Essays on Monetary and Fiscal Stabilization Policies

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Abstract
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This dissertation is a collection of three essays on the monetary and fiscal stabilization policies. Grounded in the framework of the New Keynesian model, they combine both theoretical modeling and quantitative analysis, taking into account the considerations from behavioral macroeconomics and global supply chains.

Chapter 1 considers both short-term effects and long-run consequences of alternative monetary and fiscal policies under an assumption of bounded rationality. Most of the existing analyses of the interaction between monetary and fiscal policy in the monetary literature often turn crucially on assumptions that are made about outcomes far in the future, sometimes infinitely far. This is a problematic feature of rational-expectations analyses, given the limited basis for assumptions about the distant future. By relaxing this problematic assumption regarding long-expectation, while keeping other parts as close as possible to the standard New Keynesian model, I take the approach of finite forward planning to study the interplay of fiscal transfer policies and monetary policy. In particular, this approach assumes that explicit forward planning extends only a finite distance into the future, with anticipated situations at that horizon evaluated using a value function learned from past experience. Such an approach makes announcements of future policies relevant, but avoids the debates about equilibrium selection that plague rational-expectations analyses. The combined monetary-fiscal regimes that result in stable long-run dynamics are characterized, and the effectiveness of temporary changes in either type of policy as a source of short-run demand stimulus is analyzed. The effectiveness of a coordinated change in monetary and fiscal policy is shown to be greatest when decision makers’ degree of foresight is intermediate in range (average planning horizons on the order of ten years), rather than shorter or longer.

Chapter 2, co-authored with Michael Woodford, reconsiders several issues connected with sta-
bilization policy, when the zero lower bound (ZLB) is a relevant constraint on the effectiveness of conventional monetary policy, under an assumption of bounded rationality. In particular, it assumes that decision makers only plan a finite distance into future each time they must act, and use a value function from their past experiences to estimate a continuation value for their situation at the end of the planning horizon. Forward guidance regarding future monetary policy remains relevant, even if its predicted impact is quantitatively weaker, and in particular price-level targeting continues to have advantages over purely forward-looking inflation targeting during a ZLB scenario. Moreover, recognizing that planning horizons may be relatively short for some strengthens the case for systematic price-level targeting, as opposed to temporary price-level targeting only following a ZLB scenario. Fiscal transfers can be a powerful tool to reduce the contractionary impact of an increased financial wedge during a crisis, and even make possible complete stabilization of both aggregate output and inflation under certain circumstances, but the power of such policies depends on the degree of monetary policy accommodation. We also show that a higher level of welfare is generally possible if both monetary and fiscal authorities commit themselves to history-dependent policies in the period after the financial disturbance has dissipated.

Chapter 3, co-authored with Shang-Jin Wei, studies the implications of global supply chains for the design of monetary policy, using a small-open economy New Keynesian model with multiple stages of production. Within the family of simple monetary policy rules with commitment, a rule that targets separate producer price inflation at different production stages, in addition to output gap and real exchange rate, is found to deliver a higher welfare level than alternative policy rules. As an economy becomes more open, measured by the export share, the optimal weight on the upstream inflation rises relative to that on the final stage inflation. If we have to choose among aggregate price indicators, targeting PPI inflation yields a smaller welfare loss than targeting CPI inflation alone. As the production chain becomes longer, the optimal weight on PPI inflation in the policy rule that targets both PPI and CPI inflation will also rise. A trade cost shock such as a rise in the import tariff can alter the optimal weights on different inflation variables.
Table of Contents

List of Tables ........................................................................................................ vii

List of Figures .......................................................................................................... viii

Acknowledgments .................................................................................................... xi

Chapter 1: Fiscal and Monetary Policy Interaction under Limited Foresight .......... 1

1.1 Introduction ...................................................................................................... 2

1.2 A New Keynesian DSGE Model with Finite Planning Horizon ..................... 13

1.2.1 Optimal Finite-horizon Planning for Households .................................... 14

1.2.2 Optimal Price-setting of Firms with Finite Forward Planning ................. 17

1.2.3 Monetary and Fiscal Policy ...................................................................... 20

1.2.4 Equilibrium Characterization with Common Planning Horizon ............ 21

1.2.5 Heterogeneous Agents in Terms of Planning Horizon .......................... 23

1.3 Long-run Stability of Monetary-fiscal Policy Interaction .............................. 26

1.3.1 Learning Process in the Value Function ................................................ 27

1.3.2 Equilibrium Characterization with Common Planning Horizon ............ 29

1.3.3 Heterogeneous Agent with Learning in the Value Function .................... 32

1.3.4 Determinacy Condition for Monetary Policy with Inactive Fiscal Policy ... 38
1.3.5 Stability (Determinacy) Condition with Monetary-fiscal Policy Interaction  39

1.4 Short-term Effects of Stimulative Fiscal Policy and Interaction with Monetary Policy  43

1.4.1 Lump-sum Transfer Financed by Debt with Monetary Policy under Taylor Rule  46

1.4.2 Forward Guidance and Interaction with Lump-sum Transfer  54

1.5 Long-run Dynamics of Fiscal Transfer Policy  58

1.6 Conclusion  62

1.7 Conclusion  62

Chapter 2: Stabilization Policy in a Low-Interest-Rate World: Consequences of Limited Foresight  65

2.1 Introduction  66

2.2 Output and Inflation Determination with Finite Planning Horizons  74

2.2.1 Forward Planning with a Finite Horizon  74

2.2.2 Log-Linear Approximate Dynamics  81

2.2.3 A Crisis Scenario  83

2.3 Price-Level Targeting Reconsidered  91

2.3.1 Temporary Price-level Targeting (TPLT)  93

2.3.2 Systematic Price-level Targeting (PLT Rule)  101

2.4 Coordinated Monetary and Fiscal Stabilization Policy  107

2.4.1 Fiscal Transfers and Aggregate Demand  111

2.4.2 The Dependence of Fiscal Stimulus on Monetary Accommodation  115

2.4.3 The Continuing Relevance of Forward Guidance  122

2.5 Conclusion  132

Chapter 3: Monetary Policy in an Era of Global Supply Chains  135
<table>
<thead>
<tr>
<th>Section</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Introduction</td>
<td>136</td>
</tr>
<tr>
<td>3.2 The model setting</td>
<td>141</td>
</tr>
<tr>
<td>3.2.1 Household</td>
<td>142</td>
</tr>
<tr>
<td>3.2.2 Firms</td>
<td>145</td>
</tr>
<tr>
<td>3.2.3 The Firm’s Pricing Problem</td>
<td>147</td>
</tr>
<tr>
<td>3.2.4 The Market Clearing Conditions and Equilibrium Definition</td>
<td>149</td>
</tr>
<tr>
<td>3.3 The Case of Two-stage Production</td>
<td>150</td>
</tr>
<tr>
<td>3.3.1 The Steady-state Equilibrium</td>
<td>151</td>
</tr>
<tr>
<td>3.3.2 The Flexible-price Equilibrium</td>
<td>152</td>
</tr>
<tr>
<td>3.3.3 The Sticky-price Equilibrium</td>
<td>154</td>
</tr>
<tr>
<td>3.3.4 A Utility-based Welfare Loss Function for Optimal Monetary Policy</td>
<td>155</td>
</tr>
<tr>
<td>3.3.5 Discussion on the Welfare Loss Function</td>
<td>159</td>
</tr>
<tr>
<td>3.3.6 Discussion on Value Chains and Price Stickiness</td>
<td>160</td>
</tr>
<tr>
<td>3.3.7 Effects of a Higher Import Tariff</td>
<td>161</td>
</tr>
<tr>
<td>3.4 Comparing Monetary Policy Rules</td>
<td>162</td>
</tr>
<tr>
<td>3.4.1 Model Parameters</td>
<td>163</td>
</tr>
<tr>
<td>3.4.2 Welfare Losses</td>
<td>164</td>
</tr>
<tr>
<td>3.4.3 Comparative Statics: Effects of Openness and Intermediate Goods Share</td>
<td>166</td>
</tr>
<tr>
<td>3.4.4 Effects of a Higher Import Tariff</td>
<td>169</td>
</tr>
<tr>
<td>3.4.5 Asymmetric Price Stickiness</td>
<td>172</td>
</tr>
<tr>
<td>3.4.6 Additional Loss from Sticky Monetary Policy Rules</td>
<td>173</td>
</tr>
<tr>
<td>3.5 Conclusion</td>
<td>174</td>
</tr>
</tbody>
</table>
Appendix A: Fiscal and Monetary Policy Interaction under Limited Foresight

A.1 The System of Equations for Heterogeneous Agents with No Update in the Value Function

A.2 Aggregation across the Population for Characterizing Aggregate “Trend” Variables

A.3 The System of Equations for the Equilibrium under Heterogeneous Agents with Learning in the Value Function

A.4 Proof for the Determinacy Condition and Convergence Condition with Taylor Rule and Inactive Fiscal Policy

A.5 Convergence Condition of $\Sigma \omega_h y^h$ and $\Sigma \phi_h \pi^h$ with an Endogenous Fiscal Rule

A.6 Proof for the Monotonicity of the Responses of Output and Inflation with respect to $\phi_\pi$

A.7 Proof for the Monotonicity of the Responses of Output and Inflation with respect to $\Gamma$

A.8 Forward Guidance

A.9 Proof of the Positive Interaction between Forward Guidance and Fiscal Transfer

A.10 Proof for the Property of a Long-run Stationary Equilibrium

A.11 The Fiscal Multiplier with Government Expenditure under Finite Forward Planning

Appendix B: Stabilization Policy in a Low-Interest-Rate World: Consequences of Limited Foresight

B.1 Temporary Price-level Targeting: Numerical Methods

B.2 Systematic Price-level Targeting: Numerical Methods

B.3 Output and Inflation Stabilization with an Exponential Distribution of Planning Horizons

B.4 Optimal Fiscal-Monetary Policy Coordination: Numerical Methods
Appendix C: Monetary Policy in an Era of Global Supply Chains

C.1 Equilibrium Characterization with $N$-stage of Production in a Small-open Economy

C.1.1 The Steady-state Equilibrium

C.1.2 The Flexible-price Equilibrium

C.1.3 The Sticky-price Equilibrium

C.1.4 Stage-specific employment in a small-open economy with $N$-stage production

C.2 Aggregate CPI Inflation with Two-stage Production

C.3 Stage-specific Employment with Two-stage Production

C.4 The Steady-state Share of Domestic Demand in Total Demand as to Import Tariff

C.5 Heterogeneity in Openness across Stages

C.6 $N$-stage Production in a Closed Economy

C.6.1 A Utility-based Welfare Loss Function for Optimal Monetary Policy

C.6.2 Discussion about the Terms and Coefficients in Welfare Loss Function

C.6.3 The Steady-state Equilibrium

C.6.4 The Flexible-price Equilibrium

C.6.5 The Sticky-price Equilibrium

C.6.6 Labor Demand Function in the Flexible-price Equilibrium

C.6.7 The Log-deviation of the Real Marginal Cost from the Flexible-price equilibrium

C.6.8 Stage-specific Employment Gaps with $N$-stage Production

C.7 The Closed-form Welfare Loss Function for the Cases of $N = 2$ and $N = 3$ in a Closed Economy

C.8 The Proof for a Positive Coefficient of the Output Gap in the Welfare Loss Function in the Closed Economy
# List of Tables

1.1 Calibrated values of parameters ................................. 41

2.1 Calibrated parameter values ........................................ 85

3.1 Parameter calibration .............................................. 164

3.2 Optimal alternative simple rules of monetary policy ............. 166

3.3 Alternative simple rules of monetary policy in literature .......... 166

3.4 Optimal alternative simple rules of monetary policy with higher imported tariff . 171

3.5 Optimal alternative simple rules of monetary policy with lower price stickiness in upstream production ...................................................... 172

3.6 Welfare loss for adopting old policy rules estimated in 1987 .......... 174

C.1 Optimal alternative simple rules of monetary policy with heterogeneity in export share along production chain ............................................. 224

C.2 Optimal alternative simple rules of monetary policy under trade balance ........ 243
List of Figures
1.1

Determinacy condition for {𝜙, Γ} with respect to different planning horizons . . . . 42

1.2

Output and inflation in period 0 as to the planning horizon with a one-time lumpsum transfer financed by a permanent increase in real public debt . . . . . . . . . 49

1.3

Output and inflation in period 0 as to the planning horizon with a one-time lumpsum transfer under accommodative monetary policy . . . . . . . . . . . . . . . . . 50

1.4

Output and inflation in period 0 with one-time lump-sum transfer financed by a
temporary increase in real public debt under different speeds of taxation . . . . . . 53

1.5

Output and inflation in period 0 with a one-time lump-sum transfer financed by a
temporary increase in real public debt under accommodative monetary policy . . . 54

1.6

Output and inflation in period 0 regarding the planning horizon with forward guidance and one-time lump-sum transfer financed by a permanent increase in real
public debt . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 57

1.7

Long-run dynamics with learning in the value function under a one-time debtfinanced lump-sum transfer . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 61

1.8

Long-run dynamics with learning in the value function under a one-time debtfinanced lump-sum transfer and no change in the long-run debt target . . . . . . . 62

2.1

Expenditure and rates of price increase during the crisis period, under different
assumptions about the planning horizon ℎ (in quarters) of households and firms,
when the central bank follows a strict inflation targeting policy and there is no
response of fiscal policy. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 89

2.2

Dynamics of price-level gap, output, and inflation rates with different planning
horizons ℎ under temporary price-level targeting policy . . . . . . . . . . . . . . . 100

2.3

Dynamics of price-level gap, output, and inflation rates under inflation targeting
versus temporary price-level targeting (TPLT) . . . . . . . . . . . . . . . . . . . . 102
viii


2.4 Dynamics of price-level gap, output, and inflation rates with different planning horizons \( h \) under systematic price-level targeting policy .......................... 106

2.5 Dynamics of price-level gap, output, and inflation rates under temporary price-Level targeting (TPLT) versus systematic price-level targeting (PLT Rule) .......... 108

2.6 Expenditure and rates of price increase during the crisis period, for households and firms with different planning horizons \( h \), under a policy that fully stabilizes aggregate output and inflation. Each line is for a distinct value of the mean planning horizon \( \bar{h} \). Both \( h \) and \( \bar{h} \) are in quarters. ................................. 118

2.7 Expenditure and rates of price increase during the crisis period, for households and firms with different planning horizons \( h \) (in quarters) when the central bank follows a strict inflation targeting policy. The two lines correspond to the minimal and maximal sizes of fiscal stimulus. ................................. 120

2.8 Equilibrium trajectories in the case of an elevated financial wedge for 10 quarters (panel (a)), under three alternative assumptions about policy: (i) \( b_{t+1} = 0 \) at all times, and \( \hat{t}_t = \max\{-\hat{\Delta}_t, \hat{t}_t\} \); (ii) \( b_{t+1} = 0 \) at all times, but the path \( \{\hat{t}_t\} \) is chosen optimally; or (iii) the paths of both \( \{b_{t+1}\} \) and \( \{\hat{t}_t\} \) are chosen optimally. Planning horizons extend 8 quarters into the future, and \( t \) measures quarters since the onset of the elevated financial wedge. ................................. 127

2.9 Dashed lines show the expected paths of output \((y_{\tau|t+h}^h)\) and inflation \((\pi_{\tau|t+h}^h)\) for dates \( t \leq \tau \leq t+h \), under the plans calculated by households and firms with horizon \( h = 8 \) at successive dates \( t \), in the case that both monetary and fiscal policy commitments are optimal (case (iii) from Figure 2.8). The solid lines show the predicted actual paths of output \((y_{\tau|t+h}^h)\) and inflation \((\pi_{\tau|t+h}^h)\). Both \( t \) and \( \tau \) indicate quarters since the onset of the disturbance (again shown in panel (a)). ................................. 131

3.1 Stage-of-processing producer price index and core CPI, US and Australia ........ 139

3.2 Relative weight of upstream inflation index in optimal simple rule with respect to country openness ................................................................. 168

3.3 Relative weight of upstream inflation index in optimal simple rule with respect to intermediate goods share ................................. 170

3.4 Ratio of the weight on upstream inflation index versus CPI with respect to intermediate goods share ................................. 170

A.1 Output and inflation dynamics with different planning horizons under forward guidance ................................................................. 195
A.2 Output and inflation in period 0 as to the planning horizon under forward guidance with different length of commitments ......................................................... 195

B.1 Price-gap Trajectories in “Crisis” State as a Function of Planning Horizon ........ 204

B.2 Dynamics of output and inflation under the optimal fiscal transfer policy and monetary policy with relatively short planning horizons $h$. Each line is for a distinct value of the common planning horizon $h$. Both $t$ and $h$ are in quarters. .............. 212

B.3 Dynamics of output and inflation under the optimal fiscal transfer policy and monetary policy with relatively long planning horizons $h$. Each line is for a distinct value of the common planning horizon $h$. Both $t$ and $h$ are in quarters. .............. 213

B.4 Expected paths of output and inflation in forward planning exercise under the optimal fiscal transfer policy and monetary policy with a common planning horizon of $h = 20$ quarters. ................................................................. 213

C.1 Relative welfare loss with either upstream-stage price fully flexible or downstream-stage price fully flexible with respect to country openness ......................... 245

C.2 Relative welfare loss with either higher elasticity of substitution in upstream stage or higher elasticity of substitution in downstream stage with respect to country openness .......................................................... 246
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Chapter 1

Fiscal and Monetary Policy Interaction under Limited Foresight

Yinxi Xie
1.1 Introduction

The recent economic and financial crisis has led to renewed interest in counter-cyclical fiscal policies.\(^1\) Some recent discussions of the extent to which fiscal policy can be relied upon emphasize the potential benefits of commitments to monetary accommodation of fiscal transfers.\(^2\) Some have even proposed that a “fiscally dominant” regime (i.e., passive monetary policy with active fiscal policy) would better maintain macro stability in the face of an effective lower bound on nominal interest rates.\(^3\)

Most of those discussions and analyses have been built on the hypothesis of rational expectation (RE), where the effectiveness of the analyses turns crucially on the assumptions of RE that are made about outcomes far in the future, sometimes infinitely far. Decision makers are assumed to correctly understand the economy well in the distant future, and base their decisions on their expectations regarding the infinite future. However, the assumption that agents can have such foresight is non-plausible and unrealistic. Putting too much weight on the infinite future, the assumption of RE also raises issues such as the “forward guidance puzzle” (e.g., Del Negro, Giannoni, and Patterson, 2015; Mckay, Nakamura, and Steinsson, 2016), which views a commitment to future monetary policy being too effective in stimulating output and inflation.

By relaxing this problematic assumption of RE, while keeping other parts as close as possible to the standard New Keynesian model, this paper studies the interplay of fiscal and monetary policy, including policy experiments of fiscal transfers and unconventional monetary policy (forward guidance), in both short-term effects and long-run consequences.\(^4\) In particular, I emphasize the degree to which foresight influences the effects of fiscal and monetary policy, and their interactions. More generally, this paper investigates the question of how fiscal policy and monetary policy

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\(^1\)For instance, the IMF asks for smarter use of fiscal policy in an environment of high-level public debt and low interest rates. The former managing director of IMF, Christine Lagarde, appeals for the dedicated use of fiscal policy in her April 2019 speech at US Chamber of Commerce: https://www.imf.org/en/News/Articles/2019/03/29/sp040219-a-delicate-moment-for-the-global-economy.

\(^2\)Recent examples include Ascarì and Rankin (2013), Buiter (2014), Turner (2017), and Galí (2019).

\(^3\)Among others, see Jarociński and Maćkowiak (2017).

\(^4\)In terms of fiscal policy, the major focus of this paper is to study transfer-type policy. The analysis of government expenditure, as another type of fiscal stimulus, can be found in the appendix.
jointly determine inflation and output when decision makers are boundedly rational.

Although the assumption of RE regarding the far future is strong and unrealistic, the belief that decision makers in the short run are still rational is natural. To relax the assumption concerning the long run, but retain the features of RE in the short run, I adopt the approach of finite forward planning recently developed by Woodford (2018). More specifically, instead of assuming decision makers make infinite contingent plans in each period, this approach assumes limited foresight for decision makers who look only a finite distance into the future, and use a value function learned from past experiences to evaluate situations that may be reached at the end of forward planning. That is, decision makers are still “rational” within their planning horizon but use a coarse value function for continuation values to approximate for the future beyond their planning horizon. The finite-planning-horizon model has two key components: the length of the planning horizon, and the value function used to approximate for the future beyond the planning horizon. Intuitively, as the planning horizon of decision makers is longer, or decision makers update their value function faster, the equilibrium under finite horizon planning becomes closer to the case under RE.

Note that if the decision makers can obtain an accurate state-dependent value function, the finite-horizon problem resembles the standard dynamic programming problem as in the analyses of RE. But in practice, when the real world is complex or new information is coming in, acquiring an accurate value function is computationally too costly. Thus, a simplification of the accurate state-dependent value function, that is, the value function learned from past experiences through a rule of constant-gain, would be a useful approximation, and decision makers can abstract from deriving a complex value function through finite forward planning.

In this paper, I build a New Keynesian model with finite forward planning to study the interaction between monetary policy and fiscal policy. More specifically, I characterize the combined monetary-fiscal regimes that result in stable long-run dynamics, and analyze the short-run effectiveness and long-run consequences of temporary changes in either type of policy as a source of

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5Woodford (2019) motivates the approach of finite forward planning from state-of-the-art AI programs to play games.
6The learning rule of constant-gain is also a rule in the type of error-correction.
short-run demand stimulus. I evaluate the effects of counter-cyclical fiscal stimulus such as debt-financed lump-sum transfer, and unconventional monetary policy such as forward guidance, and their interactions. Importantly, this paper emphasizes how the degree of foresight influences the effects of monetary-fiscal policies and their interactions.

The monetary-policy instrument for the central bank is represented by a rule of nominal interest rate, which is specified by the Taylor rule as in the standard New Keynesian model. Fiscal policy imposed by the government is specified by a lump-sum taxation scheme (e.g., Leeper, 1991; Woodford, 2003; Cochrane, 2019), and thus the evolution of real public debt is endogenous. The lump-sum taxation scheme has three fiscal policy instruments: one-time lump-sum transfer, the speed of taxation collections with respect to the level of real public debt, and the long-run target of real public debt.

The baseline model is first built upon the representative agent by assuming all decision makers (i.e., households and firms) share the same planning horizon. Then, I introduce heterogeneous agents to allow some fraction of the agents in the population to have short foresight while others have far foresight. For simplicity, I assume the distribution of planning horizon following an exponential distribution, in which a single parameter measures the (average) planning horizon of the whole population. Then, the aggregate behavior of the economy is “smooth” in terms of the (average) planning horizon. In this case, the characterization of the equilibrium with the endogenous evolution of real public debt can be summarized by a “hybrid” five-equation linear system. Thus, the case of heterogeneous agents can be easily compared with the standard New Keynesian three-equation system and allows us to derive closed-form analytical solutions for specific policy experiments. Methodologically, the paper develops a tractable method for the finite-forward-planning approach to analyze the dynamics of aggregate variables in the case of heterogeneous agents with an endogenous path of debt evolution.

This paper suggests that with finite forward planning, fiscal and monetary policy always jointly

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7In the appendix, I also discuss the effects of government expenditure with finite forward planning, and emphasize its interaction with monetary policy.

8The three-equation system for the standard New Keynesian model includes the IS curve, the Phillips curve, and the rule of monetary policy.
determine aggregate inflation and output, and Ricardian equivalence always breaks down. By con-
trast, the literature of the representative-agent model with rational expectations implies a limited
impact of fiscal policy in the long run, which dismisses the adoption of fiscal stimulus. In particu-
lar, under the parameterization of “Ricardian” fiscal policy as defined in Woodford (2003), that is,
when the nominal interest rate responds more than one to one to the inflation rate (“Taylor princi-
ple”) and lump-sum tax collections respond strongly enough to the government’s real public debt,
the standard New Keynesian literature suggests the output and inflation are purely determined by
monetary policy (and that Ricardian equivalence holds for fiscal policy). But with finite forward
planning, fiscal policy always matters even in this policy regime.

A unique equilibrium with finite forward planning always exists regardless of the parameter-
ization of monetary or fiscal policy rules. This uniqueness in equilibrium is a key merit of the
finite-planning-horizon model, which makes announcements of future policies relevant, but avoids
the debates about equilibrium selection that plague rational-expectations analyses. If the limiting
values of the equilibrium exist as the planning horizon approaches infinity, the finite-forward-
planning approach also provides an equilibrium-selection criterion for all possible solutions under
RE. It therefore adds to the discussion of equilibrium selection, such as “minimum-state-variable
criterion” or “E-stability criterion,” and clarifies an important issue for monetary economics and
the fiscal theory of price level (FTPL).

Two scenarios of combined monetary-fiscal regimes, depending on the degree of foresight, can
ensure long-run stability. In the language of Leeper (1991), they are “active” monetary policy (AM)
with “passive” fiscal policy (PF) or “passive” monetary policy (PM) with “active” fiscal policy
(AF). The fiscal-policy instrument (among the three) – the speed of tax collections with regard to
the government’s real public debt level – is what determines the long-run stability of equilibrium

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9 In Leeper (1991), the Ricardian fiscal policy is classified as passive fiscal policy (under the policy regime of active
monetary policy with passive fiscal policy).
10 See McCallum (1999) for “minimum-state-variable criterion,” and Evans and Honkapohja (2001) and McCallum
(2007) for the discussion of “E-stability.”
11 The definition of “active” or “passive” policy depends on whether it is forward-looking or backward-looking in
the equilibrium. Intuitively, “active” policy indicates the policy authority is free to set the policy rule depending on
past, current, or expected future variables, whereas “passive” policy indicates the authority is constrained by the active
authority’s decision in order to balance the government budget constraint.
together with monetary policy. The boundary condition for fiscal policy between the two scenarios changes little with respect to the length of the planning horizon, whereas the boundary condition of monetary policy relies heavily on the length of the planning horizon. Furthermore, as the length of the planning horizon becomes shorter, the policy space for long-run stability increases under the “AM/PF” policy regime, and decreases under the regime of the “PM/AF”.

If the government and the central bank do not have good knowledge of the actual planning horizon of the population, however, adopting a policy combination to ensure long-run stability under the regime of “AM/PF” pinned down with the hypothesis of an infinite planning horizon is more robust and safer. That is, in contrast to the recent studies proposing a “fiscally dominant” regime (e.g., Jarociński and Maćkowiak, 2017) in facing a higher probability of hitting the effective zero lower bound, the government might appreciate the policy combination of “AM/PF” to better ensure long-run stability from the concerns of limited foresight.12

In other combined monetary-fiscal policy regimes, the finite-planning-horizon model with exponential distribution of planning lengths indicates the summation across agents does not converge (in a given period). It suggests the hypothesis of exponential distribution is not appropriate in studying such parameterization. But in general, as long as the maximum planning horizon is finite, or the summation across heterogeneous agents converges, a unique equilibrium path always exists in any policy regime. This distinction is an important one with respect to the literature under RE in which the other monetary-fiscal regimes suggest multiple equilibria (by monetary policy and fiscal policy both being “passive”) or no bounded equilibrium solution (by the two policies both being “active”).

Given the policy specification that ensures long-run stability, what are the effects of temporary changes in monetary or fiscal policy as a source of short-run demand stimulus? I investigate the

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12I show analytically that, in the absence of a fiscal sector and with extremely slow learning in the value function, the Taylor principle is relaxed: that is, as the planning horizon becomes shorter, the requirement on the nominal interest rate in response to the change in inflation to ensure long-run stability is looser. But different from the literature with rational expectations or alternative approaches of modeling bounded rationality, such as incomplete information, cognitive discounting, or “Level-k” thinking (e.g., Angeletos and Lian, 2017; Gabaix, 2018; Farhi and Werning, 2019), when the Taylor principle is violated, the issue of multiple equilibria does not exist in the finite-forward-planning model. As long as the maximum planning horizon is finite, or the summation across heterogeneous agents converge (in any given period), a unique equilibrium path always exists.
effects of transfer policy, and emphasize its interaction with (conventional) monetary policy in both short-run effects and long-run consequences, as well as its interaction with unconventional monetary-policy “forward guidance.” In particular, I highlight how the degree of foresight affects the interactions between fiscal and monetary policy.

Intuitively, because monetary policy affects the economy through agents’ looking ahead, the effect of a monetary shock or a policy change decreases as decision makers become more short-sighted. By contrast, as the length of the planning horizon becomes shorter, a fiscal stimulus in the type of lump-sum transfer becomes more powerful. The intuition is that, people only incorporate the taxation in a finite future into today’s decision-making.

Before moving to the policy experiments, the key difference between short-run and long-run analysis needs to be clarified, that is, whether decision makers update their value function used as continuation values for the future beyond the planning horizon. In the short run, the value function that decision makers use is assumed to be a given one learned from the steady-state stationary equilibrium, where the steady-state value function is the fixed point of the general (constant-gain) learning process. It is valid and helpful for the study of short-term effects in which decision makers in the economy have stayed in the steady-state stationary equilibrium for a long time and do not have many experiences with those policies in the past. In the analysis of evaluating long-run consequences, however, incorporating the learning process in the value function is necessary and important, which in general dampens the stimulative effects of fiscal transfers found in the short run over time.

First, in terms of short-term effects, given conventional monetary policy specified by the Taylor rule, the three fiscal policy instruments, namely, a large one-time lump-sum transfer financed by public debt, or a slow speed of tax collections after a lump-sum transfer, or an increase in the long-run target of real public debt, can be stimulative for both output and inflation. The stimulative effects of fiscal stimulus on output increase exponentially as decision makers become more short-sighted. For instance, consider a one-time lump-sum transfer fully financed by real public debt and the public debt being kept unchanged thereafter. Quantitatively, with the size of the lump-sum
transfer being equal to one-quarter GDP, output (permanently) increases by 0.9%, if the (average) planning horizon is one-quarter as estimated in Gust, Herbst, and López-Salido (2019). In the absence of an update in the value function, the “fiscal-transfer multiplier” (defined as the discounted aggregate response of output with respect to the size of the initial lump-sum transfer) is large and equals to 0.94.

Putting monetary and fiscal policy together, I show that more accommodative monetary policy in general amplifies the stimulative effect of fiscal stimulus, but the impact of monetary policy accommodation on the fiscal effect is hump shaped with respect to decision makers’ foresight. It is the greatest when decision makers’ degree of foresight is in the intermediate range (average planning horizons on the order of 10 years), rather than shorter or longer. The intuition is that when the planning horizon is long, the equilibrium is nearly Ricardian-equivalent, and thus fiscal policy is of little effect in stimulating output and inflation. When the planning horizon is short, because monetary policy works through forward-looking behavior, it becomes ineffective and thus matters little for fiscal policy. Thus, how accommodative monetary policy is matters most for the effect of fiscal policy only when agents have an intermediate degree of foresight.

Nowadays, in a world with an equilibrium real interest rate that seems chronically low, discussion about unconventional monetary policy, such as forward guidance, has increased. As decision makers plan for shorter distances into the future, the stimulative effect of forward guidance deteriorates, which generates a demand for fiscal stimulus. More importantly, if forward guidance is accompanied by a simultaneous fiscal stimulus through lump-sum transfer, can it achieve anything that a simple summation of the two policies cannot? I analytically show a positive interaction between forward guidance and fiscal lump-sum transfer, and the aggregate effect of the two policies

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13Empirically, Gust, Herbst, and López-Salido (2019) apply the model in Woodford (2019), which does not include fiscal sector, to match with US aggregate data such as inflation and output by Bayesian estimation. Their estimation indicates the average length of the planning horizon is about one quarter, and less than 1% of the population plans for more than two years into the future. Their paper also suggests very slow learning in the decision maker’s value function, and show the approach of finite forward planning outperforms other behavioral macro models such as Angeletos and Lian (2017) and Gabaix (2018) in terms of matching with aggregate data.

14Under the assumption of no update in the value function, simply borrowing the calibrated parameters from the discounted Euler equation in McKay, Nakamura, and Steinsson (2016) implies the (average) planning horizon is around eight years. Then, the positive response of output to the one-time fiscal transfer (with the size of one-quarter GDP) is about 0.3% and the fiscal-transfer multiplier is near 0.31.
is larger than the simple summation of the two. Furthermore, the positive interaction is maximized
also when agents have an intermediate length of planning horizon.

Fiscal stimulus can be effective in the short run, whereas in the long-run, as decision makers
update their value function used to approximate for the future beyond their planning horizon, they
start to incorporate the effect of those fiscal policies from a more distant future as time passes.
Then, the short-term stimulative effect becomes transitory, and dampens over time. If a long-run
stationary equilibrium exists after the policy changes, the equilibrium in the long run will finally
converge back to the steady-state stationary equilibrium before the policy changes.\textsuperscript{15} Nevertheless,
as the speed of learning in the value function is slower, the effect of a fiscal stimulus is more
persistent.

A natural question emerges regarding how quantitatively important the fiscal transfer is after
incorporating the learning process in the value function. For illustration, still consider the one-time
lump-sum transfer fully financed by public debt with the monetary policy specified by the Taylor
rule. Borrowing the estimates of the (average) planning horizon and learning process from Gust,
Herbst, and López-Salido (2019) by matching with US aggregate data, the fiscal-transfer multiplier
is about 0.94 in the case of no update in the value function, whereas it becomes 0.27 by calibrating
to the US data with a (non-zero) constant-gain learning process.\textsuperscript{16}

\textbf{Related Literature.} This paper contributes to the increasing interest in introducing behavioral
elements into a macroeconomic model. Many of the papers in this fast-developing area provide an
explanation for the forward guidance puzzle in various forms of micro-foundation.\textsuperscript{17} Christiano,
Eichenbaum, and Trabandt (2019) give a brief survey on the new development of DSGE models
in this strand. More specifically, Angeletos and Lian (2017) relax the assumption of complete
information in the New Keynesian model, and argue the effect of forward guidance is attenuated

\textsuperscript{15}It converges to the steady-state stationary equilibrium with the same inflation and output as before the fiscal
stimulus, but could end up with a higher level of real public debt.

\textsuperscript{16}The estimation in Gust, Herbst, and López-Salido (2019) suggests a small gain in the learning process of value
function by matching with US data.

\textsuperscript{17}Instead of assuming bounded rationality, McKay, Nakamura, and Steinsson (2016) provide an explanation for
the forward guidance puzzle through incomplete markets and households’ borrowing constraints by the approach of a
Bewley-Aiyagari-Huggett structure.
without common knowledge. Farhi and Werning (2019) obtain a similar limited effect of forward guidance by replacing the hypothesis of rational expectations with “Level-k” thinking, an approach first proposed in García-Schmidt and Woodford (2019). Gabaix (2018) instead builds a behavioral New Keynesian model through cognitive discounting, and discusses the effect of fiscal stimulus by imposing an exogenous evolution of real public debt.\textsuperscript{18} Woodford (2019) first develops the framework of finite forward planning grounded on the New Keynesian model, abstracting from the fiscal sector, and provides a natural explanation for the forward guidance puzzle through limited foresight. Instead of limiting attention to the forward guidance puzzle, this paper adds to the discussion of bounded rationality in macroeconomics by focusing on the role of fiscal policy, and studies how monetary and fiscal policy determines inflation and output in both short-run effects and long-run consequences.

Notably, the aggregate behavior of the economy predicted by alternative micro-founded behavioral approaches is observationally similar to a special case in the finite-planning-horizon model by the assumption of no update in the value function and there being an exponential distribution of the planning horizon among the population. Naturally, when the distribution of the planning horizon changes, or a learning process is in the value function, the finite-forward-planning approach in this paper will have different implications with regard to alternative approaches, especially in the long-run dynamics.

This paper highlights the importance of bounded rationality in analyzing fiscal policy in an expanding literature of the fiscal theory of price level (FTPL). Most of the existing studies are built on the assumption of rational expectation. Early research includes Leeper (1991), Woodford (1996), Cochrane (2001), and seminal references on this topic. Cochrane (2019) and Canzoneri, Cumby, and Diba (2010) survey the existing development of FTPL. For recent works, Eusepi and Preston (2018) propose the scale and composition of the public debt mattering for inflation based on imperfect knowledge and learning. Hagedorn (2018) argues that allowing for heterogeneous agents and incomplete financial markets makes a big difference in contrast to a standard representative-agent

\textsuperscript{18}The online appendix of Gabaix (2019) has a brief discussion regarding a mean-reverting public debt formation.
model, and show that monetary policy and fiscal policy determine the price level together. Farmer and Zabczyk (2019) challenge the established views about what constitutes a good combination of fiscal and monetary policies by replacing the (infinite-lived) representative-agent model with the overlapping generations (OLG) model. This paper instead introduces bounded-rational agents into the New Keynesian model by replacing the hypothesis of rational expectations with finite forward planning, and shows that fiscal policy together with monetary policy determines inflation and output even in the canonical parameterization of Ricardian fiscal policy. In addition, this paper highlights the role of the degree of foresight in affecting the interaction of monetary and fiscal policy.

Whereas many works in studying the effect of fiscal policy adopt the approach of the OLG model (with rational expectations), the finite-planning-horizon model differs from the OLG model in two major aspects: (i) If the OLG model is calibrated seriously, it corresponds to the case of relatively long planning horizons, for example, sixty years; (ii) conceptually, in the finite-planning-horizon model, agents still care about the infinite future, whereas agents in the OLG model do not. Furthermore, the finite-planning-horizon model has different implications regarding long-run dynamics for a permanent increase in public debt. Given a permanent increase in government debt, agents in the finite-planning-horizon model will finally incorporate the effects of such policy changes, and the output and inflation will converge back to the initial steady-state level after a long enough time, whereas the OLG model predicts a permanent change in output and inflation.\(^\text{19}\)

Other recent studies in the literature of heterogeneous-agent New Keynesian model (HANK) allow for non-Ricardian fiscal policy through incomplete markets and borrowing constraints on consumers. Farhi and Werning (2016) provide a survey on the existing literature. This paper contributes to this literature by suggesting that, on top of assuming any physical constraint on agents in the economy, fiscal policy can be non-Ricardian just because of how decision makers form expectations about the future. If a physical constraint like a borrowing constraint is further introduced with bounded rationality, the effect of fiscal stimulus can be even stronger.

\(^\text{19}\)Ascari and Rankin (2013) show that, in the OLG model with staggered prices, if the monetary rule is kept unchanged, the long-run output and inflation will have a non-zero response to a permanent increase in public debt.
This paper adds to the literature discussing the policy interaction between monetary and fiscal policy by introducing bounded rationality, and emphasizes that the policy interaction is maximized when decision makers have an intermediate degree of foresight. Some of the works in this topic overlap with those in FTPL, including Benhabib, Schmitt-Grohé, and Martin Uribe (2001), Eusepi and Preston (2008), Leith and von Thadden (2008), Davig and Leeper (2011), Ascari and Rankin (2013), and so on.

This paper also highlights the role of fiscal stimulus and its interaction with monetary policy in the discussion of stimulative policies through short-term demand. Eggertsson and Woodford (2003, 2004) discuss the optimal monetary and fiscal policy in a liquidity trap, followed by seminal research works on this topic. A recent paper by Sims and Wu (2019) study the tools of unconventional monetary policies including quantitative easing (QE), forward guidance, and negative interest rate policy (NIRP) in a unified DSGE model. Bernanke, Kiley, and Roberts (2019) study the quantitative performance of monetary-policy strategies for a low-rate environment including various forms of the Taylor rule and price-level targeting through the Federal Reserve’s principal simulation model. Although many existing papers focus on the tools of monetary policy, Woodford and Xie (2019), in particular, explore the short-term effects for a variety of alternative monetary-fiscal policy options at the zero lower bound, including strict inflation targeting, debt-financed lump-sum transfer, government expenditure, temporary price-level targeting, and systematic price-level targeting, by assuming finite forward planning with a given value function learned from the steady-state stationary equilibrium. This paper deviates from that paper and contributes to the literature by introducing an endogenous path of debt evolution, and focuses on a broader class of fiscal policy instruments in the transfer type. More importantly, this paper not only studies the interaction of fiscal stimulus and monetary policy in the short run, but also discusses the long-run consequences of such policies by incorporating a learning process in the value function and characterizes the combined monetary-fiscal policy regimes that can ensure long-run stability.

The rest of the paper proceeds as follows: Section 1.2 introduces the basic model with finite planning horizon and policy specification; Section 1.3 incorporates the learning process in decision
makers’ value function, and characterizes the long-run stability (or determinacy) condition; Section 1.4 discusses the short-term effects of fiscal stimulus (i.e., lump-sum transfer policies), and their interaction with monetary policy under the parameterization that ensures long-run stability; in particular, Section 1.4.2 focuses on the gain from the interaction of lump-sum transfer and forward guidance; Section 1.5 evaluates the long-run consequences of those fiscal policies considered in Section 1.4; and Section 1.6 concludes the paper.

1.2 A New Keynesian DSGE Model with Finite Planning Horizon

This section lays out the New Keynesian DSGE model with finite forward planning built upon the approach developed in Woodford (2019).20 Households and firms make contingent plans for a finite distance into the future, and use a value function learned from past experiences to evaluate all possible terminal states at the ending period of the planning horizon. The central bank sets a monetary policy following the Taylor rule, and the fiscal authority (the government) specifies a fiscal policy in terms of lump-sum taxation. Intuitively, decision makers are “rational” within their planning horizon as in the standard New Keynesian model with rational expectations (e.g., Woodford, 2003; Galí, 2015), but instead of making an infinite-horizon contingent plan, they use a coarse value function from their past experiences to approximate the future beyond their planning horizon. In this aspect, decision makers are bounded rational and the model departs from the rational-expectations assumption.

In this section, I first build up the general framework with finite forward planning, and then characterize the equilibrium with log-linearization around the steady state equilibrium by assuming the value function used by decision makers is a given one learned from the steady-state stationary equilibrium. Then, Section 1.3 introduces the constant-gain learning process (an error-correction rule) of the value function over time, where the value function learned from the steady state is the fixed point of such a learning rule. For simplicity, I model the value function by considering a perturbation around the steady-state value function.

Woodford (2019) abstracts from fiscal policy (e.g., government debt or government expenditure) and focuses on the issues of monetary policy such as forward guidance puzzle and Neo-Fisher effect.

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20Woodford (2019) abstracts from fiscal policy (e.g., government debt or government expenditure) and focuses on the issues of monetary policy such as forward guidance puzzle and Neo-Fisher effect.
1.2.1 Optimal Finite-horizon Planning for Households

Instead of making an infinite-horizon state-contingent expenditure plan, an infinitely-lived household makes state-contingent plans over a fixed period $h$ by maximizing the expected utility within her planning horizon and approximating the future with a value function for continuation values. More specifically, the objective function for the representative household $i$ with planning horizon-$h$ in period $t$ is to choose a state-contingent expenditure plan $\{C_t^i\}$ for any date $t \leq \tau \leq t+h$ to maximize

$$E_	au^h [\sum_{\tau=t}^{\tau+h} \beta^{\tau-t} u(C_{\tau}^i) + \beta^{h+1} v(B_{t+h+1}^i; s_{t+h})]$$

subject to the budget constraint

$$B_{t+1}^i = (1 + i_t) [B_t^i / \Pi_t + Y_t - T_t - C_t]$$

where the parameter $\beta$ is the subjective discount factor, and the variable $B_t^i$ is the financial wealth, i.e., a one-period riskless nominal bond, carried into date $\tau$ by household $i$ deflated by the aggregate price index $P_{\tau-1}$, and $v(B_{t+h+1}^i; s_{t+h})$ is the value function that the household uses to evaluate the continuation value for each possible state $s_{t+h}$ at the ending period of planning horizon. By definition, $B_t^i$ is a real variable that is pre-determined at date $\tau - 1$. The variable $\Pi_t = P_{\tau}/P_{\tau-1}$ is gross inflation, $Y_t$ is the income of household $i$ at date $\tau$, and $T_t$ is the lump-sum tax imposed by the government. The variable $i_t$ is the interest rate of a one-period riskless nominal bond set by the central bank, and $C_t$ is the household’s consumption on the composite good, where $C_t^i = \int_0^1 (C_t^i(f))^{\alpha-1} \frac{d f}{\Pi t} \pi_t^\alpha$, and the price of the composite good is $P_t$.\footnote{The variable $C(f)$ is the consumption of the differentiated good $f$ indexed by $f \in [0, 1]$.} Operator $E_t^h [\cdot]$ refers to the expectation of a decision maker with planning horizon-$h$ in period $t$.

To focus on the household’s intertemporal decision regarding consumption and savings, I abstract endogenous labor supply from households’ decision-making. More specifically, the labor market contains an organization in which a large number of representatives bargain wage contracts with firms on behalf of households. When a given labor supply is agreed upon with a given wage,
each household is required to supply its share of the aggregate labor demanded by the representatives, and thus each household’s labor income is equal to its share of the total value $Y_t$ of composite consumption-goods production. That is, the income of household $i$ is out of her control.

The finite forward planning of household $i$ with horizon $h \geq 1$ yields the following first-order conditions for any date $t \leq \tau \leq t + h - 1$

$$u_c(C_t^i) = \beta E^i_h[(1 + i_\tau) / \Pi_{t+1} u_c(C_{t+1}^i | s_\tau)]$$

and for the ending period of planning horizon $\tau = t + h$ (or the case of $h = 0$),

$$u_c(C_{t+h}^i) = \beta (1 + i_{t+h}) v_B(B_{t+h+1}^i; s_{t+h})$$

The decision maker’s finite planning problem has two key ingredients: (i) expectation formation and (ii) the value function used for approximating continuation values at the end of planning horizon. If the household’s expectation $E^{h}_t[\cdot]$ is model-consistent and the value function $v(B_{t+h+1}^i; s_{t+h})$ is the accurate model-consistent value for the household’s continuation problem, the household’s expenditure problem regenerates the standard rational-expectations problem in which decision makers make the optimal infinite-horizon contingent plan.

In terms of expectation formation, instead of assuming model-consistent expectations, households with planning horizon-$h$ make a contingent plan for date $t$ to $t + h$. To incorporate the idea that households only plan for the finite $h$ periods, at each date $t + j$ within their planning horizon ($0 \leq j \leq h$), they are assumed to only plan forward for $h - j$ periods. Households also assume that at any date $t + j$ within their planning horizon, spending and pricing decisions made by other households and firms are made with planning horizon $h - j$. To clarify, in each period $t$, households choose a contingent plan for the following $h$ periods, but only implement the plan in the current period $t$. In the following period $t + 1$, they will re-optimize the contingent $h$-period plan, which is generally not the same plan as the one made in the previous period, and only implement the new plan in the next period $t + 1$. 

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In other words, by assumption, the expectation formation for agents with planning horizon-$h$ can be written into model-consistent expectation, that is,

$$E_t^h[Z_{\tau}\mid s_\tau] = E_t Z_{t+h-\tau}^{t+h-\tau}, \; t \leq \tau \leq t+h$$

and

$$E_t^h[Z_{\tau+1}\mid s_\tau] = E_t Z_{t+1}^{t+h-\tau}, \; t+1 \leq \tau \leq t+h$$

for any endogenous variable $Z_\tau$ at date $\tau$ and any future state $s_\tau$.

Therefore, in the forward-planning exercise, the household’s Euler equation with any planning horizon $h \geq 1$ can be translated into model-consistent expectations as

$$u_c(C_{t+h-\tau}) = \beta E_{\tau}[(1+i_{t+h-\tau})/\Pi_{\tau+1}u_c(C_{t+h-\tau-1})] \tag{1.1}$$

for any date $t \leq \tau \leq t+h-1$; for the ending period of forward planning $\tau = t+h$ (or the case of $h = 0$), it satisfies

$$u_c(C^0_{\tau}) = \beta (1+i_{\tau})v_B(B^0_{\tau+1};s_\tau) \tag{1.2}$$

The second key component in the finite-forward-planning problem is the value function used to approximate continuation values at the end of the planning horizon. Although households are sophisticated enough to make plans within their planning horizon, it is computationally too costly for them to correctly deduce the accurate (and complete) state-dependent value function as in the canonical dynamic programming problem. Instead, the value function is coarse in terms of state-dependence structure – the value function that households use is assumed to only depend on their real financial position, and households learn the value function by averaging past experiences.

For simplicity, in this section, I assume the value function that agents use to approximate for the future beyond their planning horizon is the one learned from the steady-state stationary equilibrium. This is the situation in which the economy has stayed in the steady-state stationary equilibrium for a long time, and thus households and firms have learned the correct value function.
for such environment. Then, in Section 1.3, I introduce the constant-gain learning process in the value function by considering a perturbation around the steady-state value function, where the steady-state value function is the fixed point of this learning process.

The value function learned from the steady-state stationary equilibrium can be derived as

\[ v(B) = (1 - \beta)^{-1}u(\bar{Y} - \bar{T} + (1 - \beta)B/\bar{Π}) \]

By log-linearizing the first-order condition (1.1), the finite-forward-planning exercise for households with horizon \( h \geq 1 \) at any date \( t \leq \tau \leq t + h - 1 \) satisfies

\[ c_{\tau}^{t+h-\tau} - g_\tau = E_\tau [c_{\tau+1}^{t+h-\tau-1} - g_{\tau+1}] - \sigma [\pi_{\tau}^{t+h-\tau} - E_\tau \pi_{\tau+1}^{t+h-\tau-1}] \]

and for the ending period of planning exercise \( \tau = t + h \) (or the case of \( h = 0 \)), the first-order condition (1.2) yields

\[ c_0^0 - g_\tau = -\sigma i_\tau^0 + (1 - \beta)b_{\tau+1}^0 \]

where the parameter \( \sigma = -u_c(\bar{C})/(u_{cc}(\bar{C})\bar{C}) > 0 \) is the intertemporal elasticity of substitution of households, the variable \( g_\tau \) is a demand or preference shock, and \( b_{\tau+1}^0 \) is the real financial position at the end of the planning exercise. All variables in lowercase refer to the percentage deviation from their steady-state values, namely, \( c_t = \log(C_t/\bar{C}), y_t = \log(Y_t/\bar{Y}), \pi_t = \log(\Pi_t/\bar{Π}) \), \( i_t = \log(1 + i_t/1 + \bar{i}) \). In particular, to allow a non-zero steady-state financial position, I define the deviation in real public debt as \( b_t = (B_t - \bar{B})/(\bar{Π}\bar{Y}) \).

1.2.2 Optimal Price-setting of Firms with Finite Forward Planning

The optimal pricing problem of firms with finite forward planning is similar to the standard New Keynesian model with sticky price in the style of Calvo (1983) except that, firms plan ahead.

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22The value function learned from the steady-state stationary equilibrium is derived by solving the Bellman equation; that is, \( v(B) = \max_C [u(C) + \beta v(B')] \) subject to \( B' = \beta^{-1} [B + (\bar{Y} - \bar{T} - \bar{C})\bar{Π}] \), where the stationary nominal interest rate \( \bar{i} = \beta^{-1}\bar{Π} - 1 \) has been imposed in the intertemporal budget constraint.
for only finite periods and use a value function to approximate future profits beyond their planning horizon. The expectation formation with finite forward planning is the same as in the previous section.

More specifically, a continuum of firms exists, and each firm produces a differentiated good indexed by $f \in [0, 1]$ sold in a monopolistically competitive market. As implied by the Dixit-Stiglitz (CES) preference, the demand for good $f$ is given by $Y_t(f) = Y_t(P_t^f / P_t)^{-\theta}$, where $Y_t$ is the aggregate demand for the composite good, $P_t^f$ is the price of good $f$, and $P_t$ is the price of the composite good. Each firm uses labor as the only input for producing good $f$ with a production function $Y_t(f) = A_t L_t(f)^{1/\theta}$, where $A_t$ represents the level of productivity and $L_t(f)$ is the labor hired by the firm.\(^{23}\)

Following Calvo (1983), each firm can adjust its price freely only with probability $1 - \alpha$ in any given period, regardless of the timing of the last adjustment. In other words, a measure of $1 - \alpha$ of producers in each period can reset their prices, whereas the rest $\alpha$ keep their prices unchanged. Then, the optimal pricing problem with finite planning horizon-$h$ for a firm producing good $f$ in period $t$ is given by

$$\max_{P_t^f} \hat{E}_t^f \left[ \sum_{t=1}^{t+h} (\alpha \beta)^{T-t} \Lambda_t H(P_t^f \tilde{\Pi}^{T-t} / P_\tau; Z_\tau) + (\alpha \beta)^{h+1} \tilde{v}(P_t^f \tilde{\Pi}^h / P_{t+h}; s_{t+h}) \right]$$

where $\tilde{v}(P_t^f \tilde{\Pi}^h / P_{t+h}; s_{t+h})$ is the firm’s estimate of the value of discounted real profits since date $t + h + 1$ onward (conditional on reaching state $s_{t+h}$ in at date $t + h$), $\Lambda_t = \int u_i(C_i^t)di$ is the average marginal utilities of households, $H(P_t^f \tilde{\Pi}^{T-t} / P_\tau; Z_\tau)$ is the real profits of firm $f$ at date $\tau$, $Z_\tau$ indicates the vector of real state variables that are out of the firm’s control but matter for firm profit (including $Y_\tau, \Lambda_\tau, A_\tau$, and exogenous disturbances), and $\hat{E}_t^f [\cdot]$ indicates the expectation used by firm $f$ in the planning exercise in period $t$. The definition of $\Lambda_t$ implies the assumption that shares of the firms are not traded and each household $i$ receives an equal share of firm profits.

Two more key parts remain left to fully establish the firm’s pricing problem – the value function used by firms to approximate future profits beyond the planning horizon and the labor cost of

\(^{23}\)Households own the firms and receive all the profits of the firms through dividends (with equal shares).
production. Similar to the assumption in the previous section, the value function used by firm $f$ is the one learned from the steady-state stationary equilibrium and is coarse in terms of state-dependence structure, that is, only depending on the firm’s relative price. More specifically, the steady-state value function of the firm is given by

$$\tilde{\nu}(P^f/P) = (1 - \alpha \beta)^{-1} \tilde{\Lambda}(P^f/P; \tilde{Z})$$

where the variable $\tilde{\Lambda} = u_c(\tilde{C})$ is the constant value of $\Lambda_t$ in the perfect-foresight steady state. In Section 1.3, a constant-gain learning process to update this value function is introduced for more general analysis.

The labor wage is pinned down by the idea of abstracting the decision-making of labor supply from any individual household but still maintaining the aggregate labor-supply curve as in the canonical New Keynesian model (e.g., Woodfood, 2003; Galí, 2015). As mentioned in the household’s problem, there are a lot of representatives within the organization of the labor market who bargain wages on behalf of households. For any given wage, a representative determines the number of working hours provided by households, and households must supply that number of hours and receive the same wage. Since a large number of such representatives exist, no one has any monopoly power. Therefore, the representatives will choose the number of hours $L_t$ to maximize the average utility of the households in the economy, which yields

$$\nu_L(L_t) = \Lambda_t W_t$$

Similar to the derivations in the standard New Keynesian model with Calvo-pricing rigidity, the Phillips curves implied by firms (and households) with planning horizon $h \geq 1$ at any date $t \leq \tau \leq t + h - 1$ within the forward-planning exercise are given by

$$\pi^h_{\tau} = \kappa(y^h_{\tau} - y^*_\tau) + \beta E_{\tau} \pi^{h-1}_{\tau+1}$$
and for the ending period of the planning exercise $\tau = t + h$ (or the case of $h = 0$),

$$\pi^0_\tau = \kappa (y^0_\tau - y^*_\tau)$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \xi$, $\xi = \frac{\phi - 1 + \eta \phi}{1 + (\phi - 1) \bar{\theta}}$, and $\eta = \bar{L}w_{LL}/w_L > 0$ is the Frisch elasticity of labor supply with $w = v_L(L)L$. The variable $y^*_t$ refers to exogenous supply shocks such as productivity shocks.

1.2.3 Monetary and Fiscal Policy

In this section, I specify monetary and fiscal policy. The central bank sets monetary policy reaction function following the Taylor rule, that is, $\hat{i}_t = i^*_t + \phi_{\pi, t} \pi_t$. The coefficient $\phi_{\pi, t}$ can be time-variant to incorporate the policy experiment of forward guidance, whereas in other policy experiments, for simplicity, $\phi_{\pi, t}$ is assumed to be time-invariant. Then, in a forward-planning exercise of decision makers with horizon $h \geq 0$ at time $t$, the decision makers’ expectation regarding the nominal interest rate is given by

$$i^{t+h-\tau}_t = i^*_t + \phi_{\pi, \tau} \pi^t_{\tau}$$

for any date $t \leq \tau \leq t + h$ within the planning exercise. In the case of time-varying coefficient, the coefficient $\phi_{\pi, \tau}$ is the one from the policy announcement in period $t$.

The fiscal policy is a (net) lump-sum taxation scheme; that is, the rule of tax collections $T_t$ in period $t$ is given by

$$T_t = (1 - \Gamma) \bar{T} + \Gamma \left( \frac{B_t}{\Pi_t} - \frac{B}{1 + i_t} \right)$$

24The function $w(\cdot)$ is the period-disutility of labor for households.

25The derivation of these expressions and a more detailed description of the labor market can be found in Woodford (2019).

26The nominal interest rate is assumed to only respond to inflation and not output gap, in order to have a single parameter $\phi_{\pi, t}$ measuring how accommodative the monetary policy is. This assumption is also the standard case studied in the FTPL. More generally, the policy reaction function of nominal interest rate can be extended to respond to both inflation and output gap and the major conclusions of the paper do not change.
where the variable $\bar{T} = \frac{B}{\Pi} - \frac{\bar{b}}{i_{\Pi}}$ is the lump-sum tax collection associated with the steady-state equilibrium, and $\bar{B}$ is the steady-state level of real public debt. The parameter $\Gamma$ captures how lump-sum tax collections respond to the level of real public debt as in the FTPL (e.g., Leeper, 1991; Woodford, 2003). In Section 1.4, policy experiments such as a one-time lump-sum transfer or a change in the long-run debt target are discussed in detail for the analysis of counter-cyclical fiscal stimulus.

With lump-sum taxation (and no government expenditure), the intertemporal budget constraint of the government is given by

$$B_{t+1} = (1 + i_t)[B_t/\Pi_t - T_t]$$

where $T_t$ is the net tax collections by the government in period $t$.\(^{27}\)

By log-linearization and substituting the path of tax collections into the government budget constraint, and also noting the steady-date nominal interest rate satisfies $\beta^{-1}\Pi = \bar{i} + 1$, the evolution of real public debt is given by

$$b_{t+1} = \beta^{-1}(1 - \Gamma)b_t - \beta^{-1}(1 - \Gamma)s_b\pi_t + (1 - \Gamma)s_b\bar{i}_t$$

(1.3)

where the variable $b_t = \frac{B_t - \bar{B}}{\Pi_t}$ is the deviation of real public debt from its steady-state value and $s_b = \frac{\bar{B}}{\Pi_t}$ is the relative size of steady-state real public debt relative to output.

### 1.2.4 Equilibrium Characterization with Common Planning Horizon

Now, I characterize the full equilibrium by assuming households and firms have the same planning horizon-$h$. The households are further assumed to start with the same initial financial position, and therefore they make identical decisions in each period. In Section 1.2.5, the assumption of common planning horizon is extended to heterogeneous planning horizons across decision makers in order to allow some fraction of the whole population to have short foresight and some to have long foresight.

---

\(^{27}\)The government flow budget constraints can also be written in nominal terms as $B_{t+1}P_t = (1 + i_t)[B_tP_{t-1} - T_tP_t]$. 

---

21
Goods market clearing yields $y_t^h = c_t^h$. Then, given the state variable of pre-determined real asset position \( \{b_t\} \) and exogenous disturbances in period \( t \), the equilibrium output, inflation, and nominal interest rate \( \{y_t^h, \pi_t^h, i_t^h\} \) are pinned down by the solution of the forward-planning problem with horizon-\( h \), and then the asset position \( \{b_{t+1}\} \) in the period \( t + 1 \) is given by the evolution path of real public debt.

The planning problem in period \( C \) is characterized by the system of equations as follows: For any date \( t \leq \tau \leq t + h - 1 \) in the planning exercise with \( h \geq 1 \),

\[
e^t_{\tau} + h - \tau - \sigma \left[ E_{\tau}[c^t_{\tau+1} \pi_{\tau+1}^t] - \sigma E_{\tau} \pi_{\tau+1}^{t+h-\tau-1}\right] = \tau
\]  \hspace{1cm} (1.4)

and for the ending period of planning exercise \( \tau = t + h \) (or the case of \( h = 0 \)),

\[
e^0_{t+h} - \sigma \left[ E_{\tau}[c^t_{\tau+1} \pi_{\tau+1}^t] - \sigma E_{\tau} \pi_{\tau+1}^{t+h-\tau-1}\right] = \tau
\]  \hspace{1cm} (1.5)

with the interest rate rule and the evolution of real public debt

\[
i^t_{\tau} + h - \tau = i^*_\tau + \phi_{\pi,\tau} \pi^t_{\pi^t} \pi_{\tau}^{t+h-\tau}
\]  \hspace{1cm} (1.6)

\[
b^t_{\tau+1} = \beta^{-1} (1 - \Gamma) b^t_{\tau+1} + \beta^{-1} (1 - \Gamma) s b \pi^t_{\pi^t} \pi_{\tau}^{t+h-\tau} + (1 - \Gamma) s b i^t_{\tau} \pi_{\tau}^{t+h-\tau}
\]  \hspace{1cm} (1.7)

where the variable \( b^h_{t+1} = b_t \) is the initial asset position in the planning exercise.

The system of equations (1.4)-(1.9) can be solved for the variables \( \{y_t^h, \pi_t^h, i_t^h\} \) with a unique solution under any parameterization of monetary-fiscal policy rules. Because all the decision mak-
ers have the same planning horizon, it follows that the equilibrium inflation and output are \( \pi_t = \pi^h_t \) and \( y_t = y^h_t \) with nominal interest rate \( i_t = i^*_t + \phi_{\pi,t}\pi_t \). The real public debt in period \( t + 1 \) is then given by the debt-evolution equation (1.9).

The characterization of the equilibrium implies a merit of the finite-forward-planning framework – the equilibrium of finite forward planning is always uniquely determined. Therefore, it allows us to study the situation in which the standard New Keynesian model with rational expectations indicates multiple equilibria or no bounded equilibria, which is one of the key issues in monetary economics and in the study of fiscal policy. Furthermore, if the limiting value of the equilibrium exists as the planning horizon approaches infinity \( h \to \infty \), it provides an equilibrium-selection criterion for the model with rational expectations if we consider the appropriate equilibrium under RE is the one pinned down by the limiting case of finite forward planning.

1.2.5 Heterogeneous Agents in Terms of Planning Horizon

The assumption of common planning horizon can be relaxed to accommodate heterogeneous agents. On the one hand, conceptually, in contrast to the homogeneous case with a sharp truncation of the planning horizon for the whole population, the case of heterogeneous agents allows for some fraction of the population to have short foresight and some to have long foresight. On the other hand, technically, with heterogeneous agents, the equilibrium of aggregate variables can be characterized by a similar linear-equation system as in the standard New Keynesian model, which allows easy comparison with the literature and sheds light on the role of bounded rationality in affecting the effects of fiscal and monetary policy. It also yields closed-form analytical solutions.

Suppose in each period that a \( \omega_h \) fraction of households and a \( \tilde{\omega}_h \) fraction of firms have planning horizon \( h \), where \( \Sigma_h \omega_h = 1 \) and \( \Sigma_h \tilde{\omega}_h = 1 \), respectively. Assume households and firms with planning horizon-\( h \) make decisions by assuming all other decision makers share the same planning horizon. Then, the system of equations (1.4)-(1.9) in the previous section still apply for any agent with horizon-\( h \). To simplify the analysis in the heterogeneous case, further assume that, at the beginning of each period, each household has the possibility of being a horizon-\( h \) agent with the
probability of distribution the same as the population distribution (i.i.d. in each period). It follows that, given the aggregate (average) real public debt $b_t$ known in period $t$, the group of decision makers with any horizon-$h$ as a whole start their forward planning with the same level of initial asset $b_t$. Such an assumption allows us to abstract from tracking the heterogeneous asset accumulation across agents and focus on the behaviors of aggregate variables.\textsuperscript{28} The aggregate output, inflation, and real asset position can then be defined as

$$y_t = \Sigma_h \omega_h y^h_t, \quad \pi_t = \Sigma_h \bar{\omega}_h \pi^h_t, \quad b_t = \Sigma_h \omega_h b^h_t$$

with the implicit assumption that the summations converge.

For simplicity, assume $\omega_h = \bar{\omega}_h = (1 - \rho) \rho^h$ fraction of households and firms has planning horizon $h$ with $0 < \rho < 1$. Then, the average planning horizon in the population is given by $E(h) = \rho / (1 - \rho)$, and thus this assumption allows us to have a single parameter $\rho$ to measure how forward-looking the economy is on average. The evolution of aggregate endogenous variables $\{y_t, \pi_t, \hat{i}_t, b_{t+1}\}$ can then be characterized by averaging the system of equations (1.4)-(1.9) across the population, which yields the IS equation, Phillips curve, rule of nominal interest rate, and path of debt evolution given by

$$y_t - g_t = \rho E_t (y_{t+1} - g_{t+1}) - \sigma (\hat{i}_t - \rho E_t \pi_{t+1}) + (1 - \rho)(1 - \beta)b^*_t$$

$$\pi_t = \kappa (y_t - y^*_t) + \beta \rho E_t \pi_{t+1}$$

$$\hat{i}_t = i^*_t + \phi_{\pi,i} \pi_t$$

$$b_{t+1} = \beta^{-1}(1 - \Gamma)b_t - \beta^{-1}(1 - \Gamma)\sigma \pi_t + (1 - \Gamma)\sigma b \hat{i}_t$$

\textsuperscript{28}The random possibility of being horizon-$h$ is not being incorporated into the decision maker’s forward-planning exercise, where the forward planning exercise conditional on being horizon-$h$ is the same as described in Section 1.2.1.
where \( b'_t = \psi_{b,t} b_t + \psi_{g,t} g_t + \psi_{y,t} y_t^* + \psi_{i,t} i_t^* \) with

\[
\psi_{b,t} = \beta^{-1} (1 - \Gamma) / \left[ 1 - (\phi_{\pi,t} - \beta^{-1}) s_b (1 - \Gamma) \right] \frac{(1 - \beta) k}{1 + \sigma \phi_{\pi,t} k}
\]

The time variation in the coefficients \( \{\psi_{b,t}, \psi_{g,t}, \psi_{y,t}, \psi_{i,t}\} \) only comes from the variation in \( \phi_{\pi,t} \), and if the parameter \( \phi_{\pi,t} \) in the monetary policy reaction function is time-independent, these coefficients become constant as well over time. Details of the derivation and expressions for these coefficients can be found in Appendix A.1. In the limiting case of the (average) planning horizon approaching infinity \( \rho \to 1 \), the system of equations becomes the standard New Keynesian model with fiscal policy as discussed in Woodford (2003) as long as the summation across heterogeneous agents converges.

Compared with the standard New Keynesian model, the system of equations for the finite planning horizon differs in two aspects: (i) a discount factor \( \rho \) before each expectation operator captures how far ahead decision makers on average plan for the future; (ii) the IS equation has one extra term \( (1 - \rho) (1 - \beta) b'_t \), which is composed of the current level of real public debt and exogenous disturbances. Therefore, in general, fiscal policy joint with monetary policy determines inflation and output, and Ricardian equivalence always breaks down unless the real public debt is kept constant at all times (that is, \( b_t = 0 \) for any period \( t \); defined as inactive fiscal policy). In other words, no “Ricardian fiscal policy” exists as long as the fiscal policy is active.

More specifically, the first difference as shown by the discount factor \( \rho \) before each expectation operator indicates limited effects of monetary policy. Intuitively, because monetary policy affects the economy through decision makers’ forward-looking behavior, as agents have less foresight (i.e., smaller value of \( \rho \)), the effects of temporary monetary policy changes (or monetary shocks) on output and inflation become weaker. By contrast, the second difference as captured by the extra term, \( (1 - \rho) (1 - \beta) \psi_{b,t} b_t \), indicates strong effects of fiscal transfer policy. Because the parameter \( \psi_{b,t} \) is independent of \( \rho \), as the degree of (average) foresight \( \rho \) becomes smaller, the coefficient \( (1 - \rho) (1 - \beta) \psi_{b,t} \) before \( b_t \) in the IS equation is larger. In other words, as agents are less forward-
looking, a lump-sum fiscal transfer (or taxation) has larger effects on output and inflation.

Importantly, under the assumption of the steady-state value function with no update process and an exponential distribution in the planning horizon, the above system of equations is robust to alternative approaches of modeling bounded-rationality such as incomplete information, cognitive discounting, or “Level-k” thinking (e.g., Angeletos and Lian, 2017; Gabaix, 2018; Farhi and Werning, 2019).29 In other words, the dynamics of aggregate variables predicted by alternative approaches are observationally similar to such a special case of the finite-planning-horizon model. With a different distribution of the planning horizon, or with a learning process in the value function used to approximate the future, the finite-horizon model has different implications with regard to those alternative approaches. For instance, as shown in Section 1.5, the model with finite planning horizon has different implications of the equilibrium in terms of long-run dynamics as long as a non-zero gain is present in the learning process of the value function.30

1.3 Long-run Stability of Monetary-fiscal Policy Interaction

Thus far, the analysis has been based on the assumption that the value function used to approximate the future beyond the planning horizon is learned from the steady-state stationary equilibrium, and that no learning occurs for households and firms to update their value function. Before discussing specific policy experiments in Section 1.4, first understanding under what conditions the specification of monetary and fiscal policy can ensure long-run stability, or determinacy, is essential. Then, incorporating the learning process in decision makers’ value function becomes necessary.

Thus, in this section, I introduce the learning process in the value function specified by a rule of constant gain (i.e., a type of error-correction rule), and discuss the long-run stability (determinacy)

29 Angeletos and Lian (2017) and Farhi and Werning (2019) focus on the effects of monetary policy. Based on cognitive discounting, Gabaix (2019) derives a similar modification of the equations for the standard New Keynesian model with an exogenous evolution of real public debt. The online appendix of Gabaix (2019) has a brief discussion regarding a mean-reverting public debt formation.

30 As another example, with a non-zero gain in the learning process of the value function, Woodford (2019) points out the “neo-Fisherian” fallacy implied by the finite-planning-horizon model, whereas Gabaix (2019) concludes that the economy is “neo-Fisherian” in the long-run but “Keynesian” in the short-run.
condition of monetary-fiscal policy regime. I first model the learning process by assuming all the agents sharing the same planning horizon, and characterize the equilibrium under such an assumption. Then, I introduce heterogeneous agents for analytical purposes. Methodologically, I develop a tractable method for the finite-forward-planning approach to analyze the dynamics of aggregate variables in the case of heterogeneous agents with an endogenous path of debt evolution.

1.3.1 Learning Process in the Value Function

The learning process is assumed to be a rule of constant gain (or an error-correction rule). Assume all the decision makers share the same planning horizon, and households use \( v_t(B) \) as the value function in their forward-planning exercise and \( v_t^{est}(B) \) as the estimated value function from period-\( t \) decisions. Similarly, I define \( \tilde{v}_t(r) \) and \( \tilde{v}_t^{est}(r) \) for firms, where the variable \( r \) defined as \( r = P^f/P \) is the relative price index of the firm’s product to aggregate price index. Then, the update of beliefs for the value function are assumed to follow

\[
v_{t+1}(B) = \gamma v_t^{est}(B) + (1 - \gamma) v_t(B)
\]

\[
\tilde{v}_{t+1}(r) = \tilde{\gamma}\tilde{v}_t^{est}(r) + (1 - \tilde{\gamma}) \tilde{v}_t(r)
\]

where the parameter \( \gamma \) and \( \tilde{\gamma} \) measures how fast the households and firms update their beliefs.

Now, I consider a local approximation to the dynamics implied in the above system through a perturbation around the steady-state value function, where the steady-state value function denoted as \( \{v^*, \tilde{v}^*\} \) is the fixed point of the above learning process in the situation of no exogenous disturbances and no change in fiscal and monetary policies.

First, consider a local approximation for \( v_t \). Because the household’s optimal finite-horizon plan involves the derivative \( v'(B) \) of the value function, it is parameterized as

\[
\log(v'_t(B)/v^*(\tilde{B})) = -\sigma^{-1} [v_t + \chi_t b]
\]

---

31 The technique of modeling the learning process in the value function is consistent with that in Woodford (2019).
32 The variable of \( P^f \) is the price of the firm’s product as if it can freely set prices.
Denote $C_i^j(B)$ to be the optimal expenditure plan of households under the counter-factual assumption $B_i^j = B$, and then the derivative of the estimated value function will satisfy

$$v_i^{est'}(B) = \tilde{E}_i^j[\mu_C(C_i^j(B))/\Pi_i]$$

which implies

$$log(v_i^{est'}(B)/v^*(\tilde{B})) = -\sigma^{-1}(c_i^h(b) - g_t) - \pi_t^h$$

The log-linear approximation to the optimal household plan $c_i^h(b) = c_i^h(0) + (c_i^h)'b$ allows us to express the right-hand side as a function of $b$. Note the right-hand side can be approximate as $-\sigma^{-1}(v_i^{est} + \chi_i^{est}b)$, and by equating coefficients, it gives

$$v_i^{est} = y_i^h - g_t - \chi_i^{est}b_t + \sigma \pi_t^h$$

$$\chi_i^{est} = (c_i^h)' \equiv g_h(\chi_t)$$

Therefore, the belief-updating system can be written as

$$v_{t+1} + \chi_{t+1}b = \gamma[v_i^{est} + \chi_i^{est}b] + (1 - \gamma)[v_t + \chi_t b] \quad (1.10)$$

From the households’ optimal expenditure plan, it follows that

$$g_h(\chi) = \frac{\chi}{\beta^{h+1} + \frac{1-\beta^{h+1}}{1-\beta}\chi}$$

By the equation (1.10), the variable $\chi_t$ is converging to $1 - \beta$ for any initial value $\chi_0 > 0$. Without losing generality, I then assume $\chi_t = 1 - \beta$; that is, the convergence has already occurred.

Thus, the learning dynamics of households reduces to

$$v_{t+1} = \gamma v_i^{est} + (1 - \gamma)v_t \quad (1.11)$$
where

$$v_t^{est} = y_t^h - g_t - (1 - \beta)b_t + \sigma \pi_t^h \quad (1.12)$$

Similarly, I define $\tilde{v}_t$ and $\tilde{x}_t$ for the value function of firms, and it can be shown that $\tilde{x}_t$ is converging to 1 from any initial value $\tilde{x}_0 > 0$. Without losing generality, $\tilde{x}_t$ is thus set to 1. Then, the learning dynamics of firms can be captured by one endogenous variable $\tilde{v}_t$, where the dynamics of $\tilde{v}_t$ is given by

$$\tilde{v}_{t+1} = \tilde{\gamma}v_t^{est} + (1 - \tilde{\gamma})\tilde{v}_t \quad (1.13)$$

where

$$v_t^{est} = (1 - \alpha)^{-1} \pi_t^h \quad (1.14)$$

1.3.2 Equilibrium Characterization with Common Planning Horizon

Now, I characterize the complete dynamics of (aggregate) endogenous variables $\{y_t, \pi_t, \ell_t\}$ and $\{b_t, v_t, \tilde{v}_t\}$ for horizon-$h$ decision makers with learning in the value function. In this section, I keep the assumption that all the decision makers have the same planning horizon. Monetary policy and fiscal policy are specified as in Section 1.2.3.

For horizon-$h$ decision makers in period $t$ with known state variables $\{b_t, v_t, \tilde{v}_t\}$, at any date $t \leq \tau \leq t + h$ within the planning exercise, the predicted equilibrium evolution of the endogenous variables $\{y_{\tau}^i, \pi_{\tau}^i, \ell_{\tau}^i, b_{\tau+1}^j\}_{j=t+h-\tau}$ can be written in two components

$$y_{\tau}^i = \tilde{y}_{\tau}^i + \tilde{y}_{\tau}^i, \quad \pi_{\tau}^i = \tilde{\pi}_{\tau}^i + \tilde{\pi}_{\tau}^i, \quad \ell_{\tau}^i = \tilde{\ell}_{\tau}^i + \tilde{\ell}_{\tau}^i, \quad b_{\tau+1}^j = \tilde{b}_{\tau+1}^j + \tilde{b}_{\tau+1}^j$$

where the tilde component indicates indicates the predicted value for variables under no learning in the value function; that is, $v_t = \tilde{v}_t = 0$, in all periods of the planning exercise, but taking all exogenous shocks and policy changes into consideration with beginning asset position $\tilde{b}_{t+1}^{h+1}$ (to be clarified later) in the planning. To be clear, although no learning $v_t = \tilde{v}_t = 0$ is assumed in the planning

---

33 Details of the derivation for firms are similar to those in Woodford (2018).

34 Note that the endogenous variables $\{b_{t+1}, v_{t+1}, \tilde{v}_{t+1}\}$ are pre-determined in period $t$. 
exercise for the tilde components, the beginning asset position \( \tilde{b}^{h+1} \) can be a function of \( \nu_t \) and \( \tilde{\nu}_t \). The bar component represents the discrepancy from this prediction as a result of variation in \( \nu_t \) and \( \tilde{\nu}_t \) throughout the forward planning with beginning asset position \( \tilde{b}^{h+1} \). Note that \( \{\tilde{b}^{h+1}, \tilde{b}^{h+1}\} \) satisfies the condition \( b_t = \tilde{b}^{h+1} + \tilde{b}^{h+1} \), and as long as this condition is satisfied, how specifically \( \tilde{b}^{h+1} \) and \( \tilde{b}^{h+1} \) are defined does not affect the calculation of aggregate endogenous variables \( \{y_t^h, \pi_t^h, i_t^h\} \) in period \( t \). Because all the decision makers share the same planning horizon, it follows that \( y_t = y_t^h, \pi_t = \pi_t^h \), and \( i_t = i_t^h \). The pre-determined endogenous variables \( \{b_{t+1}, \nu_{t+1}, \tilde{\nu}_{t+1}\} \) are then given by the relations (1.3), (1.11)-(1.12), and (1.13)-(1.14).

Intuitively, the bar component captures the trend of aggregate endogenous variables \( \{y_t, \pi_t\} \) resulted from the learning in the value function, whereas the tilde component captures the deviation of aggregate variables from this trend. For the deviation component, it follows from Section 1.2.4 at any date \( t \leq \tau \leq t + h - 1 \) within the planning horizon and \( h \geq 1 \),

\[
\begin{align*}
\dot{y}_t^j - g_\tau &= E_\tau [\dot{y}_{\tau+1}^j - g_{\tau+1}] - \sigma [\dot{i}_t^j - E_\tau \dot{\pi}_{\tau+1}^{j-1}] \\
\dot{\pi}_t^j &= \kappa [\dot{y}_t^j - y_\tau^*] + \beta E_\tau \dot{\pi}_{\tau+1}^{j-1}
\end{align*}
\]

where \( j = t + h - \tau \). For the ending period of forward planning \( \tau = t + h \) (or the case of \( h = 0 \)), it satisfies

\[
\begin{align*}
\dot{y}_{t+h}^0 - g_{t+h} &= -\sigma \dot{i}_{t+h}^0 + (1 - \beta) \tilde{b}_{t+h+1}^0 \\
\dot{\pi}_{t+h}^0 &= \kappa [\dot{y}_{t+h}^0 - y_{t+h}^*]
\end{align*}
\]

The rule of nominal interest rate and the evolution of real public debt in the planning exercise for the deviation component are given by

\[
\dot{i}_t^j = i_\tau^* + \phi_\pi \dot{\pi}_t^j
\]
\[
\tilde{b}_{\tau+1}^l = \beta^{-1}(1 - \Gamma) \tilde{b}_{\tau}^{l+1} - \beta^{-1}(1 - \Gamma) s_b \tilde{\pi}^l_{\tau} + (1 - \Gamma) s_b \tilde{t}^l_{\tau}
\]  
\eqref{1.20}

with the initial value \(\tilde{b}_t^{h+1}\) in period \(t\).

For the trend component, that is, fluctuations solely coming from learning in the value function, it follows for any date \(t \leq \tau \leq t + h - 1\) (with \(j = t + h - \tau\)) and \(h \geq 1\),

\[
\tilde{y}^j_{\tau} = \tilde{y}^{j-1}_{\tau+1} - \sigma [\tilde{t}^j_{\tau} - \tilde{\pi}^{j-1}_{\tau+1}]
\]  
\eqref{1.21}

\[
\tilde{\pi}^l_{\tau} = \kappa \tilde{y}^j_{\tau} + \beta \tilde{\pi}^{j-1}_{\tau+1}
\]  
\eqref{1.22}

and for the ending period of forward planning \(\tau = t + h\) (or the case of \(h = 0\)), the first-order intertemporal relation of households specified by the expression \(\eqref{1.2}\) (and a similar condition for firms) yields

\[
\tilde{y}^0_{t+h} = -\sigma \tilde{t}^0_{t+h} + (1 - \beta) \tilde{b}^0_{t+h+1} + \nu_t
\]  
\eqref{1.23}

\[
\tilde{\pi}^0_{t+h} = \kappa \tilde{y}^0_{t+h} + (1 - \alpha) \beta \tilde{v}_t
\]  
\eqref{1.24}

The rule of nominal interest rate, and the evolution of real public debt with the initial value \(\tilde{b}_t^{h+1}\), for the trend component are given by

\[
\tilde{t}^l_{\tau} = \phi_{\pi} \tilde{\pi}^l_{\tau}
\]  
\eqref{1.25}

\[
\tilde{b}_{\tau+1}^l = \beta^{-1}(1 - \Gamma) \tilde{b}_{\tau}^{l+1} - \beta^{-1}(1 - \Gamma) s_b \tilde{\pi}^l_{\tau} + (1 - \Gamma) s_b \tilde{t}^l_{\tau}
\]  
\eqref{1.26}

Because all the agents have the same planning horizon, it follows that \(y_t = y_t^h, \pi_t = \pi_t^h\), and

---

\[35\]In this section, the parameter \(\phi_{\pi,t}\) in the Taylor rule is assumed to be time-invariant, and thus I simply denote it as \(\phi_{\pi}\).

\[36\]Details of a derivation for the condition of firms can be found in Woodford (2019).
\( \hat{t}_t = \hat{t}_t^h \). Therefore, the complete system of endogenous variables \( \{y_t, \pi_t, \hat{\pi}_t, b_{t+1}, \nu_{t+1}, \nu_{t+1}\} \) with state variable \( \{b_t, \nu_t, \nu_t\} \) in period-\( t \) is characterized by the following: (1) the six equations (1.15)-(1.20) capturing the forward-looking system of the deviation component, which can be solved recursively to obtain \( \{\hat{\nu}_t^h, \pi_t^h, \hat{\nu}_t^h\} \); (2) the six equations (1.21)-(1.26) capturing the static system of the trend component, which can be solved to obtain \( \{\hat{\nu}_t^h, \pi_t^h, \hat{\nu}_t^h\} \), and thus together with the deviation component, pinning down aggregate variables \( \{y_t, \pi_t, \hat{\nu}_t\} \); (3) \( \nu_t^{est} \) and \( \hat{\nu}_t^{est} \) can be computed through the equations (1.12) and (1.14) of Section 1.3.1 as a function of \( \{y_t^h, \pi_t^h, b_t\} \) with exogenous disturbances; (4) the evolution of \( \nu_{t+1} \) and \( \nu_{t+1} \) are specified in the equations (1.11) and (1.13); and (5) the evolution of \( b_{t+1} \) is described by the evolution of real public debt (1.3).

Notably, as long as \( b_t = \bar{b}_t^{h+1} + \bar{b}_t^{h+1} \), the choice of the beginning asset position \( \{\bar{b}_t^{h+1}, \bar{b}_t^{h+1}\} \) for calculating the deviation and trend components in the planning exercise does not affect the characterization of the aggregate equilibrium process \( \{y_t, \pi_t, \hat{\pi}_t, b_{t+1}, \nu_{t+1}, \nu_{t+1}\} \). This crucial feature allows us to tractably characterize the behaviors of aggregate variables when I introduce heterogeneous agents into the model in the next section. Thus far, without loss of generality, we can simply assume \( \bar{b}_t^{h+1} = 0 \) and \( \bar{b}_t^{h+1} = b_t \).

The characterization of the equilibrium under the assumption that all decision makers share the same planning horizon suggests that with finite forward planning, the aggregate equilibrium is always uniquely determined after incorporating the learning process in the value function.

1.3.3 Heterogeneous Agent with Learning in the Value Function

To be comparable with the standard New Keynesian model with rational expectations, I introduce heterogeneous agents as in Section 1.2.5 into the finite-planning-horizon model, while incorporating the learning process in decision makers’ value function. All the discussions regarding long-run stability (or determinacy) in the following section are based on the case of heterogeneous agents for the ease of closed-form analysis.

Suppose in each period that \( \omega_h = \hat{\omega}_h = (1 - \rho)\rho^h \) fraction of households and firms have planning horizon \( h \), respectively. Similar to the assumption in Section 1.2.5, at the beginning of each
period, each household has the possibility of being a horizon-\( h \) agent with the same probability of exponential distribution as the population distribution (i.i.d. in each period). Then, given aggregate (average) real public debt \( b_t \) known in period \( t \), the group of decision makers with any horizon-\( h \) as a whole start their forward-planning exercise with initial asset \( b_t \).\(^{37}\) Furthermore, each household and firm makes forward plans by assuming all others share the same planning horizon and use the same value function.

From the linear equations (1.23)-(1.24), only the average belief of \( v_t \) and \( \tilde{v}_t \) matters for the aggregate endogenous variables such as \( \{ y_t, \pi_t, \hat{i}_t \} \), and thus I refer to these two variables as representing population averages. Then, the estimated value function in period \( t \) as specified in equations (1.12)-(1.14) can be re-written as

\[
\nu_t^{est} = \Sigma_h \omega_h [y_t^h - g_t - (1 - \beta) b_t + \sigma \pi_t^h]
\]

\[
\tilde{v}_t^{est} = \Sigma_h \tilde{\omega}_h (1 - \alpha)^{-1} \pi_t^h
\]

The general steps in this section characterizing the equilibrium with heterogeneous agents are as follows: Given the pre-determined aggregate endogenous variables \( \{ b_t, v_t, \tilde{v}_t \} \) in period \( t \), I first characterize the trend and deviation components of \( \{ y_t, \pi_t, \hat{i}_t \} \); then, I describe the evolution path of \( \{ b_{t+1}, v_{t+1}, \tilde{v}_{t+1} \} \) from period \( t \) to \( t + 1 \) by the aggregate endogenous variables \( \{ y_t, \pi_t, \hat{i}_t \} \) and \( \{ b_t, v_t, \tilde{v}_t \} \). I assume that, throughout the analysis in this section, the aggregation across agents converges, though not necessarily (always) true. In the discussion of the determinacy condition in the following Section 1.3.4, I double-check the assumption of convergence for each combined monetary-fiscal regime.

With a focus on characterizing aggregate endogenous variables \( \{ y_t, \pi_t, \hat{i}_t \} \) in the first step, as discussed in Section 1.3.2, any specific definition of beginning asset positions \( \{ \tilde{b}_t^{h+1}, \tilde{b}_t^{h+1} \} \) for “trend” and “deviation” variables does not affect the calculation of aggregate endogenous variables as long as \( b_t = \tilde{b}_t^{h+1} + \tilde{b}_t^{h+1} \). In other words, the aggregate variables are robust to the way of

\(^{37}\)Although each agent has a random probability of being horizon-\( h \) in the next period, they do not take this randomness into consideration for making forward planning.
defining \( \{ \tilde{b}^{h+1}_t, \tilde{\beta}^{h+1}_t \} \). Therefore, I impose a specific structure on the heterogeneous (beginning) asset position \( \{ \tilde{b}^{h+1}_t, \tilde{\beta}^{h+1}_t \} \) in period \( t \) for any horizon-\( h \) in a particular way to allow for easy and tractable aggregation across agents, but still preserving the aggregate variables unchanged.

More specifically, consider the forward planning exercise of “trend” variables in period \( t \). For any \( h \geq 1 \) in period \( t \), the equations (1.21)-(1.22) yield

\[
\tilde{y}^h_t (\tilde{b}^{h+1}_t) = \tilde{y}^{h-1}_{t+1} (\tilde{b}^h_t) - \sigma [\tilde{r}^{h+1}_t (\tilde{b}^{h+1}_t) - \tilde{\pi}^{h-1}_{t+1} (\tilde{b}^h_{t+1})]
\]

(1.27a)

\[
\tilde{\pi}^h_t (\tilde{b}^{h+1}_t) = \kappa \tilde{y}^h_t (\tilde{b}^{h+1}_t) + \beta \tilde{\pi}^{h-1}_{t+1} (\tilde{b}^h_{t+1})
\]

(1.27b)

\[
\tilde{v}^h_t = \phi_\pi \tilde{\pi}^h_t
\]

(1.27c)

\[
\tilde{b}^h_{t+1} = \beta^{-1} (1 - \Gamma) \tilde{b}^{h+1}_t - \beta^{-1} (1 - \Gamma) s_b \tilde{\pi}^h_t + (1 - \Gamma) s_b \tilde{v}^h_t
\]

(1.27d)

For \( h = 0 \), the equations (1.23)-(1.24) yield

\[
\tilde{y}^0_t (\tilde{b}^1_t) = -\sigma \tilde{r}^0_t (\tilde{b}^1_t) + (1 - \beta) \tilde{b}^0_{t+1} + \nu_t
\]

\[
\tilde{\pi}^0_t (\tilde{b}^1_t) = \kappa \tilde{y}^0_h (\tilde{b}^1_t) + (1 - \alpha) \beta \tilde{v}_t
\]

with

\[
\tilde{v}^0_t = \phi_\pi \tilde{\pi}^0_t
\]

\[
\tilde{b}^0_{t+1} = \beta^{-1} (1 - \Gamma) \tilde{b}^1_t - \beta^{-1} (1 - \Gamma) s_b \tilde{\pi}^0_t + (1 - \Gamma) s_b \tilde{v}^0_t
\]

Then, I define \( \tilde{b}^1_t = 0 \) for agents with horizon \( h = 0 \). For any \( h \geq 1 \), the beginning asset position \( \tilde{b}^{h+1}_t \) of “trend” variables for agents with horizon-\( h \) is defined in such a way that their asset position in the planning exercise at date \( t + 1 \) will equal the beginning asset position of agents with horizon...
\( h - 1 \) at date \( t \). That is, \( \tilde{b}^{h+1}_t \) is defined by backward induction in equations (1.27a)-(1.27d) such that \( \tilde{b}^{h+1}_{t+1} = \tilde{b}^h_t \), which is a function of only \( \nu_t \) and \( \tilde{\nu}_t \). To keep the aggregate endogenous variables being unchanged, the beginning asset position \( \tilde{b}^{h+1}_t \) of “deviation” variables in period \( t \) is defined by \( \tilde{b}^{h+1}_t = b_t - \tilde{b}^{h+1}_t \).

By aggregating across the whole population for the “trend” variables, it follows

\[
\tilde{y}_t = \rho \tilde{y}_t - \sigma [\tilde{\nu}_t - \rho \tilde{\nu}_t] + (1 - \rho)\nu_t + (1 - \rho)(1 - \beta)b^0_{t+1}
\]

(1.28a)

\[
\tilde{\nu}_t = k\tilde{y}_t + \beta \rho \tilde{\nu}_t + (1 - \rho)(1 - \alpha)\beta \tilde{y}_t
\]

(1.28b)

\[
\tilde{\nu}_t = \phi \bar{\nu}_t
\]

(1.28c)

where \( b^0_{t+1} = \psi_v \nu_t + \psi_{\tilde{\nu}} \tilde{\nu}_t \). The derivation and expressions for \( \{\psi_v, \psi_{\tilde{\nu}}\} \) can be found in Appendix A.2.

Through the system of (1.28a)-(1.28c), \( \{\tilde{y}_t, \tilde{\nu}_t\} \) can be written in \( \nu_t \) and \( \tilde{\nu}_t \); that is,\(^{38}\)

\[
B \begin{bmatrix} \tilde{y}_t \\ \tilde{\nu}_t \end{bmatrix} = \Xi \begin{bmatrix} \nu_t \\ \tilde{\nu}_t \end{bmatrix}
\]

(1.29)

Recall that the beginning asset position for the deviation component of decision makers with planning horizon-\( h \) in period \( t \) is given by \( \tilde{b}^{h+1}_t = b_t - \tilde{b}^{h+1}_t \) for \( \forall h \geq 0 \). Also note the structural equations of deviation components are the same as those specified in Section 1.2.4 with no update in the value function. By averaging across the population, the structural equations determining the

\(^{38}\)The expressions of \( B \) and \( \Xi \) are given by

\[
B = \begin{bmatrix}
1 - \rho & \sigma(\phi - \rho) \\
-\kappa & 1 - \beta \rho
\end{bmatrix},
\]

\[
\Xi = \begin{bmatrix}
(1 - \rho)(1 + (1 - \beta)\psi_v) & (1 - \rho)(1 - \beta)\psi_{\tilde{\nu}} \\
0 & (1 - \rho)(1 - \alpha)\beta
\end{bmatrix}.
\]
aggregate plans \( \{y_t, \pi_t, b_t\} \) must satisfy

\[
y_t - g_t - \bar{y}_t = \rho E_t (y_{t+1} - g_{t+1} - \bar{y}_{t+1}) - \sigma [\hat{i}_t - \bar{i}_t - \rho E_t (\pi_{t+1} - \bar{\pi}_{t+1})]
\]

\[
+(1 - \rho)(1 - \beta)(b^0_{t+1} - \bar{b}^0_{t+1})
\]

\[
\pi_t - \bar{\pi}_t = \kappa (y_t - y^*_t - \bar{y}_t) + \beta \rho E_t (\pi_{t+1} - \bar{\pi}_{t+1})
\]

\[
\hat{i}_t - \bar{i}_t = \hat{i}^*_t + \phi_\pi (\pi_t - \bar{\pi}_t)
\]

where \( b^0_{t+1} - \bar{b}^0_{t+1} = \psi_p b_t + \psi_g g_t + \psi_y y^*_t + \psi_i i^*_t \) is the same as the expression in the case of no update in the value function as in Section 1.2.4 (as showed in Appendix A.1).

Thus far, given the pre-determined state variables \( \{b_t, \nu_t, \bar{\nu}_t\} \) in period \( t \), I have characterized the system of equations capturing aggregate variables \( \{y_t, \pi_t, \hat{i}_t\} \). Then, I describe the evolution process of \( \{b_t, \nu_t, \bar{\nu}_t\} \) over time. From Section 1.3.1, the dynamics of the value-function adjustment is captured by

\[
\nu_{t+1} = \gamma \nu^* + (1 - \gamma) \nu_t
\]

\[
\bar{\nu}_{t+1} = \bar{\gamma} \bar{\nu}^* + (1 - \bar{\gamma}) \bar{\nu}_t
\]

\[
\nu^* = y_t - g_t + \sigma \pi_t - (1 - \beta) b_t
\]

\[
\bar{\nu}^* = (1 - \alpha)^{-1} \pi_t
\]

which can be re-written as

\[
\begin{bmatrix}
\nu_{t+1} \\
\bar{\nu}_{t+1}
\end{bmatrix} = \Omega \Phi \begin{bmatrix} y_t - g_t \vspace{2mm} \pi_t \end{bmatrix} + \Omega \begin{bmatrix} -(1 - \beta) \vspace{2mm} 0 \end{bmatrix} b_t + (I - \Omega) \begin{bmatrix} \nu_t \\
\bar{\nu}_t
\end{bmatrix}
\]

(1.32)
where $\Omega$ is a $2 \times 2$ diagonal matrix with diagonal elements $(\gamma, \tilde{\gamma})$ and $\Phi = \begin{bmatrix} 1 & \sigma \\ 0 & (1 - \alpha)^{-1} \end{bmatrix}$. Together with the evolution path of aggregate real public debt captured by equation (1.3), the equilibrium under heterogeneous agents has been fully characterized.

Now, I re-write the whole system of equations of the equilibrium in a more compact form, and use $\{\tilde{y}_t, \tilde{\pi}_t\}$ to substitute $\{v_t, \tilde{v}_t\}$ for capturing the “trend” evolution of the aggregate variables. By substituting (1.29) into (1.32), the “trend” variables follows that

$$\tilde{x}_{t+1} = Fx_t + G\tilde{x}_t + Hb_t$$

(1.33)

where $x_t = \begin{bmatrix} y_t - g_t \\ \pi_t \end{bmatrix}^T$, $\tilde{x}_t = \begin{bmatrix} \tilde{y}_t \\ \tilde{\pi}_t \end{bmatrix}^T$, $F = B^{-1}\Xi\Omega\Phi$, $G = B^{-1}\Xi(I - \Omega)\Xi^{-1}B$ and $H = B^{-1}\Xi\Omega\begin{bmatrix} -(1 - \beta) \\ 0 \end{bmatrix}^T$.

The system (1.30a)-(1.30c) of the “deviation” variables can be written as

$$E_t(x_{t+1} - \tilde{x}_{t+1}) = A(x_t - \tilde{x}_t) + Cb_t + Ku_t$$

(1.34)

where $u_t$ captures exogenous disturbances, and the definition of $A$, $C$, and $K$ can be found in Appendix A.1.

Also note the evolution path of real public debt (1.3) follows

$$b_{t+1} = \beta^1(1 - \Gamma)b_t + D \cdot x_t$$

(1.35)

where $D = \begin{bmatrix} 0 & (\phi - \beta^{-1})(1 - \Gamma)\phi_b \end{bmatrix}$.

Therefore, the whole system is fully characterized by the evolution of trend components (1.33), deviation components (1.34), and the evolution of real public debt (1.35). It can be summarized as
with three pre-determined variables \{\bar{y}_t, \bar{\pi}_t, b_t\} and two non-predetermined variables \{y_t, \pi_t\}. The expression of \(\Upsilon\) can be found in Appendix A.3.

1.3.4 Determinacy Condition for Monetary Policy with Inactive Fiscal Policy

Consider the first case in which fiscal policy is inactive; that is, the government bond is always in fixed supply \(B_t = \bar{B}\) (which corresponds to \(\Gamma = 1\)). In such a case, it implies \(b_t = \bar{b} = 0\) all the time. The only policy instrument is monetary policy, which determines the equilibrium path.

The system of equilibrium (1.36) can then be written into

\[
E_t \begin{bmatrix} x_{t+1} \\ \bar{x}_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} A + F & -A + G \\ F & G \end{bmatrix} \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix} + \begin{bmatrix} Ku_t \\ 0 \end{bmatrix}
\]

When the learning process is slow, i.e., \(\gamma, \tilde{\gamma} \to 0\), the “trend” variables \(\bar{x}_t\) become constant (with \(\Omega = 0, F = 0\) and \(G = I\)). Then, the entire system is determinate if and only if the equilibrium path of aggregate variables \(x_t\) is determinate. Following Blanchard and Kahn (1980), the equilibrium path of \(x_t\) is determinate if and only if the two eigenvalues of \(A\) are outside the unit circle, which is equivalent to

\[
\phi \rho > -\frac{1}{\kappa \sigma} [\beta \rho^2 - \rho (1 + \beta + \kappa \sigma) + 1] \equiv l(\rho)
\]

Because \(0 < \rho < 1\), the right-hand-side \(l(\rho)\) is strictly decreasing in \(\rho\). As the (average) length of planning horizon approaches infinity \(\rho \to 1\), it follows \(l(\rho) \to 1\), which features the boundary condition of the Taylor principle as in the standard New Keynesian model with rational expectations. As \(\rho\) becomes smaller, that is, the population is (on average) less forward-looking, the requirement on monetary policy to ensure long-run stability is relaxed. Even in the case of

\[39\text{Proofs for the determinacy condition can be found in Appendix A.4.}\]
\( \phi_{\pi} = 0 \), monetary policy can still ensure stability in the long-run equilibrium as long as agents’ foresight is short enough, that is, \( \rho \) is small enough (and the learning in the value function is really slow).

Furthermore, different from the literature of RE with multiple equilibria when the Taylor principle is violated \( \phi_{\pi} < 1 \), for the case of \( \phi_{\pi} < l(\rho) \) with finite forward planning, the summation across heterogeneous agents does not converge in any given period. It indicates the assumption of exponential distribution is not appropriate to study the policy regime under such parameterization. In fact, the aggregation across heterogeneous agents converges if and only if the condition in (1.37) is satisfied.\(^4\) In general, as long as the maximum planning horizon in the population is finite, or the summation across agents converges, a unique equilibrium path with finite forward planning always exists.

1.3.5 Stability (Determinacy) Condition with Monetary-fiscal Policy Interaction

More generally, the evolution of real public debt can be endogenous as specified in equation (1.35). Leeper (1991) categorizes monetary and fiscal policy into two groups – “active” or “passive” policy depending on whether the policy is forward-looking or backward-looking in the equilibrium. Intuitively, “active” policy indicates the policy authority is free to set the policy rule depending on past, current, or expected future variables, whereas “passive” policy indicates the authority is constrained by the active authority’s decision in order to balance the government budget constraint. In this section, I focus on the stability (or determinacy) condition in the “active/passive” language under the environment of heterogeneous agents with finite forward planning.

From the system of equations (1.36), because it has two non-predetermined variables, the equilibrium is determinate if and only if two eigenvalues of \( \Upsilon \) are outside the unit circle.

In the limiting case of \( \rho \to 1 \), it implies \( \Xi = 0 \), and the system of trend components (1.29) requires that \( \tilde{y}_t = 0, \tilde{\pi}_t = 0 \) as long as \( \phi_{\pi} \neq 1 \). In such a case, the system of equations characterizing aggregate endogenous variables \{\( y_t, \pi_t, b_t \)\} becomes the standard New Keynesian model with

\(^4\)Details can be found in Appendix A.4.
rational expectations as discussed in Woodford (2003); that is,

\[ E_t(x_{t+1}) = Ax_t + Ku_t \]

\[ b_{t+1} = \beta^{-1}(1 - \Gamma)b_t - \beta^{-1}(1 - \Gamma)s_b \pi_t + (1 - \Gamma)s_b \hat{\delta}_t \]

where the policy rule of nominal interest rate is \( \hat{i}_t = i_t^* + \phi \pi_t \). A unique (locally) bounded solution exists if and only if (i) \(|\beta^{-1}(1 - \Gamma)| < 1\) and \( \phi > 1 \), or (ii) \(|\beta^{-1}(1 - \Gamma)| > 1\) and \( \phi < 1 \).

In the language of Leeper (1991), condition (i) features active monetary policy (AM) and passive fiscal policy (PF), whereas the condition (ii) features passive monetary policy (PM) and active fiscal policy (AF).

The limiting case of the finite-planning-horizon model, however, differs from the discussion in Leeper (1991) when the standard model in the literature under RE indicates no bounded equilibrium solution (i.e., \(|\beta^{-1}(1 - \Gamma)| > 1\) and \( \phi > 1 \)) or multiple equilibria (i.e., \(|\beta^{-1}(1 - \Gamma)| < 1\) and \( \phi < 1 \)). In these two scenarios, similar to the discussion with inactive fiscal policy, the summation across heterogeneous agents does not converge, which indicates the assumption of exponential distribution is not appropriate for studying the policy regime under such parameterization. Generally, as long as the maximum planning horizon is finite, or the summation across agents converges, one unique equilibrium path in the finite-planning-horizon model always exists even in the region where the canonical model with RE indicates multiple equilibria.

Now, I focus on the more general situation of \( 0 < \rho < 1 \); that is, decision makers are not forward-looking into the infinite future. For simplicity, consider the case in which the learning in the value function is slow, i.e., \( \gamma, \tilde{\gamma} \rightarrow 0 \). From equation (1.33), the evolution of \( \tilde{x}_t \) no longer depends on \( x_t \) and becomes a constant (with \( \Omega = 0, F = 0, H = 0, \) and \( G = I \)). Without loss of

\[ 41 \text{It is verified that, under either condition (1) or condition (2) holds, the sum of } \Sigma \omega_h y^h \text{ and } \Sigma \tilde{\omega}_h \pi^h \text{ in a given period converges. More details can be found in Appendix A.5.} 
\]

\[ 42 \text{In addition, for the fiscal policy under condition (i), Woodford (2003) also names it as “Ricardian” fiscal policy.} \]
Table 1.1: Calibrated values of parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subjective discount factor</td>
<td>$\beta$</td>
<td>0.99 4% annual real interest rate (a quarter model)</td>
</tr>
<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\sigma$</td>
<td>0.5 Standard value in literature</td>
</tr>
<tr>
<td>Probability of being not able to reset prices (Calvo)</td>
<td>$\alpha$</td>
<td>0.66 An average length of price contract of 3 quarters</td>
</tr>
<tr>
<td>Response of inflation to output gap in Phillips curve</td>
<td>$\kappa$</td>
<td>0.02 Following Eggertsson and Woodford (2004)</td>
</tr>
<tr>
<td>Real public debt to output rate (steady state)</td>
<td>$s_b$</td>
<td>2.4 Implying real public debt as 60% of GDP</td>
</tr>
</tbody>
</table>

For generality, I assume $\bar{x}_t = 0$ all the time. The complete system reduces to

$$
E_t \begin{bmatrix} x_{t+1} \\ b_{t+1} \end{bmatrix} = Y_s \begin{bmatrix} x_t \\ b_t \end{bmatrix} + \begin{bmatrix} K \mu_t \\ 0 \end{bmatrix}
$$

where $Y_s$ is given by

$$
Y_s = \begin{bmatrix} A & C \\ D & \beta^{-1}(1 - \Gamma) \end{bmatrix}
$$

The equilibrium is determinant if and only if two eigenvalues of $Y_s$ are outside the unit circle. Although there is no further refined analytical solution to the determinacy condition, the boundary conditions of the determinacy condition can be mostly captured by $\phi = I(\rho)$ and $|\beta^{-1}(1 - \Gamma)| = 1$.\(^{43}\) The determinacy condition is illustrated through numerical exercises as indicated in Figure 1.1.

For quantitative analysis, the values of parameters for calibration are borrowed from Eggertsson and Woodford (2004) with a quarter model as summarized in Table 1.1. For a quarter model, the subject discount factor is set to $\beta = 0.99$, implying a 4% (annual) natural rate, the intertemporal elasticity of substitution for households is set at $\sigma = 0.5$, and the response of inflation to the output gap in the Phillips curve is set at $\kappa = 0.02$. The level of steady-state real public debt is calibrated to be 60% of annual GDP, that is, $s_b = 2.4$.

By the calibration of parameters summarized in Table 1.1, the shaded areas in Figure 1.1 show

\(^{43}\text{Note that } I(\rho) = -\frac{1}{\kappa \sigma} [\beta \rho^2 - \rho(1 + \beta + \kappa \sigma) + 1].\)
the regime of parameterization for \( \{ \phi, \Gamma \} \) in which the equilibrium reaches long-run stability (or determinacy). The four subfigures vary in different average planning horizons. The dotted lines in each subfigure (captured by \( \phi_{\pi} = l(\rho) \) and \( |\beta^{-1}(1-\Gamma)| = 1 \)) largely overlap with the solid lines and represent the thresholds of \( \phi_{\pi} \) and \( \Gamma \), respectively. Notably, the shaded areas within the dotted lines also ensure the convergence of aggregation across heterogeneous agents in a given period.\(^{44}\) For those blank areas in Figure 1.1, it can be numerically verified that under such parameterization, the aggregation across agents does not converge, which rejects the validity of assuming exponential distribution in the planning horizon for analyzing such scenarios.

From Figure 1.1, the more short-sighted decision makers are, the larger the policy space of the “AM/PF” regime with long-run stability in a reasonable range of parameterization (i.e., \( 0 \leq \Gamma \leq 1 \) and \( \phi_{\pi} > 0 \)) becomes, while the policy space for “PM/AF” becomes smaller. “A” in the figure represents “active” policy, and “P” represents “passive” policy. As shown in the bottom-right subfigure (d), when the average planning horizon is around 20 years, or \( h = 80 \) quarters, the boundary conditions are close to those in the standard New Keynesian model with rational

\(^{44}\)Details can be found in Appendix A.5.
expectations. When the planning horizon is short enough, as shown in the upper-right subfigure (b), that is, five years or $h = 20$ quarters, the requirement for long-run stability on $\phi_{\pi}$ in “AM/PF” is significantly smaller than 1.

Importantly, if the government does not have good knowledge of how forward-looking the population is, an active monetary policy with passive fiscal policy satisfying $\phi_{\pi} > 1$ and $|\beta^{-1}(1 - \Gamma)| < 1$ is robust to the length of the planning horizon; that is, the government might appreciate the policy regime of “AM/PF” to better ensure long-run stability. By contrast, some recent studies (e.g., Jarociński and Maćkowiak, 2017) propose that a “fiscally dominant” regime (“PM/AF”) would better maintain macro stability in the face of an effective lower bound on nominal interest rates.

Although Figure 1.1 shows the case under the assumption of a slow learning process in the value function, the impact of the length of the planning horizon still applies in the case with a quicker adjustment in the learning process. But a quicker adjustment in updating the value function mitigates the impact of a shorter planning horizon, and leads the boundary conditions of determinacy closer to the standard model with rational expectations.

1.4 Short-term Effects of Stimulative Fiscal Policy and Interaction with Monetary Policy

In this section, under the parameterization of the policy regime (“AM/PF”) that can ensure long-run stability as discussed in the previous section, I study the short-term effects of stimulative fiscal policy and its interaction with monetary policy. More specifically, I analyze three fiscal policy instruments in a unified framework: a one-time lump-sum transfer from government to the private sector (e.g., new financial claims on the government), the speed of tax collections by the government (with respect to the level of its real public debt), and a change in the long-run target of real public debt. In addition, I also briefly discuss the effects of unconventional monetary policy, namely forward guidance, and its interaction with fiscal stimulus, and whether fiscal policy can achieve a stimulative effect that a pure commitment on future monetary policy cannot.\footnote{Woodford (2019) shows that, in a model without a fiscal sector, the shorter the length of decision makers’ planning horizon is, the less effective the forward guidance policy is.}
Before the detailed description of policy experiments, in analyzing the short-term effects of this section, the value function decision makers use is assumed to be a given one learned from the steady-state stationary equilibrium as specified in Section 1.2. This assumption is valid and helpful to study the short-term effects of counter-cyclical fiscal policy and unconventional monetary policy, in the sense that decision makers in the economy have stayed in the steady-state stationary equilibrium for a long time and do not have much experience with such policies in the past. The following Section 1.5 relaxes this assumption by incorporating the learning process in the value function as in Section 1.3, and focuses on the long-run dynamics of these policies.

Now, consider the following policy experiment: Prior to period \( t = 0 \), the economy has stayed at the steady state for a long time with a fiscal rule as specified in Section 1.2.3; in the period \( t = 0 \), the government makes a one-time lump-sum transfer \( T_0 \) (in real value), and the government sets a new long-run target for the real public debt denoted by \( B^* \), which is characterized by \( B^* - \bar{B} = \lambda (1 + i_0)T_0 \), as well as the rule of tax collections \( \mathcal{T}_t \). In each period from \( t = 0 \) onward, net lump-sum taxes are collected in each period, which is specified by

\[
\mathcal{T}_0 = (1 - \Gamma)\bar{\mathcal{F}} + \Gamma \left( \frac{B_0}{\Pi_0} - \frac{B^*}{1 + i_0} + T_0 \right) - T_0, \quad t = 0
\]

and

\[
\mathcal{T}_t = (1 - \Gamma)\bar{\mathcal{F}} + \Gamma \left( \frac{B_t}{\Pi_t} - \frac{B^*}{1 + i_t} \right), \quad \forall t \geq 1
\]

where \( \bar{B}, \bar{\Pi}, i, \bar{\mathcal{F}} \) is the real public debt, inflation, nominal interest rate, and lump-sum taxation associated with the steady state before the one-time transfer occurs. While the first term in the rule of tax collection from \( t = 0 \) indicates the initial steady-state taxation, the second term indicates the tax collections needed to make \( B_{t+1} \) directly equal to the long-run debt target, with weight \( (1 - \Gamma) \) on the first amount and weight \( \Gamma \) on the second \((0 \leq \Gamma \leq 1)\). The parameter \( \Gamma \) captures not only how strongly lump-sum tax collections respond to the level of real public debt, but also captures how fast the real public debt will converge to the new long-run target. In the case of no lump-sum transfer \( T_0 = 0 \) and no change in long-run debt target \( B^* = \bar{B} \), the fiscal rule becomes the same one
as in Section 1.2.3.

By log-linearization and substituting the path of tax collections into the government budget constraint, the evolution of real public debt for any period \( t \geq 1 \) is given by

\[
b_{t+1} = \beta^{-1}(1 - \Gamma)b_t - \beta^{-1}(1 - \Gamma)s_b\pi_t + (1 - \Gamma)s_b\hat{\pi}_t + \Gamma b^*
\]

and for the period \( t = 0 \),

\[
b_1 = \beta^{-1}(1 - \Gamma)b_0 - \beta^{-1}(1 - \Gamma)s_b\pi_0 + (1 - \Gamma)s_b\hat{\pi}_0 + (1 - \Gamma)t^* + \Gamma b^*
\]

where the variable \( b_t = \frac{B_t - \bar{B}}{\Pi} \) is the deviation of real public debt from its initial steady-state value before the one-time transfer occurs (relative to output), \( b^* = \frac{B^* - \bar{B}}{\Pi} \) is the deviation of the long-run real debt target from the initial steady-state value of real public debt, \( t^* = \frac{T_0}{\Pi} \) is the size of a one-time lump-sum transfer relative to output, and \( s_b = \frac{\bar{B}}{\Pi} \) is the relative size of initial steady-state real public debt to output. By the definition of \( B^* \), it follows that \( b^* = \lambda t^* \). To sum up, three fiscal policy instruments exist \( \{t^*, \lambda, \Gamma\} \).

Several special situations help clarify the role of the policy instruments \( \{\Gamma, \lambda\} \). The parameter \( \Gamma \) measures how fast the real public debt converges to the long-run debt target. Intuitively, as \( \Gamma \) increases from 0 to 1, converging to the debt target takes less time. In the case of \( \Gamma = 0 \), the debt target is irrelevant, and taxation in each period is \( \tilde{T} \). Then, the path of real public debt purely depends on the path of interest rates and inflation, and there is no expectation that tax collections will ensure the real public debt under control. When \( \Gamma = 1 \), the public debt \( b_1 \) in period \( t = 1 \) directly increases from the initial steady-state value 0 to the new long-run target value \( b^* \), and remains unchanged thereafter.

To see the role of \( \lambda \), consider the case of \( \Gamma = 1 \), and in this scenario, the meaning of \( \lambda \) is straightforward – indicating the proportion of the lump-sum transfer that is financed by (long-run) real public debt. As \( \lambda \) becomes larger, it follows that a bigger proportion of the lump-sum transfer is financed by the long-run level of public debt, and the net transfer to households is larger.
For instance, in the case of $\lambda = 1$ (and $\Gamma = 1$), the lump-sum transfer is fully financed by real public debt, and the net transfer is $t^*$, whereas if $\lambda = 0$ (and $\Gamma = 1$), the lump-sum transfer is simultaneously offset by increased taxes in the same period, and thus no effect can rise.

1.4.1 Lump-sum Transfer Financed by Debt with Monetary Policy under Taylor Rule

In this section, I study the effects of counter-cyclical fiscal policies as specified in the previous section with monetary policy under the (time-invariant) Taylor rule. To be comparable with the standard New Keynesian model, I conduct the analysis by considering the case of heterogeneous agents as specified in Section 1.2.5. A similar analysis can also be done under the assumption of homogeneous agents as in Section 1.2.4, and the conclusions do not change. For simplicity, I assume no real disturbances occur.

Under the assumption of heterogeneous agents as in Section 1.2.5, the equilibrium relations in any period $t \geq 1$ are given by

$$y_t = \rho E_t y_{t+1} - \sigma (\hat{i} - \rho E_t \pi_{t+1}) + (1 - \rho)(1 - \beta)[\psi_b b_t + \psi_{b^*} b^*]$$  \hspace{1cm} (1.38)

$$\pi_t = \kappa y_t + \beta \rho E_t \pi_{t+1}$$  \hspace{1cm} (1.39)

with nominal interest rate and the evolution of real public debt satisfying

$$\hat{i}_t = \phi_{\pi} \pi_t$$  \hspace{1cm} (1.40)

$$b_{t+1} = \beta^{-1}(1 - \Gamma)b_t - \beta^{-1}(1 - \Gamma)s_b \pi_t + (1 - \Gamma)s_b \hat{i}_t + \Gamma b^*$$  \hspace{1cm} (1.41)

where the parameter $\psi_b$ and $\psi_{b^*}$ can be easily solved as in Appendix A.1, given by

$$\psi_b = \beta^{-1}(1 - \Gamma)/[1 - (\phi_{\pi} - \beta^{-1})s_b (1 - \Gamma) \frac{(1 - \beta)\kappa}{1 + \sigma \phi_{\pi} \kappa}]$$
\[
\psi_{b^*} = \Gamma / [1 - (\phi_\pi - \beta^{-1})s_b (1 - \Gamma) (1 - \beta) \kappa / (1 + \sigma \phi_\pi \kappa)]
\]

In the period \( t = 0 \), the equations of the IS and Phillips curve (1.38)-(1.39), and the evolution of real public debt (1.41) become

\[
y_0 = \rho E_0 y_1 - \sigma (i - \rho E_0 \pi_1) + (1 - \rho) (1 - \beta) [\psi_b b_0 + \psi_\pi t^* + \psi_{b^*} b^*]
\]

\[
\pi_0 = \kappa y_0
\]

\[
b_1 = \beta^{-1} (1 - \Gamma) b_0 - \beta^{-1} (1 - \Gamma) s_b \pi_0 + (1 - \Gamma) s_b \hat{\pi}_0 + (1 - \Gamma) t^* + \Gamma b^*
\]

where the initial asset position satisfies \( b_0 = 0 \), and the parameter \( \psi_{t^*} \) is given by

\[
\psi_{t^*} = (1 - \Gamma) / [1 - (\phi_\pi - \beta^{-1}) s_b (1 - \Gamma) (1 - \beta) \kappa / (1 + \sigma \phi_\pi \kappa)]
\]

To illustrate the effects of the three fiscal policy instruments \( \{t^*, \lambda, \Gamma\} \) separately and note that \( b^* = \lambda t^* \), I consider two policy experiments with closed-form solutions: (i) the case of \( \Gamma = 1 \), that is, the real public debt directly increases to the new long-run target level and stays unchanged thereafter; (ii) \( s_b = 0 \) and \( \lambda = 0 \), that is, both the steady-state level of real public debt and the long-run target of debt are zero.\(^{46}\) The first case helps clarify the role of \( \{t^*, \lambda\} \), and the second policy experiment helps to study \( \{t^*, \Gamma\} \).

**Lump-sum Transfer Financed by an Immediate Permanent Increase in Real Public Debt**

When \( \Gamma = 1 \), the real public debt \( b_t = b^* = \lambda t^* \) for \( \forall t \geq 1 \), and \( \psi_b = \psi_{t^*} = 0 \) with \( \psi_{b^*} = 1 \). Then, the solution of the equilibrium for any period \( t \geq 0 \) is time-invariant, given by

\[
y_t = \frac{(1 - \beta \rho)(1 - \rho)(1 - \beta)}{(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)} \lambda t^*
\]

\(^{46}\)Woodford and Xie (2019) study the first case of \( \Gamma = 1 \), the situation of a permanent increase in real public debt, by imposing a monetary rule of strict inflation targeting, and focus on the effects under ZLB.
\[
\pi_t = \frac{k(1-\rho)(1-\beta)}{(1-\beta\rho)(1-\rho) + \kappa\sigma(\phi_\pi - \rho)} \lambda t^*
\] 
(1.44)

By limiting the attention to the case of “AM/PF”, which implies \( \phi_\pi > l(\rho) \), and also noting that \( 0 < \rho < 1 \), the output and inflation are obviously positive in response to the lump-sum transfer, that is, \( y_t, \pi_t > 0 \). As the size of lump-sum transfer \( t^* \) becomes larger, the response of output and inflation increases. Meanwhile, the larger is the proportion of the transfer \( \lambda \) that is financed by (long-term) debt, the larger the response of output and inflation is.

First, consider how the degree of foresight influences the effect of fiscal transfers with a given parameterization of the monetary policy. Given \( \phi_\pi > 1 \), the output \( y_t \) is strictly decreasing with respect to the length of the (average) planning horizon measured by \( \rho \).\(^{47}\) In other words, as decision makers in the economy plan less distance into the future, the stimulative effect of lump-sum transfer on output becomes larger.

By the expressions (1.43) and (1.44) and the calibration of parameters from Table 1.1, Figure 1.2 shows the output and inflation in response to a one-time debt-financed lump-sum transfer in period \( t = 0 \). The (average) planning horizon in the figure is defined as \( h = \rho/(1-\rho) \) with unit in quarters. The standard New Keynesian model corresponds to the case of infinite horizon \( h \to \infty \). The two lines in Figure 1.2 indicates two different sizes of lump-sum transfer. The shape of the two lines suggests that as the length of the planning horizon decreases, a noticeable (persistent) stimulative effect of fiscal lump-sum transfer occurs in output and inflation. In particular, the response in inflation reaches the highest when the planning horizon of decision makers is around four years, or 16 quarters, whereas the response in output increases exponentially with less foresight.

Quantitatively, with the size of the lump-sum transfer being equal to one-quarter GDP as shown in the solid line, the output increases by 0.9% if the (average) planning horizon is one quarter, as estimated in Gust, Herbst, and López-Salido (2019). In this case, the “fiscal-transfer multiplier” (defined as the discounted aggregate response of output with respect to the size of the initial lump-

\(^{47}\)Note that as \( \rho \to 1 \), \(-\frac{1}{\kappa\sigma}[\beta\rho^2 - \rho(1 + \beta + \kappa\sigma) + 1] \to 1 \), and thus it requires \( \phi_\pi > 1 \) for the monotonicity of \( y_t \) with respect to \( \rho \).
Notes: $\phi_\pi = 1.5, \Gamma = 1, \lambda = 1; h$ in quarters; no real disturbances.

sum transfer) is about 0.94. If I simply borrow the calibrated parameters from the discounted Euler equation in McKay, Nakamura, and Steinsson (2016), it implies the (average) planning horizon is around eight years. Then, the positive response of output to the one-time fiscal transfer (with the size of one-quarter GDP) is about 0.3\% and the fiscal-transfer multiplier is near 0.31.

From the expressions (1.43) and (1.44), the proportion of lump-sum transfer financed by debt, which is measured by $\lambda$, has an effect similar to that of the size of lump-sum transfer $t^\ast$.

The response in output and inflation is seemingly permanent, because of the assumption of no update in the value function. But as decision makers update their value function by incorporating the effects of such counter-cyclical fiscal policies, the response in output and inflation becomes transitory.

To shed light on how monetary policy affects the effects of fiscal stimulus, expressions (1.43) and (1.44) indicate the following two propositions:

**Proposition 1** Given the equilibrium is determinant $\phi_\pi > l(\rho)$, and a one-time lump-sum transfer financed by an immediate permanent increase in real public debt $\Gamma = 1$, and $0 < \rho < 1$, it follows
that the responses of output and inflation to such a fiscal transfer are strictly decreasing with less accommodative monetary policy, that is, $\frac{\partial y}{\partial \phi_x} < 0$ and $\frac{\partial \pi}{\partial \phi_x} < 0$.

**Proposition 2** Given $l(\rho) < \phi_\pi < \frac{1}{k\sigma} + \frac{2}{1+\beta}$, and a one-time lump-sum transfer financed by an immediate permanent increase in real public debt $\Gamma = 1$, and $0 < \rho < 1$, it follows that the impact of monetary policy accommodation on the effect of fiscal transfer in stimulating output is hump-shaped with respect to the degree of foresight; that is, a unique $\tilde{\rho} \in (0, 1)$ exists such that $-\frac{\partial^2 y}{\partial \phi_x \partial \rho} > 0$ if $0 < \rho < \tilde{\rho}$, and $-\frac{\partial^2 y}{\partial \phi_x \partial \rho} < 0$ if $\tilde{\rho} < \rho < 1$. In the case of $\phi_\pi \geq \frac{1}{k\sigma} + \frac{2}{1+\beta}$, it follows $-\frac{\partial^2 y}{\partial \phi_x \partial \rho} < 0$ for any $\rho \in (0, 1)$.

Proofs of Proposition 1 and 2 can be found in Appendix A.6. Proposition 1 indicates the effect of fiscal transfer is amplified with more accommodative monetary policy. Going one step further, Proposition 2 shows that as long as the monetary policy does not respond too strongly to inflation, the impact of monetary policy accommodation on fiscal transfer in stimulating output is hump-shaped with respect to the length of the planning horizon.\textsuperscript{48} For a reasonable calibration as shown

\textsuperscript{48}Similar pattern also follows for the response of inflation, but with a different threshold.
in Table 1.1, it requires $\phi_\pi < \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta} \approx 101$, which is in the region of our major interest. The intuition of the hump-shaped relationship is that, when the planning horizon is long, the equilibrium is nearly Ricardian-equivalent, and thus fiscal policy is of little effect in stimulating output and inflation. When the planning horizon is short, because monetary policy works through forward looking, it becomes ineffective in this situation, and thus it matters little for fiscal policy.\footnote{The intuition for why no hump-shaped relationship exists for the case of $\phi_\pi > \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta}$ is that in such a situation, the monetary policy is responding so strongly to the inflation that even agents have extremely short foresight ($\rho \rightarrow 0$), the monetary policy still has a significant impact on output and inflation.}

Quantitatively, with the same size of one-time lump-sum transfer ($t^* = 1$), Figure 1.3 shows the output and inflation as to the length of the (average) planning horizon under alternative specifications of the Taylor rule. The solid line indicates a more accommodative monetary policy with $\phi_\pi = 1.5$, and the dotted line shows the case of $\phi_\pi = 2$. Obviously, as the monetary policy becomes more accommodative, the effect of lump-sum transfer is larger. But comparing the gap between the two lines shows that monetary policy matters most for the fiscal policy (in terms of stimulating output) only when decision makers have an intermediate degree of foresight, that is, they plan for about the next 10 years (or $h = 40$ quarters). The specification of monetary policy has little impact on the effects of fiscal policy in the case of short and long planning horizons.

In addition, to illustrate the roles of limited foresight for households and firms, respectively, assume households follow the distribution of $\omega_h = (1 - \rho) \tilde{\rho}^h$, whereas firms follow $\tilde{\omega}_h = (1 - \tilde{\rho}) \tilde{\rho}^h$ with $\rho, \tilde{\rho} \in (0, 1)$. Then, the solution to the policy experiment of lump-sum transfer with $\Gamma = 1$ for any period $t \geq 0$ is still time-invariant.\footnote{The solution is given by $y_t = \frac{(1 - \tilde{\rho})(1 - \rho)(1 - \tilde{\rho})}{(1 - \tilde{\rho} - \rho + \tilde{\rho} \sigma(\rho \tilde{\rho} - \rho))} \lambda t^*$ and $\pi_t = \frac{\kappa(1 - \rho)(1 - \tilde{\rho})}{(1 - \tilde{\rho} - \rho + \tilde{\rho} \sigma(\rho \tilde{\rho} - \rho))} \lambda t^*$.} For any given foresight of households $\rho < 1$, if firms become fully rational $\tilde{\rho} \rightarrow 1$, the responses of output and inflation $\{y, \pi\}$ are still positive and follow a pattern similar to the case of assuming the same distribution across households and firms. That is, even if firms are fully rational, Ricardian equivalence still breaks down and fiscal stimulus can be powerful. By contrast, for any given foresight of firms $\tilde{\rho} < 1$, if households become fully rational, $\rho \rightarrow 1$, the responses of output and inflation to the fiscal stimulus become zero. Therefore, the limited foresight of households compared with that of firms plays a more crucial role in the effect of fiscal stimulus.
Lump-sum Transfer Financed by a Temporary Increase in Real Public Debt

Due to the breakdown of Ricardian equivalence, the timing of financing the lump-sum transfer becomes important for output and inflation. To illustrate the effects of the speed of tax collections $\Gamma$, consider the case of zero steady-state real public debt $s_b = 0$ with no change in the long-run target of public debt $\lambda = 0$. Then, the evolution of real public debt is exogenously given by $b_1 = (1 - \Gamma)t^*$ and $b_{t+1} = [\beta^{-1}(1 - \Gamma)]t b_1$ for $\forall t \geq 1$. Given the path of real public debt, the system of equations (1.38)-(1.40) capturing the evolution of equilibrium from any period $t \geq 1$ can be re-written as

$$y_t = \rho E_t y_{t+1} - \sigma (i - \rho E_t \pi_{t+1}) + (1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)b_t$$

$$\pi_t = \kappa y_t + \beta \rho E_t \pi_{t+1}$$

with the rule of nominal interest rate $\hat{\iota}_t = \phi_{\pi} \pi_t$.

Conjecturing a solution of the form $y_t = \gamma y b_t$ and $\pi_t = \gamma_{\pi} b_t$ for $t \geq 1$, I then substitute $\{y_t, \pi_t\}$ into the above system of equations to get the unique solution in this form, given by

$$\gamma_y = \frac{[1 - \rho(1 - \Gamma)](1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{[1 - \rho \beta^{-1}(1 - \Gamma)][1 - \rho(1 - \Gamma)] + \kappa \sigma [\phi - \rho \beta^{-1}(1 - \Gamma)]}$$

$$\gamma_{\pi} = \frac{\kappa(1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{[1 - \rho \beta^{-1}(1 - \Gamma)][1 - \rho(1 - \Gamma)] + \kappa \sigma [\phi - \rho \beta^{-1}(1 - \Gamma)]}$$

Given the path of endogenous variables $\{y_t, \pi_t\}$ for any period $t \geq 1$, in the period $t = 0$, it can easily be solved for $\{y_0, \pi_0\}$.

By the expression of $y_t$ and $\pi_t$, given the speed of tax collections $\Gamma$, the effect of one-time lump-sum transfer since period $t = 1$ is transitory and decreases over time. As long as the speed

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51 For the real public debt to be non-explosive, I impose the assumption that $|\beta^{-1}(1 - \Gamma)| \leq 1$.

52 Given the coefficient in the Taylor rule is larger than one ($\phi_{\pi} > 1$), it follows that $\gamma_y, \gamma_{\pi} > 0$ for any $0 < \rho < 1$.

53 The output and inflation response in period 0 are given by $y_0 = \frac{\kappa(1 - \Gamma)\gamma_y}{1 + \sigma \phi_{\pi}} [\rho(\gamma_y + \sigma \gamma_{\pi}) + (1 - \rho)(1 - \beta)]$ and $\pi_0 = \frac{\kappa(1 - \Gamma)\gamma_{\pi}}{1 + \sigma \phi_{\pi}} [\rho(\gamma_y + \sigma \gamma_{\pi}) + (1 - \rho)(1 - \beta)]$. 

52
Figure 1.4: Output and inflation in period 0 with one-time lump-sum transfer financed by a temporary increase in real public debt under different speeds of taxation

Notes: $t = 1$, $\phi = 1.5$, $\lambda = 0$; $h$ in quarters; no real disturbances.

of tax collection $\Gamma$ is not extremely small, i.e., $\Gamma \geq \max\{1 - \beta, \bar{\Gamma}\}$, where $\bar{\Gamma} = \frac{[\kappa \sigma(\phi_\pi \beta - 1)]^{1/2} - 1 + \rho}{\rho}$, $\gamma_y$ and $\gamma_\pi$ is strictly decreasing with respect to $\Gamma$. Details of the proof can be found in Appendix A.7. Thus, the stimulative effect of lump-sum transfer decreases as the government collects taxes more quickly.

Taking the calibration from Table 1.1, Figure 1.4 illustrates the role of $\Gamma$ with respect to the planning horizon $h$ by the response of output and inflation in period $t = 0$. A similar pattern follows in any given period since $t = 0$. The parameter $\Gamma = 0.1$ indicates the half-life for the real public debt to converge back to zero is about 7 quarters, $\Gamma = 0.2$ indicates the half-life is around 3 quarters, and $\Gamma = 0.4$ indicates the half-life is 1 quarter. Without a change in the long-run debt target, a lump-sum transfer together with a slow speed of tax collections to repay the lump-sum transfer improves the short-term effect of fiscal stimulus.

If the coefficient in the Taylor rule is not too small, i.e., $\phi_\pi > \beta^{-1}$, then $\gamma_y$ and $\gamma_\pi$ is strictly decreasing with respect to $\Gamma$ as long as $\Gamma \geq \max\{1 - \beta, \bar{\Gamma}\}$. If the coefficient in Taylor rule satisfies $\phi_\pi < \beta^{-1}$, the condition for the monotonicity of $\{\gamma_y, \gamma_\pi\}$ with respect to $\Gamma$ is relaxed; that is, it only requires $\Gamma > 1 - \beta$. Details can be found in Appendix A.7. But note that $\phi_\pi$ can not be too small; otherwise, $\sum y_t^h$ and $\sum \pi_t^h$ do not converge. More details about the determinacy condition are discussed in Section 1.3.
Figure 1.5: Output and inflation in period 0 with a one-time lump-sum transfer financed by a temporary increase in real public debt under accommodative monetary policy

Notes: $t = 1, \Gamma = 0.025, \lambda = 0; h$ in quarters; no real disturbances.

To illustrate the impact of monetary policy accommodation on fiscal stimulus, given a low speed of tax collections $\Gamma = 0.025$ (i.e., the half-life of real public debt is around 45 quarters), Figure 1.5 shows the impact of alternative specifications of monetary policy for the effect of fiscal transfer in period $t = 0$. Similar to the discussion in the previous section, how accommodative monetary policy is matters most for the effect of fiscal policy when decision makers have an intermediate degree of foresight.

1.4.2 Forward Guidance and Interaction with Lump-sum Transfer

Now, I turn to the discussion of unconventional monetary policy, namely, “forward guidance,” and discuss how fiscal stimulus can add to the unconventional monetary policy. First, with inactive fiscal policy (i.e., the level of real public debt remains unchanged over time), the effect of forward guidance is rebated when decision makers are more short-sighted. The intuition is that, since forward guidance stimulates the output and inflation through foresight, as households and firms are less forward-looking, the stimulative effect of this unconventional monetary policy is more
limited. Therefore, the model of finite forward planning provides a natural explanation for the “forward guidance puzzle” (e.g., Del Negro, Giannoni, and Patterson, 2015). Woodford (2019) has a detailed discussion on the effect of forward guidance under finite planning horizon (which abstracts from fiscal sector). Appendix A.8 gives quantitative examples of the forward guidance policy with respect to different lengths of planning horizon.

Quantitatively, borrowing from Gust, Herbst, and López-Salido (2019), the (average) planning horizon in the US is estimated to be one quarter. With the assumption of a steady-state inflation at $\bar{\Pi} = 2\%$ and the calibration from Table 1.1, a commitment of staying at the effective zero lower bound by the central bank for $T = 10$ quarters has a limited effect in stimulating output and inflation; that is, the response of output and inflation in the period of the policy announcement for forward guidance is $1.0\%$ and $0.04\%$, respectively. As shown in Appendix A.8, the central bank’s commitment to stay longer at the effective zero lower bound is of little effect due to the short enough planning horizon. Therefore, it leaves a demand for fiscal stimulus when the (average) planning horizon is short.

More importantly, if a fiscal stimulus through lump-sum transfer is imposed simultaneously with forward guidance, can it achieve anything more than a simple summation of the two? For illustration, consider the policy experiment of the forward guidance as proposed in García-Schmidt and Woodford (2015) and Woodford (2019) together with a one-time lump-sum transfer fully financed by real public debt and the real public debt being unchanged thereafter.

Specifically, suppose prior to period $t = 0$, the economy stays at the steady-state equilibrium. The central bank announces in period $t = 0$ that from this date to some future date $t = T$, monetary policy will follow the rule of $\phi_\pi = 0$ with $i^* = i^* < 0$, and at date $t = T$, the monetary rule will revert back to the “normal” policy reaction function $\hat{i}_t = \phi_\pi \bar{\Pi}$. This policy experiment mimics the situation in which the central bank sets the nominal interest rate at the effective zero lower bound for a fixed time, and the negative $i^*$ represents that the nominal interest rate at the effective zero lower bound is smaller than the steady-state inflation rate $\bar{\Pi}$ (target rate). Further suppose a (lump-sum) fiscal stimulus simultaneously happens in period $t = 0$ as discussed in Section 1.4.1;
that is, a one-time lump-sum transfer is introduced that is fully financed by debt in period \( t = 0 \), and then the real public debt is kept constant thereafter (implying \( \Gamma = \lambda = 1 \)). For simplicity, I assume no real disturbances occur.

The fiscal policy ensures the real public debt is kept at \( b_t = b^* = t^* \) since period \( t = 0 \). Then, the equilibrium starting from period \( t = T \) is the one described in Section 1.4.1. From the expression (1.43)-(1.44), the endogenous output and inflation for any period \( t \geq T \) is given by

\[
\begin{align*}
y_t &= \frac{(1 - \beta \rho)(1 - \rho)(1 - \beta)}{(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)} t^* \\
\pi_t &= \frac{\kappa (1 - \rho)(1 - \beta)}{(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)} t^*
\end{align*}
\]

For any period \( 0 \leq t < T \), from the expression of (1.38)-(1.39) in Section 1.4.1, the system of equations capturing the evolution of the equilibrium is given by

\[
x_t = \rho ME_t x_{t+1} + Nu^* +Ns^*
\]

where \( x_t = [y_t \quad \pi_t]^T \), \( u^* = [-\sigma i^* \quad 0]^T \), and \( s^* = [(1 - \rho)(1 - \beta)b^* \quad 0]^T \).

It yields a unique solution for all \( 0 \leq t < T \), that is,

\[
x_t = x_T + [(I + \rho M + \cdots + (\rho M)^{T-t-1}] Nu^*
\]

and \( [(\rho M)^{T-t} - I] x_T + [(I + \rho M + \cdots + (\rho M)^{T-t-1}] Ns^* \)

Notably, the first term in expression (1.45) solely comes from the fiscal transfer, and the second term solely comes from the policy of forward guidance. The third and fourth terms come from the interaction of lump-sum transfer and forward guidance. The summation of the third and fourth term is proved to be always positive. Proofs can be found in Appendix A.9. That is, the unconventional
monetary policy together with a one-time debt-financed lump-sum transfer has a larger stimulative effect than a simple summation of the two.

Furthermore, the positive gain from the interaction of the two policies, that is, the summation of the third and fourth term in the expression (1.45), is not linear in terms of the planning horizon. By the calibration from Section 1.4.1, Figure 1.6 shows the effect of interaction between lump-sum transfer and forward guidance in the period $t = 0$ with respect to the (average) length of planning horizon. The solid line represents the interaction between lump-sum transfer and forward guidance, the dashed line represents the effect solely coming from forward guidance, and the dotted line represents the effect solely coming from lump-sum transfer.

From Figure 1.6, when the (average) length of the planning horizon is around five years (i.e., $h = 20$ quarters), the positive stimulative effect from policy interaction is the highest in the solid line. In either the case of too short a planning horizon or a really long horizon, the effect of the interaction is small. The intuition is similar to the discussion in Section 1.4.1, as the planning horizon increases, monetary policy becomes more effective, while fiscal policy becomes less powerful.
The amplification from monetary policy on fiscal stimulus initially plays a dominating role in the situation of a short horizon, and then the effect of policy interaction decreases due to the fiscal policy being more Ricardian-equivalent. Thus, the gain from policy interaction shows a hump shape with respect to the degree of foresight.

1.5 Long-run Dynamics of Fiscal Transfer Policy

In this section, I review the long-run consequences of those fiscal stimulus policies considered in the previous Section 1.4 (under the policy regime of “AM/PF” with long-run stability) by incorporating a learning process in decision makers’ value function as modeled in Section 1.3. Intuitively, as the decision makers adjust their value function more quickly, the behaviors of the equilibrium under the regime of “AM/PF” will be more Ricardian-equivalent. In the data, however, as suggested by Gust, Herbst, and López-Salido (2019), the learning process in the value function in the US is slow as measured by the gain in the learning process $\gamma = \tilde{\gamma} = 0.13$. That is, although the short-term effects of those policy experiments on output and inflation analyzed in Section 1.4 are rebated in the long run due to the learning in the value function, it is still quite persistent and quantitatively important.

Consider the fiscal stimulus of lump-sum transfer as specified in Section 1.4 with no real disturbances, and the monetary policy is specified by the Taylor rule. From the equation (1.36) in Section 1.3.3, which incorporates the constant-gain learning process in the value function, the system of equations characterizing the aggregate equilibrium since period $t = 1$ can be written as

$$
E_t \begin{bmatrix} x_{t+1} \\ \tilde{x}_{t+1} \\ b_{t+1} \end{bmatrix} = Y \begin{bmatrix} x_t \\ \tilde{x}_t \\ b_t \end{bmatrix} + \begin{bmatrix} B^* \\ 0 \\ \Gamma b^* \end{bmatrix}
$$
and in period $t = 0$, it gives\(^{56}\)

$$
\begin{equation}
E_0 \begin{bmatrix} x_1 \\ \tilde{x}_1 \\ b_1 \end{bmatrix} = Y \begin{bmatrix} x_0 \\ \tilde{x}_0 \\ b_0 \end{bmatrix} + B^* \begin{bmatrix} T^* \\ 0 \\ 0 \end{bmatrix} + \Gamma b^* \begin{bmatrix} 1 - \Gamma \end{bmatrix} t^*
\end{equation}
$$

Before moving to the quantitative analysis, the feature of a long-run stationary equilibrium proceeds with the following proposition:

**Proposition 3** Given the Taylor rule and lump-sum taxation scheme with fiscal stimulus, and gains in updating value function being positive $\gamma, \tilde{\gamma} > 0$, if a stationary equilibrium exists in the long-run, output gap and inflation in such an equilibrium have to satisfy $y = \pi = 0$, and the nominal interest rate satisfies $\bar{i} = 0$.

Proofs of Proposition 3 can be found in Appendix A.10. Proposition 3 indicates that if the equilibrium in the long run is stationary, the output gap and inflation have to converge back to the initial steady-state level before policy changes. The intuition is that after a long enough time with learning, agents will finally get the correct value function by incorporating the policy changes, and the long-run real rate will go back to the one before policy changes.

In terms of quantitative analysis, besides those parameters calibrated in Section 1.4.1, the average length of the planning horizon measured by $\rho$ is set to be $\rho = 0.5$ (e.g., Gust, Herbst, and López-Salido, 2019). The coefficient before inflation in the Taylor rule is set to be $\phi_\pi = 1.5$. When no update occurs in the value function $\gamma = \tilde{\gamma} = 0$, the above system of equations becomes the same one as in Section 1.4.1 for analyzing the short-term effects. Thus, the analyses in Section 1.4 are nested in the discussion with a learning process in the value function.

Figure 1.7 shows the policy of one-time debt-financed lump-sum transfer, and the real public debt is kept unchanged thereafter (i.e., the policy experiment considered in Section 1.4.1). The

\(^{56}\)The matrix $B^*$ and $T^*$ are given by

$$
B^* = \begin{bmatrix} \rho^{-1} & -\sigma(\beta \rho)^{-1} & -(1 - \rho)(1 - \beta)\psi_{\beta^*}b^* \\ 0 & (\beta \rho)^{-1} & 0 \end{bmatrix}
$$

and $T^* = \begin{bmatrix} \rho^{-1} & -\sigma(\beta \rho)^{-1} & -(1 - \rho)(1 - \beta)\psi_{\beta^*}t^* \\ 0 & (\beta \rho)^{-1} & 0 \end{bmatrix}$.
solid line in Figure 1.7 indicates the case of no update in the value function $\gamma = \ddot{\gamma} = 0$, the dashed line indicates the case of a small gain in learning of the value function $\gamma = \ddot{\gamma} = 0.13$ as estimated in Gust, Herbst, and López-Salido (2019) by matching to US data, and the dotted line indicates a large gain in learning of the value function.\textsuperscript{57}

In the case of no update in the value function, the output and inflation permanently increase due to both the one-time lump-sum transfer and the permanent increase in real public debt. As the size of the lump-sum transfer equals one-quarter GDP, the permanent increases in output and inflation are about 0.9\% and 0.04\%, respectively. The fiscal-transfer multiplier is around 0.94, which is close to one.

When there is non-zero gain of learning in the value function as shown in the dashed line and solid line, the responses of output and inflation in the initial period are similar as in the case of no update in the value function. But the effect of fiscal policy on aggregate output and inflation converges back to zero in the long run, as shown in subfigures (a) and (b). The reason is that the decision makers gradually update their value function, and the value function finally converges back to the one learned from the steady-state stationary equilibrium after a long enough time.\textsuperscript{58}

When the policy regime is set to be within the canonical policy regime of “AM/PF”, the standard New Keynesian model (e.g., Woodford, 2003) suggests fiscal policy should have no impact in determining output and inflation. By contrast, Figure 1.7 indicates fiscal policy under finite forward planning, together with monetary policy, always \textit{jointly} determines the output and inflation.

The slower the speed of updating in the value function is, the more persistent the effect of fiscal stimulus becomes, and suggests a larger fiscal-transfer multiplier. In the case of a large gain in the learning process of value function, as shown in the dotted line in Figure 1.7, the effect of lump-sum transfer disappears after nearly 75 years, or $t = 300$ quarters. In the dashed line calibrated to US data, indicating a small gain in the learning process of the value function, the effect of fiscal stimulus takes more than one hundred years to fade away. The fiscal-transfer multiplier is about

\textsuperscript{57}The gain of leaning in value function is set to be the same across households and firms, i.e., $\gamma = \ddot{\gamma}$.

\textsuperscript{58}The oscillating feature of the aggregate variables as shown in Figure 1.7 comes from the backward looking behavior through the learning in the value function.
Figure 1.7: Long-run dynamics with learning in the value function under a one-time debt-financed lump-sum transfer

Notes: $\Gamma = 1, \lambda = 1, \tau = 1, \phi = 1.5, \rho = 0.5; t$ in quarters.

0.27 for the case of a small gain of learning in the value function (dashed line), and about 0.06 for the case of a large gain (dotted line).

While Figure 1.7 shows the policy experiment with a permanent increase in the real public debt, Figure 1.8 shows the policy of a one-time lump-sum transfer *temporarily* (fully) financed by debt, and then lump-sum taxes are collected in each period to repay the initial transfer such that the long-run level of public debt does not change (i.e., the policy experiment discussed in Section 1.4.1 with a non-zero steady-state level of real public debt $s_b = 2.4$). In Figure 1.8, monetary policy is set to be the same under the Taylor rule ($\phi_\pi = 1.5$), and the speed of tax collections is set to be relatively small, namely, $\Gamma = 0.025$. Different from Figure 1.7, due to the policy specification that the long-run target of real public debt does not change, the path of real public debt as shown in subfigure (e) of Figure 1.8 converges to its long-run target zero. The subfigures (a) and (b) show that the output and inflation also converge back to zero in the long run. By comparing the three lines, we see that, as the speed of learning in the value function is slower, the effect of fiscal stimulus becomes more persistent.
Figure 1.8: Long-run dynamics with learning in the value function under a one-time debt-financed lump-sum transfer and no change in the long-run debt target

Notes: $\Gamma = 0.025$, $\lambda = 0$, $t = 1$, $\phi = 1.5$, $\rho = 0.5$, $s_p = 2.4$; $t$ in quarters.

1.6 Conclusion

This paper studies the effect of fiscal policy, specified by a lump-sum taxation scheme, in affecting output and inflation, and its interaction with monetary policy. In contrast to the standard New Keynesian model with rational expectations, fiscal policy and monetary policy always jointly determine the output and inflation under finite forward planning. Ricardian equivalence always breaks down, and “Ricardian” fiscal policy no longer exists. With an endogenous evolution of real public debt, as the length of the planning horizon becomes shorter, the policy space for long-run stability under active monetary policy with passive fiscal policy regime (“AM/PF”) increases, and the policy space of “PM/AF” decreases.\(^{59}\) The boundary condition for fiscal policy between the two scenarios almost does not change with respect to the degree of foresight, whereas the boundary condition of monetary policy rests heavily on the length of the planning horizon. More importantly, if the government and the central bank do not know the actual planning horizon of the population,\(^{59}\) as decision makers become less forward-looking, the “Taylor principle” for monetary policy becomes more relaxed.

\(^{59}\)
however, adopting a policy combination that satisfies the canonical “AM/PF” regime is more robust to ensure long-run stability.

Under the regime of “AM/PF” with long-run stability, this paper then evaluates the effect of fiscal transfers as a source of demand stimulus in both the short run and long run with an emphasis on its interaction with monetary policy. In general, as the length of the planning horizon becomes longer, the effect of monetary policy in stimulating output and inflation decreases because monetary policy works through forward-looking behavior. On the contrary, fiscal stimulus becomes much more powerful as decision makers become more short-sighted. The reason is that, agents take more near-future taxation into today’s decision-making, but not include those taxation in the far future.

Notably, more accommodative monetary policy improves the stimulative effects of fiscal stimulus. But the impact of monetary policy accommodation on the effect of fiscal policy is hump-shaped with respect to the length of the planning horizon.

In addition, the finite-planning-horizon model provides a natural explanation for the “forward guidance puzzle.” The limited effect of monetary policy in stimulating output and inflation generates a demand for fiscal stimulus. This paper suggests an unconventional monetary policy of forward guidance combined with a simultaneous fiscal stimulus such as debt-financed lump-sum transfer can reach an aggregate effect larger than a simple summation of the two. The effect of a positive interaction between the two policies is maximized also when decision makers have an intermediate degree of foresight.

In terms of long-run consequences of those fiscal stimulus through lump-sum transfers, it is initially powerful in stimulating output and inflation. As agents update their value function to incorporate the effects of such stimulus, the stimulative effect dampens over time. As the learning process in decision makers’ value function is slower, the responses of output and inflation become more persistent. Nevertheless, the quantitative analysis shows that the effect of fiscal stimulus is quite persistent even in the case of a relatively large gain in updating the value function.

In a world of widespread high-level debt and low equilibrium real interest rates in many coun-
tries, fiscal policy and government debt have become a more important issue to study. This paper inspires several directions for future research. For instance, Blanchard (2019) suggests a lower natural rate of interest in the long run. Exploring the implications of this observation with finite forward planning would be interesting in future work.
Chapter 2

Stabilization Policy in a Low-Interest-Rate World: Consequences of Limited Foresight

Yinx Xie and Michael Woodford
2.1 Introduction

The global financial crisis of a decade ago and the slow recovery from it have resulted in considerable experimentation with less-conventional approaches to stabilization policy, including both renewed interest in counter-cyclical fiscal policy and more aggressive use of “forward guidance” with regard to future monetary policy. They have also led to increased debate about more exotic proposals, such as price-level targeting as an alternative to purely forward-looking inflation targeting. There has also been reconsideration of the degree to which monetary policy should be formulated in a way that is completely independent of fiscal policy, with some arguing for the importance of a commitment to monetary accommodation of fiscal stimulus policies, or even allowing the central bank to control the size of fiscal transfers.¹

All of these policies have been the subject of a good deal of theoretical analysis, but generally under the assumption of rational expectations (RE) equilibrium – that if a novel policy is announced, people’s actions should adjust in a way that would be optimal under model-consistent predictions regarding the economy’s subsequent dynamics.

The RE assumption is always something of an idealization, but it seems particularly heroic in the case of very novel policies, like many of those recently considered, since one cannot suppose that people should know from previous experience how things should unfold under such a policy. Yet at the same time, conventional analyses of these policies lean quite heavily on assumptions that are made about what people anticipate about the future, often years in advance.

In this paper, we re-examine a number of these policies, and the issue of coordination between monetary and fiscal policies, under a more modest assumption about the degree to which people should be able to correctly foresee the future consequences of a novel policy. The approach that we take is the one proposed in Woodford (2019),² based on the architecture of state-of-the-art programs to play games of strategy such as chess or go. Our analysis assumes that in any period, both households and firms look forward from their current situations some finite distance into the future.

¹See, for example, Ascari and Rankin (2013), Buiter (2014), Turner (2017), and Galí (2019).
²A version of the model proposed in Woodford (2019) is empirically estimated in Gust et al. (2019).
to the possible situations that they can reach in the end period of foresight through some finite-horizon action plan; they use structural knowledge (including any announcements about novel government policies) to deduce the consequences of their intended actions over this horizon. For simplicity, we assume that the planning horizon is taken to be exogenously fixed.

Interim situations that someone imagines reaching in the end period of foresight are evaluated using a value function that has been learned from past experience. Crucially, we suppose that the value functions cannot be adjusted to take account of an unusual shock or a change in policy, if neither the shock nor the new policy is the one with which people have had much prior experience, though their value functions may be well-adapted to the prior environment. Under some circumstances, this kind of analysis leads to conclusions very similar to conventional RE analysis (at least, under a suitable equilibrium selection criterion), as discussed in Woodford (2019). However, a situation in which monetary policy is constrained by the zero lower bound (ZLB) for a period that may last longer than the length of many people’s planning horizons is one in which the finiteness of planning horizons can make a significant difference for the predicted macroeconomic dynamics.

We re-examine the consequences of particular combined monetary-fiscal regimes treated in the recent literature, when the ZLB is a relevant constraint on the effectiveness of conventional monetary policy, using the simple New Keynesian model with finite-horizon planning developed in Woodford (2019). Without any fiscal stimulus, we first focus on the robustness of conclusions about monetary policy to changes in the assumed degree of foresight on the part of households and firms in the economy, including strict inflation targeting, ad hoc price-level targeting (temporary price-level targeting), and systematic price-level targeting. As a complement to the monetary policy for stabilizing aggregate output and inflation, we then evaluate the effects of debt-financed government transfer policies with strict inflation targeting and the case with an accommodative monetary policy in which the interest rate target tracks the financial wedge as long as it is consistent with ZLB.\footnote{As another type of fiscal stimulus, Woodford and Xie (2019) show that the effects of government purchases depend on the length of decision makers’ planning horizons. In this paper, we focus on the discussion of fiscal transfer policies instead. In addition, in absence of the zero lower bound, Xie (2019) shows how the framework introduced in Woodford (2019) to analyze alternative monetary policies under the assumption of zero government debt at all times} Finally, we discuss the coordination in the optimal combined interest rate and
fiscal transfer policies, and emphasize the continuing relevance of forward guidance even in an economy with finite planning horizons.

Without losing generality, the fundamentals of the economy is described by a two-state Markov Chain, as in Eggertsson and Woodford (2003). For a long time prior to the period that crisis happens, the economy is assumed to have been in a stationary state in which (i) financial frictions have been negligible, (ii) the real public debt has been constant, and (iii) the inflation target has always been achieved, and that as a consequence, both households and firms have learned the value functions appropriate to that simple, stationary environment. At the period that crisis occurs, a financial disturbance occurs that increases the demand for safe assets, and causes the nominal interest rate on safe assets that would be required to maintain the inflation target to fall below the ZLB. Once the financial wedge has increased, we assume that each period there is a fixed probability of it continuing to have the same large value in the following period (i.e., the “crisis state” continues); otherwise, it reverts permanently to its previous small value (i.e., the “normal state”).

We first consider the case in which both monetary and fiscal policy remain unchanged, and monetary policy is specified by a strict inflation targeting (IT) rule, under which the central bank adjusts the interest rate as needed in order to keep inflation equal to a fixed inflation target if this is consistent with the ZLB, and if not sets the lowest interest rate possible. When planning horizons are finite, the contractionary effects of a financial crisis shock are less dramatic than in the RE analysis. In our numerical calibration, even if we assume a 10-year planning horizon for all households and firms, the contraction is only a bit more than half as severe as under the RE analysis. Nonetheless, if there is some degree of foresight, even a relatively modest financial wedge can substantially impact stabilization goals, raising the question whether tools besides conventional interest-rate policy are available to mitigate such effects.

can be extended to allow analysis of other types of fiscal transfer policies, and interactions between monetary and fiscal policy. Our focus of attention in this paper answers the particular questions about monetary-fiscal regimes, which Xie (2019) does not take up, and study the optimal combination of fiscal transfer policy and monetary policy, especially in the situations of the ZLB.

We follow the calibration proposed by Eggertsson (2011) and Woodford (2011), in which this kind of shock creates a “Great Depression” under the RE analysis.
Even in the absence of any fiscal stimulus, the contractionary effects of the financial shock can be reduced if it is possible to make credible commitments about the conduct of monetary policy after the ZLB ceases to prevent attainment of the inflation target. The effects of such “forward guidance” on aggregate demand during the crisis obviously depend on people’s being able to reason about future conditions using the information provided about the new policy. Hence, an assumption of finite planning horizons will weaken the stimulative effects of such a policy, relative to the predictions of the RE analysis of Eggertsson and Woodford (2003), as discussed in Woodford (2018). Nonetheless, as long as horizons are not too short, appropriate forward guidance should help to mitigate the effects of a financial shock despite the binding ZLB.

At least in the RE analyses, the problems resulting from the ZLB can be mitigated to an important extent if the central bank is committed to a price-level target rather than to a purely forward-looking inflation target. The policy of price-level targeting amounts to committing to do something later (after the lower bound ceases to be a binding constraint) that impedes achievement of the central bank’s stabilization goals at that later time, purely for the sake of beneficial effects of its being anticipated at an earlier date. If one doubts the degree to which the change in later policy will actually be anticipated earlier, this can greatly undermine the case for such a commitment. We accordingly examine the robustness of conclusions such as those in Eggertsson and Woodford (2003) about the advantages of commitment to a price-level target over commitment to an inflation target to different possible assumptions about the length of people’s planning horizons.

This can be illustrated by considering the effects of a commitment to keep the nominal interest rate at its lower bound until the price level is restored to a target path that grows deterministically at the target inflation rate: If the ZLB prevents the inflation target from being achieved during the crisis period, the price level will fall below this target path; the commitment then requires the interest rate to remain at the lower bound for a time even after the financial wedge becomes small again, even though this causes inflation above the rate for a time. Once the “price-level gap” has been closed, we suppose that monetary policy is used to achieve the target inflation rate thereafter.

Since policy after the gap has been closed is the same as under the IT policy, this commitment
is equivalent to a “temporary price-level target” (TPLT) of the kind proposed by Bernanke (2017). In particular, we assume that policy prior to the occurrence of the shock has been the simple IT regime, and that the value functions of households and firms are the ones appropriate to that regime, regardless of the new policy that may be announced during the crisis.

Compared with the case under the simple IT policy, the price level falls behind the target path to a much lower extent during the crisis under the TPLT commitment, in addition to being eventually returned to the target path within a few quarters of reversion to the “normal” state. The smaller price-level gap during the crisis corresponds to more successful output stabilization as well. Notably, the predicted dynamics are also more similar across different assumptions of planning horizons in the case of this policy, implying less distortions from heterogeneous responses in the case of heterogeneous planning horizons.

In order to obtain the advantages of commitment to a price-level targeting (PLT) when the ZLB binds, is it necessary for a central bank to always conduct its policy in accordance with a PLT, even when financial constraints are minimal, or does it suffice to announce a TPLT policy on an ad hoc basis only when a financial shock that causes the ZLB to bind occurs? Under the RE analysis, the two kinds of policy should achieve identical outcomes during a crisis period and the immediately following period in which the price-level gap is being closed. Since the advantages of a PLT over an IT regime are most compelling in the case of such a crisis, it might be thought that a TPLT policy offers the more prudent approach.

However, in the limited-foresight analysis, there is an advantage to a systematic PLT rule over an ad hoc commitment: this is that a different systematic approach to policy in the period before a financial crisis occurs can change the value functions that households and firms learn, and then apply in their forward planning during the crisis period. In the case of commitment to a PLT rule, we continue to assume that the value functions are not updated during a temporary period in which the ZLB binds, but we suppose that the value functions that have been learned are the ones that would be optimal under a PLT regime in which the ZLB never binds. In these latter value functions, the price-level gap is recognized as a crucial state variable (affecting the expected marginal value
Allowing households and firms to learn the value functions appropriate to a PLT regime results in less of a decline in prices (and associated with this, less contraction of output) when the ZLB binds. The difference is particularly great when planning horizons are relatively short. If we assume planning horizons of five years or more, the difference made by allowing different value functions to be learned is minimal, but the difference is considerable if the planning horizon is only a few quarters.

Some might assume that recognizing limitations on people’s ability to correctly deduce the future consequences of a new policy should reduce the theoretical benefits of commitment to policy rules, and hence favor a purely discretionary approach to policy. But this need not be the case when the planning horizon is finite. In our analysis, recognizing that planning horizons may be short reduces the predicted efficacy of ad hoc commitments in response to a special situation – and so increases the case for seeking to ensure that the “default” expectations implicit in people’s value functions are ones that help to stabilize the economy during a crisis. These “default” expectations are best shaped by systematic action in accordance with a relatively simple policy rule, since they are learned by induction from past experience, rather than being derived through deductive reasoning about a concrete current situation.

In the decades before the global financial crisis, it had become common to assume that monetary policy should be assigned responsibility for macroeconomic stabilization, with no role for cyclical variation in fiscal policy; but recognition that at times monetary policy will be constrained by the ZLB has increased attention to the benefits of using variation in the government’s budget as a tool of stabilization policy as well, at least when monetary policy is constrained by the ZLB. Finite planning horizons do increase the effectiveness of fiscal stimulus policies in one respect: they break Ricardian equivalence, and allow government transfers (or deficit spending) to stimulate aggregate demand.

In general, a debt-financed lump-sum transfer has an impact on both the output level and inflation rate; for while the present value of future taxes changes by the amount of the transfer,
changed taxes beyond households’ planning horizon are not taken into account. We show that state-contingent fiscal transfers can indeed reduce the contractionary impact of an increase in the financial wedge. A sufficiently aggressive fiscal transfer policy, combined with an interest-rate policy that tracks variations in the financial wedge as long as it is consistent with the ZLB, makes it possible to achieve complete stabilization of both aggregate economic activity and the overall rate of inflation, despite the zero lower bound, and regardless of the size of the increase in the financial wedge. Thus the existence of state-contingent transfer policies does expand the degree to which stabilization would be possible using interest-rate policy alone; and we obtain this result under conditions that would guarantee Ricardian equivalence under an assumption of rational expectations.

At the same time, we show that it would be a mistake to conclude that countercyclical transfers are so effective a tool that there is no need for a central bank to ever indicate that it would allow inflation to overshoot the bank’s long-run inflation target, nor any need for a commitment to conduct future interest-rate policy in any way different from what will best serve the bank’s goals at that future date. We find that state-contingent transfers make possible equilibria that could not be achieved using interest-rate policy alone, but that there is a limit to the stimulus that can be achieved even by massive fiscal transfers, in the absence of monetary accommodation — that is, a commitment not to raise interest rates, even if inflation overshoots its long-run target.

The reason is that, because of people’s finite planning horizons, it matters not only what happens in equilibrium, but what the central bank would be expected to do out of equilibrium; and the complete stabilization of macroeconomic aggregates actually depends on people’s understanding that the central bank is not determined to prevent over-shooting of the long-run inflation target under any circumstances. For instance, in the case of a strict inflation targeting policy, if there is an aggressively large fiscal transfer, the output gap is anticipated to close within the planning horizon, so that monetary policy is expected to be tightened in order to prevent inflation from over-shooting the target. Further increases in lump-sum transfer simply increase anticipated interest rates later in the planning horizon, preventing any further increase in desired spending.
Moreover, the important aspect of monetary policy is not what the central bank actually does during the period when the financial wedge is large (since the ZLB binds during this period); rather, it is what it leads people to believe that it would do, in the event that the ZLB were to cease to bind. Notably, the commitment to allow inflation to over-shoot increases aggregate demand even though the aggregate inflation does not ever over-shoot the target; the commitment matters only because of its effects on the (incorrect) calculations of myopic decision-makers. In this sense, commitments about the determinants of future interest-rate policy remain a crucial dimension of policy, even when aggressive use of government transfers is possible.

We also find that there is a limit to what can be achieved, even by coordinated fiscal and monetary policy, if the increase in the public debt and the monetary accommodation are both contemplated only for the period in which the financial wedge remains large, with an immediate return to both the normal level of public debt and the usual inflation target as soon as the wedge returns to a normal level; a higher level of welfare is possible, in general, if the monetary and fiscal authorities commit themselves to history-dependent policies in the period after the real disturbance has dissipated.

The paper proceeds as follows. Section 2.2 describes the New Keynesian DSGE model with finite planning horizon and the financial shocks considered in this paper. It also includes the discussion for effects of a strict inflation targeting policy with no fiscal stimulus when foresight is limited. Section 2.3 analyzes effects of price-level targeting; in particular, section 2.3.1 focuses on an ad hoc price-level targeting policy, i.e., temporary price-level targeting, and section 2.3.2 discusses a systematic price-level targeting rule. Section 2.4 introduces fiscal transfer policies, and emphasize the coordination between fiscal and monetary stabilization policy. Section 2.5 concludes the paper.
2.2 Output and Inflation Determination with Finite Planning Horizons

2.2.1 Forward Planning with a Finite Horizon

We study the consequences of limited foresight in a New Keynesian DSGE model with finite-horizon forward planning, building upon the approach developed in Woodford (2019). Households and firms make contingent plans for a finite distance into the future, and use a value function learned from past experiences to evaluate all possible terminal states in the last period of the planning horizon. Over this horizon, they use structural knowledge (including any announcements about novel central bank or government policies) to deduce the consequences of their intended actions. For simplicity, we assume that the planning horizon is taken to be exogenously fixed.

We illustrate the approach by briefly discussing here the problem of households in our model.\(^5\) As in standard New Keynesian models, we assume an economy made up of infinite-lived households, here assumed to be identical apart from possible differences in their planning horizons. But rather than assuming that each household formulates an infinite-horizon state-contingent expenditure plan, we suppose that any date \(t\), a state-contingent expenditure plan is selected only for dates between \(t\) and some date \(t + h\), a finite distance in the future. (We call \(h\) the household’s planning horizon, and in the present paper we treat this as exogenously given, though endogenizing the decision about how far into the future to plan would clearly be a desirable extension of the theory.)

Letting \(C^i_t\) be household \(i\)’s planned consumption in period \(\tau\) of a composite good (a CES aggregate of the many differentiated goods produced in the economy), we suppose that at time \(t\) the household chooses state-contingent values \(\{C^i_\tau\}\) for each of the dates \(t \leq \tau \leq t + h\) (specifying real expenditure in each of the exogenous states that may arise at any of those dates,\(^6\) given the state of the world at the time of the planning) so as to maximize the expected value (according to

\(^5\)The decision problem of price-setting firms is treated using similar methods in Woodford (2019).

\(^6\)Given the value of aggregate real expenditure \(C^i_\tau\), we suppose that the household’s purchases of each of the individual differentiated goods in period \(\tau\) is chosen so as to acquire the desired quantity of the composite good at minimum cost. This sub-problem of optimal allocation of expenditure across individual goods within a given period is a static optimization problem, the form of which is unaffected by our assumption of a finite planning horizon.
the household’s calculations at time \( t \) of an objective of the form

\[
\sum_{\tau=t}^{t+h} \beta^{\tau-t} u(C^i_{\tau}) + \beta^{h+1} v(B^i_{t+h+1}; s_{t+h}).
\]

Here the first terms represent the discounted sum of flow utilities from consumption in periods \( t \) through \( t+h \), while the final term represents the household’s estimate of the value of the discounted sum of flow utilities that it can expect to receive in later periods, if the wealth that it holds at the end of the planning horizon is \( B^i_{t+h+1} \). We allow in general for the possibility that the value assigned to the household’s continuation problem after period \( t+h \) may depend on the state of the world \( s_{t+h} \) that has been reached in period \( t+h \).

In the household’s planning exercise, it takes into account its budget constraint, and thus the way in which the value of \( B^i_{t+h+1} \) will depend on its planned level of expenditure. As in many simple New Keynesian models, we assume that there exists only a single financial asset each period, a one-period riskless nominal debt instrument, the interest rate \( i_t \) on which is also the central bank’s policy instrument. Because wealth can take this single form, the implications of the household’s choices over the planning horizon for the value of its continuation problem can be summarized by a single quantity, \( B^i_{t+h+1} \), indicating the wealth carried into period \( t+h+1 \) in the form of this riskless nominal asset.

The evolution of this quantity is determined by a flow budget constraint of the form

\[
B^i_{\tau+1} = (1 + i_\tau + \Delta_\tau) \left[ \frac{B^i_\tau}{\Pi_\tau} + Y_\tau + T_\tau - C^i_\tau \right] - \Delta_\tau \left[ \frac{B_\tau}{\Pi_\tau} + T_\tau \right]
\]

for each period \( t \leq \tau \leq t+h \). Here \( B^i_\tau \) is the value of the nominal debt held by the household that matures at date \( \tau \), deflated by the period \( \tau-1 \) price index \( P_{\tau-1} \), so that it is a predetermined

\[\text{The state } s_{t+h} \text{ refers to variables whose evolution is not under the control of the household, unlike the variable } B^i_{t+h+1} \text{ that depends on its own actions (as discussed further in the next paragraph). But the value function may depend on endogenous economic conditions, such as the general level of prices in period } t+h, \text{ and not only on exogenous states.}\]

\[\text{As usual, this price index is the minimum cost at which a unit of the composite good can be purchased in period } \tau-1.\]
real variable. This quantity must be deflated by $\Pi_\tau \equiv P_\tau / P_{\tau - 1}$, the gross inflation rate between $\tau - 1$ and $\tau$, to obtain the real value of the maturing debt in units of the period $\tau$ composite good. The term $Y_\tau$ indicates production of the composite good in period $\tau$, the value of which is received as income by the households (and treated as independent of any household decision, in the household’s forward planning exercise); and $T_\tau$ is the value of lump-sum government transfers (the same to each household), also in units of the composite good. Hence $B^l_\tau / \Pi_\tau + Y_\tau + T_\tau - C^l_\tau$ is the value of the household’s end-of-period asset balances, in units of the composite good.

Because one-period riskless nominal debt is assumed to be the only traded asset, all of the household’s saving must be held in this form. These assets earn a nominal financial yield of $i_\tau$ between periods $\tau$ and $\tau + 1$. In addition, we assume that there is an additional benefit of holding riskless claims, which we represent in (2.1) as an additional dividend equal to $\Delta_\tau$ per unit of savings held in this form. This additional dividend is intended to represent the existence of a (time-varying) safety premium as in the models of Del Negro et al. (2017) and Caballero and Farhi (2017); increases in the size of such a premium are an important reason for the lower bound on the safe nominal interest rate to become a tighter constraint during financial crises.\footnote{The only consequence of a non-zero value of $\Delta_\tau$ in our model is the introduction of a time-varying factor in the household Euler equations (2.4)--(2.5) below. The same kind of exogenous shift factor in the Euler equation could alternatively arise from exogenous variation in households’ rate of time preference, as assumed in Eggertsson and Woodford (2003). While the latter assumption would allow for a simpler and more conventional model, we believe that variation in the size of the financial wedge represented by $\Delta_\tau$ provides a more realistic picture of the kind of disturbance that is likely to give rise to the policy challenges that we address in this paper.}

The final term in (2.1) is a lump-sum effective tax on households, equal in size to the safety dividend received by households in aggregate (using the notation $B_\tau$ for the aggregate supply of public debt carried into period $\tau$). This indicates that the advantages to an individual household of holding more safe assets are at the expense of other households (as the “safety dividend” does not correspond to any additional resources created by the safe assets). We model the safety premium \{\Delta_\tau\} as an exogenous process, satisfying $\Delta_\tau \geq 0$ at all times; we do not consider in this paper the possibility of government policies that can directly affect the size of this wedge.\footnote{For example, one might well suppose that the size of the safety premium can be reduced by increasing the government supply of safe assets, either through increased government borrowing or by central-bank purchases of non-safe}
The decision problem of a household at time $t$ depends on the financial wealth $B_t^i$ that it brings into the period (a predetermined variable, and therefore known at the time of the decision about how much to spend in period $t$). It also depends on the household’s expectations about the state-contingent evolution of the variables $\{\Pi_t, Y_t, T_t, i_t, \Delta_t\}$ over periods $t \leq \tau \leq t + h$, that is, the household’s planning horizon. We assume that in their forward planning exercises, households make use of correct structural information about how the economy works (including a correct understanding of monetary and fiscal policy, taking into account any new policies that may have been announced in response to an unexpected exogenous disturbance). First, we assume a correct understanding of the state-contingent evolution of all exogenous state variables; this means that households correctly understand the current value of $\Delta_t$ (since they know the economy’s exogenous state, before undertaking forward planning), and the conditional probability of different possible future paths $\{\Delta_t\}$.

Second, households are assumed to correctly understand the rules that will determine the policy variables $\{T_t, i_t\}$ over the planning horizon. For simplicity, we restrict attention in this paper to fiscal rules under which the path of the real public debt $\{B_t\}$ is exogenously specified;\(^\text{12}\) this allows us to consider both the case of no public debt (often assumed in analyses of alternative monetary policies), and various ways in which the level of public debt might depend on the path of the financial wedge $\{\Delta_t\}$. Aggregating over households, and assuming no government purchases,\(^\text{13}\) it follows from (2.1) that the evolution of the real public debt must satisfy

$$B_{\tau+1} = (1 + i_{\tau}) \left[ B_{\tau} / \Pi_{\tau} + T_{\tau} \right]$$

(2.2)

for each of the periods $t \leq \tau \leq t + h$. This, like other structural equations of our model, is assumed

\(^{12}\)See Xie (2020) for analysis of regimes in which there is instead feedback from endogenous variables to the path of real public debt, including “active” fiscal policy regimes according to the classification of Leeper (1991).

\(^{13}\)The framework can easily be extended to allow for government purchases as well. See Woodford and Xie (2019) for analysis of how the government purchases multiplier is affected by finite planning horizons.
to be correctly understood by households. Then the assumption that fiscal policy is specified by an
exogenous process \( \{B_\tau\} \) implies that \( T_\tau \) must endogenously adjust, to ensure that (2.2) is satisfied,
in response to any changes in \( i_\tau \) by the central bank, or changes in \( \Pi_\tau \) as a result of firms’ pricing
decisions. Except in section 2.4, we shall further simplify the analysis by considering only regimes
in which \( B_\tau = 0 \) at all times, which implies that \( T_\tau = 0 \) at all times as well.

We similarly assume that households correctly understand the way in which \( i_\tau \) will be de-
termined under any contingency by the central bank’s policy. For example, if the central bank
follows a Taylor rule which requires some relation linking \( i_\tau, \Pi_\tau \) and \( Y_\tau \) to be satisfied, then the
state-contingent evolutions of those variables assumed in a household’s forward planning will nec-
essarily satisfy that relation. We further specify the monetary policies to be considered below, but
note here that any feasible policy is assumed to be subject to a zero lower bound (ZLB) constraint

\[
i_t \geq 0
\]  

(2.3)
at all times.\(^{14}\)

Finally, households are also assumed to correctly understand how the variables \( Y_\tau \) and \( \Pi_\tau \) are
determined by the decisions of households and price-setting firms respectively. However, in order
not to have to model how the economy should evolve (or anyone else should be modeling it to
evolve) beyond the horizon \( t + h \), a household with horizon \( h \) at time \( t \) must model \( Y_\tau \) and \( \Pi_\tau \) as
being determined by households and firms who do not look beyond the horizon \( t + h \) while making
their decisions at time \( \tau \). Just as the household, in its planning at time \( t \), models its own behavior
at some later date \( \tau \) as the behavior that will appear optimal to someone with a planning horizon
at that time of only \( t_h - \tau \) periods, it similarly models the behavior of other households and firms
at date \( \tau \) under the assumption that they will all have planning horizons of \( t_h - \tau \) periods. This
means that the household will model all other households as spending the same amount at time \( \tau

\(^{14}\)Whether the effective lower bound on the short-term nominal interest rate is exactly zero is not crucial to our
conclusions, though we assume that the lower bound is zero in the numerical calibration discussed below. What is
important is that there is some fixed lower bound, that constrains the set of policies that can be considered; and that it
is possible for a financial disturbance to occur that is large enough for this to become a binding constraint.
as it plans itself to spend at that time. Hence the amount of income $Y_t$ that it expects to receive in any future state will be the same as the amount $C_t^j$ that it expects to spend in that state.

Let $Y_t^j, \Pi_t^j, i_t^j$ be the (counterfactual) output, inflation, and nominal interest rate in the case that all economic units (households and firms) have a planning horizon of $j \geq 0$ periods at time $t$. Then the Euler equation for optimal forward planning requires that for any $j \geq 1$,

$$u'(Y_t^j) = \beta(1 + i_t^j + \Delta_t) E_t[u'(Y_{t+1}^{j-1})/\Pi_{t+1}^{j-1}]$$

(2.4)

while for $j = 0$,

$$u'(Y_t^0) = \beta(1 + i_t^0 + \Delta_t) v'(B_{t+1}).$$

(2.5)

In (2.5) we use the fact that in equilibrium, a household with planning horizon zero must anticipate an interest rate $i_t^0$ that leads it to choose to hold wealth $B_{t+1}^0$ equal to the exogenously specified supply of public debt $B_{t+1}$ (given that it expects other households to optimize over the same planning horizon as it does, and it expects the debt market to clear).

Thus we obtain a system of equations that can be recursively solved for the state-contingent evolution of the variables $\{Y_t^j\}$ for each possible horizon $j \geq 0$, given the state-contingent evolution of the endogenous variables $\{\Pi_t^j, i_t^j\}$ for all $j$, and the state-contingent evolution of the exogenous variables $\{\Delta_t, B_{t+1}\}$, along with any exogenous disturbances to the monetary policy rule.$^{15}$ (Equation (2.5) can be solved for the value of $Y_t^0$ in any state of the world, given the values of the other variables; then given a solution for the state-contingent evolution of $\{Y_t^0\}$, the $j = 1$ case of equation (2.4) can be solved for the value of $Y_t^1$ in any state of the world; and so on for progressively higher values of $j$.)

Modeling the optimizing decision of price-setting firms with finite planning horizons, we similarly obtain a system of equations that can be recursively solved for the state-contingent evolution of the variables $\{\Pi_t^j\}$ for each possible horizon $j \geq 0$, given the state-contingent evolution of $\{\Delta_t, B_{t+1}\}$, along with any exogenous disturbances to the monetary policy rule.$^{15}$ (Equation (2.5) can be solved for the value of $\Pi_t^0$ in any state of the world, given the values of the other variables; then given a solution for the state-contingent evolution of $\{\Pi_t^0\}$, the $j = 1$ case of equation (2.4) can be solved for the value of $\Pi_t^1$ in any state of the world; and so on for progressively higher values of $j$.)

$^{15}$The model can easily be extended to allow for exogenous disturbances to productivity, preferences, and government consumption, as treated in Woodford (2019); but in this paper, we are concerned only with possible policy responses to disturbances to the financial wedge $\Delta_t$. 

79
the endogenous variables \( \{Y_t^{j}, i_t^{j}\} \) and the state-contingent evolution of the exogenous variables. These equations, together with the monetary policy rule with which the endogenous variables must be consistent for each value of \( j \), provide a system that can be jointly solved for the state-contingent evolution of the endogenous variables \( \{Y_t^{j}, \Pi_t^{j}, i_t^{j}\} \) for each possible horizon \( j \geq 0 \), given the state-contingent evolution of the exogenous variables.

In writing the above equations, we take as given the value function \( v(B) \) that households will use in their forward planning, and similarly the value function that firms will use. In Woodford (2019), the endogenous evolution of these value functions in response to additional experience is also modeled; here, however, we abstract from this additional source of dynamics, and assume fixed value functions, that will be the same for the different policies that we consider.\(^{16}\) Our assumption is that the value functions are determined in a backward-looking way (as an inference from outcomes observed in the past), and not through a forward-looking deductive process; the whole point of the use of a value function to evaluate conditions that might be reached at the planning horizon \( t + h \) is to avoid having to reason deductively about what should happen under various contingencies beyond that date.

Thus when an unusual shock hits, and unusual policies are announced in response, the value functions that households and firms use, at least initially, will continue to be ones that they learned from macroeconomic conditions prior to either the disturbance or the new policies. Because our concern in this paper is solely with the effects of temporary policy changes in response to a transitory disturbance, we simplify the discussion by abstracting from the changes in the value functions that would eventually occur if the new conditions were to persist sufficiently long.\(^{17}\) Instead we assume that the value functions remain fixed over the scenarios that we consider below, and are

\(^{16}\)An exception is our discussion in section 2.3.2 below of the way that consistent adherence to a price-level targeting rule should eventually lead to a change in the value function. But also in that section, we do not model the dynamics of adjustment of the value function in response to a new policy, and only consider the value function that should eventually be reached in the case of an environment that remains stationary for a long enough time.

\(^{17}\)Allowing the value functions to adapt is instead critical for certain other kinds of discussions. These include consideration of the eventual effects of commitment to an interest-rate peg for a long period of time, as in Woodford (2019); empirical modeling of US economic data over a period of decades, that included significant shifts in both output and inflation trends, as in Gust et al. (2019); and analysis of the conditions under which joint monetary-fiscal policy regimes imply sustainable long-run dynamics, as in Xie (2020).
ones that represented an optimal adaptation to the stationary conditions assumed to have existed prior to the disturbance.

In most of the analyses below,\textsuperscript{18} the situation prior to the disturbance is assumed to have been one in which the government debt has been zero \((B_t = 0\) at all times); the central bank has pursued a forward-looking inflation targeting policy, setting \(i_t\) each period at the level required to ensure that \(\Pi_t = \Pi^*\), the long-run inflation target;\textsuperscript{19} and the financial wedge \(\Delta_t\) has at all times been small enough to make it possible for the central bank to achieve that target without violating the zero lower bound (2.3). In a stationary equilibrium in which these conditions always hold, the maximum attainable discounted utility for a household that enters period \(t\) with wealth \(B\) is given by

\[
v(B) = \frac{1}{1 - \beta} u(\bar{Y} + (1 - \beta)B/\bar{\Pi}),
\]

where \(\bar{Y}\) and \(\bar{\Pi}\) are the stationary values of \(Y_t\) and \(\Pi_t\).\textsuperscript{20} This is the optimal value function for households in this stationary environment; its use in a finite-horizon planning exercise in the stationary environment would result in optimal behavior, regardless of the length of the planning horizon. It is also the value function to which the adaptive process described in Woodford (2019) would converge, if such an environment were maintained for a sufficiently long time. Thus we assume the value function (2.6) for households in our analyses below; we similarly assume for firms a value function that is optimally adapted to that same stationary environment.

\subsection*{2.2.2 Log-Linear Approximate Dynamics}

As in many rational-expectations analyses, it will be convenient to approximate the solution to the model structural equations using a log-linear approximation. We linearize the model’s equa-

\textsuperscript{18}Again the discussion in section 2.3.2 is an exception, where we consider the effects of permanently conducting monetary policy in accordance with price-level targeting.

\textsuperscript{19}This target is assumed to satisfy \(\Pi^* > \beta\), so that a stationary equilibrium is possible in which this inflation rate is maintained at all times, and in this equilibrium, the ZLB constraint (2.3) is a strict inequality. Note that this will be satisfied in the case of any non-negative inflation target.

\textsuperscript{20}Under the inflation-targeting policy, the real return on assets that do not earn the safety premium will be constant and equal to the rate of time preference of households. A household’s optimal policy will then be the one given by the permanent income hypothesis: it should plan to consume a constant amount \(C = \bar{Y} + (1 - \beta)B/\bar{\Pi}\) each period, allowing it to maintain a constant wealth \(B\) indefinitely. This results in the discounted utility indicated by (2.6).
tions around a stationary equilibrium in which \( \Delta_t = 0 \) at all times, and the policy regime is the one discussed above for which the value functions of households and firms are adapted. We express the linearized equilibrium relations in terms of deviations from the stationary equilibrium values of the various state variables, using the following notation:

\[
y_j^t = \log(Y_j^t/\bar{Y}), \quad \pi_t = \log(\Pi_t/\bar{\Pi}), \quad b_t = B_t/(\bar{\Pi}\bar{Y}),
\]

\[
i_t = \log\left(\frac{1 + i_t}{1 + \bar{i}}\right), \quad \hat{\Delta}_t = \frac{\Delta_t}{1 + \bar{i}}.
\]

Here \( \bar{i} \equiv \beta^{-1}\bar{\Pi} - 1 > 0 \) is the stationary equilibrium value of the nominal interest rate.

In terms of this notation, equilibrium conditions (2.4) and (2.5) can be linearized to yield

\[
y_j^t = -\sigma(i_j^t + \hat{\Delta}_t - E_t\pi_{t+1}^{j-1}) + E_t y_{t+1}^{j-1} \tag{2.7}
\]

for each \( j \geq 1 \), and

\[
y_0^t = -\sigma(i_0^t + \hat{\Delta}_t) + (1 - \beta)b_{t+1}. \tag{2.8}
\]

Note that except for the superscripts, (2.7) has the same form as the “New Keynesian IS equation” obtained in the rational-expectations version of the model (see, e.g., Woodford, 2003, chap. 4).

Similarly, the structural relations describing optimal price-setting behavior by firms can be log-linearized to yield

\[
\pi_j^t = \kappa y_j^t + \beta E_t\pi_{t+1}^{j-1} \tag{2.9}
\]

for each \( j \geq 1 \), and

\[
\pi_0^t = \kappa y_0^t. \tag{2.10}
\]

(See Woodford, 2019, for the derivation.) Here again, it will be observed that except for the superscripts, (2.9) has the same form as the “New Keynesian Phillips curve” obtained in the rational-expectations version of the model (Woodford, 2003, chap. 3).
Finally, in terms of the deviations variables, the zero lower bound constraint can be written as

\[ \hat{y} \geq \hat{y} \]  

(2.11)

where \( \hat{y} < 0 \), meaning that the constraint does not bind when \( i_t \) is near its stationary equilibrium value \( \bar{i} \).\(^{21}\)

2.2.3 A Crisis Scenario

We consider the effects of alternative monetary and fiscal policies under the following scenario: prior to date \( t = 0 \), we suppose that the economy has for a long time been in the stationary equilibrium discussed above, in which the financial wedge has always been small, the government’s budget has been balanced each period (so that government debt has remained equal to zero), and the inflation target \( \pi^* \) has been consistently achieved. As a result, households and firms have learned the value functions that are appropriate to a stationary environment of that kind. At time \( t = 0 \), however, an unexpected financial disturbance occurs, and the economy enters a “crisis” state, in which there is a substantial financial wedge \( \hat{\Delta}_t > 0 \) between the return on safe assets (balances held at the central bank) and other assets.

We further assume the crisis state persists to the following period, whenever the economy is currently in that state, with probability \( 0 < \mu < 1 \), while with probability \( 1 - \mu \) the economy reverts back to its “normal” state, in which we suppose that the financial wedge \( \hat{\Delta}_t \) will subsequently equal zero forever after. For simplicity, we assume that the probability of exit from the crisis state is independent of the length of time already spent in that state. We further assume that the size of the financial wedge while in the crisis state is constant; thus the exogenous fundamental \( \{\Delta_t\} \) evolves according to a two-state Markov chain, as in Eggertsson and Woodford (2003). We write the constant financial wedge in the crisis state as \( \hat{\Delta}_t = -\hat{i} + \Delta \), where \( \Delta > 0 \); the latter quantity measures the degree to which the financial wedge is too large to be offset through a

\(^{21}\)If the lower bound is exactly zero, then we will have \( h_{att} = -(r^* + \pi^*)_t < 0 \), where \( r^* = \beta^{-1} - 1 \) is the stationary equilibrium real rate of return inclusive of the safety premium. This is assumed in our numerical calibration, but our qualitative results depend only on our assumption that \( \hat{i} < 0 \).
contemporaneous interest-rate reduction.\footnote{It is the fact that $\Delta > 0$ that means that the inflation target can no longer be maintained at all times, using only conventional interest-rate policy and with a balanced government budget.} It is the fact that $\Delta > 0$ that means that the inflation target can no longer be maintained at all times, using only conventional interest-rate policy and with a balanced government budget.

**Numerical calibration**

We illustrate a number of our conclusions about the effects of alternative policies under such a scenario for economic fundamentals using numerical computations. In these calculations, we calibrate the model — including our assumption about the size and persistence of the disturbance to fundamentals — largely in accordance with the parameter values proposed by Eggertsson (2011), who shows that under the assumption of rational expectations and a zero inflation target, these parameter values would imply a contraction of the size experienced by the US economy during the Great Depression, as shown by Eggertsson (2011). However, in this paper, we specify “normal” monetary policy as involving an inflation target $\pi^*$ of two percent per year, rather than a target of zero inflation, as in Eggertsson’s model of the Great Depression. This makes the zero lower bound a less severe constraint in our scenario than in the one considered by Eggertsson, since we continue to assume the same size of increase in the financial wedge as in his Depression scenario.

In our numerical calculations, the periods of our discrete-time model are identified with quarters. We set the subjective discount factor $\beta = 0.997$, the slope of the Phillips curve $\kappa = 0.00859$, and the elasticity of intertemporal substitution $\sigma = 0.862$. The shock required to account for the size of the contraction during the Great Depression is one in which $\hat{\Delta} = 0.013$,\footnote{In the notation of Eggertsson (2011), this quantity corresponds to $\Delta = -r - \pi^*$, where $r < 0$ is the natural rate of interest in the crisis state.} and the probability of staying in the crisis state is $\mu = 0.903$, so that the expected length of a crisis is about 10 quarters. In addition, we assume a long-run inflation target of 2 percent per year; that is, $\pi^* = 0.005$ in quarterly terms, which implies that the part of financial wedge that cannot be offset by monetary policy owing to the ZLB is $\hat{\Delta} = 0.005$, or two percent per year.\footnote{This is a quarterly rate; thus the assumed increase in the size of the financial wedge is a bit greater than 5 percent per annum. The natural rate of interest in the normal state is $r^* = \beta^{-1} - 1$, or slightly above 1 percent per annum; thus we assume that in the crisis state, the natural rate of interest falls to -4 percent per annum, as in Eggertsson (2011).} The calibrated parameter values

\footnote{Note that this is only half the size of $\hat{\Delta}$ in the crisis state considered by Eggertsson (2011).}
Table 2.1: Calibrated parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject discount factor</td>
<td>$\beta = 0.997$</td>
</tr>
<tr>
<td>Response of inflation to output gap in Phillips curve</td>
<td>$\kappa = 0.00859$</td>
</tr>
<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\sigma = 0.862$</td>
</tr>
<tr>
<td>Financial wedge in “crisis” state</td>
<td>$\hat{\Lambda} = 0.013$</td>
</tr>
<tr>
<td>Probability of staying in “crisis” state</td>
<td>$\delta = 0.903$</td>
</tr>
<tr>
<td>Inflation target</td>
<td>$\pi^* = 0.005$</td>
</tr>
</tbody>
</table>

are summarized in Table 2.1.

**Consequences of orthodox policy**

We first consider the consequences of a temporary large increase in the size of the financial wedge (the “crisis scenario” explained above), in the case that monetary and fiscal policy continue to be conducted as under normal conditions, which is to say as assumed above in our discussion of the stationary equilibrium prior to the occurrence of the shock. We assume that the government budget continues to be balanced each period, so that $B_{t+1} = 0$ at all times, and that the central bank continues to conduct monetary policy in accordance with a strict inflation target. The latter stipulation implies that in each period, $\hat{i}_t$ will be set as necessary to ensure that inflation is equal to the target rate ($\pi_t = 0$, in our deviations notation), if this is consistent with the ZLB; if inflation undershoots the target in any period $t$ even when the interest rate is at its lower bound, then $\hat{i}_t$ will equal $\hat{i}$ in that period (the policy as close as possible to achieving the inflation target in that period, taking as given the expected conduct of monetary policy in all future periods).

Let us first recall the analysis of such a situation under the assumption of rational expectations by Eggertsson and Woodford (2003) and Eggertsson (2011). The linearized equations of the RE model can be written in vector form as

$$z_t = A E_t z_{t+1} - \sigma a (\hat{i}_t + \hat{\Lambda}_t),$$  \hspace{1cm} (2.12)
where we define

\[
\begin{bmatrix}
H & C \\
C & C
\end{bmatrix}, \quad 1 & f^V + f \\
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

(Note that the path of public debt is irrelevant, owing to Ricardian Equivalence.) Under the assumption that \( \hat{\Delta} \) evolves according to the two-state Markov process and that \( \hat{i} \) is chosen according to the inflation targeting policy, there exists a rational-expectations solution that is also Markovian, in the sense that the vector \( z_t \) takes only two possible values: a vector \( \bar{z} \) in any period \( t \) in which the crisis state persists, and the zero vector in each period after the reversion to the normal state (in which case the inflation target is achievable each period from then on).\(^{25}\)

In the case that

\[ \kappa \sigma \mu < (1 - \mu)(1 - \beta \mu), \]

the matrix \( A \) has two positive real eigenvalues, both less than \( \mu^{-1} \), and the Markovian solution is also the unique bounded solution to the linear system (2.12). This condition holds if and only if

\[ \mu < \hat{\mu}, \tag{2.13} \]

where \( \hat{\mu} \) is a bound between zero and 1 that depends on the values of \( \kappa \sigma \) and of \( \beta \); this is the case considered by Eggertsson and Woodford (2003).\(^{26}\) In this case, the Markovian RE solution is given by

\[ z_{t} = \bar{z}_{RE} \equiv -\sigma (I - \mu A)^{-1} a \Delta << 0. \tag{2.14} \]

In this equilibrium, both output and inflation remain persistently below their target values as

\(^{25}\)Mertens and Williams (2018) call this the “target equilibrium”; it is not the only possible RE solution, even if one restricts attention to Markovian solutions. It is however the solution emphasized in the RE literature, following Eggertsson and Woodford (2003); we show below that restriction of attention to this RE solution can be justified as the limit of the unique solution associated with a model with finite planning horizons, when the length of the planning horizons is made arbitrarily long.

\(^{26}\)A Markovian rational-expectations solution can also be defined when \( \mu \) exceeds the bound (2.13), but in this case it does not correspond to the limit of an equilibrium with finite-horizon planning, as planning horizons are made arbitrarily long.
long as the crisis state continues, but return immediately to their target values as soon as the financial wedge returns to its normal (negligible) value. As Eggertsson and Woodford show in a calibrated example, this equilibrium can involve quite a severe contraction as well as substantial deflation, in response to even a few percentage points’ increase in the financial wedge. We now examine the robustness of these conclusions to allowing for finite planning horizons.

Assume again that the central bank adheres to a strict inflation targeting policy, and suppose also that there is also no government debt (so that the fiscal authority maintains a balanced budget).27 Equations (2.7) and (2.9) can then be written in vector form as

\[ I^j = A E_t z^j_{t+1} - \sigma a (i^j_t + \hat{\Delta}_t) \]  

for each \( j \geq 1 \), using the notation \( z^j_t \) for the vector \([y^j_t, \pi^j_t]'\), while (2.8) and (2.10) can be written as

\[ z^0_t = -\sigma a (i^0_t + \hat{\Delta}_t) + (1 - \beta) a b_{t+1}. \]  

Under the assumption of zero public debt, equations (2.16) imply that an expectation of strict inflation targeting requires that horizon-zero agents expect an interest rate

\[ i^0_t = \max\{-\hat{\Delta}_t, \hat{\underline{i}}\}. \]

Under the assumption that the financial wedge evolves as a two-state Markov chain, this implies that \( z^0_t = 0 \) if \( t \) is any date after the reversion to the normal state, while

\[ z^0_t = \zeta^0_t \equiv -\sigma a \Delta \ll 0 \]  

if \( t \) is any date at which the crisis state continues.

We can then use this result to solve recursively for the behavior of households and firms with

\[ ^{27}\text{This is a common assumption in New Keynesian models used for monetary policy analysis, though in models where Ricardian Equivalence would hold, it is without loss of generality. With finite planning horizons, the assumption is not innocuous, as we show in section 2.4.} \]
progressively longer planning horizons. First we observe that if \( t \) is any date after the reversion to the normal state, \( z_t^h = 0 \) for all \( h \). This can be established recursively; we first show that if \( z_t^h = 0 \) for all dates after the reversion to normal for some horizon \( h \geq 0 \), (2.15) implies that inflation targeting will require \( \bar{t}^{h+1}_t = 0 \) at any date after the reversion to normal, and hence that \( z_t^{h+1} = 0 \) as well. Then the fact that we have already shown that \( z_t^0 = 0 \) after the reversion to normal implies that \( z_t^h = 0 \) for all \( h \).

Next, consider instead dates \( t \) at which the crisis state continues, and suppose that it has already been established for some horizon \( h \) that in any crisis state, \( z_t^h = z^h \), where \( z^h \) is a vector that is negative in both elements. Then it follows from (2.15) that even if \( \bar{t}^{h+1}_t = \bar{z} \) (the most expansionary possible monetary policy that can be expected), in any crisis state the vector \( z_t^{h+1} \) will equal

\[
z_t^{h+1} = \mu A z_t^h - \sigma a \Delta << 0.
\]

Hence the ZLB will necessarily bind, and we will have \( z_t^{h+1} = z^{h+1} \) in any such state, where

\[
z^{h+1} = \mu A z^h - \sigma a \Delta << 0. \tag{2.18}
\]

It follows that the equilibrium will be Markovian,\(^\text{28}\) and that the sequence of vectors \( \{z^h\} \) characterizing the Markovian equilibrium can be computed recursively, using (2.18) together with the initial condition (2.17).

This system of equations can be recursively solved to yield

\[
z^h = -\sigma \sum_{j=0}^{h} (\mu A)^j a \Delta << 0 \tag{2.19}
\]

for any planning horizon \( h \geq 0 \). Note that the solution is well-defined for any finite \( h \); if in addition to our more general assumptions, \( \mu \) satisfies the bound (2.13), the solution has a well-defined limit

\(^\text{28}\)Note that this is not an assumption (equilibrium selection criterion), as in the case of the rational-expectations analysis; we have shown that in the case of finite-horizon planning, there is necessarily a unique solution, and that it has this property.
Figure 2.1: Expenditure and rates of price increase during the crisis period, under different assumptions about the planning horizon $h$ (in quarters) of households and firms, when the central bank follows a strict inflation targeting policy and there is no response of fiscal policy.

as $h$ is made unboundedly large. In this latter case, we find that as $h \to \infty$, $\underline{\pi}^h \to \underline{\pi}^R E$, so that the unique equilibrium with finite-horizon planning approaches the Markovian rational-expectations equilibrium discussed above. It follows that any long enough finite planning horizon will lead to outcomes similar to those in the RE analysis.

If planning horizons are only of modest length, however, the quantitative predictions of the model with finite-horizon planning are different from those of the RE analysis. Since each of the terms in the sum (2.19) is a vector with both elements negative, it is evident that both $\underline{y}^h$ and $\underline{\pi}^h$ are more negative the longer the planning horizon. This is illustrated in Figure 2.1, for the numerical parameter values listed in Table 1.

This solution tells us the value of $\underline{y}^h$ and $\underline{\pi}^h$ for each possible planning horizon $h$. These calculations are the same regardless of the distribution of planning horizons in the economy. For a given distribution of planning horizons $\{\omega_h\}$, we can then compute the predicted state-contingent evolution of aggregate output and inflation by aggregating the individual decisions of the agents with
different horizons. In the case of an exponential distribution of planning horizons, the condition required for the infinite sum $\sum_{h=0}^{\infty} \omega h^h$ to converge — and hence for there to be a well-defined equilibrium under the assumed policies — is

$$\rho \mu < \bar{\mu},$$

(2.20)

where $\bar{\mu}$ is defined as in (2.13). This is thus a weaker condition than (2.13), that requires only that the product $\rho \mu$ not be too large; it will be satisfied if either most planning horizons are not too long ($\rho$ is well below 1) or the financial disturbance is not expected to last too long ($\mu$ is well below 1), or both. In the case that it is satisfied, aggregate outcomes in the crisis state will be given by

$$z = -\sigma [I - \rho \mu A]^{-1} a \Delta << 0$$

(2.21)

Note that if (2.20) is satisfied, (2.21) is the unique solution to our model, not simply one among multiple possible solutions, as in the rational-expectations analysis. In the case that (2.13) is satisfied, the solution (2.21) approaches the RE solution specified in (2.14) as $\rho$ approaches 1; this provides a possible justification for selecting that solution in a rational-expectations analysis.

We see from Figure 2.1 that when households and firms have finite planning horizons, the contractionary and disinflationary effects of an increase in the financial wedge are less severe than in a rational-expectations analysis; the more short-sighted people are assumed to be, the milder the effects. Even if we assume a 10-year planning horizon for all households and firms (the case $h = 40$), the contraction is only slightly more than half as severe as under the RE analysis. Nevertheless, assuming some degree of foresight, the ZLB can pose a serious problem, under these assumptions about policy. (A larger increase in the financial wedge would produce a correspondingly larger contraction than those shown in the figure.) Thus it is still desirable to explore whether alternative

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29 Even under rational expectations, the contraction is only half as large in our calculations as in those of Eggertsson (2011), because the higher inflation target assumed in our calculations mean that the value of $\Delta$ is only half as large for us. Thus the contractions shown in Figure 2.1 are much smaller than a Great Depression, even when planning horizons are long.
policies can mitigate this problem.

2.3 Price-Level Targeting Reconsidered

What are the advantages of a price-level target? At least in RE analyses, the problems resulting from an effective lower bound on nominal interest rates can be mitigated to an important extent – without requiring the use of any other dimensions of policy for stabilization purposes, such as counter-cyclical fiscal policy – if the central bank is committed to a price-level target rather than to a purely forward-looking inflation target. Indeed, in the New Keynesian model of Eggertsson and Woodford (2003), an optimal state-contingent monetary policy commitment in the face of arbitrary possible shock processes can be formulated as a type of price-level targeting rule. The advantage of a price-level targeting rule over a formulation in terms of an inflation target (chosen to imply the same target rate of growth of prices as in the price-level target path) is that, when a nominal aggregate demand shortfall occurs owing to a binding lower bound on interest rates, the price-level shortfall that results will automatically require more expansionary policy later, in order to catch back up to the price-level target path; and the anticipation of this response (in an RE analysis) should stimulate demand even during the period when policy is constrained.\(^{30}\)

However, such a policy commitment amounts to committing to do something later (after the lower bound ceases to be a binding constraint) that impedes achievement of the central bank’s stabilization goals at that later time, purely for the sake of beneficial effects of its being anticipated at an earlier date; if one doubts the degree to which the change in later policy will actually be anticipated earlier, this can greatly undermine the case for such a commitment. In this section, we accordingly examine the robustness of conclusions such as those in Eggertsson and Woodford (2003) about the advantages of commitment to a price-level target over commitment to an inflation target to differing possible assumptions about the length of people’s planning horizons.

Another issue to be reconsidered in this section is the difference between a “temporary price-

\(^{30}\)For additional analyses of the potential advantages of price-level targeting in mitigating the distortions created by the zero lower bound on nominal interest rates, see for example Bernanke, Kiley and Roberts (2019) and Mertens and Williams (2019).
level target”, the kind proposed by Bernanke (2017), and a commitment to a price-level target more generally (even when the effective lower bound has not recently been a binding constraint and is not expected to bind anytime soon). Bernanke proposes that the advantages during binding-ZLB episodes of an expectation that policy will remain loose for a period after the ZLB ceases to constrain policy can be obtained without having to change how policy is conducted at other times, by committing on occasions when the ZLB is reached to keep interest rates at the lower bound thereafter for as long as may be needed to get the price level back to its (unchanged) target path; but with the central bank no longer constrained to pursue anything other than its forward-looking inflation target once the price-level gap has been closed. If one is concerned only with the effects of a single shock, of a size and magnitude that causes the ZLB to become a binding constraint, then these are the same (in the RE analysis of a linearized NK model) in the case of the “temporary PLT” commitment and a PLT rule that applies at all times. Given that, a practical policymaker might well think that a more complex policy, possibly controversial and complicated to explain to the public, should not be used routinely, but only resorted to only on an ad hoc basis when sufficiently extreme circumstances arise to warrant unusual measures – essentially treating the possibility of a temporary PLT commitment as an additional tool of discretionary policy, rather than adopting a PLT as a policy rule.

The argument that use of a temporary PLT on an ad hoc basis when the ZLB binds can achieve as much as a PLT rule, at least with regard to the mitigation of stabilization losses when the ZLB binds, depends however on RE analysis. If we assume that people have only finite planning horizons (and furthermore that at least some of them have horizons shorter than the time that may be required to eliminate the price-level gap following a binding-ZLB episode), then even if people take the temporary PLT commitment into account in their explicit forward planning, it will not change the value function that they use to evaluate the positions that they expect to be in at the end of their planning horizon. If instead the central bank were to conduct policy in accordance with a price-level target (or target path) at all times, one might expect that people should eventually learn

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31See also Hebden and López-Salido (2018) and Bernanke, Kiley and Roberts (2019) for further analysis of this proposal.
a value function appropriate to such an environment, which will be the one that takes into account the size of the price-level gap.

In this section, our analysis consider the difference that this should make for the effects of a shock that causes the ZLB to bind, under the two different types of PLT policies. An important general point that we make is that taking into account bounded rationality of the kind that we propose should strengthen the case for conducting policy in accordance with a systematic policy rule, rather than on a purely discretionary basis; instead, a high degree of confidence in people’s ability to reason deductively about the future consequences of announced policies would imply that a great deal can be achieved through skillful discretionary use of temporary policy commitments, so that this discretionary approach might well be preferred for the sake of flexibility.

2.3.1 Temporary Price-level Targeting (TPLT)

In this section, we study the effects of a commitment to keep interest rate at the ZLB until price level is restored to its trend path with constant inflation rate $\pi^*$. After the price level is restored to the trend path, the monetary policy switches back to the inflation targeting rule. We focus on the case under which the decision makers in the economy do not update their value function, since it can be simply a commitment when a rare financial shock occurs. It captures the idea of “temporary price-level target” suggested by Bernanke (2017). For simplicity, we assume that all the households and firms share the same planning horizon.

Consider a policy in which the central bank defines a price-level target path $\{P_t^*\}$ that grows deterministically at rate $\pi^*$, i.e., $\log P_{t+1}^* = \log P_t^* + \pi^*$ for all $t$, and achieves this target whenever it is consistent with the ZLB constraint, and sets $\hat{y}_t$ as low as possible otherwise. Its policy rule therefore requires that

$$\tilde{p}_t \leq 0 \quad (2.22)$$

at all times, where the price-level gap is defined as $\tilde{p}_t = \log P_t - \log P_t^*$, and that either (2.11) or (2.22) must hold with equality in each period.
Then the evolution equation for the price-level gap is given by

\[ \tilde{p}_t = \tilde{p}_{t-1} + \pi_t \]  \hspace{1cm} (2.23)

where \( \pi_t \) is again the inflation rate in excess of the target rate \( \pi^* \).

We specify the shock of financial wedge the same as in section 2.2.3. Then, the solution is also in the Markovian form: under this policy commitment, the structural equations looking forward from any date \( t \) depend only on the value of \( \tilde{p}_{t-1} \), which enters (2.23), and the current fundamental state \( \hat{\Delta}_t \) either in the “normal” or “crisis” state. Thus, once the “normal” state is reached, the solution \( z^h_t = [y^h_t \pi^h_t] \) will be of the form \( z^h_t = \tilde{z}^h(\tilde{p}_{t-1}) \) thereafter; while in the “crisis” state, the solution will be of the form \( z^h_t = \hat{z}^h(\hat{p}_{t-1}) \). Our goal is to compute the functions \( \tilde{z}^h(\hat{p}) \) and \( \hat{z}^h(\hat{p}) \) for arbitrary \( h \geq 0 \) with arbitrary values of \( \hat{p} \leq 0 \).

Once the “normal” state is reached, there exists a sequence of critical values \( \{ \tilde{p}^j \} \) for the price-level gap, as the values to be computed later, with the property that

\[ \ldots < \tilde{p}^3 < \tilde{p}^2 < \tilde{p}^1 < \tilde{p}^0 < 0 \]

and, for any horizon \( j \geq 0 \), the “price gap” \( \tilde{p}^j \) satisfies the property such that (i) if \( \tilde{p}_{t-1} \geq \tilde{p}^j \), the price-level target is expected to be reached before the end of the planning horizon (i.e., by period \( t + j \) or earlier), while (ii) if \( \tilde{p}_{t-1} < \tilde{p}^j \), the ZLB is expected to bind over the entire planning horizon, i.e., through period \( t + j \).

If \( \tilde{p}^j \leq \tilde{p}_{t-1} < \tilde{p}^{j-1} \), then decisions are the same for all horizons \( h \geq j \), i.e., for all agents such that \( \tilde{p}^h \leq \tilde{p}_{t-1} \), \( z^h_t \) depends only on the length of time until the price-level target is expected to be realized (\( j \) periods in the future), not the exact horizon of the agent. But if instead \( h < j \), \( z^h_t = \hat{z}^h \) is independent of the value of \( \tilde{p}_{t-1} \), and \( z^h_t \) depends only on the planning horizon, not the size of the current price gap.

Thus, for any price gap \( \tilde{p}_{t-1} \leq 0 \), there is a horizon \( \tau(\tilde{p}_{t-1}) \), which is a decreasing function of \( \tilde{p}_{t-1} \) (i.e., as horizon \( \tau \) is longer, the more negative the price gap is), with the property that
the price-level target is expected to be reached at \( t + \tau \) by all decision makers with horizons long enough to expect the target to be reached during their planning horizon.

For any planning horizon \( h \geq \tau(\bar{p}_{t-1}) \), we have \( z_t^h = \bar{z}(\bar{p}_{t-1}) \), where the vector of functions \( \bar{z}(\bar{p}_{t-1}) \) is horizon-independent; while for any planning horizon \( h < \tau(\bar{p}_{t-1}) \), we have \( z_t^h = \bar{z}^h \), where the sequences \( \{\bar{z}^h\} \) are independent of the price gap. Note that, for each \( h \geq 0 \), \( \bar{z}^h = \bar{z}(\bar{p}^h) \).

Hence, for any planning horizon \( h \), we have

\[
z_t^h = \bar{z}(\bar{p}_{t-1}) \quad \text{if} \quad h \geq \tau(\bar{p}_{t-1}), \quad z_t^h = \bar{z}(\bar{p}^h) \quad \text{otherwise}
\]

By stating in this way, we see that the solution can be completely described by a pair of functions \( \bar{z}(\bar{p}_{t-1}) \) and the function \( \tau(\bar{p}_{t-1}) \), which encodes the values of the sequence \( \{\bar{p}^j\} \).

Now, suppose that we change the length of the time steps in our discrete-time model, making successive “steps” only a very short period of additional time. In the continuous limit, \( \tau(\bar{p}) \) becomes a continuously decreasing function. In this limiting case, we can equivalently describe any given price gap \( \bar{p} \) using the implied length of time \( \tau(\bar{p}) \) until the price-level target is expected to be reached by any agent with a horizon equal to \( \tau(\bar{p}) \) or longer. We can rewrite the functions \( \bar{z}(\bar{p}) \) as \( \bar{z}(\tau) \) instead, and for any given \( \tau \), the ZLB binds and thus the aggregate demand and supply conditions yield

\[
\frac{d \bar{y}}{d \tau} = \sigma \rho^* + \sigma \bar{\pi}(\tau) \tag{2.24a}
\]

\[
\frac{d \bar{\pi}}{d \tau} = \gamma [\bar{y}(\tau) - \lambda \bar{\pi}(\tau)] \tag{2.24b}
\]

for all \( \tau > 0 \). Here \( \rho^* > 0 \) is the instantaneous nominal interest rate corresponding to the one-period nominal interest rate \( r^* + \pi^* \) defined by \( \rho^* \delta = (r^* + \pi^*) \), where \( r^* \) is the natural rate of interest in the “normal” state and \( \delta \) is the length of a “period” in the discrete-time model, \( \gamma > 0 \) is the slope of the continuous time Phillips curve relation corresponding to the slope \( \kappa \) in the discrete-time model, i.e., \( \gamma = \frac{\kappa}{\delta^2} \), and \( \lambda > 0 \) is the slope of the relationship between the steady-state inflation and steady-state output implied by the NK Phillips Curve, i.e., \( \lambda = \frac{(1-\beta)\delta}{\kappa} \). The system
(2.24a)-(2.24b) can be solved from boundary conditions \( \tilde{\pi}(0) = \tilde{y}(0) = 0 \).

More specifically, the system (2.24a)-(2.24b) can be expressed in matrix form:

\[
\begin{bmatrix}
\frac{d\tilde{y}}{d\tau}
\frac{d\tilde{\pi}}{d\tau}
\end{bmatrix} =
\begin{bmatrix}
0 & \sigma \\
\gamma & -\lambda\gamma
\end{bmatrix}
\begin{bmatrix}
y + \lambda\rho^*
\pi + \rho^*
\end{bmatrix}
\]

where the coefficient matrix has two real eigenvalues \( x_1 < 0 < x_2 \), i.e., the roots of \( x^2 + \lambda\gamma x - \sigma\gamma = 0 \).

The solution consistent with the boundary conditions is thus given by

\[
\begin{bmatrix}
\tilde{y}(\tau) \\
\tilde{\pi}(\tau)
\end{bmatrix} =
\begin{bmatrix}
-\lambda\rho^* \\
-\rho^*
\end{bmatrix} +
\begin{bmatrix}
\sigma x_2 \\
x_1(x_2 - x_1)
\end{bmatrix} \rho^* e^{x_1\tau} +
\begin{bmatrix}
-x_1 \\
x_2(x_2 - x_1)
\end{bmatrix} \rho^* e^{x_2\tau}
\]

(2.25)

for all \( \tau \geq 0 \). We can then integrate the solution for \( \tilde{\pi}(\tau) \) to obtain

\[
\tilde{\rho}(\tau) \equiv -\int_0^\tau \tilde{\pi}(s) \, ds = \rho^* \tau + \rho^* \frac{x_2}{x_1(x_2 - x_1)} [1 - e^{x_1\tau}] - \rho^* \frac{x_1}{x_2(x_2 - x_1)} [1 - e^{x_2\tau}]
\]

(2.26)

Notably, (2.25) implies \( \tilde{\pi}(\tau) > 0 \) for all \( \tau > 0 \), so that \( \tilde{\rho}(\tau) \) must be a monotonically decreasing continuous function. \(^{32}\) Hence, we can invert the function \( \tilde{\rho}(\tau) \) to obtain

\[
\tau(\tilde{\rho}) \equiv (\tilde{\rho})^{-1} [\tilde{\rho}] \geq 0
\]

(2.27)

for any \( \tilde{\rho} \leq 0 \). Although we cannot give an analytical expression for this function, it can be numerically computed by computing the function \( \tilde{\rho}(\tau) \) given by (2.26). Notably, the previous defined \( \tilde{\rho}^j \) in the discrete time case is equal to \( \tilde{\rho}(j) \) for any integer \( j \geq 0 \).

The solution for dynamics in the “normal” state, in the continuous limit, are then given by: for

\[^{32}\) Note that \( \tilde{\pi}(\tau) > 0 \) can be proved via

\[
\tilde{\pi}(\tau) > -\rho^* + \rho^* \frac{x_2}{(x_2 - x_1)(1 + x_1\tau)} - \rho^* \frac{x_1}{(x_2 - x_1)(1 + x_1\tau)} = 0
\]
any agent with a horizon \( h \) such that \( \tilde{p}(t) \geq \tilde{p}(h) \), where \( \tilde{p}(t) \) is the current existing price-level gap when decision is made and \( \tilde{p}(h) \) is defined in (2.26), the solution is \( \tilde{y}^h(t) = \tilde{y}(\tau(\tilde{p}(t))) \), \( \tilde{p}^h(t) = \tilde{p}(\tau(\tilde{p}(t))) \), with the functions \( \tilde{y}(\tau) \) and \( \tilde{p}(\tau) \) defined in (2.25) and \( \tau(\tilde{p}) \) given by (2.27).\(^{33}\) For any agent with a horizon \( h \) such that \( \tilde{p}(t) \leq \tilde{p}(h) \), the solution is \( \tilde{y}^h(t) = \tilde{y}(h) \) and \( \tilde{p}^h(t) = \tilde{p}(h) \).

Thus, if the economy reverts to the “normal” state at time \( T \), for an agent with a horizon \( h \) such that \( \tilde{p}(T) \geq \tilde{p}(h) \), the subsequent evolution will be expected to be:

\[
\begin{align*}
    z(t) &= \begin{cases} 
    \tilde{y}(\tau(\tilde{p}(T)) - (t - T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\
    0 & \text{for all } T + \tau(\tilde{p}(T)) \leq t \leq T + h 
    \end{cases} \\
    \tilde{p}(t) &= \begin{cases} 
    \tilde{p}(\tau(\tilde{p}(T)) - (t - T)) & \text{for all } T \leq t \leq T + \tau(\tilde{p}(T)) \\
    0 & \text{for all } T + \tau(\tilde{p}(T)) \leq t \leq T + h 
    \end{cases}
\end{align*}
\]

where the vector \( z(t) = [y(t) \pi(t)]' \).

Given that everyone has the same horizon \( h \), the actual dynamics for \( t \geq T \) will be exactly the same as expected. Even though the expected dynamics assumes that everyone’s horizon shrinks as time goes forward, and this is not true for the actual dynamics (actually everyone’s horizon continues to be \( h \) at all times), it continues to be true that \( \tilde{p}(t) \geq \tilde{p}(h) \) for all \( t \geq T \) since \( \pi(t) \geq 0 \) implies \( \tilde{p}(t) \) to be non-decreasing in \( t \). Thus, the solution for the actual dynamics continues to be given by the same formulas as are assumed in people’s forward planning.

For an agent with a horizon \( h \) such that \( \tilde{p}(T) \leq \tilde{p}(h) \), the subsequent evolution for all \( T \leq t \leq T + h \) will instead be expected to become

\[
\begin{align*}
    z(t) &= \tilde{y}(h - (t - T)) \\
    \tilde{p}(t) &= [\tilde{p}(T) - \tilde{p}(h)] + \tilde{p}(h - (t - T))
\end{align*}
\]

But the actual dynamics, if everyone has the same horizon \( h \) and \( \tilde{p}(T) \leq \tilde{p}(h) \), will be given

\(^{33}\)Here \( h \) is now a continuous length of time instead of a discrete number of periods.
by

\[ z(t) = \bar{z}(h), \quad \bar{p}(t) = \bar{p}(T) + \bar{\pi}(h) \cdot (t - T) \]

as long as it continues to be the case that \( \bar{p}(t) \leq \bar{p}(h) \). This latter inequality will hold as long as \( t \leq T + [\bar{p}(h) - \bar{p}(T)]/\bar{\pi}(h) \). To clarify, at this latter date, \( \bar{p}(t) = \bar{p}(h) \leq 0 \), so the ZLB will still be binding, and the solution calculated above will still apply. But after that finite date, we will have \( \bar{p}(t) \geq \bar{p}(h) \), and the solution in the case of \( \bar{p}(T) \geq \bar{p}(h) \) will apply from then on. In summary, the actual dynamics in the “normal” state will be given by

\[
\begin{align*}
    z(t) = & \begin{cases} 
    \bar{z}(h - (t - T)) & \text{for all } T \leq t \leq T + \frac{[\bar{p}(h) - \bar{p}(T)]}{\bar{\pi}(h)} + h \\
    0 & \text{for all } t \geq T + \frac{[\bar{p}(h) - \bar{p}(T)]}{\bar{\pi}(h)} + h 
    \end{cases} \\
    \bar{p}(t) = & \begin{cases} 
    \bar{p}(h - (t - T)) & \text{for all } T \leq t \leq T + \frac{[\bar{p}(h) - \bar{p}(T)]}{\bar{\pi}(h)} + h \\
    0 & \text{for all } t \geq T + \frac{[\bar{p}(h) - \bar{p}(T)]}{\bar{\pi}(h)} + h 
    \end{cases}
\end{align*}
\]

Now, we consider the solution while the economy is still in the “crisis” state. Instead of the differential equations specified in the system (2.24a)-(2.24b) that apply in the “normal” state, by the aggregate demand and supply equations for optimal forward planning, we have

\[
\begin{align*}
    \frac{dy}{dt} &= -\sigma \Delta^* + \sigma \pi + \nu [\bar{y} - y] \\
    \frac{d\pi}{dt} &= \gamma [y - \lambda \pi] + \nu [\bar{\pi} - \pi]
\end{align*}
\]

where \( \Delta^* > 0 \) is the instantaneous nominal rate corresponding to the one-period rate \( \Delta \), i.e., \( \Delta^* = \frac{\Delta}{\delta} \), where \( \delta \) is the length of one period, and \( \nu > 0 \) is the continuous arrival rate (Poisson rate) of transitions from the “crisis” state to the “normal” state, i.e., \( \nu = \frac{1 - \mu}{\delta} \).

Let \( y(\bar{p}, h) \) and \( \pi(\bar{p}, h) \) be the solution when the price-level gap is \( \bar{p} \) and decision makers have

98
planning horizon \( h \). Then, in a forward planning exercise, we have

\[
\frac{dy}{dt} = \frac{\partial y}{\partial \bar{p}} \frac{d\bar{p}}{dt} + \frac{\partial y}{\partial \bar{h}} \frac{d\bar{h}}{dt} = \frac{\partial y}{\partial \bar{p}} \cdot \pi(\bar{p}, h) - \frac{\partial y}{\partial \bar{h}}
\]

\[
\frac{d\pi}{dt} = \frac{\partial \pi}{\partial \bar{p}} \frac{d\bar{p}}{dt} + \frac{\partial \pi}{\partial \bar{h}} \frac{d\bar{h}}{dt} = \frac{\partial \pi}{\partial \bar{p}} \cdot \pi(\bar{p}, h) - \frac{\partial \pi}{\partial \bar{h}}
\]

so that the Euler equations give rise to a system of partial differential equations, i.e.,

\[
-\frac{\partial y(\bar{p}, h)}{\partial \bar{p}} \cdot \pi(\bar{p}, h) + \frac{\partial y(\bar{p}, h)}{\partial \bar{h}} = -\sigma \Delta^* + \sigma \pi(\bar{p}, h) + y[\bar{y}(\bar{p}, h) - y(\bar{p}, h)] \tag{2.29a}
\]

\[
-\frac{\partial \pi(\bar{p}, h)}{\partial \bar{p}} \cdot \pi(\bar{p}, h) + \frac{\partial \pi(\bar{p}, h)}{\partial \bar{h}} = \gamma [y(\bar{p}, h) - \lambda \pi(\bar{p}, h)] + \nu [\bar{\pi}(\bar{p}, h) - \pi(\bar{p}, h)] \tag{2.29b}
\]

with boundary conditions \( y(\bar{p}, 0) = \pi(\bar{p}, 0) = 0 \). The functions \( \bar{z}(\bar{p}, h) = [\bar{y}(\bar{p}, h) - \bar{\pi}(\bar{p}, h)]' \) are the functions given by

\[
\bar{z}(\bar{p}, h) = \begin{cases} 
\bar{z}(\tau(\bar{p})) & \text{if } \bar{p} \geq \bar{p}(h) \\
\bar{z}(h) & \text{if } \bar{p} \leq \bar{p}(h)
\end{cases}
\]

which have already been solved in the case of the “normal” state. We can thus numerically solve the solution of \( z(\bar{p}, h) \) through the system of (2.29a)-(2.29b). Details about the numerical method can be found in Appendix B.1.

Figure 2.2 show the full dynamics of price-level gap, output, and inflation under the temporary price-level targeting policy, respectively, with different lengths of planning horizons. In the numerical exercise, we adopt the calibration in section 2.2.3, and assume that the economy enters the “crisis” state at \( t = 0 \) and reverts to the “normal” state at \( T = 10 \), i.e., 10 quarters after crisis happens.

To compare the results of the temporary price-level targeting with the strict inflation targeting rule considered in section 2.2.3, Figure 2.3 show the comparison of full dynamics for price-level
Figure 2.2: Dynamics of price-level gap, output, and inflation rates with different planning horizons $h$ under temporary price-level targeting policy
gap, output, and inflation, respectively. In the first row of Figure 2.3, the solid line in each panel shows the response of the price-level gap $p_t = \log(P_t/P_t^*)$ under IT; the price-level gap becomes steadily more negative while the shock persists, and is then stabilized at a permanent negative level once fundamentals revert to the “normal” state. The dashed line in the first row of Figure 2.3 shows the alternative dynamics under the TPLT commitment; the price level falls behind the target path to a much lower extent during the crisis, in addition to being eventually returned to the target path within a few quarters of reversion to the normal state.

The smaller price-level gap during the crisis corresponds to more successful output stabilization as showed in the second row of Figure 2.3. Under the temporary price-level targeting rule, although there is an over-shooting for inflation and output after reversion back to the “normal” state, it is much more effective in limiting the effects of the “financial crisis” shock than the standard inflation targeting policy.

### 2.3.2 Systematic Price-level Targeting (PLT Rule)

In this section, suppose that the price-level target is not simply an *ad hoc* commitment (form of forward guidance) introduced when the “crisis” shock occurs; instead, it is followed all the time, so that people obtain extensive experience with the dynamics under a price-level target during periods when the “crisis” shock never occurs. In this case, we name the price-level targeting rule as systematic price-level targeting. Then, people should eventually be able to learn the value functions $v(\tilde{p}_{t+k})$ and $\tilde{v}(\tilde{p}_{t+k})$ for households and firms, respectively, that are correct under such a regime. Thus in such a regime, the first-order perturbations of $v(\cdot)$ and $\tilde{v}(\cdot)$ are not constants if there are shocks that occasionally cause the central bank to miss the price-level target; instead, they depend on the price-level gap at the time that the forward planning is truncated, and the value function is used to evaluate a terminal state.

The two price-level targeting policies, i.e., temporary price-level targeting and systematic price-level targeting, result in different responses during crisis due to the fact that: with finite planning horizon, no matter how credible the temporary commitment might be, pursuit of a different policy
Figure 2.3: Dynamics of price-level gap, output, and inflation rates under inflation targeting versus temporary price-level targeting (TPLT)
systematically outside of crisis periods can allow learning of different value functions by house-
holds and firms, and then matter for the behaviors during the crisis episodes. In contrast, under
the RE analysis, one can achieve the same equilibrium response to a financial shock as under a
consistently pursued price-level targeting regime simply by committing when such a shock occurs
to keep interest rate at the ZLB until the price-level target path is re-gained.

To specify the price-level targeting policy, we adopt similar notations as in section 2.3.1. The
functions of the first-order perturbation of $v(\tilde{p})$ and $\tilde{v}(\tilde{p})$ are equal to zero if $\tilde{p} = 0$, i.e., the steady
state around which we log-linearize the structural equation is also a stationary equilibrium under
the price-level targeting regime. It is the one in which the price-level target is always achieved, so
that $\tilde{p}_t = 0$ at all times.

In the “normal” state, once the correct value functions for the “normal” state have been learned,
we have

$$
\tilde{\gamma}(t) = \tilde{\gamma}(\tau(\tilde{p}(t)))
\tilde{\pi}(t) = \tilde{\pi}(\tau(\tilde{p}(t)))
\frac{d\tilde{p}(t)}{dt} = \tilde{\pi}(t)
$$

Starting from initial condition $\tilde{p}(T)$ when the economy reverts back to the “normal” state at
$t = T$, and continuing until date $t^*$ at which $\tilde{p}(t^*) = 0$, $\tilde{\gamma}(\tau)$ and $\tilde{\pi}(\tau)$ are again
given by (2.25) for all $\tau \geq 0$, and $\tau(\tilde{p})$ is the function obtained by inverting the function $\tilde{p}(\tau)$ derived in (2.26). Note
that this solution applies regardless of the planning horizon $h$, and not only when $h \geq \tau(\tilde{p}(t))$ as
in section 2.3.1; because if a agent has a horizon such that the price-level target is not expected
to be reached within the planning horizon, the value function used to evaluate the terminal state is
correct, i.e., the same valuation as would be calculated by an agent with a longer planning horizon,
which is long enough to see forward to a date at which the price-level target is achieved.

Thus, if the economy enters the “normal” state at date $T$, regardless of the planning horizon $h$,
the subsequent evolution is and is expected to be

\[
\begin{align*}
z(t) &= \begin{cases} 
\bar{z}(\tau(\hat{p}(T)) - (t - T)) & \text{for all } T \leq t \leq T + \tau(\hat{p}(T)) \\
0 & \text{for all } t \geq T + \tau(\hat{p}(T))
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\hat{p}(t) &= \begin{cases} 
\hat{p}(\tau(\hat{p}(T)) - (t - T)) & \text{for all } T \leq t \leq T + \tau(\hat{p}(T)) \\
0 & \text{for all } t \geq T + \tau(\hat{p}(T))
\end{cases} \\
\end{align*}
\]

where the vector \( z(t) = [y(t) \pi(t)]' \).

With the value functions of agents that has already adapted to the price-level targeting rule, there are no longer “two phases” of the solution for the evolution after date \( T \), i.e., no longer depending on whether \( \hat{p}(t) \) is greater or less than \( \hat{p}(h) \); instead, there is only one phase. The solution for the dynamics of \( \{y(t), \pi(t), \hat{p}(t)\} \) can be computed in a closed form using equations (2.25) and (2.26), once one has determined the value of \( \tau(\hat{p}(T)) \).

Now, we consider the dynamics of output, inflation, and price-gap in the “crisis” state. Note that the value function, which is learned and also used in the “crisis” state, is the one that is correct in the “normal” state (the only state in which people have had prior experiences from which to learn the value function). Then, the trajectory that is anticipated by a finite-horizon planner in the “crisis” state is given by paths \( \{y(t), \pi(t), \hat{p}(t), h(t)\} \), where \( h(t) \) is the remaining planning horizon in time \( t \), starting from the time at which the planning takes place. By the aggregate demand and supply equations in the optimal planning exercise, the paths \( \{y(t), \pi(t), \hat{p}(t), h(t)\} \) satisfy a system of differential equations:

\[
-\frac{dy}{dt} = -\sigma \Delta^* + \sigma \pi(t) + \nu[\bar{y}(\tau(\hat{p}(t))) - y(t)] \\
(2.30a)
\]

\[
-\frac{d\pi}{dt} = \gamma[y(t) - \lambda \pi(t)] + \nu[\bar{\pi}(\tau(\hat{p}(t))) - \pi(t)] \\
(2.30b)
\]
\[
\frac{d\tilde{\rho}}{dt} = \pi(t), \quad \frac{dh}{dt} = -1
\]  
(2.30c)

where \(\tilde{y}(t)\) and \(\tilde{\pi}(t)\) are the values if there is reversion to the “normal” state at time \(t\).

The system of (2.30a)-(2.30c) holds from time \(t = t_0\) (the time at which the planning exercise is undertaken) until \(t = t_0 + h_0\), where \(h_0\) is the actual planning horizon of the agent. It starts from initial conditions \(\tilde{\rho}(t_0)\), given by the actual dynamics up to time \(t_0\), and the agent’s actual planning horizon \(h(t_0) = h_0\) when the planning exercise is undertaken, and also satisfies the terminal conditions

\[
y(t_0 + h_0) = \tilde{y}(\tau(\tilde{\rho}(t_0 + h_0))), \quad \pi(t_0 + h_0) = \tilde{\pi}(\tau(\tilde{\rho}(t_0 + h_0)))
\]  
(2.31)

The terminal conditions (2.31) reflect the fact that the value functions \(v(\tilde{\rho})\) and \(\tilde{v}(\tilde{\rho})\) that are used when the planning process is terminated at time \(t_0 + h_0\) are the ones that would be correct under the dynamics in the “normal” state. That is, these are beliefs that would be correct if it were expected that, at time \(t_0 + h_0\), the economy will necessarily revert to the “normal” state if it has not already done so previously. Hence, the terminal beliefs are the same as the beliefs that would be jumped to in the event of a Poisson transition to the “normal” state.\(^{34}\)

Numerically, under the systematic price-level targeting rule, Figure 2.4 show the full dynamics of price-level gap, output, and inflation, respectively. In the numerical exercise, as in section 2.3.1, the economy enters the “crisis” state at \(t = 0\), and reverts back to the “normal” state at \(T = 10\). Details about the numerical method can be found in Appendix B.2.

\(h\) is in quarters, and \(t\) measures quarters since the onset of the elevated financial wedge.

To compare the results of temporary price-level targeting policy with systematic price-level targeting rule, Figure 2.5 show the comparison of full dynamics for price-level gap, output, and inflation under these two policies, respectively. The difference in the two cases reflects not any

\(^{34}\)Note that in section 2.3.1 with the temporary price-level targeting rule, an anticipated trajectory in the “crisis” state also has to satisfy equations (2.30a)-(2.30c), but instead of terminal conditions (2.31), we impose the terminal conditions \(y(t_0 + h_0) = 0, \pi(t_0 + h_0) = 0\) regardless of the value of \(\tilde{\rho}(t_0 + h_0)\). In addition to the different terminal conditions, the method used in section 2.3.1 has a more complex specification of \(\tilde{y}(\tilde{\rho}(t), h(t))\) and \(\tilde{\pi}(\tilde{\rho}(t), h(t))\); in contrast, equations (2.30a)-(2.30b) under systematic price-level targeting apply only in the case of \(h(t) \geq \tau(\tilde{\rho}(t))\), i.e., \(\tilde{\rho}(t) \geq \tilde{\rho}(h(t))\), as in section 2.3.1.
Figure 2.4: Dynamics of price-level gap, output, and inflation rates with different planning horizons $h$ under systematic price-level targeting policy
difference in the expected conduct of monetary policy during and after the crisis period, but a
difference in the value functions used in the forward planning exercises of households and firms
in the two cases. If the planning horizon is large, e.g., more than five years, i.e., \( h \geq 20 \), there
is not much difference between the two price-level targeting rules. But, if the planning horizon
is short for some portion of the people, the systematic price-level targeting rule would improve
output and inflation during crisis much better. Thus, following a systematic price-level targeting
rule, even when financial frictions are unimportant, can be more effective in limiting the effects of
the “financial crisis” shock than a temporary price-level targeting policy introduced only when the
crisis occurs.

Some might suppose that recognizing limitations on people’s ability to correctly anticipate
future consequences of a new policy should reduce the benefits from policy commitment, and
hence favor a purely discretionary approach to policy. In our analysis, instead, recognizing that
planning horizon may not be too long reduces the predicted efficacy of temporary commitments in
response to a special situation, and strengthens the case of seeking to design regimes that apply all
the time.

### 2.4 Coordinated Monetary and Fiscal Stabilization Policy

Thus far we have considered only what can be achieved through commitments regarding the
conduct of conventional interest-rate policy, assuming that neither the path of government pur-
chases nor that of the public debt are changed in response to an increase in the size of the financial
wedge. However, the global financial crisis also saw a resurgence of interest in the use of “fiscal
stimulus” as a tool of stabilization policy, owing to the constraint posed by the zero lower bound on
more aggressive use of interest-rate policy. Taking into account the finiteness of economic units’
planning horizons has important consequences for the assessment of policies of this kind as well.

An important research literature since the crisis has supported the view that fiscal stabilization
policy can be especially valuable when interest-rate policy is constrained by the lower bound. This
Figure 2.5: Dynamics of price-level gap, output, and inflation rates under temporary price-Level targeting (TPLT) versus systematic price-level targeting (PLT Rule)
literature has mainly addressed the effects of countercyclical government purchases,\textsuperscript{35} rather than government transfers, on the ground that in simple representative-agent New Keynesian models, Ricardian equivalence holds in the case of lump-sum taxes and transfers; lump-sum transfers during a crisis, if expected to be financed by future lump-sum taxes, should have no effect at all on economic activity or inflation. On the other hand, the actual fiscal stimulus packages enacted in response to the crisis consisted to an important extent of increases in government transfers (Taylor, 2018). This makes it important to further consider the potential role of countercyclical government transfers as a tool of stabilization policy.

The Ricardian Equivalence result in standard treatments depends crucially on an assumption of rational expectations on the part of all decision makers. Yet this is another case in which the grounds for assuming rational expectations are especially weak. To the extent that the fiscal stimulus package is an \textit{ad hoc} response to a single crisis, rather than an implication of a systematic policy of adjusting the government’s budget in response to the business cycle, one cannot expect that people should have rational expectations as a result of learning from experience; instead, one needs to ask what people should be able to deduce from reasoning about the predictable effects of a novel policy. Moreover, in order for it to make sense to suppose that people should anticipate the future tax increases that must result from the increased public debt occasioned by the stimulus policy, one must assume not merely that people are capable of forward planning (taking into account the policy change), but that their forward planning extends over quite a long horizon — as long as is required for the increased public debt to be fully paid off. In practice, even if one thinks that it should be predictable that the debt will be financed by future tax increases (and when and from whom those taxes should be collected), it is typically \textit{not} the case that there is any reason to expect the new public debt to be paid off within a few years — decades might well be required. This means that in order for full (or nearly full) Ricardian Equivalence to obtain, one needs to assume that most people’s planning horizons extend far into the future.

This suggests that in a model with finite planning horizons, countercyclical fiscal stimulus

\textsuperscript{35}See, for example, Eggertsson (2010), Christiano \textit{et al.} (2011), or Woodford (2011).
might be a powerful tool, and indeed one that might make it possible to stabilize the economy despite the lower bound on interest rates, and without any need to resort to commitments about future monetary policy.\textsuperscript{36} Here we consider what can be achieved by state-contingent transfer policies in the framework developed in section 2.2. We show that fiscal transfers can indeed reduce the contractionary impact of an increase in the financial wedge, and that, at least under some circumstances, a willingness to use fiscal policy with sufficient aggressiveness makes it possible to achieve complete stabilization of both aggregate economic activity and the overall rate of inflation, despite the zero lower bound, and regardless of the size of the increase in the financial wedge. Thus the existence of state-contingent transfer policies does expand the degree to which stabilization would be possible using interest-rate policy alone; and we obtain this result under conditions that would guarantee Ricardian Equivalence under an assumption of rational expectations.

At the same time, we show that it would be a mistake to conclude that countercyclical transfers are so effective a tool that there is no need for a central bank to ever indicate that it would allow inflation to overshoot the bank’s long-run inflation target, nor any need for a commitment to conduct future interest-rate policy in any way different from what will best serve the bank’s goals at that future date. We find that state-contingent transfers make possible equilibria that could not be achieved using interest-rate policy alone, but that there is a limit to the stimulus that can be achieved even by massive fiscal transfers, in the absence of monetary accommodation — that is, a commitment not to raise interest rates, even if inflation overshoots its long-run target.

We also find that there is a limit to what can be achieved, even by coordinated fiscal and monetary policy, if the increase in the public debt and the monetary accommodation are both contemplated only for the period in which the financial wedge remains large, with an immediate return to both the normal level of public debt and the usual inflation target as soon as the wedge returns to a normal level; a higher level of welfare is possible, in general, if the monetary and fiscal authorities commit themselves to history-dependent policies in the period after the real disturbance

\textsuperscript{36}Gabaix (2019) obtains a result of this kind, in the context of a “behavioral New Keynesian” model that down-weights the effects of predictable future conditions on current behavior, albeit for slightly different reasons than those considered here.
has dissipated. Thus while transfer policy can in principle be a useful addition to the arsenal of policymakers in dealing with a situation of the kind reached in late 2008, even under ideal assumptions about the precision with which fiscal policy can be adjusted to varying conditions, it should not eliminate the need for any commitment to conduct monetary policy differently in the future than would be the case in normal times.

2.4.1 Fiscal Transfers and Aggregate Demand

As explained in section 2.2, in this paper we consider only fiscal policies in which the real public debt $B_{t+1}$ is a function of the exogenous state in period $t$ (including the history of exogenous evolution of the financial wedge, through period $t$, and any information available at time $t$ about future financial wedges); but contrary to the further restriction maintained in section 2.3, we no longer require that $B_{t+1} = 0$ at all times. The implied state-contingent level of net lump-sum transfers $\tau_t$ is then given by equation (2.2). While we now allow the path of the debt to respond to shocks, we consider only policies under which the process $\{B_{t+1}\}$ remains within finite bounds with certainty for all time; this means that we consider only policies under which any increase in the public debt is eventually paid off, with certainty.\footnote{This is true regardless of how prices, interest rates, and economic activity may evolve; thus we do not consider the effects of “non-Ricardian” fiscal policy rules of the kind discussed, for example, in Woodford (2001).} Given this — together with the facts that all taxes and transfers are lump-sum and distributed equally to all households, and that there are no financial constraints (other than the “financial wedge” that allows riskless claims on the government to trade at a lower equilibrium rate of return than private debt) — our model is one in which Ricardian Equivalence would hold under an assumption of rational expectations.

Instead, if households have finite planning horizons — or even, if a sufficient number of them do — a bounded increase in the path of the real public debt (resulting from an initial increase in lump-sum transfers, followed eventually — though possibly much later — by the tax increases required to keep the debt from exploding) will increase aggregate demand. Note that the household FOCs (2.4)–(2.5) imply that real expenditure $Y_t^h$ by households with a planning horizon of $h$
periods must satisfy

$$u'(Y^h_t) = E_t[I D^{h+1-j}_{t+j} \cdot \tilde{D}^0_t v'(B_{t+h+1})],$$  \hspace{1cm} (2.32)$$

where the stochastic discount factors are defined by

$$D^j_{t+1} \equiv \beta \frac{1 + \Delta_t}{\Pi^{j}_{t+1}} \quad \text{for any } j \geq 1, \quad \tilde{D}^0_t \equiv \beta (1 + \Delta_t).$$

Now consider the effect of a fiscal policy change, that increases the planned level of $B_{t+1}$ for at least some future dates (in at least some possible states of the world), while decreasing it at no dates. If the paths of neither goods prices nor asset prices change (as would be the case under Ricardian Equivalence), then (2.32) implies that $Y^h_t$ must increase in any period $t$ with the property that $B_{t+h+1}$ is increased in at least some states that remain possible, conditional on the state at date $t$.\(^{38}\) Aggregating across households with different planning horizons, one concludes that aggregate output $Y_t$ must increase, in at least some periods; thus Ricardian Equivalence does not obtain.

The key to this result, of course, is our assumption that announcement of the policy change does not change the value function $v(B)$ used to evaluate terminal states. A household with rational expectations should instead understand that if a policy change results in a higher real public debt $B_{t+h+1}$, it must imply higher tax obligations in periods subsequent to $t + h$ (that is, beyond the planning horizon); and this should change the level of private wealth $B^j_{t+h+1}$ needed in order to ensure a given level of continuation utility. Thus the correct value function $v(B^j_{t+h+1})$ would have as another argument the aggregate supply of debt $B_{t+h+1}$. Because the value function takes account only of a coarse description of the household's situation — and because the situation that gives rise to an unusually large public debt following a financial crisis may not be similar to situations that the household has frequently encountered in the past — we suppose that households have not already learned how to take this additional state variable into account in the way that they value terminal states. Neglect of this state variable is what breaks Ricardian Equivalence. The degree to which this is quantitatively important will depend on the extent to which the time that it takes for

\(^{38}\)This follows from the fact that both $u'(Y)$ and $v'(B)$ are decreasing functions of their respective arguments.
the real public debt to return to its normal level following a shock exceeds the planning horizons
of many households.

The failure of Ricardian Equivalence adds another dimension along which government policy
can shift the equilibrium allocation of resources, possibly in ways that can improve stabilization
outcomes. This is particularly easy to see in the case of an exponential distribution of planning
horizons, \( \omega_h = (1 - \rho)^h \) for all \( h \geq 0 \), where the parameter \( 0 < \rho < 1 \) determines the mean
planning horizon \( \bar{h} \equiv \rho / (1 - \rho) \). In this case, the log-linearized aggregate demand relations (2.7)–
(2.8) can be aggregated to yield

\[
y_t = -\sigma (\breve{\gamma}_t^r + \Delta_t - \rho E_t \pi_{t+1}) + \rho E_t y_{t+1} + (1 - \rho)(1 - \beta)b_{t+1},
\]

(2.33)

where

\[
\breve{\gamma}_t^r \equiv (1 - \rho) \sum_{j=0}^{\infty} \rho^j \breve{\gamma}_t^j
\]

is an average of the interest rates expected by households with different planning horizons.\(^{39}\) The
linearized aggregate supply relations (2.9)–(2.10) can similarly be aggregated to yield

\[
\pi_t = \kappa y_t + \rho \beta E_t \pi_{t+1}.
\]

(2.34)

Note that equations (2.33) and (2.34) relating the evolution of aggregate output and inflation reduce
to the structural equations of the standard New Keynesian model under rational expectations in the
limit as \( \rho \to 1 \).

Equation (2.33) shows that variation in the level of real public debt \( b_{t+1} \) (the debt issued in
period \( t \)) shifts the aggregate-demand relation in exactly the same way as does variation in \( \breve{\gamma}_t \), the

\(^{39}\)In Woodford (2019), this equation involves \( \breve{\gamma}_t \), the actual interest-rate target of the central bank, rather than the
variable \( \breve{\gamma}_t^r \) defined here. The form (2.33) is more generally valid. In the earlier paper, monetary policy is assumed to be
categorized by a linear relationship among \( \breve{\gamma}_t \) and other aggregate variables, such as a Taylor rule \( \breve{\gamma}_t = \phi(\pi_t, y_t; s_t) \),
where \( s_t \) is an exogenous state and \( \phi \) is linear in the first two arguments. In such a case, the fact that the policy rule is
understood by all households implies that \( \breve{\gamma}_t^j = \phi(\pi_t, y_t; s_t) \) for each horizon \( j \); aggregating over the different horizons
then implies that \( \breve{\gamma}_t^r = \phi(\pi_t, y_t; s_t) = \breve{\gamma}_t \), owing to the linearity of \( \phi \). When the zero lower bound sometimes constrains
policy, as in the cases considered in this policy, \( \breve{\gamma}_t^r \) will in general no longer equal \( \breve{\gamma}_t \).
central bank’s interest-rate target. It follows that, if one is concerned solely with stabilization of the aggregate variables \( y_t \) and \( \pi_t \), there is no need to consider varying the path of the real public debt, as long as it is possible for the central bank to vary \( \hat{i}_t \) to the desired degree instead. However, when the zero lower bound is a binding constraint on interest-rate policy, the fact that the public debt can still be increased through transfer policy can effectively relax this constraint.

This allows stabilization of the aggregate economy in cases where this would not be possible under a policy that maintained \( b_{t+1} = 0 \) at all times. Note that the paths in which \( y_t = \pi_t = 0 \) at all times are consistent with both equations (2.33) and (2.34) holding at all times, if and only if

\[
-s(\hat{r}_t^r + \hat{\Delta}_t) + (1 - \rho)(1 - \beta)b_{t+1} = 0
\] (2.35)

at all times. Since everyone is assumed to understand that the central bank’s policy must conform to the lower bound \( \hat{i}_t \geq \hat{i}_z \), the interest rates expected by households must satisfy \( \hat{i}_t^r \geq \hat{i}_z \) at all times. Hence if \( \Delta_t > -\hat{i}_z \) at some time, it will not be possible to satisfy (2.35) with \( b_{t+1} = 0 \).

Instead the condition can always be satisfied if we allow fiscal transfers. Let us suppose that the central bank’s interest-rate target tracks variations in the financial wedge to the extent that this is consistent with the ZLB, i.e., that monetary policy ensures that

\[
\hat{i}_t = \max\{-\hat{\Delta}_t, \hat{i}_t\}
\] (2.36)

each period.\(^\text{40}\) Then (since the interest rate is specified as a function of the exogenous state) \( \hat{r}_t^r \) will equal \( \hat{i}_t \), and condition (2.35) will be satisfied if and only if fiscal policy is given by

\[
b_{t+1} = \frac{\sigma}{(1 - \rho)(1 - \beta)} \hat{\Delta}_t
\] (2.37)

\(^\text{40}\)More precisely, we assume that this policy is followed during a relatively brief period in which there is a non-trivial financial wedge, but that after that period the central bank reverts to a policy rule that ensures achievement of its inflation target. The latter stipulation is required in order to ensure that there should not be any long-run drift in the value functions of households and firms, allowing us to abstract from modeling the endogenous adjustment of value functions, as discussed in section 2.2. If the rule (2.36) were followed forever, then the learning process for the value functions specified in Woodford (2019) would lead to unstable dynamics, as shown in that paper for the case of a permanent zero financial wedge.
where
\[
\tilde{\Delta}_t \equiv \max \{ \hat{\Delta}_t + \hat{\Delta}, 0 \}
\] (2.38)
measures the part of the financial wedge that is not offset by interest-rate policy. If monetary policy is given by (2.36) and fiscal policy by (2.37), equilibrium will involve \( y_t = \pi_t = 0 \) at all times, regardless of the path of the financial wedge.\(^{41}\)

### 2.4.2 The Dependence of Fiscal Stimulus on Monetary Accommodation

This striking result might make it seem that there is no need for a central bank to depart from its commitment to a strict inflation-targeting policy, given that fiscal transfers can be varied to offset any effects on aggregate demand of variations in the financial wedge. Can one not simplify the tasks of both policy authorities, and communication with the public as well, by stating that the sole concern of the central bank should be to ensure that inflation remains equal to the target rate, while it is the responsibility of the fiscal authority to offset any excessive financial wedge (any positive value of \( \tilde{\Delta}_t \)) with fiscal transfers, so as to maintain a zero output gap?

It might seem from the analysis above that the central bank can commit itself to the inflation targeting policy considered in section 2.2.3 (setting \( \hat{\Delta}_t \) as needed to achieve the inflation target, or as low as possible if the target cannot be achieved), and that as long as fiscal policy is given by (2.37), the outcome will be complete stabilization of both inflation and the output gap. This would however be incorrect. It is true that the equilibrium described at the end of the previous subsection is one in which the paths of \( \hat{\Delta}_t \) and \( \pi_t \) conform to the proposed monetary policy rule; but it is not true that that equilibrium is consistent with everyone expecting that monetary policy will be conducted in accordance with that rule. In our model, because of people’s finite planning horizons, it matters not only what happens in equilibrium, but what the central bank would be expected to do out of equilibrium; and the complete stabilization of macroeconomic aggregates actually depends

\(^{41}\)It is immediately obvious from inspection of equations (2.33) and (2.34) that the asserted solution is consistent with both of these equations at all times. We show in Appendix C that this is indeed the unique equilibrium outcome, assuming a bound on the asymptotic growth rate of the excess financial wedge. The required condition holds, for example, in the case of the two-state Markov process for the financial wedge introduced in section 2.2.3 as long as (2.20) is satisfied.
on people’s understanding that the central bank is not determined to prevent over-shooting of the long-run inflation target under any circumstances.

In order to see this, we need to consider the forward plans of agents with differing planning horizons in the equilibrium in which \( y_t = \pi_t = 0 \) at all times. Substituting the monetary policy rule (2.36) into (2.15)–(2.16) yields

\[
z_t^j = A E_t z_{t+1}^j - \sigma a \tilde{\Delta}_t
\]

for each \( j \geq 1 \), and

\[
z_t^0 = -\sigma a \tilde{\Delta}_t + (1 - \beta) a b_{t+1}.
\]

In the case of an arbitrary process for the financial wedge and an arbitrary fiscal policy, this system of equations can be solved recursively to yield

\[
z_t^h = -\sigma \cdot \sum_{j=0}^{h} [A^j a] E_t \tilde{\Delta}_{t+j} + (1 - \beta) [A^h a] E_t b_{t+h+1}
\]

for any planning horizon \( h \geq 0 \). The implied solutions for the aggregates \( y_t \) and \( \pi_t \) are then obtained by averaging over the various planning horizons \( h \). If fiscal policy is given by (2.37), these equations imply \( y_t = \pi_t = 0 \); however, they do not generally imply \( y_t^h = \pi_t^h = 0 \) for each individual planning horizon.

Consider, for example, the case in which the financial wedge evolves according to a two-state Markov chain of the kind proposed in section 2.2.3. In this case, the right-hand side of (2.40) depends only on whether the economy is still in the crisis state at date \( t \), or has already returned to normal. In any period \( t \) such that the economy remains in the crisis state, the solution is given by

\[
z_t^h = z_t^h \equiv \sigma \cdot \left\{ \frac{\mu^h}{1 - \rho} [A^h a] - \sum_{j=0}^{h} \mu^j [A^j a] \right\} \cdot \Delta,
\]

where \( \Delta > 0 \) is the excess financial wedge in this state. Instead, in any period after the return
to the normal state, \( z^h_t = 0 \). Note that the solution for \( z^h \) is well-defined for any finite horizon \( h \), regardless of parameter values.

One observes that in the crisis state, the elements of \( \tilde{z}^h \) are different for different horizons \( h \). For example, when \( h = 0 \),

\[
\begin{bmatrix}
\gamma^0 \\
\pi^0
\end{bmatrix} = \sigma \frac{\rho}{1 - \rho} \begin{bmatrix} 1 \\
\kappa \end{bmatrix} \Delta >> 0.
\]

Moreover, one can show that the largest of the two positive real eigenvalues of \( A \) is equal to \( \tilde{\mu}^{-1} > 1 \), where \( \tilde{\mu} \) is the quantity introduced in (2.13). Then if \( \mu < \tilde{\mu} \), one finds that

\[\tilde{z}^h \to \tilde{z}^{RE} << 0\]

as \( h \to \infty \), where \( \tilde{z}^{RE} \) is the Markovian rational-expectations solution defined in (2.14). Thus both \( y^h \) and \( \pi^h \) are positive in the case of short enough planning horizons, while both are negative in the case of long enough horizons.

This is illustrated numerically in Figure 2.6. Here we plot \( y^h \) and \( \pi^h \) for each of the different planning horizons, for each of several different assumptions about the value of \( \rho \) (or alternatively, the mean planning horizon \( \bar{h} \)); the values of the other parameters are as in the calibration introduced in section 2.2.3. While in each case the mean value of both \( y^h \) and \( \pi^h \) is equal to zero, this averages positive values for short horizons and negative values for longer horizons. (Increasing \( \bar{h} \) increases both \( y^h \) and \( \pi^h \) for all horizons, but this is consistent with the mean values continuing to be the zero, because the distribution of weights shifts toward greater weight on the longer horizons the larger is \( \bar{h} \).)

The figure underlines the fact that an equilibrium with \( y_t = \pi_t = 0 \) depends on an understanding of the central bank’s policy that allows households and firms to anticipate that inflation would be allowed to overshoot the long-run inflation target, under some circumstances. Even though this does not occur in the equilibrium shown in the figures, it is anticipated in the forward plans of both
Figure 2.6: Expenditure and rates of price increase during the crisis period, for households and firms with different planning horizons \(h\), under a policy that fully stabilizes aggregate output and inflation. Each line is for a distinct value of the mean planning horizon \(\bar{h}\). Both \(h\) and \(\bar{h}\) are in quarters.

households and firms, as shown in the figure. It is important to note that in this solution, it is not just the households and firms with short planning horizons that must believe that such overshooting would be allowed. A household or firm with some long planning horizon \(h\) in period \(t\) (assumed to be a crisis period) anticipates that if the crisis persists until period \(t + h - j\), inflation will at that time equal \(\pi^j\) — a positive quantity, for all small enough \(j\). Since this is expected to occur with positive probability (probability \(\mu^{h-j}\)), the equilibrium with full stabilization of \(y_t\) and \(\pi_t\) depends on all households and firms believing that with positive probability a situation will be reached in which the central bank will allow inflation to overshoot its long-run target, because of the central bank’s commitment to accommodate the continuing fiscal stimulus.

The situation would be quite different if, instead, the central bank were understood to be committed to setting the interest rate required to achieve its inflation target, unless constrained by the ZLB. In that case, there would be a maximum degree of aggregate demand stimulus that could be achieved through fiscal transfers, no matter how large the transfers might be. Under strict inflation
targeting, (2.8) and (2.10) imply

\[
\pi_t^0 = -\kappa \sigma (\hat{t}_t^0 + \hat{\Delta}_t) + \kappa (1 - \beta) b_{t+1} \\
= \min \{ -\kappa \sigma (\hat{\Delta}_t) + \kappa (1 - \beta) b_{t+1}, 0 \}.
\]

In the case of the assumed two-state Markov chain for the financial wedge, this implies that as long as the crisis state persists, one will have

\[
\pi_t^0 = \pi^0 = \kappa \min \{ (1 - \beta) b_{t+1} - \sigma \Delta, 0 \}.
\]

(The corresponding value of \( \gamma^0 \) is simply this quantity without the prefactor \( \kappa \).) Thus increases in the public debt are stimulative only up to the level

\[
b^{\max} \equiv \frac{\sigma \Delta}{1 - \beta}.
\]

(In our numerical calibration, this amounts to 0.36 of annual GDP.\(^{42}\) For any level \( b_{t+1} \geq b^{\max} \), the model predicts that \( \gamma^0 = \pi^0 = 0 \).

For any longer horizon \( h \), we similarly will have \( z_t^h = z^h \) as long as the crisis state persists, where crisis values \( \{ z_t^h \} \) can be computed recursively as follows. For any \( j \geq 1 \), (2.39) implies that in any crisis period,

\[
\pi_t^j = [\kappa \beta + \kappa \sigma] z^{j-1} - \kappa \sigma \Delta
\]

if \( \hat{\gamma}_t^j \) is expected to be at the lower bound. If both elements of \( z^{j-1} \) are non-positive, this implies inflation below target, even with the interest rate at the lower bound. Hence the ZLB will bind, and we must have

\[
z_t^j = \mu A z^{j-1} - \sigma a \Delta << 0. \tag{2.41}
\]

\(^{42}\)Note that this does not mean that there would be no effect of increasing the public debt beyond 36 percent of GDP — a level that the US is already well past. It means that, under the assumptions of our calibration, there would be no effect of an increase by more than 36 percent of GDP relative to the normal steady-state level of public debt.
Figure 2.7: Expenditure and rates of price increase during the crisis period, for households and firms with different planning horizons $h$ (in quarters) when the central bank follows a strict inflation targeting policy. The two lines correspond to the minimal and maximal sizes of fiscal stimulus.

Under the assumption that $b_{t+1} \geq b^{\text{max}}$ for as long as the crisis state persists (the most favorable assumption for a stimulative effect of fiscal policy), we have shown in the previous paragraph that $z^0 = 0$; we can then show recursively using (2.41) that both elements of $z^j$ are non-positive for all $j \geq 0$. It follows that the assumption used to derive (2.41) is valid for all $j \geq 1$.

Thus under the most expansive possible fiscal policy, we will have $z^h = z^h$ as long as the crisis state persists, where the sequence $\{z^h\}$ can be computed recursively using (2.41), starting from the initial condition $z^0 = 0$. This yields the solution

$$z^h = -\sigma \sum_{j=1}^{h} (\mu A)^{j-1} a \Delta \ll 0$$

for each $h \geq 1$. Both $y^h_t$ and $\pi^h_t$ remain below their target values for all horizons $h > 0$, and more so the longer the horizon. (This is illustrated for our numerical example in Figure 2.7, which shows the values of $\bar{z}^h$ both in the case of zero fiscal stimulus\textsuperscript{43} and in the case of the maximum fiscal

\textsuperscript{43}Note that the results for $b = 0$ repeat those shown in Figure 2.1 above.
stimulus.) We see that fiscal stimulus can mitigate the contractionary and disinflationary effects of the financial disturbance, but both spending and the rate of price increase continue to fall, even with the maximum fiscal stimulus, for all horizons $h > 0$; if planning horizons extend years into the future, the fraction of the contractionary effect that can be offset using fiscal policy alone is quite modest.

Summing over the different planning horizons (again assuming an exponential distribution of horizons), the net effect on both aggregate output and aggregate inflation is necessarily contractionary. As long as (2.20) is satisfied,\(^{44}\) the weighted average of the $\{z^h\}$ is a convergent sum, and equal to

$$z = (1 - \rho) \sum_{h=0}^{\infty} \rho^h z^h = -\rho \sigma [I - \rho \mu A]^{-1} a \Delta \ll 0.$$  

It is not possible to fully stabilize either aggregate output or inflation; both necessarily fall in the crisis state. Indeed, the effects on aggregate output and inflation are similar to those obtained in the case of no fiscal response (see equation (2.21) above): they are simply both reduced by a factor of $\rho$. This means that the contractionary effects are reduced by less than half, in the case of any mean planning horizon $\bar{h}$ greater than one quarter.

Instead, it is possible to completely eliminate the contractionary effects of the increased financial wedge on both output and inflation, if an expansionary fiscal policy (an increase in the real public debt through lump-sum transfers, in an amount proportional to the excess financial wedge $\Delta$) is combined with monetary accommodation — a commitment to keep the nominal interest rate at its lower bound during the period in which the financial wedge is large, even if this causes inflation to overshoot its long-run target. A coordinated change in both monetary and fiscal policy in response to the financial disturbance can achieve more than either policy can on its own. Thus while fiscal transfers have an important contribution to make, in the case that planning horizons are finite, the availability of this additional instrument does not make monetary stabilization policy irrelevant. Moreover, the important aspect of monetary policy is not what the central bank actually

\(^{44}\)This is the condition required for both eigenvalues of $\rho \mu A$ to be less than 1, the same condition required for convergence as in the case of zero fiscal stimulus considered in section 2.2.3.
does during the period when the financial wedge is large (since the ZLB binds during this period); rather, it is what it leads people to believe that it would do, in the event that the ZLB were to cease to bind. In this sense, commitments about the determinants of future interest-rate policy remain a crucial dimension of policy, even when aggressive use of government transfers is possible.

2.4.3 The Continuing Relevance of Forward Guidance

The example considered above not only shows that the use of fiscal transfers can improve stabilization outcomes, relative to what monetary policy alone can accomplish; the results obtained might seem to make the details of monetary policy unimportant, given sufficient latitude in the way that fiscal policy can be used. If we assume a conventional objective for stabilization policy, in which the aim is to minimize the expected value of a discounted sum of squared target misses

\[
E_0 \sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda y_t^2],
\]

then it is easy to characterize an optimal joint monetary-fiscal policy in the case of an exponential distribution of planning horizons.

We have shown in this case that if (2.35) is satisfied at all times, we will have \( y_t = \pi_t = 0 \) at all times, which obviously achieves the minimum possible value of criterion (2.42). Moreover, it is possible to choose a state-contingent evolution \( \{b_{t+1}\} \) that satisfies (2.35) at all times, in the case of any assumed state-contingent evolution for \( \{\hat{i}_t\} \), as long as \( \hat{i}_t \) is a function only of the exogenous state, so that \( \hat{i}_t^e = \hat{i}_t \).\footnote{It is necessary, however, that we assume that interest rates do not adjust endogenously in response to changes in \( b_{t+1} \) in such a way as to keep \( (1-\rho)(1-\beta)b_{t+1} - \sigma \hat{i}_t^e \) constant; this is the problem with an expectation that the central bank is committed to whatever interest-rate adjustments are needed to achieve a fixed inflation target.} For example, it is not necessary for interest-rate policy to respond at all to increases in the financial wedge in order for complete aggregate stabilization to be possible; we could assume that \( \hat{i}_t = 0 \) at all times, and make the fiscal authority solely responsible for responding to variations in financial conditions.

The example suggests another strong conclusion as well: it would seem that there is no need to contemplate any deviation from what we have above called “orthodox” policy (strict inflation
targeting and zero public debt) in periods when financial wedges are small (small enough so that \( \Delta \approx 0 \)), simply because there are transitory periods in which the wedges are large. In the example discussed above, it is possible to achieve full stabilization of aggregate variables even during a “crisis,” while conducting (and being expected to conduct) policy in a completely orthodox way as soon as the economy reverts to the “normal” state. Thus there is no need for forward guidance, in the sense of a commitment to more stimulative than ordinary policy for a time even after the financial wedge is again small, for the sake of improved stabilization during the period when the wedge is large.

It would be wrong, however, to draw such conclusions. We should first note that, even if minimizing the expected value of (2.42) is the sole objective of policy, the results obtained above depend on the special assumption of an exponential distribution of planning horizons. As a simple (but instructive) alternative case, suppose instead that all households and firms have a common planning horizon. In this case, the level of the nominal interest rate is not generally irrelevant; we can determine an optimal state-contingent evolution for \( \{\hat{y}_t, C_t\} \) even under the assumption that the path of the public debt will be optimized for whatever monetary policy is chosen.

Moreover, it is not generally true that it will be optimal for either the public debt or the nominal interest to return immediately to the values associated with the long-run steady state as soon as the financial wedge becomes small again; instead, stabilization during the crisis period (which is necessarily imperfect) can be improved by a commitment to continue the anomalous policies of the crisis period for a time even after it would be possible to achieve full stabilization of both inflation and the output gap by immediately returning to policy “orthodoxy.” Thus forward guidance (both with regard to interest-rate policy and the path of the public debt) has an important role to play in improving outcomes during a crisis; indeed, we find that the optimal use of forward guidance regarding interest-rate policy in a coordinated monetary-fiscal response to the crisis is not too different than it would be if we ignored the possibility of countercyclical government transfers (as in section 2.3).

Let the common planning horizon be \( h > 0 \), and let us restrict attention to policies specified
by exogenous state-contingent paths for both \( \{i_t, b_{t+1}\} \). While not completely general, this family of possible specifications includes the kind of policy that can achieve complete stabilization in the case considered above. It also allows us to consider the possibility of returning immediately to complete stabilization of both inflation and output as soon as this is feasible, since a specification that \( i_t = -\hat{\Delta}_t, b_{t+1} = 0 \) at all times as soon as the economy returns to the normal state (a state in which it is expected that financial wedges will be small from then on) will suffice to ensure this. Within this family of policies, our goal is to choose state-contingent paths \( \{i_t, b_{t+1}\} \) so as to minimize the expected value of (2.42), where \( y_t = y_t^h, \pi_t = \pi_t^h \) at all times (because of the common planning horizon).

Let us begin by considering the optimal evolution of \( \{b_{t+1}\} \), taking as given the state-contingent path of \( \{i_t\} \). In the case of an arbitrary interest-rate policy, we can use the same methods as above to show that \( z_t^h \) will be given by (2.40), except that in the more general case the variable \( \hat{\Delta}_t \) defined in (2.38) must be replaced by \( \hat{\Delta}_t = \hat{\Delta}_t + \hat{i}_t \). Since only the evolution of the variables \( \{z_t^h\} \) for horizon \( h \) matters for the stabilization objective, it follows from this solution that the choice of \( b_{t+h+1} \) for any exogenous state \( s_{t+h} \) in period \( t + h \) affects no variables relevant to the stabilization objective other than \( z_t^h \) in the state at period \( t \) in which it is possible to reach the particular state \( s_{t+h} \) at date \( t + h \). We can then reduce the problem of choosing an optimal state-contingent evolution for \( \{b_{t+1}\} \) to a sequence of independent static problems: for any state \( s_t \) in period \( t \), choose the level of \( b_{t+h+1} \) in the states at date \( t + h \) that are possible conditional on being in state \( s_t \) so as to minimize

\[
L(z_t^h) = (\pi_t^h)^2 + \lambda (y_t^h)^2,
\]

where \( z_t^h \) is given by the generalized version of (2.40).

This is a convex minimization problem, with a unique interior solution characterized by a first-order condition. If we introduce the notation

\[
A^t a = \begin{bmatrix} \alpha_j \\ \gamma_j \end{bmatrix}
\]
for \( j \geq 0 \), then the first-order condition is given by

\[
\gamma_h \pi^h_t + \lambda \alpha_h \beta^h_t = 0.
\]

Substituting the generalization of (2.40) into this yields a linear equation (with a unique solution) for the expected public debt at the end of the planning horizon:

\[
E_t b_{t+h+1} = \frac{\sigma}{1 - \beta} \sum_{j=0}^h \frac{\lambda \alpha_h \beta^j + \gamma_h \gamma^j}{\lambda \alpha^2_h + \gamma^2_h} E_t \tilde{\Delta}_{t+j}.
\] (2.43)

Substituting this solution into the generalization of (2.40) then yields an equation for \( z^h_t \) in the case of an optimal transfer policy, but arbitrary interest-rate policy,

\[
z^h_t = \theta_t \begin{bmatrix} -\gamma_h \\ \lambda \alpha_h \end{bmatrix}, \quad \text{where} \quad \theta_t \equiv \sigma \sum_{j=0}^{h-1} \frac{\alpha_j \gamma_h - \alpha_h \gamma^j}{\lambda \alpha^2_h + \gamma^2_h} E_t \tilde{\Delta}_{t+j}.
\] (2.44)

It follows that the minimum achievable value of \( L(z^h_t) \), given interest-rate policy, will be given by

\[
L_t = \lambda (\lambda \alpha^2_h + \gamma^2_h) \theta^2_t,
\]

where \( \theta_t \) is the function of financial wedges and interest rates defined in (2.44).

Given that these results obtain regardless of the assumed interest-rate policy, the choice of an optimal interest-rate policy reduces to the choice of a state-contingent evolution \( \{\hat{i}_t\} \) subject to the lower bound (2.11) holding at all times, so as to minimize the expected value of \( \sum_{t=0}^{\infty} \beta^t \theta^2_t \), where \( \theta_t \) is given by (2.44). If the ZLB never binds, this problem will be solved by choosing \( \hat{i}_t = -\tilde{\Delta}_t \) each period, so that \( \tilde{\Delta}_t = 0 \) at all times, implying that \( \theta_t = 0 \) at all times. However, this will be possible if and only if financial wedges are never large, i.e., \( \tilde{\Delta}_t = 0 \) at all times — the same condition as is required for complete stabilization to be possible under the constraint that \( b_{t+1} = 0 \) at all times. While the availability of countercyclical fiscal transfers as an additional policy instrument reduces the losses associated with a given process \( \{\tilde{\Delta}_t\} \) for the financial wedges
not offset by contemporaneous interest-rate adjustments, it does not change the fact that complete stabilization requires (except in the case where \( h = 0 \)) that one be able to ensure that \( \dot{\lambda}_t = 0 \) at all times. As discussed in section 2.2.3, this is sometimes precluded by the ZLB.

We can also see that in general, when complete stabilization is not possible, the optimal second-best policy will involve committing to maintain \( b_{t+h} > 0 \) and/or \( \hat{i}_t < 0 \) (that is, deviation from the policies associated with the long-run steady state) even in some periods \( t \) after the financial wedge has again become small, so that an immediate return to the long-run steady state would be possible. Suppose instead that one were to have \( \hat{i}_t = 0 \) (and hence \( \dot{\lambda}_t = 0 \)) for all \( t \geq T \), where \( T \) is the (possibly random) date at which reversion to the “normal” state occurs, while \( \dot{\lambda}_t \) is instead necessarily positive (because of the ZLB) at all dates \( 0 \leq t < T \). It would then follow that at any date \( t \) (and in any state of the world at that date) at which the financial wedge remains large, \( \dot{\lambda}_t > 0 \) and \( \dot{\lambda}_{t+j} \) is also anticipated to be non-negative in all possible successor states with \( j > 0 \); hence the right-hand side of (2.43) will necessarily be positive.\(^{46}\) We can thus conclude that optimal policy would require that \( E_t b_{t+h+1} > 0 \). If we further suppose that \( t \) is a date such that the financial wedge remains large at date \( t \), but it is foreseen that it will necessarily be small at date \( t+h \), then this requires that \( b_{t+h+1} > 0 \) with positive probability even after reversion to the normal state. Furthermore, on the assumption that the economy is already in the normal state at date \( t+h \), and hence that \( \hat{i}_{t+h} = 0 \) in all possible states at that date, a policy under which \( b_{t+h+1} > 0 \) in some state \( s_{t+h} \) will also have to involve \( \pi^0_{t+h} > 0 \) in that state.\(^{47}\) Thus the optimal joint fiscal-monetary policy must also involve an understanding that inflation would be allowed to overshoot its long-run target, even in some periods \( t \geq T \). Stabilization outcomes during the period when the financial wedge remains large (and the ZLB consequently binds) are improved by committing to continue expansionary policies for a time beyond the date \( T \) at which it would be possible to again achieve complete stabilization using orthodox (and purely forward-looking) policies.

In fact, it will not generally be optimal for the central bank to set \( \hat{i}_t = 0 \) for all \( t \geq T \); it can easily

\(^{46}\)We use the fact that all elements of the vector \( a \) and the matrix \( A \) are positive, implying that \( a_j, \gamma_j > 0 \) for all \( j \geq 0 \).

\(^{47}\)This follows from (2.40), given that \( \gamma_h > 0 \).
Figure 2.8: Equilibrium trajectories in the case of an elevated financial wedge for 10 quarters (panel (a)), under three alternative assumptions about policy: (i) $b_{t+1} = 0$ at all times, and $\hat{t}_{t} = \max\{-\hat{\Delta}, \hat{\xi}\}$; (ii) $b_{t+1} = 0$ at all times, but the path $\{\hat{t}_{t}\}$ is chosen optimally; or (iii) the paths of both $\{b_{t+1}\}$ and $\{\hat{t}_{t}\}$ are chosen optimally. Planning horizons extend 8 quarters into the future, and $t$ measures quarters since the onset of the elevated financial wedge.
be optimal to set a lower level of interest rates, and even to keep the nominal interest rate at its lower bound, in the early periods following the reversion to the normal state. This is illustrated by a numerical example in Figure 2.8. In this example, all households and firms are assumed to have planning horizons extending eight quarters into the future \( h = 8 \), and the financial disturbance at date \( t = 0 \) increases the financial wedge (to the extent assumed in the numerical calibration proposed in section 2.2.3) for ten quarters. We furthermore assume for simplicity that it is known from \( t = 0 \) onward that the financial wedge will be elevated for exactly ten quarters (rather than assuming stochastic exit from the crisis state, as in the two-state Markov case), so that \( T = 10 \) with certainty.

The several panels of the figure show the (deterministic) evolution of the financial wedge, output, inflation, the nominal interest rate, and the real public debt in response to such a disturbance, under three possible assumptions about monetary and fiscal policy, which is to say about the paths of \( \{\hat{\Delta}C\} \) and \( \{b_{t+1}\} \). (Both of these evolve deterministically under all of the policies considered, since no further uncertainty is resolved after date \( t = 0 \).) In case (i), we assume that \( \hat{\Delta}C \) tracks the variation in the “natural rate of interest” (the interest rate required for stabilization of the output gap, as specified in (2.36)), and that \( b_{t+1} = 0 \) (no response of fiscal policy to the disturbance). These assumptions lead to the same outcomes as under the “orthodox” policy discussed in section 2.2.3, though here we define the policies in a way that makes them a particular case of the class of policies studied in this section. As in section 2.2.3, the consequence is output contraction and inflation below target during the crisis period, but immediate stabilization of both output and inflation as soon as the economy reverts to the normal state; the only difference from the results shown in section 2.2.3 is that here, because the expected time to the reversion to the normal state falls as time passes during the crisis state, the effects on output and inflation are both largest at the onset of the financial disturbance.

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48 We discuss further the calculations involved, and show how the results depend on the assumed planning horizon, in Appendix D.

49 Here the financial wedge, inflation and the nominal interest rate are reported in annualized terms: \( \hat{\Delta}C = 0.05 \) means a safety premium of 5 percentage points per year (and is equivalent to the value \( \hat{\Delta} = 0.013 \) given in Table 1, where the value is for a quarterly model). Output is reported as a percentage deviation from the long-run steady state level of output, and real public debt in units of years of long-run steady state real GDP.
In case (ii), we again assume that \( b_{t+1} = 0 \) at all times, but consider optimal forward guidance with respect to the future evolution of the central bank’s nominal interest-rate target \( \{ \hat{i}_t \} \). In case (iii), we instead allow the paths of both \( \{ \hat{i}_t \} \) and \( \{ b_{t+1} \} \) to be optimized. In the latter case, we see that the optimal joint fiscal-monetary commitment involves promising to maintain both \( b_{t+1} > 0 \) and \( \hat{i}_t < 0 \) for a time after the reversion to the normal state in quarter 10. The figure (panel (e)) shows that optimal policy requires an increase in the public debt (by an amount equal to nearly two years’ GDP) by at least quarter 8,\(^{50}\) which must then be maintained in quarter 9. In quarter 10, when the financial wedge has returned to zero (see panel (a)), it continues to be optimal to maintain a larger public debt than in the long-run steady state (though not as large as the debt in quarters 8 and 9); and the optimal level of the debt continues to be somewhat positive in quarters 11 and later, though much smaller than the earlier levels of debt.

The optimal joint fiscal-monetary commitment also involves keeping the nominal interest rate lower than its long-run steady-state level, for two quarters following the reversion of the financial wedge to zero. Panel (d) of the figure shows that under policy (iii), the nominal interest rate remains at the zero lower bound in quarter 10, even though it would be possible at this time to return immediately to the long-run steady state (and policy (i) would require \( \hat{i}_t = 0 \) from \( t = 10 \) onward). The nominal interest rate also remains well below its long-run steady-state level in quarter 11, though no longer at the lower bound. As in rational-expectations analyses of optimal forward guidance, a commitment to keep the interest rate “low for longer” following the reversion of the financial wedge to zero improves stabilization during earlier periods when the financial wedge is large (and the ZLB precludes complete stabilization as a result). (Even though planning horizons extend only eight quarters into the future, in this example, it remains true that expectations regarding \( \hat{i}_{10} \) affect the values of \( \theta_t \) in quarters \( t = 3 \) through 9.)

\(^{50}\)Because all households and firms are assumed to have horizon \( h = 8 \) in these calculations, the response of the public debt in quarters 0 through 7 in response to the shock has no consequences for behavior. Thus the value of \( b_{t+1} \) under the optimal policy is indeterminate in these periods; this is the meaning of the dotted line shown in the figure for those periods. (The figure shows the public debt being immediately increased at the time that the financial wedge increases, since in the periods for which there is a unique solution, the public debt should be maintained at a constant level proportional to the increase in the financial wedge. But a more gradual increase in the public debt between quarters 0 and 8 would have the same effect on equilibrium output and inflation.)
Indeed, the degree to which it is optimal to commit to keep interest rates low beyond date $T = 10$ is similar in the case when fiscal transfers are used optimally (case (iii)) as in the case where fiscal transfers cannot be used (case (ii)). The most important difference in the nature of optimal policy (and in stabilization outcomes) when fiscal policy can respond to the financial disturbance is that increasing public debt outstanding at the ends of quarters 8 and 9 can increase aggregate demand in quarters 0 and 1 — points in time at which a commitment to lower interest rates in quarter 10 or later can have no effect, because planning horizons extend only eight quarters into the future. A somewhat greater degree of stabilization in quarter 2 is also possible when fiscal transfers are used, because even though the commitment to lower interest rates in quarter 10 affects aggregate demand, the extent to which interest-rate policy can be used is limited, as the ZLB continues to bind in quarter 10. Instead, the equilibrium paths of both inflation and output are similar from $t = 3$ onward under cases (ii) and (iii). Stabilization is also superior in quarters 3 through 8 under these policies (relative to case (i)); but most of the improvement is already achieved by the use of interest-rate forward guidance alone (case (ii)).

It is also important to recognize that the superior stabilization outcomes shown in case (iii) of Figure 2.8 depend on people’s understanding that under this policy, the central bank is committed to maintaining low interest rates even if inflation and/or output overshoot their long-run target values, and even if such overshooting occurs at date $T = 10$ or later (which is to say, after complete stabilization has again become feasible). We first note that panel (c) of Figure 2.8 shows that under the optimal fiscal-monetary commitment, inflation is allowed to overshoot its long-run target value in quarters 0 through 8; this is because over this period, this degree of demand stimulus remains insufficient to raise output to its target level — some degree of over-shooting of inflation is tolerated in order not to have even greater under-shooting of output. But we further observe that this degree of stabilization depends on allowing people to believe that (under circumstances that are actually counter-factual) monetary policy would allow both output and inflation to over-shoot their targets simultaneously.

Figure 2.9 shows the paths of output and inflation that are anticipated in the forward plans
Figure 2.9: Dashed lines show the expected paths of output ($y_{\tau|t}^h$) and inflation ($\pi_{\tau|t}^h$) for dates $t \leq \tau \leq t + h$, under the plans calculated by households and firms with horizon $h = 8$ at successive dates $t$, in the case that both monetary and fiscal policy commitments are optimal (case (iii) from Figure 2.8). The solid lines show the predicted actual paths of output ($y_{\tau|t}^h$) and inflation ($\pi_{\tau|t}^h$). Both $t$ and $\tau$ indicate quarters since the onset of the disturbance (again shown in panel (a)).
of households and firms, in the case (iii) equilibrium of Figure 2.8. Here the assumed path of the financial wedge is shown again in panel (a) — because this is an exogenous variable, actual and anticipated paths coincide. In the other two panels, the paths of output and inflation that are anticipated looking forward from each date are shown. If we let \( y_{\tau|t} \) (respectively, \( \pi_{\tau|t} \)) denote the level of output (inflation) in period \( \tau \) that is anticipated during forward planning in period \( t \), then the figure shows the paths \( \{y_{\tau|t}\} \) and \( \{\pi_{\tau|t}\} \) for dates \( t \leq \tau \leq t + 8 \), looking forward from each of a succession of dates \( t \). The solid line in each panel shows the actually realized path of the variable (the paths of \( y_t = y_{t|t} \) and \( \pi_t = \pi_{t|t} \) as functions of \( t \)); the dashed lines instead show the paths of \( y_{\tau|t} \) and \( \pi_{\tau|t} \) as functions of \( \tau \), with a separate dashed line for each value of \( t \).

Note that these anticipated paths involve greater over-shooting of long-run targets than the actually realized paths do. Furthermore, the over-shooting is anticipated to extend into the period \( t \geq 10 \) in which complete stabilization would be possible. For example, both inflation and output are anticipated to over-shoot their long-run targets simultaneously in quarter 10, in the forward plans of people looking forward from quarters \( t = 2 \) through 7; both are anticipated to over-shoot simultaneously in quarter 11, by people looking forward from quarters \( t = 3 \) through 6. Thus not only does the optimal forward guidance express an intention to keep interest rates below their long-run level in these quarters, but it requires people to believe that the central bank will do so even though (in the calculations of people considering this period several quarters earlier) this is expected to lead to over-shooting of the long-run targets for both inflation and output. Despite the optimal use of counter-cyclical transfer policy, it remains valuable for the central bank to communicate that it will not quickly return to pursuit of its normal targets following a period in which the financial wedge has been so elevated as to cause the ZLB to bind.

2.5 Conclusion

In this paper, we reconsider several issues connected with stabilization policy when the zero lower bound is a relevant constraint on the effectiveness of conventional monetary policy, by relaxing the unrealistic assumption that people should be able to deductively reason about the economy’s
future evolution under a novel policy regime arbitrarily far into the future. Our analysis assumes bounded rationality, taking the approach of finite forward planning, and examines the robustness of conclusions about the consequences of particular combined monetary-fiscal regimes to changes in the assumed degree of decision makers’ foresight in the economy.

We show that, when planning horizons are finite, the contractionary effects of a financial disturbance are less dramatic than in the RE analysis. But, as long as there is some degree of foresight, even a relatively modest financial wedge can substantially impact stabilization goals. We examine the robustness of conclusions about the advantages of commitment to a price-level target over commitment to an inflation target to differing possible assumptions about the length of people’s planning horizons. Furthermore, recognizing that planning horizons may be relatively short for some strengthens the efficacy of systematic price-level targeting, as opposed to an ad hoc price-level targeting (temporary price-level targeting).

Given that Ricardian equivalence does not hold when people have finite horizons, we consider in particular the extent to which pure variation in the government’s budget balance, i.e., changes in the size of lump-sum transfers, can serve as a tool of stabilization policy. We show that fiscal transfers might be a powerful tool to reduce the contractionary impact of a financial disturbance, and make it possible to a complete stabilization of both aggregate output and inflation. But the power of fiscal transfer relies on the degree of monetary policy accommodation. A higher level of welfare is generally possible if the monetary and fiscal authorities commit themselves to history-dependent policies in the period after the real disturbance has dissipated.

There are obviously many ways in which people can fail to be perfectly correct in their beliefs, and our analysis of the detailed predictions of a specific model of bounded rationality should not be taken to imply that we believe that policymakers can count on these precise equations being perfectly descriptive of actual behavior. The spirit of the exercise is rather to propose that better policy judgments can be made if one considers the robustness of one’s conclusions to an alternative specification rather than considering only the predictions of an RE analysis. The presentation of our results give considerable emphasis to the question of the extent to which predicted outcomes
differ depending on exactly how long of planning horizons are assumed; a policy proposal will be considered more sound if it does not merely predict a good outcome under one particular parameterization, but also predicts that outcomes will not be too greatly different if the parameterization is somewhat different.
Chapter 3

Monetary Policy in an Era of Global Supply Chains

Yinxie Xie and Shang-Jin Wei
3.1 Introduction

We live in a world of supply chains. From the data of World Input-output Tables, the world gross output in 2000 was 1.97 times that of the world value added in the same year, suggesting a large role of intermediate inputs in production and supply chains in the modern economy. Many supply chains are global. International trade in intermediate goods has been growing faster than trade in final goods. The importance of supply chains has also grown over time; the ratio of gross output to value added has increased to 2.18 by the end of 2014. In this paper, we study the implications of global supply chains for the design of optimal monetary policy.

There is an active research on outsourcing and offshoring in the field of international trade, where firms purchase intermediate inputs from other firms, sometimes foreign firms, for further processing. Global supply chains are rising in importance as an increased fraction of output is produced as intermediate inputs rather than final goods or services. As important, it is accompanied by an increase in the number of production stages in many sectors (e.g., Wang et al., 2017). The role of globalization in national inflation behavior has also received increased attention.

A voluminous but separate literature in monetary economics studies optimal monetary policy. Woodford (2010) provides an excellent survey of the subject in an closed-economy setting, whereas Corsetti, Dedola, and Leduc (2010) supply an excellent survey of issues in the new open-economy macroeconomics. While central banks typically target only CPI inflation, the literature has studied whether CPI or PPI is more appropriate for monetary policy goal (e.g., De Paoli, 2009; De Gregorio, 2012). Two pioneering papers are especially worth noting. In an open-economy model featuring a single stage of production (i.e., no supply chains), Galí and Monacelli (2005) suggest that PPI is a better target, where PPI in their context is the price index for domestically produced final products. In a closed economy featuring two stages of production (i.e., there are simple national supply chains but not global supply chains), Huang and Liu (2005) demonstrate

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that the optimal simple rule should include PPI inflation as well as CPI, where PPI is the producer prices of domestically produced intermediate products. The intuition is that, in a New Keynesian model, a PPI inflation causes distortions in the allocation of productive resources, including among domestic producers of intermediate goods. Since all firms are owned by the households, the distortions associated with a PPI inflation reduce household welfare too.\textsuperscript{3}

Interactions between multi-stage production and economic openness and their implications for the design of monetary policy have not been much explored. For example, when an economy becomes more open, should the optimal weights on the upstream-sector inflation rise or fall relative to that on the final stage inflation? Should trade frictions such as a rise in the tariff rate affect the design of monetary policy?

We build a New Keynesian model that features simultaneously multi-stage production and openness. A noteworthy feature of the equilibrium is that there are separate Phillips curve relationships for each production stage that link the producer price inflation of a given stage to both the expected next-period inflation and log-deviation of that stage’s real marginal cost from the steady state. The real marginal cost term for each production stage, in turn, is a function of change in the real exchange rate (due to the openness of the economy) and a relative price gap between the production stages (due to multiple stages of production).

Following Rotemberg and Woofford (1999), Galí and Monacelli (2005), and Huang and Liu (2005), we assume that the central bank maximizes the welfare of the household which is approximated by a second-order expansion of the utility function. By making use of equilibrium conditions, we can see that the welfare loss function contains not only output gap and change in the real exchange rate, but also separate producer price inflation in each production stage, separate terms for employment fluctuations in each production stage, and the relative price gap between the production stages. Parameters describing the openness of the economy (shares of each sector’s output that are sold abroad and shares of inputs imported from abroad) appear in the welfare loss function as well.

\textsuperscript{3}Strum (2009) expands on the model with two-stage production developed in Huang and Liu (2005) and revisits the classic question of optimal commitment versus discretionary monetary policies.
Quantitatively, we estimate the nonlinear model up to the second-order expansion (of both the constraints and the welfare function), and consider a family of simple monetary rules, including (a) the classic Taylor (1993) rule that features output gap, CPI inflation, and change in the real exchange rate, (b) the Galí-Monacelli (2005) rule in which PPI inflation takes the place of CPI inflation in the Taylor rule, (c) a rule that includes separate producer price inflation in each stage of production as well as output gap and change in the real exchange rate, and (d) some variations of the previous rules that omit either output gap, change in the real exchange rate, or both. For each rule, we compute optimal weights on each term in the monetary policy rule.

Within the family of simple monetary policy rules, a rule that targets separate producer price inflation at each stage of production (as well as output gap and change in the real exchange rate) delivers a higher welfare level than alternative policy rules. As an economy becomes more open, measured by the share of exports in sales, the optimal weight on the upstream sector inflation rises relative to that on the final stage inflation.

Greater trade frictions reduce the openness of an economy. This, in the model, would dampen the optimal weight on the upstream sector inflation. However, we document a direct welfare loss associated with greater trade frictions even if the monetary policy rule adjusts optimally. In other words, the central bank cannot completely offset the negative effects of greater trade frictions. Naturally, the welfare loss would be even greater without the re-optimization by the central bank.

In general, because the optimal weights on the inflation at different production stages are not proportional to the sales weights, the PPI inflation would not be sufficient to replace these production-stage-specific inflation. At the same time, if we restrict ourselves to only consider aggregate inflation measures (PPI and CPI), targeting PPI inflation yields a smaller welfare loss than targeting CPI inflation alone (in addition to output gap and change in the real exchange rate). That is because PPI inflation puts at least some weight on the upstream sector inflation whereas CPI inflation puts none.

We also consider a general version of the model that can feature an arbitrary number of production stages (but in a closed economy). In this case, as the number of production stages increases,
the optimal weights on the upstream sector inflation as a whole relative to the final stage of production, or the optimal weight on the PPI inflation (if we only consider aggregate price index), will increase as well. This discussion is collected in an appendix.

Is it possible for countries to obtain separate producer price index for upstream and downstream sectors? It turns out that the United States, Japan, Australia, Korea and Canada already collect and report such data. For instance, the US Bureau of Labor Statistics (BLS) has considered a four-stage production process and accordingly constructed a stage-of-processing price indices in the PPI Final Demand-Intermediate Demand system. Figure 3.1 presents separate inflation paths for producer price indices at different production stages as well as core CPI for the United States (Panel A) and Australia (Panel B). It can be seen clearly that the producer price inflation in the upstream sectors and the final stage do not move together. This means that the monetary policy implied by a rule that includes separate producer price inflation would look different from the classic Taylor rule.

This paper builds on the literature on monetary policy by introducing global supply chains. Corsetti, Dedola, and Leduc (2010) provide a comprehensive survey on early literature. Galí and

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*Details about PPI Final Demand-Intermediate Demand indices can be found at https://www.bls.gov/ppi/fdidsummary.htm.*

139
Monacelli (2005) build a small-open economy New Keynesian model that features a single stage of production, and compare three alternative simple policy rules: CPI-based Taylor rule, PPI-based rule, and an exchange rate peg. De Paoli (2009) demonstrates in a model with more general parameterization but also a single stage of production and focuses on terms of trade externality in driving the optimal monetary policy. Shi and Xu (2007) build a two-country New Keynesian model with trade in vertical production, focusing on transborder spillover effect of productivity shock and the discussion of optimal money supply policy. To explain international business cycles, Huang and Liu (2007) build a two-stage production model with staggered prices. Aoki (2001), among early works, studies the optimal sectoral weights in the monetary policy rule when there are two horizontal sectors. Lombardo and Ravenna (2014) study optimal monetary policy in a two (horizontal) sectors under one stage of production with imported inputs for the tradable sector. Matsumura (2018) also studies monetary policy in a small-open economy with multiple sectors but still with only one stage of production.

Our point of departure from this set of papers is a simultaneous introduction of multi-stage production and economic openness. We pay special attention to an interaction between openness and multi-stage production and its implications for the monetary policy rule. In a closed economy setting with two stages of production, Huang and Liu (2005) show that a monetary policy rule that includes PPI inflation is preferred to targeting CPI inflation alone. This is true in our generalized model as well. In addition, we show that the degree of openness systematically affects the optimal weights on the early stage producer price inflation. We also use the model to discuss how the monetary policy rule may be affected by a sharp increase in the costs of international trade such as in a trade war. In a long appendix, we also feature an arbitrary number of production stages (in a closed economy) and investigate effects of a lengthening of the supply chains on the design of the monetary policy.

Gong, Wang, and Zou (2016) study optimal simple monetary policy rules in a two-country New Keynesian model with two stages of production. However, since labor is assumed to be only used in the first stage of production in their paper, there is no misallocation of labor across production
stages. In other words, there is no resource misallocation across production stages due to stage-specific producer price inflation. In comparison, we do allow for potential misallocation across production stages. This generalization qualitatively changes the results of the analysis.

There are other topics discussed in the literature that we do not discuss here. For example, commitment versus discretion in the monetary policy (e.g., Strum, 2008) and the role of investment goods’ prices (e.g., Basu and De Leo, 2019) can in principle be incorporated in our framework as extensions.

This paper is also related to a literature on the effects of globalization on national inflation. Auer, Levchenko, and Sauré (2017), Auer, Borio, and Filardo (2017), and Forbes (2018) study how national inflation dynamics are altered by inter-country connections through supply chains, and how the trade-offs in inflation targeting policies may be changed for central banks. Wei and Xie (2019) demonstrate how an increase in the number of production stages can lead to a weakening in the correlation between PPI and CPI inflation. However, that literature does not explore how an interaction between multi-stage production and globalization affects the design of the monetary policy.

The rest of the paper proceeds as follows: Section 3.2 introduces the basic model with global supply chains; Section 3.3 characterizes the steady-state, flexible-price, and sticky-price equilibria in the special case of two stages of production, derives an expression for the welfare loss function, and discusses the comparative statics of changes in import tariff; Section 3.4 compares several monetary policy rules via calibrations; finally, Section 3.5 concludes the paper. An appendix discusses a more general case with an arbitrary number of production stages.

### 3.2 The model setting

Consider a small-open economy New Keynesian model with an infinitely lived representative household. The household maximizes its utility through consumption and leisure. The household owns all domestic firms and receives dividends from them.

The production of a final good requires $N$-stages of production, which constitutes a vertical
production chain. In each stage of production, a large number of domestic firms produce a unit continuum of differentiated outputs, i.e., \( u \in [0, 1] \). In the first stage of production, firms use domestic labor as the only input. In each of the subsequent stages of production, intermediary inputs purchased from the previous stage (from both domestic and foreign sources) together with labor are used together for production. All production features constant returns to scale.

The firms and households take international prices of inputs and foreign demand of outputs as given (the small open economy assumption). While the firms are price-takers in the factor markets, they are assumed to be monopolistic competitive in their outputs and set the output prices in their own currency (the producer currency pricing assumption, or the PCP). In future work, alternative assumptions such as pricing to the market and the dollar currency pricing can be explored.\(^5\)

3.2.1 Household

The representative household has the following utility function and budget constraint:

\[
E \Sigma_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)]
\]

subject to:

\[P_t C_t + E_t \{ D_{t,t+1} B_{t+1} \} \leq W_t L_t + \Pi_t + B_t\]

where the variable \( C_t \) is the final consumption good, \( L_t \) is the supply of labor, \( D_{t,t+1} \) is the price of a one-period nominal bond paying off in domestic currency, \( W_t \) is the nominal wage, \( \Pi_t \) includes both firm profits and a lump-sum transfer of any government tax revenue, and \( B_t \) denotes the holding of a riskless one-period bond.

The consumption good is a composite of both domestically produced and imported final goods, i.e.,

\[C_t = \Theta \tilde{Y}_{NH,t}^\gamma \tilde{Y}_{NF,t}^{1-\gamma}\]

where \( \tilde{Y}_{NH,t} = [\int_0^1 Y_{NH,t}(u) \frac{du}{u}] \frac{\phi_t}{\pi_t} \) is a bundle of domestically produced differentiated final

\(^5\)Engel (2011) offers a detailed discussion on the implications of the local currency pricing. Mukhin (2018), Egorov and Mukhin (2019), and Gopinath et al. (2019) make a case for the dominance of US dollar pricing.
goods, and \( \tilde{Y}_{NF,i} \) is a bundle of foreign produced differentiated final goods. The parameter \( \gamma \) is the share of the household expenditure on domestically produced final goods, \( 1 - \gamma \) is the share of the expenditure on imports, and \( \theta \) is the elasticity of substitution among the differentiated final goods. The term \( \Theta = \frac{\gamma^\gamma (1 - \gamma)^{1-\gamma}}{\gamma} \) is a constant for normalization.

By the household’s expenditure minimization problem, the demand function for the final goods are

\[
Y_{NH,i}(u) = \left[ \frac{P_{NH,i}(u)}{P_{NH,i}} \right]^{\gamma} \frac{\gamma P_t}{P_{NH,i}} C_t
\]

\[
Y_{NF,i} = \frac{(1 - \gamma)P_t}{P_{NF,i}} C_t
\]

where the aggregate price index for the final consumption is \( P_t = \tilde{P}_{NH,i}^{1-\gamma} \tilde{P}_{NF,i} \), the aggregate price index for the domestic produced final goods is \( \tilde{P}_{NH,i} = (\int_0^1 P_{NH,i}(\mu)^{1-\theta} d\mu)^{1-\theta} \), and the aggregate price index for the foreign produced final goods is \( \tilde{P}_{NF,i} = T_t E_t P_{NF,i}^* \). The term \( E_t \) is the price of foreign currency in units of domestic currency, \( P_{NF,i}^* \) is the exogenous foreign price in foreign currency, and \( T_t \) is a uniform tariff on imports. An upper star * is used to denote variables in the foreign country (denominated in the foreign currency).

By the household’s utility maximization problem, we obtain the labor supply and Euler Equation, respectively, as

\[
\frac{W_t}{P_t} = \frac{V_{i,t}'}{U_{c,j}'}
\]  

and

\[
U_{c,j}' = \beta R_t E_t \left[ U_{c,j+1}' \frac{P_t}{P_{t+1}} \right]
\]

where \( R_t = \frac{1}{\varepsilon \delta_{i,1-t+1}} \) is the gross return on a one-period risk-free nominal bond in domestic currency.

Assuming a CRRA utility function, i.e., \( U(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} \) and \( V(N_t) = \frac{N_t^{1+\psi}}{1+\psi} \), the labor supply (3.1) and Euler equation (3.2) can be written in log-linearized form as

\[
w_t - p_t = \sigma c_t + \psi n_t
\]
\[ c_t = E_t(c_{t+1}) - \frac{1}{\sigma} [\hat{\gamma}_t - E_t(\pi_{t+1}) - \rho] \]  
(3.4)

where lower-case letters denote the logarithm of the respective variables, \( i_t = R_t - 1 \) is the nominal interest rate in domestic currency, \( \pi_{t+1} = p_{t+1} - p_t \) is the CPI inflation, and \( \rho = \beta^{-1} - 1 \).\(^6\)

We assume that the household has access to a complete set of (both domestic and international) state-contingent securities, and trade in the international asset market before the monetary authority chooses its policy. This timing assumption follows the convention in this literature, and ensures a risk-sharing condition that is independent of monetary policy rules (see the discussion in Senay and Sutherland, 2007, and Matsumura, 2018). Then, the intertemporal marginal rates of substitution must be equalized across countries, i.e.,

\[ \beta'(C_0^*)^{-\sigma}/P_t^* = \beta'(C_t^*)^{-\sigma}/P_0^* \Lambda = \beta'(C_t^*)^{-\sigma} E_t P_t^{-1} \]  
(3.5)

where \( \Lambda \) is the marginal utility of initial debt for the domestic household, and the risk-sharing condition implies\(^7\)

\[ C_t = \theta^* C_t^* Q_t^{1/\sigma} \]  
(3.6)

i.e.,

\[ c_t = c_t^* + \sigma^{-1} q_t + \xi \]  
(3.7)

where the variable \( Q_t = \frac{E_t P_t^*}{P_t} \) is the real exchange rate, \( \theta^* = (\Lambda P_0^*)^{-1/\sigma}/C_0^* \) is a constant (i.e., invariant across policies), and the variable \( \xi = ln \theta^* \). Note that if the asset markets cannot insure across different policies, then \( \theta^* \) (or \( \Lambda \)) will have to vary across policies.

Engel (2016) criticizes this timing assumption on the asset market and advocates using a balanced trade assumption to replace the risk-sharing condition. Since trade imbalance is a key feature of open economies and global supply chains, we choose to retain the complete market assumption and the risk-sharing condition in our baseline model. This risk-sharing condition facilitates a com-

\(^6\)It is equivalent to write \( i_t = -log E_t, D_t, t + 1 \) and \( \rho = -log \beta \).

\(^7\)Details for deriving equation (3.5) can be found in Matsumura (2018).
parison of our model with Galí and Monacelli (2005) and De Paoli (2009), where a complete asset market is also assumed.\footnote{Galí and Monacelli (2005) assume the Cole-Obstfeld parameterization, which is a knife-edge case when $\Lambda$ is exogenous to the monetary policy. This is not true in our case. Instead, we assume an exogenous $\Lambda$ only to facilitate a comparison with the previous literature.}

To acknowledge the Engel’s critique, we discuss in Appendix C.9 an alternative setup that features a balanced trade without the risk-sharing condition. With a second-order approximation of the non-linear system, we calibrate the model under a set of alternative monetary policy rules, estimate the optimal weights on the variables in each rule, and compute the associated welfare loss. A key finding is that the welfare ranking of the monetary policy rules under the balanced trade assumption is the same as under the assumption of a complete asset market and the risk-sharing condition. Based on the appendix, we suggest that the Engel’s critique may not be important for our particular research question.

We assume that the foreign consumption follows an AR(1) process, i.e., \( c_t^* = \rho c^* c_{t-1} + \epsilon_{n,t} \) with \( \rho \in (0,1) \) and \( \epsilon_{c^*} \sim N(0, \sigma_{c^*}^2) \). From the risk-sharing condition (3.7), given exogenous foreign consumption, there is an increase in consumption if and only if the real exchange rate depreciates. Under the assumption of complete international financial markets, it also implies uncovered interest parity, i.e., \( i_t - i^*_t = E_t(\Delta e_{t+1}) \), where \( e_t = \ln \mathcal{E}_t \).

3.2.2 Firms

Each final good requires \( N \)-stages of production, with a large number of domestic firms producing a unit continuum of differentiated outputs and featuring constant returns to scale at each stage. In the first stage, the production function for good \( u \in [0,1] \) is given by

\[
Y_{1H}(u) + Y^X_{1H}(u) = A_1 L_1(u)
\]

where \( A_1 \) is the productivity in stage 1 and \( L_1(u) \) is the quantity of labor employed in the production of good \( u \). The output is either sold at home \( Y_{1H}(u) \) or exported abroad \( Y^X_{1H}(u) \). The stage-1 output sold at home and its corresponding price are given by

\[
y_{1H} = \left[ \int_0^1 Y_{1H}(u) \frac{\sigma_{\theta}}{\theta} du \right]^{\frac{\theta}{\sigma}}
\]
and \( P_{1H} = \left[ \int_0^1 P_{1H}(u)^{1-\theta} du \right]^{\frac{1}{1-\theta}} \), respectively.

In each subsequent stage, the production needs to use intermediate inputs. The production in stage \( n \) (for \( n = 2, \ldots, N \)) can be viewed as a two-step process. In the first step, a firm purchases all differentiated outputs produced in the previous stage \( n-1 \) from all global sources and form a bundle of intermediate inputs. Specifically, the intermediate input bundle to be used in the production stage \( n \), i.e., \( \tilde{Y}_n \), is a bundle of two composites of stage \( n-1 \) outputs:

\[
\tilde{Y}_n = \Theta \tilde{Y}^\gamma_{(n-1)H} \tilde{Y}^{1-\gamma}_{(n-1)F}
\]

\[
\tilde{Y}_{(n-1)H} = \left[ \int_0^1 Y_{(n-1)H}(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}
\]

where \( Y_{(n-1)H}(j) \) is the amount of good \( j \) that is domestically produced in stage \( n-1 \) and purchased by the firm in stage \( n \), and \( \tilde{Y}_{(n-1)F} \) is the amounts of composite good that foreign firms produced in stage \( n-1 \). In the factor market, domestic firms are price takers in purchasing foreign composite goods \( \tilde{Y}_{(n-1)F} \), and because of the small-open economy setup, the supply of foreign composite goods is perfectly elastic in price.

The aggregate price index for the inputs in stage \( n \) is then given by \( \bar{P}_n = \bar{P}_{(n-1)H} \bar{P}_{(n-1)F} \), where \( \bar{P}_{(n-1)H} = \left[ \int_0^1 P_{(n-1)H}(u)^{1-\theta} du \right]^{\frac{1}{1-\theta}} \) and \( \bar{P}_{(n-1)F} = T_i \mathcal{E}_i P_{(n-1)F}^* \). The variable \( P_{(n-1)F}^* \) is the price of composite goods in foreign currency produced in stage \( n-1 \) by foreign firms. Note that the output price in stage \( n \) satisfies \( P_{nH} = \bar{P}_{nH} \) for \( \forall n = 1, 2, \ldots, N \).

In the second step, the firm combines the composite intermediate good with labor input to produce an output. The production function for good \( u \) in stage \( n \) is given by

\[
Y_{nH}(u) + Y^{X_{nH}}_{nH}(u) = \Theta^* A_n \tilde{Y}_n(u)^{\psi} L_n(u)^{1-\phi}
\]

where \( \Theta^* = [(1-\phi)^{1-\phi \phi}]^{-1} \) is a constant for normalization. We assume the technology in each stage following the AR(1) process \( a_{n,t} = \rho_n a_{n,t-1} + \epsilon_{n,t} \) with \( a_{n,t} = ln A_{n,t} \) and \( \rho_n \in (0, 1) \) for \( n = 1, 2, \ldots, N \). Note that \( \{\epsilon_{n,t}\}^N_{n=1} \) are i.i.d. shocks with the same normal distribution, i.e.,
\( \epsilon_n \sim N(0, \sigma^2_n) \).

Since the production of any good in stage \( n \) needs a bundle of output from the previous stage as inputs, it captures a feature of a typical input-output table in which the output from all sectors may be used as inputs into the production. In the language of Baldwin and Venables (2013), the entire manufacturing production process follows a combination of a snake and a spider patterns. At a given stage, outputs from the previous stage from all over the world are purchased to form a composite intermediate input, resembling a spider pattern. Going from one stage of production to the next, the process resembles a snake pattern.

By the small-open economy set-up, the foreign demand for domestic output in stage \( n = 1, 2, \ldots, N \) is

\[
Y_{nH}^X = \left( \frac{P_{nH}(u)}{\bar{P}_{nH}} \right) - \theta \frac{Y_{nH}^* P_{nH} E_t}{P_{nH}}
\]

where \( Y_{nH}^* \) is exogenous foreign demand and \( P_{nH}^* \) is the price for domestic produced composite goods in foreign currency. This foreign demand function can be derived from the cost minimization problem of a foreign buyer who aggregates the composite of domestic produced goods.

Similarly, the domestic demand function in stage \( n = 1, 2, \ldots, N \) is given by

\[
Y_{nH}(u) = \left( \frac{P_{nH}(u)}{\bar{P}_{nH}} \right) - \theta \frac{\tilde{Y}_{nH} P_{nH}}{P_{nH}}
\]

Note that the nominal exchange rate is not a sufficient statistics for import tariffs in a world of multi-stage production. An increase in the nominal exchange rate (i.e., a depreciation of the domestic currency) raises both the input costs and foreign demand for domestic goods simultaneously. In comparison, an increase in import tariffs only affects production cost through higher costs of imported inputs.

3.2.3 The Firm’s Pricing Problem

Firms in each stage of production are price-takers in factor markets, but are monopolistic competitors in their outputs. They are assumed to follow a Calvo pricing rule, and the probability that
firms in stage \( n \) can adjust prices freely is \( 1 - \alpha_n, \ n = 1, \ldots, N \). Then, by the law of large numbers, in each period, a fraction \( 1 - \alpha_n \) of firms in stage \( n \) can adjust prices while the rest of firms have to stay unchanged. For a firm producing good \( u \) in stage \( n \), which can set a new price in period \( t \), it chooses price \( P_{nH}(u) \) in domestic currency for its product sold both at home and in the foreign market. Its maximization problem becomes

\[
\max_{P_{nH,t}(u)} E_t \sum_{k=1}^{\infty} \alpha_n^{k-t} D_{t,k} \left[ (1 + \tau) P_{nH,t}(u) - \Psi_{n,k}(u) \right] \left[ Y_{nH,k}^d(u) + Y_{nH,k}^X(u) \right]
\]

where \( \tau \) is the subsidy to firms that corrects the distortion from monopolistic competition, \( \Psi_{n,k}(u) = \tilde{P}_{n,k} W_k^{1-\phi} / A_{n,k} \) is the nominal unit production cost for \( n = 2, \ldots, N \) and \( \Psi_{1,k}(u) = W_k / A_{1,k} \) for \( n = 1 \), \( \tilde{P}_{n,k} \) is the price for the composite of intermediate input goods at stage \( n \), and \( Y_{nH,k}^d(u) \) and \( Y_{nH,k}^X(u) \) denote the output demand from both domestic and foreign market respectively.

The optimal pricing decision is given by

\[
P_{nH,t}^o(u) = \frac{\mu}{1 + \tau} \frac{E_t \sum_{k=1}^{\infty} \alpha_n^{k-t} D_{t,k} \Psi_{n,t}(u) [Y_{nH,t}^d(u) + Y_{nH,k}^X(u)]}{E_t \sum_{k=1}^{\infty} \alpha_n^{k-t} D_{t,k} [Y_{nH,t}^d(u) + Y_{nH,k}^X(u)]}
\]

where \( \mu = \frac{\theta}{\theta - 1} \) is the markup in the market for producing outputs in each stage.\(^9\) To be abstract from the distortion generated by monopolistic competition, a subsidy is imposed such that \( 1 + \tau = \mu \).

Taking input prices as given, the cost minimization problem for the firms at stage \( n \) for \( n = 2, \ldots N \) in period \( t \) yields a factor demand function as

\[
\tilde{Y}_{n,t}^d = \phi \frac{\Psi_{n,t}}{P_{n,t}} \int_0^1 [Y_{nH,t}^d(u) + Y_{nH,k}^X(u)] du
\]

(3.8)

\[
L_{n,t}^d = (1 - \phi) \frac{\Psi_{n,t}}{W_t} \int_0^1 [Y_{nH,t}^d(u) + Y_{nH,k}^X(u)] du
\]

(3.9)

---

\(^9\)Since we assume the same elasticity of substitution among differentiated goods across the stages, it also implies the same markup across the stages.
\[
\hat{Y}_{(n-1)H,t}^d = \frac{\gamma \hat{p}_{n,t}}{P_{(n-1)H,t}} \hat{Y}_{n,t}^d 
\]

(3.10)

\[
\hat{Y}_{(n-1)F,t}^d = \frac{(1 - \gamma) \hat{p}_{n,t}}{P_{(n-1)F,t}} \hat{Y}_{n,t}^d 
\]

(3.11)

and

\[
Y_{(n-1)H,t}^d (u) = \left( \frac{P_{(n-1)H,t}(u)}{P_{(n-1)H,t}} \right)^{-\theta} \hat{Y}_{(n-1)H,t}^d 
\]

(3.12)

In the first stage of production, the firm’s pricing problem is simpler since labor is the only input. Specifically, the optimal pricing decision for a firm in stage 1 is

\[
P_{1H,t}^o (u) = \frac{E_t \sum_{\tau=1}^{\infty} \alpha_1^{\tau-t} D_{1,\tau} \Psi_{1,\tau} (u) [Y_{1,\tau}^d (u) + Y_{1H,k}^d (u)]}{E_t \sum_{\tau=1}^{\infty} \alpha_1^{\tau-t} D_{1,\tau} [Y_{1,\tau}^d (u) + Y_{1H,k}^d (u)]}
\]

where \( \Psi_{1,\tau} (u) = W_{\tau} / A_{1,\tau} \) is the unit production cost in stage 1, and the subsidy has been imposed to offset the markup.

Since labor is the only input in the first stage, the labor demand is

\[
L_{1,t}^d = \frac{\Psi_{1,t}}{W_t} \int_0^1 [Y_{1H,t}^d (u) + Y_{1H,k}^d (u)] \, du
\]

As the goods are symmetric, we drop good index \( u \) in the price variable. The aggregate price index for the outputs in stage \( n, n = 1, 2, \ldots, N \), is thus given by

\[
P_{nH,t} = [\alpha_n (P_{nH,t-1}^{1-\theta} + (1 - \alpha_n) (P_{nH,t}^o)^{1-\theta})]^{1/\theta} 
\]

(3.13)

3.2.4 The Market Clearing Conditions and Equilibrium Definition

**Equilibrium definition:** given exogenous monetary policy (the rule of nominal interest rate or nominal exchange rate \( \{i_t, \mathcal{E}_t\} \) and tariffs \( \{T_t\} \), as well as exogenous foreign demand and foreign prices \( \{C_t^*, P_{nH,t}^*, P_{nF,t}^*, Y_{nH,t}^*, Y_{nF,t}^*\}_{n=1}^{N} \), the market equilibrium consists of a set of stochastic processes
\( \{C_t, L_t\} \) for domestic households, \( \{L^d_{n,t}(u), Y_{nH,t}(u), Y^X_{nH,t}(u), P_{nH,t}(u)\}_{n=1}^{N} \) for firms \( u \in [0, 1] \) and price indices \( \{P_{nH,t}\}_{n=1}^{N} \), and wages and real exchange rate \( \{W_t, Q_t\} \), satisfying the following conditions:

1. Taking prices and wages as given, the representative household maximizes its utility.

2. Taking intermediate input goods prices, wages, and all output prices except their own’s as given, firms in each stage maximize their profits.

3. The intertemporal trade balance condition holds.

4. The labor market clears, and the goods markets clear in all production stages, i.e.,

\[
L_t = \sum_{n=1}^{N} L^d_t, \quad Y_{nH} = Y^d_{nH}, \quad Y^X_{nH} = Y^X_{nH}
\]

Note that the intertemporal trade condition is derived from a no-Ponzi-game condition in the household’s debt, which does not necessarily require trade balance in each period. The case of a balanced trade is discussed in Appendix C.9.

### 3.3 The Case of Two-stage Production

If we assume two stages of production, we can obtain a number of analytical expressions.\(^{10}\) We now characterize sequentially the steady-state, flexible-price, and sticky-price equilibria. We derive the second-order approximation of the welfare loss function for the sticky price case. With sticky prices, there is misallocation of labor across production stages. Because the terms of trade externality and the labor allocation distortions interact with each other, the real exchange rate and the relative price gaps between the production stages enter the welfare loss function.

The model lends itself well to thought experiments on how a change in openness affects the optimal monetary policy rule. This also facilitates a discussion on how a change in the import tariff

\(^{10}\) We present the results for the general case of \( N \) stages of production in Appendix C.1.
affects monetary policy. While we only consider domestic productivity shocks in this section, a broader set of stochastic shocks are considered in the numerical analysis in Section 3.4.

3.3.1 The Steady-state Equilibrium

In the steady state, \( A_1 = A_2 = 1 \), and foreign variables are kept constant. The price index satisfies \( P_{1H} = P_{2H} = 1 \), and foreign variables are kept constant. The price index satisfies \( \%_{1} = 6 \%_1 = \%_1(D) \) and \( \%_2 = 6 \%_2 = \%_2(D) \). By \( 6 \%_2 = 6 \%_2 = \%_2(W) \), from Section 3.2.2, the price indices of domestically produced goods across stages are given by \( \%_1 = \%_1 + q + Wq \) and \( \%_2 = \%_2 + q \). Therefore, the factor demand functions are given by

\[
\bar{Y}_2 = \phi W^{1-\phi}(\bar{P}_2)^{1-\phi}(Y_{2H} + X_{2H})
\]

\[
L_2 = (1 - \phi)W^{-\phi}\bar{P}_2^{-\phi}(Y_{2H} + X_{2H})
\]

\[
Y_{1H} = \frac{\gamma\bar{P}_2\bar{Y}_2}{\bar{P}_{1H}}
\]

and

\[
L_1 = Y_{1H} + X_{1H}
\]

where \( Y_{2H} = Y_{2H}P_{2H}E \), \( X_{1H} = Y_{1H}P_{1H}E \), \( Y_{2H} = \gamma CF_{2H}^{1-(1-\gamma)}\bar{P}_{2F}^{1-\gamma} \), and \( \bar{P}_{2F} = TP_{2F}E \). By backward induction, we can obtain the labor demand function in each stage of production.

The equilibrium in the steady state \( \{ C, L \} \) is then fully characterized by the labor supply equation (3.1), the risk-sharing condition (3.6), and the labor demand function \( L_1 + L_2 \) as derived above, where all price indices are a function of \( W \) and \( E \). Following Huang and Liu (2005), we set \( \psi = 0 \) to simplify expressions, which can be justified by indivisible labor (e.g., Hansen, 1985).\footnote{In the numerical analysis in Section 3.4, we use a more general value of \( \psi \) based on calibrations.} Then,
equations (3.1) and (3.6) give

\[ w = \sigma c^* + e + p^* \]

\[ c = c^* + \frac{1}{\sigma} [e + p^* - p] + \xi \]

By substituting \( w \) into \( p = \gamma \tilde{p}_{2H} + (1 - \gamma) \tilde{p}_{2F} \), together with \( \sigma c = \sigma c^* + e + p^* - p + \sigma \xi \), we obtain an expression of \( c \), which includes neither the domestic price index nor nominal exchange rate. Similarly, by substituting \( w \) into price index, together with \( L = L_1^d + L_2^d \), we obtain an expression of steady-state labor \( l \).

3.3.2 The Flexible-price Equilibrium

In the flexible-price equilibrium, \( \alpha_n = 0 \) for \( n = 1, 2 \). The optimal pricing decision for firms at stage \( n \) becomes \( P_{nH,t}^o = \Psi_{n,t} \) and thus \( P_{nH,t} = \tilde{P}_{nH,t} = P_{nH,t}^o \).

With \( \tilde{P}_{2,t} = \tilde{P}_{1H,t}^{\gamma} \tilde{P}_{1F,t}^{1-\gamma} \), stage-specific prices are given as \( P_{1H,t} = W_t/A_{1,t} \) and

\[ P_{2H,t} = W_t^{1-\phi+\gamma \phi} (T_t E_t)^{1-\gamma}\phi (P_{1F,t}^*)^{(1-\gamma)\phi} A_{1,t}^{-\gamma \phi} A_{2,t}^{-1} \]

The aggregate price for final consumption goods is \( P_t = (P_{2H,t})^\gamma (T_t P_{2F,t}^* E_t)^{1-\gamma} \), in which we have plugged the expression of \( \tilde{P}_{2F,t} \).

Similar to the analysis in the steady-state equilibrium, we have \( Y_{1H,t}^d(u) = \tilde{Y}_{1H,t}^d, Y_{2H,t}^d(u) = \tilde{Y}_{2H,t}^d, Y_{1H,t}^d = \tilde{Y}_{1H,t}^d \) and \( Y_{2H,t}^d = \tilde{Y}_{2H,t}^d \). The factor demand functions are given by

\[ \tilde{Y}_{2,t}^d = \phi \frac{W_t^{1-\phi} (\tilde{P}_{2,t})^{1-\phi}}{A_{2,t}} (Y_{2H,t}^d + Y_{2H,t}^d) \]

\[ L_{2,t}^d = (1 - \phi) \frac{W_t^{1-\phi} (\tilde{P}_{2,t})^{1-\phi}}{A_{2,t}} (Y_{2H,t}^d + Y_{2H,t}^d) \]

\[ Y_{1H,t}^d = \frac{\gamma \tilde{P}_{2,t} \tilde{Y}_{2,t}^d}{\tilde{P}_{1H,t}} \]
and

\[ L_{1,t}^d = Y_{1H,t}^d + Y_{1H,t}^x \]

where \( Y_{2H,t}^x = \frac{Y_{2H,t}^x}{P_{2H,t}^x} \), \( Y_{1H,t} = \frac{Y_{1H,t}^x}{P_{1H,t}^x} \), \( Y_{2H,t}^d = \gamma C \tilde{P}_{2H,t}^{1-(1-\gamma)} \tilde{P}_{2F,t}^{1-\gamma} \), and \( \tilde{P}_{2F,t} = T_i P_{2F,t}^* E_{t} \).

Similar to the steady-state equilibrium, the flexible-price equilibrium \( \{C_t, L_t\} \) are then fully characterized by the labor supply equation (3.1), the risk-sharing condition (3.6), and the labor demand function \( L_1^d + L_2^d \) as derived above. With the assumption of \( \psi = 0 \), the equations (3.1) and (3.6) again give

\[ w_t^f = \sigma e_t^f + e_t^f + p_t^* \]
\[ c_t^f = c_t^* + \frac{1}{\sigma} [e_t^f + p_t^* - p_t^{f*}] + \xi \]

where we denote the endogenous variables under flexible-price equilibrium with an upper symbol \( f \).

By substituting \( w_t^f \) into price index, we obtain the expressions of \( c_t^f \) and \( l_t^f \), which does not include domestic price index or nominal exchange rate. Note that, by denoting \( t_i = \ln T_i \), we have the expression of CPI index \( p_t^f \) as

\[ p_t^f = \gamma [(1 - \phi + \gamma \phi) w_t^f + (1 - \gamma) \phi (e_t^f + t_i) + (1 - \gamma) \phi p_{1F,t}^* - \gamma \phi a_{1,t} - a_{2,t}] \]
\[ + (1 - \gamma) (e_t^f + t_i + p_{2F,t}^*) \]

By substituting \( p_t^f \) into the risk-sharing condition, we obtain the natural rate of interest rate as

\[ \bar{r} = \rho + \sigma E(c_{t+1}^f - c_t^f) \]
\[ = \rho + \gamma [\gamma \rho_1 \Delta a_{1,t} + \rho_2 \Delta a_{2,t}] \]

where we treat exogenous foreign variables and import tariff as constant.
3.3.3 The Sticky-price Equilibrium

We now derive the New Keynesian Phillips curves for each stage as a function of the relative price gap and output gap, and characterize the equilibrium with sticky prices. Similar to the derivation in Galí (2015), the Phillips curve for each stage is given by

\[ \pi_{1H,t} = \beta E_t \pi_{1H,t+1} + \lambda_1 \hat{\gamma}_{1,t} \]

\[ \pi_{2H,t} = \beta E_t \pi_{2H,t+1} + \lambda_2 \hat{\gamma}_{2,t} \]

where \( \lambda_n = \frac{(1-a_n)(1-a_n)}{a_n} \) for \( n = 1, 2 \) and \( \hat{\gamma}_n \) is the log-derivation of real marginal cost from steady-state equilibrium, i.e.,

\[ \hat{\gamma}_{n,t} = \ln(\Psi_{n,t}/P_{nH,t}) - \ln(\Psi_n/P_{nH}) \]

Since \( \Psi_n \) and \( P_{nH} \) are the marginal cost and aggregate price in stage \( n \) under steady-state equilibrium, we have

\[ \hat{\gamma}_{1,t} = \sigma \hat{\epsilon}_t + \frac{1-\gamma}{\gamma} \hat{q}_t - \hat{g}_{2H,t} - a_{1,t} \]

\[ \hat{\gamma}_{2,t} = \gamma \phi \hat{g}_{2H,t} + \frac{1-\gamma}{\gamma} \hat{q}_t + (1-\phi)\sigma \hat{\epsilon}_t - a_{2,t} \]

where \( \hat{g}_{2H,t} \) is the log-deviation of relative price gap between stage-2 output price with respect to stage-1 output price from the steady-state equilibrium, i.e., \( \hat{g}_{2H,t} = \ln(P_{1H,t}/P_{2H,t}) - \ln(P_{1H}/P_{2H}) \).\(^\text{12}\)

In terms of notation, we use the variable with a hat to denote deviation from the steady-state equilibrium, and use a tilde to denote the deviation from the flexible-price equilibrium.

After log-linearizing the Euler equation of the household around the steady state and subtract-
ing the steady-state IS curve, we obtain the IS curve as

$$\hat{c}_t = E_t \hat{c}_{t+1} - \frac{1}{\sigma} [\hat{\pi}_t - E_t (\pi_{t+1})]$$

The aggregate inflation $\pi_t$ (CPI index) can be written as

$$\pi_t = \pi_{2H,t} + \frac{1 - \gamma}{\gamma} \Delta \hat{q}_t$$

The derivation of the aggregate inflation can be found in Appendix C.2.

The law of motion for the relative price gap between stage 1 and stage 2 is characterized by

$$\hat{g}_{2H,t} = \hat{g}_{2H,t-1} + \pi_{1H,t} - \pi_{2H,t}$$

The above equations together with the risk-sharing condition (3.7) fully characterize the sticky-price equilibrium.

### 3.3.4 A Utility-based Welfare Loss Function for Optimal Monetary Policy

We assume that the central bank aims to maximize the household’s utility, and represent its objective function by a second-order approximation. This follows the approach of Rotemberg and Woodford (1999), Benigno and Woodford (2006), and Galí (2015). Due to simultaneous presence of openness and multiple stages of production, the first-order terms do not cancel each others out, unlike in the standard literature. This means that the welfare loss function in our setting includes inflation for each stage of production, the relative price gap across production stages, as well as the real exchange rate and output gap. The flexible-price equilibrium is, in general, not Pareto-optimal. Only in the limit case of a closed economy would the first-order terms cancel out and the welfare loss function is left only with the second-order terms as shown in Appendix C.6. In such a case, both the steady-state equilibrium and the flexible-price equilibrium are Pareto-efficient.

Since labor is present in all stages of production and prices are sticky, there is labor misallo-
cation across production stages, which reduces the utility of the household. There is also a terms-of-trade externality since some of the domestically produced intermediate goods are exported and some imported intermediate inputs are used in domestic production.

The household’s utility function is given by

\[ E \sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)] \]

where \( U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma} \) and \( V(L_t) = \frac{L_t}{1+\psi} \).

The household’s utility function is given by

\[ E \sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)] \]

where \( U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma} \) and \( V(L_t) = \frac{L_t}{1+\psi} \).

A second-order Taylor expansion around the steady state \((C, L)\) for the period utility of consumption gives

\[ U(C_t) - U = U_t C(\hat{c}_t + \frac{1-\sigma}{2} \hat{c}_t^2) + t.i.p. \]

where \( \hat{c}_t \) denotes the log-deviation of consumption from steady state and \( t.i.p. \) stands for “terms independent of policy” following Woodford (2003).

By the labor market clearing condition, we obtain the second-order Taylor expansion around the steady state for the period utility of employment, i.e., \( V(L_t) \), as

\[ V(L_t) - V = V_L L \left\{ \sum_{n=1}^{2} \frac{L_n}{L} \left[ \bar{\bar{I}}_{n,t} + \frac{1}{2} \bar{I}_{n,t}^2 \right] \right\} + t.i.p. \]

where \( L_n/L \) is the share of labor in stage \( n \) in total labor in the steady state, as described in Section 3.3.1, with the assumption of \( \psi = 0 \).\(^{13}\)

It is useful to rewrite the employment gap in the two production stages in terms of the output

\(^{13}\)As specified for the steady-state variables in Section 3.3.1, we have \( L_1 = Y_{L}^{d}/y_{L} + \phi \) and \( L_2 = 1 - Y_{L}^{d}/L \), where \( \bar{a}_1 = y_{L}^{d}/y_{L} + \phi \) and \( \bar{a}_2 = y_{L}^{d}/y_{L} + \phi \) are the share of goods sold at home, respectively, in stages 1 and 2.
gap and the relative price gap:

\[
\hat{I}_{1,t} = [-\bar{a}_1 \phi + \bar{a}_1 \bar{a}_2 + \bar{a}_1 \sigma] \hat{c}_t + [\bar{a}_1 \phi \gamma - 1] \hat{g}_{2H,t} \\
+ \frac{1 + \bar{a}_1 - \bar{a}_1 \gamma - \bar{a}_1 \bar{a}_2 \gamma}{\gamma} \hat{q}_t + \bar{a}_1 d_{2,t} + d_{1,t} - a_{1,t} - \bar{a}_1 a_{2,t}
\]

\[
\hat{I}_{2,t} = (\bar{a}_2 - \phi \sigma) \hat{c}_t + \phi \gamma \hat{g}_{2H,t} + \frac{1 - \bar{a}_2 \gamma}{\gamma} \hat{q}_t + d_{2,t} - a_{2,t}
\]

where \(d_{n,t} = \ln(\int_0^1 \left( \frac{p_{nH,t}(u)}{p_{nH,t}} \right)^{-\theta} du)\) for \(n = 1,2\) measures the price dispersion in stage \(n\), \(\bar{a}_1 = \frac{y_{1H}}{y_{1H} + y_{2H}}\) and \(\bar{a}_2 = \frac{y_{2H}}{y_{2H} + y_{2H}}\) are the shares of goods sold, respectively, to the domestic market in stages 1 and 2 in the steady state. Details can be found in Appendix C.3.

Following Galí (2015), up to a second-order approximation around the steady state, the price dispersion term \(d_{n,t}\) for \(n = 1,2\) can be written as

\[
d_{n,t} = \frac{\theta}{2} \int_0^1 \left[ p_{nH,t}(i) - p_{nH,t} \right]^2 di \equiv \frac{\theta}{2} \text{var}\{p_{nH,t}(i)\}
\]

By Woodford (2003), the price dispersion can be re-written as a function of inflation in each stage of production, i.e.,

\[
\sum_{t=0}^{\infty} \beta^t \text{var}\{p_{nH,t}(i)\} = \lambda_n^{-1} \sum_{t=0}^{\infty} \beta^t \pi_{nH,t}^2 + t.i.p.
\]

We substitute \(\hat{I}_{n,t}\) and \(d_{n,t}\) into the period utility of employment. Since the total labor income of households are given by \(WL = \frac{PCy}{d_2} (1 - \phi) + \phi \frac{PCy}{d_2} \frac{\gamma}{\bar{a}_1}\), the steady state equilibrium implies \(WL = PC(1 - \Phi)\), where \(1 - \Phi = \frac{\gamma}{d_2} (1 - \phi) + \phi \frac{\gamma^2}{d_2 \bar{a}_1}\), and thus \(U_c C = V_L L (1 - \Phi)\).\(^{14}\) In addition, in the steady state, the labor shares in the two stages are given, respectively, by \(L_1/L = \frac{\phi}{\gamma \phi + (1 - \phi) \bar{a}_1}\) and \(L_2/L = \frac{(1 - \phi) \bar{a}_1}{\gamma \phi + (1 - \phi) \bar{a}_1}\).

By summing up \(U(C_t) - U\) and \(V(L_t) - V\), the household’s welfare loss as a fraction of the

\(^{14}\)In the small-open economy New Keynesian literature, \(\Phi\) is normally assumed to be zero due to the symmetry assumption across countries in a two-country structure model, e.g., Faia and Monacelli (2008), De Paoli (2009), or in a model of continuum of small countries, e.g., Galí and Monacelli (2005).
steady state consumption is given by

\[ W = E_0 \sum_{t=0}^{\infty} \beta^t \frac{U(C_t) - V(L_t) - (U - V)}{U_C} \]

\[ = E_0 \sum_{t=0}^{\infty} \beta^t \left( \hat{c}_t + \frac{1}{2} \{ - (1 - \sigma) \hat{c}_t^2 + (1 - \Phi) \left[ \frac{L_1}{L} (\hat{h}_{1,t} - a_{1,t} - \bar{a}_1 \bar{a}_2) \right]^2 \right. \]

\[ \left. + \frac{L_2}{L} (\hat{h}_{2,t} - a_{2,t})^2 + \frac{L_1}{L} \theta \lambda_1^{-1} \pi_1^2 H_{1,t} + \left( \frac{L_1}{L} \bar{a}_1 + \frac{L_2}{L} \bar{a}_2 \right) \theta \lambda_2^{-1} \pi_2^2 H_{2,t} \} \right\} + t.i.p. \]  

(3.14)

where

\[ \hat{h}_{1,t} = (\bar{a}_1 \Phi + \bar{a}_1 \bar{a}_2 + \bar{a}_1 \sigma) \hat{c}_t + (\bar{a}_1 \Phi \gamma - 1) \hat{g}_{2H,t} + \frac{1 + \bar{a}_1 - \bar{a}_1 \gamma - \bar{a}_1 \bar{a}_2 \gamma}{\gamma} \hat{q}_t \]

\[ \hat{h}_{2,t} = (\bar{a}_2 - \Phi \sigma) \hat{c}_t + \Phi \gamma \hat{g}_{2H,t} + \frac{1 - \bar{a}_2 \gamma}{\gamma} \hat{q}_t \]

The first-order terms can be eliminated by approximating the equilibrium conditions specified in Section 3.3.3 to a second-order expansion using the approach developed by Sutherland (2002) and Benigno and Woodford (2006). Though we do not present an explicit expression of the welfare loss function purely in second-order terms due to the complexity arising from the multi-stage production, the numerical analysis in Section 3.4 approximates the full nonlinear equilibrium in the second-order expansion.

Our setup nests several models in the existing literature as special cases. In particular, if we shut down economic openness, and assume \( N = 2, \gamma = 1, \) and \( \bar{a}_1 = \bar{a}_2 = 1, \) the expression (3.14) reproduces the welfare loss function in Huang and Liu (2005). Alternatively, if we maintain the small-open economy structure, but assume one stage of production \( (N = 1), \) and \( \bar{a}_1 = \gamma, \) and additionally impose symmetry in the foreign country, the expression (3.14) reproduces the welfare loss function in Galí and Monacelli (2005) and De Paoli (2009).\(^{15}\)

\(^{15}\)In the case of one-stage production, the results in Galí and Monacelli (2005) and De Paoli (2009) are reproduced by also recognizing that the foreign demand now is foreign final demand. It is also worth noting that De Paoli (2009) allows for a general parameterization of elasticity of substitution regarding foreign goods and domestic goods, while
To shed light on the role of the length of production chain in affecting the welfare, we derive analytical results in the case of a closed-economy (see Appendix C.6). In this case, both the steady-state equilibrium and the flexible-price equilibrium are Pareto-efficient, and the welfare loss function can be expressed in second-order terms. In particular, the stage-specific inflation terms have a direct impact, given by the expression of \( \sum_{n=1}^{N} \theta \phi^{N-n} \lambda_n^{-1} \pi_{n,t}^2 \), in the welfare loss function.

Two features of these terms deserve special attention. First, for a fixed number of production stages \( N \), assuming the same price stickiness in all stages, the coefficients before inflation in the downstream stages are larger than those in the upstream stages. Second, holding the downstream sectors constant, as one adds more upstream stages, the final stage inflation (i.e., CPI) becomes less important in the welfare loss function, while the inflation rates in the upstream stages as a whole become more important. In other words, as the production chain becomes longer, the central bank needs to care more about the inflation rates in the intermediate stages but less about the final stage inflation.

3.3.5 Discussion on the Welfare Loss Function

There are two distortions in the model, i.e., the labor allocation distortion (caused by sticky prices along the production chain) and the terms of trade externality. Those terms measuring stage-specific unemployment gaps \( \hat{h}_{1,t} \) and \( \hat{h}_{2,t} \) show up in welfare loss function because of sticky prices and misallocation of labor across production stage. The real exchange rate \( \hat{q}_t \) appears due to the terms-of-trade externality.

In an open economy with a finite elasticity of foreign demand for export, the social planner wishes to exploit a domestic monopoly power in trade. This gives rise to a terms of trade effect. As the real exchange rate \( \hat{q}_t \) and the relative price gap between production stage, \( \hat{g}_{2H,t} \), jointly enter \( \hat{h}_{1,t} \) and \( \hat{h}_{2,t} \), we see an interaction between the labor allocation distortion and the terms of trade distortion. This interaction suggests that the monetary policy discussion is not a simple sum of the results from an open-economy with one stage of production and a closed economy with two

---

we assume the elasticity to be one by taking a Cobb-Douglas form.
stages of production.

The second-order terms in the welfare loss function consist of three parts: (a) a consumption gap, and stage-specific unemployment gaps which can be written in terms of the consumption gap, (b) separate inflation terms for each production stage, and the relative price gap between production stages, and (c) the real exchange rate. The consumption gap is connected with the output gap and real exchange rate via \( \hat{\gamma} = \gamma \hat{c} + \frac{1 - \gamma^2}{\gamma} \hat{q} \).

The welfare loss function indicates that targeting CPI and PPI is not sufficient. Instead, the central bank needs to pay attention to stage-specific inflation terms along the production process as well as the price gap across the production stages. These terms will become more important as the economy becomes more open or when the number of production stages increase. The last point is elaborated in Appendix C.6.2 when we consider the case of \( N \)-stage production in a closed economy.

3.3.6 Discussion on Value Chains and Price Stickiness

A key feature studied in this paper is a vertical structure of production chain or value chain. To highlight the role of vertical structure and clarify its differences with a horizontal production structure, let us consider how a shock propagates along the production chain. The key logic was first pointed out by Huang and Liu (2001) in a closed-economy setting. The same carries over to an open-economy setting.

Let us consider a shock to the nominal wage, which may be caused by an exogenous monetary shock, and focus on a partial equilibrium in which the exchange rate is taken as fixed for simplicity. Since labor is the only input in the first stage of production, the marginal cost of the first-stage production changes immediately following the wage shock, but only a fraction of the firms in the first stage reset their prices due to price rigidity. For this reason, the first-stage output prices, which are the input prices for the second stage, only partially reflect the true change in the labor cost.

For firms in the second stage production, since they use both labor and intermediate goods for production, their marginal cost experiences a smaller change compared to the first-stage output
prices. The firms in the second stage thus have less incentive to adjust their prices even though they have the opportunity to do so. The second-stage output prices deviate from those in a flexible price equilibrium more than the first-stage output prices.

In general, when there are $N$-stages of production, the output prices of more downstream stages are more sluggish than their more upstream counterparts. In this sense, the vertical production structure creates endogenously exacerbating price rigidity moving from upstream to downstream stages along the production chain. This feature does not exist for a horizontal production structure.

3.3.7 Effects of a Higher Import Tariff

Motivated by a recent rise in international trade tensions, we study how a change in the trade policy, which alters the cost of supply chain trade, may affect the design of the monetary policy. We compare a high-tariff case with a low-tariff case. In each case, the import tariff affects the welfare loss function through its impact on the steady-state shares of the domestic demand in the total demand for domestically produced goods in the two stages of production, i.e., $\tilde{a}_1$ and $\tilde{a}_2$.

It can be shown that

$$\frac{\tilde{a}_2}{1 - \tilde{a}_2} = f_2(\ast) \cdot T^{(1 - \gamma)(1 + \phi)(1 - \frac{1}{\sigma})}$$

and

$$\frac{\tilde{a}_1}{1 - \tilde{a}_1} = f_2(\ast) \cdot \frac{1}{1 - \tilde{a}_2}$$

where $f_1(\ast)$ and $f_2(\ast)$ are functions of exogenous foreign variables. The explicit expression of $f_1(\ast)$ and $f_2(\ast)$ can be found in Appendix C.4. We proceed with the following proposition.\(^\text{16}\)

**Proposition 1:** If the relative risk aversion $\sigma = 1$, a higher import tariff does not affect the steady-state allocation, i.e., \(\frac{\partial \tilde{a}_1}{\partial \tilde{t}} = \frac{\partial \tilde{a}_2}{\partial \tilde{t}} = 0\) and \(\frac{\partial L_1/L}{\partial \tilde{t}} = \frac{\partial L_2/L}{\partial \tilde{t}} = 0\); if $\sigma > 1$, a higher import tariff will lead to a higher share of domestic demand for domestically produced goods, i.e., $\frac{\partial \tilde{a}_1}{\partial \tilde{t}}, \frac{\partial \tilde{a}_2}{\partial \tilde{t}} > 0$, and the labor share in the upstream production relative to the downstream decreases, i.e., $\frac{\partial L_1/L}{\partial \tilde{t}} < 0$

\(^{16}\)We assume that $\gamma < 1$, i.e., the share of import is not zero.
\[ \frac{\partial L_2}{\partial f} > 0. \]

### 3.4 Comparing Monetary Policy Rules

We consider a family of simple monetary policy rules. As discussed in Section 3.3, the first-order approximation for the equilibrium conditions is not enough for welfare analysis. We thus estimate the general nonlinear model specified in Section 3.2 with \( N = 2 \) and approximate the equilibrium by the second order expansion (of both the constraints as well as the welfare function). We relax the assumption of \( \psi = 0 \) and include a broader set of stochastic shocks, i.e., stage-specific productivity shocks and shocks on foreign consumption (which are the two types of shocks most commonly studied in the literature).

We consider the following set of policy rules: (a) a classic Taylor (1993) rule that is based on CPI inflation (and output gap);\(^\text{17}\) (b) a Galí-Monacelli (2005) rule that replaces CPI inflation with PPI inflation; (c) a rule that targets separate inflation terms for each production stage (i.e., stage-specific producer price indices); (d) combinations of the above with the real exchange rate; and (e) an exchange rate peg. For each rule, we examine both the case with imposed coefficients as specified in the literature (such as 1.5 and 0.5 on CPI inflation and output gap in the classic Taylor rule) and optimally estimated coefficients.

Since global supply chains have been gaining importance over the last two decades but face disruptions by recent tariff wars, we conduct comparative statics exercises on how the optimal weight on upstream inflation relative to the final stage inflation changes in response to changes in an economy’s openness. Specifically, we consider a range of openness parameter measured by the export share in sales. For each scenario, we estimate the optimal weights on the production-stage-specific inflation terms as well as on other variables. We then look at how the relative weights evolve as the degree of openness changes.

Asymmetric price stickiness along the production chain appears to be empirically relevant. Cornille and Dossche (2008) and Nakamura and Steinsson (2008) both suggest that the price con-

\(^{17}\)Henderson and McKibben (1993) have proposed a similar rule.
tracts in more upstream production stages tend to have a shorter duration than those in the finished product sectors. Gong, Wang, and Zou (2016) argue that different degrees of price stickiness in different stages would affect which price index (i.e., CPI, final-goods-based PPI, or intermediate-goods-based PPI) should be included in a simple monetary policy rule.\(^{18}\)

3.4.1 Model Parameters

We begin with the calibration of parameters for the baseline model. Each period in the model corresponds to a quarter. Following Galí and Monacelli (2005) and De Paoli (2009), the model economy is meant to resemble Canada in some key dimensions. The calibrated parameters are summarized in Table 3.1.

The subjective discount factor is set to be \(\beta = 0.99\), which implies a 4% annual real interest rate in the steady state. Following Arellano (2008) and De Paoli (2009), the inverse of intertemporal elasticity of substitution is set to be \(\sigma = 2\). The parameter in Calvo pricing in both production stages is set to be \(\alpha_1 = \alpha_2 = 0.66\), implying an average contract duration of 3 quarters. Following Benigno and Woodford (2005), the elasticity of substitution in the consumption bundle is set to be \(\theta = 10\). Consistent with Huang and Liu (2005), we set the share of intermediate goods in production to be \(\phi = 0.6\).

We set the shares of goods sold to the domestic markets in both stages to be \(\tilde{a}_1 = \tilde{a}_2 = 0.7\), implying a 30% export share of GDP (approximately the level observed for Canada since 2010). The parameters \(\tilde{a}_1, \tilde{a}_2\) are the sufficient statistics for the (exogenous) foreign demand for output in the two production stages. Following Galí and Monacelli (2005), the process of productivity shock is set to follow an AR(1) process with persistence parameter \(\rho_a = 0.66\) and standard deviation \(\sigma_a = 0.0071\), which is calibrated from Canada data. Following De Paoli (2009), the foreign consumption shock is set to an AR(1) process with persistence \(\rho_{C^*} = 0.66\) and standard deviation \(\sigma_{C^*} = 0.0129\). We normalize the import tariff in the baseline numerical exercise to be \(T = 1\)

\(^{18}\)Instead of including all stage-specific price indices in a simple monetary rule, Gong, Wang, and Zou (2016) consider a CPI-based Taylor rule, a final-goods PPI-based rule, and a intermediate-goods PPI-based rule. In other words, their rules always include one inflation index plus an output gap.
(implying a zero tariff).

### Table 3.1: Parameter calibration

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ Subjective discount factor</td>
<td>0.99</td>
<td>4% annual interest rate with a quarterly model</td>
</tr>
<tr>
<td>$\sigma$ Inverse of intertemporal elasticity of substitution</td>
<td>2</td>
<td>Standard value in literature, e.g., Arellano (2008)</td>
</tr>
<tr>
<td>$\alpha_1, \alpha_2$ Parameter in Calvo pricing</td>
<td>0.66</td>
<td>An average length of price contract of 3 quarters</td>
</tr>
<tr>
<td>$\gamma$ Share of goods purchased in domestic market</td>
<td>0.6</td>
<td>Implying 40% import share of GDP</td>
</tr>
<tr>
<td>$\theta$ Elasticity of substitution in consumption bundle</td>
<td>10</td>
<td>Following Benigno and Woodford (2005)</td>
</tr>
<tr>
<td>$\phi$ Share of intermediate goods in production</td>
<td>0.6</td>
<td>Following Huang and Liu (2005)</td>
</tr>
<tr>
<td>$\bar{a}_1, \bar{a}_2$ Share of goods selling to domestic market</td>
<td>0.7</td>
<td>Implying 30% export share of GDP</td>
</tr>
<tr>
<td>$\rho_a$ Persistency of productivity shock</td>
<td>0.66</td>
<td>Following Galí and Maonacelli (2005)</td>
</tr>
<tr>
<td>$\sigma_a$ Standard deviation of productivity shock</td>
<td>0.0071</td>
<td>Following Galí and Monacelli (2005)</td>
</tr>
<tr>
<td>$\rho c^*$ Persistency of foreign consumption shock</td>
<td>0.66</td>
<td>Following De Paoli (2009)</td>
</tr>
<tr>
<td>$\sigma c^*$ Standard deviation of foreign consumption shock</td>
<td>0.0129</td>
<td>Following De Paoli (2009)</td>
</tr>
</tbody>
</table>

### 3.4.2 Welfare Losses

The numerical estimation is conducted based on the general nonlinear model specified in Section 3.2 with $N = 2$. The equilibrium is estimated up to second order approximation (for both the constraints and the welfare loss function). We define the welfare loss $\chi$ in percentage term relative to the steady-state consumption, i.e.,

$$EX\Sigma_{t=0}^{\infty} \beta^t \left[ \frac{[C(1-\chi)]^{1-\sigma} - 1}{1 - \sigma} - \frac{L^{1+\psi} + 1}{1 + \psi} \right] = V^a$$

where $C$ and $L$ are steady-state consumption and employment, and $V^a$ is the welfare estimated from a given policy rule.

The aggregated PPI index is a sales-weighted average of producer prices index:

$$\pi_{PPI} = (1 - \omega)\pi_{1H} + \omega\pi_{2H}$$

where $\omega = \frac{P_{1h}(Y_{1h} + Y_{1h}^X)}{P_{1h}(Y_{1h} + Y_{1h}^X) + P_{2h}(Y_{2h} + Y_{2h}^X)}$ is the relative sales-weight in the upstream production stage.

We assume that neither the real marginal cost by production stage nor the relative price gap across production stages can be observed by the central bank. So they do not enter any monetary policy rule. Within the family of simple rules, the best that the central bank can do is to make
the interest rate a function of the upstream producer price inflation, the final stage producer price inflation, change in the real exchange rate, the output gap, and one-period lagged interest rate. Since the PPI inflation is a sales-weighted average of the first two terms, and the CPI inflation is a linear combination of the second and the third terms, there is no need to include these terms separately. We estimate the optimal coefficients on these variables, label this best possible rule as Policy Rule 1, and normalize its welfare loss to one. Table 3.2 reports the optimally estimated coefficients for ten different monetary policy rules. The welfare loss for each rule is expressed as relative to that under Policy Rule 1.\footnote{The welfare loss for P1 in Table 3.2 in term of steady-state consumption is 0.00319\%.} “Peg” in the table indicates a nominal exchange rate peg.

A classic Taylor rule that targets only CPI inflation and output gap (Policy Rule 2) does terribly in this economy. The welfare loss is 80% higher than Policy Rule 1. The Galí-Monacelli (2005) rule that replaces the CPI inflation with PPI inflation (Policy Rule 3) represents a significant improvement over the classic Taylor Rule in terms of a much smaller welfare loss. Still, the Galí-Monacelli rule is not as good as Policy Rule 1. That is because, with the input-output linkages across production stages, the optimal weights on the upstream sector and final stage inflation terms in Policy Rule 1 are not proportional to the relative sales of the two sectors. Including both PPI and CPI inflation (Policy Rule 4) yields a small improvement over Policy Rule 3 (but a larger improvement over Policy Rule 2). Adding the real exchange rate to Rule 4 (Policy 5) produces more noticeable improvement over Rules 2, 3, or 4. Still, Policy Rule 1 dominates Policy Rule 5.

Policy Rules 6-9 suggest that inflation measures in both the upstream stage and the final stage contain important information. Dropping either one of them from a monetary policy rule could lead to a significant increase in the welfare loss. A nominal exchange rate peg (Policy Rule 10) yields a welfare loss that is 173% higher than Policy Rule 1, which makes an exchange rate peg the worst option among the ten rules considered.

To summarize, the best simple rule would target separate producer price inflation in different stages of production and the real exchange rate (as well as the output gap). If we have to choose among aggregate price indicators, PPI targeting is superior to CPI targeting. In fact, at least with
our parameter values, including the PPI inflation moves one not too far from the best simple rule.

Table 3.2: Optimal alternative simple rules of monetary policy

<table>
<thead>
<tr>
<th></th>
<th>π_{1H}</th>
<th>π_{2H}</th>
<th>π_{PPi}</th>
<th>π_{CPI}</th>
<th>ŝ</th>
<th>̇q</th>
<th>̇l_{t-1}</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>6.3861</td>
<td>9.8675</td>
<td>0.7570</td>
<td>-1.5549</td>
<td>0.2048</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td>5.1882</td>
<td>0.0006</td>
<td>0.0215</td>
<td>1.809</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td>9.9999</td>
<td>0.1000</td>
<td>1.0441</td>
<td>1.031</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>9.9888</td>
<td>0.0009</td>
<td>0.0004</td>
<td>1.022</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P5</td>
<td>9.1138</td>
<td>0.0747</td>
<td>0.1697</td>
<td>-0.6933</td>
<td>0.1432</td>
<td>1.003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P6</td>
<td>5.2870</td>
<td>9.9965</td>
<td>0.0001</td>
<td>0.6948</td>
<td>1.009</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P7</td>
<td>4.2548</td>
<td>0.0000</td>
<td>0.8757</td>
<td>1.793</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P8</td>
<td>2.6975</td>
<td>0.0003</td>
<td>0.8327</td>
<td>1.257</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>P9</td>
<td>5.2741</td>
<td>9.9825</td>
<td>0.7004</td>
<td>1.009</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peg</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.730</td>
<td></td>
</tr>
</tbody>
</table>

Notes: PPI index (sales-weighted): \( \pi_{PPi} = (1 - \omega)\pi_{1H} + \omega\pi_{2H} \) with \( \omega = \frac{P_{hh}(\pi_{1h} + \pi_{2h})}{P_{hh}(\pi_{1h} + \pi_{2h} + \pi_{hh} + \pi_{2h})} \). 
CPI index: \( \pi_{CPI,t} = \pi_t \).

For each type of policy rule, besides optimally estimated coefficients, we also evaluate a version where the coefficients are imposed using the values suggested in the literature. Table 3.3 reports the welfare performance proposed by the original Taylor calibration (i.e., Taylor, 1993), alternative rules adopted in Galí and Monacelli (2005), and in Huang and Liu (2005). The welfare losses in the table are still reported as relative to that under Policy Rule 1 in Table 3.2. Evidently, simple monetary policy rules that target aggregate PPI, or stage-specific producer indices, outperform those targeting just the CPI index.

3.4.3 Comparative Statics: Effects of Openness and Intermediate Goods Share

A country’s openness and share of intermediate goods in production are the two most important features of global supply chains. In order to study the role of these two factors in optimal simple

Table 3.3: Alternative simple rules of monetary policy in literature

<table>
<thead>
<tr>
<th></th>
<th>π_{1H}</th>
<th>π_{2H}</th>
<th>π_{CPI}</th>
<th>ŝ</th>
<th>̇q</th>
<th>̇l_{t-1}</th>
<th>Welfare loss</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>1.5</td>
<td>0.5</td>
<td></td>
<td>5.862</td>
<td>1.166</td>
<td>2.661</td>
<td>3.843</td>
<td>Taylor (1993)</td>
</tr>
<tr>
<td>P2</td>
<td>1.42</td>
<td>1.68</td>
<td>0.04</td>
<td>1.12</td>
<td></td>
<td></td>
<td></td>
<td>Huang and Liu (2005)</td>
</tr>
<tr>
<td>P3</td>
<td>1.5</td>
<td></td>
<td></td>
<td>2.661</td>
<td></td>
<td></td>
<td></td>
<td>Galí and Monacelli (2005) – CPI based</td>
</tr>
<tr>
<td>P4</td>
<td>1.5</td>
<td></td>
<td></td>
<td>3.843</td>
<td></td>
<td></td>
<td></td>
<td>Galí and Monacelli (2005) – PPI based</td>
</tr>
</tbody>
</table>
rules, we conduct comparative statics on how the relative optimal weight on upstream sector inflation changes with respect to these two parameters.

We calibrate foreign demand in both production stages such that the shares of exports in the steady state are the same in both stages \((1 - \bar{a}_1 = 1 - \bar{a}_2 \equiv 1 - \bar{a})\), and both vary from 10% to 90%.

Figure 3.2 plots the estimated optimal weight on the upstream sector inflation relative to the sum of the coefficients for the two inflation rates in the two stages, as a function of the openness (measured by the export share, assumed to be common in both stages of production). As shown by the solid line, the optimal relative weight on the upstream sector inflation generally rises as an economy becomes more open (although the increase is not strictly monotonic). This is especially true when the economy evolves from median open to very open (e.g., an increase in openness from 0.6 to 0.9).

The intuition for this result has to do with the vertical production structure. From Section 3.3.4, the labor shares in the upstream and downstream stages in the steady state are given, respectively, by \(L_1/L = \frac{\gamma \phi}{\gamma \phi + (1-\phi)\bar{a}}\) and \(L_2/L = \frac{(1-\phi)\bar{a}}{\gamma \phi + (1-\phi)\bar{a}}\). With greater openness (i.e., a smaller \(\bar{a}\)), a higher share of total employment takes place in the upstream stage. It is therefore sensible to increase the weight on the upstream inflation relative to the downstream inflation in the monetary policy rule. One can infer that, under the classic Taylor rule (which puts zero weight on the upstream sector inflation), the welfare loss would have grown as the economy becomes more open.

In the same graph, we also plot the relative sales weight of the two sectors (the thin dotted line) and the relative value-added weight (the dashed dotted line), respectively. It is clear that the optimal weights on the stage-specific producer inflation are not proportional to either sales or value added of the sectors. This means that targeting the aggregate PPI inflation cannot achieve the same level of welfare as targeting production stage-specific producer price inflation.

We now discuss some sensitivity exercises. First, not all stages of production are equally open to international trade. To investigate the importance of this heterogeneity, we infer the degree of openness by production stage for Canada using information in the World Input-Output Database (WIOD). We then re-estimate the optimal weights on the targeting variables in each monetary
policy rule and the associated welfare loss. The results are reported in Appendix C.5. We find that the qualitative results are similar to our baseline case. This means that the observed heterogeneity in openness across production stages does not alter the basic results (at least for Canada).

Second, we explore different degrees of price stickiness across production stages. In particular, we consider two extreme cases: (i) sticky prices only in the downstream sector (but flexible prices in the upstream sector), or \( \alpha_1 = 0 \) and \( \alpha_2 = 0.66 \); and (ii) sticky prices only in the upstream sector, or \( \alpha_1 = 0.66 \) and \( \alpha_2 = 0 \). Details about this exercise can be found in Appendix C.10. Under the standard Taylor rule, greater openness leads to a smaller welfare loss in case (i), but a greater welfare loss in case (ii). The intuition is similar to that of Figure 3.2: greater openness means a greater share of the total employment in the upstream sector. Thus, a given price distortion in the upstream stage is more damaging than in the downstream stage.

Third, we study how the elasticity of substitution at each stage of production matters for the welfare, and how it relates to the degree of openness. Intuitively, a greater elasticity of substitution tends to magnify the misallocation from price stickiness. If the elasticity of substitution differs in the two stages of production, a greater substitution in the upstream stage magnifies the overall
welfare loss, and the effect becomes stronger as the economy becomes more open. Details can be found in Appendix C.10.

Fourth, we vary the share of intermediate goods, \( \phi \), and compute the optimal weights under Policy Rule 1. Figure 3.3 traces out the estimated optimal relative weight on the upstream sector inflation as a function of the share of intermediate goods in production. The estimations show that, the optimal relative weight on the upstream sector inflation goes up as intermediate goods rise in importance.

Since CPI inflation in theory is a weighted average of the final-stage producer price inflation and the real exchange rate appreciation, we compute the implied weight on the CPI inflation in the optimal rule that targets stage-specific producer price inflation rates.\(^{20}\) In Figure 3.4, we trace out the ratio of the optimal coefficient on the upstream inflation index and the sum of the coefficient on the upstream inflation and the implied coefficient on CPI as a function of the share of intermediate goods. A clear upward trend suggests that, in the optimal simple rule, the weight on CPI should decline as the intermediate goods rise in relative importance. In other words, targeting CPI alone becomes increasingly sub-optimal as supply chains rise in importance.

3.4.4 Effects of a Higher Import Tariff

Trade frictions can be thought of as a reduction in an economy’s openness. Let us consider a case of doubling the import tariff (a change to \( T = 2 \)). In such a scenario, the shares of the demand for domestic goods in the two production stages become \( \tilde{a}_1 = 0.73 \) and \( \tilde{a}_2 = 0.74 \), which are larger than those in the original calibration. The direction of the change is exactly as predicted by Proposition 1.

While the central bank cannot undo the increase in tariff directly, it can re-optimize by choosing a different set of coefficients on the variables in the monetary policy rule. We compute the new optimal weights and new welfare losses for the best simple rule, the classic Taylor rule, the Galí-Monacelli rule, a rule that includes both PPI and CPI inflation as well as the output gap and real

\(^{20}\)As shown in the expression for CPI in Section 3.3.3, the weight on final stage estimated coefficient is set to be \( \gamma \), while the weight on the estimated coefficient of exchange rate is set to be \( 1 - \gamma \).
Figure 3.3: Relative weight of upstream inflation index in optimal simple rule with respect to intermediate goods share

Figure 3.4: Ratio of the weight on upstream inflation index versus CPI with respect to intermediate goods share
exchange rate but not stage-specific producer price inflation, and finally an exchange rate peg.

The results are reported in Table 3.4. Note that the welfare losses are relative to the case of Policy Rule 1 before the tariff increase (i.e., relative to the welfare for Policy Rule 1 in Table 3.2). Comparing across different policy rules, it is still the case that Policy Rule 1 that includes stage-specific producer price inflation is the best monetary policy rule. Afterwards, including PPI inflation and the real exchange rate would beat the classic Taylor rule. The exchange rate peg and the classic Taylor yield the biggest and the second biggest welfare losses, respectively.

Recall from Proposition 1 that, a higher import tariff would reduce the optimal weight on the upstream sector inflation in the monetary policy rule relative to that on the final stage inflation. This can be confirmed in our numerical exercises. The ratio of the optimal relative weight on the upstream producer price inflation under Policy Rule 1 has changed from 0.647 in the case of $T = 1$ in Table 3.2 to 0.564 in the case of $T = 2$ in Table 3.4.\textsuperscript{21}

It is important to note that a higher tariff reduces welfare directly, as we can see from the greater welfare losses in Table 3.4 relative to their counterparts in Table 3.2, in spite of the best adjustments made by the central bank. If the central bank does not re-optimize, the welfare loss would have been even greater.

<table>
<thead>
<tr>
<th>Policy Rule</th>
<th>$\pi_{1H}$</th>
<th>$\pi_{2H}$</th>
<th>$\pi_{PPI}$</th>
<th>$\pi_{CPI}$</th>
<th>$\hat{c}$</th>
<th>$\hat{q}$</th>
<th>$\hat{f}_{1-1}$</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>5.0113</td>
<td>8.8786</td>
<td>0.2298</td>
<td>-0.7860</td>
<td>0.1629</td>
<td>1.182</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td>5.2378</td>
<td>0.0000</td>
<td>0.0007</td>
<td>2.144</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td>9.9988</td>
<td>0.1000</td>
<td>1.0509</td>
<td>1.223</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>9.3569</td>
<td>0.0487</td>
<td>0.0766</td>
<td>-0.5579</td>
<td>0.1445</td>
<td>1.188</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peg</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3.310</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PPI index (sales-weighted): $\pi_{PPI} = (1 - \omega)\pi_{1H} + \omega\pi_{2H}$ with $\omega = \frac{P_{1H}(Y_{1H} + Y_{1H}^{R})}{P_{1H}(Y_{1H} + Y_{1H}^{R}) + P_{2H}(Y_{2H} + Y_{2H}^{R})}$.

CPI index: $\pi_{CPI,t} = \pi_t$.

Since Policy Rule 1 already includes the real exchange rate, it implies that the central bank cannot offset the effects of a higher import tariff by simply changing the exchange rate. An appreciation in the domestic currency reduces the cost of imported intermediate inputs or imported

\textsuperscript{21}Note that 0.647 comes from 6.3861/9.8675 in Table 3.2 while 0.564 comes from 5.0113/8.8786 in Table 3.4.
final consumption goods, but also increases the prices of both domestically produced intermediate
goods and final goods. Since the foreign demand is price-elastic, firms will experience a reduction
in revenue from exporting.

3.4.5 Asymmetric Price Stickiness

We now consider uneven price stickiness in different stages of production. Cornille and Doss-
che (2008) and Nakamura and Steinsson (2008) argue that the duration of price contracts in the
upstream production stages is shorter than the downstream stages. For instance, Nakamura and
Steinsson (2008) document that the median price contract for finished producer goods in 1998-
2005 lasts for 8.7 months, while the median duration of price contracts for intermediate goods is
about 7.0 months.

To investigate the implications of such difference, we reduce the Calvo pricing parameter in
the first stage of production to be $\alpha_1 = 0.5$, indicating an average length of price contracts of 2
quarters.

Table 3.5: Optimal alternative simple rules of monetary policy with lower price stickiness in up-
stream production

<table>
<thead>
<tr>
<th>$\pi_{1H}$</th>
<th>$\pi_{2H}$</th>
<th>$\pi_{PPI}$</th>
<th>$\pi_{CPI}$</th>
<th>$\hat{c}$</th>
<th>$\hat{q}$</th>
<th>$\hat{t}_{t-1}$</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>3.1846</td>
<td>9.8760</td>
<td>0.0100</td>
<td>-0.5776</td>
<td>0.0328</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td>5.0515</td>
<td>0.0001</td>
<td>0.0038</td>
<td>1.889</td>
<td></td>
<td>1.889</td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td>9.9981</td>
<td>0.1001</td>
<td>1.0715</td>
<td>1.083</td>
<td></td>
<td>1.083</td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>9.8058</td>
<td>0.0240</td>
<td>-0.6126</td>
<td>0.0174</td>
<td>1.038</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peg</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3.101</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PPI index (sales-weighted): $\pi_{PPI} = (1 - \omega)\pi_{1H} + \omega\pi_{2H}$ with $\omega = \frac{\pi_{1h}(Y_{1h} + Y_{2h})}{\pi_{1h}(Y_{1h} + Y_{2h}) + \pi_{2h}(Y_{2h} + Y_{2h})}$.

CPI index: $\pi_{CPI,t} = \pi_t$.

The estimated results are shown in Table 3.5, where the welfare loss of Policy Rule 1 in the
table has been normalized to be one. The loss becomes smaller as compared to the baseline case in
Table 3.2 since the prices are less sticky overall. Furthermore, the optimal relative weight on the
upstream producer price inflation also becomes smaller.\textsuperscript{22} Intuitively, it is beneficial to put more

\textsuperscript{22}This is consistent with the findings in Gong, Wang, and Zou (2016). They argue that, when the degree of price
stickiness for intermediate-goods production is high, the central bank should follow intermediate-goods PPI-based
weight on those prices that are reset less frequently, i.e., downstream prices in this case, because resource misallocation is otherwise more severe.

3.4.6 Additional Loss from Sticky Monetary Policy Rules

If a central bank adopted a policy rule that was optimal when the economy had a lower degree of participation in the global value chains, but did not update the rule as the participation has increased, what would the additional welfare cost be?

To investigate this, we continue with Canada as the baseline economy, with the same exogenous shocks as specified in Table 3.1. We choose $\gamma = 0.67$ (to match the 33% import share in GDP in the data) and $\tilde{a}_1 = \tilde{a}_2 = 0.69$ (to match the 31% export share in the data) in 2017. Similarly, we choose $\gamma = 0.75$ and $\tilde{a}_1 = \tilde{a}_2 = 0.74$ in 1987. The data suggests that the country has become more involved in the global trade from 1987 to 2017 as both the import and export shares have grown.

By computing an optimal CPI-based Taylor rule for the period around 30 years ago, we can then estimate the welfare loss if the central bank had used the old CPI-based Taylor rule in today’s world. Table 3.6 shows the estimated welfare loss in 2017 if the central bank had continued to use an old optimal rule estimated in 1987. The welfare loss for the best new rule is normalized to be one.

The “Old Rule 1” and “Old Rule 2” refer to the optimal CPI-based Taylor rule and the optimal stage-specific PPI-based Taylor rule in 1987, respectively. From the table, we can see that the old Taylor rule (estimated optimally for 1987) generates a welfare loss in 2017. Furthermore, by comparing the two cases under “Old Rule”, if the country used to implement an optimal stage-specific PPI-based policy rule and applies to today, the welfare loss is smaller compared with the case of adopting an old CPI-based policy rule.

It is worth noting that, in estimating the old optimal CPI-based Taylor rule for 1987, we already assume two production stages. If, instead, there was a single production stage in 1987, and the world has evolved to be two production stages in 2017, then the welfare loss associated with using rule. However, in their paper, there is no labor allocation distortion between production stages since labor is assumed to be used only in the production of intermediate goods.
the old 1987 monetary policy in 2017 would be substantially bigger.

<table>
<thead>
<tr>
<th></th>
<th>$\pi_{1H}$</th>
<th>$\pi_{2H}$</th>
<th>$\pi_{CPI}$</th>
<th>$\hat{c}$</th>
<th>$\hat{q}$</th>
<th>$i_{t-1}$</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>New Rule</td>
<td>7.4579</td>
<td>9.5552</td>
<td>0.0187</td>
<td>-0.4054</td>
<td>0.1636</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Taylor Rule</td>
<td>1.5</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5.8715</td>
</tr>
<tr>
<td>Old Rule 1</td>
<td>5.6626</td>
<td>0.0001</td>
<td>0.0043</td>
<td></td>
<td></td>
<td></td>
<td>1.5217</td>
</tr>
<tr>
<td>Old Rule 2</td>
<td>7.0811</td>
<td>9.8512</td>
<td>1.3082</td>
<td>-1.8679</td>
<td>0.1636</td>
<td></td>
<td>1.1912</td>
</tr>
</tbody>
</table>

Notes: Old policy rules are estimated from Canada data in 1987, while the new policy rule is estimated from 2017. CPI index: $\pi_{CPI,t} = \pi_t$.

3.5 Conclusion

Supply chains are everywhere and are often global. This paper studies the implications of global supply chains on the design of optimal monetary policy using a small-open economy New Keynesian model with multiple stages of production. The optimal simple policy rule that produces the least welfare loss includes targeting separate producer price inflation in each production stage (in addition to output gap and real exchange rate).

Importantly, the optimal weights on the upstream sector inflation versus the final stage inflation are not proportional to the sectors’ sales or value added. As an economy becomes more open, measured by the share of exports in sales, the optimal relative weight on the upstream sector inflation will also rise. Separately, as intermediate goods become more important in the production, the optimal relative weight on the upstream sector inflation also rises. In both cases, the classic Taylor rule that targets only CPI inflation would become progressively more inferior (in the sense of an ever greater welfare loss relative to the optimal rule). As the production chain becomes longer, the optimal weights in the policy rule on the upstream sector inflation or the PPI inflation also increase.

Trade frictions can be thought of as a shock to an economy’s openness. With a higher tariff, the optimal weights on various terms in the monetary policy rule would have to change. Importantly, a higher tariff reduces the welfare directly even if the central bank re-optimizes. In particular, the negative effect of a higher tariff cannot be offset completely by a change in the real exchange rate.
If we only consider aggregate price indices in the simple monetary policy rule, then targeting aggregate PPI inflation (as well as the output gap) is superior to just targeting CPI inflation in terms of a smaller welfare loss. Adding the real exchange rate is even better. Still, no simple rule produces a smaller welfare loss than the one that includes separate producer price inflation in each production stage on top of the output gap, the real exchange rate, and the lagged interest rate.

Is it feasible in practice to obtain separate producer price inflation for different production stages? Yes, as official statistical agencies in the United States, Japan, and Australia already collect such data. For example, the US Bureau of Labor Statistics has a system of producer price indices featuring a four-stage vertical production chain (called the PPI Final Demand-Intermediate Demand indices).

Ironically, central banks use information on PPI inflation only to the extent that it helps to forecast CPI inflation. When the PPI and CPI diverge, as they often do in recent periods, central banks would ignore PPI. However, our theory suggests that a monetary policy rule that produces an even smaller welfare loss includes producer price inflation directly, and doing so becomes more important precisely when the PPI and CPI inflation rates diverge.

The research in this paper can be extended in a number of directions. First, the model adopted in our analysis assumes producer currency pricing (PCP). It may be worth exploring how results may be modified as local currency pricing or dominant dollar pricing is assumed instead. Second, one may explore a broader set of exogenous shocks than in the current paper, including shocks on foreign input prices or foreign demand along the production chain.
Bibliography


Appendix A: Fiscal and Monetary Policy Interaction under Limited Foresight

A.1 The System of Equations for Heterogeneous Agents with No Update in the Value Function

Given the assumptions imposed in Section 1.2.5, suppose in each period $\omega_j = \tilde{\omega}_j = (1 - \rho) \rho^h$ fraction of households and firms have planning horizon-$h$ with $0 < \rho < 1$. For any $h \geq 1$ in period $t$, the equations of the IS and Phillips curve yield

\[ y_t^h - g_t = E_t[y_{t+1}^{h-1} - g_{t+1}] - \sigma[i_t^h - E_t\pi_{t+1}^{h-1}] \]  
(A.1)

\[ \pi_t^h = \kappa[y_t^h - y_t^*] + \beta E_t\pi_{t+1}^{h-1} \]  
(A.2)

and for $h = 0$,

\[ y_t^0 - g_t = -\sigma i_t^0 + (1 - \beta) b_{t+1}^0 \]  
(A.3)

\[ \pi_t^0 = \kappa[y_t^0 - y_t^*] \]  
(A.4)

where the variable $b_{t+1}^0$ is the real asset position at the end of the planning horizon for households with horizon $h = 0$. The expression of the variable $b_{t+1}^0$ will be derived later.

By averaging these equations (A.1)-(A.2) across agents, it follows that

\[ y_t - g_t = \rho E_t(y_{t+1} - g_{t+1}) - \sigma(i_t - \rho E_t\pi_{t+1}) + (1 - \rho)(1 - \beta)b_t' \]
\[ \pi_t = \kappa(y_t - y^*_t) + \beta \rho \bar{E}_t \pi_{t+1} \]

with the rule of nominal interest rate and the evolution of real public debt

\[ \hat{i}_t = i^*_t + \phi_{\pi,t} \pi_t \]

\[ b_{t+1} = \beta^{-1}(1 - \Gamma) b_t - \beta^{-1}(1 - \Gamma) s_b \pi_t + (1 - \Gamma) s_b \hat{i}_t \]

where \( b_t' = b_{t+1}^0 \).

Now, I derive the expression of the variable \( b_{t+1}^0 \) as follows. Given the aggregate (average) real public debt \( b_t \) in period \( t \) and the assumptions imposed in Section 1.2.5, the decision makers with horizon \( h = 0 \) start their planning exercise with initial asset \( b_t \) in each period \( t \). Their forward planning problem is then characterized by equations (A.3)-(A.4) with the rule of nominal interest rate and the evolution of real public debt

\[ \hat{i}_t^0 = i^*_t + \phi_{\pi,t} \pi_t^0 \]

\[ b_{t+1}^0 = \beta^{-1}(1 - \Gamma) b_t - \beta^{-1}(1 - \Gamma) s_b \pi_t^0 + (1 - \Gamma) s_b \hat{i}_t^0 \]

Thus, from the four linear system of equations, the variable \( b_{t+1}^0 \) is a function of \( b_t \) and exogenous disturbances, i.e.,

\[ b_t' \equiv b_{t+1}^0 = \psi_{b,t} b_t + \psi_{g,t} g_t + \psi_{y,t} y_t^* + \psi_{i,t} i_t^* \]

where the parameter \( \psi_{b,t} \) is given by

\[ \psi_{b,t} = \beta^{-1}(1 - \Gamma) / [1 - (\phi_{\pi,t} - \beta^{-1}) s_b (1 - \Gamma) (1 - \beta) \kappa (1 + \sigma \phi_{\pi,t} \kappa) ] \]

The expression of parameters \( \{ \psi_{g,t}, \psi_{y,t}, \psi_{i,t} \} \) can be easily solved, but only the parameter \( \psi_{b,t} \) is of special interest.
The system of equations characterizing the aggregate equilibrium can be re-written as

\[
E_t \begin{bmatrix} x_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} A & C \\ D & \beta^{-1}(1 - \Gamma) \end{bmatrix} \begin{bmatrix} x_t \\ b_t \end{bmatrix} + \begin{bmatrix} K \\ 0 \end{bmatrix} \begin{bmatrix} g_t \\ y_t^* \end{bmatrix} + \begin{bmatrix} 0 \\ i_t^* \end{bmatrix}
\]

where \( x_t = \begin{bmatrix} y_t - g_t & \pi_t \end{bmatrix}^T \), \( K \) is a 2 \( \times \) 3 matrix of less interest, and the matrices \( A, C, \) and \( D \) are given by

\[
A = \begin{bmatrix} \rho^{-1} & -\sigma(\beta \rho)^{-1} & 1 & \sigma \varphi_{\pi,t} \\ 0 & (\beta \rho)^{-1} & -\kappa & 1 \end{bmatrix} = \begin{bmatrix} \rho^{-1} + \kappa \sigma(\beta \rho)^{-1} & \sigma \rho^{-1}(\phi_{\pi} - \beta^{-1}) \\ -\kappa(\beta \rho)^{-1} & (\beta \rho)^{-1} \end{bmatrix}
\]

\[
C = \begin{bmatrix} \rho^{-1} & -\sigma(\beta \rho)^{-1} \\ 0 & (\beta \rho)^{-1} \end{bmatrix} \begin{bmatrix} -(1 - \rho)(1 - \beta)\psi_b \\ 0 \end{bmatrix}
\]

\[
D = \begin{bmatrix} 0 & (\phi_{\pi,t} - \beta^{-1})(1 - \Gamma)\psi_b \end{bmatrix}
\]

### A.2 Aggregation across the Population for Characterizing Aggregate “Trend” Variables

From the system of equations (1.27a)-(1.27d) in Section 1.3.3, by aggregating across the whole population for the “trend” variables, it follows

\[
\tilde{y}_t = \rho \tilde{y}_t - \sigma[\tilde{\eta}_t - \rho \tilde{\pi}_t] + (1 - \rho)v_t + (1 - \rho)(1 - \beta)\tilde{d}_{t+1}^0 \tag{A.5a}
\]

\[
\tilde{\pi}_t = \kappa \tilde{y}_t + \beta \rho \tilde{\pi}_t + (1 - \rho)(1 - \alpha)\beta \tilde{\nu}_t \tag{A.5b}
\]

\[
\tilde{\eta}_t = \phi_{\pi} \tilde{\pi}_t \tag{A.5c}
\]
where the variable \( \tilde{b}_{t+1}^0 \) is the ending period asset position in the planning exercise of households with horizon \( h = 0 \). To derive \( \tilde{b}_{t+1}^0 \), for decision makers with \( h = 0 \), the planning exercise for “trend” components is given by

\[
\tilde{y}_t^0 = -\sigma \tilde{x}_t^0 + (1 - \beta)\tilde{b}_{t+1}^0 + \nu_t
\]

\[
\tilde{\pi}_t^0 = \kappa \tilde{y}_t^0 + (1 - \alpha)\beta \tilde{\nu}_t
\]

\[
i_t^0 = \phi \tilde{\pi}_t^0
\]

\[
\tilde{b}_{t+1}^0 = \beta^{-1} (1 - \Gamma) \tilde{b}_t^1 - \beta^{-1} (1 - \Gamma) s_b \tilde{\pi}_t^0 + (1 - \Gamma) s_b \tilde{\nu}_t^0
\]

where \( \tilde{b}_t^1 = 0 \). The above four linear equations yields \( \tilde{b}_{t+1}^0 = \psi_v \nu_t + \psi \tilde{\nu}_t \) with \( \psi_v \) and \( \psi \tilde{\nu}_t \) given by

\[
\psi_v = (1 - \Gamma) s_b [\phi \pi - \beta^{-1}] \kappa / [1 + \sigma \phi \pi \kappa - (\phi \pi - \beta^{-1})(1 - \Gamma) s_b (1 - \beta) \kappa]
\]

\[
\psi \tilde{\nu}_t = (1 - \Gamma) s_b [\phi \pi - \beta^{-1}] (1 - \alpha) \beta / [1 + \sigma \phi \pi \kappa - (\phi \pi - \beta^{-1})(1 - \Gamma) s_b (1 - \beta) \kappa]
\]

A.3 The System of Equations for the Equilibrium under Heterogeneous Agents with Learning in the Value Function

From Section 1.3.3, the system of equations for the whole equilibrium, i.e., equations (1.33), (1.34), and (1.35), can be summarized as

\[
E_t \left[ \begin{array}{c} x_{t+1} \\ \tilde{x}_{t+1} \\ b_{t+1} \end{array} \right] = \left[ \begin{array}{ccc} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{array} \right]^{-1} \left[ \begin{array}{ccc} A & -A & C \\ F & G & H \\ D & 0 & \beta^{-1} (1 - \Gamma) \end{array} \right] \left[ \begin{array}{c} x_t \\ \tilde{x}_t \\ b_t \end{array} \right] + \left[ \begin{array}{ccc} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} Ku_t \\ 0 \\ 0 \end{array} \right]
\]

\[
= \left[ \begin{array}{c} x_t \\ \tilde{x}_t \\ b_t \end{array} \right] + \left[ \begin{array}{c} Ku_t \\ 0 \\ 0 \end{array} \right]
\]
where $Y$ is given by

$$
Y = \begin{bmatrix}
A + F & -A + G & C + H \\
F & G & H \\
D & 0 & \beta^{-1}(1 - \Gamma)
\end{bmatrix}
$$

### A.4 Proof for the Determinacy Condition and Convergence Condition with Taylor Rule and Inactive Fiscal Policy

Note that the expression of $A$ is given in Section 1.3.3 as

$$
A = \begin{bmatrix} 
\rho^{-1} & -\sigma(\beta\rho)^{-1} & 1 & \sigma\phi \\
0 & (\beta\rho)^{-1} & -\kappa & 1
\end{bmatrix}
$$

The eigenvalues of $A$ satisfy the following second-order polynomial

$$
f(\lambda) \equiv (\rho\lambda)^2 - [\beta^{-1} + 1 + \kappa\sigma\beta^{-1}] (\rho\lambda) + (\beta^{-1} + \kappa\sigma\phi\beta^{-1}) = 0
$$

Then, $f(\lambda)$ has two eigenvalues outside unit circle if and only if $f(1) > 0$, which is equivalent to

$$
\phi > -\frac{1}{\kappa\sigma}[\beta\rho^2 - \rho(1 + \beta + \kappa\sigma) + 1] \equiv l(\rho). \text{ Q.E.D.}
$$

It can also be proved that the necessary and sufficient condition for the convergence of $\Sigma\omega_h y^h$ and $\Sigma\tilde{\omega}_h \pi^h$ in a given period is that two eigenvalues of $A$ are outside the unit circle.

**Proof:** For simplicity, assume no exogenous disturbances. Note that, in a given period $t$, for $\forall h \geq 1$, it follows from Section 1.3.2 that

$$
x^h = (\rho A)^{-1} x^{h-1}
$$

where $x^h = \begin{bmatrix} y^h & \pi^h \end{bmatrix}^T$.

In order for $\Sigma\omega_h y^h$ and $\Sigma\tilde{\omega}_h \pi^h$ to converge, it is equivalent to the condition that the growth rate in $x^h$ is smaller than $\rho^{-1}$ for large enough $h$. It is then equivalent to the condition that there are two
eigenvalues of $A$ outside the unit circle. Q.E.D.

A.5 Convergence Condition of $\Sigma \omega_h y^h$ and $\Sigma \omega_h \pi^h$ with an Endogenous Fiscal Rule

Following the assumption in Section 1.3.3, in period $t$, the group of households with planning horizon-$h$ start with the initial asset position $b_t$ in their planning exercise. By the definition of “trend” variables in Section 1.3.3, the aggregation across agents for the “trend” variables always converges. Then, it remains to only focus on the “deviation” components. Since the “deviation” variables are those in the case of no update in the value function throughout the planning exercise, the convergence condition of the aggregate endogenous variables is then equivalent to that in Section 1.2.5 in which there is no update in the value function.

Therefore, I focus on the convergence condition in Section 1.2.5. For simplicity, assume no real disturbances, and the planning problem for large enough $h$ in period $t$ yields (from Section 1.2.4)

$$\begin{bmatrix} x^h \\ b^{h+1} \end{bmatrix} = \begin{bmatrix} \rho A & 0 \\ D & \beta^{-1}(1-\Gamma) \end{bmatrix}^{-1} \begin{bmatrix} x^{h-1} \\ b^h \end{bmatrix}$$

(A.7)

$$= Y^* \begin{bmatrix} \rho^{-1}x^{h-1} \\ b^h \end{bmatrix}$$

where $x^h = [y^h \pi^h]^T$ and $Y^* = \begin{bmatrix} A^{-1} & 0 \\ \beta^{-1}(1-\Gamma)^{-1}DA^{-1} & [\beta^{-1}(1-\Gamma)]^{-1} \end{bmatrix}$.

In order for $\Sigma \omega_h y^h$ and $\Sigma \omega_h \pi^h$ to converge, it is equivalent to the condition that the growth rate in $x^h$ is smaller than $\rho^{-1}$ for large enough $h$. It is then equivalent to the condition that either (i) two eigenvalues of $A$ are outside the unit circle, i.e., $\phi_\pi > l(\rho)$, with $|\beta^{-1}(1-\Gamma)| < 1$, or (ii) only one eigenvalue of $A$ is outside the unit circle, i.e., $\phi_\pi < l(\rho)$, with $|\beta^{-1}(1-\Gamma)| > 1$.¹ These

¹A more rigorous method to study the convergence condition across agents is to assume $y^h = y^h h^{b+1}$ and $\pi^h = \pi^h h^{b+1}$ for any $h \geq 0$. It can be easily showed that $\gamma^0_\pi = \frac{1-\beta)[\beta^{-1}(1-\Gamma)-1]}{1+\delta(1-\gamma)(1-\beta)}$ and $\gamma^0_y = \kappa y^0_\pi$. Then, the forward-planning problem for any agent with horizon $h \geq 1$ implies that $\begin{bmatrix} y^h \\ \pi^h \end{bmatrix} = [\beta^{-1}(1-\Gamma) + (\phi_\pi - 1)]^{b+1} \begin{bmatrix} y^0 \\ \pi^0 \end{bmatrix} \mathbf{1}_{b+1}$.
two conditions are the shaded areas within the dotted lines in Figure 1.1. It can be numerically verified that, in other cases as in the blank area of Figure 1.1, the aggregation across agents does not converge.

A.6 Proof for the Monotonicity of the Responses of Output and Inflation with respect to $\phi_\pi$

Given the expressions of (1.43)-(1.44), since $\phi_\pi > l(\rho)$ and $\rho \in (0, 1)$, it is obvious that the responses of output $y$ and inflation $\pi$ are strictly decreasing with respect to $\phi_\pi$. Thus, Proposition 1 holds.

Now, let’s consider the second-order derivatives, and focus on the response of output. It follows that

$$\frac{\partial y}{\partial \phi_\pi} = -\frac{(1 - \beta \rho)(1 - \rho)(1 - \beta)\lambda t^*}{[(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)]^2} < 0$$

$$-\frac{\partial^2 y}{\partial \phi_\pi \partial \rho} = \frac{\lambda t^* \kappa \sigma (1 - \beta)}{[(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)]^2} \{-(1 + \beta - 2\beta \rho) + 2\frac{(1 - \beta \rho)(1 - \rho)(1 + \beta + \kappa \sigma - 2\beta \rho)}{(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)}\}$$

Denote $f(\rho) = 2(1 - \beta \rho)(1 - \rho)(1 + \beta + \kappa \sigma - 2\beta \rho)$ and $g(\rho) = [(1 - \beta \rho)(1 - \rho) + \kappa \sigma (\phi_\pi - \rho)](1 + \beta - 2\beta \rho)$. Since $\phi_\pi > l(\rho)$, $\beta \in (0, 1)$, and $\rho \in (0, 1)$, both $f(\rho)$ and $g(\rho)$ are strictly decreasing with respect to $\rho$. Note that $f(1) < g(1)$. Then, due to the monotonicity and continuity of $f(\cdot)$ and $g(\cdot)$, there exists a unique $\tilde{\rho} \in (0, 1)$ such that $f(\rho) = g(\rho)$ if and only if $f(0) > g(0)$, which is equivalent to $\phi_\pi < \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta}$. Otherwise, if $\phi_\pi \geq \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta}$, it follows that $f(\rho) < g(\rho)$ for any $\rho \in (0, 1)$.

Note that $-\frac{\partial^2 y}{\partial \phi_\pi \partial \rho} > 0$ is equivalent to $f(\rho) > g(\rho)$, and vice versa. Therefore, given $\phi_\pi < \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta}$, a unique $\tilde{\rho} \in (0, 1)$ exists such that $-\frac{\partial^2 y}{\partial \phi_\pi \partial \rho} > 0$ if $\rho \in (0, \tilde{\rho})$, and $-\frac{\partial^2 y}{\partial \phi_\pi \partial \rho} < 0$ if $\rho \in (\tilde{\rho}, 1)$. Otherwise, if $\phi_\pi \geq \frac{1}{\kappa \sigma} + \frac{2}{1 + \beta}$, it follows that $-\frac{\partial^2 y}{\partial \phi_\pi \partial \rho} < 0$ for any $\rho \in (0, 1)$. Q.E.D.
A.7 Proof for the Monotonicity of the Responses of Output and Inflation with respect to $\Gamma$

Note that $\beta^{-1}(1 - \Gamma) \leq 1$ and $0 \leq \Gamma \leq 1$. The expressions of $\gamma_y$ and $\gamma_\pi$ as specified in Section 1.4.1 are given by

$$
\gamma_y = \frac{[1 - \rho(1 - \Gamma)](1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{[1 - \rho\beta^{-1}(1 - \Gamma)][1 - \rho(1 - \Gamma)] + \kappa\sigma[\phi_\pi - \rho\beta^{-1}(1 - \Gamma)]} \geq 0
$$

$$
\gamma_\pi = \frac{\kappa(1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{[1 - \rho\beta^{-1}(1 - \Gamma)][1 - \rho(1 - \Gamma)] + \kappa\sigma[\phi_\pi - \rho\beta^{-1}(1 - \Gamma)]} \geq 0
$$

First, the expression of $\gamma_y$ can be re-written as

$$
\gamma_y = \frac{(1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{[1 - \rho\beta^{-1}(1 - \Gamma)](1 - \rho(1 - \Gamma)) + \kappa\sigma[\phi_\pi - \rho\beta^{-1}(1 - \Gamma)]}
$$

$$
= \frac{(1 - \rho)(1 - \beta)\beta^{-1}(1 - \Gamma)}{1 - \beta^{-1} + \beta^{-1}\kappa\sigma + \beta^{-1}\{1 - \rho(1 - \Gamma) + \kappa\sigma(\phi_\pi\beta - 1)\}^{-1}}
$$

If $\phi_\pi\beta < 1$ and note that $1 - \rho(1 - \Gamma) > 0$, then the term $1 - \rho(1 - \Gamma) + \frac{\kappa\sigma(\phi_\pi\beta - 1)}{1 - \rho(1 - \Gamma)}$ is strictly increasing with respect to $\Gamma$. Since the term $1 - \Gamma > 0$ in the numerator of $\gamma_y$ is strictly decreasing with respect to $\Gamma$, $\gamma_y$ is then strictly decreasing with respect to $\Gamma$. If $\phi_\pi\beta > 1$ instead, the term $1 - \rho(1 - \Gamma) + \frac{\kappa\sigma(\phi_\pi\beta - 1)}{1 - \rho(1 - \Gamma)}$ is strictly increasing with respect to $\Gamma$ if and only if $1 - \rho(1 - \Gamma) \geq [\kappa\sigma(\phi_\pi\beta - 1)]^{1/2}$, i.e., $\Gamma \geq \frac{[\kappa\sigma(\phi_\pi\beta - 1)]^{1/2} - 1 + \rho}{\rho} \equiv \bar{\Gamma}$. Under such a condition, $\gamma_y$ is then strictly decreasing with respect to $\Gamma$.

For $\gamma_\pi$, since the numerator of $\gamma_\pi$ is strictly decreasing with respect to $\Gamma$ and the denominator is strictly increasing with respect to $\Gamma$, it is obvious that $\gamma_\pi$ is strictly decreasing with respect to $\Gamma$.

Q.E.D.

A.8 Forward Guidance

Consider the policy experiment of forward guidance as proposed in García-Schmidt and Woodford (2015) and Woodford (2019), and the real public debt is kept at its steady-state level $\bar{B}$ at all
times. More specifically, suppose prior to period $t = 0$, the economy stays at the steady-state equilibrium. It is announced in period $t = 0$ that, from this date to some future date $t = T$, monetary policy will follow the rule of $\phi = 0$ with $i^* = i^* < 0$, and at date $t = T$, the monetary rule reverts back to the “normal” policy reaction function $\hat{i}_t = \phi \pi_t$. This policy experiment mimics the situation in which the central bank sets the nominal interest rate at the effective zero lower bound for a fixed period of time, and negative $i^*$ indicates that the nominal interest rate at the effective zero lower bound is smaller than the steady-state inflation rate $\bar{\Pi}$ (target rate).

Starting from period $t = T$ onward, the endogenous output and inflation for any period $t \geq T$ is given by $y_t = \pi_t = 0$. Then, the system of equations capturing the evolution of the equilibrium for any period $0 \leq t < T$ is given by

$$x_t = \rho M E_t x_{t+1} + Nu^*$$

where $x_t = [y_t \quad \pi_t]^T$, $u^* = [-\sigma i^* \quad 0]^T$, and $M$ and $N$ are given by

$$M = \frac{1}{1 + \kappa \sigma \phi} \begin{bmatrix} 1 & -\sigma \phi \\ \kappa & 1 \end{bmatrix}, \quad N = \frac{1}{1 + \kappa \sigma \phi} \begin{bmatrix} 1 & -\sigma \phi \\ \kappa & 1 \end{bmatrix}$$

with $\phi = 0$.

It yields a unique solution for all $0 \leq t < T$, i.e.,

$$x_t = [I + \rho M + \cdots + (\rho M)^{T-t-1}] Nu^*$$ (A.8)

By the calibration in Table 1.1 with the assumption of a steady-state inflation at $\bar{\Pi} = 2\%$, Figure A.1 shows the time-series dynamics of output and inflation in response to an interest-rate peg lasting for ten quarters. The lines in the figure indicate the average planning horizon of the decision makers from short to long. Figure A.1 shows that, given the interest-rate peg for a fixed length of periods $T = 10$, as households and firms are less forward-looking, the stimulative effect of such a commitment to an interest-rate peg policy, namely forward guidance, becomes less effective. To
Figure A.1: Output and inflation dynamics with different planning horizons under forward guidance

Notes: $T = 10$; $h$ in quarters; no real disturbances.

Figure A.2: Output and inflation in period 0 as to the planning horizon under forward guidance with different length of commitments

Notes: $h$ in quarters; no real disturbances.
see how the length of commitment of the forward guidance matters, Figure A.2 shows the responses of output and inflation in period $t = 0$ with respect to different planning horizons under alternative lengths of the commitment to an interest-rate peg, i.e., $T = 10$ and $T = 14$, respectively. From Figure A.2, when the (average) planning horizon is extremely short, i.e., one quarter as estimated by Gust, Herbst, and López-Salido (2019), the central bank’s commitment of staying longer at the effective lower bound is of little effect.

### A.9 Proof of the Positive Interaction between Forward Guidance and Fiscal Transfer

In this section, I prove the interaction between forward guidance and fiscal transfer, i.e., the summation of the third and fourth terms in the equation (1.45), to be always positive through the method of forward induction. Note that since period $t = T$, the monetary policy reverts back to the normal Taylor rule, and thus it follows $x_T = \rho M^* x_T + N^* s^*$, where $M^* = \frac{1}{1 + \kappa \sigma \phi_\pi} \begin{bmatrix} 1 - \sigma \phi_\pi & -\sigma \beta \kappa \end{bmatrix} \begin{bmatrix} 1 & \sigma \\ \kappa & 1 \end{bmatrix}$ and $N^* = \frac{1}{1 + \kappa \sigma \phi_\pi} \begin{bmatrix} 1 & -\sigma \phi_\pi \\ \kappa & 1 \end{bmatrix}$ with $\phi_\pi > 0$.

First consider the case of $T = 1$, or the period of $t = T - 1$ for any $T \geq 2$. The interaction term satisfies

$$[\rho M - I] x_T + N^* s^* \gg 0$$

$$\iff \rho M x_T + N^* s^* \gg x_T$$

$$\iff \rho [M - M^*] x_T + (N - N^*) s^* \gg 0$$

Since $M - M^* \gg 0$, $N - N^* \gg 0$, $x_T \gg 0$, and $s^* \gg 0$, it follows that the last condition in the above derivation holds.

Then, for any $T \geq 2$, I show that, in any given period $0 \leq t < T - 1$, the interaction term by a commitment for total $T$ periods is larger than that of the commitment for total $T - 1$ periods. Note
that we have

\[ ((\rho M)^{T-t-1})_{xt} + [I + \rho M + \ldots + (\rho M)^{T-t-2}] N s^* \gg ((\rho M)^{T-t-1})_{xt} + [I + \rho M + \ldots + (\rho M)^{T-t-2}] N s^* \]

\[ \iff (\rho M)^{T-t} x_T + (\rho M)^{T-t-1} N s^* \gg (\rho M)^{T-t-1} x_T \]

\[ \iff (\rho M)^{T-t-1} [\rho M x_T + N s^* - x_T] \gg 0 \]

\[ \iff (\rho M)^{T-t-1} [\rho (M - M^*) x_T + (N - N^*) s^*] \gg 0 \]

Since \( M - M^* \gg 0, M \gg 0, N - N^* \gg 0, x_T \gg 0, \) and \( s^* \gg 0, \) the last condition in the above derivation holds. Therefore, in any given period \( 0 \leq t < T - 1, \) the interaction term with a commitment for total \( T \) periods is larger than that of the commitment for total \( T - 1 \) periods. Also note that, in period \( t = T - 1 \) for any \( T \geq 2 \) and in the case of \( T = 1, \) the interaction is also positive, and thus the interaction between forward guidance and fiscal transfer is always positive.\(^2\) Q.E.D.

A.10 Proof for the Property of a Long-run Stationary Equilibrium

Before proving Proposition 3, let us first consider simpler cases. If the shock or the policy change is temporary, which makes the real public debt converging back to the original steady-state level before the shock happens or policy changes. It is obvious that the equilibrium in the long-run will converge back to the original steady-state stationary equilibrium if the long-run stationary equilibrium exists. Then, I only need to focus on the situations in which the real public debt converges to a new steady-state level in the long run. For simplicity, I first prove Proposition 3 with the fiscal policy that there is a permanent increase in real public debt and the real public debt remains unchanged thereafter (\( \Gamma = 1. \)) Then, I will show that Proposition 3 holds generally.

The monetary policy and fiscal policy are specified as in Section 1.5. Since there is only one unique (locally) bounded long-run equilibrium under the parameterization of determinacy if a long-run stationary equilibrium exists, I refer to the variables for such a stationary equilibrium by

\(^2\)A more rigorous mathematical proof can be done by expanding the matrices in the derivation.
abstracting from time index $t$.

Given $\Gamma = 1$, the equation 1.30a capturing “deviation” variables now becomes

$$y - \bar{y} = \rho(y - \bar{y}) - \sigma [\bar{i} - \bar{i} - \rho(\pi - \bar{\pi})] + (1 - \rho)(1 - \beta)b^*$$

and by equation 1.28a for “trend” variables, it gives

$$\bar{y} = \rho \bar{y} - \sigma(\bar{i} - \rho\bar{\pi}) + (1 - \rho)\nu$$

Given that there is a positive gain of learning in the value function $\gamma, \bar{\gamma} > 0$, the dynamics of the value-function adjustment yields

$$\nu = y + \sigma\pi - (1 - \beta)b^*$$

By substituting the latter two equations into the first one, it follows that

$$\bar{i} = \pi$$

which indicates that the Fisher equation must hold in the long-run in this environment if the long-run stationary equilibrium exists. Furthermore, because the monetary policy is specified by the Taylor rule, $\hat{i} = \phi\pi\pi$, it is obvious that $\pi = 0$ as long as $\phi\pi \neq 1$. Therefore, Proposition 3 holds when $\Gamma = 1$.

Now, I show that Proposition 3 holds generally. By the expressions in Section 1.3.3, the equations capturing “deviation” variables for the policy specification in Section 1.5 are given by

$$y - \bar{y} = \rho(y - \bar{y}) - \sigma [\bar{i} - \bar{i} - \rho(\pi - \bar{\pi})] + (1 - \rho)(1 - \beta)(\psi_b b + \psi_{bb^*})$$

$$\pi - \bar{\pi} = \kappa(y - \bar{y}) + \beta \rho(\pi - \bar{\pi})$$
and the equations capturing “trend” variables are given by

\[ \bar{\gamma} = \rho \bar{\gamma} - \sigma (\bar{i} - \rho \bar{\pi}) + (1 - \rho) \nu + (1 - \beta) (\psi_v \nu + \psi_\phi \bar{v}) \]

\[ \bar{\pi} = \kappa \bar{\gamma} + \beta \rho \bar{\pi} + (1 - \rho) (1 - \alpha) \beta \bar{v} \]

By substituting the latter two equations of “trend” variables \{\bar{\gamma}, \bar{\pi}\} into the first two equations, I can get two equations capturing the aggregate output \(y\) and inflation \(\pi\) that are fully composed by the aggregate variables \{\(y, \pi, b, \hat{i}, \nu, \bar{v}\}\).

Given that there is a positive gain of learning in the value function \(\gamma, \bar{\gamma} > 0\), the dynamics of the value-function adjustment yields

\[ \nu = y + \sigma \pi - (1 - \beta)b^* \]

\[ \bar{v} = (1 - \alpha)^{-1} \pi \]

Together with the debt evolution \(b = \beta^{-1} (1 - \Gamma) b - \beta^{-1} (1 - \Gamma) s_b \pi + (1 - \Gamma) s_b \hat{i} + \Gamma b^*\), the rule of nominal interest rate \(\hat{i} = \phi_\pi \pi\), there is a system of six equations characterizing the aggregate variables \{\(y, \pi, b, \hat{i}, \nu, \bar{v}\}\}, and there is a unique solution for this system.

From the expressions of \{\(\psi_b, \psi_{b*}, \psi_\nu, \psi_\phi\}\}, it can be easily verified that the solution to this system is given by

\[ y = \pi = \hat{i} = 0, b = \frac{\Gamma b^*}{1 - \beta^{-1} (1 - \Gamma)} \]

\[ \nu = -\frac{(1 - \beta) \Gamma b^*}{1 - \beta^{-1} (1 - \Gamma)}, \bar{v} = 0 \]

where I have employed the relationship that \(\frac{\psi_{b*}}{\psi_b - (1 - \beta) \psi_\nu} = -\frac{\Gamma}{1 - \beta^{-1} (1 - \Gamma)}\). Q.E.D.

A.11 The Fiscal Multiplier with Government Expenditure under Finite Forward Planning

In this section, I consider the short-term effects of a government expenditure with finite forward planning, and study how the fiscal multiplier changes with respect to the degree of foresight in the
situation of no binding ZLB. For the cases under ZLB, Woodford and Xie (2019) have showed that, as long as the government expenditure is not sufficiently large, the fiscal multiplier under ZLB is in general decreasing when decision makers are less forward-looking. But, in the case without binding ZLB, I will show that this relationship is opposite, and the effect of fiscal stimulus through government expenditure follows a pattern similar to the analysis of transfer policies.

For simplicity, I assume decision makers use the value function learned from the steady-state stationary equilibrium to approximate continuation values beyond their planning horizon, and there is an exponential distribution of planning horizon among the population. Also, the monetary policy is specified by the Taylor rule, and the government expenditure is fully financed through lump-sum taxation immediately in the same period when the expenditure is imposed.

Similar to Woodford (2011) with rational expectations, the aggregate output and inflation with finite forward planning can be characterized by

\[
y_t - g_t = \rho E_t (y_{t+1} - y_t) - \sigma (\hat{\imath} - \rho E_t \pi_{t+1})
\]

\[
\pi_t = \kappa (y_t - \Gamma g_t) + \beta \rho E_t \pi_{t+1}
\]

where \( \Gamma_g = \eta_u / (\eta_u + \eta_w) < 1 \) is the fiscal multiplier under flexible-price equilibrium, and \( g_t = (G_t - \bar{G}) / \bar{Y} > 0 \) is the log-deviation of the government expenditure relative to the steady-state level of output, and the nominal interest rate \( \hat{\imath} = \phi_\pi \pi_t \).

Consider a deterministic path of the government expenditure \( g_t = g_0 \eta^t \) with \( 0 \leq \eta < 1 \). Then, the solution of aggregate output and inflation is given by

\[
y_t = \gamma_y g_t, \quad \pi_t = \gamma_\pi g_t
\]

Note that \( \eta_u = -\bar{Y} u'' / u' > 0 \) is the negative elasticity of \( u' \) and \( \eta_w = \bar{Y} \bar{w}'' / \bar{w}' \) is the elasticity of \( \bar{w} \) with respect to increases in \( Y \). Similar to the notation in Woodford (2019), the period utility of household \( i \) is defined as \( u(C^i_t) - w(H^i_t) \), where \( C^i_t \) is the quantity consumed in period \( t \) and \( H^i_t \) is hours of labor supplied in period \( t \). As usual, \( u(\cdot) \) is an increasing, strictly concave function, and \( w(\cdot) \) is an increasing, convex function. \( \bar{w}(Y) = w(f^{-1}(Y)) \) is the dis-utility to the household of supplying a quantity of output \( Y \), and \( f \) is the production technology.
where

\[
\gamma_y = \frac{1 - \rho \eta + \frac{\sigma \kappa (\phi - \rho \eta)}{1 - \beta \rho \eta} \Gamma_g}{1 - \rho \eta + \frac{\sigma \kappa (\phi - \rho \eta)}{1 - \beta \rho \eta}}
\]

\[
\gamma_\pi = \frac{\kappa (\gamma_y - \Gamma_g)}{1 - \beta \rho \eta}
\]

It can be showed that, given \( \phi > 1 \), the fiscal multiplier \( \gamma_y \) is strictly decreasing with respect to the (average) degree of foresight \( \rho \). In other words, as decision makers are less forward-looking, the effect of the government expenditure without ZLB increases. Similar to the analysis of transfer policies, the impact of monetary policy accommodation on the fiscal multiplier (from government expenditure) is also hump-shaped with respect to the degree of foresight.
Appendix B: Stabilization Policy in a Low-Interest-Rate World:  
Consequences of Limited Foresight

B.1 Temporary Price-level Targeting: Numerical Methods

Now, we propose a numerical method for approximating solutions of the PDEs (2.29a)-(2.29b):
define a discrete grid of values for \((\tilde{p}, h)\) and let \((j, k)\) in the grid corresponding to \(\tilde{p} = \tilde{p}(j \cdot \epsilon)\)
and \(h = k \cdot \epsilon\) for some \(\epsilon > 0\), where \((j, k)\) are both non-negative integers.\(^1\)

For each integer \(j \geq 0\), the goal is to compute sequences of values \(\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}\)
for progressively higher values of \(k\). Consider the recursive computation, i.e., given the values for
\((y(j, k - 1), \pi(j, k - 1), \tilde{p}(j, k - 1))\), we compute

\[
y(j, k) = y(j, k - 1) - \sigma \Delta^\ast \epsilon + \sigma \pi(j, k - 1) \epsilon + \nu \epsilon \cdot [\tilde{y}(\tilde{p}(j, k - 1), k - 1) - y(j, k - 1)] \tag{B.1a}
\]

\[
\pi(j, k) = \pi(j, k - 1) + \gamma \epsilon [y(j, k - 1) - \lambda \pi(j, k - 1)] + \nu \epsilon \cdot [\tilde{\pi}(\tilde{p}(j, k - 1), k - 1) - \pi(j, k - 1)] \tag{B.1b}
\]

\[
\tilde{p}(j, k) = \tilde{p}(j, k - 1) - \epsilon \pi(j, k) \tag{B.1c}
\]

The idea is that a sequence \(\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}\) represents a possible trajectory \(y(t)\), \(\pi(t)\), and \(\tilde{p}(t)\) along which the economy remains in the “crisis” state, though it is not known in advance that this will be the case. Along this trajectory, \(y(t) = y(\tilde{p}(t), h(t)), \pi(t) = \pi(\tilde{p}(t), h(t))\)

\(^1\)The \(\epsilon\) measures how fine the grid is, and in the numerical calculation, we take \(\epsilon = 0.2\). The results are robust by using different values of \(\epsilon\).
for all $t$, and $\tilde{p}(t)$ and $h(t)$ evolve according to

$$\frac{d\tilde{p}(t)}{dt} = \pi(t), \quad \frac{dh(t)}{dt} = -1$$

The successive values of $k$ index the remaining horizon $h(t)$; the value of $j$ indexes the particular trajectory, which is determined by the terminal values from which one initiates the recursive computation.

For any $j \geq 0$, we start from terminal values $y(j, 0) = \pi(j, 0) = 0$ and some specified value for $\tilde{p}(j, 0)$, then iteratively apply (B.1a)-(B.1c) to compute $\{y(j, k), \pi(j, k), \tilde{p}(j, k)\}$ for progressively higher values of $k$. Thus, the complete trajectory depends on the value assumed for $\tilde{p}(j, 0) \leq 0$.

In equations (B.1a)-(B.1c), the functions $\tilde{y}(\tilde{p}(j, k - 1), k - 1), \pi(\tilde{p}(j, k - 1), k - 1)$ are given by:

$$\tilde{y}(\tilde{p}, k - 1) = \begin{cases} \tilde{y}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \tilde{p}((k - 1) \cdot \epsilon) \\ \tilde{y}((k - 1) \cdot \epsilon) & \text{if } \tilde{p} \leq \tilde{p}((k - 1) \cdot \epsilon) \end{cases}$$

$$\tilde{\pi}(\tilde{p}, k - 1) = \begin{cases} \tilde{\pi}(\tau(\tilde{p})) & \text{if } \tilde{p} \geq \tilde{p}((k - 1) \cdot \epsilon) \\ \tilde{\pi}((k - 1) \cdot \epsilon) & \text{if } \tilde{p} \leq \tilde{p}((k - 1) \cdot \epsilon) \end{cases}$$

We continue iterating the trajectories until either (i) the value $k = K$ is reached, i.e, some pre-specified upper-bound on the range of values of $k$ that are of interest, or (ii) one reaches a value of $k$ at which $\tilde{p}(j, k) \geq 0$.\(^2\) In the latter case, we stop iterating the trajectory at this point, and define

$$k^*(j) \equiv \max\{k | \tilde{p}(j, k) < 0\} + 1$$

as the largest value of $k$ for trajectory $j$.

Now, we characterize the terminal values $\{\tilde{p}(j, 0)\}$ for the different values of $j$. The $\{\tilde{p}(j, 0)\}$ is a monotonically decreasing sequence, i.e., $\tilde{p}(j + 1, 0) < \tilde{p}(j, 0) < 0$ for all $j \geq 1$. Furthermore,

\(^2\)In this numerical calculation, we take $K = 300$. 203
the sequence of values \( \{\tilde{p}(j, 0)\} \) that are considered is dense enough, i.e., successive values of \( \tilde{p}(j, 0) \) should be close enough to one another, to ensure that for every value of \( k \) of interest (i.e., a value of \( k \) for which we intend to simulate the equilibrium dynamics), there exists some \( j \) such that \( k^*(j) = k \). In the numerical exercise, given the calibration in section 2.2.3, we find there existing a \( \tilde{p}^* < 0 \) such that, for any terminal values \( \tilde{p}^* < \tilde{p}(j, 0) < 0 \), there always exists a \( k^*(j) \) as defined before, but for any terminal values \( \tilde{p}(j, 0) < \tilde{p}^* \), \( \tilde{p}(j, k) < 0 \) for \( \forall k \geq 0 \), i.e., \( \tilde{p}(j, k) \) becomes negatively exploding starting for some finite \( k \). In other words, in the latter scenario, there does not exist a \( k^*(j) \).

Figure B.1 illustrates the price trajectories with respect to planning horizons, and shows the decreasing sequence of terminal values on y-axis. Nonetheless, as long as we choose the the terminal values \( \{\tilde{p}(j, 0)\} \) to be smaller than \( \tilde{p}^* \) and make it dense enough, we will still be able to find some \( j \) such that \( k^*(j) = k \) for each \( k \leq K \).

Then, for each of the values of \( k \) of interest, we define

\[
\hat{j}(k) \equiv \max\{j|k^*(h) = k\}
\]
Now, we characterize the predicted dynamics in response to a shock. Let the planning horizon $k$ be given (one of the values of $k$ “of interest”). Consider first the dynamics for period $t < N \cdot \epsilon = T$, so that the economy remains in the “crisis state”. In any period $t$ of the simulation, we can approximate the value of $\tilde{p}(t)$ by interpolation. We start the simulation in period $t = 0$, and set $\tilde{p}(0) = 0$. The price-level gap is approximated by

$$\tilde{p}(t) = (1 - \lambda_t)\tilde{p}(j_t, k) + \lambda_t\tilde{p}(j_t + 1, k)$$

where

$$j_t \equiv \max\{j \geq j(k) | \tilde{p}(t) \leq \tilde{p}(j, k)\}$$ \hspace{1cm} (B.2a)

$$\lambda_t \equiv \frac{\tilde{p}(t) - \tilde{p}(j_t, k)}{\tilde{p}(j_t + 1, k) - \tilde{p}(j_t, k)}$$ \hspace{1cm} (B.2b)

Note that, since there exists a $j(k)$ such that $\tilde{p}(j(k), k) \geq 0$ has been computed, and there also exists at least one higher value of $j$ for which $\tilde{p}(j, k) < 0$ has also been computed, we can necessarily find such a $j_t$, and both $\tilde{p}(j_t, k)$ and $\tilde{p}(j_t + 1, k)$ will have been computed unless we find that there exist no $j$ such that $\tilde{p}(t) < \tilde{p}(j_t, k)$. The latter problem can be avoided by adding to the first of trajectories $j$ that are computed some additional trajectories starting from lower terminal values $\{\tilde{p}(j, 0)\}$.

We can then approximate the values of $y(\tilde{p}(t), h)$, $\pi(\tilde{p}(t), h)$ by linear interpolation, i.e., for all $0 \leq t \leq N - 1$,

$$y(t) = (1 - \lambda_t)y(j_t, k) + \lambda_t y(j_t + 1, k)$$ \hspace{1cm} (B.3a)

$$\pi(t) = (1 - \lambda_t)\pi(j_t, k) + \lambda_t\pi(j_t + 1, k)$$ \hspace{1cm} (B.3b)

$$\tilde{p}(t + 1) = \tilde{p}(t) + \pi(t) \cdot \epsilon$$ \hspace{1cm} (B.3c)
where the sequences of values \(\{y(j, k), \pi(j, k)\}\) for values \(j \geq j(k)\) have been computed using (B.1a)-(B.1c).

The value at \(\bar{p}(N)\) obtained from (B.3c) when \(t = N - 1\) is then the initial condition for the closed-form solution obtained for the period after reversion to the “normal” state.

B.2 Systematic Price-level Targeting: Numerical Methods

We numerically solve the system (2.30a)-(2.30c) using the same method as in section B.1. Each trajectory \(j\) is associated with a particular possible value \(\tau(\bar{p}(t_0 + h_0)) = \tau_j > 0\). We assume that the \(\{\tau_j\}\) are a monotonically increasing sequence. For remaining horizon \(k = 0\), from (2.31), we then have

\[
y(j, 0) = \bar{y}(\tau_j), \quad \pi(j, 0) = \bar{\pi}(\tau_j), \quad \bar{p}(j, 0) = \bar{\rho}(\tau_j)
\]

Starting from these initial values, we then use equations (B.1a)-(B.1c) to recursively calculate \(y(j, k), \pi(j, k), \bar{p}(j, k)\) for progressively higher values of \(k\). The process is continued for values of \(k\) up to \(k = K\), i.e., the longest horizon of interest for purposes of the simulations, or until one reaches a value of \(k\) at which \(\bar{p}(j, k) \geq 0\).

Given a numerical solution for a set of trajectories indexed by \(j\), we can compute simulated paths using the same method as in section B.1.

B.3 Output and Inflation Stabilization with an Exponential Distribution of Planning Horizons

Here we demonstrate that the combination of a monetary policy specified by (2.36) and a fiscal policy specified by (2.37) each period imply complete stabilization of aggregate output and inflation at all times, in the case of an exponential distribution of planning horizons, \(\omega_h = (1-\rho)\rho^h\) for all \(h \geq 0\). In the text, we have already shown that this monetary rule implies that the spending and price-increase decisions of households and firms with an arbitrary planning horizon \(h\) are given by equation (2.40). This is a well-defined, unique solution, independent of any assumption about
the distribution of planning horizons in the economy. There will therefore exist a well-defined, unique solution in the case of an exponential distribution of planning horizons if and only if the infinite sums

\[ z_t = (1 - \rho) \sum_{h=0}^{\infty} \rho^h z_t^h \]  

(B.4)

converge, where \( z_t^h \) is given by (2.40). This is the issue that remains to be addressed.

Let us first consider the partial sum that aggregates the decisions of only the part of the population with horizons less than or equal to \( k \) periods,

\[ z_t^{(k)} = (1 - \rho) \sum_{h=0}^{k} \rho^h z_t^h \]

for some finite \( k \). This finite sum is obviously well-defined; it remains to be determined whether the sequence \( \{z_t^{(k)}\} \) converges as \( k \) is made large.

Substituting (2.40) into this definition yields

\[ z_t^{(k)} = -\sigma(1 - \rho) \sum_{h=0}^{k} \rho^h \sum_{j=0}^{h} [A^j a] E_t \tilde{\Delta}_{t+j} + (1 - \beta)(1 - \rho) \sum_{h=0}^{k} \rho^h [A^h a] E_t b_{t+h+1} \]

\[ = -\sigma(1 - \rho) \sum_{h=0}^{k} \sum_{j=h}^{k} \rho^j [A^j a] E_t \tilde{\Delta}_{t+h} + (1 - \beta)(1 - \rho) \sum_{h=0}^{k} \rho^h [A^h a] E_t b_{t+h+1} \]

\[ = \sum_{h=0}^{k} [A^h a] E_t [(1 - \beta)(1 - \rho)\rho^h b_{t+h+1} - \sigma(\rho^h - \rho^{k+1})\Delta_{t+h}] . \]

If we further substitute the fiscal rule (2.37) into this, we obtain

\[ z_t^{(k)} = \sigma \rho^{k+1} \sum_{h=0}^{k} [A^h a] E_t \tilde{\Delta}_{t+h} . \]  

(B.5)

Thus the condition required for a well-defined solution is convergence of the sequence defined by the right-hand side of (B.5) as \( k \) becomes large.

The existence of a well-defined limit depends on the asymptotic rate of growth of the expected future excess financial wedge \( E_t \tilde{\Delta}_{t+h} \). A sufficient condition for the existence of a well-defined
solution is that $\tilde{\Delta}_t = 0$ with probability one beyond some finite future date $T$. In this case, for all $k \geq T - t$, $z_{t}^{(k)}$ is a constant multiple of $\rho^k$, and hence converges to zero as $k$ is made large (regardless of the value of $\rho < 1$).

But this is not necessary: a weaker sufficient condition is that there exists a finite constant $C > 0$ such that $E_t\tilde{\Delta}_{t+h} \leq C \cdot \mu^h$ for all $h$, where $\mu \geq 0$ is a growth factor satisfying (2.20). As discussed in the text, this bound implies that both eigenvalues of $A$ are less than $\rho^{-1}\mu^{-1}$. The partial sum $\sum_{h=0}^{k}[A^h a]$ is therefore positive, increases in $k$, and grows asymptotically with a growth factor less than $\rho^{-1}\mu^{-1}$. Hence the right-hand side of (B.5) is necessarily non-negative (since (2.38) implies that $\tilde{\Delta} \geq 0$ at all times), and bounded above by a positive sequence that converges to zero at an exponential rate as $k$ is made large.

Hence under this condition, the infinite sum in (B.4) is well-defined, and equal to

$$z_t = \lim_{k \to \infty} z_{t}^{(k)} = 0.$$ 

Thus the specified joint fiscal-monetary regime implies the existence of a well-defined unique equilibrium, in which $y_t = \pi_t = 0$ at all times, as stated in the text. Among the cases in which a well-defined equilibrium exists is the two-state Markov process for the financial wedge introduced in section 2.2.3, under the assumption that the probability $\mu$ of continuation of the crisis state satisfies (2.20), the condition already discussed in section 2.2.3 for the existence of a well-defined equilibrium in the case of a balanced-budget policy and strict inflation targeting.

### B.4 Optimal Fiscal-Monetary Policy Coordination: Numerical Methods

In this section, we propose a numerical method to compute the solutions for optimal fiscal transfer policy and interest rate policy. Assume all the agents have the same planning horizon, and the path of financial wedge is perfectly predictable, i.e., $\hat{\Delta}_t = -i + \Delta$ for $0 \leq t < T - 1$, where $\Delta > 0$ is the excess financial wedge that cannot be offset by a reduction in nominal interest rate, and $\hat{\Delta} = 0$ for all $t \geq T$. We consider the following class of policies: the fiscal policy is specified by an
exogenous path of the real public debt $b_{t+1}$, and the monetary policy is specified by an exogenous path of the nominal interest rate $\hat{i}_t$ consistent with the ZLB constraint (2.11).

The structural equations (2.7) and (2.9) can be written as

$$z_t^j = A z_{t+1}^j - \sigma a(\hat{i}_t + \hat{\Delta}_t)$$

for all $j \geq 1$, while (2.8) and (2.10) can be written as

$$z_t^0 = -\sigma a(\hat{i}_t + \hat{\Delta}_t) + (1 - \beta)ab_{t+1},$$

where $z_t^j = [y_t^j \pi_t^j]'$ and the matrices $A$ and $a$ are defined as in (2.12).

Since the path of $\hat{\Delta}_t$ is exogenously given, the policy variables $\{\hat{i}_t, b_{t+1}\}$ can be equivalently described by the sequences of $\{\hat{\Delta}_t, b_{t+1}\}$, where $\hat{\Delta}_t = \hat{i}_t + \hat{\Delta}_t$. The problem of solving optimal fiscal and monetary policy is then to choose $\{\hat{\Delta}_t, b_{t+1}\}$ for all $t \geq 0$, subject to the constraints that $\hat{\Delta}_t \geq \Delta > 0$ for all $0 \leq t < T$ and $\hat{\Delta}_t \geq \hat{\Delta}$ for all $t \geq T$, so as to minimize the welfare loss (2.42).

Now, we consider the solution to such an optimal fiscal and monetary policy problem. Let us first take the sequence of $\{\hat{\Delta}_t\}$ as given, and focus on the optimal choice of the sequence $\{b_{t+1}\}$, which is a sequence of independent static optimization problems. More specifically, for any period $t \geq 0$, we choose $b_{t+h+1}$ to minimize $(\pi_t^h)^2 + \lambda(y_t^h)^2$, where $\{y_t^h, \pi_t^h\}$ are given by

$$z_t^h = -\sigma \sum_{j=0}^{h} [A^j a] \hat{\Delta}_{t+j} + (1 - \beta) [A^h a] b_{t+h+1}.$$  \hspace{1cm} (B.6)

Denote $[A^j a] = [\alpha_j \gamma_j]'$ for each $j \geq 0$, and we have the F.O.C.s of the problem for optimal fiscal transfer policy given by

$$\begin{bmatrix} \lambda \alpha_h & \gamma_h \end{bmatrix} z_t^h = 0,$$
which yields the unique solution of \( b_{t+h+1} \) as
\[
\begin{aligned}
b_{t+h+1} &= \frac{\sigma \sum_{j=0}^{h} (\lambda \alpha_h \alpha_j + \gamma_h \gamma_j) \tilde{\Delta}_{t+j}}{(1 - \beta)(\lambda \alpha_h^2 + \gamma_h^2)}. \\
\end{aligned}
\] (B.7)

By substituting the expression of \( b_{t+h+1} \) into (B.6), under an optimal fiscal transfer policy, we obtain
\[
\begin{aligned}
z_t^h &= -\sigma \sum_{j=0}^{h-1} \left( [A^j a] - \frac{(\lambda \alpha_h \alpha_j + \gamma_h \gamma_j)}{\lambda \alpha_h^2 + \gamma_h^2} [A^h a] \right) \tilde{\Delta}_{t+j} \\
&= \left[ -\sigma \sum_{j=0}^{h-1} \theta_j \tilde{\Delta}_{t+j} \right] \\
&= \left[ \begin{array}{c} \\
\gamma_h \\
-\lambda \alpha_h \\
\end{array} \right]
\end{aligned}
\]
where \( \theta_j = \frac{\alpha_j \gamma_h - \alpha_h \gamma_j}{\lambda \alpha_h^2 + \gamma_h^2} \) for each \( 0 \leq j \leq h - 1 \). It follows that the minimized value of the objective function \( L_t \equiv (\pi_t^h)^2 + \lambda (y_t^h)^2 \) will be equal to
\[
L_t = \left[ \sigma \sum_{j=0}^{h-1} \theta_j \tilde{\Delta}_{t+j} \right] \lambda (\lambda \alpha_h^2 + \gamma_h^2).
\]

We now consider the optimal monetary policy \( \{ \tilde{\Delta}_t \} \) so as to minimize \( \sum_{t=0}^{\infty} \beta^t L_t \), subject to the constraint \( \tilde{\Delta}_t \geq \Delta > 0 \) for all \( 0 \leq t < T \) and \( \tilde{\Delta}_t \geq \hat{\Delta} \) for all \( t \geq T \). The F.O.C.s of the optimal choice of \( \tilde{\Delta}_t \) for any \( t \geq h - 1 \) are given by
\[
\begin{aligned}
\sum_{j=0}^{h-1} \beta^{-j} \left[ \sum_{i=0}^{h-1} \theta_i \tilde{\Delta}_{t-j-i} \right] \theta_j \geq 0, \quad \tilde{\Delta}_t \geq \tilde{\Delta}_t,
\end{aligned}
\] (B.8)
where at least one of these inequalities must hold with equality, and \( \tilde{\Delta}_t = \Delta \) for all \( 0 \leq t < T \) and \( \tilde{\Delta}_t = \hat{\Delta} \) for all \( t \geq T \). Instead, for any \( 0 \leq t < h - 1 \), the F.O.C.s are given by
\[
\begin{aligned}
\sum_{j=0}^{h-1} \beta^{-j} \left[ \sum_{i=0}^{h-1} \theta_i \tilde{\Delta}_{t-j-i} \right] \theta_j \geq 0, \quad \tilde{\Delta}_t \geq \tilde{\Delta}_t,
\end{aligned}
\] (B.9)

We conjecture that the solution of \( \{ \tilde{\Delta}_t \} \) to (B.8) and (B.9) has the following form: there exists a \( T^* \) such that the ZLB binds in every period up to some date \( T^* \geq 0 \), and then the ZLB never binds

210
for any dates \( t \geq T^* \), i.e., for any \( t < T^* \), \( \tilde{\Delta}_t = \Delta_t \), while \( \sum_j \beta^{-j} \left[ \sum_{i=0}^{h-1} \theta_i \tilde{\Delta}_{t-j+1} \right] \theta_j = 0 \) for all \( t \geq T^* \), and \( \tilde{\Delta}_t \to 0 \) as \( t \to \infty \).

Under this conjecture, numerically, we assume that for a large enough \( T_{max} \), \( \tilde{\Delta}_t = 0 \) for any date \( t > T_{max} \). Then, (B.8) and (B.9) give a total number of \( T_{max} - T^* + 1 \) linear equations for the periods \( T^* \leq t \leq T_{max} \) in which the ZLB is not binding, with the unknown variables \( \{ \tilde{\Delta}_t \}_{t=T^*}^{T_{max}} \). It yields a unique solution. Thus, we can start with \( T^* = 0 \) and increase the possible values of \( T^* \) until we find a value of \( T^* \) satisfying all the inequality conditions. In other words, for a given guess of \( T^* \), we have a system of linear equations to solve with a unique solution \( \{ \tilde{\Delta}_t \} \); then we check whether the solution satisfies a sequence of inequalities (in which case it is the desired solution).

Once we get the solution of \( T^* \) and the sequence of \( \{ \tilde{\Delta}_t \} \), the optimal fiscal transfer policy is accordingly pinned down by (B.7). With the solution of optimal fiscal and monetary policy, the realized paths of output and inflation are then given by (B.6), while the expected path of output and inflation are given by (2.7)-(2.10).

Figure B.2 and B.3 show the dynamics of output, inflation, nominal interest rate, and the public debt position with different planning horizons under optimal fiscal transfer policy and monetary policy. In these figures, we assume that \( T = 10 \), i.e., the economy reverts back to the “normal” state after 10 quarters. Figure B.4 illustrates the expected and realized paths of output and inflation under optimal fiscal-monetary policy with a common planning horizon \( h = 20 \). The dashed line in the figure indicates the expected output and inflation in the agents’ forward planning exercise, while the solid line indicates the realized output and inflation.

So far, we have shown the numerical methods for the solution of optimal fiscal transfer policy and monetary policy. In order to highlight the role of fiscal transfer policy, we now consider the case of optimal monetary policy but with \( b_{t+1} = 0 \) for any \( t \). In this case, the dynamics of output and inflation (B.6) are instead given by

\[
\zeta_t^h = -\sigma \sum_{j=0}^{h} [A^j a] \tilde{\Delta}_{t+j}.
\]

\(^3\)In the numerical exercise, we take \( T_{max} = 200 \).
Figure B.2: Dynamics of output and inflation under the optimal fiscal transfer policy and monetary policy with relatively short planning horizons $h$. Each line is for a distinct value of the common planning horizon $h$. Both $t$ and $h$ are in quarters.
Figure B.3: Dynamics of output and inflation under the optimal fiscal transfer policy and monetary policy with relatively long planning horizons $h$. Each line is for a distinct value of the common planning horizon $h$. Both $t$ and $h$ are in quarters.

Figure B.4: Expected paths of output and inflation in forward planning exercise under the optimal fiscal transfer policy and monetary policy with a common planning horizon of $h = 20$ quarters.
Then, the minimized value of the objective function $L_t \equiv (\pi_t^h)^2 + \lambda (y_t^h)^2$ is equal to

$$L_t = [\sigma \sum_{j=0}^{\gamma} y_j \tilde{\alpha}_{t+j}]^2 + \lambda [\sigma \sum_{j=0}^{\gamma} \alpha_j \tilde{\alpha}_{t+j}]^2.$$

The F.O.C.s for the optimal monetary policy with $b_{t+1} = 0$ for any $t$ is thus given by

$$\sum_{j=0}^{\gamma} \beta^{-j} [(\sum_{l=0}^{\gamma} y_l \tilde{\alpha}_{t-j+l}) y_j + \lambda (\sum_{l=0}^{\gamma} \alpha_l \tilde{\alpha}_{t-j+l}) \alpha_j] \geq 0, \quad \tilde{\alpha}_t \geq \tilde{\alpha}_y,$$

for any $t \geq h$, while for any $0 \leq t < h$, the F.O.C.s are given by

$$\sum_{j=0}^{\gamma} \beta^{-j} [(\sum_{l=0}^{\gamma} y_l \tilde{\alpha}_{t-j+l}) y_j + \lambda (\sum_{l=0}^{\gamma} \alpha_l \tilde{\alpha}_{t-j+l}) \alpha_j] \geq 0, \quad \tilde{\alpha}_t \geq \tilde{\alpha}_y.$$

We similarly conjecture that there exists a $T^*$ such that the ZLB binds in every period up to some date $T^* \geq 0$, and then the ZLB never binds for any dates $t \geq T^*$. With the same numerical method as in solving the optimal fiscal-monetary policy problem, we can solve for the optimal monetary policy of $\{\tilde{\alpha}_t\}$ under the assumption of $b_{t+1} = 0$ for any $t$.
Appendix C: Monetary Policy in an Era of Global Supply Chains

C.1 Equilibrium Characterization with $N$-stage of Production in a Small-open Economy

C.1.1 The Steady-state Equilibrium

We first characterize the steady state with perfect foresight. The steady state is defined as the equilibrium under non-stochastic and constant exogenous variables. Since the whole economy does not change with timing, we can ignore the timing index $t$ in all variables, and $A_n = 1$ for $n = 1, 2, \ldots, N$. The optimal pricing decision for firms at stage $n, n = 1, 2, \ldots, N$, becomes

$$P_n^{o} = \Psi = P_n = \tilde{P}_n$$

and for $n = 2, \ldots, N$, we have

$$\tilde{P}_n = \tilde{P}^\gamma_{(n-1)H} \tilde{P}^{1-\gamma}_{(n-1)F}$$

where $\tilde{P}_{(n-1)F} = TEP^*_{(n-1)F}$.

We solve for the price indices in terms of wages and derive the labor demand function. Note that $\Psi_n = \tilde{P}_n^{\phi} W^{1-\phi}$ for $n = 2, 3, \ldots, N$ with $\Psi_1 = W$. By substituting $\tilde{P}_n$, the relationship of output price index across adjacent stages is given by

$$P_n = W^{1-\phi} (\tilde{P}_n)^{\phi}$$

$$= W^{1-\phi} P_{(n-1)H}^{\gamma \phi} P^{(1-\gamma)\phi}_{(n-1)F}$$

for $n = 2, \ldots, N$ and $P_{1H} = W$.

By writing all price indices in terms of wage and exogenous variables through forward induc-
tion, we get
\[ P_{nH} = W^{(1-\phi)\frac{1-(\gamma\phi)^{n-1}}{1-\gamma \phi}+(\gamma\phi)^{n-1}}} (T\mathcal{E})^{\phi(1-\gamma)\frac{1-[(1-\gamma)\phi]^{n-1}}{1-\phi(1-\gamma)}} \Pi_{i=1}^{n-1} [P_{iF}^* \phi(1-\gamma)]^{n-i} \]  
\[ \text{(C.1)} \]
for \( n = 2, \ldots, N \) with \( P_{1H} = W \).

Since \( P_{nH}(u) = P_{nH} \) for \( u \in [0, 1] \) in steady state, by Equation (3.12), for \( n = 1, \ldots, N \), we have
\[ Y_{nH,t}^{d}(u) = \tilde{Y}_{nH,t}^{d} \]
Together with goods market clearing condition \( Y_{nH} = \tilde{Y}_{nH}^{d} \), and factor market demand function (3.9) and (3.8), for \( n = 2, \ldots, N \), we get
\[ \tilde{Y}_{n}^{d} = \phi \frac{\psi_{n}}{P_{n}} [\tilde{Y}_{nH}^{d} + Y_{nH}^{X}] \]
\[ L_{n}^{d} = (1 - \phi) \frac{\psi_{n}}{W} [\tilde{Y}_{nH}^{d} + Y_{nH}^{X}] \]
\[ \tilde{Y}_{(n-1)H}^{d} = \frac{\gamma \tilde{P}_{n}^{d}}{\tilde{P}_{(n-1)H}} \]
where \( \tilde{Y}_{nH}^{d} = \gamma C P_{nH}^{-(1-\gamma)} \tilde{P}_{nF}^{1-\gamma} \), and \( L_{1}^{d} = \frac{\psi_{1}}{W} \tilde{Y}_{1H}^{d} \). By substituting the price index and unit cost function in each stage, for \( n = 1, \ldots, N \), through backward induction, the factor demand functions for labor can be written in an implicit form as
\[ L_{n}^{d} = f (W, C, \mathcal{E}, T, P_{1F}^*, \ldots, P_{nF}^*, P_{nH}^*, \ldots, P_{NH}^*, Y_{nH}^*, \ldots, Y_{NH}^*) \]  
\[ \text{(C.2)} \]
By summing up labor demand across all stages, the total labor demand function becomes
\[ L_{n}^{d} = \sum_{n=1}^{N} L_{n}^{d} \]  
\[ \text{(C.3)} \]
Therefore, the three equations (3.1), (3.6), and (C.3) fully characterize the steady-state real wage, consumption and employment.
C.1.2 The Flexible-price Equilibrium

In this subsection, we solve for the flexible-price equilibrium similarly as for the steady-state equilibrium. In the flexible-price equilibrium, \( \alpha_n = 0 \) for \( \forall n \). The optimal pricing decision for firms at stage \( n, n = 1, 2, \ldots, N \), thus becomes

\[
P_{nH,t}^\alpha = \Psi_{n,t} = P_{nH,t} = \tilde{P}_{nH,t}
\]

and for \( n = 2, \ldots, N \), we have

\[
\tilde{P}_{n,t} = \tilde{P}^\gamma_{(n-1)H,t} \tilde{P}^{1-\gamma}_{(n-1)F,t}
\]

Similar to the steady state case, we solve for the price indices in terms of wages and productivity. Note that \( \Psi_{n,t} = \tilde{P}^\phi_{n,t} W_t^{1-\phi} / A_{n,t} \) for \( n = 2, 3, \ldots, N \) with \( \Psi_1 = W_t / A_{1,t} \). By substituting \( \tilde{P}_{n,t} \), the relationship of price index across adjacent stages is given by

\[
P_{nH,t} = W^{1-\phi} P^{\gamma \phi}_{(n-1)H,t} P_{(n-1)F,t}^{(1-\gamma)\phi}
\]

for \( n = 2, \ldots, N \) and \( P_{1H,t} = W_t / A_{1,t} \).

By writing all price indices in terms of wage through forward induction, we similarly get

\[
P_{nH,t} = W^{(1-\phi) / l-\gamma \phi} \cdot \sum_{i=1}^{n-1} \left( (\phi^{n-i}) \sum_{j=1}^{n-i} (A_{i,t})^{1-\gamma j} \right)
\]

for \( n = 2, \ldots, N \) with \( P_{1H,t} = W_t / A_{1,t} \).

Due to flexible prices, the expressions for factor market in each stage of production are exactly the same as in the steady-state case. Therefore, we can derive labor demand function in each stage,
i.e., \( n = 1, \ldots, N, \) as

\[
L_{n,t}^{fd} = f(W_t, C_t, E_t,T_t, \overline{P}_{1F,t}^*, \ldots, \overline{P}_{NF,t}^*, \overline{P}_{nH,t}^*, \ldots, \overline{P}_{NH,t}^*, Y_{nH,t}^*, \ldots, Y_{NH,t}^*, A_1,t, \ldots, A_N,t)
\]

where we denote the labor demand under flexible prices with an upper symbol \( f. \)

The total labor demand function becomes

\[
L_{t}^{fd} = \sum_{n=1}^{N} L_{n,t}^{fd}
\]  

(C.4)

The three equations (3.1), (3.6), and (C.4) fully characterize the real wage, consumption and employment in the flexible-price equilibrium, where the consumption can be written in

\[
C_t^f = f(T_t, \overline{P}_{1F,t}^*, \ldots, \overline{P}_{NF,t}^*, \overline{P}_{nH,t}^*, \ldots, \overline{P}_{NH,t}^*, Y_{nH,t}^*, \ldots, Y_{NH,t}^*, A_1,t, \ldots, A_N,t)
\]

which can be re-written in log-linearized form, i.e.,

\[
c_t^f = f(t_t, \overline{P}_{1F,t}^*, \ldots, \overline{P}_{NF,t}^*, \overline{P}_{nH,t}^*, \ldots, \overline{P}_{NH,t}^*, Y_{nH,t}^*, \ldots, Y_{NH,t}^*, a_1,t, \ldots, a_N,t)
\]

By Euler equation (3.4), the IS curve is characterized by

\[
c_t^f = E_t(c_{t+1}^f) - \frac{1}{\sigma} [\tilde{r}_t - E_t(\pi_{t+1}) - \rho]
\]

which implies the natural rate of interest as

\[
\tilde{r}_t = \rho + \sigma E_t\{c_{t+1}^f - c_t^f\}
\]  

(C.5)

C.1.3 The Sticky-price Equilibrium

We now derive New Keynesian Phillips curves for each stage as a function of the relative price gap and the output gap, and characterize the equilibrium with sticky prices. Similar to the
derivation in Galí (2015), in each stage of production \( n = 1, 2, \ldots, N \), firms’ optimal pricing decision gives

\[
\pi_{n,t} = \beta E_t \pi_{n,t+1} + \lambda_n \hat{y}_{n,t}
\]

where \( \lambda_n = \frac{(1-\beta \alpha_n)(1-\alpha_n)}{\alpha_n} \) and \( \hat{y}_n \) is the log-deviation of real marginal cost from steady-state equilibrium, i.e.,

\[
\hat{y}_{n,t} = \ln(\Psi_{n,t}/P_{nH,t}) - \ln(\Psi_n/P_{nH})
\]

where \( \Psi_n \) and \( P_{nH} \) are the marginal cost and aggregate price in stage \( n \), respectively, in the steady-state equilibrium.

Given \( P_{nH,t} = \bar{P}_{nH,t} \) for \( n = 1, 2, \ldots, N \) and the production cost function, we have for stages \( n = 2, \ldots, N \)

\[
\hat{y}_{n,t} = \gamma \phi \hat{g}_{nH,t} + (1 - \gamma) \phi \hat{g}_{nF,t} + (1 - \phi)(\hat{w}_t - \hat{p}_{nH,t}) - a_{n,t}
\]

where \( \hat{g}_{nH,t} \) and \( \hat{g}_{nF,t} \) are the log-deviation of the relative output price gap with respect to input prices from the steady-state equilibrium, i.e., \( \hat{g}_{nH,t} = \ln(\bar{P}_{(n-1)H,t}/P_{nH,t}) - \ln(\bar{P}_{(n-1)H}/P_{nH}) \), and \( \hat{g}_{nF,t} = \ln(\bar{P}_{(n-1)F,t}/P_{nH,t}) - \ln(\bar{P}_{(n-1)F}/P_{nH}) \). Since \( \bar{P}_{(n-1)H,t} = P_{(n-1)H,t} \), \( \hat{g}_{nH,t} \) also indicates the log-deviation of the relative output price gap between adjacent stages \( n \) and \( n-1 \) from the steady-state equilibrium. By the definitions of \( \hat{g}_{nH,t} \) and \( \hat{g}_{nF,t} \), we also have \( \hat{p}_{nH,t} = \hat{p}_{NH,t} + \sum_{i=n+1}^{N} \hat{g}_{iH,t} \) for \( n = 1, \ldots, N - 1 \).

Following Huang and Liu (2005), without loss of generality, we assume \( \psi = 0 \). Then, from 3.3, we have \( w_t - p_t = \sigma c_t \). By substituting \( \hat{w} \) and \( \hat{p}_{nH,t} \) into the real marginal cost function, for \( n = 2, \ldots, N - 1 \), we obtain

\[
\hat{y}_{n,t} = \gamma \phi \hat{g}_{nH,t} + (1 - \gamma) \phi \hat{g}_{nF,t} + (1 - \phi)[\sigma \hat{c}_t - (1 - \gamma) \hat{p}_{NH,t} + (1 - \gamma) \hat{p}_{NF,t} - \sum_{i=n+1}^{N} \hat{g}_{iH,t}] - a_{n,t}
\]

and for \( n = 1 \) or \( N \)

\[
\hat{y}_{N,t} = \gamma \phi \hat{g}_{NH,t} + (1 - \gamma) \phi \hat{g}_{NF,t} + (1 - \phi)[\sigma \hat{c}_t - (1 - \gamma) \hat{p}_{NH,t} + (1 - \gamma) \hat{p}_{NF,t}] - a_{N,t}
\]
\[
\dot{y}_{1,t} = \sigma \dot{e}_t - (1 - \gamma) \dot{p}_{NH,t} + (1 - \gamma) \dot{p}_{NF,t} - \sum_{i=2}^{N} \dot{g}_{iH,t} - a_{1,t}
\]

Note that \( \dot{p}_{NF,t} = \dot{e}_t \) and \( \dot{g}_{nF,t} = \dot{e}_t - \dot{p}_{nH,t} \) for \( n = 1, \ldots, N \), and \( \dot{q}_t = \gamma(\dot{e}_t - \dot{p}_{NH,t}) \). We can simplify the above system of equations by plugging in \( \dot{p}_{nF,t}, \dot{g}_{nF,t} \), and replace \( \dot{e}_t \) with the real exchange rate \( \dot{q}_t \).

After log-linearizing the Euler equation of the household around the steady state and subtracting the natural rate IS curve, we obtain the IS curve with sticky prices as

\[
\dot{e}_t = E_t \dot{e}_{t+1} - \frac{1}{\sigma} [\dot{\pi}_t - E_t(\pi_{t+1})]
\]

The law of motion for the relative price gap between stages \( n \) and \( n-1 \), for \( n = 2, 3, \ldots, N \), is characterized by

\[
\dot{g}_{nH,t} = \dot{g}_{nH,t-1} + \pi_{(n-1)H,t} - \pi_{nH,t}
\]

Given the policy rule \( \{\dot{\pi}_t, \dot{e}_t\} \), the risk-sharing condition 3.7, the IS curve, the stage-specific Phillips curves, and the law of motion for the relative price gap fully pin down the dynamic equilibrium under sticky prices.

C.1.4 Stage-specific employment in a small-open economy with \( N \)-stage production

We derive the stage-specific employment gap in terms of output gap and relative price gap. By the factor demand function (3.8), (3.9), and (3.12) in each stage, and substituting with the unit cost, for \( n = 2, 3, \ldots, N \), we have

\[
\ln L_{n,t} = \ln(1 - \phi) + \phi[\ln \tilde{P}_{n,t} - \ln W_t] - \ln A_{n,t} + \ln [Y_{nH,t}^d + Y_{nH,t}^X] + d_{n,t} \tag{C.6}
\]

where \( d_{n,t} = \ln \left( \int_0^1 \left( \frac{P_{n,t}(u)}{P_{n,t}} \right)^{-\theta} du \right) \) and \( \ln L_{1,t} = -\ln A_{1,t} + \ln [Y_{1H,t}^d + Y_{1H,t}^X] + d_{1,t} \).

By the factor demand function for intermediate goods and labor in each stage, i.e., Expression
(3.8) and (3.9), for \( n = 2, 3, \ldots, N \), we obtain

\[
\ln L_{n,t} = \ln \left( \frac{1 - \phi}{\phi} \right) + \ln \tilde{P}_{n-1,t} - \ln W_t + \ln \tilde{Y}_{n,t}^d
\]

Also, note that

\[
Y_{(n-1)H,t}^d = \frac{\gamma \tilde{Y}_{n,t}^d \tilde{P}_{n,t}}{P_{(n-1)H,t}}
\]

Then, by substituting \( \ln Y_{nH,t}^d \) using \( \ln L_{n,t} \) and \( \ln \tilde{Y}_{(n+1),t}^d \) into (C.6), we obtain via backward induction the relationship for the stage-specific employment, i.e., for \( n = 2, 3, \ldots, N \),

\[
l_{n,t} = l_{n+1,t} + d_{n,t}
\]

\[
+ F_n(\hat{e}_t, e_t, t_t, a_{1,t}, \ldots, a_{N,t}, p_{1F,t}^*, \ldots, p_{NF,t}^*, p_{N_H,t}^*, \ldots, p_{NH,t}^*, y_{nH,t}^*, \ldots, y_{NH,t}^*, s_{1,t}, \ldots, s_{N,t})
\]

where \( l_{n,t} = \ln L_{n,t} \).

### C.2 Aggregate CPI Inflation with Two-stage Production

Given exogenous foreign variables and import tariff to be constant, the aggregate CPI inflation index can be written as

\[
\pi_t = \gamma \pi_{2H,t} + (1 - \gamma) \Delta e_t
\]

Since \( \hat{q}_t = \gamma (\hat{e}_t - \hat{p}_{2H,t}) \), we have

\[
\Delta e_t = \frac{\Delta \hat{q}_t}{\gamma} + \pi_{2H,t}
\]

Then, the aggregate CPI inflation can be re-written as

\[
\pi_t = \gamma \pi_{2H,t} + (1 - \gamma) \Delta e_t = \pi_{2H,t} + \frac{1 - \gamma}{\gamma} \Delta \hat{q}_t
\]
C.3 Stage-specific Employment with Two-stage Production

We derive an explicit expression for the employment gap with two stages of production, i.e., $N = 2$. As specified in Section C.1.4, and also note that $\hat{w}_t = \gamma \hat{p}_{2H,t} + (1 - \gamma) \hat{p}_{2F,t} + \sigma \hat{c}_t$, $\hat{p}_{1H,t} = \hat{g}_{2,t} + \hat{p}_{2H,t}$, and $\hat{p}_{1F,t} = \hat{p}_{2F,t} = \hat{c}_t$, we have for the second stage\footnote{We have imposed the assumption of $\psi = 0$.}

$$\hat{l}_{2,t} = \phi[\gamma \hat{p}_{1H,t} + (1 - \gamma) \hat{p}_{1F,t} - \hat{w}_t] + \bar{a}_2[\gamma \hat{p}_{2H,t} + (1 - \gamma) \hat{p}_{2F,t} - \hat{p}_{2H,t} + \hat{c}_t]$$

$$+ (1 - \bar{a}_2)(\hat{c}_t - \hat{p}_{2H}) - a_{2,t} + d_{2,t}$$

$$= \phi[\gamma \hat{g}_{2,t} - \sigma \hat{c}_t] + \bar{a}_2[(1 - \gamma)(\hat{c}_t - \hat{p}_{2H,t})] + (1 - \bar{a}_2)(\hat{c}_t - \hat{p}_{2H,t}) - a_{2,t} + d_{2,t}$$

$$= (\bar{a}_2 - \phi \sigma)\hat{c}_t + \phi \gamma \hat{g}_{2,t} + \frac{1 - \bar{a}_2 \gamma}{\gamma} \hat{q}_t - a_{2,t} + d_{2,t}$$

where, in the first equality, we have used $Y_{1H,t} = \frac{\gamma \hat{p}_{1H,t}}{\hat{p}_{1H,t}}$, $Y_{2H,t} = \frac{\gamma \hat{p}_{2H,t} \hat{E}_t}{\hat{p}_{2H,t}}$, $Y_{2F,t} = \frac{\gamma \hat{p}_{2F,t}}{\hat{p}_{2H,t}}$, and $Y_{2H,F} = \frac{\gamma \hat{p}_{2H,F} \hat{E}_t}{\hat{p}_{2H,F}}$; the last equality uses the condition that $\hat{q}_t = \gamma(\hat{c}_t - \hat{p}_{2H,t})$.

For the first stage, the employment is given by

$$\hat{l}_{1,t} = \bar{a}_1 (\hat{p}_{2,t} + \hat{g}_{2,t} - \hat{p}_{1H,t}) + (1 - \bar{a}_1)(\hat{c}_t - \hat{p}_{1H,t}) + d_{1,t} - a_{1,t}$$

$$= \bar{a}_1 (\hat{g}_{2,t} + \hat{w}_t - \hat{p}_{2,t} - \hat{p}_{1H,t}) + (1 - \bar{a}_1)(\hat{c}_t - \hat{p}_{1H,t}) + d_{1,t} - a_{1,t}$$

$$= \bar{a}_1 \hat{l}_{2,t} + \bar{a}_1 \gamma \hat{p}_{2H,t} + (1 - \gamma)\hat{c}_t + \sigma \hat{c}_t - \hat{g}_{2H,t} - \hat{p}_{2H,t}] + (1 - \bar{a}_1) [\hat{c}_t - \hat{p}_{2H,t} - \hat{g}_{2H,t}] + d_{1,t} - a_{1,t}$$

$$= [(1 - \bar{a}_1 \sigma) \phi + \bar{a}_1 \hat{l}_{2,t} + \bar{a}_1 \sigma + [\bar{a}_1 \phi \gamma - 1] \hat{g}_{2H,t}$$

$$+ \frac{1 - \bar{a}_1 \gamma - \bar{a}_1 \hat{a}_2 \hat{c}_t + \bar{a}_1 d_{2,t} + d_{1,t} - a_{1,t} - \bar{a}_1 a_{2,t}}{\gamma} \hat{q}_t$$

where the second equality uses the condition that $\hat{l}_{2,t} = \hat{g}_{2,t} - \hat{w}_t + \hat{p}_{2,t}$. 

222
C.4 The Steady-state Share of Domestic Demand in Total Demand as to Import Tariff

We characterize how the import tariff affects the steady-state share of the domestic demand in total demand in both production stages, i.e., $\tilde{a}_1$ and $\tilde{a}_2$, as specified in Section 3.3.

By the definition of $\tilde{a}_2$, we have

$$\frac{\tilde{a}_2}{1-\tilde{a}_2} = \frac{\gamma C P}{Y_2^* P_2^* E}$$

$$= \frac{\gamma C^* (P_i^*)^{1/\sigma}}{Y_2^* P_2^*} P^{1-1/\sigma} E^{1/\sigma}$$

$$= \frac{\gamma C^* (P_i^*)^{1/\sigma}}{Y_2^* P_2^*} \{[W^{1-\phi+\phi_E} (ET)^{1-\gamma} (P_f^*)^{1-\gamma} \phi] [{ETP_{2F}^*}]^{1-\gamma} \}^{1-1/\sigma} E^{1/\sigma}$$

where the second equality uses $C = C^* (\frac{EP^*}{F})^{1/\sigma}$ and the third equality uses the condition specified in Section 3.3.1. Since $W = EP^*(C^*)^\sigma$ under the assumption of $\psi = 0$, by plugging $W$ into the expression of $\tilde{a}_2/(1-\tilde{a}_2)$, the domestic price indices all cancel out (including the nominal exchange rate) and thus we have

$$\frac{\tilde{a}_2}{1-\tilde{a}_2} = f_2(\cdot) \cdot T^{(1-\gamma)(1+\phi)(1-\frac{1}{\sigma})}$$

where $f_2(\cdot)$ is a function of exogenous foreign variables.

For the first stage, we have

$$\frac{\tilde{a}_1}{1-\tilde{a}_1} = \frac{\gamma P_2 Y_2^d}{EP_1^* Y_1^*}$$

$$= \frac{\gamma \phi P_2 (Y_2^d + Y_2^d)}{EP_1^* Y_1^*}$$

$$= \frac{\gamma \phi}{P_1^* Y_1^*} \frac{P_2 Y_2^d}{EP_2^* Y_2^*}$$

$$= \frac{\gamma \phi}{EP_1^* Y_1^*} \frac{EP^*}{EP_2^* Y_2^*}$$

$$= \frac{\gamma \phi}{EP_1^* Y_1^*} \frac{EP^*}{EP_2^* Y_2^*} \tilde{a}_2$$

$$= \frac{\gamma \phi}{EP_1^* Y_1^*} \frac{EP^*}{EP_2^* Y_2^*} \tilde{a}_2$$
where we have used \( \frac{\tilde{a}_2}{1 - \tilde{a}_2} = \frac{\gamma_{CP}}{\gamma_{2H}^* p_{2H}^*} \) in the last equality. Therefore, for \( \tilde{a}_1 \), we have

\[
\frac{\tilde{a}_1}{1 - \tilde{a}_1} = f_2(*) \cdot \frac{1}{1 - \tilde{a}_2}
\]

where \( f_1(*) \) are functions of exogenous foreign variables.

### C.5 Heterogeneity in Openness across Stages

To explore the role of heterogeneity in openness across stages, we calibrate the export share in each stage for Canada by exploiting the input-output data from World Input-Output Database (WIOD). We choose the year of 2007 as the calibration target to avoid possible contamination from the Great Recession and the European sovereign debt crisis. The shares of exports along the production chain are set to be \( \tilde{a}_1 = 0.74 \) and \( \tilde{a}_2 = 0.88 \); other parameters are as specified in Table 3.1.\(^2\)

As shown in Policy Rule 1 of Table C.1, the relative weight on the stage-specific PPI for upstream production is 0.552 (\( = 5.4553/9.8780 \)).

Table C.1: Optimal alternative simple rules of monetary policy with heterogeneity in export share along production chain

<table>
<thead>
<tr>
<th>( \pi_{1H} )</th>
<th>( \pi_{2H} )</th>
<th>( \pi_{PPI} )</th>
<th>( \pi_{CPI} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 5.4553 9.8780</td>
<td>0.1189 -0.6283 0.1577 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2 5.1281 0.0001 0.0051 1.806</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P3 9.9997 0.1000 1.0690 1.033</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P4 9.9647 0.0022 0.0024 -0.4452 0.1478 1.005</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PPI index (sales-weighted): \( \pi_{PPI} = (1 - \omega)\pi_{1H} + \omega\pi_{2H} \) with \( \omega = \frac{p_{1j}(y_{1i} + \gamma_{1i}^*)}{p_{1j}(y_{1i} + \gamma_{1i}^*) + p_{2j}(y_{2i} + \gamma_{2i}^*)} \).

CPI index: \( \pi_{CPI,t} = \pi_1 \).

\(^2\)From the WIOD dataset, the export share in intermediate goods of Canada (including goods and services) is about 26% in 2007 and the export share in final demand is about 12%. It is worth noting that the export share in intermediate goods from WIOD is the ratio calculated through gross output of intermediate goods, which is lower than the corresponding ratio in value added term.
C.6  

\textit{N-stage Production in a Closed Economy}

We consider the case of \textit{N}-stage production in a closed-economy, and focus on the effects of lengthening of production chain on welfare loss function. We can similarly characterize the equilibrium (shown in Appendix C.6.3 - C.6.5) as in the case of the open-economy model with two stages of production. In the closed-economy model, since the distortion from monopolistic competition is assumed to be corrected by a subsidy tax, the only distortion in the economy comes from sticky price. Thus, the flexible-price equilibrium is Pareto optimal and we can write each variable in the deviation from the flexible-price equilibrium. The shocks considered in this section are stage-specific productivity shocks.

\subsection*{C.6.1 A Utility-based Welfare Loss Function for Optimal Monetary Policy}

Similarly to the derivation in Section 3.3.4, the households utility function is given by

\[
E \sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)]
\]

where \( U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma} \) and \( V(L_t) = \frac{L_t^{1+\phi}}{1+\phi} \).

A second-order Taylor expansion around steady state \((C, L)\) for the period utility of consumption gives

\[
U(C_t) - U = U_c C(\hat{c}_t + \frac{1-\sigma}{2} \hat{c}_t^2) + t.i.p.
\]

where \( \hat{c}_t \) denotes the log-deviation of consumption from steady state. To write the output gap in terms of the gap between the output with sticky-price and natural output (flexible-price equilibrium), the period utility of consumption can be re-written as

\[
U(C_t) - U = U_c C(\hat{c}_t - c_t^f + \frac{1-\sigma}{2} \hat{c}_t^2 + (1-\sigma)c_t^f \hat{c}_t) + t.i.p.
\]

where \( \hat{c}_t = c_t - c_t^f \) and \( c_t^f \) is the log-deviation of consumption in the flexible-price equilibrium from the steady-state equilibrium.

225
By labor market clearing condition, we obtain the second-order Taylor expansion around steady state for the period utility of employment, i.e., $V(L_t)$, as

$$V(L_t) - V = V_L L \left\{ \sum_{n=1}^{N} \frac{L_n}{L} \left[ \tilde{l}_{n,t} + \frac{1}{2} \tilde{l}_{n,t}^2 \right] \right\} + t.i.p.$$  

where $L_n/L$ is the share of labor demand in stage $n$ in total labor demand under steady state, given by equations (C.9) and (C.10) in Appendix C.6.3, and we have imposed the assumption of $\psi = 0$. More specifically, the stage-specific labor share under steady state is given by

$$\frac{L_n}{L} = (1 - \phi) \phi^{N-n}, \quad n = 2, 3, \ldots, N$$

$$\frac{L_1}{L} = \phi^{N-1}$$

The period utility of employment can then be re-written as the gap between labor demand with sticky-price and the log-deviation of labor demand with flexible prices in each stage, i.e.,

$$V(L_t) - V = V_L L \left\{ \sum_{n=1}^{N} \frac{L_n}{L} \left[ \tilde{l}_{n,t} + \frac{1}{2} \tilde{l}_{n,t}^2 + l^f_{n,t} \tilde{l}_{n,t} \right] \right\} + t.i.p \quad (C.7)$$

where $\tilde{l}_{n,t} = l_{n,t} - l^f_{n,t}$.

As shown in Appendix C.6.8, the stage-specific employment gap in terms of output gap and relative price gap for $n = 2, 3, \ldots, N-1$ are given by

$$\tilde{l}_{n,t} = \phi \left[ \sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t \right] + \tilde{l}_{n+1,t} - \left[ \sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t \right] + d_{n,t}$$

with

$$\tilde{l}_{N,t} = \phi \left[ \tilde{g}_{N,t} - \sigma \tilde{c}_t \right] + \tilde{c}_t + d_{N,t}$$

$$\tilde{l}_{1,t} = \tilde{l}_{2,t} - \left[ \sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t \right] + d_{1,t}$$

226
where \( d_{n,t} = \ln(\int_{0}^{1} \left( \frac{p_{n,t}(a)}{F_{n,t}} \right)^{-\theta} du) \) measures the price dispersion in stage \( n \). Details can be found in Appendix C.6.8.

For simplicity, we denote

\[
\tilde{I}_{n,t} = f_{n}(\tilde{g}_{n,t}, \ldots, \tilde{g}_{N,t}) + k(n)\tilde{c}_{t} + \sum_{i=n}^{N} d_{i,t}
\]

\[
\tilde{I}_{1,t} = f_{1}(\tilde{g}_{1,t}, \ldots, \tilde{g}_{N,t}) + k(1)\tilde{c}_{t} + \sum_{i=1}^{N} d_{i,t}
\]

where \( k(n) = (N - n)(1 - \phi)\sigma + 1 - \phi\sigma \) for \( n = 2, 3, \ldots, N \), and \( k(1) = (N - 1)(1 - \phi)\sigma + 1 \).

Then, by summing up \( U(C_{t}) - U \) and \( V(L_{t}) - V \) and also noting that the efficiency of steady state implies \( U_{c}C = V_{L}L \) in the closed-economy, the first order terms all cancel out, and only the second-order terms are left. The welfare loss function as a fraction of the steady-state consumption is thus given by

\[
W = E_{0} \sum_{t=0}^{\infty} \beta^{t} \frac{U(C_{t}) - V(L_{t}) - (U - V)}{U_{c}C}
\]

\[
= -\frac{1}{2} E_{0} \sum_{t=0}^{\infty} \beta^{t} \left\{ -(1 - \sigma)\tilde{c}_{t}^{2} + \sum_{n=1}^{N} \frac{L_{n}}{L} \left[ k(n)\tilde{c}_{t} + f_{n}(\tilde{g}_{n,t}, \ldots, \tilde{g}_{N,t}) \right]^{2} + \sum_{n=1}^{N} \theta \phi^{N-n} \lambda_{n}^{-1} \pi_{n,t}^{2} \right\} \quad (C.8)
\]

where \( \frac{L_{n}}{L} = (1 - \phi)\phi^{N-n} \) for \( n = 2, 3, \ldots, N \) and \( \frac{L_{1}}{L} = \phi^{N-1} \).

Appendix C.7 shows the welfare loss function for the case of \( N = 2 \) and \( N = 3 \) without abbreviation (i.e., expanding \( \frac{L_{n}}{L}, k(n), \) and \( f_{n}(\cdot) \) for \( n \)).

In general, monetary policy cannot attain a Pareto optimal allocation except in special cases with restrictions on productivity shocks. We proceed with the following proposition.

**Proposition 2:** In the closed-economy model with \( N \) stages of production and labor being used in each production stage (i.e., \( 0 < \phi < 1 \)), there is no monetary policy that can replicate flexible price equilibrium (Pareto-optimal allocation) unless the stage-specific productivity shocks satisfy

\[
\sum_{i=1}^{n-1} \phi^{n-i-1}(\phi - 1)\Delta a_{i,t} + \Delta a_{n,t} = 0 \text{ for } n = 2, \ldots, N \text{ and for all } t.
\]
C.6.2 Discussion about the Terms and Coefficients in Welfare Loss Function

There are three main parts in the welfare loss function: (a) output gap, and terms measuring stage-specific unemployment gap written in output gap, (b) the relative price gap, and (c) terms measuring stage-specific inflation. More specifically, as showed in the expression of welfare loss function (C.8), the coefficients before output gap \( \tilde{c}_t^2 \) and the stage-specific inflation, i.e., \( \pi_{n,t}^2 \) for \( \forall n \), are all positive.\(^3\) Therefore, similar to the standard welfare loss function (e.g., Rotemberg and Woodford, 1999; Woodford, 2003), the objective for a benevolent central bank still includes stabilizing output gap and inflation (i.e., the final stage inflation corresponding to typical “inflation” in the literature).

Besides the output gap and final-stage inflation, there are many more terms included in the welfare loss function, classified by those measuring stage-specific unemployment gaps and stage-specific inflation. It suggests that the central bank should not only care about the output gap and CPI inflation, but also pay attention to the variations in PPI inflation and the gaps of the real marginal cost in the production of intermediate goods.

Importantly, as shown in the expression of welfare loss function (C.8), by aggregating the terms of output gap \( \tilde{c}_t^2 \), the coefficient before output gap is \( \sum_{n=1}^{N} L_n k(n)^2 - (1 - \sigma) \), which is a function of the production structure, and changes with the number of total production stage \( N \). In contrast, the coefficient before CPI (i.e., the final stage inflation \( \pi_{N,t}^2 \)) is a constant \( -\frac{1}{\#} \). That is to say, even if the central bank follows the Taylor Rule, or the monetary rule suggested by Huang and Liu (2005), i.e., targeting both CPI and PPI, the optimal weights before output gap and CPI (or PPI) change with the production structure of the economy.

The welfare loss function with multi-stage production indicates that targeting both CPI and PPI are not satisfactory. Instead, the central bank needs to pay attention to all stage-specific inflation along production process, especially in the case of lengthening production chain. Those terms measuring stage-specific inflation, i.e., \( \sum_{n=1}^{N} \theta \phi^{N-n} \lambda_n^{-1} \pi_{n,t}^2 \), in welfare loss function have two important implications. On the one hand, given the total number of production stages \( N \) and the price

\(^3\)Details about the proof can be found in Appendix C.8.
stickiness being the same across different stages, the coefficients before the inflation in down-
stream stages are larger compared with those in upstream stages. On the other hand, as the number
of total stages $N$ increases, there are more terms of upstream inflation included in the welfare loss
function, while the terms for downstream inflation do not change. In the latter case, the relative
importance of final stage inflation (i.e., CPI) in welfare loss function becomes smaller, while the
inflation in upstream stages becomes relatively more important. That is to say, as the production
length becomes longer, the central bank needs to care more about inflation in intermediate stages
but less on the final stage inflation (i.e., CPI).

In practice, if the stage-specific inflation cannot be attained, PPI, as a sales-weighted price
index for intermediate goods across all stages, can be a rough proxy. On the other hand, when
a PPI index is available, the information used to construct the PPI index is likely to be sufficient
to construct the stage-specific inflation. For instance, the PPI program of US Bureau of Labor
Statistics not only constructs an aggregate PPI index, but also constructs stage-specific inflation
indices in a four-stage vertical production framework with the same underlying data.\footnote{4} Their idea
in constructing this system of indices is to choose the total number of stages and assign industries
to stages of production in such a manner that simultaneously maximizes the forward goods flows
along the vertical chain while minimizing backward flows and internal goods flows within the
system.

C.6.3 The Steady-state Equilibrium

We first characterize the steady state with perfect foresight. We drop the time subscript $t$ for
all variables, and set $A_n = 1$ for $n = 1, 2, \ldots, N$. The optimal pricing decision for firms at stage $n$,$n = 1, 2, \ldots, N$, becomes

$$P^*_n = \Psi_n$$

\footnote{4Details for the stage-specific inflation indices constructed by US Bureau of Labor Statistics can be found at
https://www.bls.gov/ppi/fdidsummary.htm, or Weinhagen (2011).}
By aggregate price expression (3.13), in the steady state, we have

\[ P_n = P^*_n = \Psi_n \]

Now, we solve for the price indices in terms of the wages and derive the labor demand function. Note that \( P_n = \tilde{P}_{n+1} \) for \( n = 1, 2, \ldots, N - 1 \), and \( \Gamma_n = \tilde{P}_n^\phi W^{1-\phi} \) for \( n = 2, 3, \ldots, N \) with \( \Gamma_1 = W \). By substituting \( \tilde{P}_n \), the relationship of price index between adjacent stages is given by

\[ P_n = W^{1-\phi} (P_{n-1})^\phi \]

for \( n = 2, \ldots, N \) and \( P_1 = W \).

By rewriting all price indices in terms of wages, it comes

\[ P_n = W^{1-\phi^n} (P_1)^{\phi^{n-1}} \]

\[ = W \]

for \( n = 2, \ldots, N \) with \( P_1 = W \).

Since \( P_n(u) = P_n \) for \( u \in [0, 1] \) in the steady state, we have

\[ Y^d_{n-1,t}(u) = \tilde{Y}^d_{n,t} \]

Together with the goods markets clearing condition \( Y_{n,t} = \tilde{Y}^d_{n+1,t} \), and factor market demand function (3.9) and (3.8), for \( n = 2, \ldots, N - 1 \), we obtain

\[ \tilde{Y}^d_n = \phi \frac{\Gamma_n}{P_n} \tilde{Y}^d_{n+1} \]

\[ L^d_{n} = (1 - \phi) \frac{\Gamma_n}{W} \tilde{Y}^d_{n+1} \]

where \( Y_N = C, \tilde{Y}_N = \phi \frac{\Gamma_N}{P_N} C, L^d_N = (1 - \phi) \frac{\Gamma_N}{W} C \), and \( L^d_1 = \frac{\Gamma_1}{W} \tilde{Y}^d_2 \). By substituting the price index
and unit cost function in each stage, for \( n = 2, \ldots, N \), the factor demand functions for both labor and composite intermediate goods are given by

\[
\tilde{Y}_n^d = \phi^{N+1-n} C
\]

\[
L_n^d = (1 - \phi) \phi^{N-n} C
\]  

(C.9)

with \( L_1^d = \tilde{Y}_2^d \).

By summing up the labor demand across all stages, the total labor demand function becomes

\[
L^d = \sum_{n=2}^{N} [(1 - \phi) \phi^{N-n} C] + \phi^{N-1} C
\]

(C.10)

With the labor supply function (3.1) together with the price index, the labor supply in the steady state becomes

\[
L^\psi C^\alpha = 1
\]

(C.11)

Given \( L^d = L \), the two equations (C.10) and (C.11) fully characterize the steady-state total consumption and total employment.

C.6.4 The Flexible-price Equilibrium

In order to obtain efficient allocation in the model economy, i.e., the natural rate of the output, we solve for the flexible-price equilibrium in a similar way as in the steady state. In the flexible price equilibrium, \( \alpha_n = 0 \) for \( \forall n \), and the optimal pricing decision for firms at stage \( n \),

---

5The labor demand in each stage can be derived via backward deduction (which is helpful when taking log-linearization), i.e.,

\[
L_n^d = \phi L_{n+1}^d, \quad n = 2, \ldots, N
\]

\[
L_1^d = \frac{\phi}{1 - \phi} L_2^d
\]
\[ n = 1, 2, \ldots, N, \text{ becomes} \]
\[ P^*_{n,t} = \Gamma_{n,t} \]

By the aggregate price expression (3.13), we have
\[ P_{n,t} = P^*_{n,t} = \Gamma_{n,t} \]

Similar to the steady-state case, we solve for the price indices in terms of wages and productivity. Note that \( P_{n,t} = \tilde{P}_{n+1,t} \) for \( n = 1, 2, \ldots, N - 1 \), and \( \Gamma_{n,t} = \tilde{P}_{n,t} W_t^{1-\phi} / A_{n,t} \) for \( n = 2, 3, \ldots, N \) with \( \Gamma_1 = W_t / A_{1,t} \). By substituting \( \tilde{P}_{n,t} \), the relationship of price index across adjacent stages is given by
\[ P_{n,t} = W_t^{1-\phi} (P_{n-1,t})^\phi / A_{n,t} \]
for \( n = 2, \ldots, N \) and \( P_1 = W / A_{1,t} \).

By writing all price indices in terms of wage, we obtain
\[ P_{n,t} = W_t^{1-\phi} \prod_{g=2}^{n} A_{g,t}^{-\phi^{n-g}} \]
\[ = W \Pi_{g=1}^{n} A_{g,t}^{-\phi^{n-g}} \quad \text{(C.12)} \]
for \( n = 2, \ldots, N \).

Similar to the derivation for the steady-state case, the labor demand function in each stage in the flexible-price equilibrium is given by
\[ L^d_{n,t} = (1 - \phi) \phi^{N-n} \Pi_{g=1}^{N} A_{g,t}^{-\phi^{N-g}} C_t \quad \text{(C.13)} \]
for \( n = 2, 3, \ldots, N \) with \( L^d_{1,t} = \frac{\phi}{1-\phi} L^d_{2,t} \). Details can be found in Appendix C.6.6.
Therefore, the total labor demand is given by

\[ L_d^t = \eta \Pi_{n=1}^N \phi^{N-n} C_t \]  
(\text{C.14})

where \( \eta \) is a constant, and it is given by

\[ \eta = \sum_{n=2}^N [(1 - \phi) \phi^{N-n} C] + \phi^{N-1} \]

By the labor supply function (3.1) together with the price index, we know

\[ L^\psi C^\sigma = \Pi_{n=1}^N A_{n,t}^{\phi^{N-n}} \]  
(\text{C.15})

After taking log-deviation from the steady state for both the total labor demand (C.14) and the labor supply (C.15), we get

\[ l_f^t = c_f^t - \left[ \sum_{n=1}^N \phi^{N-n} a_{n,t} \right] \]  
(\text{C.16})

and

\[ \psi l_f^t + \sigma c_f^t = \sum_{n=1}^N \phi^{N-n} a_{n,t} \]  
(\text{C.17})

where the variables in lower case with an upper symbol \( f \) denote the log-deviation of flexible price equilibrium from the steady state.

Therefore, the log-deviation of output from the steady state under flexible prices \( c_f^t \) is given by

\[ c_f^t = \frac{1 + \psi}{\psi + \sigma} \left[ \sum_{n=1}^N \phi^{N-n} a_{n,t} \right] \]

By the Euler equation (3.4), the IS curve is characterized by

\[ c_f^t = E_t(c_{f+1}^t) - \frac{1}{\sigma} [\dot{r}_t - E_t(\pi_{N,t+1})] \]
which yields the natural rate of interest as

$$\tilde{r}_t = \hat{i}_t - E_t(\pi_{N,t+1}) + \rho$$

$$= \rho + \sigma E_t\{c^f_{t+1} - c^f_t\}$$

Given the expression of output and the process of productivity shocks, we have

$$\tilde{r}_t = \rho + \frac{\sigma(1 + \psi)}{\psi + \sigma} E_t[\sum_{n=1}^{N} \phi^{N-n} \Delta a_{n,t+1}]$$  \hspace{1cm} (C.18)

where $\Delta a_{n,t} = a_{n,t} - a_{n,t-1}$ is the growth rate of productivity in stage $n$.

### C.6.5 The Sticky-price Equilibrium

We now derive New Keynesian Phillips curves for each stage as a function of relative price gap and output gap, and characterize the equilibrium with sticky prices. Similar to the derivation in Galí (2015), in each stage of production $n = 1, 2, \ldots, N$, firms’ optimal pricing decision gives

$$\pi_{n,t} = \beta E_t \pi_{n,t+1} + \lambda_n \tilde{\gamma}_{n,t}$$

where $\lambda_n = \frac{(1-\beta a_n)(1-a_n)}{a_n}$ and $\tilde{\gamma}_n$ is the log-derivation of the real marginal cost from the flexible-price equilibrium, i.e.,

$$\tilde{\gamma}_{n,t} = ln(\Gamma_{n,t}/P_{n,t}) - ln(\Gamma^f_{n,t}/P^f_{n,t})$$

where $\Gamma^*_n$ and $P^*_n$ are, respectively, the marginal cost and aggregate price in stage $n$ in the flexible-price equilibrium.

Following Huang and Liu (2005), without a loss of generality, we assume that $\psi = 0$. Together with the labor supply function (C.15), for $n = 2, 3, \ldots, N-1$, the log-deviation of the real marginal
cost can be written as a function of relative price gap and output gap, i.e.,

\[ \tilde{W}_{C} = \tilde{W}_{C} + (1 - \tilde{q}) \tilde{c}_t \]

where \( \tilde{W}_{C} \) is the relative price gap between stage \( n \) and stage \( n - 1 \), i.e., \( \tilde{W}_{C} = \ln \left( \frac{p_{n-i,t}}{p_{n,t}} \right) - \ln \left( \frac{p_{n-i,t}}{p_{n,t}} \right) \).

Details for Expression (C.19) can be found in Appendix C.6.7.

After log-linearizing the Euler equation around the steady state and subtracting the natural rate IS curve, we obtain the IS curve with sticky prices as

\[ \tilde{c}_t = E_t \tilde{c}_{t+1} - \frac{1}{\sigma} [i_t - E_t(\pi_{N,t+1}) - r_t] \]

where \( r_t \) is the natural rate of interest.

The law of motion for the relative price gap between stage \( n \) and stage \( n - 1 \), for \( n = 2, 3, \ldots, N \), is characterized by

\[ \tilde{g}_{n,t} = \tilde{g}_{n,t-1} + \pi_{n-1,t} - \pi_{n,t} - \Delta g_{n,t}^f \]

where \( \Delta g_{n,t}^f = g_{n,t}^f - g_{n,t-1}^f \). By Equation (C.12), we have

\[ \Delta g_{n,t}^f = \sum_{i=1}^{n-1} \phi^{n-i-1}(\phi - 1) \Delta a_{i,t} + \Delta a_{n,t} \]

Give the monetary policy rule, the Phillips curve, IS curve, and the law of motion for the relative price gap fully pin down the dynamic equilibrium under sticky prices.
C.6.6 Labor Demand Function in the Flexible-price Equilibrium

Similar to the steady-state equilibrium, we derive the labor demand function in the flexible-price equilibrium. Note that, with flexible prices, 

$$\frac{P_n}{P_{n+1}} = \left(\frac{D}{D+1}\right)^{\gamma_{n+1}}$$

for \(0 \leq D \leq 1\), and we then obtain

$$Y_{n-1,t}(u) = \bar{Y}_{n,t}^d$$

Together with goods markets clearing condition \(Y_{n,t} = \bar{Y}_{n+1,t}^d\), and factor market demand function (3.9) and (3.8), for \(n = 2, \ldots, N - 1\), we obtain

$$\bar{Y}_{n,t}^d = \phi \frac{\Gamma_{n,t}}{P_{n,t}} \bar{Y}_{n+1,t}^d$$

$$L_{n,t}^d = (1 - \phi) \frac{\Gamma_{n,t}}{W_t} \bar{Y}_{n+1,t}^d$$

where \(Y_{N,t} = C_t, \bar{Y}_{N,t} = \phi \frac{\Gamma_{N,t}}{P_N} C_t, L_{N,t}^d = (1 - \phi) \frac{\Gamma_{N,t}}{W_t} C_t\), and \(L_{1,t}^d = \frac{\Gamma_{1,t}}{W_t} \bar{Y}_{2,t}^d\).

Note that \(P_{n,t} = \bar{P}_{n+1,t}^d\) for \(n = 1, 2, \ldots, N - 1\), and \(\Gamma_{n,t} = \bar{P}_{n,t}^d W_t^{1-\phi} / A_{n,t}\) for \(n = 2, 3, \ldots, N\) with \(\Gamma_1 = W_t / A_{1,t}\). By substituting the unit cost function in each stage, for \(n = 2, \ldots, N\), we obtain the labor demand in each stage as follows:

$$\bar{Y}_{n,t}^d = \phi \frac{W_t}{1 - \phi} P_{n-1,t} L_{n,t}^d$$

and thus

$$L_{n,t}^d = \phi \frac{P_{n-1,t}}{P_{n,t}} \left(\frac{W_t}{P_{n-1,t}}\right)^{1-\phi} A_{n,t}^{-1} L_{n+1,t}^d$$

$$L_{1,t}^d = \frac{1}{A_{1,t}} \frac{\phi}{1 - \phi} W_t L_{2,t}^d$$

One can derive the labor demand in each stage via backward induction (which is helpful when taking log-linearization), i.e.,

$$L_{n,t}^d = \phi L_{n+1,t}^d, \ n = 2, \ldots, N$$

236
\[ L_{1,t}^d = \frac{\phi}{1 - \phi} L_{2,t}^d \]

Note that \( L_{N,t}^d = (1 - \phi) \frac{\Gamma_{N,t}}{W_t} C_t \), which indicates

\[ L_{N,t}^d = (1 - \phi) \Pi_{g=1}^N A_{g,t}^{-\phi^{N-g}} C_t \]

Therefore, for \( n = 2, 3, \ldots, N \), we obtain the labor demand function in each stage as

\[ L_{n,t}^d = (1 - \phi) \phi^{N-n} \Pi_{g=1}^N A_{g,t}^{-\phi^{N-g}} C_t \]

with \( L_{1,t}^d = \frac{\phi}{1 - \phi} L_{2,t}^d \).

C.6.7 The Log-deviation of the Real Marginal Cost from the Flexible-price equilibrium

Note that, for \( n = 2, 3, \ldots, N \), \( \Gamma_{n,t} = \bar{P}_{n,t} W_t^{1-\phi} / A_{n,t} \) and \( \bar{P}_{n,t} = P_{n-1,t} \). The log-deviation of the real marginal cost is given by

\[ \tilde{\gamma}_{n,t} = \ln(\Gamma_{n,t}/P_{n,t}) - \ln(\Gamma_{n,t}^*/P_{n,t}^*) \]

\[ = \phi [\ln(P_{n-1,t}/P_{n,t}) - \ln(P_{n-1,t}^*/P_{n,t}^*)] + (1 - \phi) [\ln(W_t/P_{n,t}) - \ln(W_t^f/P_{n,t}^f)] \]

Denote \( g_{n,t} = \ln(P_{n-1,t}/P_{n,t}) \) and \( \tilde{g}_{n,t} = \ln(P_{n-1,t}/P_{n,t}) - g_{n,t}^f \). For \( n = 1, 2, \ldots, N - 1 \), we have

\[ \ln P_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + \ln P_{N,t} \]

\[ \iff p_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + P_{N,t} \]

Also, by the labor supply equation (C.15), by assuming \( \psi = 0 \), we have

\[ w_t - p_{N,t} = \sigma c_t \]
Therefore, for \( n = 2, 3, \ldots N - 1 \), the log-deviation of real marginal cost can be written as

\[
\tilde{\gamma}_{n,t} = \phi \tilde{g}_{n,t} + (1 - \phi) [\tilde{w}_t - \tilde{p}_{n,t}]
\]

\[
= \phi \tilde{g}_{n,t} + (1 - \phi) [\sigma \tilde{c}_t + \tilde{p}_N - \tilde{p}_{n,t}]
\]

\[
= \phi \tilde{g}_{n,t} + (1 - \phi) [\sigma \tilde{c}_t - \sum_{i=n+1}^{N} \tilde{g}_{i,t}]
\]

with \( \tilde{\gamma}_{N,t} = \phi \tilde{g}_{N,t} + (1 - \phi) \sigma \tilde{c}_t \).

Similarly, for the first stage \( n = 1 \), since \( \Gamma_1 = W_t / A_{1,t} \), we have

\[
\tilde{\gamma}_{1,t} = \tilde{w}_t - \tilde{p}_{1,t}
\]

\[
= \sigma \tilde{c}_t - \sum_{i=2}^{N} \tilde{g}_{i,t}
\]

C.6.8 Stage-specific Employment Gaps with \( N \)-stage Production

We derive the stage-specific employment gap in terms of output gap and relative price gap. By the factor demand function (3.8), (3.9), and (3.12) in each stage, and substituting with the unit cost, for \( n = 2, 3, \ldots , N \), we have

\[
\ln L_{n,t} = \ln(1 - \phi) + \phi [\ln P_{n-1,t} - \ln W_t] - \ln A_{n,t} + \ln \tilde{Y}_{n+1,t}^d + d_{n,t}
\]

\[
\iff \quad l_{n,t} = \ln(1 - \phi) + \phi [p_{n-1,t} - w_t] - a_{n,t} + ln \tilde{Y}_{n+1,t}^d + d_{n,t}
\]

where \( d_{n,t} = \ln \left( \int_0^1 \left( \frac{P_{n,t}(u)}{P_{n,t}} \right)^{-\theta} \, du \right) \) and \( l_{1,t} = -a_{1,t} + \ln \tilde{Y}_{2,t}^d + d_{1,t} \).

By the factor demand function for intermediate goods and labor in each stage, i.e., Expression (3.8) and (3.9), for \( n = 2, 3, \ldots , N \), we get

\[
l_{n,t} = \ln \left( \frac{1 - \phi}{\phi} \right) + p_{n-1,t} - w_t + \ln \tilde{Y}_{n,t}^d
\]
Note that \( \tilde{Y}_{N,t} = C_t \). Then, by substituting \( \ln(\tilde{Y}_{N,t}) \), we obtain the relationship for the stage-specific employment, i.e., for the stage of \( n = N \), via backward induction as

\[
l_{N,t} = \ln(1 - \phi) + \phi [p_{N-1,t} - w_t] - a_{N,t} + c_t + d_{N,t}
\]

for \( n = 2, 3, \ldots, N - 1 \),

\[
l_{n,t} = \ln(\phi) + \phi [p_{n-1,t} - w_t] - a_{n,t} + l_{n+1,t} - [p_{n,t} - w_t] + d_{n,t}
\]

for \( n = 1 \),

\[
l_{1,t} = -a_{1,t} + l_{2,t} - [p_{1,t} - w_t] + d_{1,t}
\]

As shown in Appendix C.6.7, for \( n = 1, 2, \ldots, N \), we have \( p_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + p_{N,t} \), and, by assuming \( \psi = 0 \), \( w_t - p_{N,t} = \sigma c_t \). The stage-specific employment can be written in terms of relative price and output as, for \( n = N \),

\[
l_{N,t} = \ln(1 - \phi) + \phi [g_{N,t} - \sigma c_t] - a_{N,t} + c_t + d_{N,t}
\]

for \( n = 2, 3, \ldots, N - 1 \),

\[
l_{n,t} = \ln(\phi) + \phi \left[ \sum_{i=n}^{N} g_{i,t} - \sigma c_t \right] - a_{n,t} + l_{n+1,t} - \left[ \sum_{i=n+1}^{N} g_{i,t} - \sigma c_t \right] + d_{n,t}
\]

for \( n = 1 \),

\[
l_{1,t} = -a_{1,t} + l_{2,t} - \left[ \sum_{i=2}^{N} g_{i,t} - \sigma c_t \right] + d_{1,t}
\]

By subtracting the corresponding equations for the flexible-price equilibrium, the stage-specific employment gap in terms of output gap and the relative price gap is given by, for \( n = 2, 3, \ldots, N - 1 \),

\[
\tilde{l}_{n,t} = \phi \left[ \sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t \right] + \tilde{l}_{n+1,t} - \left[ \sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t \right] + d_{n,t}
\]
with
\[ \bar{I}_{N,t} = \phi [\tilde{g}_{N,t} - \sigma \tilde{c}_t] + \tilde{c}_t + d_{N,t} \]
\[ \bar{I}_{1,t} = \bar{I}_{2,t} - \sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + d_{1,t} \]

Therefore, by forward induction, the stage-specific employment gap is given by, for \( n = 2, 3, \ldots, N - 1, \)
\[ \bar{I}_{n,t} = \phi [\sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + \bar{I}_{n+1,t} - \sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + d_{n,t} \]

with
\[ \bar{I}_{N,t} = \phi [\tilde{g}_{N,t} - \sigma \tilde{c}_t] + \tilde{c}_t + d_{N,t} \]
\[ \bar{I}_{1,t} = \bar{I}_{2,t} - \sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + d_{1,t} \]

C.7 The Closed-form Welfare Loss Function for the Cases of \( N = 2 \) and \( N = 3 \) in a Closed Economy

To illustrate the welfare loss function in the closed economy, we show the analytical welfare loss function for the cases of \( N = 2 \) and \( N = 3 \) without abbreviation. For the case of \( N = 2 \), by Appendix C.6.8, the stage-specific employment gap in terms of output gap and relative price gap is given by
\[ \bar{I}_{1,t} = (1 + \sigma - \sigma \phi) \tilde{c}_t + (\phi - 1) \tilde{g}_{2,t} + d_{1,t} + d_{2,t} \]
\[ \bar{I}_{2,t} = (1 - \sigma \phi) \tilde{c}_t + \phi \tilde{g}_{2,t} + d_{2,t} \]

Since \( \frac{L_1}{L} = \phi \) and \( \frac{L_2}{L} = 1 - \phi \), by plugging into Equation (C.8), the welfare loss function with \( N = 2 \) is given by
\[ W = -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \{ \sigma \tilde{c}_t^2 + \phi (1 - \phi) [\sigma \tilde{c}_t - \tilde{g}_{2,t}]^2 + \theta \lambda_2^{-1} \pi_{2,t}^2 + \theta \phi \lambda_1^{-1} \pi_{1,t}^2 \} \]

which is exactly the same as in Huang and Liu (2005).
Similarly, for the case of $N = 3$, the stage-specific employment gap in terms of output gap and relative price gap is given by

$$\tilde{I}_{1,t} = (1 + 2\sigma - 2\sigma \phi)\hat{c}_t + 2(\phi - 1)\tilde{g}_{3,t} + (\phi - 1)\tilde{g}_{2,t} + d_{1,t} + d_{2,t} + d_{3,t}$$

$$\tilde{I}_{2,t} = (1 + \sigma - 2\sigma \phi)\hat{c}_t + (2\phi - 1)\tilde{g}_{3,t} + \phi\tilde{g}_{2,t} + d_{2,t} + d_{3,t}$$

$$\tilde{I}_{1,t} = (1 - \sigma \phi)\hat{c}_t + \phi\tilde{g}_{3,t} + d_{3,t}$$

Since $\frac{L_1}{L} = \phi^2$, $\frac{L_2}{L} = \phi(1 - \phi)$ and $\frac{L_3}{L} = 1 - \phi$, by plugging into Equation (C.8), the welfare loss function with $N = 3$ is given by

$$W = -\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \{- (1 - \sigma)\hat{c}_t^2 + \phi^2 [(1 + 2\sigma - 2\sigma \phi)\hat{c}_t + 2(\phi - 1)\tilde{g}_{3,t} + (\phi - 1)\tilde{g}_{2,t}]^2$$

$$+ (1 - \phi)[(1 + \sigma - 2\sigma \phi)\hat{c}_t + (2\phi - 1)\tilde{g}_{3,t} + \phi\tilde{g}_{2,t}]^2$$

$$+ (1 - \phi)[(1 - \sigma \phi)\hat{c}_t + \phi\tilde{g}_{3,t}]^2$$

$$+ \theta \lambda_3^{-1}\pi_{3,t}^2 + \theta \phi \lambda_2^{-1}\pi_{2,t}^2 + \theta \phi^2 \lambda_3^{-1}\pi_{3,t}^2 \}$$

C.8 The Proof for a Positive Coefficient of the Output Gap in the Welfare Loss Function in the Closed Economy

The coefficient of output gap $\hat{c}_t^2$ in the welfare loss function (C.8) is given by

$$-(1 - \sigma) + \sum_{n=1}^{N} \frac{L^n}{L} k(n)^2 \equiv f$$

Note that the stage-specific labor share under the efficient steady state yields

$$\frac{L^n}{L} = (1 - \phi)\phi^{N-n}, \ n = 2, 3, \ldots, N$$
\[ \frac{L_1}{L} = \phi^{N-1} \]

and \( k(n) = (N-n)(1-\phi)\sigma + 1 - \phi \sigma \) for \( n = 2, 3, \ldots, N \), and \( k(1) = (N-1)(1-\phi)\sigma + 1 \). If \( \sigma > 1 \), then obviously \( f > 0 \); otherwise, since \( \phi < 1 \) and \( \sigma \leq 1 \), it is obvious that \( k(n) \geq 1 - \phi \sigma > 0 \) for \( \forall n \).

Therefore, in this case, we have

\[
f = -(1 - \sigma) + \sum_{n=1}^{N} \frac{L_n}{L} k(n)^2 \geq -(1 - \sigma) + \sum_{n=1}^{N} (1 - \phi) \frac{L_n}{L} k(n) = (1 - \phi) - (1 - \sigma) = \sigma (1 - \phi) > 0
\]

In other words, the coefficient on the output gap in the welfare loss function is always positive.

C.9 Trade Balance and Optimal Simple Monetary Policy Rules

Instead of imposing the risk-sharing condition as specified in Section 3.2, we now assume that the households have no access to the international asset market (i.e., they live in financial autarky). By construction, goods trade has to be balanced (and the risk-sharing condition no longer holds). Under this assumption, the aggregate expenditure must be equal to the aggregate income, i.e., \( W_t L_t = P_t C_t \).

The balanced trade condition in the steady state also requires the value of exports to equal that of imports, i.e., \( 1 = \frac{\gamma}{\bar{a}_2} (1 - \phi) + \phi \frac{\gamma^2}{\bar{a}_2 \bar{a}_1} \). By replacing the risk-sharing condition with balanced trade, we estimate the general \textit{nonlinear} model (with \( N = 2 \)) and approximate the equilibrium by a second-order expansion. The shares of goods sold in the domestic markets in the two stages are set to be \( \bar{a}_1 = \bar{a}_2 = 0.6 \) in order to satisfy the balanced trade condition. All other parameters are
the same as in Table 3.1.\footnote{Under the assumption of a balanced trade, a shock on foreign consumption is inconsequential for the domestic economy.}

Table C.2: Optimal alternative simple rules of monetary policy under trade balance

<table>
<thead>
<tr>
<th>Rule</th>
<th>(\pi_{1H})</th>
<th>(\pi_{2H})</th>
<th>(\pi_{PP})</th>
<th>(\pi_{CP})</th>
<th>(\hat{c})</th>
<th>(\hat{q})</th>
<th>(\hat{i}_{t-1})</th>
<th>Welfare loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>3.0339</td>
<td>5.0303</td>
<td>2.1428</td>
<td>-3.4481</td>
<td>0.2182</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P2</td>
<td>4.9154</td>
<td>0.0000</td>
<td>0.0001</td>
<td>1.0215</td>
<td>0.7661</td>
<td>1.040</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td>9.9707</td>
<td>0.1012</td>
<td>0.0000</td>
<td>1.055</td>
<td>1.040</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>9.9997</td>
<td>0.0001</td>
<td>0.0000</td>
<td>1.040</td>
<td>0.7661</td>
<td>1.040</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P5</td>
<td>6.0028</td>
<td>0.1431</td>
<td>2.0563</td>
<td>-3.3121</td>
<td>0.1726</td>
<td>1.011</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P6</td>
<td>5.5358</td>
<td>9.9393</td>
<td>0.0021</td>
<td>0.5837</td>
<td>1.021</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P7</td>
<td>5.5339</td>
<td>0.0002</td>
<td>0.7914</td>
<td>1.767</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P8</td>
<td>2.9431</td>
<td>0.0012</td>
<td>0.7929</td>
<td>1.301</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P9</td>
<td>5.5710</td>
<td>9.9870</td>
<td>0.5809</td>
<td>1.021</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peg</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.148</td>
</tr>
</tbody>
</table>

Notes: PPI index (sales-weighted): \(\pi_{PP} = (1 - \omega)\pi_{1H} + \omega\pi_{2H}\) with \(\omega = \frac{\Pi_{1H}(\gamma_{1H} + \gamma_{1H}^2)}{\Pi_{1H}(\gamma_{1H} + \gamma_{1H}^2) + \Pi_{2H}(\gamma_{2H} + \gamma_{2H}^2)}\).

CPI index: \(\pi_{CP, t} = \pi_t\).

With this new structure of the model, we re-estimate the optimal weights for each of the simple monetary policy rule discussed in the main text, and compute the associated welfare loss (relative to the best simple rule). Table C.2 presents the result.

By construction, Policy Rule 1 that targets the producer price inflation in all stages of production, plus the output gap and the real exchange rate, is the best rule among the family of simple rules. Given the differences in the model structure, it is not surprising that the estimated optimal weights on various variables and the numerical values of the welfare losses for the policy rules are different from those in Section 3.4.2. However, it is noteworthy that the relative welfare ordering of the simple rules is the same as before. In particular, the conventional Taylor rule (Policy Rule 2) that targets only the CPI inflation and output gap is associated with a sizable additional welfare loss, even with optimally estimated weights on the targeting variables, when compared with the best simple rule. An exchange rate peg (Policy Rule 10) produces the worst outcome among the ten policy rules considered.

Rules that allow for targeting both stage-specific producer inflation rates (Policy Rules 6 and 9) or both PPI and CPI inflation rates (Policy Rules 4 and 5) do substantially better than either the
conventional Taylor rule or the exchange rate peg, even if one forgoes the real exchange rate or even the output gap.

The fundamental intuition for these relative welfare rankings is that, with sticky prices, producer price inflation in each stage of production leads to resource misallocation. Thus, a good monetary policy rule should take into account producer price inflation in all stages of production. This intuition appears to be robust whether we use a balanced trade condition or a risk-sharing condition.

C.10 Comparative Statics: Distortions from the Stage-specific Price Stickiness and the Elasticity of Substitution

If there are different degrees of price stickiness in different stages of production, which one matters more? To shed light on this question, we consider two extreme cases: (i) let the upstream prices be fully flexible (while maintaining the Calvo parameter for the downstream sector at the baseline value), i.e., \( \alpha_1 = 0 \) and \( \alpha_2 = 0.66 \); and (ii) let the downstream prices be fully flexible (while keeping the upstream sector Calvo parameter at the baseline value), i.e., \( \alpha_1 = 0.66 \) and \( \alpha_2 = 0 \). All other parameters are the same as in Section 3.4.1. We examine how our results vary with respect to different degrees of openness. We maintain the standard Taylor rule in these exercises.

Figure C.1 traces out the welfare loss in the two cases (both relative to the benchmark case, i.e., \( \alpha_1 = \alpha_2 = 0.66 \)).\(^7\) The x-axis represents the degree of openness (or the export share). When the degree of openness is below a threshold, the price stickiness in the downstream stage produces a bigger welfare loss. However, when the economy becomes sufficiently open, the price stickiness in the upstream stage produces more welfare loss.

Since the output of the upstream stage is an input into the downstream stage, the price stickiness of the upstream stage contributes to sluggish output adjustment or resource misallocation in the

\(^7\)Notably, when the degree of openness is large, the case of only upstream-stage price being sticky can generate a higher welfare loss than the benchmark calibration with prices being sticky in both stages. The reason is that, the monetary policy reaction function is kept to be the standard Taylor rule in this exercise. When the degree of openness is large enough, the standard Taylor rule is more sub-optimal in the case of only upstream-stage price being sticky.
downstream stage. So, the deviations of the downstream labor allocation and output from the flexible-price equilibrium are greater than those of the upstream stage.

This feature by itself does not imply that the sticky prices in the upstream stage are more important for the overall welfare, because the relative importance of the two stages also depends on their relative employment shares, which in turn depend on the share of intermediate goods in the downstream production. A smaller share of the intermediate goods in the final goods production means a higher share of labor in the downstream stage. For Canada, the intermediate goods share is about 60% (inferred from the World Input-Output Table). Our calibration suggests that when the economy is not very open (including when it is closed), sticky prices in the downstream sector matters more for welfare.

From WIOD data, we calculate that 75% of the countries have an intermediate goods share less than 55%. For these countries, it is also likely the case that sticky prices in the downstream sector produce a greater welfare loss than that sticky prices in the upstream stage, as long as their degree of openness is below some threshold.

As the economy becomes more open, since the upstream sector has to produce for both the

Figure C.1: Relative welfare loss with either upstream-stage price fully flexible or downstream-stage price fully flexible with respect to country openness
world market and the downstream stage at home, the upstream sector employment occupies a progressively larger share in total employment. As a result, the distortion caused by the price stickiness in the upstream stage increases in relative importance. Eventually, when the degree of openness surpasses some threshold, sticky prices in the upstream stage generate a bigger welfare loss.

We also study how the elasticity of substitution affects the welfare at each stage of production, and how it relates to the degree of openness. In general, when the elasticity of substitution is greater, there is more misallocation. We further consider two specific cases: (i) a higher elasticity in the upstream stage, i.e., $\theta_1 = 15$ and $\theta_2 = 10$; and (ii) the opposite case of a higher elasticity in the downstream stage, i.e., $\theta_1 = 10$ and $\theta_2 = 15$. We maintain a classic Taylor rule in both cases.

Figure C.2 traces out the welfare losses in the two cases (both relative to the benchmark calibration of $\theta_1 = \theta_2 = 10$). The x-axis represents the degree of openness. For reasons similar to the discussion on heterogeneous price stickiness, an increase in the elasticity of substitution in the downstream sector produces a bigger welfare loss than an equivalent increase in the elasticity in the upstream sector as long as the degree of openness is below some threshold. As the economy
becomes more open, the labor share of the upstream sector in total employment also increases, and the gap in welfare loss between the two cases narrows. Eventually, when the degree of openness exceeds the threshold, the result flips.