably rigorous limit method of the 19th century;

and, on the other, of showing that this conception is not necessary, since the same results could be obtained by thinking about the indivisibles in Roberval's way as very thin *homogenea* whose summed areas or volumes approximated the area or volume of the continuum in question to any degree of accuracy required, so long as they were taken to be thin enough — and, eventually, with the advent of the rigorous notion of a limit in the 19th century, one could just consider the case in which these thicknesses were taken to the limit 0. In this respect, Anderson (1985) says that

> it is obvious that Roberval's procedure leading to a determination of a limit (which in the 17th century was done by omitting certain terms) was an approach to quadrature quite different from Cavalieri's calculations with collections of lines. However, Roberval himself did not seem to have considered his method very different from Cavalieri's. (p. 360)

Now, we must conclude by noting that, after these interpretations of indivisibles as very thin *homogenea*, which eventually were made rigorous by the advent of the rigorous definition of the notion of a limit, mathematicians were free to use the methods we have been discussing without having to worry about adhering to non-Aristotelian metaphysical doctrines — an attitude which is fairly ubiquitous in mathematics, not being restricted to these particular metaphysical doctrines, but having more the form of a generalized desire of mathematicians to avoid metaphysical disputes altogether. In this respect, we have statements such as Barrow's in his *Lectiones Mathematicae* of 1683:

> I sey instant or indefinite particle, for it makes no difference whether we suppose a line to be composed of points or indefinitely small linelets; and so in the same manner, whether we suppose time to be made up of instants or indefinitely minute timelets (Barrow (1916), p. 38)

and eventually that of his student Newton:

> demonstrations are shorter by the method of indivisibles but because the hypothesis of indivisibles seems somewhat harsh, an therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following Propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios, and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed

> as by the method of indivisibles; and now those principles being demonstrated, we may use them with greater safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curved lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios ... (Newton (1962), p. 29)

As we have seen, even Cavalieri himself was very political in claiming that both metaphysical conceptions regarding the constitution of continua could be regarded as a proper background to his method, albeit it seems clear that, identifying as he did the indivisibles with lower-dimensional continua, this is not strictly speaking true.

By the end of the 17th century, however, the method of indivisibles already started to lose its prestige to this other method of $infinitesimals$ and limits; and the idea of continua as constituted by an actual infinity of lower-dimensional indivisibles was replaced by dynamical intuitions of continua as the result of the motions of other thin continua in time, which were indeed around since the Medieval, if not the Ancient period.

Thus, we get to 18th century, which is the century in which mathematicians worked towards a clarification and a development of the calculus of Newton and Leibniz, thereby accepting with it the idea that integration methods should be regarded in terms of the limit notion and not in terms of lower-dimensional indivisibles. Thus, although the 18th century in a sense brought about the decline of the integration methods that relied on the assumption of infinitesimals, on the other hand, we have in the mathematics of the 18th century the beginning of the story which culminates with the rigorous presentation of the calculus by the 19th century analysts.

2. The "barren" 18th century

The mathematicians of the 18th century surely did not fail to contribute to the growth of their discipline. It is a mere case of listing the names of mathematicians living in this century — one can, roughly chronologically, start with La Hire, then mention König, Lampert, the Bernoulli family, Euler, D'Alambert, Bézout, du Châtelet, Fourier, Gergonne, Goldbach, Kästner, Lacroix, Lagrange, Laplace, de l'Hôpital, Maupertius, and end with the mighty Gauss — to realize the sheer amount of grandeur that these decades had in store for the development of mathematics in general, and of the calculus in particular.

However, the development of mathematics carried out in the 18th century had much more to do with the advancement of practical techniques that allowed these authors to develop the theory of Newton and Leibniz to a whole new level. This development is arguably what brought about the need for a revolution in the foundations of mathematics in the 19th century, which shall, in its turn, bear many fruits with respect to the history we're trying to trace in this chapter; notwithstanding this, however, we must note that this development of mathematics in the 18th century was marked by a strictly pragmatical approach that was almost exclusively based on problem-solving activities. If one goes back to the list of mathematicians that worked during the 18th century, one readily sees that they were all, with very few and localized exceptions, very skillful calculators and problem solvers, who did not seem to assign much interest to foundational issues.

A consequence of this is that there was not much happening in the 18th century that deserves mention in our historical overview of the mathematical ideas with respect to continua. In fact, pretty much all of the mathematics that was done in this century was done against a background of intuitions that sprung from the late 17th century — *viz.* either a conception of continua as composed out of infinitesimal homogenea or, what became more and more fashionable as the years went by, a dynamical conception of continua as being created by the motion of a single lower-dimensional entity, i.e., a line as the motion of a point, a surface as the motion of a curve etc.

Perhaps even more telling is the fact that these background intuitions became more and more distant as the algebraic methods grew in strength, so that much of the foundational issues being discussed in the late 18th century have to do with the applicability and reasonableness of purely algebraic expressions, such as the infamous ratio $\frac{0}{0}$, which was a nightmare for mathematicians, like Euler, who wanted to explain away infinitesimally small quantities as being merely mechanisms to deal with quantities that were actually equal to 0.[18]

This lack of any foundational concerns with respect to the nature of continua is the sense in which we have termed the XVIII "barren". In many other respects, this century was extremely fruitful, yielding many strong mathematical ideas and results. However, they do not fall under the topics with which we are concerned in this chapter, so that we might as well, for the sake of our

[18]Cf. Gray (2015).

historical reconstruction, jump straight to the 19th century, in which we shall, on the contrary, find many developments inside mathematics that are directly connected with our theme in this monograph regarding the essence of continua.

3. From continua to the continuum of real numbers

It was part and parcel of the 19th century mathematical achievement of recasting the calculus in strictly rigorous terms to present a formal characterization of the real numbers in terms of a *complete linearly ordered field*. Indeed, this characterization was such a great achievement to the eyes of 19th century mathematicians that one can even see a change in their language. Whereas earlier mathematicians talked about continua as the geometrical entities that were their object of study, after the first few constructions of the real numbers appeared and became widespread in the literature we start to see more and more mentions to "the continuum" or "the continuum of real numbers". And this socio-linguistic drive was — I believe — further fomented by the way mathematicians in the 19th century, following Grassmann and Riemann, began to thing about *any* abstract n-dimensional space as what we would now call "the n-fold Cartesian product of copies of $\mathbb{R}$", *viz.* $\mathbb{R}^n$. Couturat, writing right at the turn of the century, is very clear on this respect.

> But how do we define the continuous space of n dimensions? It is the set of points $(x_1, x_2, \ldots, x_n)$ obtained by ascribing to each coordinate $x_1, x_2, \ldots, x_n$ all real values. That definitely means that one grounds the continuity of space on the continuity of the set of real numbers. But one knows how this latter continuity is, in its turn, obtained: it is thanks to the creation of the irrational numbers in virtue of the axiom or postulate of continuity, stated as it was by Mr. Dedekind. (Couturat (1900), p. 167)

A field is a mathematical collection of entities together with two arithmetical operations corresponding to the usual addition and multiplication, which satisfy the usual requirements of associativity, commutativity, the existence of inverses and neutral elements. An ordered field is a field that is supplemented with an order relation, which in the case of the reals is further assumed to be linear. Thus, we can think of these "new" purely mathematical real numbers as indeed composing an infinite line which is "full" of points, each corresponding to one of the numbers. This notion of the "fullness" of the line of real numbers was, however, the notion that eluded rigorous mathematical characterization for many centuries.

The first obvious property that could be related to this notion of the "fullness" of the real number line is what became known as *density*. Given two real numbers x, y, there is always another real number z, such that $x < z < y$. However, it is easy to realize that this property is also satisfied by the rational numbers. This fact surely is a trivial matter, but although the existence of irrational numbers was known at least since the time of the ancient Greeks,[19] we had to wait until the 19th century for mathematicians clearly to realize that one needed a stronger property to characterize the real numbers in opposition to the rational numbers. Density just won't do since, although the real numbers are definitely dense, so are the rational numbers. And, in the 19th century, mathematicians began to ask the important question: what is it about the former that distinguishes them from the latter? In Dedekind's words, one must ask the question:

> In what then does this continuity consist? Everything must depend on the answer to this question, and only through it shall we obtain a scientific basis for the investigations of all continuous domains. By vague remarks upon the unbroken connection in the smallest parts obviously nothing is gained; the problem is to indicate a precise characteristic of continuity that can serve as the basis for valid deductions. (Ewald (1999), p. 771)

In this respect, it is interesting to mention Hobson's 1907 work, which was the first English account of these new mathematical ideas that had been around for the last 30 years of the 19th century. In it, he says that

> [b]efore the development of analysis was made to rest upon a purely arithmetical basis, it was usually considered that the field of operations was the continuum given by our intuition of extensive magnitude especially of spatial or temporal magnitude, and of the motion of bodies through space.
>
> The intuitive idea of continuous motion implies that, in order that a body may pass from one position A to another position B, it must pass through every intermediate position in its path. An attempt to answer the question, what is

[19] Sometimes this discovery is credited to Hippassus of Metapontum along with a story of how this discovery led him to drown at sea.

meant by every intermediate position, reveals the essential difficulties of this question, and gives rise to a demand for an exact theoretical treatment of continuous magnitude.

The implication in the idea of continuous magnitude shews that, between A and B, other positions A', B' exist, which the body must occupy at definite times; that between A', B' other such positions exist, and so on. The intuitive notion of the continuum and that of continuous motion, negate the idea that such a process of subdivision can be conceived of as having a definite termination. The view is prevalent that the intuitional notions of continuity and of continuous motion are fundamental and *sui generis*; and that they are incapable of being exhaustively described by a scheme of specification of positions. Nevertheless, the aspect of the continuum as a field of possible positions is the one which is accessible to Arithmetic Analysis, and with which alone Mathematical Analysis is concerned. That property of the intuitional continuum, which may be described as unlimited divisibility, is the only one that is immediately available for use in Mathematical thought; and this property is not sufficient for the purposes in view, until it has been supplemented by a system of axioms and definitions which shall suffice to provide a complete and exact description of the possible positions of points and other geometrical objects which can be determined in space. Such a scheme constitutes an abstract theory of spatial magnitude. ((1907), p. 52)

This difference between the real and rational number systems eventually became known as the *completeness* of the real numbers and there are many equivalent properties that do the trick of pinning down this difference; but, perhaps, the most famous property to be considered is the existence of a lowest upper bound to every set of real numbers that is bounded above.[20] Other equivalent properties are, for instance, the claim that every Cauchy sequence of real numbers converges to a real number, which is the one that appears in the work of Cantor and that has a history dating back to Bolzano in the early 19th century; or the perhaps more modern property that the intersection of every collection of nested intervals of real numbers is non-empty. First, though, in order

[20]A collection of real numbers said to be bounded above if there is a real number greater than or equal to any member of the collection.

to understand these properties, one must define what one is to mean by the mathematical notions involved in the statement of these properties.

3.1. An abstract account of the main notions involved in the mathematical discussions regarding continua in the 19th century. The purpose of this section — which is to be thought of as a fairly technical intermission — is to present a fully abstract and contemporary account of some mathematical notions that will be important for understanding the historical developments of the mathematical ideas in the 19th century. We shall begin with the notion of a *sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$. The current definition is that such a sequence is to be any function from the natural numbers to real numbers. Then, a *fundamental* or a *Cauchy sequence*[21] is a sequence $(x_n)_{n\in\mathbb{N}}$ such that, for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, if $m, n > n_0$, then

$$|x_n - x_m| < \epsilon,$$

where $|x_n - x_m|$ denotes the absolute value of the difference of these real numbers. Now, certain sequences might have another related property, *viz.* that they *converge* to a certain real number x. This notion of convergence — which is intimately related to the notion of a limit — was rigorously defined for the first time only well into the 19th century. In current mathematics, it is defined by means of the ϵ-n_0 style of definition that emerged from these efforts, to mean that, given and $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that, if $n > n_0$, then

$$|x_n - x| < \epsilon.$$

An example of a Cauchy sequence is the well-known sequence whose general term can be written as

$$x_n = \sum_{i=0}^{n} \frac{1}{2^i}.$$

This sequence is famous because it captures the purely mathematical structure behind many of Zeno's paradoxes of motion. We can show that this sequence converges to the number 2, which is essentially the modern resolution to these paradoxes. On the other hand, we can consider the

[21]A more historically precise name for this concept would be "Bolzano sequence" or at least "Bolzano-Cauchy sequence", since its first appearance in the literature is in Bolzano's 1817 *Rein Beweis*.

sequence whose general term is given by

$$x_n = \sum_{i=0}^{n} \frac{a_i}{10^i},$$

where each a_i is the i-th digit in π's decimal expansion. This sequence is also Cauchy and, as in the case of the former sequence, this latter sequence is indeed composed only out of rational terms. However, whereas the first sequence converges to a rational number, *viz.* 2, the latter sequence does *not* converge to a rational number, but to π, which is known to be irrational since the ancient times. Now, it was a striking result of the developments in 19th century analysis to understand that supplementing the limits of all Cauchy sequences of rational numbers that do not converge to rational numbers is enough to obtain the kind of completeness that is required to distinguish the real numbers from the merely rational numbers; and, thus, we can understand the "completeness" of the real numbers, in opposition to the "incompleteness" of the merely rational numbers, as the claim that every Cauchy sequence of real numbers converges to a real number. More precisely, nowadays we define two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ to be equivalent if, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for every $n > n_0$,

$$|a_n - b_n| < \epsilon.$$

Then, since this is an equivalence relation, we can define the real numbers to be the very equivalence classes into which this relation partitions the set of Cauchy sequences.

Another important notion for the study of real numbers is that of a *least upper bound* of a given bounded above collection X of real numbers. This is just the real number u such that $x \leq u$, for every $x \in X$ and $u \leq v$, for every real number v such that the first property holds. For instance, consider the collections X and Y consisting of the terms of the first and the second sequence introduced in the last paragraph, respectively. We know that 2 is an upper bound for X; that is, 2 satisfies the first property of the definition for a least upper bound. It so happens that, for any other upper bound, it will necessarily be greater than 2, so that 2 is indeed the least upper bound for X. Moreover, we know that 4 is an upper bound for Y and that 3 is not, so that the least upper bound of this collection, if it is to exist, must lie between 3 and 4. Indeed, the least upper bound of Y will *not* be a rational number, even though every element Y is, for it is, as one might easily suspect, π.

More generally, given a bounded above collection X of real numbers, one can consider a decreasing sequence $(u_n)_{n\in\mathbb{N}}$ of upper bounds. This sequence is going to be Cauchy and, therefore, if one assumes that every Cauchy sequence converges, one can consider u to be the limit of such a sequence. Well, this u is going to turn out to be the least upper bound for X. From this proof-sketch, one can readily see that the claim that every Cauchy sequence of real numbers converges implies the claim that every collection of real numbers that is bounded above has a least upper bound.

Conversely, consider a given Cauchy sequence $(x_n)_{n\in\mathbb{N}}$. The collection of all the terms of such a sequence is bounded — both above and below — so that, if we assume that such sets have always a least upper bound, it will also have a greater lower bound — which is the dual notion to the least upper bound concept. Indeed, one has only to consider the collection of numbers of the form $-x_n$, for x_n in the original sequence, so that the least upper bound for this collection will be the greatest lower bound for the original collection. Thus, we call, for each $n \in \mathbb{N}$, l_n the greatest lower bound for the set of terms of the original sequence starting with the n-th term. Note that we will have

$$l_0 \leq l_1 \leq \cdots$$

and also all these lower bounds are bounded by any upper bound of the original sequence, so that the set of these lower bounds has a least upper bound u. The claim is that this number will be the limit of the original sequence, so that the existence of least upper bounds for bounded collections of real numbers — and of greater lower bounds, which follows from it — also implies the convergence of Cauchy sequences of real numbers, so that we can conclude that the properties are indeed equivalent.

A final property that is equivalent to both previously seen properties that characterize the completeness of the real numbers is the assumption that the intersection of any collection of nested closed intervals is non-empty. This property is perhaps the most intuitive of them all and maybe for this reason is is the one that is most common in current mathematical presentations of this topic. The idea behind it is that the completeness of the real numbers can be characterized by the assumption that even in the case where we consider a collection of nested closed intervals whose length goes to zero, still in this case their intersection will not be empty — actually, in this case it will be a singleton. This property implies, say, the existence of a least upper bound for a non-empty bounded above set X of real numbers because, assuming the upper bound u guaranteed by the assumption is *not* the least upper bound we are looking for, we can consider again a decreasing sequence of upper bounds $(u_n)_{n\in\mathbb{N}}$ for this set and an increasing sequence $(v_n)_{n\in\mathbb{N}}$ of lower bounds to the set of upper bounds of this set. Then, we shall have a sequence of nested intervals $[v_n, u_n]$ and the real number assumed to exist by the nested interval property — which will be unique, since the size of these intervals goes to zero — is going to turn out to be the least upper bound for X. Conversely, given a sequence of nested intervals of the form $[a_n, b_n]$, we can conclude from their "nestedness" that

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1 \leq b_0.$$

Then, each b_i is an upper bound for the set of the a_i's, so that this set has by assumption a least upper bound u. We would like to see that this u is in every interval, so that it is indeed in their intersection. For that, it is enough to note that, for any $n \in \mathbb{N}$, since u is an upper bound for the set of a_i's, $a_n \leq u$ and that, since u is the *least* such upper bound for the set of a_i's and that b_n is another upper bound for this set, we have $u \leq b_n$, so that indeed, $u \in [a_n, b_n]$ for any $n \in \mathbb{N}$. Hence, we see that this nested interval property is equivalent to both the aforementioned properties.

Now, this notion of a "complete linearly ordered field", which is emblematically exemplified by the real numbers, began to be called in the 19th century "the continuum" — a name that not only stuck, but did so because, as was mentioned before, it seemed to provide both a mathematically rigorous and intuitively satisfactory description of a continuous line of points that could be identified with the formal structure behind not only the obvious 1-dimensional case, but also, through the work of mathematicians like Grassmann and Riemann, as the foundation behind the general case of an n-dimensional continuous manifold or, as it is more commonly know in current mathematics, an n-dimensional vector space, which is the starting point whence one constructs geometrical spaces, by introducing affine properties with the definition of a particular inner product on the elements of the space and, eventually, by introducing a metric on this space, which will allow one to talk about lengths, areas, volumes etc.[22] Indeed, already in the introduction to his *Ausdehnungslehre*, Grassmann is clear to state that

> [a]s an example of an extensive magnitude, the best we can choose is the limited line (segment), whose elements are essentially separated from one another and precisely thereby constitute the line as an extension. (Grassmann (1878), p. XXVII)

[22]Cf. e.g. Grassmann (1878), p. 21, and Riemann (1868).

And Bell (2006) is very clear to claim that

> Cantor seems to have become convinced by this time that the essential nature of a continuum is fully reflected in the properties of sets of points - a conviction which was later to give birth to abstract set theory. In particular a continuum's key properties, Cantor believed, resided in the range of powers of its subsets of points. Since the power of a continuum of any number of dimensions is the same as that of a linear continuum, the essential properties of arbitrary continua were thereby reduced to those of a line. (Bell (2006), p. 159)

However, besides this fundamental axiomatic characterization of the reals, one can introduce other concepts with the help of which one can describe properties of this arithmetized continuum that were out of reach for the plain axiomatic characterization, e.g. the notion of continuity of functions or that of a closed or open interval, boundaries, accumulation points etc. What we have in mind here are the standard *topological* concepts. Nowadays, mathematicians say that a *topology* has been introduced over some given set X when one — perhaps arbitrarily — chooses a certain collection $\mathfrak{T}$ of subsets of X to be called "the open subsets" of X. In fact, this choice is not wholly arbitrary, for the collection chosen must satisfy the following axioms:

1. $\emptyset \in \mathfrak{T}$ and $X \in \mathfrak{T}$;
2. for every sequence of sets $O_i \in \mathfrak{T}$, $\bigcup_{i \in I} O_i \in \mathfrak{T}$;
3. for every pair $O_1, O_2 \in \mathfrak{T}$, $O_1 \cap O_2 \in \mathfrak{T}$.

But, as long as they are satisfied, the choice has no further constraints. Then, one defines the notion of a *closed set* as a set whose complement is open. And one can really feel the degree of abstraction of the current state of mathematics when one remembers that mathematicians even use the word "clopen" to refer to sets which are both open and closed under some given topology.

With the help of this axiomatic definition of a topology, we can define other topological notions. For now, we shall define three other notions that will appear further on when we look at the birth of topology. First, we need to define the notion of an *interior point* of a set $A \subseteq X$. It will simply be a point $x \in A$ such that, there is $O \in \mathfrak{T}$ such that $x \in O \subseteq A$. The second notion, we would like to define is that of a *neighboring point* to A. A point $x \in X$ will be a neighboring point of A if, for all $O \in \mathfrak{T}$, $x \in O$ implies $O \cap A \neq \emptyset$. Note that it is enough for each $O \cap A$ to have only a single point, which might even be x itself. If it is the case that each $O \cap A$ has points which are *distinct from* x, we say that x is an *accumulation point*, which is the third notion we shall consider.

Like much of the mathematics whose history we are concerned in this chapter, this notion of an "open set" has gained in the course of the 20th century a high degree of abstraction; so much so that modern accounts of topology are filled with weird, arbitrary and highly counterintuitive examples and counterexamples, of which we might even mention a few. For instance, given an arbitrary set X, the simplest — usually called "trivial" — topology one can define over X is the topology containing only the empty set and X itself as open sets. This is not a very interesting topology and its role in mathematical presentations is essentially that of providing a "coarsest" topology to every set and thus to serve as a counterexample to some putative general properties. Besides this "coarsest topology", one can define over X what is called by mathematicians "the discrete topology", which is essentially the finest topology one can define over X, *viz.* the one that makes every subset of X open. This notion of "discrete topology" has an intuitive foundation in the fact that it is the only topology in which every singleton subset of X is open, so that X itself can be seen as a — perhaps huge — union of all the open disconnected sets of the form $\{x\}$, for $x \in X$. This is for topologists the highest degree of disconnectedness, whence its connection to the notion of a discrete set.

Thus, it will probably not come as a surprise that over the set $\mathbb{R}$ of real numbers one can define a myriad of different topologies, each having its own different characteristic properties. For instance, we can consider over $\mathbb{R}$ any of the following topologies:

(Trivial topology) $$\mathfrak{T}_C = \big\{\emptyset, \mathbb{R}\big\}$$

(Discrete topology) $$\mathfrak{T}_D = \mathcal{P}(\mathbb{R})$$

(Cofinite topology) $$\mathfrak{T}_{cof} = \big\{\emptyset\big\} \cup \big\{A \subseteq \mathbb{R} \mid \mathbb{R} \setminus A \text{ is finite}\big\}$$

(Sorgenfrey topology) $$\mathfrak{T}_S = \big\{A \subseteq \mathbb{R} \mid \forall x \in A\, \exists\, \epsilon > 0 \text{ such that } [x, x + \epsilon[\, \subseteq A\big\}$$

(Usual topology) $\mathfrak{T} = \{A \subseteq \mathbb{R} \mid \forall x \in A\ \exists \epsilon > 0 \text{ such that }]x-\epsilon, x+\epsilon[\ \subseteq A\}$.

For any one of these, we shall have a completely different ascription of topological properties to the real numbers. For instance, given $x \in \mathbb{R}$, $\{x\}$ is neither open nor closed in $\mathfrak{T}_C$; but it is both open and closed in $\mathfrak{T}_D$; moreover, it is not open, but it is closed in $\mathfrak{T}_{cof}$, $\mathfrak{T}_S$ and in $\mathfrak{T}$. $[0, 1[$ is neither open nor closed in $\mathfrak{T}$; but it is open *and* closed in $\mathfrak{T}_S$. Therefore, whenever we are interested in the topological properties of some mathematical set we must stipulate what is the topology with respect to which we are considering these properties; and, in the case of "the continuum" of real numbers, at least at the turn of 19th to the 20th century and when one is concerned with the intuitive picture one has of it, the topology to consider is arguably the one we termed "the usual topology". Thus, in what follows it is the one we shall have in mind.

This topology over the real number line is essentially the rigorous mathematical characterization of the "monstruous doctrine" first introduced by Bolzano[23] to understand the contact of distinct continuous entities. In an abstract setting, one can define the notion of the *boundary* of some given set $A \subseteq \mathbb{R}$ as the set of all those $x \in \mathbb{R}$ such that, for all $O \in \mathfrak{T}$, $x \in O$ implies

$$O \cap \big(A \setminus \{x\}\big) \neq \emptyset \text{ and } O \cap \big(A^{\complement} \setminus \{x\}\big) \neq \emptyset,$$

i.e., as the set of points which are accumulation points for both A and $A^{\complement}$. Then, by taking $\mathfrak{T}$ to be the usual topology of the real line, one gets out of this definition all the intuitive cases of boundaries. Furthermore, from the one dimensional case, one can easily abstract to higher dimensions by considering over $\mathbb{R}^n$ the product topology given by all the unions of sets of the form

$$O_1 \times O_2 \times \cdots \times O_n,$$

with all $O_i \in \mathfrak{T}$.

However, these definitions have some well known mathematical quirks that render it highly unintuitive, despite all its clarity and rigor. The main discomfort our intuition has with respect to this system of definitions is one that shows up when one tries to define contact in this setting. Let us introduce the notation ∂A for the boundary of A. Then, we can say that two regions $A, B \subseteq \mathbb{R}^n$

[23]Cf. Bolzano (1851), §66. This is, as we shall see in the next chapter, essentially the doctrine that Brentano abhorred, and whose own account was meant to overturn. In fact, the pretty detracting terminology mentioned in the text here to characterize this doctrine is Brentano's.

touch if there is a non-empty $X \subseteq \mathbb{R}^n$ such that

$$X \subseteq \partial A \cap \partial B,$$

i.e., if their boundaries share a non-empty common part. However, the notion of contact is usually related to circumstances in which the two regions that are supposed to be in contact are truly "distinct", or in set-theoretical terminology disjoint. But under these assumptions we cannot claim that A and B "share a boundary" or something to the effect that X — the region of contact — is a part of both A and B. It can be a part of one of them or of the other, but there is no principled reason for choosing either; and if one wants to stick to one's democratic ideals and claim that X is part of *neither* region, then one ends up with a situation in which there is something that is essentially "close" to both regions and that indeed causes these regions to be "in contact", but that cannot, as a matter of principle, be recognized as a part of either region — which is just as mind-boggling as either one of the arbitrary exclusive ascriptions of the boundary to only one of the regions.

Despite the last paragraph, though, we shall not be concerned with these quirks here. They are fairly well known and our goal of this chapter lies elsewhere. But before we start looking at the history of this conception of the continuum, there is one more aspect of this formal characterization of the continuum that is worth going into. When we talked about the notion of the "completeness" of the real numbers, we talked about it being possibly characterized by the statement that any Cauchy sequence of real numbers converges to a real number; and this notion of convergence was defined in the ϵ-n_0 style that is characteristic of modern calculus or real analysis. However, there is also a more abstract take on this notion of the convergence of sequences that hinges only on topological notions and does not make use of the metrical notions required by the ϵ-n_0 style definition.

Indeed, in modern mathematical presentations of general topology, it is common for one to define the convergence of some arbitrary sequence $(x_n)_{n \in \mathbb{N}}$ in some arbitrary topological space X with some given topology $\mathfrak{T}$ as the claim that there is a certain $x \in X$ such that, given $O \in \mathfrak{T}$ with $x \in O$, there is $n_0 \in \mathbb{N}$ such that $x_n \in O$, for every $n \geq n_0$. Moreover, one can define the concept of a *basis* for the topology $\mathfrak{T}$ as a set $\mathcal{B}$ of open sets such that every $O \in \mathfrak{T}$ can be written as a union of elements of $\mathcal{B}$. Then, an equivalent definition of this convergence is the analogous claim for every $B \in \mathcal{B}$ instead of every $O \in \mathfrak{T}$. The reason for going into these details here is to understand how the original ϵ-n_0 definition is merely an instance of this more general definition when one considers a basis for the usual topology on $\mathbb{R}$ to be the one composed of the "open balls" in $\mathbb{R}$. More generally, for any finite dimension n, we can consider the subsets of $\mathbb{R}^n$ given by

$$B(x;\epsilon) = \left\{ y \in \mathbb{R}^n \mid \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2} < \epsilon \right\},$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ an arbitrary point. The value

$$(*) \qquad d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$$

is known as the "usual metric" on $\mathbb{R}^n$ and $B(x;\epsilon)$ is just an instance of the very general concept of "an open ball with radius ϵ in some metric space". Now, going back to the simple example of the real line $\mathbb{R}$, the metric is simply given by

$$(*') \qquad d(x,y) = |x - y|,$$

so that one can easily see how substituting this in the general definition of convergence of a sequence in a metric space will yield the original more intuitive definition in terms of ϵ-n_0.

The goal of this formal discussion is to make clear how much structure is behind the "usual" or "purely mathematical" conception of the continuum of real numbers and how this structure is not something that is inherent to the real numbers themselves, but rather arise from more or less arbitrary assumptions regarding the topology that is supposed to be introduced on top of the mere set of real numbers; and, in particular, from the assumption that this topology is to be the one generated by the open balls given by the metric $(*')$ — or $(*)$ in the case of higher-dimensional continua. There is nothing in the mere collection of all these numbers into a set $\mathbb{R}$ that will automatically introduce into this set the topological properties with which we have been concerned in the latter paragraphs; much on the contrary, they are arbitrary choices and there are many other choices to be made, each yielding different topological properties to "the continuum".

3.2. The history of the formalization of the continuum. Now, we shall have a look at the process that took place in the 19th century and that had as a consequence the establishment of this canonical notion of "the continuum of real numbers" as a complete linearly ordered field. This process starts in the midst of the struggle to introduce rigor into the notions of limit, of a function, its continuity, its derivative, its integral, i.e., the notions that had been in place inside the mathematical community for some 200 years, at least since the full introduction of the calculus by Newton and Leibniz, but for some of them since even before.

The first appearance of the sort of properties that were used to characterize the real numbers in opposition to the merely rational is in the work of Bolzano, more precisely in his 1817 paper in which he gives the first rigorous formal definition of the notion of the *continuity* of a real-valued function of a real argument and a proof of what became known as the "intermediate value theorem for continuous functions". In it, Bolzano proves a certain property which is not, as it is sometimes claimed in the literature,[24] exactly the same as the characterization of the completeness of the real numbers in terms of the least upper bound property, but a very close property easily seen to be equivalent to it: Let $M \subseteq \mathbb{R}$ be non-empty. If there exists $u \in \mathbb{R}$ such that, for all $y < u$, $y \in M$, then there is $U \in \mathbb{R}$ such that U is the greatest real number such that, for all $y < U$, $y \in M$.

Indeed, it is very easy to show that the two properties — Bolzano's and the existence of a lowest upper bound — are equivalent. If the claims in the antecedent of the Bolzano's property hold, then U is just the least upper bound of the set M and, conversely, if we have $P \subseteq \mathbb{R}$ that is non-empty and bounded above, we consider the set

$$M = \{x \in \mathbb{R} \mid \exists y \in P \text{ such that } y \geq x\}.$$

This set satisfies the antecedent in Bolzano's property and therefore we can conclude the existence of U, which will be the least upper bound of P.

However, it must be noted that in his 1817 paper Bolzano did not conceive of his property as a true *characterization* of the real numbers, but merely as a property they happened to satisfy; and, indeed, it seems that, notwithstanding his incredible foresight regarding the role of analytical or purely logical definitions in the context of the notions relating to the calculus, he seems not to have even considered in 1817 the possibility or necessity of providing such a characterization.

In an unpublished manuscript entitled *Functionslehre*, probably written in the 1820's, he does indeed propose an interesting characterization of the reals in terms of what he calls "measuring fractions". However, this characterization is still fairly incipient and did not find its way into a large enough audience to be considered as relevant in the history we are concerned with.

[24]Cf., for instance, Grabiner (1981), p. 74.

3.2.1. *Weierstraß.* Weierstraß, on the other hand, is a central, albeit enigmatic, figure in the history we're trying to trace in this chapter. He is commonly credited with being the driving force behind the process of presenting an analytic foundation to the calculus and his name is intimately tied to the ϵ-δ style of definitions in analysis that are the roots behind the topological approach to the real numbers — and, consequently, to the higher-dimensional spaces that eventually became associated with the Cartesian products $\mathbb{R}^n$ — in terms of the "usual topology" that comes, as we saw above, from the simple metric given by

$$d(x, y) = |x - y|.$$

Nonetheless, his ideas on these issues have not been published in his lifetime — and most of them have not been published at all, while others have been printed as the notes taken by some of his students[25] during his famous courses at the university of Berlin.[26] According to Dugac (1976),

> Weirstrass exposed in a cycle, usually of two years, the totality of his mathematical edifice, whose schema was the following: The theory of analytic functions; The theory of elliptic functions; Applications of the theory of elliptic functions; The theory of Abelian functions. (Dugac (1976), p. 8)

Therefore, if one is to do justice to Weierstraß' real influence in the history we are trying to trace, one must look at the few redactions we have of these lectures in order to identify in them his true contributions to the transition towards the manifold-theoretic conception of the continuum.

The first thing we note in these lectures is, indeed, Weierstraß' desire to present his mathematical ideas on the concepts of the differential and integral calculus involving functions of one or more real variables merely in terms of algebraic methods, foregoing all the usual geometrical intuitions that have been behind these mathematical notions since their early introduction in the 17th century. In this respect, we must note, however, that Weierstraß is not a true pioneer, in that he must be firmly located in a tradition that goes back a few decades at least all the way to the works of Bolzano and

[25]Weierstraß had many notable students in his years as a professor in Berlin, including Schawrtz, Mittag-Leffer, Cantor, Thomae and Husserl, the last of which was also a student of Brentano and might have introduced some of Weierstraß' mathematical ideas to the latter philosopher. The usual sources for the content of these courses are the notes taken by Hurwitz, published as Weierstraß (1988) and some works published by students of Weierstraß that claim to give an account of some of the mathematical ideas presented in these courses. The main ones to consider are Kossak (1872) and Pinterele (1880). Dugac (1973) has published some excerpts of unpublished notes by other students such as Schwartz, Hettner and Thieme.

[26]Cf. Ulrich (1989) and Dugac (1973).

Cauchy. Indeed, for example, although Weierstraß' name became very tightly associated with the ϵ-δ style of definitions in analysis, these definitions are to be found in their complete form already in the aforementioned works of Bolzano and Cauchy. In this sense, therefore, Weierstraß must be considered less of a pioneer discoverer and more of a very apt teacher and publicizer.

Notwithstanding this, there are some true innovations to be found in Weierstraß' lectures that pertain to the topic we are trying to elucidate here. The first we should note is his algebraic account of the real numbers, which was developed at least as early as the Summer semester of 1878, but much more likely before that, since it figures in a very completed form in Adolf Hurwitz' redaction of the course given by Weierstraß in this semester.[27] In this course we can see Weierstraß talking about numbers as composed of parts, in a way that is akin to what today we would call an abstract approach to expansions in an arbitrary base. The most common example of this is the usual base 10 expansion, in which a certain number x can be written as a sum

$$x = \sum_{n \in \mathbb{Z}} h_n \cdot 10^n,$$

in which each h_n is numeral between 0 and 9. Indeed, any real number can be written like this; and the rational numbers have the special property that their expansion in any base is finite, whereas the irrational numbers have infinite expansions.

In a sense, this way of thinking about real numbers is very modern, in that it is exclusively formal and symbolic. As Tweddle says is his recent reconstruction of Weierstraß' method,

> [i]n modern language, one could consider Weierstrass's real numbers to be bounded (possibly infinite) sums of positive rational numbers. (Tweddle (2011), p. 57)

However, although this can be recast in terms of a contruction of the real numbers, as Tweddle proves by actually carrying this construction in a formal and rigorous way, from the original sources it is hard to think of Wierstraß' conception of the real numbers as properly a *construction* of these numbers in terms of the rational numbers; it seems to be much more of an algebraic approach to understanding the essential difference between the rational and irrational numbers — which seems indeed to be in accord with the mathematician's own ideas regarding the scope and methodology behind the newly created discipline of mathematical analysis. Hence, his original approach, by the very reason that it is essentially algebraic and formal, does not touch upon the notion of completeness

[27] Cf. Weierstraß (1988).

that we have been discussing and its relation to the notion of continua and, in particular, to the geometric linear continuum of real numbers upon which the higher-dimensional continuous spaces $\mathbb{R}^n$ are built.

Besides having introduced a strictly algebraic conception of the real numbers, Weierstraß has contributed to the processes we are trying to describe here by using in his courses some of the topological notions we have discussed and, more importantly, by providing for these notions some fairly rigorous definitions. However, we must note that the scope of these notions still was not the full abstract one of their modern counterparts, since these notions were restricted in Weierstraß' investigations to subsets of real numbers or of the higher-dimensional Euclidean spaces $\mathbb{R}^n$. For instance, in the course of 1886 we have the following definition of a "closed set":

> If all the positions[28] for which each closeness [*Nähe*] has an infinite numbers of positions belonging to the defined point-set, then one calls this point-set *closed*. (Weierstraß (1988b), p. 66)

It is, indeed, a perfectly rigorous, albeit a little clumsy, definition that is in accord with the modern content for this notion. However, we must note that, as we shall see below, by this time Cantor had already published his own definitions of this particular notion, as well as the ones for many other notions that play a foundational role in point-set topology.

There are other topological notions that have been used by Weierstraß in these lectures. For instance, in the redaction by Hurwitz of the lectures given in 1878, we find a metric definition of a *neighborhood* in $\mathbb{R}^n$:

> If $x_1, x_2 \ldots x_n$ are the variables and $a_1, a_2 \ldots a_n$ a place in their region — which is to mean that $x_1 = a_1, x_2 = a_2 \ldots x_n = a_n$ is in its system of values — then $x_1', x_2' \ldots x_n'$ is a place in the neighborhood δ of $a_1, a_2 \ldots a_n$, if $|x_1' - a_1| < \delta, |x_2' - a_2| < \delta \ldots |x_n' - a_n| < \delta$. (Weierstraß (1988a), p. 83)

It is interesting to note here that the metric used by Weierstraß in this definition is what we nowadays call the 1-metric on $\mathbb{R}^n$, in opposition to the 2-metric, which is the more common one, and the one we discussed above. But we shall talk more about this below, when we talk about the work of Jordan.

Then, Weierstraß goes on to define what he calls "a continuum":

[28] An element of one of the spaces $\mathbb{R}^n$ is called by Weierstraß "a position" (*ein Stelle*).

> If a is a place in the defined region x', and if all the places in a sufficiently small neighborhood of a are contained in this region, then the x' form a continuum. (*Ibid.*)

This notion is precisely the notion of an open set one gets by considering the usual topology for $\mathbb{R}$ and thinking about the open balls as the neighborhoods of this definition.

After these definitions are presented, he goes on to state and prove both that every bounded subset of the real numbers has a greater lower bound and a least upper bound, and a proposition that has come to be known as the Bolzano-Weiertraß theorem. In modern terminology, this theorem states that every bounded infinite set of real numbers has an accumulation point. Weierstraß' statement of this theorem is the following:

> In each discrete region of a manifold that has an infinite number of places there is at least one place which is distinguished by the fact that in every neighborhood of it, no matter how small, there is an infinity of places from the region. (*Ibid.*, p. 86)

In this statement, Weierstraß does not use the expression "accumulation point", but the property he claims the distinguished point has is a property that is only slightly stronger than the one we used above to define our abstract notion of an accumulation point; and, indeed, it is not only so uncommon to find in modern textbooks definitions of an accumulation point by means of this stronger property instead of the property we used here, but also we shall see that this is the property used by Cantor in his own definition, which is the one that became famous in the end of the 19th century.

3.2.2. *Heine.* The first truly modern construction of the real numbers is to be ascribed — it is true that only by a matter of months — to Heine's paper "Die Elemente der Funktionenlehre" that appeared in Crelle's Journal in 1872. The meaning of it being "truly modern" is the fact that in this characterization we already have, not only what is in essence everything that is to be required of such a construction by today's standards of rigor, but also the algebraic style that has been omnipresent in later characterizations.

Heine's construction is indeed essentially the same as Cantor's, which we shall discuss later. And just like the more famous mathematician's construction, Heine assigns to each Cauchy sequence, which he calls merely "a sequence", of rational numbers a real number and, thus, he gets all the irrational numbers as correspondents of certain Cauchy sequences. We saw above how this style of construction has to make use of equivalence classes, since there are many Cauchy sequences to which the same real number is to be assigned. However, by 1872 this notion of equivalence class was not at all discussed and one can hardly blame Heine, as one can hardly blame Cantor, for failing to provide this final touch of rigor into his construction.

Nonetheless, everything else is there. There are perfectly good definitions of the operations, of the order relation and there is even a hint towards the use of equivalence classes, since Heine defines two (Cauchy) sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ to be "simply and always equal" (*nur und immer gleich*)

> when the sequence of numbers $a_1 - b_1$, $a_2 - b_2$, etc. is an elementary[29] one. (Heine (1872), p. 175)

This is precisely the relation which yields the equivalence classes that are identified with the real numbers, so that in this respect we might even say that Heine's construction is much closer to the modern one than Cantor's.

3.2.3. *Dedekind's construction of the reals.* Perhaps the most famous construction of the real numbers in History is arguably Dedekind's construction in terms of cuts, which was first presented in his *Stetigkeit und irrationale Zahlen* of 1872. He states his intentions right at the start of his booklet in the following way:

> In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. [...] The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous

[29] Heine defines an elementary sequence of numbers to be a sequence in which the terms a_n get smaller than any given positive number as n grows.

> expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. Among these, for example, belongs the above-mentioned theorem, and a more careful investigation convinced me that this theorem, or any one equivalent to it, can be regarded in some way as a sufficient basis for infinitesimal analysis. It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. (Dedekind (1901), p. 1)

However, to do so would require a much more complicated construction than the ones leading to the other number systems. The idea behind it is to start with the rational numbers and then to "fill the gaps", so to speak, between them by introducing the irrational numbers. For that, we need to define the notion of a *Dedekind cut* to be a partition of the rational number line into two sets A and B, such that, for every rational numbers $q \in A$ and $r \in B$, $q < r$. For instance, we can consider

$$A = \{p \in \mathbb{Q} \mid p \leq \frac{1}{2}\}$$

and

$$B = \{p \in \mathbb{Q} \mid p > \frac{1}{2}\},$$

or, more generally, we can consider, for every $q \in \mathbb{Q}$,

$$A_q = \{p \in \mathbb{Q} \mid p \leq q\}$$

and

$$B_q = \{p \in \mathbb{Q} \mid p > q\}.$$

However, we can also consider other kinds of cut, which are not determined by a rational number. For instance, we can consider

$$A = \{p \in \mathbb{Q} \mid p^2 \leq 2\}$$

and

$$B = \{p \in \mathbb{Q} \mid p^2 > 2\}.$$

In this example, we note that there is no $q \in \mathbb{Q}$ such that $p \leq q$ for every $p \in A$ and $p > q$ for every $p \in B$. These are the interesting kinds of cuts because they are the ones that will give rise to the irrational numbers, such as the irrational number defined by our last example: $\sqrt{2}$.

Dedekind's idea was to $define$ the real number line $\mathbb{R}$ as being the collection of all these possible cuts on the rational numbers, so that, for instance, $\sqrt{2}$ would be $identified$ with the last cut we referred to in the previous paragraph, and analogously for every other irrational number. In his own words:

> In the preceding section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle:
>
> "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions." (*Ibid.*, p. 5)

Thus, he can say that

> [i]n this property that not all cuts are produced by rational numbers consists the incompleteness or discontinuity of the domain R of all rational numbers.
>
> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as $different$ or $unequal$ always and only when they correspond to essentially different cuts. (*Ibid.*, p. 7)

With this definition, Dedekind was then able to show how one can define the linear order relation and the operations of addition and multiplication, which satisfy under these definitions the usual axioms for a field.

Finally — and this is the important part of the whole construction —, Dedekind was able to show that the ordered field he defined in terms of his cuts was indeed complete — or, in Dedekind's words, that it “has continuity”. This completeness or continuity property is, as we saw, arguably the characteristic property of the real numbers in their relation with the rationals and it can be characterized by many equivalent formal statements. For Dedekind this was given by the principle mentioned above, according to which every cut on the real numbers is produced by a single real number.

This property is very closely related to Bolzano's property and, since we saw that the latter is equivalent to the least upper bound principle, by showing that Dedekind's property implies Bolzano's property, we will show that it implies the least upper bound property. And, indeed, if M is such that the antecedent of Bolzano's property holds, then the pair $(M, M^{\complement})$ is a cut in the real numbers, so that, by Dedekind's property, there is a real number α that produces this cut, i.e., such that $x \leq \alpha$, for every $x \in M$ and $y > \alpha$, for every $y \in M^{\complement}$. Thus, we can conclude that this α is in fact the U we need to prove Bolzano's property.

On the other hand, if we assume the least upper bound property, given a certain cut (A_1, A_2) in the real numbers, A_1 is bounded by any element of A_2, so that we can conclude that there is a least upper bound u for A_1. This u is, then, going to be the required α of Dedekind's property. Thus, we see that Dedekind's property is, indeed, equivalent to the least upper bound property and that it can very well be seen as a proper characterization of the completeness of the real numbers.

This construction of the real numbers is very well-known, but there is a much less known mathematical contribution of Dedekind that is nonetheless important for the picture we are trying to paint in this chapter, *viz.* the topological investigations that compose the manuscript entitled “General theorems about spaces”, which was first published by Emily Noether in Dedekind's *Werke*, i.e., Dedekind (1930-32). In this manuscript, which is dated by Ferreirós (1999) to the period between 1863-1869, we find many of the foundational definitions that will constitute the basis of the modern theory of point-set topology.

First, we have straight away a definition of an open set, which is called a “body” (*Körper*) by Dedekind, and the related definition of an inner point:

> A system of points p, p' ... forms a body, if for every point p one can determine a length δ such that all points, whose distance from p is less than δ, also belong to the system P. The points p, p' ... lie inside P. (Dedekind (1930-32), v.2, p. 23)

This is indeed, an incredibly early and precise definition of what is arguably, as our previous discussion hopefully made clear, the most basic foundational notion of point-set topology. What is most interesting for us here, however, is how this abstract notion of an open set has been largely forgotten — or, more precisely, has remained largely unknown for many of the next decades —, only being reclaimed as a fundamental notion in point-set topology well into the 20th century.

After defining what one is to call an open set, Dedeking goes on to prove that, what we now call the "open ball", *viz.* the set

$$B(p;r) = \{x \in P \mid d(x,p) < r\},$$

is indeed open. And, then, he defines an *outer point* of some set P as a point p such that there is an $r > 0$ such that the open ball $B(p;r)$ does not contain inner points of P. With the two notions of an inner point and an outer point, Dedekind can, then, define the notion of a *boundary point* of some set P to be a point which is neither an inner point, nor an outer point of P. It is easy to check that this definition is, indeed, equivalent to the one we provided in our abstract presentation above.

3.2.4. *Cantor.* When one thinks about the creation of set theory by Cantor, one readily thinks about the definition of the notion of the "power" or "cardinality" of a given set by means of the equivalence relation that is supposed to hold between any two sets which can be put in a one-to-one correspondence and the subsequent — now almost mathematical commonsense — results that the rational and the algebraic numbers have the same cardinality as the natural numbers, which is strictly smaller than the cardinality of all real numbers and, therefore, than the cardinality of the irrational numbers. However, one tends to give less importance to the other notions involved in his incipient set-theory — in particular to notions that will play a significant role in the development of the mathematical theory of point-set topology. Nowadays, this theory is commonly described as the theory which deals with the mathematical formalizations of notions such as "continuity" and "closeness"; and we saw above how its basic notions are thought of by current mathematicians. Now, however, we must note how this theory essentially began, despite Dedekind's early efforts which remained mostly unknown, with the definition of what is currently known as an "accumulation point" and with the formalization of the abstract notion of a "neighborhood" by means of the axiomatic definition of the notion of an "open set".

Even though the last two notions are still very intuitive in Cantor's work and only relate to sets of real numbers — or, in other words, to subsets of the real line —, the first precise definition of a "derived set" as the set of the accumulation points of a given set is introduced by him already in one of his early papers about trigonometric series, *viz.* Cantor (1872). In it, he says that

> [w]hen a point-set is given inside a finite interval, so a second point-set is with it in general given, and with the latter a third etc., which are essential to the comprehension of the nature of the first point-set.
>
> To define these derived point-sets, we must first talk about the notion of a *limit point* ['*accumulation point*'] of some point-set.
>
> By the 'limit point of a point-set P' I call a point from the line in such a position that in its neighborhood there is an $infinite$ number of points from P, where it might happen that it is itself outside the set. Under 'the neighborhood of a point' let us understand it here to be any interval that has the point *in its interior*. Then, it is easy to prove that a ['bounded'] point-set composed out of an infinity of points has at least *one* limit point.
>
> It is now a determined relation between any point in the line and a given point-set P, whether this point is or is not a limit point of this set, and it is thus *conceptually* given together with the point-set P the set of all its limit points, which I would like to denote with the sign P' and call 'the *first derived set* from P'.
>
> If the point-set P' is not merely constituted of a finite number of points, then it has analogously a derived point-set P'', I call it the *second derived from* P [*sic*]. One finds through ν such a transition to the concept of the ν-th derived point-set $P^{(\nu)}$ from P. ((1932), pp. 97-98)

These definitions are indeed identical in content, if not in form, to the contemporary textbook definitions. They constitute the first rigorous public foray into this new mathematical theory of point-set topology and they are very telling of Cantor's conception of "the continuum of real numbers" as simply a collection of a non-denumerable number of points, from which one could consider subsets P having derived sets etc. But Cantor surely was not content with merely assuming the existence of this non-denumerable number of entities composing this manifold[30] of real numbers; he went on to provide a construction for these entities that is just as rigorous as Dedekind's, equivalent to it, but wholly different in style.

Already in 1872, Cantor knew that the real numbers were intimately related to the limits of Cauchy sequences of real numbers. The current way of constructing the real numbers in terms of Cauchy sequences is to define each $x \in \mathbb{R}$ to be an equivalence class of Cauchy sequences under the equivalence relation given by

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \Longleftrightarrow \text{for every } \epsilon > 0 \text{ there is } n_0 \in \mathbb{N} \text{ such that } |a_n - b_n| \leq \epsilon, \text{ if } n > n_0$$

and, then, to define the operations by means of the term-by-term operations on rational numbers and the linear order relation by making

$$(a_n)_{n\in\mathbb{N}} \leq (b_n)_{n\in\mathbb{N}} \Longleftrightarrow \text{ there is } n_0 \in \mathbb{N} \text{ such that } a_n \leq b_n, \text{ for every } n > n_0.$$

Now, although Cantor was not in possession of the concept of an equivalence class — albeit he would certainly have understood it and perhaps realized its role in the definition of the real numbers, had he been presented to it —,[31] in §1 of his 1872, he presented an account of what he called the different fields of "number-quantities", which amounts, as we shall see to a quasi-rigorous definition of the real numbers. Indeed, he says

> When I talk about a number-quantity in general, then what happens is first that there is a law-given sequence of rational numbers
>
> (1) $$a_1, a_2, \ldots, a_n, \ldots$$
>
> that has the property that the difference $a_{n+m} - a_n$ becomes infinitely small with the change of n, no matter which positive integer m is, or with other words,

[30] *Mannigfaltichkeit* was Cantor's original term for what is today commonly known as a "set".

[31] One must note that, even at such an early date, Cantor had perhaps, just like Heine, an inkling of the need for equivalence classes, since he says, perhaps a little confusingly that

> [a]lthough the domains B and C are here in a sense covered together, it is essential to the theory laid out here that [...] one grasps the conceptual difference between the domains B and C, in that the equality between two number-quantities b, b' in B does not imply their identity, but only expresses a certain relation that holds between the sequences to which they are related. ((1932), p. 95)

> that for a any given (positive, rational) number ϵ, there is an integer n_1 so that $|a_{n+m} - a_n| < \epsilon$, when $n \geq n_1$ and m is a positive integer.
>
> I express this property of the sequence (1) by the words: '*The sequence (1) has a determined limit b.*'
>
> [...] and from the fact that we connect with the sequence (1) a certain sign b, follows that for different such sequences one can also build different signs $b', b'', b''', \ldots$. (*Ibid.*, pp. 92-93)

Then, Cantor goes on to define very precisely the operations on these new "number-quantities" and notes that, just as we build the domain B of the limit-points $b, b', b'', \ldots$, we can for every Cauchy sequence of elements of B consider *its* limit point c and, therefore, we can define a new domain C of number-quantities. Cantor, however, is quick to note that

> [w]hereas the domains A and B are related so that, to every a there is a b, but one cannot conversely identify a certain a for every b, it turns out that, just as to every b there is c, so conversely for every c there is a b. (*Ibid.*, p. 95)

Thus, although surely Cantor goes on to consider indefinite repetition of this construction technique to arbitrary domains L, we have here a clear indication of a statement of the completeness of the domain B which will be identified with the real numbers — even if it is to remain unproved and unrelated to the usual statements of completeness that we talked about before. Indeed, form an intuitive standpoint, this passage seems to be recognizing the fact that, once we introduce the limits of all possible Cauchy sequences of rational numbers, we have so-to-speak "filled all the gaps" left by the rational numbers and, thereby, arrived at a domain that is indeed complete, for any repetition of the construction method laid out will not yield any more number-quantities than the ones in B.

Now, with this fairly rigorous formal definition of the continuum of real numbers, Cantor could then take the final step of identifying all higher-dimensional continuous spaces as collections of n-tuples of real numbers, or as we would call it today, as the n-fold Cartesian product of copies of $\mathbb{R}$. This is very clearly stated in the following passage from 1883b.

> Thus it is left for me nothing more than, with the help of the concept of a real number defined in §9 to search a preferably general pure arithmetical notion of a point-continuum. As a foundation, as it could not be otherwise, I will make use of

> the n-dimensional flat *arithmetic* space G_n, i.e., the collection of all value-systems
>
> $$(x_1 \mid x_2 \mid \ldots \mid x_n),$$
>
> in which each x can take independently of all the others *any real* number-value from $-\infty$ to $+\infty$. Each particular value-system of this sort I call an *arithmetical* point of G_n. The distance between any two points will be defined through the expression
>
> $$\left| \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + \ldots (x_n' - x_n)^2} \right|$$
>
> and under an *arithmetical* set P contained in G_n we can represent any collection of points from the space G_n which is given by a rule. (*Ibid.*, p. 192)

The above characterization of higher-dimensional spaces as n-tuples of real numbers is definitely one of the earliest really modern characterizations of these objects. Indeed, a thorough such characterization will be only available in Germany with Schläfli's work *Theorie der vielfachen Kontinuität* of 1901. However, the idea behind it predates Cantor and has its roots in Riemann's idea of thinking about these spaces as multiply extended magnitudes and in Grassmann algebraic characterization of the vectors which would be later identified with Cantor's n-tuples. In fact, even before the 1880's Cantor already thought about these higher-dimensional spaces in these terms, for he is then already conducting investigations regarding the cardinality of these spaces in relation to his discovery of the different infinite cardinalities; and the interesting thing to note in this respect is that, by 1878, it was clear to him that, for any dimension n, the collection of all n-tuples of real numbers has the same cardinality as the set of real numbers, i.e., that these two sets can be put in a one-to-one correspondence. Indeed, he says that

> in posing myself the question whether a continuous manifold of n dimensions can be correlated injectively and completely with a continuous manifold of a single dimension, so that each element from the first corresponds to one and *only* one element of the second, I found that this question must be answered in the affirmative. (*Ibid.*, pp. 121-122)

This is one of the original groundbreaking cardinality results by Cantor that composed the revolution in mathematical thought that resulted in the establishment of set theory as the foundation of mathematics at the turn of the 19th to the 20th century. However, this result is particularly interesting to our topic here, not only because it is the least known of these results, but also because of a very interesting exchange of ideas between Cantor and Dedekind that followed the 1877 letter in which the former stated this result to the latter.

Before Cantor's work, it was assumed throughout the mathematical community as a fairly trivial fact that there was an essential difference between spaces of different dimensions, in that the full characterization of an element of two such spaces would at face value require a different number of coordinates, that depended exclusively on the dimension of the space — whence, e.g., the discomfort of mathematicians in the 17th century to accept the idea of continua as composed out of lower-dimensional indivisibles. However, by showing that, for any $n \in \mathbb{N}$, there is a one-to-one correspondence between the set of all n-tuples of real numbers and the set of real numbers themselves, Cantor showed that one can characterize any such n-tuple simply by the real number corresponding to it via the given one-to-one correspondence; thus, showing that, independently of n, we need only one coordinate to characterize any n-fold Cartesian product of copies of $\mathbb{R}$, i.e. any continuous space regardless of its dimension. This story is very clearly told by Cantor himself in a letter to Dedekind of 1877, in which he presents a full and irreproachable proof of this result, that takes into account comments and suggestions made by Dedekind in earlier correspondences about this topic. In this letter, he says that

> [f]or several years I have followed with interest the efforts that have been made, building on Gauss, Riemann, Helmholtz, and others, towards the clarification of all questions concerning the ultimate foundations of geometry. It struck me that all the important investigations in this field proceed from an unproven presupposition which does not appear to me self-evident, but rather to need a justification. I mean the presupposition that a ρ-fold extended continuous manifold needs ρ independent real coordinates for the determination of its elements, and that for a given manifold this number of coordinates can neither be increased nor diminished. This presupposition had become my view as well, and I was almost convinced of its correctness. The only difference between my standpoint and all others was that I regarded that presupposition as a theorem which stood in great need of a proof; and I refined my standpoint into a question that I presented to several

colleagues, in particular at the Gauss Jubilee in Göttingen. The question was the following:

'Can a continuous structure of ρ dimensions, where $\rho > 1$, be related one-to-one to a continuous structure of one dimension so that to each point of the former there corresponds one and only one point of the latter?'

Most of those to whom I presented this question were extremely puzzled that I should ask it, for it is quite self-evident that the determination of a point in an extension [*Ausgedehntheit*] of ρ dimensions always needs ρ independent coordinates. But whoever penetrated the sense of the question had to acknowledge that a proof was needed to show why the question should be answered with the 'self-evident' no. As I say, I myself was one of those who held it for the most likely that the question should be answered with a no —— until quite recently I arrived by rather intricate trains of thought at the conviction that the answer to that question is an unqualified yes. Soon thereafter I found the proof which you see before you today.

So one sees what wonderful power lies in the ordinary real and irrational numbers, that one is able to use them to determine uniquely the elements of a ρ-fold extended continuous manifold with a single coordinate. I will only add at once that their power goes yet further, in that, as will not escape you, my proof can be extended without any great increase in difficulty to manifolds with an infinitely great dimension-number, provided that their infinitely-many dimensions have the form of a simple infinite sequence.

Now it seems to me that all philosophical or mathematical deductions that use that erroneous presupposition are inadmissible. Rather the difference that obtains between structures of different dimension-number must be sought in quite other terms than in the number of independent coordinates —— the number that was hitherto held to be characteristic. (Ewald (1999), pp. 859-860)

This result seems to obliterate the background assumption that two spaces with different dimensions are somehow incommensurable or deeply and essentially different — and indeed this was the conclusion drawn by Cantor himself in the last paragraph of the aforementioned letter to Dedekind.

However, in his reply to Cantor's letter, Dedekind, albeit accepting the rigor of Cantor's proof and hence the correctness of his result, raises an interesting point that must be taken into consideration when one is evaluating this radical conclusion of Cantor. He says

> I declare (despite your theorem, or rather in consequence of reflections which it stimulated) my conviction or my faith (I have not yet had time even to make an attempt at a proof) that the dimension-number of a continuous manifold remains its first and most important invariant, and I must defend all previous writers on this subject. To be sure, I gladly concede that the constancy of the dimension-number is thoroughly in need of proof, and so long as this proof has not been furnished one may doubt. But I do not doubt this constancy, although it appears to have been annihilated by your theorem. For all authors have clearly made the tacit, completely natural presupposition that in a new determination of the points of a continuous manifold by new coordinates, these coordinates should also (in general) be *continuous* functions of the old coordinates, so that whatever appears as continuously connected under the first set of coordinates remains continuously connected under the second. Now, for the time being I believe the following theorem: 'If it is possible to establish a reciprocal, one-to-one, and complete correspondence between the points of a continuous manifold A of a dimensions and the points of a continuous manifold B of b dimensions, then this *correspondence itself*, if a and b are *unequal*, is necessarily *utterly discontinuous*.' This theorem would also explain what happened with the first proof of your theorem, namely the incompleteness of the proof; the relation which you then wished to establish (by decimal fractions) between the points of a p-fold structure and the points of a unit interval would have been (if I do not deceive myself) *continuous*, if only it had also contained *all* points of the unit interval; similarly it seems to me that in your present proof the *initial* correspondence between the points of the ρ-interval (whose coordinates are all irrational) and the points of the unit interval (also with irrational coordinates) is, in a certain sense (smallness of the alteration) as continuous as possible; but to fill up the gaps, you are compelled to admit a frightful, dizzying discontinuity in the correspondence, which dissolves everything

> to atoms, so that every continuously connected part of the one domain appears in its image as thoroughly decomposed and discontinuous. (*Ibid.*, pp. 863-864)

Dedekind's main point here is that, although Cantor does indeed present a one-to-one correspondence between a space of an arbitrary number of dimensions and the set of all real numbers, this correspondence is *discontinuous*; and, Dedekind claims, any recasting of the elements of some n-dimensional space into a new coordinate system must be a *continuous* transformation.

Continuity of functions. So far, following the terminology that dates back to Aristotle and the Medievals, we have focused our attention on continuous *spaces* and their study by modern mathematicians. However, with the advent of the calculus, a new use began to be made of the *word* "continuous" in reference to functions. It is true that it took some time for mathematicians to come up with a rigorous account of this notion of the continuity of functions[32] and that the use of the expression "continuous functions" had, until the work of Bolzano, Cauchy and Weierstrass in the 19th century, an intimate connection with the notion of a "continuous space" in that a real valued function of a real argument — which was the paradigmatic example of a function — was thought to be continuous if its graph was a continuous line in the sense of continuity that relates to spaces. A good example of this is, e.g., Lacroix's definition of the continuity of a function — or, in the terminology of the time, of a function's satisfying of the law of continuity —, which we take from the fifth edition of his *Traité élémentaire de calcul différentiel et de calcul intégral* from 1837, i.e., 20 years after Bolzano's paper in which he presents the first definition of continuity of functions with a contemporary flavor, failing to get to the current definition by a little detail.[33] Lacroix's definition goes like this:

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[32]Indeed, they even had to come up with a definition of the general notion of a function itself, so that they could even realize the need for an independent characterization of continuous functions in opposition to *dis*continuous ones, a distinction which was overseen for more than a century, during which mathematicians implicitly assumed that every function was continuous.

[33]Bolzano's original definition goes as follows:

> According to a correct *definition*, the expression *that a function fx varies according to the law of continuity for all values of x inside or outside certain limits* means just that, *if x is some such value, the difference $f(x+\omega) - fx$ can be made smaller than any given quantity provided ω can be taken as small as we please.* (Bolzano (1817), p. 230)

Unfortunately, this definition is not completely precise. In fact, we can come up with an example of a function that is *discontinuous* in our modern terminology, but satisfies Bolzano's 1817 definition, *viz.* the function given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Later on, in his unpublished manuscript *Funktionslehre*, probably written in the 1820's, we see Bolzano realizing this problem and correcting his definition so as to require that the difference $f(x+\omega) - fx$ not only be small, but also remain small for all positive $\omega' < \omega$. The corrected definition is as follows:

> One must understand by the law of continuity that which is observed in the description of lines by motion, and according to which the consecutive points of the same line succeed each other without any interval. (Lacroix (1837), p. 88)

In this quote, it is very clear how well into the 19th century, the notion of continuity for functions was very much connected with the notion of the continuity of a space or geometrical figure.

With the work of the 19th century analysts, though, we arrived at the current definition of a continuous real-valued function f of a real variable x. Now, we say that such a function f is continuous at some point a in its domain if, given some $\epsilon > 0$, there is a $\delta > 0$ such that

$$|a - x| < \delta$$

necessarily implies

$$|f(a) - f(x)| < \epsilon.$$

However, just as in the case of the convergence of sequences, mathematicians later on recognized that they could introduce a very general topological definition of the continuity of functions between any two topological spaces, so that these classical definition would follow as the particular case in which both the domain and the codomain of the function were the real numbers with its usual topology given by the metric

$$d(x, y) = |x - y|.$$

Thus, more generally, we say that a certain function $f : X \to Y$ between two topological spaces X, Y with topologies $\mathfrak{T}_X$ and $\mathfrak{T}_Y$, respectively, is continuous at some $x \in X$ if, for every $U \in \mathfrak{T}_Y$, there is $O \in \mathfrak{T}_X$ such that $f(O) \subseteq U$.

With this formal definition, the continuity of functions was thoroughly incorporated into the newborn discipline of point-set topology as a notion that is independent of the continuity of spaces or geometrical figures. Thus, in this context, we can understand the Cantor-Dedekind discussion

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> If a [...] function Fx of one or more variables is so constituted that the variation it undergoes when one of its variables passes from a determinate value x to the different value $x + \Delta x$ diminishes *ad infinitum* as Δx diminishes *ad infinitum* – if, that is, Fx and $F(x+\Delta x)$ (the latter of these at least from a certain value of the increment Δx and all smaller values) are measurable [i.e. positive non-zero real numbers], and the absolute value of the difference $F(x + \Delta x) - Fx$ becomes and remains less than any given fraction $\frac{1}{N}$ if one takes Δx small enough, and however smaller one may let it become: then I say that *the function Fx changes continuously with respect to the value x* [...]. (Bolzano (1930), p. 14)

And this is indeed equivalent to our current definition.

regarding the invariance of the dimension number of a given space, and in particular Dedekind's unwillingness to accept Cantor's conclusion that this dimension number is wholly unimportant for the a full characterization of a given space, as a discussion regarding the importance of the topological properties of a space in its characterization. And it seems that Dedekind's position is the one that prevailed, so that one now recognizes these topological properties as important for such a characterization and, therefore, one sees the dimension number of a given space as a true topological invariant. In fact, after reading Dedekind's comments, even Cantor stepped back from his radical conclusion, saying that

> [i]n the conclusion of my letter of 25 June I unintentionally gave the appearance of wishing by my proof to oppose altogether the concept of a p-fold extended continuous manifold, whereas all my efforts have rather been intended to clarify it and to put it on the correct footing. [...] I am also of your opinion that if we require that the correspondence be continuous, then only structures with the same number of dimensions can be related to each other one-to-one; and in this way we can find an invariant in the number of independent coordinates, which ought to lead to a definition of the dimension-number of a continuous structure. (Ewald (1999), p. 864)

He even attempted to find a proof of the impossibility of having a continuous one-to-one correspondence between spaces of different dimensions, but failed to do so rigorously.

Indeed, immediately after these issues were made public by the publication of Cantor (1878), a number of mathematicians, including Cantor, began to work on proofs of such impossibility. In particular, we can mention Lüroth's, Jürgens's, Thomae's and Netto's attempts, the first two of which aimed at proving such impossibility for lower dimensional cases and the latter two of which worked, as did Cantor, with general proofs that were later found to contain mistakes.[34] An

[34] For a fuller historical account of this historical development, see Johnson (1979). His overall assessment of the situation — with which I agree fully — seems to be encompassed in the following passage:

> From our present vantage point the attempted proofs of dimensional invariance put forward during 1878 and 1879 do not appear very satisfactory. Having the great benefit of hindsight, we can see that the mathematicians of the period were struggling to subdue a difficult problem with inadequate weapons. Fully developed topological methods and ideas were not available to them. Their best means of attack lay in analysis. Indeed with only these means they handled their problem very skillfully. It must be conceded that the standard of rigour and critical argument among them was high, given the poverty of their methods. (p. 162)

interesting point to be made here, though, is one that relates to the fact noted above regarding the necessity of going beyond the purely set-theoretical notions and into full-blown topological notions in order to attach this problem. Indeed, it is an interesting fact that these proofs relied heavily on the intermediate value theorem, which states that if a continuous real valued function f of a real variable has values $f(a)$ and $f(b)$, for points $a, b \in \mathbb{R}$, then, for any $d \in \big(f(a), f(b)\big)$, there is $c \in (a, b)$ such that $f(c) = d$ — or a slight generalizations of this theorem.[35] This fact is interesting because it seems to show how much these investigations depend on full-blown topological concepts that were in their infancy at the time the investigations were being undertaken. Indeed, the intermediate value theorem for real-valued continuous functions of a real variable is a particular instance of a much more general topological result related to the notion of *connectedness*. This is arguably the hardest of all the "basic" topological notions, in that there are many different — and, most importantly, non-equivalent![36] — ways of characterizing it. On the other hand, it is clear that this notion is very intimately related to the notion of a continuum in that it seems to capture the intuitive property of continua that they are such that there is "no empty room" between two points of them, or that any two such points follow each other immediately in the context of the continuum of which they are elements. However, any reasonable notion of connectedness that aims to capture this intuition seems to satisfy the property that a continuous function always takes connected spaces into connected spaces. This is the case, for instance, for the most common current topological notion of connectedness, according to which a set X is connected if there are no disjoint pair of open subsets O_1, O_2 of X such that X can be thought of as $O_1 \cup O_2$. And this permanence of connectedness under continuous transformations is precisely the general result of which the intermediate value theorem for real-valued continuous functions of a real variable is a particular instance, so that we can see how the result being used in the investigations taking place just after Cantor's publication of the result which initiated the whole discussion regarding the notion of dimension foreshadows the essentially topological character of the discussion, while at the same time failing to posses the topological generality required for furnishing the tools necessary for

[35] The history of this intermediate value theorem is itself very interesting as well. Although its truth is fairly obvious and was indeed known for at least a full century, it was only with the work of Bolzano (1817), that mathematicians began to realize that a rigorous presentation of this topic required a precise mathematical definition of the notion of continuity for functions that allowed for a precise derivation of the theorem in question.

[36] Any decent general topology textbook will go over the different notions of connected, path-connected, locally-connected, simply- or multiply-connected spaces etc.

the task at hand. These, incidentally, will only be in place with the works of Brower, Urysohn and Menger in the 1910's and 1920's.

Nonetheless, starting in 1879, Cantor began to publish a series of papers on his new theory of manifolds, which contained, besides the protagonist notion of power and the counterintuitive results that it yields, many other notions that are more related to point-set topology, than to set theory *per se*. We have already mentioned his notion of a limit point and the related one of a derived set. From these, one can define the notion of a *perfect set* as a set X which is equal to its first — and therefore to every — derived set. Then, he goes on to define his own notion of connectedness — which is now sometimes referred to as the property of being *well-chained* — as the statement that, for every $x, x' \in X$, given $\epsilon > 0$, there is a finite sequence of elements $x = x_0, x_1, \ldots x_n = x' \in X$, such that each distance

$$d(x_i, x_{i+1}) < \epsilon.$$

Now, in possession of these two notions, he can go on and define the very notion of a *continuum* as any set which is perfect and connected in this sense. Indeed, this might be a much too narrow definition, for it excludes, e.g., open intervals of real numbers. Cantor himself noticed this fact and, in a footnote, made this clear by introducing the related notion of a "semi-continuum" as an imperfect, non-denumerable[37] connected set.[38]

Further on in this series of papers,[39] Cantor goes on to define the notion of a *closed set* as one which contains its first derived set and also[40] the "converse" notion of a *dense-in-itself set*, which is a set that is fully contained in its derived set — or equivalently, a set that is composed entirely of boundary or accumulation points.[41] We talked about the *density* of both the rational and real numbers by considering their property according to which for every two distinct rational or real numbers, there is always another rational or real number that lies in between these two. This is, however, an essentially geometric notion in that it requires the previous understanding of this notion of "lying between two points or numbers", which is canonically obtained by means of a previously

[37] In Cantor's terminology, "belonging to the second class".
[38] Cf. Cantor (1932), p. 207.
[39] Cantor (1932), p. 226.
[40] Cantor (1932), p. 228.
[41] This notion is related to another density notion, defined in his 1879 paper, according to which a set P of real numbers is *dense* in some interval $(\alpha \ldots \beta)$ if

> an interval $(\gamma \ldots \delta)$ inside $(\alpha \ldots \beta)$, no matter how small, always has points of P. (Cantor (1932), p. 153)

established order on these numbers. This notion of a set which is "dense-in-itself" is, however, a purely topological notion. By a topological notion we mean a notion that does not depend on the particular geometric — or metric — aspects of a certain domain, but merely on those aspects of this domain that can be characterized in terms of notions such as its derived sets etc.

Therefore, with these new definitions, we can redefine the notion of a "perfect set" as a closed dense set, whence a continuum would be a closed, dense, connected set.[42] These definitions are interesting, first, because they show how Cantor had already realized the necessity of handling the notion of a continuum by means of topological concepts, having indeed carried out the project to a very large extent, insofar as he already had clear and precise notions regarding many of these concepts. Notwithstanding this, we have here an example of how the historical course of a certain mathematical theory runs against its logical course. As we have seen, nowadays all the topological concepts are derived from the fundamental concept of an open set; in prticular, closed sets are defined to be the complements of the open sets. However, whereas Cantor was in possession by 1884 of a precise concept of a closed set which is as a matter of fact equivalent to the modern concept, since every complement of an open set does indeed contain all its accumulation points, he did *not* have a clear abstract concept of a open set, failing even to recognize the need for such an abstract concept and using the word "open" merely for the characterization of intervals of real numbers which lack their endpoints.

3.2.5. *The French reception of the manifold-theoretic conception of continua.* In the decades that followed the establishment in the German speaking mathematical community of the, now canonical, point-set topological conception of the continuum of real numbers and of the higher-dimensional continuous spaces that one can build from the former, the ideas behind this notion found fertile ground in the work of a new generation of French mathematicians, that eventually became known as "the French analysts". The arrival of these ideas in France can be traced back to Paul Tannery's *Introduction a la théorie des fonctions d'une variable* of 1886. In it, he presents a construction of the real numbers in terms of cuts, which resembles Dedekind's in form and in rigor. However, as Tannery makes clear in the preface to this work, this construction was the development of

[42]We note, incidentally, that the rational numbers, although they form a dense and connected set (in Cantor's terminology), they are not closed in the real numbers, since all the irrational numbers are accumulation points of rational numbers; thus, the rational points do *not* form a continuum in Cantor's terminology.

> an indication given by Mr. Joseph Bertrand in his excellent *Treatise of arithmetic* and that consists of constructing an irrational number by saying which are all the rational numbers that are smaller and which are the all the rational numbers which are grater than it; (Tannery (1886), p. IX)

and that he has

> learned from a citation of Mr. Cantor (*Grundlagen einer Allgemeiner Mannichfaltigkeitslehre*, p. 21) that Mr. Dedekind has developed the same idea in a writing entitled *Stetigkeit und irrationale Zahlen*; I could not have at my disposal the work of Mr. Dedekind. (*Ibid.*)

With respect to the definition of the real numbers by means of Cauchy sequences, which Tannery ascribes first to Heine, and then to Lipschitz, du Boys-Reymond and Cantor, he says that

> I find this definition more arbitrary than the one I have adopted, which permits, if an irrational number is defined, to determine its place in the scale of numbers. (*Ibid.* p. X)

This shows, I believe, an early recognition of the naturalness with which the notion of order follows from the definition of the real numbers in terms of Dedekind cuts. However, Tannery seems not to ascribe importance to the fact that, conversely, the definition of the operations on the real numbers is much more straightforward in the definition by means of Cauchy sequences.

Even the famous Poincaré, whose work gave rise to a yet different approach to topology, *viz.* the mathematical theory that is now known as algebraic topology, wrote about these point-set constructions of the real numbers. And, in the context of this study, a discussion of Poincaré's ideas on this topic is especially interesting in that they seem to be essentially where Brentano learned of these new mathematical ideas. In particular, this fact might even partially explain the lack of sophistication in Brentano's understanding. Indeed, right at the beginning of his paper on these issues, Poincaré talks about the construction of the real number in terms of the basic "filling the gaps" intuition that surely looms behind the actual rigorous mathematical constructions both in terms of Dedekind cuts and of equivalence classes of Cauchy sequences. He says:

> Let us start from the scale of integers; between two consecutive levels, let us interpose one or more intermediate levels, and then between these new levels still other levels, and so on indefinitely. We shall thus arrive at an unlimited number

> of terms, which will be the numbers we call fractional, rational or commensurable. But this is not enough; between these terms that are nonetheless already infinite, one must still interpose others, which one calls irrational or incommensurable.
>
> [...] The continuum thus conceived is nothing other than a collection of individuals arranged in a certain order, infinite in number, it is true, but exterior from each other. It is not the ordinary conception, in which one assumes between the elements of the continuum a sort of intimate connection that makes it into a whole, where the point does not have priority over the line, but the line over the point. [...] The analysts do not have less reason to define their continuum in the way the do it, since it is with respect to it that they reason after they have injected themselves with rigor. But it is enough to forewarn us that the true mathematical continuum is not the continuum of the physicists and the one of the metaphysicians. (Poincaré (1893), pp. 26-27)

Then, Poincaré goes on to talk about the introduction of the irrational numbers by means of the notion of a Dedekind cut. However, he seems to claim that this introduction is due to Kronecker, which is absolutely false. The latter mathematician is surely to be credited with the advancement of the view that the only numbers that are truly real are the natural numbers, all the other numbers being merely symbolic mechanisms for carrying out calculations — a view that does indeed harmonize well with a somewhat arbitrary construction of the more complicated number systems in terms of the natural numbers. And, indeed, in his discussion of the cuts Poincaré seems to be hinting at this state of affairs, so that his mention of Kronecker is not wholly of the mark; however, even if there is some salvaging to Poincaré's claim, crediting Kronecker with the Dedekind cuts is just plainly historically inaccurate.

The same year as Poincaré's paper saw also the publication of the second edition of Jordan's *Cours d'analyse de l'école polytechnique*. In this new edition, that appeared eleven years after the first edition, we see major changes in the book. Whereas the first edition was a simple textbook presenting the various results in the theory of the differential and integral calculus, the second edition contained a fairly lengthy addition of chapters dedicated to more fundamental issues required for the theory to be presented in the latter chapters. First, there is a fairly simple but thorough discussion about real, and in particular irrational, numbers in terms of (Dedekind) cuts as well as the further notions to be defined on top of this set of real numbers; then, after some talk about the notion of limits and a quite anachronistic discussion of infinitesimals, Jordan goes on to discuss the new notion of sets. Essentially, Jordan is thinking about sets as subsets of some Euclidean space $\mathbb{R}^n$, but under this restriction, he provides a fairly general presentation of the notions of "derived sets" and "perfect sets", which are defined in terms of the notion of a "limit point", which in its turn is defined by Jordan not in Cantor's way, but as a point which is the limit of a sequence of points in the given set.[43]

What is interesting to note in this work, however, is that, although the general notion of a topological space is still not fully in play, we do have a definition of what is called "the gap between two points p, q". Assuming, as Jordan does, that these points are taken from $\mathbb{R}^n$, he defines the gap between them to be

$$pq = |p_1 - q_1| + \cdots + |p_n - q_n|.$$

Now, this definition amounts essentially to the definition of a *metric* on $\mathbb{R}^n$, a metric that is however *dintinct* from the "usual metric" we have been considering.[44] Furthermore, just as in the current studies of metrizable topologies, we have a clear and precise definition of an *interior point* as a point p for which there is a radius ρ, such that every point whose gap to p is smaller than ρ is in the set to which p belongs.

Finally, we have in this work by Jordan essentially the proper current mathematical definition of the *boundary* of a set E as the set of points which belong both to E or to its complement and also to the derived set of either the complement of E or of E itself.

On the other hand, a much more philosophically oriented account of the mathematical ideas and constructions related to this new conception of the continuum can be found in the work of Couturat, who undoubtedly helped to forge the context in which the so-called French analysts could flourish and develop their mathematical ideas.

[43]Even today these two not quite equivalent definitions still confuse students, since both are still in use. In general, if we call a point that satisfies Cantor's definition an "accumulation point" and a point satisfying Jordan's definition a "limit point", we can say that, in general, every accumulation point is a limit point, but not conversely. E.g., consider the set $\{1\} \subseteq \mathbb{R}$. 1 is a limit point of this set, since it is the limit of the constant sequence $(1)_{n \in \mathbb{N}}$, but it is not an accumulation point.

[44]Jordan's definition is what today is called the 1-metric on $\mathbb{R}^n$, whereas the "usual metric" is called the 2-metric on $\mathbb{R}^n$. In general, for every positive integer p, we can consider the p-metric on $\mathbb{R}^n$ as the metric given by

$$d_p(x, y) = \sqrt[p]{|x_1 - y_1|^p + \cdots + |x_n - y_n|^p}.$$

Indeed, in the first book of his 1896 *L'infini mathématique*, Couturat presents a thorough and rigorous construction of the positive rational numbers, together with the usual mathematical properties and operations on them, starting from just the natural numbers and, then, in the second chapter, he extends this definition to negative rational numbers as well.

In the third chapter, he presents the usual definition of complex numbers in terms of pairs of reals and, after that, he makes an important remark. Indeed, he notes that in order to "complete" the system of numbers with irrational numbers, the method for doing so is more complicated in the sense that it requires for the definition of each irrational number, not merely — as was the case with the other construction methods before — two numbers of the system we started from, but an infinity of them. However, he says that

> [the] introduction of these numbers completes therefore the linear sequence of arithmetical numbers and makes it continuous [...]. (Couturat (1896), p. 52)

He notes that this completion can be obtained by two distinct methods: one which he ascribes to Cantor and Weirstrass and another that he ascribes to Dedekind and Tannery. The method he goes on to follow is then Tannery's.

Chapters II and III talk about how one can understand a certain line by means of the rational and real numbers. In the following quote from this part of his work, we can clearly see how the everlasting mathematical pendulum between intuition and logical rigor is leaning towards the latter side in both the work of Couturat and, by consequence of those in the mathematical context he is helping establish through his work.

> Even though we shall not admit that the idea of a continuum has an arithmetic origin, we must nonetheless keep this remarkable fact, to wit that one can construct analytically a numerical continuum that is *independent of all intuition.* An important consequence stands out: that continuity does not belong properly and exclusively to the geometrical magnitudes, and it can be *logically* conceived in the category of pure number. [...] continuity seems to be an essential characteristic of any magnitude in general, and not merely of spatial magnitudes or even of magnitudes that can be reduced to the linear type, such as time. (*Ibid.*, p. 172)

In particular, the claim that continuity is a strictly logical property relating essentially to the pure idea of a magnitude in general stands out as a clear representative of the time and context in which these ideas are being put forth. And the basis for this shift is undoubtedly provided by the rigorous mathematical constructions of the real numbers, together with Cantor's discovery of the uncountability of these numbers, which provides a clear cut reason why one should have the completeness of the real numbers whereas the rational numbers are merely dense.

Then, we have Couturat pressing on with these ideas into a proper definition of the n-dimensional Euclidean spaces, in the manner of Cantor and Riemann, a manner which is commonsensical now and which was by the time of Couturat already fairly canonical as well; moreover, we have him laying down some fairly advanced conceptions of what it is for a set to be a neighborhood of a point.

> We call n-dimensional (arithmetical) *space* the set of points $(x_1, x_2, \ldots, x_n)$ whose coordinates take any real value. [...] An n-dimensional set [which is just a subset of an n-dimensional space] is called *continuous* when each coordinate of the generating point takes, when considered independent of the others, every value of a finite interval [...]. (*Ibid.*, p. 626)
>
> We call a *neighborhood* of a point any sphere of arbitrary radius having this point as center, that is the set of points in n-dimensional space whose distance to the considered point is less than a given quantity. (*Ibid.*, p. 630)

We have seen how these truly topological notions have been already somehow — i.e., with different degrees of preciseness and rigor — present in Cantor's work and Weierstrass' work. However, it is mainly with the work of the French analysts that they have achieved the logical watertight state they now posses.

The first work to mention in this context is Borel's *Leçons sur la théorie des fonctions* of 1898. It has as its goal to present some results on the theory of analytic functions and on the theory of series, which is indeed indispensable to the study of the former theory, since the very definition of analytic functions has a deep connection to their Taylor series expansion. However, in order to present these results, Borel has to begin by introducing some set-theoretic notions that were at the time still fairly intuitive in their presentations.

He starts by assuming the existence of the mathematical continuum of real numbers:

> Hence, we know a non-denumerable set; one can note that the existence of this set results from the following theorem: *The growing numbers, which are all inferior*

> *to some fixed number, have a limit.* Indeed, such theorem is a consequence of the *definition* of the incommensurable numbers and, in any point of view one places oneself, it postulates the notion of a *continuum*. [...] [W]e will admit that the set C of numbers between 0 and 1 is *given* [...]. We shall say that the sets with the same power as C have the *power of the continuum*. (Borel (1898), pp. 15-16)

Then, Borel goes on to consider some examples of sets of this kind, *viz.* which have the power of the continuum. The first example he gives is that of a set obtained from the interval $[0, 1]$ by deleting a denumerable number of points, e.g., the rational numbers. A second example is that of the union of a denumerable collection of sets all of which have the power of the continuum and, finally, he gives the example, as we have seen above already known to Cantor, that the set of points in a square has the same power as the interval $[0, 1]$. Moreover, he goes on to summarize the Cantor-Dedekind discussion by stressing the fact that these size characterizations fail to assume the continuity of the one-to-one correspondence between the sets.

> The preceding remarks are very important since they show us that if one *abstracts from the continuity of the correspondence between two continuous sets*, there is no essential difference between a continuous set of one dimension and continuous sets of two (or three...) dimensions [...]. (*Ibid.*, p. 20)

The next definition presented by Borel is that of a "derived set", which again is not a new notion, but one that has already been defined by Cantor.

> We call the derived set of some given set the set of all those points such that in the neighborhood of any of them we find an infinite number of points from the given set. (*Ibid.*, 3p. 34)

We note here how Borel disregards Couturat's definition of a "neighborhood" and uses this notion as Cantor did, without properly defining it. Analogously, throughout his book, Borel also uses the word "interior" without definition, taking for granted that his uses relating to intervals or geometric figures is clear enough. However, after this definition of a derived set, he goes on to define the notion of a "limit point":

> We say that a point a' is a *limit point* of the given set if, no matter how small ϵ is, there is a point a [which is assumed to be in A] distinct from a' and whose

> distance from a' is less than ϵ. [...] We call a perfect set any set that is equal to its derived set. (*Ibid.*, pp. 34-35)

This last definition is essentially Cantor's definition. Borel himself notes that Jordan gives a slightly different definition, according to which a perfect set is a set that contains its derived set, not being necessarily equal to it. These two definitions disagree only when it comes to sets with isolated points, so that this distinction is indifferent when one restricts one's attention to continuous sets — in other words, to sets with no isolated points. However, in the sequel, Borel proves that the set of these isolated points cannot be larger than a denumerably infinite set.

Finally, Borel defines what it is for a set to be *dense in an interval*:

> We shall say that a set A is *dense in an interval* a, b if any interval inside a, b, no matter how small, has points of A. (*Ibid.*, p. 38)

This is a simple generalization of Cantor's definition of a set that is dense-in-itself if one notes that Cantor's requirement that every neighborhood has an infinite amount of points is equivalent to Borel's requirement of the existence of only one point if one thinks about the neighborhoods in Couturat's manner of intervals of points as "spheres of arbitrary radii". The fact, however, that Borel did indeed require in his definition of a limit point the presence of an infinite amount of points in a given neighborhood shows how these ideas were still not fully ripe in this early period.

Seven years after Borel's book, Baire published his own work on discontinuous functions, in which he proceeded with the same mathematical methods. All the definitions used, that of a "derived set" and of a "closed set" are essentially related with his definition of a "limit point", which is clearer than the previous definitions presented here in that Baire does not make use of the intuitive notion of a neighborhood in this definition, substituting it for something like Coutourat's definition, *viz.* the more rigorous notion of a segment or interval containing a certain point:

> Let us consider a set of points P on a line segment AB. We say that M is a *limit point* of the set P if every segment containing M has in its interior a point from P distinct from M. (Baire (1905), p. 13)

Other notions defined later on in this work are the notions of a "perfect set", as a set that equals its first derived set and that of a "dense-in-itself set", whose definition is exactly like Cantor's original definition. They are essentially introduced to prove the result that there are perfect sets in the real line that are nowhere dense in the interval $(0, 1)$. His example is simply the famous Cantor set, which was first described by the latter in a footnote to his (1883b)[45]; but the layout of the definitions makes the whole presentation slightly more rigorous and closer to the current state of the theory in Baire's work.

Especially interesting is his extension of his discussions to functions of multiple variables, for this extension is achieved firmly in the context of thinking about these variables as having their values in continua that constitute the multi-dimensional space Euclidean space that we currently denote by $\mathbb{R}^n$, and that was denoted then by Cantor's conventional name G_n.

3.2.6. *Hausdorff's* pièce de résistance*.* I believe the best way to end our historical account of the birth of point-set topology is to consider the first textbook covering the early results of both set theory and point-set topology, *viz.* Hausdorff's *Grundzüge der Mengenlehre* from 1914. This choice is especially significant because the publication date for this book coincides with the year that Brentano is dictating his most comprehensive discussion regarding the nature of continua. Thus, although one can be quite sure that Brentano did not have access to Hausdorff's textbook,[46] it nonetheless can be seen as a good comprehensive collection of all the ideas that were being put forth in the prior 40 or so years with respect to the questions pertaining to these new mathematical subjects. In this sense, Hausdorff's work can be seen as a strong landmark and as an accurate portrait of the state of affairs in which these new mathematical disciplines were at the time of Brentano's attempts to engage with the same problems that were being tackled by these disciplines.

A quick look at the table of contents is enough to see that everything is there. From a introductory account containing the set-theoretical operations, the concept of a sequence and of a function and that of a ring and a field, going through Cantor's theory of transfinite cardinals and ordinals and the notion of well-ordered sets, finally ending with an account of the point-set topological notions that are of interest to us, but also of the new ideas regarding measure and integration that sprung from the work of the French analysts, in particular the work of Borel and Lebesgue.

Hausdorff's book is a masterpiece not only in that it covers the full span of the new mathematical ideas relating to the notion of sets, but also because it does so with immense precision and rigor in such a way that, if one were to abstract from the fact that it could not contain many important results in the theory that were only discovered and proved many years later, it would rival any

[45]Cf. Cantor (1931), p. 207, n. 11.

[46]Brentano certainly did not read it, since he was sadly already blind at this time. However, it is also to be surmised that he had no other type of access to it and there seems to be no textual evidence against this assumption.

contemporary textbook in the subject. In particular, we can mention in this regard how the very idea of a topological space is presented and compare it to our general abstract presentation above. Indeed, Hausdorff distinguishes three ways in which one can regard sets. The first is Cantor's more abstract way in which the elements of the set are considered either independently of their relation to each other or simply with an extra order structure. The second is the metric way of Jordan, in which the elements of the set are related through a certain distance relation. However, Hausdorff says, there is a third — one might say intermediate — way of looking at sets, in which one can

> assign as the foundation of the distance of a point subsets of the space, which we call *neighborhoods* of this point; and then we can take this system of neighborhoods as the foundation of the whole theory, thus eliminating the concept of distance. (Hausdorff (1914), p. 210)

This is as clear a statement of the goal of point-set topology as any other in the mathematical literature. And to relate it to our abstract account above, we can mention that the difference is merely accidental, in that Hausdorff uses the notion of a "neighborhood" as primitive, which is not how we chose to introduce the notion of a topological space above, but it is an equivalent way, which is still used in textbooks today as a different approach that has some technical advantages. In this approach, we define what we have called an "open set" as any set that is a union of neighborhoods; and, indeed, after presenting the very same set of axioms that characterize a metric space in contemporary textbooks, Hausdorff goes on to define a *topological space* as a a space satisfying the following axioms:

> (A) To each point x, there corresponds at least one neighborhood U_x; each neighborhood U_x contains the point x.
>
> (B) If U_x, V_x are two neighborhoods of the same point x, then there is a neighborhood W_x which is a subset of both neighborhoods ($W_x \subseteq U_x \cap V_x$).
>
> (C) If the point y is in U_x, then there is a neighborhood U_y that is a subset of U_x ($U_y \subseteq U_x$).
>
> (D) For two distinct points x, y, there are two neighborhoods U_x, U_y with no points in common ($U_x \cap U_y = \emptyset$). (Hausdorff (1914), p. 213, with modernized notation)

These axioms are not the simplest way of defining a basic topological structure on a given set — our definition of open sets is arguably simpler — and also axiom (D) is now not seen as a general requirement for topological spaces, but as the defining property for a subset of "nice" topological spaces, very sensibly called Hausdorff spaces. However, they must nonetheless be regarded as the first truly abstract and fairly complete characterization of topological spaces. And Hausdorff himself is aware of this, as he goes on to prove that, while the usual metrical topology on a certain multiply extended space $\mathbb{R}^n$ is indeed a model for these axioms, one can come up with other models that are *not* obtained from a certain background metric.

After that, given a set A, he goes on to define an *inner point* to A as a point of A such that there is a neighborhood of that point that is contained in A and also a *boundary point* of A as a point in A which is not an inner point. Thus, we have in this work the first fully worked out rigorous definition of these notions in terms of an abstract, *viz.* axiomatically defined, notion of a neighborhood. Also, he goes on to define the usual notions of a limit point, of a closed set, of a set that is dense in itself and of a perfect set, and uses these definitions to prove several results pertaining to these notions. For instance, in page 228 he shows the usual relationship between closed and open sets, i.e., that a closed set is the complement of an open set and vice-versa. Currently, since point-set topology is usually presented as we have in terms of a choice of which sets are to be open, this relation is actually used to $define$ the notion of a closed set, but Hausdorff defines both of these notions independently in terms of his neighborhoods and, thus, he has to prove that this relationship actually holds. Nonetheless, it does hold, so that we are firmly inside the territory of Bolzano's "monstrous doctrine".

Another example is the proof of a generalization of the property of nested intervals, that we saw can be used to characterize the completeness of the real numbers. This generalization, which Hausdorff calls Cantor's intersection proposition, is given in page 230 and says that given a sequence $A_1 \supseteq A_2 \supseteq \cdots$ of closed compact non-empty sets,[47] its intersection is not empty. A final interesting example is the collection of two propositions stated in page 231, which taken together are word for word identical with the current definition of compactness. Indeed, together they state that a

[47]Hausdorff had already at this point defined a compact set as set such that every infinite subset of it has an accumulation point. Closed intervals of real numbers are also compact in this sense, so that this is indeed a generalization.

compact set A is precisely a set such that, if it is contained in the union of a sequence of open sets, then it is contained in the union of a finite number of open sets in this sequence.[48]

A final example to be mentioned is Hausdorff's statement and proof of the Bolzano-Weierstrass theorem which states that every bounded infinite set of real numbers has at least one accumulation point. This is done in chapter VII, §9, and it might very well be the first clear ascription of this theorem to both Bolzano and Weierstrass.

Now, the purpose of this overview is not to present a detailed account of these historical landmarks, so that we must pick and choose the best examples to paint a fairly didactic picture of this very complicated and multifaceted history. However, I must not fail to note that in Hausdorff's textbook, i.e., at the same year which Brentano is dictating his views on continua, we have even the very first discussion regarding the so-called enumerability axioms, which are properties regarding the existence of a denumerable number of things of some kind of another relating to some set, e.g., the existence of a denumerable dense subset. We have not talked about these axioms,[49] but they play a very important role in modern general topology, and their presence in Hausdorff's textbook is surely a clear indication of how far the subject had already been developed by 1914.

4. Conclusion

In this chapter we have gone through the centuries that spanned the period immediately after the end of full-blown Scholasticism, during which the doctrines of Aristotle in general, and his doctrine regarding the essence of continuity in particular, have been dominant unquestioned dogmas. We have done so looking for the driving forces inside the development of mathematics that led to a complete reformulation in the 19th century of the mathematical notion of continuity and that of the continuum of real numbers. We have termed this reformulation in the 19th century "the manifold-theoretic conception of the continuum" and saw how this conception was already extremely dominant and well developed in the mathematical community by the time Brentano was thinking about his own doctrines about continua.

The reason for the historical reconstruction of these notions is to paint a general background picture of the standard view against which Brentano was arguing in his papers about continuity

[48] These propositions are called by Hausdorff the "Borel proposition" and its converse. Hausdorff uses the word *Gebiet*, which can be translated as region, to refer to our open sets.

[49] Any decent textbook on topology will discuss these axioms.

in the early 20th century, for I believe that this background picture shall bring much clarity as to Brentano's own views by allowing us to regard them as opposed to these background assumptions.

It is very unlikely that Brentano himself had a thorough understanding of the whole mathematical literature that was devoted to these notions in the latter part of the 19th century and early 20th century and, as we noted, it is almost surely the case that he did not have access, for instance, to Hausdorff's textbook that came out in the very year he was dictating his essay on continua. There are even passages in this essay in which it seems clear that Brentano misunderstands the relevant mathematical ideas — in particular the distinction between different sizes of infinity. However, through the work of the mathematicians whose work he was acquainted with — in particular, Riemann and Poincaré are the ones he mentions in this respect —, Brentano was definitely exposed to the core of mathematical ideas that constituted the manifold-theoretic view of continua and he is surely engaged with this new tradition when he expounds his own ideas on the subject, as shall become clearer in the next chapter.

CHAPTER 3

Brentanian continua and their boundaries

People say that ideas die hard. I'm not sure sayings are to be considered true or false, but what we will see in this chapter is definitely a case study that seems to confirm this saying, specially when it relates to interesting and fruitful ideas. Although, in the decades that preceded the turn of the 20th century, the Aristotelian position — according to which continua cannot be composed out of indivisibles — seemed to be completely dead and buried by the newly established mathematical orthodoxy that, as we saw, was founded upon the account of continua in terms of the newly developed set-theoretic topological concept of a continuuous manifold, we shall see that Brentano had the courage to go against the establishment and to propose a new account that has deep roots in the original Aristotelian tradition, but that builds on it in order to create a cohesive and credible picture of the ontology of continua and their — as Brentano will think about them — inseparable boundaries.

His views on this subject are mainly to be found his *Philosophische Untersuchungen zu Raum, Zeit und Kontinuum* (Brentano (1976), English translation Brentano (1988)), more specifically in the first paper called "On what is continuous" and dictated in 1914. There is also a fairly lengthy presentation of the subject in his *Kategorienlehre* (Brentano (1968), English translation Brentano (1981)) and also shorter presentations of his account are found in his 1917 essay "Vom *ens rationis*" (Brentano (1959), pp. 238-281; English translation Brentano (1995)), in his *Deskriptive Psychologie* (Brentano (1982), English translation Brentano (1995b)), in his *Untersuchungen zur Sinnespsychologie* (Brentano (1979)), in *Von Sinnlichen und Noetischen Bewusstsein* (Brentano (1928), English translation Brentano (1981a)) and in his manuscript "Vom Unendlichen" (Brentano (1963)). One also finds a fairl early detailed presentation of Brentano's reflections on continua in his Lectures on Elementary Logics given in 1884/1885.

However, before we dive into the details of Bretano's account, we should consider a few of the background assumptions that guided his thought. First of all, we should stress the point that Brentano's main philosophical goal was to provide the philosophical foundations for a truly scientific psychology, i.e., for a rigorous study of the subjective dimension of reality, which was delimited by him as that portion of reality that is characterized by the property he called intentionality. Certainly, this is not the place for a thorough account of this notion or of the whole philosophical project that arose from its recognition as the main property of the subjective domain; nonetheless, a few remarks are definitely in order.

First, we would like to stress a fact that is commonly overlooked in the epistemological literature, *viz.* that the term "intentionality" is a technical term in Brentano's work and is *not* to be connected with the common meaning that the word "intentionality" or its corresponding terms in other languages usually have, which is related to an individual's drive to perform some action or another, usually referred to as "the intent" of the agent.

Brentano's term comes from wholly different etymological sources. It has its origins, as he points out himself in his *PES*[1], in the scholastic doctrine of relations and, in particular, in the specific class of relations that obtain between a thinking subject and the thought object, which is characterized by Brentano by the now very weird, but historically accurate, notion of "intentional inexistence". This is again a technical term that is very distanced from the usual meaning of these words. Indeed, it has nothing to do with some kind of lack of existence as the "in" prefix might imply, but to an "existence in", which is the mode of existence of the term of an intentional relation.

Well, under these assumptions one can readily understand that, for Brentano, a philosophical account of certain objects is to be carried out as an account of how these objects are intentionally presented or, in a more Brentanian terminology, present to the cognizant subject.

Thus, it is no wonder that, when it comes to the special kind of objects that we're calling continua here, we have Brentano claiming that

> it is much rather the case that every single one of our intuitions — both those of outer perception as also their accompaniments in inner perception, and therefore also those of memory — bring to appearance what is continuous. Thus in seeing we have as object something that is extended in length and breadth which at the same time shows itself clearly as allowing us to distinguish a front and rear side and thus as characterised as the two-dimensional boundary of something extended in three dimensions. And since this continuous something presents itself to us who

[1] Cf. 1973, p. 68

> see as being our primary object, we see also at the same time and as it were incidentally, our seeing itself, that is, we are conscious of ourselves as ones who see, and we find that to every part of the seen corporeal surface there corresponds a part of our seeing, so that we also, as seeing subjects, appear to ourselves as something continuously manifold. And still more, what appears to us first and foremost is rest and motion; so also persistence and gradual change appear to us as primary qualitative objects. This happens in that, whilst certainly in our perceptual presentation of the primary object we are never able to present the same place filled with two qualities simultaneously, still we are able to present it as filled with one quality as present, with another as most recently past, and with yet another as further past, whereby the transition from present to further past takes place in an entirely continuous manner. Thus once more we appear to ourselves, in seeing phenomenal qualities following each other in a temporally continuous way or in seeing them persisting continuously in time, as something that is continuously manifold. (1988, pp. 4-5)

Here, we have a passage in which Brentano talks about how this notion of continuity is present both in pretty much all our intentional objects of experience and indeed in our very temporal nature.

So, his project is to present an account of this property of continuity as it is ubiquitously present in our common experience of both the objects in the external world and of ourselves as subjective observers of this world.

This statement, however, altough essentially correct, is not fully devoid of ambiguities, lending itself to possible conciliatory interpretations such as the one put forward by Kölner and Chisholm in their introduction to Brentano (1988). Indeed, in it they claim that

> [o]nce we recognise the distinction between the mathematical and the phenomenological conceptions of continua, it should be clear that there is no conflict at all between the theories of descriptive psychology (in the sense of Brentano) and the theories of pure mathematics (in the sense of Dedekind) or the theories of mathematical physics (in the sense of Einstein). (Brentano (1988), p. xii)

Now, although this interpretation can surely be entertained as a possible way of understanding Brentano's (proto-)phenomenological approach to psychology, we must disagree with Kölner and

Chisholm. The true situation seems to be better captured by Olivier Massin in his (2018) paper entitled "Brentanian Continua". In it, he states:

> Appealing as this irenic picture may be, it is, I believe, erroneous. Brentano and Dedekind, are not talking past each other, but actually disagreeing. First, Brentano not only thinks that he is in disagreement with the classical mathematical approach to the continuum, he in fact argues that there is genuine disagreement here. Thus, he stresses from the very beginning of his investigations into the continuum that there must be a single concept of continuity that we all share, which constitutes the subject-matter of such disagreements: The question concerning the concept of continuity cannot be framed in such a way that one would call into doubt whether we do in fact possess such a concept. For otherwise we would not be able to understand ourselves when arguing about other aspects of this concept. (Brentano 1988, 1; see also Brentano 1981b, 55) Second, as the rest of his discussion makes clear, some of Brentano's objections target not only the application of the mathematical approach to continuity to the simple contents of perception, but also to the "number-continuum" itself (a point further documented by Ierna 2012).
>
> What Brentano sets out to uncover is what all continua —sensory and mathematical— have in common. Note also that Dedekind would also not welcome a restriction of his enterprise to pure mathematical continua, since he explicitly states that he intends to lay the "scientific basis for the investigation of all continuous domains" (Dedekind 1901, 5). (Massin (2018), p. 236)

His argument is that both Brentano and the mathematicians of his time were dealing with essentially the same notion of continuity as characterized by the idea of "gaplessness", so that instead of "talking past each other" they were in actual disagreement as to how to understand systematically this intuitive, and as we saw in previous chapters, old idea of thinking about continuity in terms of this notion of "gaplessness".[2]

[2] However, this notion of "gaplessness" is ambiguous and can be formally understood either as a statement of mathematical density or as a statement of true mathematical continuity, the latter of which, as we saw in chapter 2, can be cashed out in many equivalent mathematical statements, but is indeed stronger than the former.

Thus, in Massin's view there is indeed disagreement between Brentano and the mathematical community, *viz.* disagreement about how one is to cash out this traditional idea of "gaplessness", either in terms of a convoluted step by step construction of ever new intermediary objects interposed between the ones obtained from previous steps, which was the rationale behind the various formal mathematical constructions of the continuum, or in terms of something that is immediately present in all, or at least pretty much the vast majority, of our intuitions and that is, thus, to be simply extracted from it by some abstraction process.

From this point of view, we can better understand why Brentano wants to oppose his notion of continuity from the one that was being crystallized in the set-theoretical mathematical topology of his time, since this mathematical notion appeared to him as distanced from our intuitive ideas regarding continua. Thus, before we dive deeply into Brentano's account, it will be profitable to understand these Brentanian criticisms in the context of the aforementioned (proto-)phenomenological background assumptions, which will be the task of the next section.

1. Brentano's criticisms

Brentano's criticism of the mathematical constructions of the continuum that were being carried out in the late 19th century by figures like Cantor and Dedekind are intimately related to a distinction regarding the two ways through which one can acquire concepts in general, and in particular the concept of a continuum. According to Brentano's anti-rationalist account, there cannot be any *a priori* concepts[3]. Thus, any concept, according to him, is either given straight through some intuition or is constructed by means of some logical components which were usually called marks (*Merkmale*) in the German epistemological tradition dating back at least to Kant. It is Brentano's view that the mathematical constructions of continua — essentially of the continuum of real numbers — are examples of such second way of obtaining concepts, so that his criticism of such constructions is fundamentally connected to a criticism of the view according to which the notion of a continuum can and must be obtained by such a logical construction and, therefore, is equivalent to a justification of his starting point according to which continua are abundantly given to us in experience, so that an account the notion of a continuum must necessarily be obtained in these particular intuitions we have of individual continua, or in Brentano's own words,

[3]Cf. *Versuch über die Erkenntnis*, ed. by A.Kastil (1st ed. Leipzig, 1925; repr. Hamburg, 1970), Part I, pp. 26ff., 40f., 153f

> the concept of the continuous is acquired not through combinations of marks taken from different intuitions and experiences, but through abstraction from unitary intuitions. (1988, p. 4)

These ideas, of course, will have a great impact in Brentano's students, in particular Husserl, so much so that they will play a leading role in the, at that time, incipient idea of a phenomenology. However, to stay closer to the present discussion, we must note that this essentially translates Brentano's reluctancy in accepting that an intrinsically never-ending[4] interposition of intermediate terms would account for our notion of a continuum as some kind of "completely filled" extension. Indeed, he says that,

> [p]roceeding in this way [i.e., by means of the usual mathematical constructions], we should have to ascribe to the concept of continuity an origin in operations of thought both artificial and involved. This seems unacceptable from the very start, for how could this concept then be found in the possession of the simple man or even of the immature child? And further, how dubious it appears to suppose that the halvings and other divisions have been executed to an actual infinity, that they have been brought to completion, just because one can assume without absurdity that they have been executed beyond any arbitrarily determined limit. (1988, p. 3)

Turning our attention back to Brentano's criticism, then, we see that he first presents a quick account of certain attempts of constructing the mathematical continuum of real numbers — with an emphasis on Poincaré's — and, while doing so, he criticizes some "silly" attempts of performing such a construction by means, for instance, of successive halvings of the interval between 0 and 1. These are termed "silly attempts" because they are of course much less sophisticated than either of the canonical constructions – i.e., Cantor's construction by means of Cauchy sequences or Dedekind's construction by means of his cuts – in that, as Brentano rightly observes, they obviously fail even to account for the presence of the ratio $\frac{1}{3}$ in between 0 and 1.[5]

[4]Nowadays we could even say "uncountably large", although, as we shall see, Brentano did fail to grasp this essential feature of true continuity, as opposed to the mere density of the rational numbers, which are, of course, only countably infinite.

[5]The more general statement would be that, any attempt to construct a continuum by means of successive partitions of the interval $[0, 1]$ into n equal sized portions would fail to encompass fractions of the form $\frac{1}{m}$, with m not a power of n.

However, Brentano's main criticism of taking the outcome of the mathematical constructions as that which is given to our intuition as a continuum is presented in the following passage, which is applicable even to the more sophisticated mathematical constructions mentioned above and has clearly Dedekind's construction in mind.

> That one has indeed here posited something completely absurd is seen immediately if one splits the supposedly continuous series of all fractions between 0 and 1 into two parts at some arbitrary position. One of the two parts will then end with some fraction f, the second however could now start only if there were some fraction in the series which was the immediate neighbour of f, which is however not the case. With what, then, does the second series begin? With a multiplicity of fractions rather than just one? But this, too, is impossible since every fraction is distinguished from every other by a before or after in the series. But if not with a single fraction and not with a multiplicity of fractions then with what, since there is nothing to be found in the series other than fractions taken either singly or in groups? We should apparently have something that began but without having any beginning.
>
> One sees that in this entire putative construction of the concept of what is continuous the goal has been entirely missed; for that which is above all else characteristic of a continuum, namely the idea of a boundary in the strict sense (to which belongs the possibility of a coincidence of boundaries), will be sought after entirely in vain. Thus also the attempt to have the concept of what is continuous spring forth out of the combination of individual marks distilled from intuition is to be rejected as entirely mistaken, and this implies further that what is continuous must be given to us in individual intuition and must therefore have been abstracted therefrom. (1988, p. 3)

This passage is somewhat dense, but very interesting. First, it considers the open/closed background framework on top of which the construction is formulated. We shall talk more about this problem later for it is, I believe, precisely the point in which Brentano's ideas show their true strength, especially in relation to the notion of connectivity. Furthermore, this passage is interesting because the criticism it contains is intrinsically connected with Brentano's assumption that with a given continuum, there must necessarily correspond a *boundary* or *limit*, *viz.* something that marks the precise location where such continuum "starts" and "ends"; and this, we shall claim, is Brentano's main contribution to the subject.

Indeed, one might even claim that Brentano's criticism would not really amount to a thorough refutation. It's form is much better understood as a two-pronged attack: first, he hints towards some essential kind of "intuitive unnaturalness" of all the mathematical constructions of continua that were founded on what we're calling the "manifold conception" of continua; and, then, he shows how there is an alternative way of thinking about the various kinds of continua which are presented in intuition that is much more in accord with some natural assumptions about them. This alternative way, however, is essentially not new, but springs from Aristotle's conception of continua, which is itself based on the Greek philosopher's thorough denial of actual infinities and his intuition that the notion of a continuum should be intimately connected with the notion of its boundary or its limit, which is in its turn something whose being is derivative or dependent on the being of the continuum it bounds.

Under these assumptions, it would be an absurdity to attempt at a construction of some continuum by starting from its lowest-dimensional boundaries, *viz.* its points. Such a construction would amount to nothing other than a blunt metaphysical putting of the cart in front of the horses, in that it would amount to a construction of a certain entity out of other entities whose being would be highly dependent on the first entity's being to start with. In this respect, we have the following illuminating passage:

> If something continuous is a mere boundary then it can never exist except in connection with other boundaries and except as belonging to a continuum which possesses a larger number of dimensions. Indeed this must be said of all boundaries, including those which possess no dimensions at all such as spatial points and moments of time and movement: a cutting free from everything that is continuous is for them absolutely impossible. And this allows us to grasp very clearly the topsy-turvy character of the above-mentioned attempt at construction of the concept of the continuous through interpolation of fractional numbers, where every fraction is supposed to have existence without belonging to a series of fractions. (1988, p. 7)

2. Actual infinities

In the criticisms of Brentano we can distinguish two key points that serve as foundations for the whole argument, *viz.* the question regarding the ontological status of boundaries that we mentioned above and the question regarding the possibility of actual infinities. We will study the second point in this section and the first point will be discussed in the following one.

As we saw, the question whether actual infinities are metaphysically possible is one that dates back at least to Aristotle — who answered strongly in the negative — and Brentano seems to be following the Ancient Greek philosopher closely in this regard when he denies that the mathematical constructions could ground our notion of continuity, for these constructions would require an actual infinity of interposed elements in any given continuous extension. Indeed, in a text dictated in 1917 and present in the Appendix to the English translation to the *PES*, we have the following passage:

> Just as the concept of the *actu* infinite number of entities is absurd, so is the that of a thing which is infinitely small. [...] One can say of a continuum, then, only that it can be described as being a as large a finite number of actual entities as you please, but not as an infinitely large number of actual entities. (p. 354)[6]

The acceptance of actual infinities seems to go, after the mid 19th century, hand in hand with the new set-theoretic foundation of modern mathematics. Indeed, since the groundbreaking work of Cantor, that arguably established set theory as an acceptable mathematical theory, we have the establishment of different transfinite cardinalities as a mathematical fact and the study of their arithmetical and geometrical properties as part of the set-theoretical work to be done. In particular, we have, in the context of the discovery of different sizes of infinity, the revolutionary distinction between density and continuity, which is the distinction that somehow grounds the numeric distinction between the merely rational numbers and the truly continuous set of real numbers.

Before the work of those mathematicians that aimed to establish a precise formulation of this property of continuity, which distinguished the real numbers from the merely rational numbers, the notion of "continuity" seemed to be related with the property of the real numbers — or, indeed, of any continuous extension — according to which, between any two real numbers, no matter how

[6]Brentano goes on to add: "And in order to call attention to its distinctive character, we may also use the expression 'a continuous number of things', but this must be distinguished from the *actu* infinite number." This complement is not directly relevant to the point being made here, but it perhaps shows an incipient attempt by Brentano to understand the aforementioned distinction between mere density and full-blown continuity, which is arguably what he calls "its distinctive character."

close together, we can always find another real number between them. This property is, as we saw, nowadays unambiguously called "density" and it is sharply distinguished from what we call "continuity", which is characterized by all the equivalent assertions we mentioned in chapter 2, *viz.* the existence of cuts or of lowest upper bounds for bounded subsets etc.

Unfortunately, even though Brentano was by no means a complete stranger to the latest mathematical developments of his time, he nonetheless surely failed to capture their full meaning. In particular, he never clearly understood this distinction between density and continuity, and continued to think about the latter in terms of its older characterization in terms of properties that resembled more the modern notion of density then the proper modern notion of continuity per se.

One can clearly see this fact when one pays attention to an alleged proof in his manuscript entitled Descriptive Psychology of the *false* statement that

> [i]t is also possible to give a unique and mutually exhaustive pairwise coordination between the point set of a continuum and the full set of integers. (*DP*, p. 113)

In his "proof", what he actually does is to show how the set of positive integers can be put in a one-to-one correspondence with a set in which, for every point, there would be another point closer to the former than any preassigned distance. Following many older mathematicians, then, he identifies this property with the notion of continuity. However, as we discussed previously, with the development of 19th century analysis, this property became associated with the notion of density, which is a weaker notion than continuity, not every dense set being continuous in the proper sense.

I believe that this lack of proper understanding on Brentano's part unfortunately prevented him from engaging more thoroughly in the discussion regarding actual infinities, in the sense that any of the continuity properties requires a more robust acceptance of actual infinities than any merely dense set does, since one easy way of understanding what it is to be continuous certainly would require the property of being composed by a continuously or non-denumerably infinite number of points, although this requirement would by no means be sufficient since, e.g., the Cantor set is non-denumerable, but one would hardly say that it is continuous, for, as a subset of the real numbers, considered with their usual topology, the Cantor set is *not* connected and, from a measure theoretical point of view it has measure zero.

However, what seems to be the case is that Brentano believed that his account of boundaries as dependent entities would be more in synchrony with Aristotle's tradition of denying actual infinities.

This is because his account of boundaries as dependent entities would equate them with universals, which according to him have not a proper kind of existence and could correspond to many different individuals. Indeed, he says that

> [b]ecause a boundary, even when itself continuous, can never exist except as belonging to something continuous of more dimensions (indeed receives its fully determinate and exactly specific character only through the manner of this belongingness), it is, considered for itself, nothing other than a universal, to which — as to other universals — more than one thing can correspond. (1988, p. 8)

This move to consider boundaries as universals has the upshot of allowing him to consider the inner boundaries of some extended continuum as merely potential, i.e., as not being actually instantiated by a certain individual, whereas the outer boundaries would be actually instantiated. Thus, one might be able to hold the view that only the outer boundaries have actual existence and, therefore, that number of things with actual existence remains finite.

On the other hand, this subsumption of the boundaries of continua to the class of universals also allows Brentano to think about the coincidence of boundaries, a subject which is very new and interesting and that shall be further discussed below.

3. Boundaries

Before we get to the topic of coincidence of boundaries, however, we still need to discuss the second background assumption in Brentano's criticism, which is that there is an intrinsic relation between a continuum and its boundaries, according to which the reality of the latter is strictly speaking dependent on the reality of the continuum itself. In other words, for Brentano, boundaries — as, indeed, any other universal[7] — can have no independent existence and, therefore, can only exist as boundaries of a certain particular higher-dimensional continuum.

Beside the passages in the compendium about space, time and the continuum that we mentioned above, we find clear expressions of this thesis in Brentano's *Theory of categories*:

> no continuum can be built up by adding one individual point to another. And a point exists only in so far as it belongs to what is continuous; points may be joined together just to the extent that they do belong to the same continuum.

[7]This point will be made clearer later, but have already discussed a version of it in relation to Aristotle's account in chapter 1.

> But no point can *be* anything detached from the continuum; indeed, no point can be thought of apart from a continuum. (1981, p. 20)

In the special context of Brentano's aforementioned intentional — or one might say "proto-phenomenological" — approach to the ontology of the objects which are presented to us, what we have is that the only self-standing continua, besides the one-dimensional temporal continuum, are essentially the three-dimensional bodies of outer experience; all other lower-dimensional continua are to be though of as boundaries of some three-dimensional body.[8] The first, *viz.* the temporal continuum and the three-dimensional bodies of outer experience, are what Brentano calls "primary continua" in the sense that more dependent continua have their existence founded upon the — logically, as opposed to chronologically — previous existence of these primary continua. For example, besides the boundaries themselves, one might mention as secondary or dependent continua any kind of property, such as color or hardness, that is present in some primary continuum. Thus, a red parallelepiped must be understood as being composed of the primary continuum that constitutes the parallelepiped's volume and which is both bounded by the six rectangular faces which constitute its outer boundary and filled by all the rectangular inner boundaries that can be transformed into outer boundaries of its parts, were the parallelepiped to be divided; but also, it has a secondary continuum in its composition as well, which is to be identified with the red color that permeates its outer boundary and is to be regarded, according to Brentano, as something continuous, since it is just as extended as the outer boundary itself. Indeed, he says that

> the colour, too, appears to be extended with the spatial surface, whether it manifests no specific colour-differences of its own — as in the case of a red colour

[8] Or, of course, boundaries of boundaries of such bodies, for the case of one-dimensional boundaries, and boundaries of boundaries of boundaries of such bodies, for the case of points. In this sense, it is useful to mention the following clear passage from Brentano (1988):

> If something continuous is a mere boundary then it can never exist except in connection with other boundaries and except as belonging to a continuum which possesses a larger number of dimensions. (p. 7)

However, this account is not something he developed late in his life, but was indeed a point that stayed fairly unchanged, as we can attest from this earlier passage from the *Descriptive Psychology* manuscript:

> It is to be noted in this context that the one- and two-dimensional ones, like points, are only possible as boundaries, by themselves they are nothing. Everything they are, they are only in connection with the third dimension, i.e. with the physically spatial. We said earlier that a spatial point never exists without a continuum. This must still be more precisely determined to the effect that it can never exist without connection to three-dimensional spaces. (*DP*, p. 120)

> which fills out a surface uniformly — or whether it varies in its colouring — perhaps in the manner of a rectangle which begins on one side with red and ends on the other side with blue, progressing uniformly through all colour-differences from violet to pure blue in between. In both cases we have to do with a multiple continuum, and it is the spatial continuum which appears thereby as primary, the colour-continuum as secondary. (1988, p. 15)

We would like to stress here that, although our example considered a uniform color as a secondary continuum, it is clear from Brentano's passage, that he also thinks of a continuously varying colour as a possible example of a secondary continuum.

Now, this talk about dependent boundaries and secondary continua is, indeed, the main point of Brentano's account. They are the most distinctive characteristics of this account and they are the ones that most strongly relate it to the Aristotelian tradition. Moreover, their recognition brings about the possibility of describing two new and interesting properties of continua, which shall be the topic of our presentation in the next two sections. These are the notions of *plerosis* and *teleosis*, which are intuitively to be understood, respectively, as a measure of the "fullness" of a certain boundary and a measure of the "degree of change" of a certain secondary continuum.

4. Plerosis and coincidence of boundaries

As we saw, for Brentano, it is a conceptual requirement for boundaries to have their existence be dependent on the existence of some other higher-dimensional continua they somehow limit or bound. An important part of Brentano's account, though, is that this notion of "limiting some higher-dimensional continuum" does not have to be — and moreover usually is *not* —, according to him, total. Much more commonly, boundaries only bound other higher-dimensional continua in a restricted portion of the possible total number of directions that are present in the higher-dimensional space in which the bounded continuum is embedded. For instance, according to the Brentanian account, the disc that bounds the northern hemisphere of a solid sphere only does so in the north-pointing direction, but not in the south-pointing direction. Now, to make this point clearer, Brentano introduces a concept that is supposed to be a function of the number of directions in which a given boundary bounds a higher-dimensional continuum in relation to something like "the total number of directions in which the boundary could bound some higher-dimensional continuum", i.e., some kind of "measure" of the degree to which the boundary in question actually fulfills the possibility of being a boundary in every possible direction of the space that embeds the higher-dimensional continuum the boundary is a boundary of.

What we have here is a intuitive proposal of a concept that can have a much deeper mathematical significance. The background idea here is certainly something like the Jordan curve theorem, which states that any closed Jordan curve or, in other words, any closed non-self-intersecting curve on the plane divides the plane into two connected regions. Thus, one can think about this Jordan curve as a boundary of either of these two regions, or of both. In the former case, we shall say that the curve has "half plerosis" and in the latter that the curve has "full plerosis". Indeed, this theorem — which, by the way, was the center of much discussion during the turn of the 20th century — deals with the plane as the background space, which is certainly one dimension less than what Brentano considers to be the embedding space for the "usual continua of outer experience". However, by the time Brentanto is dictating his notes, Brouwer and Lebesgue have already used homology theory to prove a generalization of the Jordan curve theorem to higher dimensions.[9]

Thus, because of these theorems, the notion of the plerosis of boundaries that have one dimension less than the embedding space is very simple and amounts essentially — in the case of boundaries which are images of an injective continuous mapping from a sphere — to the statement that this boundary has full plerosis if the portions of the embedding space into which it is divided by the boundary are not actually split up by the boundary, so that the boundary is not an actual outer boundary, but simply an inner boundary[10], or half plerosis otherwise.

However, the situation is more complicated when one tries to generalize this idea to boundaries with smaller dimension when compared to the embedding space. This is because, for instance, given a line embedded into R^3, this line can be a boundary of an infinite number of different half planes,

[9]In fact, they have proved that any topological sphere in the $(n+1)$-dimensional Euclidean space R^{n+1} $(n > 0)$, i.e., the image of an injective continuous mapping of the n-sphere S_n into R^{n+1} divides the space R^{n+1} into exactly two connected components, one of which is bounded (the interior) and another which is unbounded (the exterior), and which have the topological sphere as their common boundary.

[10]In the sense of a possible place in which the total object can be divided into two parts. In this context, Brentano says that

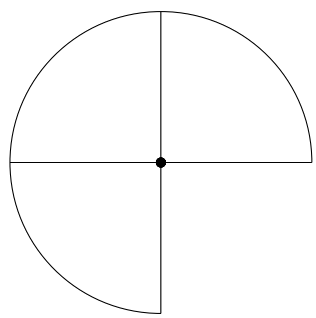
> [w]here we have to do with the interior of a continuum, every point has full plerosis, i.e. is connected in every conceivable direction with the relevant continuum. (1988, p.20)

so that ascribing to it a plerosis with the formula

$$(*) \qquad P = \frac{1}{n},$$

where n is the number of portions of the embedding space of which the boundary in question can be a boundary of, does not make any mathematical sense.

If the situation is such that, some higher dimensional continuum is, as a matter of fact, partitioned into a finite number of symmetric regions that meet at a single boundary, then one can surely ascribe a plerosis to this boundary with $(*)$. A simple example to portray this situation is a disc without one of its quadrants, as shown in the following picture:



The center point of the amputated disc is a boundary of each of the remaining three quadrants, but certainly not of the missing quadrant, so that we could ascribe to this point a plerosis of $\frac{3}{4}$.[11]

The more general case in which the higher dimensional continuum is *not* partitioned into a finite number of symmetric regions that meet at some given boundary, could be studied with the help of measure theory.[12] However, for now, we should limit our attention to the simpler and more pedagogically inclined cases in which we do indeed have such a finite partition of the embedding higher-dimensional continuum.

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[11] In this discussion, for the sake of clarity and simplicity, we are assuming the disc to be the embedding space. Otherwise, i.e., if we were considering the more realistic case of a disc embedded into the real 3-dimensional space, then we would need to consider directions that are not co-planar with the disc as well, so that we would not end up with a finite partition of the possible regions the center of the disc could be a boundary of and, therefore, could not ascribe to this point a plerosis according to $(*)$.

[12] The idea would be to ascribe to a certain boundary of dimension $n - m$ a copy of S^m and, then, to define as the plerosis of the boundary the (Lebesgue-)measure of the subset of S^m that is intersected by the directions which are actually bounded by this particular boundary, divided by the area of the n-sphere, i.e., by

$$A_n = (n+1)\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2}+1)},$$

with the gamma function as the function that takes the values $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$.

4.1. What does it mean to touch? Now, with this in mind, we can shift our attention to the most interesting consequence of this notion of plerosis, which is that it enables one to give a new and surprisingly down to earth account of what it is for two continua to *touch* each other. Under the background assumptions of the set-theoretic conception of continua, the notion of two bodies touching is a little abstract and arbitrary. To understand why, let us consider a solid 3-dimensional sphere that is cut through its equator line. According to usual set-theoretic topology, what we end up in this case are two hemispheres, one of which is closed since it contains the boundary disc, and the other open since it does *not* contain the boundary disc.[13] This type of situation, i.e., two regions A and B such that $\partial A \cap \partial B \subseteq A$, $\partial A \cap \partial B \cap B = \emptyset$ and $\partial A \cap \partial B \neq \emptyset$ is the only one in which we can talk of touching under the set-theoretical topological assumptions.

However, this situation is simply a more general statement of the position which was famously termed a "monstrous doctrine" by Brentano. The reason for doing so was that he could not accept the symmetry of this situation and the analogous situation in which we have $\partial A \cap \partial B \subseteq B$, $\partial A \cap \partial B \cap A = \emptyset$ and $\partial A \cap \partial B \neq \emptyset$, which would entail a fundamental arbitrariness to the choice of which region should contain the common boundary. [14] This problem is very accurately described in Smith (1998/99) with his example of two tangent spheres:

> Imagine two perfect spheres at rest and in contact with each other. What happens at the point where they touch? Is there a last point p_1 that belongs to the first sphere and a first point p_2 that belongs to the second? Clearly not; for then

[13]Of course, here we use the further ontological assumption that two truly distinct and separated bodies cannot contain a common part and, in particular, both halves cannot contain their common boundary at the same point in time.

[14]The talk in Bolzano and Brentano is a little bit less general than our exposition here, for they conceived only of the possibility that either A is closed and B is open or vice-versa, whereas our statement here, although it certainly encompasses this particular case, also takes into consideration cases in which neither A nor B are fully open or closed, albeit their common boundary certainly does *not* belong to both regions. Here is Brentano's famous passage against Bolzano's "monstrous doctrine":

> According to the doctrine here considered, in contrast, the divisions of the line would not occur in points, but in some absurd way behind a point and before all others of which however none would stand closest to the cut. One of the two lines into which the line would be split upon division would therefore have an end point, but the other no beginning point. This inference has been quite correctly drawn by Bolzano, who was led thereby to his monstrous doctrine that there would exist bodies with and without surfaces, the one class containing just so many as the other, because contact would be possible only between a body with a surface and another without. He ought, rather, to have had his attention drawn by such consequences to the fact that the whole conception of the line and of other continua as sets of points runs counter to the concept of contact and thereby abolishes precisely what makes up the essence of the continuum. (1988, p. 105)

> we should have to admit an indefinite number of further points between p_1 and p_2 and this would imply that the two spheres were not in contact after all. To acknowledge one of p_1 and p_2 but not the other, however, would be to countenance what is here an asymmetry of a quite peculiarly unmotivated sort. And our third alternative seems to be ruled out also. For to admit that the point where the two spheres touch belongs to neither of the two spheres seems to amount to the thesis that the two spheres do not touch after all. (Smith (1998/99), p. 108)

On Brentano's account, though, there is a very simple and intuitive definition of contact that hinges on his definition of plerosis. Since not all boundaries have full plerosis, we can think about the possibility of *coincidence* of boundaries that have only partial pleroses up to a point in which the sum of their pleroses adds up to 1 or full plerosis. On a purely extensional account, like the set-theoretical one, two boundaries which occupy the same region of space are to be identified as a single entity. This is the reason why one has to make the arbitrary decision as to whether, given a 3-dimensional region A, ∂A is to be thought of as a part of A or as a part of the complement of A, i.e., whether A is to be though of as closed or open. On the other hand, with this possibility of a coincidence of boundaries, then, one can $define$ the notion of touching as being the relation holding between two regions of space that have at least one pair of boundaries with partial pleroses which coincide at least partially. So, on this account, both A and its complement would have their boundaries as parts. But each boundary would have half plerosis in opposite directions and, thus, could coincide — which would enable one to say that A and its complement actually touch.

This new definition bypasses much of the intrinsic unintuitiveness of the set-theoretic account. As we mentioned, in this picture, we do not have to make an arbitrary decision as to whether "the common boundary" is part of the first or the second touching regions; much on the contrary, in it we don't have open regions at all (even partially), for every proper region[15] has a boundary, which in general will have half plerosis (in the case of 2-dimensional boundaries of a 3-dimensional region in 3-dimensional space) and, therefore, will be able to coincide with other boundaries having partial pleroses, thus creating the alleged contact between these two regions.

Smith (1998/99), just after introducing the problem as we saw, also presents a good account of Brentano's solution:

[15] I.e., any region that is a proper subregion of the whole embedding space.

> The picture of the world of continua and boundaries that is dictated by the above is as follows. Boundaries are full-fledged denizens of reality. They serve as objects of perception (and are perhaps the only objects of perception). But boundaries cannot exist in isolation: there are, in reality, no isolated points, lines or surfaces. Boundaries might be compared in this respect to forms or structures (for example the structure of a molecule as this is realized in a given concrete case) in that they are located in space but do not take up space. Further, both boundaries and forms or structures (and holes, and shadows; perhaps also minds or souls) are comparable to universals in that, while they require of necessity hosts which instantiate them, they can in principle be instantiated by a variety of different hosts. (See Casati and Varzi (1994)) Consider, for example, that boundary which is the surface of an apple. The whole apple can here serve as host, but so also can the apple minus core, which might have been eaten away to varying degrees from within. (*Ibid.*, p. 109)

However, it is unfornutate that he should describe the boundaries in Brentano's account as "full-fledged denizens of reality". These boundaries are for Brentano, as Smith himself recognizes,

> comparable to universals in that, while they require of necessity hosts which instantiate them, they can in principle be instantiated by a variety of different hosts.

This, however, renders boundaries as abstract entities, which in the ever more radically reistic ontological position developed by Brentano throughout his career, are *not* "full-fledged denizens of reality". The latter are only the actual psysical objects that fill actual regions of space, their boundary being analogous to properties of these objects, that have some sort of reality in that they are actual propereties of real physical objects, but are not what one might call a "full-fledged denizen of reality", since they can never exist independently from the body of which it is a boundary or a property — more generally, an instantiated universal.

4.2. The problem of internal boundaries. So far, our exposition has mainly focused on the usual notion of a boundary, which can be more precisely characterized by the expression "external boundary". These boundaries are the more abstract version of the usual common-sense "shells" of three-dimensional objects. However, as we saw, there is also concern in the Aristotelian tradition regarding these notions also with respect to what are called "internal boundaries" of continua. These are essentially, from an intuitive perspective, the various "collections of lower-dimensional entities" which have the property that, if one moves slightly in any direction from this entity, then one necessarily passes through portions of the original continuum.

Given this intuitive characterization, it might seem fairly trivial to conclude that these inner boundaries all have full plerosis, since, as we said, they seem to be boundaries of the original continuum *in all possible directions.* In Brentano's own words,

> [w]here we have to do with the interior of a continuum, every point has full plerosis, i.e. is connected in every conceivable direction with the relevant continuum. (1988, p. 20)

However, it is interesting to see that the literature on the topic is not so straightforward as one might expect, given this introduction. Indeed, we have Massin say that

> in the case of internal boundaries, Brentano's theory appears to face the following dilemma:
>
> 1. Either internal boundaries (in contrast with external boundaries) have full plerosis, but then internal contact does not consist in boundary-coincidence but in boundary-sharing.
>
> 2. Or internal contact (like external contact) consists in several coinciding boundaries, but then it is not the case that all internal boundaries have full plerosis. Does Brentano endorse the first horn of the dilemma? I do not think so [...]. (2018, p. 17)

For him, a reasonable account of what he calls "inner contact" would simply have to be analogous to the "external contact" situation we have discussed in the last section, i.e.,

> Brentano must embrace the second horn of the dilemma: inner contact, like external contact, consists in boundary-coincidence. (*Ibid.*, p. 18)

Now, as Massin himself readily recognizes, this solution seems to be in direct contradiction to Brentano's quoted statement asserting the full plerosis of any inner point of a continuum. His answer is

> that this sentence is slightly hyperbolic. Brentano should have said, more cautiously, that at every point in the interior of a continuum, there is a boundary with

> full plerosis. This more modest claim is interesting in that it does not rule out that, at every point in the interior of a continuum, there may also be boundaries with partial plerosis. (*Ibid.*)

This, indeed, seems to solve the problem and we should agree that it does go in the direction towards the right solution. However, there are further complications to entertain. In particular, if we take Massin's solution *ipsis literi*, then we must have to conclude that boundaries with partial pleroses can add up to more than 1 or *full* plerosis, which is not only against our explicit statement regarding the matter, but also against the likely intuition behind this whole talk: how can one speak of "full plerosis" if one must entertain the possibility of boundaries adding up their respective pleroses to something that is *more* than "full plerosis". And the alleged coincidence of internal boundaries assumed by Massin to solve the problem does exactly that; for if we assume his "more cautious" statement that

> at every point in the interior of a continuum, there is a boundary with full plerosis,

and the relevant claim only made possible by this logical weakening that, together with this full — i.e., 1 — plerosis boundary that is assumed to exist at every point in the interior of a continuum,[16] there might also be other coincident boundaries with partial pleroses — say, plerosis a, with $0 < a < 1$ —, then we must have to conclude that the overall plerosis in that point is $1 + a > 1$.

It should be noted, though, that Massin does indeed consider this problem, albeit very quickly and in passing when he says that

> [o]ne may worry that, if we admit internal boundaries of partial and full plerosis, we end up with too many coinciding boundaries (2018, p. 18)

and he eventually feels it is satisfyingly settled by

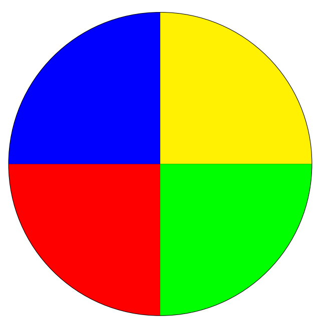
> Brentano's suggestion that coinciding boundaries may enter into part-whole relationships. *Two inner coinciding boundaries of half-plerosis form together a boundary of full-plerosis.* It is therefore not as if the inner boundary of full plerosis is a third, additional boundary coinciding with the two half-plerosis boundaries: rather, the third boundary is mereologically constituted by these two half-plerosis boundaries. Despite being spatially indivisible, some boundaries nevertheless have *plerotic parts*[,]

[16]Note here the not quite Brentanian construction "there is a boundary *at* every point". More on this later.

although he does indeed aknowledge the fact that this whole talk of "plerotic parts" is Zimmermann's and not Brentano's.

Now, one can evaluate this solution from two distinct standpoints: From a merely logico-systematic standpoint, this is surely a viable solution. It is indeed consistent and it is somehow built upon the very Brentanian notion of "coincidence of boundaries". Notwithstanding this, though, I believe that we can salvage Brentano's original claim — which is "too strong" in Massin's view — when we understand Brentano's ideas regarding multiple continua, which shall be done in the next few sections. Then, we shall come back to this question.

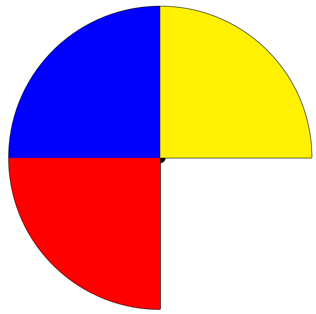
4.3. Multiple continua. This account of boundaries and their coincidence is especially suited for the talk of what Brentano calls double, or triple etc. continua. What we have in these situations is that, built upon some primary continuum, we find secondary, tertiary etc. continua that are usually to be identified with continuously varying properties that each portion of the primary continuum instantiates. For instance, we can think of a disc such that each of its quadrants is painted with a different color, as shown in the following picture:



In this case, what we have is the disc as the primary continuum, on top of which we have four secondary continua, one for each color, which are coextensive with each of the four quadrants.

If we consider only the primary continuum, then, of course, it will be clear that the center of the disc bounds each quadrant and thus has full plerosis. However, if this point is considered, not as a part of the primary continuum, but as a part of any of the secondary color continua, we would have to ascribe to it $\frac{1}{4}$ plerosis, for the total number of directions this point can bound is partitioned into 4 classes, one for each collection of directions that stay in a single quadrant.

Similarly, if one were to detach, say, the green part of the disc, then we would end up with the following figure:



in which the center point, considered as a part of the primary continuum, now only bounds three of the four quadrants and, therefore, has plerosis $\frac{3}{4}$, whereas, considered as a part of any of the remaining three color continua, it will remain with plerosis $\frac{1}{4}$. These are just simple examples that help illustrate the various types of distinctions that this notion of "plerosis" allows one to draw with respect to the various multiply continuous objects that present themselves to one's spacial intuitions.

This notion of plerosis, moreover, can play a role in elucidating the phenomena of motion and rest, and in particular the puzzles regarding the transition from motion to rest or vice-versa. For suppose one throws an object upwards. Then, this object is moving upwards with an uniformly diminishing speed up to a point where it stops and starts falling with uniformly increasing speed.

As we saw in previous chapters, ever since Antiquity, there has been a debate as to the exact moment when such transitions — from movement in one direction, to rest, to movement in the other direction — actually take place. It is not our goal here to go very deeply into the current state of this discussion, but we would like to point out that a commonly accepted answer was a set-theoretic style of answer that would ascribe perhaps a last moment for the upward motion, but no first instant of rest, or that would need to rely on an infinitesimally small, but nonetheless extended time span for the rest portion of the motion. On the other hand, with this notion of plerosis at hand, one is able to provide a much more down to earth account. Indeed, we can say that the moment of rest is literally a point in time, i.e., a true unextended boundary; and, furthermore, that it is not a single moment, but two moments with half plerosis each — *viz.*, the moment in which the upwards movement ceases and the moment in which the downwards movement begins — that coincide with each other.[17]

4.4. More on multiple continua. It is interesting to note that both Kölner and Chisholm in their introduction to Brentano (1988) and Massin in his aforementioned paper ascribe to Brentano the view that time is the only true primary continuum. Indeed, we have the former clamming that

> [t]here are, therefore, secondary and primary continua. A secondary continuum, unlike a primary continuum, is one that is founded upon another continuum. Every continuum is founded upon a temporal continuum; thus whatever is spatial is also temporal. And every qualitative continuum is founded upon a spatial continuum (*Ibid.*, pp. xii-xiii)

and the latter that

> [Brentano] thinks that time, rather than space, is the only fundamental —primary— continuum (Massin (2018), p. 8)

Brentano does in fact claim something like that; e.g. we have him say that

> [y]et still one would have to admit that if our determinations are to be of complete exactness then the temporal continuum is in the eminent sense the primary continuum if compared to the spatial. This is not only because of the limitation of the primary character of what is spatial to three of its four dimensions, which, in so far as it is primarily continuous, allow it to appear in fact only as a three-dimensional boundary of something four-dimensional. It is also because of a certain multifariously different degree of variation which, precisely because what is spatial exists in these four dimensions only as boundary, can apply to it even in an individual moment of time. Moreover, it exists in an individual moment in not quite the same way according to whether it exists as boundary of something that

[17] Brentano discusses this particular point with an example of something that ceases to be. In his words,

> [i]t was affirmed that when something that is ceases to be, then there was a last moment in which it was, but no first moment in which it was not. With a better understanding of the peculiarity of partial plerosis one should have said that in the same moment it exists in partial plerosis and that it does not exist in full plerosis, since until that moment it had existed and from that moment on however it existed no longer. (Brentano (1988), p. 21)

> is continuing to exist or of something that is gradually passing away, and, considered from the side of the past, whether it exists as something that has existed just as it is until now or has come to be what it is through gradual change. These are differences of temporal teleosis, which have significance also for the individual boundaries of time. (Brentano (1988), pp. 17-18)

As we can see, then, the situation seems to be a little more complicated in this respect. Especially when one considers that just two paragraphs before we can read Brentano say that

> [i]n addition to the temporal as such we have also put forward the spatial as such as an example of a primary continuum. (*Ibid.*, p. 17)

The way to understand this seeming discrepancy in his presentation is, however, to simply read carefully what Brentano himself has to say about this theme:

> One might see a contradiction here. Note however that we had in the first place spoken of space as conceived by the geometer who abstracts from its perseverance in time and therefore also ascribes to bodies three dimensions. If, however, he took into account the time in which it exists, then he should have to call it four-dimensional, as was noticed already by Lagrange and has been emphasised also by later thinkers. If we do not carry out this abstraction however, then we shall still have to go on affirming that the body does not appear in the fourth dimension thus accruing to it as primarily continuous, as it does in the remaining three dimensions. Rather, it appears as secondarily continuous, in that time running its course provides the primary continuum of a body which extends from the beginning to the end of time as at rest or as more or less in motion. (*Ibid.*)

Thus, the answer to the whole conundrum seems to be this: as long as we think about space as the "real space" that we live in, then we must surely consider it as a secondary continuum that is built upon the temporal continuum or, more precisely, as a three-dimensional boundary of a quadruply extended continuum, which is the true primary continuum. However, the interesting point here is that one can — and indeed usually does —[18] think about our usual three-dimensional space of experience as a continuum in its own right and, when doing so, we see that it presents several properties that are characteristic of primary continua, such as a homogeneity that in Brentano's

[18]Most usually geometers, according to Brentano.

words “is encountered throughout and as a matter of necessity, where it is present in the secondary continua only as a matter of exception.” (*Ibid.*)

Thus, if we are to be extra picky we would have to grant that, strictly speaking, space is a secondary continuum and, moreover, a mere boundary of the “true” four-dimensionaly extended continuum of reality, which would be thus the only truly primary continuum. In this sense, Massin, Kölner and Chisholm are indeed correct in saying that space is *not* a primary continuum. However, as Brentano recognizes, the abstract three-dimensional space of experience does have many properties that are to be ascribed to primary continua and they, as a matter of fact, figure in many actual cases as a basic continuum on top of which other more dependent continua — such as colour or temperature —, can be built. It is this fact, thus, that leads Brentano to incur in this (metaphysically) loose, but (practically) pregnant talk of abstract three-dimensional space as a primary continuum.

4.5. Back to the problem of coincidence of internal boundaries. Now, in possession of this notion of a secondary continuum whose existence depends on some primary “basic” continuum, we can present a solution to the problem posed by Massin regarding the coincidence of internal boundaries.

So far, we have studied Massin’s solution to this problem, which is to relax Brentano’s claim that “every [interior] point has full plerosis”, so that it is captured by the less Brentanian formulation: “at every point in the interior of a continuum, there is a boundary with full plerosis.” This version is less Brentanian simply because it seems to distinguish between a point in space and the boundaries that might occupy that point. For Brentano, however,[19] it is clear that points *are* internal boundaries and not simply formal *locuses* for such boundaries, which seems to be implied by Massin’s weaker formulation. This incongruity seems enough to reject Massin’s interpretation as a possible reconstruction of Brentano’s position, although it is, as we mentioned previously, certainly a logically possible position to support.

A more Brentanian solution seems to be the one that relies precisely on the distinction made in the latter sections between primary and secondary continua. Thus, in the rest of this section the goal is to hopefully explain precisely how this can be done.

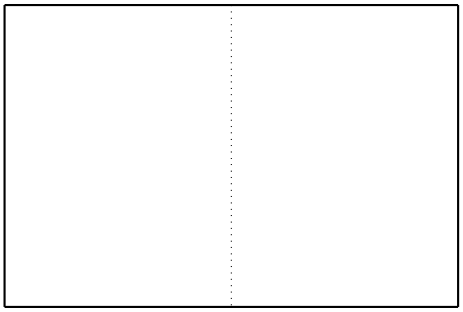
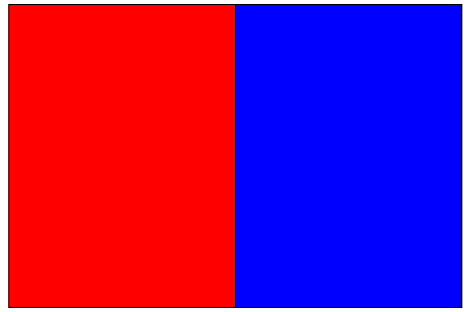
[19]As indeed it was for Aristotle, as we saw in chapter 1.

The main idea is that the Brentanian claim mentioned a couple of paragraphs ago is not to be weakened in Massin's way, but simply restricted to primary continua — in the more lax interpretation of this notion that allows for three-dimensional space to be a primary continuum. In this sense, it is certainly true, since a merely spacial continuum, being, as Brentano says, fully homogeneous just as any other primary continuum, has all internal boundaries with full *plerosis*, since they do not truly separate distinct parts of this continuum — only what one might call "abstract parts", i.e., parts that one might conceive as being distinct but that, in the context of this specific homogeneous continuum, are not really so — in opposition to actual, somehow more ontologically robust parts, that are delimited by boundaries with partial pleroses.

This is, in fact, fairly consistent with the original Aristotelian distinction studied in chapter 1 between mere coincidence of boundaries, which would imply that two distinct continua are indeed contiguous with each other — i.e., in a more modern terminology, they touch —, and true identity of boundaries, which would imply that the continua of which they are boundaries actually merge together into a new fully-fledged truly existing continuum. This is, as we saw, what one can understand by the Aristotelian expression "are continuous with each other", in opposition to mere contiguity, and thus, from a historical point of view, we can understand how a philosopher with such Aristotelian background assumptions as Brentano might have actually adhered to a position like the one we are delineating here.

So, the introduction of these secondary continua, understood as properties that somehow impregnate the more fundamental plain homogeneous regions of space, is what allows one to discern proper distinct parts in this region and it is only with respect to these that one can talk about internal boundaries with partial pleroses.

An example might make the point clearer. Let us consider, for simplicity's sake, a two-dimensional square embedded in usual $\mathbb{R}^2$ and let us consider it in a two-fold way: firstly, merely as a portion of $\mathbb{R}^2$, which might be thought to be divided in the middle by the dotted line in the picture and, secondly, as a coloured object which is painted red in the portion that is left of the dotted line and blue on its right.

Then, the dotted line, in the context of the first continuum, does *not* really divide it into two actual portions, but only marks a position in which such a division *might* occur. In this sense it has full plerosis. On the other hand, the second rectangle has two actually distinct parts that can be characterized by two different functions on two different sets of (abstractly constructed) points:[20] one for the red part and one for the blue part. Now, in this second case, in the precise place we drew the dotted line in the first rectangle, we find *two* different lines: one that is part of the *external* boundary of the blue region and one that is part of the *external* boundary of the red part. Each of these lines only bounds the region whose external boundary it is a part of in one of the possible two directions which it could do so and, therefore, to each of these lines one has to ascribe half plerosis.

However, in this example it does not even make sense to ask whether the two boundary parts we mentioned in the last paragraph somehow merge their half pleroses in order to thus "form" the dotted line. On the contrary: these three lines, albeit collocated, are completely distinct, as they are parts of different continua to start with. Only the two that are boundary parts for the secondary continua have partial pleroses and only they coincide to form a shared boundary and, thus guarantee that the two halves of the rectangle actually touch. This is what we should understand as the proper Brentanian analysis of this topic.

4.6. The aledged problem with primary continua. Now, in the already mentioned paper by Massin, he attempts to show through an analysis of the Brentanian notion of coincidence of boundaries that it cannot be logically harmonized with Brentano's account of primary continua. The section in which he talks about this is not so long, so we can quote it almost in full:

[20]We shall attempt such a construction in the next chapter.

Brentano's theory of contact as coincidence of boundaries is quite plausible in the case of things that are *in space*, such as a blue and a red book touching each other on a surface, or a blue and a red square touching each other on a line. *But can the boundaries that make up space itself coincide*? More generally, *can the boundaries constitutive of primary continua coincide*? That the boundaries of two things in space may coincide is one thing, but that the boundary of two regions of space may coincide is quite another. For boundaries to coincide, several boundaries must be at the same place. But what would it mean for a boundary that constitutes space to be at a place? How can the very boundaries making up space be located at places, since they are themselves constituent of places? We are basically saying that places are located at places.

One may retort that location is a reflexive relation (Casati and Varzi 1999, 21). Places can therefore be seen as located *at themselves*. But whatever its intrinsic merits (or problems), this proposal does not demonstrate the possibility of coincidence between the boundaries of primary continua. For even if places are located at themselves, this does not show that *two places can be located at the same place*. On the contrary: if places are (exactly) located at themselves, and if two places are exactly located at the same place, then they must be one and the same. To show this, we just need to make the additional assumption that places, if located, have only one exact location. One quick argument in favour of this assumption is that places are particulars, and particulars have only one exact location (contrary, perhaps, to universals). The argument to the effect that, if two places are located at the same place, then they are not distinct then proceeds as follows:

(1) $Coincidence$: p_1 and p_2 are exactly located at a p_3.

(2) $Reflexivity$: p_1 is exactly located at p_1. p_2 is exactly located at p_2.

(3) $Unicity$: Every place has exactly one exact location.

(4) p_1 is identical to p_3. (from 1, 2, 3)

(5) p_2 is identical to p_3. (from 1, 2, 3)

(6) p_1 is identical to p_2. (from 4, 5)

> Thanks to reflexive location we may ,find (*sic*) a way of holding that the boundaries constituting space are located, but we still do not get coincidence of boundaries. Summing up: either we hold that location is irreflexive, in which case two places can never be exactly located at the same place for the reason that places simply cannot be located. Or we accept the reflexivity of location, in which case two places cannot be exactly located at the same place for the reason that they fuse into one place. Either way, two places can never be located at the same place. The relation of coincidence, therefore, must hold between things which exist in space: it cannot hold between constituents of space. If this is right, any coincidence-based account of primary continua is doomed to fail. (Massin (2018), pp. 22-23)

As we can see, this is a very detailed presentation of what seems to be a reasonably airtight logical argument. The problem, however, is that every logical argument must start from a set of premises and Massin, for the sake of his argument here, inadvertently assumes several premises that are quite foreign to Brentano's account, most noticeably the background structural assumption that "boundaries are located at some *place*".

For Brentano, boundaries of primary spatial continua — and more specifically, 0-dimensional boundaries of such continua, called points — *are* the possible places in this primary continuum. What we have is the possibility of defining a relation of "colocation" between these boundaries, which, being an equivalence relation, logically yields an equivalence class construction of some abstract "location entity" that is just the equivalence class of all boundaries collocated with some fixed original boundary. However, this abstract entity, which is called p_3 in Massin's argument, in a notation that ensures the reader that it is actually of the same nature as the collocated boundaries p_1 and p_2, is *not* at a par with the boundaries themselves — it is an even more abstract entity. All of this confusion is, according to Massin, supposed to be justified by the fact that

> places are particulars, and particulars have only one exact location (contrary, perhaps, to universals). (*Ibid.*)

This assumption that places are particular is, however, simply in contradiction to an explicit claim by Brentano, *viz.* that

> [b]ecause a boundary, even when itself continuous, can never exist except as belonging to something continuous of more dimensions (indeed receives its fully determinate and exactly specific character only through the manner of this belongingness), it is, considered for itself, nothing other than a universal, to which—as to other universals—more than one thing can correspond. And the geometer's proposition that only one straight line is conceivable between two points, is strictly speaking false if one conceives the matter in terms of lines of incomplete plerosis whose pleroses, even though they coincide with one another, relate to different sides. (1988, p. 8)

This passage, indeed, seems to suggest that "places" — or, in Brentano's technical terminology that is based on Aristotelian assumptions, "boundaries" — are universals, which might in their turn be instantiated by a particular (outer or inner) boundary that is fully determined by the actual concrete continuum — whether primary or secondary — it bounds.

Thus, we have to disagree with Massin and say that there is no problem with Brentano's account regarding the possibility of boundaries of primary continua to coincide. This is not to say, though, that there is absolutely no difference between this coincidence in the usual case of higher-order continua and in the particular case of primary continua. In fact, there is a deep difference between them, that was already touched upon in the last section, *viz.* the fact that, whereas coincidence of boundaries in the case of higher order continua grounds the contiguity of these two continua, i.e., the fact that they touch, in the case of primary continua it grounds true fusion of the two continua into some larger new primary continuum.

This difference, however, is, according to Brentano, simply a consequence of the homogeneity of primary continua; and, in our view, instead of offering a challenge to Brentano's account, it simply makes it more clearly in sync with Aristotle's original position.

5. Are continua gunky?

In the past few decades, some of the questions that Brentano was wrestling with have been restated in terms of a notion of "gunk". In general, gunky substances can be defined as substances that have what Brentano — coming from the Aristotelian tradition — would call infinite divisibility. In other words, a gunky substance is something that can always be split, no matter how many times we have split it in the past, never reaching a final indivisible entity, like the ones assumed to ultimately constitute a continua in the set-theoretical mathematical account that, as we saw, was the one being criticized by Brentano. In David Lewis' words, a "gunky" individual is

> [a]n individual whose parts all have further proper parts. (1991, p. 20)

However, it seems that the ascription to Brentano of an account of continua that accepts some kind of "gunky substance", although surely a natural move in light of what Brentano has to say on the subject, is not, according to the current literature, so simple. Massin, for instance, in his aforementioned paper, presents an alleged dilemma that Brentano must face with his account. He says that

> Brentano's account of continua faces an unattractive dilemma:
>
> • Either continua consist entirely of boundaries, but then (1) it is impossible to destroy any part of a continuum without modifying all the rest of it (due to the essential dependence of boundaries); and (2) extension becomes impossible (due to transitivity of coincidence and the view that continuity can only stem from coincidence).
>
> • Or continua consist of atomless gunk surrounded by a bounding skin, but then (1) continuous transitions are hard to accommodate; (2) some entities —open gunk, that is, gunk in abstraction from its boundary— begin without having any beginning point. Further, the contact between the skin of the bodies and their gunky interior remains unaccounted for. (Massin (2018), p. 28)

Notwithstanding this clear analysis of the logical space covered by this question, he nonetheless fails to provide an interpretation of Brentano's view regarding it. The most he does is to refer the reader to a possible answer that is provided by Zimmerman (1996b), without claiming it to be the correct one. Indeed, he even seems to suggest that this interpretation runs into some problems, which are characterized in the second horn of the dilemma.

Now, we shall see that these problems are indeed solvable and that, as Zimmerman himself recognizes, Brentano's account, as understood by him in terms of his "bounded gunk" approach, is consistent. This approach is, for Zimmerman, to understand that

> Brentano believes that extended objects are wholes composed of two radically different kinds of parts: (1) extended parts, which, though infinitely divisible into

> extended parts within extended parts, ad infinitum, cannot be "infinitely divided" (even "in thought") into a set of simple parts; and (2) non-extended (zero-, one-, and two-dimensional) parts which are necessarily present at or along the inner and outer boundaries of every extended part. (1996b, p. 28)

This enumeration of possible classes of parts fails, however, to account for Brentano's whole position. Were this all there was to say on the matter, then this account would indeed run into the problems raised by Massin, *viz.* the difficulty of explaining continuous transitions and, especially, the difficulty involved in dealing with open regions and their supposed "contact" with their indivisible boundaries. But it so happens that, for Brentano, these two constituents of reality are not merely independent parts that normally seem to accompany each other in reality, but actually complexly related complementary parts. For instance, let us consider the second problem raised by Massin, *viz.* the "weird" open entities problem. In Brentano's picture, these entities simply would *not* have to be accounted for, so that their weird properties would not amount to a problem for his account. On the contrary, from Brentano's perspective, their presence in the usual point-set topological account is precisely what makes this theory "monstruous"; they are the entities that *cause* all the problems. Conversely, on Brentano's account, these entities do not even exist; they might be abstractly considered as objects of thought, but they cannot exist in reality in abstraction from their boundaries. In this sense, nothing can "begin without having any beginning point", as Massin correctly claims the open entities do. For any possible denizen of reality, its beginning points are always there as some of its (external) boundaries, which might be abstracted from the whole by some cognizant subject — just as one does when one considers the notion of a dog in abstraction from the particular characteristics of this particular dog one is looking at —, but never truly separated in reality from its underlying "open" gunky counterpart. Everything in the world is closed; everything has boundaries and, although one might consider a part of something in abstraction from these boundaries, that is all there is to it: a fictitious abstraction from something real and not something that can serve as a counterexample to Brentano's account of continuous reality.

Moreover, it is interesting to realize how widespread this misconception is with authors trying to interpret Brentano's account. For example, we have Smith (1996) say that

> [s]ome entities are what we might call tangential to, i.e. such as to touch or cross the exterior boundaries of, other entities. Some entities are themselves boundaries

> of other entities, though we note that the boundary of an entity may be outside the entity it bounds (as for example in the case of an open interval in the real line) (p. 290)

Although Smith surely recognizes that, according to Brentano,

> boundaries exist only as boundaries, i.e. that boundaries are dependent particulars: entities which are such that, as a matter of necessity, they do not exist independently of the entities they bound[,] (*Ibid.*, p. 295)

this seems to be simply an empty rhetorical recognition, whose consequences are not really taken into account. Note, indeed, how in the first quote, boundaries are taken to be precisely "an entity just like any other", an approach that is radically different from Brentano's very strict distinction between homogeneous portions of space and their boundaries — which are, for him, wholly dependent on the continua they bound and just as abstract as universals.

Indeed, Baumgartner and Simons (1994) say, regarding Brentano's increasingly strict ontological position, that

> In his mature work Brentano held strongly to a view which was already prefigured in the dissertation, according to which ontologists should only deal with concrete things ("the thinglike, as it really is"), so that expressions which purport to refer to something which is not a concrete thing are not "authentic" but "fictionalizing" (*fingierend*). (pp. 63-64)

In this context, we can understand that, for Brentano, the boundaries of continua are *not* one of these "thinglike" entities that "really are"; they are merely abstract parts of truly existing bodies and have an analogous being to any other universal that is instantiated by some particular truly existing body.[21]

Something analogous holds for the alleged "contact problem" between boundaries and their gunky interiors. The problem would be that, when one recognizes that reality is composed of both gunky substances and their boundaries, one would still need to define a "contact relation" between these parts which would somehow justify there not being anything else between them, or something

[21] As the authors of the last quote also note in this regard, Brentano in his 1869 essay "Auguste Comte und die positive Philosophie", in (Brentano 1968), pp. 99-133, warns against the danger of an "entity-fictionalizing mode of explanation" (p.127), which would be precisely what is involved in these accounts of gunky continua that put their boundaries on a par with the whole material bodies which are bounded by these boundaries.

of this sort. However, just as in the solution to the last problem, when we are assuming the existence of these two radically different constituents of reality, we are not conceiving of them as independent entities that enter into relations like contact with one another and thus somehow "form" reality — like two objects that are indeed in contact and thus form a composite reality; we are, on the contrary, assuming that these classes of entities have very specific dependence relations on top of which one might, e.g., logically *construct* a contact relation, that explains the usual intuitive contact intuitions we have. But note, that this contact relation holds exclusively between actual fully independent denizens of reality, *viz.* whole bodies, which are themselves closed and composed, in Brentano's account, of an inseparable communion of some gunky interior with its respective boundaries. And this communion is not to be explained in the same way that contact between bodies is supposed to be; much on the contrary, whatever relation that holds between a gunky interior and its boundaries is not "external" in the sense of a contingent relation that must be explained by means of deeper concepts — like contact between bodies is in the Brentanian account — but as something that is "internal" in the sense of something that pertains to the very nature of reality and must hold as a matter of necessity. It is not something we need to justify, but it is the very justification of real topological relations such as contact between bodies. In fact, Rush Rhees, who studied the philosophy of Brentano under Kastil, states this last thesis very clearly as follows:

> I think we must agree that it is necessary to know what is meant by the side of the one body and the side of the other before we can understand what is meant by their contact. (Erbacher, C. and Schirmer, T. (2017), p. 15)

Furthermore, this point is, not justified, but *explained* by Brentano with his distinction between something being a plurality — like a deck of cards or a space in the set-theoretical pointy conception of it — and something being a multiplicity. This last notion is used by Brentano exactly to characterize the way in which a great number of boundaries are to be identified as somehow pertaining to a given continuum in opposition to how the cards in a deck are a plurality of parts of the deck. Indeed, Brentano says that

> [w]e cannot conceive a continuum as a discrete infinite plurality, for it may be demonstrated that the latter concept involves a contradiction. But we certainly can conceive a continuum as a continuous multiplicity [*Vielheit*]. Indeed we can conceive it as a continuous multiplicity of boundaries. The boundaries do not exist

> in and for themselves and therefore no boundary can itself be an actual thing [*ein Reales*]. But boundaries stand in continuous relation with other boundaries and are real to the extent that they truly contribute to the reality of the continuum. (Brentano (1981), p. 55)

More precisely, he carachterizes the relationship between a certain continuum and its boundaries in the following way:

> every boundary is likewise a conditio sine qua non of the whole continuum. The boundary contributes to the existence of the continuum. This case of the boundary differs from that of the part: the boundary is nothing by itself and therefore it cannot exist prior to the continuum; and any finite part of the continuum could exist prior to the continuum. [...] No boundary can exist without being connected with a continuum. Therefore the continuum is also a conditio sine qua non of the boundary. But there is no specifiable part, however small, of the continuum, and no point, however near it may be to the boundary, which is such that we may say that it is the existence of that part or of that point which conditions the boundary. (*Ibid.*, p. 56)

And Rhees, who one might recognize as the first philosopher to understand and further develop Brentano's account of continua, has the following to say about this topic:

> What I want to suggest is that a surface is not something which has a relation to a body; or at any rate that it is misleading to put it in this way. It cannot properly be described as the bearer of a relation "in" which it "stands" to the body. It is better to say that it is itself something relative and essentially belongs to the body. [...] It seems to me better to say that a surface is itself something relative rather than to say that a surface is "essentially connected with" something else, chiefly because the type of "connection" to which we should wish to refer by such an expression would be just what we call "being a surface" (Erbacher, C. and Schirmer, T. (2017), pp. 25-26)

It seems, however, that precisely this confusion has creeped into the discussion with the passing of the years. Indeed, Chisholm (1983), perhaps the first contemporary paper on the issue, had already a clear conception of this distinction, recognizing it as one of the possible ways of going about defining the notion of a boundary:

> There are two ways of defining *boundary*. We could appeal to the fact that a boundary is a dependent particular — a thing which is necessarily such that it is a constituent of something. Or we could appeal to the fact that a boundary is a thing that is capable of coinciding with something that is discrete from it. One of these should be a definition and the other an axiom. (Chisholm (1983), p. 90)

Note that Chisholm does indeed start with a general notion of a "thing" and then splits this domain into two by one of the two possible definition of boundary. However, he never claims that the question as to how these two essentially distinct kinds of things interact makes any sense; on the contrary, he recognizes that the two subsets of possible things, which he calls "constituents that are not proper parts" and "proper parts", have the intrinsic relation to which we referred above, so that not even

> God could remove just the surface of a three dimensional object. (*Ibid.*, p. 91)

This surface is just not something that can exist in isolation.

It is also interesting to note that this distinction is perhaps the main distinction in Husserl's theory of parts and wholes, which surely sprang from Husserl's study under Brentano and eventually led to the creation of the mereological aproach to ontology that, in its turn, let to the interest in point-free constructions of space such as the one we are considering here. Indeed, in the very begining of his third Logcal Investigation, which is devoted to the topic of parts and wholes, Husserl says:

> The difference between 'abstract' and 'concrete' contents, which is plainly the same as Stumpf's distinction between *dependent* (non-independent) and *independent* contents, is most important for all phenomenological investigations; we must, it seems, therefore, first of all submit it to a thorough analysis. As said in my previous Investigation, this distinction, which first showed up in the field of the descriptive psychology of sense-data, could be looked on as a special case of a universal distinction. It extends beyond the sphere of conscious contents and plays an extremely important role in the field of *objects as such*. The systematic place for its discussion should therefore be in the *pure (a priori) theory of objects as*

> *such*, in which we deal with ideas pertinent to the *category of object*. (Husserl (1970), v.2, p. 3)

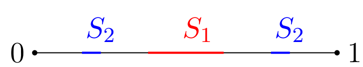
Sadly, though, Chisholm in his (1994) paper, in which he tries to present a thorough account of this distinction between dependent and independent entities, distanced himself greatly from the Brentanian account by simply abiding to the, as we said, now pretty much standard view in the literature according to which boundaries are particulars. Surely, one has to take into consideration that Chisholm's goal in this second paper was *not* to recapture Brentano's position, but to use the latter's ideas to come up with a new account that was indeed properly his own. However, I believe that, in mentioning Brentano as he was presenting his views, together with the fact that many crucial concepts in his new account do indeed resonate strongly with Brentano's position, this led to a misrepresentation in the contemporary philosophical community of Brentano's own position regarding continua and their dependent universal-like boundaries.

6. Tension between gunky spaces and measure theory

Besides the possible objections discussed above, there is one more that is to be found in the literature against the type of point-free topology we have recognized in Brentano's account. Indeed, in his (2012) paper, Arntzenius suggests the general result that any attempt to construct a countably additive measure on a point-free space would yield inconsistencies. Indeed, according to him,

> [t]he main problem is that on the topological approach to gunk one is ignoring differences that are much too large to be ignored, and the main idea of this topological approach is that differences between pointy regions that differ only on their boundaries are not real differences. This seems a fairly intuitive and reasonable idea when one considers only pointy regions which are islands, or finite collections of such islands, since their boundaries are 'small'; that is, they have measure 0. But as soon as one realizes that certain countable collections of islands, such as the Cantor Archipelago, have boundaries that have non-zero—indeed, arbitrarily large—measure, the whole idea no longer seems plausible. (Arntzenius (2012), p. 145)

The idea here is that countable additivity is not usually retained by point-free topological constructions. His counterexample is this region he calls "the Cantor Archipelago" which, although poorly defined in his aforementioned paper, is indeed a valid counterexample. A better, albeit more abstract, construction of the infringing region can be found in Russell (2008). Essentially, to carry out the construction, we start with $S_0 = [0,1]$. Then, we define S_1 to be the $\frac{1}{4}$ long interval that has the point $\frac{1}{2}$ as its middle point. S_1 then splits its complement into two equal length segments. Take the midpoint of each of these segments and define S_2 to be the union of both $\frac{1}{16}$ long intervals whose midpoints are the midpoints of these two segments composing the complement of S_1. After doing this, the complement of the union of S_1 and S_2 will have four disjoint equal sized parts. Take the midpoints of each of these parts and define S_3 as the union of all $\frac{1}{64}$ long intervals that have as their midpoints one of the mentioned four midpoints. Because the size of the ever larger collection of intervals keeps getting smaller by a factor of 2, this construction does not end in a finite number of steps, but we can take the countable union of these segments and this is what Arntzenius calls "the Cantor Archipelago":



Now, the problem is that, it seems plausible that any reasonable mereological structure would have some sort of fusion or union operator that satisfies the intuition that the fusion or union of all the Cantor sets S_n, $n > 0$, is precisely the interval S_0 we started with. However, while our intuition further stipulates that this interval has measure 1, the sum of the usual measures of each S_n, $n > 0$, would be

$$\frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}.$$

We shall not go further into the technical details of this here, but we should note that, whereas the earlier arguments against a Brentanian approach simply failed to appreciate some important details about it, this one seems to point at a truly deeper incompatibility between point-free accounts of topology and countably-additive measures.

On the other hand, if one remembers that Brentano's goal was not to understand the abstract concept of a continuous space in general but to present an account of a particular class of continuous spaces that are presented to us throughout our daily intuitions, then it might not seem too strange to restrict the respective notion of mereological union simply to a finite class of regions and to reject infinite such fusions.

7. Teleosis

Another seemingly novel notion introduced by Brentano in the context of this account of continua and boundaries is the notion of teleosis, which is introduced originally in relation to the temporal dimension that would have to be recognized in corporeal 3-dimensional objects if one were to be able to talk about change regarding these objects. Indeed, the very word "teleosis" is paraphrased by Brentano as "velocity of change". Moreover, Brentano believes that this notion of teleosis is much more important and widespread. In his view, this is

> a concept which is of the utmost importance for the understanding of the continuum and in particular for the phenomena which occur in the case of multiple continua. (1988, p. 18)

However, it seems that his own paraphrase of the concept of teleosis as "velocity of change" is, on the one hand, not quite that accurate and, on the other, indicative of the fact that this notion is not that revolutionary. Indeed, the notion of telosis does have something to do with how fast some property relating to the continuum is changing, but the way that Brentano speaks seems to imply that the teleosis of a certain continuum is not its velocity of change *per se*, but something like the inverse of this velocity. In fact, he says the following:

> Imagine a sphere which is at one moment fully at rest in a given place, at a later moment rotating within the boundaries of this place. In the first case it is clear that in regard to its temporal dimension the sphere exists in full local teleosis. [...] The local teleosis in the temporal dimension appears therefore to be incomplete in the case of a rotating sphere, and this incompleteness increases with the velocity of rotation. (*Ibid.*, p. 21)

From this passage, we can conclude that he ascribes *full* teleosis to a the stationary sphere, i.e., to the one whose "velocity of change" is zero, and *incomplete* teleosis to the one that rotates, thus having non-zero "velocity of change". Moreover, in relation to the example we considered at section 3.4.3, he says that

> [t]he moment of rest will be a moment of rest in the same sense in which a red line in a coloured surface which varies in regular fashion from pure blue to pure red and then back to pure blue is truly to be called pure red. It remains however no less correct that it is still to be distinguished from a pure red which belongs as internal boundary to a purely red surface. Indeed, certain differences will still exist in the manner in which the momentarily resting body is at rest, according to whether the motion which leads thereto and departs therefrom is subject to a more or less strong acceleration. [...] When it rests momentarily it is truly at rest, but at rest in a more incomplete teleosis than when it remains at rest for a period of time. And the completeness of teleosis in what is momentarily at rest decreases also if the motion to which it leads in infinitesimal fashion accelerates more powerfully from the very start. (*Ibid.*, p. 22)

The idea here is the following: let us consider a given primary continuum, which for simplicity will be be a line ab. Then, we can build on top of this primary continuum a plethora of different secondary continua, which can all however be formally described by a certain real-valued continuous function f on the points — i.e., on all the possible inner or outer boundaries — of the original primary continuum. More concretely, this function might stipulate the color of that particular point, or its temperature etc. Now, this function is — apart from the assumption of, at least, (usual, mathematical) piece-wise continuity — completely arbitrary, so that we can consider two radically different possibilities. First, let us assume that this function f is constant. This is, for instance, akin to the higher-dimensional case of the "purely red surface". In this case, we shall have full teleosis for every point on the line with respect to this preassigned function. Moreover, every point with full teleosis with respect to some function f must be part of a finitely extended region in which the function in question is constant, so that we can draw a parallel — indeed, a sort of equivalence — between the notion of "having full teleosis with respect to some property" and the notion of "having zero derivative with respect to some function". The latter is a mathematical characterization of the former, and the former is an intuitive appreciation of the latter.

On the other hand, we can think about the function f as varying. If this variation is smooth, then we characterize it by means of a differentiable function, i.e., by means of a function that possesses a derivative for each of the points of the primary continuum. This is the example of "a coloured surface which varies in regular fashion from pure blue to pure red and then back to pure blue". To spell it out, this case can be modeled in our simplified case of a 1-dimensional primary contiunuum by a function taking values on the interval $[0,1]$ — 0 standing for blue, 1 for red and all the numbers in between for the infinite hues in between — such that $f(a) = 0$, $f(c) = 1$ and $f(b) = 0$, where c is the midpoint between a and b — ab being a curve that cuts through the surface in the direction in which the color change is taking place.

Let us note, again, how the notions of teleosis, on the one hand, and of derivative, on the other, are inversely correlated. Having *full* teleosis is equivalent to having *zero* derivative. However, otherwise these two notions seem to serve the same role in the characterization of secondary continua, which are modeled by (piece-wise) continuous functions on the primary continuum. Indeed, a further clue that this is indeed so is the fact that, in the canonical application of the differential calculus to classical mechanics, the derivative of a function is precisely called something like its "velocity of change".[22] Therefore, since Brentano himself paraphrases his notion of teleosis as "velocity of change", it seems clear that this relation with the derivative of mathematical analysis is, if not something that Brentano had explicitly in his mind when providing his account, certainly a logical fact that allows us to say at least that, although not fully realized by Brentano himself, his account seems to rediscover the usefulness of the notion of the derivative for the study of continua, which is, for instance, the background assumption for the discipline of differential geometry.

Of course, the notion of a derivative is much more formally accurate and technically oriented than Brentano's notion of teleosis, but the latter certainly seems to have as goal to capture, in the restricted domain of continua that can be given in intuition, the same kind of property that is also captured by the much more abstract notion of a derivative.[23] This again, confirms the overall view of the project as a very interesting account given by a very capable thinker who had some, but nonetheless did *not* have much intimacy with the new mathematical developments of the century that preceded his life.

[22]In fact, in the case in which a certain function portrays the position of some particle, its derivative is *literaly* identified with its velocity.

[23]Perhaps a further constraint to consider here is that we are only considering *real valued* functions for the secondary continua. An interesting line of inquiry to pursue, might be to understand if and how this generalizes for, e.g., functions having values in $\mathbb{R}^n$ or some even more general space.

8. Conclusion

Brentano's account of continua is, as we mentioned in the beginning, essentially an account of something like "the continua that are presented to us in spatio-temporal intuition" and *not* an abstract and fully general account of what it is to be a continuum. Thus, with the help of the later developments in mathematical topology, we can characterize his account, not as a brand new conception of topology as a general discipline, but certainly as a very interesting and self-consistent account of something like the "real" or "phenomenological" topology of the "real" or "phenomenologically present" continua in spatial-temporal intuition. Going back to our discussion in the second chapter, we can further clarify this claim as the statement that Brentano's account is not at the level of the definition of a topology as the collection of all open sets satisfying some axiomatic properties, but at the level of the choice one has to make as to what is going to be the "usual topology" that we have to use in order to characterize the actual continua that are given to us in spatio-temporal intuition. And one must certainly say that, in this restricted scope, it surely succeeds as a very interesting and consistent account, having many interesting connection points with issues in ontology, such as e.g. issues regarding the notion of contact.

In this respect, we can locate Brentano's theory on the same logical region as the one proposed by, e.g., N. M. Gotts, J. M. Gooday, and A. G. Cohn (1996). In this paper, they describe their account as follows:

> The main rationale for this project is that qualitative descriptions of spatial properties and relationships, and qualitative spatial reasoning, are of fundamental importance in human thinking about the world: even where quantitative spatial data are most important (as for example in architecture, engineering and medicine), they must be attached to the components of a perceived spatial structure (of walls and floors, girders, bodily organs or cells) if we are to make use of them. RCC theory covers other qualitative aspects of spatial description and reasoning (in particular, it deals with the notion of convexity), but the topological properties and relations of spatially extended entities are fundamental to our work. The topological formalisms used by mathematicians are, in general, not well suited to the task of formalising the kinds of 'common-sense' or 'everyday' qualitative spatial description and reasoning which are our primary interest. Nevertheless, we

> must come to grips with the concepts of topology as practised by mathematicians if we are not to risk constantly 'reinventing wheels'. (1996, p. 51)

Also, a few pages later, they come back to this point, saying that

> topology as developed by mathematicians over the last century is relatively remote from the kinds of spatial reasoning employed in everyday life. Its central aims are to investigate, and prove results concerning the properties of entire classes of spaces, while ours are to find perspicuous ways to represent and reason about the topological properties and relations of spatially extended entities embedded in the space of everyday experience. (*Ibid.*, p. 53)

I believe we should take this point seriously, not to tear down the edifice of mathematical topology, but to understand that its scopes are *different* from a thorough characterization of how continua are presented in our spatio-temporal intuition, which, as this 22 year old paper seems to suggest, is still very open to further development.

A similar opinion is voiced by Russell (2008). In it, he says that

> [a] third reason to investigate gunky space is to formalize the psychology of space. Regardless of whether phenomena like perfect contact are physically or even metaphysically possible, they surely play into "common sense" spatial reasoning. Rigorously capturing this kind of reasoning is important for formal semantics, cognitive science, and artificial intelligence. (p. 250)

Moreover, I believe Brentano's account of continua can surely serve itself as a very solid starting point from whence such a development can take place. Thus, a possible interesting development of the work presented in this monograph would be an attempt at a formal reconstruction of this account as to further understand its logical relationship with the formal mereotopological theories that have been proposed in the literature. This will perhaps serve as the topic of a future paper on the issue, presenting how the Brentanian intuitions regarding these notions can serve as a background framework for interesting formal point-free versions of a topology for space.

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