Mathematicians and music: Implications for understanding the role of affect in mathematical thinking

Rena E. Gelb

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2021

Abstract

Mathematicians and music: Implications for understanding the role of affect in mathematical thinking

Rena E. Gelb

The study examines the role of music in the lives and work of 20th century mathematicians within the framework of understanding the contribution of affect to mathematical thinking. The current study focuses on understanding affect and mathematical identity in the contexts of the personal, familial, communal and artistic domains, with a particular focus on musical communities. The study draws on published and archival documents and uses a multiple case study approach in analyzing six mathematicians. The study applies the constant comparative method to identify common themes across cases. The study finds that the ways the subjects are involved in music is personal, familial, communal and social, connecting them to communities of other mathematicians. The results further show that the subjects connect their involvement in music with their mathematical practices through 1) characterizing the mathematician as an artist and mathematics as an art, in particular the art of music; 2) prioritizing aesthetic criteria in their practices of mathematics; and 3) comparing themselves and other mathematicians to musicians. The results show that there is a close connection between subjects' mathematical and musical identities. I identify eight affective elements that mathematicians display in their work in mathematics, and propose an organization of these affective elements around a view of mathematics as an art, with a particular focus on the art of music. This organization of affective elements related to mathematical thinking around the view of mathematics as an art has implications for the teaching and learning of mathematics.

Table of Contents

List of Tables

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor Alexander Karp, for all his support and guidance throughout this study. I am also very grateful to Professors Heather Gould, Sandra Okita, J. Philip Smith, and Erica N. Walker for their help as members of my dissertation committee

Thank you to Lisa Harries Schumann for her work in translating letters from the Richard Courant Papers at the New York University Archives from the original German to English.

Chapter 1: Introduction

1.1 Need for the Study

The goal of the study is to understand the role music plays in the lives and work of mathematicians. In doing so, I examine the beliefs and values of mathematicians, taking a holistic view of their lives, practices of mathematics and involvement in music. This is informative for understanding how affective elements, such as beliefs, values and identity, and involvement in the art of music, relate to mathematicians' practices of mathematics. The approach and findings of the study will contribute to the body of literature on affective elements that are constructive in mathematical thinking, and will inform mathematics education. This study is at the intersection of literature on the role of affect in mathematical thinking, the lives and beliefs of mathematicians and the relationship between mathematics/ mathematicians and the particular art of music.

It has become clear through research that affect plays an important role in students' ability to do mathematics (Bishop, 2008a and 2008b; DeBellis & Goldin, 2006; McCleod, 1988, 1989 and 1992; Norman, 1980; Resnick, 1988; Schoenfeld, 1985 and 2016; Silver, 1985; Zan, Brown, Evans & Hannula, 2006). Further, researchers recognize the importance of considering social and cultural factors, both within and outside the classroom and school environment, in the formation of affect, and a related characteristic, identity (Boaler, William & Zevenbergen, 2000; Grootenboer & Edwards-Groves, 2019; Martin, 2000; Nasir, 2002; Sfard and Prusak, 2005). Although affect is now recognized as an important factor in mathematical thinking, historically, researchers have focused only on cognitive elements in understanding mathematics learning (see Schoenfeld, 1985 and 2016; DeBellis & Goldin, 2006; Mcleod 1988). Now researchers (such as referenced above) view the study of affect as essential in understanding mathematical learning.

This shift was, in part, motivated by mathematicians', such as Hardy, Hadamard and Poincaré (see below), descriptions of their own processes of doing mathematics. These descriptions are "characterised by a strong interaction between cognitive, metacognitive, and emotional aspects" (Zan, et al., 2006, p. 115). In addition, researchers were puzzled at the fact that individuals who apparently were well equipped cognitively to solve mathematical problems were still struggling, suggesting the importance of other factors (Schoenfeld, 1985, as cited in Zan, et al., 2006).

The research is particularly necessary, in that as Bishop (2008b) and Schoenfeld (1985 and 2016) point out, qualities of affect and related characteristics such as beliefs, attitudes and values, are being taught in classrooms, for better or for worse, whether we deepen our understanding of these or not. As Schoenfeld (1985) states, "Belief systems shape cognition, even when one is not consciously aware of holding those beliefs" (p. 35).

Zan, et al. (2006) points out the need for further research in the area of affect and mathematics education. "The main efforts of research on affect in mathematics education are therefore devoted to the construction of better-founded theoretical frameworks and a broader range of methodological instruments fit to interpret students' behaviour in mathematical activities" (p. 116).

An understanding of mathematicians, learners and students from a broad, holistic perspective is instructional in understanding the nature of mathematical thinking. In particular, the current study focuses on understanding affect and mathematical identity in the context of the personal, familial, communal and artistic domains, with a particular focus on musical communities.

1.2 Purpose of the Study

The purpose of the study is to understand and document the role of music in the lives and work of 20th century mathematicians. To do this, the study seeks to describe and compare the ways in which mathematicians are involved in music, whether and how music connects to mathematicians' work in mathematics and whether and how involvement in music can be considered a part of a mathematician's identity as a mathematician.

To guide this study, the following research questions are examined:

1) What commonalities or differences are there among the ways different mathematicians involve themselves in music?
2) How do mathematicians relate their interest in music to their interest in mathematics, if at all?
3) How does involvement in music help shape mathematicians' identities as mathematicians, if at all?

1.3 Procedure

This is a qualitative study based on an examination of beliefs, values and identity, using a multiple case study approach (Merriam, 2015). The particular qualitative methodology used in this study is the multiple or comparative case study (See Merriam, 2015 and Creswell, 2007). Creswell (2007) describes case study research as:

> a qualitative approach in which the investigator explores a bounded system (a *case*) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving *multiple sources of information* (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case *description* and case-based themes. (p. 73)

This study uses a multiple case study approach since the focus is on bounded systems, each being the experiences and views with respect to music and mathematics of a mathematician in his or her own life. I conduct two stages of analysis- the within-case analysis and the cross-case analysis (Merriam, 2015). I first treat each case individually and then "seek(s) to build abstractions across cases" (Yin, 2014, as cited by Merriam, 2015, p. 234).

The study uses the multiple case study approach to allow for identification of themes across cases. Merriam (2015), quoting Miles, Huberman, and Saldaña (2014), explains, "the more cases included in a study, and the greater the variation across the cases, the more compelling an interpretation is likely to be" (p. 233). This study examines six subjects.

1.3.a Setting and Population.

The six subjects of the study are mathematicians of significant stature (as will be defined below), who lived and practiced mathematics in the 20th century, and had a demonstrated involvement in music (as will be defined below). To choose case study subjects, purposeful selection was used (Merriam, 2015) to arrive at "*information-rich cases*" (Patton, 2015, as cited by Merriam, 2015, p. 96).

The criteria for selecting the information-rich cases for this study were as follows. The first criterion is that the mathematician lived and practiced in the 20th century. This is to examine a population that could provide insight into the lives, work and education of mathematicians in current times. The second criterion is that the mathematician achieved significant stature in the field of mathematics. Significant stature is defined, for the purpose of the study, as the mathematician having been recognized as contributing in a significant way to the field of mathematics, through receiving a major mathematics prize or being recognized by one's peers. The way this criterion was applied, is that a population was formed of all mathematicians who were recipients of a leading prize in mathematics, including the Fields Medal, Wolf Prize in Mathematics and Abel Prize in Mathematics. In addition, the population contained a selection of mathematicians who headed departments of mathematics in research institutions. Certain mathematicians were included in the population because they had a demonstrated interest in music (which will be defined below) and also were recognized by their peers in research journal articles as having made significant contributions to mathematics.

To this population, the third criterion, that the subject have a demonstrated interest in music, was applied. A demonstrated interest in music is defined, for the purpose of the study, as the subject having either played a musical instrument, listened to music as a hobby and/or referred to music as a metaphor, in writings or speeches, for the practice of mathematics, on numerous occasions. All genres of music were considered.

From this pool of candidates, I selected six subjects about whom abundant, public information was available for case studies. All of the subjects have a relatively large volume of research material available about them, in order to create information-rich cases. The six case study subjects selected are: Michael Atiyah (1929-2019), Richard Courant (1888-1972), Israel Gelfand (1913-2009), Israel Glazman (1916-1968), Hans Lewy (1904-1988) and Hassler Whitney (1907-1989).

1.3.b Instruments and Data Collection.

The data for the study are collected from documents and published video or audio interviews with the subjects. The document database consists of writings and transcribed speeches by the subjects or about the subjects by family members, colleagues, students or friends. In addition, published video and audio interviews, or their transcriptions, of the subjects are included. The focus of data collection is on biographical, professional, philosophical, personal and anecdotal information about the subjects, touching on all aspects of the subjects' personal and professional lives. There is a particular emphasis on data on involvement in music. Included in the documents are introductions to technical mathematics writings that describe the philosophy of the subject as well as writings that describe the overall focus and direction of a subject's mathematical work. Not emphasized in the data collected are the subjects' technical mathematical writings themselves. For one subject, Richard Courant, data were collected from the Richard Courant Papers at the New York University ("NYU") Archives. I reviewed personal letters from and to the subject within the time period 1930-1971.

The documents mined for the database of documents for each case study are the following:

1) Biographies
2) Reviewed journal articles written by or about the subject
3) Books or book chapters in edited volumes written by or about the subject
4) Introductions authored by the subject to books authored by the subject or by others
5) Introductions authored by others to books authored by the subject
6) Published interviews with the subject, including audio or video recorded or transcribed
7) Published talks given by the subject
8) Material from university or professional websites dedicated to the subject
9) In the case of one subject, archival letters written by and to the subject

All documents in the above categories are examined for data about each of the subjects. All documents are listed in the reference section of this study.

1.3.c Data Analysis.

Data from the sources for each subject consist of:

1) The subject's personal history, education and work in mathematics
2) The history and source of the subject's interest/ education in music
3) The nature of the subject's involvement in music throughout his life
4) Clues to the subject's feelings about music in his life
5) How the subject relates to others, particularly other mathematicians, through his involvement in music
6) How the subject connects music to his work in mathematics
7) The subject's views on the process of doing mathematics
8) The subject's views on the process of mathematics education

All data collected are organized by case subject. I use the constant comparative method, proposed by Glaser and Struss (1967, as cited by Merriam, 2015, p. 201), for analyzing qualitative data as it is inductive and comparative. This consists of identifying themes as the research progresses by identifying "units" of data and continually comparing new data with existing data to arrive at categories or themes that cut across all of the data for the study.

In examining the role of music in the lives and works of mathematicians, I take a holistic view of a mathematician's experience with, practice of and philosophy of mathematics. Therefore, I investigate the full spectrum of each subject's life and works.

Based on comparisons among mathematicians and the themes, I will answer the research questions and draw conclusions about the role of music in the lives and work of mathematicians, particularly as it sheds light on the beliefs, values and identities of mathematicians. I will also discuss its implications in educating future mathematicians.

Chapter 2: Review of the Literature

The goal of the study is to understand the role music plays in the lives and work of mathematicians. In doing so, I examine the beliefs and values of mathematicians, taking a holistic view of their lives, practices of mathematics and involvement in music. This is informative for understanding how affective elements, such as beliefs, values and identity, and involvement in the art of music, relate to mathematicians' practices of mathematics. The approach and findings of the study will contribute to the body of literature on affective elements that are constructive in mathematical thinking, and will inform mathematics education. This study is at the intersection of literature on the role of affect in mathematical thinking, the lives and beliefs of mathematicians and the relationship between mathematics/ mathematicians and the particular art of music. Below is an outline of the literature I review for the purpose of the study:

- The role of affect in mathematical thinking.
- Mathematicians' views on their own beliefs and values in the practice of mathematics.
- Studies of mathematicians' lives and beliefs.
- A connection between mathematics and music.
- Recent studies on the relationship between musical involvement and mathematical achievement in students.

Where this literature comes together for the purpose of the current study, is in recognizing that an understanding of mathematicians, learners or students from a broad, holistic perspective is instructional in understanding the nature of mathematical thinking. In particular, the current study focuses on understanding affect in the context of personal, familial, communal and artistic domains. The study has broad implications for mathematics education in the classroom, the school and the larger community.

2.1 The Role of Affect in Mathematical Thinking

It has become clear through research that affect plays an important role in students' ability to do mathematics (Bishop, 2008a and 2008b; DeBellis & Goldin, 2006; McCleod, 1988, 1989 and 1992; Norman, 1980; Resnick, 1988; Schoenfeld, 1985 and 2016; Silver, 1985; Zan, Brown, Evans & Hannula, 2006). Researchers discuss the influence of affect on "mathematical thinking" (Bishop, 2006 and Schoenfeld, 2016), "problem solving" (Resnick, 1988 and McCleod, 1988) and "mathematical capability" (DeBellis & Goldin, 2006) among other related aspects.

Although affect is now recognized as an important factor in mathematical thinking, historically, researchers have focused only on cognitive elements in understanding mathematics learning (see Schoenfeld, 1985 and 2016; DeBellis & Goldin, 2006; Mcleod 1988). DeBellis and Goldin (2006) cite two reasons for this. First, this is due to a traditional belief that mathematics is a purely cognitive discipline. Second, there are numerous methodological difficulties in studying affect. These include an "absence of precise, shared language for describing mathematical affect, within a theoretical framework permitting its systematic study" (p. 131).

Now researchers (such as referenced above) view the study of affect as essential in understanding mathematical learning. This shift was, in part, motivated by mathematicians', such as Hardy, Hadamard and Poincaré (see below), descriptions of their own processes of doing mathematics. These descriptions are "characterised by a strong interaction between cognitive, metacognitive, and emotional aspects" (Zan, et al., 2006, p. 115). In addition, researchers were puzzled at the fact that individuals who apparently were well equipped cognitively to solve mathematical problems were still struggling, suggesting the importance of other factors (Schoenfeld, 1985, as cited in Zan, et al., 2006, p. 115).

Research on the affective role in mathematical thinking has direct, practical implications for the teaching of mathematics. As Bishop (2008a), when introducing his theoretical framework for studying affect in mathematical thinking, states, “So what is the problem we are trying to consider here? In one sentence the major problem seems to be: ‘How can teachers help mathematical thinking to develop in their students?’” (p. 79).

The research is particularly necessary, in that, as Bishop (2008b) and Schoenfeld (1985 and 2016) point out, qualities of affect and related characteristics such as beliefs, attitudes and values, are being taught in classrooms, for better or for worse, whether we deepen our understanding of these or not. As Schoenfeld (1985) states, “Belief systems shape cognition, even when one is not consciously aware of holding those beliefs” (p. 35).

Zan, et al. (2006) points out the need for further research in the area of affect and mathematics education. “The main efforts of research on affect in mathematics education are therefore devoted to the construction of better-founded theoretical frameworks and a broader range of methodological instruments fit to interpret students' behavior in mathematical activities” (p. 116).

Researchers in the field of affect and mathematics learning have proposed various theoretical frameworks about the interaction between cognition and affect (Bishop, 2008a and 2008b; DeBellis & Goldin, 2006; McCleod, 1988, 1989 and 1992; Op 'T Eynde, de Corte & Verschaffel, 2006; Schoenfeld, 1985 and 2016; Zan, et al, 2006). Many such researchers include affect as a key element in their models of mathematical ability, mathematical problem solving, mathematical thinking, and other related areas, as will be discussed below.

DeBellis and Goldin’s (2006) model for mathematical problem-solving competency, comprised of five elements, includes “an affective system, involving emotions, attitudes, beliefs, morals, values, and ethics” (p. 132).

Schoenfeld’s (1985) “four categories of knowledge and behavior” necessary to “‘explain’ human problem-solving behavior” include the category beliefs (p. 14). Schoenfeld (2016) interprets beliefs as “an individual’s understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior” (p. 26). In addition, Schoenfeld defines a “Belief System” as “One's ‘mathematical world view,’ the set of (not necessarily conscious) determinants of an individual's behavior: About self; About the environment; About the topic; About mathematics” (Schoenfeld, 1985, p. 15. Punctuations not in original.).

Bishop (2008a) provides a framework for understanding the development of mathematical thinking, focusing on “mathematical values” (p. 83). Bishop (2008a, p. 83) elaborates on the nature of mathematical values that he prioritizes in his research. He categorizes these into an ideological component, a sentimental (attitudinal) component, and a sociological component. The key values in the various categories are Rationalism, Objectivism, Control, Progress, Openness and Mystery.

The ecosystem of affect includes terms such as emotions, attitudes, beliefs, values, meta-affect, mathematical world view or point of view, perspective, identity and others. Different theoretical frameworks draw upon different elements of this list.

DeBellis and Goldin (2006, p. 135) drawing upon McCleod (1992), define emotions, attitudes, beliefs and values on a continuum from most fleeting to most stable. Beyond these is the concept of “meta-affect” that characterizes “emotional feelings about the emotions associated with” a certain mathematical activity (DeBellis & Goldin, 2006, p. 137).

In addition to the affective elements listed above, McLeod (1992) identifies a number of "topics that are closely identified with research on cognition, even though they have a strong connection to the affective domain" (p. 586). In this category, McLeod lists aesthetics and intuition, noting that these have been shown to play a role in mathematical problem solving and mathematical thought. These are examined in connection with the examination of affective elements for the purpose of this study.

2.1.a The Social Genesis of Affect.

Research shows the important role that social and cultural forces play in the development of affect, which in turn is fundamental to the development of mathematical thinking. Schoenfeld (2016) describes the interconnection of affective aspects and social, cultural and communal factors. Schoenfeld states, "The notion of socialization as identified by Resnick (also called enculturation—entering and picking up the values of a community or culture) is central, in that it highlights the importance of perspective and point of view as core aspects of knowledge." Schoenfeld later states, "the community to which one belongs shapes the development of one's point of view" (p. 7).

Schoenfeld (2016) further describes:

> Mathematics is an inherently social activity....The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence

with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, forthcoming). (p. 1)

Further, "Beliefs are abstracted from one's experiences and from the culture in which one is embedded" (Schoenfeld, p. 28).

In addition, Schoenfeld cites an emerging body of literature (see Schoenfeld, 2016, p. 7) that "extends the notion of constructivism from the purely cognitive sphere, where much of the research has been done, to the social sphere."

Bishop (2008a) looks at mathematical thinking as primarily influenced by social and cultural factors. Bishop's approach draws upon the following theoretical ideas:

1) "Mathematical thinking as a form of metacognition, affected by the cultural norms and values of the learner's society" (p. 81).
2) Knowledge is acquired from multiple, social sources.
3) Mathematics education has socio-cultural dimensions. Bishop identifies five levels of the socio-cultural dimension of mathematics education. These are the cultural, societal, institutional, pedagogical and individual levels. (Bishop, 2008)

Bishop frames his approach in the following three aspects: "I will start by assuming that my ideas about values regarding Mathematical thinking are: 1. Concerned with developing meta-cognition; 2. Located within the socio-cultural dimension; 3. Focused on the community of practice in the classroom" (p. 83).

2.1.b Identity.

Within affect, researchers point to identity as a driving factor in mathematics performance. Aside from the psychological/developmental and poststructural views on the formation of mathematical identity (Grootenboer, P. & Edwards-Groves, 2019), the socio-cultural view on the formation of identity is focused on by many researchers (Boaler, William & Zevenbergen, 2000; Grootenboer &

Edwards-Groves, 2019; Martin, 2000; Nasir, 2002; Sfard and Prusak, 2005). Such research suggests that social and cultural forces both within and outside of the classroom/ school environment are responsible for the formation of mathematical identities in individuals.

Sfard and Prusak (2005) define identity as related to the stories and narratives about a person that are “reifying, endorsable, and significant.” In particular, “The most significant stories are often those that imply one's memberships in, or exclusions from, various communities” (p. 16-17). Further, Sfard and Prusak state:

> Because significant narrators can count as voices of community, all of these findings corroborate the claim that designated identities are products of collective storytelling-of both deliberate molding by others and uncontrollable diffusion of narratives that run in families and communities. This assertion completes our empirical instantiation of the claim that designated identity is "a pivot between the social and the individual" aspects of learning. (In quotations, Wenger, 1998, p. 145, as cited in Sfard & Prusak, 2005, p. 21)

Boaler, William and Zevenbergen (2000) find that the types of mathematical identities students form are more decisive in mathematics outcomes than ability. Boaler, et al. (2000) show an extreme consequence of students being exposed to mathematics instruction they perceive as overly procedural, purely result focused and disconnected from their humanity. The result is that students reject the identity offered them by associating with the mathematical community they perceive. Consequently, these students discontinue their mathematics education when able to and/or perform poorly in mathematics courses. Boaler, et al. (2000) conclude, “Most students in the US schools, despite being relatively successful mathematics learners, reported disliking mathematics, not because the procedural nature denied them access to understanding, although that was important, but because their perceptions of the subject as abstract, absolute and procedural conflicted with their notions of self, of who they wanted to be” (p. 8). The strong influence the desire to identify and associate with a desirable community had on the students precluded them from adopting any affiliation with the "mathematician," as represented to them by the individuals and experiences modeled in their communities. Boaler, et al. (2000) note that high achieving mathematicians would also reject such an identity and affiliation as that being represented to the students.

Grootenboer and Edwards-Groves (2019) similarly focus on the formation of students' mathematical identities within the culture of the classroom. Grootenboer and Edwards-Groves (2019) claim identity is formed through the process of students' being led through mathematical practices and the experiences of "doing" mathematics in classrooms.

Researchers show that affect and identity are significant factors in mathematical achievement and that both are developed within multiple social frameworks of which students are part. These include the school context and the practices of teaching and learning mathematics; the personal, familial and communal contexts of the students; and the personal practices and interests in which students engage.

2.2 Mathematicians' Views on Own Beliefs and Values

As researchers were motivated to study the relationship between affect and cognition in mathematics after noting how mathematicians themselves incorporated a discussion of affect into their personal accounts of their practices of mathematics, so too I review mathematicians' own accounts of the role of affective elements in their practices of mathematics. In particular, I focus on mathematicians' views on mathematics as an art, the role of the aesthetic in the practice of mathematics, and the connection between mathematics and music.

Mathematicians make wide reference to affective aspects, in particular to the role of the aesthetic, in their discussions of their mathematical world views. The literature includes many mathematicians' reflections on the nature of their craft. I will examine the views of four renowned mathematicians of the 20th century (Poincaré, Hardy, Hadamard and von Neumann) who published works dedicated to their view of mathematical creation/ invention/ discovery (as they variously term it), where the discussion of aesthetics in mathematics plays a large role.

Particularly in the area of mathematical discovery, mathematicians and researchers acknowledge an "affective" element of doing mathematics (or "emotional sensibility," as Poincaré, 1908, terms it). Mathematicians relate this affective element to an aesthetic experience of mathematics, often referred to as a sense of beauty.

2.2.a Poincaré.

The prolific, French mathematician Jules Henri Poincaré (1854-1912) was active in the fields of non-Euclidean geometry, chaos in celestial mechanics and algebraic topology (Stillwell, 2014). At the beginning of the 20th century, Poincaré gave, before the Psychological Society in Paris, the now famous lecture entitled "Mathematical Creation." In it, Poincaré reflected on the essence of the nature of mathematics and the mathematician's brain, focusing on the key elements in mathematical creation.

In explaining his view of the essence of mathematical creation, Poincaré concludes that mathematical creation consists of making "useful" combinations, which are in a small minority compared with possible combinations, and which requires "discernment, choice." How does a mathematician come to be capable of making that choice? The mathematician has within him/her a sort of "sieve" capable of making this discernment. What is this "sieve?" Poincaré begins to explain it by objecting to the false assumption that mathematics is a discipline purely engaged in with the intellect. Rather, "emotional sensibility," awakened by "mathematical beauty," is a "true esthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility"

(p. 2047). Also, related to mathematical beauty, Poincaré mentions "the harmony of numbers and forms, of geometric elegance" (p. 2047).

Delving further, Poincaré asks what are the particular mathematical features that contain beauty and elegance and which awaken aesthetic emotion in people:

> "They are those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details. This harmony is at once a satisfaction of our esthetic needs and an aid to the mind, sustaining and guiding. And at the same time, in putting under our eyes a well-ordered whole, it makes us foresee a mathematical law. Now, as we have said above, the only mathematical facts worthy of fixing our attention and capable of being useful are those which can teach us a mathematical law. So that we reach the following conclusion: The useful combinations are precisely the most beautiful, I mean those best able to charm this special sensibility that all mathematicians know....What happens then? Among the great numbers of combinations blindly formed by the subliminal self, almost all are without interest and without utility; but just for that reason, they are also without effect upon the esthetic sensibility. Consciousness will never know them; only certain ones are harmonious, and, consequently, at once *useful and beautiful* (italics mine). They will be capable of touching this special sensibility of the geometer of which I have just spoken, and which, once aroused, will call our attention to them, and will bring them into our consciousness...Thus it is this special esthetic sensibility which plays the role of the delicate sieve of which I spoke, and that sufficiently explains why the one lacking it will never be a real creator. (p. 2048)

To Poincaré, aesthetic sensibility is a necessary characteristic in order to do mathematics, which consists primarily of making choices about what new combinations of elements will be useful. It is characterized by a feeling of harmony allowing one to embrace a whole and the details simultaneously. This harmony is a need for mathematicians as well as a guide toward what will be useful. Useful is synonymous with beautiful. All "real mathematicians" know this feeling, and a person who does not, is not capable of being a "real creator."

2.2.b Hardy.

The English mathematician Godfrey Harold Hardy (1877-1947) contributed greatly to the fields of number theory and mathematical analysis. Like Poincaré, Hardy (1940) provides a personal account of what mathematics means to him in his work "A Mathematician's Apology." Hardy describes his views of mathematics as consisting of beauty, being a creative art and dealing with patterns.

Hardy, like Poincaré, sees the aesthetic, which Hardy refers to by the name beauty, as an essential aspect of true mathematics. "Beauty is the first test: there is no permanent place in the world for ugly mathematics" (p. 2027). Hardy says that he is not able to define mathematical beauty, and justifies this with the difficulty in defining beauty of any kind. However, Hardy posits that "we may not quite know what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it" (p. 2027). This legitimizes the approach of looking to the words of mathematicians to understand what is mathematical beauty. However, Hardy does claim that the beautiful aspects of mathematics are the "mathematician's patterns," which "must be *beautiful*; the ideas, like the colours (of the painter) or words (of the poet), must fit together in a harmonious way" (p. 2027).

Hardy views mathematical beauty as accessible, claiming, “[m]ost people have some appreciation of mathematics, just as most people can enjoy a pleasant tune…” (p. 2028). Hardy points out that in games popular at his time, such as chess, bridge and even newspaper puzzles, players who enjoyed the “beauty” of the game were simply appreciating mathematics.

The other essential element of mathematics to Hardy is “seriousness.” “The best mathematics is *serious* as well as beautiful- ‘important’ if you like…” (p. 2029). However, Hardy is careful to distinguish “serious” from “useful.”

> “The ‘seriousness’ of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the ‘*significance’* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is ‘significant’ if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. (p. 2029)

Hardy’s language of “seriousness” is similar to the way Poincaré uses “useful” in that it signifies “combinations" that teach us a “mathematical law” (in Poincaré’s words) as opposed to having practical applications. Like Poincaré, Hardy views beauty (along with seriousness) as the chief assessment criterion for mathematics that achieves the goals of mathematics. “I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged” (Hardy, 1940, p. 2033).

In fact, Hardy uses the idea that mathematics is beautiful and akin to other creative arts as his justification for his life’s work, despite the fact that he claims he has never done anything “useful” (p. 2038). Hardy states, “I have added something to knowledge… and ...these somethings have a value which differs in degree only, and not in kind, from the creations of the great mathematicians, or of any of the other great artists” (p. 2038).

2.2.c Hadamard.

Jacques Salomon Hadamard (1865-1963) was a French mathematician who made contributions to the areas of analytic function theory, number theory, analytical mechanics and geometry, calculus of variations, hydrodynamics, partial differential equations, the history of mathematics, and the psychology of mathematical invention.

Hadamard (1945) wrote “The Psychology of Invention in the Mathematical Field” where he explains his goal as trying “to report and interpret observations, personal or gathered from other scholars engaged in the work of invention” (p. 133). Hadamard bases his conclusions principally on his own experience as a mathematician, since “the results of introspection, (are) the only ones I feel qualified to speak of” (p. 2).

Hadamard compares the process of invention in mathematics with that of other disciplines, including music. In explaining the phenomenon of “illumination” in mathematics, or an idea occurring to someone in the pursuit of creative work seemingly spontaneously, Hadamard gives the example of Mozart. Hadamard references the work of the psychologist Paulhan, who quotes a letter of Mozart:

> When I feel well and in a good humor, or when I am taking a drive or walking after a good meal, or in the night when I cannot sleep, thoughts crowd into my mind as easily as you could wish. Whence and how do they come? I do not know and I have nothing to do with it. Those which please me, I keep in my head and hum them; at least others have told me that I do so. Once I have my theme, another melody comes, linking itself to the first one, in accordance with the needs of the composition as a whole: the counterpoint, the part of each instrument, and all these melodic fragments at last produce the entire work. Then my soul is on fire with inspiration, if however nothing occurs to distract my

> attention. The work grows; I keep expanding it, conceiving it more and more clearly until I have the entire composition finished in my head though it may be long. Then my mind seizes it as a glance of my eye a beautiful picture or a handsome youth. It does not come to me successively, with its various parts worked out in detail, as they will be later on, but it is in its entirety that my imagination lets me hear it. (p. 16)

Mozart's description of perceiving the entire work of music "in its entirety" is similar to Poincaré's description of "those...elements (in mathematics) harmoniously disposed so that the mind without effort can embrace their totality while realizing the details" (p. 2048).

Hadamard was a great admirer of Poincaré's work on mathematical creativity and affirms Poincaré's conclusions "on account of my own auto-observation because the ideas chosen by my unconscious are those which reach my consciousness, and I see that they are those which agree with my aesthetic sense" (p. 39). That Hadamard is convinced that aesthetics is an integral part of mathematics is seen from his statement, "As a matter of fact, I consider that every mathematician, if not every scientist, would agree to that opinion. I may add that actually some of them, writing to me on the general subject of this work, spontaneously (i.e., without a question from me on that special point) expressed themselves in the same sense, in the most positive way" (p. 39).

In contrast to Poincaré and Hardy, who emphasize that the aesthetic quality of mathematics serves the purpose of helping the mathematician arrive at ideas ("combinations") that are of mathematical value, Hadamard views the role of the aesthetic quality of mathematics as guiding the mathematician to future application. Hardy states, "Concerning the fruitfulness of the future result, about which, strictly speaking, we most often do not know anything in advance, that sense of beauty can inform us and I cannot see anything else allowing us to foresee" (p. 127). Hadamard gives an illustrative example of this phenomenon:

> When Jean Bernoulli, in the eighteenth century, asked for the curve along which a small heavy body would go down from a point A to a point B in the shortest possible time, he was necessarily tempted by the beauty of that problem, so different from what had been attacked hitherto though evidently offering an analogy with those already treated by infinitesimal calculus. That beauty alone could tempt him. The consequences which "calculus of variations" i.e., the theory of problems of that kind would carry for the improvement of mechanics, at the end of the eighteenth century and the beginning of the nineteenth, could not be suspected in his time. (p. 129)

2.2.d von Neumann.

John von Neumann (1903-1957) was a Hungarian-American mathematician who made great contributions in the fields of set theory and game theory, and touched upon nearly every major branch of mathematics. Von Neumann also made strong contributions to applied areas such as quantum theory, automata theory, economics, defense planning and the stored-program digital computer.

In his work "The Mathematician," von Neumann (1947/1956) provides a personal perspective on "the nature of intellectual effort in mathematics" (p. 2053). Von Neumann describes his attempt to shed light on the mind of the mathematician, from a personal perspective, as follows, "I must admit that it is an interesting and challenging task to make the attempt and to talk to you about the nature of intellectual effort in mathematics" (p. 2053).

After justifying his claim that mathematics is like empirical sciences in that it is inextricably linked to the empirical experience of mathematicians, von Neumann then goes on to describe how mathematics as a discipline differs from the natural, empirically focused sciences. To explore this, von Neumann takes up the questions of "What is the mathematician's normal relationship to his subject? What are his criteria of success, of desirability? What influences, what considerations, control and direct his effort?" (p. 2060). Von Neumann focuses on the aspects of mathematics in these realms which are distinct from empirical subjects.

Von Neumann addresses these questions by comparing mathematics to the "most highly developed of all theoretical sciences," theoretical physics (p. 2061). Von Neumann points out that "the attitude that theoretical physics does not explain phenomena, but only classifies and correlates, is today accepted by most theoretical physicists" (p. 2026). The "criterion of success" for a theory in physics is "whether it can, by a simple and elegant classifying and correlating scheme, cover very many phenomena, which without this scheme would seem complicated and heterogeneous, and whether the scheme even covers phenomena which were not considered or even not known at the time when the scheme was evolved…" (p. 2061). In other words, "success" in theoretical physics is indicated by simplicity, elegance and what other mathematicians have referred to as "harmony."

Von Neumann characterizes the above mentioned criteria as "to a great extent of an aesthetical nature." Due to the similarities between theoretical physics and mathematics, the criteria of success in theoretical physics are very closely akin to the criteria of success in mathematics, which, according to von Neumann, "are almost entirely aesthetical" (p. 2061).

When explaining the specific aesthetic characteristics of mathematics, which are involved in "criteria of selection" and of "success", and how these go beyond even theoretical physics, von Neumann states:

> One expects a mathematical theorem or a mathematical theory not only to describe and to classify in a simple and elegant way numerous and a priori disparate special cases. One also expects 'elegance' in its 'architectural,' structural makeup. Ease in stating the problem, great difficulty in getting hold of it and in all attempts at approaching it, then again some very surprising twist by which the approach, or some part of the approach, becomes easy, etc. These criteria are clearly those of any creative art…" (p. 2062)

Due to these esthetic criteria in assessing mathematical selection (compare Poincaré in his definition of the work of the mathematician as 'choosing') and success (compare Poincaré's word 'useful' and Hardy's 'serious'), von Neumann finally distinguishes mathematics from empirical sciences by concluding that the work of the mathematician "is much more akin to the atmosphere of art pure and simple than to that of the empirical sciences" (p. 2063).

Finally, von Neumann, like others, draws on the experience of the mathematician rather than philosophical or psychological characterizations of aesthetics.

> I realize that this assertion is controversial and that it is impossible to "prove" it, or indeed to go very far in substantiating it…Suffice it to say that the aesthetical character is even more prominent (in mathematics) than in the instance I mentioned above in the case of theoretical physics. (p. 2062)

Thus, the self-reflections of Poincaré, Hardy, Hadamard and von Neumann all show views that aesthetics are integral to the process of doing "real/ useful/ serious" mathematics. They all acknowledge aesthetics as playing a crucial role in the mathematician's arriving at certain decisions which guide him/her in his/her work of "creation/ invention/ intellectual effort." To these mathematicians, the process of doing mathematics is one and the same as that of engaging in a creative art, wherein the creator is guided by an internal, possibly emotional, innate sense, of beauty. This highlights the primacy of certain affective elements in the work of mathematicians and illustrates the importance of better understanding the role affective elements, including the beliefs and values individuals hold, in mathematical thinking.

2.3 Studies of Mathematicians' Lives and Beliefs

In addition to looking at mathematicians' own characterizations of their practices of mathematics, including the beliefs and values important to them, I look at studies others have conducted about the lives and beliefs of mathematicians. In the 20th century, there have been a number of important works on the lives of mathematicians (Laisant & Fehr, 1902, as referenced in Hadamard, 1945; Albers, et al., 1985, 1990; Burton, 1999; Sinclair, 2002). These studies focus on how mathematicians became interested in their branches of study, what inspires them to pursue certain mathematical problems, how they relate to their colleagues, their creative process and their views on the nature of mathematics.

A number of studies aim to further define the aesthetic quality of mathematics (Birkhoff, 1956; Davis, Hersh & Marchisotto, 1995; Dreyfus & Eisenberg, 1986; Karp, 2008; Krull, 1987; Sinclair, 2002). Dreyfus and Eisenberg (1986) conclude that the following factors are elements of the aesthetic appeal of a mathematical solution: clarity; simplicity; brevity; conciseness; structure; power; cleverness; and surprise (p. 4). Sinclair (2002) tries to give more definition to the quality of aesthetics in mathematical work based on interviews with 70 mathematicians. Sinclair identifies five factors that "aesthetically motivate" mathematicians. These are "appeal, at both sensory and cognitive levels; surprise and paradox; identification; social 'marketing'; and, desire for 'the feeling'" (p. 72).

Studies are not limited to how mathematicians engage in their work, but also explore the lives, habits, avocations, beliefs, attitudes and relationships of mathematicians. For example, Laisant and Fehr, Eds[1] (1902) study mathematicians based on a list of interview questions (as cited in Hadamard, 1945, p. 137) related to “the mental habits and methods of work of different mathematicians" (Poincaré, 1908, p. 2047). In addition to questions about personal history, experience, work with and views on mathematics, the survey includes a list of questions on “daily habits,” such as, “what do you consider the normal amount of sleep necessary?” and “would you say that a mathematician's work should be interrupted by other occupations or by physical exercises which are suited to the individual's age and strength?” In this case, at least, an investigation into mathematicians’ extra-mathematical practices seems important to the investigator.

Of particular interest to this review, are the questions posed by Laisant and Fehr, “do artistic and literary occupations, especially those of music and poetry, seem to you likely to hamper mathematical invention, or do you think they help it by giving the mind temporary rest?” and, “what are your favorite hobbies, pursuits, or chief interests, aside from mathematics, or in your leisure time?” (as cited in Hadamard, 1945, p. 139). This seems to be an example of the approach of investigating a range of activities of the mathematician to gain insight into his/her mathematical processes.

Albers, Alexanderson and Reid (1985 and 1990) and Albers and Alexanderson (2012), through a series of studies consisting of interviews of prominent 20th century mathematicians, seek to share mathematicians’ own words on their backgrounds, life stories, worldviews, how they became interested in mathematics, how they chose their specialty, their hobbies and personal lives. A number of studies into particular groups of mathematicians have been conducted. Murray (2000) conducted extensive interviews with women who graduated from

[1] “An Inquiry into the Working Methods of Mathematicians.” Translated from L'Enteignement Mathematique, Laisant and Fehr (Eds.). Vol. IV, 1902 and Vol. VI, 1904, as cited in Hadamard (1945)

American institutions and practiced as mathematicians in the post World War II years. Murray seeks to examine how the identities of these women as mathematicians evolved through their careers.

Burton (1999a) interviewed 70 mathematicians in order to "try and unpack the attitudes, beliefs and practices of research mathematicians when engaged upon a research problem" (p. 121). Burton focused on questions related to how mathematicians feel about what mathematics is, what mathematicians are, the mathematical communities to which they claim membership, whether research is an individual pursuit or done collaboratively and how mathematicians describe the experience of "knowing" something in mathematics.

Sinclair (2002) elaborates on Burton's (1999a) study by analyzing the responses of the mathematicians for evidence of aesthetic criteria in the "evaluative, generative and motivational" aspects of their mathematical work (p. 13).

When explaining the goal of studying the lives, ways of thinking, choices and beliefs of mathematicians, researchers claim that understanding mathematicians is necessary to bridge the gap between school mathematics and professional mathematics (Brown, 1996; Burton, 1999a; Dreyfus & Eisenberg, 1986; Montano, 2014; Sinclair, 2002; Silver & Metzger, 1989). Darby and Catterall (1994) claim that when studying the role of aesthetics in mathematical work, researchers are contributing to the discussion of the importance of including the arts as a core part of the educational curriculum.

A review of the literature on mathematicians' own impressions of their practices of mathematics and studies by others of mathematicians' work processes in mathematics show the relevance of studying the lives and beliefs of mathematicians from a broad, holistic perspective, and focusing on the study of aesthetic aspects in mathematics work. I now turn to a review of the literature on two aspects of the connection between mathematics and music in particular. The first section reviews literature on the connection between mathematics/mathematicians and music. The second section reviews recent studies of a correlation or causal relationship between the study of music and mathematics performance.

2.4 Connection Between Mathematics/Mathematicians and Music

There is a long history of a documented connection between mathematics/mathematicians and music. In addition to their both being considered by practitioners as arts, there are philosophical justifications given by mathematicians for a deep linkage between music and mathematicians. Further, there is ample empirical evidence of mathematicians' involvement in the enjoyment and production of music.

Archibald (1924) suggests a few reasons why mathematics and music should be, as Helmholtz (as quoted in Archibald, 1924, p. 1) claims, "bound together."

> "Bound together?" Yes! in regularity of vibrations, in relations of tones to one another in melodies and harmonies, in tone-colour, in rhythm, in the many varieties of musical form, in Fourier's series arising in discussion of vibrating strings and development of arbitrary functions, and in modern discussions of acoustics. (p. 1)

When examining Leibniz' claim that "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers" (as quoted in Archibald, 1924, p. 1-2), Archibald (1924) asks, "Does Leibniz go too far?" Archibald qualifies this by stating "...in a very general conception of art and science, its verity may well be granted; for, in creating as in listening to music, there is no realization possible except by immediate and spontaneous appreciation of a multitude of relations of sound" (p. 1-2). This draws to mind Hardy's (1940) characterization of the mathematician as "a maker of patterns...of ideas" (p. 2033).

The English mathematician James Joseph Sylvester (1814-1897) referred to the “cultures of mathematics and music...not merely as having arithmetic for their common parent but as similar in their habits and affections...May not Music be described," Sylvester writes, "as the Mathematic of Sense, Mathematic as the Music of reason? the soul of each the same! Thus the musician feels Mathematic, the mathematician thinks Music, - Music the dream, Mathematic the working life, - each to receive its consummation from the other when the human intelligence, elevated to the perfect type, shall shine forth glorified in some future Mozart-Dirichlet, or Beethoven-Gauss - a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz!" (as quoted in Archibald, 1924, p. 2).

This musico-mathematic connection is evidenced in the eulogy Nicholas Fuss, student, son-in-law and secretary to Leonhard Euler (1707-1783), gave for his father-in-law (as quoted in Yushkevich’s, Bogolyubov’s and Mikhailov’s, Eds.). Fuss describes Euler’s enjoyment of music as follows:

> Euler’s chief relaxation was music, but even here his mathematical spirit was active. Yielding to the pleasant sensation of consonance, he immersed himself in the search for its cause and during musical performances would calculate the proportion of tones. (as cited in Pesic, 2013, p. 36)

There is widespread evidence of mathematicians’ involvement in music. Archibald (1924) describes as “widely recorded in scattered sources...the manner in which music, as an art, has played a part in the lives of some mathematicians” (p. 3). Renowned mathematicians who were actively involved in music, as listeners, players, performers, conductors and composers, include the 17th century mathematician Maupertuis, the 18th century mathematician William Herschel and the 19th-20th century mathematicians Jacobi, Grassmann, János Bolyai, De

Morgan, G. B. Mathews, Poincaré, Emile Lemoine, as well as colleagues of Archibald in the mathematics departments of the Universities of California, Chicago and Iowa, and of Cornell University (Archibald, 1924).

2.5 Mathematicians Writing About Music

It is well documented that mathematicians have written on musical topics since ancient times and have contributed prolifically to the advancement of the field.

Archibald (1924) acknowledges that “many mathematicians wrote on musical matters” (p. 3), and that “It is...not surprising that many mathematicians wrote on musical matters” (p. 2) since for two thousand years music was regarded as a mathematical science.

The ancient Pythagorean practice of viewing the study of mathematics as consisting of the four fields of geometry, arithmetic, music and astronomy (Heath, 1921, as cited in Archibald, 1924, p. 12), later became the “quadrivium” (according to Montano, 2014, p. xi, a term coined by the sixth century philosopher Boethius). The quadrivium was the classic curriculum of European universities in the middle ages through at least the 17th century. In the quadrivium curriculum, the study of music focused mainly on the mathematics of the subject, as well as mystic properties of its numbers (Archibald, 1924). The result of this multi-pronged curriculum is that classically trained scholars were studying and advancing both the fields of mathematics and music as a matter of course.

Archibald (1924) traces the twin development of mathematics and music from the time of the ancient Greeks through modern day, discussing the many mathematicians who contributed to the evolution of the science of music.[2]

The renowned mathematician Leonhard Euler (1707-1783) spent significant time thinking and writing about mathematical approaches to music, from his earliest notes and one of

[2] See Archibald for an excellent discussion on this.

his first main works, “*Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae” (*“A attempt at a new theory of music, exposed in all clearness according to the most well-founded principles of harmony”*)* (“*Tentamen*”), written in 1731 and published in 1739, to papers published over thirty years later. Euler’s seminal work, “*Tentamen*,” presents Euler’s mathematical theory of music. Topics of other works include the theory of vibrating bodies, acoustics, musical harmony, composition theory and the debate on tuning of instruments, comprising more than 30 works.[3] Within these, Euler also reveals his views on the aesthetic nature of mathematics and aesthetics in general.

Above I discussed mathematicians reflecting on and attempting to explain the aesthetic nature of mathematics. To Euler, perfection is order. “When we hear music we perceive the order which is possessed by both the simultaneous and the successive sounds” (Smith, 1960, p. 70). In fact, Euler saw mathematics as the means for augmenting beauty in any aesthetic area. “If those things which constitute order can be quantified and expressed in numbers, the treatment described herein will have great usefulness in all other matters in which beauty and order are present, as, for example, in architecture” (Smith, 1960, p. 86).

Euler justifies his interest in music theory based on his view of mathematics being the root of aesthetic pleasure. As Euler describes in a letter to Daniel Bernoulli written in May 1731:

> My main purpose was that I should study music as a part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing. (as quoted in Smith, 1960, p. 9)

[3] These include a "Physical Dissertation on Sound" (1727), ‘‘Conjecture on the Reason for Some Dissonances Generally Heard in Music’’ (1760), ‘‘On the True Character of Modern Music’’ (1764), many sections in Euler’s series of letters on natural philosophy, addressed to a German princess, the Princess of Anhalt Dessau (1768), and ‘‘On the True Principles of Harmony Represented in the Mirror of Music’’ (1773).

In fact, in a letter, Euler describes the pleasure in music as “the enjoyment of solving (mathematical) riddles” (Knobloch, 2008, p. 22).

The above review shows a long-standing connection between the fields of mathematics and music, including interest by mathematicians in studying music mathematically and appreciating music. The above also points to some potential intrinsic links between mathematics and music.

2.6 Studies on the Effect of Musical Training on Mathematics Performance

After examining views and experiences of mathematicians on the topic of a link between aesthetics, music and mathematicians, I now turn to empirical studies of psychologists, neuroscientists and educators on whether there is a correlational or causal effect of studying music on mathematics performance.

In this section I review studies that examine the relationship between music training, with a focus on instructional music, and mathematics performance. The studies in this review use a variety of methods. One approach is to examine the relationship between historical instructional music and mathematics performance, measured by an assessment or standardized test results (Bahr & Christensen, 2000; Fitzpatrick, 2006; Cox and Stephens, 2006; Kinney, 2008). A second approach is to examine the relationship between an instructional music intervention and mathematics performance, measured by a mathematical assessment (Costa-Giomi, 1999; Graziano, Peterson & Shaw, 1999; Rauscher & Zupan, 2000). I also review two meta-analyses on the topic of the impact of instructional music on mathematics performance (Hetland, 2000; Vaughn, 2000).

Researchers present theories to explain the rationale for a proposed effect of musical training on mathematics performance. Theories consist mainly of "neural connections" theories and "near transfer" theories (Hetland, 2000, p. 180).

"Neural connections" theory, proposed by Shaw and others (Shaw, 2000, as quoted in Hetland, 2000, p. 180), claims that because of neurological connections in the cortex, development of certain kinds of musical and spatial abilities are related. An outgrowth of this is the hypothesis that "music training at an early age is exercise" for brain functions such as spatial-temporal reasoning (Leng and Shaw, 1991, as quoted in Hetland, 2000, p. 180).

Parsons and colleagues (Parsons & Fox, 1995, as referenced in Hetland, 2000, p. 180) propose an alternative link between music and mathematical processes that occurs in the brain. Their "rhythm" theory suggests a neurological connection between music and spatial processes that require mental rotation, both processed in the cerebellum. Cox and Stephens (2006) also draw on research showing how musical training affects the structure of the brain.

“Near transfer” theories of the connection between music and mathematics ability are based on education theories. According to Gardner’s (as referenced in Hetland, 2000, p. 181) theory of multiple intelligences, because instrumental music requires coordination of at least six of the intelligences he defines (musical, visual-spatial, bodily-kinesthetic, logical/ mathematical, interpersonal and intrapersonal), training in music enhances these skills and also carries over into other uses of these skills. Hetland (2000) summarizes this as “Because music-making and spatial abilities are both multi-dimensional processes, we might expect a range of spatial skills to improve because of direct practice during music instruction” (p. 181).

Others propose a knowledge transfer process between music and spatial abilities in particular (Costa-Giomi, 1999; Graziano, et al., 1999; Rauscher & Zupan, 2000). Rauscher and

Zupan (2000) explain this as based on the fact that musical pieces are organized both spatially and temporally. Bahr and Christensen (2000), drawing upon Structural Analysis, claim that there are deep structural overlaps between the domains of music and mathematics, particularly in the areas of patterns and symbols. They claim that reading music involves recognition of specialized symbols and conversion of these into action and that both musicians and mathematicians learn to abstract from and comprehend complex symbolic representations.

Among the studies that assess an impact of historical instructional music training on mathematics performance, the majority of the results were significantly positive although with small effect size. Across these studies, the ages of the populations varied from fourth, sixth and ninth graders (Fitzpatrick, 2006) and middle school students (Kinney, 2008) to tenth graders (Bahr & Christensen, 2000) and high school students (Cox & Stephens, 2006). Sample sizes ranged from 85 to 15,431. The authors assessed instructional music in different ways. Measures of instructional music included knowledge of musical ideas assessed through an assessment tool (Bahr & Christensen, 2000) and enrollment in instrumental music courses in school (Cox & Stephens, 2006; Fitzpatrick, 2006; Kinney, 2008).

Among the studies that examine the relationship between instructional music and mathematics performance through a musical intervention and mathematics assessment the results were also significantly positive with small effect size for the majority of studies. The ages of the populations varied from kindergarteners (Rauscher & Zupan, 2000) to second graders (Graziano, et al., 1999) to fourth and sixth graders (Costa-Giomi, 1999). The authors provided a musical intervention- piano or keyboard lessons. The sample sizes were 62, 273 and 117 respectively.

Two meta-analyses examine the effect of instructional music training on mathematics performance. Hetland (2000) reviewed 15 independent studies and Vaughn (2000) reviewed 20 studies. Hetland (2000) conducted a main analysis combining the findings and probability levels from 15 studies (701 subjects) to test the hypothesis that active instruction in music, for periods ranging from six weeks to two years, enhances performance on spatial-temporal tasks during and immediately following (within a week of the instructional program) the music instruction for 3-12 year-old subjects. Hetland (2000) also conducted two smaller analyses that distinguish between nonspatial measures and spatial tasks. Hetland (2000) found that music instruction does appear to enhance spatial-temporal performance for preschool and elementary-aged children, at least while instruction is occurring and at least for up through two years of instruction. The effect is moderate and consistent across this population of studies. It can be generalized to similar populations of preschool and elementary-aged children, while they are engaged in similar kinds of active music programs, with or without keyboard instruments, taught in groups or individual lessons. Hetland calls it "a solid finding" (p. 220). Hetland (2000) explains that she "did not expect a larger effect for Meta-Analyses 1 and 3, because most successful educational interventions add incremental benefit rather than whole-scale change" (p. 184).

Vaughn (2000) reviewed 20 correlational studies, ten of which used data made available by the College Board on SAT math scores (between 1987-1998), and ten of which use data from comparatively small populations. In all cases, the studies compared some form of math achievement in students with and without self-selected music study. In all of the studies, students were trained to perform music either instrumentally or vocally. The comparison group consisted of students who had taken no arts courses of any kind. Vaughn (2000), when combining 20 studies with correlational designs, found a small association between the voluntary study of music and mathematics achievement. In addition, a small relationship was demonstrated when six studies were combined, showing that music training enhances math performance. However, Vaughn (2000) warns that "while it is conceivable that the music education received by the students in these studies actually led to improvement in math performance, other explanations for this correlation have not been ruled out" (p. 154).

The above studies point to a direct connection between mathematics and music and are useful in proposing theoretical explanations, rooted in neurology and psychology, for such a connection. However, the findings of the studies are limited in that they examine a very narrow aspect of musical involvement and over a relatively short time frame. It will be useful to conduct research that examines a wider aspect of musical involvement and takes a more holistic view of both the musical involvement and mathematical performance. For this reason, the current study, based on a case study approach, will add to the understanding of a relationship between musical involvement and mathematical thinking.

2.7 Conclusion

The literature cited above shows that affect is an important part of mathematical thinking and the relationship between the two warrants deeper study. It also shows that mathematicians recognize the important role of affective elements including factors "with a strong connection to the affective domain" (McLeod, 1992, p. 586), such as the aesthetic, intuition and personal taste, in their practices of mathematics. Further, the literature points to the great value to be gained from studying the lives of mathematicians from a holistic view, examining their non-mathematical activities, interests, behaviors and beliefs. The literature shows that there is a long history of a connection between mathematics/ mathematicians and the arts, particularly music. Within the literature, there are propositions of the nature of and reasons for this close connection. Finally, there is a recent body of literature studying the correlation between musical training and mathematical performance. Where this literature comes together for the purpose of the current study, is in recognizing that an understanding of mathematics practitioners and students from a broad, holistic perspective is instructional in understanding the nature of mathematical thinking. In particular, the current study focuses on understanding affect and mathematical identity in the contexts of the personal, familial, communal and artistic domains, with a particular focus on musical communities. Further, it extends the investigation into the nature of a connection between mathematics/ mathematicians and music, however, it does so by taking a holistic view of mathematicians' lives and beliefs. Finally, the study looks at a connection between musical involvement and mathematical thinking using a social and affective lens, rather than a neurological lens. In doing so, the study has implications for mathematics education in the classroom, the school and the larger community. Further, in doing so, I aim to contribute to "better-founded theoretical frameworks and a broader range of methodological instruments fit to interpret students' behaviour in mathematical activities" (Zan, et al., 2006, p. 116), identified by Zan, et al. as an important goal of mathematics education research.

Chapter 3: Methodology

3.1 Purpose of the Study

The purpose of the study is to understand and document the role of music in the lives and work of 20th century mathematicians. To do this, the study seeks to describe and compare the ways in which mathematicians are involved in music, whether and how music connects to mathematicians' work in mathematics and whether and how involvement in music can be considered a part of a mathematician's identity as a mathematician. The study examines mathematicians' beliefs related to both mathematics and music as part of this analysis. The study will contribute to an understanding of how affective elements, such as beliefs, values and identity, and involvement in the art of music, relate to mathematicians' practices of mathematics. The study will have implications for the education of future mathematicians.

To guide this study, the following research questions are examined:

1) What commonalities or differences are there among the ways different mathematicians involve themselves in music?
2) How do mathematicians relate their interest in music to their interest in mathematics, if at all?
3) How does involvement in music help shape mathematicians' identities as mathematicians, if at all?

3.1.a Theoretical Framework of the Study.

For the purpose of this study, I will focus on the more stable aspects of affect, such as beliefs, values and meta-affect, that demonstrate themselves in a consistent manner in subjects' mathematical engagement. Further, I examine identity, viewing it as a highly stable component of affect and influenced by one's social, cultural and communal context, as described by numerous researchers (see Chapter 2).

For the purpose of this study, I use Sfard and Prusak's (2005) definition of identity, emphasising the social, cultural and communal contexts for the formation of identity. In addition, I draw upon Bishop's (2008a) sociocultural perspective of mathematical thinking (described above). However, in addition, I add to the location within the socio-cultural dimension, the ontogenetic development dimension, described by Bishop. ("Ontogenetic development includes individuals' personal life histories, socially determined, which furnish the knowledge with which to interpret stimuli; this development includes participation in multiple overlapping communities" (p. 82).) Further, I add to the focus, Bishop's "individual" dimension of mathematics education, which is concerned with "individual learners' values and goals regarding mathematics, and mathematical thinking, which can differ markedly, and which do not necessarily follow the teachers' values and goals" (p. 83). This is used to examine the role of music, an activity in the sphere of the subjects' lives outside the mathematics classroom, in their mathematical practices.

The current study's investigation of factors outside of the classroom- learners' artistic, cultural and communal engagements formed personally, with their families and with communities outside the classroom, contributes to the literature on understanding how multiple social factors influence the development of affect that learners bring to the study of mathematics. Bishop (2008a) notes "we can never observe mathematical thinking─we can only observe what we assume to be its products, namely mathematical ideas and processes. But we can observe what conditions and contexts might have been responsible for the products of mathematical thinking, which brings us rather closer to the social context" (p. 79). In the present study, the "conditions and contexts" that I examine are the subjects' involvement in music in various ways. I connect these to the subjects' beliefs, values and identities that emerge in their practices of mathematics. In this way, I aim to contribute to the body of literature that tries to understand a broad range of elements that contribute to the development of mathematical thinking and have implications for mathematics education.

3.2 Procedure

This is a qualitative study based on an examination of beliefs, values and identity, using a multiple case study approach. As in qualitative research, as described by Merriam (2015), the methodology for this study is characterized by the following four aspects: "the focus is on process, understanding, and meaning; the researcher is the primary instrument of data collection and analysis; the process is inductive; and the product is richly descriptive" (p. 15).

A qualitative approach is most appropriate for this study since the focus is on gaining first hand information from mathematicians about their views and beliefs about the role of music in their lives. The study aims to use the words of the subjects or of individuals close to the subjects to gain insights into the subjects' beliefs and experiences with music and mathematics. This is in line with how Merriam (2015) explains the goal of qualitative research: "qualitative researchers are interested in how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences" (p. 15). Since the study is prioritizing the subjects' own words and experiences, and is putting the mathematicians' words into the contexts of their personal histories and experiences, a qualitative approach is most appropriate. In addition, the process of the study is inductive- the goal of the study is to gain understanding based on the experiences of the subjects as reflected in their behavior and as described in their own words.

The particular qualitative methodology used in this study is the multiple or comparative case study (See Merriam, 2015 and Creswell, 2007). Creswell (2007) describes case study research as:

> a qualitative approach in which the investigator explores a bounded system (a *case*) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving *multiple sources of information* (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case *description* and case-based themes. (p. 73)

This study uses a multiple case study approach since the focus is on bounded systems, each being the experiences and views with respect to music and mathematics of a mathematician in his or her own life. I conduct two stages of analysis- the within-case analysis and the cross-case analysis (Merriam, 2015). I first treat each case individually and then "seek(s) to build abstractions across cases" (Yin, 2014, as cited by Merriam, 2015, p. 234).

The study uses the multiple case study approach to allow for identification of themes across cases. Merriam (2015), quoting Miles, Huberman, and Saldaña (2014), explains, "the more cases included in a study, and the greater the variation across the cases, the more compelling an interpretation is likely to be" (p. 233). This study examines six subjects.

3.3 Setting and Population

The six subjects of the study are mathematicians of significant stature (as will be defined below), who lived and practiced mathematics in the 20th century, and had a demonstrated involvement in music (as will be defined below). To choose case study subjects, purposeful selection was used. According to Merriam (2015), "Purposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (p. 96). Patton (2015, as cited by Merriam), specifies these as "*information-rich cases*" (p. 96).

The criteria for selecting the information-rich cases for this study were as follows. The first criterion is that the mathematician lived and practiced in the 20th century. This is to examine a population that could provide insight into the lives, work and education of mathematicians in current times. The second criterion is that the mathematician achieved significant stature in the field of mathematics. Significant stature is defined, for the purpose of the study, as the mathematician having been recognized as contributing in a significant way to the field of mathematics, through receiving a major mathematics prize or being recognized by one's peers. In order to apply a simple and transparent standard, the way this criterion was applied, is that a population was formed of all mathematicians who were recipients of a leading prize in mathematics, including the Fields Medal, Wolf Prize in Mathematics and Abel Prize in Mathematics. In addition, the population contained a selection of mathematicians who headed departments of mathematics in research institutions. Certain mathematicians were included in the population because they had a demonstrated interest in music (which will be defined below) and also were recognized by their peers in research journal articles as having made significant contributions to mathematics.

To this population, the third criterion, that the subject have a demonstrated interest in music, was applied. A demonstrated interest in music is defined, for the purpose of the study, as the subject having either played a musical instrument, listened to music as a hobby and/or referred to music as a metaphor, in writings or speeches, for the practice of mathematics, on numerous occasions. All genres of music were considered.

Out of the 142 winners of the Fields Medal, Wolf Prize and Abel Prize through the year 2019, there are 114 unique winners. Of these, 43 demonstrated interest in music, about which information was available in public sources. To this 43, I added an additional 13 mathematicians who demonstrated interest in music, and about which there was information available in public sources. Based on the above criteria, a pool of 56 candidates for case studies was created. Of these 56 candidates, only six mathematicians had ample public information available about their musical involvement. Since the methodological choice for this study was to rely on public information available about subjects, with the goal of creating information-rich cases, I prioritized the availability of abundant information about the subjects in their selection. This resulted in a limitation of the study which I will discuss further in Chapter 5.

The six case study subjects selected are: Michael Atiyah (1929-2019), Richard Courant (1888-1972), Israel Gelfand (1913-2009), Israel Glazman (1916-1968), Hans Lewy (1904-1988) and Hassler Whitney (1907-1989), described further in Table 1.

Table 1: Case study subjects and selection criteria.

20th Century Mathematician Subject	Major Mathematics Prize Received	Leadership of Mathematics Research Institution	Involvement in Music
Michael Atiyah	Fields Medal (1966) Abel Prize (2004)	Master of Trinity College, Cambridge (1990–1997)	Co-author of paper on mathematical beauty (Zeki, Romaya, Benincasa & Atiyah, 2014). Connoisseur of classical music. Referred to music as a metaphor for the practice of mathematics.
Richard Courant		Founder and Director of the Mathematical	Lifelong amateur piano player. Played regularly in

		Institute at University of Göttingen (1928-1933). Founder and Director of The Courant Institute of Mathematical Sciences at NYU.	self-organized chamber groups. Connoisseur of classical music. Referred to music as a metaphor for the practice of mathematics.
Israel Gelfand	Wolf Prize (1978)		Connoisseur of classical music. Referred to music as a metaphor for the practice of mathematics.
Israel Glazman		Chairman of Department of Mathematical Physics at Kharkov (1955) Founded Department of Functional Analysis and Applied Mathematics at the Institute of Low Temperature.	Professional level violinist. Referred to music as a metaphor for the practice of mathematics.
Hans Lewy	Wolf Prize(1986)		Professional level violinist. Lifelong violin player. Played regularly in chamber and orchestral groups.
Hassler Whitney	Wolf Prize(1983)		Professional level pianist. Lifelong piano player. Played regularly in chamber and orchestral groups.

3.4 Instruments and Data Collection

The data for the study are collected from documents and published video or audio interviews with the subjects. The document database consists of writings and transcribed speeches by the subjects or about the subjects by family members, colleagues, students or friends. In addition, published video and audio interviews, or their transcriptions, of the subjects are included. The focus of data collection was on biographical, professional, philosophical, personal and anecdotal information about the subjects, touching on all aspects of the subjects' personal and professional lives. There is a particular emphasis on data on involvement in music. Included in the documents are introductions to technical mathematics writings that describe the philosophy of the subjects as well as writings that describe the overall focus and direction of a subject's mathematical work. Not emphasized in the data collected are the subjects' technical mathematical writings themselves. For one subject, Richard Courant, data were collected from the Richard Courant Papers at the New York University Archives. I reviewed all personal letters from and to the subject within the time period 1930-1971.

The documents mined for the database of documents for each case study are the following:

1) Biographies
2) Reviewed journal articles written by or about the subject
3) Books or book chapters in edited volumes written by or about the subject
4) Introductions authored by the subject to books authored by the subject or by others
5) Introductions authored by others to books authored by the subject
6) Published interviews with the subject, including audio or video recorded or transcribed
7) Published talks given by the subject

8) Material from university or professional websites dedicated to the subject
9) In the case of one subject, archival letters written by and to the subject

All documents in the above categories were examined for data about each of the subjects. All documents are listed in the reference section of this study.

3.5 Data Analysis

Data from the sources for each subject consist of:

1) The subject's personal history, education and work in mathematics
2) The history and source of the subject's interest in/ education in music
3) The nature of the subject's involvement in music throughout his life
4) Clues to the subject's feelings about music in his life
5) How the subject relates to others, particularly other mathematicians, through his involvement in music
6) How the subject connects music to his work in mathematics
7) The subject's views on the process of doing mathematics
8) The subject's views on the process of mathematics education

All data collected were organized by case subject. Data analysis for a qualitative study is "inductive and comparative in the service of developing common themes or patterns or categories that cut across the data" (Merriam, 2015, p. 297). Further, qualitative data analysis happens at the same time as data collection. Merriam suggests using the constant comparative method, proposed by Glaser and Struss (1967, as cited by Merriam, 2015, p. 201), for analyzing qualitative data as it is inductive and comparative. This consists of identifying themes as the research progresses by identifying "units" of data and continually comparing new data with existing data to arrive at categories or themes that cut across all of the data for the study.

The constant comparative method was employed in this study in the following way. I reviewed all the documents related to a specific subject and then prepared a case study on that subject, before moving on to a next subject. In preparing the case study, I identified “units” of data related to the first subject. These took the form of topics into which information about each subject could be organized. The case study was prepared according to these topics. Then, I moved on to the next subject seeking to organize the case study into similar topics. As Ir came across information that did not fit into an existing topic, I created additional topics. Then, I went back to the first subject and reorganized the case study according to the expanded set of topics. I continued this process in preparing case studies of all six subjects until arriving at a method of organization for all cases.

The topics into which I ultimately organized the individual case studies are the following:

1) Brief biographical sketch
2) The subject’s experience with music
3) The subject’s mathematics education
4) The subject’s professional practice of mathematics
5) The subject’s philosophy of mathematics
6) The subject’s work in mathematics education

While conducting the research, I identified commonalities among the subjects within each topic. These commonalities become the themes that cut across the subjects. I went back and forth between the data on each subject to then code quotes, behaviors or anecdotes about each subject according to these themes. In this process, I arrived at new themes. I ultimately arrived at a collection of themes that seemed most representative of all of the subjects. These themes provide insight into the beliefs, values and identities the subjects have in common that provide answers to the research questions.

In examining the role of music in the life and works of mathematicians, I took a holistic view of a mathematician's experience with, practice of and philosophy of mathematics. Therefore, I investigated the full spectrum of each subject's life and works.

Using the constant comparative method, I identified the following themes on the subjects' relationship with music and mathematics:

1) Engaging in music socially, particularly with mathematician colleagues and students
2) Music being a lifelong companion, along with mathematics.
3) Viewing the practice of mathematics as an art, prioritizing beauty in the practice of mathematics and drawing upon the art of music as a metaphor for the practice of mathematics

In addition, I also looked for themes in the way the subjects practiced mathematics, in order to gain a holistic view of the subjects. I used these themes to understand any further connections between subjects' involvement in music and practice of mathematics. The themes related to the practice of mathematics that emerged in common to the subjects are the following:

4) Desiring to be collaborative and social in practice of mathematics
5) Desiring to connect mathematics to other areas of mathematics and recognizing the "unity of mathematics"
6) Being committed to mathematics education, often viewing it as a moral imperative
7) Emphasizing personal initiative in learning and understanding mathematics
8) Using exploration and play in practice of mathematics
9) Being able to seek out the "big picture" in mathematics and to simplify complex ideas

10) Valuing delving into specific examples to gain deeper understanding of mathematics
11) Believing that mathematics must remain connected to its applications

The connection between the themes in the subjects' practice of mathematics and the way they engage in music is explored below.

Based on comparisons among mathematicians and the themes, I will answer the research questions and draw conclusions about the role of music in the lives and work of mathematicians, particularly as it sheds light on beliefs, values and the identities of mathematicians. I will also discuss its implications in educating future mathematicians.

3.6 Validity and Reliability

In examining the issues of validity and reliability in qualitative studies, Firestone (1987, as cited in Merriam, 2015) claims that the qualitative study satisfies these by providing "the reader with a depiction in enough detail to show that the author's conclusion 'makes sense'" (p. 238). This study employs four techniques commonly used to enhance validity, reliability and generalizability (Merriam, 2015). These are triangulation, adequate engagement in data collection, rich, thick language and multiple case study.

Triangulation improves the internal validity of a study by using "multiple methods, multiple sources of data, multiple investigators, or multiple theories to confirm emerging findings" (Denzin, 1978, as cited in Merriam, 2015, p. 244). This study uses multiple sources of data. Data consist of documents written by the subject and about the subject, as well as published interviews, in different contexts and at different time periods of the subject's life.

Adequate engagement in data collection is another technique for improving validity and reliability. This is useful "when you are trying to get as close as possible to participants' understanding of a phenomenon" (Merriam, 2015, p. 246). According to Merriam, an indication that a researcher has collected enough data is when “the data and emerging findings...feel saturated; that is, you begin to see or hear the same things over and over again, and no new information surfaces as you collect more data” (p. 246). The data for this study feel saturated since themes started to repeat as more data were reviewed and eventually no new themes emerged as I continued to review all the data for the study.

Generalizability, in the traditional sense, is not present in this study due to its qualitative nature. As in qualitative research, this study explores a phenomenon with the goal of generating new knowledge and understanding, rather than testing theory. However, an alternative framework for thinking about generalizability for this study is reader, or user generalizability, which Merriam (2015) calls, “probably the most common understanding of generalizability in qualitative research” (p. 256). This “involves leaving the extent to which a study’s findings apply to other situations up to the people in those situations. The person who reads the study decides whether the findings can apply to his or her particular situation” (p. 256). This fits well with Eisner’s (1998, as cited in Merriam, 2015) approach, in which he views the purpose of qualitative research findings as expanding the reader’s understanding of the possible.

Reader generalizability is supported in this study through rich, thick description (Gilbert Ryle, 1949, as cited in Merriam, 2015, p. 256). Rich, thick description, used today, “refers to a description of the setting and participants of the study, as well as a detailed description of the findings with adequate evidence presented in the form of quotes from participant interviews, field notes, and documents” (Merriam, p. 257). This study prioritizes the subjects’ own words in conveying their experiences and makes an effort to provide a holistic view, including multiple aspects of subjects’ lives and experiences.

Finally, this study supports generalizability through the multiple case study approach. According to Miles, Huberman, and Saldaña (2014, as cited in Merriam, 2015), this approach intrinsically strengthens "the precision, the validity, and the stability of the findings" by providing multiple, comparative examples (p. 233).

3.7 Translation Issues

This study involves the analysis of archival letters originally written in German. Portions of the letters relating to the subject's involvement in music were translated by a professional translator. I also used the online translation software www.deepl.com to translate these sections. I then compared the translations from the two sources, in order to check the reliability of the translation. The original German text of the letters is provided in the appendix, while the English translation is provided in the body of the text.

Chapter 4: Results

4.1 Overview

Below I address the research questions based on case studies of six 20th century mathematicians. These are Michael Atiyah (1929-2019), Richard Courant (1888-1972), Israel Gelfand (1913-2009), Israel Glazman (1916-1968), Hans Lewy (1904-1988) and Hassler Whitney (1907-1989).

Michael Atiyah is best known for his work in algebraic topology and the codevelopment of topological *K*-theory and the Atiyah–Singer index theorem, along with Isidore Singer. Atiyah received the Fields Medal (1966) and Abel Prize (2004), along with Isidore M. Singer.

Richard Courant studied under David Hilbert at Gottingen and served as Hilbert's assistant. Courant took over for Felix Klein, upon Klein's retirement, as a Professor at the famous university at Gottingen. Courant founded and served as Director of the Mathematics Institute at Gottingen. Later, Courant established the Courant Institute of Mathematical Sciences at New York University.

Israel Gelfand studied under Andrei Kolmogorov at Moscow State University. Gelfand was a member of many distinguished academies of science and a recipient of the Wolf Prize in Mathematics (1978), the AMS Steele Prize (2005) and numerous other mathematical prizes.

Israel Glazman studied at Odessa University under Mark Krein and later chaired the department of Mathematical Physics at the University of Kharkov. Later Glazman organized a Department of Functional Analysis and Applied Mathematics at the Institute of Low Temperature.

Hans Lewy, a student of and assistant to Courant at Gottingen, also studied under Hilbert, Born, Noether and Franck, as well as Hadamard in Paris. Lewy received the Steele Prize (1979) and the Wolf Prize in Mathematics (1986).

Hassler Whitney studied Physics and Music at Yale and received his PhD in Mathematics at Harvard, studying under George David Birkhoff. Whitney received the U.S. National Medal of Science (1976), the Wolf Prize in Mathematics (1983) and the Steele Prize (1985).

The six subjects achieved significant stature in the field of mathematics and were involved in music and/or spoke or wrote about music in relation to mathematics. The first and second research questions below explore in what ways the subjects were involved in music and relate music to their work as mathematicians. The third research question explores whether and how subjects view their mathematical identities as bound up with musical identities.

The study finds that the way the subjects are involved in music is personal and social, connecting them to communities of other mathematicians. The results further show that the subjects connect their musical and mathematical identities in that subjects compare themselves and other mathematicians to musicians and characterize the mathematician as an artist, of which the musician is a subset. Further, the results show that the beliefs and values of the subjects in their practices of mathematics are analogous to the beliefs, values and behaviors they express in their involvement in music. Engagement in music and mathematics is a way to hone and express their aesthetic interests and is a way to express the core values that underlie their mathematics practices.

Appendix B contains a portion of the full case study for each subject- the portion providing a brief biographical sketch and professional overview. The remaining sections of the individual case studies for the subjects are interwoven into the responses to the research questions below.

4.2 The First Research Question

1. What commonalities or differences are there among the ways different mathematicians involve themselves in music?

All of the subjects were selected because they had a demonstrated involvement in music. The ways subjects were involved in music includes through playing an instrument, participating in chamber or orchestral music, being a connoisseur of classical music and through using music as a metaphor for the practice of mathematics. Three of the subjects (Glazman, Lewy and Whitney) were trained as serious musicians, two of which (Glazman and Lewy) considered alternative careers in music. Another subject, Courant, was a lifelong amateur musician. Courant, Lewy and Whitney played music throughout their lives. They shared the playing of instrumental music with their families, colleagues and students. Gelfand and Atiyah, while not having played a musical instrument, were great connoisseurs and admirers of classical music. They shared their appreciation of musical recordings with colleagues and students. Gelfand and Atiyah, along with Courant, reference music widely as a metaphor for the practice of mathematics and draw explicit connections between music and mathematics (this last point will be explored further in response to the second research question).

Glazman was a talented violinist and chose a career in mathematics over a career in music. Lyubich and Tkachenko (1997) explain:

> He (Glazman) had studied the art of violin playing under P.S. Stolyarskii, at the famous Odessa Musical School which gave David Oistrakh and Boris Goldstein to the world. Glazman was the favorite student of Stolyarskii who presented him with the thing he

> cherished most of all, his violin....it took I. Glazman some time to make his final choice between mathematics and music. (p. 4)

Lewy was a serious musician who continued to play music throughout his life. At five years old, Lewy took up the violin after, of his own initiative, he began joining in on his older sister's piano lessons. He learned to play the violin, viola and piano "with mastery" (Lewy, Helen, 2002, p. xxi) and also learned to play the clarinet and the cello. As a pre-teen Lewy studied with a well-known concert violinist, and as a teen played regularly with chamber music groups (Lewy, Helen, 2002). Lewy made two public appearances on the violin at age fifteen. One of the reviewers called Lewy a "virtuoso." In describing the experience in a letter to his parents, Lewy states, "by the second (piece) I could hardly wait to start playing" (Lewy, Helen, 2002, p. xxii).

In Gottingen, while studying at the University, after making the decision to pursue a career in mathematics, Lewy played with the city orchestra and participated in weekly musical evenings at friends' houses, including participating in chamber evenings at the home of Lewy's advisor in mathematics, Richard Courant. Lewy also composed many string quartets (Lewy, Helen, 2002). According to Helen Lewy (2002), Lewy "made music almost every day of his life," (p. xxiii) and played piano until the end of his life.

In commenting on the choice Lewy made as a young man to pursue a career in mathematics rather than music, Lewy's wife, Helen Lewy (2002) states, "Mathematics and music remained the twin passions of his life, and, in the end, I believe he felt he had been lucky enough to have had the best of both worlds" (p. xxiii). According to Helen Lewy, music was "always the companion of the math in his life" (Lewy, Helen., 2002, p. xxi).

Whitney also was a proficient musician. Whitney showed an early interest in music, and was influenced early by his aunt, an accomplished pianist. According to Kendig (2018), "His aunt's influence went beyond the piano, for often the house was filled with chamber music, and that evolved into one of Hassler's great loves. Later, he began to study the violin and eventually the two of them would play violin-piano sonatas" (p. 18).

Whitney took piano lessons through high school. During college, Whitney majored in music along with physics and decided to master the violin so he could play the Beethoven quartets that so moved him. At Yale, Whitney won "best composition of the year." Whitney also learned to play the viola (Kendig, 2018, p. 297).

Whitney shared music with his family. Whitney's first wife studied music as an undergraduate and two of Whitney's children were interested in music. Whitney exposed his children to Sunday opera. Whitney's daughter Carol has a Ph.D. in ethnomusicology and plays multiple instruments. Whitney's daughter Molly plays the flute (Kendig, 2018). Whitney attended the summer chamber music camp held at Bennington College in Vermont each summer along with his daughter.

Whitney regularly hosted chamber group sessions in his home, inviting colleagues to participate. Whitney played music throughout his life. In Whitney's later years he would "end each day by playing on the piano Chopin's Berceuse in d-flat major, Op. 57, a set of sixteen variations on a hauntingly beautiful four measure melody" (Kendig, 2018, p. 309). Also, "He (Whitney) was especially attracted to Bach and throughout his life worked to master a variety of his keyboard compositions" (p. 297). Commenting on Whitney's taste in music, Kendig notes that Whitney was "fairly conservative...In his earlier years he would always listen to operas on Sundays—Wagner was a favorite. He intensively practiced Bach on the piano, and in chamber music we usually played Mozart, Beethoven, Brahms, or Schubert. In his later years, he developed a special fondness for Chopin, and also greatly appreciated Debussy" (p. 295).

Whitney commented to Kendig, "Music continually runs through my head" (Kendig, 2018, p. 22).

Courant was self taught in the piano while a university student. Courant was introduced to music by a fellow student and it became a social activity for him. Reid (1996) describes, "Julius Stenzel, (a fellow student and violinist) encouraged him (Courant) to accompany them to chamber music concerts. At some point during this period, he rented a piano and taught himself to play. Stenzel also introduced Richard into a lively circle of young artists, musicians, writers, and students" (p. 12). In Gottingen, Courant "practic(ed) regularly and tr(ied) to concentrate on fundamentals" (p. 18).

Courant shared his love of music with his wife Nina Runge, a musician, daughter of the mathematician Carl Runge from Gottingen. Aleksandrov and Oleinik (1975) describe Richard and Nina Courant as "profound and serious experts and connoisseurs of music" (p. 165). The Courants also shared this appreciation of music with their children. The Courants hosted regular chamber music evenings in their home with mathematician colleagues and students, in which the children participated. One of Courant's daughters became a professional musician (Lax, 2003).

Courant used piano playing as a recreation, even when not playing chamber music in groups. Nina Courant describes that, "at home he (Courant) played a great many Bach fugues (as quoted in Reid, 1996, p. 181). Further, piano "had been a great source of pleasure for him (Courant) in the past, a way of transcending conflicts and disappointments" (Lax, 2003, p. 92).

Gelfand did not play a musical instrument, as far as it appears from available sources, but he was a great lover of music and certainly spent many recreational hours listening to classical music. Gelfand's student, Dusa McDuff (2013, February), recalls Gelfand sharing his love of classical music with her. "Once he (Gelfand) took me shopping: he bought and gave me all the good classical records he could find. Very cheap, they contained wonderful performances by Russian musicians that he thought I should hear. He said that this too was 'teaching me mathematics'" (p. 163).

Atiyah himself did not play a musical instrument. However, he did enjoy listening to classical music as a recreation. After suffering a mild stroke in his late years, Atiyah describes that he "spent the rest of the day listening to Bach" (Farmelo, 2018, Time: 2:40). In addition, he was very familiar with classical music as he references it widely as an analogy to mathematics, as will be shown below.

4.2.a Social engagement in music.

The subjects' involvement in music was highly social. In addition to their personal dedication to playing and listening to music, the subjects shared music with their families and used music as a social connection with other mathematicians, colleagues and students. As will be shown below, in the particular cases of Courant, Lewy and Whitney, one can see the roles that regularly hosted, chamber music evenings and a shared love of music played in connecting mathematicians and fostering mathematics work. These evenings were occasions to welcome newcomers into a mathematical community, deepen bonds between mathematicians and discuss mathematics. As will further be shown below, in the cases of Gelfand and Whitney one can see how a mathematician's appreciation of music, when shared with a student, is a way for the mathematician to deepen the social and mathematical connection with the student.

The **Courants** regularly hosted chamber music evenings in their home in which Courant and Nina played, as well as students, colleagues and friends. These social evenings were very memorable to people at Gottingen. Reid (1996) describes, “at the Courants' house there was an unending succession of musical evenings, to which some students were always invited” (p. 128). Courant, in particular, used these musical evenings as a way to engage with his students. Fritz John (as quoted in Reid, 1996) describes, “he (Courant) was always inviting students to musical evenings at his house. I was not musical, but I went anyway” (p. 132). Aleksandrov and Oleinik (1975) describe, “among the regular visitors to the Courant home, Hans Lewy was an excellent violinist, and the young geometer Cohn-Vossen a pianist. Chamber music flourished in the house and was a source of great joy, both to the Courants and to their numerous regular visitors” (p. 165). Reid (1996) describes, “Courant also made an effort to draw Emil Artin, the other outstanding newcomer, into the inner circle of Gottingen. Artin was an Austrian with talents and interests in art and music as well as mathematics...Although...the mathematical interests of Artin were quite different from his own, Courant welcomed the young visitor, invited him for musical evenings at his home (where Artin, whose music was as pure and rigorous as his mathematics, shuddered at Courant's untutored approach to the keyboard), and saw that Klein and Hilbert met him and learned about his work” (p. 87).

Courant also used these musical evenings as a way of connecting to colleagues. Alexandroff (Alexsandrov[4] & Oleinik, 1975) recalls his farewell to Gottingen party hosted by the Courants. “There was a great crowd present; in addition to the host and hostess and the usual guests at the Courants' soirees, there was Emmy Noether and several other young mathematicians. The Schubert trio in E flat major was played by Cohn-Vossen (piano), Lewy (violin), and Nina Courant (cello)” (p. 168).

Upon moving from Gottingen to a new home in New Rochelle, New York, the Courants continued their tradition. “Shortly after moving to New York: A friend from Germany meeting

[4] Paul Alesandroff is alternatively referred to as P.S. Aleksandrov in the literature.

him (Courant) inquired, "Ah, Courant, and will you be playing quartets again in this country?" "In this country not quartets," Courant replied. "In this country, Octets!"" (Reid, 1996, p. 166).

Nirenberg (as quoted in Jackson, 2002) describes, "very often on the weekend he (Courant) would invite some graduate students to his home, which was in New Rochelle.... Courant was a great lover of music, as was his whole family. They often played chamber music at home, and sometimes I attended concerts" (p. 442).

In particular, one observes the role music played in Courant's connection with other mathematicians through personal letters Courant exchanged with mathematicians of his time. A number of personal letters from the Richard Courant Archives at New York University testify to this. Below are excerpts from letters Courant exchanged with his close friends, the Russian mathematician Paul Alexandroff and the Gottingen mathematician James Franck and his wife Ingrid Franck. (See Appendix A for the original German texts of the letters.)

Letter from Alexandroff to Courant, November 17, 1936:

> I would love to finally receive a report from either you or Nina about your life on the "other side of the great water." By the way, the music life of Moscow, which has always been lively, is now particularly intense: quite a few first-rate German conductors are constantly performing in Moscow, among them Kleiber, Klemperer and more. Actually, Kleiber has accepted the offer of a permanent position in Moscow as conductor of one of our local orchestras. (Alexandroff, 17 November 1936)

Letter from Alexandroff to Courant, upon Courant's upcoming 80th birthday, January 2, 1968:

> All these days I think very, very often also about Nina. And all the days and evenings spent in your house, all the music, everything, everything in all its details is so very vivid to me.

> On your birthday I will listen to Schubert's Trout Quintet and his Trio in E-flat major and also Beethoven's Trio in C-minor (Opus 1), which has always been the symbol of youth to me. So, dear Simplicial Cat[5], at your 80 years please remain young for many, many years. (Alexandroff, 2 January 1968)

Letter from Courant to Ingrid Franck (wife of James Franck), May 6, 1934:

> Send some news to Göttingen about how you personally are doing. How is the piano going? (Courant, R., 6 May 1934)

Letter from Courant to James Franck, February 22, 1934:

> Nina and Ernst are doing well. Ernst is already at the top of his class here. Nina is making a lot of music, and that successfully, and is recovering from the pressure of the past year, but is not worried about a Göttingen existence again through late fall.
>
> Born is doing pretty well, too…
>
> But enough for today! This letter is supposed to make it with the *Europa*! It makes me especially happy that Ingrid is playing piano. Unfortunately, I am not. (Courant, R., 22 February 1934)

Personal letters also testify to the role music played in the connection between the Courants and their son, Ernst. Ernst sent letters containing the following passages to his parents.

> As a child: "In the piano I can play a little song for three voices, two with the right hand and one with the left one." (Courant, E., 17 November 1933)

As an adult, Ernst writes to Nina:

> Yesterday evening I heard Menuhin in Philadelphia; he played the Schumann concerto and the Brahms concerto. That was beautiful! He really can play violin. The Schumann

[5] The German here is "Simplizialkater." ("Kater" is the masculine form of Katze, cat.) This seems to have been a nickname Alexandroff used for Courant.

concerto seemed quite beautiful, and the one by Brahms is, as you know, also very beautiful. And he played both wonderfully. In addition, there was Brahms's “Variations on a Theme by Haydn,” which are also quite beautiful. All in all, it was an experience, full of honey. On the other hand, Maurice Maréchal played cello here on Thursday and did almost nothing but show off how good his technique was. At least I noticed very little else. (Courant, E., January ND)

Lewy, as noted above, participated in the chamber music evenings at the Courant home. Lori Courant Lax (2013), Richard Courant’s daughter, describes, “Hans Lewy...studied with my father, Richard Courant, and often came to the chamber music evenings of my mother, Nina. On one occasion he brought a string trio he had written, and my mother and her friends tried to play it...one movement of it, (is) a romantic, gracious waltz” (p. 157).

Whitney, like Courant, shared music with his family, students and colleagues by hosting weekly musical evenings at his home. He also participated in community musical events. While at the Institute for Advanced Studies in Princeton, Whitney hosted a weekly chamber group at his home made up of members of the Institute. These were an opportunity to make music, but also to have mathematical discussions with other mathematicians and physicists in an informal setting. Among the participants in Whitney’s ongoing weekly chamber group was Henri Cartan (Kendig, 2018). Whitney actively sought out colleagues to join his music making evenings. Kendig (2018), Whitney’s biographer and younger colleague at the Institute for Advanced Studies, describes how upon arriving at the Institute:

> The first time I checked my Institute mailbox I discovered a handwritten note signed Hassler Whitney saying that he’d found out I played the cello, and was anxious to meet me and talk over music in town. So I went to his office and knocked on the door...I told

> him I had gotten this note in my mailbox from him, and in less than a second his entire demeanor changed. With a big smile and sparkling eyes, he invited me to take a seat in his office. He explained that in all his 13 years at the Institute (it was then 1965), this was the very first time there was a violinist, violist, and cellist among the members. Since he played the violin, we could have an Institute string quartet! It would meet weekly. (p. 251)

These music-making evenings were an opportunity to discuss mathematics in a playful, informal setting. Kendig (2018) describes:

> Although the centerpiece of each music-making evening was playing music, what happened at the end of each evening ended up having even larger implications for me. At the end of each session all four players, together with Hassler's wife Mary, would sit around a large table generously supplied with cheese, crackers, nonalcoholic liquid refreshments, and usually a cake. It all made for a very pleasant time. All four of us players had mathematical background...With these backgrounds, Hass found it very natural to bring up little brain teasers. There were lots of little questions like "where should you be so the gravitational forces of the earth and moon cancel out?" Or "if there were no atmosphere on earth, how long would it take for a meteoroid at rest 50,000 miles up to fall to the earth?" But a simple-sounding question could quickly turn into a discussion about what was and wasn't assumed. How would the moon's gravity influence the meteoroid's journey? (p. 252)

Whitney also thought about musical puzzles. Kendig (2018) describes, "For music aficionados: In printed music, a movable clef's center of symmetry or 'leftward pointer' points by definition to middle C...During a break in an evening of string quartets, Hass mirthfully asked, ‘In printed music, the notes are always moved up and down to represent pitch. But why not keep all the notes on the same line and put the correct clef before each note?’” (p. 296)

In addition, Whitney played with an orchestra in Princeton, and encouraged his colleague Kendig to join as well. Whitney and Kendig, under Whitney’s influence, also attended the summer chamber music camp held at Bennington College in Vermont each summer, where professional musicians coached amateur chamber players.

Kendig (2018) describes the impact of Whitney’s and his shared involvement in music. “Hassler did not readily make close friends, but because he and I both shared a strong love of classical music, the situation was different. Through the medium of string quartets, we mutually made and enjoyed our kind of music, and this music had a direct connection to both our hearts” (p. 251).

Whitney also was a connoisseur of musical performances. Kendig (2018) describes Whitney sharing with him a favorite recording:

> Had I ever heard Debussy’s string quartet? I hadn’t. He (Whitney) had a recording, and played some of each of the four movements. The slow movement, in particular, clearly moved him, and I remember him comparing it to moonlight reflecting off a lake’s quiet ripples. I had never quite seen this side of him. A little later he volunteered that Debussy wrote a lot of wonderful piano music and that, in particular, Walter Gieseking was a uniquely gifted interpreter of Debussy’s piano works. He said I ought to listen to his recordings. (p. 271)

A comparison between the subjects shows their involvements in music were social and lifelong. Their involvements include playing instruments, sharing music with their families, colleagues and students, playing in chamber or orchestral groups, being a connoisseur of classical music and using music as a metaphor for mathematics (as will be discussed below). While four of the subjects played instruments, all were connoisseurs of classical music. Of the subjects that played instruments, three engaged in a highly social manner with other mathematicians in playing in chamber groups. These social meetings served to forge new connections and strengthen existing ones between mathematicians. In addition, in one case, these led to engaging in mathematical discussions in an informal, "playful" manner. Even subjects who did not play musical instruments, used a love of classical music as a way of connecting with and forging a mathematical relationship with other mathematicians. Finally, it appears that all of the subjects' engagement with music was consistent and lifelong.

4.3 The Second Research Question

2. How do mathematicians relate their interest in music to their interest in mathematics, if at all?

Above was discussed that subjects used their involvement in music as a way of building connections and communities with mathematician colleagues and students, which sometimes led to the opportunity to exchange mathematical ideas in an informal environment.

4.3.a Music as a metaphor for the art of mathematics.

A second way the subjects connect their interest in music to their interest in mathematics is through drawing upon the art of music as a metaphor and analogy for the practice of mathematics. In their writings and speeches, subjects view mathematics as an art and themselves as artists, and therefore find common characteristics between the practice of art and mathematics. These common characteristics include prioritizing beauty, "aesthetic sense," intuition and personal judgement, or taste, as will be described further below. In particular, the art of music is most frequently referenced by the subjects in connection with mathematics.

In addition to drawing on music as a metaphor for mathematics, subjects draw upon music in the teaching of mathematics in a very direct, explicit way, and imply an inherent connection between music and mathematics/ mathematicians.

The majority of the subjects (Atiyah, Courant, Gelfand and Glazman) describe mathematics as an art and draw upon the art of music as a metaphor and analogy for the practice of mathematics in their writings, speeches and interviews. Lewy and Whitney do not reference music as a metaphor for mathematics, however, do demonstrate the prioritization of artistic characteristics, including beauty, "aesthetic sense," intuition and personal judgement, or taste, in their mathematics practices. The nature of Lewy's and Whitney's writings is such that it is limited in scope to mostly their mathematical work, and does not discuss their philosophies of mathematics. This is with the exception of Whitney's ample writing on his philosophy of education, which reflects his values on the characteristics important in mathematics.

Atiyah provides a scientific connection between mathematicians' and artists' appreciation of beauty. One of Atiyah's most widely read articles (Zeki, Romaya, Benincasa & Atiyah, 2014) investigates the question of the nature of mathematical beauty. The study concludes that the experience of beauty "derived from so abstract an intellectual source as mathematics" evokes the same physical and neurological response as the experience of beauty from other sources, such as music or art. Further, the study examines "the extent to which the experience of beauty is bound to that of "understanding" (Zeki, et al., 2014), meaning, that the greater the mathematicians' understanding, the greater his/ her experience of beauty. Zeki, et al., explain, "Mathematical and artistic beauty have been written of in the same breath by mathematicians and humanists alike, as arousing the 'aesthetic emotion.' This implies that there is a common and abstract nature to the experience of beauty derived from very different sources" (Zeki, et al., 2014).

Indeed, this fits in very well with Atiyah's philosophy that beauty and the aesthetic sense play a primary role in the practice of mathematics. According to Atiyah (1983), "It would be difficult to overestimate the subjective importance of these aesthetic criteria in the mind of the working mathematician" (p. 30).

Atiyah also is clear that the practice of mathematics is an art and directly compares the practice of mathematics to composing music. Atiya (as quoted in Cruz Morales, 2019) states:

> So, invention is selection. I select things that appeal to me. I am an artist. You have all the possible musical notes you can write. Why should I write some of those? There are millions of possibilities of musical notes. Taking some is composing, is invention; we regard this as invention. All the possibilities are there. Inventing a symphony is a human creation. So, what is the difference in creating a beautiful piece of music and creating a beautiful mathematical theorem? I think it is the same. (p. 21)

Another example of Atiyah comparing the mathematician to the artist is below (as cited in Cruz Morales, 2019):

> We choose a problem which we think is interesting if it has features that appeal to us. It must be hard. You will see some beauty and some form emerging from it, like an artist. It must be deep; you can put a lot of adjectives but at the end it is difficult to describe. It is the decision of the individual artist or mathematician and that is personal. You can give a hundred reasons but this does not define it completely. All you can do is to say that you view beauty in what you see. (p. 21)

Further, Atiyah (as quoted in Prada Oscar, 2016) states:

> We were unsure if the word was correctly used but as mathematicians we know what we mean by beauty and I think the beauty in mathematics is comparable to the beauty in music. (p. 29)

Atiyah acknowledges that the perception of beauty in mathematics, as in music, is a skill honed with experience and understanding. Atiyah (2018) states:

> To appreciate mathematical beauty may require, as in music, extensive education and training, and it is always a subjective judgment. (p. x)

Intuition plays an important role in Atiyah's mathematics work. Atiyah describes:

> The crazy part of mathematics is when an idea appears in your head. Usually when you're asleep, because that's when you have the fewest inhibitions. The idea floats in from heaven knows where. It floats around in the sky; you look at it, and admire its colors. It's just there. And then at some stage, when you try to freeze it, put it into a solid frame, or make it face reality, then it vanishes, it's gone. But it's been replaced by a structure, capturing certain aspects, but it's a clumsy interpretation. (Roberts, 2016)

Courant similarly prioritized beauty, intuition and personal judgement in the practice of mathematics, refers to mathematics as an art and compares it to music. Regarding Courant's selection of mathematical investigation, Courant's student and colleague, Friedrichs (as quoted in Reid, 1996), explains, "Courant would never tackle a problem just in order to solve it. The tools used to solve the problem had to appeal to him too. He could never separate them from the problem. They were always part of the deal. If the problem was beautiful, the tools had to be beautiful too. Otherwise, there was no appeal at all" (p. 181).

Courant (Courant, Robbins & Stewart, 1996) writes, "Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality” (p. xv).

Courant refers to mathematics as an art. In Courant’s introduction to his textbook on Calculus (Courant & Fritz, 1989), he writes, “Two of these processes, differentiation and integration, became the core of the systematic Differential and Integral Calculus... Yet, to gain mastery of the powerful art appeared at first a formidable task” (p. v).

Courant directly draws on music as a metaphor for mathematics. Reid (1996) writes:

> That same year Courant was asked by an interviewer what in mathematics he saw as most likely to be productive for the scientist. "I don't want to emphasize or advocate future mathematical activities because of my personal taste," he said. "It is the same as in music. Even now I have difficulty appreciating Bartok or more modern music to any extent. Yet my grandchildren sit at the piano and play such pieces as a matter of course. They don't know there is any difference, and so will it be with respect to the attitude of the younger generation toward scientific subjects, such as computing and computers, or outlying fields of topology or logic. (p. 302)

Similarly, Courant (1964) writes:

> The question "What is mathematics?" cannot be answered meaningfully by philosophical generalities, semantic definitions or journalistic circumlocutions. Such characterizations also fail to do justice to music or painting. No one can form an appreciation of these arts without some experience with rhythm, harmony and structure, or with form, color and composition. For the appreciation of mathematics actual contact with its substance is even more necessary. (p. 42)

Further, Courant (1960) writes:

> You cannot explain in a few words what mathematics is to somebody who does not already know something about it. In the same way, you cannot tell what music is to someone who has never listened to a symphony or an opera. You cannot define what painting is to someone who has never looked at a painting. A certain amount of experience with a subject is necessary before you can sit back and talk *about* it. (p. 32)

Courant sees intuition as an essential aspect of mathematics. In his preface to his textbook on differential and integral calculus, Courant (1988) explains, "My aim is..., without loss of rigour and precision, to give due credit to intuition as the source of mathematical truth" (p. 51). In his introduction to another textbook, Courant (Courant & Fritz, 1989) describes the aim as "to exhibit the interaction between mathematical analysis and its various applications and to emphasize the role of intuition" (p. vi).

Gelfand similarly prioritizes beauty in mathematics and directly compares the practice of mathematics to music. Gelfand (2006b) says in a formal speech, that:

> Many people consider mathematics to be a boring and formal science. However, any really good work in mathematics always has in it: beauty, simplicity, exactness, and crazy ideas. This is a strange combination. I understood earlier that this combination is essential in classical music and poetry, for example. But it is also typical in mathematics. (p. xv)

Further, Gelfand (2006a) responds to the question of "what is mathematics?" in the following way:

> The combination of these four things: beauty, exactness, simplicity and crazy ideas is just the heart of mathematics, the heart of classical music. Classical music is not only the music of Mozart, or Bach, or Beethoven. It is also the music of Shostakovich, Schnitke, Shoenberg (the last one I understand less). All this is classical music. And I think, that all

> these four features are always present in it. For this reason, as I explained in my talk, it is not by chance that mathematicians like classical music. They like it because it has the same style of psychological organization. (p. xiiv)

Gelfand further refers to his view of mathematics as an art when describing his process of learning mathematics independently as a youth. “This early period (ages 12-18) formed my style of doing mathematics. The subject of my studies varied, of course, but the artistic form of mathematics that took root at this time became the basis of my taste in choosing problems that continue to attract me right up to the present time” (as cited in Retakh & Sosinsky, 1991, p. 21).

Gelfand also prioritizes intuition. Retakh (2013, January) states, “Gelfand always was surrounded by numerous collaborators attracted by his legendary intuition and the permanent flow of new ideas: in every decade he was establishing a new area of research” (p. 25).

Glazman is recognized as prioritizing beauty in his mathematical work, describes mathematics as an art and treats the process of teaching mathematics as analogous to the process of teaching music. Lyubich (Lyubich & Tkachenko, 1997), Glazman’s collaborator, writes, “It should be noted that the criterion of beauty in mathematics was of exceptional importance for Israel Glazman” (p. 4) and directly connects this to Glazman’s musical training. Lyubich and Tkachenko (1997) state:

> It is easy to surmise that it (the importance of beauty in mathematics) was intrinsically connected with his outstanding musical talent...When, finally, he became a professional mathematician, his profound musical talent revealed itself in his works. (p. 4)

Glazman makes a connection between music and the practice of mathematics in a slightly different way than the previously discussed subjects do. In Glazman’s preface to his textbook on *Finite-Dimensional Linear Analysis*, Glazman directly states that he modeled the pedagogical approach of the book on that of Spohr's book of violin instruction- *Violinschule*, a school of violin-playing. The first paragraph of the preface is dedicated to discussing "The composer Ludwig Spohr, who lived at the turn of the 18th century" and a brief description of Spohr's musical accomplishments (Glazman & Lyubich, 1969, p. v). Then, Glazman overtly describes his book as modelled after Spohr's textbook for violin. "His (Spohr's) was the first such work, which aimed at the same time at the development of the pupil's technical ability and at his artistic education" (p. v). Glazman states, "Our book presents itself as an attempt to construct a course analogous in character and purpose to those schools of violin playing" (p. vi). There is no doubt that Glazman had the same dual goals of "technical ability and artistic education" in mind in his mathematics textbook.

Finally, Timotin (1998) writes about Glazman's textbook on *Finite-Dimensional Linear Analysis*, "The book is a unique mathematical gem, whose wonderful esthetics is easily recognized by its reader" (p. 426).

Lewy was extremely private about his personal life and there are very few writings by or about Lewy on his philosophy of mathematics. The most revealing account of Lewy's view of art and mathematics is by his wife, Helen Lewy (2002), in "The Music in Hans Lewy's Life" (p. xxi). This is based on her observing his lifelong devotion to music. Helen Lewy describes that "Mathematics and music remained the twin passions of his life, and, in the end, I believe he felt he had been lucky enough to have had the best of both worlds" (p. xxiii). Further, one sees how intuition played a role in Lewy's mathematics from an observation by Lewy's colleague, Nirenberg. Nirenberg (as cited in Reid, 1991) stated about Lewy, "He seemed to think in a different way from most mathematicians I have met. He would have, somehow, fantastic ideas, and sometimes these ideas would come out in a paper of three or four pages. And these papers would have enormous influence on later development" (p. 263). One can only deduce from others' descriptions of the values he reflected in his mathematics to what extent Lewy gave importance to esthetics in his mathematics. I argue that since many of the characteristics ascribed to Lewy's mathematics- creativity, inventiveness, originality and a search for deep understanding (as will be seen below)- are the same as those of the other subjects described in this study, who do explicitly refer to the value of aesthetics in mathematics, aesthetics influenced Lewy's work as well.

Whitney wrote and spoke prolifically on the education of young and school age children, which he focused on during his "second career". The writings and accounts about Whitney's practice of mathematics focus on his quest for deep understanding, his emphasis on getting to the root of ideas through examples and his view of mathematics as playful, exploratory and fun (as will be seen below). He received great intellectual pleasure from mathematics and exploring hypothetical mathematical puzzles. However, there is very little written by or about Whitney on his practice in doing mathematics prior to when he began writing about education in his later years. Whitney's focus on intuition is seen in his biographer's, Kendig, description. Kendig (2018) describes Whitney as:

> ...very individualistic and seemed to have some internal sensor that sounded off when he'd stumble upon something having the smell of special significance. His energy, focus, and dogged persistence would then kick in. His innate originality seemed to be enhanced in that he mostly avoided areas that had been plowed through by others. (p. 198)

Whitney's intuition worked in tandem with his "dogged determination." Kendig (2018) describes:

> Although the weak embedding theorem ensures that any smooth manifold of dimension m can be differentiably embedded in R2m+1, there was a lot of evidence that it could actually fit in the smaller space R2m. Going down this one dimension proved to be a hard nut to crack, but Whitney's intuition was so strong on this that he spent nearly eight years, on and off, trying to establish it. (p. 175)

The areas of focus of mathematics Whitney emphasizes in his writings and talks on education were firsthand experience, free exploration, self-driven investigation, responsibility and intuition (as shown below). Similarly, as with regard to Lewy, I argue that the characteristics Whitney displayed in his mathematics and the values he describes are similar to those of the other subjects of this study who recognize a strong influence of aesthetics on their mathematics processes. Therefore, it is highly likely that Whitney's work was influenced by aesthetic criteria as well.

4.3.b Intrinsic connection between music and mathematics/ mathematicians.

In further evaluating the nature of a connection between mathematics and music, one sees evidence that subjects consider there being an intrinsic connection between music and mathematics.

First, Gelfand makes an explicit connection between music and mathematics teaching. Gelfand includes a section in his textbook *Algebra* (Gelfand & Shen, 1993) entitled "The well-tempered clavier," in which he illustrates mathematical concepts related to incommensurate numbers and irrational roots in the context of music. However, beyond the explicit mathematical connection, Gelfand makes an implicit connection, demonstrating his view (see next paragraph) that music is a component of mathematics education. Notably, the final problem at the end of the section is: “Problem 221. Find a recording of Bach’s *Well-Tempered Clavier* and enjoy it” (p. 91).

Gelfand’s view that music is implicitly a component of mathematics education is shown in the anecdote shared above by Gelfand’s student McDuff. McDuff (2013, February) recalls, “Once he (Gelfand) took me shopping: he bought and gave me all the good classical records he could find. Very cheap, they contained wonderful performances by Russian musicians that he thought I should hear. He said that this too was ‘teaching me mathematics’” (p. 163).

This idea, of music being an intrinsic component of mathematics education, is further illustrated in the case of Courant. According to Courant's son-in-law, Jerome Berkowitz, who was a student of Courant’s at NYU, "I think that very much of my education occurred right here (NYU), and I think that many of the people feel that way. The education was not just mathematical. It was political and social and musical, too" (Reid, 1996, p. 303-304).

While Glazman (Glazman & Lyubich, 1969) considers that “artistic education” is equally important as the “development of the pupil’s technical ability” in mathematics education (p. v), this is still in line with the view described by subjects above that the mathematician is an artist (see examples above). However, Lyubich and Tkachenko (1997) imply a deeper, more intrinsic, connection between music and mathematics in the work of Glazman, in their statement that:

> It should be noted that the criterion of beauty in mathematics was of exceptional importance for Israel Glazman. It is easy to surmise that it was intrinsically connected with his outstanding musical talent...When, finally, he became a professional mathematician, his profound musical talent revealed itself in his works. (p. 4)

Gelfand is very direct about an implicit connection between music and mathematics when he states that “many mathematicians enjoy serious music” (Gelfand, 2006b, p. xv).

Gelfand provides a reason: "because it has the same style of psychological organization" (Gelfand, 2006a, p. xiiv). Gelfand (2006a) further posits that the relationship between mathematics and the arts is based on the languages they provide for understanding abstract notions.

> There is also another side of the similarity between mathematics and classical music, poetry, and so on. These are languages to understand many things. For example, in my lecture I discussed a question which I will not answer now, but I have the answer: Why did great Greek philosophers study geometry? They were philosophers. They learned geometry as philosophy. Great geometers followed and follow the same tradition—to narrow the gap between vision and reasoning. (p. xiiv)

4.3.c Conclusion.

Subjects relate music to mathematics in a number of ways. Music provides social connections with other mathematicians and serves to strengthen mathematical communities, as discussed in response to research question one. In addition, music serves as a metaphor for the practice of mathematics in multiple ways. Inasmuch as subjects consider mathematics to be an art and themselves artists, mathematics shares characteristics with art in general and music in particular. These include prioritizing beauty, "aesthetic sense," intuition and personal judgement, or taste. Finally, there is evidence that some subjects consider an inherent, or implicit connection between music and mathematics/ mathematicians.

4.4 The Third Research Question

3. How does involvement in music help shape mathematicians' identities as mathematicians, if at all?

Above was shown the ways in which the subjects were involved in music and how the subjects' involvement in music connected to their practice of mathematics. The next question is in what ways music formed a part of the subjects' identities as mathematicians.

4.4.a Subjects' characterizations as musicians/ composers

There are some explicit statements by subjects that imply a direct connection between a subject's musical and mathematical identity. Gelfand seemed to view himself as having the characteristics of a musician in his mathematical work.

As described above, Gelfand, according to his student McDuff (2013, February), was "obviously thinking of himself as Mozart" from the play Mozart and Salieri (p. 163). Further, Gelfand's wife, Tatiana Alekseyevskaya (Gelfand) (2020b), compares Gelfand to a composer or conductor in his mode of mathematical work, and quotes Gelfand as saying he would come back in a future life as a composer. Alekseyevskaya (Gelfand) states:

> Once, a mathematician who never worked with Gelfand asked me why most of Gelfand's papers were written jointly with co-authors and what Gelfand's role could be in so many such publications. As his co-author of this book, and from being present for many years during his collaborative work with people, I would compare Gelfand's role in his joint works with that of a composer or a conductor. Many times Gelfand himself said that, if he could have been born again, he would have become a composer. I would say that he actually was one. (p. xiv)

As seen above, Gelfand states that "many mathematicians enjoy serious music" (Gelfand, 2006b, p. xv), implying an intrinsic connection between the two. Glazman models mathematics education after musical education. Lyubich and Tkachenko (1997), when referring to Glazman, state, "When, finally, he became a professional mathematician, his profound musical talent revealed itself in his works" (p. 4). All of this implies that musical involvement was deeply connected to mathematical work.

Further, one sees reflections of subjects' mathematical style in their behavior in music, and vice versa. Friedrichs (Lax, 2003), discussing a mathematical work of Courant, states, "It is true that there were some passages in which matters of rigor were taken somewhat lightly, but the essence came through marvelously. I was reminded of this effect much later, when I heard Courant play some Beethoven piano sonata. There were also some difficult passages which he somehow simplified; but the essence carried over wonderfully" (p. 93).

Similarly, Nina Courant (as quoted in Reid, 1996) comments on her impression of Courant's piano playing when she and Courant were young.

> I was just amazed at the chutzpah with which he tackled anything that was attractive to him...You know what chutzpah is? Nerve! He never suffered from any inhibitions because something was too difficult for him, but tried everything. He somehow succeeded in making a piece intelligible, getting the spirit of it, no matter how imperfect his playing was. On the other hand, really practicing, finding a method of improving something that was too difficult for him-that he couldn't do. But I learned something from him. I made up my mind that pieces imperfectly played are not spoiled by this fact as long as one understands them. I learned not to be afraid of them. Why think of the listener? We played for ourselves alone! (p. 56)

This is reflected in Courant's student Lax's (as quoted in Reid, 1996) description of Courant's unconventional approach to mathematics. Lax describes Courant as:

> A very original guy...The way he wrote was like nothing anybody else wrote. He hated that style of stating a theorem which goes, Let M be a manifold, X a differential structure,

Y a vector field, and so on. He would really want to describe first what the problem was about and how one goes about attacking it. In fact, perhaps Courant went rather more in the other direction of not having enough theorems. But- as he said- "Most people have too many." (p. 279)

Above one sees examples of where subjects' musical involvement seems to be a part of a mathematical identity. This is seen in the way subjects characterize themselves, or writers about the subjects characterize them, as musicians/ composers. This is also seen in the say subjects' mathematical behavior is reflected in their musical behavior, and vice versa. In response to research question number two, it was seen that some subjects view the mathematician as having the identity of the artist, of which the musician is a subset. In that context it was discussed that characteristics of the artist include prioritization of beauty, the aesthetic, intuition and personal judgement, or taste, and that the subjects incorporate these characteristics in their practices of mathematics.

I now examine further how subjects view the connection between musical involvement and mathematical work. I do this by examining the beliefs, values and identities, components of affect, of the subjects as revealed in their mathematical practices. Then, I will propose how these beliefs, values and identities are related to music.

4.5 Themes- Beliefs, Values and Identities of Subjects as Mathematicians

In examining the data on the subjects, I identified a number of themes related to the subjects' beliefs and values related to the practice of mathematics. This is in an attempt to gain a holistic view on the subjects. The themes identified on the subjects' practices of mathematics are the following:

1) Collaborative and social

2) Belief in the unity of mathematics- making connections between ideas
3) Commitment to mathematics education- making social connections
4) Emphasizing personal initiative in learning and understanding mathematics- reliance on own judgement
5) Importance of exploration and play
6) Focus on big picture
7) Emphasizing use of specific examples- hands on
8) Commitment to mathematics and its applications

4.5.a Collaborative and social.

The subjects' mathematics practice was characterized by collaborating with students and colleagues, throughout their lives.

Atiyah (Raussen & Skau, 2005) recalls:

When I was a student, I learned things by going to lectures and reading books— after that I read very few books. I would talk with people; I would learn the essence of analysis by talking to Hörmander or other people...Interacting with other people is of course essential: if you move into a new field, you have to learn the language, you talk with experts; they will distill the essentials out of their experience. I did not learn all the things from the bottom upward; I went to the top and got the insight into how you think about analysis or whatever. (p. 230)

Further, according to Atiyah (as cited in Bartocci, 2011):

Most of my work has been carried out in close and extended collaboration with mathematical colleagues. I find this the most congenial and stimulating way of carrying on research. The hard abstruseness of mathematics is enlivened and mollified by human

> contact. In addition the very diversity of the fields in which I have engaged has made it essential to work with others. I have indeed been fortunate in having had so many excellent mathematicians as my friends and collaborators. (p. 207-208)

Students often became collaborators with Atiyah (as cited in Prada Óscar, 2016):

> So, I would collaborate with them and they would also have their own individual personality and mathematical tastes... They would be going in slightly different directions, which is very good. You get to broaden – some more with analysis, some geometry, some more with topology – and that way you learn with these 20-year-old students because they become more expert... When you start off, you learn something, but when you're teaching, you don't have much time to go back and study so you have to learn in a different way and one way to learn is through your students, in collaborating with your students. (p. 25)

Courant relished his work with students, who often became his lifelong collaborators and colleagues. According to Courant (Courant and Hilbert, 1962), "Throughout all my career I have had the rare fortune to work with younger people who were successively my students, scientific companions and instructors. Many of them have long since attained high prominence and yet have continued their helpful attitude" (p. viii).

Retakh (2013, January) describes how "Gelfand always was surrounded by numerous collaborators" (p. 25).

As seen above, Glazman and Lewy collaborated on their mathematical work, and Lewy enjoyed discussing mathematics with colleagues at his chamber music evenings.

4.5.b Belief in the unity of mathematics- making connections between ideas.

In Atiyah's (as cited in Bartocci, 2011) own words:

> [...] my mathematical interests have gradually shifted from field to field, starting with algebraic geometry and ending up with theoretical physics. On the other hand the change was never a deliberate or discontinuous one. It was simply that the problems I studied naturally led me in new directions, frequently into quite foreign territory. Moreover the link between the different areas was an organic one, so that I could not discard the old ideas and techniques when moving into a new field – they came with me. (p. 207-208)

Lax (2003) said about Courant, "What is remembered? His insistence on the fundamental unity of all mathematical disciplines and on the vital connection between mathematics and other sciences. The name he gave to his institute— Institute of Mathematical Sciences—expressed his attitude" (p. 94).

Gelfand recalls how at age 12, during his self-directed mathematical study, "a barrier came tumbling down, and mathematics became one. To this day I see the various branches of mathematics, together with mathematical physics, as a unified whole" (Retakh & Sosinsky, 1991, p. 23).

4.5.c Mathematics education- making social connections.

The subjects were all committed to mathematics education, some seeing it as a moral imperative. The subjects are described as dedicated, effective and egalitarian educators. Educating students was a way for subjects to themselves be lifelong learners and collaborators, as well as to fulfill a moral imperative.

In Atiyah's words, "If you are a teacher and you have a great student, what is your duty? It is to encourage him, to help him" (Farmelo, 2018, time: 20:39). Atiyah actively engaged with students and found it to be a generative part of his mathematics growth. Atiyah considered advising PhD students “a very positive experience and I enjoyed that” (as cited in Prada Óscar, 2016, p. 25). Atiyah describes, “you learn from your students, the very good students” (p. 24).

Courant was a disciple of Felix Klein, who was very concerned with the connection between research and education as well as school mathematics in Germany, and drew upon his philosophy of and commitment to education. Courant wrote many textbooks and actively collaborated with publishers, beginning with his good friend Springer who ran the Springer-Verlag publishing company in Germany to publish “self-contained monographs and advanced textbooks in which the author's research is combined with a broad survey” (Aleksandrov & Oleinik, 1975, p. 158). Courant founded and edited many monographs on mathematics as well as Klein’s lectures on the history of mathematics in the nineteenth century (Aleksandrov & Oleinik, 1975). At NYU, Courant offered a general course for mathematics teachers as well as for graduate mathematics students, which he called "Elementary Mathematics From a Higher Viewpoint," after Felix Klein’s lecture series (Reid, 1996, p. 189).

Courant emphasized the importance of teaching along with research. According to Aleksandrov and Oleinik (1975), “Courant was convinced that for a scholar, teaching is as important as research. In a speech in 1965 in connection with the move of the Courant Institute to a new building, he said ‘Our earliest ideals remain unchanged — to devote oneself to youth’” (p. 159).

A lifelong passion of Gelfand’s was his Correspondence School of Mathematics aimed at school aged children, which he established first at Moscow State University and then transplanted to Rutgers University. The free school was based on a series of pamphlets, books and problems authored by Gelfand and other professional mathematicians, to complement the school curriculum, and provided teaching assistants to give individualized and educational feedback to students' work (Retakh, 2013, January, p. 26). In describing the motivation for founding the correspondence school, Gelfand (1990) writes, "(in the Soviet Union) there are simply not enough teachers throughout the country who can show all the students how wonderful, how simple and how beautiful the subject of mathematics is" (p. v). Gelfand's (1990) stated philosophy was that "The most important thing a student can get from the study of mathematics is the attainment of a higher intellectual level" (p. vi). Further, Gelfand (2020) stated, "Read the book at your own pace and return to it if you need to. The main "rule" is to enjoy reading and drawing figures, and to see the beauty of geometry as if you were playing at discovering a new world" (p. xxi).

Gelfand's stated view of a "professional" is one who "must be able to teach at elementary school, to give talks interesting to Harvard faculty, and everything in between" (Retakh, 2013, February, p. 165). Gelfand galvanized mathematicians from a variety of backgrounds at his weekly mathematics seminar at Moscow State University, which continued at Rutgers University (Gelfand & Lepowsky, 1993). Gelfand's mathematical seminar, a "mathematical stock exchange," (Retakh, 2013, January, p. 25), was targeted to "high school students, decent undergraduates, bright graduates, and outstanding professors" (Retakh, 2013, January, p. 26). In the words of his student, Retakh (2013, January), "the seminar was a reflection of Gelfand's passion to teach, as he tried to teach everyone and everywhere" (p. 26). Gelfand's feeling toward his students is expressed in his rebuke to a mathematician who took pride that a certain accomplished mathematician was his former student. "Gelfand reacted immediately, "You cannot say 'my former student'. This is like saying 'my former son'" (Retakh, 2013, February, p. 164). In fact, another student of Gelfand observed that, for Gelfand, the occupations of mathematician and teacher were “inseparable. His way of doing mathematics always involved close personal interaction with his innumerable students and collaborators” (Zelevinsky, 2013, January, p. 48).

Glazman placed “great importance” on teaching mathematics (Lyubich & Tkachenko, 1997, p. 5). According to Lyubich and Tkachenko (1997), “his (Glazman’s) lectures combined simplicity and clarity with depth and content. The manner of presentation was brilliant, often paradoxical. His response to a question was instant, and his connection with his audience was incessant. In Glazman's opinion the student's ability to solve problems was of paramount importance” (p. 5).

Glazman’s approach toward teaching mathematics is recognized as being widely accessible and cultivating the independence of the student. In addition, Glazman’s approach emphasizes the aesthetic experience of mathematics. Perhaps Glazman’s greatest gift to education is his “exquisite” (Timotin, 1998, p. 426) work, co-authored with Y. Lyubich, *Finite-Dimensional Linear Analysis*. It is "linear algebra through problems" (Timotin, 1998, p. 426), a book containing more than 2400 problems, arranged in such a way as to guide the student to producing all of the proofs. It is described by Halmos (1980) as “an unusual one (book) (I don't know of any others of its kind), and, despite some faults, it is a beautiful and exciting contribution to the problem literature… The book is not expository prose, however; perhaps it could be called expository poetry” (p. 521-522).

According to Halmos (1980):

> The really new idea in the book is its sharp focus: this is really a book on functional analysis, written for an audience who is initially not even assumed to know what a matrix is. The ingenious idea of the authors is to present to a beginning student the easy case, the transparent case, the motivating case, the finite-dimensional case, the purely algebraic

case of some of the deepest analytic facts that functional analysts have discovered... A beautiful course could be given from this book (I would love to give it), and a student brought up in such a course could become an infant prodigy functional analyst in no time. (p. 521-522)

According to Helen Lewy (Lewy, Helen, 2002), Lewy showed great devotion to his students. Helen Lewy describes, “during his extended stay in Europe, especially in Italy, it was his custom to spend a great deal of his time socially, as well as mathematically… with the students.” Also, “I do not think he would object, if, to honor the students he loved to be with, I pass along some facts about the music which was always the companion of the math in his life (p. xxi).

Lewy’s student D. Kindelehrer (2002) describes further, “On retirement, he (Lewy) insisted that available funding should be dedicated to the support of younger scientists and declined to apply for any more grants.” Also, “always attentive to the encouragement of young people he returned to Minneapolis to participate in the 1984/1985 Institute of Mathematics and its Applications Program on Continuum Physics and Partial Differential Equations” (p. xix).

Whitney devoted significant energy during his “second career” (Kendig, 2018, p. 236), to the mathematics education of the very young. Whitney served as president of the International Council of Mathematics Instruction (ICMI), taught in elementary school classrooms, ran workshops for teachers, gave lectures, including at Bank Street College for Teachers (Lax, 1989), and wrote (Kendig, 2018).

Whitney saw the mathematics education of the very young as a moral imperative. According to Whitney (1989), mathematics education is about developing future adults that will “help mankind towards keeping us and the planet as a well functioning whole. The commitment involves a real sense of responsibility- this is moral, spiritual, ethical, rather than intellectual" (p. 9).

Whitney's motivation was his witnessing his young daughter's education in elementary school in 1967. His reaction was "There were so many dead eyes in the classroom...it seemed that here was a group of child slaves being told what to do, led by a master slave who got orders from some lesson plan that had to be followed!" (as cited in Kendig, 2018, p. 235). This was the antithesis of Whitney's experience with mathematics (Kendig, 2018).

Whitney viewed an essential part of doing mathematics as taking responsibility for one's own thinking, and mathematics education as training responsible future adults. He phrased this in a moral context and saw the alternative as having dangerous consequences.

According to Hechinger (1986), Whitney views learning mathematics as "finding one's way through problems of new sorts, and taking responsibility for the results" (Hechinger, F. M., 1986).

According to Whitney (1973), with a test centered focus of education, "By the end of the year, the children may be making better scores. But there will be a growing sense that most children are losing their creative identity; this will show up especially in later years" (p. 295).

A great emphasis of Whitney's educational philosophy is that successful learning is self-initiated and self-directed, based on the innate, extraordinary learning capability of young children, that it is the goal to keep alive throughout life. Further, the child/individual is his/her own best teacher.

Whitney states, "The great need for the children is to return to their wonderful preschool learning, when they were full of vitality and curiosity, exploring their environment, observing a myriad of interconnections, and learning complex concepts and skills like communicating verbally and nonverbally, beyond what any of us adults do, and without any formal teaching." (as quoted by Lax, A., 1989, p. 3).

According to Whitney, the child is primary, and subject matter is secondary, in the education of the child. The focus of the teacher shall be on providing the child with experiences that will allow the child to explore and ultimately arrive at his/her own understanding. Emphasis on formalization will hamper this.

Whitney (1973) states, "In brief, our focus has been too much on the subject matter, not enough on the child himself" (p. 283). "The real goal is not for the child to learn particular answers; it is for him to grow in powers of finding answers, or rather, of exploration into processes" (p. 287). "What is desired on the part of the student is exploration *towards* answers; much better, exploration of the *general subject*" (p. 293). "The child must grasp the relationships in his own way first, then find some way in which to express it. Later the expression can take the form we have chosen" (p. 287).

Further, Whitney (1987) states, "Thus, a primary aim of studying mathematics must be for students to grow in their own natural reasoning powers, especially in domains where precise reasoning is valid. The growth must include creative and critical thinking, increasing control over their work, seeing connections with related matters, and raising communication skills" (p. 230).

Whitney was democratic in his approach to mathematics education. Whitney (1985) states, "...creative mathematical work is not just the privilege of a few geniuses; it can be a natural activity of any of us with sufficient desire and freedom. Our preschool exploration of our surroundings is later hampered by the growing pressures and loss of freedom, not by loss of ability" (p. 1). Whitney (1987) further expresses his belief in children's ability to learn mathematics and encourages future research on how to improve mathematics educational outcomes. “There is a large and very important field awaiting the researchers: What are students capable of if given the chance to explore situations? We badly need to show their powers; you can help!” (p. 238).

4.5.d Emphasizing personal initiative in learning and understanding mathematics-reliance on own judgement.

The subjects emphasize the role of personal initiative and reliance on one’s own judgement in achieving mastery in mathematics. According to Atiyah (Raussen & Skau, 2005), “My fundamental approach to doing research is always to ask questions. You ask ‘Why is this true?’...you simply have to have an inquisitive mind!” (p. 230). Further, “it is always better to find your own problem, asking your own questions, rather than getting it on a plate from your supervisor. If you know where a problem comes from, why the question has been asked, then you are halfway toward its solution” (Atiyah, 2008, p. 1002). Further, Atiyah (Roberts, 2016) describes, “I always try to dig behind the scenes, so if I have a formula, I understand why it’s there. And understanding is a very difficult notion” (second to last paragraph).

According to Lewy, active and self-initiated mastery is not only desirable but fundamental to the work of a mathematician. Lewy (as cited in Tukey, et al., 1990) states, “I think the desire to fill in the details is an ethical matter. There is a certain fanatic adherence to the truth that is necessary in mathematics; and if a person doesn’t have that, he shouldn’t go into mathematics, in my opinion, as a profession” (p. 183).

Lewy (1986) further describes this self-initiative and reliance on personal judgement when discussing mathematics research:

> We older mathematicians and the majority of our younger colleagues believe it to be absolutely unavoidable to convince ourselves personally of the correctness of the results of others if we want to use them. But today there exist published papers based on results whose examination would require the study of more than 5000 printed pages, a task which is admittedly beyond human capability...But should we mathematicians, so far having been proud to recognize no other authority than our own intellect, also resign ourselves to accept upon trust the printed words as the truth? (p. ixv)

Whitney and Gelfand were self taught in mathematics, as described in the case studies in Appendix B. Courant describes the enormous personal effort required in his education at Breslau as "the basis for my learning algebra" (Courant, 1962, May 9). Above was seen that Glazman's textbook is based on an approach of the students proving all theorems. Above was also seen that Whitney's view of mathematics education is that it must cultivate the individual judgement and ability to take responsibility of the students.

4.5.e Importance of exploration and play.

Freedom and exploration are very important in the mathematical work of the subjects. These play a very big part in how Atiyah (Atiyah, as cited in Connes & Kouneiher, 2019) gets his ideas:

> Some people may sit back and say, I want to solve this problem and they sit down and say, "How do I solve this problem?" I don't. I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring up ideas; things emerge and I follow them up. Or I see something which connects up with something else I know about, and I try to put them together and things develop. I have practically never started off with any idea of what I'm going to be doing or where it's

> going to go. I'm interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics. (p. 1661)

Courant's close student and colleague, Friedrichs, commented on Courants' prioritization of intuition, esthetics and play in mathematics:

> ...In fact, within mathematics proper Courant has always fought against overemphasis of the rational, logical, legalistic aspects of this science and emphasized the inventive and constructive, esthetic and even playful on the one hand, and on the other hand those pertaining to reality. How mathematics can retain these qualities when it invades other sciences is an interesting and somewhat puzzling question. (Reid, C., 1996, p. 296)

Gelfand describes the style of his self-directed pursuit of mathematics as a youth as "pure experimentation" (Retakh & Sosinsky, 1991, p. 24).

With regard to Lewy, Reid (1991) states, "many have noted...playfulness...in his work" (p. 265).

Whitney's style was characterized by exploration. Kendig (2018) describes Whitney's method of learning mathematics independently as a child as that Whitney would "guess and play around with problems that he found interesting. There was never any time pressure, and if he made mistakes, he would eventually find them and fix things up" (p. 235). Whitney (1985) stated:

> Research is, basically, exploring a situation. The situation itself may be vague, changeable, largely unknown at first. As when you enter a new land, "explore" means move around, try out, look close and look afar; in general, get familiar with the surroundings. For the child, examine a leaf and twig; for the mathematician, try some

> special cases in detail. In general, play with what you see there; you are building up both detailed knowledge and an inner sense of how mathematical ideas work. These "insights" will come from this inner sense in large part. (p. 3)

In a very tangible way, Whitney conveyed his core values about the practice of mathematics in a musical composition in the form of a round to the anonymous "Butterfly Poem." "This poem fits in perfectly with his strong belief that life is no spectator sport—that you live and learn through involvement. That applied to children learning math as well: Let them actively test, explore, make mistakes, find, and correct them. It's what he did from his earliest years—using his wings to break out and fly" (Kendig, 2018, p. 306).

Whitney's "dogged pursuit" of mathematics where he would spend years pursuing a topic, described above, is reflected in an anecdote provided by Kendig. Kendig (2018) describes Whitney's passion and persistence in playing music. "It was a little after midnight, and we'd had an especially long and intense evening playing string quartets. As the players were about to put their instruments away, I jokingly quipped 'Let's play Death and the Maiden'—well-known as one of Schubert's longest and most demanding quartets. To everyone's surprise, Hass pounced on the idea: 'Let's do it!' It was hard to dampen Hassler's enthusiasm, and we played the first movement. After that, Hass turned the page, obviously expecting to continue. We wound up playing it all, and Hass dropped me off at my apartment somewhat after 1 a.m." (p. 6).

4.5.f Focus on big picture.

Being able to seek out the "big picture" in mathematics and to simplify complex ideas is a trait characteristic of the subjects. According to Atiyah (Raussen & Skau, 2005) deep understanding is essential for making mathematics simple enough to pass on to the next generation. Atiyah states:

> Many complicated things get simple when you have the right point of view. The first proof of something may be very complicated, but when you understand it well, you readdress it, and eventually you can present it in a way that makes it look much more understandable—and that's the way you pass it on to the next generation!... Mathematics does depend on a sufficiently good grasp, on understanding of the fundamentals so that we can pass it on in as simple a way as possible to our successors. (p. 229)

Students were charmed by Atiyah's can-do attitude. Atiyah's student, Segal (1999), states:

> But probably the most fundamental thing I learnt was an attitude: for Michael, no part of mathematics worth knowing was so technical or remote that one could not be put completely in the picture by the right twenty-minute account. He was wonderful at keeping to the high ground and avoiding the mire: talking to him, one always felt a failure if one needed to use a blackboard to explain something. (p. lix)

Courant emphasized grasping essential concepts and seeking out the bigger picture. Courant's student and collaborator Lewy (as cited in Reid, 1996) states:

> Courant had the attitude that the technical details would take care of themselves. That inspired young men. It is especially important in analysis because, as analysis is a very old subject and a vast subject, it takes a long time to get to the point where one understands where the problems lie...Courant was able to make a young man feel that one could just break through. (p. 92)

Similarly, Courant's collaborator Hildebrandt (as cited in Reid, 1996) states:

> Technical difficulties-even if he was quite ignorant-never frightened him… He made one feel that any mathematical task is solvable provided that it is based on a sound and

> convincing idea. After having spoken with him about a problem, I found myself always very confident that-in the end-the solution would be within reach. (p. 305)

Courant (1988) himself writes in his introduction to his textbook: “I do not promise to save the reader the trouble of thinking, but I do seek to lead the way straight to useful knowledge, and aim at making the subject easier to grasp, not only by giving proofs step by step, but also by throwing light on the interconnections and purposes of the whole” (p. vi).

Regarding Gelfand, Zelevinsky (2013, January) states, “I have never met any other mathematician with such an ability to see the “big picture” and always go to the heart of the matter, ignoring unnecessary technicalities. He had an uncanny ability to ask the “right” questions and to find unexpected connections between different mathematical fields” (p. 48).

Glazman had the ability to get to the root of complex situations. According to Lyubich (1969):

> When somebody talked to him (Glazman) about complicated infinite-dimensional constructions, he usually asked: “And how does that look in the two-dimenstional case?” Often enough that shocking question helped to a better understanding of the mathematical situation. All the mathematical activity of this unforgettable man of exceptional talent was directed toward recognizing the simple elements in complicated matters. (p. viii)

4.5.g Emphasizing use of specific examples- hands on.

The subjects emphasize the importance of delving into specific examples to grasp the bigger picture and gain an intuitive understanding of mathematics.

Atiyah (2008) focuses on examples to understand mathematics. “To get to grips with it (a problem), there is no substitute for a hands-on approach. You should investigate special cases and try to identify where the essential difficulty lies” (p. 1002).

Courant (1964) saw the use of examples as essential.

> Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustration of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particularized substance below. (p. 43)

Nirenberg (2002) describes Lewy's process as "these ideas often occurred when Lewy was considering a very particular problem, sometimes in a paper of only a few pages. On reading his papers, time after time, one is struck by some new ingenious, but simple looking, idea or technique" (p. xiv).

Whitney focused on specific examples to gain deep understanding. Whitney (1979) states, "what I consider a basic principle in mathematical research: If a problem is difficult, don't be afraid to get your hands dirty. That is, make up various examples, simple and complex, and test out what happens, in a rather complete manner" (p. 700).

An illustration of Whitney's work with examples in gaining a deep understanding of mathematics is given by Kendrig (2018). After witnessing Whitney explore examples and counterexamples to try to disprove a new theorem to which he was introduced, Kendig observes:

> He'd taken the theorem to the mat, wrestled it, and the theorem won. I had known about that result for at least two years, but I realized that in about 15 minutes he had gained a deeper appreciation of it than I'd ever had. Whitney looked at theorems new to him in terms of a series of concrete examples, and typically developed his own theorems the same way—working from a series of concrete examples. (p. 256)

4.5.h Commitment to mathematics and its applications.

All of the subjects show commitment to the marriage of mathematics with its applications. All of the subjects practice applied mathematics at some point in their professional career (see case studies in Appendix B), and some subjects (Atiyah, Courant, Gelfand) are very vocal in their belief that mathematics must remain connected to its applications in the sciences.

Connes and Kouneiher (2019) state, “We could describe Atiyah’s journey in mathematics by saying he spent the first half of his career connecting mathematics to mathematics and the second half connecting mathematics to physics” (p. 1661). In addition, they state, “Atiyah has been influential in stressing the role of topology in quantum field theory and in bringing the work of theoretical physicists to the attention of the mathematical community” (p. 1666). Hitchin (2019) states, “Atiyah, along with colleagues such as Raoul Bott and Is Singer, played an enormous role in introducing new ideas and encouraging and teaching physicists to study quantum field theory from new points of view” (p. 1839).

Courant (1962, May 9) describes his interest in the applications of mathematics:

> So, I was always very much interested in those aspects of mathematics that provided some bridge to, (or criticism of) physicists. And, well, I was influenced by the seminars and my contact with Hilbert, and my great scientific seriousness at the beginning. ... And I also was personally influenced by Klein...So I became quite interested, again, in this aspect of the connection between the theory of functions of complex variables, and hydrodynamics, or electrodynamics, electrostatics.

Courant sees connections with applications, as well as intuition, as essential aspects of mathematics. In his preface to his textbook on differential and integral calculus, Courant (1988) explains:

> My aim is to exhibit the close connection between analysis and its applications and, without loss of rigour and precision, to give due credit to intuition as the source of mathematical truth. The presentation of analysis as a closed system of truths without reference to their origin and purpose has, it is true, an aesthetic charm and satisfies a deep philosophical need. But the attitude of those who consider analysis solely as an abstractly logical, introverted science is not only highly unsuitable for beginners but endangers the future of the subject; for to pursue mathematical analysis while at the same time turning one's back on its applications and on intuition is to condemn it to hopeless atrophy. (p. vi)

In fact, Courant's professional focus while developing the Mathematics Institute at Gottingen and the Institute of Mathematical Sciences at NYU was to connect the study of mathematics with its applications. Courant admired the rich collaboration between mathematics and physics at Gottingen and tried to reinvent this at the Courant Institute at NYU. According to Reid (1996), "Courant's own personal conception of Gottingen (was of) a place where no distinction was made between mathematics and its applications and where advanced students and faculty were like a family" (p. 185). According to the committee appointed to evaluate establishing an institute for advanced training in mathematics at NYU, "The program (at NYU) envisages research integrated to a much higher degree with teaching than is ordinarily done in American graduate schools. In this respect, as well as in emphasis on the connection between mathematics and applications, and in emphasis on teamwork, [it] differs distinctly from the average graduate school program. It is hoped that the development of the New York University Group will stimulate similar developments in other graduate schools" (Reid, 1996, p. 254).

Similarly, Gelfand built bridges between mathematics and science. "The mutual influence of Gelfand and specialists in modern quantum and relativistic physics was extremely fruitful.

The intercommunication with them was of greatest interest to him, and he acquired most important new ideas from these contacts" (Bhat, 2011).

Gelfand was active in the application of mathematics to biology. Gelfand founded a biological seminar at Moscow State University which functioned for about 20 years, where he led discourses and an exchange of ideas with biologists, physicians and mathematicians. "Gelfand's biological work is characterized by the same clarity in posing problems, the ability to find non-trivial new approaches, and the combination of concreteness and breadth of general concepts that distinguish his mathematical research" (Russian Mathematical Surveys, 1974).

Gelfand states, "I learned the importance of applied mathematics from Gauss. I think that the greatness of Gauss came in part because he had to deal with real-world problems like astronomy and so on and that Gauss admired computations" (Gefland, 2006b, p. xxi).

Glazman also worked with the applications of mathematics. Later in his career Glazman became more interested in applied mathematics, and organized a Department of Functional Analysis and Applied Mathematics at the Institute of Low Temperatures, and attracted "outstanding young mathematicians" (Lyubich & Tkachenko, 1997, p. 4). In addition, "The monograph by Akhiezer and Glazman tremendously enhanced expansion of operator theory ideas into modern mathematics and physics" (Lyubich & Tkachenko, 1997, p. 1).

Lewy worked closely with Courant on *The Courant–Friedrichs–Lewy (CFL) Condition*, a paper with importance to physics.

Whitney expressed many of his views on the practice of mathematics through his work in school education later in his career. To Whitney, a crucial element in mathematics education is for the student to experience a natural connection between mathematics and the physical world. "My most important message here is toward the educational process: You learn math best by *doing* it; especially, by *needing* it in some situation. Especially in early grades, “dirtying your hands” becomes “act out the story” (Whitney, 1979, p. 700). In addition, “Mathematics grew up because of its enormous power in applications. It is very important for children to experience this from the outset” (Whitney, 1973, p. 291). Further, “In a whole structure, things tie together, become real and meaningful, and relate to further domains. Our cutting off all extraneous matters from ‘pure mathematics’ leaves it bare, devoid of anything that can be grasped by minds not yet used to such abstraction” (Whitney, 1987, p. 239).

Above are demonstrated the themes in beliefs and values that the subjects show in their practice of mathematics. I propose that there is a connection between the attitudes and beliefs characteristic of the subjects’ work in mathematics and the nature of musical involvement. In this way, involvement in music is not tangential to the subjects’ mathematical work, but is another means of expressing the same beliefs and values essential to the mathematicians’ work in mathematics. In such a way, these practices in music and mathematics may be considered to be a part of a mathematician’s identity.

The themes described above with respect to the subjects’ practice of mathematics are all related to a few core values. The first three themes: 1) Collaborative and social; 2) Belief in the unity of mathematics- making connections between ideas; and 3) Commitment to mathematics education- making social connections are related to making connections. These are connections with other mathematicians and students and between ideas and fields. The next three themes: 4) Emphasizing personal initiative in learning and understanding mathematics- reliance on own judgement; 5) Importance of exploration and play; and 6) Focus on big picture are related to viewing mathematics as an art. This involves using intuition, exploration, play and personal judgement. The last two themes: 7) Emphasizing use of specific examples- hands on; and 8)

Commitment to mathematics and its applications are related to being grounded in examples and applications. These are the core values that characterize the subjects' practices of mathematics.

In the case of many of the subjects, these core values are reflected in the ways they engage in music. The subjects involve themselves in music socially; as an art form; and are grounded in the actual, physical practice of making and listening to music. I suggest that these core values- making connections, viewing mathematics as an art and being grounded in examples and applications- also underlie musical involvement in general, and the way the subjects demonstrated involvement in music in particular.

The ways these three core values relate to music are as follows.

1) Making connections: Music is social. The ways the subjects engage with music are social. Being social is a way of making connections. The subjects engage with other mathematicians around music through playing together and sharing recordings. Kendig credits music for bringing Whitney and himself together. Gelfand and Courant related to students through introducing them to music. Courant was attracted to certain mathematicians based on a musical connection. Courant's letters to other mathematicians show a connection around music. Whitney talked with other mathematicians about musical problems. Music is a way the subjects make connections with other mathematicians and between musical ideas and mathematical ideas (by using music and art as a metaphor for the practice of mathematics). Similarly, as mathematicians, all of the subjects valued collaboration, some in a very visible way. Courant, Gelfand, Atiyah collaborated routinely with students and colleagues, being very productive this way. Atiyah and Gelfand learn from colleagues about other areas of mathematics and about

other fields, like physics and biology. Their high value on educating the next generation is also an example of the social aspect of mathematics.

2) Viewing mathematics as an art: Music is aesthetic. As was shown above, the subjects place a high value on the aesthetic aspects of mathematics. Aesthetics is an inherent component of music. The practice of music trains the aesthetic sense, and hones the appreciation of artistic characteristics, such as beauty, intuition, personal judgement and playfulness. Atiyah and Courant state that the aesthetic sense grows stronger when cultivated through familiarity with content, whether in music or mathematics. Atiyah, Courant and Gelfand consider the aesthetic sense fundamental to the practice of mathematics. All the subjects show, or are recognized as having, a high prioritization of beauty in the practice of mathematics.

3) Being grounded in examples and applications: Music is immersive. Music is something that involves one's whole body (when playing an instrument) and senses (when listening), and absorbs one's focus. It also engages the mental capacity (as described by Euler's son-in-law, Fuss and Helmholtz in Chapter 2). Music is something to contemplate, distract, share and engage with. The act of composing music and the physical act of playing music are by their nature applications of the aesthetic and structural principles that underlie the art of music making. The subjects involve themselves in music through playing together in ensembles (Courant, Lewy, Whitney, perhaps Glazman). They involve themselves in the process of making music. Whitney states, "Music continually runs through my head" (Kendig, 2018, p. 22). Music is a recreation and distraction. For Courant, music is, "a way of transcending conflicts and disappointments" (Lax, 2003, p. 92). Atiyah spends an afternoon listening to Bach after

suffering a stroke. Courant, Lewy and Whitney play music almost every day throughout their lives. Similarly, mathematics involves deep concentration and being absorbed in a problem. Whitney works on a problem for eight years. Atiyah dreams about mathematics. Both music and mathematics consume and absorb the attention and focus of the practitioner. Subjects advocate “active involvement” in mathematics, “getting one’s hands dirty”, “experimentation” and “play”, which is also immersive.

Therefore, it is arguable, that the beliefs and values that underlie the subjects’ involvement in mathematics are the same as those that underlie their involvement in music. These common beliefs and values comprise an identity of practice that is common to the mathematicians’ practices of mathematics and music, and thus the two can be considered related. This interpretation of identity is consistent with Sfard and Prusak’s (2005) description of identity as related to aspects of a person’s behavior that are “reifying, endorsable, and significant” (p. 16-17). Further, it is in line with Sfrad and Prusak’s (2005) claim that identity is “‘a pivot between the social and the individual’ aspects of learning” (in quotations, Wenger, 1998, p. 145, as cited in Sfard & Prusak, 2005, p. 21).

4.6 Conclusion

As shown above, the mathematical identity of many of the subjects is bound up with his musical identity. This is seen in the subjects’ characterizations of the mathematician as an artist, of which the musician is a subset. This is also seen in the direct characterization of subjects as composers and musicians, and in the ways subjects’ mathematical traits reveal themselves in their musical involvement and vice versa. Finally, this is seen in the way the beliefs and values of the subjects in their practices of mathematics are analogous to the beliefs and values they bring to their involvement in music. In addition to the subjects’ involvement in music fostering connections with mathematicians and being a way to extend their aesthetic impulses, it is also a way to express the core affective elements that underlie their mathematical practices.

Chapter 5: Summary, Discussion and Recommendations

The study aimed to explore the role of music in the lives and work of mathematicians. In doing so, I identified some strong similarities between the ways mathematicians engage in both mathematics and music. These are, engaging socially in both music and mathematics, engaging in music as a lifelong activity and "companion" to mathematics, viewing the practice of mathematics as an art, prioritizing aesthetics and beauty, and utilizing common affective elements in both the practices of mathematics and music. The results show that there is a close connection between subjects' mathematical and musical identities.

The findings of the study show that musical involvement served to accomplish a number of things that related to mathematicians' practices of mathematics. These are:

1) Musical involvement served to strengthen social bonds and created a community of mathematicians around a shared interest of music. It also allowed for active, social engagement with fellow mathematicians in the form of playing and enjoying music.
2) Musical involvement served as a "parallel" means of engaging a set of beliefs, values, behaviors and priorities, also used in mathematical practice. In a way, musical involvement was an alternative "outlet" for exercising a set of affective elements similar to those used in the practice of mathematics.
3) Musical involvement served as an artistic outlet in utilizing, refining and honing artistic qualities, including aesthetics, artistry and intuition, also used in mathematics.

5.1 Affective Aspects of Mathematical Thinking

The study identified, in addition to the value of aesthetics, eight values common to the mathematicians in their practice of mathematics, and related these to musical involvement. The values identified as important to the subjects are consistent with the values identified by other leading mathematicians of the 20th century, discussed in Chapter 2. Poincaré (1908) insists that mathematics is not "a discipline purely engaged in with the intellect." He also describes the importance of viewing mathematics as a "well-ordered whole" and mentions the "special (aesthetic) sensibility that all mathematicians know" (p. 2047-2048). Poincaré and Hadamard (1945) both invoke music as a metaphor for mathematics, with Hadamard giving the example of Mozart composing. Hardy (1940) states that "significant" mathematical ideas "can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas" (p. 2029). Hardy and von Neumann (1947/1956) describe mathematics as an art, and mathematicians comparable to "other great artists" (Hardy, 1940, p. 2038).

Further, the values identified in this study are similar to Bishop's (2008a) values that underlie productive mathematical thinking and should be reinforced in classroom teaching. Bishop (2008a), based on his socio-cultural theory of mathematical thinking, identified six values, in three components, underlying mathematical thinking (p. 84-86). Table 2, below, maps the values identified in this study with Bishop's framework of values for mathematical thinking.

Table 2: Mapping of Bishop's (2008a) six values and current study's values.

Bishop (2008a): Components of Mathematical Values and Related Values	Current study: Values Common to Subject Mathematicians
1) Valuing Rationalism means emphasising argument, reasoning, logical analysis, and explanations, arguably the most relevant value in Mathematics education.	4) Emphasizing personal initiative in learning and understanding mathematics- reliance on own judgement.
2) Valuing Objectism means emphasising objectifying, concretising, symbolising, and applying the ideas of Mathematics.	5) Importance of exploration and play.
3) Valuing Control means emphasising the power of Mathematical knowledge through the mastery of rules, facts, procedures and established criteria.	6) Focus on big picture. 8) Commitment to mathematics and its applications.

4) Valuing Progress means emphasising the ways that Mathematical ideas grow and develop, through alternative theories, development of new methods and the questioning of existing ideas.	2) Belief in the unity of mathematics- making connections between ideas. 7) Emphasizing use of specific examples- hands on.
5) Valuing Openness means emphasising the democratisation of knowledge, through demonstrations, proofs and individual explanations.	1) Collaborative and social. 3) Commitment to mathematics education- making social connections.
6) Valuing Mystery means emphasising the wonder, fascination, and mystique of Mathematical ideas.	Prioritization of beauty and aesthetics.

While the values identified in the study reinforce the values identified by Bishop (2008a), the study builds on the work of Bishop by suggesting an alternative organization of the values. Whereas Bishop organizes the values around a "three component analysis of culture" (proposed by White, 1959, as cited in Bishop, 2008a, p. 84), this study suggests organizing values around connecting mathematics to an artistic involvement in music. In this way, I organize mathematical values around the three categories of:

1) Making connections: Music is social. This comprises values such as being collaborative and social, belief in the unity of mathematics- making connections between ideas, and commitment to mathematics education- making social connections.
2) Viewing mathematics as an art: Music is aesthetic. This comprises values such as emphasizing personal initiative in learning and understanding mathematics- reliance on own judgement, importance of exploration and play, focus on a big picture, relying on intuition and prioritizing aesthetics and beauty.
3) Being grounded in examples and applications: Music is immersive. This comprises values such as emphasizing use of specific examples- hands on, and commitment to

mathematics and its applications. In particular, this relates to how mathematicians connect their mathematics to their experience. The mathematics of the subjects is one that must be in tune with their intuition, their related interests (biology in the case of Gelfand and physics in the cases of Atiyah, Courant and Glazman) and their experiences of reality. Their mathematics is immersive in the sense that it connects them to their world and their interests and is highly relevant in every sense of the word.

The usefulness of this categorization is that it draws directly on the characterization of mathematics as an art. This study shows that the subjects hold the belief that mathematics is an art, and view their work as that of the artist. Atiyah (Zeki, et al., 2014) claims that the experiences of beauty in music and in mathematics are the same. Courant invokes the metaphor of music amply in his discussions of the nature of mathematics. Lyubich (Lyubich & Tkachenko, 1997) claims that Glazman's "profound musical talent revealed itself in his works" (p. 4). Such an organization emphasizes the artistic nature of the practice of mathematics and allows values of mathematical thinking to be viewed as directly related to such a characterization. In addition, the characterization of mathematics as an art presents a useful organization of affective elements for use in a school context. This characterization connects affective elements more deeply to students experiences and draws upon their similar experiences in artistic pursuits.

5.2 Framework for Connection Between Mathematics and Musical Involvement

The study also suggests a new understanding of a connection between musical involvement and mathematics. While some researchers (cited in Chapter 2) try to explain a connection between musical involvement and mathematics based on a neurological framework, (Cox & Stephens, 2006; Hetland, 2000), and others based on a "transfer" of spatial skills (Costa-Giomi, 1999; Graziano, et al., 1999; Hetland, 2000; Rauscher & Zupan, 2000) the findings of this study point to an alternative approach, in line with the socio-cultural theory of the development of values in mathematical thinking. This study points to a connection between mathematical practice and musical involvement based on shared values between the two domains and through musical involvement reinforcing affective elements that contribute to productive mathematical beliefs.

5.2.a Social Musical Involvement.

The findings of the study emphasized a communal and social aspect to subjects' musical involvement. While all of the subjects demonstrated musical involvement, the nature of how they engaged in shared musical experiences varied among subjects. For some, these were chamber group evenings with friends and colleagues. For others, these were childhood familial experiences in addition to an adult experience. For one subject, it is not documented that he played socially specifically with mathematician colleagues. Two of the six subjects did not play in instrumental groups with communities of colleagues, but did enjoy or share listening experiences. All but one of the subjects are documented as having shared social, musical experiences with other mathematicians, forming communities of mathematician-musicians. It is interesting to note the range in the ways the subjects engaged in a social aspect of music, including sharing with family, colleagues, non-colleagues and through instrumentation versus listening. Further research is warranted to better understand the social musical experiences that contribute to the musical culture that relates to mathematical values.

5.2.b Cognitive Musical Involvement.

I also consider a cognitive connection between music and mathematics/ mathematicians, as described in Chapter 3. According to Euler, listening to music is an opportunity to feel "the enjoyment of solving (mathematical) riddles" (Knobloch, 2008, p. 22), and Euler's son-in-law, Nicholas Fuss, described that Euler would “calculate the proportion of tones” (from Yushkevich’s, Bogolyubov’s and Mikhailov’s (eds.) work, as cited in Pesic, 2013, p. 36) while listening to music. Archibald (1924) agrees with Lebniz’ claim that "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers" (as quoted in Archibald, 1924, p. 1-2).

For Euler and Leibniz, involvement in music was a mathematical activity, engaging their cognitive capacity, exercising their interests in discovering proportions and “solving riddles.” This relates to Hardy’s characterization of the mathematician as “a maker of patterns...made with ideas” (Hardy, 1940, p. 2027). To Hardy, identifying proportions and patterns is the heart of what mathematicians do. The description of Euler’s engagement with music suggests that listening to, composing or playing music exercises the mathematician’s key cognitive skills and interests- that of discovering and uncovering patterns. While this study does not reveal that the subjects found a cognitive connection with music such as described here, the question of to what extent music fulfills such a cognitive role for mathematicians in modern times warrants further investigation.

5.3 Limitations of the Study

This study is an opening into the world of exploring the role of music in the lives and work of mathematicians, with implications for the education of aspiring mathematicians and mathematics students. The study suffers a limitation. The subjects are a snapshot of mathematicians who represent only a small part of the wide variety of individuals who practiced mathematics in the 20th century (and of course, before that), currently practicing and aspiring mathematicians and future mathematicians. In order to provide a deeper understanding of the role of music in the lives and works of mathematicians, studies must explore the range of individuals who comprise the population of historically and currently practicing mathematicians and aspiring and future mathematicians.

The limitation of the study is due to the methodological choice and multiple factors that affect the availability of public information about subjects. The current study relies on information in the public domain about mathematicians who practiced in the 20th century, primarily from the earlier part, prioritizes subjects with abundant public information available and applies a simplified criterion in selecting these mathematicians. Of the 114 unique winners of the Fields Medal, Abel Prize in Mathematics and Wolf Prize in Mathematics through 2019, only two women are represented. The year 2014 saw the first woman, South American and person of Indian origin to win the Fields Medal. Although there are non-prize winning women mathematicians and mathematicians of additional sociocultural and geographic backgrounds to the ones studied, who achieved distinction in mathematics in the earlier part of the 20th century, either their interest in music was not strong or there is scarce public information about such an interest.

One may speculate about what a study similar to this one would look like in one hundred years from now, when the trend of increasing diversity among mathematicians participating in the must public and recognized arenas will be more advanced. For now, a deeper understanding of the topic necessitates the study of individuals from a broad and full range of demographic, sociocultural, geographic, racial, ethnic and educational backgrounds, and with gender diversity. Using alternative methodologies, in particular interview based methods, will allow the researcher to access subjects without abundant information on their involvement in music and/or practices of mathematics already in the public domain.

5.4 Implications for Mathematics Education

The findings of the study have implications for mathematics education. I have shown that the subjects hold eight beliefs and values related to the practice of mathematics being artistic in nature, relying on intuition, aesthetic judgements and the appreciation of beauty. These values manifest themselves in both the mathematicians' practices of mathematics and involvement in music.

The study raises a number of questions. One question is, are the findings of the study applicable only to exceptional mathematicians, such as the subjects studied here? Can the conclusions of the study be applied to students and/or practitioners of mathematics at all ranges of skill and experience?

Although the study examines exceptional mathematicians, the conclusions, that there are beliefs and values that underlie mathematicians' practices that correspond to a characterization of the practice of mathematics as an art, have implications for mathematics students and practitioners at all levels. The findings provide a guide to students, teachers and practitioners of mathematics of an approach to mathematics that has correlated with success for a group of mathematicians. This approach can be emulated, adopted and cultivated by students and practitioners.

A second question is, are the findings of a relationship between mathematical involvement and musical involvement relevant only when the musical involvement is to a very advanced degree, such as is the case with the subjects of the study? To what extent is there a connection between mathematics and music when the musical involvement is more casual or at a more elementary level? Also, does a relationship between mathematics and musical involvement exist for types of music other than classical?

Again, the study points to an approach to mathematics as a musical art. Both musical and mathematical appreciation grow with experience. It is also the case that music and mathematics as an art are accessible and appreciable at various degrees at various stages of involvement. With an orientation that mathematics is an artistic endeavor, potentially, even those with an elementary appreciation of music can still find an experience of mathematics that shares some of those artistic qualities.

The findings of the study have some specific implications for the teaching of mathematics, at the primary, secondary and post-secondary levels, in multiple ways. The findings of the study:

1) **Provide an alternative characterization of the identity of the mathematician to that offered to many students in schools.** The study offers the characterization of the identity of the mathematician as that of the artist, particularly a musician. It points out that such a self-characterization is central to the mathematician's work. The study, like many mathematicians, rejects the characterization of mathematics as overly procedural, purely result focused and disconnected from the practitioner's humanity. In particular, mathematics must draw on the aesthetic senses of the individual- intuition, perception of beauty, taste, personal judgement and creativity. It should be social, aesthetic and immersive of one's humanity- an opposite characterization of mathematics to that portrayed in many school environments. Such a portrayal, experience and modelling of the practice of mathematics and the nature of the mathematician should be offered to students in schools from the earliest to the most advanced levels in order to build a productive mathematical identity and in order to cultivate mathematical expertise and train future mathematicians.

2) **Provide eight affective elements connected to musical experience to be prioritized in school teaching and culture.** The eight affective elements identified as central to mathematicians' practices of mathematics emerge from the characterization of the mathematician as an artist. They are connected to the mathematicians' experiences with their involvement in music and are reinforced for them through musical involvement. They are deeply personal and are avenues to the enjoyment of mathematics and music. Such beliefs and attitudes, aside from being reflected in curriculum, should be fostered through classroom practices and culture, and should be reflected in educational goals and assessments. Such values as collaboration, connecting mathematics within and outside the field, sharing knowledge, personal discovery, exploration and experimentation, focusing on a big picture along with the details, applying mathematics to specific examples and varying situations should be reflected in classroom processes and curricular goals. Since they draw upon aesthetic experiences, in particular involvement in music, these beliefs will be familiar to students and can be understood through students' own connections with personal experience. Such values and beliefs cannot be limited to the classroom alone. They must be fostered and encouraged in the culture of the school, and ideally, connected to passions and interests that rely on similar values, beliefs and modes of expression, that students practice on their own or with communities.

3) **Provide suggestions for curricula for primary, secondary and post-secondary courses.** Particularly in the middle, secondary and post-secondary classrooms, where testing objectives and performance standards threaten to overtake process focused learning, it is important to consider how curricula focused on fostering beliefs and values conducive to mathematical thinking may be developed. All of the subjects are described

as highly effective and beloved teachers of mathematics. Many of the subjects authored textbooks for use at the secondary, post-secondary and graduate level. These, along with the subjects' own words on their philosophies of mathematics education, are excellent resources for the development of curricula for content courses that reflect the mathematical values described here. This is certainly an avenue worth further investigation. In addition, the study provides suggestions for the development of a high level course on the history of mathematics. Such a course may be developed using some of the approaches and findings of the study, examining the beliefs and attitudes that underlie mathematical practices of mathematicians and examining the nature of the identities they hold. At the primary level, the values identified here naturally fit with educational priorities for young learners. The values identified may help reinforce the values being promoted in current practice. This is particularly valuable as young learners are being first introduced to the experience of what it means to do mathematics and forming their identities as mathematics learners.

5.5 Recommendations

This study is only the introduction to a vast potential inquiry into the topic of mathematics/ mathematicians and music in current times. There is much more research which can be done. Some further studies are suggested below.

A first avenue will be to understand if the findings of this study hold, and how they compare to findings of studies, in the contexts of a wide variety of mathematicians from other demographic, geographic, racial, ethnic and sociological backgrounds and with gender diversity. For this, it is advised to use an interview based method to access mathematicians about whom there is not yet much publicly available information. Some of the mathematicians in the current study had female students and/or children/ children-in-law who shared their interest in music and went on to professions in mathematics or music. These may be of interest for part of a future study.

A second avenue of research will be to study mathematicians with little to no musical involvement and to compare the beliefs and values that drive their mathematical practices with the subjects, of this study or another, who have musical involvement. It will also be of interest to use a survey or questionnaire approach to understand the prevalence of musical involvement among practicing mathematicians and to examine how the mathematicians view this involvement and connect this with their beliefs and values on the practice of mathematics.

A third avenue would be to place a similar study to the one conducted in the context of mathematics classrooms. Along the lines of studies on students' and teachers' affect, one could design a study to answer whether teachers/ students who have a history of musical involvement, compared to teachers/ students with no musical involvement, hold different beliefs and values about the nature of mathematics, including on mathematics as an art, the aesthetic nature of mathematics, on collaboration as an important element of mathematics, on applications of mathematics and on the unity of mathematics.

A fourth avenue of interest would be to conduct an inverse study, examining the mathematical practices of musicians/ composers. One might study the mathematical background and involvement of such musicians, their views and beliefs about mathematics and music, and whether and how they incorporate mathematics into their work as musicians. One might examine a broad range of musical genres. In particular, there is an existing literature on the connection between jazz improvisation and mathematics (see, for example, Hassanpour and Leslie, 2008), upon which one can build.

Finally, I return to the research question Whitney (Whitney, 1985) proposed to researchers: “What are students capable of if given the chance to explore situations? We badly need to show their powers; you can help!” (p. 238).

References

Afendikov, A. L. & Brushlinskii, K. V. (2009). I.M. Gelfand and applied mathematics. *Russian Mathematical Surveys, 64*, 1149-1156. http://dx.doi.org/10.1070/RM2009v064n06ABEH004660

Albers, D., Alexanderson, G., & Reid, C. (1985). *Mathematical people*. Boston: Birkhauser.

Albers, D., Alexanderson, G., & Reid, C. (1990). *More mathematical people*. Orlando, FL: Harcourt Brace Jovanovich, Inc

Albers, D. & Alexanderson, G. (2012). *Fascinating mathematical people: Interviews and memoirs.* Princeton: Princeton University Press. Doi: https://doi.org/10.1515/9781400839551

Alexandroff, Paul. (1935, May 30). [Letter to Richard Courant]. Richard Courant Papers (MC 150, Box 2, Folder 1), New York University Archives, New York University, New York, NY. 13 November 2019.

Alexandroff, Paul. (1936, November 17). [Letter to Richard Courant]. Richard Courant Papers (MC 150, Box 2, Folder 1), New York University Archives, New York University, New York, NY. 13 November 2019.

Alexandroff, Paul. (1968, January 2). [Letter to Richard Courant]. Richard Courant Papers (MC 150, Box 1, Folder 11), New York University Archives, New York University, New York, NY. 13 November 2019.

Aleksandrov, P.S. & Oleinik, O.A. (1975). In memory of Richard Courant, *Russian Mathematical Surveys, 30*, 157-178. https://iopscience-iop-org.ezproxy.cul.columbia.edu/article/10.1070/RM1975v030n04AEH001516/pdf

Archibald, R. (1924). Mathematicians and Music. *The American Mathematical Monthly, 31*(1), 1-25. doi:10.2307/2298868

Atiyah, M. (1978). The unity of mathematics. *Bulletin of the London Mathematical Society, 10*(1), 69–76. https://doi-org.ezproxy.cul.columbia.edu/10.1112/blms/10.1.69

Atiyah, M. (1982). What is geometry?, *The Mathematical Gazette, 66*(437), 179–184. https://www.jstor.org/stable/3616542

Atiyah, M. (1983). Identifying progress in mathematics. In T. Hägerstrand (Ed), *The Identification of progress in learning: (outgrowth of an international and multidisciplinary colloquium; Colmar, 24-26 March 1983)* (pp. 24–41). Cambridge a.o: Cambridge University Press. https://archive.org/details/identificationof0000unse/page/24/mode/2up

Atiyah, M. (2005). Mathematics: Art and science. *Bulletin (New Series) of the American Mathematical Society, 43*(1), 87–88. Article electronically published on October 7, 2005. https://www.ams.org/journals/bull/2006-43-01/S0273-0979-05-01095-5/S0273-0979-05 01095-5.pdf

Atiyah, M. (2008). VIII.6 Advice to a young mathematician. In T. Gowers (Ed.), *The Princeton Companion to Mathematics* (pp. 1000 – 1004). Princeton University Press. http://assets.press.princeton.edu/chapters/gowers/gowers_VIII_6.pdf

Atiyah, M. (2010). The art of mathematics, *Notices of the American Mathematical Society, 57*(1), 8. https://www.ams.org/staff/jackson/fea-atiyah.pdf

Atiyah, M. (2013). Some personal reminiscences. In J. Fauvel, R. Flood, & R. Wilson, R. (Eds.), *Oxford figures: Eight centuries of the mathematical sciences* (pp. 324-334). Oxford Scholarship Online. https://oxford-universitypressscholarship-com.ezproxy.cul.columbia.edu/view/10.1093/acprof:oso/9780199681976.001.0001/acprof-9780199681976

Atiyah, M. (2018). Preface. In J. Kouneiher (Ed.), *Foundations of mathematics and physics one century after Hilbert* (pp. vii-xi). Springer. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-3-319-6483-2%2F1.pdf

Bahr, N., & Christensen, C. (2000). Inter-domain transfer between mathematical skill and musicianship. *Journal of Structural Learning and Intelligent Systems, 14*(3), 187- 197

Bailhache, P. (1997). Music translated into mathematics: Leonhard Euler. (J. Monzo, Trans.). Paper for the conference *Problems of Translation in the 18th Century*. Nantes, France: The Center Francois Viete. Retrieved from: http://www.tonalsoft.com/monzo/euler/euler-en.aspx#text25

Bartocci, C. (2011). Michael F. Atiyah: Mathematics' deep reasons. In C. Bartocci, R. Betti, A. Guerraggio & R. Lucchetti (Eds.), *Mathematical lives* (pp. 197–208). Springer. https://link-springer-com.ezproxy.cul.columbia.edu/chapter/10.1007%2F978-3-642-1366-1_28

Bhat, R. (2011). A short biography of Israel Moiseevich Gelfand. *Resonance, 2011*, 104. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/10.1007/s12045-011-005-2.pdf

Birkhoff, G. (1956). Mathematics of aesthetics. In J. Newman (Ed.), *The world of mathematics,* (Vol. 4, 2185-2197). New York: Simon and Schuster.

Bishop, A. J. (2008a). Teachers' mathematical values for developing mathematical thinking in classrooms: Theory, research and policy. *The Mathematics Educator, 11*(1), 79-88. https://www.researchgate.net/publication/251676085_Teachers'_Mathematical_Values_fr_Developing_Mathematical_Thinking_in_Classrooms_Theory_Research_and_Policy

Bishop, A. J. (2008b). Mathematics teaching and values education– An intersection in need of research. *ZDM: the international journal on mathematics education, 31*(1), 231-238. DOI: 10.1007/978-0-387-09673-5_16 https://www.researchgate.net/publication/226231596_Mathematics_Teaching_and_Valus_Education_-_An_Intersection_in_Need_of_Research

Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics education.* Kluwer Academic Publishers. https://link-springer-com.ezproxy.cul.columbia.edu/book/10.1007%2F978-94-009-26578

Boaler, J., Wiliam, D. & Zevenbergen, R. (2000). The construction of identity in secondary mathematics education. In J.F. Matos & M. Santos (Eds.), *Proceedings of the Second International Mathematics Education and Society Conference* (pp. 192–202). Lisbon: Centre de Investigação em Educação da Faculdade de Ciências da Universidade de Lisboa.

Bogolyubov, N.N., Gindikin, S.G., Kirillov, A.A., Kolmogorov, A.N., Novikov, S.P., Faddeev, L.D.(1983). Izrail' Moiseevich Gelfand (on his seventieth birthday). *Russian Mathematical Surveys, 38*, 145-163. http://dx.doi.org/10.1070/RM1983v038n06ABEH003469

Brown, S. (1996). Towards humanistic mathematics education. In A.J. Bishop et al. (Eds.), *International handbook of mathematics education* (1289-1321). The Netherlands: Kluwer Academic Publishers.

Burton, L. (1999a). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*(2), 121-143.

Burton, L. (1999b). Mathematics and their epistemologies - and the learning of mathematics. Retrieved on March 2, 2001, from http://www.fmd.uniosnabrueck.de/ebooks/erme/cermel-proceedings/cermel_contentsl.html.

Burton, L. & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education, 32*(4), 429-453.

Chern, Shing-Shen. (1994). Hassler Whitney (23 March 1907-10 May 1989). *Proceedings of the American Philosophical Society 138*(3), 464-467. https://www.jstor.org/stable/986754 https://www.jstor.org/stable/986754?seq=1

Connes, A. & Kouneiher, J. (2019). Sir Michael Atiyah, a knight mathematician. *Notices of the American Mathematical Society, 66*(10), 1660-1671. https://www-ams-org.ezproxy.cul.columbia.edu/journals/notices/201910/rnoti-p1660.pdf.

Costa-Giomi, E. (1999). The effects of three years of piano instruction on children's cognitive development. *Research in Music Education, 47*(3), 198-212.

Cox, H., & Stephens, L. (2006). The effect of music participation on mathematical achievement and overall academic achievement of high school students. *International Journal of Mathematical Education in Science and Technology, 37*, 757-763.

Courant, Ernst. (ND, January 23). [Letter to Nina Courant]. Richard Courant Papers (MC 150, Box 1, Folder 8), New York University Archives, New York University, New York, NY. 13 November 2019.

Courant, Ernst. (1933, November 17). [Letter to Richard Courant]. Richard Courant Papers (MC 150, Box 1, Folder 4), New York University Archives, New York University, New York, NY. 13 November 2019.

Courant, Richard. (1934, February 22). [Letter to James Franck (presumed)]. Richard Courant Papers (MC 150, Box 1, Folder 9), New York University Archives, New York University, New York, NY. 13 November 2019.

Courant, Richard. (1934, May 6). [Letter to Ingrid Franck (presumed)]. Richard Courant Papers (MC 150, Box 1, Folder 9), New York University Archives, New York University, New York, NY. 13 November 2019.

Courant, Richard. (1936, March 9). [Letter to Paul Alexandroff]. Richard Courant Papers (MC 150, Box 2, Folder 1), New York University Archives, New York University, New York, NY. 13 November 2019.

Courant, R., New York University, College of Engineering. (1941, Summer). *Advanced methods in applied mathematics: lecture course* [Lecture notes]. https://babel.hathitrust.org/cgi/pt?id=mdp.39015000995798&view=1up&seq=13

Courant, R. (1950). *Dirichlet's principle, conformal mapping, and minimal surfaces*. Interscience Publishers. https://www.springer.com/gp/book/9781461299196

Courant, R. (1960). Mathematics: Its uses and rewards. *Challenge, 8*(7), pp. 29-37. https://www.jstor.org/stable/40718169.

Courant R. & Hilbert, D. (1962). *Methods of mathematical physics: Partial differential equations, volume 2*. Wiley-VCH. https://onlinelibrary-wiley-com.ezproxy.cul.columbia.edu/doi/pdf/10.1002/978352761734

Courant, R. (1962, May 9). *Richard Courant* [Interview transcript]. Interviewed by Thomas S. Kuhn & Mark Kac. American Institute of Physics, College Park, MD. www.aip.org/history-programs/niels-bohr-library/oral-histories/4562

Courant, R. (1964). Mathematics in the modern world. *Scientific American, 211*(3), 41-49. https://www-scientificamerican-com.ezproxy.cul.columbia.edu/magazine/sa/1964/09-01/

Courant, R. (1965). Professor Richard Courant's acceptance speech for the Association's Distinguished Service Award. *The American Mathematical Monthly, 72*(4), 377-379. https://www.jstor.org/stable/2313496

Courant, R. (1981). Reminiscences from Hilbert's Göttingen. *The Mathematical Intelligencer 3*, 154–164. https://doi-org.ezproxy.cul.columbia.edu/10.1007/BF03022974

Courant, R. (1988). *Differential and integral calculus, volume 1* (2nd edition) (E. J. McShane, Trans.). Interscience Publishers. (Original work published 1937). https://onlinelibrary-wiley-com.ezproxy.cul.columbia.edu/doi/pdf/10.1002/9781118033321.fmatter

Courant Lax, L. (2013). Appendix A. Hans Lewy's recovered string trio. In C. A. de Moura & C.S. Kubrusly (Eds.), *The Courant–Friedrichs–Lewy (CFL) Condition* (pp. 157-159). Springer: New York. https://link-springer-com.ezproxy.cul.columbia.edu/book/10.1007%2F978-0-8176-8394-8

Courant, R. & Fritz, J. (1989). *Introduction to calculus and analysis, volume 1*. Springer-Verlag. (Original work published 1965). https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-1-4613-895-2%2F1.pdf

Courant, R., Robbins, H., & Stewart, I. (1996). *What is mathematics?: An elementary approach to ideas and methods* (2nd ed.). London: Oxford University Press.

Courant R. & Hilbert, D. (2004). *Methods of mathematical physics: Partial differential equations, volume 1*. Wiley-VCH: Singapore https://onlinelibrary-wiley-com.ezproxy.cul.columbia.edu/doi/pdf/10.1002/978352761710

Creswell, J.W. (2007). *Qualitative inquiry and research design: Choosing among five approaches.* Thousand Oaks, CA: Sage Publications.

Cruz Morales. J. A. (2019). Last interview with Sir Michael Atiyah, *EMS Newsletter, March 2019*, 21-22. https://www.ems-ph.org/journals/show_pdf.php?issn=1027-488X&vol=3&iss=111&ran=4

D'Ambrosio, U. (1989). The visits of Hassler Whitney to Brazil. *Humanistic Mathematics Network Newsletter 4*, 8. http://scholarship.claremont.edu/hmnj/vol1/iss4/4

Darby, J., & Catterall, J. (1994). The arts and learning. *Teachers College Record, 96*(2), 299-328.

Davis, P., Hersh, R. and Marchisotto, E. (1995). *The mathematical experience*, Birkhäuser: Boston, MA.

DeBellis V. A. & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*(2), 131-147. https://www-jstor-org.ezproxy.cul.columbia.edu/stable/25472118?pq-origsite=summon&seq=1#metadata_info_tab_contents

Dreyfus, T. & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the learning of mathematics, 6*(1), 2-10.

EGCPM.com & Tatiana V. Gelfand. (2020). *Extended Gelfand correspondence program in mathematics*. https://www.egcpm.com/

Farmelo, G. (2018). *The universe speaks in numbers – Interview 2, Michel Atiyah, 1 Nov 2018*

[Video], https://grahamfarmelo.com/the-universe-speaks-in-numbers-interview-2/

Fitzpatrick, K. R. (2006). The effect of instrumental music participation and socioeconomic status on Ohio fourth-, sixth-, and ninth-grade proficiency test performance. *Journal of Research in Music Education, 54*(1), 73-84.

Fowler, G. (1989). Obituary: Hassler Whitney, geometrician: He eased 'mathematics anxiety'. *New York Times (12 May 1989),* B10. https://www.nytimes.com/1989/05/12/obituaries/hassler-whitney-geometrician-he-eased-mathematics-anxiety.html

Fritz, J. (1989). Preface. In R. Courant and J. Fritz *Introduction to calculus and analysis, volume 2*. Springer-Verlag. (Original work published 1974). https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/10.1007%2F978-1-4613 8958-3.pdf

Gelfand, I. M. (1989). Two archetypes in the psychology of man. Kyoto Inamori Foundation Lecture. http://www.israelmgelfand.com/talks/kyoto.html

Gelfand, I. M. (1990). Preface. In I. M. Gelfand, E. G. Glagoleva & A. A. Kirilov, *The method of coordinates* (L. Cohn & D. Soonke, Trans.). Birkhauser. (Original work published 1966). https://archive.org/details/I.M.GelfandE.G.GlagolevaA.A.Kirilov/mode/2up

Gelfand, I. M. (2006a). Talk given at the dinner at Royal East restaurant on September 3, 2003. In P. Etingof, V. Retakh & I. M. Singer (Eds.), *The unity of mathematics: In honor of the ninetieth birthday of I.M. Gelfand* (pp. xiiv-xiv). Birkhauser: Boston https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-0-8176-4467-3%2F1.pdf

Gelfand, I. M. (2006b). Mathematics as an adequate language. In P. Etingof, V. Retakh & I. M. Singer (Eds.), *The unity of mathematics: In honor of the ninetieth birthday of I.M. Gelfand* (pp. xv-xxii). Birkhauser: Boston https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-0-8176-4467-3%2F1.pdf

Gelfand, I. M. (2020). Introduction. In I. M. Gelfand & T. Alekseyevskaya (Gelfand), *Geometry* (pp. xvii-xxi). Birkhäuser: Basel. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-1-0716-029-7%2F1.pdf

Gelfand, T. (2020a). Preface for the series of books written by Israel Gelfand for high-school students. In I. M. Gelfand & T. Alekseyevskaya (Gelfand), *Geometry* (pp. ix-x). Birkhäuser: Basel. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-1-0716-029-7%2F1.pdf

Gelfand, T. (2020b). Preface. In I. M. Gelfand & T. Alekseyevskaya (Gelfand), *Geometry* (pp. xi-xv). Birkhäuser: Basel. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-1-0716-029-7%2F1.pdf

Gelfand, I. & Lepowsky, J. (1993). Preface. In L. Corwin, I. Gelfand, & J. Lepowsky (Eds.), *The Gelfand mathematical seminars, 1990-1992* (pp. vii-viii). Springer Science+Business Media: New York. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/bfm%3A978-1-4612-0345-2%2F1.pdf

Gelfand, I. M. & Shen, A. (1993). *Algebra*. Birkhauser. https://archive.org/details/algebra_gelfand/page/n91/mode/2up

Glazman, I. M. & Lyubich, Y. I. (1969). Finite-dimensional linear analysis. Nauka, Moscow. (English translation: MIT Press, Cambridge Mass.-London, 1974). https://archive.org/details/finitedimensiona0000glaz/page/n9/mode/2up

Graziano, A., Peterson, M., & Shaw, G. (1999, March). Enhanced learning of proportional math through music training and spatial-temporal reasoning. *Neurological Research, 21*, 139-152.

Grootenboer, P. & Edwards-Groves, C. (2019). Learning mathematics as being stirred into mathematical practices: An alternative perspective on identity formation. *ZDM Mathematics Education 51*, 433–444. https://doi-org.ezproxy.cul.columbia.edu/10.1007/s11858-018-01017-5 https://link-springer-com.ezproxy.cul.columbia.edu/article/10.1007/s11858-018-01017-5

Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly, 87*(7), 519-524. https://www-jstor-org.ezproxy.cul.columbia.edu/stable/2321415?origin=crossref&seq=1#metadata_info_tab_contents

Hannula, M.S., DiMartino, P., Pantziara, M., Zhang, Q., Morselli, F., HeydMetzuyanim, E., Lutovac, S., Raimo Kaasila, R., Middleton, J.A., Jansen, A. & Goldin, G.A. (2016). Attitudes, beliefs, motivation, and identity in mathematics education. In Hannula, M.S., DiMartino, P., Pantziara, M., Zhang, Q., Morselli, F., Heyd-Metzuyanim, E., Lutovac, S., Raimo Kaasila, R., Middleton, J.A., Jansen, A. & Goldin, G.A. (Eds.) *Attitudes, Beliefs, Motivation and Identity in Mathematics Education: An Overview of the Field and Future Directions* (pp. 1–35). London: SpringerOpen. https://doi.org/10.1007/978-3-319-32811-9_1

Hannula M., Morselli F., Erktin E., Vollstedt M. & Zhang QP. (2017). Topic Study Group No. 28: Affect, Beliefs and Identity in Mathematics Education. In G. Kaiser (Eds.), *Proceedings of the 13th International Congress on Mathematical Education* (507-510). ICME-13 Monographs. Cham: Springer.

Hardy, G.H. (1940). *A mathematician's apology.* In J. Newman (Ed.), *The world of mathematics,* (Vol. 4, 2027-2038). New York: Simon and Schuster.

Hassanpour, N. & Leslie, G. (2008). A game theoretical model for musical interaction. *International Computer Music Conference Proceedings, 2008,* 1-4.

Hechinger, F. M. (1986) About Education: Learning math by thinking. *New York Times, June 10, 1986*, C1. https://www.nytimes.com/1986/06/10/science/about-education-learning-math-by-thinkin.html

Hetland, L. (2000). Learning to make music enhances spatial reasoning. *Journal of Aesthetic Education, 34(3/4)*, 179-238.

Heyd-Metzuyanim, E. (2017). Identity as a nexus of affect and discourse in mathematical learning. *For the Learning of Mathematics, 37*(3), pp. 33-38. Association Stable URL: https://www.jstor.org/stable/10.2307/26548469 https://www-jstor-org.ezproxy.cul.columbia.edu/stable/pdf/26548469.pdf?refreqid=excelior%3Af1ecf827214df860ac15f35330017076

Hitchin, N. (2019). Memories of Sir Michael Atiyah. *Notices of the American Mathematical Society, 66*(11), 1834-1852. https://www.ams.org/journals/notices/201911/201911FullIssue.pdf

Jackson, A. (2002). Interview with Louis Nirenberg. *Notices of the AMS 49*(4), 441-449. https://www.ams.org/notices/200204/fea-nirenberg.pdf

Kendig, K. (2018). *Never a dull moment: Hassler Whitney, mathematics pioneer*. Providence: American Mathematical Society. DOI: https://doi.org/10.1017/mag.2019.132

Kilpatrick, J. (2020). *Hassler Whitney, New York, New York 1907 - Princeton, New Jersey 1989*. History of ICMI. https://www.icmihistory.unito.it/portrait/whitney.php

Kinderlehrer, D. (2002). Hans Lewy, a brief biographical sketch. In D. Kinderlehrer (Ed.), *Hans Lewy selecta, volume 1* (pp. xvii-xx). Birkhauser: Boston

King, J. (1992). *The art of mathematics.* New York: Plenum Press.

Kinney, D. W. (2008). Selected demographic variables, school music participation, and achievement test scores of urban middle school students. *Journal of Research in Music Education, 56*(2), 145-161.

Knobloch, E. (2008). Euler transgressing limits: The infinite and music theory. *Quaderns d'Història de l'Enginyeria, 9,* 9-24.

Krull, W. (1987). The aesthetic view point in mathematics. *The Mathematical Intelligencer, 9*(1), 48-52.

Lax, A. (1989). Hassler Whitney 1907–1989: Some recollections 1979–1989. *Humanistic Mathematics Network Newsletter 4,* 2–7. http://scholarship.claremont.edu/hmn j/vol1/iss4/3

Lax, P. (2003). Richard Courant January 8, 1888–January 27, 1972. In National Academy of Sciences *Biographical Memoirs: Volume 82* (pp. 78-97). Washington, DC: The National Academies Press. https://doi.org/10.17226/10683.

Lewy, Hans. (1986). Crisis in mathematics. In D. Kinderlehrer (Ed.), *Hans Lewy selecta, volume 1* (pp. lxiii-lxvi) in Hans Lewy Selecta, reprinted in Volume 1, 2002. Birkhauser: Boston.

Lewy, Helen. (2002). The music in Hans Lewy's life. In D. Kinderlehrer (Ed.), *Hans Lewy selecta, volume 1* (pp. xxi-xxiii) in Hans Lewy Selecta, reprinted in Volume 1, 2002. Birkhauser: Boston.

Lyubich, Y. & Tkachenko, V. (1997). Israel Glazman, mathematician and personality. In I. Gohberg & Y. Lyubich (Eds.), *New results in operator theory and its applications, The Israel M. Glazman memorial volume* (pp. 1-6). Springer: Basel. https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/10.1007%2F978-3-0348-8910-0.pdf

Martin, D. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency.* Mahwah, NJ: Routledge.

McDuff, D. (2013, February). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part II (pp. 162-171). *Notices of The AMS, 60*(2). https://www.ams.org/notices/201301/rnoti-p24.pdf

McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education, 19*(2), 134-141. https://www-jstor-org.ezproxy.cul.columbia.edu/stable/749407?pq-origsite=summon&sq=1#metadata_info_tab_contents

McLeod, D.B. (1989). The role of affect in mathematical problem solving. In D. McLeod and V.M. Adams (Eds.) *Affect and Mathematical Problem Solving: A New Perspective*, (pp. 20-36). Springer Verlag: New York. https://link-springer-com.ezproxy.cul.columbia.edu/book/10.1007%2F978-1-4612-3614 6

McLeod, D.B. (1992). Research on affect in mathematics education: a reconceptualization. In D.A. Grouws (Ed.) *Handbook of Research on Mathematics Learning and Teaching*, pp. 575-596. New York: MacMillan. https://www.academia.edu/1812539/Research_on_affect_in_mathematics_education_A_econceptualization

Merriam, S.B. (2015). *Qualitative research: A guide to design and implementation.* San Francisco: John Wiley & Sons

Montano, U. (2014). *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics.* Mexico City, Mexico: Springer

Nasir, N. S. (2002). Identity, goals, and learning: Mathematics in cultural practice. *Mathematical Thinking and Learning, 4*(2-3), 213-247. DOI: 10.1207/S15327833MTL04023_6

Nirenberg, L. (2002). Comments on some of Hans Lewy's work. In D. Kinderlehrer (Ed.), *Hans Lewy selecta, volume 1* (pp. xivi-xiix) in Hans Lewy Selecta, reprinted in Volume 1, 2002. Birkhauser: Boston.

Norman, D.A. (1980). Twelve issues for cognitive science. *Cognitive Science* (4), 1-32. https://www.sciencedirect.com/science/article/abs/pii/S036402138180002X

O'Connor, J.J. & Robertson, E.F. (2000, July). *Richard Courant*. MacTutor. https://mathshistory.st-andrews.ac.uk/Biographies/Courant/

O'Connor, J. J. & Robertson, E. F. (2014, November). *Michael Francis Atiyah*. MacTutor. https://mathshistory.st-andrews.ac.uk/Biographies/Atiyah/

Op 'T Eynde, P., de Corte, E. & Verschaffel, L. (2006). Accepting emotional complexity: A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics, 63*(2), 193-207. https://www-jstor-org.ezproxy.cul.columbia.edu/stable/pdf/25472122.pdf?refreqid=excelior%3Ad81fc0cb93483fa0e7cb47925036b10b

Osborne, H. (1984). Mathematical beauty and physical science. *British Journal of Aesthetics, 24*(4), 291-300.

Pesic, P. (2013). Euler's musical mathematics. *The Mathematical Intelligencer, 35*(2), 35–43. https://doi.org/10.1007/s00283-013-9369-5

Pesic, P. (2014). *Music and the making of modern science*. MIT Press Scholarship Online. DOI: 10.7551/mitpress/9780262027274.001.0001

Poincaré, H. (1908/1956). Mathematical creation. In J. Newman (Ed.), *The world of mathematics* (Vol. 4, 2041-2050). New York: Simon and Schuster.

Prada Óscar, G. (2016). Interview with Sir Michael Atiyah. *EMS Newsletter, December 2016*, 22-30. doi: 10.4171/NEWS/102/6. http://www.ems-ph.org/journals/show_pdf.php?issn=1027-488X&vol=12&iss=102&ran=6

Rauscher, F. H. & Zupan, M. A. (2000). Classroom keyboard instruction improves kindergarten children's spatial-temporal performance: A field experiment. *Early Childhood Research Quarterly, 15*(2), 215-228.

Raussen, M. & Skau, C. (2005). Interview with Michael Atiyah and Isadore Singer. *Notices of the American Mathematical Society, 52*(2), 223-231. http://www.ams.org/notices/200502/comm-interview.pdf

Reid, C. (1991). Hans Lewy·1904-1988. In P. Hilton, F. Hirzebruch & R. Remmert (Eds.), *Miscellanea mathematica* (pp. 259-267). Springer-Verlag https://link-springer-com.ezproxy.cul.columbia.edu/chapter/10.1007/978-3-642-76709-8_16

Reid, C. (1996). *Courant*. New York: Copernicus.

Resnick, L. (1988). Treating mathematics as an ill-structured discipline. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 32–60). Reston, VA: National Council of Teachers of Mathematics. https://files.eric.ed.gov/fulltext/ED299133.pdf

Roberts, S. (2016). Michael Atiyah's imaginative state of mind. *Quanta Magazine, March 3, 2016.* https://www.quantamagazine.org/michael-atiyahs-mathematical-dreams-20160303/

Retakh, V. (2013, January). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part I (pp. 24-49). *Notices of The AMS, 60*(1). https://www.ams.org/notices/201301/rnoti-p24.pdf

Retakh, V. (2013, February). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part II (pp. 162-171). *Notices of The AMS, 60*(2). https://www.ams.org/notices/201301/rnoti-p24.pdf

Retakh, V. S. & Sosinsky, A. B. (1991). A talk with Professor I. M. Gelfand. *Quantum, January/February 1991*, 20-26. http://www.israelmgelfand.com/talks/quantum_interview.pdf

Rowe, D.E. (2015). Transforming tradition: Richard Courant in Göttingen. *The Mathematical Intelligencer, 37*, 20–29. https://doi-org.ezproxy.cul.columbia.edu/10.1007/s00283-014-9522-9

Russian Mathematical Surveys. (1974). Izrail' Moiseevich Gelfand (on his sixtieth birthday). *Russian Mathematical Surveys, 29*, 3-5. http://dx.doi.org/10.1070/RM1974v029n01ABEH001276

Schoenfeld, A.H. (1985). *Mathematical Problem Solving.* Academic Press, San Diego. https://www-sciencedirect-com.ezproxy.cul.columbia.edu/book/9780126288704/matheAtical-problem-solving

Schoenfeld, A.H. (1991). Learning to think mathematically. *The Journal of Education, 196*(2), 1-38. Stable URL: https://www.jstor.org/stable/10.2307/26612611 https://www-jstor-org.ezproxy.cul.columbia.edu/stable/pdf/25472122.pdf?refreqid=excelior%3Ad81fc0cb93483fa0e7cb47925036b10b

Schoenfeld, A.H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (reprint). *Journal of Education, (196)*2 p. 2-38.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14-22. Stable URL: https://www.jstor.org/stable/3699942 https://www-jstor-org.ezproxy.cul.columbia.edu/stable/pdf/3699942.pdf?refreqid=excelsor%3A946c1a3a17084e736d05a141c6e75b6e

Segal, G. B. (1999). Being a graduate student of Michael Atiyah. *Asian Journal of Mathematics, 3*(1), lix.

Silver, E.A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions, in E.A. Silver (ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 247-266). Lawrence Erlbaum.

Silver, E. & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D.B. McLeod and V.M. Adams (Eds.), *Affect and mathematical problem solving* (59-74). New York: Springer-Verlag.

Simons Center for Geometry and Physics. (November 7, 2011). *Sir Michael Atiyah interview* [Video]. http://media.scgp.stonybrook.edu/video/redirect.php?f=scgp_interview-atiyah_qtp.mp4w=800&h=600

Sinclair, N. (2002). *Mindful of beauty: The roles of the aesthetic in the doing and learning of mathematics* (Doctoral dissertation). Retrieved from ProQuest Dissertations Publishing. NQ69394. https://search-proquest-com.tc.idm.oclc.org/pqdtglobal/docview/305454597/53AD2AB4C9C64B3CPQ/3?accountid=14258

Smith, C.S. (1960). *Leonhard Euler's 'Tentamen Novae Theoriae Musicae': A translation and commentary* (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses Global. https://tc.idm.oclc.org/login?url=https://search-proquest-com.tc.idm.oclc.org/docview/301839227?accountid=14258 Order No. 6002844

Stewart, I. (2009, November 8). Israel Gelfand obituary. *The Guardian*. https://www.theguardian.com/science/2009/nov/08/israel-gelfand-obituary

Stillwell, J. (2014). Review of *Henri Poincaré. A Scientific Biography*. *Notices of the AMS, 61*(4), 378-383.

Tabachnikov, S. (2013, February). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part II (pp. 162-171). *Notices of The AMS, 60*(2). https://www.ams.org/notices/201301/rnoti-p24.pdf

Timotin, D. (1998). Reviewed work: New results in operator theory and its applications, Operator theory: Advances and applications, by I. Gohberg, Yu. Lyubich. *Journal of Operator Theory, 40*(2), 425-427. www.jstor.org/stable/24714712

Tukey, J. W., Albers, D. J., Alexanderson, G. L., & Reid, C. (1990). Hans Lewy. In J. W. Tukey, D. J. Albers, G. L. Alexanderson & C. Reid (Eds.), *More mathematical people: Contemporary conversations* (pp. 181-194). Boston: Harcourt Brace Jovanovich https://archive.org/details/moremathematical0000unse/page/n7/mode/2up

Vaughn, K. (2000). Music and mathematics: Modest support for the oft-claimed relationship. *Journal of Aesthetic Education, 34(3/4)*, 149-166.

Vershik, A. (2013, January). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part I (pp. 24-49). *Notices of The AMS, 60*(1). https://www.ams.org/notices/201301/rnoti-p24.pdf

Vishik, M.I., Kolmogorov, A.N., Fomin, S.V. & Shilov, G.E. (1964). Izrail' Moiseevich Gelfand (on his fiftieth birthday). *Russian Mathematical Surveys, 19*(163), 163-180. http://dx.doi.org/10.1070/RM1964v019n03ABEH001151

von Neumann, J. (1947/1956). The mathematician. In J. Newman (Ed.), *The World of mathematics* (2053-2065). New York: Simon and Schuster.

Vorob'ev, A. I. (2008). Israel Moiseevich Gelfand's seminar on biology. *Russian Journal of Developmental Biology, 39*(6), 374–376. ISSN 1062-3604 https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/10.1134/S1062360408060131.pdf

Weaver, W. (1965). Award for distinguished service to Professor Richard Courant. *The American Mathematical Monthly, 72*(1), 1-2. https://www.jstor.org/stable/2312987

Web of Stories. (N.D.). *Beauty in Mathematics* [Video]. In Michael Atiyah, mathematician [Video]. https://www.webofstories.com/play/michael.atiyah/90.

Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer, 22*(3), 37-41.

Whitney, H. (1973). Are we off the track in teaching mathematical concepts? In A. G. Howson (Ed.), *Developments in mathematical education: Proceedings of the second international congress on mathematical education* (pp. 283-296). Cambridge University Press, Cambridge, UK.

Whitney, H. (1979). Mathematical education: Comment on the division of the plane by lines. *American Mathematical Monthly 86*(8), 700. DOI 10.2307/2321306. MR1539143. https://www.jstor.org/stable/2321306

Whitney, H. (1985). Letting research come naturally, *Mathematical Chronicle 14,* 1–19. http://www.thebookshelf.auckland.ac.nz/docs/Maths/PDF2/mathschron014-001.pdf

Whitney, H. (1987). Coming alive in school math and beyond. *Educational Studies in Mathematics 18*(3), 229–242. http://www.jstor.com/stable/3482607.

Whitney, H. (1988). Moscow 1935: Topology moving toward America, A century of mathematics in America. Reprinted in J. Eells and D. Toledo (Eds.), Hassler Whitney collected papers, volume 1: Contemporary mathematicians, 1992 (pp. 1-21). Birkhäuser: Boston, MA https://link-springer-com.ezproxy.cul.columbia.edu/content/pdf/10.1007%2F978-1-4612 2972-8.pdf

Whitney, H. (1989). Education is for the students' future (draft), *Humanistic Mathematics Network Newsletter 4*, 9–12. http://scholarship.claremont.edu/hmnj/vol1/iss4/5.

Zan, R., Brown, L., Evans, J., & Hannula, M.S. (2006). *Affect in mathematics education: An introduction. Educational Studies in Mathematics, 63*(2), 113-121. https://www-jstor-org.ezproxy.cul.columbia.edu/publisher/springer https://www-jstor-org.ezproxy.cul.columbia.edu/stable/pdf/25472116.pdf?refreqid=excelior%3A117fbe414960e5199e1cdbf1323f0560

Zelevinsky, A. (2013, January). In V. Retakh (Ed.), Israel Moiseevich Gelfand, Part I (pp. 24-49). *Notices of The AMS, 60*(1). https://www.ams.org/notices/201301/rnoti-p24.pdf

Zeki S., Romaya J.P., Benincasa D.M.T. & Atiyah M.F. (2014). The experience of mathematical beauty and its neural correlates. *Frontiers in Human Neuroscience, 8*(68). doi: 10.3389/fnhum.2014.00068. https://www.frontiersin.org/articles/10.3389/fnhum.2014.00068/full

Zund, J. D. (2000, February). Lewy, Hans (20 October 1904-23 August 1988). *American National Biography.* https://www-anb-org.ezproxy.cul.columbia.edu/view/10.1093/anb/9780198606697.001.001/anb-9780198606697-e-1302442;jsessionid=36E468BBF3A5B9D574C14AF2F3C78F4

Appendix A: Texts of Excerpts of Letters from the Richard Courant Archive

Alexandroff, Paul. Letter to Richard Courant. 17 November 1936:

"Ich würde mich sehr freuen, endlich von Dir oder von Nina ein Bericht über Euer Leben 'jenseits des grossen Wassers' zu bekommen. Übrigens ist das von je her rege Musik-Leben von Moskau jetzt besonders intensiv: mehrere erstklassige deutsche Dirigenten treten fortwährend in Moskau auf, darunter Kleiber, Klemperer u.a. Kleiber hat überhaupt den ständigen Ruf nach Moskau als Dirigent eines der hiesigen Orchester angenommen."

"Und ich vermisse nach wie vor die vielen Lieder, die Nina mit Deiner Begleitung sang (vor allem den "Leiermann" und "Nur wer die Sehnsucht kennt") das Es-dur Trio von Schubert und das cis-moll Quartett von Beethoven (das Einstein-Quartett nach der Terminologie von Bessel-Hagen)."

Alexandroff, Paul. Letter to Richard Courant. 2 January 1968:

"Sehr, sehr viel denke ich diese ganze Tage auch an Nina. Und alle die Tage und Abende in Eurem Hause, alle die Musik, Alles, Alles ist mir in allen Einzelheiten so ganz lebendig.

An Deinem Geburtstage werde ich mir das Forellenquintett und das Es-Dur-Trio von Schubert vorspielen und auch das c-moll-Trio (op.1) von Beethoven, was für mich von jeher das Symbol der Jugend ist. Also, lieber Simplizialkater, bleibe mit Deinen 80 Jahren noch viele, viele Jahre jung."

Courant, Richard. Letter to Ingrid Franck. 6 May 1934:

"Gieb doch mal nach Göttingen eine Nachricht, wie es Dir persönlich geht. Was macht das Klavier?"

Courant, Richard. Letter to James Franck. 22 February 1934:

"Nina und Ernst geht es gut. Ernst ist jetzt hier in seiner Klasse schon ganz obenauf. Nina macht viel und erfolgreich Musik, und erholt sich von den Druck des letzten Jahres, hat aber keine Sorgen vor dem wieder in Göttingen Existieren bis zum Spätherbst.

Auch Born geht es recht gut...

Nun Schluss für heute! Dieser Brief soll noch mit der Europa mit! Dass Ingrid Klavier spielt, freut mich ganz besonders. Ich tue es leider nicht."

Courant, Ernst. Letter to Nina Courant. 23 January ND:

"Gestern abend habe ich in Philadelphia Menuhin gehört, der das Schumannkonzert und das Brahmskonzert spielte. Das war schön! Der kann wirklich Geige spielen. Das Schumannkonzert scheint ganz schön zu sein, und das von Brahms, wie Du weisst, ist auch sehr schön. Und er hat beide wundervoll gespielt. Ausserdem gab es die Brahmsschen Variationen über ein Thema von Hadyn, die auch ganz schön sind. Alles in alles, es war ein Erlebnis, voll von Honig. Dahingegen spielte hier Donnerstag eben Maurice Maréchal Cello, und tat fast nicht als zeigen, wie gut seine Technik war. Wenigstens merkte ich sehr wenig anderes."

Appendix B: Individual Case Studies

Brief biographical sketch and professional overview of the subjects

Michael Atiyah (1929-2019) is best known for his work in algebraic topology and the codevelopment of topological *K*-theory and the Atiyah–Singer index theorem, along with Isidore Singer. Atiyah received the Fields Medal (1966) and Abel Prize (2004), along with Isidore M. Singer. The Abel Prize recognized "their discovery and proof of the index theorem, bringing together topology, geometry, and analysis, and for their outstanding role in building new bridges between mathematics and theoretical physics. Indeed, his work has helped theoretical physicists to advance their understanding of quantum field theory and general relativity" (Connes & Kouneiher, 2019, p. 1660). Atiyah was professionally affiliated with Trinity College, Cambridge, Oxford and the Institute of Advanced Study at Princeton. Atiyah identified himself most with the area of geometry.

Atiyah was born in London, England. Atiyah was schooled in Egypt through the age of 16. Atiyah attributes his love and preference for geometry to "an old-fashioned but inspiring teacher who had graduated from Oxford in 1912" and to geometry's "elegant synthetic proofs." Atiyah "became, and remained, primarily a geometer" (Connes & Kouneiher, 2019, p. 1661).

Before attending Trinity College, Cambridge, Atiyah did two years of national service. At Trinity, Atiyah made the choice to focus on mathematics over other sciences, in particular, chemistry. According to Atiyah:

> (Chemistry was) Lists of facts, just facts, you had to memorize a vast amount of material. Organic chemistry was more interesting, there was a bit of structure to it. But inorganic chemistry was just a mountain of facts in books like this. It's true that in mathematics you don't really need an enormous memory. You can work most things out for yourself,

remember a few principles. If you're good at that, then it comes easily. If you want to do other things, you've got to work hard to learn a lot of facts. There was one reason, I think. But I enjoyed thinking, I'm good at it, and will continue with it. (as quoted in O'Connor & Robertson, 2014, November)

Atiyah also did his Doctorate at Trinity College under William V. D. Hodge. Atiyah studied as a fellow at the Institute for Advanced Study in Princeton during 1955-56. Atiyah lectured at Cambridge until 1961 when he moved to Oxford until 1969. Following this, Atitya spent three years at Princeton, before returning to Oxford until 1990. At this time, Atiyah became Master of Trinity College, Cambridge and Director of the Isaac Newton Institute for Mathematical Sciences at Cambridge.

Atiyah served as President of the Royal Society from 1990 to 1995 and President of the London Mathematical Society from 1974-76, of which he was one of the founders (Raussen & Skau, 2005), receiving its De Morgan Medal in 1980. Atiyah was knighted in 1983 and made a member of the Order of Merit in 1992.

Richard Courant (1888-1972) studied under David Hilbert at Gottingen and served as Hilbert’s assistant. Courant took over for Felix Klein, upon Klein’s retirement, as a Professor at the famous university at Gottingen. Courant founded and served as Director of the Mathematics Institute at Gottingen. Later, Courant established the Courant Institute of Mathematical Sciences at New York University. Courant received the MAA’s “Award for Distinguished Service” as a recognition for efforts on behalf of education in the mathematical sciences. Courant published numerous monographs and books, including many textbooks. Courant identified himself most with the area of analysis.

Courant was born in Lublinitz, Germany (now Lubliniec, Poland), and was raised in Breslau. Courant received little education prior to studying at the König-Wilhelm Gymnasium for high school. During this time, due to family hardship, Courant supported himself by tutoring. Following this, Courant studied at University of Breslau, intending to focus on physics. “Because of the weakness of the physics faculty he gravitated toward mathematics” (Lax, 2003, p. 80). Courant (1962, May 9) describes the mathematics education he received at Breslau, which required much personal initiative:

> The best teacher, the most successful teacher I had at Breslau, was typical of the point of view of education, was a man in algebra. His great success as a teacher was really due to the fact that he didn't teach things very well. He came to the platform. There was a blackboard. In his right hand he had some chalk, in the left hand he had a wet sponge. He turns his back to the audience, and he mumbled something towards the blackboard, and scribbled something in small letters on the blackboard, cover it up with his body. And as he roved along he erased what he had written. And then the student always had to try to snatch a few words. Then there was an enormous task after class, one sat there for another half hour to try to put together the pieces. If one succeeded, one really had learned enormously much. This was the basis for my learning algebra.

By 1907, Courant moved to Gottingen to study at the university, where he became Hilbert’s assistant. Courant (1962, May 9) describes his decision to pursue mathematics at Gottingen:

> I really wanted to do physics, but I was very frustrated by my experience with the physicists in Breslau. And trying to read a little physics was also not so easy. ...In Göttingen I got into the hands of some older mathematicians of whom you know. Toeplitz

> was my mentor first, and then Fred (Hart). I was a completely green, naive boy. I came there two weeks or three weeks before the semester started just to prepare myself for the possibility to follow classes. Then I got in the Hilbert-Minkowski seminar on the mathematical foundations of physics. ... I didn't know anything, I didn't know any electro-dynamics. For these two weeks I spent all my time studying out of Föppl and also looking through some classical papers. So I could at least at the beginning follow. Then I (gave some talks) which were successful. ... And then Haar (Hilbert's assistant) shortly afterwards left Göttingen to take care of his vineyards ... Then Hilbert couldn't find anybody, and I was suddenly in the vicinity of Hilbert and that was decisive for me.

Courant, studying under Hilbert, obtained his doctorate from Göttingen in 1910. Courant served in the German army during WWI. Following this, in 1919, Courant returned to Gottingen where he served at privatdozent, then professor, taking over for Felix Klein. In 1922 Courant founded Gottingen's new Mathematics Institute and later led the construction of a dedicated building, and became the Institute's director. Courant was expelled from Gottingen under the Nazis in 1933. Courant emigrated to the United States where he was offered a position in the department of mathematics at NYU where his task was to build a graduate program. Courant eventually succeeded in forming the "Institute of Mathematical Sciences" at NYU and led the construction of a dedicated building, later known as the Courant Institute. During WWII, Courant served on the Applied Mathematics Panel through the U.S. Office of Scientific Research and Development (OSRD) (Lax, 2003). From 1948 onwards the Courant Institute published the "Communications in Pure and Applied Mathematics", "which is one of the best journals in the world on mathematical physics and analysis in the widest sense" (Aleksandrov & Oleinik, 1975, p. 161).

Inspired by Klein's idea of a book on mathematics for teachers, Courant wrote What is Mathematics with a broadened view of his audience as "the educated layman" (Reid, 1996, p. 207). According to Niels Bohr (as cited in Aleksandrov & Oleinik, 1975), "every physicist is indebted to Courant for his profound penetration into mathematical methods which he gave us, for the understanding of nature and the physical world" (p. 159). This remark is considered to be a response to the many books which Courant wrote which "taught physicists the principles of mathematical science as an instrument for physical investigations" (Aleksandrov & Oleinik, 1975, p. 159).

Israel Gelfand (1913-2009) studied under Andrei Kolmogorov at Moscow State University. Gelfand was a member of many distinguished academies of science and a recipient of the Wolf Prize in Mathematics (1978), the AMS Steele Prize (2005) and numerous other mathematical prizes. Gelfand was professionally affiliated with Moscow State University and Rutgers University. Gelfand is the author of more than 500 works, including numerous text books.

Gelfand was born in a small town in Ukraine. He was only able to complete formal schooling through age 16. Gelfand's study of mathematics in his early years (ages 12-19) was informal and independently directed. Gelfand taught himself mostly from mathematical textbooks acquired rarely and with difficulty. By 16, Gelfand was living in Moscow with relatives while mostly unemployed due to political circumstances. Gelfand worked at the Lenin library where he had access to mathematical books and students of mathematics with whom he could discuss his emerging ideas. Gelfand describes the style of his pursuit of mathematics as "pure experimentation" (Retakh & Sosinsky, 1991, p. 24). Gelfand found himself exploring questions he later was able to put into the framework of the field of mathematics. Gelfand recalls how at age 12 he drew up a table of ratios of the lengths of chords to the lengths of arcs, and only much later realized that he was basically drawing up trigonometric tables (Retakh & Sosinsky, 1991). Gelfand recalls, “When I discovered (at age 15) that the sine can be expressed algebraically as a series, a barrier came tumbling down, and mathematics became one. To this day I see the various branches of mathematics, together with mathematical physics, as a unified whole” (Retakh & Sosinsky, 1991, p. 23).

Due to Gelfand’s independent study of mathematics he ended up, by age 19, as a graduate student in mathematics at Moscow State University studying under Andrei Kolmogorov. At Moscow State University Gelfand was appointed an Assistant Professor at the Mathematics Department. Three years later, Gelfand defended his PhD thesis at Moscow State University studying under Andrei Kolmogorov. Gelfand states, “I learned even more (than from other mathematicians) from Andrei Nikolayevich Kolmogorov. In particular, I learned that a true mathematician nowadays must be a philosopher of nature” (Retakh & Sosinsky, 1991, p. 26).

Gelfand taught throughout most of his career. He taught at Moscow State University from 1932-1990. From 1989 to 1990, Gelfand was a Visiting Professor at Harvard University and at MIT. Following immigrating to the United States in 1990, Gelfand worked as a Distinguished Professor at the Department of Mathematics at Rutgers University.

An important feature of Gelfand as a mathematician and of his philosophy of mathematics is his dedication to applications for mathematics. In this way he was also highly collaborative with physicists, biologists and medical doctors. Bhat (2011) states, “The mutual influence of Gelfand and specialists in modern quantum and relativistic physics was extremely fruitful. The intercommunication with them was of greatest interest to him, and he acquired most important new ideas from these contacts.” In particular, Gelfand was active in the application of mathematics to biology. Gelfand founded a biological seminar at Moscow State University which functioned for about 20 years, where he led discourses and an exchange of ideas with biologists, physicians and mathematicians (Russian Mathematical Surveys, 1974).

An almost legendary initiative of Gelfand was the weekly mathematics seminar he established at Moscow State University, which ran for nearly 50 years, and also continued at Rutgers University (Gelfand & Lepowsky, 1993).

Gelfand established The Correspondence School of Mathematics for middle and high school students at Moscow State University in 1964, and later imitated it at Rutgers University. The goal of the school was "to provide a quality mathematical education to motivated and talented students, mostly from smaller towns and rural areas, who did not have access to specialized mathematical education available at major scientific centers" (Tabachnikov, 2013, February, p. 166). The school was free to students. Gelfand was also among the founders of the Moscow Mathematical Olympiads (Retakh, 2013, January).

Israel Glazman (1916-1968) studied at Odessa University under Mark Krein and later chaired the department of Mathematical Physics at the University of Kharkov. Later Glazman organized a Department of Functional Analysis and Applied Mathematics at the Institute of Low Temperature.

Glazman was born in Odessa and graduated from Odessa University where his teacher was Mark Krein. After a short period working in Odessa University, Glazman was drafted into the army where he served as an artillery officer at the front lines and took part in heavy military actions in Stalingrad, Kursk, and Kiev during WWII. In 1946, Glazman went to Kharkov and worked at the Department of Mathematical Physics, of which he became Chairman in 1955 (Lyubich & Tkachenko, 1997). In his later career, Glazman began to focus more on applied mathematics. He organized a Department of Functional Analysis and Applied Mathematics at the Institute of Low Temperature, where he built a team of several gifted young mathematicians (Timotin, 1998). The KGB persecuted Glazman for Zionist activities (defined as any attempt of displaying Jewish cultural life) which resulted in his death in 1968 (Lyubich & Tkachenko, 1997). There is not much information about Glazman's work, however, Timotin (1998) states, "The research work of Israel Glazman, though probably less known world wide, is of great significance. His range of interests was large: spectral theory of concrete operators, analytic operator functions, ordinary differential equations, mathematical physics, control theory, etc" (p. 426). Glazman's mathematical work was strongly influenced by the ideas and work of D. Hilbert, R. Courant, H. Weil, M. Krein and von Neumann (Lyubich & Tkachenko, 1997). There is limited biographical information on Glazman.

Hans Lewy (1904-1988), a student of and assistant to Courant at Gottingen, also studied under Hilbert, Born, Noether and Franck, as well as Hadamard in Paris. Lewy was professionally associated with Berkeley. Lewy received the Steele Prize (1979) and the Wolf Prize in Mathematics (1986). He also was elected a member of numerous academies of science. Lewy's mathematics work focused mainly on partial differential equations (Nirenberg, 2002).

Lewy was born in Breslau, Germany (now Wrocław, Poland), where he was educated before moving to the University at Gottingen to study mathematics in 1922, upon the advice of a teacher. Lewy studied violin from the age of five and was considered a prodigy. He performed publicly with an orchestra at the age of 15 and received excellent reviews. Lewy was considered to have a promising future as a musician. Lewy actively chose between a career in mathematics and music, finally choosing mathematics due to the greater freedom and variety available from a career in mathematics. Lewy studied mathematics with great independence and initiative. While a new and relatively young student at Gottingen, when Lewy found himself in lectures beyond his comprehension, he spent time between lectures in the library consulting various books to fill in the gaps in his knowledge (Tukey, Albers, Alexanderson & Reid, 1990). By contrast, Lewy describes fellow students at the time "who would believe that they understood what was going on in mathematics… because, to them, to understand mathematics meant the same as to understand a novel. They didn't realize that it meant an active mastery of the subject." (Tukey, et al., 1990, p. 182).

Lewy earned his PhD from Gottingen while studying with Courant as well as the likes of Hilbert, Born, Noether and Franck. Lewy served as an assistant to Courant as well as a *privatdozent* (lecturer) at the University of Gottingen. In 1929 Lewy studied mathematics in Rome and then Paris under a Rockefeller Foundation fellowship, where he studied with Hadamard, among others. In 1933 Lewy took a position at Brown University, funded by the Duggan Foundation. Finally, in 1935 he moved to Berkeley where he remained for over fifty years.

During WWII, Lewy worked at the Aberdeen Proving Grounds as part of the University of California contingent and the Office of Naval Research in New York. Lewy's mathematics work focused mainly on partial differential equations (Nirenberg, 2002). Lewy received the Steele Prize and the Wolf Foundation prize. He also was elected a member of numerous academies of science. (For further biographical information, see Kinderlehrer, D., 2002.)

Hassler Whitney (1907-1989) studied Physics and Music at Yale and received his PhD in Mathematics at Harvard, studying under George David Birkhoff. Whitney was professionally affiliated with the Institute for Advanced Study at Princeton and Harvard. Whitney received the

U.S. National Medal of Science (1976), the Wolf Prize in Mathematics (1983) and the Steele Prize (1985). Whitney's mathematics focused on the area of topology.

Whitney was born in New York City. Whitney was home schooled for much of his childhood, and used the flexibility and freedom of homeschooling to pursue his interests independently. Whitney focused on working with mechanical objects, mathematics and music (Kendig, 2018). Whitney would read *Popular Science* and perform mechanical challenges laid out in the magazine. Whitney would get excited by finding "regularities" (p. 23) and patterns in mathematical explorations, like when he noticed patterns of sums of digits of multiples of integers other than 9. Whitney describes this as "my first little bit of mathematical research" (p. 23).

Kendig (2018) describes Whitney's method of learning mathematics as a child as that Whitney would "guess and play around with problems that he found interesting. There was never any time pressure, and if he made mistakes, he would eventually find them and fix things up" (p. 235). In describing how such self-initiated explorations motivated Whitney mathematically, Kendig (2018) describes, "Over a half-century later he looked back on the experience as distinctly formative, and that somehow he was like a little toy train car that—perhaps accidentally—got placed on tracks and started moving forward" (p. 23).

Whitney studied Physics and Music at Yale University. Whitney's mathematics education through undergraduate school was mainly informal. As an undergraduate, the only mathematics course Whitney took was a graduate level course in complex variables, which he chose to satisfy his curiosity about questions about complex variables he had from his informal exploration of curves defined by various polynomials. Whitney was praised by the Professor of this course, "the top man in the Yale math department" (Kendig, 2018, p. 40), for his independent study of mathematics. The Professor questioned students in the class where they had studied mathematics. Whitney's response, given that he had taken no formal mathematics courses, was, hesitatingly:

> 'I—I—I studied by myself.' The Professor immediately went into roaring mode: 'You studied by yourself! THAT'S the way to learn! Look at Riemann—how did he learn mathematics? He studied by himself. Look at Weierstrass1 —how did he learn? He studied by himself. That's how you really learn mathematics, studying by yourself!' (Kendig, 2018, p. 38)

In Whitney's first formal mathematics course, he submitted a highly original and inventive solution to a homework assignment to which the Professor responded, "Mr. Whitney, you ought to be studying real variables with Wilson. I have never before seen an example worked out like this. This is pure genius!" (Kendig, 2018, p. 39)

Based on the positive response to his mathematics Whitney received at Yale, and recognizing his poor ability to memorize things, while spending three weeks during the summer of 1930 in Gottingen, Whitney made the decision not to pursue his PhD in Physics at Harvard, but rather to pursue it in Mathematics and was accepted to do so at Harvard (Kendig, 2018).

Whitney received his PhD in Mathematics from Harvard, studying under George David Birkhoff, in 1932. Whitney taught at Harvard and Princeton as a National Research Council Fellow, and then joined the faculty of Harvard in 1933 where he remained until 1952. From 1943-1945 Whitney was active as a member of the Mathematics Panel of the National Defense Research Committee, where he did meaningful work in the area of more accurate missile sights for fighter planes. Whitney "worked out the several differential equations that tied together various rapidly changing distances and angles" used in air-to-air rocketry problems, and "...pursued his equations' implications for a wide variety of situations, and what he discovered led him to conclusions basic to making improvements in aiming at moving targets" (Kendig, 2018, p. 193).

From 1952-1977, Whitney was a professor of mathematics at the Institute of Advanced Study at Princeton. A few years before retiring from the Institute of Advanced Study, Whitney embarked on a second career in Mathematics Education, which would occupy him until his death in 1989. During this period Whitney served as president of the International Commission on Mathematical Instruction (ICMI).

Whitney's mathematics focused on the area of topology. Whitney served as editor of the *American Journal of Mathematics* and *Mathematical Review* and, among other honors, received the U.S. National Medal of Science, the Wolf Foundation Prize and the Steele Prize.