Brauer class over the Picard scheme of curves

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ABSTRACT

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We study the Brauer classes rising from the obstruction to the existence of tautological line bundles on the Picard scheme of curves. We establish various properties of the Brauer classes for families of smooth curves. We compute the period and index of the Brauer class associated with the universal smooth curve for a fixed genus. We also show such Brauer classes are trivialized when we specialize to certain generalized theta divisors. If we consider the universal totally degenerate curve with a fixed dual graph, using symmetries of the graph, we give bounds on the period and index of the Brauer classes. As a result, we provide some division algebras of prime degree, serving as candidates for the cyclicity problem. As a byproduct, we re-calculate the period and index of the Brauer class for universal smooth genus g curve in an elementary way. We study certain conic associated with the universal totally degenerate curve with a fixed dual graph. We show the associated conic is non-split in some cases. We also study some other related geometric properties of Brauer groups.

Contents

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Chapter 1

Introduction

1.1 Motivation

Let F be a field. Let A be a finite dimensional associative F-algebra. We say A is a central simple algebra, if there exists a finite field extension F'/F, such that $A \otimes_F F'$ is isomorphic to the matrix algebra $M_n(F')$. In this case, we call F' a splitting field of A, and we call $n = \sqrt{\dim_F(A)}$ the degree of A. If we can choose $F' \subset A$, with F'/F Galois and $\mathrm{Gal}(F'/F) \cong \mathbb{Z}/n\mathbb{Z}$, then we call A a cyclic algebra. One can show cyclic algebras have very explicit description, see [GS17, 2.5.3].

In [ABGV11], the authors mentioned an important open problem in the theory of central simple algebras:

Problem 1. *Given a prime number p, construct a non-cyclic division algebra of degree p over field F.*

Albert-Brauer-Hasse-Noether shown such examples do not exist if F is a local or global field, see [GS17, 6.5.6]. Saltman shown such examples do not exist if F is the function field of an l-adic curve for $l \neq p$, see [Sal07].

There are various candidates for non-cyclic algebras of prime degree, many of them are of algebraic flavor.

Now let's take a cohomological point of view. Note that the set of central simple algebras over F modulo Morita equivalence, equipped with tensor product, form a group $\mathrm{Br}(F) = \mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}(F), \mathbb{G}_m)$. Many properties of the algebra can be read off from its class in the Brauer group. For example, Wedderburn's theorem shown that every central simple algebra is isomorphic to a matrix algebra over a unique division algebra, the degree of the division algebra is an important invariant of the division algebra, called its index. Given a central simple algebra A, one can show its index $\mathrm{ind}(A)$ only depends on its class $[A] \in \mathrm{Br}(F)$. Also we call the order of $[A]$ in $\mathrm{Br}(F)$ the period of A, denoted by $\mathrm{per}(A)$. We refer to [GS17, 4.5] for the details.

We study certain Brauer classes that come from geometry, and we determine the period and index of the Brauer classes. As a result, we add new candidates to the list of prospectively non-cyclic algebras of prime degree. We also answer various geometrical questions naturally raised in the study of the Brauer classes.

1.2 Structure of the thesis

In this thesis, we study various properties of a certain Brauer class over the Picard scheme of curves.

In chapter 2, we introduce the Brauer class associated with smooth curves. We describe the class as an obstruction class to the existence of a tautological line bundle. We show the class is described by a nice Brauer–Severi scheme: the symmetric product of the curve. We show the Brauer class is related to the Picard torsor of the curve in a subtle way.

In chapter 3, we show the Brauer class associated with the universal genus g curve has period and index $g-1$. The proof uses some nontrivial known results: specialization of tame fundamental group and Franchetta's theorem. We study the restriction of the Brauer class to certain explicit subvarieties of the Picard scheme. We show the Brauer class restricts to zero on the function field of the generalized theta divisors of semistable rank 2 vector bundles on the universal curve.

In chapter 4, we study the Brauer class associated with the universal totally degenerate curve of a fixed dual graph. We associate the class to certain finite group cohomology class. We read off the period and index of the Brauer class from symmetries of the graph. By specialization, we reprove that the period and index of the Brauer class associated with the smooth universal genus g curve are $g-1$. This proof is self-contained and elementary. We give some examples of division algebras of prime degree. We also show involution of second kind exist for such Brauer classes.

In chapter 5, we work over a field k_0 of characteristic zero. We study properties of the universal totally degenerate curve with a fixed dual graph Γ. Denote this curve by $X_{\Gamma,k_0}/k_{\Gamma,k_0}$. By taking Stein factorization of the normalization, we get a conic $X^{\nu}_{\Gamma,k_0}/\mathrm{Spec}(\mathrm{H}^0(X^{\nu}_{\Gamma,k_0},\mathcal{O}))$. The triviality of the conic implies the triviality of the Picard torsor $\mathrm{Pic}^{1,1,\ldots,1}_{X_{\Gamma,k_0}/k_{\Gamma,k_0}}$. We show in some interesting cases, the conic is nontrivial. The two key inputs are the universal property of the universal curve and class field theory.

In chapter 6, we collect some miscellaneous geometric properties observed in the study of the Brauer class. Note that the $(2g-2)$-th symmetric product of a smooth genus g curve over its Picard scheme is only a Brauer-Severi scheme away from a point. We generalize the situation and study the purity of Brauer class associated with some "non-flat Brauer–Severi scheme". We also include a Lefschetz theorem for the Brauer group.

Chapter 2

The Brauer class associated with a smooth curve

2.1 Setup

Let k be a field. Let $g \geq 3$ be an integer. Let C be a proper smooth genus g curve over k. Consider the following presheaf on site $(\mathrm{Sch}/k)_{\acute{e}t}$,

$$P_{C/k} \colon (\mathrm{Sch}/k)_{\acute{e}t} \to \mathrm{Sets}, \qquad T \mapsto \mathrm{Pic}(C \times_k T).$$

Denote the étale sheafification of $P_{C/k}$ by $\mathcal{P}ic_{C/k}$. The functor $\mathcal{P}ic_{C/k}$ is represented by a scheme, denoted by $\mathrm{Pic}_{C/k}$. The scheme $\mathrm{Pic}_{C/k}$ is a disjoint union of smooth varieties, the connected components are parameterized by the relative degree of line bundles that it represents. Let $\mathrm{Pic}^{2g-2}_{C/k}$ be the degree $2g-2$ component of its Picard scheme.

Consider the following functor from $(\mathrm{Sch}/k)^{opp}$ to $\mathcal{S}ets$,

$$D^{2g-2}_{C/k} : T \mapsto \left\{ \begin{array}{c} \text{Relative effective Cartier divisors} \\ \text{on } C \times_k T \text{ of relative degree } 2g-2 \end{array} \right\}.$$

The functor $D^{2g-2}_{C/k}$ is represented by a scheme $\mathrm{Div}^{2g-2}_{C/k} \cong \mathrm{Sym}^{2g-2}_{C/k}$, see [Bos90, 9.3.3].

We have a morphism

$$\pi_{C/k}\colon \mathrm{Div}^{2g-2}_{C/k} \to \mathrm{Pic}^{2g-2}_{C/k}$$

defined by $D \mapsto \mathcal{O}_{C\times_k T}(D)$ on T-points. It follows from general fact of representable functors that the formation of "Pic" and "Div" commute with base change. If k'/k is a field extension, then the map $\pi_{C_{k'}/k'}\colon \mathrm{Div}^{2g-2}_{C_{k'}} \to \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ is the base change of $\pi_{C/k}$ along k'/k.

Let k' be a finite Galois extension of k, such that $C_{k'}$ has a k' point $c\colon \mathrm{Spec}(k') \to C_{k'}$. In this case, the curve $C_{k'}/k'$ has a rational point. By [Bos90, 8.1.4], there exists a line bundle $\mathcal{L}_0$ on $C_{k'} \times \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$, and an isomorphism

$$\alpha_0\colon (c \times 1_{\mathrm{Pic}^{2g-2}_{C_{k'}/k'}})^*\mathcal{L}_0 \cong \mathcal{O}_{\mathrm{Pic}^{2g-2}_{C_{k'}/k'}},$$

such that the pair $\left(\mathrm{Pic}^{2g-2}_{C_{k'}/k'}, (\mathcal{L}_0, \alpha_0)\right)$ represents the "rigidified Picard functor"

$$T \mapsto \left\{(\mathcal{L}, \alpha) \,\middle|\, \mathcal{L} \in \mathrm{Pic}^{2g-2}(C_{k'} \times T),\ \alpha\colon (c \times 1_T)^*\mathcal{L} \cong \mathcal{O}_T \right\} / \sim .$$

We call $\mathcal{L}_0$ the tautological line bundle on $C_{k'} \times \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ that is trivialized along c.

Note that the canonical divisor ω_C represents a k-rational point $\{\omega_C\}$ on $\mathrm{Pic}^{2g-2}_{C/k}$. Let $S = \mathrm{Pic}^{2g-2}_{C/k} \backslash \{\omega_C\}$ be its complement. Let $\pi'\colon C_{k'} \times \mathrm{Pic}^{2g-2}_{C_{k'}/k'} \to \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ be projection onto the second factor. Over $S_{k'}$, we know $\mathbb{P}(\pi'_*\mathcal{L}_0) := \mathrm{Proj}(\mathrm{Sym}^\bullet((\pi'_*\mathcal{L}_0)^\vee)) \cong \mathrm{Div}^{2g-2}_{C_{k'}/k'}$, see [Bos90, 8.2.7]. Note that $(\pi'_*\mathcal{L}_0)_{S_{k'}}$ is a locally free sheaf of rank $g-1$, so $\pi_{C/k}\colon \pi^{-1}_{C/k}(S) \to S$ is a Brauer-Severi scheme, and it gives a class $\alpha \in H^2(S, \mathbb{G}_m) = H^2(\mathrm{Pic}^{2g-2}_{C/k}, \mathbb{G}_m)$, see 2.4.1 for detail. Here the equality is given by purity of Brauer group, see [Gro68b, 6.1] and [Ces17, 1.1]. We will study this class carefully in the rest of this chapter.

2.2 Description of cohomology classes

For a general scheme, the étale cohomology group $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$ cannot be calculated by colimit of Čech cohomology. However, every class can be represented on a Čech cover by some data. We explain this in detail in this section.

Here is the idea: Cohomology classes are locally trivial, see [Sta18, 01FW], thus for any given cohomology class, we may trivialize it on some Čech over. Note that the Čech to derived functor cohomology sequence converges, thus we may calculate the cohomology classes in the total complex of Čech complex. The terms in total complex of Čech double complex are exactly the data we'll describe.

Let X be a scheme, let $\mathcal{U} = \{U_i \xrightarrow{\phi_i} X\}_{i \in I}$ be an étale cover of X. Denote $U_i \times_X U_j$ by U_{ij}, denote $U_i \times_X U_j \times_X U_k$ by U_{ijk}, etc.

Given line bundles $\mathcal{L}_{ij}$ on U_{ij} and isomorphisms $\alpha_{ijk} \colon \mathcal{L}_{ij}|_{U_{ijk}} \otimes \mathcal{L}_{jk}|_{U_{ijk}} \cong \mathcal{L}_{ik}|_{U_{ijk}}$ such that $\alpha_{ikl} \circ (\alpha_{ijk} \otimes 1) = \alpha_{ijl} \circ (1 \otimes \alpha_{jkl})$, we denote such assignment by datum $D := (\mathcal{U}, \mathcal{L}, \alpha)$. Given data D and D', we say D' is a refinement of D, if $\mathcal{U}'$ is a refinement of $\mathcal{U}$, and there exist isomorphisms $\psi_{ij} \colon \mathcal{L}_{ij}|_{U'_{ij}} \to \mathcal{L}'_{ij}$, such that $\alpha'_{ijk} \circ (\psi_{ij}|_{U_{ijk}} \otimes \psi_{jk}|_{U_{ijk}})|_{U'_{ijk}} = \psi_{ik}|_{U'_{ijk}} \circ \alpha_{ijk}|_{U'_{ijk}}$. We identify two datum if they have a common refinement.

There is a natural abelian group law on the set of equivalence classes of data on X. Given data $D = (\mathcal{U}, \mathcal{L}, \alpha), D' = (\mathcal{U}', \mathcal{L}', \alpha')$, we pick a common refinement $\mathcal{U}''$ of $\mathcal{U}, \mathcal{U}'$, the group law is defined to be $D + D' = (\mathcal{U}'', \mathcal{L}|_{\mathcal{U}''} \otimes \mathcal{L}'|_{\mathcal{U}''}, \alpha|_{\mathcal{U}''} \otimes \alpha'|_{\mathcal{U}''})$. Denote the group of equivalence classes of datum by Z_X.

Now for each i, for a fix the choice of a line bundles $\mathcal{N}_i$ on U_i, let

$$\alpha_{can} \colon \mathcal{N}_i|_{U_{ijk}} \otimes \mathcal{N}_j^{-1}|_{U_{ijk}} \otimes \mathcal{N}_j|_{U_{ijk}} \otimes \mathcal{N}_k|^{-1}_{U_{ijk}} \to \mathcal{N}_i|_{U_{ijk}} \otimes \mathcal{N}_k|^{-1}_{U_{ijk}}$$

be the canonical identification. The datum $(\mathcal{U}, \mathcal{N}_i \otimes \mathcal{N}_j^{-1}, \alpha_{can})$ form a subgroup B_X of Z_X as we change the choice of $\mathcal{N}_i$s. Denote the quotient Z_X/B_X by D_X.

Proposition 2.2.1. *Let X be a scheme, then there is a group isomorphism*

$$\eta_X \colon D_X \to \mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m).$$

Proof. Let $\mathbb{G}_m \to \mathcal{I}^\bullet$ be an injective resolution. Let $\mathcal{I}^\bullet_{\leq 0} = \{\mathcal{I}^0 \to \mathcal{I}^0/\mathbb{G}_m\}$ be the canonical truncation in non-positive degrees. Let $\mathcal{U} = \{U_i \overset{\phi_i}{\to} X\}_{i\in I}$ be an étale covering of X. We denote the Čech resolution of the complex $\mathcal{I}^\bullet_{\leq 0}$ associated with the covering $\mathcal{U}$ by

$$\mathcal{I}^\bullet_{\leq 0} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet_{\leq 0}).$$

Let

$$\phi\colon \mathcal{I}^\bullet_{\leq 0} \to \mathcal{I}^\bullet$$

be the natural inclusion. It induces a morphism of double complexes

$$\mathcal{C}(\mathcal{U}, \phi)\colon \mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet_{\leq 0}) \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet).$$

Taking global sections, we get a morphism of double complexes

$$E_0(\mathcal{U}, \phi)\colon E_0^{\bullet,\bullet} := C^\bullet(\mathcal{U}, \mathcal{I}^\bullet_{\leq 0}) \to {E'_0}^{\bullet,\bullet} := C^\bullet(\mathcal{U}, \mathcal{I}^\bullet).$$

This induces a morphism of their associated spectral sequences. On the E_1 page we have

$$\begin{array}{ccc}
\prod_{i,j,k} \mathrm{H}^0(U_{ijk}, \mathbb{G}_m) & \prod_{i,j,k} \mathrm{Pic}(U_{ijk}) & 0 \\
\prod_{i,j} \mathrm{H}^0(U_{ij}, \mathbb{G}_m) & \prod_{i,j} \mathrm{Pic}(U_{ij}) & 0 \\
\prod_{i} \mathrm{H}^0(U_{i}, \mathbb{G}_m) & \prod_{i} \mathrm{Pic}(U_{i}) & 0
\end{array}$$

$$\downarrow E_1(\mathcal{U}, \phi)$$

$$\begin{array}{ccc}
\prod_{i,j,k} \mathrm{H}^0(U_{ijk}, \mathbb{G}_m) & \prod_{i,j,k} \mathrm{Pic}(U_{ijk}) & * \\
\prod_{i,j} \mathrm{H}^0(U_{ij}, \mathbb{G}_m) & \prod_{i,j} \mathrm{Pic}(U_{ij}) & * \\
\prod_{i} \mathrm{H}^0(U_{i}, \mathbb{G}_m) & \prod_{i} \mathrm{Pic}(U_{i}) & \prod_{i} \mathrm{H}^2(U_{i}, \mathbb{G}_m)
\end{array}$$

Note that the first two columns of the spectral sequence are naturally isomorphic via $E_1(\mathcal{U}, \phi)$, and the differential on E_1 page is vertical, so ϕ induces an isomorphism of

$$E_2(\mathcal{U}, \phi)\colon E_2^{p,q} \to {E'}_2^{p,q}$$

for the first two columns $q = 0, 1$.

By five-lemma applied to

$$E_2^{0,1} \to E_2^{2,0} \to F^1 E_\infty^2 \to E_2^{1,1} \to E_2^{3,0},$$

the map $E_\infty(\mathcal{U}, \phi)$ induces an isomorphism of the second filtrated piece of the abutment of $\{E_2^{p,q}\}$ and $\{E_2'^{p,q}\}$. By construction, the spectral sequence $\{{E'}_2^{p,q}\}$ abuts to $\mathrm{H}^i(X, \mathbb{G}_m)$. Let's denote the abutment of the spectral sequence $\{E_2^{p,q}\}$ by V, and denote the isomorphism on second filtrated piece of cohomology by

$$\psi\colon F^1V \xrightarrow{\sim} F^1\mathrm{H}^2(X,\mathbb{G}_m).$$

(1) Let's construct datum from cohomology class. Given a class $a \in \mathrm{H}^2(X,\mathbb{G}_m)$, take a covering $\mathcal{U}$ fine enough so that the edge morphism maps a to $0 \in E_2'^{0,2} = \prod_i \mathrm{H}^0(U_i,\mathbb{G}_m)$. Then $a \in F^1\mathrm{H}^2(X,\mathbb{G}_m)$. By the isomorphism ψ, we may calculate a in the abutment F^1V, hence a is represented by an element in $\widetilde{a}$ in $Z^2(\mathrm{Tot}^\bullet(E_0^{\bullet,\bullet}))$, the group of 2-cocycles of the total complex of $E_0^{\bullet,\bullet}$.

Let's write the group law of $\mathcal{I}^0$ multiplicatively, and denote the map $\mathcal{I}^0 \to \mathcal{I}^0/\mathbb{G}_m$ by $t \mapsto \bar{t}$. Then $\widetilde{a}$ can be written as

$$(s_{ijk}, q_{ij}) \in C^2(\mathcal{U},\mathcal{I}^0) \oplus C^1(\mathcal{U},\mathcal{I}^0/\mathbb{G}_m).$$

The cocycle condition on $\widetilde{a}$ tells us

$$s_{jkl}\cdot s_{ikl}^{-1}\cdot s_{ijl}\cdot s_{ijk}^{-1} = 1,$$

$$\overline{s_{ijk}} = q_{jk}\cdot q_{ik}^{-1}\cdot q_{ij}.$$

In the following argument, for convenience, we identify an line bundle and its associated $\mathbb{G}_m$-torsor, thus tensor product of line bundles is identified with addition in $\mathrm{H}^1(-,\mathbb{G}_m)$.

Let $\mathcal{L}_{q_{ij}}$ be the $\mathbb{G}_m$-torsor (subsheaf) in $\mathcal{I}^0|_{U_{ij}}$ locally generated by liftings of q_{ij}^{-1}. Identify $\mathcal{L}_{q_{ij}} \otimes \mathcal{L}_{q_{jk}}$ with $\mathcal{L}_{q_{ij}\cdot q_{jk}}$ canonically as subsheaves of $\mathcal{I}^0$. Consider the isomorphism $\phi_{s_{ijk}}\colon \mathcal{I}^0|_{U_{ijk}} \to \mathcal{I}^0|_{U_{ijk}}$, defined by multiplying sections with s_{ijk}^{-1}. This isomorphism induces isomorphism of subsheaves $\alpha_{ijk}\colon \mathcal{L}_{q_{ij}\cdot q_{jk}} \cong \mathcal{L}_{q_{ik}}$.

Condition on s_{ijk} implies

$$\alpha_{jkl}\circ(\alpha_{ijk}\otimes 1) = \alpha_{ijl}\circ(1\otimes\alpha_{jkl}),$$

so we get required datum $\eta_X(a) := (\mathcal{U},\mathcal{L},\alpha)$ associated with the class a. If (s_{ijk}, q_{ij}) is a coboundary, then there exists $(s_{ij}, q_i) \in C^1(\mathcal{U}, \mathcal{I}^0) \oplus C^0(\mathcal{U}, \mathcal{I}^0/\mathbb{G}_m)$ such that

$$s_{ijk} = s_{ij} \cdot s_{jk} \cdot s_{ik}^{-1},$$

$$\overline{s_{ij}} = q_{ij} \cdot q_i \cdot q_j^{-1}.$$

Then the datum $(\mathcal{U}, \mathcal{L}_{q_{ij}}, \alpha_{ijk})$ lies in B_X, because it is equivalent to $(\mathcal{U}, \mathcal{L}_{q_i} \otimes \mathcal{L}_{q_j}^{-1}, \alpha_{can})$ via isomorphism induced by multiplying sections of $\mathcal{I}^0|_{U_{ij}}$ by s_{ij}^{-1}. Hence we get a well-defined map $\eta_X \colon \mathrm{H}^2(X, \mathbb{G}_m) \to D_X$.

(2) Let's construct cohomology class from datum $(\mathcal{U}, \mathcal{L}, \alpha)$. By the exact sequence

$$\mathrm{H}^0(U_{ij}, \mathcal{I}^0) \to \mathrm{H}^0(U_{ij}, \mathcal{I}^0/\mathbb{G}_m) \to \mathrm{H}^1(U_{ij}, \mathbb{G}_m) \to 0,$$

we can pick section $q_{ij} \in \Gamma(U_{ij}, \mathcal{I}^0/\mathbb{G}_m)$ representing $\mathcal{L}_{ij}$. Identify $\mathcal{L}_{q_{ij}} \otimes \mathcal{L}_{q_{jk}}$ with $\mathcal{L}_{q_{ij} \cdot q_{jk}}$ as before. Then the element $(\alpha_{ijk}(1), q_{ij}) \in C^2(\mathcal{U}, \mathcal{I}^0) \oplus C^1(\mathcal{U}, \mathcal{I}^0/\mathbb{G}_m)$ is a cocycle, it represents some cohomology class in $F^1V \cong F^1\mathrm{H}^2(X, \mathbb{G}_m)$.

If $(\mathcal{U}, \mathcal{L}, \alpha) \in B_X$, after possibly enlarging $\mathcal{I}^\bullet$, we can pick $q_i \in \Gamma(U_i, \mathcal{I}^0)$ such that the subsheaf $q_i^{-1} \cdot \mathbb{G}_m$ of $\mathcal{I}_0$ represents $\mathcal{N}_i$. Then $q_{ij} = \overline{q_i \cdot q_j^{-1}}$, and

$$(\alpha_{ijk}(1), q_{ij}) \in Z^2(\mathrm{Tot}^\bullet(E_0^{\bullet,\bullet})) \subseteq C^2(\mathcal{U}, \mathcal{I}^0) \oplus C^1(\mathcal{U}, \mathcal{I}^0/\mathbb{G}_m)$$

is the coboundary of

$$(q_i \cdot q_j^{-1}, \overline{q_i}) \in Z^1(\mathrm{Tot}^\bullet(E_0^{\bullet,\bullet})) \subseteq C^1(\mathcal{U}, \mathcal{I}^0) \oplus C^0(\mathcal{U}, \mathcal{I}/\mathbb{G}_m).$$

Thus we get a homomorphism $D_X = Z_X/B_X \to F^1\mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(X, \mathbb{G}_m)$. It is clear by the construction that this homomorphism and previously defined η_X are inverse to each other. $\square$

2.3 The Brauer class as obstruction

Consider the functor

$$P^{2g-2}_{C/k}\colon (\mathrm{Sch}/k)_{\acute{e}t} \to \mathrm{Sets}, \qquad T \mapsto \mathrm{Pic}^{2g-2}(C \times_k T).$$

Its étale sheafification $\mathcal{P}ic^{2g-2}_{C/k}$ is represented by a line bundle $\mathcal{L}$ on some étale cover U of the Picard scheme $\mathrm{Pic}^{2g-2}_{C/k}$. We use the description of cohomology class by data (see 2.2) to give an explicit description of the obstruction map $\mathrm{d}_2^{0,1}$, see 2.3.1. We show the obstruction class $\mathrm{d}_2^{0,1}(1)$ coincides with the class of Brauer-Severi scheme $\mathrm{Sym}^{2g-2}_{C/k} \to \mathrm{Pic}^{2g-2}_{C/k}$ (away from the point ω_C), see 2.4.2.

First let's explain the general setup for the obstruction map to the existence of a tautological line bundle. Let $f\colon X \to S$ be a proper flat morphism of schemes with geometrically reduced and geometrically connected fibers. We denote the Picard functor $\mathcal{P}ic_{X/S}\colon (\mathrm{Sch}/S)_{\acute{e}t} \to \mathrm{Ab}$ to be the sheafification of the functor $T \mapsto \mathrm{Pic}(X \times_S T)$. The sheaf $\mathcal{P}ic_{X/S}$ can be identified as the sheaf $R^1 f_* \mathbb{G}_{m,X}$, and is representable by our assumption on f, see [Bos90, 8.1].

Lemma 2.3.1. *With the above setup, there exists an exact sequence*

$$\mathrm{H}^1(X, \mathbb{G}_m) \to \mathcal{P}ic_{X/S}(S) \xrightarrow{\mathrm{d}_2^{0,1}} \mathrm{H}^2(S, \mathbb{G}_m)$$

coming from Leray spectral sequence $\mathrm{H}^p(S, R^q f_* \mathbb{G}_m) \Rightarrow \mathrm{H}^{p+q}(X, \mathbb{G}_m)$. Let U be an étale cover of S, let $\mathcal{N}$ be a line bundle on $X \times_S U$, such that $(U, \mathcal{N})$ represents an element in $\mathcal{P}ic_{X/S}(S)$.

Denote the base change of projection $\mathrm{pr}_i\colon U \times_S U \to S$ along $X \to S$ by

$$r_i\colon X \times_S U \times_S U \to X \times_S U,$$

similarly we define

$$p_i\colon X \times_S U \times_S U \times_S U \to X \times_S U,$$

$$p_{ij} \colon X \times_S U \times_S U \times_S U \to X \times U \times_S U.$$

Denote base change of f still by f.

Let $\mathcal{R}$ be a line bundle on $U \times_S U$ such that $r_1^*\mathcal{N} \cong r_2^*\mathcal{N} \otimes f^*\mathcal{R}$.

Consider the following commutative diagram:

$$\begin{tikzcd}
p_{12}^*(r_1^*\mathcal{N}) \arrow[r, "\sim"] \arrow[d, equal] & p_{12}^*(r_2^*\mathcal{N} \otimes f^*\mathcal{R}) \arrow[r, equal] & p_{23}^*(r_1^*\mathcal{N}) \otimes f^*p_{12}^*\mathcal{R} \arrow[d, "\sim"] \\
p_{13}^*(r_1^*\mathcal{N}) \arrow[d, "\sim"] & & p_{23}^*(r_2^*\mathcal{N} \otimes f^*\mathcal{R}) \otimes f^*p_{12}^*\mathcal{R} \arrow[d, equal] \\
p_{13}^*(r_2^*\mathcal{N} \otimes f^*\mathcal{R}) \arrow[r, equal] & p_3^*\mathcal{N} \otimes f^*p_{13}^*\mathcal{R} & p_3^*\mathcal{N} \otimes f^*p_{23}^*\mathcal{R} \otimes f^*p_{12}^*\mathcal{R} \arrow[l, "b"']
\end{tikzcd}$$

Because $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally, by projection formula,

$$\mathrm{Hom}(f^*p_{12}^*\mathcal{R} \otimes f^*p_{23}^*\mathcal{R}, f^*p_{13}^*\mathcal{R}) = \mathrm{Hom}(p_{12}^*\mathcal{R} \otimes p_{23}^*\mathcal{R}, p_{13}^*\mathcal{R}).$$

The isomorphism b gives an element $\beta \in \mathrm{Hom}(p_{12}^*\mathcal{R} \otimes p_{23}^*\mathcal{R}, p_{13}^*\mathcal{R})$. Then $\mathrm{d}_2^{0,1}(U, \mathcal{N})$ is represented by the datum $(U, \mathcal{R}, \beta)$.

Proof. Embed $\mathbb{G}_{m,X}$ into an injective sheaf $\mathcal{I}_X$ on X. Consider the short exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & f_*(\mathcal{I}_X/\mathbb{G}_{m,X}) \arrow[r] & R^1f_*\mathbb{G}_{m,X} \arrow[r] & 0 \\
0 \arrow[r] & f_*\mathbb{G}_{m,X} \arrow[r] \arrow[u] & f_*\mathcal{I}_X \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Take Cartan-Eilenberg resolution of the diagram as sheaves on S. Then view the diagram as an exact sequence of complex of sheaves, where the first and second rows are placed in degree 0 and -1. The map $\mathrm{d}_2^{0,1}$ is the connecting homomorphism from the third column to the first column

$$\delta \colon \mathrm{H}^0(S, R^1f_*\mathbb{G}_{m,X}) \to \mathrm{H}^1(S, f_*\mathbb{G}_{m,X}[1]) = \mathrm{H}^2(S, f_*\mathbb{G}_{m,X}).$$

Let $\mathcal{U} = \{U \to X\}$ be the cover of X. Take a morphism of Čech resolution into the Cartan-Eilenberg resolution that induces identity on the diagram. The connecting homomorphism can be computed by Čech resolution, as long as we can represent an element by a Čech cocycle. Let $\mathbb{G}_{m,S} \to \mathcal{I}_S$ be an embedding of $\mathbb{G}_{m,S}$ into an injective sheaf. We take $\mathcal{I}_X$ to be an injective sheaf containing $f^*\mathcal{I}_S$. Since $\mathcal{F} \to f_*f^*\mathcal{F}$ is injective whenever f is a surjection, we know $\mathcal{I}_S \to f_*\mathcal{I}_X$ is an injection. Hence we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & f_*\mathcal{I}_X/\mathcal{I}_S \arrow[r] & \frac{f_*(\mathcal{I}_X/\mathbb{G}_{m,X})}{\mathcal{I}_S/\mathbb{G}_{m,S}} \arrow[r] & R^1f_*\mathbb{G}_{m,X} \arrow[r] & 0 \\
0 \arrow[r] & f_*\mathbb{G}_{m,X} \arrow[r] & f_*\mathcal{I}_X \arrow[r] \arrow[u, two heads] & f_*(\mathcal{I}_X/\mathbb{G}_{m,X}) \arrow[r] \arrow[u, two heads] & R^1f_*\mathbb{G}_{m,X} \arrow[r] \arrow[u, "\sim"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,S} \arrow[r] \arrow[u, "\sim"] & \mathcal{I}_S \arrow[r] \arrow[u, hook] & \mathcal{I}_S/\mathbb{G}_{m,S} \arrow[r] \arrow[u, hook] & 0 &
\end{tikzcd}$$

So we may replace the previous short exact sequence of complexes by the quasi-isomorphic diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_S/\mathbb{G}_{m,S} \arrow[r] & f_*(\mathcal{I}_X/\mathbb{G}_{m,X}) \arrow[r] & \frac{f_*(\mathcal{I}_X/\mathbb{G}_{m,X})}{\mathcal{I}_S/\mathbb{G}_{m,S}} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I}_S \arrow[r] \arrow[u] & f_*\mathcal{I}_X \arrow[r] \arrow[u] & f_*\mathcal{I}_X/\mathcal{I}_S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Hence we can calculate the connecting homomorphism δ by the total complex of the Čech resolution of its columns. Take Čech resolution of the third column, we have short exact sequence of sheaves

$$0 \to \mathcal{C}^\bullet(\mathcal{U}, f_*\mathcal{I}_X/\mathcal{I}_S) \to \mathcal{C}^\bullet\left(\mathcal{U}, \frac{f_*(\mathcal{I}_X/\mathbb{G}_{m,X})}{\mathcal{I}_S/\mathbb{G}_{m,S}}\right) \to \mathcal{C}^\bullet(\mathcal{U}, R^1f_*\mathbb{G}_{m,X}) \to 0$$

As usual, we calculate the connecting homomorphism δ is by taking differential of a lift. Then one conclude by interpreting (as in previous section) line bundles as subsheaves of injective sheaves, morphism between line bundles as multiplying sections of the injective sheaves. $\square$

2.4 Relation with class of Brauer-Severi variety

We define the Brauer class associated with a Brauer-Severi scheme. We show the obstruction class of descending universal line bundle to $C \times_k \mathrm{Pic}^{2g-2}_{C/k}$ and Brauer class of $\mathrm{Sym}^{2g-2}_{C/k} \to \mathrm{Pic}^{2g-2}_{C/k}$ (away from the point ω_C) are related.

Let S be a scheme, we recall that a Brauer-Severi scheme over S is a proper morphism $f\colon P \to S$, which étale locally over S, is projectivization of a locally free sheaf. Brauer-Severi schemes give rise to classes in $\mathrm{H}^2_{et}(S, \mathbb{G}_m)$, see [Gro68a] for details.

Definition 2.4.1. *Let $f\colon P \to S$ be a Brauer-Severi scheme over connected scheme S. The associated Brauer class α_P is defined to be $\mathrm{d}_2^{0,1}(1)$, where $\mathrm{d}_2^{0,1}$ is the transgression map in the Leray spectral sequence for f, and 1 is the ample generator in $\mathrm{H}^0(S, \mathcal{P}ic_{P/S}) \cong \mathbb{Z}$.*

Lemma 2.4.2. *Let C be a smooth curve over k. Let $S = \mathrm{Pic}^{2g-2}_{C/k} - \{\omega_C\}$. Let $g\colon P = \mathrm{Div}^{2g-2}_{C/k} \times_{\mathrm{Pic}^{2g-2}_{C/k}} S \to S$ be the Brauer-Severi scheme. Let $f\colon C \times S \to S$ be the projection. Let $\mathrm{d}_2^{0,1}$ be the transgression map for the Leray spectral sequence of f. Then the Brauer class and obstruction class are equal, i.e.,*

$$\alpha_P = \mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^{2g-2}_{C/k}}).$$

Proof. Suppose the universal object is represented by a line bundle $\mathcal{L}$ on $C \times_k T$, where $T \to S$ is an étale cover. Let's denote all the base change of f to T by f_T, etc. We know $P_T = (\mathrm{Div}^{2g-2}_{C/k})_T \cong \mathrm{Proj}(\mathrm{Sym}^\bullet((f_{T,*}\mathcal{L})^\vee))$, see [Bos90, 8.2.7].

Let's denote the projection maps $P \times_S T \times_S T$ to its i,j factors by q_{ij}, let's denote the projection maps $C \times T \times_S T$ to its i,j factors by p_{ij}, and the projection on triple products of T over S to its factors by pr_{ij}. Suppose we have isomorphisms

$$\phi\colon p_{12}^*\mathcal{L} \to p_{13}^*\mathcal{L} \otimes p_{23}^*\mathcal{R},$$

$$\beta\colon pr_{12}^*\mathcal{R} \otimes pr_{23}^*\mathcal{R} \to pr_{13}^*\mathcal{R}$$

satisfying coycle condition, then

$$\mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^{2g-2}_{C/k}}) = (T, \mathcal{R}, \beta).$$

Push forward the isomorphism ϕ along p_{23} we get

$$p_{23,*}(\phi)\colon p_{23,*}p_{12}^*\mathcal{L} \xrightarrow{\sim} p_{23,*}p_{13}^*\mathcal{L} \otimes \mathcal{R}.$$

It induces isomorphism on the corresponding projective bundles

$$\begin{tikzcd}
 & \mathbb{P}(p_{23,*}p_{12}^*\mathcal{L}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbb{P}(p_{23,*}p_{13}^*\mathcal{L} \otimes \mathcal{R}) \arrow[d, "\wr"] & \\
P \times_S T \times_S T \arrow[r, equal] & \mathbb{P}(pr_1^* f_{T,*}\mathcal{L}) & \mathbb{P}(pr_2^* f_{T,*}\mathcal{L} \otimes \mathcal{R}) \arrow[r, equal] & P \times_S T \times_S T
\end{tikzcd}$$

and corresponding pulling isomorphisms of $\mathcal{O}(1)$:

$$q_{12}^*\mathcal{O}_{P_T}(1) \cong q_{13}^*\mathcal{O}_{P_T}(1) \otimes q_{23}^*\mathcal{R}.$$

Hence the corresponding $\mathrm{d}_2^{0,1}(\mathcal{O}_{P_T}(1))$ is represented by the same datum $(T, \mathcal{R}, \beta)$. $\square$

Remark 2.4.3. *We know there is an isomorphism $\phi\colon \mathrm{Pic}^{2g-2}_{C/k} \cong \mathrm{Pic}^0_{C/k}$ given by $\mathcal{L} \mapsto \omega_C^{-1} \otimes \mathcal{L}$. Then the same argument shows the obstruction to existence of universal line bundle on $C \times_k \mathrm{Pic}^0_{C/k}$ is also given by α_P.*

2.5 Description by Torsor

The degree 1 component $\mathrm{Pic}^1_{C/k}$ of the Picard scheme is a $\mathrm{Pic}^0_{C/k}$-torsor. The action is given by $\mathrm{Pic}^0_{C/k} \times \mathrm{Pic}^1_{C/k} \to \mathrm{Pic}^1_{C/k}$, $(\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes \mathcal{M}$.

Let k'/k be a finite Galois extension such that $C_{k'}$ has a k' rational zero cycle of degree

1, denoted by c_0. Denote $\mathrm{Gal}(k'/k)$ by G. Then the class $[\mathrm{Pic}^1_{C/k}] \in \mathrm{H}^1(G, \mathrm{Pic}^0_{C_{k'}/k'})$ is represented by the 1-cocycle

$$\sigma \mapsto \mathcal{O}_{C_{k'}}(c_0 - \sigma(c_0)).$$

In general one can take the zero cycle c_0 to be a k' rational point of $C_{k'}$, but for the simplicity of our proof, later we will choose c_0 be certain non-effective zero cycle of degree 1.

Let's simplify the situation by autoduality of smooth curves. Consider the embedding $C_{k'} \to \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$, $c \mapsto \mathcal{O}_C(c - c_0 + \omega_C)$. This morphism factors as:

$$C_{k'} \to \mathrm{Pic}^1_{C_{k'}/k'} \to \mathrm{Pic}^{2g-2}_{C_{k'}/k'}.$$

It induces isomorphism of group schemes via pullback:

$$\mathrm{Pic}^0_{\mathrm{Pic}^{2g-2}_{C_{k'}/k'}} \xrightarrow{\sim} \mathrm{Pic}^0_{\mathrm{Pic}^1_{C_{k'}/k'}} \xrightarrow{\sim} \mathrm{Pic}^0_{C_{k'}/k'}$$

The second isomorphism follows from auto-duality of curves, see [Mum65, 6.9]. This isomorphism descends to k. The first isomorphism holds because both torsors $\mathrm{Pic}^1_{C_{k'}/k'}$ and $\mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ are identified via translation. This isomorphism also descends to k, since elements in Pic^0 of an abelian variety are translation invariant, see [Mum08, II.8].

Let's explain the relation between α and $[\mathrm{Pic}^1_{C/k}]$. Our goal is to identify $\alpha \in \mathrm{Br}(\mathrm{Pic}^{2g-2}_{C/k})$ as a class in $\mathrm{H}^1(G, \mathrm{Pic}(\mathrm{Pic}^{2g-2}_{C_{k'}/k'}))$. Let

$$i \colon \mathrm{Pic}^0_{C_{k'}/k'} \cong \mathrm{Pic}^0(\mathrm{Pic}^{2g-2}_{C_{k'}/k'}) \to \mathrm{Pic}(\mathrm{Pic}^{2g-2}_{C_{k'}/k'})$$

be the isomorphism post-composed with inclusion.

Lemma 2.5.1. *Let X be a geometrically reduced and geometrically connected scheme over k (so that the invertible functions are nonzero constants). Let x be a k-ration point of X. Let k'/k be a finite Galois extension with Galois group G. Take a 1-cocycle of G in $\mathrm{Pic}(X_{k'})$, denote it by $\sigma \colon g \mapsto \mathcal{L}_g$. For each $g \in G$, fix a nonzero section s_g of the fiber $\mathcal{L}_g|_P$. Pick isomorphism $\alpha_{g_1,g_2} \colon \mathcal{L}_{g_1g_2} \cong \mathcal{L}_{g_1} \otimes g_1^*\mathcal{L}_{g_2}$, such that $\alpha_{g_1,g_2}(s_{g_1g_2}) = s_{g_1} \otimes g_1^* s_{g_2}$. Then we* *have a well defined homomorphism*

$$\phi_P\colon \mathrm{H}^1(G,\mathrm{Pic}(X_{k'})) \to \mathrm{H}^2(X,\mathbb{G}_m)$$
$$\sigma \mapsto (X_{k'},\{\mathcal{L}_g\},\{\alpha_{g_1,g_2}\})$$

Proof. Given another set of nonzero sections s'_g. Let $s'_g = a_g s_g$ with $a_g \in k'^*$. The corresponding isomorphism are $\alpha'_{g_1,g_2} = a^{-1}_{g_1g_2}(a_{g_1}\cdot g_1^*(a_{g_2}))\alpha_{g_1,g_2}$. Then datum $(X_{k'},\{\mathcal{L}_g\},\{\alpha_{g_1,g_2}\})$ and $(X_{k'},\{\mathcal{L}_g\},\{\alpha'_{g_1,g_2}\})$ are identified via isomorphism $a_g\mathrm{id}\colon \mathcal{L}_g \to \mathcal{L}_g$. So we have well defined map $Z^1(G,\mathrm{Pic}(X_{k'})) \to \mathrm{H}^2(X,\mathbb{G}_m)$. This is a homomorphism by construction. Note that $B^1(G,\mathrm{Pic}(X_{k'}))$ is mapped to B_X. So we get natural homomorphism $\mathrm{H}^1(G,\mathrm{Pic}(X_{k'})) \to D_X = \mathrm{H}^2(X,\mathbb{G}_m)$. □

Remark 2.5.2. *The rational point gives a canonical choice of "α" in the datum $(\mathcal{U},\mathcal{L},\alpha)$. So we can associate Brauer classes directly to* 1*-cocycles valued in* Pic*. Here is another way to see this: the existence of rational point implies* $\mathrm{d}_2^{1,1} = 0$*, see [Sko07, 1.1]. So* $F^1\mathrm{H}^2(X,\mathbb{G}_m)/\mathrm{Br}(k) = H^1(G,\mathrm{Pic}(X_{k'}))$*. Note that existence of rational point implies* $\mathrm{Br}(k)$ *is a summand of* $\mathrm{H}^2(X,\mathbb{G}_m)$*. So* $\mathrm{H}^1(G,\mathrm{Pic}(X_{k'}))\oplus\mathrm{Br}(k) \cong F^1\mathrm{H}^2(X,\mathbb{G}_m)$*. The lemma above gives the first inclusion. The splitting* $\mathrm{H}^2(X,\mathbb{G}_m) \to \mathrm{H}^1(G,\mathrm{Pic}(X_{k'}))$ *is given by mapping datum to* 1*-cocycle:* $(\mathcal{U},\mathcal{L},\alpha)\mapsto(\mathcal{U},\mathcal{L})$.

Proposition 2.5.3. *Let* $\mathrm{H}^1(G,i)\colon \mathrm{H}^1(G,\mathrm{Pic}^0_{C_{k'}/k'}) \to \mathrm{H}^1(G,\mathrm{Pic}(\mathrm{Pic}^{2g-2}_{C_{k'}/k'}))$ *be the map induced by* i*. Let* ϕ_{ω_C} *be the map associated to the rational point,* $\omega_C\in\mathrm{Pic}^{2g-2}_{C/k}$*, see 2.5.1. Then*

$$\phi_{\omega_C}\left(\mathrm{H}^1(G,i)([\mathrm{Pic}^1_{C/k}])\right) = \alpha.$$

Proof. Let $p\colon C^{2g-2}_{k'} \to \mathrm{Sym}^{2g-2}_{C_{k'}/k'}$ be the quotient map, let $q\colon C^{2g-2}_{k'} \to C_{k'}$ be projection onto first factor. Let c_1 be a k' point of $C_{k'}$, let D_{c_1} be the divisor $p_*q^*(c_1)$. Denote the projection $\mathrm{Sym}^{2g-2}_{C_{k'}/k'} \to \mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ by π. Since $\mathcal{L}_{c_1} := \mathcal{O}_{\mathrm{Sym}^{2g-2}_{C_{k'}/k'}}(D_{c_1})$ restricts to $\mathcal{O}(1)$ on each fiber over $S_{k'} = \mathrm{Pic}^{2g-2}_{C_{k'}/k'} - \{\omega_C\}$, we know $\mathcal{E}_{c_1} = \pi_*\mathcal{L}_{c_1}$ is a vector bundle over $S_{k'}$ whose projectivization descends to the Brauer-Severi scheme $\pi^{-1}(S)\to S$. By see-saw lemma, $\mathcal{L}_{\sigma(c_1)} = \mathcal{L}_{c_1}\otimes\pi^*\mathcal{N}_\sigma$ for some $\mathcal{N}_\sigma\in\mathrm{Pic}(\mathrm{Pic}^{2g-2}_{C_{k'}/k'})$. The datum of the Brauer class

α can be calculated by $\mathrm{d}_2^{0,1}(\mathrm{Pic}^{2g-2}_{C_{k'}/k'}, \{\mathcal{L}_{\sigma(c_1)}\}) = (\mathrm{Pic}^{2g-2}_{C_{k'}/k'}, \{\mathcal{N}_\sigma\}, \beta)$. By 2.5.2, we know α restricts to $0 \in \mathrm{Br}(k)$ by the rational section and it suffices to show there exist isomorphisms

$$\mathcal{L}_{\sigma(c_1)} \cong \mathcal{L}_{c_1} \otimes \pi^* i(\mathcal{O}_{C_{k'}}(\sigma(c_1) - c_1)).$$

Let C_{c_0} be the image of $C_{k'}$ in $\mathrm{Pic}^{2g-2}_{C_{k'}/k'}$ via embedding $c \mapsto c - c_0 + \omega_C$.

By auto-duality of Jacobian, it suffices to show for each σ, there exists some choice of c_0, such that

$$\mathcal{L}_{\sigma(c_1)}|_{C_{c_0} \times C_{k'}} \cong \mathcal{L}_{c_1}|_{C_{c_0} \times C_{k'}} \otimes \pi^* \mathcal{O}_{C_{c_0}}(\sigma(c_1) - c_1))|_{C_{c_0} \times C_{k'}}.$$

By projection formula, it suffices to show

$$\mathcal{E}_{\sigma(c_1)}|_{C_{c_0}} \cong \mathcal{E}_{c_1}|_{C_{c_0}} \otimes \mathcal{O}_{C_{c_0}}(\sigma(c_1) - c_1).$$

Let $S = C_{c_0} \times C_{k'}$ be the family of curves over C_{c_0}. Let's denote

$$\begin{aligned}
\mathcal{E} &= p_{1,*}\mathcal{O}_S(C_{c_0} \times \{\omega_C\} + \Delta - C_{c_0} \times \{c_0\}), \\
\mathcal{E}' &= p_{1,*}\mathcal{O}_S(C_{c_0} \times \{\omega_C\} + \Delta - C_{c_0} \times \{c_0\} - C_{c_0} \times \{c_1\}).
\end{aligned}$$

We claim by careful choice of c_0, we have $R^1 p_{1,*}\mathcal{E}' = 0$, hence we have exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{O}_{C_{k'}}(c_1) \to 0.$$

It suffices to find c_0 over the algebraic closure of k. By cohomology and base change, it suffices to find a zero cycle c_0 of degree one such that $\mathcal{O}(c_0)$ has no section and $\mathcal{O}(c_0 + p - q)$ has no section for any pair of geometric points p, q, where q runs through all the points of C and p has two choices (c_1 or $\sigma(c_1)$). This is always possible, since the bad loci for choice of such c_0 in $\mathrm{Pic}^1_{C/k}$ is dominated by two surface $C \times C$, while $\mathrm{Pic}^1_{C/k}$ is g dimensional, and we assumed $g \geq 3$.

Note that the in general, given an exact sequence of sheaves on a scheme X,

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

with $\mathcal{L}$ a line bundle, let $\pi \colon \mathbb{P}(\mathcal{E}) \to X$ be the projection, we have the following formula, see [Ful98, B.5.6]:

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathbb{P}(\mathcal{E}')) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}}(1) \otimes \pi^*\mathcal{L}.$$

Thus come back to the previous situation, we have

$$\mathcal{E}_{c_1}|_{C_{c_0}} \otimes \mathcal{O}_{C_{k'}}(-c_1) \cong \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Similarly,

$$\mathcal{E}_{\sigma(c_1)}|_{C_{c_0}} \otimes \mathcal{O}_{C_{k'}}(-\sigma(c_1)) \cong \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

So we have the desired relation

$$\mathcal{E}_{\sigma(c_1)}|_{C_{c_0}} \cong \mathcal{E}_{c_1}|_{C_{c_0}} \otimes \mathcal{O}_{C_{c_0}}(\sigma(c_1) - c_1).$$

□

Chapter 3

The universal genus g curve

3.1 Setup

Let k_0 be an algebraically closed field, let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ be the moduli stack of smooth genus g curves defined over k_0. Let k be the function field of $\mathcal{M}_g$. Let C/k be the generic fiber of the universal family of curves over $\mathcal{M}_g$. Let's denote the abelian variety $\mathrm{Pic}^0_{C/k}$ by A. Let k^s be the separable closure of k. Let's denote $\mathrm{Gal}(k^s/k)$ by G. We have the exact sequence

$$0 \to \mathrm{Pic}^0_{A/k}(k^s) \to \mathrm{Pic}(A_{k^s}) \to \mathrm{NS}(A_{k^s}) \to 0.$$

Taking Galois cohomology we get boundary map

$$c\colon \mathrm{H}^0(G, \mathrm{NS}(A_{k^s})) \to \mathrm{H}^1(G, \mathrm{Pic}^0_{A/k}(k^s))$$

Let W be the divisor on $\mathrm{Pic}^{g-1}_{C/k}$, given by the image of C^{g-1} in $\mathrm{Pic}^{g-1}_{C/k}$ under the morphism of summing up the points. Fix any k^s point $a \in \mathrm{Pic}^{g-1}(C_{k^s})$, we have an isomorphism $t_a\colon (\mathrm{Pic}^0_{C/k})_{k^s} \to (\mathrm{Pic}^{g-1}_{C/k})_{k^s}$ given by translation. Let $W_a := t_a^* W$ be the pullback of the theta divisor to A_{k^s}. Then it is clear that the image of W_a in $\mathrm{NS}(A_{k^s})$ is G-invariant, since for any $\sigma \in G$, we have:

$$W_a - \sigma^* W_a = t_a^* W - t_{\sigma(a)}^* (\sigma^* W) = t_a^* W - t_{\sigma(a)}^* W.$$

By theorem of the square, line bundle associated to $t_a^* W - t_{\sigma(a)}^* W$ is tranlation invariant on A, hence belongs to $\mathrm{Pic}^0_{A/k}(k^s)$, see [Mum08, 2.6]. Hence the class of $W_a \in \mathrm{NS}(A_{k^s})$ is G-invariant. Theorem of square also shows the class is independent of a, so we denote the class of W_a by Θ. It is shown in [PS99, 4.4] that the image of Θ under the connecting homormophism has a nice description

Proposition 3.1.1. *We have* $c(\Theta) = (g-1)[\mathrm{Pic}^1_{C/k}]$.

3.2 Some lemmas

Let m be an integer invertible in k_0. Taking étale cohomology of Kummer sequence over C, we know $A[m] \cong \mathrm{H}^1(C, \mu_m)$. This is also canonically isomorphic to $\mathrm{H}^1(A, \mu_m)$ via autoduality of Jacobian of curves, see [Mum65, 6.9]. Note that we have the Weil pairing (see [Mum08, IV.20] or [CS86, V.16])

$$\langle , \rangle \colon A[m] \times A^\vee[m] \to \mu_m.$$

We also have the cup product and trace map

$$\cup \colon \mathrm{H}^1(C, \mu_m) \times \mathrm{H}^1(C, \mu_m) \to \mathrm{H}^2(C, \mu_m^{\otimes 2}),$$

$$\mathrm{tr} \colon \mathrm{H}^2(C, \mu_m^{\otimes 2}) \to \mu_m.$$

Note that any line bundle $\mathcal{L}$ on an abelian variety A gives an isomorphism

$$\phi_{\mathcal{L}} \colon A \to A^\vee, \ a \mapsto T_a^* \mathcal{L} \otimes \mathcal{L}^{-1},$$

and is defined up to numerical equivalence. In case A is the Jacobian of a curve, the map ϕ_Θ induced by $\Theta \in \mathrm{NS}(A_{k^{sep}})$ is the map of autoduality, see [CS86, VII.6.6]. We show the cup product coincides with the Weil pairing:

Lemma 3.2.1. $\langle a, \phi_\Theta(b)\rangle = \mathrm{tr}(a \cup b)$.

Proof. Note that for a line bundle $\mathcal{L} \in A^\vee[m]$, the pullback $[m]^*\mathcal{L} = \mathcal{L}^{\otimes m}$ is trivial. Taking character of automorphy gives a map

$$A^\vee[m] \to \mathrm{H}^1(A[m], \mu_m) = \mathrm{Hom}(A[m], \mu_m),$$

see [Mum08, IV.20].

Note that the multiplication by m map $A \xrightarrow{m} A$ is a $A[m]$-torsor over A, hence corresponds to a class in $\gamma \in \mathrm{H}^1(A, A[m])$. Given any homomorphism $f\colon A[m] \to \mu_m$, we may reduce structure of principal bundle to get an element $\mathrm{H}^1(A, f)(\gamma) \in \mathrm{H}^1(A, \mu_m)$. Hence evaluation at γ gives a map

$$\mathrm{Hom}(A[m], \mu_m) \to \mathrm{H}^1(A, \mu_m) \cong A^\vee[m].$$

By description of cup product and trace map, see [Del77, Exp 1, 6.2.3] and [Del77, Exp 5, 3.4], our claim reduces to show the composition $A^\vee[m] \to \mathrm{H}^1(A[m], \mu_m) \to A^\vee[m]$ is identity. But this is clear as taking factors of automorphy of a torsion line bundle and constructing μ_m-torsor from factor of automorphy are inverse maps. $\square$

Lemma 3.2.2. *Let R be a ring such that 2 is not a zero divisor. Let n be an integer. Let J be the standard symplectic form on $R^{2n} = \left(\bigoplus_{i=1}^n Rx_i\right) \oplus \left(\bigoplus_{j=1}^n Ry_j\right)$, so that $J(x_k) = y_k$ and $J(y_k) = -x_k$ for all k. Let*

$$\mathrm{Sp}(2n, R) = \{A \in \mathrm{Gl}(2n, R) | A^t J A = J\}$$

be the symplectic group. Then any $\mathrm{Sp}(2n, R)$-invariant 2-form N can be written as rJ for some $r \in R$.

Proof. This proof follows from [Hug]. Consider matrices A_k and $B_{k,l}$ such that

$$A_k(x_k) = y_k,\ A_k(y_k) = -x_k,\ A_k(x_l) = x_l,\ A_k(y_l) = y_l,\ (l \neq k)$$

$$B_{k,l}(x_k) = x_l,\ B_{k,l}(x_l) = x_k,\ B_{k,l}(y_k) = y_l,\ B_{k,l}(y_l) = y_k,$$

$$B_{k,l}(y_s) = y_s,\ B_{k,l}(x_s) = x_s,\ (s \neq k,l)$$

The matrices A_k and $B_{k,l}$ are symplectic. Let E_k be the submodule generated by x_k, y_k, we have $E_k = \mathrm{Ker}(A_k^2 + I)$. Also note that $A_k^T = A_k^{-1}$ and $B_{k,l}^T = B_{k,l}$. Since N is invariant under full symplectic group, we get

$$\begin{aligned} A_k^{-1}NA_k = &\quad A_k^T N A_k = & N, \\ B_{k,l}NB_{k,l} = &\ B_{k,l}^T N B_{k,l} = & N. \end{aligned}$$

By the first equation, $A_k^{-1}NA_k(x_k) = N(x_k)$, also note that $A_k(x_k) = y_k$, so

$$N(y_k) = A_k N(x_k).$$

Similarly,

$$N(x_k) = -A_k N(y_k).$$

In particular, $A_k^2 N(x_k) = -N(x_k)$ and $A_k^2 N(y_k) = -N(y_k)$. Hence $N(x_k), N(y_k) \in E_k$, and so there are numbers a_k, b_k, c_k, d_k such that they can be written down with respect to the basis $\{x_i, y_j\}$ as

$$N(x_k) = a_k x_k + b_k y_k,\ N(y_k) = c_k x_k + d_k y_k.$$

Applying A_k to either of these equations and using $N(x_k) = -AN(y_k)$ or $N(y_k) = AN(x_k)$, we have $a_k = d_k$ and $c_k = -b_k$. Applying $B_{k,l}$ to these formulas for each k,l shows that $a_k = a_l = a$ for all k,l and $b_k = b_l = b$ for all k,l. Thus we get $N = aI + bJ$. Since N is skew symmetric and R is 2-torsion free, it follows that $a = 0$. $\square$

3.3 Facts on fundamental group

Let X be a connected finite type Deligne-Mumford stack. Let $\overline{x}$ be a geometric point, then $\pi_1(X,\overline{x})$ is defined to be the automorphism group (endowed with profinite topology) of the fiber functor $F_{\overline{x}}\colon \mathrm{Fét}(X) \to \mathrm{Sets}$. Here $\mathrm{Fét}(X)$ is the Galois category of finite étale covers of X by Deligne-Mumford stacks, see [LO10, A] for details. We also recall some facts about tame fundamental group, we refer to [GM71, 2.4] for details. Suppose $\overline{X}$ is a compactification of X, such that boundary is divisor with normal crossings. Let $\overline{x}$ be a geometric point of X, then we consider the Galois category of finite étale covers of X tamely ramified over $\overline{X}\backslash X$, denoted by $\mathrm{Fét}^{tame}(X)$. The tame fundamental group $\pi_1^{tame}(X, \overline{X}\backslash X, \overline{a})$ is defined to be the automorphism group of fiber functor $F_{\overline{x}}$. When the choice of compactification is clear, we simply write $\pi^{tame}(X,\overline{x})$. Since $\mathrm{Fét}^{tame}(X)$ is a full subcategory of $\mathrm{Fét}(X)$, the tame fundamental group is a quotient of fundamental group, see [Sta18, 0BN6].

Lemma 3.3.1. *Let η be the generic point of $\mathcal{M}_g$, then $\pi_1(\eta,\overline{\eta}) \to \pi_1(\mathcal{M}_g,\overline{\eta})$ is a surjection. (Thus $\mathrm{Gal}(k^{sep}/k) = \pi_1(\eta,\overline{\eta}) \to \pi_1(\mathcal{M}_g,\overline{\eta}) \twoheadrightarrow \pi_1^{tame}(\mathcal{M}_g,\overline{\eta})$ is surjection.)*

Proof. Let $\overline{\mathcal{M}_g}$ be the compactification of $\mathcal{M}_g$ by semi-stable curves, the boundary divisor is normal crossing, let $\pi_1^{tame}(\mathcal{M}_g,\overline{\eta})$ be defined as before. Since $\mathcal{M}_g$ is smooth irreducible, we argue as in [Tag 0BQI]. □

Assume we are working over fields with all roots of unity, we identify μ_m with $\mathbb{Z}/m\mathbb{Z}$ for m prime to the characteristic of the base field. We have a primitive skew-symmetric pairing on $H^1(C,\mathbb{Z}/m\mathbb{Z})$, induced by cup product and trace map, or via Weil pairing and principal polarization given by theta divisor, see 3.2.1.

Lemma 3.3.2. *For any odd prime $l \neq \mathrm{char}(k_0)$, and any positive integer n, the monodromy action $\pi_1(\mathcal{M}_g,\overline{\eta})$ on $H^1(C,\mathbb{Z}/l^n\mathbb{Z})$ gives surjection $\pi_1(\mathcal{M}_g,\overline{\eta}) \to \mathrm{Sp}(2g,\mathbb{Z}/l^n\mathbb{Z})$.*

Proof. Case I. When $\mathrm{char}(k_0) = 0$, this is shown by transcendental methods. By Lefschetz principle, it suffice to work with $k_0 = \mathbb{C}$. We give a sketch of the proof. For details, see [FM12, 10] and [ACG11, XV]. Recall the analytic moduli stack $\mathcal{M}_g^{an}$ can be viewed as $\{\text{Complex Structure on } S\}/\mathrm{Diffeo}^+(S) = \mathrm{Teich}(S)/\mathrm{Mod}(S)$, where $\mathrm{Teich}(S)$ is the Teichmüller space of complex structures on a genus g real surface S, and $\mathrm{Mod}(S)$ is mapping class group of S. We know $\mathrm{Teich}(S)$ is homeomorphic to a ball, so we may identify $\pi_1(\mathcal{M}_g^{an})$ with $\mathrm{Mod}(S)$. Note that $H^1(C,\mathbb{Z}/l^n\mathbb{Z})$ is finite, so the action $\pi_1(\mathcal{M}_g^{an})$ on $H^1(C,\mathbb{Z}/l^n\mathbb{Z})$ factors through $\pi_1(\mathcal{M}_g,\overline{\eta})$, the profinite completion of $\pi_1(\mathcal{M}_g^{an})$. Note that $\mathrm{Mod}(S)$ maps surjectively onto $\mathrm{Sp}(2g,\mathbb{Z})$, see [FM12, 6.4]. Also note that, the reduction map $\mathrm{Sp}(2g,\mathbb{Z})\to\mathrm{Sp}(2g,\mathbb{Z}/k\mathbb{Z})$ is surjective for any integer k, see [NS64, 1]. So we have the commutative diagram

$$\begin{tikzcd}
\pi_1(\mathcal{M}_g^{an}) \arrow[r] \arrow[dr] & \pi_1(\mathcal{M}_g,\overline{\eta}) \arrow[d] \\
& \mathrm{Sp}(2g,\mathbb{Z}/l^n\mathbb{Z})
\end{tikzcd}$$

Thus the surjectivity $\pi(\mathcal{M}_g,\overline{\eta})\to\mathrm{Sp}(2g,\mathbb{Z}/l^n\mathbb{Z})$ is implied.

Case II. When $\mathrm{char}(k_0)>0$. Let $\Lambda(k_0)$ be the Cohen ring of k_0, namely a discrete valuation ring with uniformizer p and residue field k_0, see [Sta18, 0328]. Let η and s be the generic point and special point of $S=\mathrm{Spec}(\Lambda(k_0))$. Let $\mathcal{M}_{g,S}\to S$ be the moduli stack of smooth genus g curves over S, let $\overline{\mathcal{M}}_{g,S}$ be its compactification by stable curves. Let $\overline{p},\overline{q}$ be geometric points in fiber of $\overline{\mathcal{M}}_{g,S}\to S$ over η and s, such that $\overline{p}$ specializes to $\overline{q}$. We have following commutative diagram, where the horizonal arrows are given by monodromy action, and vertical maps are given by specialization of fundamental group and specialization of cohomology.

$$\begin{tikzcd}
\pi_1^{tame}(\mathcal{M}_{g,S}|_{\overline{\eta}},\overline{p}) \arrow[r, "a"] \arrow[d, "b"] & \mathrm{Sp}(2g,\mathbb{Z}/l^n\mathbb{Z}) \arrow[d, equal] & \subset & \mathrm{Aut}(H^1(C_{\overline{\eta}},\mathbb{Z}/l^n\mathbb{Z})) \arrow[d, equal] \\
\pi_1^{tame}(\mathcal{M}_g,\overline{q}) \arrow[r, "c"] & \mathrm{Sp}(2g,\mathbb{Z}/l^n\mathbb{Z}) & \subset & \mathrm{Aut}(H^1(C_{\overline{s}},\mathbb{Z}/l^n\mathbb{Z}))
\end{tikzcd}$$

The map a is surjection, since we are over a characteristic zero field $\kappa(\overline{\eta})$, where all ramifications are tame, so $\pi_1^{tame}(\mathcal{M}_{g,S}|_{\overline{\eta}},\overline{p})=\pi_1(\mathcal{M}_{g,S}|_{\overline{\eta}},\overline{p})$, we return to the first case. The map b is surjection, by [LO10, A.12]. Thus c is surjection and the monodromy action on special fiber is full. $\square$

Proposition 3.3.3. $\mathrm{NS}(A_{k^{sep}})^{\mathrm{Gal}(k^{sep}/k)}=\mathbb{Z}\cdot\Theta$

Proof. Let l be an odd prime number different from $\mathrm{char}(k_0)$, let n be a positive integer. Consider the map $\lambda_n\colon \mathrm{NS}(A_{k^{sep}}) \mapsto \mathrm{Hom}_{\mathbb{Z}}(\wedge^2 A[l^n], \mu_{l^n})$, induced by Weil pairing and the map $\phi\colon \mathrm{NS}(A_{k^{sep}}) \to \mathrm{Hom}(A_{k^{sep}}, A^{\vee}_{k^{sep}})$, see [Mum08, IV.20]. Since k_0 is algebraically closed, we may pick a primitive l^n-th root of unity and fix an isomorphism μ_{l^n} with $\mathbb{Z}/l^n\mathbb{Z}$. Note that there's compatible Galois action on both sides of λ_n. The cup product and trace map are Galois invariant. By 3.2.1, we know $\lambda_n(\Theta)$ is a Galois invariant primitive symplectic form on $H^1(C, \mathbb{Z}/l^n\mathbb{Z})$, hence the $\mathrm{Gal}(k^{sep}/k)$-action on right hand side factors through $\mathrm{Sp}(2g, \mathbb{Z}/l^n\mathbb{Z})$.

By 3.3.2, we know $\mathrm{Gal}(k^{sep}/k) \to \mathrm{Sp}(2g, \mathbb{Z}/l^n\mathbb{Z})$ is surjective. Thus by 3.2.1 and 3.2.2, an element $s \in \mathrm{NS}(A_{k^{sep}})^{\mathrm{Gal}(k^{sep}/k)}$ can be written as $c_n \cdot \lambda_n(\Theta)$ for some $c_n \in \mathbb{Z}/l^n\mathbb{Z}$. Let $c = \varprojlim c_n$, let $\lambda = \varprojlim \lambda_n\colon \mathrm{NS}(A_{k^{sep}}) \to \mathrm{Hom}_{\mathbb{Z}_l}(\wedge^2 T_l(A), \mathbb{Z}_l(1))$, then

$$\lambda(s) = c \cdot \lambda(\Theta).$$

Note that $\mathrm{NS}(A_{k^{sep}}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{Hom}(\wedge^2 T_l(A), \mathbb{Z}_l(1))$ is injection, see [Mum08, IV.18.3], so we know $\lambda(s)$ and $\lambda(\Theta)$ are linear dependent over $\mathbb{Z}$. But $\mathrm{NS}(A_{k^{sep}})$ is finite free, so $\lambda(s) = c \cdot \lambda(\Theta)$ implies s and Θ differ by an integer multiple. Note that $(\Theta)^g/g! = 1$ and $(s)^g/g! \in \mathbb{Z}$, see [Mum08, III.16], so the only possibility is s being an integral multiple of Θ. ☐

3.4 Period of the Brauer class

Proposition 3.4.1. *Let C/k be the universal genus g curve. The class α (see 2.1) associated to $\mathrm{Sym}^{2g-2}_{C/k} \to \mathrm{Pic}^{2g-2}_{C/k}$ has period $g-1$.*

Proof. Let k^s be the separable closure of k, and let $G = \mathrm{Gal}(k^s/k)$. Consider the short exact sequence

$$0 \to \mathrm{Pic}^0(\mathrm{Pic}^0_{C_{k^s}/k^s}) \to \mathrm{Pic}(\mathrm{Pic}^0_{C_{k^s}/k^s}) \to \mathrm{NS}(\mathrm{Pic}^0_{C_{k^s}/k^s}) \to 0,$$

The associated long exact sequence gives

$$\begin{tikzcd} \mathrm{NS}(\mathrm{Pic}^0_{C_{k^s}/k^s})^G \arrow[r, "c"] & H^1(G, \mathrm{Pic}^0(\mathrm{Pic}^0_{C_{k^s}/k^s})) \arrow[r, "\psi"] \arrow[d, equal] & H^1(G, \mathrm{Pic}(\mathrm{Pic}^0_{C_{k^s}/k^s})). \\ & H^1(G, \mathrm{Pic}^0(C_{k^s})) & \end{tikzcd}$$

By construction, α is a degree $g-1$ Brauer class, so $(g-1)\alpha = 0$. We show the $n\alpha \neq 0$ for any $1 \leq n \leq g-2$.

By the previous lemma, we have $\mathrm{NS}(\mathrm{Pic}^0_{C_{k^s}/k^s})^G \cong \mathbb{Z} \cdot \Theta$.

Its generator Θ maps to $c(\Theta) = (g-1)[\mathrm{Pic}^1_{C/k}]$, see [PS99, 4].

Suppose $n\alpha = 0$ for some $1 \leq n \leq g-2$, then $\psi(n[\mathrm{Pic}^1_{C/k}]) = n\alpha = 0$. By exactness of the maps we know $n[\mathrm{Pic}^1_{C/k}] = m(g-1)[\mathrm{Pic}^1_{C/k}]$ for some $m \in \mathbb{Z}$. This gives a contradiction, since the period of $[\mathrm{Pic}^1_{C/k}]$ is $2g-2$, see [Sch03, 5.1]. $\square$

Remark 3.4.2. *The index of α is also $g-1$. Because this class is represented by a degree $g-1$ Brauer-Severi variety, so* $\mathrm{ind}(\alpha) | g-1$*, however* $g-1 = \mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$.

3.5 The class restricted to subvarieties

We probe the class $\alpha \in \mathrm{Br}(\mathrm{Pic}^{2g-2}_{C/k})$ by studying its restriction to subvarieties of the Picard scheme $\mathrm{Pic}^{2g-2}_{C/k}$. Note that C/k is the universal genus g curve, and in general that there does not exist many explicit subvarieties over the universal objects. We'll first construct such subvarieties, then study the restriction of the Brauer class to it. We choose the subvarieties to be the theta divisors of certain semistable rank 2 vector bundles. We show the Brauer class restricts to 0 in the function field of such subvarieties.

3.5.1 Semistable rank 2 vector bundle on the universal curve

Lemma 3.5.1. *There exist semistable sheaves of rank* 2 *and degree* $2g-2$ *on the universal curve* C.

Proof. Let ω_C be the cotangent bundle of C. Let $D \in |\omega_C|$ be an effective canonical divisor on C. Then D is a prime divisor, as the Picard group of C is generated by the class of D, see [Sch03, 5.1]. Hence D is a finite integral scheme over $\mathrm{Spec}(k)$, and $\Gamma(D, \mathcal{O}_D)/k$ is a degree $2g-2$ field extension.

Consider the map $\rho\colon \Gamma(C, \omega_C) \to \Gamma(D, \omega_C|_D) = \Gamma(D, \mathcal{O}_D)$. Pick $u \in \Gamma(D, \mathcal{O}_D)\backslash\{0\}$, let $u\rho$ be the map obtained by composing ρ with multiplication by u. Let's denote the sum of ρ and $u\rho$ by $(\rho, u\rho) : \omega_C \oplus \omega_C \xrightarrow{(\rho,u\rho)} \mathcal{O}_D$. This is a surjective sheaf homomorphism, as ρ is. Let $\mathcal{E}_u = \mathrm{Ker}(\rho, u\rho)$, then $\mathcal{E}_u$ is a locally free rank 2 sheaf on C with $\deg(\mathcal{E}_u) = 2\deg(\omega_C) - \deg(\mathcal{O}_D) = 2g-2$. The slope of $\mathcal{E}_u$ is $\mu(\mathcal{E}_u) = \frac{2g-2}{2} = g-1$. We calculate the slope of subsheaves and find suitable u such that $\mathcal{E}_u$ are semistable.

Let $\mathcal{L}$ be a subsheaf of $\mathcal{E}_u$. As C is smooth and $\mathcal{E}_u$ is locally free, we know $\mathcal{L}$ is locally free.

1. If $\mathrm{rank}(\mathcal{L}) = 2$, then $\deg(\mathcal{L}) = \deg(\mathcal{E}_u) - \deg(\mathcal{E}_u/\mathcal{L}) \leq \deg(\mathcal{E}_u)$, so $\mu(\mathcal{L}) \leq \mu(\mathcal{E}_u)$.

2. If $\mathrm{rank}(\mathcal{L}) = 1$, since $\mathrm{Pic}(C) = \mathbb{Z} \cdot \omega_C$, we may write $\mathcal{L} = \omega^{\otimes k}$.

   (a) If $k \leq 0$, $\mu(\mathcal{L}) \leq 0 \leq \mu(\mathcal{E}_u)$.

   (b) If $k \geq 1$, we may pick a nonzero section t of $\omega_C^{\otimes(k-1)}$. The section gives embedding $\omega_C \overset{t}{\hookrightarrow} \omega_C^{\otimes k} \hookrightarrow \mathcal{E}_u$, so $h^0(\mathcal{E}_u) \geq h^0(\omega_C) = g$. Note that $\mathcal{E}_u = \mathrm{Ker}(\rho, u\rho)$, so $h^0(\mathcal{E}_u) = 2h^0(\omega_C) - \dim_k(\mathrm{Im}(\Gamma(\rho, u\rho)))$, where Γ is the functor of taking global section of sheaves on C. Thus, in order that $\mathcal{E}_u$ is semistable, it suffices to pick u such that $\dim_k(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g+1$.

Consider the family of coherent sheaves $\mathcal{E}_u$ parameterized by u, over the vector space $\Gamma(D, \mathcal{O}_D)$. We know the rank is lower semi-continuous function, see [Har77, 12.7.2]. Hence

$\dim_k(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g+1$ is an open condition for $u \in \Gamma(D, \mathcal{O}_D) - \{0\} \cong k^{2g-2} - \{0\}$. As k is infinite, the k-points in k^{2g-2} are Zariski dense. Thus the existence of such a u is equivalent to showing the open set $\{u \in \mathbb{A}_k^{2g-2} - \{0\} | \dim_k(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g+1\}$ is not empty. Thus it suffices to show such a u exists after base change to the algebraic closure $\overline{k}$.

Let's choose D such that $D_{\overline{k}} = \sum_{i=1}^{2g-2} P_i$, where P_i are distinct points (this is possible since we work over infinite fields. Then

$$\Gamma(D_{\overline{k}}, \mathcal{O}_{D_{\overline{k}}}) = \oplus_{i=1}^{2g-2} H^0(P_i, \mathcal{O}_{P_i}) \cong \overline{k}^{2g-2}.$$

We show there exists $u = (\lambda_i)_{i=1}^{2g-2}$ with $\lambda_i \in \overline{k}^*$, such that $\dim_k(\mathrm{Im}(\Gamma(\rho, u\rho))) \geq g+1$. In the following part of the proof, for simplicity, let's assume $k = \overline{k}$.

Let's fix non-vanishing local sections of ω_C at P_i, denoted by ω_i. Then

$$\Gamma(\rho, u\rho) \colon \Gamma(C, \omega_C)^{\oplus 2} \to \oplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$$

is given by $(\omega_a, \omega_b) \mapsto (f_a(P_i) + \lambda_i f_b(P_i))_i$, where $f_a(P_i) = \frac{\omega_a(P_i)}{\omega_i(P_i)}, f_b(P_i) = \frac{\omega_b(P_i)}{\omega_i(P_i)}$.

Pick a basis $\{\Omega_i\}_{i=1}^g$ of $\Gamma(C, \omega_C)$. The map $\Gamma(\rho, u\rho) \colon \Gamma(C, \Omega_C)^{\oplus 2} \to \oplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$ can be expressed by the $(2g-2) \times 2g$ matrix $M = [M_L | M_R]$, where

$$M_L = \begin{bmatrix} \frac{\Omega_1(P_1)}{\omega_1(P_1)} & \frac{\Omega_2(P_1)}{\omega_1(P_1)} & \cdots & \frac{\Omega_g(P_1)}{\omega_1(P_1)} \\ \frac{\Omega_1(P_2)}{\omega_2(P_2)} & \frac{\Omega_2(P_2)}{\omega_2(P_2)} & \cdots & \frac{\Omega_g(P_2)}{\omega_2(P_2)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\Omega_1(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} & \frac{\Omega_2(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} & \cdots & \frac{\Omega_g(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} \end{bmatrix}_{(2g-2) \times g}$$

and

$$M_R = \begin{bmatrix} \frac{\lambda_1\Omega_1(P_1)}{\omega_1(P_1)} & \frac{\lambda_1\Omega_1(P_1)}{\omega_2(P_1)} & \cdots & \frac{\lambda_1\Omega_g(P_1)}{\omega_1(P_1)} \\ \frac{\lambda_2\Omega_1(P_2)}{\omega_2(P_2)} & \frac{\lambda_2\Omega_2(P_2)}{\omega_2(P_2)} & \cdots & \frac{\lambda_2\Omega_g(P_2)}{\omega_2(P_2)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda_{2g-2}\Omega_1(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} & \frac{\lambda_{2g-2}\Omega_2(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} & \cdots & \frac{\lambda_{2g-2}\Omega_g(P_{2g-2})}{\omega_{2g-2}(P_{2g-2})} \end{bmatrix}_{(2g-2) \times g}$$

Note that M_L describes the map $\Gamma(\rho) \colon \Gamma(C, \omega_C) \to \oplus_{i=1}^{2g-2} \Gamma(P_i, \mathcal{O}_{P_i})$. This map fits into the exact sequence $0 \to k \cong \Gamma(C, \omega_C - \sum P_i) \to \Gamma(C, \omega_C) \overset{\Gamma(\rho)}{\to} \oplus_{i=1}^{2g-2} \mathrm{H}^0(C, \mathcal{O}_{P_i})$, thus $\mathrm{rank}(M_L) = \dim \mathrm{Im}(\Gamma(\rho)) = g - 1$.

After rearranging the ordering of P_i and replacing Ω_i by suitable linear combinations, we may assume

$$M_L = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} \\ 0_{(g-1)\times 1} & X \end{bmatrix}$$

Let $L_1 = \mathrm{diag}_{i=1}^{g-1}(\lambda_i)$ be the diagonal matrix, let $L_2 = \mathrm{diag}_{i=g}^{2g-2}(\lambda_i)$, let $L_u = \mathrm{diag}(L_1, L_2)$. Then we may write

$$M = \begin{bmatrix} 0_{(g-1)\times 1} & I_{g-1} & 0_{(g-1)\times 1} & L_1 \\ 0_{(g-1)\times 1} & X & 0_{(g-1)\times 1} & L_2 X \end{bmatrix}$$

Thus $\mathrm{rank}(M) = \dim(\mathrm{Im}(\Gamma(\alpha, u\alpha))) \geq g + 1$ if $\mathrm{rank}((L_2^{-1}L_1 - I)X) \geq 2$. Let's choose λ_i such that $\lambda_i \lambda_{g-1+i} \neq 1$, so that $L_2^{-1}L_1 - I$ is invertible. We are done if we know $\mathrm{rank}(X) \geq 2$.

Consider the short exact sequence

$$0 \to \omega\left(-\sum_{i=g}^{2g-2} P_i\right) \to \omega_C \overset{\phi}{\to} \omega_C|_{\sum_{i=g}^{2g-2} P_i} \to 0,$$

then $\dim_k \Gamma(C, \omega_C(-\sum_{i=g}^{2g-2} P_i)) = g - \mathrm{rank}(\Gamma(\phi)) = g - \mathrm{rank}(X)$.

Consider the short exact sequence

$$0 \to \omega\left(-\sum_{i=1}^{g-1} P_i\right) \to \omega_C \overset{\psi}{\to} \omega_C|_{\sum_{i=1}^{g-1} P_i} \to 0.$$

Since $\Gamma(\psi)$ corresponds to I_{g-1}, we know $\mathrm{rank}(\Gamma(\psi)) = g-1$, so $h^0(C, \omega_C(-\sum_{i=1}^{g-1} P_i)) = 1$. Note that $\sum_{i=1}^{2g-2} P_i$ is a canonical divisor, so

$$h^0\left(C, \omega_C\left(-\sum_{i=1}^{g-1} P_i\right)\right) = h^0\left(C, \mathcal{O}_C\left(\sum_{i=g}^{2g-2} P_i\right)\right) = 1.$$

By Riemann–Roch,

$$h^0\left(C,\omega_C\left(-\sum_{i=g}^{2g-2}P_i\right)\right)-h^0\left(C,\mathcal{O}_C\left(\sum_{i=g}^{2g-2}P_i\right)\right)=g-1+1-g=0.$$

So $\dim_k\Gamma(C,\omega_C(-\sum_{i=g}^{2g-2}P_i))=g-\mathrm{rank}(X)=1$, thus $\mathrm{rank}(X)=g-1\geq 2$ as $g\geq 3$. $\square$

3.5.2 The generalized theta divisor

Let $\mathcal{E}$ be a semistable sheaf on C obtained as in the previous lemma. We know $\chi(\mathcal{E})=\deg(\mathcal{E})+2(1-g)=0$. Note that $\mathrm{Pic}^{2g-2}_{C/k}\cong\mathrm{Pic}^0_{C/k}$, by translation via canonical divisor, for simplicity we work over $\mathrm{Pic}^0_{C/k}$. Let's recall how the theta divisor of a vector bundle on C is constructed.

We define a closed subscheme of $\mathrm{Pic}^0_{C/k}$ as following. Let $U\to\mathrm{Pic}^0_{C_k/k}$ be an étale cover, such that the tautological line bundle is represented by $\mathcal{L}$ on $U\times_k C$. Let $\pi\colon U\times_k C\to U$ be the projection. Let's choose canonical divisors $\{D_i\}_{i=1}^m$ on C with disjoint support, let $Z=\sum_{i=1}^m D_i$. Let $\mathcal{E}_U$ be the pullback of $\mathcal{E}$ via $\mathrm{pr}_2\colon U\times_k C\to C$, similarly, let Z_U be the pullback of Z.

Consider the short exact sequence on $U\times_k C$

$$0\to(\mathcal{E}_U\otimes\mathcal{L})(-Z_U)\to\mathcal{E}_U\otimes\mathcal{L}\to(\mathcal{E}_U\otimes\mathcal{L})|_{Z_U}\to 0.$$

Take m be large enough so that $h^0(C_{\kappa(x)},(\mathcal{E}_U\otimes\mathcal{L})_x(-Z_x))=0$ for any $x\in U$. (We can do this because a section of $(\mathcal{E}_U\otimes\mathcal{L})_x(-Z_x)$ gives an injection $H^0(C_{\kappa(x)},\mathcal{O}_{C_{\kappa(x)}}(Z_x))\to H^0(C_{\kappa(x)},(\mathcal{E}_U\otimes\mathcal{L})_x)$. The left hand is unbounded in m but the right hand side is bounded.) Then $h^1(C_{\kappa(x)},(\mathcal{E}_U\otimes\mathcal{L})_x(-Z_x))=\chi((\mathcal{E}_U\otimes\mathcal{L})_x(-Z_x))$ is constant, see [Mum08, II.5.Cor 1]. Thus $R^1\pi_*((\mathcal{E}_U\otimes\mathcal{L})(-Z_U))$ is locally free by [Mum08, II.5.Cor 2]. Since $Z_U\to U$ is flat, for the same reason, $\pi_*((\mathcal{E}_U\otimes\mathcal{L})|_{Z_U})$ is locally free. Take the long exact sequence for π_*, we get exact sequence of sheaves on U,

$$0\to\pi_*(\mathcal{E}_U\otimes\mathcal{L})\to\pi_*((\mathcal{E}_U\otimes\mathcal{L})|_{Z_U})\xrightarrow{\delta}R^1\pi_*\left((\mathcal{E}_U\otimes\mathcal{L})(-Z_U)\right)\to R^1\pi_*(\mathcal{E}_U\otimes\mathcal{L})\to 0.\quad (☎)$$

Let η be the generic point of U. Since $\mathcal{E}$ is semistable and $\chi(\mathcal{E}) = 0$, by [Ray82, 1.6.2] we know $h^0(C_\eta, \mathcal{E}_\eta \otimes \mathcal{L}_\eta) = 0$ and $h^1(C_\eta, \mathcal{E}_\eta \otimes \mathcal{L}_\eta) = \chi(\mathcal{E}_\eta \otimes \mathcal{L}_\eta) - h^0(C_\eta, \mathcal{E}_\eta \otimes \mathcal{L}_\eta) = 0 + 2\deg(\mathcal{L}_\eta) - 0 = 0$. Since cohomology commutes with flat base change, see [Har77, III.9.3], we know $(R^i\pi_*(\mathcal{E}_U \otimes \mathcal{L}))|_\eta = H^i(C_\eta, \mathcal{E}_\eta \otimes \mathcal{L}_\eta)$. Note that $\pi_*((\mathcal{E}_U \otimes \mathcal{L})|_{Z_U})$ is torsion free, so $\pi_*(\mathcal{E}_U \otimes \mathcal{L}) = 0$ and δ is injection.

Thus δ is an injection of locally free rank $2\deg(Z)$ sheaves. Let's denote

$$\det(\delta)\colon \det(\pi_*((\mathcal{E}_U \otimes \mathcal{L})|_{Z_U})) \to \det(R^1\pi_*(\mathcal{E}_U \otimes \mathcal{L})(-Z_U))$$

by $\det(\delta)\colon \mathcal{F}_U \to \mathcal{G}_U$. This gives us a section $s_U\colon \mathcal{O}_U \to \mathcal{G}_U \otimes \mathcal{F}_U^{-1}$.

Lemma 3.5.2. *The line bundle $\mathcal{G}_U \otimes \mathcal{F}_U^\vee$ and section s_U descends along the cover $U \to \mathrm{Pic}^0_{C/k}$. Thus the vanishing locus of s_U descends to a well defined Cartier divisor $T \subset \mathrm{Pic}^0_{C/k}$.*

Proof. As $U \to \mathrm{Pic}^0_{C/k}$ is étale cover, all descent data are effective, it suffices to give the descent datum.

Note that on $U \times_P U \times_k C$, both $p_{23}^*\mathcal{L}$ and $p_{12}^*\mathcal{L}$ are tautological line bundles. By see-saw lemma, there exists a line bundle $\mathcal{N} \in \mathrm{Pic}(U \times_P U)$ such that $p_{23}^*\mathcal{L} \cong p_{13}^*\mathcal{L} \otimes p_{12}^*\mathcal{N}$. Fix an isomorphism $f\colon p_{23}^*\mathcal{L} \cong p_{13}^*\mathcal{L} \otimes p_{12}^*\mathcal{N}$. Consider the cartesian diagram of flat morphisms,

$$\begin{tikzcd} U \times_k U \times_k C \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & U \times_k C \arrow[d, "\pi"] \\ U \times_k U \arrow[r, "p_2"] & U \end{tikzcd}.$$

Since cohomology commutes with flat base change, we have

$$\begin{aligned} p_2^*\mathcal{G}_U &= p_2^*\left(\bigwedge^{2\deg(Z)} R^1\pi_*((\mathcal{E}_U \otimes \mathcal{L})(-Z_U))\right) \\ &= \bigwedge^{2\deg(Z)} R^1 p_{12,*}(p_{23}^*(\mathcal{E}_U \otimes \mathcal{L})(-Z_U)) \\ &= \bigwedge^{2\deg(Z)} R^1 p_{12,*}(p_{13}^*\mathcal{L} \otimes p_{12}^*\mathcal{N} \otimes p_3^*(\mathcal{E}(-Z))) \end{aligned}$$

By projection formula, this is isomorphic via f to

$$\bigwedge^{2\deg(Z)} \big(R^1 p_{12,*}(p_{13}^*\mathcal{L} \otimes p_3^*(\mathcal{E}(-Z))) \otimes \mathcal{N}\big) = p_1^*\mathcal{G}_U \otimes \mathcal{N}^{2\deg(Z)}$$

Let's denote the isomorphism by

$$a_f\colon p_2^*\mathcal{G}_U \cong p_1^*\mathcal{G}_U \otimes \mathcal{N}^{2\deg(Z)},$$

similarly we have

$$b_f\colon p_2^*\mathcal{F}_U \cong p_1^*\mathcal{F}_U \otimes \mathcal{N}^{2\deg(Z)}.$$

Let $c_f = a_f \otimes b_f^{-1}\colon p_2^*(\mathcal{G}_U \otimes \mathcal{F}_U^{-1}) \to p_1^*(\mathcal{G}_U \otimes \mathcal{F}_U^{-1})$. Then one can check $p_{13}^*(c_f) = p_{23}^*(c_f) \circ p_{12}^*(c_f)$. So the line bundle $\mathcal{G}_U \otimes \mathcal{F}_U^{-1}$ descends to a line bundle $\mathcal{M}$ on $\mathrm{Pic}^0_{C/k}$. By the naturality of connecting homomorphism δ, one can show s_U descends to a section of $\mathcal{M}$, denoted by s. Then the vanishing locus T of s is a well defined Cartier divisor of $\mathrm{Pic}^0_{C/k}$. $\square$

Lemma 3.5.3. *The divisor T is numerically equivalent to 2Θ, where Θ is the numerical class of usual theta divisor, see 3.1.*

Proof. Let's pick $U = \mathrm{Pic}^0_{C/k} \times_k \overline{k}$, so U is an abelian variety, and has trivial Todd class. By the determinantal description of T, see (☎), we know $c_1(\mathcal{O}_U(T_U)) = -c_1(R^1\pi_*(\mathcal{E} \otimes \mathcal{L}))$. Thus by [Ful98, 3.2.3], we have

$$\mathrm{ch}(-R^1\pi_*(\mathcal{E}_U \otimes \mathcal{L})) = 0 + c_1(\mathcal{O}_U(T_U)) + \text{higher chern class.}$$

By Grothendieck–Riemann–Roch theorem, see [Ful98, 15.2]

$$\mathrm{ch}(-R^1\pi_*(\mathcal{E}_U \otimes \mathcal{L})) = \pi_*\left(\mathrm{ch}(\mathcal{E}_U \otimes \mathcal{L}) \cdot p_2^*\left(1 + \frac{\omega_C}{2}\right)\right).$$

Let κ be a square root of $\omega_{C_{\overline{k}}}$. Note the usual theta divisor satisfies

$$\mathrm{ch}(-R^1\pi_*(\kappa_U \otimes \mathcal{L})) = 0 + c_1(\mathcal{O}_U(\Theta_U)) + \text{higher chern class.}$$

By Grothendieck–Riemann-Roch theorem,

$$\mathrm{ch}(-R^1\pi_*(\kappa_U \otimes \mathcal{L})) = \pi_*\left(\mathrm{ch}(\kappa_U \otimes \mathcal{L}) \cdot p_2^*\left(1 + \frac{\omega_C}{2}\right)\right).$$

Note that $\mathcal{E}_U$ is numerically equivalent to $\kappa_U \oplus \kappa_U$, see [Ati57, I.5.3]. The above equations show $T_U \overset{num}{=} 2\Theta_U$, thus $T \overset{num}{=} 2\Theta$, since numerical equivalence is preserved by change of base fields. $\square$

Lemma 3.5.4. *The subscheme T is reduced and irreducible.*

Proof. Recall T is an effective divisor, write it as a sum of prime divisors $T = \sum_{i=1}^k r_i D_i$. Note that $\mathrm{NS}(\mathrm{Pic}^0_{C_{k'}/k'})^G = \mathbb{Z} \cdot \Theta$, so $D_i \overset{num}{=} n_i\Theta$ for $n_i \in \mathbb{Z}_{>0}$ and $T \overset{num}{=} (\sum_{i=1}^k r_i n_i)\Theta$. By the previous lemma, $T \overset{num}{=} 2\Theta$. The only possible cases are $k = 1, r_1 = 1, n_1 = 2$ or $k = 1, r = 2, n_1 = 1$ or $k = 2, r_1 = r_2 = 1, n_1 = n_2 = 1$. In the first case T is reduced. The other two cases imply existence of a divisor on $\mathrm{Pic}^0_{C/k}$ whose numerical class is Θ, this is not true. Otherwise the following exact sequence

$$\mathrm{Pic}(\mathrm{Pic}^0_{C/k}) = \mathrm{Pic}(\mathrm{Pic}^0_{C_{k'}/k'})^G \to \mathrm{NS}(\mathrm{Pic}^0_{C_{k^s}/k^s})^G \xrightarrow{c} H^1(G, \mathrm{Pic}^0(\mathrm{Pic}^0_{C_{k^s}/k^s}))$$

implies $c(\Theta) = 0$. However $c(\Theta) = [\mathrm{Pic}^{g-1}_{C/k}] \neq 0$, see 3.4.1. $\square$

Lemma 3.5.5. *Let $T_U = T \times_{\mathrm{Pic}^0_{C/k}} U$. The generic rank of $\pi_{T_U,*}(\mathcal{E}_{T_U} \otimes \mathcal{L}_{T_U})$ is 1.*

Proof. Recall by (☎), we need to check the corank of δ is 1. If at a point $t \in T$, corank of $\delta|_t$ is at least 2, then $\det(\delta_t) \subset \mathfrak{m}_t^2$, thus t will lie in the singular locus of T. But T is reduced, so it is generically regular. $\square$

3.5.3 Triviality of Brauer class

We show the vanishing of Brauer class by showing the corresponding universal line bundle descends.

Lemma 3.5.6. *Let X be a scheme. Let*

$$h\colon U \to X$$

be an étale cover of X.

Let $\mathcal{F}$ be a coherent sheaf on U. Assume there exists a line bundle $\mathcal{N} \in \mathrm{Pic}(U \times_X U)$, such that there exists isomorphism (let's denote the projection maps $U \times_X U \to U$ by p_i)

$$\phi\colon p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F} \otimes \mathcal{N},$$

and isomorphism (let's denote the projection maps $U \times_X U \times_X U \to U \times_X U$ by p_{ij})

$$\beta\colon p_{12}^*\mathcal{N} \otimes p_{23}^*\mathcal{N} \to p_{13}^*\mathcal{N},$$

satisfying (let's denote q_i are the projection maps $U \times_X U \times_X U \to U$)

$$(1_{q_3^*\mathcal{F}} \otimes \beta) \circ (p_{23}^*\phi \otimes 1_{p_{12^*\mathcal{N}}}) \circ (p_{12}^*\phi) = p_{13}^*\phi,$$

then the scheme

$$a_{\mathcal{F}}\colon P' := \mathbb{P}(\mathcal{F}) = \mathrm{Proj}(\mathrm{Sym}^\bullet(\mathcal{F}^\vee)) \to U$$

descends to a scheme

$$a\colon P \to X.$$

Proof. Recall in general, given a scheme S, a line bundle $\mathcal{L}$ on S and a coherent sheaf $\mathcal{F}$ on S, there exists a unique isomorphism $f\colon A := \mathbb{P}(\mathcal{F}) \cong B := \mathbb{P}(\mathcal{F} \otimes \mathcal{L})$ such that $f^*\mathcal{O}_B(1) \cong \mathcal{O}_A(1) \otimes \mathcal{L}$. In our case, the isomorphisms ϕ provide the covering datum $\mathbb{P}(\phi)$, and the isomorphism β similarly provides descent datum for descending P' along $U \to X$. Effectiveness of the descent datum follows from [Bos90, 6.1.7]. $\square$

Lemma 3.5.7. *Keep the notation as in the previous lemma. Let*

$$f\colon Y \to X$$

be a scheme over X.

Let q_{ij} be the projection maps from $U \times_X U \times_X Y$ to its factors.

Let $\mathcal{G}$ be a coherent sheaf on $U \times_X Y$ such that there exists isomorphism

$$\psi\colon q_{13}^*\mathcal{G} \xrightarrow{\sim} q_{23}^*\mathcal{G} \otimes q_{12}^*\mathcal{N},$$

which satisfy cocycle condition compatible with β (as we previously assumed for $\mathcal{F}$).

Let

$$r\colon Y \times_X P' \to P'$$

be the projection onto second factor and

$$s\colon Y \times_X P' \to Y \times_X U$$

the base change of $a_{\mathcal{F}}$.

Then $s^*\mathcal{G} \otimes r^*\mathcal{O}_{P'}(-1)$ descends to a coherent sheaf on $Y \times_X P$.

Proof. We have diagram

$$\begin{tikzcd}
 & Y \times_X P' \arrow[rr, "r"] \arrow[dl] \arrow[dd, "s"] & & P' \arrow[dl] \arrow[dd] \\
Y \times_X P \arrow[rr] \arrow[dd] & & P \arrow[dd] & \\
 & Y \times_X U \arrow[rr] \arrow[dl] & & U \arrow[dl] \\
Y \arrow[rr] & & X &
\end{tikzcd}$$

Let $p_i'\colon P' \times_P P' \to P'$ be the projections. Let $q_{i3}'\colon P' \times_P P' \times_X Y \to P' \times_X Y$,

$q'_{12}\colon P' \times_P P' \times_X Y \to P' \times_P P'$ be the projections. Let $t\colon P' \times_P P' \to U \times_X U$ be the product of structure morphism. Then by assumption on $\mathcal{G}$, we have isomorphism

$$t^*\psi\colon {q'_{13}}^*(s^*\mathcal{G}) \xrightarrow{\sim} {q'_{23}}^*(s^*\mathcal{G}) \otimes {q'_{12}}^* t^*\mathcal{N}.$$

On the other hand by assumption on $\mathcal{F}$, we know

$${p'_1}^*\mathcal{O}_{P'}(1) = {p'_2}^*\mathcal{O}_{P'}(1) \otimes t^*\mathcal{N}.$$

Apply ${q'_{12}}^*$ to both sides we get (note that $r \circ q'_{i3} = p' \circ q'_{12}$)

$${q'_{13}}^*(r^*\mathcal{O}_{P'}(1)) = {q'_{23}}^*(r^*\mathcal{O}_{P'}(1)) \otimes {q'_{12}}^* t^*\mathcal{N}.$$

Thus we have the following canonical isomorphism T_ψ :

$$\begin{aligned}
&{q'_{13}}^*(s^*\mathcal{G} \otimes r^*\mathcal{O}_{P'}(-1)) \\
\cong\ &{q'_{23}}^*(s^*\mathcal{G}) \otimes {q'_{12}}^*(t^*\mathcal{N}) \otimes {q'_{12}}^*(t^*\mathcal{N}^{-1}) \otimes {q'_{23}}^*(r^*\mathcal{O}_{P'}(-1)) \\
\cong\ &{q'_{23}}^*(s^*\mathcal{G} \otimes r^*\mathcal{O}_{P'}(-1))
\end{aligned}$$

The cocycle condition for ψ implies T_ψ also satisfies cocycle condition, thus gives the descent datum. Hence $s^*\mathcal{G} \otimes r^*\mathcal{O}_{P'}(-1)$ descends to $P \times_X Y$, by effective of descend for quasi-coherent sheaves, see [Bos90, 6.1]. $\square$

Let $\pi_{T_U} : T_U \times_k C \to T_U$ be the projection. Let $\widetilde{T}_U = \mathbb{P}(\pi_{T_U,*}(\mathcal{E}_{T_U} \otimes \mathcal{L}_{T_U}))$. By 3.5.6, the scheme $\widetilde{T}_U \to T_U$ descends to a scheme $\widetilde{T} \to T$, along the covering $T_U \to T$.

We have diagram

$$
\begin{tikzcd}
& C \times \widetilde{T}_U \arrow[rr, "r"] \arrow[dl] \arrow[dd, "s"] & & \widetilde{T}_U \arrow[dl] \arrow[dd] \\
C \times \widetilde{T} \arrow[rr] \arrow[dd] & & \widetilde{T} \arrow[dd] & \\
& C \times T_U \arrow[rr] \arrow[dl] & & T_U \arrow[dl] \\
C \times T \arrow[rr] & & T &
\end{tikzcd}
$$

Proposition 3.5.8. *The Brauer class* $\alpha \in \mathrm{Br}(\mathrm{Pic}^0_{C/k})$ *restricts to* $0 \in \mathrm{Br}(\widetilde{T})$.

Proof. By 2.4.2, it suffices to show there exists a tautological line bundle on $C \times_k \widetilde{T}$. Let's apply 3.5.7 to the case where we take f to be $C \times_k T \to T$, we take h to be $T_U \to T$, and we take $\mathcal{F} = \pi_{T_U,*}(\mathcal{E}_{T_U} \otimes \mathcal{L}_{T_U})$, take $\mathcal{G} = \mathcal{L}_{T_U}$ to be the tautological line bundle on $T_U \times_k C$. Then we know the line bundle $s^*\mathcal{L}_{T_U} \otimes r^*\mathcal{O}_{\widetilde{T}}(-1)$ is a tautological line bundle over $C \times_k \widetilde{T}_U$ and descends to $C \times \widetilde{T}$ by 3.5.6. $\square$

Corollary 3.5.8.1. *The Brauer class restricts to* 0 *in* $\mathrm{Br}(f.f.(T))$.

Proof. Note that $\pi_{T_U,*}(\mathcal{E}_{T_U} \otimes \mathcal{L}_{T_U})$ has generic rank 1, so the projectivization $\widetilde{T} \to T$ is a isomorphism at generic point of T, thus α restrict to 0 in $\mathrm{Br}(f.f.(\widetilde{T})) = \mathrm{Br}(f.f.(T))$. $\square$

Remark 3.5.9. *It is not clear if the Brauer class restricted to the singular variety* Θ *is* 0*, as the Brauer group of a singular variety does not necessarily embed into the Brauer group of its function field.*

Chapter 4

The universal totally degenerate curve

4.1 Preliminaries

We review some facts and introduce some notations that will be repeatedly used in following sections.

4.1.1 Torus and its character module

We recall some basic properties of tori, for details, see [Mil17].

Let k be a field, let k^s be its separable closure. Let $G = \mathrm{Gal}(k^s/k)$ be the absolute Galois group of k. Let X be a scheme over k. For any $\sigma \in G$, we denote the morphism $1_X \times \mathrm{Spec}(\sigma)\colon X_{k^s} \to X_{k^s}$ by σ_X.

Let T be a group scheme over k. We say T is a torus if $T_{k^s} \cong (\mathbb{G}_{m,k^s})^r$. The character module of T is a right $\mathbb{Z}[G]$-module, denoted by $\mathrm{X}(T)$, whose underlining abelian group is $\mathrm{Hom}(T_{k^s}, \mathbb{G}_{m,k^s}) \cong \mathbb{Z}^r$. The G-action on $\mathrm{X}(T)$ is given by $(\sigma, \chi) \mapsto \sigma(\chi) = \sigma_{\mathbb{G}_m} \circ \chi \circ \sigma_T^{-1}$. We say the torus T is split if $T \cong \mathbb{G}_{m,k}^r$. Let k' be an extension of k that splits T, then the Galois action on $\mathrm{X}(T)$ factors through $\mathrm{Gal}(k'/k)$. By Galois descent, one can show the contravariant functor $T \mapsto \mathrm{X}(T)$ induces an equivalence from category of tori over k to the category of continuous right G-representations on finite rank free $\mathbb{Z}$-modules.

4.1.2 Totally degenerate curves

We recall some basic notion about nodal curves and their dual graphs.

Let k be a field, let $\overline{k}$ be its algebraic closure. Let X be a k-scheme, such that $X_{\overline{k}}$ is a connected union of smooth rational curves with nodal singularities. We call such X a totally degenerate curve. (Note that it suffices to check the nodal condition over k^s, because the nodes in $X_{\overline{k}}$ are the singular subschemes cut out by Fitting ideals: the construction of Fitting ideals commute with arbitrary base change, see [Sta18, Tag 0C3I], thus a node in $X_{\overline{k}}$ descends to a node in X_{k^s} under the homeomorphism $X_{\overline{k}} \to X_{k^s}$.)

Recall the dual graph of X, denoted by Γ_X, is a graph whose vertices V_X are the irreducible components of X_{k^s}. When two components intersect, for each intersection point, we assign an edge between the corresponding vertices. We denote the edge set by E_X, and denote the set of edges connected to vertex v by $E_X(v)$. If for all $v \in V_X$, we have $|E_X(v)| \geq 3$, then X is a Deligne-Mumford stable curve, see [ACG11, X.3].

The components and nodes of X_{k^s} may not be defined over k, but by Galois descent, the Galois orbit of the nodes or components are defined over k. There is a $\mathrm{Gal}(k^s/k)$-action on the graph Γ_X, hence a Galois action on $H_1(\Gamma_X, \mathbb{Z})$. Note that X is geometrically connected, thus $p_a(X) = h^1(X, \mathcal{O}_X) = h^1(X_{k^s}, \mathcal{O}_{X_{k^s}})$ [Sta18, 02KH]. An argument by taking normalization shows $p_a(X)$ coincides with

$$g(\Gamma_X) = \mathrm{rank}_{\mathbb{Z}} H_1(\Gamma_X, \mathbb{Z}) = |E_X| - |V_X| + 1.$$

4.1.3 Moduli of totally degenerate curves

We recall some facts about moduli stack of totally degenerate nodal curves, for details, see [ACG11, XII.10].

Let $\Gamma = (V, E)$ be a graph, with $|E(v)| \geq 3$ for each $v \in V$. Let k_0 be an infinite field, for example, we may assume $k_0 = \mathbb{C}$. Consider the moduli stack $\overline{\mathcal{M}}_{g(\Gamma)}$ of stable genus $g(\Gamma)$ curves over k_0. It has a locally closed substack $\mathcal{D}_\Gamma$, parameterizing families of totally degenerate curve with dual graph Γ. For any finite set S, let $\mathcal{M}_{0,S}$ be the moduli stack of stable genus 0 curves with S-labeled marked points. This is an irreducible smooth k_0 variety. Let $\mathcal{M}_\Gamma = \prod_{v\in V} \mathcal{M}_{0,E(v)}$, there is an obvious $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$. The $\mathcal{D}_\Gamma$ can be naturally identified with $[\mathcal{M}_\Gamma/\mathrm{Aut}(\Gamma)]$ as the image of the clutching morphism $c\colon \mathcal{M}_\Gamma \to \overline{\mathcal{M}}_{g(\Gamma)}$.

4.1.4 Universal curve associated with a graph

We keep the notations as in the previous paragraph.

Consider the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$. If the action is generically free, then $\mathcal{D}_\Gamma$ has an open substack represented by a scheme D_Γ°, and there exists a universal family of totally degenerate nodal curves with dual graph Γ over D_Γ°. Take the generic fiber, we obtain a totally degenerate nodal curve X over $k = \mathrm{frac}(D_\Gamma^\circ)$. Let k' be the function field of $\mathcal{M}_\Gamma$, then k'/k is a Galois extension with $\mathrm{Gal}(k'/k) = \mathrm{Aut}(\Gamma)$, and $X_{k'}$ is a union of $\mathbb{P}^1_{k'}$s with k'-rational nodes and dual graph Γ.

Definition 4.1.1. *We call a graph $\Gamma = (V, E)$ is good if*

1. *$|E(v)| \geq 3$ for all $v \in V$,*
2. *the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$ is generically free.*

For a good graph Γ, we call X/k the universal curve associated with Γ.

Let's briefly discuss when the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma$ is generically free. (As in the arguments below, one see this is an inherent property of the graph, and does not depend on k_0.)

Lemma 4.1.2. *If $|E(v)| \geq 4$ for each $v \in V$, then the $\mathrm{Aut}(\Gamma)$-action on $\overline{\mathcal{M}}_\Gamma$ is generically free.*

Proof. For $n \geq 4$, a point in $\mathcal{M}_{0,n}$ represents a smooth rational curve with n marked points. The marked points determine $6 \cdot \binom{n}{4}$ cross ratios. Since k_0 is an infinite field, we may pick a

k_0-point P of $\mathcal{M}_\Gamma = \prod_{v\in V} \mathcal{M}_{0,E(v)}$, such that the cross ratios on all the components are all different. Note that the $\mathrm{Aut}(\Gamma)$-action on $\mathcal{M}_\Gamma = \prod_{v\in V} \mathcal{M}_{0,E(v)}$ is given by permutation on the factors and automorphism on each factor. The components are distinguished by the cross ratios. On each component, for a set $S \subset \mathbb{P}^1(k_0)$ of more than 3 points with general cross ratios, the only automorphism of $\mathbb{P}^1_{k_0}$ that keeps S is identity. Thus the $\mathrm{Aut}(\Gamma)$-action on P is free. Finally, note that being free under a finite group action is an open condition. $\square$

Remark 4.1.3. *The condition "$|E(v)| \geq 4$ for each $v \in V$" is sufficient but not necessary. For example, if Γ is the complete bipartite graph $K_{3,4}$, we can also show the $\mathrm{Aut}(\Gamma)$-action on $\overline{\mathcal{M}}_\Gamma$ is generically free.*

4.1.5 Picard scheme of curves

We recall some facts about the Picard scheme of curves, for details, see [Bos90, 8].

Let X be a geometrically reduced and connected proper curve over k. Consider the functor $P_{X/k}\colon \mathrm{Sch}/k \to \mathrm{Sets},\ T \mapsto \mathrm{Pic}(X\times T)/\mathrm{pr}_2^*\mathrm{Pic}(T)$. We define the Picard functor $\mathcal{P}ic_{X/k}$ to be the étale sheafification of $P_{X/k}$. We know $P_{X/k}$ is in general not representable, but $\mathcal{P}ic_{X/k}$ is always represented by a smooth k-group scheme $\mathrm{Pic}_{X/k}$. Note that the representability of $P_{X/k}$ means there exists a tautological line bundle on $X \times \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}$, and the representability of $\mathcal{P}ic_{X/k}$ means there exist a tautological line bundle after some étale base change $U \to \mathrm{Pic}_{X/k}$. The functor $P_{X/k}$ is representable when X admits a k-section. Thus, if k'/k is a field extension such that $X(k') \neq \emptyset$, then we can take $U = \mathrm{Pic}_{X/k} \times_k k'$.

Let $\mathrm{Pic}^0_{X/k}$ be the identity component of $\mathrm{Pic}_{X/k}$. When X be a totally degenerate nodal curve, one can show $\mathrm{Pic}^0_{X/k}$ is a torus. More precisely, let $\nu\colon X^\nu_{k^s} \to X_{k^s}$ be the normalization map, let $\pi\colon X_{k^s} \to \mathrm{Spec}(k^s)$ be the structure map. Apply $\mathrm{R}\pi_*$ to $0 \to \mathbb{G}_m \to \nu_*\mathbb{G}_m \to Q = \nu_*\mathbb{G}_m/\mathbb{G}_m \to 0$, we get

$$0 \to \mathbb{G}_{m,k^s} \to \mathbb{G}_{m,k^s}^{|V_X|} \to \mathbb{G}_{m,k^s}^{|E_X|} \to \mathrm{Pic}_{X_{k^s}/k^s} \xrightarrow{d} \mathrm{Pic}_{X^\nu_{k^s}/k^s} \cong \mathbb{Z}^{|V_X|} \to 0,$$

here d assigns a line bundle to the degree of its restriction on each component. Then

$\mathrm{Pic}^0_{X_{k^s}/k^s} = \ker(d)$ descends to the torus $\mathrm{Pic}^0_{X/k}$. Its functor of points represents line bundles whose pullback to the normalization $X^{\nu}_{k^s}$ has degree 0 on each component, we call such line bundles multidegree zero line bundles.

To sum up, descending the above exact sequence, we get an exact sequence of tori

$$0 \to \mathbb{G}_m \to T_1 \to T_2 \to \mathrm{Pic}^0_{X/k} \to 0.$$

Taking characters, we get:

Proposition 4.1.4. *If X is a totally degenerate curve, then there is canonical* $\mathrm{Gal}(k^s/k)$-*module isomorphism*

$$\mathrm{X}(\mathrm{Pic}^0_{X/k}) \cong H_1(\Gamma_X, \mathbb{Z}).$$

Proof. Apply $\mathrm{X}(-)$ to the above exact sequence, we get the simplicial chain complex that calculates the homology of Γ_X. It is easy to check this is an $\mathrm{Gal}(k^s/k)$-isomorphism. $\square$

4.1.6 Functoriality of Leray spectral sequence

Lemma 4.1.5. *Given a commutative diagram of morphisms of schemes.*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & Y \arrow[d, "c"] \\ Z \arrow[r, "d"] & W \end{tikzcd}$$

Let G, H be group schemes over $\mathrm{Spec}(\mathbb{Z})$. *They represent étale sheaves G_X on X and H_Y on Y. Let $h\colon H \to G$ be a homomorphism, it induces natural homomorphisms*

$$h^*_{ij}\colon H^i(W, R^j c_* H_Y) \to H^i(Z, R^j b_* G_X).$$

Then there exist a morphism from Leray spectral sequence for (c, H_Y) to the Leray spectral sequence for (b, G_X), compatible with all h^*_{ij}.

Proof. Recall we may calculate Leray spectral sequences by the double complex associated with Cartan-Eilenberg resolutions. One show there exists natural morphism between corresponding double complexes. $\square$

4.2 The Brauer class

Let Γ be a good graph. Fix an infinite field k_0. Let $\mathcal{D}_\Gamma$ be the moduli stack of totally degenerate curves over k_0 with dual graph Γ. Let $k = \mathrm{frac}(\mathcal{D}_\Gamma)$ be its function field, let X/k be the universal curve. Let k'/k be the $\mathrm{Aut}(\Gamma)$-Galois extension that splits X. Let's denote $\mathrm{Aut}(\Gamma)$ by G.

We introduce the Brauer class α that will be studied later. We define an obstruction class α_d and an extension class α_e. We show they are related by $j(\alpha_e) = \alpha_d, i(\alpha_d) = \alpha$ under natural injections

$$H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k})) \overset{j}{\hookrightarrow} H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m)) \overset{i}{\hookrightarrow} H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m).$$

4.2.1 Cohomological description

Consider the morphism $\pi\colon X \times \mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{X/k}$ and structure map $s\colon \mathrm{Pic}^0_{X/k} \to \mathrm{Spec}(k)$. The Leray spectral sequence $\mathrm{R}s_*\mathrm{R}\pi_*\mathbb{G}_m \Rightarrow \mathrm{R}(s\pi)_*\mathbb{G}_m$ implies the following low term short exact sequence:

$$0 \to \mathrm{Pic}(\mathrm{Pic}^0_{X/k}) \to \mathrm{Pic}(X \times \mathrm{Pic}^0_{X/k}) \to \mathrm{Pic}_{X/k}(\mathrm{Pic}^0_{X/k}) \overset{\mathrm{d}_2^{0,1}}{\to} H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m),$$

see [Bos90, 8.1.4] for details. Then $\alpha = \mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X/k}})$ is a cohomological Brauer class over $\mathrm{Pic}^0_{X/k}$. It is clear from above exact sequence that α is the obstruction to existence of a tautological line bundle on $X \times \mathrm{Pic}^0_{X/k}$.

Definition 4.2.1. *We call $\alpha \in H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m)$ the Brauer class associated with the graph Γ.*

The following proposition shows the class lies in the Azumaya Brauer group.

Proposition 4.2.2. *Let $f\colon T \to S$ be a projective, flat and finitely presented morphism, with connected geometric fibers, and S is quasi-compact, then* $\mathrm{d}_2^{0,1}(\mathrm{Pic}_{T/S}(S)) \subset \mathrm{Br}(S)$.

Proof. See [Gir71, 4.9.1]. □

4.2.2 Galois descent description

Note that $X_{k'}(k') \neq \emptyset$, so tautological line bundles exists on $X_{k'} \times \mathrm{Pic}^0_{X_{k'}/k'}$. Two tautological line bundles differ by tensoring pullback of a line bundle on $\mathrm{Pic}^0_{X_{k'}/k'}$. Note that $\mathrm{Pic}^0_{X_{k'}/k'} \cong (\mathbb{G}_{m,k'})^{g(\Gamma)}$ is the affine scheme of a unique factorization domain, it has only trivial line bundles. Hence the tautological line bundle on $X_{k'} \times \mathrm{Pic}^0_{X_{k'}/k'}$ is unique up to isomorphism, denote it by $\mathcal{L}$.

Let's denote $X \times \mathrm{Pic}^0_{X/k}$ by Y. Note that by uniqueness, tautological line bundles pulls back to tautological lines bundle under base change. For each $\sigma \in G$, fix an isomorphism $\phi_\sigma\colon \mathcal{L} \cong \sigma_Y^*\mathcal{L}$. (If $\mathcal{L}$ descends, we may alter ϕ_σ by scaling isomorphism with $b_\sigma \in H^0(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m)$, such that $\varphi_\sigma = b_\sigma \cdot \phi_\sigma$ satisfies cocycle condition $\varphi_{\sigma\tau} = \sigma_Y^*\varphi_\tau \circ \varphi_\sigma$.) Fix a rational section s of $\mathcal{L}$. Take the isomorphism $r_\sigma\colon \mathrm{pr}_{2,*}(\mathcal{L}^{-1} \otimes \sigma_Y^*\mathcal{L}) \cong \mathcal{O}_{\mathrm{Pic}^0_{X_{k'}/k'}}$ such that $r_\sigma(s^{-1} \otimes \sigma_Y^* s) = 1$. Note that the isomorphism ϕ_σ gives a non-vanishing section $s_\sigma \in H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathrm{pr}_{2,*}(\mathcal{L}^{-1} \otimes \sigma_Y^*\mathcal{L}))$. Consider the cocycle

$$\alpha_d\colon (\sigma,\tau) \mapsto r_\sigma(s_\sigma(1)) \cdot r_{\sigma\tau}^{-1}(s_{\sigma\tau}(1)) \cdot \sigma^*(r_\tau(s_\tau(1))).$$

Then the $\{\phi_\sigma\}$s can be modified to descent datum if and only if the class of α_d is zero in $H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m))$. Since Galois descent of coherent sheaves is effective, this is the obstruction to existence to tautological line bundles on $X \times \mathrm{Pic}^0_{X/k}$.

Definition 4.2.3. *We call $\alpha_d \in H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m))$ the descent obstruction class associated with Γ.*

By the Hochschild-Serre spectral sequence, we have natural map

$$i\colon H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k}, \mathbb{G}_m)) \to H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m).$$

Proposition 4.2.4. *The map i is an injection.*

Proof. The low term short exact sequence for the Hochschild-Serre spectral sequence shows the kernel is generated by the image of $H^1(G, \mathrm{Pic}^0(X_{k'}))$. But

$$H^1(G, \mathrm{Pic}^0(X_{k'})) = H^1(G, \{1\}) = 0.$$

$\square$

Proposition 4.2.5. *We have $i(\alpha_d) = \alpha$.*

Proof. By the triple spectral sequence associated to the composition the functors

$$\mathrm{Sh}(X_{k'} \times \mathrm{Pic}^0_{X_{k'}/k'}) \overset{\mathrm{pr}_{2,*}}{\to} \mathrm{Sh}(\mathrm{Pic}^0_{X_{k'}/k'}) \overset{\Gamma(-)}{\to} \mathrm{Mod}_{k[G]} \overset{(-)^G}{\to} \mathrm{Mod}_k,$$

there exists natural morphism, see [Sta18, 08BI],

$$\begin{tikzcd} t\colon \mathrm{Pic}_{X/k}(\mathrm{Pic}^0_{X/k}) \arrow[r, hook] & H^0(G, \mathrm{Pic}_{X_{k'}/k'}(\mathrm{Pic}^0_{X_{k'}/k'})) \arrow[d, "\sim"] & \\ & H^0(G, H^1(\mathrm{Pic}^0_{X_{k'}/k'}, \mathrm{pr}_{2,*}\mathbb{G}_m)) \arrow[r] & H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m)) \end{tikzcd}$$

such that $i \circ t = \mathrm{d}_2^{0,1}$. By the previous description of cocyle, we have $t(1_{\mathrm{Pic}^0_{X/k}}) = \alpha_d$, thus $i(\alpha_d) = \alpha$. $\square$

Remark 4.2.6. *Note that the tautological line bundles on $X \times \mathrm{Pic}^0_{X/k}$ may not be unique, as the torus $\mathrm{Pic}^0_{X/k}$ may have nontrivial line bundles. Running the Hochschild-Serre spectral sequence shows $\mathrm{Pic}(\mathrm{Pic}^0_{X/k}) = H^1(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m))$.*

4.2.3 Extension class description

Recall that $\mathrm{Pic}^0_{X/k}$ fits into the exact sequence of tori, see 4.1.5

$$0 \to \mathbb{G}_m \to T_1 \to T_2 \to \mathrm{Pic}^0_{X/k} \to 0.$$

Taking character modules, we get

$$0 \to \mathrm{X}(\mathrm{Pic}^0_{X/k}) \to \mathrm{X}(T_2) \to \mathrm{X}(T_1) \to \mathrm{X}(\mathbb{G}_m) \to 0,$$

its extension class is a class $\alpha_e \in \mathrm{Ext}^2_G(\mathbb{Z}, \mathrm{X}(\mathrm{Pic}^0_{X/k})) = H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k}))$.

Definition 4.2.7. *We call $\alpha_e \in H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k}))$ the extension class associated with Γ.*

Proposition 4.2.8. *We have the short exact sequence*

$$0 \to \mathrm{X}(\mathrm{Pic}^0_{X/k}) \to H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m) \to \mathbb{G}_{m.k'} \to 0,$$

where the first map is inclusion, and second map is evaluating at identity section $e \in \mathrm{Pic}^0_{X/k}$.

Proof. Fix an isomorphism $\mathrm{Pic}^0_{X_{k'}/k'} \cong \mathbb{G}^g_{m,k'} \hookrightarrow \mathbb{A}^g_{k'}$. Suppose the coordinate on $\mathbb{A}^g_{k'}$ are given by $T_1, \ldots, T_g$ such that $\mathrm{Pic}^0_{X_{k'}/k'} \cong \mathrm{Spec}(k'[T_1^{\pm 1}, \ldots, T_g^{\pm 1}])$. Given any regular section of $s \in H^0(\mathrm{Pic}^0_{\mathrm{X_{k'}/k'}}, \mathbb{G}_m)$ such that $s(e) = 1$, let $v_i(s)$ be the valuation of s along T_i. Then $s' = \prod T_i^{-v_i(s)} s$ is a rational section on $\mathbb{A}^g_{k'}$, regular and nonvanishing on the complement of codimension at least 2 subset. By Hartogs's theorem, see [Sta18, Tag 031T], we know $s' \in H^0(\mathbb{A}^2_{k'}, \mathbb{G}_m) = \mathbb{G}_{m,k'}$. Then $s'(e) = \prod_i T_i(e)^{-v_i(s)} s(e) = 1$ implies $s' = 1$, hence $s = \prod_i T_i^{v_i(s)}$, thus the exactness in middle. The right and left exactness are obvious. □

Proposition 4.2.9. *The natural map induced by inclusion*

$$j\colon H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k})) \to H^2(G, H^0(\mathrm{Pic}^0_{X_{k'}/k'}, \mathbb{G}_m))$$

is an injection.

Proof. Consider long exact sequence of Galois cohomology of the short exact sequence 4.2.8, the kernel is generated by $H^1(G, \mathbb{G}_m)$, which is trivial by Hilbert 90. □

Next we show $j(\alpha_e) = \alpha_d$. First we describe the class α_e.

Let T_3 be the image of T_1 in T_2, we have short exact sequences

$$\text{(a) } 0 \to \mathbb{G}_m \to T_1 \to T_3 \to 0 \quad \text{and} \quad \text{(b) } 0 \to T_3 \to T_2 \to \mathrm{Pic}^0_{X/k} \to 0.$$

Let M be a G-module, let $C^\bullet(G, M)$ be the canonical resolution by inhomogeneous cochains $C^i(G, M) = \mathrm{Map}(G^i \to M)$. Let δ be the connecting homomorphism on cohomology groups.

Proposition 4.2.10. *The class α_e can be calculated in the following two ways.*

1. *Apply $C^\bullet(G, \mathrm{Hom}(-, \mathrm{X}(\mathrm{Pic}^0_{X/k})))$ to $(b), (a)$. Then $\alpha_e = \delta \circ \delta(1_{\mathrm{X}(\mathrm{Pic}^0_{X/k})})$.*
2. *Apply $C^\bullet(G, \mathrm{Hom}(\mathrm{X}(\mathbb{G}_m), -))$ to $(a), (b)$. Then $\alpha_e = \delta \circ \delta(1_{\mathrm{X}(\mathbb{G}_m)})$.*

Proof. The Ext class can be calculated by either the first or second variable in $\mathrm{Hom}(-, -)$. □

Remark 4.2.11. *We will use the the first and second calculation in 4.2.12 and 4.3.4.*

In order to calculate the descent obstruction a_d, we first explain how to describe line bundles on $X_{k'} \times \mathrm{Pic}^0_{X_{k'}/k'}$. More precisely, we describe the map

$$T_{2,k'}(\mathrm{Pic}^0_{X_{k'}/k'}) \to \mathrm{Pic}^0_{X_{k'}/k'}(\mathrm{Pic}^0_{X_{k'}/k'}).$$

Let $X^\nu_{k'}$ be the normalization of $X_{k'}$. Denote $\mathrm{Pic}^0_{X_{k'}/k'}$ by P. Consider the following $2|E|$-tuple: $\{r_{v,e}\}_{v \in V, e \in E(v)}$, $r_{v,e} \in \Gamma(P, \mathbb{G}_m)$. Take the trivial line bundle on normalization $X^\nu_{k'} \times P$. Fix a nowhere vanishing section s for the trivial line bundle. For an edge e with endpoints v, w, we denote the k'-section corresponding to $e \in E$ on component v of $X^\nu_{k'}$ by $Q_{v,e}$. We identify the trivial line bundle over $Q_{v,e} \times P$ and $Q_{w,e} \times P$ by $r_{v,e} \cdot s|_{Q_{v,e} \times P} = r_{w,e} \cdot s|_{Q_{w,e} \times P}$. We get a multidegree zero line bundle on $X \times P$. By the construction, we get same line bundle if we scale $r_{v,e}, r_{w,e}$ by the same function. Hence we get the map $(\prod_{e \in E} \mathbb{G}_m \times \mathbb{G}_m / \mathbb{G}_m)(P) \cong T_{2,k'}(P) \to P(P)$. This map is surjection as we may read the gluing datum on normalization.

Proposition 4.2.12. *We have $j(a_e) = a_d$.*

Proof. Consider the following natural transformation of functors from category of tori to abelian groups, induced by the equivalence between the category of Galois modules and tori:

$$\mathrm{Hom}_G(\mathrm{X}(-), \mathrm{X}(\mathrm{Pic}^0_{X/k})) \to ((-_{k'})(\mathrm{Pic}^0_{X_{k'}/k'}))^G.$$

The class α_e is calculated by iterated δ-homomorphisms of $1_{\mathrm{X}(\mathrm{Pic}^0_{X/k})}$ on the left hand side.

Consider the short exact sequences of G-modules

$$0 \to \mathbb{G}_m(P) \to T_{1,k'}(P) \to T_{3,k'}(P) \to 0$$

and

$$0 \to T_{3,k'}(P) \to T_{2,k'}(P) \to P(P) \to 0.$$

The natural transformation commutes with δ-homomorphisms. Hence in order to show $i(\alpha_e) = \alpha_d$, it suffices to show the obstruction class α_d is also calculated by applying iterated δ-homomorphism to 1_P for the $C^\bullet(G, -)$ resolutions.

Fix any datum $\{r_{v,e}\}$ that represents the tautological line bundle. Let's still denote $P = \mathrm{Pic}^0_{X_{k'}/k'}$. Let $Z = X \times P$ and $Z^\nu = X^\nu \times P$. An isomorphism $\phi_\sigma \colon \mathcal{L} \cong \sigma_Z^* \mathcal{L}$ is determined by an isomorphism on their pullbacks to Z^ν. Since $\mathcal{L}$ is trivialized on Z^ν, the isomorphism $\phi_\sigma|_{Z^\nu}$ is multiplication by $u_{v,\sigma} = \sigma_P^*(r_{\sigma^{-1}v,\sigma^{-1}e})/r_{v,e} \in H^0(P, \mathbb{G}_m)$ on the v-component of Z^ν (independent of choice of $e \in E(v)$). So the isomorphism is represented by $\{u_{v,\sigma}\} \in T_{1,k'}(P)$. Then calculate $\phi_\sigma|_{Z^\nu} \cdot \phi_{\sigma\tau}|_{Z^\nu}^{-1} \cdot \sigma_{Z^\nu}^*(\phi_\tau|_{Z^\nu})$. This is multiplication by elements in $\Gamma(P, \mathbb{G}_m)$ on each component. Since it is the pullback of $\phi_\tau \cdot \phi_{\sigma\tau}^{-1} \cdot \sigma_Z^*(\phi_\tau)$ from Z and Z is connected, the multiplication on each component are the by same element in $\Gamma(P, \mathbb{G}_m)$, denoted by $c_{\sigma,\tau}$. Then $(\sigma, \tau) \mapsto c_{\sigma,\tau}$ is the descent obstruction class α_d as defined in 4.2.2. □

4.3 Period and index

Let S be a scheme, recall for a class $\beta \in \mathrm{Br}(S)$, the period of β is the least $n \in \mathbb{Z}_{>0}$ such that $n\beta = 0$. The index of β is the greatest common divisor of rank of Azumaya algebras representing the class. Note Azumaya algebras of rank m corresponds to Brauer-Severi schemes of dimension $m-1$. (Base change an Azumaya algebra to $\mathcal{E}nd(\mathcal{F})$, then descend $\mathbb{P}(\mathcal{F})$ to get the Brauer Severi scheme. Conversely, base change a Brauer-Severi scheme to $\mathbb{P}(\mathcal{F})$, then descent $\mathcal{E}nd(\mathcal{O}(1)^{\oplus m})$ to get the Azumaya algebra.) If S is a regular scheme, let

$\mathrm{frac}(S)$ be the function field of S, then we have injection $\mathrm{Br}(\mathrm{frac}(S)) \to \mathrm{Br}(S)$, they share the same period and index, see [AW14, 6.1]. In this case, we will not distinguish the Brauer class in $\mathrm{Br}(S)$ or in $\mathrm{Br}(\mathrm{frac}(S))$.

4.3.1 Upper bounds on index

Let X be a stable curve over a field k, with genus at least 2. As explained in 4.2.1, there exists a Brauer class $\alpha_X = \mathrm{d}_2^{0,1}(\mathrm{Pic}^0_{X_k/k}) \in \mathrm{Br}(\mathrm{Pic}^0_{X/k})$, which is the obstruction class to existence of a universal line bundle on $X \times \mathrm{Pic}^0_{X/k}$.

Theorem 4.3.1. *We have* $\mathrm{ind}(\alpha_X)|g-1$.

Proof. Note there is a canonical isomorphism $\mathrm{Pic}^0_{X/k} \cong \mathrm{Pic}^{2g-2}_{X/k}$ by tensoring with canonical line bundle. Let $S = \mathrm{Pic}^0_{X/k} - \{\omega_C\}$. The descent obstruction to the universal line bundle gives the Brauer class on $\mathrm{Br}(\mathrm{Pic}^{2g-2}_{X/k}) = \mathrm{Br}(S)$. The equality is given by purity of Brauer group, see [Gro68b, 7] and [Ces17] for details. Over S, a tautological line bundle $\mathcal{L}$ on $X_{k^s} \times \mathrm{Pic}_{X_{k^s}/k^s}$ does not necessarily descend, but $\mathbb{P}(\pi_{2,*}\mathcal{L})$ descends to the Brauer-Sever scheme $\mathrm{Div}^{2g-2}_{X/k}$, see [Bos90, 8.1] for details. For a Brauer-Severi scheme $f\colon P \to S$, consider the Leray spectral sequence associated to the sheaf $\mathbb{G}_m$. The Brauer class is defined by $\mathrm{d}_2^{0,1}(1)$, where $\mathrm{d}_2^{0,1}\colon H^0(S, \mathrm{Pic}_{P/S}) \cong \mathbb{Z} \to H^2(S, \mathbb{G}_m)$ is the transgression map. One can show the Brauer class of the Brauer-Severi scheme coincides with the Brauer class α, see section 2.3 or [Gir71, V.4] for detail. By Riemann-Roch, the rank of $\pi_*\mathcal{L}$ over S is $g-1$, so P/S has relative dimension $g-2$, thus $\mathrm{ind}(\alpha) = |g-1$. ☐

For totally degenerate nodal curves, we have extra criterion. Let Γ be a good graph, let X/k be the universal curve. Let α be the Brauer class associated with graph Γ. Suppose Γ_0 is an $\mathrm{Aut}(\Gamma)$-invariant subgraph, with e_0, v_0 be the number of edges and vertices of Γ_0.

Theorem 4.3.2. *We have* $\mathrm{ind}(\alpha)|e_0$.

Proof. By the description in 4.1.4 and Galois descent, the nodes corresponding to edges in Γ_0 descend to a union of closed points $P_1 \cup \cdots \cup P_s$ over k. Let $\kappa(P_i)$ be the fraction field of P_i. Since $X_{\kappa(P_i)}$ has a $\kappa(P_i)$-rational point, the universal line bundle exist over

$X_{\kappa(P_i)} \times \mathrm{Pic}^0_{X_{\kappa(P_i)}/\kappa(P_i)}$ exists, hence the Brauer class is zero in $\mathrm{Br}(\mathrm{Pic}^0_{X/k} \times_k \kappa(P_i))$. Hence by [GS17, 4.5.8], we know $\mathrm{ind}(\alpha) | [\kappa(P_i)\colon k]$ for each i, thus $\mathrm{ind}(\alpha)|e_0$. □

Theorem 4.3.3. *We have* $\mathrm{ind}(\alpha)|2v_0$.

Proof. By the description in 4.1.4 and Galois descent, the components corresponding to vertices in Γ_0 descend to an integral scheme X_0 over k. Let X_0^ν be the normalization of X_0, let $X_0^\nu \xrightarrow{a} S_0 \xrightarrow{b} \mathrm{Spec}(k)$ be the Stein factorization, see [Sta18, Tag 03GX], where a is a conic bundle and b is a union of closed points $Q_0 \cup \cdots \cup Q_r$. Since $X_0^\nu \to S_0$ is a conic bundle, for each i, X_0^ν has infinitely many degree $2\deg(Q_i)$ points over k, hence does X_0 and X. Then we argue as in the last theorem. □

4.3.2 Bounds on period

Let Γ be a good graph. Let X/k be the universal curve associated with graph Γ. Let $G = \mathrm{Aut}(\Gamma)$. Recall that $\mathrm{X}(\mathrm{Pic}^0_{X/k})$ can be identified with $H_1(\Gamma, \mathbb{Z})$ as G-module, see 4.1.4.

Let σ be a period m element in G. Let L be a loop in Γ with no repeated vertices. Suppose L is fixed by σ and we can write $L = \cup_{i=0}^{m-1} \sigma^i(ve)$, where v is a vertex in L and e is the union of edges connecting v and $\sigma(v)$. Let $\langle\sigma\rangle \subset G$ be the subgroup generated by σ.

Theorem 4.3.4. *View* $H_1(\Gamma, \mathbb{Z})$ *as a* $\langle\sigma\rangle$*-module. If the submodule generated by* L *is a direct summand, then* m *divides* $\mathrm{per}(\alpha)$.

Proof. Consider the restriction map $H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k})) \to H^2(\langle\sigma\rangle, \mathrm{X}(\mathrm{Pic}^0_{X/k}))$. Since $\mathbb{Z} \cdot L$ is a summand of $\mathrm{X}(\mathrm{Pic}^0_{X/k})$, we have projection map $H^2(\langle\sigma\rangle, \mathrm{X}(\mathrm{Pic}^0_{X/k})) \to H^2(\langle\sigma\rangle, \mathbb{Z})$. By 4.2.4 and 4.2.9, we know $i \circ j\colon H^2(G, \mathrm{X}(\mathrm{Pic}^0_{X/k})) \to H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m)$ is injection, thus it suffices to show α_e maps to a generator in $H^2(\langle\sigma\rangle, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$.

We use $\delta \circ \delta(1_{\mathrm{X}(\mathbb{G}_m)})$ to calculate α_e, see 4.2.10. An inverse image of $1 \in \mathrm{X}(\mathbb{G}_m)$ in $\mathrm{X}(T_1) \cong \mathbb{Z}^{|V|}$ can be taken to be characteristic function χ_v. Take differential we get 1-cocycle

$$\phi\colon \sigma^i \mapsto \chi_{\sigma^i(v)} - \chi_v$$

in $\mathrm{X}(T_3)$. We take an inverse image of ϕ in $\mathrm{X}(T_2) = \mathbb{Z}^{|E|}$. For example, we take

$$\sigma^i \mapsto \sigma^0(e) + \cdots + \sigma^{i-1}(e).$$

Take differential again, the 2-cocycle can be written as

$$(\sigma^i, \sigma^j) = \begin{cases} 0 & \text{if } \ i+j<n \\ 1 \cdot L & \text{if } \ i+j \geq n \end{cases}$$

Thus image of α in $H^2(\langle\sigma\rangle, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ is the canonical generator, hence $m|\mathrm{per}(\alpha)$. $\square$

Theorem 4.3.5. $\mathrm{per}(\alpha)$ *divides* $|G| = |\mathrm{Aut}(\Gamma)|$.

Proof. Note α is the image of α_e, and $|\mathrm{Aut}(\Gamma)|$ annihilates α_e in group cohomology. $\square$

4.3.3 Functorial property

Let k be a field, let $f\colon X \to Y$ be a morphism of proper geometrically connected k-curves. Let $\phi\colon \mathrm{Pic}^0_{Y/k} \to \mathrm{Pic}^0_{X/k}$ be the natural morphism induced by pullback. Consider the following commutative diagram

$$\begin{tikzcd} Y \times \mathrm{Pic}^0_{Y/k} \arrow[d, "p_1"] & X \times \mathrm{Pic}^0_{Y/k} \arrow[l, "f\times 1"'] \arrow[r, "1\times\phi"] \arrow[d, "p_2"] & X \times \mathrm{Pic}^0_{X/k} \arrow[d, "p_3"] \\ \mathrm{Pic}^0_{Y/k} \arrow[r, equal] & \mathrm{Pic}^0_{Y/k} \arrow[r, "\phi"] & \mathrm{Pic}^0_{X/k} \end{tikzcd}$$

Consider the Leray spectral sequence associated to the projections in the columns and the sheaf $\mathbb{G}_m$. By functoriality of Leray spectral sequence, we have the following commutative diagram, where the map $\mathrm{d}_2^{0,1}$s are denoted by δ_i:

$$\begin{tikzcd} \mathrm{Pic}^0_{Y/k}(\mathrm{Pic}^0_{Y/k}) \arrow[r, "\phi\circ-"] \arrow[d, "\delta_1"] & \mathrm{Pic}^0_{X/k}(\mathrm{Pic}^0_{Y/k}) \arrow[d, "\delta_2"] & \mathrm{Pic}^0_{X/k}(\mathrm{Pic}^0_{X/k}) \arrow[l, "-\circ\phi"'] \arrow[d, "\delta_3"] \\ H^2(\mathrm{Pic}^0_{Y/k}, \mathbb{G}_m) \arrow[r, "\sim"] & H^2(\mathrm{Pic}^0_{Y/k}, \mathbb{G}_m) & H^2(\mathrm{Pic}^0_{X/k}, \mathbb{G}_m) \arrow[l, "\phi^*"'] \end{tikzcd}$$

Theorem 4.3.6. *Let* $\alpha_X = \delta_3(1_{\mathrm{Pic}^0_{X/k}})$, $\alpha_Y = \delta_1(1_{\mathrm{Pic}^0_{Y/k}})$*. Then* $\phi^*(\alpha_X) = \delta_2(\phi) = \alpha_Y$*.*

Proof. This follows from the above commutative diagram. □

4.3.4 Subgraph

Let Γ_1 be a good graph. Let $\Gamma_2 \subset \Gamma_1$ be a subgraph, such that Γ_2 is also a good graph and Γ_2 is $\mathrm{Aut}(\Gamma_1)$-stable. Let X_i/k_i be universal curve of Γ_i.

Lemma 4.3.7. *There exists a map* $t\colon \mathrm{Spec}(k_1) \to \mathrm{Spec}(k_2)$ *and a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[d] & (X_2)_{k_1} \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d] \\ \mathrm{Spec}(k_1) & \mathrm{Spec}(k_1) \arrow[l, "\sim"'] \arrow[r, "t"] & \mathrm{Spec}(k_2) \end{tikzcd}$$

where the right square is cartesian, the left square is a composition of partial normalization and closed immersion.

Proof. Note that we have natural surjective morphism $\mathrm{Fgt}\colon \mathcal{M}_{\Gamma_1} = \prod_{v \in V_1} \mathcal{M}_{0,E_1(v)} \to \prod_{v \in V_2} \mathcal{M}_{0,E_2(v)} \cong \mathcal{M}_{\Gamma_2}$ induced by forgetting vertices and forgetting nodes. Note that there exists universal family of curves with marked nodes $\mathcal{Y}_i \to \mathcal{M}_{\Gamma_i}$, and there exist a morphism $\mu\colon \mathcal{Y}_2 \times_{\mathcal{M}_{\Gamma_2}} \mathcal{M}_{\Gamma_1} \to \mathcal{Y}_1$ given by partial normalization and closed immersion. It is easy to see Fgt is $\mathrm{Aut}(\Gamma_1)$-$\mathrm{Aut}(\Gamma_2)$-equivariant. Taking quotient we get morphism $\mathcal{D}_{\Gamma_1} \to \mathcal{D}_{\Gamma_2}$. Similarly by equivariance of μ, we get morphism $[\mathcal{Y}_2/\mathrm{Aut}(\Gamma_2)] \times_{\mathcal{D}_{\Gamma_2}} \mathcal{D}_{\Gamma_1} \to [\mathcal{Y}_1/\mathrm{Aut}(\Gamma_1)]$. Note that $[\mathcal{Y}_i/\mathrm{Aut}(\Gamma_i)] \to [\mathcal{M}_{\Gamma_i}/\mathrm{Aut}(\Gamma_i)]$ is the universal family $\mathcal{X}_i \to \mathcal{D}_i$, hence we get the following diagram of universal families:

$$\begin{tikzcd} \mathcal{X}_1 \arrow[d] & \mathcal{D}_{\Gamma_1} \times_{\mathcal{D}_{\Gamma_2}} \mathcal{X}_2 \arrow[l, "t"'] \arrow[r] \arrow[d] & \mathcal{X}_2 \arrow[d] \\ \mathcal{D}_{\Gamma_1} & \mathcal{D}_{\Gamma_1} \arrow[l, "\sim"'] \arrow[r] & \mathcal{D}_{\Gamma_2} \end{tikzcd}$$

The X_1/k_1, $(X_2)_{k_1}/k_1$ and X_2/k_2 are the generic fibers of the columns. □

Theorem 4.3.8. *Let α_i be the Brauer class associated with Γ_i. Then α_2 maps to α_1, under the natural map induced by base change and restriction:* $\mathrm{Br}(\mathrm{Pic}^0_{X_2/k_2}) \to \mathrm{Br}(\mathrm{Pic}^0_{(X_2)_{k_1}/k_1}) \to \mathrm{Br}(\mathrm{Pic}^0_{X_1/k_1})$.

Proof. By functoriality of Leray spectral sequence, we see α_2 restricts to $\mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{(X_2)_{k_1}/k_1}})$. Then apply 4.3.6 to $(X_2)_{k_1} \to X_1$, we see $\mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{(X_2)_{k_1}/k_1}})$ maps to α_1. □

Corollary 4.3.8.1. *We have* $\mathrm{ind}(\alpha_1)|\mathrm{ind}(\alpha_2)$.

Proof. The follows since Azumaya algebras pulls back to Azumaya algebras of the same rank. □

4.3.5 Covering

Let k be a field, let X_1, X_2 be curves over k. Given a finite flat morphism $f\colon X_2 \to X_1$, we may consider the Brauer classes $\beta_1 = \mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X_2/k}})$ and $\beta_2 = \mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X_1/k}})$.

Proposition 4.3.9. *We have* $\frac{\mathrm{per}(\beta_2)}{\gcd(\mathrm{per}(\beta_2),\deg(f))} = \mathrm{per}(\deg(f)\cdot\beta_2)|\mathrm{per}(\beta_1)$.

Proof. It suffices to show β_1 maps to $\deg(f)\cdot\beta_2$ by some homomorphism. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Pic}^0_{X_2/k} \times X_2 \arrow[r] \arrow[d] & \mathrm{Pic}^0_{X_2/k} \times X_1 \arrow[r] \arrow[d] & \mathrm{Pic}^0_{X_1/k} \times X_1 \arrow[d] \\
\mathrm{Pic}^0_{X_2/k} \arrow[r, "\sim"] & \mathrm{Pic}^0_{X_2/k} \arrow[r, "\mathrm{Nm}"] & \mathrm{Pic}^0_{X_1/k}
\end{tikzcd}$$

Here Nm is the morphism induced by $\mathcal{L} \mapsto \det(f_*\mathcal{L}) \otimes \det(f_*\mathcal{O}_{X_2})^{-1}$. Let $\phi\colon \mathrm{Pic}^0_{X_1/k} \to \mathrm{Pic}^0_{X_2/k}$ be the map induced by pullback of line bundles. By functoriality of Leray spectral sequence, we have the following commutative diagram, where $\mathrm{d}_2^{0,1}$ maps are denoted by ϵ_i.

$$\begin{tikzcd}
\mathrm{Pic}^0_{X_2/k}(\mathrm{Pic}^0_{X_2/k}) \arrow[d, "\epsilon_1"] & \mathrm{Pic}^0_{X_1/k}(\mathrm{Pic}^0_{X_2/k}) \arrow[l, "\phi\circ-"'] \arrow[d, "\epsilon_2"] & \mathrm{Pic}^0_{X_1/k}(\mathrm{Pic}^0_{X_1/k}) \arrow[l, "-\circ\mathrm{Nm}"'] \arrow[d, "\epsilon_3"] \\
\mathrm{Br}(\mathrm{Pic}^0_{X_2/k}) & \mathrm{Br}(\mathrm{Pic}^0_{X_2/k}) \arrow[l, "\sim"'] & \mathrm{Br}(\mathrm{Pic}^0_{X_1/k}) \arrow[l, "\mathrm{Nm}^*"']
\end{tikzcd}$$

Then $\mathrm{Nm}^*(\beta_1) = \epsilon_1(\mathrm{Nm} \circ \phi) = \epsilon_3(\deg(f) \cdot 1_{\mathrm{Pic}^0_{X_1/k}}) = \deg(f) \cdot \beta_2$. $\square$

Remark 4.3.10. *Let Γ_1, Γ_2 be graphs such that Γ_2 is covering of Γ_1, let α_i be the associated Brauer classes. Suppose the universal curve of Γ_1 has a flat cover whose dual graph is Γ_2, then we have induced morphism $\mathcal{D}_{\Gamma_1} \to \mathcal{D}_{\Gamma_2}$. We may consider the curve $\mathcal{X}_1 \to \mathcal{Y} = \mathcal{X}_2 \times_{\mathcal{D}_{\Gamma_2}} \mathcal{D}_{\Gamma_1}$. Let $\alpha' = \mathrm{d}_2^{0,1}(1)$ be the Brauer class associated with $\mathcal{Y}/\mathcal{D}_{\Gamma_1}$. Apply the previous proposition, we get $\mathrm{per}(\deg(f) \cdot \mathrm{per}(\alpha'))|\mathrm{per}(\alpha_1)$. But it is not clear if $\mathrm{per}(\deg(f) \cdot \mathrm{per}(\alpha_2))|\mathrm{per}(\alpha_1)$.*

4.4 Examples

We give some examples where the period and index of Brauer class can be determined. In all these examples period to equal index.

Example 4.4.1. *Let $g \geq 3$ be an integer. Let Γ be the graph with $g-1$ vertices $v_1, \ldots, v_{g-1}$, and two edges e_i, f_i between $v_i v_{i+1}$ for $i = 1, \ldots g-2$ and $v_{g-1}v_1$ for $i = g-1$, then $g(\Gamma) = g$ (See Fig. 4.4.1 for $g = 5$). Let α be the Brauer class associated with graph Γ. Note the g loops $f_1 \ldots f_{g-1}$, $e_i f_i$ form a basis for $H_1(\Gamma, \mathbb{Z})$ as $\mathbb{Z}/(g-1)\mathbb{Z}$-module, with rotation action on indices. Then apply 4.3.4 to the loop $f_1 \ldots f_{g-1}$, we know $g-1|\mathrm{per}(\alpha)$. By 4.3.1, $\mathrm{ind}(\alpha)|g-1$, so $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha) = g-1$.*

Remark 4.4.2. *Let $g \geq 3$ be an integer. Let $\mathcal{M}_g$ be the moduli stack of smooth genus g curves. Let k be its function field, let $X \to k$ be the universal curve. Let $\alpha = \mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X/k}})$ be the Brauer class. Then since period drops by specialization, the previous example shows $g-1|\mathrm{per}(\alpha)$, thus $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha) = g-1$.*

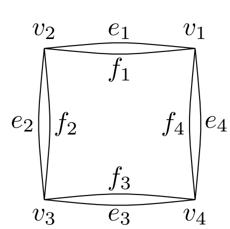

Fig 4.4.1

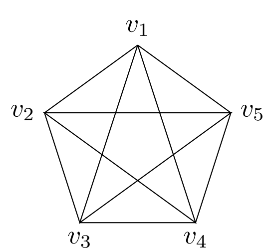

Fig 4.4.3.i

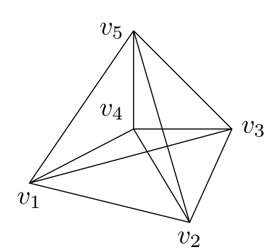

Fig 4.4.3.ii

Example 4.4.3. *Let* $\Gamma = K_5$ *be the complete graph with* 5 *vertices. Label the vertices of the graph by* $v_1, \ldots, v_5$*, then* $\mathrm{Aut}(\Gamma) = S_5$ *acts by permutation on vertices. In this case, the loops* $v_{i-1}v_iv_{i+1}(i = 1, \ldots, 5), v_1v_2v_3v_4v_5$ *form a basis of* $H_1(\Gamma, \mathbb{Z})$*. Take* $\sigma = (12345) \in S_5$*, then* $L = v_1v_2v_3v_4v_5$ *is a cycle class fixed by permutation* σ*, and is a direct summand. Let* α *be the Brauer class associated with* Γ*, then by 4.3.4, we have* $5|\mathrm{per}(\alpha)$*. Also* $g(\Gamma) = 6$*, by 4.3.1 we conclude* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha) = 5$.

On the other hand, although the loop $L' = v_1v_2v_3$ *is invariant under* $\sigma = (123)$*, it does not generate a summand of* $H^1(\Gamma, \mathbb{Z})$ *as* $\langle\sigma\rangle$*-module, so we cannot conclude* $3|\mathrm{per}(\alpha)$.

Example 4.4.4. *Let* Γ *be the graph of complete graph* K_4*, with edge doubled along a simple length four loop (Fig 4.4.4.i). Let* α *be the associated Brauer class. Consider the automorphism* $\sigma \in \mathrm{Aut}(\Gamma)\colon v_i \mapsto v_{5-i}$*, but switch the edges between* v_1v_4, v_2v_3 *(we may view this automorphism as* 180 *degree rotation in Fig 4.4.4.ii). Then apply 4.3.4 to the loop of* v_2v_3*, we can show* $2|\mathrm{per}(\alpha)$*. Since* $g(\Gamma) = 7$*, we know* $\mathrm{ind}(\alpha)|6$*. Apply 4.3.2 to* $\Gamma_0 = \Gamma$*, we see* $\mathrm{ind}(\alpha)|10$*, thus* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha) = 2$.

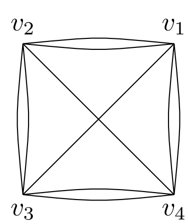

Fig 4.4.4.i

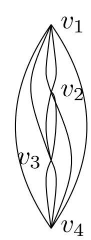

Fig 4.4.4.ii

Example 4.4.5. *Let* Γ *be the graph with vertices* $v_1, \ldots, v_4; w_1, \ldots, w_4$*, with edges as in the following diagram. This graph admits Fig 4.4.1 as an* $\mathrm{Aut}(\Gamma)$*-invariant subgraph. Apply 4.3.4 and 4.3.8, we know* $\mathrm{per}(\alpha) = 4$*. By 4.3.8.1, we also have* $\mathrm{ind}(\alpha) = 4$.

Example 4.4.6. *Let* $\Gamma = K_{3,4}$ *be the complete bipartite graph. Let* α *be the associated Brauer class. Since* $\mathrm{g}(\Gamma) = 6$*, we know* $\mathrm{per}(\alpha)|g - 1 = 5$*. On the other hand, by 4.3.5,* $\mathrm{per}(\alpha)$ *divides* $|\mathrm{Aut}(K_{3,4})| = |S_3 \times S_4| = 4! \times 3!$*. Then* $\mathrm{per}(\alpha)|(5, 4! \times 3!) = 1$*, hence the Brauer class is trivial.*

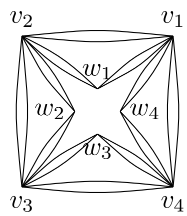

Fig 4.4.5

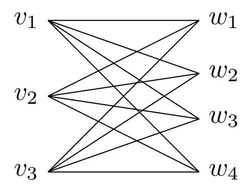

Fig 4.4.6

4.5 Further Discussion

4.5.1 Period and index

Let k_0 be a field. Let Γ be a good graph, let X/k be the associated curve and α be the associated Brauer class. The period can be calculated by group cohomology, thus $\mathrm{per}(\alpha)$ is independent of choice k_0. We may ask

Question 4.5.1. *Does* $\mathrm{ind}(\alpha)$ *depend on the field* k_0*?*

We show the index does not depend on k_0, and in general:

Theorem 4.5.2. *Let k_0'/k_0 be an extension of algebraically closed fields. Let l be a subfield of k_0. Let V be a variety over l. Let K be the fraction field of V. Let α be a class in* $\mathrm{Br}(K)$*. Let α_{k_0} and $\alpha_{k_0'}$ be its restriction to* $\mathrm{Br}(K_{k_0})$ *and* $\mathrm{Br}(K_{k_0'})$*. Then* $\mathrm{ind}(\alpha_{k_0}) = \mathrm{ind}(\alpha_{k_0'})$.

Proof. Let P be the Brauer-Severi variety over K corresponding to α. Let $L \subset P_{k_0}$ be the twisted linear subvariety with dimension $\mathrm{ind}(\alpha_{k_0}) - 1$. Since L is a closed subvariety cut out by finitely many equations, there exists a finite type algebra $A \subset k_0$ over l, such that L is defined over A, so that there exists a scheme L_A over A such that $L = (L_A)_{l_0}$.

By generic flatness of finite type morphism to integral schemes, see [Sta18, 052A], we can pick a maximal ideal $\mathfrak{m}$ of A, such that $(L_A)_{A/\mathfrak{m}}$ is a twisted linear subvariety of $P_{A/\mathfrak{m}}$, one can see 6.1.1 for details. Then we know $\mathrm{ind}(\alpha_{A/\mathfrak{m}})|\mathrm{ind}(\alpha_{k_0})$. On the other hand, it is general fact that index drop after field extension, hence $\mathrm{ind}(\alpha_{k_0})|\mathrm{ind}(\alpha_{A/\mathfrak{m}})$. Thus $\mathrm{ind}(\alpha_{A/\mathfrak{m}}) = \mathrm{ind}(\alpha_{k_0})$. Since we naturally have $A \subset k_0 \subset k_0'$ the same argument shows $\mathrm{ind}(\alpha_{A/\mathfrak{m}}) = \mathrm{ind}(\alpha_{k_0'})$. Hence $\mathrm{ind}(\alpha_{k_0}) = \mathrm{ind}(\alpha_{k_0'})$. $\square$

In all the examples above, as long as we can determine period and index, $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$. We may ask if this is true in general.

Question 4.5.3. *Is it always true that* $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$*?*

Here is an example where the author cannot determine the period and index.

Example 4.5.4. *Let* Γ *be the graph of truncated icosahedron(soccer), with adjacent edges of hexagons doubled. Apply 4.3.4 to the hexagon containing* v_1, v_2, v_3*, we know* $3|\mathrm{per}(\alpha)$*. Apply 4.3.4 to the loop* v_2v_4*, we know* $2|\mathrm{per}(\alpha)$*. Apply 4.3.4 to the loop* $v_1v_4v_5v_6v_7$*, one know* $5|\mathrm{per}(\alpha)$*. Thus* $30|\mathrm{per}(\alpha)$*. In this case* $g = 61$*, so the possible period and index could be* $(30, 30), (30, 60)$ *or* $(60, 60)$*.*

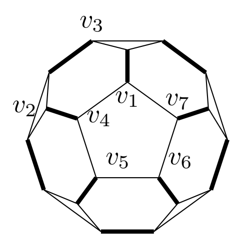

Fig 4.5.4

4.5.2 Involution of second kind

We gave some information about the period and index of the Brauer class. One may ask if we can say more about refined structure of the corresponding division algebras. For example, one could ask if the division algebras corresponding to the Brauer classes have involutions of second kind?

Definition 4.5.5. *Let* k *be a field with nontrivial involution* $\sigma \in \mathrm{Aut}(k)$*. Let* D *be a central simple algebra over* k*. We say* D *has an involution of second kind if* σ *extends to an involution of* D*, or there exists a* k^σ*-automorphism* $\widetilde{\sigma}\colon D \cong D^{opp} \otimes_{k,\sigma} k$*, such that* $\widetilde{\sigma}^2 = 1_D$*, where* 1_D *is the canonical identification* $D \cong (k \otimes_{\sigma,k} (k \otimes_{\sigma,k} D)^{opp})^{opp}$

Theorem 4.5.6. *Having involution of second kind is equivalent to* $\mathrm{Cores}_{k/k^\sigma}([D]) = 0 \in \mathrm{Br}(k^\sigma)$*.*

Proof. See [KMRT98, 1.3] for details. $\square$

Note that $\mathrm{Pic}^0_{X/k}$ has natural involution given by $\sigma = -1_{\mathrm{Pic}^0_{X/k}} : [\mathcal{L}] \mapsto [\mathcal{L}^{-1}]$. Consider the diagram

$$\begin{tikzcd} X \times \mathrm{Pic}^0_{X/k} \arrow[r, "1\times\sigma"] \arrow[d] & X \times \mathrm{Pic}^0_{\mathrm{X/k}} \arrow[d] \\ \mathrm{Pic}^0_{X/k} \arrow[r, "\sigma"] & \mathrm{Pic}^0_{X/k} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pic}^0_{X/k}(\mathrm{Pic}^0_{X/k}) \arrow[d, "\mathrm{d}_2^{0,1}"] & \mathrm{Pic}^0_{X/k}(\mathrm{Pic}^0_{X/k}) \arrow[l, "-\circ\sigma"'] \arrow[d, "\mathrm{d}_2^{0,1}"] \\ \mathrm{Br}(\mathrm{Pic}^0_{X/k}) & \mathrm{Br}(\mathrm{Pic}^0_{X/k}) \arrow[l, "\sigma^*"'] \end{tikzcd}$$

By functoriality of Leray spectral sequence, $\sigma^*\mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X/k}}) = \mathrm{d}_2^{0,1}(\sigma) = \mathrm{d}_2^{0,1}(-1_{\mathrm{Pic}^0_{X/k}}) = -\mathrm{d}_2^{0,1}(1_{\mathrm{Pic}^0_{X/k}})$. Then $\mathrm{Res}_{k/k^\sigma} \circ \mathrm{Cores}_{k/k^\sigma}(\alpha) = \sigma^*(\alpha) + \alpha = 0$. But this does necessarily imply $\mathrm{Cores}_{k/k^\sigma}(\alpha) = 0$. However in our case, due to the moduli interpretation of the involution on Picard scheme, involution of second kind exists under some mild assumption.

First let's describe the isomorphisms between central simple algebras. For the split case, we have

Theorem 4.5.7 (Skolem-Noether)**.** *Let k be a field, let V, V' be vector spaces over k of same dimension, then* $\mathrm{Isom}(V, V') \cong \mathrm{Isom}(\mathrm{End}(V), \mathrm{End}(V'))$*, given by* $g \mapsto (f \mapsto g \circ f \circ g^{-1})$

In general, we have

Lemma 4.5.8. *Let k be a field. Let D, D' be isomorphic central simple algebras. Let k' be a splitting field of D, D', fix some isomorphism $\phi\colon D \to \mathrm{End}(V), \phi'\colon D' \cong \mathrm{End}(V')$, where V, V' are some k'-vector spaces. Then $\mathrm{Isom}(V, V')^{\mathrm{Gal}(k'/k)} \cong \mathrm{Isom}(D, D')$. The isomorphism is given by $g \mapsto (f \mapsto \phi'^{-1}(g \circ \phi(f) \circ g^{-1}))$*

Proof. Note that by Galois descent, we have $\mathrm{Isom}(D, D') \cong \mathrm{Isom}(\mathrm{End}(V), \mathrm{End}(V'))^{\mathrm{Gal}(k'/k)}$, see [GS17, 2.3.8]. $\square$

Let k be a field, let C be a smooth genus g curve over k. Let k^s be the separable closure of k. Let K be the function field of $\mathrm{Pic}^0_{C/k}$. Let $\mathcal{L}$ be the tautological line bundle on C_K. Let $s\colon C_K \to K$ be the structure map. Let $\sigma\colon K \to K$ be the field automorphism induced by the canonical involution on $\mathrm{Pic}^0_{C/k}$. Then we know after base change to k^s, the Brauer class $\alpha \in$

$\mathrm{Br}(K)$ is represented by the Brauer-Severi variety of $\mathbb{P}(s_*(\omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}))$. Hence α is represented by the central simple algebra associated descended from $\mathrm{End}(H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}))$, whose opposite algebra is $\mathrm{End}(H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s})^\vee)$, and $\sigma^*\alpha \in \mathrm{Br}(K)$ is represented by the central simple algebra descended from $\mathrm{End}(H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}^{-1}))$.

Theorem 4.5.9. *If there exists a* $\mathrm{Gal}(k^s/k)$*-invariant perfect pairing*

$$\Phi\colon H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}) \times H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}^{-1}) \to Kk^s,$$

then the Brauer class $\alpha \in \mathrm{Br}(K)$ *has an involution of second kind extending* σ*.*

Proof. Given such a perfect pairing, we naturally have isomorphism

$$\sigma \in \mathrm{Isom}(H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s})^\vee, H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}_{k^s}^{-1}))^{\mathrm{Gal}(k^s/k)}.$$

By lemma 4.5.8 , the pairing induces an isomorphism $D^{opp} \cong K \otimes_{\sigma,K} D$, where D is the central simple algebra representing the Brauer class. To show the isomorphism is an involution, it suffices to show $H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}) \to H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes \mathcal{L}^{-1})^\vee \to H^0(C_{Kk^s}, \omega_{C_{Kk^s}} \otimes (\mathcal{L}^{-1})^{-1})^{\vee\vee}$ coincides with identity under canonical identification. This is formal. $\square$

To show the existence of such a perfect pairing, first we prove a linear algebra lemma:

Lemma 4.5.10. *Let k be an algebraically closed field. Let V, W, U be k-vector spaces. Let $B\colon V \times W \to U$ be a bilinear pair with no zero-divisors, namely $B(v,w) \neq 0$ if $v \neq 0$ and $w \neq 0$. If $\dim_k V = \dim_k W$, then there exists a linear form $\sigma\colon U \to k$ such that $\sigma \circ B\colon V \times W \to k$ is a perfect pairing.*

Proof. Since the bilinear pairing B has no zero divisor, and k is algebraically closed, the pairing B induces a morphism $\beta\colon \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(U)$ such that $\beta^*\mathcal{O}(1) = \mathcal{O}(1,1)$. Note that finding an linear form σ, up to scaling, is equivalent to pin down a hyperplane $H \subset \mathbb{P}(U)$. The pairing $\sigma \circ B$ is perfect if and only if H does not contain $\beta(t \times \mathbb{P}(W))$ for any $t \in \mathbb{P}(V)$. Since $\beta^*\mathcal{O}(1) = \mathcal{O}(1,1)$, the morphism β restricted to $t \times \mathbb{P}(W)$ is a linear embedding for any $t \in \mathbb{P}(V)$.

Let $S = \cup_{t\in\mathbb{P}(V)}\{H \in \mathbb{P}^\vee(U) | H \supseteq \beta(t \times \mathbb{P}(W))\}$ be the locus where the bilinear form on $V \times W$ induced by the hyperplane is not perfect. Let $Z \subset \mathbb{P}^\vee(U) \times \mathbb{P}(V)$ be the incidence correspondence $Z = \{(H,t)|H \supset \beta(t\times\mathbb{P}(W))\}$, denote the two projections on $\mathbb{P}^\vee(U)\times\mathbb{P}(V)$ by pr_1 and pr_2. Then by definition we have $S = \mathrm{pr}_1(Z)$. Note that Z is a closed subvariety of $\mathbb{P}^\vee(U) \times \mathbb{P}(V)$ (since Z can be explicitly cut out by conditions on coefficients of linear forms), thus S is a closed subvariety of $\mathbb{P}^\vee(U)$. Then note that pr_2 is a smooth fibration in projective spaces, whose fibers are hyperplanes in $\mathbb{P}(U)$ containing the subspace $\beta(t\times\mathbb{P}(W))$. The relative dimension of pr_2 is $\dim_k U - 1 - \dim_k W$. Hence the dimension of Z is

$$\dim_k V - 1 + \dim_k U - 1 - \dim_k W = \dim_k U - 2.$$

Note that S is the image of Z under pr_1, we know $\dim_k S \leq \dim_k Z < \dim_k \mathbb{P}(U)$ thus $\mathbb{P}^\vee(U)\backslash S \neq \emptyset$. For any choice of hyperplane H in the Zariski open subset $\mathbb{P}^\vee(U)\backslash S$, the defined linear form $\sigma \circ B \colon V \times W \to k$ is a perfect pairing. $\square$

Consider the pairing

$$H^0(C_K, \omega_{C_K} \otimes \mathcal{L}) \times H^0(C_K, \omega_{C_K} \otimes \mathcal{L}^{-1}) \to H^0(C_K, \omega_{C_K}^{\otimes 2}) \to K,$$

we know geometrically this pairing has no zero divisors, since tensor product of nonzero sections is nonzero. So there exists an Zariski-open subset of hyperplanes in $\mathbb{P}^\vee(H^0(C, \omega_{C_K}^{\otimes 2}))$ such that the pairing induced from the hyperplanes are perfect. Note that K is an infinite field, so rational points are Zariski dense in the projective space, take a pairing corresponds to a K point, such pairing is obviously $\mathrm{Gal}(Kk^s/K) = \mathrm{Gal}(k^s/k)$-invariant.

Theorem 4.5.11. *Let k_0 be a field, let C is a smooth irreducible curve over field k_0, then the central simple algebra at the generic point of $\mathrm{Pic}^0_{C/k}$, representing the Brauer class (of the obstruction to existence of a universal line bundle) has an involution of second kind extending the natural involution on $\mathrm{Pic}^0_{C/k}$.*

Remark 4.5.12. *It is not clear if the Azumaya algebra over the whole punctured Picard scheme admits an involution of second kind.*

Remark 4.5.13. *We need the irreducibility of the curve to ensure that the pairing* $H^0(C, \omega \otimes \mathcal{L}) \otimes H^0(C, \omega \otimes \mathcal{L}^{-1}) \to H^0(C, \omega_C^{\otimes 2})$ *has no zero divisors. But the condition can be dropped in many cases, as long as* $\omega_C \otimes \mathcal{L}$ *does not allow sections vanishing on certain components. For example in 4.4.3, let* s *be a section of* $\omega_C \otimes \mathcal{L}$ *vanishing on exactly one component, if this were a zero divisor, there exists a section* s' *vanishing on four other component such that* $ss' = 0$*, then the nonvanishing section of* s' *on the other component has* 4 *zeros, which is forces to vanish, since* $\omega \otimes \mathcal{L}$ *has degree* 2 *on each components. Similarly one show any zero-divisor* s *cannot vanish on exactly two components, otherwise any* t *with* $st = 0$ *vanishes on three components, which forces* t *has three zeros on non-vanishing components.*

4.5.3 Cyclicity

We say a little bit about cyclicity.

Theorem 4.5.14. *If* g *is even, then the Brauer class in 4.4.1 are cyclic.*

Proof. Note that $\mathrm{Gal}(k'/k) = \mathrm{Aut}(\Gamma_g) = D_{g-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{g-1}$. Take the invariant subfield k'' of $(\mathbb{Z}/2\mathbb{Z})^{g-1}$. Then k'/k'' is an extension of degree 2^{g-1} and k''/k is a dihedral extension. The Brauer class $\alpha \in \mathrm{Br}(k)$ restricts to $0 \in \mathrm{Br}(k')$, since the universal curve splits over k'. This implies the class $\mathrm{Res}_{k''/k}(\alpha) = 0$ in $\mathrm{Br}(k'')$, as we know α has odd degree $g-1$ and $0 = \mathrm{Cores}_{k'/k''} \circ \mathrm{Res}_{k'/k''}(\mathrm{Res}_{k''/k}\alpha) = 2^{g-1}\mathrm{Res}_{k''/k}(\alpha)$. Then we know the Brauer class is cyclic since dihedral algebras are known to be cyclic [RS82, 1]. $\square$

The cyclicity of the last example maybe expected from the symmetries of the graph. In 4.4.3, the automorphism graph is symmetric group S_5. In 4.5.4, the graph has automorphism A_5 has a normal subgroup. we may ask:

Question 4.5.15. *Are the Brauer classes in 4.4.3 or 4.5.4 cyclic?*

Chapter 5

Conic associated with the universal curve

5.1 Setup

In this chapter, unless otherwise specified, all fields are of characteristic zero. For totally degenerate curves and related concepts, we use the terminologies defined in 4.1. We use the term "normalization" for usual normalization of integral scheme. Now we explain what we mean by "normalization in families". We only use this term for families of totally degenerate stable curves with fixed dual graph. Since the moduli stack of families such curves is smooth, the normalization of the universal stable nodal curve is also fiberwise normalization. We define the "normalization in families" for such a family over a general base by taking pullback of the normalized universal family.

We call a smooth genus zero curve a smooth conic, since the sections of its anti-canonical bundle embeds it as a conic in $\mathbb{P}^2$. We say the conic is split if it has a rational point.

Let Γ be a good graph (e.g, graph with $\deg(v) \geq 4$ for each vertex $v \in V$). Fix an algebraically closed field k_0. Let $X_{\Gamma,k_0}/k_{\Gamma,k_0}$ be the universal totally degenerate curve with dual graph Γ (for families of curves defined over k_0), where k_{Γ,k_0} is the fraction field of

$\mathcal{D}_{\Gamma,k_0} = [\mathcal{M}_{\Gamma,k_0}/\mathrm{Aut}(\Gamma)]$, and $\mathcal{M}_{\Gamma,k_0} = \prod_{v \in V} \mathcal{M}_{0,E(v),k_0}$. After base change to the function field of $\mathcal{M}_{\Gamma,k_0}$, the curve X_{Γ,k_0} becomes a nodal union of smooth rational curves with dual graph Γ.

Let $\nu\colon X^{\nu}_{\Gamma,k_0} \to X_{\Gamma,k_0}$ be the normalization of X_{Γ,k_0}. Let K_{Γ,k_0} be the ring of global sections of X^{ν}_{Γ,k_0}. Consider the Stein factorization $X^{\nu}_{\Gamma,k_0} \to \mathrm{Spec}(K_{\Gamma,k_0}) \to \mathrm{Spec}(k_{\Gamma,k_0})$.

$$\begin{tikzcd}
X^{\nu}_{\Gamma,k_0} \arrow[r, "\nu"] \arrow[d, "\text{conic}"] & X_{\Gamma,k_0} \arrow[d] \\
\mathrm{Spec}(K_{\Gamma,k_0}) \arrow[r] & \mathrm{Spec}(k_{\Gamma,k_0})
\end{tikzcd}$$

Since geometrically X^{ν}_{Γ,k_0} is a union of smooth conics, by properties of Stein factorization, see [Har77, 11.3], we know X^{ν}_{Γ,k_0} is a smooth conic over $\mathrm{Spec}(K_{\Gamma,k_0})$. We ask:

Question 5.1.1. *Is the conic $X^{\nu}_{\Gamma,k_0}/K_{\Gamma,k_0}$ split? Or, is the corresponding class $\beta \in \mathrm{Br}(K_{\Gamma,k_0})[2]$ zero?*

In the following discussion, we make the following assumptions:

1. $\mathrm{Aut}(\Gamma)$ acts transitively on the vertices and edges of Γ.

2. In the graph Γ, the number of edges emitting from each edge is an even number.

The first assumption shows the geometrical points of $\mathrm{Spec}(K_{\Gamma,k_0})$ are in one orbit, hence K_{Γ,k_0} is a field. If this assumption is not satisfied, then after Stein factorization, we get a conic over disconnected space $\mathrm{Spec}(K_{\Gamma,k_0}) = \cup_{i\in I}\mathrm{Spec}(k_i)$, which can be reduced to the study on each connected component. If the second assumption is not satisfied, then the conic $X^{\nu}_{\Gamma,k_0} \to \mathrm{Spec}(K_{\Gamma,k_0})$ has the subscheme of nodes as an odd degree zero-cycle on X^{ν}_{Γ,k_0}, which naturally forces the conic to split. We show in some cases, the conic is nonsplit. Hence there exists an interesting 2-torsion Brauer class. Specifically, we show

Theorem 5.1.2. *If $\Gamma = K_5$ is the complete graph with 5 vertices. Let $X_{\Gamma,k_0}/k_{\Gamma,k_0}$ be the universal curve with dual graph Γ. Let $K_{\Gamma,k_0} = \Gamma(X^{\nu}_{\Gamma,k_0}, \mathcal{O}_{X^{\nu}_{\Gamma,k_0}})$. Then the conic $X^{\nu}_{\Gamma,k_0} \to \mathrm{Spec}(K_{\Gamma,k_0})$ is nonsplit.*

Here is the idea of the proof: Suppose the conic splits, then the conic has rational sections. Theses sections can be defined over a finite type algebra over $\mathbb{Q}$. Then we may specialize the section to some number field. We show the existence of rational section on the specialized universal curve implies that every conic over the number field obtained by Stein factorization of totally degenerate curve splits. Then we find a counterexample to this assertion, which leads to a proof by contradiction. We work out the details in the rest of the chapter.

5.2 Reduction to number field

We slightly generalize our situation. Note that the moduli stack of totally degenerate curves can be defined over an arbitrary field, we don't necessarily need to work over an algebraically closed field. Let $\mathcal{M}^{\circ}_{\Gamma,\mathbb{Q}}$ be the open subscheme of $\mathcal{M}_{\Gamma,\mathbb{Q}} = \prod_{v\in V}\mathcal{M}_{0,E(v),\mathbb{Q}}$ with free $\mathrm{Aut}(\Gamma)$-action. The moduli stack of curves over $\mathbb{Q}$ with dual graph Γ is $\mathcal{D}_{\Gamma,\mathbb{Q}} := [\mathcal{M}_{\Gamma,\mathbb{Q}}/\mathrm{Aut}(\Gamma)]$. It has an open subscheme $\mathcal{D}^{\circ}_{\Gamma,\mathbb{Q}} := [\mathcal{M}^{\circ}_{\Gamma,\mathbb{Q}}/\mathrm{Aut}(\Gamma)]$, and universal curve over $\mathcal{D}^{\circ}_{\Gamma,\mathbb{Q}}$ is denote by $\mathcal{X}^{\circ}_{\Gamma,\mathbb{Q}}$. Let $\mathcal{X}^{\circ,\nu}_{\Gamma,\mathbb{Q}}$ be the normalization of the universal curve and $\mathcal{X}^{\circ,\nu}_{\Gamma,\mathbb{Q}} \to \mathcal{D}^{\circ,\prime}_{\Gamma,\mathbb{Q}}$ be the Stein factorization. Then $\mathcal{X}^{\circ,\nu}_{\Gamma,\mathbb{Q}} \to \mathcal{D}^{\circ,\prime}_{\Gamma,\mathbb{Q}}$ is a conic over $\mathcal{D}^{\circ,\prime}_{\Gamma,\mathbb{Q}}$. Let $X^{\nu}_{\Gamma,k_0}/K_{\Gamma,k_0}$ be the generic fiber of $\mathcal{X}^{\nu}_{\Gamma,k_0}/\mathcal{D}^{\circ,\prime}_{\Gamma,k_0}$. We have the following commutative diagram of schemes

$$\begin{tikzcd}
X^{\nu}_{\Gamma,k_0} \arrow[r] \arrow[d] & \mathrm{Spec}(K_{\Gamma,k_0}) \arrow[r] \arrow[d] & \mathrm{Spec}(k_{\Gamma,k_0}) \arrow[d] \\
\mathcal{X}^{\circ,\nu}_{\Gamma,k_0} \arrow[r] & \mathcal{D}^{\circ,\prime}_{\Gamma,k_0} \arrow[r] & \mathcal{D}^{\circ}_{\Gamma,k_0}
\end{tikzcd}$$

Suppose $X^{\nu}_{\Gamma,k_0} \to \mathrm{Spec}(K_{\Gamma,k_0})$ is a trivial conic, then $X^{\nu}_{\Gamma,k_0} \to \mathrm{Spec}(K_{\Gamma,k_0})$ has sections. Let $s\colon \mathrm{Spec}(K_{\Gamma,k_0}) \to X^{\nu}_{\Gamma,k_0}$ be a section. Since $X^{\nu}_{\Gamma,\mathbb{Q}}$ is a finite type scheme over $\mathbb{Q}$, and $X^{\nu}_{\Gamma,k_0} = \left(X^{\nu}_{\Gamma,\mathbb{Q}}\right)_{k_0}$, we know there exists a finite type $\mathbb{Q}$-subalgebra $A \subset k_0$ over which s is defined. Thus s is the base change of a section $s_A\colon \mathrm{Spec}(K_{\Gamma,\mathbb{Q}} \otimes_{\mathbb{Q}} A) \to X^{\nu}_{\Gamma,\mathbb{Q}} \times_{\mathbb{Q}} A$. Take a maximal ideal $\mathfrak{m}$ of A, then $M := A/\mathfrak{m}$ is a finite extension of $\mathbb{Q}$, see [AM16, 7.8]. The section s_A specializes to a section $s_{\mathfrak{m}} := \mathrm{Spec}(K_{\Gamma,\mathbb{Q}} \otimes_{\mathbb{Q}} M) \to X^{\nu}_{\Gamma,M}$, this is a section on the generic fiber of the universal family $\mathcal{X}^{\circ,\nu}_{\Gamma,M} \to \mathcal{D}^{\circ,\prime}_{\Gamma,M}$.

Note that we have functoriality for normalization in families and Stein factorization in our case.

Lemma 5.2.1. *Let $f\colon X \to Y$ be a proper flat family of totally degenerate curves. Let $T \to Y$ be a morphism, let $f_T\colon X_T \to T$ be the pullback. Let ν be the normalization in families, then $(X^\nu)_T \cong (X_T)^\nu$ and $(f_*\mathcal{O}_{X^\nu})_T = f_{T,*}\mathcal{O}_{X_T^\nu}$.*

Proof. The first assertion follows since the normalization in families is defined by pulling back the universal family. The second assertion follows since cohomology commutes with flat base change. ☐

Proposition 5.2.2. *If the generic conic $X^\nu_{\Gamma,M}/K_{\Gamma,M}$ splits, then for any field extension M'/M, and any totally degenerate curve X over M' with dual graph Γ, the conic*

$$X^\nu \to \operatorname{Spec}(\Gamma(X^\nu, \mathcal{O}_{X^\nu}))$$

splits.

Proof. Since $\left(X^\nu_{\Gamma,M}\right)_{M'} = X^\nu_{\Gamma,M'}$, we may assume $M = M'$. And denote the closed point in $\mathcal{D}_\Gamma$ corresponding to X by x. We know $\mathcal{D}_\Gamma$ is a smooth Deligne-Mumford stack, as it is explicitly presented by $[\mathcal{M}_\Gamma/\operatorname{Aut}(\Gamma)]$. By [Sta18, 0DR0], we know there exists a flat morphism $\phi\colon \operatorname{Spec}(M[[t_1,\dots,t_d]]) \to \mathcal{D}_\Gamma$, where $d = \dim(\mathcal{D}_\Gamma)$, such that the closed point maps to x. Flatness implies the generic point η_1 of $T := \operatorname{Spec}(M[[t_1,\dots,t_d]])$ is mapped to the generic point η of $\mathcal{D}_\Gamma$. Then the rational section of the generic associated conic over $\mathcal{D}_\Gamma$ pulls back to a rational section of the associated conic over η_1. We may connect the associated conic of X to the generic associated conic, by a successive specialization of discrete valuation rings:

$$M((t_1,\dots,t_{d-1}))[[t_d]],$$

$$M((t_1,\dots,t_{d-2}))[[t_{d-1}]],$$

$$\dots$$

$$M[[t_1]].$$

The rational point on the generic fiber is inherited by taking closure in each specialization step. □

In the next section, in some specific cases, we construct totally degenerate curve with dual graph Γ such that the associated conic is nonsplit. The idea is to pick certain nontrivial conic C and clutch certain points to get a totally degenerate curve over M with dual graph Γ, such that the associated conic C is nontrivial. We work out the reverse engineering process in the following section.

5.3 The clutching method

We work out the details of "clutching". Let k_1 be a field, let k_2 be a finite extension of k_1. Let $n = [k_2 \colon k_1]$. Let C be a conic over k_2. Let $i_P \colon P \hookrightarrow C$ be a closed point with residue field k_3. Let $m = [k_3 \colon k_2]$. Let k_4 be an intermediate field between k_1 and k_3, such that $[k_3 \colon k_4] = 2$. Here k_3 does not necessarily contain k_2. Let k_5/k_1 be the normal closure of k_3/k_1. To sum up, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spec}(k_5) \arrow[r] & \mathrm{Spec}(k_3) \arrow[r, "m\colon 1"] \arrow[d, "2\colon 1"] & \mathrm{Spec}(k_2) \arrow[d, "n\colon 1"] \\
 & \mathrm{Spec}(k_4) \arrow[r] & \mathrm{Spec}(k_1)
\end{tikzcd} .$$

Consider the following pushout diagram in the category of schemes (Sch/$\mathbb{Z}$), obtained locally by taking spectrum of fiber product of rings, the existence is guaranteed by [Sch05, 3.4], the key fact we use is that i_P is a closed immersion.

$$\begin{tikzcd}
\mathrm{Spec}(k_3) \arrow[r, "i_P"] \arrow[d, "2\colon 1"] & C \arrow[d] \\
\mathrm{Spec}(k_4) \arrow[r] & Y
\end{tikzcd}$$

Proposition 5.3.1. *The curve Y is a totally degenerate curve over k_1 with dual graph $\Gamma = (V, E)$. Let's denote $G_i = \mathrm{Gal}(k_5/k_i)$. Then the vertex set be identified with $V = G_1/G_2$.*

The edge set can be identified with $E = G_1/G_4$. The set of directed edges can be identified with $E' = G_1/G_3$. There are n vertices and each vertex lets out m edges. The identifications are natural in the sense that we have the map of G_1-sets:

$$\begin{tikzcd} E' = G_1/G_3 \arrow[r] \arrow[d] & G_1/G_2 = V \\ E = G_1/G_4 & \end{tikzcd},$$

where the horizontal map assigns a directed edge with its source, and the vertical map is given by forgetting the direction of directed edges.

Proof. The key is to show that the previous pushout diagram is also a cartesian diagram. This follows from the next lemma 5.3.2. Then the identification follows form the following observations: For a Galois extension K/k of fields with Galois group G, let H be a subgroup of G, then there is a canonical isomorphism of G-sets $\phi_H \colon \mathrm{Hom}_k(K^H, K) \to G/H$ given by extending an inclusion $\sigma \colon K^H \to K$ to any k-automorphism of K. Finally note that $\mathrm{Hom}_k(K^H, K)$ is just the set of K-points of $\mathrm{Spec}(K^H)$. $\square$

Lemma 5.3.2. *Let A, B be commutative rings. Let I be an ideal of A. Let $B \to A/I$ be a subring. Let $C = A \times_{A/I} B$ be the fibred product of rings. Then $A/I = A \otimes_C B$, where $C \to A$ and $C \to B$ are given by projections.*

Proof. We construct a map $f \colon A \otimes_C B \to A/I$ given by $(a, b) \mapsto \overline{ab}$, this is easily seen to be a surjective ring homomorphism. Consider another map $g \colon A/I \to A \otimes_C B$ given by $\overline{a} \mapsto a \otimes 1$. This map is well defined because $i \otimes 1 = i(1 \otimes 1) = (1 \otimes 1) \cdot 0 = 0$. It is easy to check f, g are inverse maps to each other, hence we have desired isomorphism. $\square$

Example 5.3.3. *Let k_5 be a field with faithful S_5 action. Let $k_1 = k^{S_5}$ be the invariant subfield. Let S_i be the subgroup of S_5 of permuting the first i vertices. Let $k_2 = k_5^{S_4}$, let $k_4 = k_5^{S_3 \times (45)}$ and let $k_3 = k_5^{S_3}$. Then the dual graph of $\Gamma = (V, E)$ has 5 vertices, 10 edges and exactly one edge between any two vertices. Hence $\Gamma = K_5$ the complete graph with 5 vertices.*

Let $\Gamma = (V, E)$ be a graph with transitive $\mathrm{Aut}(\Gamma)$-action on vertices and edges. Let's pick an edge e_0 with vertices v_0, v_1. Let $G_1 = \mathrm{Aut}(\Gamma)$. Let k_5 be a field with faithful $\mathrm{Aut}(\Gamma)$-action. Let $k_1 = k_5^{G_1}$. Let $G_2 = \mathrm{Stab}(\{v_0\})$, $G_4 = \mathrm{Stab}(\{e_0\})$ and $G_3 = \mathrm{Stab}(\{v_0\}) \cap \mathrm{Stab}(\{e_0\})$.

Proposition 5.3.4. *Suppose there exists an index 2 element in $\mathrm{Br}(k_5^{G_3}/k_5^{G_2})$, then there exist a totally degenerate curve over k_1, with dual graph Γ such that the Stein factorization gives a nontrivial conic.*

Proof. Pick a conic $C/k_5^{G_2}$ corresponding to an index 2 Brauer class in $\mathrm{Br}(k_5^{G_3}/k_5^{G_2})$. Then $C \times_{k_5^{G_2}} k_5^{G_3} \cong \mathbb{P}^1_{k_5^{G_3}}$. Since we work over infinite fields, we may take a closed point Q of C with residue field $k_5^{G_3}$. Take the pushforward scheme Y of $Q \to C$ and $Q \to \mathrm{Spec}(k_5^{G_2})$. Then Y is the totally degenerate curve over $k_5^{G_1}$ satisfying required conditions. $\square$

By the previous discussion, we can show non-triviality of the conic $X^{\nu}_{\Gamma,k_0}/K_{\Gamma,k_0}$ in 5.1.2 if: For any number field M, we can find an field extension M'/M with faithful S_5 action, such that $M \subset (M')^{S_5}$, and $\mathrm{Br}((M')^{S_3}/(M')^{S_4})[2] \neq 0$.

Note that for number fields, the index of Brauer class coincides with period, see [ABGV11, 4.4.1]. If such a 2-torsion exists, then it is represented by a non-split conic over $(M')^{S_4}$. Since this conic splits over $(M')^{S_3}$, we may pick a closed point with residue field $(M')^{S_3}$ in the conic. By clutching method we obtain a totally degenerate curve with dual graph K_5 whose associated conic does not split, which implies the generic conic does not split.

Note that faithful S_5 extension can be obtained by taking splitting field of degree 5 polynomials with general coefficients. Hence it suffices to know for general extension of number fields L/K, if there exists a 2-torsion element in $\mathrm{Br}(K) \to \mathrm{Br}(L)$. This can be clarified by class field theory. The key lemma is made explicit in the following section.

5.4 A lemma on number fields

Lemma 5.4.1. *Let G be a finite group. Let N/K be a Galois extension with $\mathrm{Gal}(N/K) \cong G$. Let H be a subgroup of G. Let $L = N^H$. Suppose there exists an element g_0 in G such that* *the g_0-orbit of G/H are all in even size, then* $\mathrm{Br}(L/K)[2] := \ker(\mathrm{Br}(K) \to \mathrm{Br}(L))[2] \neq 0$.

Proof. Let's denote the set of places in K and L by S_K and S_L. Recall that the local-global principal for Brauer group gives us the following diagram of exact sequences, see [GS17, 6.5.4]

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(K) \arrow[r] \arrow[d] & \oplus_{v\in S_K}\mathrm{Br}(K_v) \arrow[r, "\oplus\mathrm{inv}_v", "\cong"'] \arrow[d] & \oplus_{v\in S_K}\mathbb{Q}/\mathbb{Z} \arrow[r, "\sum"] \arrow[d, "\phi"] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "{[L\colon K]}"] & 0 \\
0 \arrow[r] & \mathrm{Br}(L) \arrow[r] & \oplus_{w\in S_L}\mathrm{Br}(L_w) \arrow[r, "\oplus\mathrm{inv}_w", "\cong"'] & \oplus_{v\in S_K}\mathbb{Q}/\mathbb{Z} \arrow[r, "\sum"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

If a place $w \in S_L$ lies over place $v \in S_K$, we write $w|v$, and use e, f to denote the corresponding inertia degree and degree of residue field extension. We know for a fixed $v \in S_K$, it is true that $\sum_{w|v} e(w|v)f(w|v) = [L\colon K]$. The maps $\mathrm{inv}_v\colon \mathrm{Br}(K_v) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ are the invariant map from local class field theory. The map ϕ sends

$$(0, \dots, 1_v, \dots, 0) \in \oplus_{v\in S_K}\mathrm{Br}(K_v)$$

to

$$\sum_{w|v}(0, \dots, e(w|v)f(w|v) \cdot 1_w, \dots, 0) \in \oplus_{w\in S_L}\mathrm{Br}(L_w).$$

It suffices to find two places v_1, v_2 in K, such that for $w|v_i$, the residue field degree $f(w|v_i)$ is even. Then the class $(0, \dots, (1/2)_{v_1}, \dots, 0, \dots, (1/2)_{v_2}, \dots, 0)$ gives the desired two torsion class in $\ker(\mathrm{Br}(K) \to \mathrm{Br}(L))$.

Let $\mathfrak{p}$ be a maximal ideal in $\mathcal{O}_K$ that is unramified in $\mathcal{O}_L$. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ be the primes in L lying over $\mathfrak{p}$, let $\mathfrak{Q}$ be a prime lying over $\mathfrak{p}$ in $\mathcal{O}_N$. Let $G_{\mathfrak{Q}} = \langle g_0 \rangle$ be the decomposition group of $\mathfrak{Q}$. Then $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ corresponds to the double cosets $G_{\mathfrak{Q}}\backslash G/H$, via $G_{\mathfrak{Q}}gH \mapsto g\mathfrak{Q} \cap L$, or equivalently, the primes $\{\mathfrak{P}_i\}$ corresponds to the g_0 orbits of G/H, and the residue field degrees corresponds to the length of the orbits, see [Neu99, I.9].

Suppose there exists an element $g_0 \in G$ such that all g_0-orbits of G/H are of even size. By Chebotarev's theorem [Neu99, VII.13.4], the primes in $\mathcal{O}_N$ whose Frobenius elements are conjugate to g_0 has positive Dirichlet density. Hence there are infinitely many primes

$\mathfrak{Q} \in \mathcal{O}_N$ whose decomposition group $G_{\mathfrak{Q}}$ is isomorphic to $\langle g_0 \rangle$. Pick any two such primes $\mathfrak{Q}_1, \mathfrak{Q}_2$ lying over different primes in $\mathcal{O}_K$, then $\mathfrak{Q}_i \cap \mathcal{O}_K$ determine the desired places of K. $\square$

Remark 5.4.2. *Here is an instance where the lemma holds: Suppose some* 2*-power order element* $g_0 \in G$ *such that* $g_0 \notin \bigcup_{g \in G} g^{-1}Hg$*, then the condition in the lemma is satisfied, since the only possibility to have an odd size* g_0*-orbit of* G/H *is an to have an orbit of length* 1*. This means* $g_0 gH = gH$*, or equivalently* $g_0 \in g^{-1}Hg$ *for some* $g \in G$*. Specifically, such condition holds if* G *has an order* 2^s *element but* H *does not.*

Remark 5.4.3. *Let* $G = S_4, H = S_3$*. Let* K *be a number field with faithful* S_4*-action, then* $\mathrm{Br}(K^{S_4}/\mathrm{K}^{\mathrm{S}_3})[2] \neq 0$*, since* S_4 *has an order* 4 *element but* S_3 *does not.*

Remark 5.4.4. *In general, for an even degree extension* L/K*, it is possible that* $\mathrm{Br}(L/K)[2] = 0$*. Let* K *is a totally imaginary field, let* N/K *be a unramified extension with Galois group* A_4*, let* H *be an order* 2 *subgroup in* G*, let* $L = N^H$*. The local map for* $\mathrm{Br}(K) \to \mathrm{Br}(L)$ *are multiplication by the size of orbits. At each local place, the Frobenius element has order* $1, 2, 3$*. If the Frobenius element has order* $1, 3$*, then there is odd size orbit; in case the Frobenius element has order* 2*, there is also size* 1 *element by 5.4.2 since all the order* 2 *elements in* A_4 *are conjugate.*

Chapter 6

Miscellaneous Properties

6.1 Extension of Brauer classes

We show the extension of Brauer class by purity has some compatibility in certain geometric situation, see 6.1.10. For details on intersection theory, we refer to [Ful98, 8.1.1].

Let k a field, recall the Chow ring $\mathrm{CH}^*(\mathbb{P}^n_k)$ is isomorphic to $\mathbb{Z}[H]/(H^{n+1})$, where H is the hyperplane class. If $x = cH^i \in \mathrm{CH}^i(\mathbb{P}^n_k)$, we define the degree of x to be c. We have $\deg(x) \cdot \deg(y) = \deg(x \cdot y)$ for any $x \in \mathrm{CH}^i(\mathbb{P}^n_k), y \in \mathrm{CH}^j(\mathbb{P}^n_k)$.

Lemma 6.1.1. *Let k be a field, let X be subscheme of $\mathbb{P}^n_k$. Suppose the Hilbert polynomial P of X has degree m. Then $P_X(d) \geq \binom{m+d}{d}$. The equality holds if and only if X is a linearly embedded $\mathbb{P}^m_k$.*

Proof. Here "$\geq$" means inequality holds when the variable is large enough. We only need to show the lemma when X is reduced and irreducible, since for a reduced top dimensional irreducible component Y of X, we know their Hilbert polynomial P_Y and P_X has the same degree and $P_Y \leq P_X$. The case $n = 1$ is trivial. For $n > 1$, if X is contained in a hyperplane, then we reduce to the $n-1$ case. Now suppose X is not contained in a hyperplane in $\mathbb{P}^n_k$.

If X is the whole space, then we are done, since $P_{\mathbb{P}^n_k}(d) = \binom{n+d}{d}$. If X is not the whole space, and is not contained in any linear subspace, we show $\deg(X) \neq 1$. If X

has codimension 1, this is clear since degree 1 hypersurfaces are hyperplanes. If X has codimension $r+1$, take projection $f\colon \mathrm{Bl}_L\mathbb{P}^n_k \to \mathbb{P}^{n-r}_k$ from a linear subspace of dimension r disjoint from X, then f_*X is a divisor. Let H a line in $\mathbb{P}^{n-r}_k$. Then we know $X \cdot f^*H = \deg(X)$, as can be calculated in $\mathbb{P}^n_k$ after blow down. If $\deg(X) = 1$, then by projection formula, $1 = f_*(X \cdot f^*H) = f_*X \cdot H$. This implies that f_*X is a hyperplane and X is contained in the hyperplane $f^{-1}(f(X))$, contradiction. Hence X has degree at least 2. Then the leading coefficient of P_X will be at least $2/m!$, which implies the strict inequality $P_X(d) > \binom{m+d}{d}$. $\square$

Lemma 6.1.2. *Let R be a discrete valuation ring, let $Z = \mathrm{Spec}(R)$. Let $f\colon X \to Z$ be a projective morphism such that the generic fiber X_η and the special fiber X_s are both Brauer-Severi varieties. Suppose an ample line bundle $\mathcal{L}$ on X restricts to the same degree on both fibers. Then the closure $\overline{X}_\eta$ restricts to a linearly embedded subspace of X_s.*

Proof. Since the morphism $\overline{X}_\eta \to Z$ is dominant, it is flat by [Har77, 9.7]. Hence the Hilbert polynomial of X_η and $(\overline{X}_\eta)_s$ are equal by [Har77, 9.9]. Denote $\dim_{\kappa(\eta)}(X_\eta) = a$, $\dim_k(X_s) = b$.

In the following calculation of intersection number, we freely base change to an algebraically closed field, then take hyperplane divisors over the base change of Brauer-Severi varieties.

Let H_s be the hyperplane class of X_s, let H_η be the hyperplane class of X_η. By our assumption, there exists a positive integer k such that $c_1(\mathcal{L}|_{X_s}) = kH_s, c_1(\mathcal{L}|_{X_\eta}) = kH_\eta$. Suppose the cycle class $[(\overline{X}_\eta)_s] = m(H_s)^{b-a}$, we want to show $m = 1$.

Since intersection number with an ample line bundle is determined by the leading coefficient of the Hilbert polynomial, by flatness, we have $(\mathcal{L}|^a_{X_\eta} \cdot X_\eta)_{X_\eta} = (\mathcal{L}|^a_{X_s} \cdot (\overline{X}_\eta)_s)_{X_s}$. So $k^a = (k^a H_s^a \cdot mH_s^{b-a}) = k^a \cdot m$. Thus $m = 1$. Let $P^{H_s}_{(\overline{X}_\eta)_s}(d)$ be the Hilbert polynomial of $(\overline{X}_\eta)_s$ for the hyperplane class H_s, then $P^{H_s}_{(\overline{X}_\eta)_s}(kd) = P^{\mathcal{L}|_{X_s}}_{(\overline{X}_\eta)_s}(d) = P^{\mathcal{L}|_{X_\eta}}_{X_\eta}(d) = \binom{a+kd}{kd}$, so $P^{H_s}_{(\overline{X}_\eta)_s}(d) = \binom{a+d}{d}$. Thus $(\overline{X}_\eta)_s$ is a subspace linearly embedded in X_s, by 6.1.1. $\square$

Lemma 6.1.3 (Cohomological purity)**.** *Let S be a scheme, let $i\colon Z \to X$ be a closed immersion of smooth S-schemes, and let $\mathcal{F}$ be a locally constant torsion sheaf on $X_{ét}$ whose* *torsion is prime to* $\mathrm{char}(X)$*. Denote* $c = \mathrm{codim}_X(Z)$*. Then*

$$R^n i^! \mathcal{F} = \begin{cases} 0, & n \neq 2c \\ i^*\mathcal{F}(-c), & n = 2c. \end{cases}$$

Let $U = X \backslash Z$*, let* $j \colon U \to X$ *be the open immersion, the above implies that*

$$(R^n j_*) j^* \mathcal{F} = \begin{cases} \mathcal{F}, & n = 0 \\ 0, & n \neq 0, 2c-1. \end{cases}$$

Proof. See [Mil80, 5.1]. □

Lemma 6.1.4. *Let* X, Y *be smooth varieties over* k*. Let* $f \colon Y \to X$ *be a projective morphism. Let* $Z \subset X$ *be a smooth closed subvariety, let* $U = X \backslash Z$*.*

$$\begin{tikzcd} Y_Z \arrow[r, "i'"] \arrow[d, "f_Z"] & Y \arrow[d, "f"] & Y_U \arrow[l, "j'"'] \arrow[d, "f_U"] \\ Z \arrow[r, "i"] & X & U \arrow[l, "j"'] \end{tikzcd}$$

Let $i \colon Z \to X$ *be the closed immersion and* $j \colon U \to X$ *be the open immersion. Denote their base change along* f *by* i', j'*. Let* f_Z, f_U *be the base change of* f *along* i, j*. Suppose* $Y_U \to U$*,* $Y_Z \to Z$ *are both Brauer-Severi schemes of positive relative dimension. Suppose the codimension of* Z *in* X *and* Y_Z *in* Y *are both at least* 2*. Then for any* $n \in \mathbb{Z}_{>0}$ *prime to* $\mathrm{char}(k)$*, we have*

$$R^2 f_* \mu_n = \underline{\mathbb{Z}/n}_X.$$

Proof. We use $\mathrm{R}f_*$ to denote functors between derived categories, we use $R^i f_*$ to denote the i-th cohomology of $\mathrm{R}f_*$, etc. When there's a canonical isomorphism between two objects, and the canonical isomorphism is clear from context, we write $=$ for simplicity.

Consider the short exact sequence

$$0 \to j_! \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}}_X \to i_* \underline{\mathbb{Z}}_Z \to 0.$$

Apply $\mathrm{R}\mathcal{H}\mathrm{om}_X(-, \mathrm{R}f_*\mu_n)$ to this sequence, we get a distinguished triangle in $D^b(X)$:

$$\mathrm{R}\mathcal{H}\mathrm{om}_X(i_*\underline{\mathbb{Z}}_Z, \mathrm{R}f_*\mu_n) \to \mathrm{R}\mathcal{H}\mathrm{om}_X(\underline{\mathbb{Z}}_X, \mathrm{R}f_*\mu_n) \to \mathrm{R}\mathcal{H}\mathrm{om}_X(j_!\underline{\mathbb{Z}}_U, \mathrm{R}f_*\mu_n).$$

For any testing sheaf T, apply $\mathrm{Hom}_X(T, -)$ to each term in the above sequence. By Yoneda lemma and properties of adjoint functors, we know

$$\begin{aligned} \mathrm{R}\mathcal{H}\mathrm{om}_X(i_*\underline{\mathbb{Z}}_Z, \mathrm{R}f_*\mu_n) &= i_*\mathrm{R}i^!\mathrm{R}f_*\mu_n, \\ \mathrm{R}\mathcal{H}\mathrm{om}_X(\underline{\mathbb{Z}}_X, \mathrm{R}f_*\mu_n) &= \mathrm{R}f_*\mu_n, \\ \mathrm{R}\mathcal{H}\mathrm{om}_X(j_!\underline{\mathbb{Z}}_U, \mathrm{R}f_*\mu_n) &= \mathrm{R}j_*j^*\mathrm{R}f_*\mu_n. \end{aligned}$$

So we have a distinguished triangle

$$i_*\mathrm{R}i^!\mathrm{R}f_*\mu_n \to \mathrm{R}f_*\mu_n \to \mathrm{R}j_*j^*\mathrm{R}f_*\mu_n.$$

Taking cohomology we get an exact sequence

$$H^2(i_*(\mathrm{R}i^!\mathrm{R}f_*\mu_n)) \to R^2f_*\mu_n \xrightarrow{\phi} H^2(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n). \tag{6.1}$$

We'll show the map ϕ in (6.1) is an isomorphism and prove the lemma.

Step 1. ϕ is injective.

By cohomological purity for the pair (Y_Z, Y), we know $R^n i'^! \mu_n = 0$ for $n = 0, 1, 2$. By the spectral sequence $R^p f_{Z,*}(R^q i'^! \mu_n) \Rightarrow (\mathrm{R}f_{Z,*}\mathrm{R}i'^!\mu_n)^{p+q}$, we know

$$H^2(\mathrm{R}f_{Z,*}\mathrm{R}i'^!\mu_n) = 0.$$

We know $\mathrm{R}i^!\mathrm{R}f_*$ is right adjoint to f^*i_*. By proper base change we have $f^*i_* = i'_*f_Z^*$. Note that $\mathrm{R}f_{Z,*}\mathrm{R}i'^!$ is right adjoint to $i'_*f_Z^*$. By uniqueness of adjoint functor, we know

$$\mathrm{R}i^!\mathrm{R}f_* = \mathrm{R}f_{Z,*}\mathrm{R}i'^!.$$

Together with exactness of i, we know

$$H^2(i_*(\mathrm{R}i^!\mathrm{R}f_*\mu_n)) = i_*H^2(\mathrm{R}i^!\mathrm{R}f_*\mu_n) = i_*H^2(\mathrm{R}f_{Z,*}\mathrm{R}i'^!_*\mu_n) = 0$$

hence the map ϕ in (6.1) is injective.

Step 2. ϕ is surjective. We already know $R^2f_*\mu_n \xrightarrow{\phi} H^2(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n)$ is injective. It suffices to show the cardinality of stalk on the right hand side is no greater than cardinality of stalk on the left hand side.

By assumption $f_U : Y_U \to U$ is a Brauer-Severi scheme, so

$$\begin{aligned} f_{U,*}\mu_n &= \mu_n, \\ R^1f_{U,*}\mu_n &= 0, \\ R^2f_{U,*}\mu_n &= \underline{\mathbb{Z}/n}_U. \end{aligned}$$

Consider the spectral sequence

$$R^pj_*R^qf_{U,*}\mu_n \Rightarrow H^{p+q}(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n).$$

By cohomological purity for (Z, X), the E_2 page can be explicitly written down as

$$\begin{tikzcd}
j_*R^2f_{U,*}\mu_n = \underline{\mathbb{Z}/n}_X \arrow[drr, "\mathrm{d}_2^{0,2}"] & & & . \\
j_*R^1f_{U,*}\mu_n = 0 & R^1j_*R^1f_{U,*}\mu_n = 0 & R^2j_*R^1f_{U,*}\mu_n = 0 & \\
j_*f_{U,*}\mu_n = \underline{\mathbb{Z}/n}_X & R^1j_*f_{U,*}\mu_n = 0 & R^2j_*f_{U,*}\mu_n = 0 & R^3j_*f_{U,*}\mu_n
\end{tikzcd}$$

Thus $H^2(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n) = \ker(\mathrm{d}_3^{0,2})$. Note that $\ker(\mathrm{d}_3^{0,2})$ is a subsheaf of $\underline{\mathbb{Z}/n}_X$, so the cardinality of stalk of $H^2(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n)$ is at most n. By proper base change, the fibers of $R^2f_*\mu_n$ all has cardinality n. So the injection ϕ in (6.1) is indeed an isomorphism and forces

$\ker(\mathrm{d}_3^{0,2}) = \underline{\mathbb{Z}/n}_X$. Hence

$$R^2 f_* \mu_n = H^2(\mathrm{R}j_*\mathrm{R}f_{U,*}\mu_n) = \underline{\mathbb{Z}/n}_X.$$

$\square$

Remark 6.1.5. *In the proof above, we need* $(n, \mathrm{char}(k)) = 1$ *so that the assumptions on cohomological purity for étale cohomology holds.*

Lemma 6.1.6. *In the same setting as 6.1.4, assume furthermore that* $\mathrm{char}(k) = 0$. *Then any line bundle* $\mathcal{L}$ *on* Y *restricts to the same degree on each geometric fiber.*

Proof. Apply f_* to the Kummer sequence on Y:

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

We get a morphism of sheaves $R^1 f_* \mathbb{G}_m \xrightarrow{\psi} R^2 f_* \mu_n \cong \underline{\mathbb{Z}/n}_X$.

For any geometric point $\overline{x}$ of X, the map on stalk $\psi_{\overline{x}}$ sends a germ represented by line bundle $\mathcal{L}$ to $\overline{\deg(\mathcal{L}_{\overline{x}})} \in \mathbb{Z}/n$. Consider the following diagram of cospecialization maps.

$$\begin{tikzcd}
(R^1 f_* \mathbb{G}_m)_{\overline{\eta}} \arrow[r, "\psi_{\overline{\eta}}"] & (R^2 f_* \mu_n)_{\overline{\eta}} \arrow[r, equal] & \mathbb{Z}/n \\
(R^1 f_* \mathbb{G}_m)_{\overline{x}} \arrow[r, "\psi_{\overline{x}}"] \arrow[u, "\mathrm{cosp}_1"] & (R^2 f_* \mu_n)_{\overline{x}} \arrow[r, equal] \arrow[u, "\mathrm{cosp}_2"', "\sim"] & \mathbb{Z}/n \arrow[u, "h"]
\end{tikzcd}$$

We know the cosp_2 is an isomorphism as $R^2 f_* \mu_n$ is a constant sheaf, see 6.1.4. It is not clear if cosp_1 is also isomorphism, or equivalently, if h is the identity map. If this is true, then the degree of line bundle over $\overline{x} \in Z$ and $\overline{\eta}$ are the same modulo any n, take n large enough, we are done. But this is not true in general, see 6.1.7.

However, since cosp_2 is isomorphism, for any n prime to $\mathrm{char}(k)$, we know $n | \deg(\mathcal{L}_{\overline{\eta}})$ if and only if $n | \deg(\mathcal{L}_{\overline{x}})$. If $\mathrm{char}(k) = 0$, the above implies $\deg(\mathcal{L}_{\overline{\eta}}) = \pm \deg(\mathcal{L}_{\overline{x}})$. So $\deg(\mathcal{L}_{\overline{\eta}}) = \deg(\mathcal{L}_{\overline{x}})$ holds for an ample line bundle $\mathcal{L}$. For a general line bundle, we can write it as the difference of two ample line bundles. $\square$

Remark 6.1.7. *Let's consider the family $V(tT_0^2+tT_1^2+tT_2^2) \subset \mathbb{P}^2 \times \mathbb{A}^1$, here t is coordinate of $\mathbb{A}^1$ and T_i are homogeneous coordinates of $\mathbb{P}^2$. The map h in above proof is not identity. Take $n = 3$, as $\mathcal{O}(1)$ restricts to degree $1, 2$ on the special fiber and generic fiber, we know h is multiplication by 2. In this case, we know $R^2 f_* \mu_3$ is still a constant sheaf, but the cospecialization map does not preserve degree. This is not an counterexample to our lemma since the assumptions on smoothness and codimension are not satisfied.*

Remark 6.1.8. *If $\mathrm{char}(k) = p > 0$, by the same argument, we can only claim $\deg(\mathcal{L}_{\overline{x}}) = p^l \cdot \deg(\mathcal{L}_{\overline{\eta}})$. It is not clear if the lemma is true in general.*

Lemma 6.1.9. *Let X be a regular scheme, $k(X)$ be its function field, then the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is an injection.*

Proof. See [Mil80, 2.22]. □

Proposition 6.1.10. *In the same setting as 6.1.4, denote Brauer classes of $Y_U \to U$ and $Y_Z \to Z$ by α_U, α_Z. By purity of Brauer group, there exists a unique class $\alpha \in \mathrm{Br}(X)$ such that $\alpha|_U = \alpha_U$. Then $\alpha|_Z = \alpha_Z$.*

Proof. By shrinking X and Z, we can find a filtration $Z \subset Z_1 \subset X$ with Z_1 smooth and $\mathrm{codim}_{Z_1}(Z) = 1$. Let $\mathcal{L}$ be an ample line bundle on Y. By 6.1.4 it has same degree on each geometric fiber.

Let η, η_1 be the generic point of Z, Z_1. Let's denote the Brauer class of $Y_\eta \to \eta, Y_{\eta_1} \to \eta_1$ by α_η, α_{η_1}. By 6.1.2, we can extend the Brauer-Severi scheme $Y_{\eta_1} \to \eta_1$ to a Brauer-Severi scheme over $\mathrm{Spec}(\mathcal{O}_{Z_1,Z})$ by taking closure. The Brauer-Severi scheme obtained by taking closure gives a class $\alpha'_{\eta_1} \in \mathrm{Br}(\mathrm{Spec}(\mathcal{O}_{Z_1,Z}))$.

Since $\alpha'_{\eta_1}|_{\eta_1} = \alpha_{\eta_1} = \alpha_U|_{\eta_1} = \alpha|_{\eta_1} = (\alpha|_{\mathrm{Spec}(\mathcal{O}_{Z,Z_1})})|_{\eta_1}$, we see $\alpha'_{\eta_1} = \alpha|_{\mathrm{Spec}(\mathcal{O}_{Z,Z_1})}$ by 6.1.9. Thus $\alpha'_{\eta_1}|_\eta = (\alpha|_{\mathrm{Spec}(\mathcal{O}_{Z,Z_1})})|_\eta = \alpha|_\eta$. By 6.1.2 and 6.1.4, on the special fiber obtained by taking closure, we have linear embedding. By Châtelet's theorem, see [GS17, 5.3.3], we know $\alpha_\eta = \alpha'_{\eta_1}|_\eta$. Thus $\alpha_Z|_\eta = \alpha_\eta = \alpha'_{\eta_1}|_\eta = \alpha|_\eta = (\alpha|_Z)|_\eta$. By 6.1.9, we know $\alpha_Z = \alpha|_Z$. □

Example 6.1.11. *The class α in 2.1 maps to 0 by pullback along section $[\omega_C]$ to $\mathrm{Br}(k)$.*

By [Bos90, 8.2.7], the class $\alpha|_{[\omega_C]}$ is represented by the projective bundle $\mathbb{P}(\pi_*\omega_C)$, where $\pi\colon C \to \mathrm{Spec}(k) = [\omega_C] \in \mathrm{Pic}^{2g-2}_{C/k}$ is the structure map. In case $\mathrm{char}(k) = 0$, the claim directly follows from proposition above. In case $\mathrm{char}(k) \neq 0$, we can lift the family to charcateristic 0, since the construction involved in α can be carried out over $\mathrm{Spec}(\mathrm{W}(k))$. Or one directly see this since α_Z is killed by $g-1$ as α is, and α_Z is killed by g as it is represented by Brauer-Severi scheme of relative dimension $g-1$.

6.2 Lefschetz theorem for Brauer group

Let k be an algebraically closed field. Let X be a smooth projective variety over k of dimension at least 3.

Proposition 6.2.1. *Let* $\mathrm{char}(k) = 0$. *Let Z be a smooth ample divisor of X. If* $\dim(X) \geq 4$, *then the map*

$$\mathrm{Br}(X) \to \mathrm{Br}(Z)$$

is an isomorphism. If $\dim(X) = 3$, *then there exists a smooth ample divisor $Z \subset X$ such that* $\mathrm{Br}(X) \to \mathrm{Br}(Z)$ *is an injection.*

Proof. By our assumption X and Z are regular schemes. Since the Brauer group of a regular scheme is a torsion group, see [Mil80, IV.2.6], it suffices to prove injectivity (surjectivity) for n-torsion elements on both sides. Consider the commutative diagram obtained by Kummer sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(X)/n\mathrm{Pic}(X) \arrow[r] \arrow[d, "f"] & H^2(X,\mu_n) \arrow[r] \arrow[d, "g"] & \mathrm{Br}(X)[n] \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathrm{Pic}(Z)/n\mathrm{Pic}(Z) \arrow[r] & H^2(Z,\mu_n) \arrow[r] & \mathrm{Br}(Z)[n] \arrow[r] & 0
\end{tikzcd}$$

we know if $\dim(X) \geq 4$, then g is an isomorphism; if $\dim(X) = 3$, then g is an injection, see [Laz04, 3.1.17] and [Laz04, 3.1.18]. Also note by [Gro65, XII.3.6], if $\dim(X) \geq 4$, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is isomorphism. Hence if $\dim(X) \geq 4$, then $\mathrm{Br}(X)[n] \cong \mathrm{Br}(Z)[n]$ by five lemma.

If $\dim(X) = 3$, by Noether-Lefschetz theorem, there exists a smooth ample divisor Z such that $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$, see [Jos95, 5.1], hence f is an isomorphism. Suppose there exists a nonzero class $\alpha \in \mathrm{Br}(X)[n]$ which restricts to $0 \in \mathrm{Br}(Z)[n]$, take a lift $\alpha' \in H^2(X, \mu_n)$ of α. Then $g(\alpha')$ maps to 0 in $\mathrm{Br}(Z)[n]$, hence $g(\alpha') \in \mathrm{Pic}(Z)/n\mathrm{Pic}(Z)$. Then $\alpha' - f^{-1}(g(\alpha'))$ is also an inverse image of α, which maps to $0 \in H^2(Z, \mu_n)$, but this contradicts with the injectivity of g. $\square$

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