Dynamic Pricing and Demand Shaping: Theory and Applications in Online Assortments, Ride Sharing and Smart Grids

Shuangyu Wang

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ABSTRACT

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Shuangyu Wang

This dissertation consists of three papers in revenue management: on-line assortment optimization with reusable resources, spatial distribution of surge price under incentive compatible assignment for drivers and optimal price rebates for demand response under power flow constraints.

In Chapter 2, we study an on-line assortment optimization problem of substitutable products with fixed reusable capacities. At any time, a potential user with her preference model (possibly adversarially chosen) arrives to the selling platform and the platform offers a subset of products from the available set of products to the user. The user selects a product with probability given by her preference model, uses it for a random duration, which is distributed according to a distribution that only depends on the product selected, and generates revenue to the seller. The revenue contribution depends on the product selected and the actual usage time of this user. The goal of the seller is to find a policy for determining the assortment offered to each arrival to maximize the expected cumulative revenue over a time horizon.

We find that a simple myopic policy offering the available assortment that maximizes the expected revenue from a single user at her arrival time provides a good approximation for the problem. In particular, we show that the myopic policy is 1/2-competitive, i.e., the expected cumulative revenue of the myopic policy is at least 1/2 times the expected cumulative revenue of an optimal clairvoyant policy that has full information about the adversarially chosen user sequence, including their preference models and arrival epochs. The proof is based on partitioning the expected revenue of optimal clairvoyant policy into two parts and a coupling argument that allows us to bound the two parts in terms of the

expected revenue of the myopic policy.

In Chapter 3, we study the surge pricing problem on a ride sharing platform when there is a demand shock to the traffic network. The goal of the platform is to maximize the revenue by setting the prices over the network and the assignments between drivers and riders. In particular, we model the city as a continuous two dimensional network with exogenous arrivals of baseline riders, available drivers and demand shocks. We consider the demand shock only exists in a short time scale, so the rider chooses to request the ride or not depending on their willingness to pay and the price quoted to them, and the driver accepts any price to provide service. Since drivers can see the price distribution on driver app, they only accept the assignment from the locations that are incentive compatible for them. Thus, the price change at one location may affect the operations over the network and the platform must consider the incentive of drivers when assigning them.

We develop a model for this surge pricing problem and show the structural properties of an optimal solution. Once the prices at the location with demand shock is determined, we can determine the optimal prices on other part of the network. Then, the optimal assignments between riders and drivers can be determined analytically. The surge pricing problem reduces to one that only depends on the price at the location with demand shock. We then extend our model by including strategic behavior of riders, using throughput as objective, dealing with multiple demand shocks, un-constraining the price and considering movement time. We also conduct numerical experiments to study the properties of the model which can not be explored analytically.

In Chapter 4, we study the demand response problem of computing price rebates to offer to the customers to reduce the consumption in the presence of power flow constraints and transmission losses on the distribution grid. In particular, we employ alternating current power flow model for the power flow constraints with transmission loss. However, the demand response problem with alternating current power flow constraints is known as a non-convex problem, which is in-tractable to solve. To overcome this, we apply a semidefinite relaxation of alternating current power flow model to obtain a convex approximation for the problem. At the same time, to handle the uncertainty in the power reduction of customers, we use sample average to approach the expected cost and linear injection approximation to estimate the impact of uncertainty in the power reduction. Based on these relaxations and approximations, we propose an efficient iterative heuristic to solve the near-optimal offer price under alternating current power flow constraints and transmission losses. We conduct a substantial amount of numerical tests on our heuristic and compare its performance with other popular models. The result shows that our iterative heuristic leads to a significant reduction in the rebates that one needs to offer to shed a certain demand than the solution which does not consider full transmission loss in its model.

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To my parents

Chapter 1

Overview

Revenue management has drawn more and more attention in various industries, such as on-line retailing and ride-sharing platforms, that provide products or services with limited inventory. Because the inventory is limited and customers usually have different preferences for products, the seller can improve its revenue by using a well-designed pricing scheme and assortment offering strategy. Some methods for revenue management include discriminative pricing, assortment optimization and so on. In particular, discriminative pricing charges a customer dependent price and assortment optimization decides the product set or the ranking of products shown to customer to enhance its potential income. These decisions are made based on the preference of the customer and the inventory level. This thesis works on three different types of problems in revenue management, in particular, on-line assortment optimization with reusable resources, spatial distribution of surge price under incentive compatibility of drivers and optimal price rebates for demand response under power flow constraints.

In Chapter 2, we show our work for on-line assortment optimization with reusable resources. In many applications, such as hotel booking, physical storage, clouding computing, on-line food ordering and so on, by offering different choice sets under different situations, the seller can reserve some low-inventory items for future customers with high preference and display the high-inventory items at this time to improve its cumulative revenue in a long period. Especially for those business that discriminative pricing is restricted, a good assortment offering policy plays a key role in their revenue improvement.

To develop a well designed policy, some critical factors, including the state of platform, the preference model of current customer arrival and the information about future customer arrivals, are considered in deciding the choice set. In particular, when the seller has partial or no information about future incoming arrivals, the problem is an on-line problem. This type of on-line problem draws much of attention from researchers, as the platform is blurry of how to balance the revenue in current period with the unknown future arrival users under its limited inventory.

The problem becomes more complex when dealing with the reusable products. When the products dealt with are non-reusable, once customers purchase one product, the inventory is removed from the platform permanently. In this case, the platform only needs to track the inventory level of each product to monitor the system state, which reduces the dimensionality of the problem. When the products are reusable, the platform not only needs to record the inventories on-hand, but also needs to track the state of all in-use products since these busy units become available in a future time.

In our work, we consider an online assortment optimization problem where we have n substitutable products with fixed reusable capacities $c_1, \ldots, c_n$. In each period t, a user with some preferences (potentially adversarially chosen) arrives to the seller's platform who offers a subset of products S_t, from the set of available products. The user selects product $j \in S_t$ with probability given by the preference model and uses it for a random number of periods, $\tilde{t}_j$ that is distributed i.i.d. according to some distribution that depends only on j generating a revenue $r_j(\tilde{t}_j)$ for the seller. The goal of the seller is to find a policy that maximizes the expected cumulative over a finite horizon T. Our main contribution in this work is to show that a simple myopic policy (where we offer the myopically optimal assortment from the available products to each user) provides a good approximation for the problem. In particular, we show that the myopic policy is 1/2-competitive, i.e., the expected cumulative revenue of the myopic policy is at least 1/2 times the expected revenue of an optimal policy that has full information about the sequence of user preference models and the distribution of random usage times of all the products. In contrast, the myopic policy does not require any information about future arrivals or the distribution of random usage times. The analysis is based on a coupling argument that allows us to bound the expected revenue of the optimal algorithm in terms of the expected revenue of the myopic policy.

To prove this result, we employ a coupling argument to bound the expected revenue of the optimal clairvoyant policy in terms of the expected revenue of the myopic policy. We firstly present the coupling argument analysis for the case of exponential usage time distributions. For this case, we show a relatively easy coupling between the sample paths for the myopic policy and the optimal clairvoyant policy. However, the argument does not extend to general usage time distributions. For general usage time distributions, we design a novel coupling method and charging scheme between the myopic policy and the optimal clairvoyant policy to bound the expected revenue of the optimal clairvoyant policy.

We would like to emphasize that even with full-information about the user sequence, computing the optimal clairvoyant policy explicitly is intractable due to the curse of dimensionality arising from random user choices and random usage times. Golrezai et al. [39] use a linear programming relaxation with full-information about the user sequence as the benchmark for their on-line policy where resources are perishable. On the other hand, our method does not require the explicit form of the clairvoyant policy or even the problem formulation for the clairvoyant policy. Our coupling analysis is valid for any policy, including the optimal clairvoyant policy.

In Chapter 3, we study the problem of spatial distribution of surge price on ride-sharing platform under incentive compatibility of drivers. Compared to traditional transportation industries, ride-sharing platforms manage the drivers in a different way that it does not employ any drivers, instead, drivers are self-employed and they have their own preferences of working. The platform operates to dynamically match the available drivers, who have their de-centralized response to the platform.

One difficulty that the platform usually encounters is that the amount of ride requests can experience a temporary increase when there is a rain or a sports game is going to end. We refer this phenomenon as demand surge. When demand surge happens, it creates imbalance between drivers and riders in a region such that the number of available drivers is not enough to serve all ride requests. This is the main problem of this chapter to model the demand surge problem and return a solution.

In practice, two common operational tools that platform can use to handle demand surge are the price charged to riders and assignment policy between drivers and riders. The price adjusts the number of effective riders and incentivizes the relocation of the drivers. The first role has been drawn much attention in practice. When demand surge happens, the ride-sharing platform increases the price around surge location such that the number of effective riders reduces to a level that can be served by nearby available drivers. The second role of the pricing profile, together with the assignment policy, is also very important to serve the demand surge that it can increase the number of available drivers around the demand surge location by relocating drivers there. In particular, since the drivers have access to the price distribution on driver app, they have incentive of serving the high price area to improve their short term income. As long as the price difference is enough to compensate their dis-utility of relocation, platform is able to assign drivers to pick up riders at different locations and to increase the availability of drivers in demand surge area. We call this as the incentive compatible assignment that the drivers are only willing to work at a place that maximizes their effective income. By designing a pricing profile with spatial structure and the assignment policy across the network, the platform can utilize more available drivers across the network to digest the demand surge.

Our outcome of this chapter is to construct a model for such a surge pricing problem with the objective of maximizing the revenue for platform. Following the model, we show the algorithm of constructing the optimal pricing profile and the assignment policy. Resulted from the solution, the optimal pricing policy gives highest price on demand surge location, decreases by the dis-utility of drivers for relocation, and remains constant at the baseline price. The intersection between the pricing policy and baseline price is the boundary of surge region and the inside area of this boundary is the surge region. Within the surge region, we keep using the drivers on lowest priced location to serve the riders at the highest priced places until we serve all riders or we are out of available drivers. The remaining drivers are matched with local riders. We show that the matching rate, i.e., the probability of drivers who get a ride, is 1 inside the surge region. Outside the surge region, the drivers are never relocated and they are matched with the riders in the same area. We also discuss 5 extensions to the main problem. At last, we conduct computational experiments to discuss the properties of the problem that can not be studied analytically.

In Chapter 4, we discuss the optimal price rebates in demand response, which is an application of revenue management in the electricity market. In electricity market, the users pay the forward price which is determined in advance, whereas the utility company pays the real-time price when it needs to fulfill the shortage of supply. This creates a problem for the utility company that when its supply can not fulfill the overall demand on the user ends, that it must pay the real-time price to buy electricity, which is usually a higher price when there is more demand over the market. This happens more frequently when the regenerative energy is more and more engaged into the grid, as it reduces the stability of energy supply than the transitional energy sources. Demand response is one way to reduce the loss from this side. By signing contracts with users in advance, the utility company has the right to offer a price rebate to users to reduce their load. The load reduction and rebate structure are specified in the contact. The benefit of demand response is that the utility company can use a small cost to diminish large cost from buying utility on the real-time market. In this chapter, we show a heuristic to solve the sub-optimal price rebates for demand response under power flow constraints and transmission loss of the distribution grid.

We model the transmission loss using the AC power flow constraints. However, optimal power flow problem with AC power flow constraints is known as a non-convex problem, which can not be solved directly by general optimization method. To handle this difficulty, we use the SDP relaxation for the AC power flow constraints to relax the problem into convex optimization. Another difficulty in the problem is that the actual reduction of end users given the price rebates are uncertain. We use sample average to approximate the uncertainty and adjust the power flow constraints for each sample. Following these ideas, we develop an iterative heuristic for the offer price optimization. This heuristic achieves a good numerical performance in our computational study as it can return the price rebates efficiently and save a substantial amount of cost to fulfill the shortfall compared to other power flow constraints without considering the transmission loss.

In sum, all these three works give some innovative models or methods for revenue management applications in different areas. Chapter 2, Chapter 3 and Chapter 4 are self-contained and independent of each other.

Chapter 2

On-line Assortment Optimization with Reusable Resources

2.1 Introduction

Assortment optimization is an important problem that arises in a broad set of applications including online advertising, recommendations and e-retailing where the goal of the decision-maker is to select a subset of products from the available universe to offer to the user to maximize the expected revenue or reward. For any given subset S of offered products, the selection of the user depends on his or her random preference over the set of products including the no-purchase or exit option. We model this random selection using a *choice-model* that for any offer set S, specifies the probability that the user selects product $j \in S \cup \{0\}$ (where 0 refers to the no-purchase or exit option). Several parametric choice models have been studied in the literature including multinomial logit (MNL) model [52, 67, 56], the nested logit model [87, 57, 26, 36], Markov chain based model [15] and the mixture of multinomial logit model [58] (see [83, 48, 10] for a detailed overview of these models).

In this paper, we consider an online assortment problem where we are given n substitutable products with fixed capacities or inventories $c_1, \ldots, c_n$. Users with different choice models arrive sequentially. For each user, the seller offers a subset S of the available products satisfying certain constraints, the user selects a random product $j \in S \cup \{0\}$ with probability given by his or her choice model and uses it for a random amount of time, $\tilde{t}_j$ and returns it to the platform generating revenue $\boldsymbol{r}_j(\tilde{t}_j)$ for the seller. The goal of the platform or the seller is to design a policy to offer assortments to the users so that the overall expected revenue is maximized. This fits the setting of classical online product allocation or revenue management. However, unlike traditional settings where the capacity or inventory of any product decreases whenever any user selects that product, the products are reusable in our setting. Such setting arises commonly in many applications including cloud computing, physical storage and other sharing economy applications.

This problem has been considered in the literature in settings with non-reusable products as well as reusable products. Golerezai et al. [39] consider the online assortment problem with fixed product capacities for the case of non-reusable products (i.e. inventory of a product decreases whenever any user selects that product). They give an inventory balancing based algorithm that is $(1-1/e)$-competitive for adversarial arrivals in the limit of capacities going to infinity. For the case of capacities being equal to one, their algorithm is $1/2$-competetive. They also give a near-optimal algorithm for the case of stochastic i.i.d. arrivals where in each period there is a user whose type is sampled i.i.d. from a known distribution over user types. Ma and Simchi-Levi [53] consider a more general setting where the seller can make joint assortment and pricing decisions and obtain similar guarantees in the adversarial and stochastic arrivals cases for non-reusable capacities. Most closely related to our setting is the work of Rusmevichientong et al. [71]. They consider the setting of stochastic arrivals with known distribution of user types and reusable products and give a $1/2$-approximation for the problem.

We consider an adversarial model for the sequence of user types. Here, user type refers to the choice model of the user which is revealed to the platform when the user arrives. We make the following assumptions about choice probabilities and usage time distributions.

Assumption 2.1. *For any user type z, assortments $S \subseteq T \in \mathcal{S}$ and $i \in S$, $\phi^z(i,S) \geq \phi^z(i,T)$.*

Assumption 2.2. *For every product j, the usage time distribution depends only on j and not on the user type.*

Assumption 2.1 is mild and without much loss of generality. In fact, all random utility based choice models including multinomial logit (MNL), nested logit and mixture of MNLs satisfy Assumption 2.1. Assumption 2.2 is fairly reasonable in many settings where while the choice of the product depends on the user type but the usage time depends only on the product. For instance, consider a make-to-order setting where user selects a product from the offered assortment. Once the user makes the selection, a dedicated server makes the product for the user. In such a setting, the busy time of the server depends only on the product. While there are settings where the above assumption does not hold, we show that without Assumption 2.2, it is not possible to obtain any constant factor competitive algorithm for the case of adversarial arrivals.

The revenue to the seller, $\boldsymbol{r}_j(\tilde{t}_j)$ when product j is used for some random time, $\tilde{t}_j$, could be a general function of the usage time. In particular, we can model fixed revenue for every use as well as revenue which is an affine function of usage time with fixed component and per-unit usage time component. Let r_j denote the expected revenue of product j where the expectation is taken over the random usage time of product j, i.e.,

$$r_j = \mathbb{E}_{t_j \sim F_j}[\boldsymbol{r}_j(t_j)],$$

where F_j is the cdf of usage time distribution of product j.

Our Contributions. Our main contribution is to show that a myopic policy provides a good approximation for the online assortment optimization problem with reusable products.

Myopic Policy. For each user, the myopic policy offers an assortment $S \in \mathcal{S}$ from the set of available products that maximizes the expected revenue from that user. More specifically, suppose user at time t has type z_t and let $\mathcal{I}_t$ be the set of products available to the myopic policy at time t. Then the myopic policy offers assortment S_t where

$$S_t \in \text{ argmax } \left\{ \sum_{i \in S} r_i \cdot \phi^{z_t}(i, S) \mid S \subseteq \mathcal{I}_t, S \in \mathcal{S} \right\},$$

where ϕ^z is the choice model for user type z, $\phi^z(i, S)$ is the probability that user type z selects product i given assortment S and r_i is the expected revenue when product i is selected where the expectation is taken over the random usage time of product i. Recall that we assume the usage time distribution only depends on the product and is not dependent on user type. Therefore, the myopic policy only needs the expected revenue, r_j from product j if it is selected and does not need any further information about the usage time distribution. Further, the optimal set S_t can be found using any black-box algorithm for static assortment optimization.

We show that this myopic policy is 1/2-competitive. In other words, the expected revenue of the myopic policy is at least 1/2 times the expected revenue of an optimal policy that has full information about the sequence of user types and product usage distributions (although not the choice realizations and the realization of usage times). We refer to this as the *clairvoyant benchmark*. More generally, when the (possibly constrained) static assortment optimization problem at each stage can only be solved up to within an α factor of the optimal, our myopic policy is $\alpha/2$ competitive.

We also show that if the usage time distribution depends on the user type, then there is no online algorithm that can obtain a constant factor competitive ratio as compared to our clairvoyant benchmark for the case of adversarial arrivals. We would like to note that Rusmevichientong et al. [71] consider the case where usage time distributions can depend on the user type. However, they consider the setting of stochastic arrivals with known distribution of user types and give a 1/2-approximation compared to the optimal dynamic programming solution as opposed to the clairvoyant benchmark.

Challenges and New Techniques. We would like to note that even with full-information about the sequence of users and the usage time distribution, computing an optimal policy is intractable due to the curse of dimensionality. Golrezai et al. [39] use a LP based upper bound as a benchmark for the case of non-reusable products. One of the challenges in extending the results to the case of reusable products is the lack of a good LP based upper bound. A natural LP formulation based on the one in [39] has an unbounded integrality gap and therefore, is not a useful benchmark for the problem.

In order to prove the competitive ratio bound, we use a coupling argument instead to upper bound the expected revenue of an optimal algorithm with full information in terms of the expected revenue of the myopic policy. If the usage time distributions satisfy the *Increasing Failure Rate (IFR)* property, it is relatively easy to design such a coupling. However, that coupling does not extend to general distributions. One of the main contributions of this work is designing a novel *queue-based coupling* that allows us to relate the expected revenue of an optimal policy to the expected revenue of the myopic policy for any usage time distributions.

2.1.1 Other Related Work

There is a considerable amount of literature on dynamic assortment optimization problems with non-reusable products starting with [11], which studied the problem of dynamic assortment optimization for a stochastic arrival model where users choose according to a multinomial logit choice model (see [77], [51], [35] and [82]) and the user type is drawn i.i.d. from a stationary distribution. Chan and Farias [22] considered a stochastic depletion framework for non-stationary environments which includes the assortment planning problem under random arrivals. They gave a 1/2 competitive myopic policy for this general framework. More recently, [74] and [85] considered other closely related models for online product allocation with stochastic arrivals. We refer the reader to [39] for a more detailed review.

For revenue management with reusable products and random usage times, [50] first studied a product independent demand model where the users do not exhibit any choice behavior and the goal is to design a policy to maximize the average revenue in an infinite horizon setting. Owen and Levi [65] extended this model to include user choices and also study the infinite horizon setting. Chen et al. [23] considered a related problem of control admission for a system with multiple units of a single product which can be reserved in advance for time intervals determined by users arriving according to a multi-class Poisson process.

Product allocations problems also closely relate to online matching problems and often generalize the classical online bipartite matching problem (see [46]). In this seminal work, they showed that matching arriving users (the unknown vertices) based on a random RANKING over all products (vertices on the known side of the graph) gives the best possible competitive guarantee of $(1-1/e)$. The analysis was considerably simplified in [13], [28], [38] and extended to more general settings in [2], [61]. There is also a rich body of literature on online matching with random arrivals [27], [55], [45], [32] and stochastic rewards [60], [62]. We refer the interested reader to [59] for a more detailed review.

Outline. The rest of the chapter is organized as follows. In Section 2.2, we present the competitive ratio analysis for the case of IFR usage time distributions. In Section 2.3, we present the competitive ratio analysis for general usage time distributions. In Section 2.4, we present a family of examples to show that no online algorithm can have a constant competitive ratio for adversarial arrivals if the usage time distributions depend on the user type. In Section 2.5, we use the survey data from New York Health Club to present the numerical performance of myopic policy. Finally, in Section 2.6 we summarize our results and mention some directions for future work.

2.2 Competitive Ratio for IFR Usage time Distributions

In this section, we consider the special case when the usage time distribution for each product satisfies the *Increasing Failure Rate* (IFR) property and present the competitive ratio analysis for the myopic policy. For any product j with usage time distribution cdf, F_j

and pdf f_j, the failure rate denoted as $h_j(t)$ is given by

$$h_j(t) = \frac{f_j(t)}{1 - F_j(t)}.$$

For any $t \in \mathbb{R}_+$, $h_j(t)$ can be interpreted as the conditional density conditioned on the usage time being at least t. The distribution satisfying the IFR property implies that h_j is increasing in t. A large class of distributions including exponential, Poisson and geometric distributions satisfy the IFR property. Therefore, it is an important class of distributions to study.

Our analysis, as we mention earlier, is based on a coupling argument that upper bounds the expected revenue of the clairvoyant optimal in terms of the expected cumulative revenue of the myopic policy at each period t. While our results hold for general usage time distributions, we would like to present the case of IFR usage time distributions first as the coupling for IFR distributions is easier to design. For the sake of simplicity, we also assume that revenue, r_j of each product j is constant and does not depend on usage times. In the next section, we extend our results to the case of general usage time distributions and general revenue functions.

We would like to note that we can assume without loss of generality, capacity $c_i = 1$ for all products i. Guarantees for the case of unit inventory leads to stronger results that generalize to the case of arbitrary inventories. We can consider a unit inventory setting where for each i we have c_i identical products, each with capacity of 1, to replace c_i units of product i in the original instance. Now the clairvoyant algorithm knows all arrivals in advance and thus knows these copies represent the same product. Therefore, OPT remains unchanged and an algorithm for the unit capacity case can be used for arbitrary capacity levels without loss in guarantee.

Let us first introduce some notations. For any $t = 1, \ldots, T$, let z_t denote the user type at time t. Let $\omega = (\omega_1, \ldots, \omega_T)$ denote the sample path that specifies the random preference realizations of all users $z_1, \ldots, z_T$ and the random usage times of all products. For any

$t = 1, \ldots, T$, let $\omega_{[t]} = (\omega_1, \ldots, \omega_t)$ refer to the restriction of sample path ω until time t .

From hereon, we refer to the myopic policy as ALG and the clairvoyant optimal as OPT. Let $\mathcal{I}_t(\omega)$ ($\mathcal{O}_t(\omega)$ respectively) denote the set of available products in ALG (OPT respectively) at time t on sample path ω. Also, let $S_t(\omega)$ ($S_t^*(\omega)$ respectively) denote the assortment offered by ALG (OPT respectively) at time t to user type z_t. We would like to emphasize again that at any time t, the clairvoyant benchmark knows the full sequence of user types $z_1, \ldots, z_T$ but not the realizations of preferences of future users or the random future product usage times. Therefore, $S_t^*(\omega)$ is a function of $z_1, \ldots, z_T$ and $\omega_{[t-1]} = (\omega_1, \ldots, \omega_{t-1})$. In contrast, the online algorithm does not have any information about the future user types and $S_t(\omega)$ is a function of only user types $z_1, \ldots, z_t$ and $\omega_{[t-1]} = (\omega_1, \ldots, \omega_{t-1})$. More specifically, since ALG is a myopic policy, $S_t(\omega)$ is just the myopically optimal subset of $\mathcal{I}_t(\omega)$ that maximizes the expected revenue for user type z_t, i.e.,

$$S_t(\omega) = \mathsf{argmax}_{S \subseteq \mathcal{I}_t(\omega)} \, R(S, z_t),$$

where for any assortment S,

$$R(S, z_t) = \sum_{i \in S} r_i \cdot \phi^{z_t}(i, S),$$

is the expected revenue of assortment S for user type z_t. Let $j_t(\omega) \in S_t(\omega) \cup \{0\}$ be the product selected by user z_t in ALG, and let $j_t^*(\omega) \in S_t^*(\omega) \cup \{0\}$ be the product selected by user z_t in OPT at time t on sample path ω. Note that product 0 refers to the do-nothing or exit option.

We show that for any T and any sequence of user types $z_1, \ldots, z_T$, if the usage time distribution for each product satisfies the IFR property, the expected cumulative revenue of ALG is at least $1/2$ times the expected cumulative revenue of OPT where the expectation is taken over the random preferences and random usage times. In particular, we have the following theorem.

Theorem 2.1. *Suppose the usage time distribution for every product satisfies the IFR property. Then for any time T and sequence of user types $z_1, \ldots, z_T$,*

$$\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right] \geq \frac{1}{2} \cdot \mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t^*(\omega), z_t)\right].$$

To prove the above theorem, we try to upper bound the expected revenue in OPT in terms of the expected revenue of ALG. In particular, we bound the expected revenue of $S_t^*(\omega)$ by bounding the expected revenue from products in $S_t^*(\omega) \cap \mathcal{I}_t(\omega)$ and $S_t^*(\omega) \setminus \mathcal{I}_t(\omega)$ separately. For the purpose of analysis, we label each unit of every product differently so that each unit can be treated as a different product. Therefore, we can assume without loss of generality the initial inventory $c_i = 1$ for all products i. The following lemma bounds the total expected revenue in OPT from products that were available in ALG during periods OPT offered them.

Lemma 2.2. *For any sequence of user types $z_1, \ldots, z_T$, any $t = 1, \ldots, T$ and any sample path ω,*

$$\sum_{j \in S_t^*(\omega) \cap \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega)) \leq R(S_t(\omega), z_t)$$

Proof. From Assumption 2.1 (substitutability of the choice model), we have that

$$\phi^{z_t}(j, S_t^*(\omega)) \leq \phi^{z_t}(j, S_t^*(\omega) \cap \mathcal{I}_t(\omega)).$$

Therefore,

$$\begin{aligned}
\sum_{j \in S_t^*(\omega) \cap \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega)) &\leq \sum_{j \in S_t^*(\omega) \cap \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega) \cap \mathcal{I}_t(\omega)) \\
&= R(S_t^*(\omega) \cap \mathcal{I}_t(\omega), z_t) \\
&\leq R(S_t(\omega), z_t),
\end{aligned}$$

where the last inequality follows from the choice of $S_t(\omega)$ which is the revenue maximizing subset of $\mathcal{I}_t(\omega)$ for user type z_t. □

The following lemma bounds the total expected revenue in OPT from products that were unavailable in ALG during periods OPT offered them. Here, we present the bound under the assumption that the usage time distribution for every product satisfies the IFR property. We extend the argument for general distributions in the following section.

Lemma 2.3. *Suppose the usage time distribution for every product satisfies the IFR property. Then for any T and sequence of user types $z_1, \ldots, z_T$,*

$$\mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega)) \right] \leq \mathbb{E}_\omega \left[\sum_{t=1}^{T} R(S_t(\omega), z_t) \right].$$

Proof. Consider any ω such that $j_t^*(\omega) \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)$. For brevity, let us refer to $j_t^*(\omega)$ as j_t^*. At time t, j_t^* is not available in ALG on sample path ω. Therefore, it is in use at time t and must have been selected in ALG at some previous time period, say $t-\tau$ for some $\tau \geq 1$. Therefore, ALG received revenue, $r_{j_t^*}$ at $(t-\tau)$. We charge the revenue, $r_{j_t^*}$ that OPT collects at time t to the revenue that ALG collected at time $t-\tau$. However, we need to guarantee that the revenue in ALG at time $t-\tau$ is not charged multiple times. We do this by coupling the usage times of j_t^* in ALG and OPT that guarantees that j_t^* becomes available in ALG before it becomes available in OPT.

Coupling of Usage Times. Let $\tilde{L}$ denote the random usage time of j_t^*. Since the usage time distribution is IFR, we know that for any $u \geq 0$, $P(\tilde{L} - \ell \geq u | \tilde{L} \geq \ell)$ is decreasing in ℓ. This implies that

$$P(\tilde{L} - \tau \geq u | \tilde{L} \geq \tau) \leq P(\tilde{L} \geq u).$$

Therefore, we can create a coupling between the residual time of $j_t^*(\omega)$ starting from period t in ALG with the usage time in OPT as follows. Let $\tilde{u}$ be a random sample from uniform$(0,1)$. Set the residual usage time of j_t^* in ALG to be $F^{-1}(\tilde{u} | \tilde{L} \geq \tau) - \tau$ and the usage time in OPT to $F^{-1}(\tilde{u})$ where F is the cumulative density function for usage time distribution of product $j_t^*(\omega)$. Since F is IFR,

$$F^{-1}(\tilde{u}|\tilde{L} \geq \tau) - \tau \leq F^{-1}(\tilde{u}),$$

for any $\tau \geq 0$ and $0 \leq \tilde{u} \leq 1$. Therefore, with this coupling, j_t^* becomes available in ALG before it becomes available in OPT. Consequently, we charge the revenue, $r_{j_t^*}$ collected by ALG at time $t - \tau$ only once. This implies on any sample path of choice realizations, every revenue collected by ALG is charged at most once. Therefore,

$$\mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \mathbb{1}(j = j_t^*(\omega)) \right] \leq \mathbb{E}_\omega \left[\sum_{t=1}^{T} r_{j_t(\omega)} \right] = \mathbb{E}_\omega \left[\sum_{t=1}^{T} R(S_t(\omega), z_t) \right]. \tag{2.1}$$

Consequently, we have

$$\begin{aligned} \mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega)) \right] &\leq \mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega) \setminus \mathcal{I}_t(\omega)) \right] \\ &= \mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \mathbb{1}(j = j_t^*(\omega)) \right] \\ &\leq \mathbb{E}_\omega \left[\sum_{t=1}^{T} R(S_t(\omega), z_t) \right], \end{aligned}$$

where the first inequality follows from Assumption 2.1 and the last inequality follows from (2.1). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 We have that,

$$\begin{aligned}
\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t^*(\omega), z_t)\right] &= \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right] \\
&= \mathbb{E}_\omega\left[\sum_{t=1}^{T}\left(\sum_{j\in S_t^*(\omega)\cap\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega)) + \sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right)\right] \\
&\le \mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right] + \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right] \\
&\le 2\cdot\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right],
\end{aligned}$$

where the second last inequality follows from Lemma 2.2 and the last inequality follows from Lemma 2.3. □

2.3 General Usage Time Distributions

In this section, we extend the competitive ratio analysis for the myopic policy for general usage time distributions and general revenue functions (as functions of usage times) and show that the myopic policy is 1/2-competitive in general. Similar to the previous section, we assume without loss of generality that $c_i = 1$ for all products i. We have the following theorem, in particular.

Theorem 2.4. *Suppose for every product j, the usage time is distributed according a distribution that only depends on j. Then for any sequence of user types $z_1, \ldots, z_T$, the expected cumulative revenue of the myopic policy is at least $1/2$ times the expected cumulative revenue of the clairvoyant optimal that knows the full sequence, i.e.,*

$$\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right] \ge \frac{1}{2}\cdot\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t^*(\omega), z_t)\right].$$

The proof proceeds along similar lines as in the case of IFR usage time distributions. As in the case of IFR usage time distribution, the key challenge is to bound the total expected revenue in OPT from products that are unavailable in ALG at the time they are offered in OPT. For the case of IFR usage distributions, we use a coupling and a charging argument in Lemma 2.3 to show that the total expected revenue from such products is upper bound by the total expected revenue of ALG. However, the coupling in Lemma 2.3 does not work for the general distributions. In particular, the coupling of the usage time for a product j missing in ALG at some time t when user z_t selects j in OPT at time t, guarantees that j becomes available in ALG before it is available in OPT for the case of IFR usage time distributions ensuring that the revenues in ALG are not charged multiple times. However, the guarantee does not necessarily hold for the case of more general distributions and we may end up charging the same revenue of ALG multiple times.

New Coupling Technique. In order to address this issue, we introduce a new coupling between the usage times in ALG and OPT. In particular, we introduce *coupling queues* to specify the coupling of usage times between sample paths in ALG and OPT. We maintain n queues, $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ with usage time samples for each of the n products. The queues are initially empty and for each product j we initialize a sequence of T i.i.d. samples of usage times in $\mathcal{H}_j$.

We use the following coupling between the usage times for any product j in ALG and OPT. Whenever product j is selected in ALG by some user, we sample a usage time $\tilde{L}_j$ for product j from the corresponding distribution and push sample $\tilde{L}_j$ to (the bottom of) queue $\mathcal{Q}_j$. Whenever product j is selected in OPT by any user, we first check if queue $\mathcal{Q}_j$ is empty. If queue $\mathcal{Q}_j$ is not empty we pop a sample from $\mathcal{Q}_j$ for the usage time for this selection (i.e., we use the sample at the top of $\mathcal{Q}_j$ and delete the sample from the queue). Otherwise, we give OPT the next unused sample from $\mathcal{H}_j$.

Lemma 2.5. *For any time $t = 1, \dots, T$ and any $j = 1, \dots, n$, whenever a user selects product j in OPT the usage time distribution given by the above coupling is i.i.d. according to the usage time distribution for product j.*

Proof. When $\mathcal{Q}_j$ is not empty, the sample that is popped from the queue was originally picked independently from all previous samples and according to the true distribution, and added to the queue unconditionally. Any other samples that might have been added to the queue subsequent to adding this sample do not affect the sample, and it is popped before them. Therefore, the popped sample is i.i.d. according to the true distribution. When $\mathcal{Q}_j$ is empty, samples are chosen from $\mathcal{H}_j$ and the claim holds directly. $\square$

We are now ready to bound the total expected revenue in OPT from products that are unavailable in ALG at the time they are offered in OPT. In particular, we have the following lemma analogous to Lemma 2.3.

Lemma 2.6. *For any usage time distributions and sequence of user types* $z_1, \ldots, z_T$,

$$\mathbb{E}_\omega \left[\sum_{t=1}^{T} \sum_{j \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)} r_j \cdot \phi^{z_t}(j, S_t^*(\omega)) \right] \leq \mathbb{E}_\omega \left[\sum_{t=1}^{T} R(S_t(\omega), z_t) \right].$$

Proof. Consider any ω such that $j_t^*(\omega) \in S_t^*(\omega) \setminus \mathcal{I}_t(\omega)$. Let us refer to $j_t^*(\omega)$ as j_t^* for brevity. At time t, j_t^* is not available in ALG on sample path ω. Therefore, it is in use at time t and must have been selected in ALG at some previous time period, say $t - \tau$ for some $\tau \geq 1$. Let $\tilde{L}$ be the random usage time that ALG sampled for j_t^* at time $(t - \tau)$ and pushed on the queue $\mathcal{Q}_{j_t^*}$ corresponding to product j_t^*. Since j_t^* is still in use by ALG we get $\tilde{L} \geq \tau$. Using this and the fact that OPT is able to select j_t^* at time t, we have that the sample $\tilde{L}$ must exist on the queue up to time t (but may be popped at t). Therefore, $\mathcal{Q}_{j_t^*}$ is non empty before user arrives at t.

Hence, when OPT selects j_t^* at time t, we pop a sample from $\mathcal{Q}_{j_t^*}$. Suppose the sample popped by OPT was generated and added to the queue by ALG at time $t' \leq t - \tau$. We charge the revenue earned by OPT for this selection at time t to the revenue earned by ALG for using j_t^* at time t'. Observe that the charging is unique since each sample on the queue is used at most once by OPT, and we only charge to ALG when the corresponding sample is used by OPT. We would also like to note that the revenue from a product can now even depend on the usage time duration of the product. The queue based coupling ensures that every usage time dependent revenue of ALG on any sample path is charged at most once by selections of OPT that are unavailable in ALG at the time they are offered in OPT. Therefore,

$$\mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\mathbb{1}(j=j_t^*(\omega))\right]\leq\mathbb{E}_\omega\left[\sum_{t=1}^{T} r_{j_t(\omega)}\right]=\mathbb{E}_\omega\left[\sum_{t=1}^{T}R(S_t(\omega),z_t)\right]. \quad (2.2)$$

Simplifying as before, we get

$$\begin{aligned}\mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j,S_t^*(\omega))\right] &\leq \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j,S_t^*(\omega)\setminus\mathcal{I}_t(\omega))\right]\\ &= \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\mathbb{1}(j=j_t^*(\omega))\right]\\ &\leq \mathbb{E}_\omega\left[\sum_{t=1}^{T}R(S_t(\omega),z_t)\right],\end{aligned}$$

where the first inequality follows from Assumption 2.1 and the last inequality follows from (2.2). □

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4 We have that

$$\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t^*(\omega), z_t)\right] = \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right]$$
$$= \mathbb{E}_\omega\left[\sum_{t=1}^{T}\left(\sum_{j\in S_t^*(\omega)\cap\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega)) + \sum_{j\in S_t^*(\omega)\setminus\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right)\right]$$
$$\leq \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\cap\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega))\right] + \mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right]$$
$$\leq \mathbb{E}_\omega\left[\sum_{t=1}^{T}\sum_{j\in S_t^*(\omega)\cap\mathcal{I}_t(\omega)} r_j\cdot\phi^{z_t}(j, S_t^*(\omega)\cap\mathcal{I}_t(\omega))\right] + \mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right]$$
$$\leq 2\cdot\mathbb{E}_\omega\left[\sum_{t=1}^{T} R(S_t(\omega), z_t)\right],$$

where the third inequality follows from Lemma 2.6 and the second last inequality follows from Assumption 2.1. □

Tightness of $1/2$ Competitive Ratio. In this part, we give an example to show that $1/2$ competitive ratio is tight for myopic policy in our problem setting, i.e., the competitive ratio of myopic policy can not be any constant higher than $\frac{1}{2}$. Suppose platform has two products $\{1, 2\}$, with inventory $c_1 = 1, c_2 = 1$. Usage time for product j is distributed as exponential distribution with mean $1/\lambda_j$. Price r_j for product j is deterministic and independent of usage time. Consider the following two types of users:

1. Choice model of type 1 user:

$$\phi^1(j, \{j\}) = p, \phi^1(0, \{j\}) = 1-p, p\in[0,1], \forall j\in\{1,2\},$$
$$\phi^1(1, \{1,2\}) = q, \phi^1(2, \{1,2\}) = p-q, \phi^1(0, \{1,2\}) = 1-p, q\in[0,p].$$

2. Choice model of type 2 user:

$$\phi^1(2,S) = 1, \phi^1(0,S) = 0, \forall S \ s.t. \ 2 \in S,$$
$$\phi^1(2,S) = 0, \phi^1(0,S) = 1, \forall S \ s.t. \ 2 \notin S.$$

The user arrival sequence contains two users, a type 1 user followed by a type 2 user, with inter-arrival time $\epsilon > 0$. For this problem, the competitive ratio of myopic policy approaches 1/2 when we take $\epsilon \to 0, p \to 1, r_2 \to r_1$ with $r_2 > r_1$.

When the type 1 user arrives, if platform offers $\{1\}$, the expected revenue collected from this user is pr_1 based on this user's choice model. If $\{2\}$ is offered, the expected revenue is pr_2. If $\{1,2\}$ is offered, the expected revenue becomes $qr_1 + (p-q)r_2$. Since we have $r_2 > r_1$ and $p \in [0,1], q \in [0,p]$ in this example, myopic policy offers $\{2\}$ to the first user which maximizes the expected revenue. Then, with probability p, the first user selects product 2. When the first user selects product 2, let $\tilde{L}_2$ be the usage time of first user. Because the second user selects product 2 for sure if it is offered and leave without purchasing if product 2 is not offered, if $\tilde{L}_2 \geq \epsilon$, then product 2 can not be offered to the second user and then she leaves without purchasing; if $\tilde{L}_2 < \epsilon$, platform is able to offer product 2 to second user and receive r_2 from her. Thus, the expected value of myopic policy is

$$\begin{aligned} & \mathbb{P}\left(\tilde{L}_2 \geq \epsilon\right) r_2 + \mathbb{P}\left(\tilde{L}_2 < \epsilon\right) 2r_2 && (2.3) \\ = \quad & e^{-\lambda_2 \epsilon} r_2 + (1 - e^{-\lambda_2 \epsilon}) 2r_2 && (2.4) \\ = \quad & r_2 + (1 - e^{-\lambda_2 \epsilon}) r_2. && (2.5) \end{aligned}$$

(2.3) is formulated by the law of total expectation and the analysis in previous paragraph. (2.4) is derived from the cumulative density function of $\tilde{L}_2$. (2.5) is from rearranging the terms in (2.4).

If first user does not select product 2, she leaves without purchasing. Myopic policy then offers $\{2\}$ to the second user. The revenue in this case is simply r_2. Consequently, the expected value E_M of myopic policy is given by

$$E_M = p\left(r_2 + (1 - e^{-\lambda_2 \epsilon})r_2\right) + (1-p)r_2 = r_2 + p\left(1 - e^{-\lambda_2 \epsilon}\right) r_2.$$

On the other hand, consider the policy that offers $\{1\}$ to first user and $\{2\}$ to the second user. It takes the advantage of the information of inter-arrival time and choice model of future user. The expected value of this policy E_C is $E_C = pr_1 + r_2$. Thus, the performance ratio of the two polices is,

$$\frac{E_M}{E_C} = \frac{r_2 + p\left(1 - e^{-\lambda_2 \epsilon}\right) r_2}{pr_1 + r_2}. \tag{2.6}$$

Since we have freedom to choose ϵ, p, r_1, r_2 as long as they satisfy $\epsilon > 0,\ \alpha_2 > \alpha_1,\ p \in [0,1]$, we take the limit $\epsilon \to 0, \alpha_2 \to \alpha_1, p \to 1$. Because the formula in (2.6) is continuous in these parameters, we have

$$\frac{\alpha_2 + p\left(1 - e^{-\lambda_2 \epsilon}\right) \alpha_2}{p\alpha_1 + \alpha_2} \to \frac{1}{2},$$

i.e., the competitive ratio of myopic policy can not be any constant higher than $\frac{1}{2}$.

2.4 User Type dependent Usage Times: Bad Examples

Recall that we show that our myopic policy is 1/2-competetive for the case where the usage time distributions depend only on the product and not on the user types. In this section, we consider the case where the product usage time distributions could depend on the user type. We show that in this case, there is no online algorithm with a constant competitive ratio in the adversarial arrival model.

It suffices for us to consider a single product, so for a user arriving at time t, let S_t denote the random usage duration. Even for the special case of online matching with a single reusable product we have the following upper bound on the competitive ratio of any online algorithm.

Theorem 2.7. *For online matching with a single reusable product, if the random usage durations depend on the user, no online (randomized) algorithm can have competitive ratio* *better than* $O\big(\frac{\log n}{n}\big)$*, where* n *is the number of users.*

Before we prove the above lemma, consider first the special case of algorithms that always match a user to some available product if such a matching is possible. Suppose we have a single unit of a single product with reward 1 and the following arrival sequences,

- Sequence A: A single user with usage time duration ∞ (never returns the product).
- Sequence B: A user with usage duration ∞, followed by n users that return the product right away i.e., $\mathbb{P}(S_t = 0) = 1$ for all $t \in \{2, \ldots, n+1\}$.

In order to be competitive on sequence A, the algorithm must match the arrival with the only available product. Consequently, even on sequence B the algorithm matches the product to the first user and earn a net reward of 1. An optimal offline algorithm would earn total reward n on sequence B hence, an online algorithm that always matches an arriving costumer if possible can never have competitive ratio better than $O\big(\frac{1}{n}\big)$. For the general case, consider the following family of arrival sequences and subsequent lemma.

- Sequence $C(i)$: n^i users, each with identical usage duration distribution where the item is either returned immediately with probability $p_i = 1 - \frac{1}{n^i}$ or never returned i.e., $\mathbb{P}(S(i) = 0) = p_i = 1 - \frac{1}{n^i}$ and $\mathbb{P}(S(i) = \infty) = 1 - p_i = \frac{1}{n^i}$.

Lemma 2.8. *For any given* i*, if an algorithm generates*

$$\textit{Expected Revenue} \geq \frac{1 - p_i^{\alpha n^i}}{1 - p_i},$$

on arrival sequence $C(i)$*, then the probability that the product is consumed forever after the last arrival is at least* $1 - p_i^{\alpha n^i}$*.*

Proof. Given a randomized algorithm, fix a random seed r for the algorithm and let $\alpha(r) \in [0, 1]$ denote the fraction of users that are to be matched by the algorithm if possible (since all users are identical it does not matter which ones are matched, only the total number).

The expected total reward R of the algorithm given seed r is,

$$\mathbb{E}[R|r] = (1-p_i) + p_i(1-p_i)2\cdots + p_i^{\alpha(r)n^i-1}\alpha(r)n^i = \frac{1-p_i^{\alpha(r)n^i}}{1-p_i}.$$

Where the last equality follows by summing up the arithmetic-geometric progression. Observe that for $n \to \infty$, $1-p_i^{\alpha(r)n^i} \to 1-e^{-\alpha(r)}$ and thus, $\mathbb{E}[R|r] \to (1-e^{-\alpha(r)})n^i$.

Now, the probability that the product is consumed forever (extinguished) conditioned on r is

$$\mathbb{P}(\text{Product Extinguished} \mid r) = 1-p_i^{\alpha(r)n^i}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(\text{Product Extinguished}) &= \mathbb{E}_r[1-p_i^{\alpha(r)n^i}] \\ &= \sum_r \mathbb{P}(r)(1-p_i^{\alpha(r)n^i}).\end{aligned}$$

Moreover,

$$\mathbb{E}[R] = \sum_r \mathbb{P}(r)\frac{\mathbb{P}(\text{Product Extinguished} \mid r)}{1-p_i} = \frac{\mathbb{P}(\text{Product Extinguished})}{1-p_i},$$

which gives us the desired. □

Corollary 2.9. *For any given i, the maximum expected revenue generated by any algorithm (online or offline) for sequence $C(i)$ is $\frac{1-p_i^{n^i}}{1-p_i} = \Theta(n^i)$.*

Proof. Follows from observing that matching the product to every user possible maximizes the expected revenue and using Lemma 2.8. □

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7 Consider the following n sequences $D(i) = \{C(1), \ldots, C(i)\}$ for $i \in [n]$ that begin with n users arriving from sequence $C(1)$ followed by n^2 users from sequence $C(2)$ and so on in order till $C(i)$. For any sequence $D(i)$ the maximum possible expected revenue is $\Theta(n^i)$, since it is lower bounded by $(1-p_i^{n^i})n^i = \Omega(n^i)$ (matching only the users in $C(i)$ and ignoring earlier users) and at most, $\left(\sum_{k=1}^{i} n^i\right) = O(n^i)$.

We prove by contradiction. Consider a β competitive online algorithm and assume $\beta = \Omega\left(\frac{\log n}{n}\right)$ (otherwise we are done). For any such algorithm subjected to arrival sequence $D(1)$, from Corollary 2.9 and Lemma 2.8 we have that the probability the product is available after all arrivals is at most $1-\beta(1-p_1^n) \to 1-\beta(1-1/e)$. Similarly, in order to be β competitive on sequence $D(2)$ where the maximum expected profit is $\Theta(n^2)$, the expected reward generated from the $C(2)$ part of sequence $D(2)$ must be at least $\beta(1-p_2^{n^2})n^2$ as the contribution from arrivals $C(1)$ is at most $\Theta(n) = o(\beta n^2)$ for $\beta = \Omega\left(\frac{\log n}{n}\right)$). Therefore, the probability of product surviving after all arrivals from $C(2)$ conditioned on product surviving after the first n arrivals from $C(1)$ is at most $1-\beta(1-p_2^{n^2}) \to 1-\beta(1-1/e)$. Thus, the probability of product surviving after all arrivals in $D(2)$ is at most $(1-\beta(1-1/e))^2$. More generally, it follows that the probability of product surviving arrivals from sequence $D(i)$ is at most $(1-\beta(1-1/e))^i$. Therefore, on sequence $D(n)$ there is at most a $(1-\beta(1-1/e))^{n-1}$ probability that the product survives until the first arrival from sequence $C(n)$. Hence, the expected revenue on $D(n)$ is,

$$O\left(\max\left\{(1-\beta(1-1/e))^{n-1}, \frac{1}{n}\right\} n^i\right).$$

Therefore, the competitive ratio β of the algorithm must satisfy,

$$\beta \leq O\left(\max\left\{(1-\beta(1-1/e))^{n-1}, \frac{1}{n}\right\}\right).$$

Since the RHS is $O(\frac{1}{n})$ for $\beta = \frac{2\log n}{n}$ we have a contradiction. Hence, $\beta = O\left(\frac{\log n}{n}\right)$. Note that a more refined argument can be used to further tighten the log factor. $\square$

2.5 Numerical Experiment

In this section, we empirically compare the performance of myopic policy with an offline benchmark. The data is from customer surveys of New York Health Club (NYHC) in a published teaching case from Columbia Business School [54]. The survey contains $1,000$ potential customers of their willingness to pay for 6 different work-out time slots: $6-9$ a.m., 9 a.m. - noon, noon - 2 p.m., $2-5$ p.m., $5-9$ p.m. and 9 p.m. - midnight. For the number in each cell of Table 2.1, it represents the dollar amount that the questionnaire answerer is willing to pay for the membership of using the gym in that specific time period.

Client	$6-9$ a.m.	9 a.m. - noon	noon - 2 p.m.	$2-5$ p.m.	$5-9$ p.m.	9 p.m. - midnight
1	18	50	41	76	69	41
2	66	14	86	62	71	46
3	60	43	43	26	91	58

Table 2.1: Example of Data in NYHC Survey

We use MNL fit for the choice model of each user arrival and assume the user uses the time slot for a random amount of days and a fixed capacity for each time slot. In the following sections, we show the details of the experiments.

2.5.1 Experiment Setup

In this section, we introduce the setup of our experiments. We consider a sequence of 1000 customer arrivals with MNL choice models over time horizon $[0, T]$. Each customer has her choice model for the different time slots with MNL parameters fit from the data in [54]. For each customer arrival, the club knows the choice model of the customer upon her arrival and decides an assortment of time slots offered to the customer. The capacity of each time slot j is c_j. In our experiments, we use $c_j = C,\ j = 1, ..., 6$. Different of assuming $C = 1$ in proof, we use different values for C. Then, the customer selects a product from the offering or exit with no purchase based on her choice model. If the user chooses a time slot j, she pays a fixed price r_j and uses that time slot for a random number of days following distribution uniform$(0, t^{\max})$. The $t^{\max}$ is same for all products $j = 1, ..., 6$, i.e., all usage times are uniform$(0, t^{\max})$. The user returns the unit after she finishes her usage. We use Monte Carlo simulation to estimate the expected value of myopic policy, then compare it to the benchmark.

Choice Model Parameters. In the dataset [54], w_{jt} is given in each cell to represent the willingness to pay of t-th arrival customer for product j. We use $\{w_{jt}\}$ to fit MNL model to estimate the choice probability of t-th arrival customer. Given the prices $r_1, ..., r_6$ for all products and the assortment $S_t \subseteq \{1, ..., 6\}$ offered to customer, the choice probability is given by

$$\phi(j, S_t) = \frac{e^{(w_{jt} - r_j)/\mu}}{\sum_{j \in S_t} e^{(w_{jt} - r_j)/\mu} + 1}, \forall j \in S_t; \ \ \phi(j, S_t) = 0, \forall j \notin S_t,$$

where μ is the variance of customer utility, which is estimated from the sample variance of the willingness to pay data. In our experiment, we use the average value of willingness to pay for product j as the price r_j.

Arrival Process. In this experiment, we consider $T = 1000$ arrival users. Each user arrives at time t indexed by her row number in the survey.

Usage Time. We use uniform random usage time for each product. In particular, each customer uses time slot for a random number of days following distribution uniform$(0, t^{\max})$. We use a same distribution for all products $j = 1, ..., 6$.

2.5.2 Benchmark

In this section, we construct the benchmark policy in this experiment. The benchmark used in our main conclusion is the optimal clairvoyant policy with full information of the arrival sequence. However, the value of this benchmark is difficult to compute. In particular, we need to solve the benchmark value using dynamic programming, which is under the curse of dimensionality. In this experiment, we have a long customer sequence with a large state space for the platform. Thus, using optimal clairvoyant policy as benchmark is impractical for the experiments.

Inspired by the work in [39], we construct an off-line benchmark that uses the full information of the arrival sequence. The value of the benchmark is determined by the following linear programming,

$$
\begin{array}{llcll}
\max_y & \sum_{t=1}^{T} \sum_{S \in \mathcal{S}_t} \sum_{j \in \mathcal{N}} \; r_j \phi^t(j,S) y^t(S) & & & \\
\text{s.t.} & \sum_{k=1}^{t} \sum_{S \in \mathcal{S}_k} \; y_k(S) \phi^k(j,S) \bar{F}_j(t-k) & \leq & c_j, & \forall j \in \mathcal{N}, \forall t \in [T] \\
& \sum_{S \in \mathcal{S}_t} \; y_t(S) & = & 1, & \forall t \in [T] \\
& y_t(S) \geq 0. & & &
\end{array}
\tag{2.7}
$$

In Program 2.7, $\mathcal{S}_t$ is the set of all feasible assortments and $y_t(S)$ is the decision variable, which is the weight on a particular assortment S for customer arrival at time t. The objective is the total expected revenue collected from the arrival sequence. The first constraint in (2.7) is that at any time t, the expected number of busy product j is less or equal to capacity c_j. The second and third constraints require that $\{y_t(S), S \in \mathcal{S}_t\}$ must be a probability vector so that the total weight of the assortments offered to customer is 1. Since this is an off-line benchmark, (2.7) relaxes the feasibility constraint from the assortment must be feasible at the time offered to the expected number of busy units of product j is lower than c_j at any time. Consider that we only have 6 products in total and the possible choices of $\mathcal{S}_t$ is at most $2^6 = 64$, it is numerically easy to solve (2.7). Thus, we choose the value of (2.7) as our benchmark.

2.5.3 Results

In this section, we present the result of our numerical experiments. The value of the off-line benchmark can be deterministically solved given the parameters. To get the value of the myopic policy, we repeat simulating the process of offering myopic policy for $N = 10,000$ times and record its average revenue as the approximation for the myopic policy. In the experiment, the tunable parameters are $t^{\max}$ and C. In Table 2.2, we show the values chosen for $t^{\max}$ and C,

The results for the 16 groups of parameters are given in the Table 2.3 below.

$t^{\max}$	30	120	300	600
C	1	2	5	10

Table 2.2: Values of $t^{\max}, C$ in Experiment

$O^{\max}\backslash C$	1	2	5	10
30	85.42%	96.32%	99.98%	100.00%
120	94.58%	92.20%	96.52%	99.96%
300	98.16%	97.68%	95.88%	96.58%
600	99.23%	99.52%	98.62%	97.69%

Table 2.3: Ratio of Myopic Policy over Benchmark for Different Capacities and Usage Times

2.5.4 Discussion

In Table 2.3, when we fix capacity level C, the ratio is not monotone in the expected usage time (as we assume the usage time is uniformly distributed). The ratio decreases with $t^{\max}$ first and then increases with $t^{\max}$. When usage time is short, it is reasonable to offer the products in greedy way since the units are returned quickly. When the usage time is long, myopic policy can still get a good performance since the benchmark is not able to achieve an outperforming revenue from the customers. If $t^{\max} = \infty$, i.e., the products are non-reusable, as long as the platform can sell all the products within the time horizon, it is the optimal policy. Since the customers are not adversarially chosen in this experiment and we only have 10 or fewer capacities for each product in our experiments, myopic policy can achieve a good performance if the usage time is long.

When the usage time distribution is fixed, the ratio decreases with capacity C first and then increases with C. When the capacity is low, the myopic policy can get a total expected revenue close to the benchmark as the benchmark itself could not achieve an outstanding performance. When the capacity is high, the platform actually should offer the products greedily and the myopic policy is a good solution for the problem.

We need to emphasize that since the arrival sequence is fixed in our experiments instead of being adversarially chosen, the performance of myopic policy could be worse if we set the future arrival customers adaptively against the previous realizations in myopic policy. On the other hand, when the arrival sequence is not adversarially chosen, myopic policy can achieve a better performance than 1/2 competitive ratio. In Table 2.3, myopic policy can achieve at least 90% of the benchmark value in most experiments. In particular, when the capacity C increases to 10, the performance ratio of myopic policy never drops below 96%. This is consistent with the intuition that when the capacity is not scarce, myopic policy is a good and simple policy to choose.

2.6 Conclusions

In this paper, we consider an online assortment optimization problem with resusable resources or products under an adversarial arrival model. Under the assumption, that product usage time distributions do not depend on the user type, we show that a myopic policy is 1/2-competitive. In other words, the policy that offers a myopically optimal assortment to every user from the set of available products achieves an expected revenue that is at least 1/2 times the expected revenue of a clairvoyant algorithm that has full information about the sequence of user types. For the case of reusable capacities, we do not have a good upper bound (LP based or otherwise) for the clairvoyant optimal which makes the comparison with the benchmark challenging. We present a novel stack-based coupling technique that allows us to relate the expected revenue of the clairvoyant optimal to the expected revenue of the myopic policy. This coupling is algorithmic and might be of independent interest.

The assumption that product usage time distribution does not depend on user type is fairly reasonable and satisfied in many settings. We also show that if the assumption is not satisfied, there is no online algorithm that can be constant-factor competitive as compared to our clairvoyant benchmark. Therefore, the assumption is necessary to get any non-trivial performance guarantee for the case of adversarial arrivals.

An interesting open question is to study whether we can obtain results analogous to the online assortment problem with non-reusable capacities. In particular, a $(1 - 1/e)$-competitive algorithm in the adversarial arrivals model and better than $(1 - 1/e)$-approximation for the stochastic arrivals model, both for the setting of large capacities.

Chapter 3

Spatial Distribution of Surge Price under Incentive Compatible Assignment for Drivers

3.1 Introduction

Ride-sharing platforms have experienced extraordinary growth in the last few years. It works as a dynamic marketplace that matches available drivers to sequential ride requests that arise over time. When the platform decides its pricing and matching policy for current demand, both future driver availability or ride requests are not known to the platform. In some extreme cases, the number of ride requests could increase in sudden, which can not be all served by nearby drivers at normal price. For examples, the amount of ride requests can experience a temporary surge when there is a rain or a sports game just ends. We refer this phenomenon as *demand surge*.

In practice, two common operational tools that platform can use to handle demand surge are the price charged to riders and assignment policy between drivers and riders. The real time price serves a twofold role to the two sides of the market. It adjusts the number of effective riders, i.e., the riders who actually request ride, and incentivizes the relocation of drivers to regions of high price and demand. The first role has been drawn much attention in practice.

The second role of the price, together with the assignment policy, is also an important handle to address the demand surge. It can increase the number of available drivers around the demand surge location by relocating drivers there. In particular, since the drivers have access to the price distribution over the network, they have incentive to serve in the high price area. However, there is dis-utility in relocation and since drivers are strategic, they would trade off the gains from high price with the dis-utility in relocation. Therefore, we require assignments to be incentive compatible.

The goal is to design a spatial price distribution and matching policy to maximize several different performance measures while modeling both drivers and riders as strategic agents. Our particular focus is on scenario of a demand shock added on the baseline demand and study the problem in short time scale.

3.1.1 Our Contributions

We consider a fluid approximation of the network. The arrival process of drivers and riders is exogenous. We assume in the baseline case there is sufficient supply to satisfy all demand. We consider the scenario of a demand shock and model it as a short-lived increase in the arrival rate of riders at a particular location. We present the details of the model in Section 3.3. The goal of the platform is to decide prices at all locations such that certain performance measures (throughput/revenue) are maximized when riders and drivers are strategic.

Our main contributions are the following:

1. **Structure of Pricing Policy**. We show the following results for optimal pricing policy.

    (a) Optimal pricing policy can be completely determined by the prices at surge locations. For instance, if there is one demand surge, it is a single parameter problem.

(b) In particular, for the one demand surge problem, the price is highest at surge location and decreases as we move away from the surge that depends on the dis-utility function, until the price equals the baseline price. Figure 3.1 represents an example. We refer the locations where $p > p_b$ as surge region.

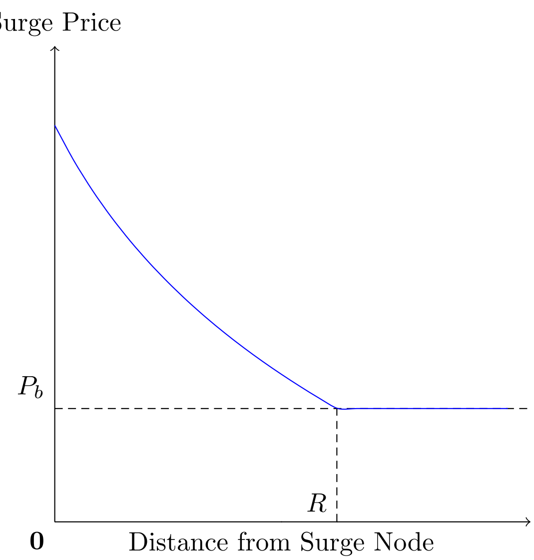

Figure 3.1: Revenue Maximization Surge Prices as a Function of Distance from Demand Surge Node **0**, Price Constrained: $p(x) \geq p_b$

(c) Since the policy depends on a small number of parameters (equal to the number of surge locations), we propose an algorithm to compute them efficiently.

2. **Structure of Assignment Policy**. We also show the optimal incentive compatible assignment assigns all excess supply of drivers in the surge region to the riders at the surge location. Therefore, all drivers in surge region are matched to demand in surge region. We refer this as *star assignment*. All drivers in surge region are matched to the demand. The drivers outside surge region do not relocation. For this assignment, we assume drivers can relocate instantaneously even though their is a dis-utility.

Extensions. We discuss several extensions and show that the optimal policy has similar structure.

1. In particular, we consider rider relocation and multiple demand shocks. We extend our structural optimal results to these cases and give an efficient algorithm to find the optimal policy.

2. We also consider the case that relocation is not instantaneous. The pricing policy satisfies similar structure as before. However, optimal assignment is not a star assignment, but needs to be computed by solving an linear programming(LP).

We also conduct numerical experiments for several variants of our model discussed above, and discuss the insights from the solution. For instance, for the model where relocation is not instantaneous, we observe that the assignment is a *hopping assignment* where general relocation of excess supply is forwards surge location but the drivers may be matched to riders before reaching the surge location.

3.1.2 Literature Review

Our work focuses on the design of price distribution and matching policy on ride-sharing platform under demand shock. It sits in the general areas that discuss the revenue management under limited capacity and assignment of supply to demand on two-sided platform. In this part, we discuss the literature in the streams of these areas that are related to our work.

Firstly, a set of papers discuss general peak-load pricing problem that charges higher prices during peak periods of demand, see [75], [25] and [34]. The motivation of these papers is to increase revenue by relocating demand from the peak period to the off-peak period. Also, the value of dynamic prices in systems that experience congestion has been extensively studied in the literature [21], [5] and [47]. Banerjee et al. [68] consider a network of queues with stationary demand arrival process and a closed network for drivers to describe their arrival processes and actions, and use Markov chain to model the change of system status. The authors show that static pricing is as good as state-contingent pricing policy asymptotically, and state-contingent pricing policy is more robust with respect to inaccurate parameters.

Our work is also related to the rapidly increasing literature that explores the design and operations of on-line marketplaces. Allon et al. [3] study the role of a platform in improving the operational efficiency of large-scale service marketplaces. More recent works also provide insights about how platform impacts the behavior of users. For example, Benjaafar et al. [9] research the affect of product-sharing platform on the decision of individual to own. In [41], [4] and [44], the authors discuss how the reduction of search cost can result inefficiencies of online matching markets. We refer readers to [69] and [72] for a summary of works in two-sided markets.

Next, we discuss the literature on ride-sharing platforms. Gurvich et al. [40] study the cost of self-scheduling capacity in a news vendor type model in which the firm chooses the amount of agents it recruits and selects a compensation level in each period. Cachon et al. [17] consider a single node model that matches the demand and supply in high and low demand scenarios and analyze various compensation schemes in a setting that the platform considers the long-term and short-term incentives of drivers. The authors show that a state dependent pricing policy is better than a unique price for both scenarios, and fixed commission contracts can be nearly optima. Besides state dependent pricing policy, how stochasticity in market conditions affects the pricing and compensation decisions of platform is also extensively discussed in [40], [80] and [7].

There is another set of papers that explore the problem of matching supply with demand on ride-sharing platform. Feng et al. [33] compare the waiting time performance of on-demand matching versus traditional street hailing matching. Hu and Zhou [42] consider a dynamic matching problem as well as the structure of optimal policies. Besides them, [66] develop a heuristic to determine the assignment between drivers and potential riders based on a continuous linear program that maximizes the number of matches in a network. They also establish its asymptotic optimality. Afeche et al. [1] discuss how platforms can optimally accept ride requests and reposition drivers within a two-location network without pricing. Yang et al. [88] consider a ride-sharing service motivated model that the agents compete for time changing but location based resources. Since the incentive of drivers is considered in our model, we point out that several papers, such as [42] and [3], explore the process for matching supply to demand when capacities are exogenous and all participants have preferences for being matched.

The works that are most closely related to ours are those that study pricing on ride-sharing platform with spacial considerations. Castillo et al. [19] take space into account and point out that surge pricing can help to avoid an inefficient situation that the earnings of drivers are low due to long pick-up times. Bimpikis et al. [12] focus on pricing for steady-state conditions in a network. In their model, drivers behave in equilibrium and decide whether and when to provide service as well as where to relocate. Buchholz [16] structurally estimates a spatial equilibrium model to understand the welfare cost of taxi fare regulations. In sum, these papers focus on the equilibrium of ride-sharing system and analyze spatial pricing policy under time invariant status.

Outline. The rest of the chapter is organized as follows. We introduce the specifics of the model and problem in Section 3.2. We present the structure of optimal policy in Section 3.3. We discuss the extensions of our model in Section 3.4 - 3.7, and we present numerical study in Section 3.9. We conclude our findings in Section 3.10.

Notations. In our model, we use $\mathbb{V}$ to present the space driver and riders live in. We use bold character to represent a location in $\mathbb{V}$ or a set. In particular, we use lowercase letter to present a location, e.g., $\boldsymbol{x} \in \mathbb{V}$, and uppercase to represent a set. We use regular character to present a scalar number or a scalar function.

3.2 Our Model and Problem Formulation

In this section, we introduce our model and assumptions, and present the problem formulation.

Arrival Process. We consider a fluid arrival process for drivers with an exogenous, time invariant rate $\mu(\boldsymbol{x})$ at any $\boldsymbol{x} \in \mathbb{V}$. Similarly, we consider a fluid arrival process for riders, with rate $\lambda(\boldsymbol{x})$. We assume

Assumption 3.1.

$$\lambda(\boldsymbol{x}) \leq \mu(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{V}.$$

This is a reasonable assumption in practice. If Assumption 3.1 is violated, we can handle it by reforming it to a new problem with Assumption 3.1 satisfied. Next, we assume

Assumption 3.2 (Integrable Condition)**.**

$$\int_{\boldsymbol{x}\in\mathbb{V}} \mu(\boldsymbol{x})d\boldsymbol{x} < \infty, \int_{\boldsymbol{x}\in\mathbb{V}} \lambda(\boldsymbol{x})d\boldsymbol{x} < \infty,$$

and λ and μ are finite everywhere in $\mathbb{V}$.

Assumption 3.2 is an integrability conditions that ensures demand, supply, revenue and other metrics are bounded. We refer to $\lambda(\boldsymbol{x})$ as baseline demand at $\boldsymbol{x}$. As we consider the arrivals of drivers and riders as fluids, we assume instantaneous abandon of unmatched units.

Assumption 3.3. *All unmatched riders or drivers are abandoned instantaneously.*

Utility of Riders. Riders have willingness-to-pay(WTP) given by a cumulative density function F_v same for everyone. Therefore, at any location $\boldsymbol{x}$, if the price is $p(\boldsymbol{x})$, there is a rate of $\lambda(x)\bar{F}_v(p(\boldsymbol{x}))$ riders to request rides. Since we have $\mu(\boldsymbol{x}) \geq \lambda(\boldsymbol{x})$ from Assumption 3.1, there is sufficient supply to serve demand at location $\boldsymbol{x}$. Due to Assumption 3.3, remaining riders instantaneously leave and drivers are abandoned of rate $\mu(\boldsymbol{x}) - \lambda(x)\bar{F}_v(p(\boldsymbol{x}))$. Next, define $p_b := \arg\max p\bar{F}_v(p)$, and we assume

Assumption 3.4. *$p\bar{F}_v(p)$ is weakly decreasing on $p \geq p_b$.*

Assumption 3.4 is not very restrictive, it holds, for example, if F_v has an increasing hazard-rate, i.e., $h_F(x) = f_v(x)/\bar{F}_v(x)$ is an increasing function in x, which is quite general. Assumption 3.4 is important to prove our main result Theorem 3.4.

Utility of Drivers. We start this part by introducing the dis-utility function c. For any $\boldsymbol{x} \in \mathbb{V}$, the dis-utility in repositioning the driver at $\boldsymbol{x}$ to $\boldsymbol{y}$ is modeled using function $c(\boldsymbol{x}, \boldsymbol{y})$. In particular, c measures the magnitude of unhappiness of driver location. We assume c is non-negative and satisfies the triangle inequality,

$$c(\boldsymbol{x}, \boldsymbol{y}) + c(\boldsymbol{y}, \boldsymbol{z}) \geq c(\boldsymbol{x}, \boldsymbol{z}), \ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}. \tag{3.1}$$

We discuss the numerical result that relaxes (3.1) in the Section 3.9. We also assume the relocation completes instantaneously. We relax this instantaneous relocation assumption in Section 3.7.

Next, we define the utility of driver at $\boldsymbol{x}$ relocated to $\boldsymbol{y}$ as

$$U(\boldsymbol{x}, \boldsymbol{y}) = p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y}), \tag{3.2}$$

i.e., the price at the relocated location minus the dis-utility of location.

Incentive Compatibiltiy Constraint. In assignment policy, platform can relocate a driver only if it is incentive compatible for the driver. In particular, for $\forall \boldsymbol{x} \in \mathbb{V}$, the best response for driver at $\boldsymbol{x}$ is to relocate to $\boldsymbol{y}$, where $\boldsymbol{y} \in \arg\max_{\boldsymbol{y}} U(\boldsymbol{x}, \boldsymbol{y})$.

For the above best response model, the drivers assume a probability 1 of being matched if platform relocates them to a different location $\boldsymbol{y}$. We will see later that this assumption holds for relocation inside the surge region. Then, we define the incentive compatibility constraint for assignment as that platform must assign driver at $\boldsymbol{x}$ to a location in $\arg\max_{\boldsymbol{y}} U(\boldsymbol{x}, \boldsymbol{y})$.

Since platform can relocate drivers, we denote $\nu(\boldsymbol{x})$ as the arrival rate of drivers after relocation. Then, the rate of actual rides at $\boldsymbol{x}$ becomes $\min\Big(\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \nu(\boldsymbol{x})\Big)$.

Objective of Platform. We assume a fixed proportion of ride price as commission fee for the platform in our model. Therefore, the goal of platform is to maximize the revenue, which is also commission maximal.

If the demand is at the baseline level $\lambda(\boldsymbol{x})$, the optimal pricing policy is to charge the baseline price p_b everywhere. However, if there is a demand surge, the pricing policy is not p_b any more and we discuss its solution in the following Section 3.3.

3.3 Surge Pricing Policy

In this section, we show the formulation of the surge pricing problem and state its structural properties. We discuss the baseline problem without demand shock first, and then extend our analysis to problem with demand shock.

3.3.1 Optimal Policy of Baseline Problem

Under the baseline demand, we have $\lambda(\boldsymbol{x}) \leq \mu(\boldsymbol{x})$ for all locations. Since we have enough drivers to serve all riders, there should be no benefit to relocate drivers. To solve the baseline problem, we firstly introduce an auxiliary problem,

$$\begin{aligned} \max_{p(\boldsymbol{x})} \int_{\mathbb{V}} p(\boldsymbol{x})\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} \\ s.t.\ \ p^{\max} \geq p(\boldsymbol{x}) \geq p^{\min}. \end{aligned} \tag{3.3}$$

where decision variable $p(\boldsymbol{x})$ is the price charged to riders at $\boldsymbol{x}$. The objective is the overall revenue rate from all ride requests. Since parameter $\lambda(\boldsymbol{x})$ is unchanged in time invariant, the actual revenue is revenue rate multiplied by a fixed time length. Thus, it is equivalent to maximize the revenue rate in Problem (3.3). The upper bound $p^{\max}$ and lower bound $p^{\min}$ denote a reasonable range for the price. For instance, $p^{\min}$ can be set at 0 and $p^{\max}$ can be set at the support value of willingness to pay distribution F_v.

We claim Problem (3.3) gives an upper bound to the baseline problem. This is because Problem (3.3) relaxes all the assignment constraints of baseline problem, and the revenue rate collected at location $\boldsymbol{x}$ for baseline problem is capped by $p(\boldsymbol{x})\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)$ due to that the number of actual rides can not surpass the ride requests at any location $\boldsymbol{x}$. Next, we solve Problem (3.3), and claim its solution is feasible for the baseline problem. Therefore it is also an optimal solution for the baseline problem.

Problem (3.3) is actually a point-wise maximization problem for each $p(\boldsymbol{x})$ after exchanging the maximization and integral operations. We can rewrite the objective function in Problem (3.3) as,

$$\int_{\mathbb{V}} \lambda(\boldsymbol{x}) \left(\max_{p(\boldsymbol{x})} p(\boldsymbol{x}) \bar{F}_v\big(p(\boldsymbol{x})\big) \right) d\boldsymbol{x}.$$

The problem is then to maximize $p\bar{F}_v(p)$ where the solution is p_b. Thus, the optimal solution to Problem (3.3) is $p(\boldsymbol{x}) \equiv p_b$. As the price is same everywhere, it is incentive compatible to retain all drivers at their original locations. Hence, it is the optimal solution for baseline problem as well. Due to that $p(\boldsymbol{x}) = p_b$ is the optimal pricing policy to baseline problem, we denote p_b as the baseline price in Assumption 3.4.

3.3.2 Optimal Surge Pricing

In this section, we present our surge pricing problem. We consider demand shock as additional ride arrival rate over the baseline demand. In particular, we use Λ to denote the extra potential riders requesting cars at location $\mathbf{0}$, and Λ is also exogenous and time invariant in its life period. Here, we emphasize that Λ is a point mass on $\mathbf{0}$ with the same unit of $\int_{\mathbb{V}} \lambda(\boldsymbol{x}) d\boldsymbol{x}$. We model the demand shock as a point mass for the brevity of the model and result. The additional riders also behave strategically with the willingness-to-pay F_v and they are abandoned if unmatched.

In the section, we focus on the case that demand surge only happens at one location $\mathbf{0}$. In Section 3.5, we present the results of the problem that demand surges appear at multiple locations.

Due to the existence of demand surge, the platform does not have enough drivers at $\mathbf{0}$ to serve riders, i.e., $(\lambda(\mathbf{0}) + \Lambda)\bar{F}_v(p(\mathbf{0})) \leq \mu(\mathbf{0})$ is violated, and the baseline policy is not optimal either. Our surge pricing problem can be formulated as follows

$$\max_{\Gamma,\gamma,\nu,r,\nu_{\mathbf{0}},p} \quad p(\mathbf{0}) \min\Big(\nu_{\mathbf{0}}, \Lambda \bar{F}_v\big(p(\mathbf{0})\big)\Big) + \int_{\mathbb{V}} p(\boldsymbol{x}) \min\Big(\nu(\boldsymbol{x}), \lambda(\boldsymbol{x}) \bar{F}_v\big(p(\boldsymbol{x})\big)\Big) d\boldsymbol{x} \quad (3.4)$$

$$s.t. \quad \nu(\boldsymbol{x}) = \int_{\mathbb{V}} \Gamma(\boldsymbol{y}, \boldsymbol{x}) d\boldsymbol{y},\ \forall \boldsymbol{x} \in \mathbb{V} \quad (3.5)$$

$$\mu(\boldsymbol{x}) = \int_{\mathbb{V}} \Gamma(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y} + \gamma(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathbb{V} \quad (3.6)$$

$$\nu_{\mathbf{0}} = \int_{\mathbb{V}} \gamma(\boldsymbol{x}) d\boldsymbol{x}, \quad (3.7)$$

$$(\Pi): \quad r(\boldsymbol{x}) \geq p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y}),\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V} \quad (3.8)$$

$$\Gamma(\boldsymbol{x}, \boldsymbol{y}) \Big(r(\boldsymbol{x}) - \big(p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y})\big) \Big) = 0,\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V} \quad (3.9)$$

$$\gamma(\boldsymbol{x}) \Big(r(\boldsymbol{x}) - \big(p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\big) \Big) = 0,\ \forall \boldsymbol{x} \in \mathbb{V} \quad (3.10)$$

$$\Gamma(\boldsymbol{x}, \boldsymbol{y}), \gamma(\boldsymbol{x}), \nu(\boldsymbol{x}), r(\boldsymbol{x}), \nu_{\mathbf{0}} \geq 0,\ \forall \boldsymbol{x} \in \mathbb{V} \quad (3.11)$$

$$p^{\max} \geq p(\boldsymbol{x}) \geq p^{\min}, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}, \quad (3.12)$$

where the decision variables are $\Gamma(\boldsymbol{x}, \boldsymbol{y})$, relocation flow of drivers from $\boldsymbol{x}$ to $\boldsymbol{y}$; $\gamma(\boldsymbol{x})$, relocation flow of drivers from $\boldsymbol{x}$ to demand surge location $\mathbf{0}$; $\nu(\boldsymbol{x})$, arrival rate of drivers at $\boldsymbol{x}$ after relocation of drivers; $r(\boldsymbol{x})$, maximum utility for drivers at location $\boldsymbol{x}$; $\nu_{\mathbf{0}}$, arrival rate of drivers at $\mathbf{0}$ after relocation; $p(\boldsymbol{x})$, the price at location $\boldsymbol{x}$. The objective is the overall revenue rate including the revenue earned from the baseline demand and the demand surge.

For the constraints, (3.5), (3.6) and (3.7) are flow balance equations, measuring the inflow and outflow of drivers at each location. We measure the flow into demand surge location $\mathbf{0}$ separately because Λ is a point mass of riders at $\mathbf{0}$, the integral of $\gamma(\boldsymbol{x})$ has the same unit with Λ. (3.8), (3.9) and (3.10) are the incentive compatibility constraints of drivers. Under optimality, $r(\boldsymbol{x}) = \max_{\boldsymbol{y}} p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y})$ is the maximum utility for drivers at $\boldsymbol{x}$. Since a driver at location $\boldsymbol{x}$ only accepts rides from locations that provide the maximum utility, there is driver relocation from $\boldsymbol{x}$ to $\boldsymbol{y}$ only if $r(\boldsymbol{x}) = p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y})$, i.e., the utility of location $\boldsymbol{y}$ is highest for drivers at point $\boldsymbol{x}$. If $r(\boldsymbol{x}) > p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y})$, the utility of location $\boldsymbol{y}$ is not highest for drivers at $\boldsymbol{x}$, then there is no driver relocated from $\boldsymbol{x}$ to $\boldsymbol{y}$. Thus, at least one of $\Gamma(\boldsymbol{x}, \boldsymbol{y})$ and $r(\boldsymbol{x}) - (p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y}))$ is 0 and so is $\gamma(\boldsymbol{x})$ and $r(\boldsymbol{x}) - (p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y}))$. If the solution to $\max_{\boldsymbol{y}} p(\boldsymbol{y}) - c(\boldsymbol{x}, \boldsymbol{y})$ is unique and achieved at $\boldsymbol{y}$, all drivers at $\boldsymbol{x}$ only provide service at $\boldsymbol{y}$.

Since Problem Π is a functional optimization problem with infinite dimensions, there is no general method to solve it. Thus, we need to develop structural properties for the problem to simplify it. Next, we present these properties for the optimal pricing and assignment policy, and the algorithm of solving Problem Π.

3.3.3 Structure of Optimal Solution and Algorithm

We start from the following three propositions of the main Problem Π. These properties can simplify the problem by shrinking its feasible region.

Proposition 3.1 (Flow Exclusiveness). *There exists an optimal solution for the surge pricing Problem* Π *that satisfies*

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k}) = 0, \ \forall \boldsymbol{x_i} \neq \boldsymbol{x_j}, \boldsymbol{x_j} \neq \boldsymbol{x_k} \in \mathbb{V}.$$

We present the proof in Appendix A.1. Proposition 3.1 eliminates solutions with drivers repositioning into a specific node and drivers repositioning out of that same node simultaneously, i.e., we only consider the solutions that any node can only be at most a driver supplier or a driver receiver, and still achieve the same optimal value. We denote a node as a driver supplier if there are drivers relocated from this location to other locations and as a driver receiver if there are drivers relocated from other locations to this location. Next, we present another proposition which shapes the optimal pricing policy.

Proposition 3.2 (Minimal Incentive Compatibility). *There exists an optimal solution for the surge pricing Problem* Π *that satisfies*

$$\forall \boldsymbol{x_i}, \boldsymbol{x_j} \in \mathbb{V}, \ p(\boldsymbol{x_i}) \geq p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}).$$

We present the proof in Appendix A.2. According to Proposition 3.2, the price at any location defines an upper and lower bound for the prices on all other locations. Notice that in the original problem, pricing policy satisfying $p(\boldsymbol{x_i}) < p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j})$ is still feasible, as long as the prices are in the interval $[p^{\min}, p^{\max}]$. However, under this pricing policy, we are not able to retain drivers at location $\boldsymbol{x_i}$ as this assignment is not incentive compatible for drivers. If we increase $p(\boldsymbol{x_i})$ to $p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j})$, we can still relocate drivers at location $\boldsymbol{x_i}$, achieving a same assignment as that for $p(\boldsymbol{x_i}) < p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j})$. Furthermore, when $p(\boldsymbol{x_i}) = p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j})$, we can also retain drivers at location $\boldsymbol{x_i}$. In this way, we increase the number of feasible assignments and improve the overall revenue.

Another proposition directly resulted from Proposition 3.1 and Proposition 3.2 is that we only need to consider those solutions that move drivers from nodes at baseline demand level into nodes with demand surge.

Proposition 3.3 (Star Assignment)**.** *There exists an optimal solution for the surge pricing Problem* Π *that satisfies*

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) = 0, \ \textit{for}\ \boldsymbol{x_i} \neq \boldsymbol{x_j} \ \textit{and}\ \boldsymbol{x_j} \neq \mathbf{0}.$$

We present the proof in Appendix A.3. Proposition 3.3 states that there is an optimal solution where every relocation is to the surge node. The shape of resulted assignment likes a star as the relocated drivers flow into the demand surge location from all directions and the remaining drivers stay at their locations. Proposition 3.3 reduces the dimension of the feasible region dramatically by limiting the relocation flows. Both Proposition 3.1 and Proposition 3.3 are about the structures of optimal assignment structure and driver flows.

We emphasize that the original Problem Π may have multiple optimal solutions and Proposition 3.1, 3.2 and 3.3 guarantee that there exists an optimal solution that satisfies them. Our goal is to find an optimal solution (both pricing and assignment) that satisfies above structural properties.

We make an additional assumption that $p^{\min} = p_b$ in this section. Under this assumption, we show the optimal pricing policy is

$$p(\boldsymbol{x}) = \max\Big(p_b, p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\Big),\ \forall \boldsymbol{x} \in \mathbb{V}.$$

Therefore, $p(\boldsymbol{x})$ is completely determined by $p(\mathbf{0}$ and we only need to determine $p(\mathbf{0})$ for the optimal solution. Figure 3.2 represents an example for the optimal pricing policy when $\mathbb{V} = [0, \infty)$ and demand surge happens at $\mathbf{0}$. The surge region (locations with price higher than p_b) is $x \in [0, R]$.

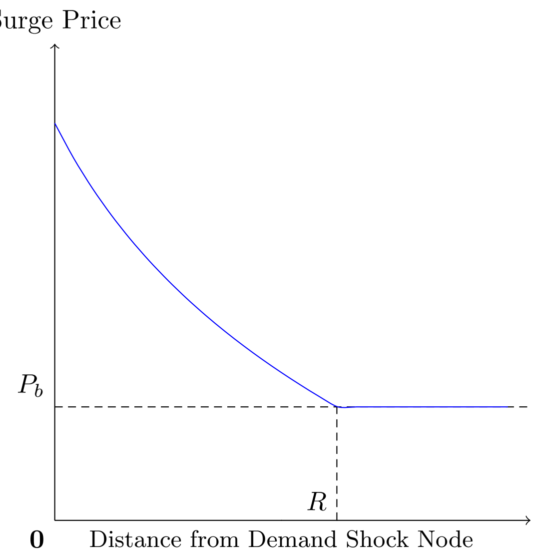

Figure 3.2: Revenue Maximization Surge Prices as a Function of Distance from Demand Surge Node $\mathbf{0}$, Price Constrained: $p(x) \geq p_b$

Under this pricing policy, we maximize the revenue by assigning drivers in the following procedure. Within surge region,

1. Assign drivers at demand shock location $\mathbf{0}$ and relocate the extra drivers $\mu(\boldsymbol{x}) - \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)$ to $\mathbf{0}$;

2. If the demand surge is not completely served, assign drivers at the lowest price location to $\mathbf{0}$, until the demand surge is processed or the surge region is out of available drivers;

3. Retain the remaining drivers at their original locations.

Outside surge region, platform matches the drivers with effective riders at same location. The details of assignment policy is in the following Algorithm 3.1.

Algorithm 3.1 Optimal Assignment under Pricing Policy (3.13)

$\tilde{\Lambda} = \Lambda \bar{F}_v\big(p(\mathbf{0})\big), \tilde{\mu} = \int_{\mathbb{S}} \mu(\boldsymbol{x})d\boldsymbol{x}, \hat{\mu} = \int_{\mathbb{S}} \mu(\boldsymbol{x}) - \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x}$
if $\tilde{\Lambda} \geq \tilde{\mu}$ **then**
  Set $\nu_{\mathbf{0}} = \tilde{\mu}, \nu(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \mathbb{S}$
else if $\tilde{\mu} > \tilde{\Lambda} \geq \hat{\mu}$ **then**
  Set $\nu_{\mathbf{0}} = \tilde{\Lambda}$
  Denote $\hat{p}$ as the solution p of $\int_{\{\boldsymbol{x}|p(\boldsymbol{x})\leq p, \boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} = \tilde{\Lambda} - \hat{\mu}$
  for $\boldsymbol{x} \in \mathbb{S}$ **do**
    if $p(\boldsymbol{x}) > \hat{p}$ **then**
      Set $\nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)$
    else if $p(\boldsymbol{x}) \leq \hat{p}$ **then**
      Set $\nu(\boldsymbol{x}) = 0$
    end if
  end for
else if $\tilde{\Lambda} < \hat{\mu}$ **then**
  Set $\nu_{\mathbf{0}} = \hat{\mu}, \nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \forall \boldsymbol{x} \in \mathbb{S}$
end if
for $\boldsymbol{x} \notin \mathbb{S}$ **do**
  Set $\nu(\boldsymbol{x}) = \mu(\boldsymbol{x})$
end for

Where we denote $\mathbb{S} := \{\boldsymbol{x}|p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\}$ as surge region. Therefore, we have the following theorem.

Theorem 3.4. *If* $p^{\min} = p_b$*, the optimal pricing policy of Problem* Π *is in form of,*

$$p(\boldsymbol{x}) = \max\Big(p_b, p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\Big), \ \forall \boldsymbol{x} \in \mathbb{V}. \tag{3.13}$$

Assignment policy determined by Algorithm 3.1 is an optimal assignment of Problem Π.

We present the proof in Appendix A.4.

We emphasize that Theorem 3.4 requires assumption $p^{\min} = p_b$. This condition ensures that we can apply the monotonicity of $p\bar{F}_v(p)$ specified in Assumption 3.4 to construct the optimal pricing policy. We discuss the relaxation of this condition in Section 3.6.

Algorithm for Optimal Policy. By Theorem 3.4, the problem is reduced to deciding the price at the demand shock location. However, even though this is a single variable problem, we do not know a closed form analytical solution for the optimal value. The dependence of the total revenue rate on $p(\mathbf{0})$ is not in tractable form. In particular, we do not know of a closed form for the integral of revenue over $\mathbb{S}$ as a function of $p(\mathbf{0})$. We also need to compare the values of $\tilde{\mu}, \tilde{\Lambda}, \hat{\mu}$ but they are defined implicitly. Thus, we conduct a line search for $p(\mathbf{0})$ from p_b to $p^{\max}$ and pick the value that maximizes total revenue rate.

Algorithm 3.2 Algorithm for Problem Π with Price Constrained: $p^{\min} = p_b$

1: Set $M = 1000,\ \delta = \frac{p^{\max} - p_b}{M}$
2: **for** $i = 0 : M$ **do**
3: Set $p(\mathbf{0}) = p_b + i\delta$
4: Apply Theorem 3.4 to read off the pricing and assignment policy
5: Compute the overall revenue rate $R(i)$ under the given pricing and assignment policy
6: **end for**
7: Return $p^*(\mathbf{0}) = p_b + (\arg\max_i R(i))\delta$.

Another property of our policy is that all the drivers within surge region are matched with riders. We conclude this result as that the matching probability is always 1 inside the surge region. Therefore a relocated driver is always matched.

Proposition 3.5. *For $\tilde{\Lambda}, \hat{\mu}$ defined in Algorithm 3.1, $p^*(\mathbf{0})$ is the optimal solution in Algorithm 3.2, we must have*

$$\hat{\mu} \le \tilde{\Lambda}, \tag{3.14}$$

which implies the matching probability is 1 in surge region $\mathbb{S}$.

The matching probability is 1 in $\mathbb{S}$ is because when $\hat{\mu} \le \tilde{\Lambda}$, we have the following assignment for $\boldsymbol{x} \in \mathbb{S}$:

1. If $\tilde{\mu} \le \tilde{\Lambda}$, we have

$$\nu_{\mathbf{0}} = \tilde{\mu} \le \tilde{\Lambda}, \nu(\boldsymbol{x}) = 0 \le \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big).$$

2. If $\hat{\mu} \le \tilde{\Lambda} < \tilde{\mu}$, we have

$$\nu_{\mathbf{0}} = \tilde{\Lambda}, \nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \forall p(\boldsymbol{x}) > \hat{p};\ \nu(\boldsymbol{x}) = 0 \le \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \forall p(\boldsymbol{x}) \le \hat{p}.$$

We present the proof in Appendix A.5. Consequently, we always have $\nu_{\mathbf{0}} \leq \tilde{\Lambda},\ \nu(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)$, i.e., more effective riders than drivers in surge region.

When we define the utility function for drivers, we use the price instead of the price times matching probability in utility function as we assume a probability 1 of being matched if platform relocates driver to a different location. Proposition 3.5 validates this assumption.

3.4 Strategic Rider Relocation

In this section, we discuss the case that riders are also strategic in relocation. We show that we can reuse the propositions above to solve this new problem. First of all, we define the effective cost of riders for relocation analogue to the utility of drivers.

Effective Cost of Riders. We start this part by introducing the dis-utility function w for riders. For any $\boldsymbol{x} \in \mathbb{V}$, the dis-utility in repositioning the rider at $\boldsymbol{x}$ to $\boldsymbol{y}$ is modeled using function $w(\boldsymbol{x}, \boldsymbol{y})$, which measures the magnitude of unhappiness of rider location. We assume w is non-negative and satisfies the triangle inequality. We also assume the relocation completes instantaneously same as the drivers.

Next, we define the effective cost of rider at $\boldsymbol{x}$ relocated to $\boldsymbol{y}$ as

$$C(\boldsymbol{x}, \boldsymbol{y}) = p(\boldsymbol{y}) + w(\boldsymbol{x}, \boldsymbol{y}), \tag{3.15}$$

i.e., the price at $\boldsymbol{y}$ plus the dis-utility of relocation.

Incentive Compatibiltiy Constraint. When platform assignment relocates a rider, it is feasible only if it is incentive compatible for this rider. In particular, for $\forall \boldsymbol{x} \in \mathbb{V}$, the best choice for rider at $\boldsymbol{x}$ is to relocate to $\boldsymbol{y}$, where $\boldsymbol{y} \in \arg\min_{\boldsymbol{y}} C(\boldsymbol{x}, \boldsymbol{y})$. Then, we define the incentive compatibility constraint for rider assignment as that platform must assign rider at $\boldsymbol{x}$ to a location in $\arg\min_{\boldsymbol{y}} C(\boldsymbol{x}, \boldsymbol{y})$. After relocation, this rider contributes to the revenue at $\boldsymbol{y}$ if he requests ride and gets matched.

Pricing Problem Formulation with Strategic Rider Relocation. In this part, we show the propositions and theorems for solving the problem with strategic rider relocation. The problem is now formulated as,

$$
\begin{aligned}
& \max && p(\mathbf{0})\min(\nu_{\mathbf{0}},\hat{\Lambda}) + \int_{\mathbb{V}} p(\boldsymbol{x})\min\Big(\nu(\boldsymbol{x}),\hat{\lambda}(\boldsymbol{x})\Big)d\boldsymbol{x} && (3.16)\\
& s.t. && \Lambda\bar{F}_v\big(q(\mathbf{0})\big) = \Lambda_0 + \int_{\mathbb{V}} \Delta(\boldsymbol{x})d\boldsymbol{x}, && (3.17)\\
& && \lambda(\boldsymbol{x})\bar{F}_v\big(q(\boldsymbol{x})\big) = \alpha(\boldsymbol{x}) + \int_{\mathbb{V}} A(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y},\ \forall \boldsymbol{x}\in\mathbb{V} && (3.18)\\
& && \hat{\Lambda} = \Lambda_0 + \int_{\mathbb{V}} \alpha(\boldsymbol{x})d\boldsymbol{x}, && (3.19)\\
& && \hat{\lambda}(\boldsymbol{x}) = \Delta(\boldsymbol{x}) + \int_{\mathbb{V}} A(\boldsymbol{y},\boldsymbol{x})d\boldsymbol{y},\ \forall \boldsymbol{x}\in\mathbb{V} && (3.20)\\
(\Pi_r): & && q(\boldsymbol{x}) \le p(\boldsymbol{y}) + w(\boldsymbol{x},\boldsymbol{y}),\ \forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{V} && (3.21)\\
& && \Lambda_0\Big(p(\mathbf{0}) + w(\mathbf{0},\mathbf{0}) - q(\mathbf{0})\Big) = 0, && (3.22)\\
& && \Delta(\boldsymbol{x})\Big(p(\boldsymbol{x}) + w(\mathbf{0},\boldsymbol{x}) - q(\mathbf{0})\Big) = 0,\ \forall \boldsymbol{x}\in\mathbb{V} && (3.23)\\
& && \alpha(\boldsymbol{x})\Big(p(\mathbf{0}) + w(\boldsymbol{x},\mathbf{0}) - q(\boldsymbol{x})\Big) = 0,\ \forall \boldsymbol{x}\in\mathbb{V} && (3.24)\\
& && A(\boldsymbol{x},\boldsymbol{y})\Big(p(\boldsymbol{y}) + w(\boldsymbol{x},\boldsymbol{y}) - q(\boldsymbol{x})\Big) = 0,\ \forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{V} && (3.25)\\
& && A(\boldsymbol{x},\boldsymbol{y}),\Delta(\boldsymbol{x}),\alpha(\boldsymbol{x}),q(\boldsymbol{x}),\hat{\lambda}(\boldsymbol{x}),\Lambda_0,\hat{\Lambda} \ge 0,\ \forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2 &&\\
& && \text{Constraints (3.5) - (3.12).} &&
\end{aligned}
$$

Here, the additional decision variables and constraints model the relocation of riders and the corresponding incentive compatibility constraints. In particular, Λ_0 denotes the relocation of riders from demand shock location $\mathbf{0}$ to $\mathbf{0}$; $\Delta(\boldsymbol{x})$ denotes the relocation of riders from demand shock location $\mathbf{0}$ to a non-shock location $\boldsymbol{x}$; $\alpha(\boldsymbol{x})$ denotes the relocation of riders from a non-shock location $\boldsymbol{x}$ to demand shock location $\mathbf{0}$; $A(\boldsymbol{x},\boldsymbol{y})$ denotes the relocation of riders from a non-shock location $\boldsymbol{x}$ to another non-shock location $\boldsymbol{y}$; $q(\boldsymbol{x})$ is the minimum effective cost of requesting a ride for riders at $\boldsymbol{x}$. (3.17), (3.18), (3.19) and (3.20) denote the flow balance equations of riders. (3.21), (3.22), (3.23), (3.24) and (3.25) are incentive compatibility constraints that there is rider relocated from $\boldsymbol{x}$ to $\boldsymbol{y}$ only if the effective cost at $\boldsymbol{y}$ is minimum for riders at $\boldsymbol{x}$.

3.4.1 Structure of Optimal Solution for Problem Π_r

We have similar results for the problem with strategic rider relocation in addition to Proposition 3.1, 3.2 and 3.3.

Proposition 3.6. *There exists an optimal solution for Problem Π_r that satisfies the following constraints*

1. *Flow Exclusiveness for Riders:*

$$\begin{aligned}&A(\boldsymbol{x_i},\boldsymbol{x_j})A(\boldsymbol{x_j},\boldsymbol{x_k})=0,\\&\Delta(\boldsymbol{x_j})A(\boldsymbol{x_j},\boldsymbol{x_k})=0,\\&A(\boldsymbol{x_i},\boldsymbol{x_j})\alpha(\boldsymbol{x_j})=0,\\&\Delta(\boldsymbol{x_j})\alpha(\boldsymbol{x_j})=0,\ \ \forall\boldsymbol{x_i}\neq\boldsymbol{x_j},\boldsymbol{x_j}\neq\boldsymbol{x_k}\in\mathbb{V}.\end{aligned}$$

2. *Minimal Incentive Compatibility for Riders:*

$$p(\boldsymbol{x_i})\geq p(\boldsymbol{x_j})-w(\boldsymbol{x_j},\boldsymbol{x_i})\ \forall\boldsymbol{x_i},\boldsymbol{x_j}\in\mathbb{V}.$$

3. *Star Assignment for Riders:*

$$\Delta(\boldsymbol{x_i})=0,\ \textit{for}\ \ \boldsymbol{x_i}\neq\boldsymbol{0};\ A(\boldsymbol{x_i},\boldsymbol{x_j})=0,\ \textit{for}\ \ \boldsymbol{x_i}\neq\boldsymbol{x_j}.$$

Then, we have the following theorem.

Theorem 3.7. *If $p^{\min}=p_b$, the optimal pricing policy for Problem Π_r is in form of,*

$$p(\boldsymbol{x})=\max\Big(p_b,p(\boldsymbol{0})-c(\boldsymbol{x},\boldsymbol{0}),p(\boldsymbol{0})-w(\boldsymbol{0},\boldsymbol{x})\Big),\ \forall\boldsymbol{x}\in\mathbb{V}.\tag{3.26}$$

We denote $\mathbb{S}:=\{\boldsymbol{x}|p(\boldsymbol{x})=p(\boldsymbol{0})-c(\boldsymbol{x},\boldsymbol{0})\}\cup\{\boldsymbol{x}|p(\boldsymbol{x})=p(\boldsymbol{0})-w(\boldsymbol{0},\boldsymbol{x})\}$ as surge region.

Given pricing policy (3.26), the procedure of constructing the assignment policy is same as the procedure presented after Algorithm 3.1. However, the explicit algorithm involves tedious discussions. Therefore we will not show this discussion in this paper.

3.5 Multiple Demand Shocks

In this section, we extend previous analysis for one demand shock to the problem with multiple demand shocks. Without loss of generality, we show our results for $m = 2$ demand shocks. The case of more demand shocks can be generalized from $m = 2$ demand shocks. In $m = 2$ demand shocks problem, we denote that the first demand shock arrives at location $\boldsymbol{x_1}$ and the second demand shock arrives at location $\boldsymbol{x_2}$ on the network. We formulate the problem as,

$$
\begin{aligned}
\max_{\Gamma,\gamma_1,\gamma_2,\nu,r,\nu_1,\nu_2,p}\; & p(\boldsymbol{x_1})\min\Big(\nu_1,\Lambda_1\bar{F}_v\big(p(\boldsymbol{x_1})\big)\Big)+p(\boldsymbol{x_2})\min\Big(\nu_2,\Lambda_2\bar{F}_v\big(p(\boldsymbol{x_2})\big)\Big)\\
& +\int_{\mathbb{V}}p(\boldsymbol{x})\min\Big(\nu(\boldsymbol{x}),\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)\Big)d\boldsymbol{x}\\
s.t.\;\; & \nu(\boldsymbol{x})=\int_{\mathbb{V}}\Gamma(\boldsymbol{y},\boldsymbol{x})d\boldsymbol{y},\;\forall \boldsymbol{x}\in\mathbb{V}\\
& \mu(\boldsymbol{x})=\int_{\mathbb{V}}\Gamma(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}+\gamma_1(\boldsymbol{x})+\gamma_2(\boldsymbol{x}),\;\forall \boldsymbol{x}\in\mathbb{V}\\
& \nu_1=\int_{\mathbb{V}}\gamma_1(\boldsymbol{x})d\boldsymbol{x},\\
& \nu_2=\int_{\mathbb{V}}\gamma_2(\boldsymbol{x})d\boldsymbol{x},\\
& r(\boldsymbol{x})\geq p(\boldsymbol{y})-c(\boldsymbol{x},\boldsymbol{y}),\;\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{V}\\
& \Gamma(\boldsymbol{x},\boldsymbol{y})\Big(r(\boldsymbol{x})-(p(\boldsymbol{y})-c(\boldsymbol{x},\boldsymbol{y}))\Big)=0,\;\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{V}\\
& \gamma_1(\boldsymbol{x})\Big(r(\boldsymbol{x})-(p(\boldsymbol{x_1})-c(\boldsymbol{x},\boldsymbol{x_1}))\Big)=0,\;\forall \boldsymbol{x}\in\mathbb{V}\\
& \gamma_2(\boldsymbol{x})\Big(r(\boldsymbol{x})-(p(\boldsymbol{x_2})-c(\boldsymbol{x},\boldsymbol{x_2}))\Big)=0,\;\forall \boldsymbol{x}\in\mathbb{V}\\
& \Gamma(\boldsymbol{x},\boldsymbol{y}),\gamma_1(\boldsymbol{x}),\gamma_2(\boldsymbol{x}),\nu(\boldsymbol{x}),r(\boldsymbol{x}),\nu_1,\nu_2\geq 0,\;\forall \boldsymbol{x}\in\mathbb{V}\\
& p^{\max}\geq p(\boldsymbol{x})\geq p^{\min},\forall \boldsymbol{x}\in\mathbb{V}.
\end{aligned}
\tag{3.27}
$$

In Problem (3.27), we add the revenue contribution from the additional demand shock to the objective. We also add the flow balance equation of drivers and the incentive compatibility constraints for the location with additional demand shock.

3.5.1 Structure of Optimal Solution for Problem (3.27)

We show that the Proposition 3.1, 3.2 and 3.3 are still valid for Problem (3.27). In particular, we have

Proposition 3.8. *There exists an optimal solution for Problem* (3.27) *that satisfies the following constraints*

1. *Flow Exclusiveness with Multiple Shocks:*

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k}) = 0, \ \forall \boldsymbol{x_i} \neq \boldsymbol{x_j}, \boldsymbol{x_j} \neq \boldsymbol{x_k} \in \mathbb{V}.$$

2. *Star Assignment with Multiple Shocks:*

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) = 0, \ \textit{for}\ \boldsymbol{x_i} \neq \boldsymbol{x_j} \ \textit{and}\ \boldsymbol{x_j} \notin \{\boldsymbol{x}_1, \boldsymbol{x}_2\}.$$

3. *Minimal Incentive Compatibility with Multiple Shocks:*

$$p(\boldsymbol{x_i}) \geq p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}), \ \forall \boldsymbol{x_i}, \boldsymbol{x_j} \in \mathbb{V}.$$

Based on these propositions, we have the following theorem.

Theorem 3.9. *If $p^{\min} = p_b$, the optimal pricing policy of Problem (3.27) is in form of,*

$$p(\boldsymbol{x}) = \max\Big(p(\boldsymbol{x_1}) - c(\boldsymbol{x}, \boldsymbol{x_1}), p(\boldsymbol{x_2}) - c(\boldsymbol{x}, \boldsymbol{x_2}), p_b\Big), \forall \boldsymbol{x} \in \mathbb{V} \quad (3.28)$$

$$s.t. \quad |p(\boldsymbol{x_1}) - p(\boldsymbol{x_2})| \leq \min\Big(c(\boldsymbol{x_1}, \boldsymbol{x_2}), c(\boldsymbol{x_2}, \boldsymbol{x_1})\Big). \quad (3.29)$$

We denote $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$ as surge region, where $\mathbb{S}_1 := \{\boldsymbol{x} | p(\boldsymbol{x}) = p(\boldsymbol{x_1}) - c(\boldsymbol{x}, \boldsymbol{x_1})\}$ and $\mathbb{S}_2 := \{\boldsymbol{x} | p(\boldsymbol{x}) = p(\boldsymbol{x_2}) - c(\boldsymbol{x}, \boldsymbol{x_2})\}$.

We emphasize that in Theorem 3.9, Constraint (3.29) creates additional restriction on the prices for demand surge locations such that they can not be freely searched as in its individual demand surge problem. The platform needs to find the optimal $p(\boldsymbol{x_1})$ and $p(\boldsymbol{x_2})$ jointly.

Next, we discuss the assignment policy for Problem (3.27). The procedure of constructing the assignment policy is same as the procedure presented after Algorithm 3.1. However, we may not have an explicit assignment in some cases. Denote $\mathbb{S}_b = \mathbb{S}_1 \cap \mathbb{S}_2$, i.e., $\mathbb{S}_b$ is the intersection of two surge regions. If $\int_{\boldsymbol{x} \in \mathbb{S}_b} \mu(\boldsymbol{x}) d\boldsymbol{x} = 0$, i.e., $\mathbb{S}_b$ is a 0-measure set for μ, then this intersection has no contribution to the total revenue. The two surge regions can make their assignments independently following Algorithm 3.1.

However, when $\int_{\boldsymbol{x} \in \mathbb{S}_b} \mu(\boldsymbol{x}) d\boldsymbol{x} > 0$, drivers in $\mathbb{S}_b$ can serve both demand shocks. We need to solve Problem (3.27) for a given pricing policy to get the assignment.

In sum, the platform needs to search the values of $p(\boldsymbol{x_1})$ and $p(\boldsymbol{x_2})$ under the constraint $|p(\boldsymbol{x_1}) - p(\boldsymbol{x_2})| \leq \min \Big(c(\boldsymbol{x_1}, \boldsymbol{x_2}), c(\boldsymbol{x_2}, \boldsymbol{x_1}) \Big)$ and select the best pair. The algorithm that processes this method is stated below.

Algorithm 3.3 Algorithm of Solving $m = 2$ Demand Shocks Problem with $p^{\min} = p_b$

Set $M = 1000,\ \delta = \frac{p^{\max} - p_b}{M}$
for $i_1 = 0 : M$ **do**
  for $i_2 = 0 : M$ **do**
    Set $p(\boldsymbol{x_1}) = p_b + i_1 \delta,\ p(\boldsymbol{x_2}) = p_b + i_2 \delta$
    if $|p(\boldsymbol{x_1}) - p(\boldsymbol{x_2})| \leq \min \Big(c(\boldsymbol{x_1}, \boldsymbol{x_2}), c(\boldsymbol{x_2}, \boldsymbol{x_1}) \Big)$ **then**
      Apply Theorem 3.9 to read off the pricing policy
      Given pricing policy, solve Problem (3.27) for assignment and overall revenue rate $R(i_1, i_2)$
    end if
  end for
end for
Return $(i_1^*, i_2^*) = \arg\max_{(i_1, i_2)} R(i_1, i_2),\ p^*(\boldsymbol{x_1}) = p_b + i_1^* \delta,\ p^*(\boldsymbol{x_2}) = p_b + i_2^* \delta$

Figure 3.3 gives an example of the optimal pricing policy when $\mathbb{V} = (-\infty, \infty)$ and demand shocks are at location $\boldsymbol{x_1}$ and Shock $\boldsymbol{x_2}$.

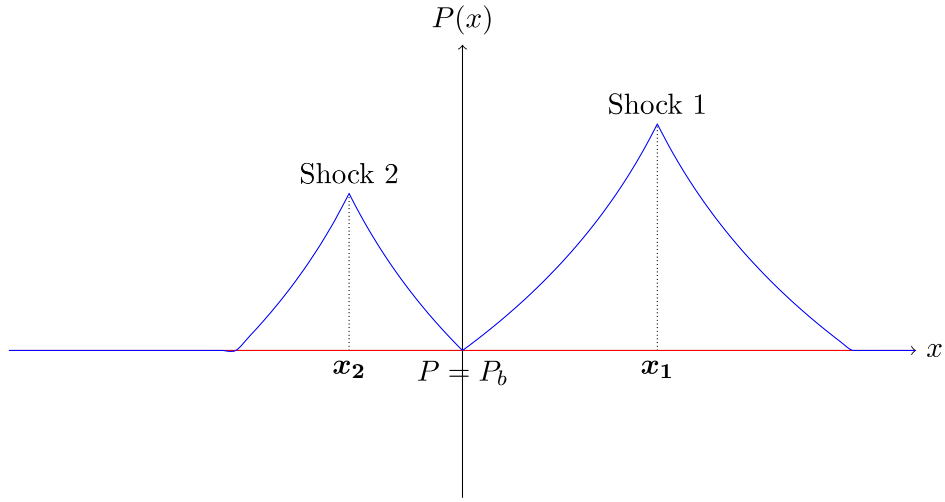

Figure 3.3: Optimal Pricing Function p under $m = 2$ Demand Shocks, Price Constrained: $p^{\min} = p_b$

When there are m demand shocks in general, the pricing policy in Theorem 3.9 becomes

$$p(\boldsymbol{x}) = \max\left(\max_{i\in[m]}\left(p(\boldsymbol{x_i}) - c(\boldsymbol{x}, \boldsymbol{x_i})\right), p_b\right), \forall \boldsymbol{x} \in \mathbb{V}$$
$$s.t.\ \ |p(\boldsymbol{x_i}) - p(\boldsymbol{x_j})| \leq \min\left(c(\boldsymbol{x_i}, \boldsymbol{x_j}), c(\boldsymbol{x_j}, \boldsymbol{x_i})\right),\ \forall i, j \in [m].$$

Then, we search for the values of $\{p(\boldsymbol{x_1}), ..., p(\boldsymbol{x_m})\}$ in Algorithm 3.3 and pick the best one.

3.6 Unconstrained Pricing Problem

In this section, we extend our analysis to the unconstrained pricing problem, i.e., $p^{\min}$ is strictly lower than the baseline price p_b. Without loss of generality, we use $p^{\min} = 0$ in this part. The result is valid for any $p^{\min} > 0$.

The formulation of unconstrained pricing problem is same as Problem Π, we do not duplicate it here for brevity. For unconstrained pricing problem, Proposition 3.1, 3.2 and 3.3 are still valid. However, as $p\bar{F}_v(p)$ is not decreasing on $[p^{\min}, p^{\max}]$, Theorem 3.4 fails for $p^{\min} = 0$.

In particular, for each $\boldsymbol{x}$ which is not a demand shock location, we need to make a separate decision of whether to include it in surge region. If so, the price at $\boldsymbol{x}$ must be $p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$ following Proposition 3.2 and incentive compatibility constraint. If not, platform can freely choose $p(\boldsymbol{x})$ under the constraints in Problem Π. When $p^{\min} = p_b$, the optimal prices of these two choices coincide to same value $p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$ due to the monotonicity of $p\bar{F}_v(p)$. However, when $p^{\min} = 0$, if drivers are not relocated to the demand shock node, the optimal $p(\boldsymbol{x})$ may be different from $p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$. Thus, the optimal pricing policy in Theorem 3.4 is invalid. We show a counter-example about this in Appendix B.1.

Next, we discuss the solution for unconstrained pricing problem under different cases.

3.6.1 Norm Induced Distance Metric c

When dis-utility function c is a norm induced distance metric, we have the following results.

$\mathbb{V} = [0, \infty)$, Demand Surge at 0. When space $\mathbb{V} = [0, \infty)$, demand surge happens at origin, or $\mathbb{V}$ is rotational symmetric with respect to the demand shock node, the problem is reduced to a one-dimensional space. Under this setting, we make an additional assumption in this section.

Assumption 3.5. *$p\bar{F}_v(p)$ is weakly increasing on $p \leq p_b$.*

Together with Assumption 3.4, we in fact assume $p\bar{F}_v(p)$ is uni-modular on $[p^{\min}, p^{\max}]$ in this section. Then, we have

Theorem 3.10. *If $p^{\min} = 0$, the optimal pricing policy with distance metric c and $\mathbb{V} = [0, \infty)$ is in form of,*

$$p(x) = p(0) - c(x, 0),\ \forall x \leq l;\ p(x) = \min\Big(p_b, p(l) + c(l, x)\Big),\ \forall x \geq l, \tag{3.30}$$

where l satisfies $p(0) - c(l, 0) \leq p_b$. We denote $\mathbb{S} := \{x \leq l\}$ as surge region. Under the pricing policy (3.30)*, the assignment determined by Algorithm 3.1 is optimal.*

Figure 3.4 shows an example of this pricing policy.

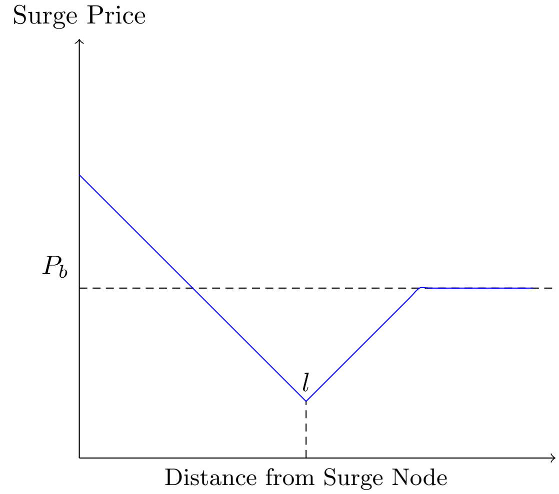

Figure 3.4: Surge Prices p as a Function of Distance from Surge Node, Price Unconstrained: $p^{\min} = 0$

We present the proof in Appendix A.6. In Theorem 3.10, the optimal pricing function is determined by two parameters, the price at demand shock location $p(0)$ and boundary l of surge region. Different from Problem Π with $p^{\min} = p_b$ that the boundary of surge region is intrinsically determined by $p(\mathbf{0})$, for unconstrained pricing problem, the boundary l of surge region needs to be decided separately. We need to search for both of $p(0)$ and l for optimal policy. Also, the optimal pricing policy for the unconstrained pricing problem can set price lower than p_b at some locations.

When $\mathbb{V}$ is rotational symmetric with respect to the demand shock node, we lose the structure of pricing policy in Theorem 3.10. In particular, Proposition 3.1, 3.2 and 3.3 are still valid, and the pricing and assignment policy within the surge region are still same as those in Theorem 3.10. However, we do not any efficient way to determine surge region. This is the same situation when c is a general function.

3.7 Non-instantaneous Relocation

In this section, we discuss the problem when the relocation of driver is not instantaneous. For brevity of the model, we use discrete space to present our results. In particular, we have the following problem formulation.

$$
\begin{aligned}
\max_{\Gamma_{ij},\nu_i,r_i,p_i} \quad & \sum_i p_i \min\left(\nu_i, \lambda_i \bar{F}_v(p_i)\right) \\
s.t. \quad & \nu_i = \sum_j \Gamma_{ji}, \ \forall i \in [n] \\
& \mu_i = \sum_j \Gamma_{ij}, \ \forall i \in [n] \\
& r_i \geq p_j - c_{ij}, \ \forall i,j \in [n] \\
& \Gamma_{ij}\left(r_i - (p_j - c_{ij})\right) = 0, \ \forall i,j \in [n] \\
& \Gamma_{ij}, \nu_i, r_i \geq 0, \ p^{\max} \geq p_i \geq p^{\min}, \forall i,j \in [n],
\end{aligned}
\tag{3.31}
$$

where c_{ij} is the dis-utility for relocating from i to j. We have n locations in total and the demand shock is at location 1. To keep the problem concise, we redefine $\lambda_1 = \lambda_1 + \Lambda$, i.e., λ_1 includes the riders from baseline demand and demand surge.

Then, we consider relocation time in Problem (3.31). In particular, we denote t_{ij} as the relocation time from i to j. If we relocate drivers from i to j, these drivers arrive at j at t_{ij} units of time after they leave i. The revenue generated from these drivers is recognized t_{ij} units of time later. Then, we formulate the problem with relocation time as the following.

$$
\begin{aligned}
\max_{p_i,\Gamma_{ij},r_i} \quad & \int_0^\tau \sum_j p_j \min\left(\sum_i \Gamma_{ij}\mathbf{1}(t \geq t_{ij}), \lambda_j \bar{F}_v(p_j)\right) dt \\
s.t. \quad & \mu_i = \sum_{j=1}^n \Gamma_{ij}, \ \forall i \in [n] \\
& r_i \geq p_j - c_{ij}, \ \forall i,j \in [n] \\
& \Gamma_{ij}\left(r_i - (p_j - c_{ij})\right) = 0, \ \forall i,j \in [n] \\
& \Gamma_{ij}, r_i \geq 0, \ \forall i,j \in [n] \\
& p_i^{\max} \geq p_i \geq p_i^{\min}, \ \forall i \in [n],
\end{aligned}
\tag{3.32}
$$

where τ is life time duration for demand surge. In the objective of Problem (3.32), if the platform relocates drivers from region i to pick up a rider in region j, the platform loses the riders at j in the first t_{ij} units of time. The indicator $\mathbf{1}(t \geq t_{ij})$ identifies the beginning time of driver relocation flow Γ_{ij} contributing to the total revenue.

Without loss of generality, we assume $t_{ij} \leq \tau, \forall (i,j)$. If this is not true for a pair of (i,j), then the driver relocation flow from i to j has no contribution to the total revenue because they arrive after the demand surge ends. Then, we have,

Theorem 3.11. *If $p^{\min} = p_b$, the optimal pricing policy of Problem* (3.32) *is in form of,*

$$p_i = \max(p_b, p_1 - c_{i1}), \ \forall i \in [n]. \tag{3.33}$$

We denote $\mathbb{S} := \{i | p_i = p_1 - c_{i1}\}$ as surge region.

To determine the assignment policy, we first notice that the relocation can only happen within the surge region. Outside the surge region, relocation is not incentive compatible and we match all drivers with riders at same location. For the assignment within surge region, we have the following lemma.

Lemma 3.12. *The value of Problem* (3.32) *remains same if we require at most one of the following two inequalities*

$$\nu_i > \lambda_i \bar{F}_v(p_i), \ \ \sum_{j \neq i} \Gamma_{ji} > 0,$$

is true for any location i in surge region.

We present the proof in Appendix A.7. Then, for each location i in surge region, we have,

1. If $\sum_{j \neq i} \Gamma_{ji} = 0$, i.e., there is no driver relocated into i, all the effective riders at node i are served immediately. The revenue collected from location i in time window $[0, \tau]$ is

$$\tau p_i \min\left(\Gamma_{ii}, \lambda_i \bar{F}_v(p_i)\right),$$

which is same as

$$\tau p_i \min\left(\sum_j \Gamma_{ji}, \lambda_i \bar{F}_v(p_i)\right) - p_i \sum_j t_{ji}\Gamma_{ji},$$

because $\sum_{j\neq i}\Gamma_{ji} = 0$ and $t_{ii} = 0$ in this case.

2. If $\sum_{j\neq i}\Gamma_{ji} > 0$, then we have $\nu_i \leq \lambda_i \bar{F}_v(p_i)$ from Lemma 3.12, i.e., we have less available drivers than the amount of effective riders at location i. Then, all the drivers are matched and the revenue collected at location i is

$$\tau p_i \nu_i - p_i \sum_j t_{ji}\Gamma_{ji},$$

where $p_i \sum_j t_{ji}\Gamma_{ji}$ is the revenue loss caused by relocation time. Since $\nu_i = \sum_j \Gamma_{ji} \leq \lambda_i \bar{F}_v(p_i)$, the revenue above can also be written as

$$\tau p_i \min\left(\sum_j \Gamma_{ji}, \lambda_i \bar{F}_v(p_i)\right) - p_i \sum_j t_{ji}\Gamma_{ji}. \tag{3.34}$$

Thus, we can always use (3.34) to represent the revenue collected at location i. Then, we can write the assignment problem within surge region as,

$$\begin{aligned}
\max_{\Gamma_{ij}} \ & \sum_i \left(\tau p_i \min\left(\nu_i, \lambda_i \bar{F}_v(p_i)\right) - \sum_j p_i t_{ji}\Gamma_{ji}\right) \\
s.t. \ & \nu_i = \sum_{i=1}^{n} \Gamma_{ji}, \ \forall i \in [n] \\
& \mu_i = \sum_{j=1}^{n} \Gamma_{ij}, \ \forall i \in [n] \\
& \Gamma_{ij}\left(r_i - (p_j - c_{ij})\right) = 0, \ \forall i, j \in [n] \\
& \Gamma_{ij}, \nu_i \geq 0, \ \forall i, j \in [n].
\end{aligned} \tag{3.35}$$

Problem (3.35) is a LP for $\{\Gamma_{ij}\}$. Then, we propose the following algorithm to solve Problem (3.32).

Algorithm 3.4 Algorithm for Problem (3.32) with $p^{\min} = p_b$

Set $M = 1000,\ \delta = \frac{p^{\max} - p_b}{M}$
for $i = 0 : M$ **do**
    Set $p_1 = p_b + i\delta$
    Apply Theorem 3.11 to read off the pricing policy
    Solve the optimal assignment from Problem (3.35)
    Compute the total revenue $R(i)$ under the given pricing and assignment policy
end for
Return $i^* = \arg\max_i R(i),\ p_1^* = p_b + i^*\delta$ and the assignment policy

Hopping Assignment. For Problem (3.32) with relocation time, we need to solve Problem (3.35) to find the assignment. We conduct numerical experiments in Section 3.9 and the result shows that the optimal assignment for problem with relocation time is a hopping assignment. Figure 3.5 shows an example for hopping assignment.

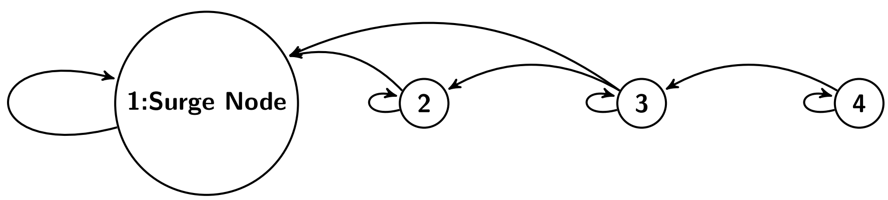

Figure 3.5: Hopping Assignment for Problem (3.32) with Movement Time

In Figure 3.5, each arrow represents a non-zero driver relocation flow. We find there is driver relocation flow from non-surge node to non-surge node, which is excluded in the problem without relocation time. Therefore, to maximize the revenue when there is relocation time, the relocation of excess supply is forwards surge location but the drivers may be matched to riders before reaching the surge location.

3.8 Throughput Maximization

In this section, we discuss the problem that the platform aims to maximize overall throughput instead of the revenue. In particular, we show how to apply the propositions and algorithms stated in previous sections to solve the throughput maximization problem, and we also discuss the differences with revenue maximization. First of all, we formulate the throughput maximization problem as,

$$\begin{aligned} & \min\left(\nu_{\mathbf{0}}, \Lambda \bar{F}_v\big(p(\mathbf{0})\big)\right) + \int_{\mathbb{V}} \min\left(\nu(\boldsymbol{x}), \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)\right) d\boldsymbol{x} \\ & \text{Constraints (3.5) - (3.12).} \end{aligned} \tag{3.36}$$

3.8.1 Structure of Optimal Solution for Problem (3.36)

Since the throughput objective is increasing in ν_0 and $\nu(\boldsymbol{x})$ and decreasing in $p(\boldsymbol{x}$, and the constraints are same as the revenue maximization problem, we can use the same proofs for Proposition 3.1, 3.2, 3.3 and Theorem 3.4 to show they are also valid for the throughput optimization problem. In particular, we have,

Theorem 3.13. *The throughput maximization pricing policy is in form of,*

$$p(\boldsymbol{x}) = \max\left(p^{\min}, p_{\mathbf{0}} - c(\boldsymbol{x}, \mathbf{0})\right), \forall \boldsymbol{x} \in \mathbb{V}. \tag{3.37}$$

We denote $\mathbb{S} := \{\boldsymbol{x} | p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\}$ *as surge region. Under the pricing policy* (3.37)*, the assignment determined by Algorithm 3.5 is optimal.*

Algorithm 3.5 Optimal Assignment for Problem (3.36) under Pricing Policy (3.37)

---

for $\boldsymbol{x} \in \mathbb{S} \setminus \{\mathbf{0}\}$ **do**
  Set $\nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)$
end for
Set $\nu_{\mathbf{0}} = \int_{\mathbb{S}} \mu(\boldsymbol{x}) - \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big) d\boldsymbol{x}$
for $\boldsymbol{x} \notin \mathbb{S}$ **do**
  Set $\nu(\boldsymbol{x}) = \mu(\boldsymbol{x})$
end for

We emphasize that Algorithm 3.1 shown in Theorem 3.4 is also an optimal solution for this throughput problem. Here, we have multiple optimal assignment policy because for locations with different prices, there is no difference in their throughput contribution. As assignment policy uses up all effective riders or drivers is optimal.

Consequently, the throughput problem is also reduced to a single variable problem to decide $p(\mathbf{0})$. Since the maximum throughput contributed from the entire surge region is the minimum of effective riders and drivers within surge region and this value can be achieved by the Algorithm 3.5, Problem (3.36) reduces to

$$\begin{aligned} \max\ & \left(\min \left(\Lambda \bar{F}_v \big(p(\mathbf{0})\big) + \int_{\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x}, \int_{\mathbb{S}} \mu(\boldsymbol{x}) d\boldsymbol{x} \right) + \int_{\mathbb{V}\backslash\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x} \right) \\ s.t.\ & p(\boldsymbol{x}) = \max \left(p^{\min}, p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}) \right),\ \forall \boldsymbol{x} \in \mathbb{V} \\ & \mathbb{S} = \{\boldsymbol{x} | p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\}, \\ & p^{\max} \geq p(\mathbf{0}) \geq p^{\min}. \end{aligned} \tag{3.38}$$

The objective in Problem (3.38) is the optimal throughput inside surge region plus the throughput outside surge region. The constraints are the definitions for $p(\boldsymbol{x})$ and $\mathbb{S}$, where $p(\mathbf{0})$ is the only decision variable. Next, we write the objective in Problem (3.38) as

$$\min \left(\Lambda \bar{F}_v \big(p(\mathbf{0})\big) + \int_{\mathbb{V}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x}, \int_{\mathbb{S}} \mu(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{V}\backslash\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x} \right), \tag{3.39}$$

by moving $\int_{\mathbb{V}\backslash\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x}$ into the min operator. We claim that the first term in (3.39) is decreasing with $p(\mathbf{0})$ whereas the second term is increasing with $p(\mathbf{0})$. To show that, when $p(\mathbf{0})$ increases, $p(\boldsymbol{x})$ increases, then $\bar{F}_v(p(\mathbf{0}))$ and $\bar{F}_v(p(\boldsymbol{x}))$ decrease. Therefore the first term in (3.39) decreases. Also, $\mathbb{S}$ enlarges when $p(\mathbf{0})$ increases. In the area that $\mathbb{S}$ expands, we use $\mu(\boldsymbol{x})$ to replace $\lambda(\boldsymbol{x})\bar{F}_v(p(\boldsymbol{x}))$ in the second integral in (3.39). Since $\mu(\boldsymbol{x}) \geq \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big)$, the second term increases with $p(\mathbf{0})$.

Consequently, the optimal $p(\mathbf{0})$ of Problem (3.38) is the solution of

$$\Lambda \bar{F}_v \big(p(\mathbf{0})\big) + \int_{\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v \big(p(\boldsymbol{x})\big) d\boldsymbol{x} = \int_{\mathbb{S}} \mu(\boldsymbol{x}) d\boldsymbol{x}, \tag{3.40}$$

i.e., the $p(\mathbf{0}$ that equals the amount of effective riders and drivers inside surge region. Next, we present the algorithm for solving Problem (3.36).

Algorithm 3.6 Algorithm for Throughput Maximization Problem (3.36)

Define $f(p) = \Lambda \bar{F}_v(p) + \int_{\mathbb{S}} \lambda(\boldsymbol{x}) \bar{F}_v\Big(\max\big(p^{\min}, p - c(\boldsymbol{x}, \mathbf{0})\big)\Big) d\boldsymbol{x} - \int_{\mathbb{S}} \mu(\boldsymbol{x}) d\boldsymbol{x}$, where $\mathbb{S} = \{\boldsymbol{x} | p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\}$,
$\epsilon = 0.001,\ a = p^{\min},\ b = p^{\max}$
if $f(a) \leq 0$ **then**
    Return $p^*(\mathbf{0}) = a$
else if $f(b) \geq 0$ **then**
    Return $p^*(\mathbf{0}) = b$
else
    while $|f(\frac{a+b}{2})| > \epsilon$ **do**
        if $f(\frac{a+b}{2}) > 0$ **then**
            Set $a = \frac{a+b}{2}$
        else
            Set $b = \frac{a+b}{2}$
        end if
    end while
    Return $p^*(\mathbf{0}) = \frac{a+b}{2}$
end if

Algorithm 3.6 applies bisection method to determine the solution of (3.40). Once $p^*(\mathbf{0})$ is determined, we can apply Theorem 3.13 to read off the optimal pricing and assignment policy. Figure 3.6 represents an example for the optimal pricing policy of throughput and revenue maximization problems with same parameters when $\mathbb{V} = [0, \infty)$ and demand surge happens at 0.

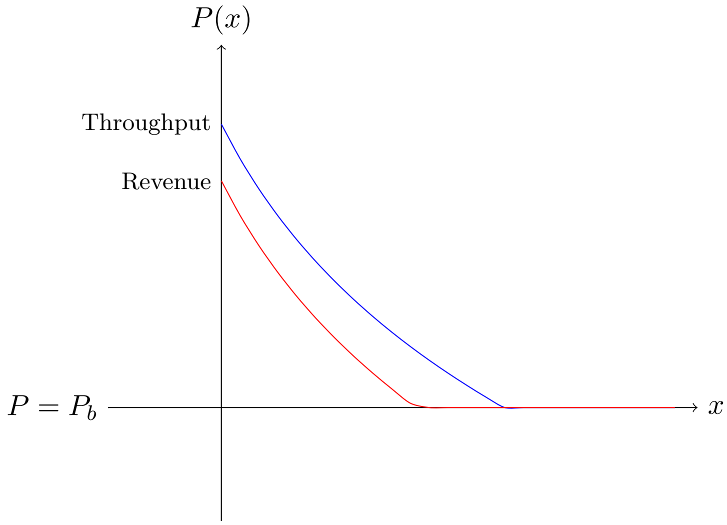

Figure 3.6: Optimal Price Function p for Revenue and Throughput Maximization, Demand Surge at 0, Price Constrained: $p^{\min} = p_b$

These two pricing policies yield same structure of $p(\boldsymbol{x}) = \max\Big(p_b, p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\Big)$, but the throughput optimal pricing curve is at a higher position than the revenue optimal pricing curve. This is because the throughput maximization solution serves all effective riders. However, the revenue maximization solution may have some effective riders unserved. Since the number of excess riders inside surge region is decreasing with $p(\mathbf{0})$, the optimal $p(\mathbf{0})$ for revenue maximization is lower.

3.9 Numerical Study

In this section, we use the model of Problem (3.31) to conduct numerical analysis for the surge pricing problem.

Experiment Setup. In this part, we set $\mathbb{V}$ as a discrete space with $n = 7$ locations and demand surge happens at node 1. The dis-utility function c is defined as $c_{ij} = C \cdot \frac{|i-j|}{n}$. The arrival rate of available drivers is $\mu_i = 10, \forall i \in [n]$ and the arrival rate of baseline riders is $\lambda_i = 3, \forall i \in [n]$. The willingness to pay function of all riders is $F_v(p) = \frac{p}{p^{\max}}$, where $p^{\min} = 0, p^{\max} = 100$. Then, the baseline price p_b is 50.

Comparation between Unconstrained and Constrained Pricing Problems. In Table 3.1 and 3.2, we present the optimal values and prices on all locations of the unconstrained pricing problem and constrained pricing problem when demand shock has size of $\Lambda = 13, 33, 53, 73, 93, 113, 143$ and dis-utility coefficient is $C = 50$.

Λ	Obj	p_1	p_2	p_3	p_4	p_5	p_6	p_7
13	7750.0	50.0	50.0	50.0	50.0	50.0	50.0	50.0
33	12736.0	50.6	43.5	50.0	50.0	50.0	50.0	50.0
53	17670.0	51.4	44.2	37.1	44.2	50.0	50.0	50.0
73	22498.4	52.8	45.6	38.5	31.4	38.5	45.6	50.0
93	27159.1	54.4	47.3	40.1	33.0	25.9	33.0	40.1
113	31706.1	55.6	48.5	41.4	34.2	27.1	19.9	27.1
143	38298.1	59.3	52.2	45.0	37.9	30.7	23.6	16.5

Table 3.1: Optimal Values and Prices when $p^{\min} = 0$

Λ	Obj	Ratio to Table 3.1	p_1	p_2	p_3	p_4	p_5	p_6	p_7
13	7750.0	100.0%	50.0	50.0	50.0	50.0	50.0	50.0	50.0
33	12581.7	98.8%	57.1	50.0	50.0	50.0	50.0	50.0	50.0
53	16653.0	94.2%	64.3	57.1	50.0	50.0	50.0	50.0	50.0
73	21244.7	94.4%	64.3	57.1	50.0	50.0	50.0	50.0	50.0
93	23836.7	87.8%	70.4	63.2	56.1	50.0	50.0	50.0	50.0
113	27484.7	86.7%	71.4	64.3	57.1	50.0	50.0	50.0	50.0
143	31519.6	82.3%	74.5	67.4	60.2	53.1	51.5	50.0	50.0

Table 3.2: Optimal Values and Prices when $p^{\min} = p_b$

From Table 3.1 and 3.2, we find that when the size of demand shock increases, the optimal value of $p^{\min} = 0$ outperforms more and more than that of $p^{\min} = p_b$. The optimal prices on some locations can be lower than p_b when $p^{\min} = 0$, which is consistent with our analysis.

We also find that when the size of demand shock increases to a severe high level, e.g., 5 times more than the total available drivers in our experiment, the service rate within surge region drops below 1. In this case, platform sacrifices some riders in surge region with baseline demand, and relocate these drivers to demand surge location.

Market Clear Heuristic. In Section 3.3, we propose a market clear heuristic that the pricing function is set at the value that all effective riders get served. In this part, we show the numerical performance for the market clear heuristic. In addition to the parameters in experiment setup, we use different values for $\{\lambda_i\}$ and $\{\mu_i\}$ to conduct the experiments. We parameterize λ_i, μ_i, and size of demand shock Λ as

$$\mu_i = \beta\lambda_i, \forall i \in [n]; \quad \Lambda = \lambda_1 \frac{\Lambda}{\lambda_1} = \lambda_1 \Lambda_{scale}.$$

In Table 3.3, we show the performance of market clear heuristic for values of $\beta = 1, 3, 6, 10$ and $\Lambda_{scale} = 23, 63, 143$ in column *Obj of Heuristic %*, which is percentage of heuristic value to the optimal value.

β	Λ_{scale}	Obj of Heuristic %	β	Λ_{scale}	Obj of Heuristic %
1	23	99.2%	6	23	100.0%
1	63	96.0%	6	63	100.0%
1	143	93.5%	6	143	99.8%
3	23	100.0%	10	23	100.0%
3	63	99.9%	10	63	100.0%
3	143	98.7%	10	143	100.0%

Table 3.3: Percentage of Optimal Value Achieved by Clear-market Heuristic

In Table 3.3, we find that market clear heuristic can achieve more than 93% of the optimal value in our experiments. When β is high or Λ_{scale} is low, i.e., the number of available drivers is ample or the size of the demand shock is small, the heuristic performs well.

Convex Dis-utility Function c. In this part, we conduct experiment to the case when c does not satisfy triangle inequality (3.1). In particular, we consider the case that dis-utility function c is strictly convex,

$$c(\boldsymbol{x}, \boldsymbol{y}) = 10||\boldsymbol{x} - \boldsymbol{y}||^2/7.$$

We also change the driver arrival rate μ_i and number of nodes in space $\mathbb{V}$. These numbers together with the experiment results are shown in Table 3.4, 3.5 and 3.6

i	p_i	$\lambda_i \bar{F}_v(p_i)$	μ_i	ν_i	Γ_{ij}	1	2	3	4	5	6
1	52.8	7.6	6	7.6	1	6	0	0	0	0	0
2	51.4	1.0	2	1.0	2	1.6	0.4	0	0	0	0
3	50.0	2	4	3.5	3	0	0.5	3.5	0	0	0
4	50.0	3	6	6.0	4	0	0	0	6.0	0	0
5	50.0	4	8	8.0	5	0	0	0	0	8.0	0
6	50.0	5	10	10.0	6	0	0	0	0	0	10.0

Table 3.4: Solution with Quadratic Dis-utility Function, $\Lambda = 10$

i	p_i	$\lambda_i \bar{F}_v(p_i)$	μ_i	ν_i	Γ_{ij}	1	2	3	4	5	6
1	63.9	13.0	6	13.4	1	6.0	0	0	0	0	0
2	57.0	0.9	2	1.6	2	2.0	0	0	0	0	0
3	52.8	1.9	4	2.5	3	4.0	0	0	0	0	0
4	51.4	3.0	6	4.1	4	1.4	1.6	2.5	0.5	0	0
5	50.0	4.0	8	4.4	5	0	0	0	3.6	4.4	0
6	50.0	5.0	10	10.0	6	0	0	0	0	0	10.0

Table 3.5: Solution with Quadratic Dis-utility Function, $\Lambda = 30$

i	p_i	$\lambda_i \bar{F}_v(p_i)$	μ_i	ν_i	Γ_{ij}	1	2	3	4	5	6
1	69.2	17.2	6	17.2	1	6.0	0	0	0	0	0
2	62.3	0.8	2	0.8	2	2.0	0	0	0	0	0
3	56.9	1.7	4	3.9	3	4.0	0	0	0	0	0
4	52.8	2.8	6	3.1	4	5.2	0.8	0	0	0	0
5	51.4	3.9	8	5.4	5	0	0	3.9	3.1	1.0	0
6	50.0	5.0	10	5.6	6	0	0	0	0	4.4	5.6

Table 3.6: Solution with Quadratic Dis-utility Function, $\Lambda = 50$

The results show that the optimal assignment policy is hopping assignment, not star assignment any more. The pricing policy is not consistent to the structure in Theorem 3.4 either.

3.10 Conclusion

In this chapter, we consider fluid model and give tractable approach to compute optimal prices and assignment to match supply with demand during surge scenarios under reasonable assumptions.

Problem is challenging because it is non-convex and has infinite number of decisions variables. We first show that there exists an optimal solution that satisfies some nice structural properties. Based on these properties, we design an optimal pricing and matching policy. The optimal pricing policy is determined by the price at surge node, and the price decreases with the dis-utility of driver relocation until it reaches baseline price. The optimal assignment policy relocates excess drivers to demand surge node within surge region and retains the remaining drivers at their original locations.

We extend our results in a number of directions, including strategic rider relocation, multiple demand shocks, unconstrained pricing policy, non-instantaneous relocation time, and throughput maximization, and we obtain similar qualitative results.

Chapter 4

Near-Optimal Price Rebates for Demand Response under Power Flow Constraints

4.1 Introduction

Due to an increasing integration of renewable sources such as wind and solar power on the grid, the supply uncertainty in the electricity market has increased significantly. Demand-side participation has, therefore, become essential to maintain a real-time energy balance in the grid. There are several ways to increase the demand-side participation for the real-time energy balance including time of use pricing, real-time pricing for smart appliances, interruptible demand-response contracts and real-time price rebates and incentives. Typically, an electric utility buys the forecast day-ahead load in the day-ahead market and pays the shortfall (if the actual demand turns out to be higher) on the real-time market. However, if the supply in the real-time market is scarce, the real-time prices can be very high and the utility is exposed to the high prices. Such a scenario can arise often if a significant fraction of power is generated by highly uncertain sources such as wind and solar plants. Since end customers, including residential and most commercial customers, do not pay the real-time prices, the demand does not adjust to the real-time prices and the utility is exposed to the price shocks. With price rebates or interruptible load contracts, the utility company has the option of offering financial incentives to the customers to reduce their demand in such scenarios.

Interruptible load contracts have been studied extensively in the literature as an approach to demand response, both from the perspective of optimal execution of contracts (see Oren and Smith [64]. Caves et al. [20]) and also design and pricing (see Fahrioglu and Alvarado [30], Kamat and Oren [43], Tan and Varaiya [79], Oren [63], Bhattacharta et al. [84]) and more recently, Bitar and Low [14], Roozbehani et al. [70]. We refer readers to the survey by Baldick et al. [8] that provides a good overview of the literature.

In this chapter, we consider an alternative approach where the utility can offer real-time price incentives or rebates to consumers to reduce the power consumption. The goal is to compute the prices (or rebates) to offer to different consumers to reduce the power consumption such that the required reduction can be achieved in minimum possible cost. Since the AC power flow model allows us to model the transmission losses in the distribution network, we can optimize over *both* the reduction in demand and the reduction of transmission losses in the network. Our main contribution is that we propose an efficient iterative heuristic to solve the offer price optimization problem under AC power flow constraints, and we show that the AC formulation leads to a significant reduction in the rebates that one needs to offer in order to shed a certain demand.

The main challenge in this problem arises from the non-convexity of AC power flow constraints and also the uncertainty in price elasticity of the demand. To solve the problem, we developed our heuristic which gives an iterative procedure to compute the offer prices to minimize the total expected cost using a sample average approximation (SAA) to deal with the stochastic optimization problem (4.2), and a semidefinite programming (SDP) based relaxation to deal with the non-convexity of AC power flow constraints. We conduct numerical experiments to compare the performance of our heuristic with other optimization approaches including using DC power flow model or no power flow model at all. Our computational study shows that the performance of our AC power flow based heuristic is significantly better than the other approaches. Unlike the DC power flow constraints, the AC power flow constraints model transmission losses. Therefore, we can optimize the offer prices based on the topology of the grid and leverage both the actual load reduction as well as the reduction in the transmission losses.

The rest of this chapter is organized as follows. In Section 4.2, we give a full mathematical formulation to the demand response problem. In Section 4.2.1, we define the sample average approximation method. In Section 4.2.2, we introduce the remaining model notations and the SDP relaxation for the AC power flow constraints. Next, we present the iterative heuristic for the offer price optimization problem (4.8) in Section 4.3 and show some other power flow models for compare in Section 4.4. Finally, we present the computational study in Section 4.5 and our conclusion in 4.6.

4.2 Problem Definition

In this section, we will give a mathematical formulation for this demand response problem. Let $\mathcal{K} := \{1, 2, ..., K\}$ denote the set of buses, $\mathcal{G} \subseteq \mathcal{K}$ denote the set of generator buses, $\mathcal{C} \subseteq \mathcal{K}$ denote the set of demand buses. Without loss of generality, we assume $\mathcal{G} \cap \mathcal{C} = \emptyset$. Let $\mathcal{N} \subseteq \mathcal{K} \times \mathcal{K}$ denote the set of transmission lines. Let $P_k^g + jQ_k^g$ denote the generation at bus $k \in \mathcal{G}$, they are the decision variables in this problem. Let $\bar{P}_i^c + j\bar{Q}_i^c$ denote the *nominal* load at demand bus $i \in \mathcal{C}$, i.e. the demand in the absence of any rebates. They are given quantities. For each demand bus, we are given the response (or supply) function $R_i(\gamma_i)$, that specifies the mean reduction of load at bus i at any given offer price (or rebate) γ_i. We assume that the actual demand reduction is random, and is given by

$$\tilde{R}_i(\gamma_i) = R_i(\gamma_i) + \epsilon_i,$$

where ϵ_i is a mean zero random variable with a known distribution. We assume that the distribution of ϵ_i does not depend on the rebate γ_i. We allow for the error distributions at different demand buses to be possibly different. The total expected payment at offer price $\mathbf{\Gamma}$ is given by $\mathbb{E}[\sum_i \gamma_i(R_i(\gamma_i)+\epsilon_i)] = \sum_i \gamma_i R(\gamma_i)$. Meanwhile, the actual payment is $\sum_i \gamma_i \tilde{R}_i(\gamma_i)$ and it achieves a total reduction of $\sum_i \tilde{R}_i(\gamma_i)$ on the demand load. The total reduction on the load plus the reduction on transmission loss contributes the total reduction in the power generation. Let $P^g(0) = \sum_{k\in\mathcal{G}} P_k^g(\mathbf{0},\mathbf{0})$ be the total power generation when there is no rebate and $P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}) = \sum_{k\in\mathcal{G}} P_k^g(\mathbf{\Gamma},\boldsymbol{\epsilon})$ be the total power generation when the rebate vector is $\mathbf{\Gamma}$ and the realization of uncertainty vector is $\boldsymbol{\epsilon}$. Thus, the total reduction over the grid is $P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon})$.

We are also given a power reduction target D and we pay a shortfall penalty λ per unit whenever we are not able to meet target reduction. If $P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon})$ is greater or equal to D, we achieve our target at the cost of $\sum_i \gamma_i \tilde{R}_i(\gamma_i)$. However, if it is smaller than D, we will suffer from the shortfall penalty of $\lambda\big(D - (P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}))\big)$ of not fulfilling the target. Thus, the total expected cost of the demand response program for offer prices $\mathbf{\Gamma}$ is given by

$$\sum_{i=1}^K \gamma_i R_i(\gamma_i) + \lambda \mathbb{E}_{\boldsymbol{\epsilon}}\big[\big(D - (P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}))\big)_+\big], \tag{4.1}$$

where $(y)_+ = \max\{y, 0\}$. Again, $P^g(0) = \sum_{k\in\mathcal{G}} P_k^g(\mathbf{0},\mathbf{0})$ denotes the total generation (or injection) without any price rebate, and $P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}) = \sum_{k\in\mathcal{G}} P_k^g(\mathbf{\Gamma},\boldsymbol{\epsilon})$ is the total generation when the price rebate is $\mathbf{\Gamma} = (\gamma_1,\ldots,\gamma_K)$ and the load at each demand bus i is $(P_i^c - R_i(\gamma_i) - \epsilon_i + jQ_i^c)$. From (4.1), it follows that the offer price optimization problem can be formulated as the following stochastic optimization problem,

$$\begin{array}{ll} \min_{\mathbf{\Gamma}} & \sum_{i=1}^K \gamma_i R_i(\gamma_i) + \lambda \mathbb{E}_{\boldsymbol{\epsilon}}\big[\big(D - (P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}))\big)_+\big] \\ \text{s.t.} & P^g(0), P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}) \text{ satisfy AC power flow constraints.} \end{array} \tag{4.2}$$

Note that the power flow constraints for an AC power grid are non-convex and optimizing over these constraints is NP-hard in general [49]. Therefore, solving (4.2) to compute offer prices is computationally hard even for very simple supply functions R_i. In the following sections, we will introduce how to use the SDP relaxation of problem (4.2) to solve the demand response problem.

4.2.1 Sample Average Approximation

The first step is to estimate the expected value in the objective function by using sample average approximation(SAA), i.e., we approximate the expectation by an average over a set of samples. Thus, after applying SAA, (4.2) becomes to the following optimization problem,

$$\begin{array}{ll} \min_{\mathbf{\Gamma}} & \sum_{i=1}^{K} \gamma_i R_i(\gamma_i) + \frac{\lambda}{M} \sum_{n=1}^{M} \left(D - (P^g(0) - P^g(\mathbf{\Gamma}, \boldsymbol{\epsilon}^{(n)}))\right)_+ \\ \text{s.t.} & P^g(0), P^g(\mathbf{\Gamma}, \boldsymbol{\epsilon}^{(n)}) \text{ satisfy AC power flow constraints,} \end{array} \tag{4.3}$$

where $\boldsymbol{\epsilon}^{(1)}, \ldots, \boldsymbol{\epsilon}^{(M)}$ are M IID samples of stochastic error vector $\boldsymbol{\epsilon}$. SAA will be used to deal with the expectation in objective function all the time.

4.2.2 SDP Formulation for AC Power Flow Constraints

The second step of solving (4.2) is to introduce the SDP relaxation for the AC power flow constraints. Our SDP relaxation model is based on the work of Lavaei and Low (2012). As we stated before, the non-convexity of this problem is from non-convex AC power flow constraints. Lavaei and Low (2012) derived the circuit model of the power network through replacing every transmission line and transformer by their equivalent Π models. In this circuit model, let y_{kl} denote the effective admittance between buses k and l, and y_{kk} denotes the admittance-to-ground at bus k. Let $\boldsymbol{Y} \in \mathbb{C}^{K \times K}$ represent the admittance matrix of the distribution network in this equivalent circuit model, where for each $(k, l) \in \mathcal{N}$, $Y_{kl} = -y_{kl}$ if $k \neq l$, and $y_{kk} + \sum_{m \in N_k} y_{km}$ otherwise (N_k denotes the set of all buses that are directly connected to bus k). Let $\boldsymbol{V} = (V_1, ..., V_K)$ denote the complex voltage vector and $\boldsymbol{I} = (I_1, ..., I_K)$ denote the complex current vector. Then, it follows that the power injected into the grid at bus k is given by

$$\begin{aligned} S_k &= \text{Re}(V_k I_k^*) + j\text{Im}(V_k I_k^*) \\ &= \text{Re}(\boldsymbol{V}\boldsymbol{e}_k\boldsymbol{e}_k\boldsymbol{Y}\boldsymbol{V}^*) + j\text{Im}(\boldsymbol{V}\boldsymbol{e}_k\boldsymbol{e}_k\boldsymbol{Y}\boldsymbol{V}^*) \\ &= \boldsymbol{X}^\top\boldsymbol{Y}_k\boldsymbol{X} + j\boldsymbol{X}^\top\bar{\boldsymbol{Y}}_k\boldsymbol{X} = \text{Tr}(\boldsymbol{Y}_k\boldsymbol{W}) + j\text{Tr}(\bar{\boldsymbol{Y}}_k\boldsymbol{W}), \end{aligned}$$

where

$$\begin{aligned} Y_k &:= e_k e_k^T \boldsymbol{Y} \\ \boldsymbol{Y}_k &:= \frac{1}{2}\begin{bmatrix} \text{Re}\left\{Y_k + Y_k^T\right\} & \text{Im}\left\{Y_k^T - Y_k\right\} \\ \text{Im}\left\{Y_k - Y_k^T\right\} & \text{Re}\left\{Y_k + Y_k^T\right\} \end{bmatrix} \\ \bar{\boldsymbol{Y}}_k &:= -\frac{1}{2}\begin{bmatrix} \text{Im}\left\{Y_k + Y_k^T\right\} & \text{Re}\left\{Y_k - Y_k^T\right\} \\ \text{Re}\left\{Y_k^T - Y_k\right\} & \text{Im}\left\{Y_k + Y_k^T\right\} \end{bmatrix} \\ \boldsymbol{X} &:= \begin{bmatrix} \text{Re}\left\{\mathbf{V}\right\}^T & \text{Im}\left\{\mathbf{V}\right\}^T \end{bmatrix}^T \\ \boldsymbol{W} &:= \boldsymbol{X}\boldsymbol{X}^T, \end{aligned}$$

and $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_K$ are standard unit vectors in $\mathbb{R}^K$. Thus, the power injection constraints are given by

$$\begin{aligned} \text{Tr}(\boldsymbol{Y}_k\boldsymbol{W}) &= \begin{cases} P_k^g & k \in \mathcal{G}, \\ -P_k^c & k \in \mathcal{C}, \end{cases} \\ \text{Tr}(\bar{\boldsymbol{Y}}_k\boldsymbol{W}) &= \begin{cases} Q_k^g & k \in \mathcal{G}, \\ -Q_k^c & k \in \mathcal{C}, \end{cases} \end{aligned} \tag{4.4}$$

And

$$P_k^{\min} \leq P_k^g \leq P_k^{\max},\ Q_k^{\min} \leq Q_k^g \leq Q_k^{\max},\ k \in \mathcal{G}. \tag{4.5}$$

In addition to the power injection constraints, there are upper and lower bounds on the magnitude $|V_k|^2 = \text{Re}(V_k)^2 + \text{Im}(V_k)^2$ at bus k, on the magnitude $|V_l - V_k|^2$ of the difference in voltage between bus l and m, re-formulated as

$$\begin{aligned} &(V_k^{\min})^2 \leq \mathrm{Tr}(\boldsymbol{M}_k \boldsymbol{W}) \leq (V_k^{\max})^2, \\ &\mathrm{Tr}(\boldsymbol{M}_{lm} \boldsymbol{W}) \leq (\Delta V_{lm}^{\max})^2, \end{aligned} \tag{4.6}$$

where

$$\boldsymbol{M}_k := \begin{bmatrix} \boldsymbol{e}_k \boldsymbol{e}^T & 0 \\ 0 & \boldsymbol{e}_k \boldsymbol{e}^T \end{bmatrix}$$
$$\boldsymbol{M}_{lm} := \begin{bmatrix} (\boldsymbol{e}_l - \boldsymbol{e}_m)(\boldsymbol{e}_l - \boldsymbol{e}_m)^T & 0 \\ 0 & (\boldsymbol{e}_l - \boldsymbol{e}_m)(\boldsymbol{e}_l - \boldsymbol{e}_m)^T \end{bmatrix}.$$

There are line constraints on the voltage, there are also constraints on the real and apparent power being carried on a line (l, m). The constraint that the real power $P_{lm} \leq P_{lm}^{\max}$ and the magnitude of the apparent power $|S_{lm}| \leq S_{lm}^{\max}$, can be re-formulated as

$$\begin{aligned} &\mathrm{Tr}\{\mathbf{Y}_{lm} \boldsymbol{W}\} \leq P_{lm}^{\max}, \\ &\mathrm{Tr}\{\mathbf{Y}_{lm} \boldsymbol{W}\}^2 + \mathrm{Tr}\{\overline{\mathbf{Y}}_{lm} \boldsymbol{W}\}^2 \leq (S_{lm}^{\max})^2, \end{aligned} \tag{4.7}$$

where

$$Y_{lm} := (\bar{y}_{lm} + y_{lm}) e_l e_l^\top - y_{lm} e_l e_m^\top$$
$$\mathbf{Y}_{lm} := \frac{1}{2} \begin{bmatrix} \mathrm{Re}\left\{Y_{lm} + Y_{lm}^T\right\} & \mathrm{Im}\left\{Y_{lm}^T - Y_{lm}\right\} \\ \mathrm{Im}\left\{Y_{lm} - Y_{lm}^T\right\} & \mathrm{Re}\left\{Y_{lm} + Y_{lm}^T\right\} \end{bmatrix}$$
$$\overline{\mathbf{Y}}_{lm} := -\frac{1}{2} \begin{bmatrix} \mathrm{Im}\left\{Y_{lm} + Y_{lm}^T\right\} & \mathrm{Re}\left\{Y_{lm} - Y_{lm}^T\right\} \\ \mathrm{Re}\left\{Y_{lm}^T - Y_{lm}\right\} & \mathrm{Im}\left\{Y_{lm} + Y_{lm}^T\right\} \end{bmatrix}.$$

The definition $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{X}^\top$ is equivalent $\boldsymbol{W} \succeq 0$, i.e. $\boldsymbol{W}$ is symmetric positive semidefinite, and $\mathrm{Rank}\{\boldsymbol{W}\} = 1$. Any rank-1 positive semidefinite matrix $\boldsymbol{W}$ that satisfies (4.4)-(4.7) represents a feasible power flow.

Relaxing the rank constraint on $\boldsymbol{W}$, we obtain an SDP relaxation of the power flow constraints. The resulting constraint set is the actual one we will use to solve the AC power flow demand response problem. Lavaei and Low [49] show that the above relaxation is exact if the distribution network is a tree. Sojoudi and Lavaei [73] extend the results to several other classes of networks where the above SDP formulation is exact. However, in general, it is NP-hard to optimize over the power flow constraints.

4.3 Offer Price Optimization Problem under AC Power Flow Constraints

In this section, we present our heuristic for the offer price optimization problem under AC power flow constraints. The last step to formulate the whole program is that we need to connect the objective function (4.3) to the voltage matrix $\boldsymbol{W}$ we defined above. When applying pricing policy $\boldsymbol{\Gamma}$ and a given error $\boldsymbol{\epsilon}$ to the demand response problem, we have $P_k^g(\boldsymbol{\Gamma}, \boldsymbol{\epsilon}) = \text{Tr}\{\boldsymbol{Y}_k \boldsymbol{W}(\boldsymbol{\epsilon})\}$ for $k \in \mathcal{G}$. Notice that $\boldsymbol{W}(\boldsymbol{\epsilon})$ is a function of $\boldsymbol{\epsilon}$. For $i \in \mathcal{C}$, we have $P_i^c(\gamma_i) = P_i^c - R_i(\gamma_i) - \epsilon_i$. Since $-P_i^c(\gamma_i) = \text{Tr}\{\mathbf{Y}_i \boldsymbol{W}(\boldsymbol{\epsilon})\}$, we can add a constraint that links the $\boldsymbol{\Gamma}$ with $\boldsymbol{W}$, $-P_i^c - \text{Tr}\{\boldsymbol{Y}_i \boldsymbol{W}(\boldsymbol{\epsilon})\} + R_i(\gamma_i) + \epsilon_i = 0$.

Then, the optimization problem can be written as,

$$
\begin{aligned}
\min_{\mathbf{\Gamma},\boldsymbol{W}} & \sum_{i=1}^{K} \gamma_i R_i(\gamma_i) + \frac{\lambda}{M}\sum_{n=1}^{M}\left(D - (P^g(0) - P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}^{(n)}))\right)_+ \qquad (4.8)\\
\text{s.t. } & P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}^{(n)}) = \sum_{j\in\mathcal{G}} \mathrm{Tr}\{\boldsymbol{Y}_j\boldsymbol{W}\}\\
& P_k^{\min} - P_k^c \le \mathrm{Tr}\{\boldsymbol{Y}_k\boldsymbol{W}\} \le P_k^{\max} - P_k^c \ \ \forall k\in\mathcal{G}\\
& Q_k^{\min} - Q_k^c \le \mathrm{Tr}\{\overline{\boldsymbol{Y}}_k\boldsymbol{W}\} \le Q_k^{\max} - Q_k^c, \ \ \forall k\in\mathcal{G}\\
& (V_k^{\min})^2 \le \mathrm{Tr}\{\boldsymbol{M}_k\boldsymbol{W}\} \le (V_k^{\max})^2, \ \ \forall k\in\mathcal{K}\\
& (\mathrm{Tr}\{\boldsymbol{Y}_{lm}\boldsymbol{W}\})^2 + (\mathrm{Tr}\{\overline{\boldsymbol{Y}}_{lm}\boldsymbol{W}\})^2 \le (S_{lm}^{\max})^2, \ \ \forall (l,m)\in\mathcal{N}\\
& \mathrm{Tr}\{\boldsymbol{Y}_{lm}\boldsymbol{W}\} \le P_{lm}^{\max}, \ \ \forall (l,m)\in\mathcal{N}\\
& \mathrm{Tr}\{\boldsymbol{M}_{lm}\boldsymbol{W}\} \le (\Delta V_{lm}^{\max})^2, \ \ \forall (l,m)\in\mathcal{N}\\
& -P_i^c - \mathrm{Tr}\{\boldsymbol{Y}_i\boldsymbol{W}\} + R_i(\gamma_i) + \epsilon_i^{(n)} = 0, \ \ \forall i\in\mathcal{C}\\
& \boldsymbol{W} \succeq 0, \quad \forall i\in\{1,...,M\}.
\end{aligned}
$$

Where the decision variable $\boldsymbol{W}$ is still dependent on the error term $\boldsymbol{\epsilon}^{(n)}$, and we are going to talk how to simplify and solve it in the following sections.

4.3.1 Linear Supply Function

Recall that we are given a load reduction target D and the random load reduction at demand bus i as a function of the price rebate γ_i is $R_i(\gamma_i)+\epsilon_i$. In general, function $R_i(\gamma_i)$ can admit any form to reflect customer's preference. For simplicity, we take linear form for the supply function, as it can reflect consumer's reference that they are willing to reduce more load at a higher rebate price. At the same time, we assume ϵ_i is independent with each other and we can generate samples of the random vector $\boldsymbol{\epsilon}$. Therefore, the supply function turns to $\tilde{R}_i(\gamma_i) = a_i\gamma_i + \epsilon_i$, where a_i is a positive constant. Following these assumptions, our objective function becomes to

$$\sum_{i=1}^{K} a_i \gamma_i^2 + \frac{\lambda}{M} \sum_{n=1}^{M} \left(D - \left(P^g(0) - P^g(\mathbf{\Gamma}, \boldsymbol{\epsilon}^{(n)})\right)\right)_+. \tag{4.9}$$

4.3.2 Linear Approximation for Injected Power

Even though we have applied sample average approximation and SDP relaxation model to the original problem, in optimization problem (4.8), since $M \approx 100$ to 1000 and $\boldsymbol{W}$ is dependent on the error term $\boldsymbol{\epsilon}^{(n)}$ in (4.8), this problem is still intractable even for very small networks, e.g. a network with $K = 30$ buses. To solve this dilemma, we construct an affine function $\hat{P}(\mathbf{\Gamma}, \boldsymbol{\epsilon})$ to approximate the power generation $P^g(\mathbf{\Gamma}, \boldsymbol{\epsilon})$ function in a small neighborhood of a given rebate vector $\mathbf{\Gamma}$.

In particular, we define

$$P^g(\mathbf{\Gamma}, \boldsymbol{\epsilon}) \approx \hat{P}(\mathbf{\Gamma}, \boldsymbol{\epsilon}) = P^g(\mathbf{\Gamma}, \mathbf{0}) + \sum_i \pi_i^{\mathbf{\Gamma}} \epsilon_i, \tag{4.10}$$

where $P^g(\mathbf{\Gamma}, \mathbf{0})$ denotes the total generation when the offer price is $\mathbf{\Gamma}$ and the stochastic error $\boldsymbol{\epsilon} = \mathbf{0}$. The sensitivity coefficients $\boldsymbol{\pi}^{\mathbf{\Gamma}} = (\pi_i^{\mathbf{\Gamma}})$ depend on the offer price $\mathbf{\Gamma}$ and denote the change in total injected power per unit change in the demand at bus i. Thus, $\boldsymbol{\pi}^{\mathbf{\Gamma}}$ can be interpreted as the dual variables corresponding to the real power balance constraint in the optimal power flow problem of implementing offer price Γ. Algorithm 4.1 describes the computation of the linear approximation of the total power generation at any offer price $\mathbf{\Gamma}$. We conduct numerical experiments to compare the error of the linear approximation for a set of test networks and observe that the difference between the true injected power and the linear approximation is at most, 2% across all test networks. Therefore, the linear approximation provides a good approximation under reasonable bounds on standard deviation of the stochastic errors.

Algorithm 4.1 Linear approximation of Power Generation

1: **Input**: Offer price $\mathbf{\Gamma}$.
2: Compute a solution $\boldsymbol{W} \succeq 0$ to (4.4)-(4.7) that minimizes total power generation with demand at each node $k \in \mathcal{K}$, $P_k^c - R_k(\gamma_k)$.
3: $\pi_k^{\mathbf{\Gamma}}$: optimal dual variable for real power balance constraint at bus $k \in \mathcal{K}$.
4: **Return**:

$$\hat{P}(\mathbf{\Gamma}, \boldsymbol{\epsilon}) = P^g(\mathbf{\Gamma}, \mathbf{0}) + \sum_k \pi_k^{\mathbf{\Gamma}} \epsilon_k.$$

4.3.3 Optimization Heuristic

Now we are ready to describe our iterative heuristic to compute the offer prices $\mathbf{\Gamma}$ using a linear approximation of the total power generation (4.10). For the convenience of description, we assume that for the whole distribution network there is a single node where power is injected from ISO into the distribution network. As mentioned earlier, a direct SAA approach (4.8) is computationally intractable, since we need $W(\boldsymbol{\epsilon})$ to satisfy the constraints for each sample $\boldsymbol{\epsilon}$ and the number of samples is large even for small networks. Therefore, instead of the direct SAA approach, we start with an initial offer price, $\mathbf{\Gamma}^0$ from solving the demand response problem with deterministic supply functions and then we can formulate the optimization problem to compute the offer prices in the next iterate as follows,

$$\begin{aligned} \min_{\mathbf{\Gamma}, \boldsymbol{W}} \quad & \textstyle\sum_{i \in \mathcal{K}} a_i \gamma_i^2 \\ & + \tfrac{\lambda}{M} \textstyle\sum_{n=1}^{M} \left(D - P^g(0) + P^g(\mathbf{\Gamma}, \mathbf{0}) + \sum_{i \in \mathcal{K}} \pi_i^0 \epsilon_i^{(n)} \right)_+ \\ \text{s.t.} \quad & \boldsymbol{W} \succeq 0 \text{ satisfies (4.4)-(4.7) with } P_i^c + jQ_i^c = \\ & \bar{P}_i^c - a_i \gamma_i + j\bar{Q}_i^c \text{ for } i \in \mathcal{C}, \end{aligned} \tag{4.11}$$

where $\{\boldsymbol{\epsilon}^n : n = 1, \ldots, M\}$ denotes M samples of the random vector $\boldsymbol{\epsilon}$. Note that the constraints in (4.11) no longer depend on $\boldsymbol{\epsilon}$; consequently, there is a single positive semidefinite variable $\boldsymbol{W}$. Therefore, we can solve the above problem efficiently. Note also that the coefficients π_i are constants in (4.11). We solve (4.11) to compute the offer prices in the next iterate. We then compute the linear approximation for the power generation at the new offer prices and continue this iterative procedure until it converges to a fixed point. We describe the details in Algorithm 4.2.

Algorithm 4.2 Offer Price optimization Heuristic, AC-PF-DR

Initialize: $t := 0$, $\delta := 1$, offer prices $\mathbf{\Gamma}^0$.
while $(\delta > 0.001)$ **do**
  Call **Algorithm 4.1** to compute coefficients $\pi_i^{\mathbf{\Gamma}^t}$.
  Solve (4.11) to compute $\mathbf{\Gamma}^{t+1}$.
  $\delta = \max_i \frac{|\Gamma_i^{t+1} - \Gamma_i^t|}{\Gamma_i^t}$.
  $t := t + 1$.
end while
Return: Γ^t.

On small instances of distribution networks, we conduct numerical experiments showing that the algorithm converges quickly. However, the above procedure may not converge in a reasonable time or may not even converge in general, we may need to modify this algorithm for concrete instances. We will talk about the modification in the computational study section.

4.4 Alternative Power Flow Constraints

In this section, we describe some alternative power flow models which will be used as comparisons to the performance of our iterative heuristic.

4.4.1 DC Power Flows

The DC power flow model is constructed by linearizing the AC power flow equations. Let $\boldsymbol{\theta}$ denote the vector of phase angles of voltage at all the buses. Under typical operating conditions, the angle difference $|\theta_l - \theta_m|$ for any transmission line $(l, m) \in \mathcal{N}$ is small ($\ll 10$ degrees). Therefore, $\sin(\theta_l - \theta_m) \approx (\theta_l - \theta_m)$ and $\cos(\theta_l - \theta_m) \approx 1$. For all transmission lines $(l, m) \in \mathcal{N}$, we assume that the resistance is nearly zero and also the magnitude of voltage is 1 p.u. at all buses. Furthermore, we can, without loss of generality, assume that $\theta_1 = 0$. With these approximations, we can formulate the power flow constraints as $\boldsymbol{P} = \boldsymbol{B}\boldsymbol{\theta}$, where $\boldsymbol{P} \in \mathbb{R}^{K-1}$ is the vector of power injections for buses $2, \ldots, K$, $\boldsymbol{\theta} \in \mathbb{R}^{K-1}$ is the vector of nodal voltage angles, and $\boldsymbol{B} \in \mathbb{R}^{(K-1)\times(K-1)}$ is the network admittance matrix. The constraint for bus 1 is linearly dependent on the other constraints and can therefore, be eliminated. In the DC approximation of the power flow constraints, the injected power at each node sums up to 0. There is no transmission loss under the DC framework and the cost on power loss in the objective function vanishes. However, we need to monitor the differences in phase angles as large phase angle difference could cause instability in the power network.

Let $\boldsymbol{N} \in \{0, -1, 1\}^{K\times K}$ be the bus-line incidence matrix for the distribution network, and let ρ denote the upper bound on allowed angle difference on any link. Then the DC power flow offer price optimization problem can be formulated as follows.

$$
\begin{aligned}
\min_{\boldsymbol{\Gamma},\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)}),\boldsymbol{P}(\boldsymbol{\epsilon}^{(n)})} \quad & \textstyle\sum_{i\in\mathcal{K}} a_i\gamma_i^2 + \\
& \textstyle\frac{\lambda}{M}\sum_{n=1}^{M}\left(D-(P^g(0)-P^g(\boldsymbol{\Gamma},\boldsymbol{\epsilon}^{(n)}))\right)_+ \\
\text{s.t.} \quad & \boldsymbol{P}(\boldsymbol{\epsilon}^{(n)}) = \boldsymbol{B}\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)}) \\
& P^g(\boldsymbol{\Gamma},\boldsymbol{\epsilon}^{(n)}) = \textstyle\sum_{k\in\mathcal{G}} P_k(\boldsymbol{\epsilon}^{(n)}) \\
& -P_k(\boldsymbol{\epsilon}^{(n)}) = P_k^c - a_k\gamma_k - \epsilon_k^{(n)}, \forall k \in \mathcal{C} \\
& \left\|\boldsymbol{N}\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)})\right\|_\infty \le \rho \\
& P_k^{\min} \le P_k(\boldsymbol{\epsilon}^{(n)}) \le P_k^{\max}, \forall k \in \mathcal{G},
\end{aligned}
\tag{4.12}
$$

where the notation $\boldsymbol{P}(\boldsymbol{\epsilon}^{(n)})$, $\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)})$ and $P^g(\boldsymbol{\Gamma},\boldsymbol{\epsilon}^{(n)})$ emphasizes that the phase angles, the power on lines, and the overall power generation is a function of the stochastic error sample $\boldsymbol{\epsilon}^{(n)}$. Note that angle difference constraint can also be modeled as a penalty term $\eta\mathbb{E}_{\boldsymbol{\epsilon}}\left(\boldsymbol{N}\boldsymbol{\theta}(\boldsymbol{\epsilon})-\rho\right)_+$ in the objective. Unlike the original AC power flow model, the DC model (4.12) can be solved efficiently as it is actually a quadratic program. We need to emphasize that since we assume that the resistance on transmission lines is zero, the formulation (4.12) is not able to model transmission losses.

4.4.2 DC$^\beta$ Power Flows

Since the DC model does not model any transmission loss over the grid, we can expect that the optimal solution from DC model above will achieve a total reduction of D on the load when penalty is high, so that the actual reduction over the grid will be greater than D. If the transmission loss is a large proportion, this is a great waste in the incentive, as the company overpays the rebates to customer. Therefore, we find an interesting modified DC model for this purpose, as it is able to include partial transmission loss into the optimization. We name it DC$^\beta$ model. This model is actually a two-step method. Firstly, we solve the OPF problem without any rebate. Then, define:

$$\beta = \frac{P^g(0)}{\sum_{k=1}^{K} P_k^c} - 1. \tag{4.13}$$

Here, β is the percentage of transmission loss over the grid when there is no rebate. Then, in the second step, we use the same β value as our estimation of the percentage of transmission loss over the grid when there exists rebates. Thus, the total reduction estimation in this modified DC model becomes to,

$$(1+\beta)\sum_{i=1}^{K} P_i^c - (1+\beta)\sum_{i=1}^{K}(P_i^c - \tilde{R}_i(\gamma_i)) = (1+\beta)\sum_{i=1}^{K} \tilde{R}_i(\gamma_i).$$

Thus, we add our estimate of reduction in transmission loss $\beta\sum_{i=1}^{K} \tilde{R}_i(\gamma_i)$ into the total reduction. We claim that this estimation will only underestimate the true reduction in transmission loss, which means that it will never incur a short-fall penalty due to we overestimate the reduction on the transmission loss. This is because suppose β_1 is the real value of the percentage of transmission loss over the grid when we implement the offer prices $\mathbf{\Gamma}$. Then, the actual total reduction is,

$$\begin{aligned} & (1+\beta)\textstyle\sum_{i=1}^{K} P_i^c - (1+\beta_1)\sum_{i=1}^{K}(P_i^c - \tilde{R}_i(\gamma_i)) \\ = \; & (\beta-\beta_1)\textstyle\sum_{i=1}^{K} P_i^c + (1+\beta_1)\sum_{i=1}^{K} \tilde{R}_i(\gamma_i). \end{aligned}$$

Here, the actual reduction in transmission loss is $(\beta-\beta_1)\sum_{i=1}^K P_i^c+\beta_1\sum_{i=1}^K \tilde{R}_i(\gamma_i)$. Because the load is reduced, we have $\beta_1 \le \beta$. Then, since the actual reduction can not surpass the original load, we have $\sum_{i=1}^K \tilde{R}_i(\gamma_i) \le \sum_{i=1}^K P_i^c$, which leads to $(\beta-\beta_1)\sum_{i=1}^K \tilde{R}_i(\gamma_i) \le (\beta-\beta_1)\sum_{i=1}^K P_i^c$. Finally, we have

$$\beta\sum_{i=1}^K \tilde{R}_i(\gamma_i) \le (\beta-\beta_1)\sum_{i=1}^K P_i^c + \beta_1\sum_{i=1}^K \tilde{R}_i(\gamma_i),$$

i.e., we underestimate the reduction in transmission loss. Then, our problem in step two is:

$$\begin{aligned}
\min_{\mathbf{\Gamma},\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)}),\boldsymbol{P}(\boldsymbol{\epsilon}^{(n)})} \quad & \textstyle\sum_{i\in\mathcal{K}} a_i\gamma_i^2 + \frac{\lambda}{M}\sum_{n=1}^M \left(D-(1+\beta)(P^g(0)-P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}^{(n)}))\right)_+ \\
\text{s.t.} \quad & \boldsymbol{P}(\boldsymbol{\epsilon}^{(n)}) = \boldsymbol{B}\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)}) \\
& P^g(\mathbf{\Gamma},\boldsymbol{\epsilon}^{(n)}) = \textstyle\sum_{k\in\mathcal{G}} P_k(\boldsymbol{\epsilon}^{(n)}) \\
& -P_k(\boldsymbol{\epsilon}^{(n)}) = P_k^c - a_k\gamma_k - \epsilon_k^{(n)}, \forall k\in\mathcal{C} \\
& \left\|\boldsymbol{N}\boldsymbol{\theta}(\boldsymbol{\epsilon}^{(n)})\right\|_\infty \le \rho \\
& P_k^{\min} \le P_k(\boldsymbol{\epsilon}^{(n)}) \le P_k^{\max}, \forall k\in\mathcal{G}.
\end{aligned}$$

This problem is actually a quadratic program, which is efficient to solve. Thus, we have a modified DC model which considers the saving in transmission loss and keeps the model simple.

4.4.3 No Network

We also consider an offer price optimization approach without any power flow constraints. In this approach, we assume that for any given offer price $\mathbf{\Gamma}$, the total load reduction is $\sum_{i\in\mathcal{K}} a_i\gamma_i+\epsilon_i$, without taking any power flow model or transmission losses into account. The following offer price optimization problem without power flows

$$\min_{\Gamma}\sum_{i=1}^K a_i\gamma_i^2 + \frac{\lambda}{M}\sum_{n=1}^M\left(D-\sum_{i=1}^K\left(a_i\gamma_i+\epsilon_i^{(n)}\right)\right)_+$$

can be solved efficiently since it is a quadratic program.

It is easy to find that this method will not give a higher total cost than the DC model (4.12), as they have same objective but this program has no constraints. However, since this method considers nothing about the network, the implementation of the rebates may be very instable.

4.5 Computational Study

Up to now, we have given all the mathematical formulations of this problem and the algorithms to solve the problem. In this section, we are going to talk about the numerical experiments we conducted for the problems. We have four different price rebates optimization . Γ^{DR} is the solution of price rebates problem for no network model. Γ^{DC} is the solution of price rebates problem for DC model. $\Gamma^{DC^{\beta}}$ is the solution of price rebates problem for DC$^{\beta}$ model. Γ^{AC} is the solution of price rebates problem for AC model, solved by Algorithm 4.2.

After solving the rebates, we compare the performance of our AC power flow offer price optimization heuristic with the other three approaches by comparing two separate costs,

$$\begin{aligned} \text{DR-cost} : & \sum_{i \in \mathcal{K}} \gamma_i R_i(\gamma_i), \\ \text{Shortfall-penalty} : & \frac{\lambda}{M} \sum_{n=1}^{M} \left(D - P^g(0) + P^g(\mathbf{\Gamma}, \mathbf{0}) + \sum_{i \in \mathcal{K}} \pi_i^{\mathbf{\Gamma}} \epsilon_i^{(n)} \right)_+, \end{aligned}$$

where $\mathbf{\Gamma} \in \{\mathbf{\Gamma}^{DR}, \mathbf{\Gamma}^{DC}, \mathbf{\Gamma}^{DC-\beta}, \mathbf{\Gamma}^{AC}\}$. To be specific, After we get the offer prices $\mathbf{\Gamma}$, we can compute the DR-cost directly. Then, we use OPF to implement the demand reduction induced by $\mathbf{\Gamma}$ assuming a deterministic supply function, to get $P^g(\mathbf{\Gamma}, \mathbf{0})$ and $\{\pi_i^{\mathbf{\Gamma}}\}$, then use the formula above to compute the final shortfall-penalty. The experimental procedure is described in Algorithm 4.3.

Algorithm 4.3 Computational Experiment

1: Compute
$$\mathbf{\Gamma}^{DR}, \mathbf{\Gamma}^{DC}, \mathbf{\Gamma}^{DC-\beta}, \mathbf{\Gamma}^{AC}.$$
2: Pick $\mathbf{\Gamma} \in \{\mathbf{\Gamma}^{DR}, \mathbf{\Gamma}^{DC}, \mathbf{\Gamma}^{DC-\beta}, \mathbf{\Gamma}^{AC}\}$.

3: Compute **DR-cost**
$$\sum_{i\in\mathcal{K}} \gamma_i R_i(\gamma_i).$$
4: Use OPF to implement $\mathbf{\Gamma}$, compute $P^g(\mathbf{\Gamma}, \mathbf{0})$ and $\{\pi_i^{\mathbf{\Gamma}}\}$.

5: Sample M values: $\boldsymbol{\epsilon}^{(1)}, \ldots, \boldsymbol{\epsilon}^{(M)}$.

6: Compute **Shortfall-Penalty** as
$$\frac{\lambda}{M} \sum_{n=1}^{M} \left(D - P^g(0) + P^g(\mathbf{\Gamma}, \mathbf{0}) + \sum_{i\in\mathcal{K}} \pi_i^{\mathbf{\Gamma}} \epsilon_i^{(n)}\right)_+.$$

We want to emphasize that in Algorithm 4.3, we solve all the OPF problems by both Matpower and the SDP relaxation of the problem. Matpower is an application of first order method, therefore it can not guarantee to get the global optimal solution. Then, its cost gives an upper bound to the true optimal cost. On the other hand, since the SDP relaxation is not tight, it only provides a lower bound on the total cost. However, for all the instances in our computational study, the SDP relaxation has a rank one optimal solution which implies that the relaxation is tight for our instances, and luckily, the result from Matpower is same as that from SDP relaxation. Therefore, the comparison in Algorithm 4.3 is accurate in our experiments. We would like to emphasize that this is not the case in general. Computing the exact total cost for OPF problem is known to be NP-hard in general [49].

4.5.1 Setup

We select the standard IEEE test cases as main networks to run the experiments. We use the following values for the parameters in the experiments described in Tables 4.1 and 4.2: $\lambda = 10, 100$, $\rho = 15$ in degrees and target load reduction $D \in \{2\%, 5\%, 10\%, 15\%, 20\%, 25\%\}$ of the total active load. $a_i's$ are the unit payment of reduction at each bus. To make the test more realistic, we generate $a_i's$ as I.I.D Gaussian with $\mu = 1, \sigma^2 = 0.1$. If a_i is less than

0, we regenerate the a_i. Once they are generated, we will not change them anymore in the whole procedure. We use MATLAB with the cvx package to solve the SDPs on a 12-core server.

4.5.2 Modified Algorithm

Algorithm 4.2 for Γ^{AC} works well for small networks such as 14-bus and 30-bus. However, for larger networks (e.g. 300-bus case), each iteration can take about 15 minutes and the algorithm is not guaranteed to converge in less than a hundred iterations. Therefore, as mentioned before, we consider a modified algorithm. In this modified method, we still get the offer price $\mathbf{\Gamma}^0$ by using the deterministic supply function in the demand response problem. Then, instead of running a loop, we assume that the values of $\{\pi_i\}$ are unchanged. In the modified approach, we just solve one single SDP instead of iteratively solving a sequence of SDPs for a convergent solution. We have tested that, for large networks, this algorithm completes in a much shorter time, and the gap between this sub-optimal solution with the iterative heuristic solution is negligible.

4.5.3 Main Results

We conducted extensive numerical experiments on a large set of test instances and parameters. Not surprisingly, they gave us similar implications about the experiment. Therefore, we present our result tables for only the IEEE 57-bus network here. A full version of test result can be found on our Website.

Before we show the tables, there are some additional notations we need to introduce. In Table 4.1 and Table 4.2, reduction is total reduction over the grid as the sum of reduction at nodes and reduction in transmission loss; CPU time(s) is CPU time used to compute the offer prices; (DR-AC)/AC is relative difference of total costs between DR model and AC model; (DC-AC)/AC is relative difference of total costs between DC model and AC model; (DC$^{\beta}$-AC)/AC is relative difference of total costs between DC$^{\beta}$ model and AC model.

Now, we are ready to display the tables.

D	25.02	62.54	125.08	187.62	250.16	312.70
DR model:						
DR-cost	14.84	92.72	375.36	860.78	939.66	939.66
Shortfall penalty	0.00	0.00	0.00	0.00	470.94	1096.34
Total cost	14.84	92.72	375.36	860.78	1410.60	2036.00
Reduction	26.36	65.67	130.55	194.80	203.07	203.07
CPU time(s)	0.69	0.56	0.64	0.72	0.73	0.51
DC model:						
DR-cost	14.84	92.72	375.36	860.78	939.66	939.66
Shortfall penalty	0.00	0.00	0.00	0.00	470.94	1096.34
Total cost	14.84	92.72	375.36	860.78	1410.60	2036.00
Reduction	26.36	65.67	130.55	194.80	203.07	203.07
CPU time(s)	1.08	0.97	0.98	0.90	0.86	0.92
DC^{β} model	$\beta =$	0.009				
DR-cost	14.57	91.07	368.54	844.88	954.40	954.41
Shortfall penalty	0.00	0.00	0.00	0.00	455.94	1081.34
Total cost	14.57	91.07	368.54	844.88	1410.35	2035.74
Reduction	26.13	65.09	129.39	193.08	204.57	204.57
CPU time(s)	0.97	1.19	0.76	1.09	0.76	0.83
AC model:						
DR-cost	13.34	83.97	343.82	796.73	968.70	968.72
Shortfall penalty	0.00	0.01	0.04	0.12	443.44	1068.75
Total cost	13.34	83.98	343.85	796.85	1412.14	2037.48
Reduction	25.02	62.54	125.08	187.61	205.82	205.82
CPU time(s)	166.06	342.00	891.67	985.40	842.62	794.06
(DR-AC)/AC	11.2%	10.4%	9.2%	8.0%	-0.1%	-0.1%
(DC-AC)/AC	11.2%	10.4%	9.2%	8.0%	-0.1%	-0.1%
(DC^{β}-AC)/AC	9.2%	8.4%	7.2%	6.0%	-0.1%	-0.1%

Table 4.1: Comparison of AC, DR, DC and DC^{β} models at $\rho = 15°$, $\lambda = 10$ on IEEE 57-bus test case

D	25.02	62.54	125.08	187.62	250.16	312.70
DR model:						
DR-cost	14.84	92.72	375.36	860.78	1583.84	2618.46
Shortfall penalty	0.00	0.00	0.00	0.00	0.00	0.00
Total cost	14.84	92.72	375.36	860.78	1583.84	2618.46
Reduction	26.36	65.67	130.55	194.80	258.49	321.79
CPU time(s)	0.62	0.53	0.64	0.47	0.76	0.66
DC model:						
DR-cost	14.84	92.72	375.36	860.78	1583.84	2618.46
Shortfall penalty	0.00	0.00	0.00	0.00	0.00	0.00
Total cost	14.84	92.72	375.36	860.78	1583.84	2618.46
Reduction	26.36	65.67	130.55	194.80	258.49	321.79
CPU time(s)	0.80	1.01	0.69	0.76	0.87	1.28
DC^{β} model	$\beta =$	0.009				
DR-cost	14.57	91.07	368.54	844.88	1552.94	2564.76
Shortfall penalty	0.00	0.00	0.00	0.00	0.00	0.00
Total cost	14.57	91.07	368.54	844.88	1552.94	2564.76
Reduction	26.13	65.09	129.39	193.08	256.22	318.96
CPU time(s)	0.80	0.94	1.33	0.87	1.11	1.11
AC model:						
DR-cost	13.34	83.99	343.90	796.94	1478.78	2465.27
Shortfall penalty	0.00	0.00	0.01	0.03	0.05	0.08
Total cost	13.34	83.99	343.91	796.97	1478.83	2465.35
Reduction	25.02	62.55	125.09	187.63	250.17	312.71
CPU time(s)	154.91	372.66	1024.36	1007.63	984.60	970.12
(DR-AC)/AC	11.2%	10.4%	9.1%	8.0%	7.1%	6.2%
(DC-AC)/AC	11.2%	10.4%	9.1%	8.0%	7.1%	6.2%
(DC^{β}-AC)/AC	9.2%	8.4%	7.2%	6.0%	5.0%	4.0%

Table 4.2: Comparison of AC, DR, DC and DC^{β} models at $\rho = 15°$, $\lambda = 100$ on IEEE 57-bus test case

The following paragraphs cover the main properties of our heuristics and models based on the computational results in our experiments.

The offer prices $\boldsymbol{\gamma}$ are capped at approximately $\frac{\lambda}{2}$. This follows from the first order condition: objective function $f \sim (a_i\gamma_i^2 + \lambda \times \max(D - a_i\gamma_i, 0))$ and $\frac{\partial f}{\partial \gamma_i} \sim (2a_i\gamma_i - a_i\lambda)$. We can see that in Table 4.1, when $\lambda = 10$, total reduction is capped in columns 5 and 6. In Table 4.2, when $\lambda = 100$, the total reduction is not capped by λ any more.

When the reduction is not capped by λ, the cost of the AC power flow based heuristic is significantly lower than the other three for all values of the target demand reduction D. The DC model and no network model compute identical solutions where the total demand reduction at the demand buses is equal to the target D. Since these two approaches do not account for transmission losses, the actual reduction in total power generation is greater than D. These two approaches end up paying more rebate than needed to meet the target.

On the other hand, the AC power flow based heuristic achieves the target demand reduction through a combination of reduction at load buses and reduction in transmission losses, since lower cumulative power needs to be transmitted. This is because the AC power flow models the transmission losses in the optimization phase. Therefore, the total payments for the AC power flow based heuristic are smaller, especially for large network whose transmission loss has a larger proportion of total generation.

However, in some large network instances which are not listed here, the cost of DC model is significantly higher than the cost of no network model because the angle difference constraint is binding. Therefore, the total cost needs to be higher in DC model.

Maximum angle difference is significantly less than $15°$ for many test cases. Notice that this is the phase angle difference from the OPF implementation of given offer prices from all models. In fact, we can see that the angle difference constraint of $15°$ in the DC optimization model can possible be tight even when maximum angle difference for the OPF implementation of the DC model rebates is $4°$.

The DC$^\beta$ model we introduced uses a simple trick to estimate the transmission loss in DC model, like $\beta = 0.009$ in the previous two tables. Therefore, as the consequence of considering partial transmission loss, the total reduction in this model is less than the original DC model, but still higher than the AC model, ending up with a total cost between original DC model and AC model. However, this model is much less complex than the AC model heuristic, and we can achieve a better result by carefully adjusting the value of β.

4.6 Conclusion

In this chapter, we consider a new price rebate approach to demand-response problem under power flow constraints. We consider an AC power flow based model that allows us to model transmission loss and therefore, optimize the offer prices to achieve the target demand reduction through a combination of reduction at load buses and reduction in transmission loss. This is important since the DC power flow based model does not consider the transmission loss and can not account for its reduction at the offer-price optimization stage. However, the AC power flow based offer price optimization problem is non-convex, and therefore, intractable and hard to solve. We propose an iterative algorithm to compute price rebates to achieve the required demand reduction with minimum possible cost. We conducted computational study to compare the performance of our iterative method with other demand response models or heuristics. Our results show that our iterative heuristic performs significantly better than the DC power flow based models or model without any power flow constraints, which are not able to account for the savings in transmission losses. Therefore, there is significant value in using an AC power flow based model for demand-response optimization. It is important to note that our iterative heuristic is exact only when the electric grid instance satisfies the numerical requirement in [49]. When the requirement is not satisfied, our iterative heuristic only computes a lower bound of the optimal cost. The problem of how to design a provably near-optimal algorithm for these instances is an interesting open question.

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Appendices

Appendix A

Proofs for Chapter 3

A.1 Proof of Proposition 3.1

Proof. If $\exists \boldsymbol{x_i} \neq \boldsymbol{x_j}, \boldsymbol{x_j} \neq \boldsymbol{x_k} \in \mathbb{V}$, such that

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) > 0,\ \Gamma(\boldsymbol{x_j}, \boldsymbol{x_k}) > 0, \tag{A.1}$$

then we have driver relocation from $\boldsymbol{x_i}$ to $\boldsymbol{x_j}$ and from $\boldsymbol{x_j}$ to $\boldsymbol{x_k}$. Because of the compatible incentive requirement for driver relocation, we have

$$p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) \geq p(\boldsymbol{x_i}),\ p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k}) \geq p(\boldsymbol{x_j}). \tag{A.2}$$

By linking these two inequalities, we have

$$p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) \geq p(\boldsymbol{x_i}). \tag{A.3}$$

From triangle inequality (3.1), we have

$$c(\boldsymbol{x_i}, \boldsymbol{x_k}) \leq c(\boldsymbol{x_i}, \boldsymbol{x_j}) + c(\boldsymbol{x_j}, \boldsymbol{x_k}). \tag{A.4}$$

Next, we discuss three different cases. Firstly, if $c(\boldsymbol{x_i}, \boldsymbol{x_k}) < c(\boldsymbol{x_i}, \boldsymbol{x_j}) + c(\boldsymbol{x_j}, \boldsymbol{x_k})$ is true, use (A.2), we have,

$$p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) \leq p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) < p(\boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_k}).$$

Thus, $\boldsymbol{x_k}$ offers a higher effective price to drivers at $\boldsymbol{x_i}$ than $\boldsymbol{x_j}$. The platform should not assign any drivers to move from $\boldsymbol{x_i}$ to $\boldsymbol{x_j}$. It contradicts the incentive compatibility of drivers.

Secondly, if $p(\boldsymbol{x_j}) < p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k})$, use (A.2) and (A.4), we also have,

$$p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) < p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) \leq p(\boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_k}).$$

Thus, $\boldsymbol{x_k}$ offers a higher effective price to drivers at $\boldsymbol{x_i}$ than $\boldsymbol{x_j}$. The platform should not assign any drivers to move from $\boldsymbol{x_i}$ to $\boldsymbol{x_j}$. It contradicts the incentive compatibility of drivers.

Finally, if $c(\boldsymbol{x_i}, \boldsymbol{x_k}) = c(\boldsymbol{x_i}, \boldsymbol{x_j}) + c(\boldsymbol{x_j}, \boldsymbol{x_k})$ and $p(\boldsymbol{x_j}) = p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k})$ are true, we have,

$$p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) = p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) = p(\boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_k}).$$

Then drivers at $\boldsymbol{x_i}$ are indifferent of going to $\boldsymbol{x_j}$ or going to $\boldsymbol{x_k}$. Because of $p(\boldsymbol{x_j}) = p(\boldsymbol{x_k}) - c(\boldsymbol{x_j}, \boldsymbol{x_k})$, drivers at $\boldsymbol{x_j}$ are indifferent of staying at $\boldsymbol{x_j}$ or going to $\boldsymbol{x_k}$. Then, if $\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) < \Gamma(\boldsymbol{x_j}, \boldsymbol{x_k})$, we can send $\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})$ amount of drivers from $\boldsymbol{x_i}$ to $\boldsymbol{x_k}$ directly, canceling $\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})$ amount of drivers in and out of node $\boldsymbol{x_j}$. If $\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) \geq \Gamma(\boldsymbol{x_j}, \boldsymbol{x_k})$, we can send $\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k})$ amount of drivers from $\boldsymbol{x_i}$ to $\boldsymbol{x_k}$ directly, canceling $\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k})$ amount of drivers in and out from node $\boldsymbol{x_j}$. The new constructed driver flow is feasible and it achieves same net driver flows and objective value as before. More importantly, it satisfies

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k}) = 0.$$

This is true for any $\boldsymbol{x_i} \neq \boldsymbol{x_j}, \boldsymbol{x_j} \neq \boldsymbol{x_k} \in \mathbb{V}$, thus we can require

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j})\Gamma(\boldsymbol{x_j}, \boldsymbol{x_k}) = 0,\ \forall \boldsymbol{x_i} \neq \boldsymbol{x_j}, \boldsymbol{x_j} \neq \boldsymbol{x_k} \in \mathbb{V},$$

and it does not change the value of Program 3.4. □

A.2 Proof of Proposition 3.2

Proof. Suppose we have

$$\exists \boldsymbol{x_i}, \boldsymbol{x_j} \in \mathbb{V},\ s.t.\ p(\boldsymbol{x_i}) < p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}).$$

Then,

$$\max_{\boldsymbol{x} \in \mathbb{V}} p(\boldsymbol{x}) - c(\boldsymbol{x_i}, \boldsymbol{x}) \geq p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}) > p(\boldsymbol{x_i}).$$

Because location $\boldsymbol{x_i}$ does not offer the maximum effective price for drivers at location $\boldsymbol{x_i}$, all drivers at location $\boldsymbol{x_i}$ leave. Denotes set $\mathbb{I}$ as

$$\mathbb{I} := \arg\max_{\boldsymbol{x} \in \mathbb{V}} p(\boldsymbol{x}) - c(\boldsymbol{x_i}, \boldsymbol{x}),$$

i.e., $\mathbb{I}$ contains the destinations that are incentive compatible for drivers at location $\boldsymbol{x_i}$ to relocate. By Proposition 3.1, since there exists drivers moving out of location $\boldsymbol{x_i}$, there is no driver moving into location $\boldsymbol{x_i}$. Thus, all riders at location $\boldsymbol{x_i}$ are unserved and it has no contribution to the total revenue. Next, we reset the price at location $\boldsymbol{x_i}$ to

$$p(\boldsymbol{x_i}) = \max_{\boldsymbol{x} \in \mathbb{V}} p(\boldsymbol{x}) - c(\boldsymbol{x_i}, \boldsymbol{x}).$$

In this way, we can still have the same relocation of drivers at location $\boldsymbol{x_i}$ as before because the original assignment is still incentive compatible,

$$p(\boldsymbol{x_i}) = p(\boldsymbol{x_k}) - c(\boldsymbol{x_i}, \boldsymbol{x_k}), \forall \boldsymbol{x_k} \in \mathbb{I}.$$

By applying Proposition 3.1 again, all other driver relocations which do not involve $\boldsymbol{x_i}$ are still incentive compatible. Since the prices at other locations are unchanged, there is no new destination added into set $\mathbb{I}$ except for location $\boldsymbol{x_i}$ itself. At this time, location $\boldsymbol{x_i}$ is still unserved and it contributes nothing to the total revenue. In this way, we have the same objective value as before at this new price of $p(\boldsymbol{x_i})$, and it satisfies

$$p(\boldsymbol{x_i}) = \max_{\boldsymbol{x} \in \mathbb{V}} p(\boldsymbol{x}) - c(\boldsymbol{x_i}, \boldsymbol{x}) \geq p(\boldsymbol{x_j}) - c(\boldsymbol{x_i}, \boldsymbol{x_j}).$$

□

A.3 Proof of Proposition 3.3

Proof. Suppose we have

$$\exists \boldsymbol{x_i} \neq \boldsymbol{x_j} \text{ and } \boldsymbol{x_j} \neq \boldsymbol{0},\ s.t.\ \Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) > 0.$$

Then there are some drivers moving into location $\boldsymbol{x_j}$. By Proposition 3.1, there is no driver moving out of location $\boldsymbol{x_j}$. All its original drivers $\mu(\boldsymbol{x_j})$ are still at location $\boldsymbol{x_j}$. Denote set $\mathbb{K}$ as

$$\mathbb{K} := \{\boldsymbol{x_k} | \Gamma(\boldsymbol{x_k}, \boldsymbol{x_j}) > 0\},$$

i.e., the set of locations that supplies drivers to location $\boldsymbol{x_j}$. $\mathbb{K}$ is not empty as $\boldsymbol{x_i} \in \mathbb{K}$. Because of the incentive must be compatible for these drivers, we have

$$p(\boldsymbol{x_j}) \geq p(\boldsymbol{x_k}) + c(\boldsymbol{x_k}, \boldsymbol{x_j}), \forall \boldsymbol{x_k} \in \mathbb{K}.$$

By Proposition 3.2, we can require that

$$p(\boldsymbol{x_j}) \leq p(\boldsymbol{x_k}) + c(\boldsymbol{x_k}, \boldsymbol{x_j}), \forall \boldsymbol{x_k} \in \mathbb{K}.$$

Thus, we get

$$p(\boldsymbol{x_j}) = p(\boldsymbol{x_k}) + c(\boldsymbol{x_k}, \boldsymbol{x_j}), \forall \boldsymbol{x_k} \in \mathbb{K}.$$

Consequently, the drivers moving from $\boldsymbol{x_k}$ to $\boldsymbol{x_j}$ are indifferent of staying at $\boldsymbol{x_k}$ or moving to $\boldsymbol{x_j}$. Next, we send back all the relocated drivers moving into node $\boldsymbol{x_j}$ to construct a new assignment. For location $\boldsymbol{x_j}$, since $\boldsymbol{x_j} \neq \mathbf{0}$ and all its original drivers $\mu(\boldsymbol{x_j})$ are still at location $\boldsymbol{x_j}$, it still has enough drivers to serve all riders. This is because our Assumption 3.1 for the baseline demand

$$\lambda(\boldsymbol{x_j})\bar{F}_v\big(p(\boldsymbol{x_j})\big) \leq \lambda(\boldsymbol{x_j}) \leq \mu(\boldsymbol{x_j}).$$

Thus, it has the same revenue contribution $p(\boldsymbol{x_j})\lambda(\boldsymbol{x_j})\bar{F}_v\big(p(\boldsymbol{x_j})\big)$ as before. For all locations in $\mathbb{K}$, the prices are unchanged and they get more drivers to serve their riders, which may offer a higher contribution to the total revenue. At the same time, the prices and assignments at all other locations are still feasible since we do not change any price. Then, we improve the solution with condition

$$\Gamma(\boldsymbol{x_i}, \boldsymbol{x_j}) = 0.$$

satisfied for $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$. □

A.4 Proof of Theorem 3.4

Proof. Firstly, we show the pricing formula (3.13) in Theorem 3.4 is optimal. For any feasible solution to Program 3.4, we apply Proposition 3.2 such that it satisfies

$$p(\boldsymbol{x}) \geq p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}),\ \forall \boldsymbol{x} \in \mathbb{V}.$$

Since $p^{\min} = p_b$, then $p(\boldsymbol{x}) \geq p_b$, $\forall \boldsymbol{x} \in \mathbb{V}$. Combine it with the inequality above, we have

$$p(\boldsymbol{x}) \geq \max\Big(p_b, p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})\Big),\ \forall \boldsymbol{x} \in \mathbb{V}.$$

We denote set $\mathbb{K}$ as

$$\mathbb{K} := \{\boldsymbol{x} \in \mathbb{V} | p(\boldsymbol{x}) > \max\Big(p_b, p(\boldsymbol{0}) - c(\boldsymbol{x}, \boldsymbol{0})\Big)\}.$$

It is trivial to see that $\boldsymbol{0} \notin \mathbb{K}$. If $\mathbb{K}$ is empty, we do not need to do any change, the optimal pricing function is already achieved. If $\mathbb{K}$ is not an empty set, then $\forall \boldsymbol{x_k} \in \mathbb{K}$, we have

$$p(\boldsymbol{x_k}) > \max\Big(p_b, p(\boldsymbol{0}) - c(\boldsymbol{x_k}, \boldsymbol{0})\Big) \geq p(\boldsymbol{0}) - c(\boldsymbol{x_k}, \boldsymbol{0}).$$

Thus, location $\boldsymbol{x_k}$ can not supply drivers to demand shock location $\boldsymbol{0}$ because the price difference is not incentive compatible for relocation. Also, since $\boldsymbol{x_k}$ is not a demand shock location, by Proposition 3.3, there should be no drivers relocated to $\boldsymbol{x_k}$ and no drivers relocated from $\boldsymbol{x_k}$ to other non-shock locations. Thus, the available drivers at $\boldsymbol{x_k}$ are from its original drivers

$$\nu(\boldsymbol{x_k}) = \mu(\boldsymbol{x_k}).$$

Since $\lambda(\boldsymbol{x_k}) \leq \mu(\boldsymbol{x_k})$, its revenue contribution is $p(\boldsymbol{x_k})\lambda(\boldsymbol{x_k})\bar{F}_v\big(p(\boldsymbol{x_k})\big)$. This is true for all locations in $\mathbb{K}$. Next, we set the prices for locations in $\mathbb{K}$ to

$$p(\boldsymbol{x_k}) = \max\Big(p_b, p(\boldsymbol{0}) - c(\boldsymbol{x_k}, \boldsymbol{0})\Big), \forall \boldsymbol{x_k} \in \mathbb{V}.$$

Since $p\bar{F}_v(p)$ is a decreasing function on $[p_b, p^{\max}]$, the new price $p(\boldsymbol{x_k})$ is lower than its previous price and it increases its revenue contribution $p(\boldsymbol{x_k})\lambda(\boldsymbol{x_k})\bar{F}_v\big(p(\boldsymbol{x_k})\big)$. The only thing we need to show is $\nu(\boldsymbol{x_k}) = \mu(\boldsymbol{x_k})$ is still feasible for $\boldsymbol{x_k} \in \mathbb{K}$. Once this part is proved, then the assignment within $\mathbb{K}$ is unchanged. Moreover, since the assignment within $\mathbb{K}$ is unchanged, it will not affect the assignment within $\mathbb{R} \setminus \mathbb{K}$. Because the prices for locations in $\mathbb{R} \setminus \mathbb{K}$ are unchanged, the assignment within $\mathbb{R} \setminus \mathbb{K}$ remains same as before and so does its revenue contribution. Next, we show $\nu(\boldsymbol{x_k}) = \mu(\boldsymbol{x_k})$ is feasible under the new prices.

If a location $\boldsymbol{x_k}$ in $\mathbb{K}$ has to supply drivers to another location $\boldsymbol{x_j}$ at the new prices, we have

$$p(\boldsymbol{x_k}) = \max\left(p_b, p(\mathbf{0}) - c(\boldsymbol{x_k}, \mathbf{0})\right) < p(\boldsymbol{x_j}) - c(\boldsymbol{x_k}, \boldsymbol{x_j}) \tag{A.5}$$

$$= \max\left(p_b, p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0})\right) - c(\boldsymbol{x_k}, \boldsymbol{x_j}) \tag{A.6}$$

$$= p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0}) - c(\boldsymbol{x_k}, \boldsymbol{x_j}) \tag{A.7}$$

(A.5) and (A.6) are due to the incentive compatibility for relocating drivers from $\boldsymbol{x_k}$ to $\boldsymbol{x_j}$ and the definition of pricing formula (3.13). (A.7) is because if $p(\boldsymbol{x_j}) = p_b$, due to (A.5), we have

$$p(\boldsymbol{x_j}) = p_b > \max\left(p_b, p(\mathbf{0}) - c(\boldsymbol{x_k}, \mathbf{0})\right) + c(\boldsymbol{x_k}, \boldsymbol{x_j}) \geq p_b + 0 = p_b,$$

which is a contradiction. So we must have $p(\boldsymbol{x_j}) = p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0})$ and $p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0}) > p_b$. Next, by triangle inequality (3.1) and (A.5), (A.6), (A.7), we have

$$\begin{aligned} p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0}) - c(\boldsymbol{x_k}, \boldsymbol{x_j}) &\leq p(\mathbf{0}) - c(\boldsymbol{x_k}, \mathbf{0}) \\ &\leq \max\left(p_b, p(\mathbf{0}) - c(\boldsymbol{x_k}, \mathbf{0})\right) \\ &< p(\mathbf{0}) - c(\boldsymbol{x_j}, \mathbf{0}) - c(\boldsymbol{x_k}, \boldsymbol{x_j}), \end{aligned}$$

which is a contradiction. Thus, any location $\boldsymbol{x_k}$ in $\mathbb{K}$ does not have to supply drivers to another location and it is feasible for them to keep all their drivers. By using the exact same proof, we can also show that any location $\boldsymbol{x_k}$ in $\mathbb{K}$ does not have to receive drivers from another location. Thus, $\nu(\boldsymbol{x_k}) = \mu(\boldsymbol{x_k}), \forall \boldsymbol{x_k} \in \mathbb{K}$ is feasible under the new prices. But, its total revenue contribution is improved because of the monotonicity of $p\bar{F}_v(p)$. At the same time, the revenue contribution from $\mathbb{R} \setminus \mathbb{K}$ is unchanged. Thus, we improve our objective value by resetting its price function to pricing formula (3.13) and this is true for any feasible solution. Thus, the pricing formula (3.13) in Theorem 3.4 is optimal.

Secondly, given the pricing formula (3.13), we need to show the assignment constructed by Algorithm 3.1 in Theorem 3.4 is optimal. We prove this by two steps. Firstly, we show the assignment is feasible. Then, we show it is optimal by claiming it achieves an upper bound value for Program 3.4 under pricing formula (3.13). In Theorem 3.4, there are two types of assignments. The first type is to match drivers at location $\boldsymbol{x}, \forall \boldsymbol{x} \in \mathbb{V}$ to local riders at same location, and the second type is to relocate drivers at location $\boldsymbol{x}$ in surge region to the demand shock location $\mathbf{0}$. We only need to show these two types of assignments are incentive compatible for drivers.

In the first part, we have showed that any location $\boldsymbol{x_k}$ in $\mathbb{K}$ does not have to supply drivers to another location under pricing formula (3.13). By using the exact same proof, we can show that for $\forall \boldsymbol{x} \in \mathbb{V}$, the drivers at location $\boldsymbol{x}$ do not have to move to other locations, i.e., it is incentive compatible to keep the drivers at $\boldsymbol{x}$. Thus, the first type of assignment is feasible. By definition of surge region $\mathbb{S}$, we have

$$p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}), \forall \boldsymbol{x} \in \mathbb{S}.$$

Since $p(\boldsymbol{x})$ is the maximum effective price for drivers at location $\boldsymbol{x}$, $p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$ is also the maximum effective price for drivers at location $\boldsymbol{x}$, and it is incentive compatible to relocate drivers to $\mathbf{0}$. Thus, the second type of assignment is also feasible. The assignment in Theorem 3.4 is feasible. Secondly, we show it is optimal. In fact, the optimal value of Program 3.4 is capped by both the rider side and driver side. In particular, outside the surge region $\mathbb{S}$, the revenue contribution on these places is upper bounded by the potential demand,

$$\int_{\boldsymbol{x} \notin \mathbb{S}} p(\boldsymbol{x}) \min \Big(\mu(\boldsymbol{x}), \lambda(\boldsymbol{x}) \bar{F}_v\big(p(\boldsymbol{x})\big) \Big) d\boldsymbol{x} \leq \int_{\boldsymbol{x} \notin \mathbb{S}} p(\boldsymbol{x}) \lambda(\boldsymbol{x}) \bar{F}_v\big(p(\boldsymbol{x})\big) d\boldsymbol{x}.$$

Then, this upper is achieved in Theorem 3.4 by setting $\nu(\boldsymbol{x}) = \mu(\boldsymbol{x}), \forall \boldsymbol{x} \notin \mathbb{S}$. This is because the price satisfies

$$p(\boldsymbol{x}) > p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$$

outside the surge region by the definition of surge region. Thus, it is not incentive compatible to relocate them to the demand shock location $\mathbf{0}$. By Proposition 3.3, all the drivers stay at their original location. Because $\mu(\boldsymbol{x}) \geq \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \forall \boldsymbol{x} \notin \mathbb{S}$, the solution can achieve the revenue upper bound.

Then inside the surge region $\mathbb{S}$, we have the following discussions. If $\tilde{\Lambda} \geq \tilde{\mu}$, then all the drivers in $\mathbb{S}$ are not enough to serve the effective riders at the demand shock location, which has the highest price under pricing formula (3.13). Thus, the revenue inside surge region is upper bounded by the driver side $\tilde{\mu}p(\mathbf{0})$. This is achieved by setting

$$\nu_{\mathbf{0}} = \tilde{\mu},\ \nu(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \mathbb{S}.$$

Next, if $\hat{\mu} \leq \tilde{\Lambda} < \hat{\mu}$, then we have enough drivers to serve the effective riders at the demand shock location, but not enough for all effective riders inside the surge region $\mathbb{S}$. Then, the revenue inside surge region is upper bounded by the driver side, capturing the riders at the locations with highest prices. Then, the upper bound for the revenue is,

$$\begin{aligned} & p(\mathbf{0})\tilde{\Lambda} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})>\bar{p},\boldsymbol{x}\in\mathbb{S}\}} p(\boldsymbol{x})\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} \\ s.t.\ \ & \tilde{\Lambda} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})>\bar{p},\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} = \tilde{\mu} \end{aligned} \tag{A.8}$$

In fact, $\hat{p}$ defined in Algorithm 3.2 is exactly the solution of (A.8). By the definition of $\hat{p}$, we have,

$$\int_{\{\boldsymbol{x}|p(\boldsymbol{x})\leq\hat{p},\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} = \tilde{\Lambda} - \hat{\mu}$$

Then, use the definition above, we verify $\hat{p}$ solves (A.8).

$$\begin{aligned} & \tilde{\Lambda} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})>\hat{p},\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} \\ = & \hat{\mu} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})>\hat{p},\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})\leq\hat{p},\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} \\ = & \hat{\mu} + \int_{\{\boldsymbol{x}|\boldsymbol{x}\in\mathbb{S}\}} \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x} = \tilde{\mu}. \end{aligned}$$

In sum, the following value

$$p(\mathbf{0})\tilde{\Lambda} + \int_{\{\boldsymbol{x}|p(\boldsymbol{x})>\hat{p},\boldsymbol{x}\in\mathbb{S}\}} p(\boldsymbol{x})\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x}$$

is an upper bound for revenue inside surge region and it is achieved by setting

$$\nu_{\mathbf{0}} = \tilde{\Lambda};\ \nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big),\ \forall \boldsymbol{x} \in \{\boldsymbol{x}|p(\boldsymbol{x}) > \hat{p}\},\ \nu(\boldsymbol{x}) = 0,\ \forall \boldsymbol{x} \in \{\boldsymbol{x}|p(\boldsymbol{x}) \le \hat{p}\}$$

inside surge region $\mathbb{S}$. Finally, if $\tilde{\Lambda} < \hat{\mu}$, then the number of drivers is higher than the number of effective riders inside the surge region $\mathbb{S}$, that means we have enough drivers to serve all the effective riders. Then, the revenue contributed from surge region is upper bounded by the rider side

$$p(\mathbf{0})\tilde{\Lambda} + \int_{\boldsymbol{x}\in\mathbb{S}} p(\boldsymbol{x})\lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big)d\boldsymbol{x}.$$

This upper bound is achieved by setting

$$\nu_{\mathbf{0}} = \hat{\mu},\ \nu(\boldsymbol{x}) = \lambda(\boldsymbol{x})\bar{F}_v\big(p(\boldsymbol{x})\big), \forall \boldsymbol{x} \in \mathbb{S}.$$

To conclude, the assignment in Theorem 3.4 can achieve a value which is an upper bound for every feasible assignment given the pricing formula (3.13). Combine with its feasibility, the assignment in Theorem 3.4 is optimal given the pricing formula (3.13). Theorem 3.4 is proved. □

A.5 Proof of Proposition 3.5

Proof. To prove this proposition, it is equivalent to show that if $\bar{p}(\mathbf{0})$ and the corresponding $\tilde{\Lambda}\big(\bar{p}(\mathbf{0})\big)$ and $\hat{\mu}\big(\bar{p}(\mathbf{0})\big)$ determined in Theorem 3.4 satisfy

$$\tilde{\Lambda}\big(\bar{p}(\mathbf{0})\big) < \hat{\mu}\big(\bar{p}(\mathbf{0})\big),$$

then $\bar{p}(\mathbf{0})$ is not the optimal solution for Program 3.4. To show this claim, we fix $\bar{p}(\mathbf{0})$ and its surge region $\bar{\mathbb{S}}$ that leads to $\tilde{\Lambda}\big(\bar{p}(\mathbf{0})\big) < \hat{\mu}\big(\bar{p}(\mathbf{0})\big)$. It is sufficient to show that $\exists \epsilon > 0$, such that if we set $p(\mathbf{0}) = \bar{p}(\mathbf{0}) - \epsilon$, platform can collect more revenue from region $\bar{\mathbb{S}}$. Now, suppose the price at demand shock location drops from $\bar{p}(\mathbf{0})$ to $\bar{p}(\mathbf{0}) - \epsilon$, all the prices in $\bar{\mathbb{S}}$ drop. If we have

$$\tilde{\Lambda}\big(\bar{p}(\mathbf{0}) - \epsilon\big) < \hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon\big),$$

then based on the assignment algorithm in Theorem 3.4, all the effective riders in $\bar{\mathbb{S}}$ get served. Thus, the revenue collected in $\bar{\mathbb{S}}$ is

$$\begin{aligned}&\big(\bar{p}(\mathbf{0}) - \epsilon\big)\Lambda \bar{F}_v\big(\bar{p}(\mathbf{0}) - \epsilon\big) + \\ &\int_{\boldsymbol{x}\in\bar{\mathbb{S}}} \max\Big(\bar{p}(\mathbf{0}) - \epsilon - c(\boldsymbol{x}, \mathbf{0}), p_b\Big)\lambda(\boldsymbol{x})\bar{F}_v\Big(\max\big(\bar{p}(\mathbf{0}) - \epsilon - c(\boldsymbol{x}, \mathbf{0}), p_b\big)\Big)d\boldsymbol{x}.\end{aligned} \tag{A.9}$$

On the other hand, the revenue collected in $\bar{\mathbb{S}}$ when $\bar{p}(\mathbf{0})$ is the price at demand shock node is

$$\bar{p}(\mathbf{0})\Lambda \bar{F}_v\big(\bar{p}(\mathbf{0})\big) + \int_{\boldsymbol{x}\in\bar{\mathbb{S}}} \max\Big(\bar{p}(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}), p_b\Big)\lambda(\boldsymbol{x})\bar{F}_v\Big(\max\big(\bar{p}(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}), p_b\big)\Big)d\boldsymbol{x}. \tag{A.10}$$

Since $p\bar{F}_v(p)$ decreases on $p \geq p_b$, terms in (A.9) are point-wise larger than terms in (A.10). Thus, $\bar{p}(\mathbf{0}) - \epsilon$ yields a better solution, and $\bar{p}(\mathbf{0})$ is not optimal. Consequently, if there exists one $\epsilon > 0$ satisfying

$$\tilde{\Lambda}\big(\bar{p}(\mathbf{0}) - \epsilon\big) < \hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon\big),$$

then the proposition is proved.

On the other hand, if $\forall \epsilon > 0$ and we always have

$$\tilde{\Lambda}\big(\bar{p}(\mathbf{0}) - \epsilon\big) \geq \hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon\big),$$

we claim that we can still find a $\epsilon > 0$ such that $\bar{p}(\mathbf{0}) - \epsilon$ yields a better solution than $\bar{p}(\mathbf{0})$. Pick a positive sequence $\{\epsilon_1, ..., \epsilon_n, ...\}$ such that $\lim_{n\to\infty} \epsilon_n = 0$. Because $\epsilon_n > 0$, we have

$\tilde{\Lambda}\big(\bar{p}(\mathbf{0}) - \epsilon_n\big) \geq \hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon_n\big)$. Then, the revenue for platform is at least,

$$\begin{aligned}&\big(\bar{p}(\mathbf{0}) - \epsilon_n\big)\hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon_n\big) + \\ &\int_{\boldsymbol{x}\in\bar{\mathbb{S}}} \max\Big(\bar{p}(\mathbf{0}) - \epsilon_n - c(\boldsymbol{x},\mathbf{0}), p_b\Big)\lambda(\boldsymbol{x})\bar{F}_v\Big(\max\big(\bar{p}(\mathbf{0}) - \epsilon_n - c(\boldsymbol{x},\mathbf{0}), p_b\big)\Big)d\boldsymbol{x}.\end{aligned} \tag{A.11}$$

(A.11) is a lower bound because we give up the redundant riders at the demand shock node, which is the highest loss. Then, as $n \to \infty$, since $\hat{\mu}$ is an integral of a function without mass point, the first term in (A.11) is continuous in ϵ_n, and it goes to the limit $\bar{p}(\mathbf{0})\hat{\mu}\big(\bar{p}(\mathbf{0})\big)$. Since we have

$$\hat{\mu}\big(\bar{p}(\mathbf{0})\big) > \tilde{\Lambda}\big(\bar{p}(\mathbf{0})\big) = \Lambda\bar{F}_v\big(\bar{p}(\mathbf{0})\big),$$

the limit $\bar{p}(\mathbf{0})\hat{\mu}\big(\bar{p}(\mathbf{0})\big)$ is strictly higher than $\bar{p}(\mathbf{0})\Lambda\bar{F}_v\big(\bar{p}(\mathbf{0})\big)$. Then, there must exist an ϵ_{n_0} such that

$$\big(\bar{p}(\mathbf{0}) - \epsilon_{n_0}\big)\hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon_{n_0}\big) > \bar{p}(\mathbf{0})\Lambda\bar{F}_v\big(\bar{p}(\mathbf{0})\big). \tag{A.12}$$

The second term in (A.11) is greater than the second term in (A.10) for any $\epsilon_n > 0$, which is also true for ϵ_{n_0}. Thus, when the price at demand shock is $\bar{p}(\mathbf{0}) - \epsilon_{n_0}$, the lower bound of its maximum possible revenue in $\bar{\mathbb{S}}$ is

$$\begin{aligned}&\big(\bar{p}(\mathbf{0}) - \epsilon_{n_0}\big)\hat{\mu}\big(\bar{p}(\mathbf{0}) - \epsilon_{n_0}\big) + \\ &\int_{\boldsymbol{x}\in\bar{\mathbb{S}}} \max\Big(\bar{p}(\mathbf{0}) - \epsilon_{n_0} - c(\boldsymbol{x},\mathbf{0}), p_b\Big)\lambda(\boldsymbol{x})\bar{F}_v\Big(\max\big(\bar{p}(\mathbf{0}) - \epsilon_{n_0} - c(\boldsymbol{x},\mathbf{0}), p_b\big)\Big)d\boldsymbol{x}.\end{aligned}$$

This value is strictly greater than the revenue collected in $\bar{\mathbb{S}}$ when $\bar{p}(\mathbf{0})$ is the price at demand shock location denoted in (A.10), because of (A.12) and monotonicity of $p\bar{F}_v(p)$. Thus, we find $\bar{p}(\mathbf{0}) - \epsilon_{n_0}$ yields a better solution than $\bar{p}(\mathbf{0})$. In sum, $\bar{p}(\mathbf{0})$ is not the optimal solution for Program 3.4, which concludes this proof. □

A.6 Proof of Theorem 3.10

Proof. First of all, we state Lemma A.1 and assume it is true at the moment to prove Theorem 3.10. We give the proof of Lemma A.1 right after this proof.

Lemma A.1. *Under the problem setting specified in Theorem 3.10, for any* $\boldsymbol{x}, \boldsymbol{y}$ *satisfy*

$$p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}), c(\boldsymbol{y}, \mathbf{0}) + c(\boldsymbol{y}, \boldsymbol{x}) = c(\boldsymbol{x}, \mathbf{0}),$$

we have

$$p(\boldsymbol{y}) = p(\mathbf{0}) - c(\boldsymbol{y}, \mathbf{0}).$$

Lemma A.1 implies that under the problem setting of Theorem 3.10, if $\boldsymbol{x}$ supplies drivers to demand shock node $\mathbf{0}$, any location on the line segment between $\mathbf{0}$ and $\boldsymbol{x}$ is incentive compatible to supply drivers to demand shock node $\mathbf{0}$. In particular, on the ray network, if x satisfies $p(x) = p(0) - c(x, 0)$, any point y on the line segment between origin 0 and x has the property of

$$p(y) = p(0) - c(y, 0)$$

i.e., y is also incentive compatible to supply drivers to location 0. Denote l as the furthest point from 0 that satisfies $p(l) = p(0) - c(l, 0)$, i.e.,

$$l = \arg\max x \;\; s.t. \;\; p(x) = p(0) - c(x, 0).$$

Because of Lemma A.1, any point x on line $[0, l]$ has $p(x) = p(0) - c(x, 0)$. At the same time, by definition of l and Proposition 3.2, we have

$$p(x) > p(0) - c(x, 0), \forall x > l.$$

Thus, for location $x > l$, it can not supply drivers to demand shock location under incentive constraint. Then, by Proposition 3.3, for any $x > l$, the drivers only serve local effective riders at same location. Thus, the prices for locations $x > l$ should optimize local revenue under incentive compatibility constraint. By Proposition 3.2, for $x > l$, we have $p(x) \leq p(l)+c(l,x)$. Due to the uni-modularity of $p\bar{F}_v(p)$ specified in Assumption 3.5, and $p(l) \leq p_b$, the optimal price is

$$\min\Big(p_b, p(l) + c(l,x)\Big),\ \forall x \geq l.$$

As c is a norm induced distance metric, the pricing profile $p(x)$ for $x \geq l$ is feasible with the assignment because the price difference for $x_1, x_2 \geq l$ is at most $|c(x_1,l)-c(x_2,l)| = c(x_1,x_2)$, which is incentive compatible for retaining all drivers at their original places.

Next, we show l satisfies $p(l) \leq p_b$. If l violates the property, then for $x > l$, it needs to satisfy

$$p(x) > p(0) - c(x,0) = p(0) - c(l,0) - c(l,x) = p(l) - c(l,x),$$

by the definition of l. However, as $p(l) > p_b$, we have $x > l$ such that $c(l,x) \leq p(l) - p_b$. Thus, we have

$$p(x) > p(l) - c(l,x) = p(0) - c(x,0) \geq p_b.$$

Then, following the same analysis in the proof of Theorem 3.4 about the optimality of pricing formula (3.13), for all these x specified above, setting their pricing profile at $\max\Big(p(0) - c(x,0), p_b\Big)$ is a strictly better solution to the total revenue due to the monotonicity of $p\bar{F}_v(p)$ on $p \geq p_b$. Thus, in the optimal solution, we must have $p(l) \leq p_b$.

The optimality of the assignment stated in Theorem 3.10 can be proved using the same proof of the optimality of the assignment stated in Theorem 3.4. We have showed that all the assignments stated in Theorem 3.10 are incentive compatible when we prove the optimality of the pricing function above.

□

A.7 Proof of Lemma 3.12

Proof. If $\nu_i > \lambda_i \bar{F}_v(p_i)$ and $\sum_{j \neq i} \Gamma_{ji} > 0$ are both true for a location i, we have drivers relocated to node i and the amount of drivers is more than the amount of effective riders at location i. Then, we have unmatched drivers at location i. Since we also have drivers relocated into i, we send these relocated drivers coming from their source locations with the longest relocation time until $\nu_i = \lambda_i \bar{F}_v(p_i)$ or $\sum_{j \neq i} \Gamma_{ji} = 0$. Since the revenue contribution from location i is $p_i \lambda_i \bar{F}_v(p_i)$, it is unchanged when we send back these relocated drivers. Thus, the value of Problem (3.32) remains same. □

A.8 Proof of Lemma A.1

Proof. Because of Proposition 3.2, the price $p(\boldsymbol{y})$ satisfies

$$p(\boldsymbol{x}) + c(\boldsymbol{x}, \boldsymbol{y}) \geq p(\boldsymbol{y}) \geq p(\mathbf{0}) - c(\boldsymbol{y}, \mathbf{0}). \tag{A.13}$$

Then, we have

$$\begin{aligned} p(\boldsymbol{x}) + c(\boldsymbol{x}, \boldsymbol{y}) &= p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0}) + c(\boldsymbol{x}, \boldsymbol{y}) && \text{(A.14)} \\ &= p(\mathbf{0}) - c(\boldsymbol{y}, \mathbf{0}) - c(\boldsymbol{y}, \boldsymbol{x}) + c(\boldsymbol{x}, \boldsymbol{y}) && \text{(A.15)} \\ &= p(\mathbf{0}) - c(\boldsymbol{y}, \mathbf{0}) && \text{(A.16)} \end{aligned}$$

(A.14) is because of $p(\boldsymbol{x}) = p(\mathbf{0}) - c(\boldsymbol{x}, \mathbf{0})$. (A.15) is due to $c(\boldsymbol{y}, \mathbf{0}) + c(\boldsymbol{y}, \boldsymbol{x}) = c(\boldsymbol{x}, \mathbf{0})$. (A.16) results from the symmetry of a distance metric. Combine (A.16) with (A.13), we have $p(\boldsymbol{y}) = p(\mathbf{0}) - c(\boldsymbol{y}, \mathbf{0})$. □

Appendix B

Numerical Examples for Chapter 3

B.1 Counter-example of Theorem 3.10 for General c

Proof. In this part, we construct a counter example which shows that Theorem 3.10, in particular, the pricing formula (3.30) is not optimal. Consider a discrete network with three nodes $\{0, 1, 2\}$ on a line. The dis-utility function c has values

$$c(0,1) = 20,\ c(0,2) = 30,\ c(1,2) = 20.$$

Thus, c can not be a norm induced distance metric since we set $0, 1, 2$ on a line, but $c(0,1) + c(1,2) \neq c(0,2)$. The arrival rates of drivers at each node are

$$\mu_0 = 40,\ \mu_1 = 40,\ \mu_2 = 40,$$

and the arrival rates of potential riders at each nodes are

$$\lambda_0 = 100,\ \lambda_1 = 10,\ \lambda_2 = 0.$$

Then, the demand shock happens at node 0. The willingness to pay function F_v is defined as $F_v(p) = \frac{p}{p^{\max}}$ for $p \in [p^{\min}, p^{\max}]$ with $p^{\min} = 0, p^{\max} = 100$. If we use the pricing formula

(3.30) in Theorem 3.10, there are three possible values for l. If $l = 0$, the surge region only contains the node 0 itself. The optimal price for node 0 is $p_0 = 60$ to maximize the revenue in surge region under the constraint for pricing formula (3.30) in Theorem 3.10. Then, the optimal prices for node 1 and 2 are $p_1 = p_2 = 50$. The total revenue contribution is

$$p_0\mu_0 + p_1\lambda_1\bar{F}_v(p_1) = 2400 + 250 = 2650.$$

Next, if $l = 1$, we do not need to consider this case because it is dominated by $l = 2$. Since $\lambda_2 = 0$, there is no opportunity cost to relocate drivers at node 2, so we can always benefit from setting a low price at node 2 and relocating the drivers there to other places. Thus, if $l = 2$, suppose the price at node 0 is p, then the price at node 1 and node 2 are $p - 20$ and $p - 30$. Because the actual number of rides is capped by the demand, the revenue is upper bounded by

$$p\lambda_0\bar{F}_v(p) + (p - 20)\lambda_1\bar{F}_v(p - 20), \tag{B.1}$$

if all the effective riders are served. By plugging the parameters, (B.1) becomes

$$-1.1p^2 + 114p - 240,$$

with optimal solution $p^* = 51.8$ and optimal value 2713.6. Thus, the optimal value given by using pricing formula 3.30 is upper bounded by 2713.6.

However, if we give up using pricing formula (3.30), we can achieve a better value. Considering the surge region as $\{0, 2\}$, i.e., surge region skips node 1, we set the prices as

$$p_0 = 50, p_1 = 40, p_2 = 20.$$

Thus, it is incentive compatible to relocate all drivers at node 2 to node 0 and keep the drivers at nodes 0 and 1 at their original locations. The revenue of this solution is

$$50 \cdot \min(50, 80) + 40 \cdot 6 = 2740,$$

beating the previous value of 2713.6. Since this price profile violates pricing formula (3.30), Theorem 3.10 is not valid any more. □