Jones grading from symplectic Khovanov homology

Zhechi Cheng

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2020

Abstract

Jones grading from symplectic Khovanov homology

Zhechi Cheng

Symplectic Khovanov homology is first defined by Seidel and Smith as a singly graded link homology. It is proved isomorphic to combinatorial Khovanov homology over any characteristic zero field by Abouzaid and Smith.

In this dissertation, we construct a second grading on symplectic Khovanov homology from counting holomorphic disks in a partially compactified space. One of the main theorems asserts that this grading is well-defined. We also conclude the other main theorem that this second grading recovers the Jones grading of Khovanov homology over any characteristic zero field, through showing that the Abouzaid and Smith's isomorphism can be refined as an isomorphism between doubly graded groups. The proof of the theorem is carried out by showing that there exists a long exact sequence in symplectic Khovanov homology that commutes with its combinatorial counterpart.

Table of Contents

List of Figures

Acknowledgements

First and foremost, I sincerely thank my advisor, Mohammed Abouzaid, for his continuous support of my research. Mohammed is always willing to share his profound knowledge and insights, whenever help is needed during my study. This thesis would never have come to fruition without Mohammed's patience and enthusiasm.

I also want to express my gratitude to my dissertation committee. I thank Kristen Hendricks, Mikhail Khovanov, Francesco Lin, and Robert Lipshitz for agreeing to join my dissertation defense committee and giving helpful comments on an earlier draft of the thesis.

I learned a lot from the graduate courses in Columbia. I thank Mohammed Abouzaid, Robert Lipshitz, Melissa Liu, Andrei Okounkov, and Duong Phong for all the wonderful lectures. In particular, I am grateful to Robert, from whom I learned the beauty of topology through our discussions on various topics in low-dimensional topology.

I benefited a lot from discussing with researchers in the symplectic geometry and low-dimensional topology community. I thank the group in Columbia, including Akram Alishahi, Robert Castellano, James Cornish, Nate Dowlin, Ailsa Keating, Linh Truong, Sara Venkatesh, Mike Wong, and Jingyu Zhao, for sharing their knowledge. I would also like to thank researchers around the world, including Matthew Hedden, Adam Levine, Ciprian Manolescu, Sucharit Sarkar, Ivan Smith, Jiajun Wang, and Chris Woodward for helpful discussions regarding my research.

I also appreciate the administrative staff of the Columbia University math department, including Shay Bailey, Terence Cope, Deniz Macleod, Crispina Pincus, Alenia Reynoso, and Nathan Schweer for maintaining the department.

Finally, I would like to thank my family for their selfless support and immovable faith in my research, no matter how obscure my research might appear to them.

Dedication

To my parents Fuxiang Cheng and Minjun Huang.

Chapter 1: Introduction and background

1.1 Background

In [1], Seidel and Smith defined a link invariant $Kh_{symp}(L)$. It is the Lagrangian intersection Floer homology of two Lagrangians in a symplectic manifold $\mathcal{Y}_n$. The manifold $\mathcal{Y}_n$ is built through taking a fiber of the restriction of the adjoint quotient map $\chi : \mathfrak{sl}_{2n}(\mathbb{C}) \to Conf^0_{2n}(\mathbb{C})$ to a nilpotent slice $\mathcal{S}_n$.

Each link L in S^3 can be realized as a braid closure $\beta_L \in Br_n$ and $\beta_L \times id \in Br_{2n}$ gives a path in the base $\mathbb{C}^{2n-1}$. The parallel transport induces a symplectomorphism $(\beta_L \times id)_*$of $\mathcal{Y}_n$ to itself. There is a distinguished Lagrangian submanifold $\mathcal{K}$ given by iterated vanishing cycles and $Kh_{symp}(L)$ is defined to be the following singly graded Floer homology group

$$Kh^*_{symp}(L) = HF^*(\mathcal{K}, (\beta_L \times id)_*(\mathcal{K})) \tag{1.1.1}$$

called *symplectic Khovanov homology*. There is a conjectural relation between Kh^*_{symp} and the combinatorial Khovanov homology $Kh^{*,*}$:

Conjecture 1.1.1 (Seidel-Smith, [1])**.** *For any link $L \subset S^3$, $Kh^k_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L)$.*

This was proved over any characteristic zero field $\mathbf{k}$ by Abouzaid and Smith, see [2]. For the case of non characteristic zero fields, only a few example have been computed for which the theories are isomorphic, see [1, Proposition 55] for the case of trefoil and the methods of [1] apply more generally to the $T_{2,n}$ torus link.

1.2 Defining weight grading

The rest of the paper relies on the result of Abouzaid-Smith that $Kh^k_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L)$ is true for characteristic zero fields, so we will assume **k** has characteristic zero, unless noted otherwise. Abouzaid and Smith construct an endomorphism ϕ of $CF^*(\mathcal{K}, (\beta_L \times id)_*(\mathcal{K}))$, which induces a generalized eigenspace decomposition of $HF^*(\mathcal{K}, (\beta_L \times id)_*(\mathcal{K}))$, see also [3]. The eigenvalues give an additional grading on $HF^*(\mathcal{K}, (\beta_L \times id)_*(\mathcal{K}))$, called the *weight grading*. One of the main theorems of this paper is the proof of the following result which appeared as a conjecture in [2]:

Theorem 1.2.1. *The relative weight grading on $Kh^*_{symp}(L)$ is independent of the choice of link diagram.*

The result will enable us to bigrade $Kh^*_{symp}(L)$. The computation in [3] shows that the relative weight grading coincide with the Jones grading for unlinks. The weight grading is only well-defined as a relative grading because an auxiliary choice will need to be made on each Lagrangian called *equivariant structure*. Changing such structures will shift the overall weight grading by a constant. Abouzaid and Smith point out that for unlink diagrams, there are canonical choices of equivariant structures such that the weight grading of each generator equals to its homological grading. But the author presently does not know of a canonical choice of equivariant structures in an arbitrary diagram to make it a well-defined absolute grading.

The construction of the relative weight grading is worked out in the framework of Manolescu's Hilbert scheme point of view. The space $\mathcal{Y}_n$ is an open subscheme of $Hilb^n(A_{2n-1})$, the n-th Hilbert scheme of the Milnor fibre of A_{2n-1}-surface singularity. One of the advantages of this point of view is that instead of working with braid closures, we can work with *bridge diagrams*, which are diagrams obtained by breaking the link diagrams into n overcrossing α arcs and n undercrossing β arcs. These arcs give two Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ in $\mathcal{Y}_n \subset Hilb^n(A_{2n-1})$. It is shown in [4] that $Kh^*_{symp}(L) = HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ for a specific type of diagram, called a *flattened braid diagram*. Through out this paper, we always assume that our bridge diagrams are oriented.

Independence of choices of bridge diagrams of Kh^*_{symp} is proved by Waldron in [5, Section 6] and Hendricks-Lipshitz-Sarkar in [6, Section 7]. A downside of the Hilbert scheme viewpoint is that the absolute grading is only constructed for a flattened braid diagram but not preserved under some Reidemeister moves. So we will also prove the following theorem, as a refinement of Waldron and Hendricks-Lipshitz-Sarkar's result to an absolute grading or a rephrasing of Seidel-Smith's result in the bridge diagram point of view.

Theorem 1.2.2. *[Seidel-Smith [1]] If we denote w to be the writhe of the diagram, rot to be its rotation number and h_0 be the homological grading of a distinguished generator. Then*

$$Kh^*_{symp}(L) = HF^{*+rot+w+h_0}(\mathcal{K}_\alpha, \mathcal{K}_\beta) \tag{1.2.1}$$

is a link invariant.

In constructing weight grading, we have a choice of equivariant structure $c_{\mathcal{K}} \in CF^0(\mathcal{K}, \mathcal{K})$ for each Lagrangian L, see [3, Definition 2.10] and Definition 4.1.4. According to [3, Lemma 2.12] and the fact that $HF^*(\mathcal{K}, \mathcal{K})$ is one dimensional for our Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, the choices of equivariant structures will serve as overall grading shifts. So we can rephrase Theorem 1.2.1 as

Theorem 1.2.3. *Given two equivalent bridge diagrams and a pair of equivariant structure on Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ associated to one of the diagrams, there exist a pair of equivariant structure on the Lagrangians $\mathcal{K}'_\alpha$ and $\mathcal{K}'_\beta$ associated to the other bridge diagram, such that the weights on the two Floer groups are equal.*

In other words, if we treat choices of equivariant structures as overall weight grading shifts on weight gradings, we could always define a relative weight grading for a link which does not depend on the diagram we choose. The advantage of the bridge diagram point of view is that we can now prove that this grading is an invariant of bridge diagrams instead of braids, and thus we can mimic the proof in [6]. It is not hard to see that any two bridge diagrams of the same link are connected by a series of

- Isotopies
- Handleslides
- (De)stabilizations

using the terms of [6, Section 7.4]. Isotopies and handleslides are Hamiltonian isotopies of Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, see [1, Lemma 43] so they can be dealt with together. Invariance under Hamiltonian isotopies can be proved using the fact that the Floer product preserves the weight grading, see [3, Equation 3.94], [7, Remark 4.4] and Proposition 4.2.1.

For stabilization invariance of the grading, there is a technical issue when we attempt to mimic the proof of stabilization invariance of Kh_{symp} in [6, Section 7.4]. We will point out an outline of a direct proof of stabilization invariance in the thesis and the technical issue within the proof. The proof could lead to a proof of invariance of *reduced symplectic Khovanov homology* of pointed links, which is stated as a conjecture in [4]. The stabilization invariance will not be used in the rest of the thesis, but instead Theorem 1.2.3 will be proved as a corollary of Theorem 1.3.2.

1.3 Recovering Jones grading

After defining the second grading, we prove that this bigrading recovers the Jones grading (or quantum grading in some contexts) of Khovanov homology. Abouzaid-Smith's purity result leads to a computation for unlink:

Proposition 1.3.1 (Abouzaid-Smith, [3])**.** *If $L \subset S^3$ is an unlink, there exists a choice of equivariant structures on Lagrangians such that for any element $x \in Kh^k_{symp}(L)$, $wt(x) = k$.*

It is well-known that Khovanov homology of an unlink of n components is $V^{\otimes n}$, where V is supported in $(i, j) = (0, 1), (0, -1)$ that i stands for homological grading and j stands for Jones grading. So if we compute $i - j$ we see V is supported in $(i - j, j) = (-1, 1), (1, -1)$. This example leads to the following theorem:

Theorem 1.3.2. *Symplectic Khovanov homology and Khovanov homology are isomorphic as bigraded vector spaces over any characteristic zero field, where the gradings are related by* $k = i - j$ *and* $wt = -j + c$*, where* k *is the homological grading,* wt *is the weight grading and c is a correction term of the relative weight grading.*

To prove this theorem, we show that Abouzaid-Smith's a long exact sequence of symplectic Khovanov homology groups, see [2, Equation 7.9] decomposes with respect to the weight grading. In other words, if we fix a weight grading wt_1 of the first group, the only non-trivial map can happen between a single weight grading wt_2 of the second group and wt_3 of the third group.

$$\ldots \to HF^{*,wt_1}(L_+) \to HF^{*,wt_2}(L_0) \to HF^{*+2,wt_3}(K_\infty) \to HF^{*+1,wt_1}(L_+) \to \ldots \quad (1.3.1)$$

The proof of Theorem 1.3.2 is a proof based on bridge diagrams of links, such that it does not depend on the stabilization invariance of the weight grading. In fact, after showing that weight grading relatively recovers Jones grading for any bridge diagram, we conclude Theorem 1.2.3.

1.4 Organization

The paper is organized as follow: We review the moduli spaces that we use to build the weight grading in Chapter 2. We recall the definition of the invariant through the Hilbert scheme point of view, construct an absolute grading on the Floer group and prove Theorem 1.2.2 in Chapter 3. We then define the weight grading in Chapter 4 and prove its Hamiltonian invariance, and we give an outline of a proof of stabilization invariance, which could lead to a direct proof of Theorem 1.2.3. In Chapter 5, we show that there exist a long exact sequence of symplectic Khovanov homology and then prove Theorem 1.3.2 and Theorem 1.2.3 in Chapter 6.

Chapter 2: Geometry generalities

2.1 Hochschild cohomology

We start with the case of $\mathcal{A}_\infty$-algebra. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded $\mathcal{A}_\infty$-algebra over a field $\mathbf{k}$ and $\mathcal{A}[d]$ be a copy of $\mathcal{A}$ with all gradings shifted downward by d. It is equipped with a product structure

$$\mu^d : \mathcal{A}^{\otimes d} \to \mathcal{A}[2-d] \tag{2.1.1}$$

that satisfies the $\mathcal{A}_\infty$ relation, i.e.

$$\mu^1\mu^1 = 0 \tag{2.1.2}$$

$$\mu^2(\mu^1, -) + \mu^2(-, \mu^1) + \mu^1\mu^2 = 0 \tag{2.1.3}$$

and so forth.

Then we can define the Hochschild cochain complex as

$$CC^d(\mathcal{A}, \mathcal{A}) = \prod_{s\geq 0} Hom_d(\mathcal{A}[1]^{\otimes s}, \mathcal{A}) \tag{2.1.4}$$

We can define a differential

$$\delta : CC^{d-1}(\mathcal{A}, \mathcal{A}) \to CC^d(\mathcal{A}, \mathcal{A}) \tag{2.1.5}$$

$$\begin{aligned}(\delta\sigma)^d(a_d, \ldots, a_1) = {} & \Sigma_{i,j}(-1)^{|\sigma|-1}\mu^{d-j+1}(a_d, \ldots, \sigma^j(a_{i+j}, \ldots, a_{i+1}), \ldots, a_1) \\ & + \Sigma_{i,j}(-1)^{|\sigma|}\sigma^{d-j+1}(a_d, \ldots, \mu^j(a_{i+j}, \ldots, a_{i+1}), \ldots, a_1)\end{aligned} \tag{2.1.6}$$

The cohomology $H^*(CC^d, \delta) = HH^*(\mathcal{A}, \mathcal{A})$ is called Hochschild cohomology of the $\mathcal{A}_\infty$-algebra $\mathcal{A}$.

Now we turn to $\mathcal{A}_\infty$-categories. All concepts are analogous, when we keep track of the source and domain of morphisms. Let $\mathcal{A}$ be an $\mathcal{A}_\infty$-category over a field $\mathbf{k}$. Given $X_i \in Ob(\mathcal{A})$, $\mathcal{A}$ is equipped with a series of products

$$\mu^d : \bigotimes Hom_{\mathcal{A}}(X_i, X_{i+1}) \to Hom_{\mathcal{A}}(X_1, X_{d+1})[2-d] \tag{2.1.7}$$

which satisfy the same relation as Equation 2.1.2.

We define the Hochschild cochain complex of degree r as the space of linear maps

$$\sigma^d : \bigotimes Hom_{\mathcal{A}}(X_i, X_{i+1}) \to Hom_{\mathcal{A}}(X_1, X_{d+1})[r-d] \tag{2.1.8}$$

. The differential is given by the same formula as Equation 2.1.5, except that the inputs are sequences of morphisms.

Now we have the Hochschild cohomology $HH^*(\mathcal{A}, \mathcal{A})$ of an $\mathcal{A}_\infty$-category. In this paper we do not want to study the cohomology in general. Instead, we only look at an important class in the cohomology: an *nc-vector field* in [3, Definition 2.3].

Definition 2.1.1. *An nc-vector field is a cocycle $b \in CC^1(\mathcal{A}, \mathcal{A})$*

Keep in mind that b consists a series of morphisms of different degrees and different number of inputs. But we are interested in its linear part $b^1 = Hom_0(X_1, X_2)$, which is an endomorphism that preserves the grading.

2.2 Seidel-Solomon moduli spaces

In this section, we study the moduli space Seidel and Solomon used in [7] to define the q-intersection number. It is also used in [3] to define the weight grading. Let $\mathcal{R}^{k+1}_{(0,1)}$ be the moduli space of holomorphic classes of closed unit disks with the following additional data:

- two marked points z_0 and $z_1 \in (0,1)$.

- $k+1$ boundary punctures at $p_0 = 1$ and k others placed counter-clockwise.

This space has a natural compactification of Deligne-Mumford type with the following codimension one boundary components:

- Some boundary marked points move together. This will provide a boundary disk bubble. Notice that the interior marked points could lie on either component. When two are on the same component, we have

$$\mathcal{R}^{k-i+1}_{(0,1)} \times \mathcal{R}^{i} \tag{2.2.1}$$

  where $\mathcal{R}^i$ is the moduli space of disks whose domains consist of the closed unit disk with $k+1$ boundary punctures and no interior marked point, while the case of different components will lead to

$$\mathcal{R}^{k-i+1}_{1} \times \mathcal{R}^{i}_{1} \tag{2.2.2}$$

  where the subindex 1 in $\mathcal{R}^i_1$ means a single interior marked point.

- Two interior marked points move together, i.e. $z_1 \to 0$. This will give a sphere bubble carrying both interior marked points:

$$\mathcal{R}^{k+1}_{1} \times \mathcal{M}_{0,3} \tag{2.2.3}$$

  where $\mathcal{M}_{0,3}$ is the moduli space of spheres of 3 interior marked points.

Before we study maps from these domains to a symplectic manifold, we first introduce some hypotheses on our manifold M. We start with a smooth projective variety $\bar{\bar{M}}$ with three reduced effective divisors D_0, D_∞ and D_r. Let $\bar{M} = \bar{\bar{M}} \backslash D_\infty$ and $M = \bar{M} \backslash (D_r \cup D_0)$. We will mostly study $\bar{M}$ so for most of the cases we will not talk about D_∞ later and thus D_0 and D_r actually means their restriction in $\bar{M}$ or M depending on the context. We assume:

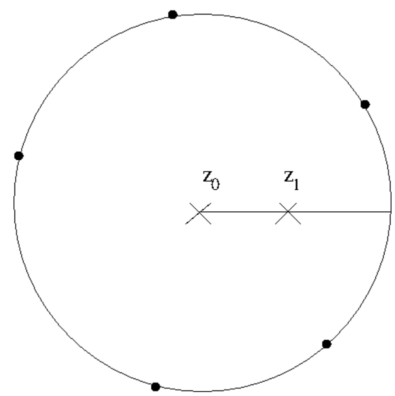

Figure 2.1: A representative of $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$. $z_0 = 0$ and z_1 lies on $(0, 1)$.

- (H-1) the union of $D_0 \cup D_\infty \cup D_r$ supports an ample divisor D with strictly positive coefficients of each of these three divisors.
- (H-2) D_∞ is nef.
- (H-3) $\bar{M}$ admits a meromorphic volume form which is non-vanishing on M, holomorphic along D_r and having simple poles along D_0.
- (H-4) Each irreducible component of D_0 moves in $\bar{M}$, with base locus containing no rational curves.

With (H-4), we let D_0' be a divisor linearly equivalent to D_0 through moving it. Then D_0 and D_0' share no irreducible component by hypothesis. One can think D_0' as a push-off of D_0 along its normal direction within a small neighborhood and $B_0 = D_0 \cap D_0'$, which is a subvariety of complex codimension 2. The hypothesis above are crucial to computing the Gromov-Witten invariants of $\bar{M}$, see [3, Lemma 6.8]. We further denote the Kähler form $\omega_{\bar{\bar{M}}}$ in the class Poincaré dual to D and J for the natural complex structure on $\bar{\bar{M}}$, $\bar{M}$ and M.

Then we can study pseudo-holomorphic maps with respect to J from domains in $\mathcal{R}^{k+1}_{(0,1)}$ to the symplectic manifold $\bar{M}$. Fix Lagrangians $L_0, \ldots, L_k$ and $x_i \in L_i \cap L_{i-1}$ (where $x_0 \in L_0 \cap L_k$). We can define $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$ to be the moduli space of finite energy maps

$$u : \mathbb{D} \to \bar{M} \tag{2.2.4}$$

with homology class

$$[u] \cdot [D_r] = 0 \tag{2.2.5}$$

and

$$[u] \cdot [D_0] = 1 \tag{2.2.6}$$

The boundary marked points p_i are mapped to x_i, respectively. The arc between p_i and p_{i+1} is mapped to a Lagrangian L_i. The interior marked points z_0 and z_1 are mapped to D_0 and D_0', respectively. If we have a complex volume form ita on $\bar{M}$ and L_i are exact oriented Lagrangians with spin structures, then we can define an absolute grading $deg(x_i)$ for each x_i. We know the expected dimension of the moduli space through the following proposition:

Proposition 2.2.1. *[3, Lemma 3.16] The virtual dimension of* $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$ *is*

$$k - 1 + deg(x_0) - \sum_1^k deg(x_i) \tag{2.2.7}$$

Proof. We start with the Fredholm index of the required disk

$$2 + deg(x_0) - \sum_1^k deg(x_i) \tag{2.2.8}$$

. Having $k+1$ boundary marked point will increase the virtual dimension by $k+1$, while at two interior marked point will decrease this number by 4. By considering all these contributions, we get the desired result. ☐

It is also important to study boundary strata of $\bar{\mathcal{R}}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$. First we claim that the term in the Equation 2.2.2 is empty.

Proposition 2.2.2. *In the boundary strata of* $\bar{\mathcal{R}}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$*, the component lying over* $\mathcal{R}^{k-i+1}_1 \times \mathcal{R}^i_1$ *is empty, i.e. two interior marked point cannot lie on different components of the domain.*

Proof. If they are on distinct components of the domain,since two interior marked points are mapped to D_0 and D_0', respectively, both components have non-trivial intersection with D_0. So by positivity of intersection, we conclude that the intersection number of any map in $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$ with D_0 is bigger than 1, a contradiction. □

The rest of the cases that could contribute to boundary and these spaces are regular, see [3, Lemma 3.18, 3.19]. There are disc bubbles for $z \in (0, 1)$.

$$\coprod_x \bar{\mathcal{R}}^{k-k_1+1}_{(0,1)}(x_0; x_k, \ldots, x_{i+k_1+1}, x, x_i \ldots, x_1) \times \mathcal{R}^{k_1+1}(M|x; x_{i+k_1}, \ldots, x_{i+1}) \tag{2.2.9}$$

and

$$\coprod_x \mathcal{R}^{k_1+1}_{(0,1)}(M|x_0; x_k, \ldots, x_{i+k_1+1}, x, x_i \ldots, x_1) \times \bar{\mathcal{R}}^{k-k_1+1}(x; x_{i+k_1+1}, \ldots, x_i) \tag{2.2.10}$$

where M indicates that disks are mapped to M not $\bar{M}$, i.e. they are disjoint from D_0 and, of course, D_r. There could also be sphere bubbles. For simplicity, we assume that the virtual dimension is less than 3. Together with regularity, we conclude that in the top dimension only one of the components, either sphere or disk, could be non-constant. Thus the top dimension strata are

$$\mathcal{M}_1(\bar{M}|1) \times_{\bar{M}} \mathcal{R}^{k+1}_1(M|x_0; x_k, \ldots, x_1) \tag{2.2.11}$$

$$B_0 \times_{\bar{M}} \mathcal{R}^{k+1}_1(\bar{M}; (1,0)|x_0; x_k, \ldots, x_1) \tag{2.2.12}$$

where $(1, 0)$ indicates the intersection number with D_0 is 1 while D_r is 0.

It is important for us to understand moduli spaces like $\mathcal{R}^{k+1}_1(M|x_0; x_k, \ldots, x_1)$. Consider the following evaluation map to the interior marked point:

$$ev : \mathcal{R}^{k+1}_1(M|x_0; x_k, \ldots, x_1) \to M \tag{2.2.13}$$

For a Lagrangian $L \in C^*(M)$, the moduli space above will be 0-dimensional when we ask its interior marked points being mapped to L, and thus it represents the class $ev^*(L)$. Counting such 0-dimensional moduli space will give rise to a map

$$CO : C^*(M) \to CC^*(\mathcal{F}(M), \mathcal{F}(M)) \tag{2.2.14}$$

the cochain $L \in C^*(M)$ is mapped to an element in $\otimes_1^k x_i \to x_0$.

When these moduli spaces are 0-dimensional, counting these two moduli spaces defines a Hochschild cochain. It is not hard to see Equation 2.2.11 computes

$$CO(GW_1) \tag{2.2.15}$$

while Equation 2.2.12 computes

$$co(B_0) \tag{2.2.16}$$

We now can use the moduli spaces above to define an element b_D in $CC^1(\mathcal{F}(M), \mathcal{F}(M))$, a nc-vector field. First, we define $\tilde{b}_D \in CC^*(\mathcal{F}(M), \mathcal{F}(M))$ through counting disks in $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$:

$$\tilde{b}^k_D(x_k, \ldots, x_1) = \sum_{x_0} \#\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1) x_0 \tag{2.2.17}$$

. This expression is not necessarily closed, but if we consider the sum

$$b_D = \tilde{b}_D + CO(gw_1) + co(\beta_0) \tag{2.2.18}$$

we have the following proposition

Proposition 2.2.3. *[3, Proposition 3.20] The formula $b_D = \tilde{b}_D + CO(gw_1) + co(\beta_0)$ given by Equation 2.2.18 is closed.*

Proof. This is a closed because the boundary strata of $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$ will contribute two kinds of terms in $\delta\tilde{b}_D$. The disk bubbles contribute to $\tilde{b}^i_D \circ \mu^j$ or $\mu^j \circ \tilde{b}^i_D$, these vanish by A_∞

relations. The sphere bubbles contribute to $CO(GW_1) + co(B_0)$, by Equation 2.2.11 and Equation 2.2.12 and these terms are cancelled by the boundary of the two additional terms in b_D in Equation 2.2.18. Thus b_D is indeed a closed element in the Hochschild cochain. $\square$

Chapter 3: A cylindrical reformulation of Kh^*_{symp}

We will briefly review the definition of Seidel-Smith's symplectic Khovanov homology [1] in Section 3.1. Then we will turn to Manolescu's reformulation [4] with Hilbert schemes in Section 3.2. Then we will give a cylindrical reformulation of Kh^*_{symp} with Mak-Smith's cylindrical category in Section 3.3 and Section 3.4. Lastly, we will discuss the absolute homological grading in the Hilbert schemes set up in Section 3.5. The construction of this chapter is true for any field $\mathbf{k}$ of any characteristic.

3.1 Symplectic Khovanov homology from nilpotent slices

We start this chapter with a brief review of Seidel-Smith's original construction. They started their construction with the total space $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then consider the following nilpotent matrix with Jordan block size (n, n),

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

.

We can construct a slice S_n given by all matrices with the form

$$\begin{bmatrix} a_1 & 1 & 0 & \dots & 0 & b_1 & 0 & \dots & 0 \\ a_2 & 0 & 1 & \dots & 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & \dots & 1 & b_{n-1} & 0 & \dots & 0 \\ a_n & 0 & 0 & \dots & 0 & b_n & 0 & \dots & 0 \\ c_1 & 0 & 0 & \dots & 0 & d_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_n & 0 & 0 & \dots & 0 & d_n & 0 & \dots & 0 \end{bmatrix}$$

.

This is a space whose tangent space is complementary to the orbit in $\mathfrak{g}$, so it is called a *nilpotent slice* in [1].

Proposition 3.1.1. *With a nilpotent slice S_n above, the restriction of the characteristic map χ to S_n is a symplectic fiber bundle*

$$\chi|_{S_n} : S_n \to Conf_0^{2n}(\mathbb{C}) \tag{3.1.1}$$

Explicitly a matrix is mapped to its eigenvalues $\tau = (\tau_1, \dots, \tau_{2n})$, and they call its fiber at τ, $\mathcal{Y}_{n,\tau}$. The Floer homology in the definition of Kh^*_{symp} take place in $\mathcal{Y}_{n,\tau}$. Each path $\gamma \in Conf_0^{2n}(\mathbb{C})$, which connects a set of eigenvalues to another, it induces a parallel transport

$$h_\gamma : \mathcal{Y}_{n,\gamma(0)} \to \mathcal{Y}_{n,\gamma(1)} \tag{3.1.2}$$

It is worth pointing out that they did some rescaling through subtracting some Liouville vector fields on the isotopy and called it h_γ^{resc} because of the Kähler. The original parallel transport might not converge. First they defined a Kähler form $\Omega = -dd^c\phi$ on S_n, with ϕ a nice-behaved height function on S_m, so that we could define Kähler forms on fibers $\mathcal{Y}_{n,\tau}$ through restriction, so that it behaves well at the infinity. With this Kähler form Ω and the rescaling, for any Lagrangian submanifold $\mathcal{K} \subset \mathcal{Y}_{n,\gamma(0)}$, its parallel transport image $h_\gamma^{resc}(\mathcal{K})$ is well-defined up to Lagrangian isotopy.

In order to construct the Lagrangians, Seidel and Smith use a technique called *iterated relative vanishing cycle*. The rough idea is that $\mathcal{Y}_{n,\tau}$ can be locally identified with a smooth fiber of

$$\mathcal{Y}_{n-1,\bar{\tau}} \times \mathbb{C}^3 \to \mathbb{C} \tag{3.1.3}$$

where $\bar{\tau}$ is τ with first two coordinates removed and the map sends (x, u, v, z) to $u^2 + v^2 + z^2$.

Then if we consider a path γ from $\tau = (\tau_1, \tau_2, \ldots, \tau_{2n})$ to $\tau' = (0, 0, \tau_3, \ldots, \tau_{2n})$, the (rescaled) parallel transport will send a Lagrangian $\mathcal{K}$ to a product like Lagrangian $\mathcal{K}' \times S^2$ if we locally identify $\mathcal{Y}_{n,\tau'}$ with a product like space $\mathcal{Y}_{n-1,\bar{\tau}} \times S$.

So if we perform this construction, eventually we reduce the case to $\mathcal{Y}_{1,\pm 1}$, which is

$$S = \{(u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + z^2 = 1\} \in \mathbb{C}^3 \tag{3.1.4}$$

. If we think of this space as a level set of $\mathbb{C}^3 \to \mathbb{C}$, the preimage of the unit circle will provide us with a Lagrangian S^2. By induction, we now have a Lagrangian L in $\mathcal{Y}_{n,\tau}$, which is a product of S^2.

As for a link L, we choose a braid representation $\beta \in br_n$, and express it as a product of elementary generators σ_i and σ_i^{-1}. For each of the elementary braids σ_i, consider a path that swaps μ_i and $\mu_i + 1$ clockwise; and counterclockwise for its inverse σ_i^{-1}. We could think of the braid $\beta \times id \in br_{2n}$ acting on $Conf^{2n}(\mathbb{C})$, which induces a new Lagrangian $h_\beta(L)$ from the (rescaled) parallel transport. Thus Seidel and Smith obtain a link invariant through the following proposition.

Proposition 3.1.2. *[1, Theorem 1] $HF^*(\mathcal{K}, h_\beta(\mathcal{K}))$ is invariant under Markov moves on β.*

Now we can state the following definition of symplectic Khovanov homology,

Definition 3.1.3. *Let β be a braid and L_β be its braid closure, then the symplectic Khovanov homology group of L_β is*

$$Kh^*_{symp}(L_\beta) = HF^*(\mathcal{K}, h_\beta(\mathcal{K})) \tag{3.1.5}$$

The group above can be absolutely graded with a suitable correction term. Seidel and Smith give an absolute grading in section 6 of their paper [1], We will discuss this correction term in Hilbert scheme setup in the Section 3.5

3.2 Hilbert scheme reformulation

In the previous section, we defined a link invariant $Kh^*_{symp}(L)$. But it is defined through braid group actions on a high dimensional space. To make it easier to study, we need to use Manolescu's reformulation, [4], which is easier to visualize and more similar to other low dimensional invariants, such as Heegaard Floer homology.

Let us start with Hilbert schemes of points on surfaces. We choose our algebraically closed field to be $k = \mathbb{C}$ and X to be a complex variety. The Hilbert scheme of n points on X, $Hilb^n(X)$ is defined to be closed 0-dimensional subschemes of X of length n. An important part of this variety is a subvariety consisting of n distinct points. But its diagonals with points colliding together are pretty complicated. However, we have the following *Hilbert-Chow morphism* from [8]:

Proposition 3.2.1. *The Hilbert-Chow morphism π is a natural morphism from the Hilbert scheme of n points on X to the n-fold symmetric product of X such that*

$$\pi(Z) = \sum_{x \in X} length(Z_x)[x] \tag{3.2.1}$$

Moreover, if X is complex 1-dimensional, then π is an isomorphism. If X is complex 2-dimensional, then π is a resolution of singularities and $Hilb^n(X)$ is smooth.

The 1-dimensional case corresponds to examples like Heegaard Floer homology, which is defined on the symmetric product of a complex 1-dimensional surface. And in the latter case, we would expect that symplectic Khovanov homology is taking place in the Hilbert scheme of n points on a complex 2-dimensional surface. We know that Hilbert schemes and symmetric products of the same surface only differ from their diagonals so we could mimic some proofs in Heegaard Floer homology to our case, specifically the neck-stretching technique.

Now we illustrate how to construct our complex surface. Consider the following complex surface

$$S = \{(u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + p(z) = 0\} \in \mathbb{C}^3 \tag{3.2.2}$$

This is isomorphic to a fibre of nilpotent slice of block size $(1, 2n-1)$ with eigenvalues p_i in Seidel-Smith's construction. We can also think of this as the Milnor fibre of the A_{2n-1}-surface singularity. Their invariant is defined in the nilpotent slice $\mathcal{Y}_n$ of block size (n, n), which is proved to be an open subscheme of $Hilb^n(S)$, the Hilbert scheme of n points on S. Denote its complement

$$D_r = Hilb^n(S) \backslash \mathcal{Y}_n \tag{3.2.3}$$

which is a complex co-dimension 1 subvariety and, in fact, a relative Hilbert subscheme given by all elements with length less than n.

To fully characterize $\mathcal{Y}_n$, consider a projection $i : S \to \mathbb{C}$ such that $i(u, v, z) = z$, then

$$\mathcal{Y}_n = \{I \in Hilb^n(S) | i(I) \text{ has length } n\} \tag{3.2.4}$$

Manolescu proves in [4, Proposition 2.7] that $\mathcal{Y}_n$ is biholomorphic to the space $\mathcal{Y}_{n,\tau}$ obtained by a fibre of nilpotent slice by Seidel-Smith in [1].

Now we need to assign some Lagrangians to the given link. To do that, we fix a link $L \subset \mathbb{R}^3$. A *bridge diagram* D for L is a triple $(\vec{\alpha}, \vec{\beta}, \vec{p})$, where $\vec{p} = (p_1, p_2, \ldots, p_{2n})$ are $2n$ distinct points in $\mathbb{R}^2$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ are n pairwise disjoint embedded arcs and $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ are also pairwise disjoint embedded arcs such that $\partial(\cup\alpha_i) = \partial(\cup\beta_i) = \{p_1, \ldots, p_{2n}\}$ and if we let the β arcs surpass the α arcs at the intersections in $\mathbb{R}^3$, we get L.

From now on, we identify $\mathbb{R}^2$ used in describing link diagrams with $\mathbb{C}$ and let $p(z) = (z - p_1) \ldots (z - p_{2n})$.

For each arc α_i or β_i, we can build a Lagrangian sphere Σ_{α_i} or Σ_{β_i} in S through the following equation:

$$\Sigma_{\alpha_i} = \{(u, v, z) \in S | z \in \alpha_i, u, v \in \sqrt{-p(z)}\mathbb{R}\} \tag{3.2.5}$$

$$\Sigma_{\beta_i} = \{(u,v,z) \in S | z \in \beta_i, u,v \in \sqrt{-p(z)}\mathbb{R}\} \tag{3.2.6}$$

For each interior point of the arc, we have an S^1, while end points give us single points. So Σ_{α_i} and Σ_{β_i} are copies of S^2 and it is easy to see that they are Lagrangians for an appropriate choice of Kähler form

These sphere enable us to build two Lagrangians in $Hilb^n(S)$, and in fact in $\mathcal{Y}_n$ by

$$\mathcal{K}_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \ldots \times \Sigma_{\alpha_n} \tag{3.2.7}$$

$$\mathcal{K}_\beta = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \ldots \times \Sigma_{\beta_n} \tag{3.2.8}$$

Definition 3.2.2. *The symplectic Khovanov homology of the bridge diagram D is defined to be $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ in $\mathcal{Y}_n = Hilb^n(S) \backslash D_r$.*

It is worth pointing out that $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ do not intersect transversely. The interstections of these spheres are points in the fibers of the end points of the α arcs and β arcs and a S^1 in the fibers of interior intersections of those arcs. So in the Hilbert scheme, we could have some tori as intersection. To deal with this, we should perturb one of the Lagrangians, see [4, section 6.1], or use Floer theory with clean intersections, like [9]

Because $\mathcal{K}_\beta$ is a product of $\Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \ldots \times \Sigma_{\beta_n}$, it will be easier to just perturb each Σ_β in our complex surface S. Let $N = \Sigma_\alpha \cap \Sigma_\beta$ be the portion of the intersection consisting of copies of S^1 intersection in S and V be a neighborhood of N. Then according to Weinstein [10], we use the standard height function on N as a Morse-Smale function that is required by the context. Then Σ_β can be isotopied into Σ'_β such that they are identical outside of V and intersects $\Sigma_\beta \cap V$ exactly at the maximum and the minimum of our height function. And the resulting Floer chain $CF^*(\Sigma_\alpha, \Sigma'_\beta)$ will be identical to $CF^*(\Sigma_\alpha, \Sigma_\beta)$.

From now on, we always regard $\mathcal{K}_\beta$ as the original Lagrangian perturbed, so it will intersect transversely with $\mathcal{K}_\alpha$, unless otherwise noted. So intersections of the Lagrangians at midpoints of curves α_i and β_j now give the intersection of Σ_{α_i} and Σ_{β_j} at two points instead of a circle. The contribution to CF^* remains the same whether one works with perturbed Lagrangians or Floer theory with clean intersections.

Proposition 3.2.3. *[4, Theorem 1.2] $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is isomorphic to $Kh^*_{symp}(L)$.*

With this isomorphism in mind, whenever we refer to $Kh^*_{symp}(L)$ in the rest of the paper, we will always be working with the Hilbert scheme setup, unless noted otherwise.

3.3 Mak-Smith's Fukaya-Seidel cylindrical category

We need cylindrical reformulation to achieve the goal of the paper. To formulate the cylindrical reformulation, we need to modify the chain complex which defines symplectic Khovanov homology following the strategy of Mak-Smith, [11]. If we use the standard complex structure on S, and it induces an almost complex structure $J_S^{[n]}$ on $Hilb^n(S)$, and we use the standard complex structure j on $\mathbb{D}$. Normally, when we define products in Fukaya category of $Hilb^n(S)$, we consider maps $u : \mathbb{D} \to Hilb^n(S)$ in $\mathcal{R}^{n+1}(x_0; x_n, \dots, x_1)$, with $x_0 \in L_0 \cap L_d$ and $x_i \in L_{i-1} \cap L_i$ such that

$$Du^{0,1} = 0, \tag{3.3.1}$$

$$u(\partial_i \mathbb{D}) \in L_i, \tag{3.3.2}$$

$$\lim_{z \to w_i} u(z) = x_i, \tag{3.3.3}$$

where w_i are boundary mark points of $\mathbb{D}$ and $\partial_i \mathbb{D}$ is the boundary between w_{i-1} and w_i with $w_{-1} = w_n$.

We need to recall the domain-dependent family $(J_z)_{z \in \mathbb{D}}$ of almost complex structures on $Hilb^n(S)$ used in [11]. We use $\mathcal{J}^{cyl}$ to represent families of almost complex structures satisfying the following conditions:

- $J_z = j_z^n$ away from the big diagonal Δ for some ω-tame almost complex structure j_z on S. Here j_z^n on $S^{\times n}$ naturally descend to $Sym^n(S)$ which can be identified with $Hilb^n(S)$ away from the diagonal.

- $J_z = J_S^{[n]}$ when $z \in mk(\mathbb{D})$, which are the points mapped to the diagonal Δ. Here $J_S^{[n]}$ is J_S^n smoothly extended to the diagonal.

One thing worth pointing out is that the family of almost complex structures are defined on $Hilb^n(S)\backslash\Delta$ and only on diagonal when z is mapped to the diagonal. Even though it is not defined everywhere, it is enough to make sense of the holomorphic equation (3.14) for every $z \in \mathbb{D}$. To keep track of points in $mk(\mathbb{D})$ mapped to the diagonal, we consider disks with not only $(n+1)$ boundary marked points but also h interior marked points $mk(\mathbb{D})$ to record the points mapped to the diagonal. So we can now define $\mathcal{R}^{n+1,h}(x_0; x_n, \ldots, x_1)$ to be maps $u : \mathbb{D} \to Hilb^n(S)$ such that

$$u^{-1}(\Delta) = mk(\mathbb{D}), \#mk(\mathbb{D}) = h \tag{3.3.4}$$

$$Du^{0,1} = 0 \text{ with respect to } J_z \tag{3.3.5}$$

$$\text{the intersection multiplicity with } \Delta \text{ equals the multiplicity of } z \in mk(S) \tag{3.3.6}$$

$$u(z) \in L_i \text{ for } z \in \partial_i\mathbb{D} \tag{3.3.7}$$

$$\lim_{z \to w_i} u(z) = x_i \tag{3.3.8}$$

Mak and Smith proved the following propositions in their paper

Proposition 3.3.1 (Regularity)**.** *[11, Lemma 2.15] With a generic choice of the family of almost* *complex structure $J_z \in \mathcal{J}^{cyl}$ on $Hilb^n(S)$, any solution $u \in \mathcal{R}^{n+1,h}(x_0; x_n, \ldots, x_1)$ satisfying Equation 3.3.4-Equation3.21 is regular.*

Proposition 3.3.2 (Compactness)**.** *[11, Proposition 2.33] For a sequence of solutions $u_k \in \mathcal{R}^{n+1,h}(x_0; x_n, \ldots, x_1)$. Suppose there exists $T > 0$ such that*

$$sup_{\mathbb{D}_k - \nu(mk(\mathbb{D}_k))} \|Du_k\|_g < T \tag{3.3.9}$$

for all k. Then there exists u_∞ such that a subsequence of u_k converges to u_∞

This new moduli space would enable us to define an $\mathcal{A}_\infty$-category on $Hilb^n(S)$, but in order to study symplectic Khovanov homology, we should only consider disks away from a divisor D_r. The intersection number of between a disk and the diagonal cannot be less than the topological intersection number given by pairing the disk class $[u] \in H_2(Hilb^n(S), L)$ and the diagonal class $\Delta \in H_{n-2}(Hilb^n(S))$, because of positivity of intersection. Let us call this number $ind(u)$. This number $ind(u)$ can be further reduced to the intersection number within $Sym^n(\mathbb{C})$ if we consider the standard projection $\pi_S : S \to \mathbb{C}$. Let $B(\bar{S})$ be the real blow-up of $\bar{S}$ at the boundary punctures and then we can define

$$\bar{G}(u) = (Sym^n(\pi_S) \circ \pi_{HC} \circ u, id) : B(\bar{S}) \to Sym^n(\mathbb{C}) \times B(\bar{S}) \tag{3.3.10}$$

, and the intersection of the disk u with the diagonal Δ

$$int(u) = [\bar{G}(u)] \cdot [\Delta_{\mathbb{C}} \times B(\bar{S})] \tag{3.3.11}$$

.

We can then conclude:

Proposition 3.3.3. *[11, Lemma 2.21] For every $u \in \mathcal{R}^{n+1,h}(x_0; x_n, \ldots, x_1)$, we have*

$$int(u) \geq h \tag{3.3.12}$$

and the equality holds if and only if the image of u is disjoint from D_r.

So, to exclude disks that intersect Δ, we only need to consider disks with the correct number of inner marked points, equaling to the intersection number $int(u)$ given by topological datum of u. Mak-Smith define their cylindrical Fukaya-Seidel category $\mathcal{FS}^{cyl,n}(\pi_S)$ as follows:

Definition 3.3.4. *[11, Definition 2.2.11] An object of $\mathcal{FS}^{cyl,n}(\pi_S)$ is given by $L_1 \times \ldots \times L_n$, a product of Lagrangians on S such that whose projection under $\pi_S : S \to \mathbb{C}$ is pairwise disjoint. The $\mathcal{A}_\infty$-structure is given by*

$$\mu^d(x_d, \ldots, x_1) = \sum(\#u \in \mathcal{R}^{n+1,int(u)}(x_0; x_n, \ldots, x_1))x_0 \tag{3.3.13}$$

Remark 3.3.5. *We simplified the construction of Mak-Smith a little bit by omitting some discussions of Hamiltonian isotopies, because we only need compact Lagrangians in our case. Keep in mind if we have non-compact Lagrangians, we need to apply some admissible Hamiltonian isotopies at the non-compact end to define the correct $\mathcal{A}_\infty$ map. Moreover, we omitted the sign convention in the definition and this could be done by replacing x_i with some choices of orientations 0_{x_i} and do a signed count of rigid moduli space after diving out some symmetry, according to Mak and Smith in [11, Definition 2.2.11].*

The following proposition of Mak-Smith relates $\mathcal{FS}^{cyl,n}(\pi_S)$ to Fukaya-Seidel category $\mathcal{FS}^n(\pi_{\mathcal{Y}})$ of $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}$ with respect to the standard projection:

Proposition 3.3.6. *[11, Proposition 4.12 and 5.16] If $S = A_{2n-1}$, then there is a fully faithful essential surjective embedding from $D^\pi\mathcal{FS}^n(\pi_{\mathcal{Y}})$ to $D^\pi\mathcal{FS}^{cyl,n}(\pi_S)$.*

When we apply the result above to symplectic Khovanov homology that we consider compact Lagrangians given by matching paths $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, we have the following:

Corollary 3.3.7. *With a generic family of almost complex structure J_z on $\mathcal{Y}_n$, the chain complex $CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ with differentials given by counting solutions of $\mathcal{R}^{2,int(u)}(x_0; x_1)$ also computes the symplectic Khovanov homology of the given link.*

From now on, we use Mak-Smith's new moduli space $\mathcal{R}^{2,int(u)}(x_0;x_1)$ to compute symplectic Khovanov homology, unless noted otherwise.

3.4 Cylindrical reformulation

In this section, we study the cylindrical reformulation of holomorphic maps $u:\mathbb{D}\to Hilb^n(S)$. Specifically, such holomorphic maps u have a one-to-one correspondence with pairs of holomorphic maps $v:\Sigma\to S$ and a branched cover $\pi:\Sigma\to\mathbb{D}$, if we equip $Hilb^n(S)$ with the almost complex structure $J_S^{[n]}$ induced from a complex structure J_S on S. This construction in symplectic Khovanov homology was first presented in [6]. However these almost complex structures are not generic enough to achieve transversality, see an erratum [12]. With the construction in the previous section, we are ready to construct the cylindrical reformulation.

In [13], Lipshitz defines the cylindrical reformulation when S is a real 2-dimensional surface. We want to generalize it to the case when S is 4-dimensional. The only non-trivial thing here is that $\pi_{HC}:Hilb^n(S)\to Sym^n(S)$ is an homeomorphism when S is 2-dimensional but a resolution of singularities when S is of higher dimensions. Even with such differences, the rough idea of cylindrical reformulation is true as follow:

Lemma 3.4.1 (Weak tautological correspondence)**.** *Each map $u:\mathbb{D}\to Hilb^n(S)$ satisfying all Equation 3.3.4-3.3.8 uniquely determines a k-branched cover $\pi_\Sigma:\Sigma\to\mathbb{D}$ and a map $v:\Sigma\to S$ such that $\pi_{HC}\circ u=v\circ\pi_\Sigma^{-1}$.*

The map v can be made holomorphic, if we pull back the standard complex structure j on $\mathbb{D}$ via π_Σ to Σ. Then equation 3.3.5 will translate into a holomorphic condition for v. But with complex structures above, the moduli space of v will not cut out transversely. To fix this, we need to use Mak-Smith's new moduli space $\mathcal{R}^{2,int(u)}(x_0;x_1)$ instead. There are a few additional ingredients if we want to strictly phrase the cylindrical reformulation of this moduli space.

First, we should consider inner marked points on Σ that are lifts of $mk(\mathbb{D})$ of the branched cover π_S. These points should generically be away from branched points, so the number of those points gives the information of the number of elements of $mk(\mathbb{D})$ and thus gives the information whether the original map u intersects with D_r.

Second, with this k-branched cover π_Σ, each boundary puncture z_i now corresponds to k punctures $z_{i,j}$, where $j = 1, \ldots, k$. A similar statement is true for boundary segments $\partial_{i,j}$.

Last, we only consider Lagrangians given by matching paths, see section 3.2, as follow

$$\mathcal{K}_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \ldots \times \Sigma_{\alpha_n} \tag{3.4.1}$$

$$\mathcal{K}_\beta = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \ldots \times \Sigma_{\beta_n} \tag{3.4.2}$$

and $x_i = x_{i,1} \times \ldots \times x_{i,k}$ also being the product of $x_{i,k} \in \Sigma_{\alpha_{n_k}} \cap \Sigma_{\beta_{n_k}}$

Proposition 3.4.2 (Tautological correspondence)**.** *There is a one-to-one correspondence between solutions $u : \mathbb{D} \to \mathcal{Y}_n = Hilb^n(S) \backslash D_r \in \mathcal{R}^{2,int(u)}(x_0; x_1)$ and pairs of holomorphic maps $(\pi_\Sigma, v) : \Sigma \to \mathbb{D} \times S$ satisfying the following conditions:*

$$\pi \textit{ is a } k-\textit{branched covering such that } mk(\Sigma) = \pi^{-1}(mk(\mathbb{D})) \textit{ are branched points}. \tag{3.4.3}$$

$$\#mk(\Sigma) = \#mk(\mathbb{D}) = int(u) \tag{3.4.4}$$

$$Dv^{0,1} = 0 \textit{ with respect to } j_z \textit{ on } S\textit{, where } J_z = j_z^n \tag{3.4.5}$$

$$v(z) \in \Sigma_{\alpha_{n_j}} \textit{ for } z \in \partial_{0,j}\Sigma \tag{3.4.6}$$

$$v(z) \in \Sigma_{\beta_{n_j}} \textit{ for } z \in \partial_{1,j}\Sigma \tag{3.4.7}$$

$$\lim_{z\to z_{i,j}} = x_{i,j} \tag{3.4.8}$$

$$\pi_{HC} \circ u = v \circ \pi_{\Sigma}^{-1} \tag{3.4.9}$$

Proof. On one hand, it is easy to see that for each u satisfying 3.3.4-3.3.8 determines a unique pair (π_Σ, v) satisfying the above properties. Equation 3.4.4 ensures that the map u is disjoint from the divisor D_r.

On the other hand, a pair (π_Σ, v) gives a continuous map $u : \mathbb{D} \to Hilb^n(S)$, with the correct boundary condition 3.3.7 and 3.3.8. 3.4.5 proves 3.3.5, when we compare the holomorphic equations. 3.4.3 gives 3.3.4, when we relate the branched points via the branched covering map $\pi_\Sigma : \Sigma \to \mathbb{D}$. 3.4.4 shows that the disk u is disjoint from D_r. 3.3.6 only requires that we count number of branched points with multiplicity. □

Now we can instead count holomorphic maps $v : \Sigma \to S$ with the correct number of interior marked points to compute the disk contribution to the differentials in Kh^*_{symp}, so we call this moduli space $\mathcal{R}^{cyl,2,int(u)}(x_0; x_1)$.

3.5 Absolute grading

In Manolescu's reformulation, there is actually an absolute grading discussion of the Hilbert scheme view point of a flattened braid diagram as follow:

Proposition 3.5.1. *[4, Theorem 3.5] If we denote the writhe of the braid $\beta \in Br_n$ to be w, then $HF^{*+n+w}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is a link invariant.*

This is mainly a rephrasing of Seidel-Smith's main theorem in [1]. Later in [14], a combinatorial discussion of this point of view through grid diagrams is discussed by Droz and Wagner. But their grid diagram are limited to the ones obtained from a flattened braid diagram. In fact, we can make any bridge diagram into their grid diagram. Simply by starting with any (oriented)

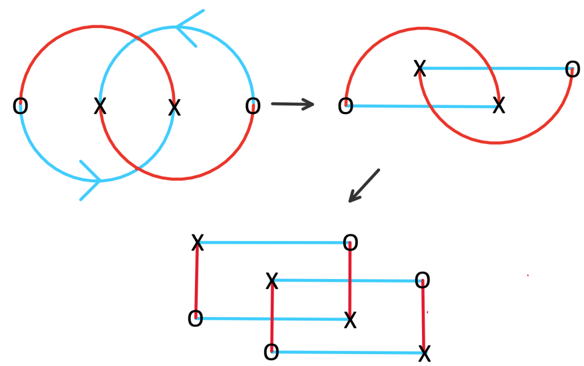

Figure 3.1: Obtaining a grid diagram from a bridge diagram of Hopf link

bridge diagram, and mark each endpoint with X and O with the rule that we will only go from X to O along α-curves and O to X along β-curves. Then with some isotopies (and stabilization if necessary) we can bring this diagram into a grid such that each row and column contains exactly one O and X and β-curves are vertical and α-curves horizontal. It is important to point out that in their paper, their homological grading is $P(x)$ or $\bar{P}(x)$, which are the same projective grading as the homological grading defined by Manolescu.

In [14], the authors define an absolute grading for any grid diagram:

Proposition 3.5.2. *[14, Theorem 3] If we denote $rot(D)$ to be the rotation number of the diagram, and $w(D)$ to be the total writhe, let x_0 be the generator whose coordinates are all X marked points then*

$$P(x) = h(x) - h(x_0) - rot(D) - w(D) \tag{3.5.1}$$

is an absolute grading for the homological grading.

We can generalize the quantities above to any bridge diagram as follow:

- Firstly, the writhe $w(D)$ of the diagram can be computed independently from grid diagram, with adding signs at each intersection.

- Secondly, the rotation number $rot(D)$ of a bridge diagram is the sum of the contributions of all Seifert circles with $+1$ for each counterclockwise component and -1 for each clockwise component.

- Lastly, the distinguished generator x_0 can also be assigned without referring to a grid diagram. We choose x_0 to be one of the generators with all coordinates at the end points such that at all corners, it goes from a β-curve to an α-curve.

Thus we conclude the following proposition from Droz and Wagner's argument:

Proposition 3.5.3. *With the assigned distinguished generator x_0, writhe w and rotation number rot of the diagram, $h(x_0)+w(D)+rot(D)$ is the correction term for computing symplectic Khovanov homology with a bridge diagram, and it equals to $n+w$ if the diagram is presented as a flattened braid diagram.*

Remark 3.5.4. *In the example of Hopf link above, $w(D)=-2$ and $rot(D)=2$. $w(D)+rot(D)=0$ so in the absolute grading, we set the homological grading of the distinguished generator that has both coordinate at X to be 0. It is not hard to compute that the gradings of elements that has one O and one X is 2, while the one with two Os is 4. The two intersections give a subcomplex of dimension 4 with dimension 1 on grading 1 and 3 and dimension 2 on grading 2. As for differential, there are differentials given the disks which project down to the left semi disk of the left circle and the right semi disk of the right circle. These differentials cancels the grading 1 and 3 generators with the two grading 2 generators with one coordinate at X and the other at O. We get $Kh^*_{symp}(L)=\boldsymbol{k}_0\oplus\boldsymbol{k}_2\oplus\boldsymbol{k}_2\oplus\boldsymbol{k}_4$, which is the same as the Khovanov homology of Hopf link with the relation $k=i-j$, whose homology consists of one dimensional subgroups on bigrading $(0,0)$, $(0,-2)$, $(-2,-4)$, and $(-2,-6)$.*

Chapter 4: Weight grading on Kh^*_{symp}

In this chapter, we follow the idea of Abouzaid-Smith in [3] and construct a second grading, the *weight grading* on $Kh^*_{symp}(D)$ of a bridge diagram, in Section 4.1.

To prove it is independent of the choice of bridge diagrams, we need to define some elementary moves in bridge diagrams. Luckily, by some combinatorial argument, it is not hard to see that any two bridge diagrams of the same link are connected by a series of

- Isotopies of arcs α_i and β_i rel endpoints.
- Handleslides α_i (or β_i) over another arc α_j (or β_j respectively).
- (De)stabilizations: add two p_i to one of the arcs, breaking it into 3 new arcs(and the reverse operation for destabilizations).

A sketch of the combinatorial to understand these equivalences is that we can relate these moves on planar link diagrams and Reidemeister moves: isotopies induce both Reidemeister I and II moves, handleslides induces Reidemeister III moves and (de)stabilizations preserve the planar link diagrams but change the number of endpoints.

We will combine the proof of isotopies and handleslides in Section 4.2, thanks to the following proposition:

Proposition 4.0.1. *Isotopies of arcs rel endpoints and handleslides on a bridge diagram induce Hamiltonian isotopies on Lagrangians.*

Proof. If we treat $\mathcal{Y}_n$ as a Lefschetz fibration $\pi_{\mathcal{Y}} : \mathcal{Y}_n \to \mathbb{C}$, the isotopies of α_i and β_i rel endpoints in the base $\mathbb{C}$ obviously induce Hamiltonian isotopies of $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, respectively. By [1, Lemma 48], handleslides also induce Hamiltonian isotopies of $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$. □

Combine with the fact that the Floer homology group will remain invariant under Hamiltonian isotopies, we know symplectic Khovanov homology remains isomorphic when we perform isotopies and handleslides.

Later in Sections 4.3-4.5, we will discuss the stabilization invariance and its sign conventions. We will attempt to give a proof stabilization invariance in Sections 4.3-4.4, but we will point out a technical difficulty in Section 4.3. In the end of the chapter, we will discuss orientations in Section 4.5. The results in Sections 4.3-4.5 will not be used in later chapters so readers can skip these sections. As a result, the weight grading is now only defined for bridge diagrams of a link, instead of the link itself. However, we will prove it is a link invariant that coincides with Jones grading in Chapter 6.

Remark 4.0.2. *The relative weight grading can be defined with any field* k *of any characteristic. Since we do not have a direct proof of stabilization invariance and the arguments of Chapter 6 only work in a characteristic zero field, we require our field* k *to be characteristic zero from now on.*

4.1 Constructing the weight grading

In this section, we define the weight grading wt on Kh^*_{symp}. The brief idea of constructing such a grading is building an automorphism, more precisely, the linear term of a *non-commutative vector field*

$$\phi : CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \to CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \tag{4.1.1}$$

which preserves the homological grading and commutes with the differential. Then it will induce an automorphism Φ on homology. If x is an eigenvector of eigenvalue λ, then we define $wt(x) = \lambda$.

To define ϕ, we need to work with a partially compactified space $\bar{\mathcal{Y}}_n$. Following the hypothesis in Section 2.2, we specify the following geometric information:

- Let $Z = \bar{\bar{A}}_{2n-1}$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $2n$ points. It admits a Lefschetz fibration structure over $\mathbb{P}^1$ and let F_∞ be its fiber at ∞ and s_0, s_∞ be two sections. The original surface is simply

$$A_{2n-1} = \bar{\bar{A}}_{2n-1} \backslash (F_\infty \cup s_0 \cup s_\infty) \tag{4.1.2}$$

- Let the intermediate space be

$$\bar{A}_{2n-1} = \bar{\bar{A}}_{2n-1} \backslash (F_\infty) \tag{4.1.3}$$

.

- Through taking Hilbert scheme, the projective variety is $\bar{\bar{M}} = Hilb^n(Z)$.
- D_0 is the divisor of subschemes whose support meets $s_0 \cup s_\infty$.
- D_∞ is the divisor of subschemes whose support meets F_∞.
- D_r is the relative Hilbert scheme of the projection $Z \to \mathbb{P}^1$, which is a divisor supported on a compactification of the complement of $\mathcal{Y}_n$.
- $\bar{\mathcal{Y}}_n = \bar{\bar{M}} \backslash D_\infty = Hilb^n(\bar{A}_{2n-1})$.

We also need to work with some chains in $\bar{\mathcal{Y}}_n$. There are Chern one spheres in $\bar{\mathcal{Y}}_n$. But when we compute the Gromov-Witten invariant strict to $\mathcal{Y}_n$, we have

Proposition 4.1.1. *[3, Lemma 6.8]* $GW_1|_{\mathcal{Y}_n} = 0 \in H^2(\mathcal{Y}_n)$.

When working in the complement of the diagonal, GW_1 is in fact the Poincaré dual of $\cup\pi^{-1}(p_i) \times Hilb^{n-1}(A_{2n-1})$, where p_i are those basepoints for singular fibers. $GW_1 = 0 \in H^2(\bar{\mathcal{Y}}_n)$ and thus it bounds a bounding cochain gw_1. It is not hard to see that we can choose gw_1 to be dual to $\cup\pi^{-1}(\gamma_{p_i}) \times Hilb^{n-1}(A_{2n-1})$, where γ_{p_i} is a path connect p_i to the ∞. The fact that $H^1(Hilb^n(\bar{A}_{2n-1}))$ is trivial guarantee the choice of a null homology gw_1 of GW_1 is homologically unique.

Let d_0 be the a divisor on $\bar{A}_{2n-1}$ supported on two sections we added at ∞. Away from the diagonal, D_0 coincides with $d_0 \times Hilb^{n-1}(\bar{A}_{2n-1})$. We push D_0 slightly to the normal direction to get D_0', and think this as $d_0' \times Hilb^{n-1}(\bar{A}_{2n-1})$. d_0 and d_0' do not intersect in $\bar{A}_{2n-1}$, but after taking the product with the Hilbert scheme, we need the following proposition about the intersection of divisors:

Proposition 4.1.2. *[3, Lemma 6.7] $B_0 = D_0 \cap D_0'$ is homologous to a locally finite cycle supported on $D_0^{sing} \cup (D_0 \cap D_r)$*

Thus the chain β_0 can be chosen to connect B_0 to $D_0^{sing} \cup (D_0 \cap D_r)$ by $d_0 \times d_t \times Hilb^{n-2}(\bar{A}_{2n-1})$, where d_t is the isotopy that connect d_0 and d_0'.

We needed a holomorphic volume form to give an absolute grading on Floer groups. Abouzaid and Smith proved the following:

Proposition 4.1.3. *[3, Lemma 6.3] There is a holomorphic volume form on $Hilb^n(\bar{A}_{2n-1})$ with poles contains in $D_0 \cup D_\infty$ and simple poles on D_0.*

Now we are ready to introduce our automorphism Φ. We start with the element

$$b = \tilde{b} + CO(gw_1) + co(\beta_0) \in CC^1(\mathcal{F}(M), \mathcal{F}(M)). \tag{4.1.4}$$

In the formula, the main part $\tilde{b}$ counts holomorphic disks in $\mathcal{R}^{k+1}_{(0,1)}(\mathbf{x}_1, \ldots, \mathbf{x}_k; \mathbf{y})$. But $\tilde{b}$ it is not closed in $CC^1(\mathcal{F}(\mathcal{M}), \mathcal{F}(\mathcal{M}))$. To make it closed, we add two additional components $CO(gw_1)$ and $co(\beta_0)$, see [3, Proposition 3.20] and Proposition 2.2.3.

We are interested in its linear part $b^1|_{CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)} : CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \to CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$. b^1 will be a chain map if $b^0 = 0$. This is not true in general. But we can consider a weaker case where we only require that $b^0|_L = 0$, for any Lagrangian L. We need the idea of the equivariant structures

Definition 4.1.4. *An equivariant object is a pair (L, c), with $L \in Ob(\mathcal{F}(M))$ and $c \in CF^0(L, L)$, with $dc = b^0|_L$.*

By definition, given an (exact) Lagrangian L, the obstruction to the existence of an equivariant structure c is given by $b^0|_L \in HF^1(L, L) \cong H^1(L)$, and the set of choices when this vanishes is an affine space equivalent to $H^0(L)$. In our case, the Lagrangians are a product of S^2, thus $H^1(L) \cong \{0\}$ and $H^0(L) \cong \mathbf{k}$.

Since we do not assume that b^0 vanishes, for an arbitrary cocycle $b \in CC^1(\mathcal{F}(M), \mathcal{F}(M))$ and equivariant objects $(\mathcal{K}_\alpha, c_\alpha)$ and $(\mathcal{K}_\beta, c_\beta)$, the first order term b^1 of b is not always a chain map. However, we can define a chain map

$$\phi(x) = b^1(x) - \mu^2(c_\alpha, x) + \mu^2(x, c_\beta) \tag{4.1.5}$$

. It induces an endomorphism Φ on $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$. If we consider the (generalized) eigenspace decomposition, the eigenvalue of the generalized eigenvector x will be its weight grading denoted as $wt(x)$. A priori, the weight grading is indexed by elements of the algebraic closure $\bar{\mathbf{k}}$.

We learn from [3, Lemma 2.12] that since $HF^0(\mathcal{K}, \mathcal{K}) \cong \mathbf{k}$, changing equivariant structures changes the weights by an overall shift

$$\Phi_{(\mathcal{K}_\alpha, c_\alpha),(\mathcal{K}_\beta, c_\beta)} = \Phi_{(\mathcal{K}_\alpha, c_\alpha + s_\alpha),(\mathcal{K}_\beta, c_\beta + s_\beta)} + (s_\alpha - s_\beta)id \tag{4.1.6}$$

So we have the following definition of the relative weight grading:

Definition 4.1.5. *Let Φ be the endomorphism constructed above on $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ and a be an eigenvector of Φ. The relative weight grading $wt(a)$ is defined to be the eigenvalue of a. This construction relies on an auxiliary choice of equivariant structures on $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, but different choices of such structures will only change all gradings by a fixed number and thus $wt(a)$ as a relative grading is independent of choices of equivariant structures.*

Remark 4.1.6. *With given equivariant structures on $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, we can compute an absolute weight grading. But at the time of writing this paper, the author does not know the choices that would give a correction term independent of the link diagram.*

Then we can rephrase Abouzaid-Smith's purity result as follow:

Proposition 4.1.7. *[3, Proposition 6.11] Let D be a bridge diagram without any crossings and all* *marked points are exactly* $(2k-1)$ *for* $k=-n+1,\ldots,n$ *on the real axis. Then we can choose* c_α *on* $\mathcal{K}_\alpha$ *and* c_β *on* $\mathcal{K}_\beta$ *such that for any* $x\in HF^*(\mathcal{K}_\alpha,\mathcal{K}_\beta)$*, we have* $wt(x)=gr(x)$*.*

If we recall the Khovanov homology of an unlink of k-component U_k is

$$Kh^{*,*}(U_k)=\bigotimes^{k}(\mathbf{k}_{(0,1)}\oplus\mathbf{k}_{(0,-1)}). \tag{4.1.7}$$

With the choice of the equivariant structures by Abouzaid and Smith, we have

$$Kh^{*,*}_{symp}(U_k)=\bigotimes^{k}(\mathbf{k}_{(1,1)}\oplus\mathbf{k}_{(-1,-1)}) \tag{4.1.8}$$

This proposition is a special case of our main theorem with crossing number equals to 0, if we relate gradings (h,wt) on symplectic Khovanov homology and (i,j) on Khovanov homology with the formula:

$$i=h-wt \tag{4.1.9}$$

$$j=-wt \tag{4.1.10}$$

4.2 Hamiltonian isotopy invariance

We can now prove the Hamiltonian isotopy invariance, which includes the cases of isotopies and handleslides of bridge diagrams by Proposition 4.0.1.

In the [3, Section 3], Abouzaid-Smith point out a very important fact that the weight grading is compatible with Floer products, without actually phrasing and proving the precise statement. This fact is crucial in the proof of Hamiltonian invariance. We prove the following version:

Proposition 4.2.1. *Let* $\mathcal{K}_0$*,* $\mathcal{K}_1$*,* $\mathcal{K}_2$ *be compact Lagrangians given by crossingless matchings in* $\mathcal{Y}_n$*. For any eigenvector* $\alpha\in HF^*(\mathcal{K}_1,\mathcal{K}_2)$ *and* $\beta\in HF^*(\mathcal{K}_0,\mathcal{K}_1)$*, we have*

$$wt(\mu^2(\alpha,\beta))=wt(\alpha)+wt(\beta) \tag{4.2.1}$$

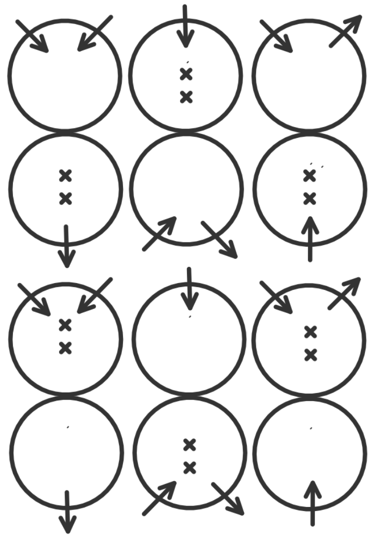

Figure 4.1: 6 possible boundary bubbles. The incoming arrows are inputs and the outgoing ones are outputs

Proof. Consider the boundary strata of the moduli space $\bar{\mathcal{R}}^2_{(0,1)}(x_0; x_1, x_2)$, see Equation 2.2.9 and Equation 2.2.10 . We can exclude sphere bubbles through a correct choice of bounding cycle. There are still 6 kinds of degeneration (see Figure 4.1) of this moduli space. Three degenerations shown in the first row compute

$$\phi(\mu^2(x_1, x_2)) \tag{4.2.2}$$

$$\mu^2(x_1, \phi(x_2)) \tag{4.2.3}$$

$$\mu^2(\phi(x_1), x_2) \tag{4.2.4}$$

The three degenerations shown in the second row compute

$$\mu^1\phi^2(x_1, x_2) \tag{4.2.5}$$

$$\phi^2(\mu^1(x_1), x_2) \tag{4.2.6}$$

$$\phi^2(x_1, \mu^1(x_2)) \tag{4.2.7}$$

If we pass to homology, those terms with μ^1 will vanish and thus we have the following relation by counting all the boundary components of 1 dimensional moduli space $\bar{\mathcal{R}}^2_{(0,1)}(x_0; x_1, x_2)$

$$\phi(\mu^2(x_1, x_2)) = \mu^2(x_1, \phi(x_2)) + \mu^2(\phi(x_1), x_2) \tag{4.2.8}$$

This is equivalent to

$$wt(\mu^2(x_1, x_2))\mu^2(x_1, x_2) = \mu^2(x_1, wt(x_2)x_2) + \mu^2(wt(x_1)x_1, x_2) = (wt(x_1) + wt(x_2))\mu^2(x_1, x_2). \tag{4.2.9}$$

, which proves the result. □

Remark 4.2.2. *By studying the similar set up with more boundary marked points, we can generalize the result above to higher Floer products. Specifically, when $n = 3$, we have $wt(\mu^3(x_1, x_2, x_3)) = wt(x_1) + wt(x_2) + wt(x_3)$.*

Now, we generalize Proposition 4.1.7, which is [3, Proposition 6.11] as follow

Proposition 4.2.3. *For any unlink diagram, the weight grading of each generator coincides with its homological grading, with the correct choice of equivariant structures.*

Proof. Proposition 4.3 implies that if we arrange all the end points of D to $(2k-1)$ for $k = -n+1, \dots, n$ on the real axis and all α-curves lie in the upper half of the plane while β-curves in the lower half, then the theorem is true. They call this collection of α or β curves a *crossingless matching*.

Next we need to consider more general diagrams. First, it is easy to see that moving based points around can be achieved by a symplectomorphism f of $\mathcal{Y}_n$ thus we can choose the new c'_α and c'_β such that $f^*(c'_\alpha) = c_\alpha$ $f^*(c'_\beta) = c_\beta$.

Then we need to consider crossings being added to the diagram, all the crossings can be created from the following ways, see Figure 4.2:

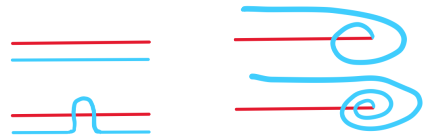

Figure 4.2: Crossings creating moves. The left column is a finger move and the right column is a winding at the end point.

- A *finger move* that isotopy one of the arc to the other creating a pair of crossings.
- A *winding at the end point* that applies a half-twist to an arc at the end point.

We can apply these moves one at a time, say α_1, and we can arrange α_1 only intersects with α'_1 at the end point. After locally perturbing other α_i to α'_i such that α_i only intersects with α'_i at end points, it is easy to see that $\vec{\alpha}$ and $\vec{\alpha}'$ form a diagram with no crossing. From Proposition 4.1.7, we know there exists a choice of equivariant structures such that homological grading and weight grading coincide on $HF(\mathcal{K}_\alpha, \mathcal{K}_{\alpha'})$. The isomorphism between $HF(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ and $HF(\mathcal{K}_{\alpha'}, \mathcal{K}_\beta)$ is given by taking product with an element in $HF^0(\mathcal{K}_\alpha, \mathcal{K}_{\alpha'})$, which should have weight grading 0 as well. Together with the fact that weight grading is preserved with taking product with an element of weight grading 0 from Proposition 4.2.1, we finish the proof. □

Combining the two propositions above, while keeping in mind the fact that β-curve and its image $\phi(\beta)$ after isotopy or handle slide form an unlink, we have the following corollary:

Corollary 4.2.4. *Let $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ be two Lagrangians given by α and β arcs in $\mathcal{Y}_n$ and ψ be a Hamiltonian isotopy. There is a canonical class $c_\psi \in HF^0(\mathcal{K}_\beta, \psi(\mathcal{K}_\beta))$ such that $\mu^2(c_\psi, \bullet)$ is an isomorphism between $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ and $HF^*(\mathcal{K}_\alpha, \psi(\mathcal{K}_\beta))$. The map ψ induces a bijection ψ^* between equivariant structures of $\mathcal{K}_\beta$ and $\psi(\mathcal{K}_\beta)$ such that $wt(c_\psi) = 0$ for the pair of equivariant structures. For any equivariant structures $c_{\mathcal{K}_\alpha}$ and $c_{\mathcal{K}_\beta}$ and any elements $x \in HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ if we equip $\psi(\mathcal{K}_\beta)$ with $\psi^*(c_{\mathcal{K}_\beta})$, we have*

$$wt(\mu^2(id, x)) = wt(x) \tag{4.2.10}$$

Proof. $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ and $HF^*(\mathcal{K}_\alpha, \psi(\mathcal{K}_\beta))$ are isomorphic via taking product with an element c_ψ in $HF^0(\mathcal{K}_\beta, \psi(\mathcal{K}_\beta))$. Because β-arcs and $\psi(\beta)$-arcs form an unlink, we conclude from the previous proposition that with a correct choice of equivariant structures $c_{\mathcal{K}_\beta}$ and $c_{\psi(\mathcal{K}_\beta)}$, we have $wt(c_\psi) = 0$. Thus taking product with c_ψ preserves the weight grading.

As for the part about equivariant structures, if we change $c_{\mathcal{K}_\beta}$ and $c_{\psi(\mathcal{K}_\beta)}$ by the same amount, the overall grading shifts will cancel each other. Thus we can pair equivariant structures on $\mathcal{K}_\beta$ with $\psi(\mathcal{K}_\beta)$ such that $wt(c_\psi) = 0$ for each pair. □

This corollary implies that if we endow $\psi(\mathcal{K}_\beta)$ with the equivariant structure $\psi^*(c_{\mathcal{K}_\beta})$, the isomorphism $id^* : HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \to HF^*(\mathcal{K}_\alpha, \psi(\mathcal{K}_\beta))$ preserves the weight grading.

4.3 Stabilization invariance I: Neck-stretching setup

The argument for stabilization invariance is more complicated. There is a technical issue in this section that I will point out in Warning 4.3.8. The proof in Sections 4.3-4.5 is not complete without a more careful study of the almost complex structures. The goal of the next few sections is to provide a strategy of proof of the following conjecture:

Conjecture 4.3.1. *Suppose D and D' are 2 bridge diagrams for $L \subset \mathbb{R}^3$ and D' is obtained by stabilizing at the part of an arc that is adjacent to the non-compact region. For any choice of equivariant structures $c_{\mathcal{K}_\alpha}$ and $c_{\mathcal{K}_\beta}$ on $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, we can also find $c_{\mathcal{K}'_\alpha}$ and $c_{\mathcal{K}'_\beta}$ on $\mathcal{K}'_\alpha$ and $\mathcal{K}'_\beta$ such that the identification of $(CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta), \delta)$ and $(CF^*(\mathcal{K}'_\alpha, \mathcal{K}'_\beta), \delta')$ preserves the endomorphism ϕ and thus the weight grading.*

Remark 4.3.2. *This conjecture is true after we prove Theorem 1.2.1 in Chapter 6 that the relative weight grading is independent of the choice of bridge diagram. We mark it as a conjecture for now because the author does not have a direct proof of the statement.*

To prove the stabilization invariance, we want to use a neck-stretching argument akin to the one in [6], but to avoid the same trouble of genericity of moduli spaces as Hendricks, Lipshitz and Sarkar encountered in [12], we need to adapt Mak-Smith's modification of moduli spaces to the moduli space we used in Section 2.2. Remember we used the moduli space $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$, with two inner marked points z_0 and z_1 mapping to divisors D_0 and D_0', to define the weight grading. To get enough genericity for cylindrical reformulation, we need to consider the following moduli space with h additional inner marked points $w_1, \ldots, w_h \in mk(\mathbb{D})$ and a generic family of almost complex structures $J_z \in \mathcal{J}^{cyl}$.

Definition 4.3.3. *Define $\mathcal{R}^{k+1,h}_{(0,1)}(x_0; x_k, \ldots, x_1)$ to be the moduli space of holomorphic maps $u : \mathbb{D} \to \bar{\mathcal{Y}}_n$ satisfying the following:*

$$u(z_0) \in D_0 \textit{ and } u(z_1) \in D_0' \tag{4.3.1}$$

$$u^{-1}(\Delta) = mk(\mathbb{D}), \#mk(\mathbb{D}) = h \tag{4.3.2}$$

$$Du^{0,1} = 0 \textit{ with respect to } J_z \tag{4.3.3}$$

$$\textit{the intersection multiplicity with } \Delta \textit{ equals the multiplicity of } z \in mk(S) \tag{4.3.4}$$

$$u(z) \in L_i \textit{ for } z \in \partial_i \mathbb{D} \tag{4.3.5}$$

$$\lim_{z \to w_i} u(z) = x_i \tag{4.3.6}$$

Then we can define the map ϕ^{cyl} in the same way using $\mathcal{R}^{k+1,h}_{(0,1)}(x_0; x_k, \ldots, x_1)$ instead of $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$. We can conclude the following proposition from the same argument we did for Corollary 3.3.7:

Proposition 4.3.4. *[11, Proposition 8.9] The map Φ^{cyl} and Φ induced by ϕ^{cyl} and ϕ on homology are isomorphic.*

Proof. The proof is akin to the proof of Corollary 3.3.7. The main idea is to use the total Fukaya category. The map ϕ^{cyl} defined by counting $\mathcal{R}^{k+1,h}_{(0,1)}$ is quasi-isomorphic to a map $\bar{\phi}^{cyl}$ defined by counting the same domain but with J independent of the interior marked points. This new map counts holomorphic maps $u : \mathbb{D} \to \mathcal{Y}_n$ with the additional condition that $u(z) \in \Delta$ if and only if $z \in mk(\mathbb{D})$. Thus the inner marked points are decorative and the map $\bar{\phi}^{cyl}$ is quasi-isomorphic to ϕ with counting holomorphic disks with no inner marked point. □

To apply a neck-stretching argument, we need to work with the cylindrical reformulation off the moduli space above. We have the following proposition:

Proposition 4.3.5 (cf Proposition 3.4.2)**.** *There is a one-to-one correspondence between solutions* $u : \mathbb{D} \to \bar{\mathcal{Y}}_n \in \mathcal{R}^{2,int(u)}_{(0,1)}(x_0; x_1)$ *and pairs of holomorphic map* $(\pi_\Sigma, v) : \Sigma \to \mathbb{D} \times \bar{S}$ *satisfying the following conditions:*

$$\pi \textit{ is a } k-\textit{branched covering such that } mk(\Sigma) = \pi^{-1}(mk(\mathbb{D})) \textit{ are branched points.} \quad (4.3.7)$$

$$\#mk(\Sigma) = \#mk(\mathbb{D}) = int(u). \quad (4.3.8)$$

$$\textit{There is a lift of } z_0,\ \pi_\Sigma(\bar{z}_0) = z_0,\ v(\bar{z}_0) \in d_0. \quad (4.3.9)$$

$$\textit{There is a lift of } z_1,\ \pi_\Sigma(\bar{z}_1) = z_1,\ v(\bar{z}_1) \in d_0'. \quad (4.3.10)$$

$$Dv^{0,1} = 0 \textit{ with respect to } j_z \textit{ on } S\textit{, where } J_z = j_z^n. \quad (4.3.11)$$

$$v(z) \in \Sigma_{\alpha_{n_j}} \textit{ for } z \in \partial_{0,j}\Sigma. \quad (4.3.12)$$

$$v(z) \in \Sigma_{\beta_{n_j}} \textit{ for } z \in \partial_{1,j}\Sigma. \tag{4.3.13}$$

$$\lim_{z \to z_{i,j}} = x_{i,j}. \tag{4.3.14}$$

$$\pi_{HC} \circ u = v \circ \pi_{\Sigma}^{-1}. \tag{4.3.15}$$

Proof. The proposition is similar to Proposition 3.4.2, except that we have two other inner marked points z_0 and z_1 that are mapped to certain divisors according to equation (4.18). Generically, we can work away from the diagonal so $z_0 \in D_0$ is equivalent to $z_0 \in d_0 \times Hilb^{n-1}(\bar{S})$, which means one of the coordinate of z_0 must lie in d_0, that is why we translate (4.18) into (4.26) and (4.27). If we have one more preimages of z_0 to be in d_0, then the intersection of such a disk and D_0 is more than 1, which contradicts the assumption that the disk intersect with divisor D_0 only once. □

We call this new moduli space $\mathcal{R}^{2,int(u),cyl}_{(0,1)}(x_0; x_1)$. There are two types of inner marked points. We call the ones representing the intersections with divisor d_0 and d_0' *type* 1 and the ones representing the branched points *type* 2. Keep in mind that to make sure our disks are disjoint from the divisor D_r, we only count disks with the number of type 2 inner marked points equaling to the number of branched points $mk(\mathbb{D})$.

Remark 4.3.6. *This cylindrical reformulation and the construction in [6] are different in the following ways:*

- *We count disks in $\bar{S}$ instead of S so we would have some spherical classes.*
- *We have some inner marked points to handle. Type 1 inner marked points are essential because they record the points that go to the partial compactification divisor d_0 and its linear equivalence d_0'. Type 2 inner marked points are almost just decorative in the part of $v : \Sigma \to \bar{S}$ because the only restriction it impose is that the domain dependent J should*

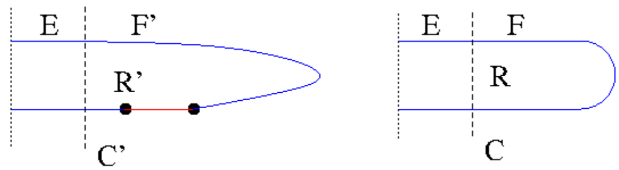

Figure 4.3: The neck-stretching region in proving stabilization invariance. The rest of the knot is on the left of the left-most dashed line. C and C' are the cut, R and R' are two bounded regions and E, F are different sides of the cut.

equal to J_S at those marked points but not decorative in general because we do not know the transversality of such a moduli space with type 2 inner marked points.

Now we want to apply the neck-stretching argument to our maps in $\mathcal{R}^{2,int(u),cyl}_{(0,1)}(x_0; x_1)$ that is similar to the one in [6, Subsection 7.4.1]. We want to compare a diagram D and its stabilization diagram D' obtained from breaking a beta arc β_n into consequently three arcs β'_n, α_{n+1}, β_{n+1}.

First, we set up some notation for the neck-stretching. Let $\bar{S}$ and $\bar{S}'$ be the surfaces before and after the stabilization, such that S' has 2 more singular fibers than S. Let us assume we stablize the β_n arc at a point p, so we get α_{n+1}, β'_n and β'_{n+1} in D'. Denote the two additional marked points as $p_{2n+1} = \beta'_n \cap \alpha_{n+1}$ and $p_{2n+2} = \alpha_{n+1} \cap \beta'_{n+1}$, and these are the only two intersection with α_{2n+1} and β arcs. Since p_i indeed lies in the $z = 0$ plane, we abuse notation such that p_i is also the intersection of corresponding Lagrangian spheres in $\bar{S}$ (or $\bar{S}'$ respectively.) Those arcs divide the plane $\mathbb{C}$ into many regions. Let us call the bounded region with p at its boundary R in $\bar{S}$ (and R' in $\bar{S}'$).

We use some isotopies and handeslides to arrange our bridge diagram to look like Figure 4.3. Then consider a vertical cut C, which separates α'_{n+1} from the rest of the α arcs and orthogonally intersects β'_n and β'_{n+1} in one point each and disjoint from the rest of the arcs. Let F' be unbounded component of $\mathbb{C}\backslash C$ with α'_{n+1} and F be the corresponding component of the diagram of D. The other component is denoted as E in both diagrams, since they are identical. Furthermore, let $S_L = S'_L = p^{-1}(E)$, $S'_R = p^{-1}(F')$ and $S_R = p^{-1}(F)$. Moreover, let $\sigma'_n = \Sigma_{B'_n} \cap i^{-1}(C)$ and $\sigma'_{n+1} = \Sigma_{B'_{n+1}} \cap i^{-1}(C)$ be topologically circles. There is a vector field $\vec{R}$ on $i^{-1}(C)$ given by the stretching, i.e. a lift of vector field $\frac{\partial}{\partial y}$ on C to $\bar{S}'$. Starting with any point q in σ'_n, we travel through the vector field and end with a point on σ'_{n+1}. We call this $\vec{R}$-chord γ_q. This chord records the asymptotic condition of the disk about the cut C.

The generators are easily identified. A given generator in D determines uniquely a generator in D' by adding one of p_{2n+1} or p_{2n+2}, because in D' there has to be a coordinate in α'_{n+1}, but there is already one coordinate at one of β'_n or β'_{n+1} thus only one of it can be added to form a generator in D'.

Following closely the notation of Hendricks-Lipshitz-Sarkar, we give our $\mathbb{D}$ strip-type coordinate $\mathbb{R} \times [0,1]$ such that our maps in $(\pi_\Sigma, v) \in \mathcal{R}^{2,int(u),cyl}_{(0,1)}(x;y)$ have coordinate

$$(\pi_\Sigma, v) : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times [0,1] \times \bar{S}, (\mathbb{R} \times \{0\} \times (\Sigma_{\alpha_1} \cup \ldots \Sigma_{\alpha_n})) \cup (\mathbb{R} \times \{1\} \times (\Sigma_{\beta_1} \cup \ldots \Sigma_{\beta_n}))) \tag{4.3.16}$$

It is useful to talk about the projected domain. First, each holomorphic map $u \in \mathcal{R}^{2,int(u),cyl}_{(0,1)}(x,y)$ determines a domain in

$$D(u) = (\pi_S \circ v)_*([X, \partial X]) \in H_2(S, (\cup\Sigma_{\alpha_i}) \cup (\cup\Sigma_{\beta_i})) \tag{4.3.17}$$

and then it will be easier if we further reduce the dimension of the target space by projecting the last factor $\bar{S}$ to $\mathbb{C}$ when we treat this as a Lefschetz fibration and thus we can further project $D(\phi)$ down to $\mathbb{C}$ get the projected domain $i_*(D(\phi)) \in H_2(\mathbb{C}, (\cup\alpha_i) \cup (\cup\beta_i))$. Then we can call the number in each homology class its multiplicity. Since our map $i \circ \pi_S \circ \psi$ is holomorphic, the projected domain should only have non-negative multiplicities, compare [15, Section 3]. The multiplicity at the non-compact region is clearly 0. Thus we have the following proposition:

Proposition 4.3.7. *[6, Lemma 7.21] The projected domain of any holomorphic disk in* $(\mathcal{Y}_n, \mathcal{K}_\alpha, \mathcal{K}_\beta)$ *or* $(\mathcal{Y}_{n+1}, \mathcal{K}'_\alpha, \mathcal{K}'_\beta)$ *has multiplicity 0 or 1 at* R*, respectively* R'*.*

Now if we consider a one-parameter family J_T of almost complex structures with a long neck along C. As the neck stretching parameter $T \to \infty$, the limit of the holomorphic map $u_T : X \to$

$\mathbb{R} \times [0,1] \times S'$, as $T \to \infty$. approaches

$$v_\infty^F : (\Sigma_F, \partial\Sigma_F) \to (\mathbb{R} \times [0,1] \times i^{-1}(F'), (\mathbb{R} \times \{0\} \times \Sigma_{A'_{n+1}}) \cup (\mathbb{R} \times \{1\} \times \Sigma_{B_n^F})) \quad (4.3.18)$$

$$v_\infty^E : (\Sigma_E, \partial\Sigma_E) \to (\mathbb{R}\times[0,1]\times i^{-1}(E'), (\mathbb{R}\times\{0\}\times(\Sigma_{A_1}\cup\ldots\cup\Sigma_{A_n}))\cup(\mathbb{R}\times\{1\}\times(\Sigma_{B_1}\cup\ldots\cup\Sigma_{B_{n-1}}\cup\Sigma_{B_n^E}))) \quad (4.3.19)$$

such that Σ_F has three boundary punctures and Σ_E has $2n+1$ boundary punctures. There are punctures p_E and p_F on each part, whose projection to S' is asymptotic to the same $\vec{R}$-chord γ in $i^{-1}(C)$. After filling the punctures, the map $\pi_{\mathbb{R}\times[0,1]} \circ v_\infty^F$ is a diffeomorphism, while $\pi_{\mathbb{R}\times[0,1]} \circ v_\infty^E$ gives a degree n branched cover.

But there is an issue in the discussion as follow:

Warning 4.3.8. *In fact, we should work with Mak-Smith's framework to ensure transversality and consider a one-parameter family J_T of domain dependent almost complex structures in $\mathcal{J}^{cyl}$. The domain dependent almost complex structures in $\mathcal{J}^{cyl}$ depend on a choice of complex structure on S that defines the Hilbert scheme. Given a generic family of complex structures j_T on S with a long neck at C, let $J_{T,z}$ be a one-parameter family of domain dependent almost complex structures such that $J_{T,z} = j_T^n$ near the marked points. Thus the process above as $T \to \infty$, not only we change our domain dependent almost complex structure $J_{T,z}$, we also change the complex structure j_T on S simultaneously.*

For now, let us assume doing the procedure above will not cause any transversality issue and the limit curves are well-behaved. The gluing theorem should be similar to the argument in [6] except that we have some inner marked points. To get a correct disk count, we need to consider all the possible distribution of inner marked points to two domains Σ_E and Σ_F. To be more specific, the sum of numbers of type 1 marked point should be 2 and the sum of numbers of type 2 marked point should be h. The compactness theorem is also similar to [6] except that we now have to deal with the convergency of inner marked points. This is already dealt with by Mak-Smith in [11, Proposition 2.31].

The main observation about the neck-stretching argument is the following:

Lemma 4.3.9. *For any curve v_∞^E asymptotic to a $\vec{R}$-chord γ, there is a unique curve v_∞^F asymptotic to the same $\vec{R}$-chord γ that lies totally in S'.*

Proof. Consider the intersection of $i^{-1}(F)$ and $\{u = 0\}$. This is an (open) Riemann surface $\mathcal{H}$. If we equip $\mathcal{H} \subset S'$ by standard complex structure through pulling back through the projection

$$\pi : S' \to \mathbb{C} \tag{4.3.20}$$

the complex structure i on $\mathbb{C}$, for any chord γ, there is a holomorphic disk v_∞^F asymptotic to γ by boundary injectivity. Furthermore, the Maslov index of these disks is 1, so transversality is obtained automatically, see [16, Theorem 2]. And due to S^1-symmetry of S', by applying the action on these disks, we get a disk for each chord. For uniqueness, we first use S^1 symmetry to move γ into the Riemann surface $\mathcal{H}$. Fix an arbitrary curve v_∞^F asymptotic to γ. Consider its projection to u-plane, since $\pi_u(\gamma) = 0$, the boundary of our domain should be sent to a bounded subset of $\mathbb{R} \cup i\mathbb{R}$. Together with the open mapping principle we claim that $\pi_u \circ v_\infty^F$ has to be a constant, i.e. this disk totally lies in $\mathcal{H}$ and could only be the disk we have just constructed. □

Remark 4.3.10. *If we do the neck-stretching argument on D instead of D', there is a unique curve with a single puncture that asymptotic to each $\vec{R}$-chord on the right hand side.*

With such a lemma, we can outline the proof of invariance of Kh^*_{symp} as a singly-graded link homology, which will come into handy in the following section. We want to show that the count of $\mathcal{R}^{2,h,cyl}_{(0,1)}(x;y)$ in D and $\mathcal{R}^{2,h',cyl}_{(0,1)}(x';y')$ in D' are the same. If the holomorphic map does not go to the modified side F, then it is easy to identify those maps. If the holomorphic map goes to F, by Proposition 4.3.7, the multiplicity on R is exactly 1.

These maps with multiplicity 1 on R can be identified through the following procedure. First, topologically, for a pair of disk in D and D' that only differ on the right-hand side, it is easy to see that they should have the same number of branched points (counted with multiplicity) and thus $h = h'$ in the definition of the moduli spaces.

We start with the diagram D. From Remark 4.3.10 any map v contributes to $\mathcal{R}^{2,h,cyl}_{(0,1)}(x;y)$ is uniquely determined by the punctured curve on the left hand side. Because the left hand side of D and D' are the same, the curve counts v^E_∞ are the same for both diagrams. With Lemma 4.3.9, we know that in D' there is a unique curve v^F_∞ asymptotic to the same chord. Gluing these two pieces together yields a curve in D', and the number of type 2 inner marked points always match because we only count disks with the number of inner marked points that equals h.

4.4 Stabilization invariance II: A strategy of proof

Now we are ready to show a strategy of proof of Conjecture 4.3.1. It leads to a proof if we can deal with the issue in Warning 4.3.8. We start this section with some basic facts.

First, the space we are working with is slightly different than [6], because we are dealing with $\bar{S}$ instead of S. The cutting region $C \subset S$ is no longer $\mathbb{R} \times \mathbb{R} \times S^1$ but $\mathbb{R} \times S^2$. The original R-chords remain the same but we added two copies of $\mathbb{R} \times pt$ to the cutting region which will result in potentially two additional R-chords along these two lines. Since the Lagrangians intersect C away from the additional lines, the R-chords that disks can converge to will remain in S instead of $\bar{S}$.

Through compactifying the fibers, we changed some geometric conditions of our manifold. Precisely, we adds some sphere classes into $\mathcal{Y}_n$, and this will give rise to some Maslov discs in $\bar{\mathcal{Y}}_n$ through the following proposition

Proposition 4.4.1. *There are stable Maslov index 0 disks whose boundary are on compact Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ in $\bar{\mathcal{Y}}_n$ and their intersections with D_0 are always nontrivial.*

Proof. For a generic almost complex structure, there is no Maslov index 0 disk in $\mathcal{Y}_n$, because such disk contributes to $\mathcal{R}^2(x, y)$, which has expected dimension -1 . Thus the only possible Maslov 0 disks will intersect the divisor D_0 we use to compactify the space. To see the existence of such disks, a candidate is given by a constant disk sliced with a sphere at an exceptional fiber. □

The reason we care for those disks is that the weight grading is induced by the following endomorphism

$$b = \tilde{b} + CO(gw_1) + co(\beta_0) \tag{4.4.1}$$

requiring us to count discs with interior marked points connecting generators with same degree.

We want to set up the intersection numbers between those limit curves and the divisor d_0. Since we are working with the cylindrical set up, the intersection number of u with D_0 is the same as v with d_0, because D_0 coincides with $d_0 \times Hilb^n(S)$ away from diagonal. Considering the spaces those curves live in, we want to break d_0 into two halves $Dd_L = d_0 \cap S_L$ and $D_R = d_0 \cap S_R$. The only obstacle is that there is a boundary puncture on each of v_∞^E and v_∞^F asymptotic to a chord in the cutting region C. However, in this case, the chords are away from the divisor d_0 and thus the intersection number with the divisor makes sense to count this intersection number in a compact region away from the puncture.

Lemma 4.4.2. *With the hypothesis that the intersection $D_0 \cap u = 1$, one of the intersections $D_L \cap v_\infty^E$ or $D_R \cap v_\infty^F$ is 0.*

Proof. Positivity of intersection and the fact that $D_0 \cap u = 1$ implies that $d_0 \cap v = 1$. The intersection number of v on both sides with D_L and D_R should sum to 1. Thus one of them must be 0. □

We can further rule out one of the cases through the following proposition:

Lemma 4.4.3. *If v_∞^E is obtained from a Maslov index 0 disk and $D_L \cap v_\infty^E = \emptyset$, v_∞^E is a constant map.*

Proof. $D_L \cap v_\infty^E = \emptyset$ implies that the image $\pi_{\mathbb{R}\times[0,1]} \circ v_\infty^E(\mathbb{D})$ lies totally in S_L not in the compactified space $\bar{S}_L$. From Remark 4.3.10 his punctured curves determines a Maslov-0 disk in S. The count of such Maslov-0 disks should be 0 by Proposition 4.4.1. □

We are now ready to sketch the strategy of proving stabilization invariance:

Strategy of proof of Conjecture 4.3.1. Let's review the endomorphism we use in the Chapter 3 to define the grading. In Equation 4.1.4

$$b = \tilde{b} + CO(gw_1) + co(\beta_0) \tag{4.4.2}$$

with each term counting different holomorphic disks connecting generator **x** and **y** with $deg(\mathbf{x}) = deg(\mathbf{y})$ to achieve the right dimension.

For a map $u \in \mathcal{R}^{2,int(u),cyl}_{(0,1)}(x, y)$, we can study its projected domain which will have multiplicity 0 or 1 in the region R or R', by Proposition 4.3.7.

The multiplicity 0 maps are naturally identified without even referring to neck-stretching, because disks don't travel to the region that we modified by stabilization.

We focus our discussion on multiplicity 1. We start with the stabilized diagram D'. Because of the disks we count has Maslov index 0 and its intersection with D_0 is exact 1, so there is at most one component that will intersect D_0. We know from Proposition 4.4.1, either the part in S_L is constant, which mean the disk lies totally in S_R or $D_L \cap v^E_\infty \neq \emptyset$. This addresses the possible break of the two type 1 inner marked points. In the first case, both points are on the right of the cut, while in the second case, both points are on the left. Lemma 4.4.2 prevents type 1 inner marked points to be on different sides.

About type 2 inner marked points, we can show that, if two disks only differ on the right hand side, then the number of branched points (counted with multiplicity) should be the same. Thus, after gluing in both cases, the number of type 2 inner marked points always match because we only count disks with number of inner marked points equaling to the number of branched points.

If a disk in S_L is trivial, then it should be a disk connect a generator to itself. In this case, the disks that are counted here are those Maslov-0 disks given by sphere bubbles on the two additional singular fibers. Since it maps a generator to itself, its overall effect to b is simply adding some multiple of the identity map and thus won't influence the result of invariance of relative grading.

If $D_L \cap v^E_\infty \neq \emptyset$, we apply neck-stretching, such a disk u yields v^E_∞ and v^F_∞. From Lemma 4.4.2, we see that the only disks that will contribute non-trivially are those disks with $D_R \cap v_\infty^F = \emptyset$. Namely, D_R lives in S instead of its compactification $\bar{S}$. So we do not have to worry about the chords by adding ∞ to fibers. The count of such punctured disks in S_R asymptotic to each chord ρ is 1 according to Lemma 4.3.9. This implies that the left part disks v_∞^E has the same count as the original disks u. The left part of the stabilized and destabilized diagrams D are easily identified and thus the contributions are the same. □

4.5 Stabilization invariance III: Orientations

We will discuss some orientations that we omitted in the previous sections, but it is important in the proof of stabilization invariance. The proof is similar to [6, Section 7], but Hendricks, Lipshitz and Sarkar used $\mathbb{F}_2$ as their base ring. However, we are working with a characteristic 0 field, so we need to address how disks are identified with correct signs. This is used in both the proof of invariance of Kh^*_{symp} as a singly-graded link homology in Hilbert scheme point of view, that we outlined in the end of Section 4.3 and the stabilization invariance of the relative weight grading that we showed in Section 4.4. We will outline some of the argument here.

As a singly graded group itself, Seidel and Smith give a brief discussion at [1, Section 4D], stating that we can find coherent orientations of each generator if Lagrangians intersect transversely. Further discussions on the topic are in [2, Section 5.6] and [3, Section 4.4], that they discuss an orientation convention for crossingless matchings explicitly, whereas in later one they discuss orientations of moduli spaces for weight gradings.

To orient the moduli spaces, we need a canonical trivialization of the determinant line bundle, which is given by a canonical choice of spin structures on Lagrangians. The spin structures exist because the obstruction $H^1(\mathcal{K}) = 0$ for Lagrangians as products of 2-spheres. We use product spin structures and thus orientation lines can also be treated as product.

In the proof stabilization invariance, we need specify our choice in the stabilized diagram D' from a given spin structure on Lagrangian constructed in D, so that the contributions of moduli spaces are the same with signs. The difference of D and D' is that we break arc β_n into arcs β'_n,

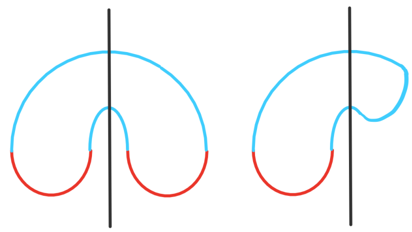

Figure 4.4: Two different unknot diagrams. The black lines indicate the cuts of the neck-stretching region.

α_{n+1} and β'_{n+1}, while creating two more singular fibers. Notice that the right side surface F can be treated as the case of $\mathcal{Y}_1$, so we could assign spin structures just like Abouzaid-Smith did for the crossingless matching in [3, Section 4.4]. There is an overall sign choice but there is a map $p : S \to S'$ creating those two singular fibres and $p(\Sigma_{\beta_n}) = \Sigma_{\beta'_n} \cup \Sigma_{\alpha_{n+1}} \cup \Sigma_{\beta'_{n+1}}$, so we can require the pullback of the sum of spin structures on $\Sigma_{\beta'_n}$, $\Sigma_{\alpha_{n+1}}$ and $\Sigma_{\beta'_{n+1}}$ is the given spin structure on Σ_{β_n}.

For disks that contribute to differentials in Section 4.4. Consider a generator x in D and its corresponding generatorx' in D', that they only differ from an additional point p_{2n+1} or p_{2n+2} in the right surface F. We could treat the right side F as part of our complex surface $\mathcal{Y}_1$, which is case of $n = 1$. In this case [2] gives us a canonical choice of $o_{p_{2n+1}}$ and $o_{p_{2n+2}}$. Tensoring o_x with either $o_{p_{2n+1}}$ or $o_{p_{2n+2}}$ determines a choice of $o_{x'}$ in D'. To show that the holomorphic disks that only differ from the right hand side in the stabilization area contribute to the same sign in D and D', we need the following local computation.

Abouzaid and Smith showed that the disks in the compact region correspond to the same sign in two diagrams in Figure 4.4. The corresponding punctured disks in two diagrams have the same sign in gluing. Thus, no matter what diagram have on the left hand side, as long as the choice of $o_{p_{2n+1}}$ or $o_{p_{2n+2}}$ remains the same, splicing a punctured disk with a right hand side punctured disk in either diagram will yield the same sign.

For the case of the weight grading. There are two arguments with sign involved. First in the neck-stretching argument, we are splicing punctured disks with the same bigons that are computed in the last paragraph. Disks spliced with different right hand side should be identified with sign from the same local computation.

The other one is for disks that is constant on the left hand side, which correspond to the spherical bubbles on the right hand side. As in [3, Lemma 4.6], whether the involution σ changes the sign of the Moduli space with 2 interior marked points depending on whether $\frac{\mu(\mu+1)}{2} + \frac{(k-1)(k-2)}{2} + k$ is odd or not, where μ is the Maslov grading and $(k+1)$ is the number of boundary marked points.

We can still use $\mathcal{Y}_1$ to locally compute the stabilized region F. There are two disks for each generator p_{2n+1} and p_{2n+2}. Specifically, we can use the left diagram of Figure 4.4. We have $k = 1$ at both points, and $\mu = 1$ at p_3 while $\mu = 2$ at p_4, if we name the end points p_1 to p_4 from left to right. The quantity $\frac{\mu(\mu+1)}{2} + \frac{(k-1)(k-2)}{2} + k$ is even for both cases and thus they add up to the same sign.

Chapter 5: Long exact sequence of $Kh^{*,*}_{symp}$

In Abouzaid and Smith's proof of the equivalence of symplectic Khovanov homology and its combinatorial sibling as singly graded groups, they construct a long exact sequence via an exact triangle of bimodules, see [2, Equation 7.9]:

$$\ldots \to HF^{k+n+w}(L_+) \to HF^{k+n+w}(L_0) \to HF^{k+n+w+2}(L_\infty) \to HF^{k+n+w+1}(L_+) \to \ldots \tag{5.0.1}$$

where L_+ is a link with a positive crossing and L_0 and L_∞ are links given by 0 or ∞ resolutions. The goal of this chapter is to give an explicit construction of such a long exact sequence with the framework of bridge diagrams that preserves the weight grading, just like the combinatorial Khovanov homology. For conciseness, we omit the discussion of the absolute grading unless noted otherwise, but keep in mind that, as in Section 3.5, all the terms n and w above in braid diagrams can be determined by the terms $h(x_0) + w(D) + rot(D)$ which consist of the homological grading of a distinguished generator, the writhe, and the rotation number of the bridge diagram.

We use the following local diagrams for the computation: The α curves in different pictures are the same and so are the other curves outside of this tangle, but with different blue curves these pictures represent L_+, L_0 and L_∞ respectively. If we name the blue curves β, γ and δ, see Figure 5.1, then we have the following exact sequence:

Proposition 5.0.1. *[2, Proposition 7.4] If we have α, β, γ and δ curves presented like Figure 5.1 locally and β, γ and δ are the same apart from this region, then we have the following exact sequence*

$$\ldots \xrightarrow{c_1} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \xrightarrow{c_2} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\gamma) \xrightarrow{c_3} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\delta) \xrightarrow{c_1} \ldots \tag{5.0.2}$$

In particular, there are elements $c_1 \in CF^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in CF^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in CF^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$

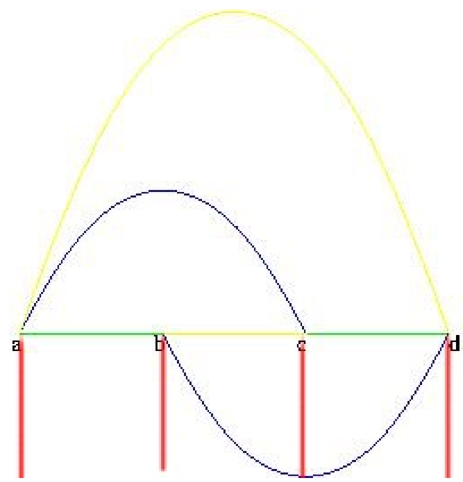

Figure 5.1: Exact triangle of Lagrangians. The red, blue, green and yellow are α, β, γ and δ curves

such that the maps above are Floer products with the corresponding elements.

Proof. Abouzaid-Smith prove that there is an exact triangle among the identity bimodule, the cup-cap bimodule and the bimodule representing a half-twist τ. Evaluating at $HF^*(\bullet, \mathcal{K}_\gamma)$, we have an exact triangle between $\mathcal{K}_\gamma$, $\mathcal{K}_\delta$ and a one-sided module that is equivalent to $\mathcal{K}_\beta$, such that the maps connecting those terms are Floer products with some elements $c_1 \in CF^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in CF^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in CF^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$. Evaluating these one-sided modules at $\mathcal{K}_\alpha$, we have the desired long exact sequence □

Now we need to show that this long exact sequence preserves the relative weight grading in the sense that each element summing to c_i has the same weight grading so that c_i has a well defined weight grading, and moreover the weight gradings of c_1, c_2 and c_3 sum to 0. With a closer look into the diagram, we have the following observation:

Lemma 5.0.2. *After a small perturbation, each two of β, γ and δ form a bridge diagram for an unlink of $n-1$ components without any crossing.*

Proof. In the region shown in Figure 5.1, each two of β, γ and δ form an unknot. In the other regions not shown in the figure, β, γ and δ share the same arcs (and there are $(n-2)$ such arcs). Perturbing γ and δ slightly away from β to different sides (so they don't intersect in the interior as well), each pair of arcs form an unknot component as well. Thus we have 1 crossingless unknot component inFigure 5.1 and crossingless (n-2) unknot components off that region. Together, each two of β, γ and δ form a bridge diagram for an unlink of $n-1$ components without any crossing. □

From the computations of [2], we have

Corollary 5.0.3. *With some grading shifts*

$$CF^*(\mathcal{K}_\beta, \mathcal{K}_\gamma) \cong CF^*(\mathcal{K}_\gamma, \mathcal{K}_\delta) \cong CF^*(\mathcal{K}_\delta, \mathcal{K}_\beta) \cong \bigotimes^{n-1} H^*(S^2) \tag{5.0.3}$$

and thus all generators are cocycles.

Proposition 5.0.4. *Fix a choice of equivariant structures on Lagrangians, there is a well-defined weight grading for c_1, c_2 and c_3.*

Proof. From Corollary 5.0.3, we know each of the Floer groups above can be made such that weight grading equals to homological grading. Because of the Hamiltonian isotopy invariance, we know that small perturbation of β, γ and δ-arcs will not change the relative weight grading. Together with the observation that choices of equivariant structures will only apply an overall grading shift to all weight gradings, thus we know the weight grading of any element in CF^n must be the same. If we fix a set of equivariant structures on $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$, we have a well-defined weight grading for each c_i from its homological grading plus the effect a grading shift from changing the equivariant structure from the standard one. □

Before we prove that $wt(c_1) + wt(c_2) + wt(c_3) = 0$. Recall the following lemma from Seidel:

Lemma 5.0.5. *[17, Lemma 3.7] A triple $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$ form an exact triangle of Lagrangians if and only if there exist $c_1 \in CF^1(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in CF^0(\mathcal{K}_\gamma, \mathcal{K}_\beta)$, $c_3 \in CF^0(\mathcal{K}_\delta, \mathcal{K}_\gamma)$, $h_1 \in HF^0(\mathcal{K}_\gamma, \mathcal{K}_\beta)$, $h_2 \in HF^0(\mathcal{K}_\beta, \mathcal{K}_\delta)$ and $k \in HF^{-1}(\mathcal{K}_\beta, \mathcal{K}_\beta)$ such that*

$$\mu^1(h_1) = \mu^2(c_3, c_2) \tag{5.0.4}$$

$$\mu^1(h_2) = -\mu^2(c_1, c_3) \tag{5.0.5}$$

$$\mu^1(k) = -\mu^2(c_1, h_1) + \mu^2(h_2, c_2) + \mu^3(c_1, c_3, c_2) - e_{\mathcal{K}_\beta} \tag{5.0.6}$$

Lemma 5.0.6. *For any choice of equivariant structures on Lagrangians, the sum of weight gradings $wt(c_1) + wt(c_2) + wt(c_3) = 0$.*

Proof. We know μ^1 vanishes on all the Floer groups above. If we know that $h_1 = 0$ and $h_2 = 0$, then the equation (5.6) becomes $\mu^3(c_1, c_3, c_2) = e_{\mathcal{K}_\beta}$. Together with the fact weight grading is compatible with Floer product, see Remark 4.2.2, we know that $wt(c_1) + wt(c_2) + wt(c_3) = wt(e_{\mathcal{K}_\beta})$. From [3, Lemma 4.10], no matter which equivariant structure we choose on $\mathcal{K}_\beta$, the identity always has the weight grading 0.

To see $h_1 = 0$ and $h_2 = 0$, we need to look into the absolute grading. We claim that $HF^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $HF^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$ are supported in odd degrees. This is because pairing γ with β gives a flattened braid diagram for braid σ with has writhe 1. The number of strands is even and thus this group is supported in odd degrees. Similarly, pairing β with δ gives σ^{-1}, this also gives an odd degree supported group. Notice that the third map c_3 should have degree 2 in the absolute grading case, thus the exact triangle is actually between $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta[2]$. But shifting the grading of Floer groups by 2 does not change the fact that h_1 and h_2 have even degrees. Thus, they must be 0. □

Thus we conclude the following,

Proposition 5.0.7. *The long exact sequence Equation 5.0.2*

$$\ldots \xrightarrow{c_1} HF^{*,wt_1}(\mathcal{K}_\alpha, \mathcal{K}_\beta) \xrightarrow{c_2} HF^{*,wt_2}(\mathcal{K}_\alpha, \mathcal{K}_\gamma) \xrightarrow{c_3} HF^{*,wt_3}(\mathcal{K}_\alpha, \mathcal{K}_\delta) \xrightarrow{c_1} \ldots \tag{5.0.7}$$

decomposes with respect to weight gradings.

Proof. From Proposition 5.4, we know that there is a well-defined weight grading for c_1, c_2 and c_3. If we start with the first group $HF^{*,wt_1}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$, the only non-trivial map will be at weight grading $wt_1 + wt(c_2)$ because the weight grading is compatible with Floer products. The same goes for $wt_3 = wt_1 + wt(c_3) + wt(c_2)$. The next weight grading should be $wt_1 + wt(c_2) + wt(c_3) + wt(c_1) = wt_1$, which is exactly where we started with the first group. □

Chapter 6: Isomorphism to $Kh^{*,*}$

Now we have enough to prove our main theorems.

Theorem 6.0.1. *Symplectic Khovanov homology, graded by (k, wt), and Khovanov homology, graded by (i, j), are isomorphic as bigraded vector spaces over any characteristic zero field, where the gradings are related by $k = i - j$ and $wt = -j + c$, with an ambiguity of a grading shift c on relative weight grading.*

Our isomorphism is a graded refinement of the isomorphism in [2].

Proposition 6.0.2. *[2, Theorem 7.5] For any link L, we have an isomorphism Φ such that*

$$\Phi : Kh^*_{symp}(L) \to Kh^*(L) \tag{6.0.1}$$

This is an isomorphism with only information on the homological grading. But Abouzaid-Smith's result implies that the long exact sequence in Equation 5.0.2 commutes with the corresponding long exact sequence in Khovanov homology.

Lemma 6.0.3. *Fix a link diagram L_+ and its resolutions L_0 and L_δ at one of the crossings. We represent their corresponding bridge diagrams with (α, β), (α, γ) and (α, δ)-arcs respectively such that α β, γ and δ are locally shown in Figure 5.1. The isomorphism Φ is compatible with the exact sequence Equation 5.0.2, i.e. the following diagram commutes*

$$\begin{array}{ccccccccc} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\gamma) & \longrightarrow & HF^{*+2}(\mathcal{K}_\alpha, \mathcal{K}_\delta) & \longrightarrow & HF^{*+1}(\mathcal{K}_\alpha, \mathcal{K}_\beta) & \longrightarrow & HF^{*+1}(\mathcal{K}_\alpha, \mathcal{K}_\gamma) & \longrightarrow & HF^{*+3}(\mathcal{K}_\alpha, \mathcal{K}_\delta) \\ \downarrow{\Phi} & & \downarrow{\Phi} & & \downarrow{\Phi} & & \downarrow{\Phi} & & \downarrow{\Phi} \\ Kh^*(L_0) & \longrightarrow & Kh^{*+2}(L_\infty) & \longrightarrow & Kh^{*+1}(L_+) & \longrightarrow & Kh^{*+1}(L_0) & \longrightarrow & Kh^{*+3}(L_\infty) \end{array}$$

where the first line is the exact sequence of Equation 5.0.2 and the second line is the exact sequence for combinatorial Khovanov homology with grading $i - j$.

Proof. In the construction of Abouzaid-Smith's isomorphism, they show that the cup functors of Khovanov and symplectic Khovanov are identified with the isomorphism in the arc algebra. Cap functors in both cases are just adjoint to the corresponding cup functors. Thus the maps given by applying cap-cup functor in the second square commute with the isomorphism. The third group is constructed as the mapping cone in either case, and thus the naturality of Φ should give the information of other squares. □

Corollary 6.0.4. *The isomorphism Φ also commutes with the exact sequences of resolving a negative crossing.*

We compared the bigradings of two theories in the end of Section 4.1 for unlinks:

Proposition 6.0.5. *[3, Proposition 6.11] The isomorphism Φ preserves the weight grading if D is crossingless diagram, with grading correspondence $k = i - j$ and $wt = -j$.*

Proof of Theorem 6.0.1. We only need to prove that for any fixed Jones grading j_0, Φ is also an isomorphism with $wt = -j_0 + c$ with some grading shift c,

$$\Phi : Kh^{*,-j_0}_{symp}(L) \to Kh^{*,j_0}(L) \tag{6.0.2}$$

We prove that this statement is true for any link (bridge) diagram through an induction on number of crossings. The base case for unlinks is proved with Proposition 6.0.5.

Now we assume that L_+ is a link diagram with n crossings, where as its resolutions L_0 and L_∞ have $(n-1)$ crossings. Let us assume we are doing resolutions at a positive crossing, if all crossings are negative, a similar argument can be applied for the exact sequences induced by doing resolution at a negative crossing. From the inductive assumption, Φ are bigraded isomorphisms for L_0 and L_∞. Let us fix a Jones grading j_1 on $Kh(L_0)$. The maps in the long exact sequence will be trivial unless the Jones grading j_2 on $Kh(L_\infty)$ is $j_1 - 3v - 2$, where v is the signed count of crossings between this arc and other components, and j_3 on $Kh(L_+)$ to be $j_1 + 1$. Thus we can decompose our commutative diagram with respect to the Jones grading:

$$\begin{array}{ccccccccc}
HF^{*,-j_1}(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+2,-j_2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) & \xrightarrow{c_3} & HF^{*+1,j_3'}(\mathcal{K}_\alpha,\mathcal{K}_\beta) & \xrightarrow{c_1} & HF^{*+1,-j_1}(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+3,-j_2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) \\
\Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \downarrow\Phi \\
Kh^{*,j_1}(L_0) & \longrightarrow & Kh^{*+2,j_2}(L_\infty) & \longrightarrow & Kh^{*+1,j_3}(L_+) & \longrightarrow & Kh^{*+1,j_1}(L_0) & \longrightarrow & Kh^{*+3,j_2}(L_\infty)
\end{array}$$

where the weight grading j_3' is given by $-j_2+wt(c_3)$. The first, second, fourth and fifth columns are all isomorphisms, so by the five lemma we conclude that the third column is also an isomorphism and thus we know Φ is also a bigraded isomorphism for L_+. As for the grading correspondence, because the c_i all have fixed weight grading after specifying the choice of equivariant structures, if we change j_1 by any number k, we change j_3 also by k. As for the first row, if $-j_1$ is changed into $-j_1 - k$, this will result in j_3' shifting by $-k$ as well. This is enough to show that as a relative grading, the weight grading recovers the Jones grading as a relative grading.

□

As a corollary of the theorem above, we can also conclude Theorem 1.2.1 that the relative weight grading is independent of the choice of link diagrams.

Proof of Theorem 1.2.1. For any two bridge diagrams D and D' representing a link L, relative weight grading wt and wt' can be defined on D, and respectively D'. Theorem 6.0.1 indicates that both wt and wt' coincide with $-j$ with as relative gradings. Thus wt and wt' are the same as relative gradings. □

References

[1] P. Seidel and I. Smith, "A link invariant from the symplectic geometry of nilpotent slices," Duke Math. J., vol. 134, no. 3, pp. 453–514, 2006.

[2] M. Abouzaid and I. Smith, "Khovanov homology from Floer cohomology," J. Amer. Math. Soc., vol. 32, no. 1, pp. 1–79, 2019.

[3] ——, "The symplectic arc algebra is formal," Duke Math. J., vol. 165, no. 6, pp. 985–1060, 2016.

[4] C. Manolescu, "Nilpotent slices, Hilbert schemes, and the Jones polynomial," pp. 1–43, 2004.

[5] J. W. Waldron, "An invariant of link cobordisms from symplectic khovanov homology," ArXiv:0912.5067,

[6] K. Hendricks, R. Lipshitz, and S. Sarkar, "A flexible construction of equivariant Floer homology and applications," J. Topol, vol. 9, no. 4, pp. 1153–1236, Oct. 2015.

[7] P. Seidel and J. P. Solomon, "Symplectic cohomology and q-intersection numbers," Geom. Funct. Anal., vol. 22, no. 2, pp. 443–477, 2012.

[8] H. Nakajima, "Lectures on Hilbert schemes of points on surfaces," Amer. Math. Soc, vol. 18, 1999.

[9] M Pozniak, "Floer homology, Novikov rings and clean intersections," Amer. Math. Soc, no. 196, pp. 199–181, 1999.

[10] A. Weinstein, "Lagrangian submanifolds and Hamiltonian systems," Annals of Math, vol. 98, pp. 377–410, 1973.

[11] C. Y. Mak and I. Smith, "Fukaya-Seidel categories of Hilbert schemes and parabolic category $\mathcal{O}$," ArXiv:1907.07624,

[12] K. Hendricks, R. Lipshitz, and S. Sarkar, "Correction to the paper "a flexible construction of equivariant Floer homology and applications"," ArXiv:1810.03049,

[13] R. Lipshitz, "A cylindrical reformulation of Heegaard floer homology," Geom. Topol., vol. 10, pp. 955–1097, 2006.

[14] J. M. Droz and E. Wagner, "Grid diagrams and Khovanov homology," Algebr. Geom. Topol., vol. 9, no. 3, pp. 1275–1297, 2009.

[15] P. Ozsváth and Z. Szabó, "Holomorphic disks and topological invariants for closed three-manifolds," Annals of Math, vol. 159, no. 3, pp. 1027–1158, 2004.

[16] H. Hofer, V. Lizan, and J.-C. Sikorav, "On genericity for holomorphic curves in four-dimensional almost-complex manifolds," J. Geom. Anal., vol. 7, no. 1, pp. 149–159, 1997.

[17] P. Seidel, "Fukaya categories and picard-lefschetz theory," Zurich Lectures in Advanced Mathematics, 2008.