On the Kähler Ricci flow, positive curvature in Hermitian geometry and non-compact Calabi-Yau metrics

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Abstract

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In this thesis, we study three problems in complex geometry. In the first part, we study the behaviour of the Kähler-Ricci flow on complete non-compact manifolds with negative holomorphic curvature. We show that Kähler-Ricci flow converges to a Kähler-Einstein metric when the initial manifold admits a suitable exhaustion function, thus improving upon a result of D. Wu and S.T. Yau [1]. These results are partially obtained in [2] and partly obtained in joint work with S. Huang, M.-C. Lee and L.-F. Tam in [3].

In the second part of this thesis, we introduce a new Kodaira-Bochner type formula for closed $(1,1)$-form in non-Kahler geometry. Based on this new formula, We propose a new curvature positivity condition in non-Kahler manifolds and proved a strong rigidity type theorem for manifolds satisfying this curvature positivity condition. We also find interesting examples non-Kahler manifolds satisfying the curvature positivity condition in a class of manifolds called Vaisman manifolds. These results are obtained in [4].

In the third part of this thesis, we study the degenerations of asymptotically conical Calabi-Yau manifolds as the Kahler class degenerates to a non-Kahler class. Under suitable hypothesis, we prove the convergence of asymptotically conical Calabi-Yau metrics to a *singular* asymptotically conical Calabi-Yau current with compactly supported singularities. Using this, we construct singular asymptotically conical Calabi-Yau metrics on non-compact singular varieties and

identify the topology of these singular metrics with the singular variety. We also give some interpretations of these asymptotically conical Calabi-Yau metrics from the point of view of physics. These results are obtained in joint work with T. Collins and B. Guo in [5].

Table of Contents

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Dedication

I dedicate this thesis to my grandmother and late grandfather.

Chapter 1: Introduction and Background

1.1 Overview

This thesis is concerned with metrics with special curvature properties in complex geometry. It is a longstanding theme in differential and complex geometry that existence of special metrics with special curvature properties reveal a lot about the topology and algebraic geometry of the underlying space. The existence and analysis of these special metrics can be often reduced to studying the behaviour of certain nonlinear elliptic and parabolic PDEs, an important example is the complex Monge-Ampere equation, which is the primary tool in the solution of the Calabi conjecture due to Yau [6]. We will also explore some variations of this theme in this thesis, in particular, the elliptic and parabolic complex Monge-Ampere equation will play a big role in this thesis.

More specifically, this thesis is concerned with the study of three problems in complex geometry. In Chapter 2, we will study the behaviour of the Kähler-Ricci flow on complete manifolds with negative holomorphic sectional curvature. In [7], Wu and Yau proved the existence of Kähler-Einstein metrics on complete Kähler manifolds of negative holomorphic sectional curvature assuming the curvature is bounded, it is natural to ask whether the Kähler-Ricci flow converges to this Kähler-Einstein metric on such manifolds and whether one can recover their results using the Kähler-Ricci flow. In the fist part of this Chapter, we prove the following theorem

Theorem 1.1.1 (Tong [2])**.** *Let (M, ω_0) be a complete Kähler manifold with bounded curvature and satisfying*

$$H_{\omega_0}(\eta) \leqslant -\kappa|\eta|^4 \tag{1.1}$$

for some $0 < \kappa$. Then the normalized Kähler-Ricci flow starting at any metric which is uniformly *equivalent to ω_0 exists for all time and converges smoothly to a complete Kähler-Einstein metric with negative scalar curvature.*

This also gives a simplified proof of the theorem of Wu and Yau using Kähler-Ricci flow. In the second half of the Chapter, we will replace the bounded curvature assumption by a weaker assumption of the existence of an exhaustion function.

Theorem 1.1.2 (Huang-Lee-Tam-Tong, [3])**.** *If we replace the bounded curvature assumption in Theorem 1.1.1 with the following assumption.*

- *There exists a smooth exhaustion function $\rho \geqslant 1$ such that*

$$\limsup_{\rho\to\infty}\left[\frac{|\partial\rho|_\omega}{\rho}+\frac{|\sqrt{-1}\partial\bar{\partial}\rho|_\omega}{\rho}\right]=0.$$

Then the same conclusion as Theorem 1.1.1 holds.

This allows us to improve upon the theorem of Wu and Yau by allowing for the possibility of unbounded curvature. In order to prove this theorem, we will also develop some techniques for starting the Kähler-Ricci flow on complete Kähler manifolds with *unbounded* curvature.

In Chapter 3, we introduce a new Kodaira-Bochner type formula for closed $(1,1)$-form for non-Kähler manifolds. Based on this new formula, We propose a new curvature positivity condition in non-Kahler manifolds, which we call Q-nonnegativity (see Chapter 2 for precise definition) and proved a rigidity theorem for such manifolds.

Theorem 1.1.3 (Tong)**.** *Suppose (X, J, g) is a compact Hermitian manifold which is Q-nonnegative, then any class in the $(1,1)$ Bott-Chern cohomology $H^{1,1}_{BC}(X)$ contains a unique representative which is parallel respect to the Bismut connection.*

We also find interesting examples non-Kahler manifolds satisfying the Q-nonegativity condition in a class of manifolds called Vaisman manifolds.

In Chapter 4, we study the degenerations of asymptotically conical Calabi-Yau manifolds as the Kahler class degenerates to a non-Kahler class. The main application of this work is that we are able to construct many examples of singular Calabi-Yau metric on quasi-projective algebraic varieties which are asymptotic to a cone on a complete end. These singular metrics are constructed by first resolving the singularities of the quasi-projective variety, solving a degenerate complex Monge-Ampere equation on the resolution, which gives the singular Calabi-Yau metric at the level of a Kahler current. We then go on to show that the metric geometry of this Calabi-Yau current is homeomorphic to the quasi-projective variety itself.

1.2 Background and Convention

Before we move on to the next 3 chapters, we will fix some conventions that we will use throughout this thesis.

Let (X, J, g) be a compact Hermitian manifold, then in local holomorphic coordinates, we can write $\omega = \sum_{i,j} g_{i\bar{j}}\sqrt{-1}dz^i \wedge d\bar{z}^j$.

On the holomorphic tangent bundle, there is a natural connection called the Chern connection, which in holomorphic coordinate is

$$\nabla_i X^j = \partial_i X^j + \Gamma_{il}^j X^l \tag{1.2}$$

$$\nabla_{\bar{i}} X^j = \partial_{\bar{i}} X^j \tag{1.3}$$

where the connection coefficients are given by

$$\Gamma_{ki}^j = g^{j\bar{m}}\partial_k g_{\bar{m}i}$$

The Torsion tensor of the Chern connection is $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and it satisfies $T(JX,Y) = T(X,JY)$. Hence the torsion has no $(1,1)$-component, and it is entirely determined by its $(2,0)$-component. In local holomorphic coordinates, this is given by

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$$

If the torsion is zero, then we say the metrics is Kähler, this is if and only if the Kähler form ω is closed.

The curvature of the Chern connection satisfies the symmetry

$$R(X, Y, W, Z) = R(JX, JY, W, Z) = R(X, Y, JW, JZ)$$

this implies the curvature has no $(2, 0)$ or $(0, 2)$ components and the entire curvature tensor is determined by the $(1, 1)$-part, which in holomorphic coordinates is given by

$$R_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}\Gamma^{l}_{ik}.$$

The Chern connection can be characterized as the unique connection such that is Hermitian i.e $\nabla g = 0$ and satisfy $\nabla^{0,1} = \bar{\partial}_{T^{1,0}M}$ where $\nabla^{0,1}$ is the $(0, 1)$ part of the connection and $\bar{\partial}_{T^{1,0}M}$ is the d-bar operator on holomorphic vector bundles.

Chapter 2: Kähler-Ricci flow and negative holomorphic sectional curvature

Suppose X is a complex manifold and ω a Hermitian metric on X. Then the holomorphic sectional curvature of ω is defined to be

$$H_\omega(\eta) = R(\eta, \bar{\eta}, \eta, \bar{\eta}) \tag{2.1}$$

and we say that the holomorphic sectional curvature is bounded above (or below) by κ if

$$H_\omega(\eta) \leqslant (\geqslant)\kappa|\eta|^4$$

for all $\eta \in T^{1,0}X$and for convenience, sometime we will just write $H_\omega \leqslant (\geqslant)\kappa$.

The holomorphic sectional curvature is an important quantity in complex geometry, it occurs very naturally in a Schwartz lemma [8] which plays an important role in the theory of hyperbolicity, in fact the Schwarz lemma implies that manifolds with negative holomorphic sectional curvature are hyperbolic. A longstanding conjecture by Kobayashi states that hyperbolic manifolds have an ample canonical bundle. Motivated by this conjecture, Yau conjectured that manifolds with negative holomorphic sectional curvature have ample canonical bundle.

In 2015, Wu and Yau [1, 9, 7] proved this conjecture by showing the existence of Kähler-Einstein metrics with negative scalar curvature on Kähler manifold (X, ω) with negative holomorphic sectional curvature, and they also established the same result in the non-compact case. Their result sheds light on the deep relationship between holomorphic sectional curvature and Ricci curvature. The key input in their approach is to solve a complex Monge-Ampere equation by method of continuity, and the key input is a crucial use of a Schwartz lemma, which they use to establish apriori estimates along the method of continuity. A different approach to study the existence of

Kähler-Einstein metrics is using the Kähler-Ricci flow. In this situation, the Kähler-Ricci flow apparoach has been explored by Nomura in [10], where he reproves the results in [1, 9] using the Kähler-Ricci flow.

In Section 2.1, we will study the behaviour of the Kähler-Ricci flow on manifold of negative holomorphic sectional curvature. As a by-product, we recover the results of [7] by showing the convergence of the normalized Kähler-Ricci flow to a complete Kähler-Einstein metric. One advantage of this approach is that it relies only on elementary maximum principle and we are able avoid some of the more sophisticated arguments used in [7]. This approach is also interesting from the point of view of parabolic PDEs because the equation obtained (2.9), is a complex monge ampere equation with a background metric which is time dependent, hence the nonlinear parabolic theory on manifolds developed in [11] does not apply. In Section 2.2, we extend the result to allow for the possibility that the curvature of the initial metric is unbounded, along the way, we establish some methods for starting the Kähler-Ricci flow from complete Kähler metrics with unbounded curvature, which may be of independent interest. The content of this section is obtained in joint work in [3].

2.1 The case of bounded curvature

In this section, we show that the Kähler-Ricci flow on a complete manifold with negative holomorphic sectional curvature and bounded curvature always exists for all time and converges to a Kähler-Einstein metric of negative holomorphic curvature, thereby recovering the result of Wu-Yau [7]. The main theorem of this section is the following

Theorem 2.1.1. *Let (M, ω_0), is a complete Kähler manifold with bounded curvature, that is*

$$\sup_M |Rm(\omega_0)| \leqslant B$$

for some constant $B > 0$, and suppose that the holomorphic sectional curvature of ω_0 is bounded *above by a negative constant,*

$$H_{\omega_0}(\eta) = R_{\omega_0}(\eta, \bar{\eta}, \eta, \bar{\eta}) \leqslant -\kappa|\eta|^4$$

for some $\kappa > 0$. Then the normalized Kähler-Ricci flow starting at ω_0 exists for all time and converges smoothly to a complete Kähler-Einstein metric ω_{KE} of negative scalar curvature on M which satisfies

1. $Ric(\omega_{KE}) = -\omega_{KE}$
2. $C^{-1}\omega_0 \leqslant \omega_{KE} \leqslant C\omega_0$
3. $|\nabla^l_{\omega_{KE}} Rm(\omega_{KE})|_{\omega_{KE}} \leqslant C_l$

for constants C and C_l, for $l = 0, 1, \ldots$ depending only on B and κ.

From the rest of Section 2.1, let us assume ω_0 is a complete Kähler metric of bounded curvature and whose holomorphic sectional curvature is bounded above by a negative constant κ.

2.1.1 Lower bound on the evolving metric

In this section, we prove a key estimate on the lower bound of the evolving metrics along the Kähler-Ricci flow.

We note that the short time existence of the Ricci flow when (M, ω_0) is complete and has bounded curvature was established by Shi in [12]. In this case, one can obtain a Ricci flow solution for short time which also has bounded curvature and satisfied his derivative estimates. The uniqueness of this solution is proved by Chen and Zhu in [13].

From now on, let $(M, \omega(t))$ denote the solution of the normalized Kähler-Ricci flow with initial metric ω_0 obtained from Shi's estimates. That is, $\omega(t)$ satisfies

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t) \tag{2.2}$$

$$\omega(0) = \omega_0 \tag{2.3}$$

and the metrics $\omega(t)$ are all complete with bounded curvature, are uniformly equivalent to the initial metric ω_0 and satisfies Shi's derivative estimates.

Before we proceed to the main content of this section, we need a version of the maximum principle on non-compact manifolds along the Ricci flow. This can be found for instance in [14], but we will prove a version that we need here.

Proposition 2.1.1. *Suppose $(M, g(t))_{t\in[0,T]}$ is a complete solution of the Kähler-Ricci flow with bounded curvature. Then for any C^2 function f which is bounded above on $M \times [0, T]$, either*

1. $\sup_{M\times[0,T]} f(x, t) = \sup_M f(x, 0)$
2. *There exist a sequence of points (x_k, t_k), such that*

$$\lim_{k\to\infty} f(x_k, t_k) = \sup_{M\times[0,T]} f \qquad \lim_{k\to\infty} |\nabla f|(x_k, t_k) = 0$$

$$(\frac{\partial}{\partial t} - \Delta)f(x_k, t_k) \geqslant -\varepsilon_k$$

for some sequence $\varepsilon_k \to 0$.

Proof. From [15, p. 26] we know that there exist a proper smooth function $\rho : M \to \mathbb{R}$ and a constant C such that $|\nabla\rho|_{\omega_0} \leqslant C$, $\rho(x) \geqslant Cr_{\omega_0}(x)$ and $|\nabla^2\rho|_{\omega_0} \leqslant C$ where $r_{\omega_0}(x)$ is the distance function from some point.

Then, if $\sup_{M\times[0,T]} f(x, t) = \sup_M f(x, 0)$, we are done, so suppose $\sup_{M\times[0,T]} f(x, t) > \sup_M f(x, 0)$. Pick points (x_k, t_k) to achieve the maximum of the function $f(x, t) - \frac{1}{k}\rho(x)$, which has a maximum since f is bounded above and $\rho(x) \geqslant Cr(x)$ goes to infinity at infinity. Then it is easy to see that $t_k \neq 0$ for k large, so we have

$$\lim_{k\to\infty} f(x_k, t_k) = \sup_{M\times[0,T]} f \qquad |\nabla f|(x_k, t_k) = \frac{1}{k}|\nabla\rho|_{g(t_k)}(x_k)$$

$$(\frac{\partial}{\partial t} - \Delta)f(x_k, t_k) \geqslant \frac{1}{k}(\frac{\partial}{\partial t} - \Delta)\rho(x_k) = -\frac{1}{k}\Delta_{g(t_k)}\rho(x_k)$$

since the metrics are uniformly equivalent along the Ricci flow, then $|\nabla\rho|_{g_0}\leqslant C$ implies $|\nabla\rho|_{g(t)}\leqslant C$. We only need to show that $|\Delta_{g(t)}\rho|\leqslant C$

$$\begin{aligned}|\nabla^2_{g(t)}\rho - \nabla^2_{g(0)}\rho| = |(\Gamma_{g(t)} - \Gamma_{g(0)})^k_{ij}\partial_k\rho)|\leqslant C|\Gamma_{g(t)} - \Gamma_{g(0)}|\leqslant C\int_0^t|\dot{\Gamma}|(s)\,ds \\ \leqslant C\int_0^t|\nabla\mathrm{Ric}(g(s))|\,ds \leqslant C\int_0^t \frac{1}{s^{1/2}}\,ds \leqslant C\end{aligned}$$

and since $|\nabla^2_{g(0)}\rho|\leqslant C$, we conclude that $|\nabla^2_{g(t)}\rho|\leqslant C$ is always bounded, which means $|\Delta_{g(t)}\rho|\leqslant C$. □

Now we are ready to prove the key estimate on the lower bound of the evolving metrics $\omega(t)$. This is an adaptation of the Schwarz lemma calculation of Wu and Yau in [1]. The original calculation is based on Yau's generalized Schwarz lemma [16] and Royden's improvement [8] to the setting of holomorphic sectional curvature.

Proposition 2.1.2. *Along the flow, we have*

$$\omega_0 \leqslant \max(n, \frac{2n}{(n+1)\kappa})\omega(t) \tag{2.4}$$

for all time t, as long as the flow exists.

Proof. We compute the evolution of the quantity $S = \mathrm{tr}_{\omega(t)}\,\omega_0$, adopting the notation that $\omega_0 = \sqrt{-1}\hat{g}_{i\bar{j}}\,dz^i \wedge d\bar{z}^j$ and $\omega(t) = \sqrt{-1}g_{i\bar{j}}\,dz^i \wedge d\bar{z}^j$, and we put a hat over all quantities and objects associated to the metric ω_0

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) S &= \frac{\partial}{\partial t} g^{i\bar{j}} \hat{g}_{i\bar{j}} - g^{k\bar{l}} \partial_k \partial_{\bar{l}} (g^{i\bar{j}} \hat{g}_{i\bar{j}}) \\
&= -g^{i\bar{l}} g^{k\bar{j}} (\frac{\partial}{\partial t} g_{k\bar{l}}) \hat{g}_{i\bar{j}} + g^{k\bar{l}} g^{i\bar{j}} \hat{R}_{k\bar{l}i\bar{j}} - g^{i\bar{n}} g^{m\bar{j}} R_{m\bar{n}} \hat{g}_{i\bar{j}} - g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{n}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}} \\
&= g^{i\bar{l}} g^{k\bar{j}} g_{k\bar{l}} \hat{g}_{i\bar{j}} + g^{k\bar{l}} g^{i\bar{j}} \hat{R}_{k\bar{l}i\bar{j}} - g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{n}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}} \\
&= S + g^{k\bar{l}} g^{i\bar{j}} \hat{R}_{k\bar{l}i\bar{j}} - g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{n}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}}
\end{aligned}
$$

By Royden's lemma. [8, pg. 552] we have

$$
g^{k\bar{l}} g^{i\bar{j}} \hat{R}_{k\bar{l}i\bar{j}} \leqslant -\frac{n+1}{2n} \kappa S^2
$$

so

$$
\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) S \leqslant S - \frac{n+1}{2n} \kappa S^2 - g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{n}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}} \tag{2.5}
$$

and it follows

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) \log S &= \frac{\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) S}{S} + \frac{|\nabla S|_{g^2}}{S^2} \\
&\leqslant 1 - \frac{n+1}{2n} \kappa S + \frac{|\nabla S|_{g^2}}{S^2} - \frac{g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{l}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}}}{S}
\end{aligned}
$$

we need the following algebraic identity, which was one of the steps in Yau's proof of the Calabi conjecture. The proof can be found in [6, p.349]

$$
\frac{|\nabla S|_{g^2}}{S^2} - \frac{g^{k\bar{l}} g_{m\bar{n}} \hat{g}_{i\bar{j}} \hat{\nabla}_k g^{i\bar{l}} \hat{\nabla}_{\bar{l}} g^{m\bar{j}}}{S} \leqslant 0
$$

we obtain the following differential inequality for S.

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)\log S \leqslant 1 - \frac{n+1}{2n}\kappa S \tag{2.6}$$

Since S is bounded above by our assumption, we can apply the maximum principle to this inequality and get a uniform estimate for S up to any time,

$$S \leqslant \max(n, \frac{2n}{(n+1)\kappa})$$

and this implies

$$\omega_0 \leqslant \max(n, \frac{2n}{(n+1)\kappa})\omega(t)$$

giving the desired lower bound for the metrics $\omega(t)$. □

2.1.2 Upper bound for the metric

In this section we will provide an upper bound for the metric. To do this, we need to define a potential φ and write the Kähler-Ricci flow equation as an equation of the potential φ. Define

$$\varphi(x,t) := e^{-t}\int_0^t e^s \log\frac{\omega(s)^n}{\omega_0^n}ds \tag{2.7}$$

differentiating this equation gives

$$\frac{\partial}{\partial t}\varphi = \log\frac{\omega(t)^n}{\omega_0^n} - \varphi \tag{2.8}$$

Then if we let $\tilde{\omega}(t) = e^{-t}(\omega_0 - i\partial\bar{\partial}\log\omega_0^n) + i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi$, by (2.8), $\tilde{\omega}$ satisfies the ODE,

$$\frac{\partial}{\partial t}\tilde{\omega}(t) = -\mathrm{Ric}(\omega(t)) - \tilde{\omega}(t)$$

and since $\tilde{\omega}(0) = \omega_0$, we can conclude from uniquess of solutions to ODE that $\tilde{\omega}(t) = \omega(t)$. Hence the potential φ satisfies the equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{(e^{-t}(\omega_0 - i\partial\bar{\partial}\log\omega_0^n) + i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi)^n}{\omega_0^n} - \varphi \tag{2.9}$$

Proposition 2.1.3. *There exist a constant $C > 0$ depending only on B and κ in theorem 2.1.1, such that*

$$|\varphi| \leqslant C \qquad \textit{and} \qquad |\dot{\varphi}| \leqslant Cte^{-t}$$

for any t.

Proof. We compute the evolution of $\dot{\varphi}$,

$$\begin{aligned}
\frac{\partial}{\partial t}\dot{\varphi} &= \frac{\partial}{\partial t}\left(\log\frac{\omega^n(t)}{\omega_0^n} - \varphi\right) = \tilde{g}^{i\bar{j}}\frac{\partial}{\partial t}\tilde{g}_{i\bar{j}} - \dot{\varphi} \\
&= \operatorname{tr}_{\omega(t)}\left[\frac{\partial}{\partial t}(e^{-t}(\omega_0 - i\partial\bar{\partial}\log\omega_0^n) + i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi)\right] - \dot{\varphi} \\
&= \Delta_{\omega(t)}\dot{\varphi} - e^{-t}\operatorname{tr}_{\omega(t)}(\omega_0 - i\partial\bar{\partial}\log\omega_0^n) - \dot{\varphi}
\end{aligned}$$

we can rewrite this as

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)(e^t\dot{\varphi}) = -\operatorname{tr}_{\omega(t)}(\omega_0 + \operatorname{Ric}(\omega_0)) \tag{2.10}$$

by the previous section we know that $\operatorname{tr}_{\omega(t)}\omega_0$ is bounded. Also by our assumptions, we know that the curvature tensor at the initial time is bounded so $|\operatorname{Ric}(\omega_0)|_{\omega_0}$ and $\operatorname{tr}_{\omega(t)}\omega_0$ are both bounded, which gives a bound

$$|\operatorname{tr}_{\omega(t)}\operatorname{Ric}(\omega_0)| \leqslant C$$

so the right hand side of (2.10) is bounded

$$\left|\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)(e^t\dot{\varphi})\right| \leqslant C \tag{2.11}$$

we can apply the maximum principle to this and get

$$|e^t\dot{\varphi}|\leqslant Ct \implies |\dot{\varphi}|\leqslant Cte^{-t}$$

then integrating this gives the bound on the potential φ. □

Corollary 2.1.1. *There exist a constant $C > 0$, depending only on the constants B and κ in theorem 2.1.1, such that*

$$C^{-1}\omega_0 \leqslant \omega(t) \leqslant C\omega_0$$

for any t, as long as the flow exists.

Proof. From the equation (2.8), we know that

$$\omega(t)^n = e^{\varphi+\dot{\varphi}}\omega_0^n$$

and by our proposition, we know that $\varphi + \dot{\varphi}$ is bounded for all time, hence we have

$$\omega(t)^n \leqslant C\omega_0^n$$

and from the previous section, we also have

$$\omega_0 \leqslant C\omega(t)$$

If we choose an orthonormal frame on the tangent space such that $(g_0)_{i\bar{j}} = \delta_{ij}$ and $g_{i\bar{j}} = \lambda_i\delta_{ij}$, then the inequalities can be written as

$$\prod_{i=1}^{n}\lambda_i \leqslant C$$

and

$$\frac{1}{\lambda_i} \leqslant C$$

so combining them we get

$$\lambda_j = \frac{\prod_{i=1}^n \lambda_i}{\prod_{i\neq j} \lambda_i} \leqslant C^n$$

this gives the upper bound for λ_i, which gives an upper bound for $\omega(t)$. □

From this, the long time existence and convergence can be obtained from standard estimates of the Kähler-Ricci flow. For example, we can apply local estimates in [17] applied to the holomorphic coordinate systems constructed in [7] to obtain the result. For the sake of completeness, we will provide an argument here, the argument is elementary in that it uses only the maximum principle.

2.1.3 Long Time Existence

In this section, we show the flow exists for all time and the curvature and its covariant derivatives are all bounded. The estimates are purely local and uses only the maximum principle.

Proposition 2.1.4. *The flow $(M, g(t))$ exists for all time, and there exist constants C_l depending only on the constants B and κ from theorem 2.1.1 such that*

$$\sup_M |\nabla^l_{g(t)} Rm(g(t))|_{g(t)} \leqslant C_l$$

for any $t \in [1, \infty)$.

Proof. Suppose the flow exists on a maximal time interval $[0, T_{\max})$, it suffices for us to bound the curvature of $\omega(t)$ are bounded up to the final time $T_{\max}$. Then by Shi's theorem, we can then extend the flow past time $T_{\max}$ and we get our desired bounds on the covariant derivatives of curvature.

First we need an estimate on the derivative of the metric, such estimates were used by Yau in his solution of the Calabi conjecture [6].

Consider the following quantity

$$S = |\hat{\nabla} g|^2_g = g^{i\bar{l}} g^{n\bar{j}} g^{k\bar{m}} \hat{\nabla}_i g_{\bar{j}k} \hat{\nabla}_{\bar{l}} g_{\bar{m}n} = |T|^2_g$$

where $\hat{\nabla}$ is the connection with respect to some fixed reference metric $\hat{g}$, and $T_{ik}^l = \Gamma_{ik}^l - \hat{\Gamma}_{ik}^l$ is the difference of the connections. S satisfy the following identity

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)S &= -|\nabla T|^2 - |\overline{\nabla} T|^2 + S - 2\mathrm{Re}(\langle g^{a\bar{b}} \nabla_a \hat{R}_{i\bar{b}k}^l, T_{ik}^l \rangle) \\
&= -|\nabla T|^2 - |\partial_{\bar{k}} T_{ij}^l|^2 + S - \hat{\nabla}\hat{R} \star T - \hat{R} \star T \star T \\
&= -|\nabla T|^2 - |\mathrm{Rm}(g(t)) - \mathrm{Rm}(\hat{g})|^2 + S - \hat{\nabla}\hat{R} \star T - \hat{R} \star T \star T \qquad (2.12)
\end{aligned}
$$

and

$$
(\frac{\partial}{\partial t} - \Delta)\,\mathrm{tr}_{\hat{g}}\, g = -\,\mathrm{tr}_{\hat{g}}\, g - g^{k\bar{l}} \hat{R}_{k\bar{l}}^{i\bar{j}} g_{i\bar{j}} - g^{n\bar{m}} g^{k\bar{l}} \hat{g}^{i\bar{j}} \hat{\nabla}_k g_{i\bar{m}} \hat{\nabla}_{\bar{l}} g_{n\bar{j}} \qquad (2.13)
$$

see [18] or [19] for the calculations. By Shi's theorem, we know the flow must exist for at least some definite amount of time $t \in [0, \varepsilon)$ where ε depends only on the constant B in theorem 2.1.1. So we can now fix $\hat{g} = g(\varepsilon/2)$ for instance, then by Shi's theorem, the curvature of $\hat{g}$ and its covariant derivatives are all bounded by constants depending only on the initial curvature bound B.

Then the identity (2.12) gives

$$
(\frac{\partial}{\partial t} - \Delta)S \leqslant S - \hat{\nabla}\hat{R} \star T - \hat{R} \star T \star T \leqslant C(S+1)
$$

and from (2.13),

$$
(\frac{\partial}{\partial t} - \Delta)\,\mathrm{tr}_{\hat{g}}\, g \leqslant M - M^{-1}S
$$

so we have

$$
(\frac{\partial}{\partial t} - \Delta)(S + A\,\mathrm{tr}_{\hat{g}}\, g) \leqslant (C - AM^{-1})S + MA + C
$$

Letting $A = 2CM$ and applying the maximum principle, we obtain that $S \leqslant C$ for some uniform constant C.

Now we bound the curvature, the norm of the curvature tensor satisfies an inequality, [19, Lemma 2.12]

$$(\frac{\partial}{\partial t} - \Delta)|\mathrm{Rm}| \leqslant C(|\mathrm{Rm}|^2 + 1) \tag{2.14}$$

for C depending only on dimension. And from (2.12),

$$(\frac{\partial}{\partial t} - \Delta)S \leqslant D - |\mathrm{Rm}(g(t))|^2$$

so we have

$$(\frac{\partial}{\partial t} - \Delta)(|\mathrm{Rm}| + NS) \leqslant C(|\mathrm{Rm}|^2 + 1) + N(D - |\mathrm{Rm}(g)|^2) \tag{2.15}$$

$$= (C - N)|\mathrm{Rm}(g)|^2 + C + ND \tag{2.16}$$

picking $N = 2C$. and applying the maximum principle gives the desired bound on cuvature. Then the bounds on covariant derivatives of curvature follow by Shi's estimates. □

2.1.4 Convergence

Proposition 2.1.5. *There exist constants $C_l' > 0$ depending only on B and κ from theorem 2.1.1 such that*

$$|g(t)|_{C^l(M, g(1))} \leqslant C_l'$$

for all $t \in [1, \infty)$.

Proof. The proof follows a similar method as the one used to bound the derivative of the metric under the Kähler-Ricci flow in [18]. We will denote $\hat{g} = g(1)$ and put a hat on all quantities associated to $\hat{g}$. Consider the quantities

$$T_{ij}^k = \Gamma_{ij}^k - \hat{\Gamma}_{ij}^k$$

and define

$$(M_l)^k_{i_1\ldots i_{l-1}ij} = \nabla^{l-1}_{i_1\ldots i_{l-1}} T^k_{ij}$$

and

$$S_l = |M_l|^2_g$$

we would like to obtain bounds for all the S_l, this is the content of the following lemma.

Lemma 2.1.1. *For every $l \geqslant 0$, there exist constants D_l so that the quantities S_l satisfies the following inequality*

$$(\frac{\partial}{\partial t} - \Delta)S_l \leqslant -S_{l+1} + D_l(S_l + 1) \tag{2.17}$$

the constant $D_l > 0$ depend only on C_l from proposition 2.1.4 and C in corollary 2.1.1. Then there exist constant $B_l \geqslant 0$ depending only on D_l such that

$$S_l \leqslant B_l.$$

for all $t \in [1, \infty)$

Proof. We prove this by induction on l, for $l = 0$, this is true by the calculation in [18]. Suppose the lemma is true for any $l < k$, then we compute

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\Delta)S_k &= \sum_{l=1}^{k-1} g^{i_1\bar{j}_1}\dots R^{i_l\bar{j}_l}\cdots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}g^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad + g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}R^{p\bar{q}}g^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad + g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}R^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad - g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}g^{r\bar{s}}R_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad + kS_k + \langle\frac{\partial}{\partial t}M_k, M_k\rangle + \langle M_k, \frac{\partial}{\partial t}M_k\rangle - |\nabla M_k|^2 - |\overline{\nabla}M_k|^2 \\
&\quad - \langle\overline{\Delta}M_k, M_k\rangle - \langle M_k, \Delta M_k\rangle && (2.18) \\
& && (2.19)
\end{aligned}
$$

where $\Delta = g^{a\bar{b}}\nabla_{\bar{b}}\nabla_a$ and $\overline{\Delta} = g^{a\bar{b}}\nabla_a\nabla_{\bar{b}}$ is its conjugate. Then we have

$$
\begin{aligned}
\Delta M_k &= \overline{\Delta}M_k + g^{a\bar{b}}[\nabla_{\bar{b}}, \nabla_a]M_k && (2.20) \\
&= \overline{\Delta}M_k - \sum_{l=1}^{k-1} R^c_{i_l}(M_k)^m_{i_1\dots i_{l-1}ci_{l+1}\dots i_{k-1}pr} \\
&\quad + R^c_p(M_k)^m_{i_1\dots i_{k-1}cr} + R^c_p(M_k)^m_{i_1\dots i_{k-1}pc} - R^m_c(M_k)^c_{i_1\dots i_{k-1}pr} && (2.21)
\end{aligned}
$$

and in particular we have

$$
\begin{aligned}
\langle M_k, \Delta M_k\rangle &= \langle M_k, \overline{\Delta}M_k\rangle + \sum_{l=1}^{k-1} g^{i_1\bar{j}_1}\dots R^{i_l\bar{j}_l}\cdots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}g^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad + g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}R^{p\bar{q}}g^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad + g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}R^{r\bar{s}}g_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} \\
&\quad - g^{i_1\bar{j}_1}\dots g^{i_{k-1}\bar{j}_{k-1}}g^{p\bar{q}}g^{r\bar{s}}R_{m\bar{n}}(M_k)^m_{i_1\dots i_{k-1}pr}\overline{(M_k)^n_{j_1\dots j_{k-1}qs}} && (2.22)
\end{aligned}
$$

so combining the computations above, we have

$$(\frac{\partial}{\partial t} - \Delta)S_k = -|\nabla M_k|^2 - |\overline{\nabla} M_k|^2 - 2\mathrm{Re}\langle(\frac{\partial}{\partial r} - \overline{\Delta})M_k, M_k\rangle + kS_k \tag{2.23}$$

$$\leqslant -S_{k+1} - 2\mathrm{Re}\langle(\frac{\partial}{\partial r} - \overline{\Delta})M_k, M_k\rangle + kS_k \tag{2.24}$$

next we compute,

$$(\frac{\partial}{\partial t} - \overline{\Delta})M_k = (\frac{\partial}{\partial t} - \overline{\Delta})\nabla^{k-1}T \tag{2.25}$$

$$= \nabla^{k-1}(\frac{\partial}{\partial t} - \overline{\Delta})T + R \star \nabla^{k-1}T + \sum_{j=1}^{k-1} \nabla^j R \star \nabla^{k-1-j}T \tag{2.26}$$

$$= -\nabla^{k-1}(g^{a\bar{b}}\nabla_a \hat{R}^l_{i\bar{b}p}) + R \star \nabla^{k-1}T + \sum_{j=1}^{k-1} \nabla^j R \star \nabla^{k-1-j}T \tag{2.27}$$

$$= -\nabla^{k-1}(\hat{\nabla}^{\bar{b}} \hat{R})_{i\bar{j}p} + \nabla^{k-1}(T \star \hat{R}) + R \star \nabla^{k-1}T + \sum_{j=1}^{k-1} \nabla^j R \star \nabla^{k-1-j}T \tag{2.28}$$

$$= \hat{R} \star \nabla^{k-1}T + R \star \nabla^{k-1}T + Q(T, \ldots, \nabla^{k-2}T, R, \ldots, \nabla^{k-1}R, \hat{R}, \ldots, \hat{\nabla}^k \hat{R}) \tag{2.29}$$

where Q is some expression involving only the quantities in the bracket, in particular this term is bounded by the induction hypothesis. So by the induction hypothesis, we have

$$|(\frac{\partial}{\partial t} - \Delta)M_k| \leqslant C(|\nabla^{k-1}T|+1) = C(|M_k|+1) \tag{2.30}$$

then from (2.24), we get

$$(\frac{\partial}{\partial t} - \Delta)S_k \leqslant -S_{k+1} + C_k(S_k + \sqrt{S_k}) \tag{2.31}$$

$$\leqslant -S_{k+1} + C_k(S_k + 1) \tag{2.32}$$

this proves the first part of the lemma. Next, from this we have

$$(\frac{\partial}{\partial t} - \Delta)(S_k + AS_{k-1}) \leqslant (C_k - A)S_k + C_k + AC_{k-1}(S_{k-1} + 1)$$

and by choosing $A = C_k + 1$ and using the induction hypothesis that S_{k-1} is bounded, we get

$$(\frac{\partial}{\partial t} - \Delta)(S_k + AS_{k-1}) \leqslant -S_k + C_k + AC_{k-1}(B_{k-1} + 1)$$

applying the maximum principle, we get the bounds $S_l \leqslant B_l$, where we can set B_l to be $C_k + AC_{k-1}(B_{k-1} + 1)$. □

So we have uniform bounds on S_k, which means bounds on all derivatives of T in the holomorphic directions. But we claim that actually all derivatives of T are bounded. To see this, note that the identity $\nabla_{\bar{i}} T^l_{jk} = \hat{R}^l_{\bar{i}jk} - R^l_{\bar{i}jk}$ allows us to get rid of a derivative in the antiholomorphic direction and replace it by curvature terms, which we know are bounded and have bounded derivatives. So to bound the mixed derivatives, we can first commute the derivatives to move the antiholomorphic ones in front and then use this identity and the fact that the curvature and all its derivatives are bounded. From this we can conclude that $|T|_{C^k(M,g(t))} \leqslant C_k$ it follows that $|T|_{C^k(M,\hat{g})} \leqslant C_k$, so we get uniform bounds on the C^k norms of T with respect to the fixed background metric. Then from the following identity

$$T^k_{ij} g_{\bar{m}k} = \hat{\nabla}_i g_{\bar{m}j}$$

we can get uniform bounds on the C^k norm of g. □

Now we can prove the main theorem of this section.

Proof of theorem 2.1.1. By (2.1.3), we know that $\varphi(t)$ convergences in C^0 to some limit φ_∞, which is continuous, and $\dot{\varphi}$ converges to 0 uniformly. Now from our definition of φ, we have

$$\varphi(x,t) := e^{-t} \int_0^t e^s \log \frac{\omega(s)^n}{\omega_0^n} ds$$

and since $\omega(s)$ is bounded in C^k norm by our previous lemma, so the right hand side is bounded in C^k norm and we get uniform bounds of $|\varphi|_{C^k(M,g_0)}$ for all k. Hence the convergence of φ to φ_∞ is actually in C^∞. So we can take a limit of the following equation

$$\dot{\varphi} = \log \frac{[e^{-t}(\omega_0 - i\partial\bar{\partial}\log\omega_0^n) + i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi]^n}{\omega_0^n} - \varphi$$

and we get

$$0 = \log \frac{(i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi_\infty)^n}{\omega_0^n} - \varphi_\infty$$

hence $\omega_\infty = i\partial\bar{\partial}\log\omega_0^n + i\partial\bar{\partial}\varphi_\infty$ is Kähler-Einstein and the bounds on the curvature and its covariant derivatives follows from the C^∞ convergence of the metrics. □

2.2 The case of unbounded curvature

In this section, we will extend the results of the last section to the case of possibly unbounded curvature assuming the existence of an exhaustion function. This is based on joint work with S. Huang, M.C. Lee and L.F. Tam [3]. We first give a rather general condition for a normalized Kähler-Ricciflow to converge to a Kähler-Einstein metric.

Theorem 2.2.1. *Suppose there is a complete noncompact Hermitian metric h on a complex manifold M^n compatible with the complex structure J such that the torsion T and the holomorphic sectional curvature H_h satisfies*

$$H_h + \frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \leqslant -k \tag{2.33}$$

for some $k > 0$. Then any long-time solution of normalized Kähler-Ricciflow $g(t)$ will converge in C^∞_{loc} to the unique Kähler-Einsteinmetric $g_\infty = -Ric(g_\infty)$. In particular, there is no Ricci flat Kählermetric on M compatible with the same complex structure J.

Here $\widehat{\nabla}$ is the derivative with respect to the Chern connection of h. See [20] for more details on the Chern connection, its torsion and curvature. See also [21] for a related assumption on the Hermitian metric.

By the theorem, to obtain a Kähler-Einstein metric, in some cases it is sufficient to obtain a longtime solution to the Kähler-Ricciflow. In this respect, we will prove the following:

Theorem 2.2.2. *Let (M^n, g_0) be a complete noncompact Kählermanifold and h be a fixed complete Hermitian metric on M such that the following hold.*

(i) There exists a smooth exhaustion function $\rho \geqslant 1$ such that

$$\limsup_{\rho\to\infty}\left[\frac{|\partial\rho|_h}{\rho}(1+|\widehat{\nabla}g_0|_h)+\frac{|\sqrt{-1}\partial\bar{\partial}\rho|_h}{\rho}\right]=0;$$

(ii) the holomorphic sectional curvature of h and torsion of h satisfy

$$H_h+\frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h\leqslant -k$$

for some constant $k \geqslant 0$;

(iii) there exists $\alpha > 1$ such that on M, $\alpha^{-1}g_0 \leqslant h \leqslant \alpha g_0$, $|\widehat{T}|_h \leqslant \alpha$;

Then there is $\beta(n,\alpha) > 0$ such that the Kähler-Ricciflow has a complete solution $g(t)$ on $M \times [0,+\infty)$ with $g(0) = g_0$ and satisfies

$$\beta h \leqslant g(t)$$

on $M \times [0,+\infty)$.

Remark 2.2.1. It is known that if M has bounded curvature, then it will support an exhaustion function ρ with bounded gradient and Hessian [12, 22]. Hence if h is uniformly equivalent to a complete Hermitian metric with bounded Riemannian curvature and bounded torsion, then condition (i) in the theorem will be satisfied. See also a recent result in [23]. Hence condition (i) is more general than the condition that the curvature is bounded for Kähler metrics.

Putting the two theorems together we get the following generalization of the theorem of Wu and Yau.

Corollary 2.2.1. *Let (M, ω_0) be a complete Kähler manifold satisfying $H_{\omega_0} \leqslant -\kappa$ for some $\kappa > 0$ and the following condition*

- *There exists a smooth exhaustion function $\rho \geqslant 1$ such that*

$$\limsup_{\rho \to \infty} \left[\frac{|\partial \rho|_\omega}{\rho} + \frac{|\sqrt{-1}\partial\bar{\partial}\rho|_\omega}{\rho} \right] = 0.$$

Then the normalized Kähler-Ricci flow exists for all time and converges smoothly to a unique complete Kähler-Einstein metric with negative scalar curvature.

The main difficulty in this setting is with starting the flow, since the short-time existence theory of Shi does not apply when the initial curvature is unbounded. In order to start the Kähler-Ricci flow, we will use the techniques developed in [24] where we run the Chern-Ricci flow on a conformally changed manifold constructed from a large open set of the original manifold, and then take a limit to obtain a complete Ricci flow. The complete flow will then be obtained as a limit when we perform the conformal change on larger and larger open sets. The key estimate will be an estimate on the uniform existence time for Chern-Ricci flow with holomorphic sectional curvature bounded above, which we establish in Section 2.2.3. The estimate follows from a generalization of the Schwartz lemma to the Hermitian case, which we will present in Section 2.2.2.

2.2.1 A short time existence lemma

In this section, we review the short-time existence for the complete Chern-Ricci flow for Hermitian manifolds with bounded geometry. We consider Hermitian metrics because as we shall see, we will need to work with Hermitian metrics later on, since a conformally changed Kähler metric may no longer be Kähler. Let (M^n, g_0) be a complete noncompact Hermitian manifold with complex dimension n. In the following, connection and curvature will be referred to the Chern connection and curvature with respect to the Chern connection. When the torsion vanishes, the Chern connection coincides with the Levi-Civita connection. For basic facts on the Chern connection and curvature of Hermitian manifolds, we refer readers to [20] for example. The Chern-Ricci flow is the following

$$\begin{cases} \frac{\partial}{\partial t} g_{i\bar{j}} = & -R_{i\bar{j}}; \\ g(0) = & g_0. \end{cases} \tag{2.34}$$

where $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g(t))$ is the Chern-Ricci curvature of $g(t)$. This equation is equivalent to the following parabolic complex Monge-Ampère equation:

$$\begin{cases} \frac{\partial}{\partial t}\psi = & \log \dfrac{(\omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega_0^n}; \\ \psi(0) = & 0. \end{cases} \tag{2.35}$$

More precisely, if $g(t)$ is a solution to (2.34), let

$$\psi(x,t) = \int_0^t \log\left(\frac{\omega^n(x,s)}{\omega_0^n(x)}\right) ds \tag{2.36}$$

where $\omega(t)$ and ω_0 are the associated (1,1) forms of $g(t)$, g_0 respectively. Then ψ satisfies (2.35). One can see that $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi$. Conversely, if ψ is a smooth solution to (2.35) so that $\omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0$, then $\omega(t)$ defined by the above relation satisfies (2.34). We will say that ψ is the solution of (2.35) corresponding to the solution $g(t)$ of (2.34).

Let us recall the following definition of bounded geometry:

Definition 2.2.1. Let (M^n, g) be a complete Hermitian manifold. Let $k \geqslant 1$ be an integer and $0 < \alpha < 1$. g is said to have bounded geometry of order $k+\alpha$ if there are positive numbers r, κ_1, κ_2 such that at every $p \in M$ there is a neighbourhood U_p of p, and local biholomorphism ξ_p from $D(r)$ onto U_p with $\xi_p(0) = p$ satisfying the following properties:

(i) the pull back metric $\xi_p^*(g)$ satisfies:

$$\kappa_1 g_e \leqslant \xi_p^*(g) \leqslant \kappa_2 g_e$$

where g_e is the standard metric on $\mathbb{C}^n$;

(ii) the components $g_{i\bar{j}}$ of $\xi_p^*(g)$ in the natural coordinate of $D(r) \subset \mathbb{C}^n$ are uniformly bounded in the standard $C^{k+\alpha}$ norm in $D(r)$ independent of p.

(M, g) is said to have bounded geometry of infinity order if instead of (ii) we have for any k, the k-th derivatives of $g_{i\bar{j}}$ in $D(r)$ are bounded by a constant independent of p. g is said to have bounded geometry of infinite order on a compact set Ω if (i) and (ii) are true for all k for all $p \in \Omega$.

In [24], it has been shown that when (M, g_0) has bounded geometry of infinite order, the Monge-Ampère equation (2.35) and hence the Chern-Ricci flow equation (2.34) has a short time solution on M.

Lemma 2.2.1 (see [25, 24])**.** *Let (M^n, g_0) be a complete noncompact Hermitian metric. Suppose g_0 has bounded geometry of infinite order, then* (2.34) *has a solution $g(t)$ on $M \times [0, S]$ for some $S > 0$ and there is a constant $C > 0$ such that $C^{-1}g_0 \leqslant g(t) \leqslant Cg_0$.*

2.2.2 A Schwartz lemma for Chern-Ricci flow

Now we will prove a Schwartz lemma for the Chern-Ricci flow. The calculations are analogous to the ones in Proposition 2.1.2, except now we only assume the metrics are Hermitian and hence we have to take care of the torsion terms.

Let (M^n, g_0) be a Hermitian manifold, and let $g(t)$ be a solution of the Chern-Ricci flow with initial metric $g(0) = g_0$ and h is another Hermitian metric on M. Now we wish to obtain some a-priori estimates for $g(t)$. First we list some evolution equations which are related to the Chern-Ricci flow. Let

$$\Delta u := g^{i\bar{j}} u_{i\bar{j}}.$$

The following lemma concerns the evolution equation for the lower bound on evolving metric g with respect to fixed metric h while in [20, Proposition 3.1], Tosatti-Weinkove considered the upper bound of g.

Lemma 2.2.2. *Let $\Lambda = \mathrm{tr}_g\, h = g^{i\bar{j}} h_{i\bar{j}}$. Then the evolution equation of Λ is given by*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda = \mathbf{(I)} + \mathbf{(II)} + \mathbf{(III)}. \tag{2.37}$$

where

$$
\begin{aligned}
(\mathbf{I}) &= -h_{k\bar{l}}g^{i\bar{j}}g^{p\bar{q}}\Psi^k_{pi}\overline{\Psi^l_{qj}} + 2Re\left[g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\Psi^{\bar{s}}_{\bar{l}\bar{q}}(T_0)_{pi\bar{s}}\right];\\
(\mathbf{II}) &= g^{l\bar{k}}g^{j\bar{i}}g^{q\bar{p}}h_{j\bar{k}}(T_0)_{\bar{p}\bar{i}r}\left[\hat{T}_{ql\bar{s}}h^{r\bar{s}} - (T_0)_{ql\bar{s}}g^{r\bar{s}}\right]\\
&\quad + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\left[\hat{\nabla}_p(T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}}(T_0)_{pi\bar{q}}\right];\\
(\mathbf{III}) &= g^{i\bar{j}}g^{p\bar{q}}\hat{R}_{p\bar{q}i\bar{j}}.
\end{aligned}
$$

Here T_0 and $\hat{T}$ are the torsion of metric g_0 and h respectively and

$$\Psi^k_{ij} := \hat{\Gamma}^k_{ij} - \Gamma^k_{ij}.$$

where $\hat{\Gamma}, \Gamma$ are the Chern connections of h and g respectively.

In particular,

$$(\mathbf{I}) \leqslant h_{p\bar{r}}h_{c\bar{q}}h^{k\bar{a}}g^{s\bar{r}}g^{c\bar{d}}g^{i\bar{j}}g^{p\bar{q}}(T_0)_{si\bar{a}}(T_0)_{\bar{d}\bar{j}k}.$$

Moreover, evolution equation of $\log\Lambda$ *is given by:*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\Lambda = (\mathbf{IV}) + \Lambda^{-1}\Big[(\mathbf{II}) + (\mathbf{III})\Big] \tag{2.38}$$

with

$$
\begin{aligned}
(\mathbf{IV}) &\leqslant \Lambda^{-1}h_{p\bar{r}}h_{c\bar{q}}h^{k\bar{a}}g^{s\bar{r}}g^{c\bar{d}}g^{i\bar{j}}g^{p\bar{q}}(T_0)_{si\bar{a}}(T_0)_{\bar{d}\bar{j}k}\\
&\quad + 2\Lambda^{-2}Re\Big[h_{p\bar{r}}g^{a\bar{r}}g^{i\bar{l}}g^{p\bar{q}}(T_0)_{ia\bar{l}}\partial_{\bar{q}}\Lambda\Big].
\end{aligned}
$$

Proof.

$$\partial_t\,\mathrm{tr}_g\, h = g^{i\bar{q}}g^{p\bar{j}}h_{i\bar{j}}R_{p\bar{q}}.$$

$$
\begin{aligned}
\Delta \operatorname{tr}_g h &= g^{i\bar{j}} g^{p\bar{q}} \nabla_{\bar{q}} \nabla_p h_{i\bar{j}} \\
&= g^{i\bar{j}} g^{p\bar{q}} \nabla_{\bar{q}} \left(\Psi^k_{pi} h_{k\bar{j}} \right) \\
&= g^{i\bar{j}} g^{p\bar{q}} \left[\left(R^k_{p\bar{q}i} - \hat{R}^k_{p\bar{q}i} \right) h_{k\bar{j}} + \Psi^k_{pi} \Psi^{\bar{l}}_{\bar{q}\bar{j}} h_{k\bar{l}} \right].
\end{aligned}
$$

Using the fact that the torsion T of $g(t)$ satisfies $T_{ij\bar{k}} = (T_0)_{ij\bar{k}}$, we have

$$
\begin{aligned}
R_{p\bar{q}i\bar{l}} &= R_{i\bar{l}p\bar{q}} - \nabla_p T_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} T_{pi\bar{q}} \\
&= R_{i\bar{l}p\bar{q}} - \nabla_p (T_0)_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} (T_0)_{pi\bar{q}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
g^{i\bar{j}} g^{p\bar{q}} h_{k\bar{j}} R^k_{p\bar{q}i} &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} R_{p\bar{q}i\bar{l}} \\
&= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[R_{i\bar{l}p\bar{q}} - \nabla_p (T_0)_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} (T_0)_{pi\bar{q}} \right] \\
&= g^{i\bar{j}} g^{k\bar{l}} h_{k\bar{j}} R_{i\bar{l}} - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\nabla_p (T_0)_{\bar{q}\bar{l}i} + \nabla_{\bar{l}} (T_0)_{pi\bar{q}} \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\Lambda \\
&= -h_{k\bar{l}}g^{i\bar{j}}g^{p\bar{q}}\Psi^k_{pi}\overline{\Psi^l_{qj}} + g^{i\bar{j}}g^{p\bar{q}}\hat{R}_{p\bar{q}i\bar{j}} + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\left[\nabla_p(T_0)_{\bar{q}\bar{l}i} + \nabla_{\bar{l}}(T_0)_{pi\bar{q}}\right] \\
&= -h_{k\bar{l}}g^{i\bar{j}}g^{p\bar{q}}\Psi^k_{pi}\overline{\Psi^l_{qj}} + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\left[\Psi^r_{pi}(T_0)_{\bar{q}\bar{l}r} + \Psi^{\bar{s}}_{\bar{l}\bar{q}}(T_0)_{pi\bar{s}}\right] \\
&\quad + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\left[\hat{\nabla}_p(T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}}(T_0)_{pi\bar{q}}\right] + g^{i\bar{j}}g^{p\bar{q}}\hat{R}_{p\bar{q}i\bar{j}} \\
&= -h_{k\bar{l}}g^{i\bar{j}}g^{p\bar{q}}\Psi^k_{pi}\overline{\Psi^l_{qj}} + 2\mathrm{Re}\left[g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\Psi^{\bar{s}}_{\bar{l}\bar{q}}(T_0)_{pi\bar{s}}\right] \\
&\quad + g^{l\bar{k}}g^{j\bar{i}}g^{q\bar{p}}h_{j\bar{k}}(T_0)_{\bar{p}\bar{i}r}\left[\hat{T}_{ql\bar{s}}h^{r\bar{s}} - (T_0)_{ql\bar{s}}g^{r\bar{s}}\right] \\
&\quad + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}h_{k\bar{j}}\left[\hat{\nabla}_p(T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}}(T_0)_{pi\bar{q}}\right] + g^{i\bar{j}}g^{p\bar{q}}\hat{R}_{p\bar{q}i\bar{j}}.
\end{aligned}
$$

From this, the first part of the lemma follows. Thus,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\log\Lambda &= \Lambda^{-1}\left(\frac{\partial}{\partial t}-\Delta\right)\Lambda + \Lambda^{-2}g^{i\bar{j}}\partial_i\Lambda\,\partial_{\bar{j}}\Lambda \\
&= \frac{1}{\Lambda}\left[(\mathbf{I}) + \frac{1}{\mathrm{tr}_g\, h}|\partial\Lambda|^2\right] + \frac{1}{\Lambda}\Big[(\mathbf{II}) + (\mathbf{III})\Big] \\
&=(\mathbf{IV}) + \frac{1}{\Lambda}\Big[(\mathbf{II}) + (\mathbf{III})\Big].
\end{aligned}
$$

In case that g_0 is Kähler, it was shown by Yau [6] that the first bracket term is nonpositive. In the Hermitian case, we follow a generalization of this argument in [20]. For any tensor C, we consider the following nonnegative quantity:

$$
\begin{aligned}
K &:= h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \left(\Psi^k_{pi} - \frac{\delta^k_i}{\mathrm{tr}_g\, h} \partial_p \Lambda + C^k_{pi} \right) \left(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{\delta^{\bar{l}}_{\bar{j}}}{\mathrm{tr}_g\, h} \partial_{\bar{q}} \Lambda + C^{\bar{l}}_{\bar{q}\bar{j}} \right) \\
&= h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi^k_{pi} \Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{1}{\mathrm{tr}_g\, h} |\partial \, \mathrm{tr}_g\, h|^2 + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} \left(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{\delta^{\bar{l}}_{\bar{j}}}{\mathrm{tr}_g\, h} \partial_{\bar{q}} \Lambda \right) \\
&\quad + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^{\bar{l}}_{\bar{q}\bar{j}} \left(\Psi^k_{pi} - \frac{\delta^k_i}{tr_g h} \partial_p \Lambda \right) + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\mathbf{I}) + \frac{1}{\Lambda} |\partial \Lambda|^2 &= -K + 2\mathrm{Re} \left[g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \Psi^{\bar{s}}_{\bar{l}\bar{q}} (T_0)_{pi\bar{s}} \right] \\
&\quad + 2\mathrm{Re} \left[h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} \Psi^{\bar{l}}_{\bar{q}\bar{j}} \right] \\
&\quad + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}} - 2\Lambda^{-1} \mathrm{Re} \left[h_{k\bar{l}} g^{i\bar{l}} g^{p\bar{q}} C^k_{pi} \partial_{\bar{q}} \Lambda \right] \\
&= -K + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}} - 2\Lambda^{-1} \mathrm{Re} \left[h_{k\bar{l}} g^{i\bar{l}} g^{p\bar{q}} C^k_{pi} \partial_{\bar{q}} \Lambda \right] \\
&\quad + 2\mathrm{Re} \left[\Psi^{\bar{l}}_{\bar{q}\bar{j}} g^{p\bar{q}} g^{i\bar{j}} (C^k_{pi} h_{k\bar{l}} + g^{k\bar{r}} h_{p\bar{r}} (T_0)_{ik\bar{l}}) \right].
\end{aligned}
$$

Therefore, if we choose the tensor C to be

$$
C^q_{pi} = -g^{k\bar{r}} h^{q\bar{l}} h_{p\bar{r}} (T_0)_{ik\bar{l}}
$$

then the last term vanished. Hence,

$$
\begin{aligned}
(\mathbf{IV}) &\leqslant \Lambda^{-1} h_{p\bar{r}} h_{c\bar{q}} h^{k\bar{a}} g^{s\bar{r}} g^{c\bar{d}} g^{i\bar{j}} g^{p\bar{q}} (T_0)_{si\bar{a}} (T_0)_{\bar{d}\bar{j}k} \\
&\quad + 2\Lambda^{-2} \mathrm{Re} \Big[h_{p\bar{r}} g^{a\bar{r}} g^{i\bar{l}} g^{p\bar{q}} (T_0)_{ia\bar{l}} \partial_{\bar{q}} \Lambda \Big].
\end{aligned}
$$

The estimate (**I**) follows the same line by considering a simpler quantity

$$
K = h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} (\Psi^k_{pi} - C^k_{pi})(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - C^{\bar{l}}_{\bar{q}\bar{j}}).
$$

□

In [1], Wu-Yau made use of the Royden's Lemma [8] which relates the holomorphic sectional curvature with a bisectional curvature quantity. We follows their idea and generalize the Royden's Lemma for Hermitian metrics. In the following, $|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h(x)$ at a point x is defined as:

$$|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h = \max|\widehat{\nabla}_{\bar i}\widehat{T}_{jl\bar k}| \tag{2.39}$$

where the maximum is taken over all unitary frames e_i of h at x. Define $|\widehat{\nabla}_{\bar\partial}T_0|_h$ similarly.

Lemma 2.2.3. *Let (M, h) be a Hermitian manifold and g be another Hermitian metric on M. Suppose that the holomorphic sectional curvature of h at x is bounded above by $\kappa(x)$. Suppose $\kappa(x) \leqslant \kappa_0$. Then we have*

$$g^{i\bar j}g^{k\bar l}\hat R_{i\bar jk\bar l} \leqslant \left(\frac{n+1}{2n}\kappa + \frac{1}{2}|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h\right)(\mathrm{tr}_g\, h)^2 + \frac{1}{2}\kappa_0\left[-\frac{1}{n}(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right].$$

Proof. Following the proof in [8] without appealing the symmetry of $\hat R$, we can deduce that at x,

$$g^{i\bar j}g^{k\bar l}\hat R_{i\bar jk\bar l} + g^{i\bar j}g^{k\bar l}\hat R_{i\bar lk\bar j} \leqslant \kappa(tr_g h)^2 + \kappa g^{i\bar j}g^{k\bar l}h_{k\bar j}h_{i\bar l}.$$

By the "Kähler" identity for the Chern curvature, e.g. see [20]. We have

$$\begin{aligned}
g^{i\bar{j}}g^{k\bar{l}}\hat{R}_{i\bar{j}k\bar{l}} &= g^{i\bar{j}}g^{k\bar{l}}(\hat{R}_{i\bar{l}k\bar{j}} - \hat{\nabla}_i\hat{T}_{\bar{j}\bar{l}k}) \\
&= \frac{1}{2}g^{i\bar{j}}g^{k\bar{l}}(\hat{R}_{i\bar{l}k\bar{j}} + \hat{R}_{i\bar{j}k\bar{l}} - \hat{\nabla}_i\hat{T}_{\bar{j}\bar{l}k}) \\
&\leqslant \frac{\kappa}{2}(tr_g h)^2 + g^{i\bar{j}}g^{k\bar{l}}\left(\frac{\kappa}{2}h_{k\bar{j}}h_{i\bar{l}} - \frac{1}{2}\hat{\nabla}_i\hat{T}_{\bar{j}\bar{l}k}\right) \\
&\leqslant \frac{1}{2}(\kappa(x)-\kappa_0)\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar{j}}g^{k\bar{l}}h_{kj}h_{i\bar{l}}\right] + \frac{1}{2}(\mathrm{tr}_g\, h)^2|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \\
&+ \frac{1}{2}\kappa_0\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar{j}}g^{k\bar{l}}h_{kj}h_{i\bar{l}}\right] \\
&\leqslant \frac{1}{2}(\kappa(x)-\kappa_0)(1+\frac{1}{n})(\mathrm{tr}_g\, h)^2 + \frac{1}{2}(\mathrm{tr}_g\, h)^2|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \\
&+ \frac{1}{2}\kappa_0\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar{j}}g^{k\bar{l}}h_{kj}h_{i\bar{l}}\right] \\
&= \left(\frac{n+1}{2n}\kappa + \frac{1}{2}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h\right)(\mathrm{tr}_g\, h)^2 + \frac{1}{2}\kappa_0\left[-\frac{1}{n}(\mathrm{tr}_g\, h)^2 + g^{i\bar{j}}g^{k\bar{l}}h_{kj}h_{i\bar{l}}\right]
\end{aligned}$$

□

Combining this with Lemma 2.2.2, we have the following corollary. This is the key estimate will allow us to start the Chern-Ricci flow even when the curvature is not bounded.

Corollary 2.2.2. *With the same assumptions and notation as in Lemma 2.2.2. Suppose the holomorphic sectional curvature of h is bounded above by* $\kappa(x)$ *at x and suppose* $\frac{n+1}{2n}\kappa(x)+\frac{1}{2}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|(x) \leqslant \kappa_0$ *for some* $\kappa_0 \geqslant 0$ *for all x. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda \leqslant c(n)\left(\Lambda^4|T_0|_h^2 + \Lambda^3(|T_0|_h|\widehat{T}|_h + |\widehat{\nabla}_{\bar{\partial}}T_0|_h) + \Lambda^2\kappa_0\right)$$

for some constant $c(n) > 0$ *depending only on n.*

To get a C^0 estimate, it is useful to consider the Chern scalar curvature of $g(t)$ which gives us information on the derivatives of the volume form.

Lemma 2.2.4. *Under Chern-Ricci flow*

$$\partial_t g = -Ric$$

the Chern scalar curvature $R = g^{i\bar{j}}R_{i\bar{j}}$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) R = |Ric|^2 \geqslant \frac{1}{n}R^2.$$

Proof.

$$\begin{aligned}\partial_t R &= \partial_t(g^{i\bar{j}}R_{i\bar{j}}) \\ &= R^{i\bar{j}}R_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}(\partial_t \log\det g) \\ &= |\mathrm{Ric}|^2 + \Delta R.\end{aligned}$$

The inequality can be observed by taking a coordinate chart at p such that $g_{i\bar{j}} = \delta_{ij}$ and $R_{i\bar{j}} = \lambda_i\delta_{ij}$. Then it follows immediately by Cauchy inequality. □

We now need a version of the maximum principle that applies without the bounded curvature assumption. .

Lemma 2.2.5. *Let (M^n, h) be a complete noncompact Hermitian manifold satisfying condition:* **(a1)***There exists a smooth positive real exhaustion function ρ such that $|\partial\rho|_h^2 + |\sqrt{-1}\partial\bar{\partial}\rho|_h \leqslant C_1$. Suppose g_0 is another Hermitian metric uniformly equivalent to h and $g(t)$ is a solution to the Chern-Ricci flow with initial metric $g(0) = g_0$ on $M \times [0, S)$. Assume for any $0 < S_1 < S$, there is $C_2 > 0$ such that*

$$C_2^{-1}h \leqslant g(t)$$

for $0 \leqslant t \leqslant S_1$. Let f be a smooth function on $M \times [0, S)$ which is bounded from above such that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leqslant 0$$

on $\{f > 0\}$. Suppose $f \leqslant 0$ at $t = 0$, then $f \leqslant 0$ on $M \times [0, S)$.

Proof. For any $\varepsilon > 0$, if $\sup_{M\times[0,T]}(f - \varepsilon\rho - 2\varepsilon C_1C_2t) > 0$, then there is (x_0, t_0) with $t_0 > 0$ such that $f - \varepsilon\rho - 2\varepsilon C_1C_2t \leqslant 0$ on $M \times [0, t_0]$ and $f - \varepsilon\rho - 2\varepsilon C_1C_2t = 0$ at (x_0, t_0). In particular,

$f(x_0,t_0)>0$. Hence at (x_0,t_0), we have

$$0 \leqslant \left(\frac{\partial}{\partial t}-\Delta\right)(f-\varepsilon\rho-2\varepsilon C_1C_2 t)<0,$$

which is impossible. Since ε is arbitrary, this completes the proof. $\square$

Next we give a local estimate on the Chern scalar curvature's lower bound. Note that here we do not need global bounds on the Hessian and gradient of the exhaustion function ρ. The estimate only depends on those bounds on a compact set.

Lemma 2.2.6. *Suppose h is a fixed Hermitian metric with a smooth positive real exhaustion function ρ and $g(t)$ is a solution to the Chern-Ricci flow on $M\times[0,S]$ with $g(t)\geqslant \alpha^{-1}h$ for some $\alpha>1$. Then for any $0<r_1<r_2$, there exists $C>0$ depending only on n,α and $\sup_{U_{r_2}}(|\partial\rho|+|\sqrt{-1}\partial\bar{\partial}\rho|)$ such that for any $x\in U_{r_1}$ and $t\in[0,S]$, we have*

$$R(x,t)\geqslant -\max\left\{C[(r_2-r_1)^{-2}+1],\ \sup_{\rho(y)<r_2} R_-(y,0)\right\}.$$

Here R_- is the negative part of R and $U_r=\{x\in M:\ \rho(x)<r\}$.

Proof. Let φ be a cutoff function on $\mathbb{R}$ such that $\varphi\equiv 1$ on $(-\infty,1]$, vanishes outside $(-\infty,2]$ and satisfies $\varphi^{-1}|\varphi'|^2\leqslant 100$ and $\varphi''\geqslant -100\varphi$. Define

$$\Phi(x)=\varphi\left(\frac{\rho(x)+r_2-2r_1}{r_2-r_1}\right).$$

When the function ΦR achieves its local minimum at (x_0,t_0) in which we may assume $R(x_0,t_0)<0$

and $t_0 > 0$, it satisfies the following.

$$\begin{aligned}
0 &\geqslant \left(\frac{\partial}{\partial t} - \Delta\right)(\Phi R) \\
&= \Phi\left(\frac{\partial}{\partial t} - \Delta\right) R - R\Delta\Phi - 2Re\left(g^{i\bar{j}}\partial_i\Phi\, \partial_{\bar{j}} R\right) \\
&\geqslant \frac{1}{n}\Phi R^2 - R\left[\frac{\varphi''}{(r_2 - r_1)^2}|\partial\rho|^2 + \frac{\varphi'}{r_2 - r_1}\Delta\rho - 2\frac{(\varphi')^2}{(r_2 - r_1)^2\varphi}|\partial\rho|^2\right] \\
&\geqslant \frac{1}{n}\Phi R^2 + CR[(r_2 - r_1)^{-2} + 1].
\end{aligned}$$

Hence, at its minimum point (x_0, t_0),

$$\Phi R \geqslant -C[(r_2 - r_1)^{-2} + 1].$$

The conclusion follows by the minimum principle. □

2.2.3 Existence of the Chern-Ricci flow

In this section, we will discuss the existence of the Chern-Ricci flow starting from a Hermitian metric with holomorphic sectional curvature bounded from above. We will give an estimate on the existence time. More generally, we will consider initial metric which is uniformly equivalent to a Hermitian metric with holomorphic sectional curvature bounded from above.

Lemma 2.2.7. *Let (M^n, g_0) be a Hermitian metric with bounded geometry of infinite order. Suppose g_0 is uniformly equivalent to a Hermitian metric h with holomorphic sectional curvature and torsion satisfying: $H_h(x)$ bounded above by $\kappa(x)$ and $\frac{n+1}{n}\kappa(x) + |\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h(x) \leqslant \kappa_0$ for some $\kappa_0 \geqslant 0$ for all x, so that*

$$\alpha^{-1}h \leqslant g_0 \leqslant \alpha h,$$

for some $\alpha > 1$. Then the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S]$ with the following properties:

(i) There is a constant $c = c(n) > 0$ so that

$$S \geqslant \frac{1}{3c(n\alpha+1)^3\mathfrak{s}} =: S_1$$

where

$$\mathfrak{s} = \sup_M \left(|T_0|_h^2 + |T_0|_h|\widehat{T}|_h + |\widehat{\nabla}_{\overline{\partial}} T_0|_h + \kappa_0\right)$$

where T_0 is the torsion of g_0, $\widehat{T}$ is the torsion of h and $\widehat{\nabla}$ is the derivative of h with respect to the Chern connection; and

(ii) $g(t)$ is uniformly equivalent to h with

$$\operatorname{tr}_g h \leqslant \left(\frac{1}{(n\alpha+1)^{-3} - 3c\mathfrak{s}t}\right)^{\frac{1}{3}}$$

on $M \times [0, S_1]$.

Proof. If $\mathfrak{s} = \infty$, then there is nothing to be proved. Suppose $\mathfrak{s} < \infty$, then by Lemma 2.2.1, there is a maximal $S > 0$ such that the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S)$ so that $g(t)$ is uniformly equivalent to g_0 on $[0, S']$ for all $S' < S$. Let $\Lambda = \operatorname{tr}_{g(t)} h$. By Corollary 2.2.2,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda &\leqslant c_1 \left(\Lambda^4 |T_0|_h^2 + \Lambda^3(|T_0|_h|\widehat{T}|_h + |\widehat{\nabla}_{\overline{\partial}} T_0|_h) + \Lambda^2\kappa_0\right) \\ &\leqslant c_1 \left(\Lambda + 1\right)^4 \mathfrak{s}\end{aligned}$$

on $M \times [0, S]$. Here and below c_i will denote positive constants depending only on n. Let

$$v(t) = \left(\frac{1}{(n\alpha+1)^{-3} - 3c_2\mathfrak{s}t}\right)^{\frac{1}{3}}$$

Then $v(t)$ is defined on $[0, S_1)$ with $S_1 = 1/\left[3c_2(n\alpha + 1)^3\mathfrak{s}\right]$, with

$$\frac{dv}{dt} = c_2\mathfrak{s}v^4$$

and $v(0) \geqslant (\Lambda + 1)|_{t=0}$. Suppose $S < S_1$. Since Λ and v are bounded on $[0, S']$ for all $0 < S' < S$, by Lemma 2.2.5 as in the proof of [24, Theorem 4.2], one can conclude that

$$\Lambda \leqslant v(t) - 1$$

on $M \times [0, S)$. In particular,

$$h \leqslant c_3(v(t) - 1)g(t) \tag{2.40}$$

If $S < S_1$, then $v(t) \leqslant C_1 < \infty$ on $[0, S]$ for some C_1. Hence $\Lambda \leqslant C_1$ on $M \times [0, S)$.

On the other hand, since g_0 has bounded geometry of infinite order, by Lemma 2.2.6, we conclude that $R(x, t) \geqslant -C_2$ on $M \times [0, S)$ for some C_2. Since

$$\frac{\partial}{\partial t}\left(\log\frac{\det(g(t))}{\det(h)}\right) = -R \leqslant C_2,$$

we conclude that $\det(g(t)) \leqslant C_3 \det(h)$. Together with (2.40), we conclude that

$$C_3^{-1}g_0 \leqslant g(t) \leqslant C_3 g_0$$

on $M \times [0, S)$ for some $C_3 > 0$. Here we have used the fact that g_0 is uniformly equivalent to h. Using the fact that g_0 has bounded geometry of infinite order and by the local estimates of [17], $g(t)$ can be extended to be a solution of the Chern-Ricci flow which is uniformly equivalent to g_0 beyond S. Hence we have $S \geqslant S_1$. This proves (i).

(ii) Follows from (2.40).

□

Let (M^n, h) be a complete noncompact Hermitian manifold. From now on, we work with the following assumptions:

(**a**) There exists smooth exhaustion $\rho \geqslant 1$, and constant $\beta > 0$ such that

$$|\partial\rho|_h + |\sqrt{-1}\partial\bar{\partial}\rho|_h \leqslant \beta\rho$$

if ρ is large enough.

(**b**) The holomorphic sectional curvature at x is bounded from above by $\kappa(x)$, and the torsion of h is such that

$$\frac{n+1}{n}\kappa + |\widehat{\nabla}_{\overline{\partial}}\widehat{T}|_h \leqslant \kappa_0$$

for some $\kappa_0 \geqslant 0$.

Theorem 2.2.3. *Let (M^n, h) be a complete Hermitian metric as above. Let g_0 be another Hermitian metric. Suppose the following are true:*

(i) $\alpha^{-1}g_0 \leqslant h \leqslant \alpha g_0$ *and* $|\widehat{T}|_h \leqslant \alpha$ *for some* $\alpha > 1$*;*

(ii) $|T_0|_h^2 + |\widehat{T}|_h|T_0|_h + |\widehat{\nabla}_{\overline{\partial}}(T_0)|_h \leqslant \beta$*; and*

(iii) $|\partial\rho|_h|\widehat{\nabla}g_0|_h \leqslant \beta\rho$ *for* ρ *large enough.*

There exist constants c_1 depending only on n and c_2 depending only on n, α such that there is a solution $g(t)$ for the Chern-Ricci flow on $M \times [0, S)$ with $g(0) = g_0$, where

$$S = \frac{1}{3c_1(n\alpha+1)^3\mathfrak{s}}$$

and $\mathfrak{s} = \kappa_0 + c_2\beta(1+\beta)$*. Moreover,*

$$\operatorname{tr}_g h \leqslant v(t) - 1$$

where

$$v(t) = \left(\frac{1}{(n\alpha+1)^{-3} - 3c_1\mathfrak{s}t}\right)^{\frac{1}{3}}.$$

on $M \times [0, S)$.

We want to apply Lemma 2.2.7. However, in general it is not true that g_0 has bounded geometry of all order, we cannot apply Lemma 2.2.7 directly to obtain a solution of the Chern-Ricci flow. We now proceed as in [24, 26] to construct a Hermitian approximation.

Let $\tau \in (0, \frac{1}{8})$, $f : [0,1) \to [0, \infty)$ be the function:

$$f(s) = \begin{cases} 0, & s \in [0, 1-\tau]; \\ -\log\left[1 - \left(\dfrac{s-1+\tau}{\tau}\right)^2\right], & s \in (1-\tau, 1). \end{cases} \tag{2.41}$$

Let $\varphi \geqslant 0$ be a smooth function on $\mathbb{R}$ such that $\varphi(s) = 0$ if $s \leqslant 1-\tau+\tau^2$, $\varphi(s) = 1$ for $s \geqslant 1-\tau+2\tau^2$

$$\varphi(s) = \begin{cases} 0, & s \in [0, 1-\tau+\tau^2]; \\ 1, & s \in (1-\tau+2\tau^2, 1). \end{cases} \tag{2.42}$$

such that $\dfrac{2}{\tau^2} \geqslant \varphi' \geqslant 0$. Define

$$\mathfrak{F}(s) := \int_0^s \varphi(\tau) f'(\tau) d\tau.$$

From [26], we have:

Lemma 2.2.8. *Suppose* $0 < \tau < \frac{1}{8}$*. Then the function* $\mathfrak{F} \geqslant 0$ *defined above is smooth and satisfies the following:*

(i) $\mathfrak{F}(s) = 0$ *for* $0 \leqslant s \leqslant 1-\tau+\tau^2$.

(ii) $\mathfrak{F}' \geqslant 0$ *and for any* $k \geqslant 1$, $\exp(-k\mathfrak{F})\mathfrak{F}^{(k)}$ *is uniformly bounded.*

(iii) For any $1-2\tau < s < 1$*, there is* $\tau > 0$ *with* $0 < s-\tau < s+\tau < 1$ *such that*

$$1 \leqslant \exp(\mathfrak{F}(s+\tau) - \mathfrak{F}(s-\tau)) \leqslant (1+c_2\tau); \quad \tau \exp(\mathfrak{F}(s_0-\tau)) \geqslant c_3\tau^2$$

for some absolute constants $c_2 > 0, c_3 > 0$.

Fix $0 < \tau < \frac{1}{8}$. For any $\rho_0 > 0$, let U_{ρ_0} be the component of $\{x| \ \rho(x) < \rho_0\}$ which contains a fixed point and ρ is the positive exhaustion function mentioned above. Hence U_{ρ_0} will exhaust M as $\rho_0 \to \infty$.

Let $\rho_i > 1$ be a sequence increasing to $+\infty$, let $F^{(i)}(x) = \mathfrak{F}(\rho(x)/\rho_i)$. Let $g_{0,i} = e^{2F^{(i)}}g_0$. In the following, $F^{(i)}$ will be denoted simply by F if there is no confusion.

Then $(U_{\rho_i}, g_{0,i})$ is a complete Hermitian metric, (e.g. see [27]) and $g_{i,0} = g_0$ on $\{\rho(x) < (1 - \tau + \tau^2)\rho_0\}$. Moreover, the new manifold has a very nice structure.

Lemma 2.2.9 ([24])**.** *For each $\rho_i > 1$ sufficiently large, $(U_{\rho_i}, g_{0,i})$ has bounded geometry of infinite order.*

In the following, we will estimate the torsion and the holomorphic sectional curvature after performing conformal change.

Lemma 2.2.10. *Let g_0 and h be as in Theorem 2.2.3. For $i \to \infty$, let $g_{0,i}$ be as in Lemma 2.2.9 and $h_i = e^{2F}h$ for the corresponding F. Let $T_{0,i}$ be the torsion of $g_{0,i}$. Then there is a constant $c(n, \alpha)$ depending only on n and α so that as $i \to \infty$, there*

(i) $|T_{0,i}|^2_{h_i} \leqslant c\beta(1 + \beta)$;

(ii) $|T_{0,i}|_{h_i}|\hat{T}^{(i)}|_{h_i} \leqslant c\beta(1 + \beta)$;

(iii) $|\widehat{\nabla}^{(i)}_{\bar{\partial}}T_{0,i}|_{h_i} \leqslant c\beta(1 + \beta)$, *where $\widehat{\nabla}^{(i)}$ is derivative with respect to the Chern connection of h_i;*

(iv)

$$\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i}(x) \leqslant \kappa_0 + c\beta(1 + \beta)$$

where $\kappa_i(x)$ is the upper bound of holomorphic sectional curvature of h_i at x and T_i is the torsion of h_i.

Proof. In the following, c_i will denote a positive constant depending only on n, α.

(i)

$$
\begin{aligned}
(T_{0,i})_{pk\bar{q}} &= \partial_p(e^{2F}(g_0)_{k\bar{q}}) - \partial_k(e^{2F}(g_0)_{p\bar{q}}) \\
&= 2e^{2F}(F_p(g_0)_{k\bar{q}} - F_k(g_0)_{p\bar{q}}) + e^{2F}(T_0)_{pk\bar{q}} \\
&= 2e^{2F}\rho_0^{-1}\mathfrak{F}'\left(\rho_p(g_0)_{k\bar{q}} - \rho_k(g_0)_{p\bar{q}}\right) + e^{2F}(T_0)_{pk\bar{q}}.
\end{aligned}
\tag{2.43}
$$

Hence

$$
|T_{0,i}|^2_{h_i} \leqslant c_1\beta(1+\beta).
$$

This proves (i). The proof of (ii) is similar using the assumption $|\widehat{T}|_h \leqslant \alpha$.

(iii)

$$
\begin{aligned}
\widehat{\nabla}^{(i)}_{\bar{l}}(T_{0,i})_{pk\bar{q}} &= \widehat{\nabla}^{(i)}_{\bar{l}}[2e^{2F}\rho_0^{-1}\mathfrak{F}'\left(\rho_p(g_0)_{k\bar{q}} - \rho_k(g_0)_{p\bar{q}}\right) + e^{2F}(T_0)_{pk\bar{q}}] \\
&= 2e^{2F}\rho_0^{-1}\mathfrak{F}'(\rho_{p\bar{l}}(g_0)_{k\bar{q}} - \rho_{k\bar{l}}(g_0)_{p\bar{q}}) \\
&\quad + (2e^{2F}\rho_0^{-2}\mathfrak{F}'' + 4e^{2F}\rho_0^{-2}(\mathfrak{F}')^2)(\rho_p\rho_{\bar{l}}g_{k\bar{q}} - \rho_k\rho_{\bar{l}}g_{p\bar{q}}) \\
&\quad + 2e^{2F}\mathfrak{F}'\rho_0^{-1}\left(\rho_p\widehat{\nabla}^{(i)}_{\bar{l}}(g_0)_{k\bar{q}} - \rho_k\widehat{\nabla}^{(i)}_{\bar{l}}(g_0)_{p\bar{q}}\right) \\
&\quad + 2e^{2F}\rho_0^{-1}\mathfrak{F}'\rho_{\bar{l}}(T_0)_{pk\bar{q}} + e^{2F}\widehat{\nabla}^{(i)}_{\bar{l}}(T_0)_{pk\bar{q}}
\end{aligned}
\tag{2.44}
$$

Using the fact that

$$
(\widehat{\Gamma}^{(i)} - \widehat{\Gamma})^l_{pq} = 2F_p\delta^l_q = 2\rho_0^{-1}\mathfrak{F}'\rho_p\delta^l_q
$$

and hence

$$
\begin{aligned}
\widehat{\nabla}^{(i)}_{\bar{l}}(g_0)_{k\bar{q}} &= (\widehat{\nabla}^{(i)}_{\bar{l}} - \widehat{\nabla}_{\bar{l}})(g_0)_{k\bar{q}} + \widehat{\nabla}_{\bar{l}}(g_0)_{k\bar{q}} \\
&= -2\rho_0^{-1}\mathfrak{F}'\rho_{\bar{l}}(g_0)_{k\bar{q}} + \widehat{\nabla}_{\bar{l}}(g_0)_{k\bar{q}}
\end{aligned}
\tag{2.45}
$$

We may further infer that (iii) is true using the assumption $|\partial\rho|_h|\widehat{\nabla}g_0|_h \leqslant \beta\rho$ and equivalence of g_0 and h.

Now we examine the holomorphic sectional curvature after conformal change. Let $e_1 \in T^{1,0}U_R$ be such that $|e_1|_{h_i} = 1$, $|e_1|_h = e^{-F}$. Let $\kappa(x)$ be the upper bound of the holomorphic sectional curvature of h at x

$$\begin{aligned}\hat{R}_{1\bar{1}1\bar{1}} &= -\partial_1\partial_{\bar{1}}(e^{2F}h_{1\bar{1}}) + e^{-2F}h^{p\bar{l}}\partial_1(e^{2F}h_{1\bar{l}})\cdot\partial_{\bar{1}}(e^{2F}h_{p\bar{1}}) \\ &= -\partial_1(e^{2F}\partial_{\bar{1}}h_{1\bar{1}} + 2e^{2F}h_{1\bar{1}}F_{\bar{1}}) \\ &\quad + e^{-2F}h^{p\bar{l}}\left(e^{2F}\partial_1 h_{1\bar{l}} + 2\hat{h}_{1\bar{l}}F_1\right)\left(e^{2F}\partial_{\bar{1}}h_{p\bar{1}} + 2\hat{h}_{p\bar{1}}F_{\bar{1}}\right) \\ &= e^{2F}\tilde{R}_{1\bar{1}1\bar{1}} - 2\hat{h}_{1\bar{1}}F_{1\bar{1}} \\ &\leqslant e^{-2F}\kappa - 2F_{1\bar{1}} \\ &\leqslant e^{-2F}\kappa + c_2(\beta+\beta^2).\end{aligned} \tag{2.46}$$

Estimate $|\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i}$ in a similar way as above, we may conclude that

$$\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i} \leqslant e^{-2F}\kappa_0 + c_3\beta(1+\beta).$$

From this (iv) is true.

□

Now we are able to construct a solution of the Chern-Ricci flow on M.

Proof of Theorem 2.2.3. For each sufficiently large ρ_i, $(U_{\rho_i}, g_{0,i})$ has bounded geometry by Lemma 2.2.9. By Lemma 2.2.10, using the notation in the lemma, we have:

$$\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i} \leqslant \kappa_0 + c(\beta+\beta^2) =: \kappa_{0,i}.$$

Let

$$\mathfrak{s}_i := \sup_M \left(|T_{0,i}|^2_{h_i} + |T_{0,i}|_{h_i}|\widehat{T}_i|_{h_i} + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_{0,i}|_{h_i} + \kappa_{0,i}\right).$$

Then by Lemma 2.2.10,

$$\mathfrak{s}_i \leqslant \kappa_0 + c\beta(1+\beta) =: \mathfrak{s}.$$

By Lemma 2.2.7, there is a solution $g_i(t)$ on $U_{\rho_i} \times [0, S)$ with initial metric $g_{0,i}$ where

$$S = \frac{1}{3c_1(n\alpha + 1)^3 \mathfrak{s}}$$

for some constant $c_1 = c_1(n)$. Moreover, g_i is uniformly equivalent to $g_{0,i}$ and

$$\operatorname{tr}_{g_i} h_i \leqslant v(t) - 1 \tag{2.47}$$

on $U_{\rho_i} \times [0, S)$ where

$$v(t) = \left(\frac{1}{(n+1)^{-3} - 3c_1 \mathfrak{s} t} \right)^{\frac{1}{3}}.$$

Fix any compact subset $K \subset M$ and any $S' \in (0, S)$. Then for sufficiently large i, $g_i(t)$ is a solution of the Chern-Ricci flow defined on $U_{\rho_i} \supset U_{2r} \supset U_r \supset K$ for some large $r > 0$. By Lemma 2.2.6, for any $(x, t) \in K \times [0, S']$,

$$R_{g_i(t)} \geqslant -\max\left\{ C(n, \alpha, \beta, S', r), \sup_{\rho(y) < 2r} R_-(y, 0) \right\}$$

where we have used the fact that $h_i = h$ on U_{2r} for sufficiently large ρ_i. In particular, it is bounded from below uniformly. Since

$$\frac{\partial}{\partial t}\left(\log \frac{\det g_i(t)}{\det h} \right) = -R_{g_i(t)} \leqslant C(n, K, \alpha, \beta, g_0, S', h),$$

so on $K \times [0, S']$,

$$C(n, \alpha, \beta, S')h \leqslant g(t) \leqslant C(n, K, \alpha, \beta, g_0, S', h)h.$$

By the local estimate of the Chern-Ricci flow [17], for any $k \in \mathbb{N}$, there is $C(n, k, g_0, h, \beta, \alpha, K, S')$ such that for any $(x, t) \in K \times [0, S']$,

$$|\hat{\nabla}^k g_i(t)|_h \leqslant C(n, k, g_0, h, \beta, \alpha, K, S').$$

By taking diagonal subsequence and using Arzelà-Ascoli theorem, we may obtain a limiting solution of $g(t)$ defined on $M \times [0, S)$. The conclusion on $\mathrm{tr}_g\, h$ follows from (2.47). This completes the proof of the theorem. □

Next we want to apply Theorem 2.2.3 to obtain long-time solution for Kähler-Ricciflow.

Theorem 2.2.4. *Let (M, g_0) be a complete Kählermanifold and h is a fixed complete Hermitian metric on M such that the following hold:*

(i) There exists smooth exhaustion $\rho \geqslant 1$ such that

$$\limsup_{\rho\to\infty}\left[\frac{|\partial\rho|_h}{\rho}(1+|\widehat{\nabla}g_0|_h)+\frac{|\sqrt{-1}\partial\bar{\partial}\rho|_h}{\rho}\right]=0;$$

(ii) the holomorphic sectional curvature of h and torsion of h satisfy

$$H_h+\frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h\leqslant -k$$

for some constant $k \geqslant 0$.

(iii) $\exists\alpha > 1$ such that on M, $\alpha^{-1}g_0 \leqslant h \leqslant \alpha g_0$, $|\widehat{T}|_h\leqslant \alpha$;

Then there is $\beta(n, \alpha) > 0$ such that the Kähler-Ricciflow has a complete solution $g(t)$ on $M \times [0, +\infty)$ with $g(0) = g_0$ and satisfies

$$\beta h \leqslant g(t)$$

on $M \times [0, +\infty)$.

Proof. By Theorem 2.2.3 and the assumptions, one can apply the theorem with β arbitrarily small. Hence one can find solution $g_i(t)$ to the Chern-Ricci flow with $g_i(0) = g_0$ on $M \times [0, T_i]$ with $T_i \to \infty$. Moreover, $\mathrm{tr}_g\, h \leqslant c(n, \alpha)$. Using the local estimate of scalar curvature in Lemma 2.2.6 as in the proof of Theorem 2.2.3, the results follow. □

2.2.4 Existence of the Kähler-Einstein metric

In this section, we discuss the existence of the Kähler-Einsteinmetric on M via the Kähler-Ricciflow. We have the following:

Theorem 2.2.5. *Suppose there is a complete Hermitian metric h on M compatible with the complex structure J such that*

$$H_h + \frac{n}{n+1}|\widehat{\nabla}_{\overline{\partial}}\widehat{T}|_h \leqslant -k \tag{2.48}$$

for some $k > 0$. Then any longtime solution of normalized Kähler-Ricciflow $g(t)$ will converge locally uniformly in C^∞ to a KählerEinstein metric $g_\infty = -Ric(g_\infty)$. In particular, there is no Ricci flat Kählermetric on M compatible with the same complex structure J.

Combining with Theorem 2.2.4, we have the following results which generalize the result by Wu-Yau [7]:

Corollary 2.2.3. *Let (M^n, g_0) be a complete noncompact Kählermanifold with holomorphic sectional curvature bounded above by a negative constant. Suppose M supports an exhaustion function with uniformly bounded gradient and uniformly bounded complex Hessian, which is the case if M has bounded curvature. Then M^n support a unique Kähler Einstein metric with negative scalar curvature.*

We also want to discuss metrics which are uniformly equivalent to g_0 as in the previous corollaries. The following is an immediate consequence of Theorem 2.2.4 and Theorem 2.2.5.

Corollary 2.2.4. *Let (M^n, g_0) be a complete Kählermanifold and h is a fixed complete Hermitian metric on M such that g_0, h satisfy the assumptions in Theorem 2.2.4 with $k > 0$. Then M supports a unique Kähler-Einstein metric with negative scalar curvature.*

Let us prove Theorem 2.2.5. Here we do not have a good exhaustion function for h. However, the distance function $d(x, t)$ in a Ricci flow behaves well. First using the idea by Chen [28], we have the following:

Lemma 2.2.11. *Let $(M^m, g(t))$ be a complete noncompact solution to the Ricci flow on $M \times [0,T]$ with $0 < T < \infty$, where $m \geqslant 2$ is the real dimension of M. Let Q be a smooth function so that*

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leqslant -\alpha Q^2 + \beta$$

for some $\alpha, \beta > 0$ at the point where $Q > 0$. Then

$$tQ(x,t) \leqslant \frac{1 + \sqrt{1 + 4\alpha\beta T^2}}{2\alpha}.$$

on $M \times (0,T]$.

Proof. Let $x_0 \in M$, and let $r_0 > 0$ be small enough so that:

$$\mathrm{Ric}(x,t) \leqslant (m-1)r_0^{-2}$$

for $x \in B_t(x_0, r_0)$, $t \in [0,T]$. By [29] (see also [28]), we then have

$$\left(\frac{\partial}{\partial t} - \Delta\right) d_t(x, x_0) \geqslant -\frac{5(m-1)}{3} r_0^{-1} \tag{2.49}$$

whenever $d_t(x,x_0) \geqslant r_0$ in the sense of barrier, where $d_t(x,x_0)$ is the distance function from x_0 with respect to $g(t)$. In the following, argue as in [30], we may assume that $d_t(x,x_0)$ to be smooth when applying maximum principle. We consider the function

$$u(x,t) = t\varphi\left(\frac{1}{Ar_0}\left[d_t(x,x_0) + \frac{5(m-1)t}{3r_0}\right]\right) Q(x,t)$$

A is sufficiently large so that $Ar_0 >> \frac{5(m-1)T}{3r_0}$, and φ is a fixed smooth nonnegative non-increasing function such that $\varphi \equiv 1$ on $(-\infty, \frac{1}{2}]$, vanishes outside $[0,1]$ and satisfies $|2\frac{(\varphi')^2}{\varphi}| + |\varphi''| \leqslant c_1$ for some absolute constant. Note that u also depends on A. However,

$$u(x_0,t) = tQ(x_0,t).$$

if $Ar_0 \geqslant \frac{10(m-1)T}{3r_0}$.

If $u \leqslant 0$, then we are done. Suppose the function $u > 0$ somewhere, then there exists (x_1, t_1) with $0 < t_1 \leqslant T$ so that u attains its maximum at (x_1, t_1). We have at (x_1, t_1) we have

$$0 \leqslant \left(\frac{\partial}{\partial t} - \Delta\right) u; \quad \nabla Q = -\frac{\nabla \varphi}{\varphi} Q.$$

Suppose $d_{t_1}(x_1, x_0) < r_0$, then $u(x, t) = tQ(x, t)$ near (x_1, t_1) provided Ar_0 is large enough. Then we have at (x_1, t_1)

$$\begin{aligned} 0 \leqslant & \left(\frac{\partial}{\partial t} - \Delta\right) u \\ = & t_1 \left(\frac{\partial}{\partial t} - \Delta\right) Q + Q \\ \leqslant & -\alpha t_1 Q^2 + \beta t_1 + Q \end{aligned}$$

and so

$$0 \leqslant -\alpha u^2 + u + \beta T^2$$

which implies

$$u(x_0, t) \leqslant u(x_1, t_1) \leqslant \frac{1 + \sqrt{1 + 4\alpha\beta T^2}}{2\alpha}. \tag{2.50}$$

for $t \in [0, T]$.

Suppose $d_{t_1}(x_1, x_0) \geqslant r_0$, then at (x_1, t_1),

$$
\begin{aligned}
0 &\leqslant \left(\frac{\partial}{\partial t} - \Delta\right) u \\
&= Qt\left(\frac{\partial}{\partial t} - \Delta\right)\varphi + \varphi\left(\frac{\partial}{\partial t} - \Delta\right)(Qt) - 2t\langle\nabla\varphi, \nabla Q\rangle \\
&\leqslant Qt\varphi'\frac{1}{Ar_0}\left[\left(\frac{\partial}{\partial t} - \Delta\right)d_t(x,p) + \frac{5}{3}(m-1)r_0^{-1}\right] \\
&\quad + |\varphi''|\frac{1}{(Ar_0)^2}tQ + \varphi\left(-\alpha t Q^2 + \beta t + Q\right) + 2tQ\frac{1}{(Ar_0)^2}\cdot\frac{(\varphi')^2}{\varphi} \\
&\leqslant -\alpha t\varphi Q^2 + \varphi Q + \beta t\varphi + c_1 Qt\frac{1}{(Ar_0)^2}.
\end{aligned}
$$

Multiply both the inequality by $t\varphi = t_1\varphi$, we have

$$
0 \leqslant -\alpha u^2 + \left(1 + \frac{c_1 T}{A^2 r_0^2}\right)u + \beta T^2.
$$

Hence we have

$$
u(x_0,t) \leqslant u(x_1,t_1) \leqslant \frac{1 + \frac{c_1 T}{A^2 r_0^2} + \sqrt{\left(1 + \frac{c_1 T}{A^2 r_0^2}\right)^2 + 4\alpha\beta T^2}}{2\alpha}. \tag{2.51}
$$

for $0 < t \leqslant T$. Let $A \to \infty$ together with (2.51) and the fact that x_0 is any point in M, we conclude the lemma is true. □

As an application of the lemma, we can prove the following uniqueness of complete Kähler-Einsteinmetric. Here we do not assume the curvature is bounded, see also [7]. Note that the result can be also obtained by using elliptic theory.

Proposition 2.2.1. *Suppose ω_1 and ω_2 are complete noncompact KählerEinstein metrics on M with $Ric(\omega_i) = -\omega_i$ for $i = 1,2$. Then $\omega_1 = \omega_2$ on M.*

Proof. Let Let $\widetilde{\omega}_1(t) = (t+1)\omega_1$ and $\widetilde{\omega}_2(t) = (t+1)\omega_2$. Then both $\widetilde{\omega}_1, \widetilde{\omega}_2$ are solutions to the Kähler-Ricciflow on $M \times [0, +\infty)$. Define $F(x,t) = F(x)$ to be the function

$$
F(x,t) = \log\left[\frac{\widetilde{\omega}_2^n}{\widetilde{\omega}_1^n}\right]^{\frac{1}{n}} = \log\left[\frac{\omega_2^n}{\omega_1^n}\right]^{\frac{1}{n}}
$$

which is independent of t. The function F is independent of $t > 0$ but we treat it as a function over the Kähler-Ricciflow. Let Δ be the Laplacian of $\widetilde{\omega}_1$. Then it satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) F &= \frac{1}{t+1}\left(1 - \frac{1}{n}\operatorname{tr}_{\omega_1}\omega_2\right)\\ &\leqslant \frac{1}{t+1}\left(1 - e^F\right)\\ &\leqslant -\frac{1}{4}F^2\end{aligned}$$

whenever $F > 0$ on $M \times [0, 1]$.

Apply Lemma 2.2.11 on $M \times [0, 1]$, $tF \leqslant 4$. In particular, $F(x)$ is bounded from above uniformly on M. By interchanging ω_1 and ω_2, we conclude that F is a bounded function on M. Let Δ_1 be the Laplacian of ω_1, we have as above

$$-\Delta_1 F \leqslant 1 - e^F.$$

By the generalized maximum principle [31], we conclude that $F \leqslant 0$. Interchanging the roles of ω_1 and ω_2, we can prove similarly that $F \geqslant 0$. Hence $F = 0$. So $\partial\bar{\partial}F = 0$ and $\omega_1 = \omega_2$ because they are Kähler-Einstein. □

Now we are ready to prove Theorem 2.2.5.

Proof of Theorem 2.2.5. Let $g(t)$ and h as in the Theorem. Let $\Lambda = \operatorname{tr}_g h$. By Corollary 2.2.2, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda \leqslant -c_1 k \Lambda^2,$$

for some c_1 depending only on n. By Lemma 2.2.11 with $\beta = 0$, we conclude that

$$\Lambda(x, t) \leqslant \frac{1}{c_1 k t} \tag{2.52}$$

on $M \times (0, \infty)$.

On the other hand, let $R(x,t)$ be the scalar curvature of $g(t)$ at x and let $R_-(x,t)$ be its negative part. For any $\varepsilon > 0$, let $f = \frac{1}{2}\left((R^2+\varepsilon^2)^{\frac{1}{2}} - R\right)$. Note that if $\varepsilon \to 0$, then $f \to R_-$. Using the fact that

$$\left(\frac{\partial}{\partial t} - \Delta\right) R \geqslant \frac{1}{n} R^2,$$

direct computations show that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leqslant -\frac{1}{n} f(f - 2c_1\varepsilon) \leqslant -\frac{1}{n}(f - c_1\varepsilon)^2 + c_2\varepsilon$$

for some absolute constant $c_1 > 0$ and $c_2 > 0$ depending only on n. By Lemma 2.2.11, we conclude that

$$t(f - c_1\varepsilon) \leqslant \frac{n}{2}\left(1 + \sqrt{1 + \frac{4c_2\varepsilon}{n}}\right).$$

on $M \times (0, \infty)$. Let $\varepsilon \to 0$, we conclude that

$$tR(x,t) \geqslant -n. \tag{2.53}$$

Since

$$\frac{\partial}{\partial t} \log\left(\frac{\det g(t)}{\det h}\right) = -R \leqslant \frac{n}{t}$$

we conclude for any bounded open set Ω, there is a constant C_1 depending only on $\Omega, g(1), h, n$ such that

$$\frac{\det g(t)}{\det h} \leqslant C_1 t^n$$

on $\Omega \times [1, \infty)$. Combining this with (2.52), we conclude that

$$C_2^{-1} t h \leqslant g \leqslant C_2 t h$$

on $\Omega \times [1, \infty)$ for some constant $C_2 > 0$ depending only on $\Omega, g(1), h, k, n$.

Consider the normalized metric

$$\widetilde{g}(x,s) = e^{-s}g(x,e^s).$$

Then we have

$$\frac{\partial}{\partial s}\widetilde{g} = -\mathrm{Ric}(\widetilde{g}) - \widetilde{g} \tag{2.54}$$

on $M \times [0,\infty)$, and

$$C_2^{-1}h \leqslant \widetilde{g}(s) \leqslant C_2 h \tag{2.55}$$

on $\Omega \times [0,\infty)$.

In the following, let $\omega(t)$, $\widetilde{\omega}(s)$ be the Kählerforms of $g(t)$, $\widetilde{g}(s)$ respectively. By [19, Theorem 2.17], we conclude that for any bounded open set in M and $\ell \geqslant 0$, there is a constant C_3 depending only on Ω, $g(1)$, h, k, n and ℓ such that

$$||\widetilde{\omega}(s)||_{C^\ell(\Omega,\widetilde{g}_0)} \leqslant C_3. \tag{2.56}$$

On the other hand, let

$$\widetilde{\varphi}(x,s) = e^{-s}\int_0^s e^\tau \log\left(\frac{(\widetilde{\omega}(\tau))^n}{(\widetilde{\omega}(0))^n}\right) d\tau.$$

Then

$$\widetilde{\omega}(s) = e^{-s}\widetilde{\omega}(0) - (1-e^{-s})\mathrm{Ric}(\widetilde{\omega}(0)) + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi}(s). \tag{2.57}$$

Moreover,

$$\begin{cases} \frac{\partial}{\partial s}\widetilde{\varphi} = & \log\left(\frac{\widetilde{\omega}^n}{(\widetilde{\omega}(0))^n}\right) - \widetilde{\varphi} \ \text{ in } M \times [0,\infty); \\ \widetilde{\varphi}(0) = & 0 \ \text{ on } M. \end{cases} \tag{2.58}$$

Denote $\partial_s\widetilde{\varphi}$ by $\widetilde{\varphi}'$ etc., and let $\widetilde{R}$ be the scalar curvature of $\widetilde{g}$, then

$$\begin{aligned}\widetilde{\varphi}'' + \widetilde{\varphi}' &= - \widetilde{R} - n \\ &= - e^s R(g(e^s)) - n \\ &= - e^s \left(R(g(e^s)) + e^{-s} n\right) \\ &\leqslant 0\end{aligned} \tag{2.59}$$

by (2.53). Hence $\widetilde{\varphi}' + \widetilde{\varphi}$ is non-increasing and $\widetilde{\varphi}' + \widetilde{\varphi} \leqslant 0$ because $\widetilde{\varphi}' + \widetilde{\varphi} = 0$ at $s = 0$. On the other hand, by (2.56) and (2.58), we conclude that for any bounded open set Ω, there exists $s_i \to \infty$ such that

$$(\widetilde{\varphi}' + \widetilde{\varphi})(s_i)$$

converge uniformly in C^∞ norm in Ω. By the monotonicity of $\widetilde{\varphi}' + \widetilde{\varphi}$, we conclude that $\widetilde{\varphi}' + \widetilde{\varphi}$ converges in C^∞ norm in Ω to some function.

By (2.59), we have

$$(e^s\widetilde{\varphi}')' \leqslant 0,$$

and so $\widetilde{\varphi}' \leqslant 0$ because $\widetilde{\varphi}' = 0$ at $s = 0$. Combine this with (2.56) and (2.57), we conclude that $\widetilde{\varphi}$ also converges in C^∞ norm to some function $\widetilde{\varphi}_\infty$. Hence $\widetilde{\varphi}'$ also converge in C^∞ norm to some function. However, by (2.55) we conclude that φ is bounded from below. This implies that $\widetilde{\varphi}' \to 0$ as $s \to \infty$. Moreover, $\widetilde{\omega}(s) \to \widetilde{\omega}_\infty$ in C^∞ norm in Ω as $s \to \infty$ with

$$\widetilde{\omega}_\infty = -\mathrm{Ric}(\widetilde{\omega}(0)) + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi}_\infty.$$

Note that $\widetilde{\omega}_\infty$ is a Kählerform of a Kählermetric by (2.55). Moreover,

$$\widetilde{\varphi}_\infty = \log\left(\frac{\widetilde{\omega}^n}{(\widetilde{\omega}(0))^n}\right).$$

Taking $\partial\bar{\partial}$ to both sides, we conclude that

$$\mathrm{Ric}(\widetilde{\omega}_\infty) = -\widetilde{\omega}_\infty.$$

Suppose $\bar{\omega}$ is a Ricci flat metric compatible with the same complex structure of h. Then $\omega(t) = \bar{\omega}$ is a steady solution of the Kähler-Ricciflow. By the convergence of normalized Kähler-Ricciflow, $t^{-1}\omega(t)$ converges to a Kähler Einstein metric on M which is impossible since $t^{-1}\omega(t) \equiv t^{-1}\bar{\omega}$ converges to a zero tensor on M. This completes the proof. □

Chapter 3: Positivity conditions in non-Kähler geometry

3.1 Preliminaries

A general principle in Kähler geometry has been that positive curvature conditions impose important geometric and topological constraints on the manifold. A manifestation of this is the solution of the Frankel conjecture by Siu-Yau and Mori [32, 33], and its generalization by Mok [34], which can be viewed as a uniformization theorem for Kähler manifolds with nonnegative bisectional curvature. In recent years, the study of non-Kähler Hermitian geometry has received a lot of attention, partly due to its connection with heterotic string theory [35, 36, 37, 38, 39, 40]. It is then natural to look for curvature positivity conditions in non-Kähler geometry which can lead to significant geometric and topological consequences. In one such direction, Ustinovskiy [41] proposes to study the uniformization of Hermitian manifolds with nonnegative Griffiths curvature by using a geometric flow which preserves the positivity of the Griffiths curvature. As shown by Fei and Phong [42], the Hermitian curvature flow which he identified as positive curvature preserving is precisely the same as the flows motivated by string theory introduced in [40].

In this chapter, we introduce a new type of curvature positivity condition for a Hermitian manifold involving a tensor Q, which is different from the Chern curvature. This tensor Q is less positive than the Chern curvature, hence the corresponding positivity conditions formulated using Q are generally stronger than the corresponding positivity of the Chern curvature. We prove a general Bochner-Kodaira type formula, in which this tensor appears naturally and we show that under a certain Q-nonnegativity condition, every class in $\alpha \in H^{1,1}_{BC}(X)$ can be represented by a closed $(1,1)$-form which is parallel with respect to the Bismut connection. This can be seen as a generalization of a theorem by Howard-Smyth-Wu [43] to the non-Kähler setting. Perhaps surprisingly, Q also arises in several different contexts in non-Kähler geometry: on one hand, it can be interpreted as a combination of the form $\partial\bar{\partial}\omega$ with the $(1,1)$-part of the curvature of the Bismut connection. On the other hand, it acquires an additional symmetry on Vaisman manifolds. We discuss some corresponding applications, which suggest that Q may be very useful for future investigations in non-Kähler geometry.

3.1.1 Positivity condition

We are interested in a 4-tensor Q, which is defined by the following

$$Q_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - g^{p\bar{q}}T_{kp\bar{j}}\overline{T_{lq\bar{i}}} \tag{3.1}$$

It turns out that this tensor naturally appears in a Bochner formula for closed $(1,1)$-forms and has a nice connection with the Bismut connection.

Remark 3.1.1. It is also worth noting that the contraction of Q in the first two coordinates coincides with the evolution term in the pluriclosed flow defined by Streets and Tian in [44], the same term also appears in the Anomaly flow in dimension three which is studied by Phong, Picard, Zhang in [40].

We now state the positivity condition that we are interested in, this new positivity condition is expressed in terms of the Q tensor.

Definition 3.1.1. We say (X, J, g) is *Q-nonnegative* if in any orthonormal frame (i.e where $g_{i\bar{j}} = \delta_{ij}$) and for any list of real numbers $\lambda_1, \ldots, \lambda_n$, we have

$$\sum_{k,m=1}^{n} Q_{m\bar{m}k\bar{k}}(\lambda_k^2 - \lambda_k\lambda_m) \geqslant 0 \tag{3.2}$$

Remark 3.1.2. If Q satisfies an additional symmetry $Q_{i\bar{j}k\bar{l}} = Q_{k\bar{l}i\bar{j}}$, then this expression simplifies

$$2\sum_{k,m=1}^{n} Q_{m\bar{m}k\bar{k}}(\lambda_k^2 - \lambda_k\lambda_m) = \sum_{k,m=1}^{n} Q_{m\bar{m}k\bar{k}}(\lambda_k - \lambda_m)^2 \tag{3.3}$$

This is the case when the metric is Kähler, in that case $Q = R$ and the condition reduces to the nonnegative quadratic orthogonal bisectional curvature condition. (see [45])

Remark 3.1.3. A simple observation is that if a product manifold $M \times N$ is Q-nonnegative, then each of its components must be Q-nonnegative.

Example 3.1.1. All Bismut-flat manifold are Q-nonnegative. Indeed, In it is shown in [46] that these manifold have to be pluriclosed, and from the discussion in section 3.2, it follows that on these manifolds, Q is equal to the curvature of the Bismut connection which is 0, hence Bismut-flat manifolds are all Q-nonnegative. These manifolds has been classified in [47], and their universal covers must a product of a compact semisimple Lie group with a real vector space.

We will give several other examples where the positivity condition holds in Section 3.

We now state our main theorem regarding manifolds with Q-nonnegative Q tensors. Our theorem implies that on a Q-nonnegative manifold, one can always solve the Poincare-Lelong equation for a closed $(1,1)$-form up to the addition of a Bismut parallel $(1,1)$-form.

Theorem 3.1.2. *Suppose* (X, J, g) *is a compact Hermitian manifold which is* Q*-nonnegative, then any closed* $(1,1)$*-form with constant trace is parallel with respect to the Bismut connection. In particular, any class in the Bott-Chern cohomology* $H^{1,1}_{BC}(X)$ *contains a representative which is parallel respect to the Bismut connection.*

Our theorem can be viewed as a non-Kähler generalization of a theorem of Howard-Smyth-Wu [43], which says for a compact Kähler manifold with non-negative quadratic orthogonal bisectional curvature, any class in $H^{1,1}(X)$ contains a parallel representative given by the unique harmonic form in that class. This turned out to be a key ingredient of the resolution of the generalized Frankel conjecture by Mok [34]. The key to the proof of the theorem is a Bochner-Kodaira type identity for closed $(1,1)$-form, which generalizes the identity in [43]. In the Kähler case, such an identity was also used by Mok-Siu-Yau [48] to study the solution Poincare-Lelong equation in relation to the uniformization conjecture for non-compact Kähler manifolds. We refer the readers to [49, 50] and the references therein for subsequent developments in that direction.

3.2 Q and the Bismut connection

On a Hermitian manifold, there is another canonical connection called the Bismut connection, this is also sometimes called the Strominger connection or the Strominger-Bismut connection in literature. This connection was first written down by physicists (see [51, 52]) and Bismut rediscovered and used it in [53] to prove an index formula for non-Kähler manifolds. This connection has received a lot of attention recently because of its relation to the study of non-Kähler Calabi-Yau manifolds, and the Hull-Strominger system arising from physics, see [36] and [54] where the Bismut connection is used in an crucial way. It is also a natural connection to in the study of pluriclosed metrics via the geometric flow as defined by Streets and Tian [44, 55].

Definition 3.2.1 (Bismut connection)**.** The Bismut connection is the connection on $T^{1,0}M$ which in local holomorphic coordinates is given by

$$\nabla^+_i X^j = \partial_i X^j + \Gamma^j_{li} X^l \tag{3.4}$$

$$\nabla^+_{\bar{i}} X^j = \partial_{\bar{i}} X^j + g^{j\bar{m}} \overline{T_{im\bar{l}}} X^l \tag{3.5}$$

where $\Gamma^j_{ki} = g^{j\bar{m}} \partial_k g_{\bar{m}i}$ and $T_{im\bar{l}} = \partial_i g_{m\bar{l}} - \partial_m g_{i\bar{l}}$.

Remark 3.2.1. One can check that this connection respects both the metric and the complex structure (i.e $\nabla^+ g = \nabla^+ J = 0$) and the corresponding Torsion tensor $T^+(X, Y, Z) = g(\nabla^+_X Y - \nabla^+_Y X - [X, Y], Z)$ is skew-symmetric. These properties uniquely characterizes the Bismut connection.

The curvature of the Bismut connection does not satisfy the same symmetries as the curvature of the Chern connection. Thus in general, the curvature has a decomposition into $(2, 0), (1, 1)$ and $(0, 2)$-parts, and we compute the $(1, 1)$-part of the curvature of the Bismut connection in terms of the curvature and torsion of the Chern connection.

Lemma 3.2.1. *The* $(1, 1)$*-part of the curvature of the Bismut connection is given by*

$$B_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} - R_{i\bar{j}k\bar{l}} + R_{i\bar{l}k\bar{j}} - T_{ki\bar{\gamma}} \overline{T^{\gamma}_{jl}} - g^{\gamma\bar{\kappa}} T_{i\gamma\bar{l}} \overline{T_{j\kappa\bar{k}}} \tag{3.6}$$

where B is the curvature of the Bismut connection and R is the curvature of the Chern connection.

Proof. The connection coefficients of the Bismut connection is given by

$$\mathcal{A}^l_{ik} = \Gamma^l_{ki}$$

$$\mathcal{A}^l_{\bar{j}k} = g^{l\bar{q}}\overline{T_{jq\bar{k}}}$$

and the $(1,1)$-part of the curvature is given by

$$B_{i\bar{j}m}{}^{k} = \partial_i \mathcal{A}^k_{\bar{j}m} - \partial_{\bar{j}} \mathcal{A}^k_{im} + \mathcal{A}^k_{il}\mathcal{A}^l_{\bar{j}m} - \mathcal{A}^k_{\bar{j}l}\mathcal{A}^l_{im} \tag{3.7}$$

$$= g^{k\bar{l}}\nabla_i \overline{T_{jl\bar{m}}} + R_{m\bar{j}i}{}^{k} - T^k_{il}\mathcal{A}^l_{\bar{j}m} + \mathcal{A}^k_{\bar{j}l}T^l_{im} \tag{3.8}$$

$$= g^{k\bar{l}}R_{i\bar{l}m\bar{j}} - R_{i\bar{j}m}{}^{k} + R_{m\bar{j}i}{}^{k} - g^{l\bar{n}}T^k_{il}\overline{T_{jn\bar{m}}} + g^{k\bar{n}}\overline{T_{jn\bar{l}}}T^l_{im} \tag{3.9}$$

$$\tag{3.10}$$

lowering the last index gives the result. □

Proposition 3.2.1. *The following formula holds*

$$Q_{i\bar{j}k\bar{l}} = B_{k\bar{l}i\bar{j}} + \partial_{\bar{l}}T_{ik\bar{j}} - \partial_{\bar{j}}T_{ik\bar{l}} \tag{3.11}$$

where $B_{k\bar{l}i\bar{j}}$ is the $(1,1)$ part of the curvature of the Bismut connection.

proof of Proposition 3.2.1. In local coordinates where $g_{i\bar{j}} = \delta_{ij}$ and $\partial_i g_{j\bar{l}} = \frac{1}{2}T_{ij\bar{l}}$, we compute

$$\partial_{\bar{l}}T_{ki\bar{j}} - \partial_{\bar{j}}T_{ki\bar{l}} = \partial_{\bar{l}}\partial_k g_{i\bar{j}} - \partial_{\bar{l}}\partial_i g_{k\bar{j}} - \partial_k\partial_{\bar{j}} g_{i\bar{l}} + \partial_{\bar{j}}\partial_i g_{k\bar{l}} \tag{3.12}$$

$$= R_{i\bar{l}k\bar{j}} + R_{k\bar{j}i\bar{l}} - R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} - \sum_{\gamma} T_{ki\bar{\gamma}}\overline{T_{jl\bar{\gamma}}} \tag{3.13}$$

$$= B_{i\bar{j}k\bar{l}} - Q_{k\bar{l}i\bar{j}} \tag{3.14}$$

□

Remark 3.2.2. The extra term $\partial_{\bar{l}}T_{ik\bar{j}} - \partial_{\bar{j}}T_{ik\bar{l}}$ is the components of the tensor $\partial\bar{\partial}\omega$, and the previous lemma implies that if ω is pluriclosed, then Q is simply the $(1,1)$ part of the curvature of the Bismut connection.

3.3 Q and a Bochner-Kodaira identity

In this section, we prove Theorem 3.1.2. The key is a Bochner-Kodaira type formula for $(1,1)$-forms, from which the tensor Q naturally emerges.

In the Kähler case, this formula is well-known and is intimately related to the uniformization of manifolds with nonnegative bisectional curvature. It first appeared in [56], and is used in [43] to deduce a splitting theorem for manifolds with nonnegative bisectional curvature, which was one of the key imput in the solution of the generalized Frankel conjecture by Mok [34].

Theorem 3.3.1. *The following formula holds for ρ a real closed $(1,1)$-form with $g^{i\bar{j}}\rho_{i\bar{j}} = const$,*

$$g^{i\bar{j}}\partial_i\partial_{\bar{j}}|\rho|^2 = 2|\nabla^+\rho|^2 + 2Q_{i\bar{j}k\bar{l}}(g^{i\bar{j}}(\rho^2)^{k\bar{l}} - \rho^{i\bar{j}}\rho^{k\bar{l}}) \tag{3.15}$$

where $Q_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - g^{p\bar{q}}T_{kp\bar{j}}\overline{T_{lq\bar{i}}}$.

Proof. In local holomorphic coordinates, we have $\rho = i\rho_{k\bar{j}}\, dz^k \wedge d\bar{z}^j$ where $\overline{\rho_{k\bar{j}}} = \rho_{j\bar{k}}$. By the closedness of ρ, we have

$$\nabla_i\rho_{k\bar{j}} = \nabla_k\rho_{i\bar{j}} + T^m_{ki}\rho_{m\bar{j}} \tag{3.16}$$

and

$$\nabla_{\bar{j}}\rho_{i\bar{l}} = \nabla_{\bar{l}}\rho_{i\bar{j}} + \overline{T^n_{lj}}\rho_{i\bar{n}} \tag{3.17}$$

we compute

$$\nabla_i \nabla_{\bar{j}} \rho_{k\bar{l}} = \nabla_i \nabla_{\bar{l}} \rho_{k\bar{j}} + \nabla_i [\overline{T_{lj}^m} \rho_{k\bar{m}}] \tag{3.18}$$

$$= \nabla_{\bar{l}} \nabla_i \rho_{k\bar{j}} + [\nabla_i, \nabla_{\bar{l}}] \rho_{k\bar{j}} + \overline{T_{lj}^m} \nabla_i \rho_{k\bar{m}} + [\overline{R_{j\bar{i}l}{}^m} - \overline{R_{l\bar{i}j}{}^m}] \rho_{k\bar{m}} \tag{3.19}$$

$$= \nabla_{\bar{l}} \nabla_k \rho_{i\bar{j}} + \nabla_{\bar{l}} [T_{ki}^m \rho_{m\bar{j}}] + [\nabla_i, \nabla_{\bar{l}}] \rho_{k\bar{j}} + \overline{T_{lj}^m} \nabla_i \rho_{k\bar{m}} + [\overline{R_{j\bar{i}l}{}^m} - \overline{R_{l\bar{i}j}{}^m}] \rho_{k\bar{m}} \tag{3.20}$$

$$= \nabla_{\bar{l}} \nabla_k \rho_{i\bar{j}} + T_{ki}^m \nabla_{\bar{l}} \rho_{m\bar{j}} + \overline{T_{lj}^m} \nabla_i \rho_{k\bar{m}} + [R_{i\bar{l}k}{}^m - R_{k\bar{l}i}{}^m] \rho_{m\bar{j}} + [\overline{R_{j\bar{i}l}{}^m} - \overline{R_{l\bar{i}j}{}^m}] \rho_{k\bar{m}} \tag{3.21}$$

$$- R_{i\bar{l}k}{}^m \rho_{m\bar{j}} + R_{i\bar{l}}{}^{\bar{n}}{}_{\bar{j}} \rho_{k\bar{n}} \tag{3.22}$$

$$= \nabla_{\bar{l}} \nabla_k \rho_{i\bar{j}} + T_{ki}^m \nabla_{\bar{l}} \rho_{m\bar{j}} + \overline{T_{lj}^m} \nabla_i \rho_{k\bar{m}} - R_{k\bar{l}i}{}^m \rho_{m\bar{j}} + R_{i\bar{j}}{}^{\bar{m}}{}_{\bar{l}} \rho_{k\bar{m}} \tag{3.23}$$

differentiating $|\rho|^2$ once, we get

$$\partial_{\bar{j}} |\rho|^2 = \nabla_{\bar{j}} (g^{p\bar{q}} g^{k\bar{l}} \rho_{k\bar{q}} \rho_{p\bar{l}}) = g^{p\bar{q}} g^{k\bar{l}} (\nabla_{\bar{j}} \rho_{k\bar{q}} \rho_{p\bar{l}} + \nabla_{\bar{j}} \rho_{p\bar{l}} \rho_{k\bar{q}}) = 2 g^{p\bar{q}} g^{k\bar{l}} \nabla_{\bar{j}} \rho_{k\bar{q}} \rho_{p\bar{l}} \tag{3.24}$$

taking the divergence and using our previous calculation for $\nabla_i \nabla_{\bar{j}} \rho_{k\bar{q}}$ gives

$$\Delta |\rho|^2 = 2 |\nabla \rho|^2 + 2 g^{p\bar{q}} g^{k\bar{l}} g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \rho_{k\bar{q}} \rho_{p\bar{l}} \tag{3.25}$$

$$= 2 |\nabla \rho|^2 + 2 \langle \partial \bar{\partial} \operatorname{tr} \rho, \rho \rangle + 2 g^{p\bar{q}} g^{k\bar{l}} [g^{i\bar{j}} (T_{ki}^m \nabla_{\bar{q}} \rho_{m\bar{j}} + \overline{T_{qj}^m} \nabla_i \rho_{k\bar{m}})] \rho_{p\bar{l}} \tag{3.26}$$

$$+ 2 g^{k\bar{l}} \operatorname{tr}_{12}(R)^{\bar{m}p} \rho_{k\bar{m}} \rho_{p\bar{l}} - 2 R^{\bar{l}p\bar{j}m} \rho_{m\bar{j}} \rho_{p\bar{l}} \tag{3.27}$$

We now specialize to a coordinate system where $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\rho_{i\bar{j}} = \lambda_i \delta_{i\bar{j}}$, and use the assumption $g^{i\bar{j}} \rho_{i\bar{j}} = const$,

$$\Delta |\rho|^2 = 2 |\nabla_i \rho|^2 + \sum (\cdots) + 2 \sum_{k,m=1}^{n} R_{m\bar{m}k\bar{k}} (\lambda_k^2 - \lambda_k \lambda_m) \tag{3.28}$$

where

$$\sum(\cdots) = 2\sum_{k,m,i=1}^{n}[T_{ki\bar{m}}\nabla_{\bar{k}}\rho_{m\bar{i}} + \overline{T_{ki\bar{m}}}\nabla_{i}\rho_{k\bar{m}}]\lambda_k \tag{3.29}$$

$$= 2\sum_{i,k,m=1}^{n}[T_{ki\bar{m}}\nabla_{\bar{k}}\rho_{m\bar{i}} + \overline{T_{ki\bar{m}}}\nabla_{k}\rho_{i\bar{m}}]\lambda_k + 2\sum_{i,k,m=1}^{n}\overline{T_{ki\bar{m}}}T_{ki\bar{m}}\lambda_m\lambda_k \tag{3.30}$$

we can complete the square for the gradient term and we get

$$\Delta|\rho|^2 = 2\sum_{i,k,m=1}^{n}|\nabla_k\rho_{i\bar{m}} + T_{ki\bar{m}}\lambda_k|^2 + 2\sum_{i,k,m=1}^{n}|T_{ki\bar{m}}|^2\lambda_m\lambda_k \tag{3.31}$$

$$-2\sum_{k,i,m=1}^{n}|T_{ki\bar{m}}|^2\lambda_k^2 + 2\sum_{k,m=1}^{n}(R_{m\bar{m}k\bar{k}}(\lambda_k^2 - \lambda_k\lambda_m)) \tag{3.32}$$

$$= 2\sum_{i,k,m=1}^{n}|\nabla_k\rho_{i\bar{m}} + T_{ki\bar{m}}\lambda_k|^2 - 2\sum_{k,m=1}^{n}\left(R_{m\bar{m}k\bar{k}} - \sum_{i=1}^{n}T_{ki\bar{m}}\overline{T_{ki\bar{m}}}\right)\lambda_k\lambda_m \tag{3.33}$$

$$+2\sum_{k,m=1}^{n}\left(R_{m\bar{m}k\bar{k}} - \sum_{i=1}^{n}T_{ki\bar{m}}\overline{T_{ki\bar{m}}}\right)\lambda_k^2 \tag{3.34}$$

$$= 2\sum_{i,k,m=1}^{n}|\partial_i\rho_{k\bar{m}} - \Gamma_{ki\bar{m}}\lambda_m - T_{ik\bar{m}}\lambda_k|^2 + 2\sum_{k,m=1}^{n}\left(R_{m\bar{m}k\bar{k}} - \sum_{i=1}^{n}T_{ki\bar{m}}\overline{T_{ki\bar{m}}}\right)(\lambda_k^2 - \lambda_k\lambda_m) \tag{3.35}$$

$$= 2|\nabla^+\rho|^2 + 2\sum_{k,m=1}^{n}Q_{m\bar{m}k\bar{k}}(\lambda_k^2 - \lambda_k\lambda_m) \tag{3.36}$$

where $Q_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - g^{p\bar{q}}T_{kp\bar{j}}\overline{T_{lq\bar{i}}}$. □

Corollary 3.3.1. *Suppose (X, J, g) is Q-nonnegative, then any closed real $(1,1)$-form ρ with constant trace is parallel with respect to the Bismut connection. Moreover, $T(X, Y, \bar{Z}) = 0$ if $X, Y, \bar{Z}$ are in different eigenspaces of ρ corresponding to eigenvalues $\lambda_X, \lambda_Y, \lambda_Z$ and $\lambda_X + \lambda_Y - \lambda_Z \neq 0$.*

Proof. If (M, J, g) is Q-nonegative, then we have

$$\Delta|\rho|^2 \geqslant 2|\nabla^+\rho|^2 \geqslant 0$$

and by the maximum principle, we must have $|\rho|^2 = const$ which implies $\Delta|\rho|^2 = 0$ so the right hand side of equation 3.15 must be identically 0, hence we have $\nabla^+\rho = 0$. If we work in coordinates where $g_{i\bar{j}} = \delta_{ij}$ and $\rho_{i\bar{j}} = \lambda_i \delta_{ij}$, then $\nabla^+ \rho = 0$ and ρ being closed implies that

$$T_{ki\bar{m}}(\lambda_k + \lambda_i - \lambda_m) = 0 \tag{3.37}$$

from which the second part of the proposition follows. □

Proof of Theorem 3.1.2. The first statement follows by Corollary 3.3.1, we will prove the second statement that every class in $H^{1,1}_{BC}(X)$ contains a Bismut parallel representative. Recall that

$$H^{1,1}_{BC}(X) := \frac{\{\alpha \in \Omega^{1,1} | d\alpha = 0\}}{i\partial\bar{\partial}C^{\infty}(X, \mathbb{C})},$$

hence this statement amounts to showing that for any closed $(1,1)$-form α, there exist a smooth function u such that $\nabla^+(\alpha + i\partial\bar{\partial}u) = 0$. Let α be a real closed $(1,1)$-form. In [57], Gauduchon proved that every Hermitian metric is conformally equivalent to a Gauduchon metric, so let $\hat{\omega} = e^f \omega$ be a Gauduchon metric in the conformal class of ω. By the Gauduchon property, we know the equation $i\partial\bar{\partial}u \wedge \hat{\omega}^{n-1} = f\hat{\omega}^n$ has a solution iff $\int_X f\hat{\omega}^n = 0$. Let u solve the equation

$$n\frac{\sqrt{-1}\partial\bar{\partial}u \wedge \hat{\omega}^{n-1}}{\hat{\omega}^n} = \operatorname{tr}_{\hat{\omega}} \alpha + ce^{-f} \tag{3.38}$$

where c is the constant given by

$$c = -\frac{\int_X (\operatorname{tr}_{\hat{\omega}} \alpha)\hat{\omega}^n}{\int_X e^{-f}\hat{\omega}^n}$$

Setting $\rho = \alpha - \sqrt{-1}\partial\bar{\partial}u$, then $d\rho = 0$ and $\operatorname{tr}_{\omega} \rho = \operatorname{tr}_{\omega} \alpha - \Delta_{\omega} u = c$, hence by Corollary 3.3.1, we have $\nabla^+ \rho = 0$. If α is not real then we can write $\alpha = u + iv$ where u, v are the real and imaginary parts of α, then u, v are both real closed $(1,1)$ forms and we can apply the above argument to both u and v. □

Corollary 3.3.2. *If (X, J, g) is non-Kähler, Q-nonnegative and $H^{1,1}_{BC} \neq 0$. Then the holonomy of ∇^+ is contained in a subspace $U(m) \times U(n-m) \subset U(n)$.*

Proof. Since $H^{1,1}_{BC} \neq 0$, by Theorem 3.1.2, there exist a class $0 \neq [\rho] \in H^{1,1}_{BC}$ containing a Bismut flat representative ρ, and furthermore, since X is non-Kähler, ρ is not a multiple of g. Hence if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of ρ with respect to g, then there exist at least two distinct eigenvalues, and since both g and ρ is flat with respect to the Bismut connection, it follows that the eigenspaces of ρ are invariant under the holonomy ∇^+, and we have our result. □

3.4 Q and Vaisman manifolds

In this section, we study this positivity condition on a class of Locally conformally Kähler manifolds called Vaisman manifolds. These manifolds were first introduced by Vaisman [58] as an important class of complex non-Kähler manifold which he called generalized Hopf manifolds. By [59, 60], a large class of these manifolds can be viewed as a Sasakian manifolds equipped with a Sasakian automorphism. For more on these manifolds, we refer the reader to [59, 60] and the references therein.

Definition 3.4.1. (1) A compact Hermitian manifold (M, J, g) is called *locally conformally Kähler* if there exist an closed form θ such that

$$d\omega = \theta \wedge \omega \tag{3.39}$$

The 1-form θ is called the *Lee form*.

(2) A locally conformally Kähler manifold is a *Vaisman manifold* if $\nabla^{LC}\theta = 0$, where ∇^{LC} is the Levi-Civita connection of (M, J, g).

Remark 3.4.1. A locally conformally Kähler metric can always locally be written as a Kähler metric times a conformal factor, and the coverse is clearly true as well. Indeed, since the Lee form θ is closed, by the Poincare lemma, it is locally exact, hence locally, we can always write $\theta = -df$ for some locally defined functin f, then $e^f\omega$ is a Kähler metric since $d(e^f\omega) = e^f df \wedge \omega + e^f d\omega = e^f(\theta + df) \wedge \omega = 0$. Globally, any LCK manifold admits a Kähler covering. Indeed, on any cover $\pi : \tilde{M} \to M$ such that $H^1(\tilde{M}, \mathbb{R}) = 0$, the Lee form on the cover $\pi^\star\theta$ is globally exact, and the same argument says that $\pi^\star\omega$ is globally conformal to a Kähler metric.

From this discussion, we see that a locally conformally Kähler metric can locally be described by two functions, a Kähler potential for the Kähler metric φ and a conformal factor e^f. On a Vaisman manifolds, if θ is exact and $\theta = -df$, then in [61] Verbitsky showed that the metric ω can be expressed by $e^{-f} i\partial\bar{\partial} e^f$. Hence up to a cover, a Vaisman metric can be described by the data of a single potential function φ and the metric is given by $\omega = \varphi^{-1} i\partial\bar{\partial}\varphi$. Motivated by this, Ornea and Verbitsky introduced a more general class of metrics called LCK metrics with potential in [62].

Definition 3.4.2. A locally conformally Kähler manifold (M, J, g) has a potential if it has a Kähler covering $\pi : \tilde{M} \to M$ with a global positive Kähler potential $\varphi \in C^\infty_{>0}(\tilde{M})$ such that $\varphi^{-1} i\partial\bar{\partial}\varphi = c\pi^\star\omega$ for some constant c.

It turns out that on this class of manifolds, the Q-tensor enjoys an extra symmetry that the Chern curvature does not have.

Proposition 3.4.1. *Suppose a Hermitian manifold (M, J, g) is an LCK manifold with potential, then Q has the additional symmetry $Q_{i\bar{j}k\bar{l}} = Q_{k\bar{l}i\bar{j}}$.*

To prove this, first we need to compute the change of the Q tensor under a conformal change

Lemma 3.4.1. *If g is a Kähler metric and $\tilde{g} = e^f g$, then we have*

$$\tilde{Q}_{i\bar{j}k\bar{l}} = e^f \left[R_{i\bar{j}k\bar{l}} - g_{k\bar{l}} f_{i\bar{j}} - f_k f_{\bar{l}} g_{i\bar{j}} - |\partial f|^2 g_{k\bar{j}} g_{i\bar{l}} + f_k f_{\bar{j}} g_{i\bar{l}} + f_i f_{\bar{l}} g_{k\bar{j}} \right] \tag{3.40}$$

Proof. By straightforward computation, we have

$$\tilde{\Gamma}^k_{ij} = \tilde{g}^{k\bar{l}} \partial_i \tilde{g}_{j\bar{l}} = e^f g^{k\bar{l}} \partial_i (e^f g_{j\bar{l}}) = \Gamma^k_{ij} + f_i \delta^k_j$$

since we assumed g is Kähler, we have $\Gamma^k_{ij} = \Gamma^k_{ji}$, so

$$\tilde{T}^k_{ij} = f_i \delta^k_j - f_j \delta^k_i \implies \tilde{T}_{ij\bar{k}} = e^f (f_i g_{j\bar{k}} - f_j g_{i\bar{k}})$$

and

$$\tilde{R}_{i\bar{j}k}{}^{m} = -\partial_{\bar{j}}\tilde{\Gamma}^{m}_{ik} = -\partial_{\bar{j}}(\Gamma^{m}_{ik} + f_i\delta^{m}_{k}) = R_{i\bar{j}k}{}^{m} - f_{i\bar{j}}\delta^{m}_{k}$$

loweing indices gives

$$\tilde{R}_{i\bar{j}k\bar{l}} = e^{f}(R_{i\bar{j}k\bar{l}} - f_{i\bar{j}}g_{k\bar{l}})$$

substituting into the formula $\tilde{Q}_{i\bar{j}k\bar{l}} = \tilde{R}_{i\bar{j}k\bar{l}} - \tilde{g}^{p\bar{q}}\tilde{T}_{kp\bar{j}}\bar{\tilde{T}}_{lq\bar{i}}$ the result. □

proof of Proposition 3.4.1. Suppose g satisfies $g = c\varphi^{-1}dd^{c}\varphi$, then by Lemma 3.4.1 with $g = i\partial\bar{\partial}\varphi$ and $f = -\log\varphi + \log c$, we get that Q is given by

$$Q_{i\bar{j}k\bar{l}} = c\varphi^{-1}\left[R^{i\partial\bar{\partial}\varphi}_{i\bar{j}k\bar{l}} + \frac{\varphi_{k\bar{l}}\varphi_{i\bar{j}}}{\varphi} - \frac{\varphi_{k\bar{l}}\varphi_{i}\varphi_{\bar{j}}}{\varphi^2} - \frac{\varphi_{i\bar{j}}\varphi_{k}\varphi_{\bar{l}}}{\varphi^2} - |\partial\log\varphi|^2_{i\partial\bar{\partial}\varphi}\varphi_{k\bar{j}}\varphi_{i\bar{l}} + \frac{\varphi_{i\bar{l}}\varphi_{k}\varphi_{\bar{j}}}{\varphi^2} + \frac{\varphi_{k\bar{j}}\varphi_{i}\varphi_{\bar{l}}}{\varphi^2}\right] \tag{3.41}$$

from which we can read off the symmetry $Q_{i\bar{j}k\bar{l}} = Q_{k\bar{l}i\bar{j}}$. □

All Vaiman manifolds are LCK with potential, and the Vaisman manifolds are charaterized by the following condition.

Theorem 3.4.1 ([63])**.** *A compact LCK manifold with potential (M, J, g) is Vaisman if and only if $|\theta| = const$.*

Remark 3.4.2. Let (M, J, g) be LCK with potential, then if we pull-back the metric g to its Kähler cover and write $\pi^{\star}\omega = c\frac{i\partial\bar{\partial}\varphi}{\varphi}$ for some potential φ, then the condition $|\theta| = const$ is equivalent to $|\partial\log\varphi|^2_{\varphi^{-1}i\partial\bar{\partial}\varphi} = 1$.

Theorem 3.4.2. *A Vaisman manifold (M, J, g) is Q-nonnegative if the corresponding Kähler metric $\tilde{\omega}$ on its Kähler cyclic cover has nonnegative quadratic orthogonal bisectional curvature.*

Proof. Suppose M is a Vaisman manifold and $\tilde{M} \to M$ be a Kähler cyclic cover with nonnegative quadratic orthogonal bisectional curvature. Since Vaiman manifolds are LCK with potential, by Proposition 3.4.1, we know that on a Vaisman manifiold Q satisfy the symmetry $Q_{i\bar{j}k\bar{l}} = Q_{k\bar{l}i\bar{j}}$. Moreover, in normal coordinates for $i\partial\bar{\partial}\varphi$, i.e where $\varphi_{i\bar{j}} = \delta_{i\bar{j}}$, and for $m \neq k$, equation 3.41

reduces to

$$Q_{m\bar{m}k\bar{k}} = c\varphi^{-1}\left[R^{i\partial\bar{\partial}\varphi}_{m\bar{m}k\bar{k}} + \frac{1}{\varphi} - \frac{\varphi_k\varphi_{\bar{k}} + \varphi_m\varphi_{\bar{m}}}{\varphi^2}\right] \tag{3.42}$$

$$\geqslant c\varphi^{-1}\left[R^{i\partial\bar{\partial}\varphi}_{m\bar{m}k\bar{k}} + \frac{1 - |\partial\log\varphi|^2_{\varphi^{-1}i\partial\bar{\partial}\varphi}}{\varphi}\right] \tag{3.43}$$

and by the Vaisman condition, we have $|\partial\log\varphi|^2_{\varphi^{-1}i\partial\bar{\partial}\varphi} = 1$. So in the those coordinates, for any $\lambda_1, \ldots, \lambda_n$, we get

$$\sum_{m,k} Q_{m\bar{m}k\bar{k}}(\lambda_m - \lambda_k)^2 \geqslant c\varphi^{-1}\sum_{m,k} R^{i\partial\bar{\partial}\varphi}_{m\bar{m}k\bar{k}}(\lambda_m - \lambda_k)^2 \geqslant 0$$

□

Example 3.4.3. A diagonal Hopf surface is one which can be written as $M_{\alpha,\beta} = \mathbb{C}^2 \setminus (0,0)/\sim$ where $(z_1, z_2) \sim (\alpha z_1, \beta z_2)$ for $|\alpha| = |\beta| < 1$. These manifold are diffeomorphic to $S^3 \times S^1$ and are non-Kähler as $b_1 = 1$. They also admit a Vaisman metric given explicitly by

$$\omega_{M_{\alpha,\beta}} = \frac{4\sqrt{-1}\partial\bar{\partial}|z|^2}{|z|^2} \tag{3.44}$$

this metric is Q-nonegative by Proposition 3.4.2. In fact this metric is pluriclosed and Q is identically Q. One can check that $h^{1,1}_{BC} = 1$. Thus Theorem 3.1.2 implies there exist a Bismut parallel $(1,1)$-form ρ, the two eigenspaces of this form are then Bismut parallel subspaces of $T^{1,0}$, hence this gives a splitting of the holomorphic tangent bundle with respect to the Bismut connection. We remark that that Gauduchon and Ornea constructed Vaisman metrics on all class 1 Hopf surfaces in [64], however the Vaiman metrics on the non-diagonal Hopf surfaces are not Q-nonnegative. It would be interesting to know if the non-diagonal Hopf surfaces admit other metrics that are Q-nonnegative.

Example 3.4.4. A above construction of a diagonal Hopf surface can be generalized to higher dimensions. Define $M_\alpha = \mathbb{C}^n \setminus (0, \ldots, 0)/\sim$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy $|\alpha_1| = \cdots = |\alpha_n| < 1$

and $(z_1, \ldots, z_n) \sim (\alpha_1 z_1, \ldots, \alpha_n z_n)$, then

$$\omega_{M_\alpha} = \frac{4\sqrt{-1}\partial\bar{\partial}|z|^2}{|z|^2} \tag{3.45}$$

is still Q-nonnegative by Proposition 3.4.2. In the higher dimensional case, the metrics are no longer pluriclosed and Q does not vanish identically.

Chapter 4: Degenerations of non-compact Calabi-Yau metrics

4.1 Motivation and introduction

In this chapter we study the degenerations of *non-compact* Calabi-Yau manifolds, and obtain the existence of Calabi-Yau metrics on certain *non-compact, singular* varieties.

Recall a Calabi-Yau manifold is a Kähler manifold which is also Ricci flat. Following Yau's solution of the Calabi conjecture, it's known that on a compact Kähler manifold with trivial canonical bundle, there exists a unique Calabi-Yau metric in every Kähler class. The degenerations of compact Calabi-Yau metrics are a widely studied subject and have been studied quiet extensively in recent years. [65, 66, 67, 68, 69].

In this chapter, we will study the degenerations of Calabi-Yau metrics on complete *non-compact* Calabi-Yau manifolds which are asymptotic to cones. Complete, non-compact Calabi-Yau manifolds were first constructed by Tian-Yau in [70, 71], and a plethora of examples are now known to exist. A particular subset of these are Calabi-Yau manifolds which are asymptotic to a cone at infinity, these are sometimes called *asymptotically conical Calabi-Yau manifolds*. Asymptotically conical Calabi-Yau manifolds are of fundamental importance, since they arise as blow-up limits at the singular points in the limit of a non-collapsing family of Kähler-Einstein manifolds (or more generally Kähler manifolds with bounded Ricci curvature). The conical asymptotics should be regarded here as akin to the non-collapsing condition in the setting of compact Calabi-Yau manifolds.

The first analytic construction of asymptotically conical Calabi-Yau manifolds was given in [71] and [72, 73], and the construction has been further refined by the work of many authors, see [74, 75, 76, 77, 78, 79, 80, 81, 82] and the references therein. One nice improvement given by these refinements is that, in analogy with Yau's theorem in the compact case [6], one is able to produce an asymptotically conical Ricci-flat Kähler metric in every suitable Kähler class on an asymptotically conical Kähler manifold X. In particular, this yields families of degenerating asymptotically conical Ricci-flat Käher metrics as the classes approach a non-Kähler class, and one can then ask what properties limits of these spaces possess.

Let us describe the set-up under consideration and state our main theorems. The terminologies used in this section will be explained in the next section. Let (X, J, ω, Ω) be an open Kähler manifold with trivial canonical bundle, with only one end which is asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate $\nu > 0$. Consider a linear family of ν-almost compactly supported Kähler classes $[\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_\nu(X)$ for $t \in (0,1]$. Suppose $[\alpha_0]$ satisfies the following assumption.

Assumption 1. $[\alpha_0]$ *contains a semi-positive form* α_0*, and there exists* $\varepsilon_0 > 0$ *and a* $\psi \in PSH(X, \alpha_0)$ *such that* $\alpha_0 + i\partial\bar{\partial}\psi \geqslant \varepsilon_0\omega$ *for some Kähler form* ω *on* X*. Furthermore, assume that* ψ *is smooth away from a compact analytic subvariety* $V \subset X$*, and* $V = \{\psi = -\infty\}$.

Remark 4.1.1. We expect that Assumption 1 essentially always applies, possibly after weakening the semi-positivity assumption. In fact, in analogy with the main result of [69], we expect that

$$V = \bigcup_{Y \subset X : \int_Y \alpha_0^{\dim Y} = 0} Y$$

where the union is taken over compact, irreducible analytic subvarieties. We will prove this in a large class of examples; see the discussion in Section 4.3.1.

In [78], it is proved that for $t \in (0,1]$ there exists a unique asymptotically conical Ricci-flat Kähler metric $\omega_{t,CY} \in [\alpha_t]$ satisfying the complex Monge-Ampère equation

$$\omega^n_{t,CY} = i^{n^2}\Omega \wedge \bar{\Omega}$$

The first main theorem of this Chapter is the following,

Theorem 4.1.1. *Let $0 < \nu < 2n$ and consider a linear family of ν-almost compactly supported Kähler classes $[\omega_t] = (1-t)[\alpha_0] + t[\omega] \in H^{1,1}_{\nu}(X,\mathbb{R})$ for $t \in (0,1]$. Suppose $[\alpha_0]$ satisfies Assumption 1. Let $\omega_{t,CY}$ be the asymptotically conical Calabi-Yau metrics in $[\omega_t]$. Then, as $t \to 0$, the Ricci-flat Kähler metrics $\omega_{t,CY}$ converge in $C^{\infty}_{loc}(X \setminus V)$ to an incomplete Ricci-flat Kähler metric $\omega_{0,CY}$ on $X \setminus V$ satisfying*

$$\omega^n_{0,CY} = i^{n^2}\Omega \wedge \bar{\Omega}. \tag{4.1}$$

Moreover, we have

1. *$\omega_{0,CY}$ extends across V as a positive current with locally bounded potentials and* (4.1) *holds globally in the sense of Bedford-Taylor [83].*

2. *$\omega_{0,CY}$ is asymptotically conical at infinity and, outside of a compact set $K \subset X$, $\omega_{0,CY}$ satisfies $|\nabla^k(\omega_{0,CY} - \omega_C)|_{\omega_C} = O(r^{-\nu-k})$, where $r(x) = \mathrm{dist}(x_0, x)$ is the distance to a fixed point with respect to the conical Kähler metric ω_C.*

3. *$\omega_{0,CY}$ is unique in the sense that, if ω is any closed positive current in the class $[\omega_{0,CY}]$ with locally bounded potentials, which is smooth on $X \setminus V$, asymptotically conical at any rate $\delta > 0$, and satisfying* (4.1) *on X in the sense of Bedford-Taylor, then $\omega = \omega_{0,CY}$.*

The reader may wish to compare this result with the analogous result in the compact case [69, Theorem 1.6].

A natural way to construct examples where Theorem 4.1.1 applies is to consider resolutions of singular varieties. We consider a set-up as follows: let (X_0, Ω) be a normal, log-terminal, Gorenstein variety with K_{X_0} trivial, and suppose that X_0 has compactly supported singularities and admits a crepant resolution of singularities $\pi : (X, \Omega) \to (X_0, \Omega)$. Suppose that $L \to X_0$ is an ample line bundle on X_0 (see Section 4.5 for the definition of ampleness in this context). Let $[\alpha_0] = \pi^* c_1(L) \in H^2(X, \mathbb{R})$ and suppose that (X, J, ω, Ω) and $[\omega_t] = (1-t)[\alpha_0] + t[\omega] \in H^{1,1}_{\nu}(X,\mathbb{R})$ is a family of Kähler classes satisfying the same hypothesis as in Theorem 4.1.1. (In particular $[\alpha_0]$ satisfies Assumption 1)

Theorem 4.1.2. *In the situation above the singular Ricci-flat current $\omega_{0,CY}$ descends to a Ricci-flat Kähler current on X_0 and satisfies*

1. $\omega_{0,CY}$ *is a smooth Ricci-flat Kähler metric on* $\pi^{-1}(X_0^{reg})$.

2. $\omega_{0,CY}$ *is a Kähler current on* X_0*, (i.e.* $\omega_{0,CY} \geqslant \omega$ *for some smooth Kähler form* ω *on* X_0*)*

3. $\overline{(X_0^{reg}, \omega_{0,CY})}$ *is homeomorphic to* X_0.

4. $(X, \omega_{t,CY})$ *Gromov-Hausdorff converges to* X_0 *with the distance function induced by* $\omega_{0,CY}$.

A couple of remarks are in order concerning the assumptions of Theorem 4.1.2

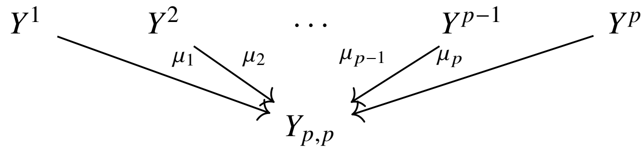
Remark 4.1.2. 1. Theorem 4.1.2 requires that Assumption 1 to hold for the class $[\alpha_0]$. As pointed out in Remark 4.1.1, we expect that in this situation that we can always take $V = \pi^{-1}(X_0^{sing})$, and we will prove this is a large number of cases in Lemma 4.3.1. Although we don't actually need to assume this for the proof of Theorem 4.1.2.

2. In many cases where Theorem 4.1.2 applies, we will take $L = \mathcal{O}_{X_0}$. This can be done when X_0 is affine, which is a natural setting for studying Calabi-Yau varieties with isolated singularities.

We apply these results to study several classes of examples. Let us briefly describe one particular class. Consider the quasi-homogeneous affine variety

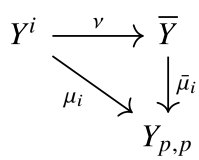
$$Y_{p,q} := \{xy + z^p - w^q = 0\} \subset \mathbb{C}^4,$$

where without loss of generality we can assume $p \leqslant q$. $Y_{p,q}$ is normal, Gorenstein and log-terminal, and by [84] $Y_{p,q}$ admits a conical Calabi-Yau metric if and only if $q < 2p$. A result of Katz [85] says that the $Y_{p,p}$ admits p inequivalent small (and hence crepant) resolutions resolutions $\mu_i : Y^i \to Y_{p,p}$ (and if $p \neq q$ then no small resolution exists). We therefore have the following picture



with each pair Y^i, Y^j related by a flop. When $p = 2$, this is the well-known example of the Atiyah flop [86]. In Section 4.6 we apply our results to this setting.

Corollary 4.1.1. *Let Y^i be a small resolution of the $Y_{p,p}$ singularity, and let ω_0 denote the Calabi-Yau metric on $Y_{p,p}$. Let $[\omega_t] := (1-t)[\alpha_0] + t[\omega]$ be any linear family of Kähler classes on Y^i, where $[\alpha_0] \in H^{1,1}(Y^i, \mathbb{Q})$ is not Kähler. Then for all $t > 0$ there is an asymptotically conical Calabi-Yau metric $\omega_{t,CY} \in [\omega_t]$. Furthermore, there is a partial resolution $\bar{\mu}_i : \overline{Y} \to Y_{p,p}$ and a map $\nu : Y^i \to \overline{Y}$ such that the following diagram commutes*



As $t \to 0$, $\omega_{t,CY}$ converge in $C^\infty_{loc}(Y^i \backslash \mathrm{Exc}(\nu))$ to an incomplete, asymptotically conical Calabi-Yau metric $\overline{\omega}$ on $\overline{Y}_{reg}$ and (Y^i, ω_t) Gromov-Hausdorff converges to $\overline{(\overline{Y}_{reg}, \overline{\omega})}$ which is homeomorphic to $\overline{Y}$. Furthermore, if $[\alpha_0] = 0$ then $\overline{Y} = Y_{p,p}$, $\bar{\mu}_i$ is the identity, and $\bar{\omega} = \omega_0$ the Calabi-Yau metric on $Y_{p,p}$. In particular, when $[\alpha_0] = 0$, for any i, j the flop from Y^i to Y^j is continuous in the Gromov-Hausdorff topology in the sense that

$$(Y^i, \cdot\omega_{t,CY}) \xrightarrow{GH} (Y_{p,p}, \omega_0) \xleftarrow{GH} (Y^j, \cdot\omega_{t,CY})$$

A second general class of examples we consider gives rise to the following specific example. Let X be a del Pezzo surface of degree $d \geqslant 2$, and let $\tilde{X} = \mathrm{Bl}_p X$ be the blow-up at a point $p \in X$. Assume that $\tilde{X}$ is Fano (and if $d = 8$ assume that $\tilde{X}$ is toric). Then the canonical cone

$$C := \mathrm{Spec} \bigoplus_{m \geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$$

admits a conical Calabi-Yau metric [87, 88, 89, 90, 84]. Then we prove

Corollary 4.1.2. *In the above setting, there is an asymptotically conical Calabi-Yau metric on the relative spectrum* $Z := \underline{Spec}\,(K_X \otimes \mathfrak{m}_p)$ *which is asymptotic at infinity to the conical Calabi-Yau metric on* C.

The metric on Z is constructed as a limit of asymptotically conical Calabi-Yau metrics on a small resolution, and we again obtain a Gromov-Hausdorff covergence statement; see Section 4.6 for a complete discussion.

We will explain a speculative picture in which that space Z can be viewed as a cobordism between Sasakian manifolds; in this case, the link of the A_1 singularity (topologically $S^2 \times S^3$) and the link of the cone C (topologically $\#(9-d+1)S^2 \times S^3$). The Calabi-Yau metric on Z upgrades this to a cobordism of Sasaki-Einstein manifolds. In this picture the volume of the geodesic spheres can be viewed as a sort of Morse function.

The examples above all come from (partial-)resolutions of Calabi-Yau cones. Our theorem can also yield examples where the complex structure at ∞ is not biholomorphic to the asymptotic cone.

One class of examples come from the following. Let X be an asymptotically conical Calabi-Yau manifold, then by [91], there exist a normal Stein space Y with finitely many isolated singularities and there is a holomorphic map $\pi : X \to Y$ with connected fibers, is an biholomorphim outside the singularities of Y and $\pi^\star \mathcal{O}_Y = \mathcal{O}_X$. The map π contracts the maximal compact analytic subset of X and Y is called the Remmert reduction of X. Since Y is a stein space, it properly embeds into $\mathbb{C}^N$ for some N sufficiently large. The singularities of Y are rational [78, Theorem A.2], and hence Cohen-Macaulay, and since K_X is trivial and Y is normal, it follows that K_Y is trivial and Y is Gorenstein. Hence π is a crepant resolution of Y.

Corollary 4.1.3. *Assuming Assumption 1 holds for* $[\alpha_0] = 0$*, applying our theorem with* $[\alpha_0] = 0 \in H^2(X, \mathbb{R})$*,* $\omega_{0,CY}$ *descends to a singular CY current on* Y *and the AC Calabi-Yau metrics* $\omega_{t,CY}$ *in the classes* $t[\omega] \in H^2(X, \mathbb{R})$ *Gromov-Haussdorff coverge to the Remmert reduction* Y.

The outline of this chapter is as follows. In Section 4.2 we discuss some basic properties of asymptotically conical Kähler manifolds, and state two main propositions (Propositions 4.2.3, and 4.2.4). We give the proof of Theorem 4.1.1 assuming these two propositions. In Section 4.3 we discuss the construction of good background metrics, and prove Proposition 4.2.3. In Section 4.4 we prove some a priori estimates and deduce Proposition 4.2.4, completing the proof of Theorem 4.1.1. In Section 4.5 we use L^2 estimates to prove Theorem 4.1.2. Finally, in Section 4.6 we explain examples where Theorems 4.1.1 and 4.1.2 are applicable, and discuss a speculative Morse theoretic picture.

4.2 Preliminaries

In this section, we discuss some preliminary background material we will need about asymptotically conical Calabi-Yau manifolds and we will reduce the Proof of Theorem 4.1.1 to two propositions which we will prove in the next two sections. Our main reference is [78].

4.2.1 Asymptotically conical Kähler manifolds

We review here some basic definitions and an existence theorem for asymptotically conical Calabi Yau metrics from [78].

Definition 4.2.1. (A) An open *Kähler cone* (C, J_C, ω_C, g_C) is a Riemannian cone (C, g_C) with smooth link L that is additionally equipped with a complex structure J_C such that the Kähler form is $\omega_C = i\partial\bar{\partial}r_C^2$ where r_C is the distance function from the tip of the cone.

(B) A *Calabi-Yau cone* $(C, J_C, \omega_C, g_C, \Omega_C)$ is a Kähler cone with an additional holomorphic volume form Ω_C such that $\omega_C^n = i^{n^2}\Omega_C \wedge \bar{\Omega}_C$.

Definition 4.2.2. (A) A Kähler manifold (X, J, g, ω) is called *asymptotically conical* if there exist a Kähler cone (C, J_C, g_C, ω_C) and a diffeomorphism $\Phi : C \setminus B_R(o) \to X \setminus K$ for some $K \subset\subset X$ and o is the vertex of the cone C, and $\nu > 0$ such that the following hold

$$|\nabla^k(\Phi^*J - J_C)|_{g_C} + |\nabla^k(\Phi^*\omega - \omega_C)|_{g_C} = O(r_C^{-\nu-k}), \quad \forall k \in \mathbb{N}$$

where the covariant derivatives are taken with respect to g_C. We say that X *asymptotic to* C *with rate* ν.

(B) We say that an open Calabi-Yau manifold (X, J, ω, Ω) is *asymptotic to the Calabi-Yau cone* $(C, J_C, \omega_C, \Omega_C)$ *with rate* ν if (X, J, g, ω) is asymptotic to the Kähler cone (C, J_C, g_C, ω_C) with rate ν, and, in addition

$$|\nabla^k(\Phi^*\Omega - \Omega_C)|_{g_C} = O(r_C^{-\nu-k})$$

Remark 4.2.1. 1. On any asymptotically conical Kähler manifold, we can always find a smooth function $r : X \to \mathbb{R}_{\geqslant 0}$ satisfying $r = r_C \cdot \Phi^{-1}$ away from some compact set K where r_C is the radial distance on the cone C, and furthermore, r satisfies: $|\nabla r| + r|\nabla^2 r| \leqslant C$. We will call such an r a *radius function*.

2. In fact, it is shown in [78, Lemma 2.14] that $\Phi^*J - J_C$ always decays at the same rate as $\Phi^*\Omega - \Omega_C$, so it suffices just to assume $|\nabla^k(\Phi^*\Omega - \Omega_C)|_{g_C} = O(r^{-\nu-k})$, and $|\nabla^k(\Phi^*J - J_C)|_{g_C} = O(r^{-\nu-k})$ follows automatically.

3. We will often say (X, J, g, ω) is an *asymptotically conical Kähler manifold* if it is asymptotic to some Kähler cone (C, J_C, g_C, ω_C) at some rate $\nu > 0$ by some map Φ. We will therefore often suppress the map Φ, with the understanding that all asymptotics are measured with respect to the diffeomorphism Φ. Furthermore, when Φ is implicit, we will often abuse notation and write ω_C, J_C, Ω_C in place of $(\Phi^{-1})^*\omega_C$, $(\Phi^{-1})^*J_C$, $(\Phi^{-1})^*\Omega_C$.

4. On an asymptotically conical Kähler manifold with rate ν we will often refer to a $(1, 1)$ form α being asymptotically conical. By this we mean that there is a compact set K such that, on $X \backslash K$ the form α defines an asymptotically conical Kähler metric with rate ν.

We now quote two versions of the $\partial\bar{\partial}$-lemma which hold on asymptotically conical Calabi-Yau manifolds, see [78] for a proof.

Proposition 4.2.1 ($\partial\bar{\partial}$-lemma, [78], Corollary A.3)**.** *Suppose X is an asymptotically conical Kähler manifold with trivial canonical bundle, then*

1. *If α is an exact real $(1,1)$-form on X, then $\alpha = i\partial\bar{\partial}u$ for some smooth function u.*

2. *If $\dim_{\mathbb{C}} X > 2$, then if α is an exact real $(1,1)$-form on $X \setminus K$ for some compact subset K, then there exist a compact set K' containing K such that $\alpha = i\partial\bar{\partial}u$ on $X \setminus K'$.*

Proposition 4.2.2 (Quantitative $\partial\bar{\partial}$-lemma, [78], Theorem 3.11)**.** *Suppose X is an asymptotically conical Kähler manifold with Ric $\geqslant 0$, then there exist $\varepsilon_0 > 0$, such that for any η an exact $(1,1)$-form with $\eta \in C^{\infty}_{-\varepsilon}(X)$ for $0 < \varepsilon < \varepsilon_0$, then $\eta = i\partial\bar{\partial}u$ for $u \in C^{\infty}_{2-\varepsilon}$.*

4.2.2 Kähler classes on AC Kähler manifolds

We recall the definition of a ν-almost compactly supported class, this is defined in [78], but our definition is slightly different.

Definition 4.2.3. Let X be an asymptotically conical Kähler manifold, then for any class $[\alpha] \in H^2(X,\mathbb{R})$, we say that

1. $[\alpha]$ is a Kähler class if it contains a positive real $(1,1)$-form $\alpha > 0$

2. $[\alpha]$ is a ν-almost compactly supported class if it contains a real $(1,1)$-form ξ satisfying $|\nabla^k \xi| = O(r^{-\nu-k})$

and we will denote the set of all ν-almost compactly supported classes by $H^{1,1}_{\nu}(X)$.

Remark 4.2.2. Definition 4.2.3 is slightly more restrictive than the definition given in [78] where it is only required that the form ξ be defined away from a compact set. But by the second part of Lemma 4.2.1, the condition in [78] implies our condition in the case when X has trivial canonical bundle and $\dim_{\mathbb{C}} X > 2$.

In [78], it is shown that if $[\alpha]$ is a ν-almost compactly supported and Kähler, then one can always construct an asymptotically conical Kähler form $\omega \in [\alpha]$ with $|\nabla^k(\omega - \omega_C)| = O(r^{-\nu-k})$. We will recall this construction below in Section 4.3.

4.2.3 Weighted Hölder spaces and solvability of Poisson's equation

Let us recall some useful Holder spaces defined on asymptotically conical manifolds and some basic theorems regarding the solvability of Poisson equations, which will be useful for us later on. For a detailed treatment of these material, see [92, 93].

Definition 4.2.4. Let X be a AC Kähler manifold as above.

1. We define the $C^{k,\alpha}_{-\gamma}(X)$ norm of a function as follows

$$\|u\|_{C^{k,\alpha}_{-\gamma}} = \sum_{j=0}^{k} \sup_X |r^{\gamma+j}\nabla^j u| + [\nabla^k u]_{C^{\alpha}_{-\gamma-k-\alpha}}$$

   where r is a radius function and

$$[\nabla^k u]_{C^{\alpha}_{-\gamma-k-\alpha}} = \sup_{x\neq y, d(x,y)\leqslant\delta}\left[\min(r(x),r(y))^{\gamma+k+\alpha}\frac{|\nabla^k u(x) - \nabla^k u(y)|}{|d(x,y)|^{\alpha}}\right]$$

   where $\delta > 0$ is the convexity radius of X, and $|\nabla^k u(x) - \nabla^k u(y)|$ is defined by parallel transporting $\nabla^k u(x)$ along the minimal geodesic from x to y.

2. We define $C^{\infty}_{-\gamma}(X)$ to be the intersection of $C^{k,\alpha}_{-\gamma}(X)$ over all $k \geqslant 0$.

3. We will also often use the following space $C^{\infty}_{-\gamma}(X \setminus V)$, which we define to be the space of functions $u \in C^{\infty}_{loc}(X \setminus V)$ such that $(1-\chi)u \in C^{\infty}_{-\gamma}(X)$, where χ is a cutoff function with compact support that is equal to 1 in a neighborhood of V. Where V is the compact analytic subset coming from Assumption 1.

The space $C^{k,\alpha}_{-\gamma}(X)$ consisting of all functions with finite $C^{k,\alpha}_{-\gamma}(X)$ norm is then a Banach space, and it follows that the Laplace operator $\Delta : C^{k+2,\alpha}_{-\gamma+2}(X) \to C^{k,\alpha}_{-\gamma}(X)$ is a bounded map between the two Banach spaces. There is a well-developed Fredholm theory for the Laplace operator on these Banach spaces on an asymptotically conical manifold (see, e.g. [93]), which we summarize below.

Definition 4.2.5. Let (C, g_C) be a Riemannian cone of real dimension n over a smooth compact manifold L^{n-1}, then we denote the set of *exceptional weights* of the cone C,

$$P = \left\{ -\frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda} : \lambda \text{ is an eigenvalue of } \Delta_{L^{n-1}} \right\}.$$

These correspond to the growth rates of homogenous harmonic functions on the cone (C, g_C).

The following theorem summarizes Fredholm theory on an asymptotically conical manifold

Theorem 4.2.1 ([93], Theorem 6.10)**.** *Suppose (X, g) is an asymptotically conical Kähler manifold of dimension $2n$. Consider the mapping*

$$\Delta : C^{k+2,\alpha}_{-\gamma}(X) \to C^{k,\alpha}_{-\gamma-2}(X) \tag{4.2}$$

and let P be the set of exceptional weights of the asymptotic cone (C, g_C). Then:

1. *The operator (4.2) Fredholm if $-\gamma \notin P$.*

2. *The operator (4.2) is surjective if $-\gamma \in (2-2n, \infty) \setminus P$*

3. *The operator (4.2) is injective if $-\gamma \in (-\infty, 0) \setminus P$*

Remark 4.2.3. We note that $P \cap (2-2n, 0) = \emptyset$, hence (4.2) is an isomorphism for all $-\gamma \in (2-2n, 0)$.

Now we state a general theorem regarding the solvability of the complex Monge-Ampère equation on an asymptotically conical Kähler manifold, which is proved in [78].

Theorem 4.2.2 ([78], Theorem 2.4)**.** *Let (X, J, ω) be a open Kähler manifold asymptotic to a Kähler cone (C, J_C, ω_C) with rate $\nu > 0$, and suppose $f \in C^{\infty}_{-\gamma-2}(X)$, then following Complex Monge-Ampere equation then admits a solution*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^f \omega^n$$

with $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0$ and

1. *If $\gamma + 2 > 2n$, then we can take $\varphi \in C^{\infty}_{2-2n}$ and φ is the unique solution in C^{∞}_{2-2n}.*

2. *If $\gamma + 2 \in (2, 2n)$ then we can take $\varphi \in C^{\infty}_{-\gamma}$ and φ is the unique solution in $C^{\infty}_{-\gamma}$.*

3. *If $\gamma + 2 \in (0, 2)$ and $-\gamma$ is not an exceptional weight, we can take $\varphi \in C^{\infty}_{-\gamma}$.*

4.2.4 Proof of Theorem 4.1.1

We breakdown the proof of Theorem 4.1.1 in the following two propositions, and we will give the proof of Theorem 4.1.1 assuming these results. We will prove Proposition 4.2.3 in Section 4.3 and Proposition 4.2.4 in Section 4.4. Theorem 4.1.2 will be proved in section 4.5.

Proposition 4.2.3 (Constructing background metrics). *Suppose $\nu > 0$, and let (X, J, ω, Ω) be an asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate ν. Suppose that $-\nu \in (-2n, 0)$ and $-\nu + 2$ is not an exceptional weight. Suppose $[\alpha_t] = (1 - t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_{\nu}(X)$ is a linear family of Kähler classes in $H^{1,1}_{\nu}$ for $t \in (0, 1]$, and suppose that $[\alpha_0] \in H^{1,1}_{\nu}$ has a semi-positive representative α_0. Then there exists $\varepsilon > 0$, a compact set $K \subset X$ and a smooth family of real $(1, 1)$-forms $\hat{\omega}_t \in [\alpha_t]$ for $t \in [0, \varepsilon]$ satisfying the following:*

1. *$\hat{\omega}_t > 0$ for all $t \in (0, \varepsilon]$.*

2. *$\hat{\omega}_0 \geqslant 0$ and $\hat{\omega}_0 = \alpha_0$ on a compact set $K \subset\subset X$. (In fact, we can choose this compact set K to be as large as we like)*

3. *On $X \backslash K$ there holds $|\nabla^k(\hat{\omega}_t - \omega_C)|_{g_C} \leqslant Cr^{-\nu-k}$ for all $t \in [0, \varepsilon]$ for a constant C independent of t.*

4. *There exist $\gamma > 0$ such that, on $X \backslash K$ the Ricci potentials $f_t = \log \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\hat{\omega}_t^n}$ satisfy the asymptotics $|\nabla^k f_t| \leqslant Cr^{-\gamma-2-k}$ uniformly in t.*

Proposition 4.2.4 (A priori estimates). *Let (X, J, ω, Ω) be asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate $\nu > 0$, and $H^{1,1}_{\nu}(X) \ni [\alpha_t] = (1 - t)[\alpha_0] + t[\alpha_1]$ is a linear family of Kähler classes for*

$t \in (0,1]$ satisfying Assumption 1, and let $\hat{\omega}_t \in [\alpha_t]$ be the forms constructed in Proposition 4.2.3. Let φ_t be the solution of the complex Monge-Ampère equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}_t^n (= i^{n^2}\Omega \wedge \bar{\Omega}) \tag{4.3}$$

obtained from Theorem 4.2.2. Then the following estimates hold uniformly in t

1. *$|\varphi_t| \leqslant C$.*
2. *φ_t is uniformly bounded in $C^\infty_{loc}(X \setminus V)$.*
3. *There exist a compact subset $K \subset X$ containing V such that the following estimate hold outside of K*

   $$|\nabla^k \varphi_t| \leqslant C r^{-\gamma-k}$$

   for C independent of t.

Now we prove Theorem 4.1.1 given the above two propositions

Proof of Theorem 4.1.1. Let $[\alpha_t] = (1-t)[\alpha] + t[\varepsilon\omega]$, then by Proposition 4.2.3, we can construct a sequence of background metrics $\hat{\omega}_t \in [\alpha_t]$ satisfying the properties stated in the Proposition. Then using these as background metrics, we can write down a family of complex Monge-Ampere equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}_t^n (= i^{n^2}\Omega \wedge \bar{\Omega})$$

then by the Theorem 4.2.2, the equations are solvable for $t > 0$, and Proposition 4.2.4 applies to the family of solutions φ_t. Once we have the a priori estimate, it's then clear that by taking a subsequence, we can take a limit $\varphi_{t_i} \to \varphi_0$ in $C^\infty_{loc}(X \setminus V)$, which satisfies the equation

$$(\hat{\omega}_0 + i\partial\bar{\partial}\varphi_0)^n = i^{n^2}\Omega \wedge \bar{\Omega} \tag{4.4}$$

smoothly away from the analytic set V. Moreover, φ_0 is a bounded by the uniform C^0 estimate of φ_t, hence $\hat{\omega}_0 + i\partial\bar{\partial}\varphi_0$ extends as a non-negative current on X by [94], and it does not charge any analytic subsets, so the equation (4.4) holds globally. From Proposition 4.2.3 (2), and Proposition 4.2.4 (3), we see that ω_{φ_0} is asymptotically conical. It only remains to establish the incompleteness and uniqueness statements of ω_{φ_0} in Theorem 4.1.1. The incompleteness of ω_{φ_0} follows from the diameter bound in Lemma 4.4.2, while the uniqueness is established in Theorem 4.4.1 □

4.3 Background metrics

The goal of this section is to prove Proposition 4.2.3, which constructs a family of "good" background metrics $\hat{\omega}_t \in [\alpha_t]$ whose Ricci potentials decay faster than quadratically. Indeed, it is easy to construct $\omega_t \in [\alpha_t]$ satisfying only the first two conditions of Proposition 4.2.3. However, the proof of the a priori estimates of Proposition 4.2.4 depends crucially on the additional decay of the Ricci potentials. This idea is used in [78] (see also [95, Prop. 4.2.6]).

From now on we fix an open Calabi-Yau manifold (X, J, Ω) asymptotic to some Calabi-Yau cone $(C, J_C, \Omega_C, \omega_C, g_C)$ at rate $\nu > 0$. In the following proposition, we summarize a construction of asymptotically conical Kähler (semipositive) forms in almost compactly support classes, which is based on [78].

Proposition 4.3.1. *Suppose $[\alpha] \in H^{1,1}_\nu(X)$ contains a (semi-)positive form α, then there exist a (semi-)positive form $\omega \in [\alpha]$ which agrees with α in a compact set K and satisfies the asymptotics $|\nabla^k(\omega - \omega_C)| = O(r^{-\nu-k})$ for $r \gg 1$.*

Proof. This follows from construction in [78, Theorem 2.4].

□

Proposition 4.3.2. *Suppose that $(X, J, \Omega, \omega_t, g_t)_{t\in[0,1]}$ are a smooth family of data which is asymptotic to the cone $(C, J_C, \Omega_C, \omega_C, g_C)$ at the rate $-\nu \in (-2, 0)$. Suppose that for $t \in (0, 1]$, ω_t are asymptotically conical Kähler metrics and ω_0 is asymptotically conical and semi-positive $(1, 1)$ form. Let f_t, $t \in [0, 1]$ be the Ricci potentials of ω_t, defined by $e^{f_t} = \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\omega_t^n}$, and suppose there is a compact set $K \subset X$ so that on $X\backslash K$, f_t satisfy the following asymptotics:*

1. $|f_t| \leqslant Cr^{-\beta}$

2. $|\nabla^k f_t|_{g_C} \leqslant Cr^{-\beta-k}$

where C is independent of t and $\nu \leqslant \beta < 2n-2$ and $-\beta+2$ is not an exceptional weight.

Then there exist $\varepsilon > 0$ and a family of functions u_t for $t \in [0,\varepsilon]$ such that the following are satisfied

1. There exist a compact subset $K \subset X$ such that supp$(u_t) \subset X \setminus K$

2. $\omega_t + i\partial\bar{\partial}u_t > 0$ *on supp*(u_t)

3. $|\nabla^k u_t|_{g_C} \leqslant Cr^{-\beta+2-k}$

4. $|\nabla^k \frac{\partial u_t}{\partial t}|_{g_C} \leqslant Cr^{-\beta+2-k}$

5. Away from a compact set K, we have

$$(\omega_t + i\partial\bar{\partial}u_t)^n = e^{f_t - f_t'}\omega_t^n = e^{-f_t'} i^{n^2}\Omega \wedge \bar{\Omega}$$

where $|\nabla^k f_t'| \leqslant Cr^{-2\beta-k}$ *outside a compact set K.*

where the constant C is independent of t. In particular, this means if we set $\omega_t' = \omega_t + i\partial\bar{\partial}u_t$, then ω_t' converges to ω_C at the same rate as ω_t, but the Ricci potentials f_t' of ω_t' decays a rate of -2β.

Proof. We can essentially follow the same procedure as in [78, Lemma 2.12]. First we want to solve the equation

$$\Delta_{\omega_t}\hat{u}_t = 2f_t$$

for $t \geqslant 0$, away from a compact set while controlling of the growth of the solutions.

We now fix a standard cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ with

$$\chi(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ 1 & \text{for } x \geqslant 2 \end{cases}$$

and satisfy $0 \leqslant \chi \leqslant 1$, $|\chi'| \leqslant 2$, $|\chi''| \leqslant 5$. Then we define $\zeta_R : X \to \mathbb{R}$ by setting $\zeta_R(x) = \chi(\frac{r(x)}{R})$, and let $\hat{g}$ be any metric on X. Then set

$$\bar{g}_t = (1 - \zeta_R)\hat{g} + g_t$$

Since ω_0 is semi-positive and asymptotically conical we can choose R sufficiently large so that $\bar{g}_0$ is an asymptotically conical Riemannian metric. Then for all $t \in [0, 1]$, g_t defines a background metric and for $t \in (0, 1]$, this metric is equal to the ω_t away from a compact set.

If $-\beta + 2$ is not an exceptional weight, then $\Delta_{\bar{g}_t} : C^\infty_{-\beta+2} \to C^\infty_{-\beta}$ is surjective by Theorem 4.2.1, so we can always solve the equation

$$\Delta_{\bar{g}_t} \hat{u}_t = 2\zeta_R f_t$$

for $\hat{u}_t \in C^\infty_{-\beta+2}$. In fact, by the Implicit Function Theorem [96, Proposition 4.2.19], we can find a family of smoothly varying solutions for $t \in [0, \varepsilon)$, and such that following bounds hold uniformly for small t.

1. $|\nabla^k \hat{u}_t| \leqslant Cr^{-\beta+2-k}$
2. $|\nabla^k \frac{\partial \hat{u}_t}{\partial t}| \leqslant Cr^{-\beta+2-k}$

If we set $u_t = \zeta_S \hat{u}_t$ then u_t is supported on $\mathrm{supp}(\zeta_S)$, and then we have

$$\begin{aligned}
|i\partial\bar{\partial}u_t| &\leqslant |\zeta_S||\partial\bar{\partial}\hat{u}_t| + |\hat{u}_t||\partial\bar{\partial}\zeta_S| + 2|\partial\zeta_S||\partial u_t| \\
&\leqslant C\zeta_S r^{-\beta} + Cr^{-\beta+2}|\partial\bar{\partial}\zeta_S| + Cr^{-\beta+1}|\nabla\zeta_S| \\
&\leqslant Cr^{-\beta}(\zeta_S + r|\nabla\zeta_S| + r^2|i\partial\bar{\partial}\zeta_S|) \\
&\leqslant Cr^{-\beta}(\zeta_S + S|\nabla\zeta_S| + S^2|i\partial\bar{\partial}\zeta_S|)
\end{aligned}$$

but since $\zeta_S(x) = \chi(\frac{r(x)}{S})$, we see that

$$|\nabla\zeta_S| = S^{-1}|\chi'\nabla r| \leqslant CS^{-1}$$

and

$$|i\partial\bar{\partial}\zeta_S|\leqslant S^{-2}|\chi''||\nabla r|^2+|\chi'|S^{-1}|i\partial\bar{\partial}r|\leqslant CS^{-2}$$

where we used that $r|i\partial\bar{\partial}r|\leqslant C$. So we have

$$|i\partial\bar{\partial}u_t|\leqslant Cr^{-\beta}(\zeta_S+C)$$

and $i\partial\bar{\partial}u_t$ is supported on the support of ζ_S. Hence for S sufficiently large, we can ensure that $\omega_t+i\partial\bar{\partial}u_t>0$ on the supp(u_t).

Away from the compact set K, we have

$$\begin{aligned}\frac{(\omega_t+i\partial\bar{\partial}u_t)^n}{\omega_t^n}&=1+f_t+O(|i\partial\bar{\partial}u_t|^2)\\&=1+f_t+O(r^{-2\beta})\end{aligned}$$

so setting $f_t'=f_t-\log\frac{(\omega_t+i\partial\bar{\partial}u_t)^n}{\omega_t^n}$, we have

$$(\omega_t+i\partial\bar{\partial}u_t)^n=e^{f_t-f_t'}\omega_t^n$$

and $f_t'=f_t-\log(1+f_t+O(r^{-2\beta}))$ has the desired asymptotics. □

Remark 4.3.1. If $-\beta+2$ is an exceptional weight, we can apply the proposition with $\beta+\varepsilon$ in place of β for ε arbitrarily small (since the exceptional weights are discrete). We can then repeatedly apply Proposition 4.3.2 to improve the decay of Ricci potential for a family of metrics until we obtain the decays we need.

The two previous propositions combined proves Proposition 4.2.3.

Proof of Proposition 4.2.3. By Proposition 3.1, we can find a semi-positive form $\omega_0\in[\alpha_0]$ satisfying the asymptotics $|\nabla^k(\omega_C-\omega_0)|=O(r^{-\nu-k})$ and a metric $\omega_1\in[\alpha_1]$ satisfying the same asymptotics, then if we write ω_t by linearly interpolating between ω_0 and ω_1, then clearly ω_t are positive for $t>0$ and satisfy the desired asymptotics, and the Ricci potentials f_t satisfy

$|\nabla^k f_t| \leqslant C(1+r)^{-\nu-k}$. If $\nu > 2$, then we can take $\gamma = \nu$ and we are done, otherwise, we can apply Proposition 4.3.2 repeatedly to improve the asymptotics of the Ricci potentials until they decay faster than quadratically. □

4.3.1 Kähler currents and Null loci in the asymptotically conical case

Before proceeding we would like to briefly discuss Assumption 1. Recall that if (X, ω) is compact Kähler and $[\alpha] \in \overline{K}$ is a nef class with $\int_X \alpha^n > 0$, then, by results of Demailly-Păun [97] there is a function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ such that

$$\alpha + \sqrt{-1}\partial\bar{\partial}\psi \geqslant \varepsilon\omega$$

for some $\varepsilon > 0$, ψ is smooth on the complement of an analytic subset Z, and $\{\psi = -\infty\} = Z$. Furthermore, by results of Collins and Tosatti [69] ψ can be chosen so that the analytic subvariety Z is given by

$$\mathrm{Null}(\alpha) := \bigcup_{\int_V \alpha^{\dim V} = 0} V$$

where the union is taken over irreducible analyitic subvarieties $V \subset X$. We expect that a similar result holds in the asymptotically conical setting. We make the following conjecture

Conjecture 4.3.1. *Suppose $[\alpha] \in H^{1,1}_\nu(X, \mathbb{R})$ is a limit of ν-almost compactly support Kähler classes. Then there is a function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\alpha + \sqrt{-1}\partial\bar{\partial}\psi \geqslant \varepsilon\omega$ for some asymptotically conical Kähler form ω. Define*

$$Null(\alpha) := \bigcup_{\int_V \alpha^{\dim V} = 0} V \tag{4.5}$$

where the union is taken over all compact, irreducible, analytic subvarieties $V \subset X$. Then $Null(\alpha)$ is an analytic subvariety, and ψ can be chosen so that ψ is smooth on $X \backslash Null(\alpha)$ and

$$\{\psi = -\infty\} = Null(\alpha).$$

At a purely moral level, the reason that non-compact analytic subvarieties should not enter into the definition of $\mathrm{Null}(\alpha)$ in the asymptotically conical setting is that, at least when $[\alpha]$ admits a semi-positive representative, Proposition 4.3.1 yields the existence of a form $\hat{\alpha} \in [\alpha]$ which is asymptotically conical. Thus, if V is a non-compact subvariety, then $\int_V \hat{\alpha}^{\dim V} = +\infty$. Of course, this is purely moral reasoning, since the integral $\int_V \hat{\alpha}^{\dim V}$ is not independent of the representative of $[\alpha]$.

Lemma 4.3.1. *Conjecture 4.3.1 holds when, $[\alpha]$ is semi-positive and the cone at infinity is quasi-regular.*

Recall that the cone $(C, J_C, \Omega_C, \omega_C, g_C)$ is quasi-regular if the holomorphic vector field $r_C \frac{\partial}{\partial r_C} - \sqrt{-1} J_C \left(r_C \frac{\partial}{\partial r_C} \right)$ integrates to define a $\mathbb{C}^*$ action.

Proof. By a result of Conlon-Hein [80], building on work of Li [81], if $(X, J, \Omega, \omega, g)$ is asymptotically conical Calabi-Yau with quasi-regular Calabi-Yau cone at infinity, then there is a complex, projective orbifold M without codimension 1 singularities, and a orbidivisor D with positive normal orbibundle such that $M = X \cup D$, and $-K_M = q[D]$ for some $q \geqslant 1$. Furthermore, every Kähler form on X is cohomologous to the restriction of a Kähler form on M, and the restriction map $H^{1,1}(M) \to H^2(X)$ is surjective. Let $[\omega_t] = (1-t)[\alpha_0] + t[\omega_0] \in H^{1,1}(X)$ be a family of ν-almost compactly supported Kähler classes for $t \in (0, 1]$ such that $[\alpha_0]$ is semi-positive. In fact, according to [78, Proposition 2.5] all Kähler classes on X are 2-almost compactly supported, so the assumption of almost compact support can be dropped. Let $[\hat{\omega}], [\hat{\alpha}_0] \in H^{1,1}(M)$ be such that $[\hat{\omega}]$ is Kähler, and $[\hat{\omega}]|_X = [\omega_0], [\hat{\alpha}_0]|_X = [\alpha_0]$. Since α_0 is semi-positive, and D has positive normal bundle, the argument in the proof of [80, Theorem A] shows that we can find a constant $C > 0$ so that $[\hat{\alpha}_0] + C[D]$ is semi-positive, and positive in a neighborhood of D. Furthermore, since $D|_D$ is positive, after possibly increasing C we can assume that

$$\int_M ([\hat{\alpha}_0] + C[D])^n > 0$$

Let $\pi : \overline{M} \to M$ be a resolution of singularities, obtained by blowing up smooth centers. Since X

is smooth, and M has only codimension 2 singularities, we can assume that $\pi|_X$ is an isomorphism, and that π is an isomorphism at the generic point of D. Let E denote the exceptional divisor of π, and let $\bar{D} = \pi^{-1}(D)$ be the total transform of D. Now we have

$$\pi^*[\hat{\alpha}_0] + C[\bar{D}]$$

is nef, and big by Demailly-Păun [97]. By the results of [97] and the Collins and Tosatti [69] there is a Kähler current in $\pi^*[\hat{\alpha}_0] + C[\bar{D}]$ which is smooth on the complement of $\mathrm{Null}(\pi^*[\hat{\alpha}_0] + C[\bar{D}])$. Let $Y \subset \bar{M}$ be an irreducible analytic subvariety of dimension $p > 0$. If $Y \cap \pi^{-1}(D) = \emptyset$, then

$$\int_Y (\pi^*[\hat{\alpha}_0] + C[\bar{D}])^p = \int_{\pi(Y)} \alpha_0^p$$

and so $Y \subset \mathrm{Null}(\pi^*[\hat{\alpha}_0] + C[\bar{D}])$ if and only if $\pi(Y) \subset \mathrm{Null}([\alpha_0])$. Now suppose that $Y \cap \pi^{-1}(D) \cap \pi^{-1}(X) \neq \emptyset$. Let $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u$ be the smooth semi-positive representative of $[\hat{\alpha}_0] + C[D]$ which is positive in a neighborhood of D. Then, since π is an isomorphism at the generic point of Y we have

$$\int_Y (\pi^*[\hat{\alpha}_0] + C[\bar{D}])^p = \int_{Y\setminus(E\cap Y)} [\pi^*(\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u)]^p + \int_{\pi(Y)} (\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u)^p > 0,$$

where the last inequality follows from the fact that $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u \geqslant 0$ and there is a neighborhood of $\pi(Y) \cap D$ where $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u > 0$. Thus we have

$$\mathrm{Null}(\pi^*[\hat{\alpha}_0] + C[\bar{D}]) \cap (\pi^{-1}(D))^c = \pi^{-1}\left(\mathrm{Null}([\alpha_0])\right).$$

Since $\pi : \bar{M}\setminus\pi^{-1}(D) \to X$ is an isomorphism, the result follows.

□

4.4 A priori estimates

In this section, we prove Proposition 4.2.4. Let us first recall the general setup of the proposition. Let (X, J, ω, Ω) be an asymptotically conical Calabi-Yau manifold which is asymptotic to the Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate $\nu > 0$, and $[\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_\nu$ for $t \in [0,1]$ is a family of ν-almost compactly supported classes such that $[\alpha_t]$ is Kähler for $t > 0$. Suppose $[\alpha_0]$ satisfies Assumption 1. Then let $\hat{\omega}_t \in [\alpha_t]$ for $t \in (0,1]$ be a family of asymptotically conical Kähler metrics satisfying the conclusion of Proposition 4.2.3. Then by Theorem 4.2.2, we can solve the equation

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = i^{n^2}\Omega \wedge \bar{\Omega} (= e^{f_t}\hat{\omega}_t^n)$$

for $\varphi_t \in C^\infty_{-\gamma}(X)$, the our goal in this section is to prove a priori estimates on the potentials φ_t that are uniform in t as $t \to 0$.

4.4.1 Uniform estimates

In this section, we prove a uniform bound for φ_t that is independent of t. In the compact case, such an estimate can be proved using pluripotential theory following the seminal work of Kolodziej [98], see [66]. Pluripotential methods allow one to obtain an estimate with a sharper dependence on the data of the right hand side. However, such methods are hard to adapt to the non-compact setting and no proper analogue of such estimates are known. It would be of interest to try to find extensions of the pluripotential estimates to the non-compact setting, as it would give a sharper estimates which would apply more generally to singular Calabi-Yau manifolds not admitting crepant resolutions.

Instead, we will use an idea based on the original argument of Yau [6] using the Moser iteration. However, following an idea of Tosatti [65] we perform the Moser iteration using the Calabi-Yau metrics $\omega_{\varphi_t} := \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ as background metrics. The advantage of this trick is that since the metrics ω_{φ_t} are Ricci flat and asymptotically conical, they have a uniform Sobolev inequality by results of Croke [99] and Yau [100].

Proposition 4.4.1. *The metrics ω_{φ_t} satisfy a uniform Sobolev inequality of the form*

$$\left(\int_X |u|^{\frac{2n}{n-1}} i^{n^2}\Omega\wedge\bar{\Omega}\right)^{\frac{n-1}{n}} \leqslant C\int_X |du|^2_{\omega_{\varphi_t}} i^{n^2}\Omega\wedge\bar{\Omega} \tag{4.6}$$

Proof. It suffices to prove the result for compact supported smooth functions. Results of Croke [99] and Yau [100] show that for a compactly supported function u, with $\operatorname{supp}(u)\subset\Omega$ for an arbitrary relatively compact set $\Omega\subset X$, (4.6) holds for a constant C, depending on an upper bound for the diameter of Ω, a lower bound for the volume of Ω, and a lower bound for the Ricci curvature. We only need to exploit the scale invariance of these quantities for asymptotically conical Calabi-Yau metrics. Fix a point $x_0\in X$. Since ω_{φ_t} are asymptotically conical, for R sufficiently large we have

$$\mathrm{Vol}_{\omega_{\varphi_t}}(B_R(x_0)) \sim R^{2n}\mathrm{Vol}_{\omega_C}(L)$$

where L is the link of the cone, identified with $\{r_C = 1\}\subset C$, and the volume is computed using the conical Calabi-Yau metric ω_C. Therefore, if $\omega_R = R^{-2}\omega_{\varphi_t}$, then with respect to the rescaled metric the diameter is 1, and the volume is $\mathrm{Vol}_{\omega_C}(L)$. Since (4.6) is scale invariant, the result follows. □

Proposition 4.4.2. *Given solutions φ_t to* (4.3)*, with $|\nabla^k\varphi| = O(r^{-\gamma-k})$ we have the following uniform estimate for the potential*

$$|\varphi_t| \leqslant C\|\varphi_t\|_{L^p(i^{n^2}\Omega\wedge\bar{\Omega})}$$

for any $p > \frac{2n-2}{\gamma} \geqslant 1$ and C depending on n, p, and a uniform bound on $\|e^{-f_t}-1\|_{L^q}$ for $q\in[p,\infty]$.

Proof. If we set $T_t = \sum_{k=0}^{n-1}\omega^k_{\varphi_t}\wedge\hat{\omega}_t^{n-1-k}$, then we can rewrite the equation as

$$-i\partial\bar{\partial}\varphi_t\wedge T_t = (e^{-f_t}-1)i^{n^2}\Omega\wedge\bar{\Omega}$$

multiplying both sides by $|\varphi_t|^{p-2}\varphi_t$ and integrating, we get

$$-\int_M |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t = \int_M |\varphi_t|^{p-2}\varphi_t(e^{-f_t}-1)i^{n^2}\Omega\wedge\bar{\Omega}$$

we will integrate by parts on the first term

$$\begin{aligned}
-\int_M |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t &= \lim_{R\to\infty}\left(-\int_{B_R} |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t\right)\\
&= \lim_{R\to\infty}\left((p-1)\int_{B_r} |\varphi_t|^{p-2} i\partial\varphi_t\wedge\bar{\partial}\varphi_t \wedge T_t - \int_{\partial B_R} |\varphi_t|^{p-2}\varphi_t i\bar{\partial}\varphi_t \wedge T_t\right)\\
&= \frac{4(p-1)}{p^2}\int_M i\partial|\varphi_t|^{\frac{p}{2}}\wedge\bar{\partial}|\varphi_t|^{\frac{p}{2}}\wedge T_t - \underbrace{\lim_{R\to\infty}\int_{\partial B_R}|\varphi_t|^{p-2}\varphi_t i\bar{\partial}\varphi_t\wedge T_t}_{=0 \text{ for } p>\frac{2n-2}{\gamma}}
\end{aligned}$$

Combined with the Sobolev inequality, we have

$$\left(\int_M |\varphi_t|^{p\frac{n}{n-1}} i^{n^2}\Omega\wedge\bar{\Omega}\right)^{\frac{n-1}{n}} \leqslant C\frac{np^2}{4(p-1)}\int_M |\varphi_t|^{p-1}|e^{-f_t}-1| i^{n^2}\Omega\wedge\bar{\Omega}$$

for any $p > \frac{2n-2}{\gamma}$. By Hölder's inequality, we have (below $\frac{1}{q}+\frac{1}{q'}=1$)

$$\|\varphi_t\|^p_{L^{p\frac{n}{n-1}}} \leqslant C\frac{np^2}{4(p-1)}\||\varphi_t|^{p-1}\|_{L^q}\|e^{-f_t}-1\|_{L^{q'}} = C\frac{np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^{q(p-1)}}\|e^{-f_t}-1\|_{L^{q'}} \qquad (4.7)$$

picking q such that $q = \frac{p}{p-1} > 1$, we get

$$\begin{aligned}
\|\varphi_t\|^p_{L^{p\frac{n}{n-1}}} &\leqslant \frac{C_S np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^p}\|e^{-f_t}-1\|_{L^p}\\
&\leqslant \frac{CC_S np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^p}
\end{aligned}$$

a standard Moser iteration argument gives the result.

□

Proposition 4.4.3. *For any $p > \frac{2n}{\gamma}$, we have a uniform L^p estimate of the form*

$$\|\varphi_t\|_{L^p} \leqslant C$$

for C depending on n, p and $\|e^{-f_t}-1\|_{L^{\frac{np}{n+p}}}$.

Proof. In equation (4.7), if we pick $q>1$, such that $q(k-1)=k\frac{n}{n-1}$, we get

$$\|\varphi_t\|_{L^{k\frac{n}{n-1}}}\leqslant C\frac{np^2}{4(p-1)}\|e^{-f_t}-1\|_{L^{\frac{nk}{n+k-1}}}$$

taking $p=\frac{nk}{n-1}$ gives us our result. □

Corollary 4.4.1. *The potentials φ_t are bounded in L^p uniformly in t for any $p\in(\frac{2n}{\gamma},\infty]$,*

$$\|\varphi_t\|_{L^p}\leqslant C_p$$

In particular, th potentials φ_t are uniformly bounded in C^0.

Proof. This follows by combining Proposition 4.4.2 and Proposition 4.4.3. Note that since $|f_t|\leqslant Cr^{-\gamma-2}$ outside a fixed compact set, we have an estimate $\|e^{-f_t}-1\|_{L^{\frac{np}{n+p}}}\leqslant C$ for a constant C independent of p,t for any $p>\frac{2n-2}{\gamma}$. □

4.4.2 Convergence of the metric away from the degeneracy locus

In this section, we prove an estimate for $\partial\bar{\partial}\varphi_t$ away from V, the subvariety coming from Assumption 1. Recall that by Assumption 1, there exist $\psi\in PSH(X,\alpha_0)$ which is smooth outside of V and goes to $-\infty$ near V, the idea is to use this function as a barrier function in the C^2 estimate, and this is first used by Tsuji in in [101] to study Kähler-Ricci flow. We remark that this is the only part of the Theorem that uses the current in Assumption 1.

Before we prove the estimate, we first construct a slightly more better behaved barrier function $\psi_\varepsilon\in PSH(X,\hat{\omega}_0)$ which is compactly supported. Recall that from the construction of $\hat{\omega}_0$, $\hat{\omega}_0$ is equal to α_0 on a large compact set. (which from the construction can be as large as one want)

Lemma 4.4.1. *There exist $\psi_\varepsilon\in PSH(X,\hat{\omega}_0)$ which is compactly supported and satisfy $\hat{\omega}_0+i\partial\bar{\partial}\psi_\varepsilon\geqslant\varepsilon\omega$, and is smooth outside V and goes to $-\infty$ near V.*

Proof. Recall by [78, Lemma 2.15], we know that $r^{2\kappa}$ for $\kappa \in (0,1)$ is strictly plurisubharmonic for r sufficiently large, and satisfies

$$|\nabla r^{2\kappa}| = O(r^{2\kappa-1}) \qquad |i\partial\bar{\partial} r^{2\kappa}| = O(r^{2\kappa-2})$$

Pick $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ smooth satisfy $\Psi', \Psi'' \geqslant 0$ and

$$\Psi(x) = \begin{cases} T+2 & \text{for } x < T+1 \\ x & \text{for } x > T+3 \end{cases}$$

then as in [78, Lemma 2.15], for $T \gg 1$, $\Psi(r^{2\kappa})$ is plurisubharmonic and equal to $r^{2\kappa}$ for r sufficiently large.

We set

$$\psi_\varepsilon = (1-\zeta_S)\psi + C(1-\zeta_R)\Psi(r^{2\kappa})$$

where S, C, R are chosen as follows. First we pick $S \gg 1$ large enough such that $\hat{\omega}_0 = \alpha_0$ on $\{r \leqslant S\}$ and $i\partial\bar{\partial}\Psi(r^{2\kappa}) > 0$ on $\{S \leqslant r \leqslant 2S\}$, which implies that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon = \alpha_0 + i\partial\bar{\partial}\psi \geqslant \varepsilon_0\omega$ on $\{r \leqslant S\}$. Then pick $C \gg 1$ large enough so that $Ci\partial\bar{\partial}\Psi(r^{2\kappa}) > i\partial\bar{\partial}((1-\zeta_S)\psi)$ on $\{S \leqslant r \leqslant 2S\}$. Finally, we pick $R \gg S$ such that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > 0$ on $\{R \leqslant r \leqslant 2R\}$, which is possible since for R large, we have

$$|i\partial\bar{\partial}(1-\zeta_R)\Psi(r^{2\kappa})| \leqslant |\nabla^2\zeta_R||r^{2\kappa}| + |\nabla^2\Psi(r^{2\kappa})||1-\zeta_R| + |\nabla\zeta_R||\nabla r^{2\kappa}| \leqslant CR^{2(\kappa-1)} \ll 1$$

Then $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > 0$ and $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > \varepsilon_0\omega$ on the compact set K containing V, hence there exist an $\varepsilon > 0$ such that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > \varepsilon\omega$ holds. □

Now we prove the main estimate of this section.

Proposition 4.4.4. *There are uniform constants $B, C > 0$, independent of t such that the following* *estimate holds:*

$$|\partial\bar{\partial}\varphi_t| \leqslant Ce^{-B\psi_\varepsilon}.$$

Proof. By the well-known computation of Aubin and Yau, we have

$$\Delta_{\varphi_t} \log \operatorname{tr}_\omega \omega_{\varphi_t} \geqslant -A \operatorname{tr}_{\varphi_t} \omega$$

where A is a lower bound for the bisectional curvatures of ω. Then if we pick $N \gg B$ sufficiently large, we have

$$\begin{aligned}\Delta_{\varphi_t}\left(\log \operatorname{tr}_\omega \omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t\right) &\geqslant (B\varepsilon - A) \operatorname{tr}_{\varphi_t} \omega - B \operatorname{tr}_{\varphi_t} \hat{\omega}_0 + N \operatorname{tr}_{\varphi_t} \hat{\omega}_t - Nn \\ &\geqslant C\left(\frac{\omega^n}{ni^{n^2}\Omega \wedge \bar{\Omega}} \operatorname{tr}_\omega \omega_{\varphi_t}\right)^{\frac{1}{n-1}} - Bn\end{aligned}$$

since ψ_ε goes to $-\infty$ near V and the function $\log \operatorname{tr}_\omega \omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t$ goes to 0 at infinity, either $\log \operatorname{tr}_\omega \omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t$ is always non-positive, in which case we are done, or maximum is achieved in the interior, and applying the maximum principle gives

$$\operatorname{tr}_\omega \omega_{\varphi_t} \leqslant Ce^{B(\sup \psi_\varepsilon - \psi_\varepsilon)}$$

from which the estimate follows. ◻

Remark 4.4.1. This argument is the only place where we used the Kähler current in Assumption 1. In the situation where $[\alpha_0] = \pi^* c_1(L)$ where $\pi : X \to X_0$ is a crepant resolution of a singular Calabi-Yau variety with compactly supported singularities and $L \to X_0$ is an ample line bundle on X_0, the above C^2 estimate can be replaced by the argument in Lemma 4.5.1, and the convergence holds away from $\pi^{-1}(X_0^{sing})$. In that case we do not need the Kähler current in Assumption 1 to prove Theorem 4.1.1.

The higher order estimates follow from the standard methods of Yau [6, 18, 17].

Proposition 4.4.5 (Higher order estimates)**.** *We have a uniform estimate*

$$\|\varphi_t\|_{C^{k,\alpha}_{loc}(K)} \leqslant C(K,k,\alpha)$$

for any $K \subset\subset X \setminus V$ and C independent of t.

Proof. This follows from the local estimates in [17]. □

Corollary 4.4.2. *The metrics ω_{φ_t} converge after passing to a subsequence in $C^\infty_{loc}(X \setminus V)$ to a possibly incomplete metric ω_{φ_0} on $X \setminus V$, which is uniformly equivalent to ω_C at infinity.*

So far, we've shown the first two parts of Proposition 4.2.4, in the next section we prove decay estimates for φ_t.

4.4.3 Decay estimates

In this section, we prove uniform decay estimates for φ_t. We use the method of Moser iteration with a weight, similar to the technique used in [74, Chap 8]. However, as in Section 4.4.1, we use the Ricci flat metrics ω_{φ_t}, exploiting the uniform control of the Sobolev constants.

Recall that $r : X \to \mathbb{R}_{>0}$ is a radius function such that $|\nabla r| + r|i\partial\bar{\partial} r| \leqslant C$, and it's not hard to see that we can also assume that $r = const$ on a compact set K containing the singular set V.

Definition 4.4.1. We define the following weighted L^p norms,

$$\|u\|_{L^p_\delta(i^{n^2}\Omega\wedge\bar{\Omega})} = \left(\int_X |ur^\delta|^p r^{-2n} i^{n^2} \Omega \wedge \bar{\Omega}\right)^{\frac{1}{p}}$$

Remark 4.4.2. Notice if we let $p \to \infty$, then the L^p_δ norms converge to the L^∞_δ norm given by $\|u\|_{L^\infty_\delta} = \sup_X |ur^\delta|$

Proposition 4.4.6. *For any $\delta < \gamma$, we have a uniform bound of the form*

$$\|\varphi_t\|_{L^p_\delta(\omega^n_{\varphi_t})} \leqslant C$$

for any $p \in (0, \frac{2n}{\delta}]$, and constant depending on p, δ.

Proof. If $p = \frac{2n}{\delta}$, then this is simply the $L^{\frac{2n}{\delta}}$ norm, which is bounded if $\delta < \gamma$ by Proposition 4.4.3. If $p < \frac{2n}{\delta}$, then

$$\int_X |\varphi r^\delta|^p r^{-2n}\omega_{\varphi_t}^n \leqslant \left(\int_X |\varphi_t|^{pq}\right)^{\frac{1}{q}} \left(\int_X r^{\frac{q}{q-1}(\delta p - 2n)}\omega_{\varphi_t}^n\right)^{\frac{q-1}{q}}$$

the first term is bounded if $q > \frac{2n}{\gamma p}$ by Proposition 4.4.3, and the second term is finite if $q < \frac{2n}{\delta p}$, so we just need to pick $q \in (\frac{2n}{\gamma p}, \frac{2n}{\delta p})$ with $q > 1$, which is possible since $p < \frac{2n}{\delta}$. □

Proposition 4.4.7. *For any $\delta < \gamma$, $p > 1$, we have*

$$\|\varphi_t r^\delta\|^p_{L^{p\frac{n}{n-1}}(r^{-2n}\omega_{\varphi_t}^n)} \leqslant \frac{Cp^2}{p-1}\left(\|\varphi_t r^\delta\|^{p-1}_{L^{p-1}(r^{-2n}\omega_{\varphi_t}^n)} + \|\varphi_t r^\delta\|^p_{L^p(r^{-2n}\omega_{\varphi_t}^n)}\right)$$

for C depending on the Sobolev constant of ω_{φ_t}, δ and the dimension n.

Proof. We use the same method as in [74, Proposition 8.6.7], but using the Calabi-Yau metrics ω_{φ_t} as the background metrics. The reason is because the metrics ω_{φ_t} are Ricci-flat and hence have a uniform Sobolev inequality. First we set

$$T_t = \sum_{k=0}^{n-1} \omega_{\varphi_t}^k \wedge \hat{\omega}_t^{n-1-k}.$$

If $q - p\gamma < -2n + 2$, then Stoke's theorem gives the following two identities

$$\begin{aligned}
0 &= \int_X i\partial\left(r^q|\varphi_t|^{p-2}\varphi_t\bar{\partial}\varphi_t \wedge T_t\right) \\
&= (p-1)\int_X r^q|\varphi_t|^{p-2} i\partial\varphi_t \wedge \bar{\partial}\varphi_t \wedge T_t + q\int_X r^{q-1}|\varphi_t|^{p-2}\varphi_t i\partial r \wedge \bar{\partial}\varphi_t \wedge T_t + \int_X r^q|\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t
\end{aligned}$$

and

$$\begin{aligned}
0 &= -\int_X i\bar{\partial}\left(r^{q-1}|\varphi_t|^p i\partial r \wedge T_t\right) \\
&= p\int_X r^{q-1}|\varphi_t|^{p-2}\varphi_t i\partial r \wedge \bar{\partial}\varphi_t \wedge T_t + (q-1)\int_X r^{q-2}|\varphi_t|^p i\partial r \wedge \bar{\partial} r \wedge T_t + \int_X r^{q-1}|\varphi_t|^p i\partial\bar{\partial} r \wedge T_t
\end{aligned}$$

using these identities, we can obtain through integration by parts

$$\begin{aligned}\int_X |\nabla(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}})|^2_{\omega_{\varphi_t}} \omega^n_{\varphi_t} &= n\int_X i\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \bar{\partial}(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \omega^{n-1}_{\varphi_t} \\ &\leqslant n\int_X i\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \bar{\partial}(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge T_t \\ &= -\frac{np^2}{4(p-1)}\int_X \varphi_t|\varphi_t|^{p-2} r^q i\partial\bar{\partial}\varphi_t \wedge T_t \\ &\quad + \frac{mq}{4(p-1)}\int_X |\varphi_t|^p r^{q-2}[(p+q-2)i\partial r \wedge \bar{\partial} r - (p-2) r i\partial\bar{\partial} r] \wedge T_t \\ &= -\frac{np^2}{4(p-1)}\int_X \varphi_t|\varphi_t|^{p-2} r^q (e^{f_t}-1)\omega^n_{\varphi_t} \\ &\quad + \frac{mq}{4(p-1)}\int_X |\varphi_t|^p r^{q-2}[(p+q-2)i\partial r \wedge \bar{\partial} r - (p-2) r i\partial\bar{\partial} r] \wedge T_t\end{aligned}$$

where in the last equality, we used the equation $i\partial\bar{\partial}\varphi_t \wedge T_t = (e^{f_t}-1)\omega^n_{\varphi_t}$. Now we claim there also exist a uniform constant C independent of t and r such that

$$\left|\frac{[(p+q-2)i\partial r \wedge \bar{\partial} r - (p-2) r i\partial\bar{\partial} r] \wedge T_t}{i^{n^2}\Omega \wedge \bar{\Omega}}\right| \leqslant C(p+|q|)$$

recall that we chose r so that $r = const$ on a compact set K containing V, so the left hand side of the expression is 0 on K. By Corollary 4.4.2 we know that $|T_t| \leqslant C$ on $X \setminus K$ and because r is a radius function, we also $|\nabla r| + r|\partial\bar{\partial} r| \leqslant C$, putting them together, we get that the expression also holds on $X \setminus K$, hence this whole expression is bounded by the right hand side.

This then combined with the Sobolev inequality, we conclude that

$$\left(\int_X |\varphi_t|^{p\frac{n}{n-1}} r^{q\frac{n}{n-1}} \omega^n_{\varphi_t}\right)^{\frac{n-1}{n}} \leqslant \frac{Cnp^2}{4(p-1)}\int_X |e^{-f_t}-1||\varphi_t|^{p-1} r^q \omega^n_{\varphi_t} + \frac{Cq(p+q)}{4(p-1)}\int_X |\varphi_t|^p r^{q-2}\omega^n_{\varphi_t}$$

for any $\delta < \gamma$ we can set $q = 2(1-n) + p\delta$ and use the fact that $|e^{f_t}-1| \leqslant Cr^{-\gamma-2}$ to obtain,

$$\begin{aligned}\left(\int_X |\varphi_t r^\delta|^{p\frac{n}{n-1}} r^{-2n}\omega^n_{\varphi_t}\right)^{\frac{n-1}{n}} &\leqslant C\frac{p^2}{4(p-1)}\left(\int_X |\varphi_t|^{p-1} r^{p\delta-\gamma} r^{-2n}\omega^n_{\varphi_t} + \int_X |\varphi_t r^\delta|^p r^{-2n}\omega^n_{\varphi_t}\right) \\ &= C\frac{p^2}{4(p-1)}\left(\int_X |\varphi_t r^\delta|^{p-1} r^{\delta-\gamma} r^{-2n}\omega^n_{\varphi_t} + \int_X |\varphi_t r^\delta|^p r^{-2n}\omega^n_{\varphi_t}\right)\end{aligned}$$

and since $\delta < \gamma$, which means for any $p > 1$, we have

$$\|\varphi_t r^\delta\|^p_{L^{p\frac{n}{n-1}}(r^{-2n}\omega^n_{\varphi_t})} \leqslant \frac{Cp^2}{p-1}\left(\|\varphi_t r^\delta\|^{p-1}_{L^{p-1}(r^{-2n}\omega^n_{\varphi_t})} + \|\varphi_t r^\delta\|^p_{L^p(r^{-2n}\omega^n_{\varphi_t})}\right)$$

□

Corollary 4.4.3. *For any $\delta < \gamma$, we have a uniform bound of the form*

$$|\varphi_t| \leqslant Cr^{-\delta}$$

for C depending on δ.

Proof. By Proposition 4.4.6, we have a weighed L^p bound for any $p \leqslant \frac{2n}{\delta}$, combined with the previous proposition, we can use the standard Moser iteration argument starting from $p = \frac{2n}{\delta} \geqslant \frac{n}{n-1} > 1$. □

Proposition 4.4.8. *For any $\delta < \gamma$, the derivative of the solutions φ_t satisfy uniform decay estimates on $X \setminus K$,*

$$|\nabla^k \varphi_t| \leqslant Cr^{-\delta-k}$$

where $C = C(n, \delta, k)$ which doesn't depend on t.

Proof. This follows from the methods of [74, Theorem 8.6.11] verbatim. The point to note here is that the metrics ω_{φ_t} are uniformly equivalent to ω_C on the region $X \setminus K$, with bounded derivatives as well, hence the Schauder constants are uniformly controlled on far away balls. □

Proposition 4.4.9. *If $\gamma \in (0, 2n-2)$, then in fact we have*

$$|\nabla^k \varphi_t| \leqslant Cr^{-\gamma-k}$$

on $X \setminus K$, and $C = C(n, k)$ independent of t.

Proof. This follows from the same argument as in [74, Chap 8.7, Theorem A2]. □

We can now prove Proposition 4.2.4, thereby completing the proof of Theorem 4.1.1.

Proof of Proposition 4.2.4. Combine Corollary 4.4.1, Proposition 4.4.5, Proposition 4.4.3 and Proposition 4.4.9.

□

We now prove the local diameter bound, which will play an important role throughout the remainder of the chapter.

Lemma 4.4.2. *In the setting of Theorem 4.1.1, let $K \subset X$ be a compact subset containing V. Then the diameter of K with respect to the Calabi-Yau metrics $\omega_{t,CY}$ is uniformly bounded from above as $t \to 0$.*

$$Diam_{\omega_{\varphi_t}} K \leqslant C$$

Proof. It suffices to show that the sets $K_R = \{r(x) \leqslant R\}$ have bounded diameters for R sufficiently large. Recall that the metrics ω_{φ_t} are uniformly asymptotic to ω_{cone} for r large and t close to 0 by Proposition 4.4.8. Fix any two points $x, y \in K_R$, and joint them by a length minimizing geodesic $\gamma : [0, L] \to X$. We claim that γ must lie inside K_{R^2} for R sufficiently large. Note for R large, on the region $\{r(x) \geqslant R\}$ the metric ω_{φ_t} is C^∞ close to a cone metric uniformly in t, and hence for R sufficiently large, the boundary of K_R has diameter bounded by $2\pi R$. However, the distance between the boundary of K_R and K_{R^2} on the order of R^2, so it's clear that any minimizing geodesic between two points in K_R cannot leave K_{R^2}. Now consider $x_i = \gamma(2i + 1)$ and disjoint balls $B_1(x_i)$. Note that these balls have a fixed lower bound on the volume, since by Bishop-Gromov volume comparison and the asymptotically conical geometry we have

$$\mathrm{Vol}(B_1(x_i)) \geqslant \lim_{S\to\infty} \frac{\mathrm{Vol}(B_S(x_i))}{S^{2n}} = \mathrm{Vol}_{g_C}(L) =: c > 0$$

where L is the link of the cone at infinity, identified with $\{r_C = 1\}$ and g_C is the conical Calabi-Yau metric. Thus, we have

$$\sum_i \mathrm{Vol}(B_1(x_i)) \geqslant c\frac{\lfloor L \rfloor}{2}$$

where c is the non-collapsing constant. On the other hand, these balls must all lie in K_{2R}, and since the volume form of the Calabi-Yau metrics are fixed, we must have that

$$c\frac{\lfloor L \rfloor}{2} \leqslant \int_{K_{R^2}} i^{n^2}\Omega \wedge \bar{\Omega}$$

which gives us a bound for L, which is $d_{\omega_{\varphi_t}}(x, y)$. □

4.4.4 Uniqueness

In this section, we discuss the uniqueness of the Calabi-Yau currents constructed in the previous sections.

Theorem 4.4.1. *The current that we constructed ω_{φ_0} above is unique in the sense that if ω is another positive current with locally bounded potentials in the same cohomology class as ω_{φ_0} which is smooth on $X \setminus V$, asymptotically conical at infinity with any rate $\delta > 0$ and satisfies the complex Monge-Ampère equation*

$$\omega^n = \omega^n_{\varphi_0} = i^{n^2}\Omega \wedge \bar{\Omega}$$

in the Bedford-Taylor sense, then $\omega = \omega_{\varphi_0}$.

The proof is modelled after the idea introduced in [78], which relies on the following crucial Lemma proved in [78].

Lemma 4.4.3. *[78, Corollary 3.9] Suppose (X, ω) is an asymptotically conical Kähler manifold with Ric $\geqslant 0$, then for any $\varepsilon > 0$, any harmonic function $u \in C^\infty_{2-\varepsilon}(X)$ is pluriharmonic.*

The idea is to write $\omega = \omega_{\varphi_0} + i\partial\bar{\partial}\psi$ and use this lemma to improve the asymptotics of the potential function ψ by subtracting off pluriharmonic functions from it, until we are left in the case where the potential function is decaying in which case uniqueness follows from a standard integration by parts argument.

Proposition 4.4.10. *Suppose $\varphi \in PSH(X, \omega_{\varphi_0}) \cap L^\infty(X) \cap C^\infty_{-\varepsilon}(X \setminus V)$ is a function such that the current $\omega_{\varphi_0} + i\partial\bar{\partial}\varphi$ satisfies*

$$(\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n = \omega^n_{\varphi_0} = i^{n^2}\Omega \wedge \bar{\Omega}$$

in the Bedford Taylor sense, then $\varphi = 0$.

Proof.

$$\begin{aligned}
0 &= -\int_{B_R} |\varphi|^{p-2}\varphi((\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n - \omega^n_{\varphi_0}) = -\int_{B_R} |\varphi|^{p-2}\varphi i\partial\bar{\partial}\varphi \wedge \left(\sum_{k=0}^{n-1} \omega^k_{\varphi_0} \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) \\
&= \frac{4(p-1)}{p^2} \int_{B_R} i\partial(|\varphi|^{\frac{p}{2}}) \wedge \bar{\partial}(|\varphi|^{\frac{p}{2}}) \wedge \left(\sum_{k=0}^{n-1} \omega^k_{\varphi_0} \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) \\
&\quad - \int_{\partial B_R} |\varphi|^{p-2}\varphi i\bar{\partial}\varphi \wedge \left(\sum_{k=0}^{n-1} \omega^k_{\varphi_0} \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right)
\end{aligned}$$

picking $p > \frac{2n-2}{\gamma}$ and letting $R \to \infty$, we get

$$\int_X i\partial(|\varphi|^{\frac{p}{2}}) \wedge \bar{\partial}(|\varphi|^{\frac{p}{2}}) \wedge \left(\sum_{k=0}^{n-1} \omega^k_{\varphi_0} \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) = 0$$

which shows that $\varphi = 0$. □

Lemma 4.4.4. *Suppose (X, J, g) is an asymptotically conical Calabi-Yau manifold with rate $\nu > 0$, and $\eta = \eta_{i\bar{j}}$ is a asymptotically conical hermitian metric with rate $\nu > 0$ and let $u \in C^\infty_{2-\beta}$ such that $\eta^{i\bar{j}}u_{i\bar{j}} \in C^\infty_{-\kappa}$, then there exist $\tilde{u} \in C^\infty_{2-\beta-\nu}$ such that $i\partial\bar{\partial}\tilde{u} = i\partial\bar{\partial}u$.*

Proof. We have

$$g^{i\bar{j}}u_{i\bar{j}} = (g^{i\bar{j}} - \eta^{i\bar{j}})u_{i\bar{j}} + \eta^{i\bar{j}}u_{i\bar{j}} \in C^\infty_{-\min(\kappa,\beta+\nu)}$$

hence we can solve the equation $g^{i\bar{j}}\tilde{u}_{i\bar{j}} = g^{i\bar{j}}u_{i\bar{j}}$ with $\tilde{u} \in C^\infty_{2-\min(\kappa,\beta+\nu)}$ and by Lemma 4.4.3 we have $i\partial\bar{\partial}\tilde{u} = i\partial\bar{\partial}u$. □

Proof of Theorem 4.4.1. By the $\partial\bar{\partial}$-Lemma (Proposition 4.2.1), we can write $\omega = \omega_{\varphi_0} + i\partial\bar{\partial}\psi$, for $\psi \in PSH(X, \omega_{\varphi_0}) \cap L^\infty_{loc}(X) \cap C^\infty_{loc}(X \setminus V)$, then choose a cutoff χ such that χ has compact support and $\chi = 1$ on a compact set K containing V, then since $i\partial\bar{\partial}\psi = \omega - \omega_{\varphi_0} \in C^\infty_{-\varepsilon}(X \setminus V)$ for some $\varepsilon > 0$, hence by Proposition 4.2.2, we can solve $i\partial\bar{\partial}f = i\partial\bar{\partial}[(1-\chi)\psi]$ for $f \in C^\infty_\gamma$, $\gamma = 2 - \varepsilon$. Setting $\varphi = \chi\psi + f$, we have that $\varphi \in L^\infty_{loc}(X) \cap C^\infty_\gamma(X \setminus V)$ and

$$(\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n = \omega^n_{\varphi_0} = i^{n^2}\Omega \wedge \bar{\Omega}$$

If $\gamma < 0$, then we are done by Proposition 4.4.10. If $\gamma > 0$, then we proceed by the following: note that the equation above can be rewritten as

$$\Delta_{\omega_{\varphi_0}}\varphi = -(i\partial\bar{\partial}\varphi)^2 \wedge \left(\sum_{k=2}^n \binom{n}{k} \frac{(i\partial\bar{\partial}\varphi)^{k-2} \wedge \omega^{n-k}_{\varphi_0}}{\omega^n_{\varphi_0}}\right) \in C^\infty_{2\gamma-4}(X \setminus V)$$

if χ is the cutoff function as before, then we have $\Delta_{\omega_{\varphi_0}}[(1-\chi)\varphi] \in C^\infty_{2\gamma-4}(X)$ if we let $\eta = \chi\omega_{\varphi_t} + (1-\chi)\omega_{\varphi_0}$, then η is an asymptotically conical hermitian metric which is equal to ω_{φ_0} outside of a compact set, hence $\eta^{i\bar{j}}[(1-\chi)\varphi]_{i\bar{j}} \in C^\infty_{2\gamma-4}(X)$, hence we can apply Lemma 4.4.4 with $\kappa = 2(2-\gamma)$ and $\beta = 2-\gamma$, so we can solve $i\partial\bar{\partial}v = i\partial\bar{\partial}[(1-\chi)\varphi]$ with $v \in C^\infty_{\gamma-\min(2-\gamma,\nu)}$ now we can set $\tilde{\varphi} = v + \chi\varphi \in C^\infty_{\gamma-\min(2-\gamma,\nu)}(X \setminus C)$ and we can keep repeating this process with $\tilde{\varphi}$ in place of φ and $\gamma - \min(2-\gamma, \nu)$ in place of γ until are in the case where $\gamma < 0$, then we are done by Proposition 4.4.10.

□

4.5 Metric geometry of the singular Calabi-Yau

The goal of this section is to prove Theorem 4.1.2. Let us first begin with some definitions and the general setup.

Definition 4.5.1. We say that a complex analytic space X_0 is a *singular Calabi-Yau variety with compactly supported, crepant singularities*, if

- X_0 is normal singularities, Gorenstein and log-terminal,
- there is a compact set K so that $X_0 \backslash K$ is smooth,

- there exists a resolution $\pi : X \longrightarrow X_0$ such that X also has trivial canonical bundle and $\pi^*\Omega$ extends as a non-vanishing global holomorphic $(n,0)$-form on X. (By abuse of notation, we will also denote this holomorphic $(n,0)$-form by Ω)

Let X_0 be a singular Calabi-Yau variety with compactly supported, crepant singularities. Suppose that the resolution (X, J, Ω) is Kähler and it has a Kähler metric ω such that (X, J, ω, Ω) is asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ at rate ν.

Definition 4.5.2. A line bundle L on X_0 is ample if for some $k > 0$, there exist sections $s_0, \ldots, s_N \in H^0(X_0, L^k)$ such that $[s_0, \ldots, s_N]$ gives an embedding of X_0 into a finite dimensional projective space $\mathbb{C}P^N$, and denote this embedding map by ι, then we have $\frac{1}{k}[\iota^\star \omega_{FS}] = c_1(L)$.

Let us now fix L an ample line bundle on X_0. If set $[\alpha_0] = \pi^* c_1(L)$, then suppose (X, J, ω, Ω) and $[\alpha_0]$ satisfy the hypothesis of Theorem 4.1.2. Then from the previous sections, we have on X, a sequence of Calabi-Yau metrics $\omega_{\varphi_t} = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ with $[\omega_{\varphi_t}] = (1-t)[\alpha_0] + t[\alpha_1]$, which satisfy the equation

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}_t^n (= i^{n^2}\Omega \wedge \bar{\Omega})$$

and $f_t = \log \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\hat{\omega}_0^n} \in C^\infty_{-\gamma-2}(X)$, and $\varphi_t \in C^\infty_{-\gamma}(X)$.

If we fix a point $p \in \pi^{-1}(X_0^{reg})$, then by Gromov compactness, after passing to a subsequence, the pointed spaces $(X, \omega_{\varphi_{t_i}}, p)$ for $t_i \to 0$ pointed Gromov-Haussdorff converge to a limiting pointed metric space $(X_\infty, d_\infty, p_\infty)$ as $i \to \infty$. By the definition of pointed Gromov-Haussdorff convergence, the convergence can be interpreted in the following sense: If we set $Z = (X_\infty, d_\infty, p_\infty) \sqcup \bigsqcup_{t_i}(X, \omega_{\varphi_{t_i}}, p)$, then there exist a metric d_Z on Z such that

1. $d_Z|_{X_i} = d_{g_{\varphi_{t_i}}}$
2. $d_Z(\underbrace{p}_{\in X_i}, p_\infty) \to 0$
3. $B_{g_{\varphi_{t_i}}}(p, r) \subset X_i \to B_{g_\infty}(p_\infty, r) \subset X_\infty$ in the Haussdorff sense with respect to d_Z.

The asymptotically conical property of ω_{φ_t} implies that the tangent cone at ∞ is independent of t, and by Bishop-Gromov, this a uniform lower bound on volume, and we have $\mathrm{Vol}_{\omega_{\varphi_t}} B(p,r) \geqslant cr^{2n}$ where c is the volume ratio of the asymptotic cone C. Hence the regularity theory of Cheeger, Colding and also Tian [102, 103, 104, 105, 106] applies, and the limiting space admits the following structure

1. All tangent cones of X are metric cones.
2. $X = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ consists of all the points where all tangent cones are isometric to $\mathbb{R}^{2n}$.
3. $\mathcal{R}$ is an open dense set in X_∞ with a smooth metric g_∞ and complex structure J_∞ which makes it Ricci-flat Kähler manifold and $(X_\infty, d_\infty) = \overline{(\mathcal{R}, d_{g_\infty})}$. Moreover, the convergence of $(X, J, \omega_{\varphi_t}, p) \to (X_\infty, J_\infty, g_\infty, p)$ is smooth on $\mathcal{R}$ in the sense that for every $K \subset\subset \mathcal{R}$, there exist smooth maps $\eta_i : K \to X$ such that $(\eta_i^\star g_{t_i}, \eta_i^\star J)$ converges to (g_∞, J_∞) smoothly on K. (In fact, we can arrange η_i such that $d_Z(\eta_i(z), z) \to 0$ uniformly in K)
4. $\mathcal{S}$ is a closed subset of X_∞ with real Hausdorff codimension greater or equal to 4.

4.5.1 Properties of the Gromov-Hausdorff limit

In this section, we prove several preliminary propositions about the relationship between X_∞ and the Kähler current constructed from Theorem 4.1.1. In particular, we show the following:

1. ω_{φ_0} is in fact well-defined and smooth on $\pi^{-1}(X_0^{reg})$
2. There exist a locally isometric embedding of $\iota_\infty : (\pi^{-1}(X_0^{reg}), \omega_{\varphi_0}) \to (\mathcal{R}, g_\infty)$.
3. X_∞ is isometric to the metric completion $\overline{(\pi^{-1}(X_0^{reg}), \omega_{\varphi_0})}$
4. ι_∞ is a bijective local isometry between X_0^{reg} and $\mathcal{R}$.

One of the key ingredients is the local diameter bound Lemma 4.4.2, which we apply with $V = \pi^{-1}(X_0^{sing})$.

Proposition 4.5.1. *The family of metrics ω_{φ_t} has a uniform lower bound*

$$\omega_{\varphi_t} \geqslant \frac{1}{C}\hat{\omega}_0 \tag{4.8}$$

Proof. By the standard Schwartz lemma calculation, we have

$$\Delta_{\omega_{\varphi_t}} \log \operatorname{tr}_{\omega_{\varphi_t}} \pi^\star \omega_{FS} \geqslant -4 \operatorname{tr}_{\omega_{\varphi_t}} \pi^\star \omega_{FS}$$

and for any other Kähler metric $\hat{\omega}$, one also has

$$\Delta_{\omega_{\varphi_t}} \log \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega} \geqslant -C \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}$$

with C depending only on the upper bound for the holomorphic bisectional curvature of $\hat{\omega}$. Recall from the construction of $\hat{\omega}_0$ in Propositions 4.3.1 and 4.3.2 that $\hat{\omega}_0$ can be taken to be equal to $\frac{1}{k}\pi^\star\omega_{FS}$ on a compact set K containing $\pi^{-1}(X_0^{sing})$, and is a genuine non-degenerate, asymptotically conical Kähler metric outside of K, so we can apply the first inequality inside K and the second outside K to get a uniform estimate

$$\Delta_{\omega_{\varphi_t}} \log \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0 \geqslant -C \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0. \tag{4.9}$$

Since $\omega_{\omega_{\varphi_t}} = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$, taking trace gives

$$n = \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_t + \Delta_{\varphi_t}\varphi_t,$$

and we also know that for t reasonably small $\hat{\omega}_t \geqslant c\hat{\omega}_0$ holds for some small constant c uniformly in t as $t \longrightarrow 0$, which means we have

$$n \geqslant c \operatorname{tr}_{\varphi_t} \hat{\omega}_0 + \Delta_{\varphi_t}\varphi_t.$$

Combining this with (4.9), we have

$$\Delta_{\omega_{\varphi_t}}\left(\log \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - A\varphi_t\right) \geqslant \left(\frac{Ac}{2} - C\right) \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - An$$

since $\log \operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - A\varphi_t$ converges to the constant $\log n$ at spacial infinity, if the maximum is attained at infinity, then we automatically have a uniform bound that we wanted. So we can assume the maximum is achieved in the interior, and applying the maximum principle to the equation above, and we obtain

$$\operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0 \leqslant Ce^{A(\varphi_t - (\varphi_t)_{min})}$$

which gives a uniform upper bound for $\operatorname{tr}_{\omega_{\varphi_t}} \hat{\omega}_0$. □

Corollary 4.5.1. *On $X \setminus \pi^{-1}(X_0^{sing})$, we have*

$$C^{-1}\hat{\omega}_0 \leqslant \omega_{\varphi_t} \leqslant Ce^{f_0}\hat{\omega}_0$$

where $e^{f_0} = \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\hat{\omega}_0^n}$ is bounded uniformly away from $\pi^{-1}(X_0^{sing})$. In particular, this implies that ω_{φ_0} is smooth on $\pi^{-1}(X_0^{reg})$, and on X_0 it is a Kähler current since it dominates $\hat{\omega}_0$.

Proof. The lower bound on ω_{φ_t} is the content of the previous lemma, and from that and the fact that $\omega_{\varphi_t}^n = i^{n^2}\Omega \wedge \bar{\Omega} = e^{f_0}\hat{\omega}_0^n$, the corollary follows immediately. □

Corollary 4.5.2. *The maps $\pi_i : (X, \omega_{\varphi_{t_i}}, p) \to (X_0, \hat{\omega}_0, p)$ are has bounded derivative, hence it is uniformly lipschitz and we can pass to a continuous surjective map from the Gromov-Haussdorff limit $\pi_\infty : (X_\infty, d_{X_\infty}, p_\infty) \to X_0$. Furthermore, for any $q \in X_0^{reg}$, the preimage $\pi_\infty^{-1}(q)$ consists of a single point.*

Proof. The fact that the maps have bounded derivative follows from the estimate (4.8), and from this it follows from an Arzela-Ascoli type argument that after passing to a subsequence, the projection maps π_i limit to a continuous surjective map $\pi_\infty : X_\infty \to X_0$. The map π_∞ can be characterized in the following way: if we fix $h_i : (X, \omega_{\varphi_{t_i}}) \to X_\infty$ an ε_i-isometry for $\varepsilon_i \to 0$, then for any sequence of points $q_i \in X$ with $\pi(q_i) \to q \in X_0$, and $h_i(q_i) \to q_\infty \in X_\infty$, we have $\pi_\infty(q) = q_\infty$.

To see that the preimage of $\pi^{-1}(q)$ for $q \in X_0^{reg}$ consists of a single point, suppose for contradiction that it consisted of two points $q_1, q_2 \in X_\infty$ with $d_{X_\infty}(q_1, q_2) = d > 0$ and $\pi_\infty(q_1) = \pi_\infty(q_2) = q \in X_0^{reg}$, then from the construction of π_∞, there exist a sequences of points $q_1^i, q_2^i \in X$ such that $\pi_i(q_1^i) \to q$ and $\pi_i(q_2^i) \to q$ and $h_i(q_1^i) = q_1$ and $h_i(q_2^i) = q_2$. Then from the fact that $\pi_i(q_1^i) \to q$ and $\pi_i(q_2^i) \to q$ and $q \in X_0^{reg}$, we know that $q_1^i \to \pi^{-1}(q)$ and $q_2^i \to \pi^{-1}(q)$ in X since π is a resolution of singularities of X_0, and $g_{t_i} \to g_\infty$ smoothly in a neighborhood of q, it follows that $d_{g_{t_i}}(q_1^i, q_2^i) \to 0$ as $i \to \infty$. But we also have

$$d_{X_\infty}(h_i(q_1^i), h_i(q_2^i)) - \varepsilon_i \leqslant d_{g_{t_i}}(q_1^i, q_2^i)$$

since h_i is an ε_i-isometry. This is a contradiction, because $d_{X_\infty}(h_i(q_1^i), h_i(q_2^i)) - \varepsilon_i \to d > 0$ by our assumption. □

Proposition 4.5.2. *There is an embedding $i_\infty : (X_0^{reg}, \omega_{\varphi_0}, p) \hookrightarrow (\mathcal{R}, g_\infty, p)$, which is a locally isometric embedding, and $\pi_\infty \circ \iota_\infty = id$.*

Proof. We can simply take $\iota_\infty = \pi_\infty^{-1}|_{X_0^{reg}}$, which is well-defined by the previous proposition. It's clear that the image of ι_∞ is contained in the regular set $\mathcal{R} \subset X_\infty$ and that it is continuous, so it suffices to show that this map is a local isometry. To see this, we note that if $q \in X_0^{reg}$, then there exist an $\varepsilon > 0$ such that $B_{g_{t_i}}(q, \varepsilon) \subset X_0^{reg}$ for all $i \gg 1$. It follows from the diameter estimate (c.f. Lemma 4.4.2) that the points $h_i(\pi^{-1}(q))$ are uniformly bounded in X_∞, hence after passing to a subsequence, it converge to some point $q_\infty \in X_\infty$, it's clear that $q_\infty = \iota_\infty(q)$ since $\pi_i(q) = q$. Since the points $\pi^{-1}(q) \in X_i$ have a uniform harmonic radius lower bound, hence $(B_{g_{t_i}}(\pi^{-1}(q), \varepsilon), g_{\varphi_{t_i}}) \xrightarrow{C^\infty} (B_{g_\infty}(q_\infty, \varepsilon), g_\infty)$ and by the smooth convergence of $g_{\varphi_t} \to g_{\varphi_0}$, we also have $(B_{g_{t_i}}(\pi^{-1}(q), \varepsilon), g_{\varphi_{t_i}}) \xrightarrow{C^\infty} (B_{g_{\varphi_0}}(q, \varepsilon), \omega_{\varphi_0})$, it is then clear from the construction of π_∞ that it maps $(B_{g_\infty}(q_\infty, \varepsilon), g_\infty)$ isometrically onto $(B_{g_{\varphi_0}}(q, \varepsilon), \omega_{\varphi_0})$. □

The following Proposition follows from the same arguments as in [107]. We include a proof here for the convenience of the reader.

Proposition 4.5.3. *The subset $E = \mathcal{R} \setminus \iota_\infty(X_0^{reg}) \subset \mathcal{R}$ is an analytic subset, hence of real codimension bigger than or equal to 2, and moreover $\overline{(X_0^{reg}, g_\infty)} = X_\infty$.*

Proof. It suffices to show that the holomorphic maps $\pi : (X, \omega_{\varphi_t}, p) \to X_0 \subset (\mathbb{C}P^N, \omega_{FS}, p)$ limits to a holomorphic map $\pi_\infty|_{\mathcal{R}}$: $(\mathcal{R}, J_\infty, g_\infty) \to X_0 \subset \mathbb{C}P^N$. Assuming for now that this is the case, then $\mathcal{R} \setminus \iota_\infty(X_0^{reg}) = \pi_\infty|_{\mathcal{R}}^{-1}(X_0^{sing})$. Since $X_0^{sing} \subset X_0$ is an analytic set, if $\pi_\infty|_{\mathcal{R}}$ is holomorphic, then $\pi_\infty|_{\mathcal{R}}^{-1}(X_0^{sing}) = E \subset \mathcal{R}$ is an analytic subset, and since analytic subsets have real codimension 2, it follows that $X_\infty \setminus X_0^{reg} \subset X_\infty$ has Haussdorff codimension at least 2, and by [104, Theorem 3.7], we have $\overline{(X_0^{reg}, g_\infty)} = X_\infty$.

Now we show that $\pi_\infty|_{\mathcal{R}}$ is holomorphic. Consider the holomorphic maps $\pi : (X, \omega_{\varphi_t}, p) \to X_0 \subset (\mathbb{C}P^N, \omega_{FS}, p)$, since $(X, \omega_{\varphi_t}, p)$ Gromov-Haussdorff converge to X_∞, by Cheeger-Colding theory [103], for any $K \subset\subset \mathcal{R}$ containing p, there exist maps $\iota_{t_i} : K \to (X, \omega_{\varphi_{t_i}})$ such that $\iota_{t_i}^\star g_{t_i} \to g_\infty$ and $\iota_{t_i}^\star J \to J_\infty$ in the smooth topology, and we also get a sequence of holomorphic maps $\pi_i = \pi \circ \iota_{t_i} : (K, \iota_{t_i}^\star g_{t_i}, \iota_{t_i}^\star J) \to X_0 \subset \mathbb{C}P^N$. Furthermore, if we regard these maps as harmonic maps, then we have

$$|d\pi_i|^2_{\omega_{\varphi_t}, \omega_{FS}} = \mathrm{tr}_{\omega_{\varphi_t}} \pi_i^\star \omega_{FS} \leqslant C$$

hence by the regularity theory of harmonic maps ([108]), we have uniform C^∞ estimates on the maps $\|\pi_i^l\|_{C^{k,\alpha}(K)} \leqslant C_K$, for some constant C_K independent of i, which allows us to extract a limit of the maps $\pi_i : K \to \mathbb{C}P^N$ to a map $\pi_\infty : K \to \mathbb{C}P^N$ and since the convergence of the maps are smooth, and the convergence of the metrics $\iota_{t_i}^\star g_{t_i} \to g_\infty$ and the complex structures $\iota_{t_i}^\star J \to J_\infty$ are all smooth, it follows that the holomorphicity of the maps π_i passes to the limit, and hence the map π_∞ is holomorphic. □

Proposition 4.5.4. *In fact we have $\mathcal{R} = \iota_\infty(X_0^{reg})$.*

Proof. The proof is the same as in [107, Lemma 2.2]. □

4.5.2 Identification of X_0 with the geometry of singular Calabi-Yau

In this section, we identify the geometry of the singular Calabi-Yau current $X_\infty = \overline{(X_0^{reg}, g_\infty)}$ with the variety X_0 itself. This result is the analogue of the result in [67], where the similar thing was shown in the compact case, our proof follows the approach in [67], adapted to the non-compact case. The idea is based on ideas developed in [109] together with a new gradient estimate for the potential φ_t with respect to the Calabi-Yau metrics ω_{φ_t}.

A gradient bound for φ_0

The goal of this section is to prove the following estimate

Proposition 4.5.5. *The following bound hold*

$$\sup_{\pi^{-1}(X_0^{reg})} |\nabla_{\omega_{\varphi_0}} \varphi_0| \leqslant C$$

Proposition 4.5.6. *If we set let $v_t = \varphi_t - t\dot{\varphi}_t$, then we have a uniform estimate*

$$\sup_X |v_t| \leqslant C$$

Proof. Recall from the construction of $\hat{\omega}_t$ (Proposition 4.3.2) that

$$\begin{aligned}\hat{\omega}_t &= \omega_t + i\partial\bar{\partial}u_t \\ &= (1-t)\omega_0 + t\omega_1 + i\partial\bar{\partial}u_t\end{aligned}$$

where $\omega_0 = \pi^\star \omega_{X_0}$ and ω_{X_0} is a Kähler metric on X_0. So we have

$$\begin{aligned}\Delta_{\varphi_t}\varphi_t &= n - \operatorname{tr}_{\varphi_t} \hat{\omega}_t \\ &= n - (1-t)\operatorname{tr}_{\varphi_t} \omega_0 - t\operatorname{tr}_{\varphi_t} \omega_1 - \Delta_{\varphi_t} u_t.\end{aligned}$$

Recall that by the construction of $\hat{\omega}_t$, Proposition 4.3.1, we have

$$\log \frac{(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n}{\hat{\omega}_t^n} = f_t \in C^\infty_{-\gamma-2}.$$

for some $0 < \gamma < 2n - 2$. Differentiating the equation, we have

$$\Delta_{\varphi_t}\dot{\varphi}_t = \dot{f}_t - \mathrm{tr}_{\varphi_t}\frac{\partial}{\partial t}\hat{\omega}_t + \mathrm{tr}_{\hat{\omega}_t}\frac{\partial}{\partial t}\hat{\omega}_t \in C^\infty_{-\gamma-2}(X) \tag{4.10}$$

so we have $\dot{\varphi} \in C^\infty_{-\gamma}(X)$ for $t > 0$.

If we differentiate the equation $(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = i^{n^2}\Omega \wedge \bar{\Omega}$ with respect to t, we obtain another expression for $\Delta_{\varphi_t}\dot{\varphi}_t$

$$\Delta_{\varphi_t}\dot{\varphi}_t = -\Delta_{\varphi_t}\dot{u}_t + \mathrm{tr}_{\varphi_t}(\omega_0 - \omega_1) \tag{4.11}$$

The equations (4.10) and (4.11) imply that v_t satisfy the two equations

$$\Delta_{\varphi_t}v_t = n - \mathrm{tr}_{\varphi_t}\,\omega_0 - \Delta_{\varphi_t}(u_t - t\dot{u}_t) \tag{4.12}$$

and

$$\Delta_{\varphi_t}v_t = n - \mathrm{tr}_{\varphi_t}\,\hat{\omega}_t - t(\dot{f}_t - \mathrm{tr}_{\varphi_t}\frac{\partial}{\partial t}\hat{\omega}_t + \mathrm{tr}_{\hat{\omega}_t}\frac{\partial}{\partial t}\hat{\omega}_t) \tag{4.13}$$

From the first equation and Proposition 4.5.1, we see that $|\Delta_{\varphi_t}\dot{\varphi}_t| \leqslant C$ uniformly in t. From the second equation we see that $|\Delta_{\varphi_t}v_t| \leqslant Cr^{-\gamma-2}$ away from a compact set K, so we have a uniform bound $|\Delta_{\varphi_t}v_t|_{L^p} \leqslant C$ for $p > \frac{2n}{\gamma+2}$. Since $v_t \in C^\infty_{-\gamma}$, we can do integrate by parts to get

$$\begin{aligned} -\int_X |v_t|^{p-2}v_t i\partial\bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1} &= \lim_{R\to\infty}(p-1)\int_{B_R}|v_t|^{p-2}i\partial v_t \wedge \bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1} \\ &\quad - \lim_{R\to\infty}\left(\int_{\partial B_R}|v_t|^{p-2}v_t i\bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1}\right) \\ &= \frac{4(p-1)}{p^2}\int_X i\partial|v_t|^{\frac{p}{2}} \wedge \bar{\partial}|v_t|^{\frac{p}{2}} \wedge \omega_{\varphi_t}^{n-1} \end{aligned}$$

the boundary term goes away when $p > \frac{2n-2}{\gamma}$ since $|\nabla^k v_t| = O(r^{-\gamma-k})$. Hence we get

$$\int_X |\partial |v_t|^{\frac{p}{2}}|^2 \omega_{\varphi_t}^n = -\frac{np^2}{4(p-1)} \int_X |v_t|^{p-2} v_t (\Delta_{\varphi_t} v_t) \omega_{\varphi_t}^n$$

combined with the Sobolev inequality, one gets

$$\left(\int_X |v_t|^{p\frac{n}{n-1}} i^{n^2} \Omega \wedge \bar{\Omega} \right)^{\frac{n-1}{n}} \leqslant C \frac{np^2}{p-1} \int_X |v_t|^{p-1} |\Delta_{\varphi_t} v_t| i^{n^2} \Omega \wedge \bar{\Omega}$$

applying Holder, we get

$$\|v_t\|_{L^{p\frac{n}{n-1}}} \leqslant C \frac{np^2}{p-1} \|\Delta_{\varphi_t} v_t\|_{L^{\frac{np}{n+p+1}}}$$

hence for $p > \frac{2n}{\gamma}$, we have

$$\|v_t\|_{L^p} \leqslant C_p \tag{4.14}$$

where C_p depends on $\|\Delta_{\varphi_t} v_t\|_{L^{\frac{np}{n+p}}}$. and also for $p > \frac{2n-2}{\gamma}$

$$\|v_t\|^p_{p\frac{n}{n-1}} \leqslant C \frac{np^2}{p-1} \|v_t\|^{p-1}_{L^p} \|\Delta_{\varphi_t} v_t\|_{L^p}$$

we can then apply Moser iteration to this to get the estimate

$$\|v_t\|_{L^\infty} \leqslant B_p \|v_t\|_{L^p} \leqslant B_p C_p$$

where C_p is the constant from (4.14) and B_p depends only on the L^p norm of $\|\Delta_{\varphi_t} v_t\|_{L^p}$. □

Corollary 4.5.3. *For any compact set $K \subset\subset \pi^{-1}(X_0^{reg})$, we have an estimate*

$$|v_t|_{C^{k,\alpha}(K)} \leqslant C(K, k, \alpha)$$

uniformly in t as $t \to 0$.

Proof. This follows from the equation (4.12) and the fact that ω_{φ_t} and the right hand side of the equation is uniformly bounded in $C^{\infty}_{loc}(\pi^{-1}(X_0^{reg}))$. $\square$

Proposition 4.5.7. *We also have the following local uniform gradient estimate for v_t.*

$$\sup_K |\nabla_t v_t| \leqslant C_K$$

for any $K \subset\subset X$.

Proof. By the Bochner formula, we have

$$\begin{aligned}\Delta_{\varphi_t}|\nabla v_t|^2_{g_{\varphi_t}} &= |\nabla\nabla v_t|^2_{g_{\varphi_t}} + |\partial\bar{\partial} v_t|^2_{g_{\varphi_t}} - 2\mathrm{Re}(\nabla v_t \cdot \nabla \operatorname{tr}_{\varphi_t} \omega_0) - 2\mathrm{Re}(\nabla v_t \cdot \nabla\Delta_{\varphi_t}(u_t - t\dot{u}_t)) \\ &\geqslant -2|\nabla v_t|^2_{g_{\varphi_t}} - |\nabla \operatorname{tr}_{\omega_{\varphi_t}} \omega_0|^2_{g_{\varphi_t}} - |\nabla\Delta_{\varphi_t}(u_t - t\dot{u}_t)|^2_{g_{\varphi_t}}\end{aligned}$$

we also have from (4.9),

$$\Delta_{\omega_{\varphi_t}} \operatorname{tr}_{\omega_{\varphi_t}} \omega_0 \geqslant -C + c_0|\nabla \operatorname{tr}_{\omega_{\varphi_t}} \omega_0|^2$$

If we set $H_t = |\nabla v_t|^2_{g_{\varphi_t}} + A \operatorname{tr}_{\omega_{\varphi_t}} \omega_0$, then $H_t \geqslant 0$ and satisfies

$$\Delta_{\varphi_t} H_t \geqslant -H_t - C$$

We can apply Moser iteration to this, since ω_{φ_t} has uniform Ricci bounds and volume lower bound, this then gives the estimate

$$\|H_t\|_{L^\infty(B_{g_\infty,R}(p))} \leqslant C\|H_t\|_{L^2(B_{g_\infty,2R}(p))} \tag{4.15}$$

for R sufficiently large. Note that for R sufficiently large ω_{φ_t} converge uniformly in C^∞ to g_∞ on the region $B_{g_\infty,2R}(p) \setminus B_{g_\infty,R}(p)$, hence we can also choose cutoff functions with uniformly controlled gradients and standard Moser iteration gives the inequality. Now it suffices to show that

$\|H_t\|_{L^2(B_{g_\infty,2R})}$ is bounded.

$$\begin{aligned}\int_{B_{2R}} |H_t|^2 &\leqslant \|H_t\|_{L^\infty(B_{2R})}\int_{B_{2R}} |H_t| \\ &\leqslant C\|H_t\|_{L^2(B_{4R})}\|H_t\|_{L^1(B_{2R})} \\ &\leqslant C(\|H_t\|_{L^2(B_{2R})}+\|H_t\|_{L^2(B_{4R}\setminus B_{2R})})\|H_t\|_{L^1(B_{2R})}\end{aligned}$$

if R is sufficiently large, then $B_{4R} \setminus B_{2R}$ doesn't contain any of $\pi^{-1}(X_0^{sing})$, hence $\|H_t\|_{L^2(B_{4R}\setminus B_{2R})}$ is uniformly bounded in t on $B_{4R} \setminus B_{2R}$ by the Corollary above. So we have

$$\|H_t\|^2_{L^2(B_{2R})} \leqslant C(\|H_t\|_{L^2(B_{2R})}+C)\|H_t\|_{L^1(B_{2R})}$$

hence either $\|H_t\|_{L^2(B_{2R})}$ is bounded by 1 and we are done, or we get the bound

$$\|H_t\|_{L^2(B_{2R})} \leqslant C\|H_t\|_{L^1(B_{2R})} \tag{4.16}$$

so it suffices to prove an L^1 bound for H_t on compact sets.

Choose cutoff function η such that $\eta = 1$ on B_{2R} for all t, then

$$\begin{aligned}\int_{B_{2R}} H_t &\leqslant \int_X \eta^2 H_t \\ &\leqslant \int_X \eta^2 |\nabla v_t|^2 \omega^n_{\varphi_t} + C \\ &\leqslant -\int_X \eta^2 v_t(\Delta_{\varphi_t} v_t) + 2\int_X \eta|\nabla\eta||\tilde{v}_t||\nabla\tilde{v}_t|+C\end{aligned}$$

and so we have

$$\int_{B_{2R}} H_t \leqslant C\int_X (\eta^2 + |\nabla\eta|^2)v_t^2 \leqslant C$$

which gives us the L^1 bound, combined with (4.16) and (4.15), we get

$$\|H_t\|_{L^\infty(B_R)} \leqslant C$$

as desired. ◻

Proposition 4.5.8. *For any $x \in \iota_\infty(X_0^{reg})$, we have a bound*

$$|\dot{\varphi}_t(x)| \leqslant C$$

for some constant C potentially depending on the point x.

Proof. Fix $x \in \iota_\infty(X_0^{reg})$, then fix a ball $B_{g_\infty,\varepsilon}(x) \subset \iota_\infty(X_0^{reg})$ on which the metrics g_{φ_t} converge smoothly to g_∞, also fix a set $K \subset\subset X$ containing all of $\pi_\infty^{-1}(X_0^{sing})$ and also $B_{g_\infty,\varepsilon}(x)$. Then by the Green's formula representation formula, we have

$$\begin{aligned}
\dot{\varphi}_t(x) &= -\int_X \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) \\
&= -\int_{B_{g_\infty,\varepsilon}(x)} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) - \int_{K\setminus B_{g_\infty,\varepsilon}(x)} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) \\
&\quad - \int_{X\setminus K} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y)
\end{aligned}$$

where $G_t(x,y)$ is the positive decaying Green's function on (X, ω_{φ_t}). By the estimates for Green's function [48, p.190], [110, 111], the Greens functions $G_t(x,y)$ satisfy the uniform estimates

$$C^{-1}d_t(x,y)^{2-2n} \leqslant G_t(x,y) \leqslant C d_t(x,y)^{2-2n}$$

where d_t is the distance function induced by g_{φ_t}. And since $\Delta_{\varphi_t}\dot{\varphi}_t = -\Delta_{\varphi_t}\dot{u}_t + \mathrm{tr}_{\varphi_t}(\omega_0 - \omega_1)$, this implies $|\Delta_{\varphi_t}\dot{\varphi}_t| \leqslant |\Delta_{\varphi_t}\dot{u}_t| + \mathrm{tr}_{\varphi_t}(\omega_0 + \omega_1) \leqslant C + \mathrm{tr}_{\varphi_t}\omega_1$ and we have

$$\begin{aligned}
|\dot{\varphi}_t(x)| \leqslant& \int_{B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y)d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y) + \int_{K\setminus B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y)d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \\
&+ \int_{X\setminus K} |\Delta_{\varphi_t}\dot{\varphi}_t|(y)d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \qquad (4.17)
\end{aligned}$$

and we analyze the three terms in the above formula seperately. For the first term, we note that

$\Delta_{\varphi_t}\dot{\varphi}_t$ is uniformly bounded on $B_{g_\infty,\varepsilon}(x)$, so

$$\int_{B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\int_{B_{g_\infty,\varepsilon}(x)} d_t(x,y)^{2-2n} \leqslant C$$

For the second term, observe that on $K\setminus B_{g_\infty,\varepsilon}(x)$, $d_t(x,y)^{2-2n}$ is bounded by $C\varepsilon^{2-2n}$, so

$$\int_{K\setminus B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\left(1+\int_{K\setminus B_{g_\infty,\varepsilon}(x)} \operatorname{tr}_{\varphi_t}\omega_1\right)$$

hence it suffices to bound the integral of $\operatorname{tr}_{\varphi_t}\omega_1$, to do this, we integrate by parts

$$\begin{aligned}\int_K \omega_1\wedge\omega^{n-1}_{\varphi_t} = \int_K \omega_1\wedge(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^{n-1} &= \int_K \omega_1\wedge\hat{\omega}^{n-1}_t \\ &+ \int_{\partial K}\partial\varphi_t\wedge\omega_1\wedge\left(\sum_{l=0}^{n-2}\binom{n-1}{l}\hat{\omega}^l_t\wedge(i\partial\bar{\partial}\varphi_t)^{n-2-l}\right) \\ &\leqslant C\end{aligned}$$

because φ_t and its derivatives are all bounded on the boundary of K.

The last term in (4.17) is bounded because $|\Delta_{\varphi_t}\dot{\varphi}_t|\leqslant Cd_t(x,y)^{-2-\beta}$ on $X\setminus K$, so we have

$$\int_{X\setminus K} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\int_{X\setminus K} d_t(x,y)^{-2n-\beta} \leqslant C$$

and we get our result. □

proof of Proposition 4.5.5. Note that we already know $|\nabla_{g_\infty}\varphi_0|$ is bounded and decaying at infinity, so it suffices to prove that it's bounded near $\pi_\infty^{-1}(X_0^{sing})$. Fix a compact set K containing $\pi_\infty^{-1}(X_0^{sing})$, then by Proposition 4.5.7, $|\nabla v_t|\leqslant C$, but on X_0^{reg}, v_t converges to φ_0 smoothly on compact sets, hence we get our result. □

The main goal of this gradient bound is to show the following.

Proposition 4.5.9. *For any holomorphic section $s \in H^0(X_0, L^k)$ satisfies*

$$\sup_K |s|_{h_\infty^k} \leqslant C$$

and

$$\sup_K |\nabla s|_{h_\infty^k, k\omega_{\varphi_0}} \leqslant C$$

Proof. Locally we can write $h_\infty = e^{-\varphi_0}\hat{h}_0$ where $-i\partial\bar{\partial}\log \hat{h}_0 = \hat{\omega}_0$, since $\hat{\omega}_0 = \pi^\star\omega_{FS}$ on K, we have simply $h_\infty = e^{-\varphi_0}h_{FS}$, and by the C^0 bound for φ_0, it follows that $|s|_{h_\infty^k} \leqslant C|s|_{h_{FS}^k} \leqslant C$. To see the bound for the gradient, we note

$$|\nabla_{h_\infty^k} s|^2_{h_\infty^k, k\omega_{\varphi_0}} = |\nabla_{h_{FS}^k} s + k(\partial\varphi_0)s|_{h_\infty^k, k\omega_{\varphi_0}} \leqslant |\nabla_{h_{FS}^k} s|_{h_{FS}^k, k\omega_{\varphi_0}} + k|\nabla\varphi_0|_{k\omega_{\varphi_0}}|s|_{h_\infty^k}$$

and by the gradient estimate (4.5.5) $|\nabla\varphi_0|_{k\omega_{\varphi_0}} \leqslant C$, so the second term is bounded, and by the estimate (4.8), we have $|\nabla_{h_{FS}^k} s|_{h_{FS}^k, k\omega_{\varphi_0}} \leqslant C|\nabla_{h_{FS}^k} s|_{h_{FS}^k, k\omega_{FS}} \leqslant C$ and we get the bound that we wanted. □

We will need the boundedness of $|s|_{h_\infty^k}$ and $|\nabla s|_{h_\infty^k, k\omega_{\varphi_0}}$ to make the Moser iteration argument work with cutoff functions in the next section.

L^2 estimates on X_0

The argument of this section follows in the same way as in [67], with minor modifications.

We first quote a proposition stating the existence of good cutoff functions on X_∞ from [109].

Lemma 4.5.1. *[109, Proposition 3.5] There exist cutoff functions ρ_ε on X_∞ satisfying the following*

1. $0 \leqslant \rho_\varepsilon \leqslant 1$
2. *supp*$(\rho_\varepsilon) \subset\subset \mathcal{R} = X_0^{reg}$
3. *For any compact set $K \subset\subset \mathcal{R}$, there exist $\varepsilon_K > 0$ such that for all $\varepsilon < \varepsilon_K$, we have $\rho_\varepsilon = 1$ on K.*

4. $\int_X |\nabla \rho_\varepsilon|^2 \to 0$ *as* $\varepsilon \to 0$.

We recall the following version of Hormander's L^2 estimates for the $\bar{\partial}$ equation.

Theorem 4.5.1. *[112, Cor 5.3] Let (M, ω) be a Kähler manifold. Assume M is weakly pseudo-convex. Let (L, h) be a Hermitian line bundle with curvature with (possibly) singular Hermitian metric h, and suppose*

$$-i\partial\bar{\partial}\log h + Ric(\omega) \geqslant \gamma(x)\omega$$

then for any $\beta \in \bigwedge^{0,1} \otimes L$, with $\bar{\partial}\beta = 0$, there exist a section $s \in L$ satisfying $\bar{\partial}s = \beta$ with

$$\int_M |s|_h^2 \omega^n \leqslant \int_M \frac{1}{\gamma}|\beta|^2_{h,\omega}\omega^n,$$

provided the integral on the RHS is finite.

Now we will prove a version of the above theorem on X equipped with a singular metric ω_{φ_0} that we constructed. First we fix a Hermitian metric h_0 on L such that $-i\partial\bar{\partial}\log h_0 = \hat{\omega}_0$, which is possibly by the $\partial\bar{\partial}$-Lemma.

Theorem 4.5.2. *Let $h_\infty = e^{-\varphi_0}h_0$, so $-i\partial\bar{\partial}\log h_\infty = \omega_{\varphi_0}$, and $K \subset\subset X$ a compact subset with pseudoconvex boundary. Then for any $\beta \in \bigwedge^{0,1} \otimes L^k$, with compact support and $supp(\beta) \subset X_0^{reg} \cap K$ and $\bar{\partial}\beta = 0$, there exist a section $u \in H^0(L^k)$ satisfying $\bar{\partial}u = \beta$ with*

$$\int_K |u|^2_{h_\infty^k}\omega^n_{\varphi_0} \leqslant \int_K |\beta|^2_{h_\infty^k, k\omega_{\varphi_0}}\omega^n_{\varphi_0}$$

Proof. By Assumption 1, we know that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon \geqslant \varepsilon\omega$, which implies $\hat{\omega}_0 + ti\partial\bar{\partial}\psi_\varepsilon \geqslant (1-t)\hat{\omega}_0 + t\varepsilon\omega$. By the discussion in the previous sections, we can solve

$$\omega^n_{\varphi_t} = \left((1-t)\hat{\omega}_0 + t\varepsilon\omega + i\partial\bar{\partial}\tilde{\varphi}_t\right)^n = i^{n^2}\Omega \wedge \bar{\Omega}$$

with $\tilde{\varphi}_t$ is bounded on any compact set $K \subset\subset X$, uniformly as $t \to 0$ and $\tilde{\varphi}_t \to \varphi_0$ in $L^\infty_{loc}(X)$ and in $C^\infty_{loc}(X_0^{reg})$. We pick a metric h_0 on L such that $-i\partial\bar{\partial}\log h_0 = \hat{\omega}_0$, then if we set $\tilde{h}_t = e^{-t\psi_\varepsilon - \tilde{\varphi}_t}h_0$, it satisfies

$$-i\partial\bar{\partial}\log\tilde{h}_t^k = k(\hat{\omega}_0 + ti\partial\bar{\partial}\psi_\varepsilon + i\partial\bar{\partial}\tilde{\varphi}_t) \geqslant k\omega_{\varphi_t}$$

By the previous lemma, we can always solve $\bar{\partial}u_t = \beta$, satisfying the estimate

$$\int_K |u_t|^2_{\tilde{h}_t^k}\omega^n_{\varphi_t} \leqslant \int_K |\beta|^2_{\tilde{h}_t^k, k\omega_{\varphi_t}}\omega^n_{\varphi_t} = \int_K e^{-tk\psi_\varepsilon - k\tilde{\varphi}_t}|\beta|^2_{h_0^k, k\omega_{\varphi_t}}\omega^n_{\varphi_t}$$

Since β is compactly supported on X_0^{reg}, $\omega_{\varphi_t} \longrightarrow \omega_{\varphi_0}$ on the support of β, and $e^{-tk\psi_\varepsilon} \longrightarrow 1$ in L^1_{loc}, so we have

$$\lim_{t\to 0}\int_K |\beta|^2_{\tilde{h}_t^k, k\omega_{\varphi_t}}\omega^n_{\varphi_t} = \int_K e^{-k\varphi_0}|\beta|^2_{h_0^k, k\omega_{\varphi_0}}\omega^n_{\varphi_0}$$

and since $e^{-tk\psi_\varepsilon - k\tilde{\varphi}_t}$ is bounded from below on any compact set K, it follows that

$$\int_K |u_t|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} \leqslant C\int_K e^{-tk\psi_\varepsilon - k\tilde{\varphi}_t}|u_t|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} = C\int_K |u_t|^2_{\tilde{h}_t^k}\omega^n_{\varphi_t} \leqslant C$$

hence there exist a weakly convergent subsequence $u_t \rightharpoonup u$ in $L^2(K, h_0^k)$ and the equation $\bar{\partial}u_t = \beta$ carries through the limit in the weak convergence, so we have $\bar{\partial}u = \beta$. Since the sections $u_t - u$ are holomorphic and weakly converge to 0, it follows that the convergence is smooth it happens strongly, hence we have

$$\int_K e^{-k\varphi_0}|u|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} \leqslant \int_K e^{-k\varphi_0}|\beta|^2_{h_0^k, \omega_{\varphi_0}} i^{n^2}\Omega\wedge\bar{\Omega}$$

□

Proposition 4.5.10. *The following Sobolev inequality hold for $f \in L^\infty \cap H^1(X_0^{reg}, \omega_\infty)$*

$$\left(\int_{X_0^{reg}} |f|^{2\frac{n}{n-1}}\omega^n_\infty\right)^{\frac{n-1}{n}} \leqslant C\int_{X_0^{reg}} |\nabla f|^2_{g_\infty}\omega^n_\infty$$

Proof. Without loss of generality, we can assume $f \geqslant 0$. If f is supported in X_0^{reg}, this follows from [99]. For $f \in L^\infty$, we can define $f_\varepsilon = f\rho_\varepsilon$, f_ε is supported in X_0^{reg}, then we clearly have

$\|f_\varepsilon\|_{L^2} \to \|f\|_{L^2}$, and we also have

$$\int_X |\nabla f_\varepsilon|^2 = \int_X \rho_\varepsilon^2 |\nabla f|^2 + \int_X f^2 |\nabla \rho_\varepsilon|^2 + 2\int_X f\rho_\varepsilon \langle \nabla f, \nabla \rho_\varepsilon \rangle$$

the second and third term goes to 0 as $\varepsilon \to 0$ because $\int_X |\nabla \rho_\varepsilon|^2 \to 0$, and this gives what we wanted. □

Lemma 4.5.2. *Suppose $u \geqslant 0$ is a bounded function on X_0^{reg} that satisfy*

$$\Delta_{\omega_\infty} u \geqslant -Au$$

then for $R \geqslant 1$ sufficiently large (so that $X_0^{sing} \subset B_R(p)$), we have the estimate

$$\|u\|_{L^\infty(B_R(p))} \leqslant C(A + CR^{-2})^{\frac{n}{2}} \|u\|_{L^2(B_{2R}(p))}$$

Proof. Using the Sobolev inequality above and the cutoff function, we can do Moser iteration on (X_0^{reg}, g_∞)

$$\begin{aligned}
A\int_X \eta^2 \rho_\varepsilon^2 u^{p+1} \omega_\infty^n \geqslant & \int_X \eta^2 \rho_\varepsilon^2 u^p (-\Delta u) \omega_\infty^n \\
= & \frac{4p}{(p+1)^2} \int_X \eta^2 \rho_\varepsilon^2 |\nabla u^{\frac{p+1}{2}}|^2 \omega_\infty^n + 2\int_X \rho_\varepsilon^2 \eta (\nabla \eta \cdot \nabla u) u^p \omega_\infty^n \\
& + \frac{4}{(p+1)} \int_X \eta^2 \rho_\varepsilon (\nabla \rho_\varepsilon \cdot \nabla u^{\frac{p+1}{2}}) u^{\frac{p+1}{2}} \omega_\infty^n \\
\geqslant & \frac{4p}{(p+1)^2} \int_X \eta^2 \rho_\varepsilon^2 |\nabla u^{\frac{p+1}{2}}|^2 \omega_\infty^n + 2\int_X \rho_\varepsilon^2 \eta (\nabla \eta \cdot \nabla u) u^p \omega_\infty^n \\
& - \frac{4}{(p+1)} \left(\int_X \eta^2 \rho_\varepsilon^2 |\nabla u^{\frac{p+1}{2}}|^2 \omega_\infty^n \right)^{\frac{1}{2}} \left(\int_X \eta^2 |\nabla \rho_\varepsilon|^2 u^{p+1} \omega_\infty^n \right)^{\frac{1}{2}}
\end{aligned}$$

when u is bounded, we can take a limit as ε goes to 0 and the last term will disappear, so we have

$$\begin{aligned} A\int_X \eta^2 u^{p+1}\omega_\infty^n &\geqslant \frac{4p}{(p+1)^2}\int_X \eta^2|\nabla u^{\frac{p+1}{2}}|^2\omega_\infty^n + \frac{4}{p+1}\int_X \eta(\nabla\eta\cdot\nabla u^{\frac{p+1}{2}})u^{\frac{p+1}{2}}\omega_\infty^n \\ &\geqslant \frac{3p}{(p+1)^2}\int_X \eta^2|\nabla u^{\frac{p+1}{2}}|^2\omega_\infty^n - \frac{16}{p}\int_X |\nabla\eta|^2 u^{p+1}\omega_\infty^n \end{aligned}$$

which implies

$$\int_X |\nabla\eta u^{\frac{p+1}{2}}|^2\omega_\infty^n \leqslant \frac{(p+1)^2}{p}\int_X (A\eta^2 + \frac{17}{p}|\nabla\eta|^2)u^{p+1}\omega_\infty^n$$

then by the Sobolev inquality from Proposition 4.5.10, we have for any $p > 0$,

$$\left(\int_X |\eta u|^{(p+1)\frac{n}{n-1}}\omega_\infty^n\right)^{\frac{n-1}{n}} \leqslant \frac{C(p+1)^2}{p}\int_X (A\eta^2 + \frac{17}{p}|\nabla\eta|^2)u^{p+1}\omega_\infty^n$$

by carefully choosing cutoff functions $0 \leqslant \eta_k \leqslant 1$ such that $\mathrm{supp}(\eta_k) \subset B_{(1+2^{-k})R}$, $\eta_k = 1$ on $B_{(1+2^{-k-1})R}$ and $|\nabla\eta_k|\leqslant CR^{-1}2^k$, and set $p_k = 2(\frac{n}{n-1})^k$, then for $k = 0, 1, 2, \ldots$ we have

$$\|u\|^{p_k}_{L^{p_{k+1}}(B_{(1+2^{-k-1})R})}\leqslant C(Ap_k + CR^{-2}4^k)\|u\|^{p_k}_{L^{p_k}(B_{(1+2^{-k})R})}$$

iterating gives

$$\sup_{B_R} u \leqslant C_{sob}^{\frac{n}{2}} C(2A + CR^{-2})^{\frac{n}{2}}\|u\|_{L^2(B_{2R})}$$

□

We now prove L^2 estimates for holomorphic sections of L^k.

Proposition 4.5.11. *If s is a holomorphic section of (L^k, h_∞^k), then the following estimates hold on (X_0^{reg}, kg_∞) for R large enough so that $B_R(p)$ contains all of X_0^{sing},*

$$\sup_{B_R(p)} |s|_{h_\infty^k}\leqslant C\|s\|_{L^2_{h_\infty^k, kg_\infty}(B_{2R}(p))}$$

$$\sup_{B_R(p)} |\nabla s|_{h_\infty^k, kg_\infty}\leqslant C\|s\|_{L^2_{h_\infty^k, kg_\infty}(B_{2R}(p))}$$

Proof. For a holomorphic section s, we have $\nabla_{\bar{j}} s = 0$, so $g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i s = -ns$. It follows then from standard calculations that

$$\Delta|s| \geqslant -n|s|$$

and

$$\Delta|\nabla s| \geqslant -(n+2)|\nabla s|$$

so now we can apply Lemma 4.5.2 with $u = |s|$ and $u = |\nabla s|$ to get

$$\|s\|_{L^\infty(B_R)} \leqslant C\|s\|_{L^2(B_{2R})}$$

and

$$\|\nabla s\|_{L^\infty(B_R)} \leqslant C\|\nabla s\|_{L^2(B_{2R})} \tag{4.18}$$

and it suffices to show that $\|\nabla s\|_{L^2(B_{2R})} \leqslant C\|s\|_{L^2(B_{3R})}$. We use integration by parts

$$\begin{aligned}
\int_X \eta^2\rho_\varepsilon^2|\nabla s|^2 &= \int_X \eta^2\rho_\varepsilon^2 h g_\infty^{i\bar{j}}\nabla_i s\nabla_{\bar{j}}\bar{s}\omega_\infty^n \\
&= -\int_X \eta^2\rho_\varepsilon^2 h g_\infty^{i\bar{j}}\nabla_{\bar{j}}\nabla_i s\bar{s}\omega_\infty^n - 2\int_X \nabla_{\bar{j}}(\eta^2\rho_\varepsilon^2) h g_\infty^{i\bar{j}}\nabla_i s\bar{s}\omega_\infty^n \\
&\leqslant n\int_X \eta^2\rho_\varepsilon^2|s|^2 + 2\int_X \eta\rho_\varepsilon(\rho_\varepsilon|\nabla\eta| + \eta|\nabla\rho_\varepsilon|)|s||\nabla s| \\
&\leqslant C\int_X (\eta^2 + |\nabla\eta|^2)\rho_\varepsilon^2|s|^2 + \varepsilon\int_X \eta^2\rho_\varepsilon^2|\nabla s|^2 + C\int_X \eta^2|\nabla\rho_\varepsilon|^2|s|^2
\end{aligned}$$

taking ε to 0 gives

$$\int_X \eta^2|\nabla s|^2 \leqslant C\int_X (\eta^2 + |\nabla\eta|^2)|s|^2$$

by choosing $0 \leqslant \eta \leqslant 1$ so that $\operatorname{supp}(\eta) \subset B_{4R}$ and $\eta = 1$ on B_{2R}, this gives $\|\nabla s\|_{L^2(B_{2R})} \leqslant \|s\|_{L^2(B_{4R})}$ Combined with estimate (4.18), this gives the desired estimates. □

Corollary 4.5.4. *For any holomorphic sections $s_0, s_1 \in H^0(L^k|_K)$ on K, the function $|s_i|_{h_\infty^k}$ extends as a lipshitz function on K and this function vanishes precisely on the set $\pi_\infty^{-1}(\{s_i = 0\})$. Also, $\frac{s_0}{s_1}$ extends as a locally Lipshitz function defined on the set $\{|s_1|_{h_\infty^k} > 0\}$.*

Proof. This follows immediately from Kato's inequality

$$|\nabla |s|_{h_\infty^k}|_{g_\infty} \leqslant |\nabla s|_{h_\infty^k, kg_\infty} \leqslant C$$

$$\left|\nabla \frac{s_0}{s_1}\right|_{g_\infty} \leqslant \frac{|s_1|_{h_\infty^k} |\nabla s_0|_{h_\infty^k, kg_\infty} + |s_0|_{h_\infty^k} |\nabla s_1|_{h_\infty^k, kg_\infty}}{|s_1|^2_{h_\infty^k}} \leqslant \frac{C}{|s_1|^2_{h_\infty^k}}$$

and the fact that $K = \overline{(K \cap X_0^{reg}, g_\infty)}$. □

In this section we prove that the map $\pi_\infty : X_\infty \longrightarrow X_0$ is injective, hence it is an isomorphism.

Proposition 4.5.12. *For any $p, q \in X_\infty$ with $p \neq q$ there exist an $k = k(p,q) > 0$ and $s_p, s_q \in H^0(L^k)$ such that*

$$|s_p(p)|_{h_\infty^k}, |s_q(q)|_{h_\infty^k} \geqslant \frac{2}{5}$$

and

$$|s_p(q)|_{h_\infty^k}, |s_q(p)|_{h_\infty^k} \leqslant \frac{1}{3}$$

Proof. This follows from the same argument as Proposition 3.9 in [67].

□

Proposition 4.5.13. *The map $\pi_\infty : X_\infty \longrightarrow X_0$ is an homeomorphism.*

Proof. It's clear that the map is surjective and restricts to a homeomorphism on $X_0^{reg} \subset X_\infty$, it suffices to show that is seperates points near X_0^{sing}. Given $p, q \in K$, suppose for a contradiction that $\pi_\infty(p) = \pi_\infty(q)$, then for any $k > 0$, and any two sections $s_0, s_1 \in H^0(K \cap X_0^{reg}, L^k)$, by the normality of X_0, we know that these two sections extend over the singular set to two sections of $s_0', s_1' \in H^0(\pi_\infty(K), L^k)$, hence we must have $\frac{s_0(p)}{s_1(p)} = \frac{s_0(q)}{s_1(q)}$. But if $d_{X_\infty}(p,q) > 0$, then by the previous lemma, there exist $k > 0$ and we can construct sections $s_p, s_q \in H^0(K \cap X_0^{reg}, L^k)$ such that $|s_p|_{h_\infty^k}(p), |s_q|_{h_\infty^k}(q) \geqslant \frac{2}{5} > \frac{1}{3} \geqslant |s_p|_{h_\infty^k}(q), |s_q|_{h_\infty^k}(p)$ which contradicts $\frac{s_p(p)}{s_q(p)} = \frac{s_p(q)}{s_q(q)}$.

Observe that the singular set $\mathcal{S} \subset X_\infty$ is closed and of finite diameter, from which we can see that $\pi_\infty : X_\infty \longrightarrow X_0$ is a proper map, hence closed, and this implies π_∞^{-1} is also continuous. Thus π_∞ is a homeomorphism. □

proof of Theorem 4.1.2. This is just a combination of Proposition 4.5.1, Proposition 4.5.3 and Proposition 4.5.13. □

4.6 Examples and Applications

In this section we apply Theorems 4.1.1 and Theorem 4.1.2 to study certain explicit examples of crepant resolutions.

4.6.1 Small Resolutions of Brieskorn-Pham cones

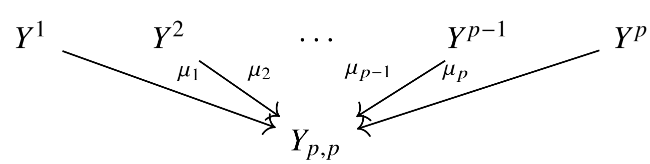
Consider the quasi-homogeneous affine varieties

$$Y_{p,q} = \{xy + z^p - w^q = 0\} \subset \mathbb{C}^4,$$

where we assume that $p \leqslant q$. These singularities, which are compound du Val of type cA_p are Gorenstein and log-terminal and by the main result of [84], $Y_{p,q}$ admits a conical Calabi-Yau metric if and only if $q < 2p$. Let r denote the radial function of the Calabi-Yau cone metric. The Euler vector field $r\frac{\partial}{\partial r}$ associated with the cone structure is given by the real part of the holomorphic vector Reeb field ξ acting on the coordinates (x, y, z, w) with weights

$$\frac{3}{2(p+q)}(pq, pq, 2q, 2p);$$

in particular, the $Y_{p,q}$ are quasi-regular Calabi-Yau cones. A result of Katz [85] says that the $Y_{p,q}$ admits a small (and hence crepant) resolution $\mu : Y \to Y_{p,p}$ if and only if $p = q$. In fact, $Y_{p,p}$ admits p inequivalent small resolutions



with each pair Y^i, Y^j related by a flop; the $\frac{p(p-1)}{2}$ flops are in correspondence with the reflections in the Weyl group of the A_{p-1} Dynkin diagram [113]. When $p = 2$, this recovers the Atiyah flop [86]. The exceptional locus of each contraction μ_j is a chain of $p-1$ rational curves with normal bundle $(-1,-1)$ intersecting transversally. Explicitly, let $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$, and write

$$z^p - w^p = \prod_{j=0}^{p-1}(z - \zeta^j w).$$

Fix $1 \leqslant \ell \leqslant p-1$ and consider the rational map $\nu_\ell : Y_{p,p} \to \mathbb{P}^1$ defined by

$$\nu_\ell(x, y, z, w) = ([x : \prod_{j=0}^{\ell}(z - \zeta^j w)]) \in \mathbb{P}^1. \tag{4.19}$$

Then a small resolution $\mu : Y \to Y_{p,p}$ (say Y^1 for concreteness) is obtained by taking the closure of the graph of

$$\nu_1 \times \cdots \times \nu_{p-1} : Y_{p,p} \to \mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(p-1)}.$$

There are also corresponding partial resolutions $\overline{Y}$ by projecting out some collection of the ν_j. Fix $1 \leqslant i \leqslant p$, and let $\overline{Y}$ be any partial resolution whose contraction $\bar{\pi} : \overline{Y} \to Y_{p,p}$ factors through ν_i. Clearly these resolutions are obtained by repeatedly blowing-up along the lines $x = z - \zeta^j w = 0$. There is a divisor E_i defined by $-E_i = \nu_i^{-1}(p)$ for a generic point $p \in \mathbb{P}^1$, and these divisors satisfy $\mathcal{O}_{\overline{Y}}(-E_i)|_{\mathrm{Exc}(\nu_i)} = \mathcal{O}_{\mathbb{P}^1}(1)$, and $\mathcal{O}_{\overline{Y}}(E_i)$ is trivial on any other component of $\mathrm{Exc}(\bar{\pi})$. Furthermore, if $\overline{Y}$ is obtained from $\nu_{i_1} \times \cdots \times \nu_{i_k}$ then $\otimes_{j=1}^k \mathcal{O}_{\overline{Y}}(-E_{i_j})^{\otimes \ell_j}$ is ample for any $\ell_j \in \mathbb{Z}_{>0}$. These statements follows straightforwardly from the corresponding statements for the blow-ups of the ambient $\mathbb{C}^4$.

Let us fix a small resolution $\mu : Y \to Y_{p,p}$. By Hartog's theorem the holomorphic Reeb vector field extends over $\mathrm{Exc}(\mu)$ and generates a holomorphic retraction onto $\mathrm{Exc}(\mu)$. Thus we have

$$H^{1,1}(Y, \mathbb{R}) = \bigoplus_{i=1}^{p-1} H^{1,1}(\mathbb{P}^1_{(i)}, \mathbb{R}) = \bigoplus_{i=1}^{p-1} \mathbb{R} \cdot [E_i]$$

By the above discussion, the classes $\sum_{i=1}^{p-1}(-t_i)[E_i]$ are Kähler on Y, provided $t_i > 0$ for all i, and semi-positive for $t_i \geqslant 0$. Each of these cohomology classes is 2-almost compactly supported. Fix a class $[\alpha_0] = \sum_{i=1}^{p-1}(-t_i)[E_i]$ where $t_i \geqslant 0$, and at least one $t_j = 0$ and let $[\omega] \in H^{1,1}(Y,\mathbb{R})$ be any Kähler class. Let $[\omega_t] = (1-t)[\alpha_0] + t[\omega]$ be a linear family of Kähler classes. Then by [77] (see also [78]) there is an asymptotically conical Calabi-Yau metric $\omega_{t,CY}$ in $[\omega_t]$ for all $t > 0$.

Since the cone at infinity is quasi-regular we can apply Lemma 4.3.1 to conclude that there is a Kähler current in $[\alpha_0]$ which is smooth on the complement of

$$V := \left\{\mathbb{P}^1_j \subset Y : \int_{\mathbb{P}^1_{(j)}} \alpha_0 = 0\right\}$$

Let $\overline{Y}$ be the partial resolution obtained by contracting V, and let $\hat{\pi} : Y \to \overline{Y}$ be the contraction map. If $[\alpha_0] \in H^{1,1}(Y,\mathbb{Q})$ then, by the preceding discussion, after rescaling we can assume that $[\alpha_0] = \pi^* c_1(L)$ for some ample line bundle $L \to \overline{Y}$. Applying Theorem 4.1.1 and Theorem 4.1.2 we obtain

Proposition 4.6.1. *In the above situation we have*

1. *$\overline{Y}_{reg}$ admits a smooth Ricci-flat metric $\bar{\omega}$, asymptotic to the Calabi-Yau metric on $Y_{p,p}$ at infinity, and with $\overline{(\overline{Y}_{reg},\bar{\omega})}$ homeomorphic to $\overline{Y}$.*

2. *As $t \to 0$ $(Y,\omega_{t,CY})$ converges in the Gromov-Hausdorff sense to $\overline{(\overline{Y}_{reg},\bar{\omega})}$.*

3. *In particular, if we take $[\alpha_0] = 0$, the flops of the $Y_{p,p}$ are continuous in the Gromov-Hausdorff sense.*

Proof. The only point which is not an immediate consequence of Theorems 4.1.1 and 4.1.2 is the third point. However, by the uniqueness part of Theorem 4.1.1, the limiting limiting Calabi-Yau metric $\bar{\omega}$ on $Y_{p,p}$ is isometric to the conical Calabi-Yau metric from [84]. Alternatively, this can be seen as follows. Let ω_c denote the Calabi-Yau metric on $Y_{p,p}$. Clearly $t\omega_{1,CY}$ is a Calabi-Yau metric in $t[\omega]$ asymptotic to $t\omega_c$. Let $\hat{\xi}$ denote the extension of the holomorphic Reeb vector field on Y, and, for $\lambda \in \mathbb{C}$ let $\varphi_\lambda : Y \to Y$ denote the λ-flow of $\hat{\xi}$. Then

$$\left(\varphi_{\frac{1}{\sqrt{t}}}\right)^* t\omega_{1,CY}$$

is Calabi-Yau, asymptotic to ω_c, and lies in the cohomology class $t[\omega]$ and hence is equal to $\omega_{t,CY}$ by the uniqueness results of [78]. From this description, and the convergence result of Theorem 4.1.1 it follows that $\omega_{t,CY}$ converges to $\mu_i^*\omega_c$ on compact sets of $Y\backslash\mathrm{Exc}(\mu_i)$. □

It's not hard to check that if a partial resolution $\overline{Y}$ is obtained by blowing up $0 < k < p-1$ lines $x = z - \zeta^j w = 0$, then $\overline{Y}$ has an isolated singularity biholomorphic to a neighborhood of the singular point in $Y_{p-k,p-k}$. More precisely, suppose for simplicity that $\overline{Y}$ is obtained by blowing-up the lines $x = z - \zeta^j w = 0$ for $0 \leqslant j \leqslant k < p-1$. Then $\overline{Y}$ has an isolated singularity biholomorphic to

$$\widetilde{Y}_{p-k,p-k} := \{xy = \prod_{j=k}^{p-1}(z - \zeta^j w)\} \subset \mathbb{C}^4$$

which is deformation equivalent to $Y_{p-k,p-k}$ and admits a conical Calabi-Yau metric by argument of [84]. The link of this singularity is topologically $(p-k-1)\#(S^2\times S^3)$ and it comes equipped with a Sasaki-Einstein metric. It was shown in [84] that the volume of these Sasaki-Einstein metrics is given by

$$\frac{2(2(p-k))^3}{27(p-k)^4} = \frac{16}{27(p-k)}$$

Thus $\overline{Y}$ yields a cobordism between $(p-k-1)\#(S^2 \times S^3)$ and $\#(p-1)(S^2 \times S^3)$. It is natural to expect that that the metric $\bar{\omega}$ on $\overline{Y}$, close to the singular point, is close to the conical Calabi-Yau metric on $\tilde{Y}_{p-k,p-k}$. At the very least, we expect

Conjecture 4.6.1. *Let $(\overline{Y}, d)$ denote the metric space obtained as the completion of $(\overline{Y}_{reg}, \bar{\omega})$. Then the tangent cone to $(\overline{Y}, d)$ at the singular point is isometric to $\widetilde{Y}_{p-k,p-k}$ equipped with its conical Calabi-Yau metric.*

Let $y \in \overline{Y}$ denote the singular point, and consider the function

$$\mathbb{R}_{>0} \ni r \mapsto v(r) := \frac{\mathrm{Vol}_{\bar{\omega}}(B_{\bar{\omega}}(y, r))}{r^6}$$

Since $(\overline{Y}, \overline{\omega})$ is Calabi-Yau, $v(r)$ is monotone decreasing by the Bishop-Gromov comparison theorem. Furthermore, assuming Conjecture 4.6.1, since $\bar{\omega}$ is asymptotic to the conical Calabi-Yau metric on $Y_{p,p}$ we have

$$\frac{16}{27(p-k)} = \lim_{r \to 0} v(r) \geqslant \lim_{r \to +\infty} v(r) = \frac{16}{27p}.$$

Note that the equality case of Bishop-Gromov already shows that if $k = 0$, then the metric is conical.

While deducing $k \geqslant 0$ in this way is not particularly interesting, this discussion holds for any asymptotically conical Calabi-Yau variety with or without singularities (indeed, a smooth, asymptotically conical Calabi-Yau variety is naturally a cobordism between the standard Sasaki-Einstein structure on the sphere and the link of the cone at infinity). Suppose $(\overline{Y}, \bar{\omega})$ is a asymptotically conical Calabi-Yau variety with asymptotic cone C_∞, and with a singular point y. Assume that a neighborhood of y is biholomorphic to a neighborhood of an isolated singular point in some quasi-homogeneous affine variety C_0 admitting a conical Calabi-Yau metric. Assuming that $\bar{\omega}$ is close to the Calabi-Yau metric on C_0 near the singularity at y, the volume ratio of geodesic balls centered at y will decrease (by Bishop-Gromov) from the volume ratio of the cone $v(C_0)$ to the volume of ratio of the cone at infinity, $v(C_\infty)$. Since these volume ratios are algebraic invariants of the singularities C_0, C_∞, this situation is obstructed in general; for example one cannot take $C_0 = Y_{p,p}$ and $C_\infty = Y_{p-k,p-k}$.

It is tempting to speculate that the volume function on Sasaki-Einstein structures could give rise to a sort of Morse function on the space of Sasaki-Einstein manifolds. For two Sasaki-Einstein manifolds S_0, S_∞ with corresponding cones C_0, C_∞ a Calabi-Yau space $(\overline{Y}, \bar{\omega})$ with an isolated singularity C_0 and cone C_∞ at infinity could be regarded as a kind of flow line of the Morse function between S_0 and S_∞. We will give further examples of this discussion below.

4.6.2 Examples from Fano manifolds

Let us next indicate how to construct examples starting from Fano manifolds with a different singular structure than the previous examples. Suppose X is a Fano manifold of dimension n. Let $\tilde{X} = \mathrm{Bl}_p X$ be the blow up of X at a point and let $\tilde{E} \subset \tilde{X}$ be the exceptional divisor. Assume in addition that that $\tilde{X}$ is Fano and $-K_{\tilde{X}}$ is base-point free. Assume that $\tilde{X}$ has a Kähler-Einstein metric, or more generally that the affine cone over $\tilde{X}$, Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$, admits a conical Calabi-Yau metric. This holds, for example, whenever $\tilde{X}$ is toric, by [90]. It is not difficult to generate examples satisfying these assumptions. For example

- Let $X = \mathbb{P}^n$, with p a torus invariant point. Then $\tilde{X} = \mathrm{Bl_p}\mathbb{P}^\mathrm{n}$ is Fano and $-K_{\tilde{X}}$ is base point free. These manifolds do not admit Kähler-Einstein metrics, as can be seen from Matsushima's obstruction. However, they are toric, and so the theorem of Futaki-Ono-Wang implies the existence of a Calabi-Yau cone metric on the affine cone $C :=$ Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$. Note that the conical Calabi-Yau structure on C need not be quasi-regular, as happens for example when $n = 2$ [114, 90, 87].

- Let X be a del Pezzo surface with $K_X^2 \geqslant 3$, and p chosen sufficiently generic so that $\tilde{X} = \mathrm{Bl}_p X$ is Fano. The global generation of $-K_{\tilde{X}}$ follows from Reider's Theorem [115]. Furthermore, a theorems of Tian-Yau [89] and Tian [88] say that X admits a Kähler-Einstein metric if $K_X^2 < 8$. If, however, $K_X^2 = 8, 9$ then X does not admit a Kähler-Einstein metric by Matsushima's obstruction [116]. On the other hand, in these latter examples, if p is chosen so that $\tilde{X}$ is toric, then the affine cone Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$ admits a conical Calabi-Yau metric thanks to results of Futaki-Ono-Wang [90] (See also [84]). In these examples the Calabi-Yau cone structure is not quasi-regular [87, 90].

Let $Y = K_{\tilde{X}}$ be the total space of the canonical bundle, and let $p : Y \to \tilde{X}$ be the projection. The pull-back p^* identifies $H^{1,1}(Y, \mathbb{R}) = H^{1,1}(\tilde{X}, \mathbb{R})$, and Y admits an asymptotically conical Calabi-

Yau metric in any Kähler class in $H^{1,1}(Y,\mathbb{R})$ [78]. Suppose $[\alpha] \in H^{1,1}(X,\mathbb{R})$ is a Kähler class, so that $p^*[\pi^*\alpha] \in H^{1,1}(Y,\mathbb{R})$ is a nef class on Y admitting a semi-positive representative. By regarding the exceptional divisor of the blow-up $\pi : \tilde{X} \to X$ as a subvariety of the zero section in Y, we get a natural codimension 2 subvariety, $E \subset Y$ (explicitly $E = p^{-1}(\tilde{E}) \cap \{$ zero section $\}$). Our goal is to show that if $[\omega_t] = (1-t)[p^*\pi^*\alpha] + t[\omega] \in H^{1,1}(Y,\mathbb{R})$ and $\omega_{t,CY}$ are conical Calabi-Yau metrics in $[\omega_t]$ then, as $t \to 0$, $(Y, \omega_{t,CY})$ Gromov-Hausdorff converges to a variety Z with an isolated, Gorenstein, log-terminal singularity which is obtained from Y by contracting E to a point. As a first step, we need to verify that Assumption 1 holds, since the failure of the cone at infinity to be quasi-regular means that Lemma 4.3.1 does not apply in general.

Lemma 4.6.1. *The cohomology class $p^*[\pi^*\alpha]$ contains a Kähler current which is smooth outside of E.*

Proof. It is a standard fact that we can choose a hermitian metric $h_{\tilde{E}}$ on $\mathcal{O}_{\tilde{X}}(\tilde{E})$ such that

$$\pi^*\alpha + \varepsilon\sqrt{-1}\partial\overline{\partial}\log h_{\tilde{E}} > \omega_{\tilde{X}} \tag{4.20}$$

for some $\varepsilon > 0$ and $\omega_{\tilde{X}}$ a Kähler form on $\tilde{X}$. Let $s_{\tilde{E}}$ denote the defining section of $\tilde{E} \subset \tilde{X}$. After scaling we may assume that $|s_{\tilde{E}}|^2_{h_{\tilde{E}}} < 1$. The current $\tilde{T} := \pi^*\alpha + \varepsilon\sqrt{-1}\partial\overline{\partial}\log|s_{\tilde{E}}|^2_{h_{\tilde{E}}}$ is a Kähler current on $\tilde{X}$ which is singular along $\tilde{E} \subset \tilde{X}$. Let $h_{\tilde{X}}$ be a negatively curved metric on $K_{\tilde{X}}$, and let s denote a coordinate on the fibers of $K_{\tilde{X}}$. We claim that

$$T = p^*\pi^*\alpha + \sqrt{-1}\partial\overline{\partial}(|s|^2_h + \varepsilon\log(p^*|s_{\tilde{E}}|^2_{h_{\tilde{E}}} + |s|^2_{h_{\tilde{X}}})) \tag{4.21}$$

is a Kähler current. This can be verified by a straightforward calculation, which we leave to the reader.

□

The next step is to show that there is a space Z, and a map $\Phi : Y \to Z$ which is an isomorphism outside E and contracts E to a point, which is an isolated, Gorenstein log-terminal singularity in

Z. Let us begin with a local description of this map and the resulting singularity. Note that the normal bundle of $E \subset Y$ is given by

$$N_{E/Y} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1))$$

which follows from $K_{\tilde{X}} = \pi^* K_X + (n-1)\tilde{E}$. There is a contraction map

$$\nu : \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1)) \to C$$

contracting the zero section of $N_{E/Y}$ to a point. Explicitly, this map is given by [117, Page 314]

$$\begin{aligned} N_{E/Y} &= \operatorname{Spec} \bigoplus_{m \geqslant 0} \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right) \\ &\to \operatorname{Spec} \bigoplus_{m \geqslant 0} H^0 \left(\mathbb{P}^{n-1}, \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right)\right) = C_0 \end{aligned}$$

Since

$$H^0 \left(\mathbb{P}^{n-1}, \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right)\right) = H^0(\mathbb{P}(N_{E/Y}), \mathcal{O}_{\mathbb{P}(N_{E/Y})}(m))$$

we see that C_0 is the affine cone over $\mathbb{P}(N_{E/Y})$ obtained by blowing down the zero section of $\mathcal{O}_{\mathbb{P}(N_{E/Y})}(-1)$. We claim that $\mathbb{P}(N_{E/Y})$ is Fano. In general, the canonical bundle of a projective bundle $\pi : \mathbb{P}(V) \to X$, where V has rank r is given by

$$K_{\mathbb{P}(V)} = \mathcal{O}_{\mathbb{P}(V)}(-r-1) \otimes \pi^*(\det V^*) \otimes \pi^* K_X.$$

Applying this formula in the current scenario yields

$$K_{\mathbb{P}(N_{E/Y})} = \mathcal{O}_{\mathbb{P}(N_{E/Y})}(-3).$$

Since $N_{E/Y}$ is a direct sum of negative line bundles, $\mathcal{O}_{\mathbb{P}(N_{E/Y})}(3)$ is ample. It follows from this that C_0 has an isolated Gorenstein, log-terminal singularity and $K_{C_0} \sim \mathcal{O}_{C_0}$ is trivial. Finally, since

$N_{E/Y} \to \mathbb{P}^{n-1}$ is a direct sum of line bundles, $\mathbb{P}(N_{E/Y})$ is toric. Therefore the result of Futaki-Ono-Wang [90] (see also [84]) says that C_0 admits a conical Calabi-Yau metric for some choice of Reeb vector field.

Next we will globalize this construction using the input of an ample line bundle L on X. First note that a section $f \in H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$ naturally induces a holomorphic function $f \in H^0(Y, O_Y)$ vanishing to order m on $\tilde{X} = \{$ zero section $\} \subset Y$. Let $f_1 \dots, f_M$ be generators of the coordinate ring $\bigoplus_{m>0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$. Since $-K_{\tilde{X}}$ is ample and globally generated, the holomorphic functions $f_1 \dots, f_M$ separate points and tangent vectors on $Y \backslash \tilde{X}$, and generate the normal bundle to $\tilde{X}$ in Y. Let L be a very ample line bundle on X, and let $\{s_0, \dots, s_N\}$ be a basis of $H^0(X, L)$. Fix coordinates $(z_1, \dots, z_n)$ on X centered at p. Up to making a linear change of coordinates we can assume that $s_0(p) \neq 0$, and near p we have

$$\frac{s_i(z)}{s_0(z)} = z_i + O(z^2) \quad 1 \leqslant i \leqslant n, \qquad \frac{s_j(z)}{s_0(p)} = O(z^2) \quad n \leqslant j \leqslant N$$

By inspection the sections $\{p^*\pi^* s_i\}_{0 \leqslant i \leqslant N}$ separate points and tangents in $\tilde{X} \backslash \tilde{E}$ and generate the normal bundle to $\tilde{E}$ in $\tilde{X}$. Now consider the map $\Phi : Y \to \mathbb{P}^N \times \mathbb{P}^M$ defined by

$$\Phi(z) := ([p^*\pi^* s_0(z) : \cdots : p^*\pi^* s_N(z)], [1 : f_1(z) : \cdots : f_M(z)]) \in \mathbb{P}^N \times \mathbb{P}^M \tag{4.22}$$

By the preceeding discussion this map is an isomorphism on $Y \backslash E$, and $\Phi(E) = [1 : 0 : \cdots : 0] \times [1 : 0 : \cdots : 0]$. Since the differential $d\Phi$ is an isomorphism on $N_{E/Y}$, the germ of Φ agrees with the contraction ν on $N_{E/Y}$. Note also that $\Phi|_{\tilde{X}} = \pi$ (composed with the imbedding X into projective space by sections of L). Let $Z = \Phi(Y)$. From the local description above Z has an isolated Gorenstein, log-terminal singularity, and $K_Z = O_Z$. The map $\Phi : Y \to Z$ is therefore a small, and hence crepant, resolution of Z. It follows from the construction that we can describe Z has the relative spectrum

$$Z = \underline{\text{Spec}}\, (K_X \otimes \mathfrak{m}_p) \to X$$

where $\mathfrak{m}_p$ is the ideal sheaf of $p \in X$. In order to apply Theorems 4.1.1 and 4.1.2 it suffices to show

Lemma 4.6.2. *In the above setting, there is an ample line bundle L' on Z such that $p^*c_1(\pi^*L) = \Phi^*c_1(L')$.*

Proof. Since Z is normal and Φ is projective with connected fibers we have $\Phi_*\mathcal{O}_Y = \mathcal{O}_Z$, and $f_1, \ldots, f_M$ extend over the singular point to global sections of $\mathcal{O}_Z$. Furthermore, there is a natural projection

$$\hat{p} : Z \to X$$

obtained by projecting from Z onto the $\mathbb{P}^N$ factor in (4.22) and we have $\pi \circ p = \Phi \circ p = \hat{p} \circ \Phi$. Thus

$$[p^*\pi^*s_0 : \cdots : p^*\pi^*s_N] = [\hat{p}^*s_0 : \cdots : \hat{p}^*s_N].$$

Combining this observation with the Segre embedding $\mathbb{P}^N \times \mathbb{P}^M \hookrightarrow \mathbb{P}^{(N+1)(M+1)-1}$ it follows that $L' := \hat{p}^*L$ is ample on Z. Since

$$p^*c_1(\pi^*L) = \Phi^*c_1(\hat{p}^*L)$$

the lemma follows. ◻

We can now conclude

Corollary 4.6.1. *With notation as above, consider the family of Kähler classes $[\omega_t] = (1-t)p^*c_1(\pi^*L) + t[\omega] \in H^{1,1}(Y, \mathbb{R})$ for $t > 0$. Let $\omega_{t,CY}$ be the asymptotically conical Kähler metrics in $[\omega_t]$. Then there is a incomplete, asymptotically conical Calabi-Yau metric $\overline{\omega}$ on Z_{reg} such that $\overline{(Z_{reg}, \bar{\omega})} = (Z, d)$ and*

$$(Y, \omega_{t,CY}) \to_{GH} (Z, d).$$

Proof. Combine Lemmas 4.6.1 4.6.2 with Theorems 4.1.1 and 4.1.2. ◻

It is again natural to conjecture

Conjecture 4.6.2. *Let (Z, d) be the metric space structure on Z induced from Y by Theorem 4.1.2. Then the tangent cone to (Z, d) at the singular point $z \in Z$ is isometric to the blow down of the zero section in $\mathcal{O}_{\mathbb{P}(V)}(-1)$ where*

$$V := \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1))$$

equipped with its conical Calabi-Yau metric.

Assuming this conjecture, the space Z can be viewed as a kind of cobordism between Sasaki-Einstein manifolds, and the speculative discussion from Section 4.6.1 can be applied in the same way.

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