Non-compact geometric flows: long time existence and type II singularities

Beomjun Choi

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ABSTRACT

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In this work, we study how solutions of certain non-compact geometric flows of fast-diffusion type interact with their asymptotic geometries at infinity. In the first part, we show the long time existence theorem to the inverse mean curvature flow for complete convex non-compact initial hypersurfaces. The existence and behavior of a solution is tied with the evolution of its tangent cone at infinity. In particular, the maximal time of existence can be written in terms of the area ratio between the initial tangent cone at infinity and the flat hyperplane. In the second part, we study the formation of type II singularity for non-compact Yamabe flow. Assuming the initial metric is conformally flat and asymptotic to a cylinder, we show the higher order asymptotics of the metric determines the curvature blow-up rates at the tip in its first singular time. We also show the singularities of such solutions are modeled on rotationally symmetric steady gradient solitons.

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To my parents

Introduction

A geometric flow is an evolution of a metric of a Riemannian manifold or a submanifold in an ambient manifold by means of a curvature. While the theory of compact flow has been well developed for various flows including the two flows we are interested, there are still numerous open questions for non-compact cases. A huge difference which makes non-compact problems both difficult and interesting is the fact that the information from the infinite region can permeate inside and affect the behavior of solutions. The two geometric flows we study in this thesis is the inverse mean curvature flow(IMCF) and the Yamabe flow. They are examples of fast-diffusion type equations and this implies we can expect a strong communication between distanced regions. This thesis is mostly based on some of the author's previous joint work [CD2] for the inverse mean curvature flow and [CDK] for the Yamabe flow, which reveal these kinds of phenomena. The paper [CH] is a joint work of the author and P.-K. Hung which is a close subsequent work of [CD2] and the paper [CD1] is a preliminary work of [CD2]. Thus those results will also be presented if they are needed.

Chapter 1, which is based on [CD2], investigates the complete non-compact inverse mean curvature flow under the convexity assumption. We show the existence of solution up to the maximal time, which could be either finite or infinite defending on the asymptotic geometry of initial hypersurface. We will show that the asymptotic geometry of the flow evolves independently by itself under the IMCF in the same time scale: let Σ_t be a convex non-compact solution. Then the tangent cone at infinity of the solution at each time, say $\mathcal{C}_t$, can be considered and we show $\mathcal{C}_t$ is a solution of IMCF. It is equivalent to say that the link of the cone $\Gamma_t := \mathcal{C}_t \cap \mathbb{S}^n$ is a solution of the IMCF in the sphere. The maximal time of existence easily follows from the maximal time of existence of Γ_t in $\mathbb{S}^n$, which can be written in terms of the area of Γ_0. Under the IMCF, the Γ_t converges to an equator of the sphere as time approaches the maximal time. In other words, $\mathcal{C}_t$ and Σ_t open up and become flat at this time. In the proof of this theorem, our main a priori estimate will be attained by taking into account of these expectations into an auxiliary test function to which the maximum principle is applied.

Chapter 2, which is based on [CDK], investigates the formation of type II singularity for complete non-compact conformally flat Yamabe flow. The Yamabe flow refers to the conformal deformation of a metric by $-Rg$ where R is the scalar curvature. i.e. $\partial_t g_{ij} = -Rg_{ij}$. In this equation, the conformal factor of the metric follows a fast-diffusion equation and it was shown that the compact flow only creates a type I singularity in its first singular time and it converges to a metric of constant scalar curvature at this time. Here the type I singularity at time $t = T$ refers the case when the curvature blow-up rate is less than that of the sphere, i.e.

$$\limsup_{t\to T^-} (T-t) \sup_{M_t} |\mathrm{Rm}| < +\infty$$

and we say the singularity is type II if if it is not type I. Under non-compact asymptotically cylindrical metric assumption, in our previous work [CD1] we showed the flow may develop a type II singularity if it satisfies a higher order asymptotic condition to the cylinder. In [CDK], we show the higher order asymptotics determine the specific type II blow-up rate of the curvature. Roughly speaking, for some small $T > 0$ if a globally conformally flat initial metric $g_0 = u_0^{\frac{4}{n+2}}\delta_{ij}$ has positive Ricci and

$$u_0^{1-m}(x) = \frac{(n-1)(n-2)}{|x|^2}\left(T - \left(\frac{\ln|x|}{A}\right)^{-\frac{1}{\gamma}} + O\big((\ln|x|)^{-\frac{1}{\gamma}-1}\big)\right), \ \text{ as } |x| \to +\infty,$$

then the solution of Yamabe flow (2.1) will develop a type II singularity at time $t = T$ with specified blow up rate given by $\limsup_{t\to T^-} (T-t)^{1+\gamma}\sup_{M_t} |Rm| = \frac{2\gamma A}{\sqrt{n(n-1)}}$. Moreover, it converges to a conformally flat rotationally symmetric steady soliton if the solution is rescaled around its tip region.

Chapter 1

Inverse Mean Curvature Flow

1.1 Introduction

A one-parameter family of imersions $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a smooth complete solution to the *inverse mean curvature flow* (IMCF) in $\mathbb{R}^{n+1}$ if each $M_t := F(\cdot,t)(M^n)$ is a smooth strictly mean convex complete hypersurface satisfying

$$\frac{\partial}{\partial t}F(p,t) = H^{-1}(p,t)\,\nu(p,t) \tag{1.1}$$

where $H(p,t) > 0$ and $\nu(p,t)$ denote the mean curvature and exterior unit normal of M_t.

This flow for compact case is well known under certain assumptions. Gerhardt [Ge1] and Urbas [Ur] showed that for smooth star-shaped compact initial hypersurface of strictly positive mean curvature, there is a unique smooth solution for all times $t > 0$. Moreover, the solution approaches to a homothetically expanding sphere as $t \to \infty$. For non-starshaped initial data it is well known that singularities may develop (See examples in [HI1] [Sm].) This happens when the mean curvature vanishes in some regions which makes the classical flow undefined.

However, in [HI1; HI2] Huisken and Ilmanen developed a level set approach to *weak variational solutions* of the flow which allows the solutions to *jump outwards* in possible regions where $H = 0$. Using the weak formulation, they gave the first proof of the *Riemannian Penrose inequality* in General Relativity. One key observation in [HI2] was the fact the Hawking mass of a 2d-surface in a 3-manifold of nonnegative scalar curvature is monotone under the weak flow, which was first discovered for classical solutions by Geroch [Ger]. Note that the Riemannian Penrose inequality was shown in more general settings by Bray [Bray] and Bray-Lee [BL] by different methods. Using similar techniques, the IMCF and other expanding flows have also been used to show geometric inequalities in various settings. For instance, see [GL; BHW; LN; MS; DF; GWW; BN].

In [HI4] Huisken and Ilmanen studied the higher regularity of solutions to the IMCF, for compact star-shaped weakly mean convex initial data of class C^1. Using star-shapedness and the ultra-fast diffusion character of the flow, they derive a bound from above on H^{-1} for $t > 0$ *which is independent of the initial curvature assumption*. This follows by a *Stampacchia iteration* argument and utilizes the *Michael-Simon Sobolev inequality*. The C^∞ regularity of solutions for $t > 0$ easily follows from the bound on H^{-1}. The estimate in [HI4] is local *in time*, but necessarily *global in space* as it depends on the area of the initial hypersurface M_0 and uses global integration on M_t. As a consequence of the techniques in [HI4] cannot be applied directly to the non-compact setting. Let us also note that the works [LW] and [Z] provide similar estimates on H^{-1} for compact star-shaped solutions of the IMCF in some negatively curved ambient spaces.

The main result of [CD2] addresses *the long time existence* of *non-compact* smooth convex solutions M_t to the IMCF embedded in *Euclidean space* $\mathbb{R}^{n+1}$. The important works by K. Ecker and G. Huisken [EH1; EH2] address the evolution of entire graphs by *mean curvature flow* and establish a surprising result: existence for all $0 < t < +\infty$ with the only assumption that the initial data M_0 is a *locally Lipschitz* entire graph and *no assumption of the growth at infinity of* M_0. This result is based on priori estimates which are *localized in space and time*. By local in time, it means the main local bound on the second fundamental form $|A|^2$ of M_t is achieved without any bound assumption on $|A|^2$ on M_0. An open question between experts in the field has been whether the techniques of Ecker and Huisken in [EH1; EH2] can be extended to the fully-nonlinear setting, in particular on entire convex graphs evolving by the powers of the Gaussian curvature flow and the *inverse mean curvature flow*. Note that Gauss curvature flow is an example of *degenerate diffusion* while the inverse mean curvature flow is the opposite, an example of *ultra-fast diffusion*. Also note the problem of the long time existence for the powers Gauss curvature flow has been established in [CDKL].

In [CDKL] they showed that similar estimates as in [EH1; EH2] which are *localized in space* can be obtained for this flow, however the methods are more involved due to the *degenerate* and *fully-nonlinear character* of the Monge-Amperé type of equation involved. However, such localized results are not expected to hold for the *inverse mean curvature flow* where the *ultra-fast diffusion* tends to cause instant propagation from spatial infinity. In fact, one sees certain similarities between the latter two flows and the well known quasilinear models of diffusion on $\mathbb{R}^n$

$$u_t = \operatorname{div}\big(u^{m-1}\,\nabla u\big). \tag{1.2}$$

Exponents $m > 1$ correspond to degenerate diffusion while exponents $m < 0$ to ultra-fast diffusion. We will see in the sequel that under the IMCF the mean curvature H satisfies an equation which is similar to (1.2) with $m = -1$. Our goal in this work is to study this phenomenon and establish the *long time existence* of complete non-compact convex hypersurfaces, the analogue of the results in [EH1; EH2] and [CDKL].

We will next state our main result in this work. The following observation motivates the formulation of our theorem.

Example 1.1 (Conical solutions of IMCF). For a solution of the IMCF Γ_t in $\mathbb{S}^n$, the *family of cones generated by* Γ_t

$$\mathcal{C}\Gamma_t := \{rx \in \mathbb{R}^{n+1} \,:\, r \geq 0,\, x \in \Gamma_t\}$$

is a solution of the IMCF in $\mathbb{R}^{n+1}$ which is smooth except from the origin. When $\Gamma_0^{n-1} \subset \mathbb{S}^n$ is a smooth strictly convex hypersurface., the results of Gerhardt [Ge3] and Makowski-Scheuer [MS] show there exists a unique solution $\Gamma_t \subset S^n$ of the IMCF in S^n with initial data Γ_0^{n-1}, which exists for time $t \in [0, T)$ with $T < \infty$ and converges to an equator, as $t \to T$. Moreover one can explicitly compute using the exponential growth of area with respect to time that $T = \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|$.

From Example 1.1 and the ultra-fast diffusive character of the equation, it is reasonable to guess that for a general convex non-compact solution with initial data M_0, its existence time is governed by the asymptotics at infinity. For a non-compact convex set $\hat{M}_0$ and the associated hypersurface $M_0 = \partial\hat{M}_0$, we recall the definition of the *blow-down*, so called *the tangent cone at infinity.*

Definition 1.1 (Tangent cone at infinity)**.** Let $\hat{M}_0 \subset \mathbb{R}^{n+1}$ be a non-compact closed convex set. For a point $p \in \hat{M}_0$, we denote the tangent cone of $\hat{M}_0$ at infinity by

$$\hat{C}_0 := \cap_{\lambda>0}\lambda(\hat{M}_0 - p).$$

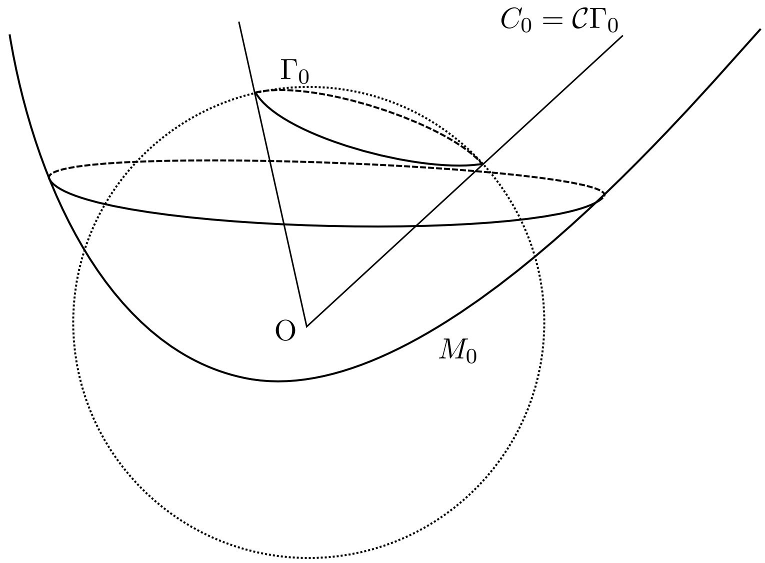

Figure 1.1: Tangent cone at infinity

We also define $C_0 := \partial\hat{C}_0$, $\hat{\Gamma}_0 := \hat{C}_0 \cap \mathbb{S}^n$, $\Gamma_0 := C_0 \cap \mathbb{S}^n$. The definition is independent of $p \in \hat{M}_0$. We say $\hat{C}_0$, C_0 the tangent cone of $\hat{M}_0$ and $M_0 = \partial\hat{M}_0$ at infinity, respectively. We say $\hat{\Gamma}_0$ and Γ_0 the link of $\hat{C}_0$ and C_0, but we will also often call them as the tangent cone at infinity.

Our main result establishes the long time existence and characterize its maximum time of existence T of the solution in terms of the size of the tangent cone at infinity Γ_0.

Theorem 1.2. *For $n \geq 2$, let $M_0^n = \partial\hat{M}_0$ be a convex non-compact embedded $C^{1,1}_{loc}$ hypersurface in $\mathbb{R}^{n+1}$. Then, there is a smooth convex solution of the IMCF, say $\{M_t\}_{t\in(0,T)}$, which converges to M_0 locally uniformly as $t \to 0$. The time of existence is given in terms of the link of tangent cone of $\hat{M}_0$ at infinity, say $\hat{\Gamma}_0 \subset \mathbb{S}^n$, by*

$$T = \ln|\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_0) \in [0,\infty]. \tag{1.3}$$

Here, $|\cdot| := \mathcal{H}^{n-1}(\cdot)$ and $P(\hat{\Gamma}) :=$ the perimeter of a convex set $\hat{\Gamma}$ in $\mathbb{S}^n$. The solution is strictly convex when $\hat{\Gamma}_0 \subset \mathbb{S}^{n-1}$ is compactly included in an open hemisphere.

Remark 1.2. Under our assumption of M_0, $\hat{\Gamma}_0$ can be an arbitrary convex set in $\mathbb{S}^n$. For a convex set $\hat{\Gamma}_0 \subset \mathbb{S}^n$ and $\Gamma_0 = \partial\hat{\Gamma}_0$, note that

$$P(\hat{\Gamma}_0) = \begin{cases} |\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has non-empty interior in } \mathbb{S}^n \\ 2|\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has empty interior in } \mathbb{S}^n. \end{cases}$$

Moreover if M_t evolves by IMCF then its tangent cone at infinity Γ_t, evolves by IMCF on $\mathbb{S}^n$ in some generalized sense and becomes flat as $t \to T$. See Remark 1.8 for this.
Finally, formula (1.3) says $T = 0$ when $P(\hat{\Gamma}_0) = |\mathbb{S}^{n-1}|$. In [CH], the author and P.-K. Hung showed that for a convex set $\hat{\Gamma}_0 \subset \mathbb{S}^n$ if $P(\hat{\Gamma}_0) = |\mathbb{S}^{n-1}|$ then $\hat{\Gamma}_0$ is either a hemisphere or a wedge

$$\hat{W}_{\theta_0} = \mathbb{S}^n \cap \left(\{(r\sin\theta, r\cos\theta) \,:\, \theta \in [0,\theta_0], \text{ and } r > 0\} \times \mathbb{R}^{n-1} \right)$$

for some $\theta_0 \in [0,\pi)$, up to an isometry of $\mathbb{S}^n$. According to the formula, $T = \infty$ when $P(\hat{\Gamma}_0) = 0$, which happens when the cone degenerates and it is lower dimensional.

Remark 1.3. Let us emphasize that Theorem 1.2 allow $H = 0$ on a possibly non-compact region of M_0 and in that case $H > 0$ instantly for $t > 0$ provided $T = T(M_0) > 0$. This is possible due to our main apriori estimate Theorem 1.4. Note that the similar phenomenon was observed for solutions to the Cauchy problem on $\mathbb{R}^n$ of the ultra-fast diffusion equation (1.2) with $m < 0$ in [DP1; DP2].

Next, we show $T = T(M_0)$ in Theorem 1.2 is the maximal time of existence. The following theorem holds not only for the solutions of our constructions, but applies to arbitrary solutions.

Theorem 1.3. *Let $M_0 = \partial\hat{M}_0$ satisfy the same assumptions as in Theorem 1.2 and $T = T(M_0) \in [0,\infty]$ be given by the formula (1.3). If $T < \infty$, then no smooth solution M_t, which locally uniformly converges to M_0 as $t \to 0+$, can be defined beyond $t > T(M_0)$. In particular, this implies non-existence of a smooth solution when $T(M_0) = 0$.*

Non-compact solutions of the IMCF in $\mathbb{R}^{n+1}$ were first considered by P. Daskalopoulos and G. Huisken in [DH], where they established the existence and uniqueness of a smooth solution to the IMCF, under the assumption that the initial hypersurface M_0 is an entire C^2 graph, $x_{n+1} = u_0(x')$ with $H > 0$, in the following two cases:

(i) M_0 has *super linear growth* at infinity and it is *strictly star-shaped*, that is $H\langle F - x_0, \nu\rangle \geq \delta > 0$ holds, for some $x_0 \in \mathbb{R}^{n+1}$;

(ii) M_0 a *convex graph* satisfying $0 < c_0 \leq H\,\langle F - x_0, e_{n+1}\rangle \leq C_0 < +\infty$, for some $x_0 \in \mathbb{R}^{n+1}$ and lies between *two round cones of the same aperture*, that is

$$\alpha_0|x'| \leq u_0(x') \leq \alpha_0|x'| + k, \qquad \alpha_0 > 0,\ k > 0. \tag{1.4}$$

In the first case, a unique smooth solution exists up to time $T = +\infty$, while in the second case a unique smooth convex solution M_t exists for $t \in [0, T]$ where $T = T(\alpha_0) > 0$ is the exact time when an evolving cone solution of the IMCF $\{x_{n+1} = \alpha(t)|x'|\}$, with $\alpha(0) = \alpha_0$ becomes flat (i.e. $\alpha(t) \to 0$). In the latter case, the solution M_t lies between two evolving round cones and becomes flat as $t \to T$. To derive a local lower bound of H up to $t < T$, a parabolic Moser's iteration argument was used in [DH] along with a variant of Hardy's inequality, which plays a similar role as the Micheal-Simon Sobolev inequality in [HI4].

Theorem 1.2 and the results in [DH] show that convex surfaces with *linear growth at infinity* have *critical behavior* in the sense that in this case the *maximal time of existence is finite* and it depends on the *behavior at infinity* of the initial data. However, while the techniques in [DH] only treat this critical linear case under the condition (1.4), Theorem 1.2 allows any behavior at infinity. Moreover, the techniques in [DH] require to assume that H is globally controlled from below a initial time, namely that $H\langle F - x_0, \nu\rangle \geq \delta > 0$ in the case of *super-linear growth* and $H\,\langle F - x_0, e_{n+1}\rangle \geq c > 0$ in the case of *linear growth.*

In this work we depart from the techniques in [DH] and [HI4] and establish an priori L^∞ bound on H^{-1} which is *local in time.* In this attempt, we develop a new method based on the maximum principle rather than the integration methods used in [DH] and [HI4]. Our key estimate in the long time existence is the following bound on H^{-1} which roughly says that one has a global bound on $(H|F|)^{-1}$ as long as a nontrivial convex cone is supporting our surface from outside.

Theorem 1.4. *Let $F : M^n \times [0, T] \to \mathbb{R}^{n+1}$, $n \geq 2$, $T > 0$, be a smooth convex closed solution of the IMCF and suppose there is $\theta_1 \in (0, \pi/2)$ for which*

$$\langle F, e_{n+1}\rangle \geq \sin\theta_1\,|F| \quad \text{on } M^n \times [0, T]. \tag{1.5}$$

Then

$$\frac{1}{H|F|} \leq C\left(1 + \frac{1}{t^{1/2}}\right) \quad \text{on} \quad M^n \times [0, T] \tag{1.6}$$

for a constant $C = C(\theta_1) > 0$.

Let us note that the assumption that M_t is a closed hypersurface *will only be used to apply maximum principle* and will not affect the application of the estimate in proving of our main non-compact result, Theorem 1.2, as we will approximate non-compact solutions by closed ones. Also, let us emphasize that our bound is independent on an initial upper bound on $(H|F|)^{-1}$. This will allow non-compact initial data to have flat regions where $H = 0$. In addition to the non-compact results stated above, our new methods lead to an equivalent estimate of the result by Ilmanen and Huisken, Theorem 1.1 in [HI4], for compact, star-shaped (not necessarily convex) solutions.

Theorem 1.5 (Theorem 1.1 in [HI4])**.** *Let $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ be a smooth closed star-shaped solution of* (1.1) *such that $M_0 := F_0(M^n)$ satisfies*

$$0 < R_1 \leq \langle F, \nu \rangle \leq R_2. \tag{1.7}$$

Then, there is a constant $C_n > 0$ depending only on n such that

$$\frac{1}{H} \leq C_n \left(\frac{R_2}{R_1}\right) \left(1 + \frac{1}{t^{1/2}}\right) R_2 \, e^{\frac{t}{n}} \tag{1.8}$$

holds everywhere on $M^n \times [0,T]$.

In fact, one expects that similar estimates as in Theorem 1.5 can be possibly derived for the IMCF in other ambient spaces, including some positively curved spaces or asymptotically flat spaces, using this new method and this generalize the results of [HI4; LW; Z]. See in [HI3] for a consequence of such an estimate when this is shown in asymptotically flat ambient spaces.

Remark 1.4. Recently, the auther and P.-K. Hung in [CH] addressed the IMCF of arbitrary convex hypersurface which allows singularities on M_0. Using the main estimate Theorem 1.4 as a key ingredient, [CH] shows the limiting tangent cone after blowing-up at a singularity also evolves by the IMCF. As a corollary, we could generalize Theorem 1.2 and obtain the following necessary and sufficient condition for existence of a smooth solution: *for an arbitrary non-compact convex M_0 with $T(M_0) > 0$, there is a smooth solution if and only if M_0 has density one everywhere. i.e.* $\Theta_0(p) = \lim_{r\to 0} \frac{|B_r(p)\cap M_0|}{\omega_n r^n} = 1$ *for all $p \in M_0$.* See [CH] for more details.

A brief outline is as follows: In Section 1.2, we introduce basic notation, evolution equations of basic geometric quantities, and prove some identities which will be useful in the upcoming sections. Section 1.3.1 is devoted to the proof of our main a priori estimate Theorem 1.4. Only assuming that the solution stays above a round cone, the estimate shows a uniform bound of $(H|F|)^{-1}$, for $t > 0$, which is independent of the initial bound. We also an alternative proof of a priori H^{-1} estimate shown in [HI4] using our maximum principle argument. This is to show how star-shapedness condition can also be incorporated in our method, but will not be used in the rest of the chapter. In Section 1.5, we prove the long time existence theorem of non-compact convex solution via an approximation argument that uses a priori estimates in Section 1.3.1. In Section 1.4, we prove the convexity of solution is preserved and show the solution become strictly convex immediately for $t > 0$ unless the lowest principle curvature λ_1 is zero everywhere initially. This will be shown for the solutions of the IMCF in a space of constant sectional curvature as this adds no difficulty in the proof but could be useful in other application.

1.2 Preliminaries and Notation

Let $\nabla := \nabla^{g(t)}$ and $\Delta := \Delta_{g(t)}$ denote the connection and Laplacian on M^n with respect to the induced metric $g_{ij}(t) = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$. Recall that on a local system of coordinates $\{x^i\}$ on M^n,

$$\frac{\partial^2 F}{\partial x^i \partial x^j} = -h_{ij}\nu + \Gamma_{ij}^k \frac{\partial F}{\partial x^k} \qquad \text{and} \qquad \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \nu}{\partial x^i} \right\rangle = h_{ij} \tag{1.9}$$

where ν denotes the exterior unit normal. We also define the operator

$$\square := \left(\partial_t - \frac{1}{H^2}\Delta\right)$$

and use it frequently as this is the linearized operator of the IMCF.

Note that the IMCF or generally curvature flows of homogeneous degree -1, have the following scaling property which can be directly checked and will be frequently used:

Lemma 1.6 (Scaling of IMCF)**.** *If $M_t^n \subset \mathbb{R}^{n+1}$ is a solution of the IMCF, then $\tilde{M}_t^n = \lambda\, M_t^n$ is again a solution for $\lambda > 0$.*

Lemma 1.7 (Huisken, Ilmanen [HI4])**.** *Any smooth solution of the IMCF* (1.1) *in $\mathbb{R}^{n+1}$ satisfies*

(1) $\partial_t g_{ij} = \frac{2}{H} h_{ij}$

(2) $\partial_t d\mu = d\mu$, *where $d\mu$ is the volume form induced from g_{ij}*

(3) $\partial_t \nu = -\nabla H^{-1} = \frac{1}{H^2}\nabla H$

(4) $\left(\partial_t - \frac{1}{H^2}\Delta\right)h_{ij} = -\frac{2}{H^3}\nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij}$

(5) $\partial_t H = \nabla_i\left(\frac{1}{H^2}\nabla_i H\right) - \frac{|A|^2}{H} = \frac{1}{H^2}\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}$

(6) $\left(\partial_t - \frac{1}{H^2}\Delta\right)H^{-1} = \frac{|A|^2}{H^2}\, H^{-1}$

(7) $\left(\partial_t - \frac{1}{H^2}\Delta\right)\langle F - x_0, \nu\rangle = \frac{|A|^2}{H^2}\, \langle F - x_0, \nu\rangle$.

Remark 1.5. If the ambient space is not $\mathbb{R}^{n+1}$, then the evolution equations of g_{ij}, $d\mu$, and ν remain the same as in $\mathbb{R}^{n+1}$, but the evolution of curvature h_{ij} is different and complicated. On a space form of sectional curvature K, the formula hugely simplifies becoming

$$\partial_t h_{ij} = \frac{1}{H^2}\Delta h_{ij} + \frac{|A|^2}{H^2} h_{ij} - \frac{2}{H^3}\nabla_i H \nabla_j H - \frac{nKh_{ij}}{H^2} \tag{1.10}$$

(See Chapter 2 in [Ge2].) In this paper we will mostly focus on the flow in Euclidean space and we will only use (1.10) in Appendix 1.4.

Using Lemma 1.7 one can easily deduce the following formulas.

Lemma 1.8. *For a fixed vector ω in $\mathbb{R}^{n+1}$, the smooth solutions of the IMCF* (1.1) *in $\mathbb{R}^{n+1}$ satisfy*

(1) $\left(\partial_t - \frac{1}{H^2}\Delta\right)|F - x_0|^2 = -\frac{2n}{H^2} + \frac{4}{H}\langle F - x_0, \nu\rangle$

(2) $\left(\partial_t - \frac{1}{H^2}\Delta\right)\langle \omega, \nu\rangle = \frac{|A|^2}{H^2}\langle \omega, \nu\rangle$

(3) $\left(\partial_t - \frac{1}{H^2}\Delta\right)\langle \omega, F - x_0\rangle = \frac{2}{H}\langle \omega, \nu\rangle$.

Proof. By (1.9) we have

$$\Delta F = g^{ij}(\partial^2_{ij}F - \Gamma^k_{ij}F_k) = g^{ij}(-h_{ij}\nu + \Gamma^k_{ij}F_k - \Gamma^k_{ij}F_k) = -H\,\nu.$$

which combined with $\partial_t F = H^{-1}\nu$ implies (3). Next,

$$\Delta|F - x_0|^2 = 2\langle \Delta F, F - x_0\rangle + 2\langle \nabla F, \nabla F\rangle = 2H\langle \nu, F - x_0\rangle + 2n$$

implies (1). Finally,

$$\begin{aligned}
\Delta\nu &= g^{ij}(\partial^2_{ij}\nu - \Gamma^k_{ij}\partial_k\nu) = g^{ij}(\partial_j(h^k_i F_k) - \Gamma^k_{ij}h^l_k F_l) \\
&= g^{ij}((\partial_j h^k_i)F_k - h^k_i h_{jk}\nu + \Gamma^l_{jk}h^k_i F_l - \Gamma^k_{ij}h^l_k F_l) \\
&= -|A|^2\nu + g^{ij}\nabla_j h^k_i F_k = -|A|^2\nu + \nabla H
\end{aligned}$$

where we used the Codazzi identity in the last equation. This implies (2). $\square$

The following simple lemma, which commonly appears in Pogorelov type computations, will be useful in the sequel when we compute the evolution of products.

Lemma 1.9. *For any C^2 functions $f_i(p,t)$, $i = 1, \dots, m$, denote*

$$w := f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}.$$

Then on the region where $w \neq 0$, we have

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) \ln|w| = \frac{(\partial_t - H^{-2}\Delta)w}{w} + \frac{1}{H^2}\frac{|\nabla w|^2}{w^2} = \sum_{i=1}^m \alpha_i \left(\frac{(\partial_t - H^{-2}\Delta)f_i}{f_i} + \frac{1}{H^2}\frac{|\nabla f_i|^2}{f_i^2} \right). \quad (1.11)$$

Proof. The lemma simply follows from

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) \ln|f| = \frac{(\partial_t - H^{-2}\Delta)f}{f} + \frac{1}{H^2}\frac{|\nabla f|^2}{f^2}. \quad (1.12)$$

□

Next two lemmas are straightforward computations and we leave their proofs for readers.

Lemma 1.10. *For any two C^2 functions f, g defined on $M^n \times (0,T)$ and any C^2 function $\psi : \mathbb{R} \to \mathbb{R}$,*

$$\Box(fg) = (\Box f)g + f(\Box g) - \frac{2}{H^2}\langle \nabla f, \nabla g \rangle$$

and

$$\Box\psi(f) = \psi'(f)\Box\psi(f) - \frac{\psi''(f)}{H^2}|\nabla f|^2$$

where $\Box := (\partial_t - H^{-2}\Delta)$.

Lemma 1.11. *If a C^2 function f is defined on a solution M_t of the IMCF and satisfies*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta \right) f = \frac{|A|^2}{H^2} f$$

then for any fixed $\beta \in \mathbb{R}$ we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta \right) f^\beta = \beta \frac{|A|^2}{H^2} f^\beta - \frac{\beta(\beta-1)}{\beta^2} \frac{|\nabla f^\beta|^2}{H^2 f^\beta}.$$

For instance, H^{-1}, $\langle \omega, \nu \rangle$ and $\langle F - x_0, \nu \rangle$ are examples of such a function f.

We finish with the following local estimate which is an easy consequence of Proposition 2.11 in [DH]. Here B_r denotes an extrinsic ball of radius $r > 0$ in $\mathbb{R}^{n+1}$.

Proposition 1.12 (Proposition 2.11 [DH]). *For a solution M_t, $t \in [0,T]$ of the IMCF, there is a constant $C_n > 0$ such that*

$$\sup_{M_t \cap B_r} H \leq C_n \max\big(\sup_{M_0 \cap B_{2r}} H,\, r^{-1}\big).$$

1.3 Speed estimate

1.3.1 Bound of $(H|F|)^{-1}$ for convex solutions

In this section, we give the proof of Theorem 1.4 which gives the main a priori estimate on which our main existence result Theorem 1.2 is based upon. Lets us first introduce some standard notation. We consider spherical coordinates with respect to the origin in $\mathbb{R}^{n+1}$, namely

$$x = (x_1, \dots, x_{n+1}) = (r\omega \sin\theta, r\cos\theta) \quad \text{ with } r \geq 0,\ \omega \in \mathbb{S}^{n-1},\ \text{ and }\ \theta \in [0,\pi]$$

which are smoothly well-defined except from the origin or x_{n+1}-axis. We will also denote by $\bar{\nabla}$ and ∇ metric-induced connections on $(\mathbb{R}^{n+1}, g_{\text{euc}})$ and $(M^n, F^* g_{\text{euc}})$, respectively. Before the proof, we need the evolution equation of the important quantity θ, defined in the ambient space as follows:

Definition 1.13. *We define*

$$\theta : \mathbb{R}^{n+1} \setminus \{0\} \to [0,\pi] \quad \text{by}\ \ \theta(x) := \arccos\left(\frac{\langle x, e_{n+1}\rangle}{|x|}\right) \tag{1.13}$$

and

$$r : \mathbb{R}^{n+1} \to [0,\infty) \quad \text{by } r(x) := |x|.$$

Moreover, we define smooth unit orthogonal vector fields

$$e_\theta(x) = e_\theta(x', x_{n+1}) := \frac{1}{|x|}\frac{\partial}{\partial\theta} = \left(\frac{x'}{|x|}\frac{\cos\theta}{\sin\theta}, -\sin\theta\right) \qquad \text{on } \mathbb{R}^{n+1} \setminus \{x_{n+1} - \text{axis}\}$$

and

$$e_r(x) := \frac{\partial}{\partial r} = \frac{x}{|x|} \qquad \text{on } \mathbb{R}^{n+1} \setminus \{0\}.$$

Though θ is not smooth at the points on the x_{n+1}-axis, note that θ^2, $\cos\theta$, and $\sec\theta$ are all smooth on $\{x_{n+1} > 0\}$.

Lemma 1.14. *On the region $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$,*

$$(\partial_t - \frac{1}{H^2}\Delta)\theta = -\frac{1}{H^2 r^2}\left(\frac{n - |\nabla r|^2}{\tan\theta}\right) + \frac{1}{H^2}\frac{|\nabla\theta|^2}{\tan\theta} + \frac{2}{H^2}\langle\frac{\nabla r}{r}, \nabla\theta\rangle + \frac{2}{H}\langle\nu, \bar{\nabla}\theta\rangle.$$

Proof. Consider a spherical coordinate chart

$$(r, \theta, (w^\alpha)_{\alpha=1\dots n-1}) \quad \text{with } r > 0,\ \theta \in (0, \pi),\ (w^\alpha) \in \mathbb{S}^{n-1}$$

around a point $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$ in $\mathbb{R}^{n+1}$, where w^α is a coordinate chart of $\mathbb{S}^{n-1}$. On this chart,

$$g_{\text{euc}} = dr^2 + r^2\, d\theta^2 + r^2 \sin^2 \theta \sigma_{\alpha\beta}\, dw^\alpha dw^\beta. \tag{1.14}$$

Also note that

$$\operatorname{grad} \theta = \frac{1}{r^2}\frac{\partial}{\partial \theta} = \frac{1}{r} e_\theta \quad \text{and} \quad \operatorname{grad} r = \frac{\partial}{\partial r} = e_r \quad \text{on } (\mathbb{R}^{n+1}, g_{\text{euc}}). \tag{1.15}$$

At a given $p \in M^n$ with $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$, let us choose a geodesic normal coordinate of M^n, say $\{y^i\}_{i=1}^n$. In this coordinate at this point,

$$\begin{aligned}
\Delta\theta = \partial_i \partial_i \theta &= \frac{\partial}{\partial y^i} d\theta \left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^i}\left(\frac{1}{r^2}\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i}\rangle\right) \\
&= -\frac{2}{r^2}\langle \frac{\partial}{\partial y^i}, e_\theta\rangle\langle \frac{\partial}{\partial y^i}, e_r\rangle + \frac{1}{r^2}\langle \bar{\nabla}_{\partial_i} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i}\rangle + \langle \frac{\partial}{\partial \theta}, -h_{ii}\nu\rangle.
\end{aligned}$$

Since $\left(\left(\frac{\partial}{\partial y^i}\right)_{i=1}^n, \nu\right)$ constitutes an orthonormal basis of $T_{F(p)}\mathbb{R}^{n+1}$,

$$\langle \frac{\partial}{\partial y^i}, e_\theta\rangle\langle \frac{\partial}{\partial y^i}, e_r\rangle + \langle \nu, e_r\rangle\langle \nu, e_\theta\rangle = \langle e_r, e_\theta\rangle = 0. \tag{1.16}$$

Therefore,

$$\Delta\theta = -\frac{H}{r}\langle \nu, e_\theta\rangle + \frac{2}{r^2}\langle \nu, e_r\rangle\langle \nu . e_\theta\rangle + \frac{1}{r^2}\sum_i \langle \bar{\nabla}_{\partial_i}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i}\rangle. \tag{1.17}$$

Claim 1.1.

$$\sum_{i=1}^n \langle \bar{\nabla}_{\partial_i}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial y^i}\rangle = \frac{\cos\theta}{\sin\theta}\left(n - (1 - \langle \nu, e_r\rangle^2) - (1 - \langle \nu, e_\theta\rangle^2)\right). \tag{1.18}$$

Proof of Claim 1.1 . By computing the Christoffel symbols from the metric (1.14), we get:

$$\bar{\nabla}_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial \theta} = -r\frac{\partial}{\partial r}, \quad \bar{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \bar{\nabla}_{\frac{\partial}{\partial w^\alpha}}\frac{\partial}{\partial \theta} = \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial w^\alpha}. \tag{1.19}$$

Suppose $\partial_i = \partial_{y^i} = a_\theta \partial_\theta + a_r \partial_r + \sum_\alpha a_\alpha \partial_{w^\alpha}$. Then $\nabla_{\partial_i}\frac{\partial}{\partial\theta} = -r a_\theta \partial_r + \frac{a_r}{r}\partial_\theta + \sum_\alpha a_\alpha \frac{\cos\theta}{\sin\theta}\partial_{w^\alpha}$ and hence

$$\begin{aligned}
\langle \nabla_{\partial_i}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y^i}\rangle &= -r a_\theta a_r + r a_\theta a_r + r^2 \sin^2\theta \frac{\cos\theta}{\sin\theta} a_\alpha a_\beta \sigma^{\alpha\beta} \\
&= \frac{\cos\theta}{\sin\theta}\left[\left|\frac{\partial}{\partial y^i}\right|^2 - \left\langle \frac{\partial}{\partial y^i}, e_r\right\rangle^2 - \left\langle \frac{\partial}{\partial y^i}, e_\theta\right\rangle^2\right].
\end{aligned}$$

The claim follows by summing this over i. □

Now $\partial_t\theta = d\theta(\partial_t F) = \frac{1}{H}\langle\nu, \operatorname{grad}\theta\rangle = \frac{\langle\nu, e_\theta\rangle}{rH}$, (1.17) and (1.18) imply

$$(\partial_t - \frac{1}{H^2}\Delta)\theta = \frac{2\langle\nu, e_\theta\rangle}{rH} - \frac{1}{(rH)^2}\left[\frac{\cos\theta}{\sin\theta}[n - (1-\langle\nu, e_r\rangle^2) - (1-\langle\nu, e_\theta\rangle^2)] + 2\langle\nu, e_r\rangle\langle\nu, e_\theta\rangle\right].$$

Hence, the lemma follows by using (1.15) and the orthonormality of $\left(\left(\frac{\partial}{\partial y^i}\right)_{i=1}^n, \nu\right)$ in the equation above. $\square$

Proof of Theorem 1.4. Using the definition (1.13), our condition (1.5) can be written as $\theta(p,t) \leq \pi/2 - \theta_1$. Setting $c := \frac{\pi - \theta_1}{\pi - 2\theta_1} > 1$, we have $c\,\theta \leq \frac{\pi}{2} - \frac{\theta_1}{2} < \frac{\pi}{2}$ and $\sec(c\theta) \leq 2\sec\theta$ for $\theta = \theta(p,t)$ on $t \in [0, T]$.

By lemma 1.10,

$$\begin{aligned}
\square\sec(c\theta) &= c\sec(c\theta)\tan(c\theta)\square\theta - \frac{1}{H^2}c^2[\sec(c\theta)\tan^2(c\theta) + \sec^3(c\theta)]|\nabla\theta|^2] \\
&= \sec(c\theta)\left[c\tan(c\theta)\square\theta - \frac{c^2}{H^2}(2\tan^2(c\theta) + 1)|\nabla\theta|^2\right].
\end{aligned}$$

After defining $\varphi := \sec(c\theta)$, Lemma 1.14 and $\frac{\nabla\varphi}{\varphi} = c\tan(c\theta)\nabla\theta$ imply

$$\begin{aligned}
\frac{\square\varphi}{\varphi} &= -\frac{c}{H^2r^2}\frac{\tan(c\theta)}{\tan\theta}(n - |\nabla r|^2 - r^2|\nabla\theta|^2) + \frac{2}{H^2}\langle\frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle + \frac{2}{H}\langle\nu, \frac{\bar{\nabla}\varphi}{\varphi}\rangle \\
&\quad - \frac{2}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2} - \frac{1}{H^2}c^2|\nabla\theta|^2 \\
&\quad (\text{since } n - |\nabla r|^2 - r^2|\nabla\theta|^2 = n - 2 \geq 0 \quad \text{and} \quad \frac{\tan(c\theta)}{\tan\theta} \geq c) \\
&\leq -\frac{c^2}{H^2r^2}(n - |\nabla r|^2) - \frac{2}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2} + \frac{2}{H^2}\langle\frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle + \frac{2}{H}\langle\nu, \frac{\bar{\nabla}\varphi}{\varphi}\rangle.
\end{aligned}$$

Note that this inequality holds on $\{x_{n+1} > 0\}$, where our solution M_t is located. Let $w := \frac{\sec(c\theta)t}{Hr} = \varphi\psi r^{-1}t$

where $\psi := H^{-1}$. Then by Lemma 1.9 and the previous inequality

$$
\begin{aligned}
&\frac{\Box w}{w} + \frac{1}{H^2}\frac{|\nabla w|^2}{w^2} \\
=&(\frac{\Box \varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla \varphi|^2}{\varphi^2}) + (\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla \psi|^2}{\psi^2}) - \frac{1}{2}(\frac{\Box r^2}{r^2} + \frac{1}{H^2}\frac{|\nabla r^2|^2}{r^4}) + \frac{1}{t} \\
=&(\frac{\Box \varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla \varphi|^2}{\varphi^2}) + (\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla \psi|^2}{\psi^2}) + (\frac{n}{H^2 r^2} - \frac{2}{H}\langle \nu, \frac{\bar\nabla r}{r}\rangle - \frac{2}{H^2}\frac{|\nabla r|^2}{r^2}) + \frac{1}{t} \\
\leq& \left[\frac{|A|^2}{H^2} + \frac{1}{t} + \frac{2}{H}\langle \nu, \frac{\bar\nabla \varphi}{\varphi}\rangle - \frac{2}{H}\langle \nu, \frac{\bar\nabla r}{r}\rangle\right] \\
&\quad + \frac{1}{H^2}\left[\frac{|\nabla\psi|^2}{\psi^2} + \frac{n}{r^2} - 2\frac{|\nabla r|^2}{r^2} - c^2\frac{n - |\nabla r|^2}{r^2} - \frac{|\nabla \varphi|^2}{\varphi^2} + 2\langle \frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle\right] \\
=&:(1) + (2).
\end{aligned}
\tag{1.20}
$$

Suppose a nonzero maximum of $w(p,t)$ on $M^n \times [0,t_1]$ is achieved at (p_0,t_0) with $t_0 \in (0,t_1]$. At this point,

$$0 = \frac{\nabla w}{w} = \frac{\nabla \psi}{\psi} + \frac{\nabla\varphi}{\varphi} - \frac{\nabla r}{r}$$

and therefore

$$\frac{|\nabla\psi|^2}{\psi^2} = \left|\frac{\nabla r}{r} - \frac{\nabla\varphi}{\varphi}\right|^2 = \frac{|\nabla r|^2}{r^2} + \frac{|\nabla \varphi|^2}{\varphi^2} - 2\langle\frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle.$$

At the maximum point, by plugging this into (2) in (1.20), $(2) = -(c^2-1)\frac{n-|\nabla r|^2}{H^2r^2}$. Therefore at the maximum point,

$$0 \leq (1) - (c^2-1)\frac{n-|\nabla r|^2}{H^2 r^2}.$$

Let us estimate terms in (1). Note that by our choice of $c > 1$,

$$\left|\frac{\bar\nabla\varphi}{\varphi}\right| = |c\tan(c\theta)\bar\nabla\theta| \leq \frac{c}{r}\tan(c\theta) = \frac{1}{r}\sin(c\theta)\sec(c\theta) \leq \frac{2}{r\cos\theta} \leq \frac{2}{r}\frac{1}{\sin\theta_1} = \frac{C}{r}$$

for some $C = C(\theta_1)$. Next, $\frac{|A|^2}{H^2} \leq 1$ from convexity and $|\bar\nabla r| \leq 1$ imply at (p_0,t_0),

$$
\begin{aligned}
0 &\leq -(n - |\nabla r|^2)\frac{c-1}{H^2r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0}. \\
&\leq -\frac{c-1}{H^2r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0} \quad (\text{since } |\nabla r|^2 \leq 1,\ n \geq 2) \\
&\leq -\frac{c-1}{2H^2r^2} + \frac{C}{2(c-1)} + \frac{1}{t_0}.
\end{aligned}
$$

Note that $0 < t_0 \leq t_1$ and $1 \leq \varphi \leq C$ on $M \times [0,t_1]$. Multiplication of $(\varphi(p_0,t_0)t_0)^2$ implies

$$w^2(p_0,t_0) = \left(\frac{\varphi t_0}{Hr}\right)^2 \leq C(t_1+1).$$

On any other point $p \in M$ at $t = t_1$,

$$\frac{1}{H^2 r^2}(p,t_1) = \left(\frac{w(p,t_1)}{t_1\varphi(p,t_1)}\right)^2 \le \frac{w^2(p_0,t_0)}{t_1^2\varphi^2(p,t_1)} \le C\left(1+\frac{1}{t_1}\right).$$

We used $\varphi \ge 1$ in the last inequality. This finishes the proof of Theorem 1.4. □

Remark 1.6. If we define $\bar{w} := \varphi\psi r^{-1}$ and follow the rest similarly, we get an estimate which includes the initial bound

$$\frac{1}{H|F|} \le C \max\left(\sup_{M_0} \frac{1}{H|F|}, 1\right).$$

1.3.2 Bound of H^{-1} for compact star-shaped solutions

The goal is this section is to give an alternative proof of Theorem 1.1 in [HI4] which will be based on the maximum principle. This is an interesting result, but will not be used in the proof of the main existence theorem. The theorem holds in any dimension $n \ge 1$.

Proof of Theorem 1.5. Since M_0 satisfies (1.7), by Proposition 1.3 in [HI4], we have

$$R_1 \le R_1\, e^{\frac{t}{n}} \le \langle F, \nu\rangle \le |F| \le R_2\, e^{\frac{t}{n}} \tag{1.21}$$

for all $0 < t < +\infty$. Let us denote $w := \langle F,\nu\rangle^{-1}$ and we will consider a function

$$Q := \frac{\varphi^{1-\epsilon}(w)\, e^{\gamma|F|^2}}{H}$$

for some function $\varphi := \varphi(w)$, constants $\gamma > 0$ and $\epsilon \in (0,1)$ which will be chosen shortly.

Direct computation shows that

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) \ln e^{|F|^2} = \big(\partial_t - \frac{1}{H^2}\Delta\big)|F|^2 = -\frac{2n}{H^2} + \frac{4}{Hw}.$$

Moreover, by (7) in Lemma 1.7 and Lemma 1.11 with $\beta = -1$,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) w = -\frac{|A|^2}{H^2} w - \frac{2}{wH^2}|\nabla w|^2$$

and hence, on $\{\varphi \ne 0\}$,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big)\ln\varphi = \frac{(\partial_t - H^{-2}\Delta)\varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2} = -\frac{|A|^2}{H^2}\frac{\varphi' w}{\varphi} - \frac{|\nabla w|^2}{H^2}\left(2\frac{\varphi'}{w\varphi} + \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}\right).$$

Inspired by the choice of φ in the well known interior curvature estimate by Ecker and Huisken in [EH2] (see also [CNS]), we define

$$\varphi(s) := \left(\frac{s}{2R_1^{-1} - s}\right). \tag{1.22}$$

For this $\varphi := \varphi(w)$, under the notation $\varphi' = \varphi'(w)$ and $\varphi'' = \varphi''(w)$, a direct computation yields

$$\frac{\varphi' w}{\varphi} = -\Big(\frac{2}{2 - wR_1}\Big) \quad \text{and} \quad 2\,\frac{\varphi'}{w\phi} + \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} = \frac{\varphi'^2}{\varphi^2}.$$

Lemma 1.9 and the computations above imply

$$\begin{aligned}
\big(\partial_t - \frac{1}{H^2}\Delta\big)\ln Q &= \left[\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla H^{-1}|^2}{H^{-2}}\right] + \gamma\left[\frac{4}{Hw} - \frac{2n}{H^2}\right] - (1-\epsilon)\left[\frac{|A|^2}{H^2}\frac{\varphi' w}{\varphi} + \frac{1}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2}\right] \\
&= -\Big(\frac{wR_1 - 2\epsilon}{2 - wR_1}\Big)\frac{|A|^2}{H^2} + \Big(\gamma\frac{4}{Hw} - \gamma\frac{2n - 4\epsilon^{-1}\gamma\,|F|^2\left|\nabla\,|F|\right|^2}{H^2}\Big) \\
&\quad - \frac{1}{H^2}\left[(1-\epsilon)\frac{|\nabla\varphi|^2}{\varphi^2} - \frac{|\nabla H^{-1}|^2}{H^{-2}} + \epsilon^{-1}\gamma^2\frac{|\nabla e^{|F|^2}|^2}{|e^{|F|^2}|^2}\right].
\end{aligned} \tag{1.23}$$

Note that we have added and subtracted the term $\frac{\epsilon^{-1}\gamma^2\left|\nabla|F|^2\right|^2}{H^2}$ in the last equality. At a nonzero critical point of Q,

$$0 = \frac{\nabla Q}{Q} = (1-\epsilon)\frac{\nabla\varphi}{\varphi} + \gamma\frac{\nabla e^{|F|^2}}{e^{|F|^2}} + \frac{|\nabla H^{-1}|^2}{H^{-2}},$$

and thus

$$\begin{aligned}
\left|\frac{\nabla H^{-1}}{H^{-1}}\right|^2 = \left|(1-\epsilon)\frac{\nabla\varphi}{\varphi} + \gamma\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2 &= (1-\epsilon)^2\left|\frac{\nabla\varphi}{\varphi}\right|^2 + 2(1-\epsilon)\gamma\left\langle\frac{\nabla\varphi}{\varphi}, \frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right\rangle + \gamma^2\left|\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2 \\
&\le ((1-\epsilon)^2 + \epsilon(1-\epsilon))\left|\frac{\nabla\varphi}{\varphi}\right|^2 + (1 + \frac{1-\epsilon}{\epsilon})\gamma^2\left|\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2 \\
&= (1-\epsilon)\frac{|\nabla\varphi|^2}{\varphi^2} + \epsilon^{-1}\gamma^2\frac{|\nabla e^{|F|^2}|^2}{e^{2|F|^2}}.
\end{aligned}$$

For a given $T > 0$, note that $\frac{R_1}{R_2 e^{\frac{T}{n}}} \le wR_1 \le 1$. It remains to choose ϵ and γ. The choice $\epsilon := \frac{R_1}{2R_2 e^{\frac{T}{n}}}$ makes the first term on RHS of the second equality in (1.23) nonpositive. Next, choose $\gamma := \frac{\epsilon}{4n}\frac{1}{(R_2 e^{\frac{T}{n}})^2} > 0$ so that $4\epsilon^{-1}\gamma|F|^2 \le n$ on M_t for $t \in [0,T]$. Combining the choices and estimates, at a nonzero spatial critical point of Q,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big)\ln Q = \frac{(\partial_t - H^{-2}\Delta)Q}{Q} + \frac{|\nabla Q|^2}{Q^2} \le \gamma\left(-\frac{n}{H^2} + \frac{4}{H\,w}\right). \tag{1.24}$$

We will now apply the maximum principle on $\hat{Q} := tQ$. Suppose that nonzero maximum of $\hat{Q}$ on $M^n \times [0,T]$ occurs at the point (p_0, t_0), which necessarily implies $t_0 > 0$. At this point, (1.24) implies

$$0 \le \big(\partial_t - \frac{1}{H^2}\Delta\big)\ln \hat{Q} \le \gamma\left(-\frac{n}{H^2} + \frac{4}{H\,w}\right) + \frac{1}{t_0} \le \gamma\left(-\frac{n}{2H^2} + \frac{8}{n}R_2^2 e^{2\frac{T}{n}}\right) + \frac{1}{t_0} \tag{1.25}$$

where the second inequality comes from

$$\frac{4}{H\,w} \le \frac{8}{nw^2} + \frac{n}{2H^2} \le \frac{8}{n}R_2^2 e^{2\frac{T}{n}} + \frac{n}{2H^2}.$$

The rest is a standard argument shown in the proof of Theorem 3.1 [EH2]. By the choices of ϵ, γ, bounds (1.21) and

$$\frac{R_1}{2R_2 e^{T/n}} \le \varphi((R_2 e^{T/n})^{-1}) \le \varphi(w) \le \varphi(R_1^{-1}) = 1,$$

we proceed and obtain, for every $(p,t) \in M^n \times (0,T]$,

$$\frac{1}{H^2}(p,t) \le C_n \left(\frac{R_2}{R_1}e^{\frac{T}{n}}\right)^{2-\epsilon} (R_2 e^{\frac{T}{n}})^2 \left(1 + \frac{1}{t}\right). \tag{1.26}$$

Now for time $t > 1$, we can alway apply this estimate starting at time $t-1$. Inequality (1.21) implies that the ratio between star-shapedness bounds from above and below remains unchanged over time. This way we can replace $(R_2 e^{\frac{T}{n}}/R_1)^{2-\epsilon}$ in the above estimate by $(R_2/R_1)^{2-\epsilon}$ after possibly enlarging the constant C_n. Since $(R_2/R_1)^{2-\epsilon} \le (R_2/R_1)^2$, the theorem follows. $\square$

1.4 Strict convexity of solutions in space form

Throughout this section, we assume that $F : M^n \times (0,T) \to (N^{n+1}, \bar{g})$ is a complete smooth convex solution of the IMCF, where $(N^{n+1}, \bar{g})$ is a *space form* of sectional curvature $K \in \mathbb{R}$, in particular which includes Euclidean space, the sphere, or hyperbolic space. As before, ν denotes the unit outward normal, H the mean curvature and h_{ij} the second fundamental form.

Suppose we have an (incomplete) smooth convex solution of the IMCF with $H > 0$ on an open set $\Omega \subset M$ for $t \in (0,T)$. Our aim is to prove Theorem 1.17, a strong minimum principle on λ_1. However by looking at the evolution of the second fundamental form h_{ij} given in (1.10), it is not clear that the convexity is preserved. To do so we need to use a viscosity solution argument and we need the following lemma shown from [BCD].

Lemma 1.15 (Lemma 5 in Section 4 [BCD])**.** *Suppose that ϕ is a smooth function such that $\lambda_1 \ge \phi$ everywhere and $\lambda_1 = \phi$ at $x = \bar{p} \in \Omega$. Let us choose an orthonormal frame so that*

$$h_{ij} = \lambda_i \delta_{ij} \ \text{ at } \bar{p} \in \Omega \qquad \text{with } \lambda_1 = \lambda_2 = \ldots = \lambda_\mu < \lambda_{\mu+1} \le \ldots \le \lambda_n.$$

We denote $\mu \geq 1$ by the multiplicity of λ_1. Then at $\bar{p}$, $\nabla_i h_{kl} = \delta_{kl}\nabla_i \phi$ for $1 \leq k, l \leq \mu$. Moreover,

$$\nabla_i \nabla_i \phi \leq \nabla_i \nabla_i h_{11} - 2\sum_{j>\mu}(\lambda_j - \lambda_1)^{-1}(\nabla_i h_{1j})^2.$$

Proposition 1.16. *For $n \geq 1$, let $F : \Omega \times (0, T) \to (N^{n+1}, \bar{g})$ be a smooth convex solution of the IMCF where $(N, \bar{g})$ is a space form. Let λ_1 denote the lowest eigenvalue of h^i_j. Then $u := \lambda_1/H$ is a viscosity supersolution of the equation*

$$\frac{\partial}{\partial t}u - \frac{1}{H^2}\Delta u + \frac{1}{H^3}\langle V, \nabla u\rangle + \left(\frac{W}{H^4}\right)u \geq 0 \tag{1.27}$$

where V is a vector field, and W is a scalar function such that

$$|W|, |V| \leq C(|\nabla H|, n) \quad \text{at each point.}$$

Proof. Using equation (1.10) in Remark 1.5, we can easily compute the evolution equation of h^i_j/H:

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\frac{h^i_j}{H} = 2\frac{|A|^2}{H^2}\frac{h^i_j}{H} - 2\frac{h^{ik}h_{kj}}{H^2} + \frac{2}{H^4}(\nabla_m H \nabla^m h^i_j - \nabla^i H \nabla_j H). \tag{1.28}$$

We will use this equation and the Lemma above to the proposition. Suppose a smooth function of space time, namely ϕ/H, touches λ_1/H from below at $(\bar{p}, \bar{t})$. At time $\bar{t}$ around $\bar{p}$, let us fix a time independent frame $\{e_i\}$ using the metric $g(t)$ as in Lemma 1.15.

Since $\phi \leq \lambda_1 \leq h^1_1$ and they coincide at $(\bar{p}, \bar{t})$, $\partial_t \phi \geq \partial_t h^1_1$ at $(\bar{p}, \bar{t})$. At this point $(\bar{p}, \bar{t})$ with the frame $\{e_i\}$, we use Lemma 1.15, equation (1.28), and the Codazzi identity $\nabla_i h_{jk} = \nabla_j h_{ik}$ to obtain

$$\begin{aligned}
\square\frac{\phi}{H} &\geq \frac{\partial}{\partial t}\frac{h^1_1}{H} - \frac{1}{H^2}\Delta\frac{h^1_1}{H} + \frac{2}{H^3}\sum_i\sum_{j>\mu}(\lambda_j - \lambda_1)^{-1}|\nabla_i h_{1j}|^2 \\
&= \square\frac{h^1_1}{H} + \frac{2}{H^3}\sum_{i\geq 1, j>\mu}(\lambda_j - \lambda_1)^{-1}|\nabla_1 h_{ij}|^2 \\
&= \frac{2\sum_j \lambda_j^2 - 2\lambda_1\sum_j \lambda_j}{H^2}\frac{\lambda_1}{H} + \frac{2}{H^4}\left[\nabla_m H \nabla_m h_{11} - |\nabla_1 H|^2 + H\sum_{i\geq 1, j>\mu}\frac{|\nabla_1 h_{ij}|^2}{\lambda_j - \lambda_1}\right] \\
&\geq \frac{2}{H^4}\left[\nabla_m H \nabla_m \phi - |\nabla_1 H|^2 + H\sum_{i\geq 1, j>\mu}\frac{|\nabla_1 h_{ij}|^2}{\lambda_j - \lambda_1}\right].
\end{aligned} \tag{1.29}$$

In the last line, we used $\lambda_1\sum_j \lambda_j \leq \sum_j \lambda_j^2 = |A|^2$ which is true for $H > 0$.

Next, note that

$$\nabla_1 H \nabla_1 H = \sum_{i,j}\nabla_1 h_{ii}\nabla_1 h_{jj} = 2\mu\nabla_1 H \nabla_1 \phi - \mu^2|\nabla_1 \phi|^2 + \sum_{i>\mu, j>\mu}\nabla_1 h_{ii}\nabla_1 h_{jj}. \tag{1.30}$$

Since $H\,\nabla\dfrac{\phi}{H} = \nabla\phi - \dfrac{\phi}{H}\,\nabla H$, we have the following for each fixed unit direction e_m

$$\nabla_m H \nabla_m \phi = H \nabla_m H \nabla_m \frac{\phi}{H} + \frac{\phi}{H} |\nabla_m H|^2. \tag{1.31}$$

We first plug (1.31) with $m = 1$ into (2.5) and then plug that into the last line of (1.29) to obtain

$$\begin{aligned} \Box\frac{\phi}{H} \geq\ & \frac{2}{H^4}\sum_{m>1} |\nabla_m H|^2 \frac{\phi}{H} + \frac{2(1-2\mu)}{H^4}|\nabla_1 H|^2 \frac{\phi}{H} \\ & + \frac{2}{H^3}\sum_{m>1} \nabla_m H \nabla_m \frac{\phi}{H} + \frac{2(1-\mu)}{H^3}\nabla_1 H \nabla_1 \frac{\phi}{H} + \frac{2\mu^2}{H^3}\frac{|\nabla_1 \phi|^2}{H} \\ & + \frac{2}{H^4}\left[H \sum_{i\geq 1, j>\mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ij}|^2 - \sum_{i>\mu, j>\mu} \nabla_1 h_{ii} \nabla_1 h_{jj} \right]. \end{aligned} \tag{1.32}$$

We now use the convexity, $\lambda_1 \geq 0$, in the proof of the following claim.

Claim 1.2. $\left[H \sum_{i\geq 1, j>\mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ij}|^2 - \sum_{i>\mu, j>\mu} \nabla_1 h_{ii} \nabla_1 h_{jj} \right] \geq 0$ *on* $\{\lambda_1 \geq 0\}$.

Assuming that the claim is true, then by taking away the good term $\dfrac{2\mu^2}{H^3}\dfrac{|\nabla_1 \phi|^2}{H}$ in (1.32), we easily conclude that (1.27) holds by choosing a vector filed V and a scalar function W as a function of ∇H accordingly. Thus it remains to show the claim.

Proof of Claim 1.2. Since $\lambda_1 \geq 0$, $H = \sum_{l\geq 1} \lambda_l \geq \sum_{l>\mu} \lambda_l$, the claim follows by:

$$\begin{aligned} H \sum_{i\geq 1, j>\mu} (\lambda_j - \lambda_1)^{-1} |\nabla_1 h_{ij}|^2 &\geq \sum_{l>\mu} \lambda_l \sum_{i>\mu} \lambda_i^{-1} |\nabla_1 h_{ii}|^2 = \sum_{i>\mu, j>\mu} \lambda_j \lambda_i^{-1} |\nabla_1 h_{ii}|^2 \\ &= \sum_{i>\mu, j>\mu} \frac{\lambda_j \lambda_i^{-1} |\nabla_1 h_{ii}|^2 + \lambda_i \lambda_j^{-1} |\nabla_1 h_{jj}|^2}{2} \\ &\geq \sum_{i>\mu, j>\mu} \nabla_1 h_{ii} \nabla_1 h_{jj}. \end{aligned} \tag{1.33}$$

□

□

Now, let $M_t \subset N^{n+1}$ be a smooth complete convex solution for $t > 0$, which could be either compact or non-compact. One expects M_t to be strictly convex, that is to have $\lambda_1 > 0$ for $t > 0$. Indeed, this follows easily by Proposition 1.16 and the strong minimum principle for nonnegative supersolutions which is a consequence of the weak Harnack inequality for viscosity solutions of (locally) uniformly parabolic equations.

Theorem 1.17. *Suppose $F : M^n \times (0,T) \to (N^{n+1}, \bar{g})$ is a smooth convex solution of the IMCF with $H > 0$ where $(N^{n+1}, \bar{g})$ is a space form. If $\lambda_1(p_0, t_0) = 0$ at some (p_0, t_0) with $0 < t_0 < T$, then $\lambda_1 = 0$ on $M^n \times (0, t_0]$.*

Proof. Since solution is smooth, $|H|$, $|\nabla H|$, and $|H^{-1}| = |\partial_t F|$ are locally bounded. Therefore, λ_1 is a nonnegative supersolution of equation (1.27) which is locally uniformly parabolic with bounded coefficients. We can apply strong minimum principle on a sequence $\{\Omega_k\}$ of expanding domains containing (p_0, t_0) such that $M^n = \cup_k \Omega_k$ and conclude that the theorem holds. □

Corollary 1.18. *Let $F : M^n \times (0,T) \to \mathbb{R}^{n+1}$ be a smooth convex complete solution of the IMCF. If $\mathcal{H}^n(\nu[M_{t_0}]) > 0$ at $t_0 \in (0,T)$, then the solution is strictly convex for $(0, t_0]$.*

Proof. If it is not, $\lambda_1 \equiv 0$ for all $M^n \times (0, t_0]$. In particular, $\mathcal{H}^n(\nu[M_{t_0}]) = \int_M K(\cdot, t_0) d\mu = 0$. This contradicts and proves the assertion. □

Remark 1.7 (Strict convexity of solutions)*.* The theorem and corollary above do not exactly explain why convexity is preserved along the IMCF since they both assume the convexity of the solution. First of all, assume $M_t^n \subset N^{n+1}$ is a smooth *compact* solution for $t \in [0,T]$, where M_0 is smooth and strictly convex. Then by considering the first time when λ_1 becomes zero, Theorem 1.17 implies that M_t is strictly convex for all time. i.e. *the strict convexity is preserved for compact solutions.* We observe next that all solutions, including non-compact ones, which will be constructed later section are obtained as a locally smooth limit of strictly convex solutions. In particular, they are at least weakly convex. Therefore one can apply Theorem 1.17 and Corollary 1.18 and conclude that they are *strictly convex.*

1.5 Long time existence of non-compact solutions

In this section we will give the proof of our main results in this work concerning the long time existence of non-compact solutions of the IMCF, Theorem 1.2 and Theorem 1.3 stated in the introduction. The proof of Theorem 1.2 is based on our main a priori estimate, Theorem 1.4 which provides an estimate of $(H\,|F|)^{-1}$ from above, in terms of the angle θ of a supporting cone from outside. Since, this estimate holds for compact surfaces, we will first construct a family of *compact convex* approximating solutions $M_{i,t} = \partial \hat{M}_{i,t}$ which is monotone in i. The results in [Ge1] and [Ur] guarantee the existence of each compact expanding solution $M_{i,t}$, for all $t \in (0, +\infty)$. However, we will see that the limit $M_t := \lim_{i\to+\infty} M_{i,t}$ is non-trivial only up to time $T = T(M_0)$. In fact, the proof of Theorem 1.3 in this section shows that the limit M_t must a hyperplane in $\mathbb{R}^{n+1}$ for $t > T$, i.e. $\cup_i \hat{M}_{i,t} = \mathbb{R}^{n+1}$. Here is where the connection between our non-compact solution M_t in Euclidean space and solutions on the sphere is revealed. Recall the notation $\Gamma_0 := C_0 \cap \mathbb{S}^n$ of the link of the tangent cone C_0 of M_0 at infinity. For each time $T - \delta < T(M_0)$, we are going to find smooth strictly convex $\Gamma_0^\delta \subset \mathbb{S}^n$ such that $\hat{\Gamma}_0 \subset\subset \hat{\Gamma}_0^\delta$ and $T_\delta := \ln|\mathbb{S}^n| - \ln|\Gamma_0^\delta| > T - \delta$. In view of the results in [MS] and [Ge3], also described in Example 1.1, for each such Γ_0^δ there is a smooth IMCF solution $\Gamma_t^\delta \subset \mathbb{S}^n$ which exists up to time T' and we can make use of $\mathcal{C}\Gamma_t^\delta$ as an outer barrier for $M_{i,t}$. Indeed, moving its vertex far away from M_0 initially, we can make $\mathcal{C}\Gamma_t^\delta$ (after an initial translation) to contain $M_{i,t}$ up to time T_δ, implying that each $M_{i,t}$ satisfies condition (1.5) in Theorem 1.4 up to time $T - \delta$ for a uniform $\theta_1 > 0$. Theorem 1.4 then leads to an upper bound on $(|F|\,H)^{-1}$ implying that the IMCF on $M_{i,t}$ is locally uniformly parabolic and the rest is straightforward. We begin with Theorem 1.2.

Proof of Theorem 1.2. Let $M_0 = \partial\hat{M}_0$ satisfy the assumptions of our theorem and let C_0, $\hat{C}_0$ be the tangent cones at infinity of M_0, $\hat{M}_0$ respectively and $\Gamma_0 = C_0 \cap \mathbb{S}^{n-1}$, $\hat{C}_0 = \hat{C}_0 \cap \mathbb{S}^{n-1}$ their links. Assume that T given by (1.3) satisfies $T > 0$, as there is nothing to prove for the case $T = 0$. Note that, if $\hat{M}_0$ contains an infinite straight line, then $\hat{M}_0$ splits off in the direction of the line by an elementary convexity argument. By repeating this, $\hat{M}_0 = \hat{N}_0 \times \mathbb{R}^k$ for some $k \geq 0$ and we can assume $\hat{N}_0$ does not contain any infinite lines. Also,

$$T(M_0) = \ln|\mathbb{S}^n| - \ln P(\hat{\Gamma}_{0,M_0}) = \ln|\mathbb{S}^{n-k}| - \ln P(\hat{\Gamma}_{0,N_0}) = T(N_0).$$

Moreover, $0 \leq k \leq n-2$ since $k = n-1$ or n imply $T(N_0) = T(M_0) = 0$. In conclusion, it suffices to show the existence of solution for $N_0^{n-k} = \partial\hat{N}_0 \subset \mathbb{R}^{n-k+1}$. Hence, we may assume, without loss of generality, that $\hat{M}_0$ *does not contain any straight lines.* In this case, *the link of the tangent cone at infinity* $\hat{\Gamma}_0$ does not contain any antipodal points and is *compactly contained in an open hemisphere*

$$H(v_0) := \{p \in \mathbb{S}^n \,:\, \langle p, v_0\rangle > 0\}$$

for some $v_0 \in \mathbb{S}^n$ (see Lemma 3.8 [MS]).

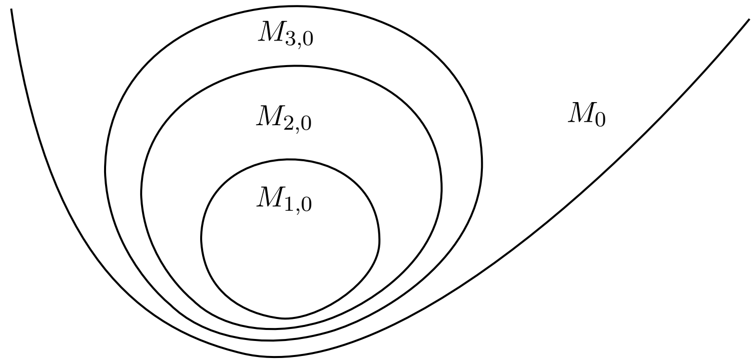

Figure 1.2: Approximation of M_0

Next, we create a sequence of strictly monotone convex compact hypersurfaces $M_{i,0}$ which approximate M_0 from inside as follows: let $\Sigma_{i,0}$ be the compact hypersurface $\Sigma_{i,0} := \partial[B_i(0) \cap \hat{M}_0]$ where $B_i(0)$ denotes a all of radius i in $\mathbb{R}^{n+1}$. To smoothen out each $\Sigma_{i,0}$ at the intersection of $\partial B_i(0)$ and M_0 we let $\Sigma_{i,s}$, $s > 0$, be the *mean curvature flow* (MCF) running from $\Sigma_{i,0}$. For a positive decreasing sequence $s_i \to 0$, let $M_{i,0} := \Sigma_{i,s_i}$. Then, $\{M_{i,0}\}$ satisfies the desired properties and $M_{i,0} \to M_0$ locally uniformly on compact sets. Moreover, $M_0 \in C^{1,1}_{\text{loc}}$ implies the mean curvature H of $M_{i,0}$ is i-uniformly bounded on every extrinsic ball of finite radius. By the results in [Ur] and [Ge3], for each $M_{i,0}$ there exists a unique smooth solution of the IMCF, $M_{i,t} = \partial \hat{M}_{i,t}$ for $t \in [0, \infty)$. $M_{i,t}$ are strictly convex (see Remark 1.7) and strictly monotone increasing in i by the comparison principle. By Proposition 1.12, the mean curvature H is locally uniformly bounded for $M_{i,t}$, i.e. given $R > 0$, there is $M > 0$ such that

$$0 < H \leq M \quad \text{on} \quad B_R(0) \cap M_{i,t} \quad \text{for all } i \text{ and } t \geq 0. \tag{1.34}$$

We define our solution by

$$M_t = \partial \hat{M}_t \quad \text{with} \quad \hat{M}_t := \overline{\cup_{i=1}^{\infty} \hat{M}_{i,t}} \quad \text{for } t \in [0, \infty).$$

This is convex by definition and it remains to prove that M_t *is (nontrivial) strictly convex smooth solution of the flow for* $t \in (0, T(M_0))$ *and converges to* M_0 *locally uniformly as* $t \to 0+$. We will need the following simple observation.

Claim 1.3. *Let* $\hat{\Gamma}_0 \subset \mathbb{S}^n$ *be a convex set which is compactly contained in an open hemisphere* $H(v_0)$. *Then there is a family of smooth, strictly convex hypersurfaces* $\{\Gamma_0^\epsilon\}_{\epsilon>0}$ *in* $\mathbb{S}^n$ *with* $\partial \hat{\Gamma}_0^\epsilon = \Gamma_0^\epsilon$ *which* *are also contained in $H(v_0)$, strictly monotone decreasing in the sense that*

$$\hat{\Gamma}^{\epsilon_1} \subset\subset \hat{\Gamma}^{\epsilon_2} \text{ for } 0 < \epsilon_1 < \epsilon_2$$

and $\cap_{\epsilon>0}\hat{\Gamma}_0^\epsilon = \hat{\Gamma}_0$. For such a sequence, $|\Gamma_0^\epsilon| = P(\hat{\Gamma}_0^\epsilon) \to P(\hat{\Gamma}_0)$.

Proof of Claim 1.3. If $\hat{\Gamma}_0$ is a single point, we may choose Γ_0^ϵ to be concentric geodesic spheres in $\mathbb{S}^n$. Thus we may assume that $\hat{\Gamma}_0$ is a closed convex set in an open hemisphere which is not a point. Define the *dual* of $\hat{\Gamma}_0$ by

$$\hat{\Gamma}_0' := \{v \in \mathbb{S}^n \mid \langle v, w\rangle \leq 0 \text{ for all } w \in \hat{\Gamma}_0\}.$$

Then, it can be easily checked that $\hat{\Gamma}_0'$ is contained in a closed hemisphere. The fact that $\hat{\Gamma}_0$ lies in $\text{int}H(v_0)$ implies $\hat{\Gamma}_0'$ has non-empty interior. i.e. $\partial\hat{\Gamma}_0'$ is a convex hypersurface. We may run *mean curvature flow* $\Gamma_{0,s}'$ starting at $\Gamma_0' = \partial\hat{\Gamma}_0'$ for a short time $s \in (0, s')$. $\{\Gamma_{0,s}'\}$ are smooth, strictly convex and monotone decreasing unless $\hat{\Gamma}_0'$ is a hemisphere (which isn't the case as $\hat{\Gamma}_0$ is not a point). We define $\hat{\Gamma}_0^\epsilon = (\hat{\Gamma}_{0,\epsilon}')'$. Then, it is known (see Chapters 9, 10 of [Ge2] and also in [Ge3]) that $\Gamma_0^\epsilon = \partial\Gamma_0^\epsilon$ is the image of $\Gamma_{0,\epsilon}'$ under the Gauss map and $\{\Gamma_0^\epsilon\}$ are smooth, strictly convex, and strictly monotone decreasing in ϵ. Since $\Gamma_{0,\epsilon}'$ converges to Γ_0' as $\epsilon \to 0$ from inside, Γ_0^ϵ converges to Γ_0 from outside. It follows that $|\Gamma_0^\epsilon| = P(\hat{\Gamma}_0^\epsilon) \searrow P(\hat{\Gamma}_0)$. □

Now fix $t_0 \in (0, T)$ an arbitrary time. By the claim, we may find a small $\epsilon_0 > 0$ such that $T^{\epsilon_0} := \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0^{\epsilon_0}| > t_0$. Since $\hat{\Gamma}_0$ is contained in the interior of $\hat{\Gamma}_0^{\epsilon_0}$, we may find a vector $v_0' \in \mathbb{R}^{n+1}$ such that $\hat{M}_{i,0} \subset \hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0^{\epsilon_0} + v_0'$. Theorem 1.4 [MS] guarantees the existence of a smooth strictly convex IMCF solution $\Gamma_t^{\epsilon_0}$ in $\mathbb{S}^n$ with initial data $\Gamma_0^{\epsilon_0}$, for $t \in [0, T^{\epsilon_0})$. Then by the comparison principle $\hat{M}_{i,t} \subset \mathcal{C}\hat{\Gamma}_t^{\epsilon_0} + v_0'$ for $t \in [0, T^{\epsilon_0})$. Since $\Gamma_t^{\epsilon_0}$ is a strictly convex solution which converges to an equator, we may find a direction $\omega_0 \in \mathbb{S}^n$ and small $\delta_0 > 0$ such that

$$\langle F - v_0', \omega_0\rangle \geq (\sin\delta_0)\,|F - v_0'| \qquad \text{for } t \in [0, t_0] \text{ on } M_{i,t}.$$

By Theorem 1.4, we have uniform bounds of $(H|F|)^{-1}$ for $M_{i.t}$ on $t \in [t_0/2, t_0]$. The conical barrier $\mathcal{C}\hat{\Gamma}_t^{\epsilon_0} + v_0'$ also shows M_t is nonempty for $t \in [0, t_0]$.

Let us choose an arbitrary point $x_0 \in M_{t_0}$. By the previous argument, we have uniform bounds of H and H^{-1} on $M_{i,t} \cap B_1(x_0)$ for $t \in [t_0/2, t_0]$. Since M_{t_0} is convex, there is a supporting hyperplane at x_0 and after an isometry, we may assume $x_0 = 0$ and the hyperplane is $\{x_{n+1} = 0\}$ and $M_{i,t}$ are located in $\{x_{n+1} \geq 0\}$ for $t \leq t_0$.

Claim 1.4. *Let $D_{r_0} = \{x' \in \mathbb{R}^n : |x'| \leq r_0\}$. Then there is small $r_0 > 0$ and $\tau_0 > 0$ such that for large $i \geq i_0$, $M_{i,t} \cap (D_{r_0}(0) \times [-r_0, r_0])$ can be written as graphs $x_{n+1} = u^{(i)}(x', t)$ on $D_{r_0} \times [t_0 - \tau_0, t_0]$ with uniformly bounded C^1 norm.*

Proof of Claim 1.4. Assume H, $H^{-1} \leq M$ on $B_1(0) \cap M_{i,t}$ for $t \in [t_0/2, t_0]$. By the bound of H^{-1} and $x_0 = 0 \in M_{t_0}$, for every $r \in (0, 1/2)$, there is $\tau > 0$ such that $M_{i,t} \cap (D_r \times [-r, r]) \neq \phi$ for large $i \geq i_0$ and $t \in [t_0 - \tau, t_0]$ (for instance, we can choose $\tau = \min(\frac{t_0}{2}, \frac{r}{2M})$). Meanwhile, $H \leq M$ implies that every point on $M_{i,t}$ has a inscribed ball of radius M^{-1}. Since $M_{i,t}$ lies in $\{x_{n+1} \geq 0\}$, by choosing r small enough compared to M^{-1}, those points in $M_{i,t} \cap (D_r \times [-r, r])$ should have uniformly bounded gradient in terms of r and M. Now, we can choose smaller r_0 (if needed) to make $M_{i,t} \cap (D_r \times [-r, r])$ a one sheeted graph over D_r. We also choose $\tau_0 = \tau(r_0)$.

$\square$

Since $M_{i,t}$ are solutions to IMCF, the graphs $u(x', t) = u^{(i)}(x', t)$ evolve by the fully nonlinear parabolic equation

$$\partial_t u = -\frac{(1 + |Du|^2)^{1/2}}{H} = -(1 + |Du|^2)^{1/2} \left[\operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{1/2}} \right) \right]^{-1}$$

and the equation is uniformly parabolic if $|Du|$, H, H^{-1} are bounded. Therefore, our estimates above show that $u^{(i)}$ are solutions of a uniformly parabolic equation on $D_{r_0} \times [t_0 - \tau_0, t_0]$ and moreover they are uniformly bounded, since $|u^{(i)}| \leq r_0$. Standard parabolic regularity theory implies the smooth subsequential convergence $u^i \to u$ on $D_{r_0/2} \times [t_0 - \tau_0/2, t_0]$. Since $M_{i,t}$ are monotone in i, this proves that $x_{n+1} = u(x', t)$ is a smooth graphical parametrization of M_t. i.e. M_t is a smooth solution of the IMCF for $t \in (0, T)$. In addition, the locally uniform convergence of M_t to M_0, as $t \to 0$, follows from the bound in Theorem 1.4 as $t^{-1/2}$ is integrable around $t = 0$.

It remains to check the strict convexity of M_{t_0}, for any $t_0 \in (0, T)$. Note that

$$\int_{M_{t_0}} \lambda_1 \ldots \lambda_n \, d\mu = \int_{M_{t_0}} K \, d\mu = \mathcal{H}^n(\nu[M_{t_0}]) = \mathcal{H}^n((\hat{\Gamma}_{t_0})').$$

Here $(\Gamma_{t_0})'$ is the *dual* of the tangent cone of $\hat{M}_{t_0}$ at infinity. On the other hand, $\hat{\Gamma}_{t_0} \subset \hat{\Gamma}^{\epsilon_0}_{t_0}$ implies $(\hat{\Gamma}^{\epsilon_0}_{t_0})' \subset (\hat{\Gamma}_{t_0})'$. $\hat{\Gamma}^{\epsilon_0}_{t_0}$ is compactly contained in an open hemisphere and hence $(\hat{\Gamma}^{\epsilon_0}_{t_0})'$ has nonempty interior. This shows that $\mathcal{H}^n((\hat{\Gamma}_{t_0})') \geq \mathcal{H}^n((\hat{\Gamma}^{\epsilon_0}_{t_0})') > 0$. By Corollary 1.18 in our Appendix, M_t is strictly convex for $t \in (0, t_0)$ and this finishes the proof. $\square$

The following simple observation says that our constructed solution in Theorem 1.2 is the smallest of all solutions with initial data M_0.

Lemma 1.19. *Let $N_t = \partial \hat{N}_t$ be a smooth solution of the IMCF and $M_t = \partial \hat{M}_t$ be a convex solution obtained from Theorem 1.2. If $\hat{M}_0 \subset \hat{N}_0$, then $\hat{M}_t \subset \hat{N}_t$ as long as both solutions exist.*

Proof. Note that the approximating sequence of convex closed hypersurfaces $M_{i,0}$ in Theorem 1.2 was strictly monotone. i.e $\hat{M}_{i,0} \subset\subset \hat{M}_{j,0}$ if $j > i$. This implies $\hat{M}_{i,0} \subset\subset \hat{N}_0$. By classical comparison principle between compact and non-compact solutions, $\hat{M}_{i,t} \subset N_t$ and hence $\hat{M}_t \subset \hat{N}_t$. $\square$

We next show that the comparison principle also holds between a non-compact solution and a conical solution which is inserted inside.

Lemma 1.20. *Let $\Gamma_0 = \partial \hat{\Gamma}_0 \subset \mathbb{S}^n$ be a smooth strictly convex hypersurface in $\mathbb{S}^n$ and Γ_t be the unique solution of the IMCF by Theorem 1.4 in [MS]. Suppose that $N_t := \partial \hat{N}_t$ is a smooth complete non-compact solution of the IMCF which converges to N_0 locally uniformly as $t \to 0$. If $\mathcal{C}\hat{\Gamma}_0 \subset \hat{N}_0$, then $\mathcal{C}\hat{\Gamma}_t \subset \hat{N}_t$ as long as the solution exists.*

Proof. Since $\mathcal{C}\Gamma_0$ is singular at the origin, we first smoothen it inside the ball $B_{1/2}(0)$, creating a smooth hypersurface $M_0 = \partial \hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0$ such that $M_0 = \mathcal{C}\Gamma_0$ outside of $B_{1/2}(0)$. Theorem 1.2 shows the existence of the smallest smooth solution M_t, for $t \in (0, \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|)$ with initial data M_0. For every $\epsilon \in (0,1)$, $\hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0 \subset \epsilon^{-1}\hat{N}_0$ implies $\hat{M}_t \subset \epsilon^{-1}\hat{N}_t$ by Lemma 1.19. i.e. $\epsilon\hat{M}_t \subset \hat{N}_t$. We want to argue that ϵM_t converges to $\mathcal{C}\Gamma_t$, as $\epsilon \to 0$, and conclude that $\mathcal{C}\hat{\Gamma}_t \subset \hat{N}_t$.

From our construction we have $H(|F|+1) \leq C$ for some $C > 0$ on M_0. By Proposition 1.12, $H|F| \leq C$ on M_t for some larger $C > 0$. Next, since $\mathcal{C}\Gamma_t$ works as a conical barrier outside, Theorem 1.4 (Remark 1.6) can be applied to the approximating compact solutions of M_t to conclude that $H|F| \leq C_\delta$ on M_t for $t \in [0, T(M_0) - \delta)$. This implies $M_t \setminus B_1(0)$ satisfies $c_\delta \leq H|F| \leq C$ for $t \in [0, T-\delta]$ and hence the same bound holds for ϵM_t with $\epsilon < 1$. Using these bounds (following a similar argument of the proof of Theorem 1.2) it is easy to pass a smooth blow-down limit $\epsilon \to 0$ outside of $B_1(0)$ and get a solution of IMCF. On the other hand, convexity implies ϵM_t converges to

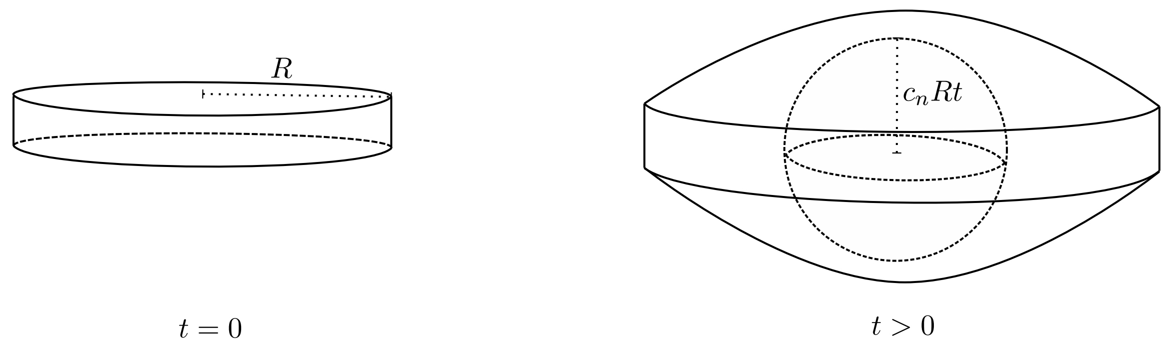

Figure 1.3: Barrier in Lemma 1.21

a cone as $\epsilon \to 0$ at each t and in particular ϵM_0 converges to $\mathcal{C}\Gamma_0$. Since Γ_t is the unique solution of the IMCF from Γ_0, the previous observation implies ϵM_t converges to $\mathcal{C}\Gamma_t$. This proves $\mathcal{C}\hat{\Gamma}_t \subset N_t$. $\square$

We shall show Theorem 1.3, which shows the solution obtained from Theorem 1.2 has the maximal time of existence. We the next barrier lemma which shows that if $\hat{M}_0$ contains a round cylinder $\hat{D}_{R,\epsilon} = B^n_R(0) \times (-\epsilon, \epsilon)$ of radius $R > 0$ and small height $\epsilon \leq R/10$, then $\hat{M}_1$ contains a whole $(n+1)$-ball $B^{n+1}_{c_n Rt}(0)$ of radius $c_n tR$, for some c_n depending only on dimension n.

Lemma 1.21. *Let $\hat{D}_{R,\epsilon} = B^n_R(0) \times (-\epsilon, \epsilon) \subset \mathbb{R}^{n+1}$ be a round cylinder of radius $R > 0$ and small height $\epsilon \in (0, R/10)$. If $\hat{D}_{R,\epsilon} \subset \hat{M}_0$, then there is small $c_n > 0$ such that $B^{n+1}_{c_n Rt}(0) \subset \hat{M}_t$ for $t \in [0, c_n]$.*

Proof. By smoothing the edges of $D_{R,\epsilon}$ (outside the ball $B^{n+1}_{R/2}(0)$) we get a smooth pancake like convex hypersurface $\Sigma_{R,\epsilon}$ which coincides with $D_{R,\epsilon}$ on $B^{n+1}_{R/2}(0)$. We can further assume that $\Sigma_{R,\epsilon}$ has the same symmetry of $D_{R,\epsilon}$, i.e. it has $O(n)$ rotational symmetry and reflection symmetry with respect to $\{x_{n+1} = 0\}$. Then, the IMCF solution $\Sigma_{R,\epsilon}(t)$ starting at $\Sigma_{R,\epsilon}$ has two points $(0, \epsilon + c(t))$ and $(0, -\epsilon - c(t))$ for each $t > 0$ and their normal vectors are e_{n+1} and $-e_{n+1}$, respectively. In view of Lemma 1.12, $c'(t) > cR$ as long as $\epsilon + c(t) < R/2$. Since $\Sigma_{R,\epsilon}(t)$ contains these two points and the disk $B^n_{R/2} \times \{0\}$, the convexity implies that $\hat{M}_t$ includes the desired ball. $\square$

Proof of Theorem 1.3. Suppose there is a smooth solution on $t \in (0, T + \tau]$ for some $\tau > 0$. We will show $\hat{M}_{T+\tau}$ contains $B_R(0)$ for all $R > 0$, which is a contradiction. The same notations in Theorem 1.2 will be used. After a translation, we assume $0 \in \hat{M}_0$.

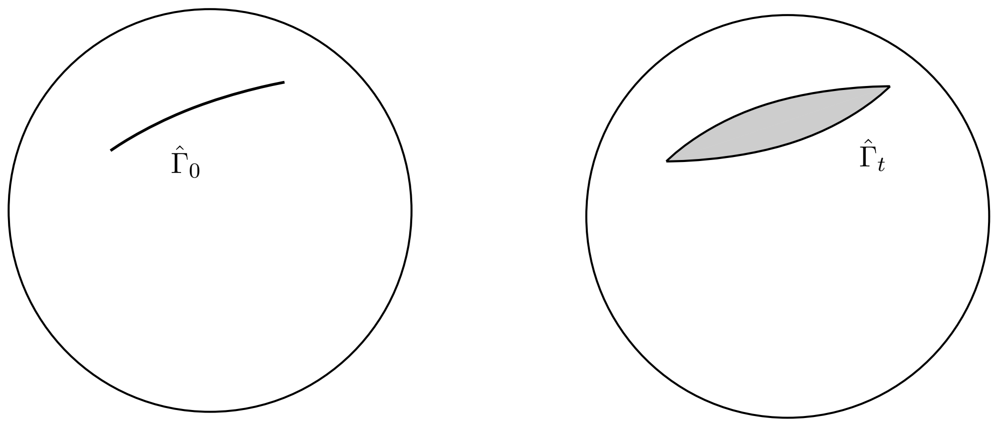

Figure 1.4: Fattening of $\hat{\Gamma}_0$ when it has empty interior

Case 1. Suppose $\hat{\Gamma}_0$ has nonempty interior in $\mathbb{S}^n$.

Using an approximation by the mean curvature flow, we may find a smooth strictly convex hypersurface $\Gamma'_0 \subset\subset \hat{\Gamma}_0$ with $T' = \ln|\mathbb{S}^{n-1}| - \ln|\Gamma'_0| \in (T, T+\tau/2)$. Let $\{\Gamma'_t\}_{t' \in [0,T')}$ be the unique smooth solution of the IMCF in $\mathbb{S}^n$ which converges to an equator as $t \to T'$. By Lemma 1.20, $\hat{M}_t$ contains $\mathcal{C}\hat{\Gamma}'_t$ for $t \in [0, T')$ and thus $\hat{M}_t$ contains a half space at $t = T'$. WLOG let's assume the half space is $\{x_{n+1} \geq 0\}$. For each $r > 0$, $\partial B_{re^{t/n}}(re_{n+1})$ is a solution of the IMCF and is contained in $\hat{M}_{T'+t}$ by avoidance principle. Note $B_{r(e^{t/n}-1)}(0) \subset B_{re^{t/n}}(re_{n+1})$. By choosing $r > 0$ arbitrary large, we get $\hat{M}_{T+\tau}$ contains arbitrary large ball centered at the origin.

Case 2. Suppose $\hat{\Gamma}_0$ has empty interior in $\mathbb{S}^n$.

After splitting out $\mathbb{R}$ factors, we may also assume $\hat{\Gamma}_0$ is compactly contained in an open hemisphere. Let's define $\hat{\Gamma}_t$ for small $t > 0$ to be the tangent cone of $\hat{M}_t$ at infinity. Since $\hat{M}_t$ is monotone, $\hat{\Gamma}_t$ is monotone increasing convex set in $\mathbb{S}^n$. Since convex set in a hemisphere is outer area(perimeter) minimizing, $P(\hat{\Gamma}_t) = |\Gamma_t| \geq 2|\Gamma_0| = P(\hat{\Gamma}_0)$ and it is increasing. If we show $\hat{\Gamma}_t$ has non-empty interior for $t > 0$ and $|\Gamma_t| = P(\hat{\Gamma}_t) \to 2|\Gamma_0| = P(\hat{\Gamma}_0)$ as $t \to 0$. Then Case 1 applied to $\hat{M}_t$ and $\hat{\Gamma}_t$ for sufficiently small $\tau > 0$ and the fact $P(\hat{\Gamma}_t)$ is monotone increasing imply a contradiction.

Note $P(\hat{\Gamma}_0) > 0$ since $T < \infty$. When $\hat{\Gamma}_0$ has empty interior, it can be checked that $\hat{\Gamma}_0 = \Gamma_0$ and is a $n-1$ dimensional convex set in some (totally geodesic) equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ with non-empty interior in $\mathbb{S}^{n-1}$. Let's say $e_{n+1} \in \hat{\Gamma}_0 \subset \mathbb{S}^{n-1}$ and $B^{\mathbb{S}^n}_{2r}(e_{n+1})$, n-dim geodesic ball of radius $2r$ in

$\mathbb{S}^{n-1}$, is contained in $\hat{\Gamma}_0$. For $\hat{M}_0$, this implies that there is a small $e_0 > 0$ such that at each point $(0, he_{n+1}) \in \hat{M}_0$ for $h \geq h_0 > 0$, a thick disk $B^n_{rh} \times (-\epsilon, \epsilon)$ centered at $(0, he_{n+1})$ could be inserted in $\hat{M}_0$ after a rotation. This implies $\hat{M}_t$ has $B^{n+1}_{c_n rht}(0, he_{n+1})$ for small $t > 0$. This proves $\hat{\Gamma}_t$ has a ball centered at e_{n+1} for $t > 0$ and hence has non-empty interior.

□

Remark 1.8. Theorems 1.2 and 1.3 indicate that it is likely true that *the tangent cone* $\Gamma_t \subset \mathbb{S}^n$ *of our solution* M_t *at infinity also evolves by the IMCF in* $\mathbb{S}^n$. Indeed, if one inserts cones which approximate $\mathcal{C}\Gamma_0$ from inside and outside and use the solutions with initial data those cones as barriers, it is not hard to see that the assertion is true when Γ_0 produces a unique classical solution of IMCF on $\mathbb{S}^n$ for $t > 0$. However, for general Lipschitz Γ_0, there might be no classical solution of the IMCF and we can only conclude that Γ_t satisfies the IMCF in some generalized limit sense. In fact, this is how a weak solution is defined in the upcoming work of the author with P.K. Hung in [CH] and it turns out that Γ_t will then be a weak solution of IMCF on $\mathbb{S}^n$ in the sense of [CH].

Chapter 2

Yamabe Flow

2.1 Introduction and preliminaries

In 1989 R. Hamilton introduced the *Yamabe flow*

$$\frac{\partial g}{\partial t} = -R\,g \tag{2.1}$$

as an approach to solve the *Yamabe problem* on manifolds of positive conformal Yamabe invariant. Here, $R(t)$ is the scalar curvature of the metric $g(t)$. Let (M, g_0) be a Riemannian manifold without boundary of dimension $n \geq 3$. When a metric $g = u^{\frac{4}{n+2}}\, g_0$ is conformal to g_0, the scalar curvature R of g is given in terms of the scalar curvature R_0 of g_0 by

$$R = u^{-1}\big(- \bar{c}_n \Delta_{g_0} u^{\frac{n-2}{n+2}} + R_0\, u^{\frac{n-2}{n+2}}\big)$$

where Δ_{g_0} denotes the Laplace Beltrami operator with respect to g_0 and $\bar{c}_n = 4(n-1)/(n-2)$. Therefore, the Yamabe flow can be expressed as a nonlinear diffusion equation of a scalar function u on a fixed manifold (M, g_0).

In the case where M is compact the long time existence and convergence of Yamabe flow is well understood. Hamilton [Ha1] himself showed the existence of the normalized Yamabe flow (which is the re-parametrization of (2.1) to keep the volume fixed) for all time; moreover, in the case when the scalar curvature of the initial metric is negative, he showed the exponential convergence of the flow to a metric of constant scalar curvature. Chow [Ch] showed the convergence of the flow, under the conditions that the initial metric is locally conformally flat and of positive Ricci curvature. The convergence of the flow for any locally conformally flat initially metric was shown by Ye [Ye].

Schwetlick and Struwe [SS] obtained the convergence of the Yamabe flow on a general compact manifold when its Yamabe energy is less than certain threshold assuming a suitable Kazdan-Warner type of condition that rules out the formation of bubbles and that is verified (via the positive mass Theorem) in dimensions $3 \leq n \leq 5$. The convergence result, in its full generality, was established by Brendle [B1] and [B2] (up to a technical assumption, in dimensions $n \geq 6$, on the rate of vanishing of Weyl tensor at the points at which it vanishes): starting with any smooth metric on a compact manifold, the normalized Yamabe flow converges to a metric of constant scalar curvature.

Although the Yamabe flow on compact manifolds is well understood, the complete non-compact case is unsettled. In this case one expects to have more types of singularities which could be either of type I or type II according to the definition below.

Definition 2.1. *Assume that a solution $g(t)$ of the Yamabe flow* (2.1) *on a Riemannian manifold has a singularity at time T. This singularity is called type I if*

$$\limsup_{t \to T^-} \, (T-t) \, \sup_M |Rm|(\cdot, t) < +\infty.$$

A singularity which is not of type I, is called type II.

The results mentioned above show that in the generic *compact* case the only singularities of the Yamabe flow are type I. Moreover the works [DS1; DKS] address the *singularity formation* of complete non-compact solutions to the conformally flat Yamabe flow whose conformal factors have *cylindrical behavior at infinity.* These singularities are all of type I.

A natural question to ask is whether the Yamabe flow *admits any singularities which are of type II in the non-compact case.* The author with P. Daskalopoulos in [CD1] presented, for the first time in the Yamabe flow, examples of complete solutions which develop a type II singularity, either at finite time $T < +\infty$ or at infinite time $T = +\infty$. Such solutions are conformally equivalent to $\mathbb{R}^n$ and their initial data has cylindrical behavior at infinity if $T < \infty$. What distinguishes our type II solutions from the type I solutions which are modeled on shrinkers, is that their initial metric has *slower second order decay rate* to the cylindrical metric than that of any other Yamabe shrinkers. In this work we study complete non-compact and conformally flat solutions of the Yamabe flow (2.1) on $\mathbb{R}^n$ which develop *type II singularity* and provide their *detailed asymptotic behavior* near the singularity.

Let us briefly discuss next the known results on the singularity formation of non-compact Yamabe flow. Even though the analogue of Perelman's monotonicity formula is still lacking for the Yamabe flow, one expects that Yamabe soliton solutions model finite and infinite time singularities. These are special solutions of the Yamabe flow (2.1) which is characterized by a metric $g = g_{ij}$ and a *potential function* P so that

$$(R-\rho)g_{ij} = \nabla_i \nabla_j P, \qquad \rho \in \{1, -1, 0\}.$$

Depending on the sign of the constant ρ, a Yamabe soliton is called a *Yamabe shrinker*, a *Yamabe expander* or a *Yamabe steady soliton* if $\rho = 1, -1$ or 0 respectively.

The classification of locally conformally flat Yamabe solitons with positive sectional curvature was established in [DS2] (c.f. also [CSZ] and [CMM]). It is shown in [DS2] that such solitons are globally conformally equivalent to $\mathbb{R}^n$ and correspond to radially symmetric self-similar solutions of the fast-diffusion equation

$$u_t = \frac{n-1}{m} \Delta\, u^{\frac{n-2}{n+2}}, \quad \text{on } \mathbb{R}^n \times [0, T) \tag{2.2}$$

satisfied by the conformal factor defined by $g_{ij} = u^{\frac{4}{n+2}} \delta_{ij}$. Here and in the sequel δ_{ij} denotes the standard metric on $\mathbb{R}^n$ and we set $m := (n-2)/(n+2)$. A complete description of those solutions is given in [DS2]. In [CSZ] the assumption of positive sectional curvature was relaxed to that of nonnegative Ricci curvature.

As mentioned above, in [DS1; DKS] the *singularity formation* of complete non-compact solutions to the conformally flat Yamabe flow with *cylindrical behavior at infinity* was studied. The singularity profiles are Yamabe shrinking solitons which are determined by the second order asymptotics at infinity of the initial data, which is matched with that of the corresponding soliton. The solutions may become extinct at the extinction time T of the cylindrical tail or may live longer than T. In the first case, the singularity profile is described by a *Yamabe shrinker* that becomes extinct at time T. This result can be seen as a stability result around the Yamabe shrinkers with cylindrical behavior at infinity. In the second case, the flow develops a singularity at time T which is described by a *singular* Yamabe shrinker slightly before T and by a matching *Yamabe expander* slightly after T.

Recently the author with P. Daskalopoulos [CD1] studied long time behavior of the complete non-compact conformally flat Yamabe flow and in particular showed the stability around the steady solitons. Such solitons are conformally equivalent to $\mathbb{R}^n$ and rotationally symmetric. They are the analogue to the bowl translating soliton of MCF or the Bryant soliton of the Ricci flow.

In this work we study the *asymptotic behavior* of type II singularities in the conformally flat non-compact case. More precisely, for a sufficiently small $T < +\infty$, we provide a condition, in terms of the second order decay rate of the initial metric $g_{\gamma,\kappa}(\cdot, 0)$ at spatial infinity, which guarantees that the Yamabe flow $g_{\gamma,\kappa}(\cdot, t)$ develops a type II singularity at time T with specified blow up rate

$$\limsup_{t\to T^-} (T-t)^{1+\gamma} \sup_M |Rm|(\cdot, t) = \kappa. \tag{2.3}$$

Moreover, we prove that after rescaling the solution $g_{\gamma,\kappa}(\cdot, t)$ around highest curvature point by $(T-t)^{-(1+\gamma)}$, it converges to a radial steady gradient soliton.

Our main result states as follows:

Theorem 2.2. *Let $g_0 = u_0^{1-m}(x)\, \delta_{ij}$ be a conformally flat metric with positive Ricci curvature. For any given $\gamma > 0$ and $A > 0$, there is $T_1 > 0$ with the following property: for any $T < T_1$, if*

i) $u_0^{1-m}(x) < \frac{(n-1)(n-2)}{|x|^2}\, T, \ \forall x \in \mathbb{R}^n$, *and*

ii) $u_0^{1-m}(x) = \frac{(n-1)(n-2)}{|x|^2} \left(T - \left(\frac{\ln|x|}{A}\right)^{-\frac{1}{\gamma}} + O\big((\ln|x|)^{-\frac{1}{\gamma}-1}\big) \right)$, *as* $|x| \to +\infty$

then the solution of Yamabe flow (2.1) *with initial data g_0 will develop a type II singularity at time $t = T$ with specified blow up rate given by*

$$\limsup_{t\to T^-} (T-t)^{1+\gamma} \sup_M |\mathrm{Rm}|(\cdot, t) = \frac{2\gamma A}{\sqrt{n(n-1)}}. \tag{2.4}$$

Moreover, after rescaling the metric around the highest curvature point by $(T-t)^{-(1+\gamma)}$, it converges to the unique radial steady gradient soliton of maximum scalar curvature $2\gamma A$.

Theorem 2.2 shows for the first time that the conformally flat radial steady soliton appears as a finite time singularity model for the Yamabe flow. In the Mean curvature flow and the Ricci flow, examples of type II singularities and their asymptotic behavior were shown in both compact and non-compact settings under rotational symmetry (c.f. [AV], [IW] for Mean curvature flow and [AIK2],[AIK1],[Wu] for Ricci flow). Let us remark that unlike in the cases mentioned above our result also includes *non-radial initial data.*

To achieve our result we first construct sharp barriers (super and sub-solutions) for given fixed γ and A. The barriers are chosen sufficiently close to each other so that they give a model solution whose blow up limit at the tip is a radial steady soliton with the curvature blow up rate (2.4). In other words this already proves the result of Theorem 2.2 if the initial metric is in between sharp barriers. When the initial metric satisfies the condition of Theorem 2.2, the solution can be located between two model solutions which are differ by a translation in cylindrical direction. Then we do further analysis to show the solution also has the same asymptotic behavior and curvature blow up rate.

The barriers will be constructed to be radially symmetric though we don't assume it for initial metrics. Note that for a radially symmetric solution $g := u^{\frac{4}{n+2}}(r,t)\,(dr^2 + r^2 dg_{S^{n-1}})$ of the Yamabe flow (2.2), it is often convenient to work in cylindrical coordinates where the metric is expressed as $g = w(s,t)\,(ds^2 + dg_{S^{n-1}})$, with $s = \ln r$. The conformal factor in cylindrical coordinates is given by

$$w(s,t) = r^2\, u^{\frac{4}{n+2}}(r,t), \qquad s = \ln r \tag{2.5}$$

and satisfies the equation

$$\frac{m}{n-1}\left(w^{\frac{n+2}{4}}\right)_t = \left(w^{\frac{n-2}{4}}\right)_{ss} - \left(\frac{n-2}{2}\right)^2 w^{\frac{n-2}{4}}, \qquad \text{on } \mathbb{R}\times[0,T] \tag{2.6}$$

with $m := \dfrac{n-2}{n+2}$.

The *outline of our paper* is as follows: In the Section 2.2 we will begin by giving the *formal matched asymptotics* of the type II singularity at time $t = T$. Based on this analysis, we will introduce in Section 2.3 the two different regions *outer and inner* and the scalings in each region. Also some notation. We will refer to the notation of this section throughout the paper. Section 2.4 deals with the *barrier construction* in the *outer* region (c.f. Propositons 2.6 and 2.7) and Section 2.5 deals with the *barrier construction* in the *inner* region (c.f. Proposition 2.8). Combining the results from Sections 2.4 and 2.5, in Section 2.6 we will glue the barriers in the inner and outer region to construct suitable *super and sub-solutions.* In Section 2.7 we will show one of our main results, Theorem 2.13, which shows the *type II convergence* of any given conformally flat Yamabe flow to the steady soliton, assuming that its initial data is trapped between our super and sub-solutions. Finally, our last section 2.8 will be devoted to the proof of Theorem 2.2. In this section, along with the barriers constructed in previous sections, we will make use of the differential Harnack inequality in [Ch] and the classification result of Yamabe solitons (c.f. [DS2; CMM; CSZ]).

2.2 Formal matched asymptotics

Here we present the formal constructions of model solutions which is based on matched asymptotic analysis. This will give our reader an intuition for our later rigorous construction.

For any given parameters $\gamma > 0$ and $A > 0$ we will construct below a family of formal rotationally symmetric solutions where the curvature blows up in the type II rate

$$\lim_{t \to T^-} (T-t)^{1+\gamma} \sup_{M_t} |\mathrm{Rm}| = \frac{2\gamma A}{\sqrt{n(n-1)}}. \tag{2.7}$$

Note that our main results Theorem 2.12 and 2.13 are not restricted on rotationally symmetric solutions, however the barriers which will be constructed in next sections are obtained from perturbations of this formal solutions which are radial.

Motivated by condition (2.7) and in order to capture at the end a stationary solution, we perform the change of variables on our solution $w(s,t)$ of (2.6) setting

$$\hat{w}(\eta,\tau) = (T-t)^{-1} w(s,t), \quad \eta = (T-t)^{\gamma} s, \;\; \tau = -\ln(T-t). \tag{2.8}$$

A direct calculation shows that $\hat{w}(\eta,\tau)$ satisfies the evolution

$$B[\hat{w}] = 0 \tag{2.9}$$

where

$$\begin{aligned} B[\hat{w}] := \partial_\tau \hat{w} - (n-1)e^{-2\gamma\tau}\left(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\frac{\hat{w}_\eta^2}{\hat{w}^2}\right) \\ - \big(\gamma\eta\,\hat{w}_\eta + \hat{w} - (n-1)(n-2)\big). \end{aligned} \tag{2.10}$$

Thus, if we assume that

$$\partial_\tau \hat{w} \qquad \text{and} \qquad e^{-2\gamma\tau}\left(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\frac{\hat{w}_\eta^2}{\hat{w}^2}\right) \tag{2.11}$$

are negligible as $\tau \to \infty$, the above equation is reduced to the following ODE in η variable

$$\gamma\eta\,\hat{w}_\eta + \hat{w} - (n-1)(n-2) = 0. \tag{2.12}$$

Solving this equation on $\eta > 0$ gives the solution $\hat{w}_0(\eta)$ given by

$$\hat{w}_0(\eta) := (n-1)(n-2)\big(1 - \kappa\,\eta^{-1/\gamma}\big) \tag{2.13}$$

for a parameter $\kappa \in \mathbb{R}$. We will assume from now on, without loss of generality, that $\kappa > 0$, although $\kappa \leq 0$ also gives a solution. Moreover, there is a family of solutions to (2.12) on $\eta < 0$, but this case is exactly symmetric to the $\eta > 0$ case which we will handle below. Indeed, the solutions given by (2.13) describe non-compact surfaces moving in positive η (hence positive s) direction, while the corresponding solutions defined on $\eta < 0$ describe a symmetric surface just moving on the opposite direction. This ansatz, namely setting

$$\hat{w}(\eta,\tau) := (n-1)(n-2)\left(1 - \kappa\,\eta^{-\frac{1}{\gamma}}\right) \quad \text{on} \quad \eta > \kappa^{\gamma} \tag{2.14}$$

approximates a solution of the equation (2.9). In fact, plugging $\hat{w}(\eta,\tau)$ given by (2.14) into (2.9), we see that the error term becomes

$$B[\hat{w}] = -(n-1)e^{-2\gamma\tau}\left(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\frac{\hat{w}_\eta^2}{\hat{w}^2}\right). \tag{2.15}$$

Writing $B[\hat{w}] = (n-1)(\hat{w}e^{\gamma\tau})^{-2}\big(\hat{w}\hat{w}_{\eta\eta} + \frac{n-6}{4}\hat{w}_\eta^2\big)$, we see that it becomes arbitrarily small in the space-time region

$$(e^{\gamma\tau}\,\hat{w})^{-1} = o(1), \qquad \text{as } \tau \to +\infty$$

which we call the *outer region* (see Figure 2.1 below). This is the region where the diffusion doesn't play an important role and the remaining advection and reaction terms are dominant.

The *inner-region* is given by

$$e^{\gamma\tau}\,\hat{w}(\eta,\tau) = O(1), \qquad \text{as } \tau \to +\infty \tag{2.16}$$

(see Figure 2.1). In this region we perform another scaling, setting

$$\bar{w}(\xi,\tau) = e^{\gamma\tau}\hat{w}(\eta,\tau), \qquad \eta = A + e^{-\gamma\tau}\xi \tag{2.17}$$

for some choice of $A > 0$, which combined with (2.8) gives

$$\bar{w}(\xi,\tau) = e^{\gamma\tau}\,\hat{w}(A + e^{-\gamma\tau}\xi,\tau) = e^{(1+\gamma)\tau}\,w(Ae^{\gamma\tau} + \xi, T - e^{-\tau}). \tag{2.18}$$

The evolution equation for $\bar{w}(\xi,\tau)$is given by

$$\begin{aligned} I[\bar{w}] :=& e^{-\gamma\tau}\Big(\bar{w}_\tau - (1+\gamma)\bar{w}\Big) - (n-1)\Big(\frac{\bar{w}_{\xi\xi}}{\bar{w}} + \frac{n-6}{4}\frac{\bar{w}_\xi^2}{\bar{w}^2}\Big) \\ &+ (n-1)(n-2) - \gamma A \bar{w}_\xi = 0 \end{aligned}. \tag{2.19}$$

Thus assuming that in this region the first term having $e^{-\gamma\tau}$ becomes negligible as $\tau \to \infty$, the equation is reduced to

$$(n-1)\Big(\frac{\bar{w}_{\xi\xi}}{\bar{w}} + \frac{n-6}{4}\frac{\bar{w}_\xi^2}{\bar{w}^2}\Big) - (n-1)(n-2) + \gamma A \bar{w}_\xi = 0\cdot \tag{2.20}$$

This can be seen as the equation satisfied by a traveling wave solution of (2.6) with speed γA. For each $A > 0$, this equation admits a τ independent entire solution $\bar{w}(\xi)$ which is unique up to translation in ξ. For such profile $\bar{w}$, the function $\bar{w}(s - \gamma A t)$ becomes a traveling wave solution of equation (2.6). From the geometric point of view, $\bar{w}(s)$ corresponds to a radially symmetric non-compact metric on $\mathbb{R}^n$ via (2.5) and solutions with different A's are the only radially symmetric *steady gradient solitons* on locally conformally flat manifolds (c.f. [CSZ; DS2]). For such a solution $\bar{w}$, the highest curvature point is at the origin (i.e. $s = -\infty$) and one may formally compute that $|\operatorname{Rm}|_{\max}(t) = \dfrac{2\gamma A}{\sqrt{n(n-1)}}$, hence leading to (2.7).

Remark 2.1. Note that a solution $\bar{w}(\xi,\tau)$ of (2.20) which also depends on τ, can be written as

$$\bar{w}(\xi,\tau) = \bar{w}_0(\xi + C(\tau)) \tag{2.21}$$

for a function $C(\tau)$, where $\bar{w}_0$ is one $\tau-$independent solution of (2.20). By plugging this into (2.19) again, we get an error term

$$e^{-\gamma\tau}\Big((1+\gamma)\bar{w} - C'(\tau)\,\bar{w}_\xi\Big) \approx 0. \tag{2.22}$$

We may choose $C(\tau)$ so that C' is small and thus the error term above vanishes appropriately as $\tau \to \infty$. Later, we will use this $C(\tau)$ to glue barriers from the two different regions, inner and outer.

Next, we will carry out a matching asymptotic analysis between the inner and outer regions and obtain a relation between $\kappa > 0$ in (2.13) and $A > 0$ in (2.19). It is known that a solution $\bar{W}(\xi)$ of (2.20) satisfies the asymptotic behavior

$$\bar{W}(\xi) \approx \frac{(n-1)(n-2)}{\gamma A}\xi + O(1), \qquad \text{as } \xi \to \infty.$$

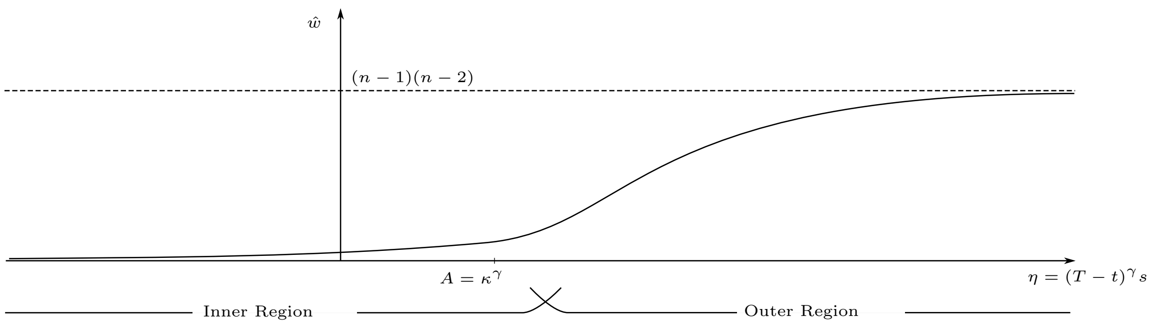

Figure 2.1: two regions and matched asymptotics

Therefore, recalling that $\eta = A + e^{-\gamma\tau}\,\xi$, our solution $\bar{w}(\xi,\tau)$ which is approximately $\bar{W}(\xi)$ satisfies

$$e^{-\gamma\tau}\,\bar{w}(\xi,\tau) \approx \frac{(n-1)(n-2)}{\gamma A}(\eta - A) + o(1), \quad (\eta - A)\,e^{\gamma\tau} \gg 1,\ \tau \gg 1.$$

On the other hand, from the outer region ansatz (2.13), by the first order Taylor approximation near $\eta = \kappa^\gamma$ we have

$$\begin{aligned}\hat{w}(\eta,\tau) &\approx (n-1)(n-2)\,\frac{\kappa}{\gamma}\,(\kappa^\gamma)^{-\frac{1}{\gamma}-1}(\eta - \kappa^\gamma) + o(1)\\ &\approx \frac{(n-1)(n-2)}{\gamma\kappa^\gamma}(\eta - \kappa^\gamma) + o(1), \qquad \text{for } \eta \text{ near } \kappa^\gamma,\ \tau \gg 1.\end{aligned}$$

Thus, we can see that these two asymptotics are matched exactly if

$$A = \kappa^\gamma.$$

To see this in another way, we can argue that if $A < \kappa^\gamma$ the two asymptotics are inconsistent as $\eta \to (\kappa^\gamma)_+$ and if $\kappa^\gamma < A$ the linearization of (2.13) near $\eta = A$ is inconsistent with the asymptotic behavior from the inner region.

2.3 Notation and different scalings

In this section we will summarize the coordinates and different scalings of our solutions, as introduced in the previous section. We will refer to the notation introduced below throughout the paper.

Coordinate systems

Since our metric $g(t)$ is conformally equivalent to the standard euclidean metric on $\mathbb{R}^n$ denoted briefly by δ_{ij}, we represent

$$g(t) = u^{1-m}(x,t)\,\delta_{ij}$$

where the scalar function $u(x,t)$ evolves by equation (2.2) under Yamabe flow.

In case when the metric is radial, it is often convenient to work in the cylindrical coordinates, that is

$$g(t) = u^{1-m}(r,t)(dr^2 + r^2 g_{S^{n-1}}) = w(s,t)(ds^2 + g_{S^{n-1}}), \quad r = |x|$$

and

$$w(s,t) = r^2 u^{1-m}(r,x), \qquad s = \ln r = \ln |x|.$$

Under this coordinate change, $w(s,t)$ evolves by equation (2.6).

Scalings

We use the following different scaling in different regions:

- In the *outer region*, the conformal factor is represented by $\hat{w}(\eta,\tau)$ and is scaled from $w(s,t)$ as follows

$$\hat{w}(\eta,\tau) = e^{\tau} w(e^{\gamma\tau}\eta, T - e^{-\tau}).$$

  The function $\hat{w}(\eta,\tau)$ evolves by the equation (2.9).

- In the *inner region*, the conformal factor is represented by $\bar{w}(\xi,\tau)$ and is scaled from previous factors as follows

$$\bar{w}(\xi,\tau) = e^{\gamma\tau}\hat{w}(A + \xi e^{-\gamma\tau}, \tau) = e^{(1+\gamma)\tau} w(Ae^{\gamma\tau} + \xi, T - e^{-\tau}).$$

  The function $\bar{w}(\xi,\tau)$ evolves by the equation (2.19).

- The above scaling change from $w(s,t)$ to $\bar{w}(\xi,\tau)$ corresponds to the following scaling change in euclidean coordinates from $u^{1-m}(x,t)$ to $\bar{u}^{1-m}(y,l)$:

$$|x|^2\, u^{1-m}(x,t) = (T-t)^{1+\gamma}\, |y|^2\, \bar{u}^{1-m}(y,l) \tag{2.23}$$

  where l is a new time variable $l = (T-t)^{-\gamma}/\gamma$ and

$$x = y\, e^{\gamma A l} = y\, e^{A(T-t)^{-\gamma}}. \tag{2.24}$$

This scaling is used only in Section 2.7. $\bar{u}(y,l)$ evolves by equation (2.65).

Relations

Let us summarize relations between different variables and functions appearing in the scalings introduced above.

- We use *three time scales*:

$$t \in [0,T), \quad \tau = -\ln(T-t), \quad l = \frac{(T-t)^{-\gamma}}{\gamma} = \frac{e^{\gamma\tau}}{\gamma}. \tag{2.25}$$

  The last one will only be used in the last section.

- The *three space scales* in cylindrical coordinates are:

$$s \in \mathbb{R}, \quad \eta := e^{-\gamma\tau} s, \quad \xi = s - A\, e^{\gamma\tau} = (\eta - A)\, e^{\gamma\tau}. \tag{2.26}$$

- The corresponding *three conformal factors* in cylindrical coordinates at the different scales are:

$$w(s,t) = e^{-\gamma\tau}\, \hat{w}(\eta,\tau) = e^{-(1+\gamma)\tau}\, \bar{w}(\xi,\tau). \tag{2.27}$$

In the *radial case*, $w(\xi,\tau)$ and $\bar{u}^{1-m}(y,l)$ given by (2.23) are related by

$$\bar{w}(\xi,\tau) = |y|^2\, \bar{u}(y,l), \qquad \xi = \ln|y|, \qquad l = \frac{e^{\gamma\tau}}{\gamma}. \tag{2.28}$$

Functions

We introduce below the functions $\hat{w}_0$, $\bar{w}_0$, $\bar{U}$, which play important roles in the paper.

- For every $A > 0$, we define $\hat{w}_0(\eta)$ to be the *outer region ansatz*

$$\hat{w}_0(\eta) := (n-1)(n-2)\left[1 - \left(\frac{\eta}{A}\right)^{-\frac{1}{\gamma}}\right] \text{ on } \eta > A, \tag{2.29}$$

  which is a solution of (2.12).

- For the same $A > 0$ as above, we denote by $\bar{w}_0(\xi)$ the *inner region approximation* which the solution of equation

$$(n-1)\Big(\frac{\bar{w}_{\xi\xi}}{\bar{w}} + \frac{n-6}{4}\frac{\bar{w}_\xi^2}{\bar{w}^2}\Big) - (n-1)(n-2) + \gamma A\, \bar{w}_\xi = 0. \tag{2.30}$$

  having asymptotic behavior

$$\bar{w}_0(\xi) = \frac{(n-1)(n-2)}{\gamma A}\xi + 0 + \frac{(n-1)(n-6)}{4\gamma A}\frac{1}{\xi} + O\Big(\frac{1}{\xi^2}\Big). \tag{2.31}$$

  This solution represents a steady gradient soliton of the flow and it is unique up to translation in ξ.

- Finally $\bar{U}$ denotes the representation of $\bar{w}_0$ on $\mathbb{R}^n$ by the following change of coordinate

$$|x|^2\, \bar{U}^{1-m}(x) = \bar{w}_0(\ln|x|). \tag{2.32}$$

2.4 Barrier construction in the outer region

Let us fix parameters $\gamma > 0$ and $A > 0$ as they appear in the curvature blow up rate of our singularity (2.7). In this section we will construct appropriate super and sub solutions in the *outer region* $(e^{\gamma\tau}\, \hat{w}(\eta,\tau))^{-1} = o(1), \qquad \text{as } \tau \to +\infty$, which will be given by

$$\{(\eta,\tau)\,|\,\eta \geq A + \xi_0\, e^{-\gamma\tau},\ \tau \geq \tau_0\}$$

for some $\xi_0 > 0$.

Recall that for a rotationally symmetric solution $u(r,t)$, $r = |x|$ of the conformally flat Yamabe flow (2.2), we perform the cylindrical change of coordinates (2.5) leading to a solution $w(s,t)$ of (2.6). As already seen in section 2.2, to capture the behavior in the outer region we perform a further change of variables (2.8) leading to a solution $\hat{w}(\eta,\tau)$ of (2.9). Assuming that near our singularity (2.11) holds, we find that the zero order behavior of $\hat{w}$ near the singularity in the outer region is given by a solution of the ODE (2.12). The general solution of (2.12) is given by (2.13) for a parameter $\kappa > 0$. Thus, setting $A := \kappa^{1/\gamma} > 0$ we define the zero order approximation of $\hat{w}$ in the outer region to be

$$\hat{w}_0(\eta) = (n-1)(n-2)\Big[1 - \Big(\frac{\eta}{A}\Big)^{-1/\gamma}\Big] \qquad \text{on } \eta > A. \tag{2.33}$$

In this section, we are going to construct sub and super solutions of equation (2.9) in the form

$$\hat{w}(\eta,\tau) = \hat{w}_0(\eta) + e^{-2\gamma\tau}(\hat{w}_1(\eta) + \theta\, \hat{w}_2(\eta)) \tag{2.34}$$

for a parameter $\theta \in \mathbb{R}$. To this end, we will choose $\hat{w}_1(\eta)$ and $\hat{w}_2(\eta)$ to be solutions of a first order linear ODE with different inhomogeneous terms. By setting

$$f_1(\eta) := -(n-1)\,\frac{(\hat{w}_0)_{\eta\eta}}{\hat{w}_0} \quad \text{and} \quad f_2(\eta) := -(n-1)\,\frac{(\hat{w}_0)^2_\eta}{\hat{w}_0^2}, \tag{2.35}$$

we will choose w_1, w_2 to be solutions of the equations

$$\begin{aligned} \gamma\eta\,(\hat{w}_1)_\eta + (1+2\gamma)\,\hat{w}_1 &= f_1(\eta) \qquad \text{on } \eta > A \\ \gamma\eta\,(\hat{w}_2)_\eta + (1+2\gamma)\,\hat{w}_2 &= f_2(\eta) \qquad \text{on } \eta > A. \end{aligned} \tag{2.36}$$

Plugging $\hat{w}(\eta,\tau)$ given by (2.34) into the equation gives that

$$\frac{e^{2\gamma\tau}}{n-1} B[\hat{w}] = \Big(\frac{(\hat{w}_0)_{\eta\eta}}{\hat{w}_0} + \theta\,\frac{(\hat{w}_0)^2_\eta}{\hat{w}_0^2}\Big) - \Big(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\,\frac{\hat{w}^2_\eta}{\hat{w}^2}\Big) \tag{2.37}$$

where $B[\cdot]$ is given by (2.10) Thus, from the proposed choice of $\hat{w}_1, \hat{w}_2$ to satisfy (2.36), one expects that $\hat{w}$ is a subsolution of equation (2.9) if $\theta < \frac{n-6}{4}$ and a supersolution if $\theta > \frac{n-6}{4}$, for all parameters $\gamma > 0$. The rest of this section is devoted to the justification of this idea which requires a rather delicate calculation. The case of parameters $\gamma \geq 1/2$ is shown in Proposition 2.6 below. As we will see in the proof Proposition 2.7 below, in the case of parameters $\gamma < 1/2$ one needs to add correction term to $\hat{w}$.

Recalling the definition of $\hat{w}_0$ in (2.33) and f_1, f_2 in (2.35) we have

$$\begin{aligned} f_1(\eta) &= +(n-1)\,\frac{\gamma+1}{\gamma^2}\,\frac{\eta^{-\frac{1}{\gamma}-2}}{A^{-\frac{1}{\gamma}} - \eta^{-\frac{1}{\gamma}}} > 0 \\ f_2(\eta) &= -(n-1)\,\frac{1}{\gamma^2}\,\frac{\eta^{-\frac{2}{\gamma}-2}}{(A^{-\frac{1}{\gamma}} - \eta^{-\frac{1}{\gamma}})^2} < 0 \end{aligned} \tag{2.38}$$

hence the explicit solutions $w_i, i = 1,2$ of (2.36) are given by

$$\hat{w}_i(\eta) = \frac{\eta_0^{2+\frac{1}{\gamma}}\,\hat{w}_i(\eta_0)}{\eta^{2+\frac{1}{\gamma}}} + \frac{1}{\eta^{2+\frac{1}{\gamma}}}\int_{\eta_0}^{\eta} \frac{f_i(x)}{\gamma} x^{1+\frac{1}{\gamma}}\,dx, \quad \eta > A. \tag{2.39}$$

We will now fix the functions $\hat{w}_1$ and $\hat{w}_2$ by fixing their values at a given point $\eta_0 > A$. While doing so, we will choose $\hat{w}_2$ to be a positive function on $\eta > A$. Indeed, since $f_2(\eta)\,\eta^{1+\frac{1}{\gamma}}$ is integrable near $\eta = \infty$, we may choose

$$\hat{w}_2(\eta) := -\frac{1}{\eta^{2+\frac{1}{\gamma}}} \int_{\eta}^{+\infty} \frac{f_2(x)}{\gamma} x^{1+\frac{1}{\gamma}}\, dx > 0$$

that is we choose $\hat{w}_2(\eta_0) := -\gamma^{-1}\, \eta_0^{-(2+\frac{1}{\gamma})} \displaystyle\int_{\eta_0}^{+\infty} f_2(x)\, x^{1+\frac{1}{\gamma}} dx$ in (2.39). For the function $\hat{w}_1$ we may choose any value as $\hat{w}_1(\eta_0)$, since we do not need to choose it to be positive. Note that by choosing $\hat{w}_2$ to be positive, the family of functions $\hat{w}_1 + \theta\, \hat{w}_2$ is monotone in $\theta \in \mathbb{R}$. To simplify the notation we will simply set from now on

$$\hat{h} := \hat{w}_1 + \theta\, \hat{w}_2. \tag{2.40}$$

We will investigate the behavior of the family $\hat{h}(\eta) := \big(\hat{w}_1 + \theta \hat{w}_2\big)(\eta)$, $\theta \in \mathbb{R}$, as $\eta \to A^+$ and $\eta \to +\infty$. We will first see that the behavior near $\eta = A^+$ is governed by $\hat{w}_2$.

Lemma 2.3. *For any linear combination $\hat{h} := \hat{w}_1 + \theta \hat{w}_2$ of the solutions $\hat{w}_1, \hat{w}_2$ chosen above we have, we have*

$$\begin{aligned}
\hat{h}(\eta) &= +\frac{\theta}{\gamma A} \frac{n-1}{(\eta - A)} + o\Big(\frac{1}{(\eta - A)}\Big) \\
\hat{h}'(\eta) &= -\frac{\theta}{\gamma A} \frac{n-1}{(\eta - A)^2} + o\Big(\frac{1}{(\eta - A)^2}\Big) \\
\hat{h}''(\eta) &= +\frac{2\theta}{\gamma A} \frac{n-1}{(\eta - A)^3} + o\Big(\frac{1}{(\eta - A)^3}\Big)
\end{aligned}$$

as $\eta \to A^+$.

Proof. By (2.39) we have

$$\hat{h}(\eta) = \frac{\eta_0^{2+\frac{1}{\gamma}} \big(\hat{w}_1 + \theta \hat{w}_2\big)(\eta_0)}{\eta^{2+\frac{1}{\gamma}}} + \frac{1}{\eta^{2+\frac{1}{\gamma}}} \int_{\eta_0}^{\eta} \frac{\big(f_1 + \theta f_2\big)(x)}{\gamma} x^{1+\frac{1}{\gamma}} dx. \tag{2.41}$$

Now, by Taylor's theorem, (2.38) and derivatives of these equations imply the following behavior as $\eta \to A^+$

$$(f_1 + \theta f_2)(\eta) = -\frac{\theta(n-1)}{(\gamma - A)^2} + O((\eta - A)^{-1})$$

and

$$(f_1 + \theta f_2)'(\eta) = 2\frac{\theta(n-1)}{(\gamma - A)^3} + O((\eta - A)^{-2}).$$

In particular we see from the above that $\hat{w}_2$ dominates as $\eta \to A^+$ and by L'Hôpital's rule on $(\eta - A)\int_{\eta_0}^{\eta} \eta^{-1}\big(f_1 + \theta f_2\big)(x)x^{1+\frac{1}{\gamma}}dx$ we obtain

$$\lim_{\eta \to A+} (\eta - A)\hat{h}(\eta) = \frac{(n-1)\theta}{\gamma A}.$$

Similarly, taking derivatives in (2.41) and using the asymptotics for $(f_1 + \theta f_2)$ and $(f_1 + \theta f_2)'$, we obtain

$$\lim_{\eta \to A+} (\eta - A)^2 \hat{h}'(\eta) = -\frac{(n-1)\theta}{\gamma A}, \quad \lim_{\eta \to A+} (\eta - A)^3 \hat{h}''(\eta) = \frac{2(n-1)\theta}{\gamma A}.$$

□

Remark 2.2. Lemma 2.3 shows that $\hat{w}(\eta, \tau)$ defined by (2.34), which is the ansatz for super and sub solutions, for $\theta > \frac{n-6}{4}$ and $\theta < \frac{n-6}{4}$ respectively, predicts the correct lower order asymptotic behavior in the outer region which is then matched with the inner region where a steady soliton comes in. Indeed, in our previous work [CD1] we showed that a steady soliton of (2.2) satisfies the asymptotics

$$\bar{w}(\xi) = \frac{(n-1)(n-2)}{\gamma A}\xi + \kappa + \frac{(n-1)(n-6)}{4\gamma A}\frac{1}{\xi} + o\Big(\frac{1}{\xi}\Big). \tag{2.42}$$

On the other hand, Lemma 2.3 yields

$$\hat{w}(\eta, \tau) = \frac{(n-1)(n-2)}{\gamma A}\big(\eta - A\big) + e^{-2\gamma\tau}\,\frac{(n-1)\theta}{\gamma A}\,\frac{1}{\eta - A} + O\big((\eta - A)^2\big).$$

Hence, if we match the inner-outer variables by setting $\eta = A + \xi e^{-\gamma\tau}$, we obtain

$$e^{\gamma\tau}\hat{w}(\eta, \tau) = \frac{(n-1)(n-2)}{\gamma A}\,\xi + \frac{(n-1)\theta}{\gamma A}\,\frac{1}{\xi} + O\Big((\xi^2 + 1)e^{-\gamma\tau}\Big).$$

Also, this suggests that in the inner region $\bar{w}$ should be $\bar{w}_0$, the translating soliton which satisfies asymptotics (2.42) with $\kappa = 0$.

We will next see that behavior of $\big(\hat{w}_1 + \theta\hat{w}_2\big)(\eta)$ as $\eta \to \infty$ is governed by $\hat{w}_1$.

Lemma 2.4. *For any linear combination $\hat{h} := \hat{w}_1 + \theta\,\hat{w}_2$ of the solutions $\hat{w}_1, \hat{w}_2$ chosen above we have, we have*

$$\begin{aligned}
\hat{h}(\eta) \;&= \frac{(n-1)(1+\gamma)}{\gamma^3}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-2}\ln\eta + o(\eta^{-\frac{1}{\gamma}-2}\ln\eta) \\
\hat{h}'(\eta) &= -\frac{(n-1)(1+\gamma)(1+2\gamma)}{\gamma^4}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-3}\ln\eta + o(\eta^{-\frac{1}{\gamma}-3}\ln\eta) \\
\hat{h}''(\eta) &= \frac{(n-1)(1+\gamma)(1+2\gamma)(1+3\gamma)}{\gamma^5}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-4}\ln\eta + o(\eta^{-\frac{1}{\gamma}-4}\ln\eta)
\end{aligned}$$

as $\eta \to \infty$.

Proof. The proof is similar as in Lemma 2.3 if we check the asymptotics

$$f_1(\eta) = \frac{(n-1)(1+\gamma)}{\gamma^2}\left(A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-2} + A^{\frac{2}{\gamma}}x^{-\frac{2}{\gamma}-2}\right) + o(x^{-\frac{2}{\gamma}-2})$$

$$f_2(\eta) = -\frac{(n-1)}{\gamma^2}\left(A^{\frac{2}{\gamma}}\eta^{-\frac{2}{\gamma}-2} + 2A^{\frac{3}{\gamma}}x^{-\frac{3}{\gamma}-2}\right) + o(\eta^{-\frac{3}{\gamma}-2})$$

and corresponding asymptotics for f_1' and f_2' as $\eta \to \infty$.

□

In the next lemma we give more precise asymptotics which will be used later when we have to add higher order terms to barrier in the case $0 < \gamma \leq \frac{1}{2}$. Notice that $\eta^{-\frac{1}{\gamma}-2}$ is a solution of the homogenous equation of (2.36) and we have constants C in the lemma below since we haven't chosen a specific $\hat{w}_1$.

Lemma 2.5. *For any linear combination $\hat{h} := \hat{w}_1 + \theta\hat{w}_2$ of the solutions $\hat{w}_1, \hat{w}_2$ chosen above, we have*

$$\begin{aligned}
\hat{h}(\eta) = &+\frac{(n-1)(1+\gamma)}{\gamma^3}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-2}\ln\eta + C\eta^{-\frac{1}{\gamma}-2} \\
&- (n-1)\frac{(n-1)(1+\theta+\gamma)}{\gamma^2}A^{\frac{2}{\gamma}}\gamma^{-\frac{2}{\gamma}-2} + o(\eta^{-\frac{2}{\gamma}-2}) \\
\hat{h}'(\eta) = &-\frac{(n-1)(1+\gamma)(1+2\gamma)}{\gamma^4}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-3}\ln\eta \\
&+ C'\eta^{-\frac{1}{\gamma}-3} + o(\eta^{-\frac{2}{\gamma}-2}) \\
\hat{h}''(\eta) = &+\frac{(n-1)(1+\gamma)(1+2\gamma)(1+3\gamma)}{\gamma^5}A^{\frac{1}{\gamma}}\eta^{-\frac{1}{\gamma}-4}\ln\eta \\
&+ C''\eta^{-\frac{1}{\gamma}-4} + o(\eta^{-\frac{2}{\gamma}-2})
\end{aligned}$$

as $\eta \to +\infty$. Here, C, C', and C'' are constants depending on the choice of $\hat{w}_1$.

Proof. Can be shown in the same manner as in Lemma 2.4.

□

We will now show that $\hat{w}(\eta,\tau)$ given by (2.34) is a sub or super - solution of equation (2.9) in the appropriate regions. We will first deal with the case of parameters $\gamma > 1/2$. The case $\gamma \leq 1/2$ is more delicate and will be considered later.

Proposition 2.6. *For any $\gamma > 1/2$ and given $\theta \neq \frac{n-6}{4}$, there exist $\tau_0 \in \mathbb{R}$ and $\xi_0 > 0$ depending on n, A, γ and θ such that the function*

$$\hat{w}(\eta,\tau) := \hat{w}_0(\eta) + e^{-2\gamma\tau}(\hat{w}_1(\eta) + \theta\hat{w}_2(\eta))$$

is a subsolution of equation (2.9) *if $\theta < \frac{n-6}{4}$ or a supersolution if $\theta > \frac{n-6}{4}$, respectively, in the region*

$$\{(\eta,\tau)\,|\,\eta \geq A + \xi_0\, e^{-\gamma\tau},\ \tau \geq \tau_0\}.$$

Proof of Proposition 2.6. As before, let us denote by

$$\hat{h} := \hat{w}_1 + \theta\hat{w}_2.$$

We need to show that

$$B[\hat{w}] < 0, \quad \text{if } \theta < \frac{n-6}{4} \quad \text{ or } \quad B[\hat{w}] > 0, \quad \text{if } \theta > \frac{n-6}{4} \tag{2.43}$$

holds in the region

Proposition 2.6 follows from the the two claims below

Claim 2.1. *For any $\gamma > 0$ and given $\theta \neq \frac{n-6}{4}$, there exist $\xi_0 > 0$ and $\delta > 0$ such that $\bar{w}(\eta,\tau)$ is a subsolution of equation* (2.9) *if $\theta < \frac{n-6}{4}$ or a supersolution if $\theta > \frac{n-6}{4}$, respectively, in the region*

$$\{(\eta,\tau)\,|\,A + \xi_0\, e^{-\gamma\tau} \leq \eta < A + \delta,\ \tau \in \mathbb{R}\,\}.$$

Proof of Claim 2.1. By Lemma 2.3, we may find $\kappa = \kappa(n, A, \gamma) > 0$ such that

$$|\hat{h}|(\eta - A),\ |\hat{h}'|(\eta - A)^2,\ |\hat{h}''|(\eta - A)^3 < \kappa\,|\theta| \tag{2.44}$$

holds on the region $A < \eta < A + 1$. Moreover, by Taylor's theorem we may choose the constant κ so that

$$\left|\hat{w}_0(\eta) - \frac{(n-1)(n-2)}{\gamma A}(\eta - A)\right| < \kappa\,(\eta - A)^2$$

and

$$\left|\hat{w}_0'(\eta) - \frac{(n-1)(n-2)}{\gamma A}\right| < \kappa\,(\eta - A) \quad \text{and} \quad \left|\hat{w}_0''(\eta)\right| < \kappa$$

hold. Using these, we get

$$\begin{aligned}\Big|\hat{w} - \frac{(n-1)(n-2)}{\gamma A}(\eta - A)\Big| &= \Big|\hat{w}_0 - \frac{(n-1)(n-2)}{\gamma A}(\eta - A) + e^{-2\gamma\tau}\hat{h}\Big| \\ &\leq \kappa\,(\eta - A)^2 + \kappa\,|\theta|\,e^{-2\gamma\tau}\frac{1}{(\eta - A)} \\ &= \Big(\kappa(\eta - A) + \frac{\kappa\,|\theta|}{((\eta - A)e^{\gamma\tau})^2}\Big)\,(\eta - A) \\ &\leq \kappa\,\Big(\delta + \frac{|\theta|}{\xi_0^2}\Big)\,(\eta - A).\end{aligned}$$

Hence by restricting to $0 < \delta < 1$ small and $\xi_0 > 0$ large, we may assume that

$$\hat{w} > \frac{(n-1)(n-2)}{2\gamma A}\,(\eta - A) > 0.$$

Using the above we find that in the considered region we have

$$\begin{aligned}\Big|\frac{\hat{w}_{\eta\eta}}{\hat{w}}\Big| \leq \frac{|\hat{w}_0''| + |e^{-2\gamma\tau}\hat{h}''|}{\hat{w}} &\leq \frac{1}{(\eta - A)}\frac{2\gamma A}{(n-1)(n-2)}\big(\kappa + \frac{|\theta|\kappa}{(\eta - A)^3 e^{2\gamma\tau}}\big) \\ &\leq \frac{\kappa}{(\eta - A)^2}\frac{2\gamma A}{(n-1)(n-2)}\Big(\delta + \frac{|\theta|}{\xi_0^2}\Big)\end{aligned}$$

and

$$\begin{aligned}\Big|\frac{\hat{w}_\eta^2}{\hat{w}^2}\Big| &= \frac{|\hat{w}_0'|^2|1 + e^{-2\gamma\tau}\hat{h}'/\hat{w}_0'|^2}{\hat{w}^2} \\ &\leq \frac{1}{(\eta - A)^2}\Big|\frac{\hat{w}_0'^2}{\hat{w}^2\,(\eta - A)^{-2}}\Big(1 + \frac{|\theta|\kappa}{(\eta - A)^2 e^{2\gamma\tau}\hat{w}_0'}\Big)^2\Big| \\ &\leq \frac{1}{(\eta - A)^2}\Big(\frac{\frac{(n-1)(n-2)}{\gamma A} + \kappa\delta}{\frac{(n-1)(n-2)}{\gamma A} - \kappa\big(\delta + \frac{|\theta|}{\xi_0^2}\big)}\Big)^2\Big(1 + \frac{|\theta|\kappa}{\xi_0^2\big(\frac{(n-1)(n-2)}{\gamma A} - \kappa\delta\big)}\Big)^2.\end{aligned}$$

Similarly, we estimate from below

$$\Big|\frac{\hat{w}_\eta^2}{\hat{w}^2}\Big| \geq \frac{1}{(\eta - A)^2}\Big(\frac{\frac{(n-1)(n-2)}{\gamma A} - \kappa\delta}{\frac{(n-1)(n-2)}{\gamma A} + \kappa\big(\delta + \frac{|\theta|}{\xi_0^2}\big)}\Big)\Big(1 - \frac{|\theta|\kappa}{\xi_0^2\,\big(\frac{(n-1)(n-2)}{\gamma A} - \kappa\delta\big)}\Big)^2.$$

Thus, for a fixed $\epsilon > 0$ to be determined later, we may find small $\delta > 0$ and large $\xi_0 > 0$ so that all

$$\Big|\frac{\hat{w}_\eta^2}{\hat{w}^2} - \frac{1}{(\eta - A)^2}\Big|,\ \Big|\frac{\hat{w}_0'^2}{\hat{w}_0^2} - \frac{1}{(\eta - A)^2}\Big|,\ \Big|\frac{\hat{w}_{\eta\eta}}{\hat{w}}\Big|,\ \Big|\frac{\hat{w}_0''}{\hat{w}_0}\Big| \leq \frac{\epsilon}{(\eta - A)^2}.$$

Recalling the formula for $B[\hat{w}]$ in (2.37) and using the triangle inequality successively we obtain

$$\Big|\frac{e^{2\gamma\tau}B[\hat{w}]}{(n-1)} - \Big(\theta - \frac{n-6}{4}\Big)\frac{1}{(\eta - A)^2}\Big| \leq \epsilon\frac{|\theta| + \Big|\frac{n-6}{4}\Big| + 2}{(\eta - A)^2}.$$

Finally, by choosing $\epsilon := \frac{1}{2}\big|\theta - \frac{n-6}{4}\big|\big(|\theta| + \big|\frac{n-6}{4}\big| + 2\big)^{-1}$ we conclude that (2.1) holds, finishing the proof of the claim.

$\square$

We will next proceed in our next claim which holds for $\gamma > 1/2$.

Claim 2.2. *For $\gamma > 1/2$ and given $\theta \neq \frac{n-6}{4}$ and $\delta > 0$, there is $\tau_0 = \tau_0(\theta, \delta, \gamma)$ such that such that $\hat{w}(\eta, \tau)$ is a subsolution of equation (2.9) if $\theta < \frac{n-6}{4}$ or a supersolution if $\theta > \frac{n-6}{4}$ on the set*

$$\{(\eta, \tau)\,|\,\eta > A + \delta,\ \tau \geq \tau_0\}.$$

Proof of Claim 2.2. For a given $\delta_0 > 0$, Lemma 2.4 and the asymptotic behavior of $\hat{w}_0$, $\hat{w}_0'$ and $\hat{w}_0''$ can be used to find a constant $\kappa > 0$ such that

$$\Big|\frac{\hat{h}}{w_0}\Big|, \Big|\frac{\hat{h}'}{w_0'}\Big|, \Big|\frac{\hat{h}''}{w_0''}\Big| < \kappa\,\frac{\ln(1+\eta)}{\eta^2} < \frac{\kappa}{A+\delta_0} \qquad \text{on } \ \eta > A + \delta_0.$$

Thus, we may start with some large τ_0 such that

$$\hat{w},\ \hat{w}_\eta,\ \hat{w}_{\eta\eta} \text{ have same the sign as } \hat{w}_0,\ \hat{w}_0',\ \hat{w}_0'' \text{ respectively}$$

on $\eta > A + \delta$ and $\tau > \tau_0$. In particular, they are nonzero. Using the formula for $B[\hat{w}]$ in (2.37), we write

$$\frac{e^{2\gamma\tau}}{(n-1)}\,B[\hat{w}] = \Big(\frac{\hat{w}_0''}{\hat{w}_0} - \frac{\hat{w}_{\eta\eta}}{\hat{w}}\Big) + \frac{n-6}{4}\Big(\frac{\hat{w}_0'^2}{\hat{w}^2} - \frac{\hat{w}_\eta^2}{\hat{w}^2}\Big) + \big(\theta - \frac{n-6}{4}\big)\frac{\hat{w}_0'^2}{\hat{w}_0^2}.$$

We will show that the first two terms become arbitrarily small in comparison with the last term, for $\tau_0 \gg 1$ (and hence $\tau \gg 1$). Indeed, we have

$$\Big|\frac{\hat{w}_{\eta\eta}}{\hat{w}} - \frac{\hat{w}_0''}{\hat{w}_0}\Big| = e^{-2\gamma\tau}\Big|\frac{\hat{w}_0''}{\hat{w}_0}\Big|\,\Big|\frac{(\hat{h}''/\hat{w}_0'') - (\hat{h}/\hat{w}_0)}{1 + e^{-2\gamma\tau}(\hat{h}/\hat{w}_0)}\Big|$$

and

$$\Big|\frac{\hat{w}_\eta^2}{\hat{w}^2} - \frac{\hat{w}_0'^2}{\hat{w}_0^2}\Big| = e^{-2\gamma\tau}\,\frac{\hat{w}_0'^2}{\hat{w}_0^2}\,\frac{\big|2((\hat{h}'/\hat{w}_0') - (\hat{h}/\hat{w}_0)) + e^{-2\gamma\tau}((\hat{h}'/\hat{w}_0')^2 - (\hat{h}/\hat{w}_0)^2)\big|}{(1 + e^{-2\gamma\tau}(\hat{h}/\hat{w}_0))^2}.$$

Thus, the asymptotics in Lemma 2.5 imply that by choosing $\tau_0 \gg 1$ we have

$$\begin{aligned} \Big|\frac{\hat{w}_{\eta\eta}}{\hat{w}} - \frac{\hat{w}_0''}{\hat{w}_0}\Big| &\leq 10\,e^{-2\gamma\tau}\,\kappa\,\Big|\frac{\hat{w}_0''}{\hat{w}_0}\Big|\,\frac{\ln(1+\eta)}{\eta^2} \\ \Big|\frac{\hat{w}_\eta^2}{\hat{w}^2} - \frac{\hat{w}_0'^2}{\hat{w}_0^2}\Big| &\leq 10\,e^{-2\gamma\tau}\,\kappa\,\Big(\frac{\hat{w}_0'^2}{\hat{w}_0^2}\Big)\,\frac{\ln(1+\eta)}{\eta^2}. \end{aligned} \tag{2.45}$$

for all $\tau > \tau_0$. In the case $\gamma > \frac{1}{2}$, by $\frac{\ln \eta}{\eta^2}$, $e^{-2\gamma\tau}$ terms in the previous estimate and asymptotics (2.4), we can make τ_0 large and conclude

$$\Big|\Big(\frac{\hat{w}_0''}{\hat{w}_0} - \frac{\hat{w}_{\eta\eta}}{\hat{w}}\Big) + \frac{n-6}{4}\Big(\frac{\hat{w}_0'^2}{\hat{w}^2} - \frac{\hat{w}_\eta^2}{\hat{w}^2}\Big)\Big| \le \frac{1}{2}\,\Big|\theta - \frac{n-6}{4}\Big|\frac{\hat{w}_0'^2}{\hat{w}_0^2} \tag{2.46}$$

on the considered region. This proves that for $\tau \ge \tau_0 \gg 1$, (2.43) holds, finishing the proof of the claim. $\square$

To finish the proof of the proposition, for any given $\gamma > 1/2$ and $\theta \ne \frac{n-6}{4}$, Claim 2.1 implies that there exists $\delta > 0$ such that such that (2.43) holds in the region $A + \xi_0\, e^{-\gamma\tau} \le \eta < A + \delta$. In addition, by Claim 2.2, there exists $\tau_0 = \tau_0(\theta, \delta, \gamma)$ such that (2.43) holds in the region $\eta > A + \delta$, $\tau \ge \tau_0$. We conclude that (2.43) holds in the whole outer region $\eta > A + \xi_0\, e^{-\gamma\tau}$ for $\tau \ge \tau_0$ finishing the proof of the proposition.

$\square$

In the case $0 < \gamma \le 1/2$, we need to add a higher order correction term in our barrier. For integers $k \ge 2$ and $0 \le l \le k$, we define the functions

$$v_{k,l}(\eta) := \eta^{-2k-\frac{1}{\gamma}}\,(\ln \eta)^l, \qquad \eta > 1. \tag{2.47}$$

They satisfy the following relation

$$(1 + 2k\gamma)\, v_{k,l} + \gamma\eta\, v_{k,l}' = \begin{cases} \gamma l\, v_{k,l-1} & \text{if } l > 0 \\ 0 & \text{if } l = 0 \end{cases} \tag{2.48}$$

and to simplify the notation we also set $v_{k,-1}(\eta) = 0$ and $v_{k,-2}(\eta) = 0$. We will show the following.

Proposition 2.7. *For any $0 < \gamma \le 1/2$ and given $\theta \ne \frac{n-6}{4}$, there exist $\tau_0 \in \mathbb{R}$, $\xi_0 > 0$, integer $N \ge 2$ and coefficients $\{c_{k,l}\}_{2\le k\le N,\, 0\le l\le k}$ such that the function*

$$\hat{w}(\eta, \tau) := \hat{w}_0(\eta) + e^{-2\gamma\tau}(\hat{w}_1(\eta) + \theta\hat{w}_2(\eta)) + \Sigma_{k=2}^N e^{-2k\gamma\tau}\Sigma_{l=0}^k c_{k,l}\, v_{k,l}(\eta)$$

is a subsolution of equation (2.9) if $\theta < \frac{n-6}{4}$ or a supersolution if $\theta > \frac{n-6}{4}$, in the region

$$\{(\eta, \tau)\,|\, \eta \ge A + \xi_0\, e^{-\gamma\tau},\ \tau \ge \tau_0\}.$$

Proof of Proposition 2.7. For the given $0 < \gamma \le 1/2$, let N denote the smallest integer making $\gamma > 1/(2N)$, namely $N := [1/(2\gamma)] + 1$. The next claim corresponds to Claim 2.2 for this case.

Claim 2.3. *For any $0 < \gamma \leq 1/2$ and any given choice of $\{c_{k,0}\}_{2\leq k\leq N}$, there are coefficients $\{c_{k,l}\}_{2\leq k\leq N, 0\leq l\leq k}$ so that for any given $\delta > 0$ and $\theta \neq \frac{n-6}{4}$, the function*

$$\hat{w}(\eta,\tau) = \hat{w}_0(\eta) + e^{-2\gamma\tau}(\hat{w}_1(\eta) + \theta\,\hat{w}_2(\eta)) + \Sigma_{k=2}^{N} e^{-2k\gamma\tau}\Sigma_{l=0}^{k} c_{k,l}\, v_{k,l}(\eta)$$

is a subsolution of equation (2.9) if $\theta < \frac{n-6}{4}$ or a supersolution if $\theta > \frac{n-6}{4}$ on the set

$$\{(\eta,\tau) \mid \eta > A+\delta,\ \tau \geq \tau_0\}$$

where $\tau_0 = \tau_0(\gamma,\theta,\delta) \gg 1$.

Proof of Claim 2.3. Let us assume $\theta < \frac{n-6}{4}$ because the other case follows similarly. Suppose $\hat{w}$ is of the form of (2.3). We will choose the coefficients $c_{k,l}$ later.

We split the operator $B[\cdot]$ given by (2.10) into linear and nonlinear parts, that is we write

$$B[\hat{w}] = I_1[\hat{w}] + (n-1)(n-2) - I_2[\hat{w}]. \tag{2.49}$$

where

$$I_1[\hat{w}] := \partial_\tau \hat{w} - \gamma\eta\,\hat{w}_\eta - \hat{w}, \quad I_2[\hat{w}] := (n-1)\Big(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\frac{\hat{w}_\eta^2}{\hat{w}^2}\Big)e^{-2\gamma\tau}.$$

Then, using (2.35), (2.36) and (2.48) we find

$$\begin{aligned}
I_1[\hat{w}] =& I_1[\hat{w}_0] + I_1[\hat{h}\,e^{-2\gamma\tau}] + \Sigma_{k=2}^{N}\Sigma_{l=0}^{k} c_{k,l}\, I_1[v_{k,l}\,e^{-2k\gamma\tau}] \\
=& -(n-1)(n-2) + (n-1)\Big(\frac{(\hat{w}_0)_{\eta\eta}}{\hat{w}_0} + \theta\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}\Big)e^{-2\gamma\tau} \\
& - \Sigma_{k=2}^{N} e^{-2k\gamma\tau}\Sigma_{l=1}^{k}\gamma l\, c_{k,l}\, v_{k,l-1}.
\end{aligned}$$

Meanwhile, using the asymptotics of $\hat{w}_0$, $\hat{w}_1 + \theta\hat{w}_2$, $v_{k,l}$ and their derivatives

$$\begin{aligned}
I_2[\hat{w}] =& (n-1)\Big(\frac{\hat{w}_{\eta\eta}}{\hat{w}} + \frac{n-6}{4}\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2} + o(\eta^{-2-\frac{2}{\gamma}})e^{-2\gamma\tau}\Big)e^{-2\gamma\tau} \\
=& (n-1)\Big(\frac{\hat{w}_{\eta\eta}}{\hat{w}_0} + \frac{n-6}{4}\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2} + o(\eta^{-2-\frac{2}{\gamma}})e^{-2\gamma\tau}\Big)e^{-2\gamma\tau} \\
=& (n-1)\Big(\frac{(\hat{w}_0)_{\eta\eta}}{\hat{w}_0} + \frac{n-6}{4}\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}\Big)e^{-2\gamma\tau} \\
& + (n-1)\,\frac{\hat{h}_{\eta\eta}e^{-4\gamma\tau} + \Sigma_{k=2}^{N} e^{-2(k+1)\gamma\tau}\,\Sigma_{l=0}^{k} c_{k,l}(v_{k,l})_{\eta\eta}}{\hat{w}_0} + o(\eta^{-2-\frac{2}{\gamma}})e^{-4\gamma\tau}. \\
=& (n-1)\Big(\frac{(\hat{w}_0)_{\eta\eta}}{\hat{w}_0} + \frac{n-6}{4}\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}\Big)e^{-2\gamma\tau} \\
& + \frac{\hat{h}_{\eta\eta}e^{-4\gamma\tau} + \Sigma_{k=2}^{N-1} e^{-2(k+1)\gamma\tau}\,\Sigma_{l=0}^{k} c_{k,l}(v_{k,l})_{\eta\eta}}{(n-2)} + o(\eta^{-2-\frac{2}{\gamma}})\,e^{-4\gamma\tau}
\end{aligned}$$

In the last line we used that $2N > 1/\gamma$. Also, $g(\eta,\tau) = o(\eta^{-2-\frac{2}{\gamma}})$ means that $\sup_{\eta>\eta',\tau>\tau'} \eta^{2+\frac{2}{\gamma}} g(\eta,\tau) \to 0$, as $\eta' \to \infty$ for any fixed τ'.

Combining the above computations yields

$$\begin{aligned}
B[\hat{w}] =&(n-1)\Big(\theta - \frac{n-6}{4}\Big)\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}e^{-2\gamma\tau} - \Sigma_{k=2}^N e^{-2k\gamma\tau}\,\Sigma_{l=1}^k \gamma l\, c_{k,l}\, v_{k,l-1} \\
&- \frac{\hat{h}_{\eta\eta}e^{-4\gamma\tau} + \Sigma_{k=2}^{N-1} e^{-2(k+1)\gamma\tau}\,\Sigma_{l=0}^k c_{k,l}\,(v_{k,l})_{\eta\eta}}{(n-2)} + o(\eta^{-2-\frac{2}{\gamma}})e^{-4\gamma\tau} \\
=&(n-1)\Big(\theta - \frac{n-6}{4}\Big)\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}e^{-2\gamma\tau} - \Big(\frac{h_{\eta\eta}}{n-2} + \Sigma_{l=1}^2 \gamma l\, c_{2,l} v_{2,l-1}\Big)e^{-4\gamma\tau} \\
&- \Sigma_{k=3}^N\Big(\frac{1}{n-2}\Sigma_{l=1}^k c_{k-1,l-1}(v_{k-1,l-1})_{\eta\eta} + \Sigma_{l=1}^k \gamma l\, c_{k,l} v_{k,l-1}\Big)\, e^{-2k\gamma\tau} \\
&+ o(\eta^{-2-\frac{2}{\gamma}})e^{-4\gamma\tau}.
\end{aligned}$$

Let us remark that $(v_{k-1,l-1})_{\eta\eta}$ can be written as a linear combination of $\{v_{k,l-1}, v_{k,l-2}, v_{k,l-3}\}$. Hence for any given $\{c_{k,0}\}_{2\le k\le N}$, there is a unique choice $\{c_{k,j}\}_{2\le k\le N,\ 1\le l\le k}$ such that

$$B[\hat{w}] = (n-1)\Big(\theta - \frac{n-6}{4}\Big)\frac{(\hat{w}_0)_\eta^2}{\hat{w}_0^2}e^{-2\gamma\tau} + o(\eta^{-2-\frac{2}{\gamma}})e^{-4\gamma\tau}.$$

Here we also used the asymptotic expansion of $h_{\eta\eta}$ as $\eta \to \infty$, namely

$$h_{\eta\eta} = (n-1)\, A^{\frac{1}{\gamma}}\,\frac{(1+\gamma)(1+2\gamma)(1+3\gamma)}{\gamma^5} v_{2,1} + C'' v_{2,0} + o(\eta^{-2-\frac{2}{\gamma}})$$

which has been shown in Lemma 2.5. Finally, we may find a large $\tau_0 = \tau_0(\delta,\gamma,\theta)$ such that $B[\hat{w}] < 0$ on the region $\eta > A + \delta$ for $\tau \ge \tau_0$. This finishes the proof of our claim.

$\square$

As we fixed $\hat{w}_1$ and $\hat{w}_2$ in the proof of Proposition 2.6, from now on let us fix $c_{k,l}$ so that Claim 2.3 holds, by choosing $c_{k,0} = 0$. Next, we give the analogue of Claim 2.1 in this case.

Claim 2.4. *For given $0 < \gamma \le 1/2$, $\theta \ne \frac{n-6}{4}$, there exist $\xi_0 > 0$ and $\delta > 0$ such that*

$$\hat{w}(\eta,\tau) = \hat{w}_0(\eta) + e^{-2\gamma\tau}(\hat{w}_1(\eta) + \theta\hat{w}_2(\eta)) + \Sigma_{k=2}^N e^{-2k\gamma\tau}\,\Sigma_{l=0}^k c_{k,l}\, v_{k,l}(\eta)$$

is is a subsolution of equation (2.9) if $\theta < \frac{n-6}{4}$ and a supersolution if $\theta > \frac{n-6}{4}$ on the region

$$\{(\eta,\tau)\mid A + \xi_0 e^{-\gamma\tau} < \eta < A + \delta,\ \tau > 0\}.$$

Proof of Claim 2.4. By rewriting $\hat{w}(\eta,\tau) = \hat{w}_0(\eta) + e^{-2\gamma\tau}\,\hat{h}(\eta,\tau)$, we have the same estimate of Proposition 2.3 and the proof is actually the same as of Claim 2.1. $\square$

The proof of the Proposition 2.7 now readily follows by combining claims 2.3 and 2.4. Let us fix $0 < \gamma \leq 1/2$ and $\theta \neq \frac{n-6}{4}$. Let $c_{k,l}$ be coefficients with $c_{k,0} = 0$ be so that Claim 2.3 holds. For that choice of $c_{k,l}$, Claim 2.4 gives the existence of $\xi_0 > 0$ and $\delta > 0$ so that $\hat{w}$ is a subsolution (supersolution) in the region $A + \xi_0 e^{-\gamma\tau} < \eta < A + \delta,\ \tau > 0$. By Claim 2.3 there exists $\tau_0 = \tau_0(\gamma, \theta, \delta)$ such that $\hat{w}$ is a subsolution (supersolution) in the region $\eta > A + \delta$, $\tau \geq \tau_0$. We conclude that $\hat{w}$ is a subsolution (supersolution) in the region $\eta > A + \xi_0 e^{-\gamma\tau}$, $\tau \geq \tau_0$. Since $\delta = \delta(\gamma, \theta)$ we also have that $\tau_0 = \tau_0(\gamma, \theta)$.

□

2.5 Barrier construction in the inner region

We will now construct the appropriate barrier in the *inner region* which is the region where

$$e^{\gamma\tau}\,\hat{w}(\eta,\tau) = O(1), \qquad \text{as } \tau \to +\infty.$$

In this region we define $\bar{w}(\xi,\tau)$ as in (2.17), that is we set $\bar{w}(\xi,\tau) = e^{\gamma\tau}\,\hat{w}(\eta,\tau)$, $\xi = (\eta - A)\,e^{\gamma\tau}$. We have seen in section 2.2 that $\bar{w}(\xi,\tau)$ satisfies the equation $I[\bar{w}] = 0$ with $I[\cdot]$ given by (2.19). Let us assume that in this region the first term in (2.19) having $e^{-\gamma\tau}$ becomes negligible as $\tau \to \infty$. Then, we expect that the solution $\bar{w}_0(\xi)$ of equation

$$\frac{(\bar{w}_0)_{\xi\xi}}{\bar{w}} + \frac{n-6}{4}\frac{(\bar{w}_0)_\xi^2}{\bar{w}^2} - (n-1)(n-2) + \gamma A\,(\bar{w}_0)_\xi = 0$$

is the leading order term for $\bar{w}(\xi,\tau)$ in this region. We are going to find super and sub solutions $\bar{w}^+$ and $\bar{w}^-$, respectively in the following form

$$\bar{w}^+(\xi,\tau) = \frac{1}{1+\epsilon}\bar{w}_0(\xi + C_1(\tau))$$
$$\bar{w}^-(\xi,\tau) = \frac{1}{1-\epsilon}\bar{w}_0(\xi + C_2(\tau)).$$

Here, $\epsilon > 0$ is a small constant and $C_1(\tau)$ and $C_2(\tau)$ are smooth functions of τ. Both will be chosen later and will depend on ξ_0 which appears in the construction of our barriers in the outer region. As we will see below, the construction is rather straightforward.

If we plug these into $I[\cdot]$, we get

$$I[\bar{w}^+] = +\epsilon\bar{w}^+_\xi + e^{-\gamma\tau}(C_1'(\tau)\,\bar{w}^+_\xi - (1+\gamma)\,\bar{w}^+)$$
$$I[\bar{w}^-] = -\epsilon\bar{w}^-_\xi + e^{-\gamma\tau}(C_2'(\tau)\,\bar{w}^+_\xi - (1+\gamma)\,\bar{w}^-).$$

We will next show the following.

Proposition 2.8. *Let $0 < \epsilon < 1$ and $\tau_0 \in \mathbb{R}$. If $|C_1'(\tau)|, |C_2'(\tau)| \leq M$ on $\tau \geq \tau_0$, then ξ_1 there exist $\tau_1 = \tau_1(\epsilon, M, \xi_1) \geq \tau_0$ such that $\bar{w}^+$ or $\bar{w}^-$ are super or sub solutions of equation (2.15) respectively, in the region $(\xi, \tau) \in (-\infty, \xi_1) \times (\tau_1, \infty)$.*

Proof. For any two functions $f(s), g(s)$ we use the notation

$$f(s) \sim g(s), \text{ as } s \to \infty \qquad \text{iff} \qquad c < \left|\frac{f(s)}{g(s)}\right| < C, \text{ for } s \gg 1$$

for some fixed constants $c > 0$, $C < +\infty$.

In this proof we will use the asymptotics for $\bar{w}_0(s)$ and $\bar{w}_0'(s)$, as $s \to \infty$, which were shown in Proposition 2.1 in [CD1] or [DS2; Hsu]. Since $\bar{w}_0(s) = e^{2s}\, \bar{U}^{1-m}(e^s)$ and $\bar{U}^{1-m}(|x|)\,\delta_{ij}$ is a smooth radial metric on $\mathbb{R}^n$, we have

$$\bar{w}_0 \sim (\bar{w}_0)_s \sim e^{2s}, \qquad \text{as } s \to \infty.$$

Moreover since $\bar{w}_0 \sim s$, $(\bar{w}_0)_s \sim 1$, as $s \to \infty$, it is clear that there is some τ_2 and c so that

$$e^{-\gamma\tau}\,(1+\gamma)\,\bar{w}_0(s) < \frac{\epsilon}{2}(\bar{w}_0)_s(s), \quad (s,\tau) \in (-\infty, c\,e^{\gamma\tau}) \times (\tau_2, \infty).$$

Now given ξ_1 and $C_1(\tau)$ and $C_2(\tau)$ satisfying the conditions in our proposition, we can find some $\tau_1 > \max(\tau_0, \tau_2)$ such that

$$\xi_1 + C_i(\tau) < c\,e^{\gamma\tau} \quad \text{and} \qquad |e^{-\gamma\tau} C_i'(\tau)| < \frac{\epsilon}{2}, \qquad \text{for } \tau > \tau_1. \tag{2.50}$$

The last two formulas and the fact that $w_0 > 0$, $(w_0)_s > 0$, imply that $I[\bar{w}^+] > 0$ and $I[\bar{w}^-] < 0$ on $(\xi, \tau) \in (-\infty, \xi_1) \times (\tau_1, \infty)$, as claimed. □

2.6 Construction of super and sub-solutions

In this section we will combine the results from Sections 2.4 and 2.5 to construct a family of super-solutions w_ϵ^+ and sub-solutions w_ϵ^- of equation (2.6) which is equivalent to the conformally flat Yamabe flow (2.2) under rotational symmetry and after the cylindrical change of variables (2.5). This will give a family of rotationally symmetric super and sub solutions of equation (2.2) which we will then be used in the next section to analyze the type II blow up behavior of any solution $u(x,t)$ of (2.2) which satisfies the assumptions of Theorem 2.12.

We begin by fixing

$$\gamma > 0, \qquad A > 0, \qquad \theta^+ > \frac{n-6}{4}, \qquad \theta^- < \frac{n-6}{4}.$$

For these choices of parameters and following the results in Section 2.4, we define the super and sub-solutions $\hat{w}^+$ and $\hat{w}^-$ *corresponding to* θ^+ *and* θ^- *respectively* in the *outer region* $\eta > A + \xi_0\, e^{-\gamma\tau}$ separately for different ranges of γ: for $\gamma > 1/2$ we set

$$\hat{w}^{\pm}(\eta,\tau) := \hat{w}_0(\eta) + e^{-2\gamma\tau}\left(\hat{w}_1(\eta) + \theta^{\pm}\,\hat{w}_2(\eta)\right)$$

while for $0 < \gamma \leq 1/2$, we add the extra correction term setting

$$\hat{w}^{\pm}(\eta,\tau) := \hat{w}_0(\eta) + e^{-2\gamma\tau}\left(\hat{w}_1(\eta) + \theta^{\pm}\hat{w}_2(\eta)\right) + \Sigma_{k=2}^{N}\Sigma_{l=0}^{k} c_{k,l} e^{-2k\gamma\tau} v_{k,l}(\eta).$$

Propositions 2.6 and 2.7, show that there exist τ_0 and $\xi_0 > 0$, such that $\hat{w}^+$ and $\hat{w}^-$ are super and sub solutions, respectively on the region $(\eta,\tau) \in (A + \xi_0 e^{-\gamma\tau}, \infty) \times [\tau_0, \infty)$. Also, following the results in the previous section 2.5, we define the prospective super and sub-solutions $\bar{w}_\epsilon^+$ and $\bar{w}_\epsilon^-$ in the *inner region* by setting

$$\begin{aligned} \bar{w}_\epsilon^+(\xi,\tau) &:= \frac{1}{1+\epsilon}\,\bar{w}_0(\xi + C_1(\tau)) \\ \bar{w}_\epsilon^-(\xi,\tau) &:= \frac{1}{1-\epsilon}\,\bar{w}_0(\xi + C_2(\tau)). \end{aligned} \tag{2.51}$$

The small constant $\epsilon \in [0,1)$ will be chosen later. Also, for some fixed ξ_1 to be determined later, let $C_1(\tau)$, $C_2(\tau)$ be smooth functions defined on $\tau \geq \tau_0$ such that

$$e^{\gamma\tau}\,\hat{w}^{\pm}(A + \xi_1\, e^{-\gamma\tau}, \tau) = \bar{w}_\epsilon^{\pm}(\xi_1, \tau). \tag{2.52}$$

Note that the functions $C_i(\tau)$ uniquely exist and are smooth because $\bar{w}_0(\cdot)$ is strictly increasing smooth function onto $(0,\infty)$ and $\hat{w}^{\pm}(A + \xi_1 e^{-\gamma\tau}, \tau)$ are positive smooth functions on $\tau \geq \tau_0$. Moreover, since $\hat{w}_2 > 0$ and $\theta^+ > \theta^-$, we have $e^{\gamma\tau}\hat{w}^+(A + \xi_1 e^{-\gamma\tau}, \tau) > e^{\gamma\tau}\hat{w}^-(A + \xi_1 e^{-\gamma\tau}, \tau)$. Therefore (2.52) and the definition of $w_\epsilon^{\pm}$ imply that $\bar{w}_0(\xi_1 + C_1(\tau)) > \bar{w}_0(\xi_1 + C_2(\tau))$. Using again that $\bar{w}_0(\cdot)$ is a strictly increasing we conclude that

$$C_1(\tau) > C_2(\tau), \qquad \tau \geq \tau_0 \tag{2.53}$$

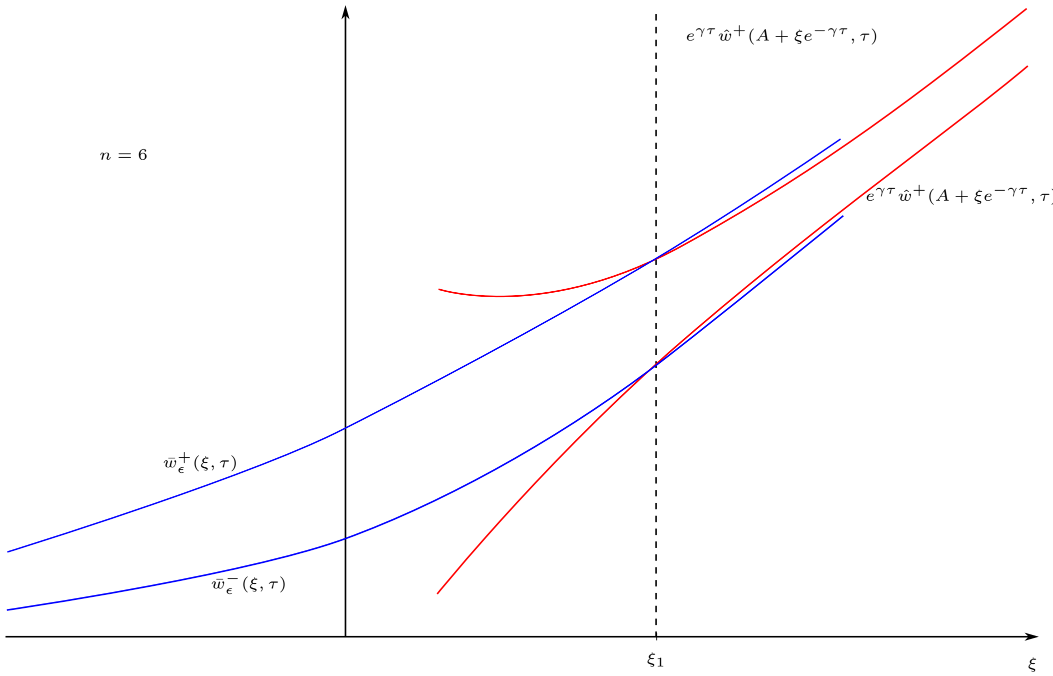

Figure 2.2: super and sub solutions

which will be used later.

It follows from the above discussion that for $\tau \geq \tau_0$, we can glue the functions $e^{\gamma\tau}\,\hat{w}^{\pm}(A+\xi e^{-\gamma\tau},\tau)$ and $\bar{w}^{\pm}_{\epsilon}(\xi,\tau)$ at $\xi=\xi_1$ to form a continuous and piecewise smooth function, namely we define

$$
\begin{aligned}
w^{+}_{\epsilon}(\xi,\tau) &:= \begin{cases} \bar{w}^{+}_{\epsilon}(\xi,\tau) & \text{if } \xi \leq \xi_1 \\ e^{\gamma\tau}\hat{w}^{+}(A+\xi e^{-\gamma\tau},\tau) & \text{if } \xi > \xi_1 \end{cases} \\
w^{-}_{\epsilon}(\xi,\tau) &:= \begin{cases} \bar{w}^{-}_{\epsilon}(\xi,\tau) & \text{if } \xi \leq \xi_1 \\ e^{\gamma\tau}\hat{w}^{-}(A+\xi e^{-\gamma\tau},\tau) & \text{if } \xi > \xi_1 \end{cases}
\end{aligned} \tag{2.54}
$$

(see Figure 2.2.)

We will show next that the functions $w^{+}_{\epsilon}(\xi,\tau)$ and $w^{-}_{\epsilon}(\xi,\tau)$ have the following properties:

Proposition 2.9. *There exist $\xi_1>0$ and $\epsilon_1>0$ such that for any $0<\epsilon<\epsilon_1$ there is a $\tau_1=\tau_1(\epsilon)$ for which the functions w^{+}_{ϵ} and w^{-}_{ϵ} given by (2.54) with $0<\epsilon<\epsilon_1$, have following properties:*

(i) $w^{+}_{\epsilon}(\xi,\tau) > w^{-}_{\epsilon}(\xi,\tau) > 0$ *on* $(-\infty,\infty)\times[\tau_1,\infty)$*;*

(ii) $w^{+}_{\epsilon}(\xi,\tau)$ *and* $w^{-}_{\epsilon}(\xi,\tau)$ *are continuous on* $(-\infty,\infty)\times[\tau_1,\infty)$ *and smooth for* $\xi\neq\xi_1$*;*

(iii) *for all* (ξ,τ) *with* $\xi \neq \xi_1$ *and* $\tau \geq \tau_1$, *they satisfy* $I[w_\epsilon^+] > 0$ *and* $I[w_\epsilon^-] < 0$, *i.e. they are super and sub-solutions, respectively;*

(iv) *at the non-smooth points* (ξ_1,τ), $\tau \geq \tau_1$, *they satisfy*

$$\lim_{\xi\to\xi_1-} \frac{\partial}{\partial\xi} w_\epsilon^+(\xi,\tau) > \lim_{\xi\to\xi_1+} \frac{\partial}{\partial\xi} w_\epsilon^+(\xi,\tau)$$

$$\lim_{\xi\to\xi_1-} \frac{\partial}{\partial\xi} w_\epsilon^-(\xi,\tau) < \lim_{\xi\to\xi_1+} \frac{\partial}{\partial\xi} w_\epsilon^-(\xi,\tau).$$

For the proof of the proposition we will need the next two lemmas.

Lemma 2.10. *For any fixed* $\xi_1 > \xi_0$, *we have*

$$e^{\gamma\tau}\,\hat{w}^\pm(A+\xi_1 e^{-\gamma\tau},\tau) \to \frac{(n-1)(n-2)}{A\gamma}\xi_1 + \frac{(n-1)\theta^\pm}{A\gamma}\frac{1}{\xi_1} \tag{2.55}$$

and

$$\partial_\xi\Big[e^{\gamma\tau}\hat{w}^\pm(A+\xi e^{-\gamma\tau},\tau)\Big]_{\xi=\xi_1} \to \frac{(n-1)(n-2)}{A\gamma} - \frac{(n-1)\theta^\pm}{A\gamma}\frac{1}{\xi_1^2} \tag{2.56}$$

as $\tau \to \infty$.

Proof. We have that $\lim_{\tau\to+\infty} e^{\gamma\tau}\hat{w}_0(A+\xi_1 e^{-\gamma\tau}) = \frac{(n-1)(n-2)}{A\gamma}\xi_1$ from Taylor's theorem on $\hat{w}_0$ at $\xi = A$ and, for $\gamma > \frac{1}{2}$, $\lim_{\tau\to+\infty} e^{-\gamma\tau}\hat{h}(A+\xi_1 e^{-\gamma\tau}) = \frac{(n-1)\theta}{A\gamma}\frac{1}{\xi_1}$ from Lemma 2.3. This proves the first statement of the lemma for $\gamma > \frac{1}{2}$. Similarly, Taylor's Theorem on $\hat{w}_0'$ and Lemma 2.3 imply the second statement for $\gamma > \frac{1}{2}$. In the case where $0 < \gamma \leq 1/2$ a statement similar to Lemma 2.3 clearly holds for the modified $\hat{h}(\eta,\tau)$ and the result follows in the same way.

□

Although we haven't chosen ξ_1 yet, we will next check that w^+ stays above w^-, for all small $\epsilon > 0$.

Lemma 2.11. *For any fixed* ξ_1 *and* $\tau_0 \in \mathbb{R}$, *there exists* $\epsilon_0 = \epsilon_0(\xi_1,\tau_0) > 0$ *such that for all* $0 < \epsilon < \epsilon_0$,

$$w_\epsilon^+(\xi,\tau) > w_\epsilon^-(\xi,\tau), \qquad \text{for } (\xi,\tau) \in (-\infty,\infty)\times[\tau_0,+\infty).$$

Proof. Since $\hat{w}_2$ is a positive function, we have $\hat{w}^+ > \hat{w}^-$ and hence by the definition (2.51) we have

$$w_\epsilon^+ > w_\epsilon^- \quad \text{on } \ \xi \geq \xi_1 \text{ and } \tau \geq \tau_0.$$

On the other hand, it is obvious from the definition of $\bar{w}_\epsilon^\pm$ that

$$\bar{w}_\epsilon^+ > \frac{1}{1+\epsilon}\,\bar{w}_0^+ \quad \text{and} \quad \frac{1}{1-\epsilon}\,\bar{w}_0^- > \bar{w}_\epsilon^- \tag{2.57}$$

in the remaining region $\xi < \xi_1$ and $\tau \geq \tau_0$. Here, $\bar{w}_0^+$, $\bar{w}_0^-$ are $\bar{w}_\epsilon^+$, $\bar{w}_\epsilon^-$ with $\epsilon = 0$. Thus, it suffices to find small $\epsilon_0 > 0$, depending on ξ_1, such that

$$\frac{\bar{w}_0^+(\xi,\tau)}{1+\epsilon_0} > \frac{\bar{w}_0^-(\xi,\tau)}{1-\epsilon_0} \qquad \text{on } \ \xi \leq \xi_1 \ \text{ and } \ \tau \geq \tau_0. \tag{2.58}$$

To this end, we will first show that if $C_{1,0}, C_{2,0}$ are defined by (2.52) when $\epsilon = 0$, then $C_{1,0} > C_{2,0}$ and, as $\tau \to +\infty$,

$$C_{0,1}(\tau) \to C_{0,1,\infty}, \quad C_{0,2}(\tau) \to C_{0,2,\infty} \quad \text{with} \quad C_{0,1,\infty} > C_{0,2,\infty}. \tag{2.59}$$

Indeed, this readily follows from the definition (2.52), $\hat{w}^+ > \hat{w}^-$ and the fact that as $\tau \to +\infty$,

$$e^{\gamma\tau}\hat{w}^\pm(A + \xi_1 e^{-\gamma\tau}, \tau) \to \frac{(n-1)(n-2)}{A\gamma}\xi_1 + \frac{(n-1)\theta^\pm}{A\gamma}\frac{1}{\xi_1},$$

with $\theta^+ > \theta^-$. This in particular implies $\bar{w}_0^+ > \bar{w}_0^-$.

To conclude (2.57), we will now use (2.59) and the fact that under the coordinate change (2.5) where $\xi = \ln r$ the functions $\bar{w}_0^\pm(\xi,\tau)$ are mapped into the functions $\bar{U}^+(r,\tau) > \bar{U}^-(r,\tau)$ given by

$$(\bar{U}^+)^{1-m}(r,\tau) := r^{-2}\bar{w}_0^+(\ln r,\tau) = e^{2C_{0,1}(\tau)}\bar{U}^{1-m}(re^{C_{0,1}(\tau)})$$
$$(\bar{U}^-)^{1-m}(r,\tau) := r^{-2}\bar{w}_0^-(\ln r,\tau) = e^{2C_{0,2}(\tau)}\bar{U}^{1-m}(re^{C_{0,2}(\tau)})$$

where under this transformation the region $\xi \leq \xi_1$ corresponds to the compact region $\{x \in \mathbb{R}^n \,|\, r = |x| \leq e^{\xi_1}\}$ of $\mathbb{R}^n$. Here, recall that $\bar{U}^{1-m}(r) = r^{-2}\,\bar{w}_0(\ln r)$. We then conclude, using (2.59) that there exists small $\epsilon_0(\xi_1, \tau_0)$ such that

$$\frac{1}{1+\epsilon_0}(\bar{U}^+)^{1-m} > \frac{1}{1-\epsilon_0}(\bar{U}^-)^{1-m} \quad \text{on } r \leq e^{\xi_1} \text{ and } \tau \geq \tau_0$$

showing (2.58). $\square$

We are now ready to proceed to the proof of Proposition 2.9.

Proof of Proposition 2.9. We have to find ξ_1, ϵ_1 and $\tau_1(\epsilon)$ for each $0 < \epsilon < \epsilon_1$. Notice that ξ_0 and τ_0 come from Proposition 2.6 and 2.7 and they are fixed throughout the proof. As long as $\tau > \tau_0$ and $0 < \epsilon < 1$, w_ϵ^+ and w_ϵ^- are well defined and part (ii) follows from their construction. For $\xi_1 > \xi_0 > 0$ to be determined later, we have $\epsilon_0(\xi_1) > 0$ from Lemma 2.11 so that part (i) is true for $0 < \epsilon < \epsilon_0$ and $\tau > \tau_0$. In summary, we may choose $\epsilon_1 \leq \epsilon_0(\xi_1)$ and any $\tau(\epsilon) > \tau_0$ for undetermined $\xi_1 > \xi_0 > 0$ so that part (i) and (ii) are always true. Before going to show (iv), let us recall asymptotic properties of $\bar{w}_0$ shown in [CD1]. As $\xi \to \infty$, we have

$$\begin{aligned}\bar{w}_0(\xi) &= \frac{(n-1)(n-2)}{A\gamma}\xi + \frac{(n-1)(n-6)}{4A\gamma}\frac{1}{\xi} + o(\xi^{-1}) \\ \bar{w}_0'(\xi) &= \frac{(n-1)(n-2)}{A\gamma} - \frac{(n-1)(n-6)}{4A\gamma}\frac{1}{\xi^2} + o(\xi^{-2}).\end{aligned} \tag{2.60}$$

Let us just check that $\lim_{\xi\to\xi_1-}(w_\epsilon^+)_\xi(\xi,\tau) > \lim_{\xi\to\xi_1+}(w_\epsilon^+)_\xi(\xi,\tau)$, as a similar argument holds for the other inequality. By the gluing condition and Lemma 2.10, we have that for $\xi_1 > \xi_0$,

$$w_\epsilon^+(\xi_1,\tau) = (1+\epsilon)^{-1}\bar{w}_0(\xi_1 + C_1(\tau)) \to \frac{(n-1)(n-2)}{A\gamma}\xi_1 + \frac{(n-1)\theta^\pm}{A\gamma}\frac{1}{\xi_1}$$

as $\tau \to \infty$. Let's assume $\epsilon < 1$. Invoking (2.60), we may choose a large $\xi_1 > \xi_0$ so that the following holds independently from ϵ

- $\limsup_{\tau\to\infty}|C_1(\tau) - \epsilon\xi_1| \leq 1$
- $\liminf_{\tau\to\infty}(1+\epsilon)^{-1}\bar{w}_0'(\xi_1 + C_1(\tau)) > \frac{(n-1)(n-2)}{(1+\epsilon)A\gamma} - \frac{(n-1)}{(1+\epsilon)A\gamma}\frac{2^{-1}(\theta^+ + \frac{n-6}{4})}{((1+\epsilon)\xi_1)^2}$.

Continuing with this choice of ξ_1, we may find small $\epsilon_1 > 0$ so that $\epsilon_1 < \min(\epsilon_0(\xi_1), 1)$ and, for all $\epsilon < \epsilon_1$,

$$\frac{(n-1)(n-2)}{(1+\epsilon)A\gamma} - \frac{(n-1)}{(1+\epsilon)A\gamma}\frac{2^{-1}(\theta^+ + \frac{n-6}{4})}{((1+\epsilon)\xi_1)^2} > \frac{(n-1)(n-2)}{A\gamma} - \frac{(n-1)}{A\gamma}\frac{\theta^+}{\xi_1^2}.$$

Since

$$\lim_{\xi\to\xi_1-}\frac{\partial}{\partial\xi}w_\epsilon^+(\xi,\tau) = (1+\epsilon)^{-1}\bar{w}_0'(\xi_1 + C_1(\tau))$$

and

$$\lim_{\xi\to\xi_1+}\frac{\partial}{\partial\xi}w_\epsilon^+(\xi,\tau) = \partial_\xi\Big[e^{\gamma\tau}\hat{w}^+(A + \xi e^{-\gamma\tau},\tau)\Big]_{\xi=\xi_1},$$

the second part of Lemma 2.10 and above observation proves (iv) for a large τ_1.

In showing (iii), we only need to check this in the inner region as we assume $\xi_1 > \xi_0$ and $\tau_1 > \tau_0$. By Proposition 2.8, it suffices to show for each fixed ξ_1 and $0 < \epsilon < \epsilon_1$ there exists $\tau_1 \gg 1$ such that

$$|C_1'(\tau)| \leq M \quad \text{and} \quad |C_2'(\tau)| \leq M, \quad \text{for } \tau \geq \tau_0 \tag{2.61}$$

for some constant M. We will actually show that $\lim_{\tau \to +\infty} C_i'(\tau) = 0$, $i = 1, 2$ which yields (2.61). Lets prove this for C_1, as the proof for C_2 is identical. Recall that

$$0 < e^{\gamma\tau}\, \hat{w}^+(A + \xi_1 e^{-\gamma\tau}, \tau) = \bar{w}_\epsilon^+(\xi_1, \tau) = \frac{1}{1+\epsilon}\bar{w}_0(\xi_1 + C_1(\tau))$$

For $\gamma > 1/2$, differentiating in τ the LHS using that $e^{\gamma\tau}\, \hat{w}^+ = e^{\gamma\tau}\, \hat{w}_0 + e^{-\gamma\tau}\, \hat{h}$, we obtain

$$\text{LHS} = \gamma\big(e^{\gamma\tau}\, \hat{w}_0 - \xi_1\, \hat{w}_0'\big)(A + \xi_1 e^{-\gamma\tau}) - \gamma\left(e^{-\gamma\tau}\hat{h} + \xi_1 e^{-2\gamma\tau}\hat{h}'\right)(A + \xi_1 e^{-\gamma\tau}).$$

Both terms converge to zero, as $\tau \to \infty$, by Taylor's theorem for $\hat{w}_0$, $\hat{w}_0'$ and the asymptotics in Lemma 2.3. The same convergence could be proven similarly for $0 < \gamma \leq 1/2$ as additional terms multiplied by $e^{\gamma\tau}$ are very small and their τ-derivatives converges to zero at the point $(A+\xi_1 e^{-\gamma\tau}, \tau)$. At the same time, for any $\gamma > 0$, if we take derivative of RHS we obtain

$$\text{RHS} = \frac{1}{1+\epsilon}\bar{w}_0'(\xi_1 + C_1(\tau))\, C_1'(\tau).$$

Since the smooth function $C_1(\tau)$ converges as $\tau \to \infty$ and hence

$$\bar{w}_0'(\xi_1 + C_1(\tau)) \to \bar{w}_0'(\xi_1 + \lim_{\tau \to \infty} C_1(\tau)) > 0$$

this concludes that $C_1'(\tau) \to 0$ as $\tau \to \infty$ and hence bounded for $\tau \gg 1$. The same argument also applies to $C_2(\tau)$. Thus (2.61) holds.

Finally, by the arguments above and Propositions 2.6, 2.7 and 2.8, we can find $\tau_1 \geq \tau_0$ which makes all the statements in our proposition true.

□

We will finish this section with the following result which is an immediate consequence of the comparison principle and Proposition 2.9.

Theorem 2.12. *Let ξ_1, ϵ_1 and $\tau_1 = \tau_1(\epsilon)$ are such Proposition 2.9 holds. Assume that a given conformally flat initial metric $g_0 = u_0^{1-m}(x)\, \delta_{ij}$ is bounded above and below by $w_\epsilon^+(\xi, -\ln T)$ and*

$w_\epsilon^-(\xi,-\ln T)$, for some $0<\epsilon<\epsilon_1$ and $0<T<e^{-\tau_1}$, via the coordinate change

$$w(\xi,\tau)=|x|^2\,u(x,t),\quad \xi=\ln|x|-A\,e^{\gamma\tau},\ \ \tau=-\ln(T-t) \tag{2.62}$$

at $t=0$. That is

$$w_\epsilon^-(\xi,-\ln T)\le\frac{|x|^2u_0^{1-m}(x)}{T^{1+\gamma}}\le w_\epsilon^+(\xi,-\ln T) \tag{2.63}$$

holds, with $\xi=\ln|x|-AT^{-\gamma}$. Then, the solution of the Yamabe flow (2.2) exists on the time interval $(0,T)$ and it is bounded between $w_\epsilon^+(\xi,\tau)$ and $w_\epsilon^-(\xi,\tau)$, that is

$$w_\epsilon^-(\xi,-\ln(T-t))\le\frac{|x|^2\,u^{1-m}(x,t)}{(T-t)^{1+\gamma}}\le w_\epsilon^+(\xi,-\ln(T-t)) \tag{2.64}$$

with $\xi=\ln|x|-A(T-t)^{-\gamma}$.

Proof. Immediate by Proposition 2.9 and the comparison principle. □

2.7 Asymptotic shape of the singularity in the inner region and geometric properties

Throughout this section we will fix $\xi_1>0$ and $\epsilon>0$ so that $w_\epsilon^+(\xi,\tau)$ and $w_\epsilon^-(\xi,\tau)$ given by (2.54) are barriers in view of Proposition 2.9 and Theorem 2.12. To simplify the notation we will denote them by $w^+(\xi,\tau)$ and $w^-(\xi,\tau)$ respectively. They are super and sub-solutions of equation (2.9) on $\mathbb{R}\times[\tau_1,\infty)$, respectively.

We will first prove that if our initial conformally flat metric of the Yamabe flow (2.2) $u_0(\cdot)$ is bounded from above and below by $w^+(\cdot,-\ln T)$ and $w^-(\cdot,-\ln T)$, for some $-\ln T\ge\tau_1$ (c.f. (2.63)), then the rescaled solution converges to a steady gradient soliton $\bar{w}_0(\xi)$, which is the unique entire solution of the equation (2.30) with asymptotic behavior (2.31) as $\xi\to\infty$.

Since we are not assuming that our solution $u(x,t)$ of (2.2) is radially symmetric, it is more convenient to work in euclidean coordinates on $\mathbb{R}^n$, rather than cylindrical coordinates. We have seen that in order to see the steady state $\bar{w}_0$ in the *inner region* one needs to perform the coordinate change (2.17) on radially symmetric solutions in cylindrical coordinates. Under the transformation (2.5) which brings us back to the plane, this change of variables corresponds the coordinate change (2.23)-(2.24) which transforms a solution $u(x,t)$ of (2.2) to a solution $\bar{u}(y,l)$ of equation

$$\partial_l\bar{u}-\frac{1+\gamma}{(1-m)\gamma}\frac{\bar{u}}{l}=\frac{n-1}{m}\Delta_y\bar{u}^m+\gamma A(y\cdot\nabla_y\bar{u})+\frac{2\gamma A}{1-m}\bar{u}. \tag{2.65}$$

We denote $\bar{U}(y)$ the steady soliton $\bar{w}_0$ in euclidean coordinates, namely

$$\bar{U}(y)^{1-m} = |y|^{-2}\bar{w}_0(\ln|y|).$$

This is the unique radial solution of

$$\frac{n-1}{m}\,\Delta u^m + \gamma A(y\cdot\nabla u) + \frac{2\gamma A}{1-m}u = 0 \tag{2.66}$$

with asymptotic behavior

$$u^{1-m}(y) = \frac{1}{|y|^2}\Big(\frac{(n-1)(n-2)}{\gamma A}\ln|y| + o(1)\Big). \tag{2.67}$$

We will next prove the following result.

Theorem 2.13. *Under the assumptions of Theorem 2.12, the rescaled solution $\bar{u}(y,l)$ converges, as $l\to+\infty$, smoothly on compact sets of $\mathbb{R}^n$ to the radial steady soliton $\bar{U}^{1-m}(y)$.*

Proof. Let $l_0 := \gamma^{-1}T^{-\gamma}$ be the initial rescaled time, corresponding to $t=0$. By Theorem 2.12, for $l > l_0 > 0$ we have

$$|y|^{-2}w^{-}(\ln|y|,\tau) \le \bar{u}^{1-m}(y,l) \le |y|^{-2}w^{+}(\ln|y|,\tau),\ l = \frac{e^{\gamma\tau}}{\gamma}. \tag{2.68}$$

These two bounds give upper and lower bounds away from zero for $\bar{u}(\cdot,l)$ on every compact set in $\mathbb{R}^n$ which are uniform in time $l \ge l_0 \gg 1$. Hence, by standard higher order regularity estimates for uniformly parabolic equations and a compactness argument, we conclude that for any sequence $l_i\to\infty$, the solutions $\bar{u}_i(y,l) := \bar{u}(y,l_i+l)$ converge, passing to a subsequence, to a limit $\bar{u}_\infty(y,l)$. The convergence is smooth on compact subsets of $\mathbb{R}^n\times\mathbb{R}$. Therefore, in view of (2.65) and the uniform local upper bound of our sequence, the limit $\bar{u}_\infty$ is a smooth eternal solution of

$$\partial_l\bar{u} = \frac{n-1}{m}\,\Delta_y\bar{u}^m + \gamma A(y\cdot\nabla_y\bar{u}) + \frac{2\gamma A}{1-m}\bar{u}. \tag{2.69}$$

To finish the proof we need to show that

$$\bar{u}_\infty(y,l) = \bar{U}(y)$$

which would also imply that our limit is unique, thus concluding that $\bar{u}(\cdot,l)\to\bar{U}$, as $l\to\infty$. To this end, we first observe that by our barrier construction (2.54), we have

$$w^{\pm}(\ln|y|,\tau) \to \frac{(n-1)(n-2)}{\gamma A}\ln|y| + \frac{(n-1)}{A\gamma}\theta^{\pm}\frac{1}{\ln|y|}$$

as $\tau \to +\infty$, uniformly on $e^{\xi_1} \le |y| \le K$, for any fixed $K > e^{\xi_1}$. In particular, this implies that our limit $\bar{u}_\infty^{1-m}$ has these bounds and thus

$$\bar{u}_\infty^{1-m}(\cdot, l) = \frac{1}{|y|^2}\Big(\frac{(n-1)(n-2)}{\gamma A}\ln|y| + o(1)\Big) \tag{2.70}$$

as $|y| \to \infty$ uniformly in $l \in \mathbb{R}$. For $\lambda > 0$, if we denote $\bar{U}_\lambda(y) := \lambda^{\frac{2}{1-m}}\bar{U}(\lambda y)$, this is again a radial solution of (2.66) with

$$\bar{U}_\lambda^{1-m}(y) = \frac{1}{|y|^2}\Big(\frac{(n-1)(n-2)}{\gamma A}\ln|y| + \ln\lambda + o(1)\Big). \tag{2.71}$$

This is just a time translation of the radial steady soliton $\bar{U}$ and they are isometric. Since on the soliton the scalar curvature $R > 0$ everywhere, the solution pointwise decreases as time increases and hence $\bar{U}_{\lambda_1} > \bar{U}_{\lambda_2}$ for $\lambda_1 > \lambda_2$. Thus we may define

$$\lambda_+ := \inf\{\lambda > 0 \mid \bar{U}_\lambda(\cdot) \ge \bar{u}_\infty(\cdot, l) \text{ for all } l\}$$
$$\lambda_- := \sup\{\lambda > 0 \mid \bar{U}_\lambda(\cdot) \le \bar{u}_\infty(\cdot, l) \text{ for all } l\}.$$

Our proof will finish if we show that $\lambda_+ = \lambda_- = 1$. Let us prove that $\lambda_+ = 1$. Since $\inf_{B(e^{\xi_1},0)} \bar{U}_\lambda \to \infty$ as $\lambda \to \infty$ (see the observation in Corollary 3.3 in [CD1]), the construction of w^+ in the inner region and (2.71) imply that we can find large $\lambda > 1$ such that $\bar{U}_\lambda(\cdot) > \bar{u}_\infty(\cdot, l)$ for all l. By (2.70) and (2.71), $\bar{U}_\lambda(\cdot) \not\ge \bar{u}_\infty(\cdot, l)$ for $\lambda < 1$. Therefore, λ_+ is a well defined number with $\lambda_+ \ge 1$. Assume that $\lambda_+ > 1$. For each $\bar{U}_{\lambda_+ - 2^{-n}}$, there is a point (x_n, l_n) with $\bar{U}_{\lambda_+ - 2^{-n}}(x_n) < \bar{U}_\infty(x_n, l_n)$. Moreover, the sequence of points $\{x_n\}$ such that $\lambda_+ - 2^{-n} > 1$ is bounded due to (2.70) and (2.71). By standard regularity estimates on the equation (2.69), we can find a subsequence of (x_n, l_n) such that

$$\bar{u}^*_{n_j}(x, l) := \bar{u}_\infty(x, l_{n_j} + l) \to \bar{u}^*_\infty(x, l)$$

smoothly on compact sets and $x_{n_j} \to x^*$.

Note that $\bar{U}_{\lambda_+}(x) \ge \bar{u}^*_\infty(x, l)$ for all l. On the other hand we have $\bar{U}_{\lambda_+}(x^*) = \bar{u}_\infty(x^*, 0)$. Hence, by the strong maximum principle, we must have $\bar{U}_{\lambda_+}(\cdot) = \bar{u}^*_\infty(\cdot, l)$, for all l. But this can't happen since (2.70), (2.71) holds and we have assumed that $\lambda_+ > 1$. By contradiction, this proves that $\lambda_+ = 1$ and $\lambda_- = 1$ can be shown similarly. This concludes the proof of our theorem. $\square$

Let us remark the following.

Remark 2.3 (Scalar curvature blow up rate near the origin). $u(x,t)$ of (2.2) represents conformally flat solution of the Yamabe flow $g(t) = u^{1-m}(x,t)\,\delta_{ij}$ on $\mathbb{R}^n$. The metric of rescaled solution $\bar{u}(y,l)$ can be written as

$$\bar{u}^{1-m}\delta_{ij} = (T-t)^{-(1+\gamma)}\,\phi_t^* g(t)$$

where ϕ_t is a one parameter family of diffeomorphisms $\phi_t(x) = e^{A(T-t)^{-\gamma}}\,x$. Therefore, Theorem 2.13 can be rephrased as convergence of the pointed manifold

$$(\mathbb{R}^n, \frac{g(t)}{(T-t)^{1+\gamma}}, 0) \to (\mathbb{R}^n, \bar{U}^{1-m}\delta_{ij}, 0)$$

in Cheeger-Gromov sense. This, in particular, implies that

$$|\mathrm{Rm}_{g(t)/(T-t)^{(1+\gamma)}}|(e^{A(T-t)^{-\gamma}}y) \to |\mathrm{Rm}_{\bar{U}^{1-m}\delta_{ij}}|(y)$$

concluding that

$$(T-t)^{1+\gamma}\,|\mathrm{Rm}_{g(t)}|(e^{A(T-t)^{-\gamma}}y) \to |\mathrm{Rm}_{\bar{U}^{1-m}\delta_{ij}}|(y)$$

and also

$$(T-t)^{1+\gamma}\,R_{g(t)}(e^{A(T-t)^{-\gamma}}y) \to R_{\bar{U}^{1-m}\delta_{ij}}(y).$$

In particular the above implies the following blow up rate of the Riemmannian and Scalar curvature at the origin

$$\lim_{t\to T-}(T-t)^{1+\gamma}|\mathrm{Rm}_{g(t)}|(0) = |\mathrm{R}m_{\bar{U}^{1-m}\delta_{ij}}|(0) = \frac{2\gamma A}{\sqrt{n(n-1)}}$$

and

$$\lim_{t\to T-}(T-t)^{1+\gamma}\,R_{g(t)}(0) = R_{\bar{U}^{1-m}\delta_{ij}}(0) = 2\gamma A.$$

We will next show that the global supremum of the curvature occurs asymptotically *at the origin* as $t \to T-$.

Proposition 2.14. *Under the assumptions of Theorem 2.12 we have*

$$\lim_{t\to T-}\Big[(T-t)^{1+\gamma}\sup_{x\in\mathbb{R}^n}|\mathrm{Rm}_{g(t)}|\Big] = \frac{2\gamma A}{\sqrt{n(n-1)}} \tag{2.72}$$

and if $\{(x_i,t_i)\}$ *are points such that* $t_i \to T$ *and*

$$\lim_{i\to\infty}\Big[(T-t_i)^{1+\gamma}|\mathrm{Rm}_{g(t_i)}|(x_i)\Big] = \frac{2\gamma A}{\sqrt{n(n-1)}} \tag{2.73}$$

then

$$y_i := e^{-A(T-t_i)^{-\gamma}} x_i \to 0, \quad \text{as } i \to \infty.$$

In other words, $(T-t)^{-\frac{1+\gamma}{2}} \operatorname{dist}_{g(t_i)}(x_i, 0) \to 0$ *due to the convergence of the metric.*

The result of this proposition follows from the following curvature estimate lemma in the outer region. This lemma is useful in the sense that it also gives the curvature blow up rate in other regions. For example, it shows that the curvature blows up in a *type I manner* near the *infinite cylindrical region.*

Lemma 2.15. *There exist* $C > 0$, $r_0 \gg 1$ *and* $0 < t_0 < T$ *such that the following holds*

$$(T-t)\, \hat{w}_0\big(A + (T-t)^\gamma \ln|y|\big)\, |\mathrm{Rm}_{g(t)}(e^{A(T-t)^{-\gamma}} y)| \le C \tag{2.74}$$

on $|y| \ge r_0$ *and* $t_0 < t < T$.

Let us first show that Lemma 2.15 implies the proposition and then finish this section by proving the lemma.

Proof of Proposition 2.14. Assuming that the Lemma 2.15 holds, then since $\hat{w}_0$ is increasing function, for each fixed $r_1 \ge r_0$ we have

$$(T-t)^{1+\gamma} \sup_{|y| \ge r_1} |\mathrm{Rm}_{g(t)}(e^{A(T-t)^{-\gamma}} y)| < \frac{C(T-t)^\gamma}{\hat{w}_0(A + (T-t)^\gamma \ln|r_1|)}$$

and, by the Taylor expansion of $\hat{w}_0(\cdot)$ at A, taking the limit $t \to T-$, yields

$$\limsup_{t \to T-} \Big[(T-t)^{1+\gamma} \sup_{|y| \ge r_1} |\mathrm{Rm}_{g(t)}(e^{A(T-t)^{-\gamma}} y)|\Big] \le \frac{C\gamma A}{(n-1)(n-2)\ln r_1}. \tag{2.75}$$

Choose $r_1 \ge r_0$ sufficiently large so that

$$\frac{C\gamma A}{(n-1)(n-2)\ln r_1} < \frac{2\gamma A}{\sqrt{n(n-1)}}.$$

Now on the remaining region $|x| \le r_1$, due to the smooth convergence of $\bar{u}^{1-m}$ to $\bar{U}^{1-m}$ on compact sets and the fact that the steady soliton $\bar{U}^{1-m}$ attains its maximum curvature at the origin we have

$$\lim_{t \to T-} \Big[(T-t)^{1+\gamma} \sup_{|y| \le r_1} |Rm_{g(t)}(e^{A(T-t)^{-\gamma}} y)|\Big] = \frac{2\gamma A}{\sqrt{n(n-1)}}$$

and also the second statement of the proposition holds, concluding the proof.

□

Proof of Lemma 2.15. For r_0 and t_0 to be chosen later, we pick and fix a point (y_1, t_1) with $|y_1| \geq r_0$ and $t_0 < t_1 < T$ and consider the following scaling of the solution

$$v^{1-m}(z,\sigma) := (e^{A(T-t_1)^{-\gamma}}|y_1|)^2 \frac{u^{1-m}(e^{A(T-t_1)^{-\gamma}}|y_1|\, z, t)}{(T-t)\, \hat{w}_0(A + (T-t_1)^\gamma \ln|y_1|)}$$

with

$$\sigma(t) := \ln\big(\frac{T-t_1}{T-t}\big)\, \frac{1}{\hat{w}_0(A + (T-t_1)^\gamma \ln|y_1|)}. \tag{2.76}$$

Since u satisfies (2.2), it follows that $v(z,\sigma)$ evolves by

$$\frac{\partial}{\partial\sigma} v = \frac{n-1}{m}\Delta_z v^m - \hat{w}_0(A + (T-t_1)^\gamma \ln|y_1|)\, v. \tag{2.77}$$

Claim 2.5. *There are $r_0 > 1$ and $t_0 > 0$ so that if $|y_1| \geq r_0$ and $t_1 \geq t_0$, then there exist positive constants c and C which are independent of (y_1, t_1) such that*

$$c \leq v^{1-m}(z,\sigma) \leq C, \qquad \textit{for } (z,\sigma) \in \mathcal{A} \times [-1, 0]$$

where $\mathcal{A}$ is the annulus $\mathcal{A} = \{z \in \mathbb{R}^n \mid 1/2 \leq |z| \leq 3/2\}$.

Lets us assume that the claim holds and finish the proof of the lemma. Since $|\hat{w}_0| \leq (n-1)(n-2)$, the equation (2.77) is uniformly parabolic on $\mathcal{A} \times [-1, 0]$, independently from choice of the point (y_1, t_1), and therefore by standard parabolic regularity estimates, we have uniform bounds for $|\nabla v|$ and $|\nabla^2 v|$ on any strictly smaller parabolic cylinder. In particular, we have

$$|\nabla v(y_1/|y_1|, 0)| \quad \text{and} \quad |\nabla^2 v(y_1/|y_1|, 0)| < C.$$

Since $v^{1-m}(\cdot, 0)\delta_{ij}$ and $u^{1-m}(\cdot, t_1)\delta_{ij}$ are isometric, we conclude that

$$\begin{aligned}(T-t_1)\, \hat{w}_0(A + (T-t_1)^\gamma \ln|y_1|)\, &\big|\mathrm{Rm}_{g(t_1)}(e^{A(T-t_1)^{-\gamma}} y_1)\big| \\ &= \big|\mathrm{Rm}_{v^{1-m}(0)\delta_{ij}}\big(\frac{y_1}{|y_1|}\big)\big| \leq C.\end{aligned}$$

The constant C is independent from (y_1, t_1). This finishes the proof of the lemma.

Proof of the Claim 2.5. Although the computations below might look intimidating, the idea is simple. Since u is trapped between the two barriers in the outer region where $|x| \geq e^{\frac{A}{(T-t)^\gamma}} e^{\xi_1}$, namely

$$\hat{w}^-((T-t)^\gamma \ln|x|, -\ln(T-t)) \leq \frac{|x|^2 u(x,t)}{T-t} \leq \hat{w}^+((T-t)^\gamma \ln|x|, -\ln(T-t)) \tag{2.78}$$

and since $\hat{w}^+$ and $\hat{w}^-$ are close to $\hat{w}_0$, we expect that different values of v are similar in the whole annulus.

Indeed, suppose $r_0 > 1$ and $t_0 \in (0,T)$ are first chosen to satisfy

$$\ln\frac{r_0}{2} > \xi_1 \quad \text{and} \quad -\ln(T-t_0) > -\ln T + (n-1)(n-2).$$

With this choice of t_0 and $0 \le \hat{w}_0 \le (n-1)(n-2)$, we see for that

$$\sigma(0) = \frac{\ln(T-t_1)-\ln(T-0)}{w_0(A+(T-t_1)^\gamma \ln|y_1|)} \le \frac{\ln(T-t_0)-\ln T}{(n-1)(n-2)} \le -\frac{(n-1)(n-2)}{(n-1)(n-2)} = -1$$

where $\sigma(t)$ is defined by (2.76). Since $u(x,t)$ is defined for $t \ge 0$, this shows that the rescaled function $v(z,\sigma)$ is well defined on $\mathcal{A} \times [-1,0]$. By choosing $r_0 > 2e^{\xi_1}$ sufficiently large and $t_0 \in (0,T)$ closer to T, we may assume that

$$\frac{1}{2}\hat{w}_0(A+\xi e^{-\gamma\tau}) \le \hat{w}^-(A+\xi e^{-\gamma\tau},\tau) \tag{2.79}$$

and

$$\hat{w}^+(A+\xi e^{-\gamma\tau},\tau) \le 2\hat{w}_0(A+\xi e^{-\gamma\tau}) \tag{2.80}$$

on $\xi \ge \ln(r_0/2)$ and $\sigma \ge -1$. This is possible because $\hat{w}^\pm = \hat{w}_0 + e^{-2\gamma t}\hat{h}$, where $\hat{h}$ is bounded away from A and satisfies 2.3 near A, when $\gamma > 1/2$. The other range $\gamma \in (0,1/2]$ is similar (see Claim 2.4). Using (2.78) we can then estimate

$$\begin{aligned} v^{1-m}(z,\sigma) &= \frac{(e^{\frac{A}{(T-t_1)^\gamma}}|y_1|)^2}{T-t}\,\frac{u^{1-m}(e^{\frac{A}{(T-t)^\gamma}}|y_1|\,e^{\frac{A}{(T-t_1)^\gamma}-\frac{A}{(T-t)^\gamma}}z,t)}{\hat{w}_0(A+(T-t_1)^\gamma\,\ln|y_1|)} \\ &\le \frac{1}{|z|^2}\,\frac{\hat{w}^+\Big(A+(T-t)^\gamma \ln\big(|y_1||z|e^{\frac{A}{(T-t_1)^\gamma}-\frac{A}{(T-t)^\gamma}}\big),\tau\Big)}{\hat{w}_0(A+(T-t_1)^\gamma\,\ln|y_1|)} \\ &\le \frac{2}{|z|^2}\,\frac{\hat{w}_0\Big(A+(T-t)^\gamma\big(\ln|y_1|+\ln|z|\big)+A\big(\frac{(T-t)^\gamma}{(T-t_1)^\gamma}-1\big)\Big)}{\hat{w}_0(A+(T-t_1)^\gamma \ln|y_1|)}. \end{aligned}$$

Since $t \le t_1$ and $|y_1| \ge r_0$, we have $|y_1||z|e^{\frac{A}{(T-t_1)^\gamma}-\frac{A}{(T-t)^\gamma}} \ge r_0 \cdot (1/2)\cdot 1 \ge e^{\xi_1}$, thus we could bound above $v^{1-m}(z,\sigma)$ using our barrier $\hat{w}^+$ of the outer region in the second line and use (2.80) in the last line. Similarly we get a lower bound

$$v^{1-m}(z,\sigma) \ge \frac{1}{2|z|^2}\,\frac{\hat{w}_0\Big(A+(T-t)^\gamma\big(\ln|y_1|+\ln|z|\big)+A\big(\frac{(T-t)^\gamma}{(T-t_1)^\gamma}-1\big)\Big)}{\hat{w}_0(A+(T-t_1)^\gamma \ln|y_1|)}.$$

If we could find constants $0 < c < 1 < C$ such that

$$c \le \frac{(T-t)^\gamma \left(\ln|y_1| + \ln|z|\right) + A\left(\frac{(T-t)^\gamma}{(T-t_1)^\gamma} - 1\right)}{(T-t)^\gamma \ln|y_1|} \le C \tag{2.81}$$

holds on $(v, \sigma) \in A \times [-1, 0]$, then since $\hat{w}_0(A + x)$ is an increasing function on $x \ge 0$, concave and $\hat{w}_0(A + 0) = 0$, we would deduce that

$$c \le \frac{\hat{w}_0\Big(A + (T-t)^\gamma \left(\ln|y_1| + \ln|z|\right) + A\left(\frac{(T-t)^\gamma}{(T-t_1)^\gamma} - 1\right)\Big)}{\hat{w}_0(A + (T-t)^\gamma \ln|y_1|)} \le C$$

which combined with the above would finish the proof of our claim.

Now let us now find the bounds (2.81). First, it is clear that

$$1 + \frac{\ln\frac{1}{2}}{\ln r_0} \le \frac{\ln|y_1| + \ln|z|}{\ln|y_1|} \le 1 + \frac{\ln\frac{3}{2}}{\ln r_0} \tag{2.82}$$

and notice that $1 + \frac{\ln\frac{1}{2}}{\ln r_0} = \frac{\ln\frac{r_0}{2}}{\ln r_0}$ is positive since $\ln\frac{r_0}{2} > \xi_1 > 0$. Thus, it suffices to prove

$$0 < \frac{A\left(\frac{(T-t)^\gamma}{(T-t_1)^\gamma} - 1\right)}{(T-t)^\gamma \ln|y_1|} < C_1 \tag{2.83}$$

by some constant $C_1 < \infty$. If we introduce $\theta := (T-t_1)^\gamma \ln|y_1| > 0$, then by the definition of σ in (2.76)

$$\frac{(T-t)^\gamma}{(T-t_1)^\gamma} = e^{-\sigma\gamma\, \hat{w}_0(A + \ln|y_1|(T-t_1)^\gamma)} = e^{-\sigma\gamma\, \hat{w}_0(A+\theta)} > 0.$$

Now $0 \le -\sigma \le 1$ and $e^{-\sigma\gamma\hat{w}_0(A+\theta)} \ge 1$ imply

$$0 < \frac{A\Big(\frac{(T-t)^\gamma}{(T-t_1)^\gamma} - 1\Big)}{(T-t)^\gamma \ln|y_1|} = A\, \frac{e^{-\sigma\gamma\hat{w}_0\,(A+\theta)} - 1}{\theta\, e^{-\sigma\gamma\hat{w}_0\,(A+\theta)}} \le A\, \frac{e^{\gamma\hat{w}_0\,(A+\theta)} - e^{\gamma\hat{w}_0\,(A+0)}}{\theta - 0}.$$

By the mean value theorem, there exists $0 < \theta_0 < \theta$ such that

$$\frac{e^{\gamma\,\hat{w}_0(A+\theta)} - e^{\gamma\,\hat{w}_0(A+0)}}{\theta - 0} = \gamma\, \hat{w}_0'(A+\theta_0)\, e^{\gamma\hat{w}_0(A+\theta_0)} \le \gamma\, \frac{(n-1)(n-2)}{\gamma A}\, e^{\gamma(n-1)(n-2)}.$$

Here, we used the facts that $\hat{w}_0(A + x)$ is a nonnegative concave function on the set $x \ge 0$ with

$$\hat{w}_0'(A) = \frac{(n-1)(n-2)}{\gamma A} \quad \text{and} \quad \hat{w}_0(A+x) \le (n-1)(n-2).$$

Combining the last two inequalities implies that (2.83) holds with

$$C_1 := (n-1)(n-2)\, e^{\gamma(n-1)(n-2)}.$$

This finished the proof of the claim and also the proof of the lemma. □

□

2.8 Proof of Theorem 2.2

In this final section, we will give the proof of Theorem 2.2 as stated in the introduction. We will need the following rigidity result for eternal solutions of conformally flat Yamabe flow.

Proposition 2.16. *Let $g(t) = u^{1-m}(x,t)\,\delta_{ij}$ be a smooth eternal solution to the conformally flat Yamabe flow* (2.2) *on $\mathbb{R}^n \times (-\infty, \infty)$, with positive Ricci and uniformly bounded sectional curvature. We further assume that u is bounded from below by a radial steady gradient soliton centered at the origin with maximum scalar curvature $2\gamma A > 0$, that is*

$$u(x,t) \geq e^{-\frac{2\gamma A(t+\xi_1)}{1-m}}\,\bar{U}(|x|e^{-\gamma A(t+\xi_1)}) \quad \text{for some } \xi_1 \in \mathbb{R}. \tag{2.84}$$

Then, $u(x,t)$ must be a radial gradient steady soliton, that is

$$u(x,t) \equiv e^{-\frac{2\gamma A(t+\xi_0)}{1-m}}\,\bar{U}(|x|e^{-\gamma A(t+\xi_0)}) \quad \text{for some } \xi_0 \leq \xi_1. \tag{2.85}$$

Proof. By the Harnack inequality for the Yamabe flow (c.f. Theorem 3.7 in [Ch]), for any 1-form X_i

$$(n-1)\Delta R + \langle \nabla R, X\rangle + \frac{1}{n-1} R_{ij} X^i X^j + R^2 \geq 0. \tag{2.86}$$

Note that since our solution exists from $t = -\infty$ we could drop R/t term from the original Harnack expression in [Ch]. This inequality, in particular implies that

$$(n-1)\Delta R + R^2 = \partial_t R > 0.$$

Claim 2.6. $R_g(0,0) = 2\gamma A$ *and* $2\gamma A = \sup R_g(x,t)$.

Proof of Claim 2.6. The proof is simple and uses that $\partial_t R > 0$ and

$$\partial_t u^{1-m}(x,t) = -R_g(x,t)\,u^{1-m}(x,t). \tag{2.87}$$

Suppose there is a point (x_0, t_0) with $R_g(x_0,t_0) > 2\gamma A$. Since $\partial_t R > 0$, a ODE comparison implies

$$u(x_0,t) = C(x_0)\,e^{-\frac{1}{1-m}R_g(x_0,t_0)t}, \quad \text{as } t \to \infty.$$

On the other hand, this contradicts to

$$u(x_0,t) \geq \bar{e}^{-\frac{2\gamma A(t+\xi_1)}{1-m}}\,\bar{u}_0(|x_0|e^{-\gamma A(t+\xi_1)}) \geq e^{-\frac{2\gamma A(t+\xi_1)}{1-m}} \inf_{|y| \leq |x_0|e^{-\gamma A(t_0+\xi_1)}} \bar{u}_0(y)$$

which holds for $t \geq t_0$. Similarly as before, if $R_g(0,0) < 2\gamma A$, then

$$u(0,t) = C_0\, e^{-\frac{1}{1-m} R_g(0,0)t}, \qquad \text{as } t \to -\infty$$

which again contradicts to (2.84). $\square$

According to the classification of conformally flat radial solitons (c.f. Propositions 1.4 and 1.5 in [DS2]) the one parameter family of solutions

$$\bar{U}_\xi(x,t) := e^{-\frac{2\gamma A(t+\xi)}{1-m}} \bar{U}(|x| e^{-\gamma A(t+\xi)}), \qquad \xi \in \mathbb{R}$$

are all possible conformally flat radial steady gradient solitons whose maximum scalar curvature is $2\gamma A$ at the origin. It also is known that these solutions attain a strict curvature maximum at the origin. Meanwhile, due to the Claim 2.6, $g(x,t)$ attains its maximum scalar curvature at an interior space-time point $(0,0)$. Furthermore, since $g(x,t)$ has positive Ricci curvature and bounded sectional curvature, Corollary 5.1 in [DS2] implies that $g(x,t)$ must be a steady gradient soliton. Also, under the nonnegative Ricci condition, such steady gradient solitons are (globally) conformally flat and radially symmetric (c.f. Theorem 3.2 and Corollary 3.3 in [CMM] or Corollary 1.6 and Remark 1.2 in [CSZ]). Since $g(x,t)$ has its maximum scalar curvature at the origin, it must be symmetric with respect to this point. In view of Liouville's rigidity theorem on conformal mappings on $\mathbb{R}^n$ with $n \geq 3$, $u(x,t)$ must be a radially symmetric function which represents a steady gradient soliton. Hence $u(x,t) = \bar{U}_{\xi_0}(x,t)$ by the classification theorem in [DS2]. $\square$

We are now in position to finally give the proof of our main result, Theorem 2.2.

Proof of Theorem 2.2. We begin by fixing ξ_1, $\epsilon \leq \epsilon_1$ and τ_1, as they appear in Proposition 2.9 and Theorem 2.12. Set $T_1 := e^{-\tau_1}$ and fix T with $0 < T < T_1$.

Claim 2.7. *There exist $\xi_a > \xi_b$ so that*

$$w_\epsilon^+(\xi - \gamma A \xi_a, -\ln T) \leq \frac{|x|^2 u_0^{1-m}(x)}{T^{1+\gamma}} \leq w_\epsilon^-(\xi - \gamma A \xi_b, -\ln T) \tag{2.88}$$

holds, under the coordinate change $\xi = \ln|x| - AT^{-\gamma}$.

Proof of Claim 2.7. For our fixed T, it is not hard to check that

$$T^{1+\gamma}\, w_\epsilon^\pm(\xi, -\ln T) = (n-1)(n-2)\left(T - (\xi/A)^{-\frac{1}{\gamma}} + O(\xi^{-\frac{1}{\gamma}-1})\right), \quad \text{as } \xi \to \infty$$

and

$$T^{1+\gamma}w_\epsilon^\pm(\xi,-\ln T)\sim e^{2\xi},\quad \text{as } \xi\to-\infty.$$

We can also check that if $\ln|x|=\xi+AT^{-\gamma}$, $|x|^2u_0^{1-m}(x)$ also satisfies these asymptotics. Indeed, Condition ii) in Theorem 1.2 implies that

$$\begin{aligned}|x|^2u_0^{1-m}(x)&=(n-1)(n-2)\left(T-\left(\frac{\xi+AT^{-\gamma}}{A}\right)^{-\frac{1}{\gamma}}+O\big((\xi+AT^{-\gamma})^{-\frac{1}{\gamma}-1}\big)\right)\\&=(n-1)(n-2)\left(T-(\xi/A)^{-\frac{1}{\gamma}}+O(\xi^{-\frac{1}{\gamma}-1})\right)\quad \text{as } \xi\to\infty\end{aligned}$$

and also

$$\lim_{\xi\to-\infty}\frac{|x|^2u_0^{1-m}(x)}{e^{2\xi}}=\lim_{\xi\to-\infty}e^{AT^{-\gamma}}u_0^{1-m}(x)=e^{AT^{-\gamma}}\,u_0^{1-m}(0)>0.$$

Using these asymptotic behaviors, we can first find $\xi_{a,0}>\xi_{b,0}$ such that (2.88) holds asymptotically (outside of compact interval in ξ). Next, we may use condition i) of Theorem 1.2 to find possibly smaller $\xi_b\le\xi_{b,0}$ so that the second inequality of (2.88) holds everywhere. Finally, by a similar argument which uses the fact $|x|^2u_0(x)$ is uniformly bounded away from zero on $|x|\ge r_0$ for all $r_0>0$, we may find larger $\xi_a\ge\xi_{a,0}$ so that the first inequality of (2.88) holds everywhere. This argument is very similar to the proof of Claim 4.4 in [CD1]. $\square$

Let $\bar u(y,l)$ be the rescaled solution obtained from $u(x,t)$ under (2.23)-(2.24). By Theorem 2.12, 2.13 and the claim, we have local uniform upper and lower bounds on $\bar u$, namely $\bar u_a\le\bar u\le\bar u_b$ where

$$\bar u_a(y,l)\to e^{-\frac{2\gamma A\xi_a}{1-m}}\bar U(ye^{-\gamma A\xi_a})=\bar U_{\xi_a}(y,0)$$

and

$$\bar u_b(y,l)\to e^{-\frac{2\gamma A\xi_b}{1-m}}\bar U(ye^{-\gamma A\xi_b})=\bar U_{\xi_b}(y,0)$$

locally uniformly in y as $l\to\infty$.

In order to show the blow up rate (2.4), we first need the following claim which asserts that Lemma 2.15 holds for our given solution.

Claim 2.8. *For our metric $g(x,t)=u^{1-m}(x,t)\,\delta_{ij}$, there exist $C>0$, $r_0\gg 1$ and $0<t_0<T$ such that the following holds*

$$(T-t)\,\hat w_0\big(A+(T-t)^\gamma\ln|y|\big)\,|\mathrm{Rm}_{g(t)}(e^{A(T-t)^{-\gamma}}y)|\le C \tag{2.89}$$

on $|y|\ge r_0$ and $t_0<t<T$.

Proof of Claim 2.8. The proof is the same as that of Lemma 2.15 except a few modifications which we point out next. Instead of (2.78), now we have

$$\hat{w}^-(\Xi_a, -\ln(T-t)) \leq \frac{|x|^2 u(x,t)}{T-t} \leq \hat{w}^+(\Xi_b, -\ln(T-t))$$

with $\Xi_a := (T-t)^\gamma(\ln|x| - \gamma A\xi_a)$, $\Xi_b := (T-t)^\gamma(\ln|x| - \gamma A\xi_b)$ and for those points (x,t) with $|x|e^{-\gamma A\xi_a} \geq e^{\frac{A}{(T-t)^\gamma}} e^{\xi_1}$.

The proof of Lemma 2.15 now applies after if choose possibly larger r_0 and t_0 now depending on ξ_a and ξ_b. To be specific, we can choose them r_0 and t_0 so that $\ln\frac{r_0}{2} \geq \xi_1 + \gamma A\xi_a$ and we have following inequalities instead of (2.79) and (2.80)

$$\frac{1}{2}\hat{w}_0(A + \xi e^{-\gamma\tau}) \leq \hat{w}^-(A + (\xi - \gamma A\xi_a)e^{-\gamma\tau}, \tau)$$

and

$$\hat{w}^+(A + (\xi - \gamma A\xi_b)e^{-\gamma\tau}, \tau) \leq 2\hat{w}_0(A + \xi e^{-\gamma\tau})$$

on $\xi \geq \ln(r_0/2)$ and $\sigma \geq -1$. The rest of the proof follows as before. $\square$

We now continue with the proof of the theorem. Due to the claim, we have for $r_1 \geq r_0$ (c.f. (2.75))

$$\begin{aligned}\sup_{\mathbb{R}^n\setminus B_{r_1}(0)} |\mathrm{Rm}_{\bar{g}_\infty(l)}| &\leq \limsup_{t\to T-}\Big[(T-t)^{1+\gamma}\sup_{|y|\geq r_1}|\mathrm{Rm}_{g(t)}(e^{A(T-t)^{-\gamma}}y)|\Big] \\ &\leq \frac{C\gamma A}{(n-1)(n-2)\ln r_1}.\end{aligned} \tag{2.90}$$

Let us consider a given sequence $l_i \to \infty$. Using the two bounds $\bar{u}_a$ and $\bar{u}_b$, we may pass to a subsequence $\bar{u}(y, l + l_i)$ and obtain a $C^\infty_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ limit $\bar{u}_\infty$ which is an eternal solution of the equation (2.69). After taking limit our two bounds imply

$$\bar{U}_{\xi_a}(y,0) \leq \bar{u}_\infty(y,l) \leq \bar{U}_{\xi_b}(y,0). \tag{2.91}$$

Now our limit $\bar{g}_\infty(y,l) = \bar{u}_\infty^{1-m}(y,l)\delta_{ij}$ has nonnegative Ricci since this is preserved along the flow and the limit under the locally conformally flat condition (c.f. [Ch]).

Our final step will be to show that $\bar{u}_\infty(y,l)$ must be one of the steady gradient solitons

$$\bar{U}_{\xi_0}(y,0) = e^{-\frac{2\gamma A\xi_0}{1-m}}\bar{U}(|y|e^{-\gamma A\xi_0}).$$

Note that the time dilation parameter ξ_0 might be different for different limits along sequences $l_i \to \infty$, but metrics with different ξ_0 represent the same soliton and thus this proves Cheeger-Gromov convergence of the metric $\bar{u}^{1-m}(y,l)\delta_{ij}$ to the same limit soliton as $l \to \infty$. Also this convergence and (2.90) proves (2.4) (c.f. Proposition 2.14).

Let us consider $u_\infty(y,l) := e^{-\frac{2\gamma A}{1-m}l}\bar{u}_\infty(ye^{-\gamma Al}, l)$. Then (2.91) turns into the inequality between eternal solutions of conformally flat Yamabe flow (2.2)

$$\bar{U}_{\xi_a}(y,l) \le u_\infty(y,l) \le \bar{U}_{\xi_b}(y,l). \tag{2.92}$$

$g_\infty(l) = u_\infty^{1-m}(y,l)\delta_{ij}$ is an eternal solution of the flow which has nonnegative Ricci curvature.

To apply Proposition 2.16 to $u_\infty(y,l)$, we need to show it has actually strictly positive Ricci curvature and uniformly bounded $|Rm|$. We first show uniform boundedness of curvature. By (2.90), $\bar{g}_\infty(l)$ has bounded curvature on $\mathbb{R}^n \setminus B_{r_1}(0)$, for some large r_1. We also have a uniform curvature bound of $\bar{g}_\infty(l)$ on $B_{r_1}(0)$ by two bounds (2.91) and interior uniformly parabolic regularity estimate of the equation (2.69). Since $\bar{g}_\infty(l)$ and g_∞ are isometric, this gives uniform bound of $|\mathrm{Rm}|$. Next, the proof for positive Ricci uses Theorem 2.17, the classification of locally conformally flat nonnegative Ricci Yamabe flow having a nontrivial null eigenvector. It solely an interesting result, so we prove it in a separate theorem. $(\mathbb{R}^n, \bar{g}_\infty(l))$ can not be flat by bounds (2.92). Also an eternal solution can not be isometric to a cylinder solution which exists up to a finite time. Hence Ricci of $\bar{g}_\infty(l)$ is positive definite everywhere by Theorem 2.17.

Finally, by Proposition 2.16, we conclude $u_\infty(y,l) = U_{\xi_0}(y,l)$ for some $\xi_0 \le \xi_a$.

□

We will finish with proving of the following result which was used above in the proof of Theorem 2.2.

Theorem 2.17. *For $n \ge 3$, let $(M, g(t))$ for $t \in (0,T)$ be a complete locally conformally flat solution of the Yamabe flow which has nonnegative Ricci and uniformly bounded Riemann curvature. If the Ricci tensor has a null eigenvector at some point (p_0, t_0), then $(M, g(t))$ is either locally isometric to flat Euclidean space or a cylinder solution $(\mathbb{R} \times S^{n-1}, f(t)(dr^2 \times g_{can}))$ where g_{can} is the round metric on S^{n-1} and $f(t) = (n-1)(n-2)(T'-t)$ for some $T' > T$.*

Proof. The uniform boundedness of the Riemann curvature tensor will only be used to apply the (strong) maximum principle. For a locally conformally flat solution of the Yamabe flow, the evolution of Ricci tensor R_{ij} is shown in Lemma 2.4 [Ch] as

$$\partial_t R_{ij} = (n-1)\Delta R_{ij} + \frac{1}{n-2} B_{ij}$$

where B_{ij} is a quadratic expression of R_{ij}. It was shown in (2.11) and (2.12) of [Ch] that, with respect to an orthonormal basis which diagonalize Ricci tensor by $R_{ij} = \lambda_i \delta_{ij}$, we have $B_{ij} = \mu_i \delta_{ij}$ where

$$\mu_i = \sum_{k,l\neq i,\, k>l} (\lambda_k - \lambda_l)^2 + (n-2)\sum_{k\neq i}(\lambda_k - \lambda_i)\lambda_i. \tag{2.93}$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of R_{ij} in an increasing order. Note that for any $1 \leq k \leq n$, we have

$$m_k := \lambda_1 + \lambda_2 + \cdots + \lambda_k = \inf\{\mathrm{Tr}_g(R_{ij}(V,V)) \mid V \subset T_pM \text{ is a subspace of dim k}\}$$

is a concave function of R_{ij}. Since the solution has nonnegative Ricci, $m_j = 0$ implies $\lambda_i = 0$ for all $i \leq j$. From equation (2.93), it is easy to check that the ODE $\partial_t R_{ij} = B_{ij}$ preserves $m_k \geq 0$ under the nonnegative Ricci condition. Therefore, we can apply the strong maximum principle (Lemma 8.1 in [Ha2]) on $m_k \geq 0$. The lemma and the continuity of m_k imply that either $m_k \equiv 0$ or $m_k > 0$ everywhere at each time $t = t'$. Furthermore, if $m_k > 0$ at $t = t'$, $m_k > 0$ for all $t > t'$. As a consequence, there is a well defined decreasing function $\hat{k}(t) \in \{0, \ldots, n\}$ such that $m_k(p,t) = 0$ if $k \leq \hat{k}(t)$ and $m_k(p,t) > 0$ if $k > \hat{k}(t)$. Since $m_k = 0$ iff $\dim(\mathrm{Null}(R_{ij})) \geq k$, we conclude that the rank of R_{ij} is constant in space and it is equal to $n - \hat{k}(t)$, which is increasing with respect to time.

Under the assumption that there is a point (p_0, t_0) where Ricci curvature has a null eigenvector, we will show that the rank of Ricci curvature is either 0 or $n-1$ for all time. By the previous argument, the Ricci tensor can't have full rank for $t \leq t_0$. Also since it is increasing, there is an interval of time (t_1, t_2) with $t_2 \leq t_0$ such that $\dim(\mathrm{Null}(R_{ij})) = k$, for some fixed $k \in \{1, \ldots, n-1, n\}$ on this time interval. If $k = 0$, then it is clear that the solution must be stationary for all time and the solution must be Ricci flat. Since on a locally conformally flat manifold the Riemann curvature tensor is determined by the Ricci tensor, this implies that the solution is locally euclidean. Next, in case where $1 \leq k \leq n-1$ we can exactly follow the argument of Lemma 8.2 [Ha2] on the time interval (t_1, t_2) to conclude that the null space of the Ricci tensor is invariant under parallel translation and also it is invariant in time. Moreover, it lies in the null space of B_{ij}. By this last property and (2.93), we see that k has to be 1 and other λ_is except λ_1 should be the same positive number (possibly different at each point). In this case, the manifold locally splits off along this parallel 1-dimensional null eigenvector distribution (see the lemma which follows after Theorem 8.3 [Ha2]) i.e. $(M, g(t))$ is locally splits $(\mathbb{R} \times N^{n-1}, dr^2 \times g^N(t))$ where $(N, g^N(t))$ is a solution of $n-1$ dimension the Yamabe flow.

Actually, it is locally isometric to a cylinder $(\mathbb{R} \times S^{n-1}, dr^2 \times g_{\text{can}}(t))$ where $g_{\text{can}}(t)$ is a round metric on the sphere. Let us fix a time t. From the previous observation that the other λs are the same, we know that $(N^{n-1}, g^N(t))$ is an Einstein manifold. i.e. $\text{Ric}^N(x) = \lambda(x)g^N$. If $n-1 \geq 3$, $\lambda \equiv$ constant could be seen by the contracted second Bianchi identity. $\nabla_j R_i^j = \frac{1}{2}\nabla_i R$ implies

$$\nabla_i \lambda = \frac{n-1}{2}\nabla_i \lambda \quad \text{or} \quad 0 = \frac{n-1}{2}\nabla_i \lambda$$

depending on the direction i. When $(M^n, g(t))$ is locally conformally flat and $(N^{n-1}, g^N(t))$ is Einstein, we directly check from the Weyl tensor of (M, g) that (N, g^N) is also locally conformally flat and a space form of positive sectional curvature. When $n = 3$, the Cotton tensor of (M, g) vanishes.

$$C_3 := C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{4}\left(\nabla_i R g_{jk} - \nabla_j R g_{ik}\right) \equiv 0.$$

This implies

$$\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik} = 0 \quad \text{and hence} \quad g^{ik}(\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) = 2\nabla_i \lambda = 0.$$

Now again λ is a positive constant and this proves the theorem. □

Remark 2.4. In addition to this, if the manifold is simply connected, the solution is globally $(\mathbb{R}^n, g_{\text{can}})$ or $(\mathbb{R} \times S^{n-1}, f(t)(dr^2 \times g_{\text{can}}))$ with $f(t) = (n-1)(n-2)(T' - t)$ for some $T' > T$. The only simply connected complete locally euclidean manifold is $(\mathbb{R}^n, g_{\text{can}})$. When it locally splits off, let us consider a smooth unit null eigenvector field of Ricci. Its dual 1-form is closed since the vector field is parallel. Since the manifold is simply connected, it is (globally) exact and the potential function will give a global splitting.

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