High-dimensional asymptotics: new insights and methods

Shuaiwen Wang

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2020
ABSTRACT

High-dimensional asymptotics: new insights and methods

Shuaiwen Wang

As an important element of statistics, linear model $y = Ax + w$ has gained a lot of attention for decades. With the emergence of new data, new problems and new techniques, it is still of great interest to study this model under different settings. In this thesis, we focus on an asymptotic framework where the number of observation $n$ is comparable to the number of variables $p$, and only a subset of $k$ components of the coefficient vector $x$ are nonzero with $k$ being comparable to $p$. The prediction, variable selection and concentration properties of several techniques are studied. Regarding variable selection, we consider a class of two stage variable selection procedures, where we generate an optimally tuned Bridge regression estimator $\hat{x} = \arg\min_x \frac{1}{2}\|y - Ax\|^2 + \gamma\|x\|^q$ [KF00] in the first stage, and threshold this estimator in the second stage. We then compare LASSO with our two stage procedures. Further we discuss the best choice of $q$ in the first stage. It turns out that the variable selection performance of such procedures depends on the estimation mean-square error (MSE) of the Bridge estimator. This motivates us to further study the estimation accuracy of Bridge estimators and compare their MSEs for different choices of $q$. The tool of approximate message passing [DMM09, BM11, BLM+15, WMZ18] enables us to characterize the limiting MSE and provide accurate comparison between different estimators. Next we move our focus to the SLOPE estimator $\hat{x} := \arg\min_x \frac{1}{2}\|y - Ax\|^2 + \gamma \sum_{i=1}^p \lambda_i |x|^{(i)}$, where $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$ are the regularization parameters and $|x|^{(1)} \geq \ldots \geq |x|^{(p)} \geq 0$ denote the components of the signal (or regression coefficients) in the decreasing
order [SBC15]. We provide an accurate comparison between the MSE of SLOPE and that of the bridge estimators. The non-separable nature of SLOPE makes it hard to characterize its limiting MSE as $p \to \infty$. Hence we first prove concentration inequalities for its MSE under finite sample and characterize the concentrated mean through a system of equations. By using the concentration results, we show SLOPE has larger MSE than LASSO in a low noise regime and larger MSE than Ridge in a large noise regime.
# Table of Contents

List of Figures v

1 Introduction 1
   1.1 Objective and Organization ................................. 1
   1.2 Asymptotic framework ..................................... 3

2 Bridge regression based two stage variable selection procedure 6
   2.1 A two stage variable selection (TVS) procedure ............ 7
   2.2 Variable selection performance of TVS ........................ 8
   2.3 Comparison between LASSO and LASSO based TVS ............ 10
   2.4 Related Work .............................................. 13

3 Estimation Accuracy of Bridge regression 16
   3.1 Introduction ................................................ 16
   3.2 Analysis of AMSE for nearly black objects ................. 18
   3.3 Analysis of AMSE in large noise scenario ................... 22
   3.4 Analysis of AMSE in large sample scenario ................. 26
   3.5 Connection between large sample scenario with results in classical settings ................................................. 29
   3.6 Debiasing .................................................... 30
      3.6.1 Implications of debiasing for LASSO ................. 30
      3.6.2 Debiasing and Sure Independence Screening .......... 33
List of Figures

2.1 Comparison of AFDP-ATPP curve between LASSO and two-stage LASSO. Here we pick the setting $\delta = 0.8$, $\epsilon = 0.3$, $\sigma \in \{0.5, 0.22, 0.15\}$, $p_G = \delta_1$. For two-stage LASSO, we use optimal tuning $\lambda^*_1$ in the first stage. All the curves are calculated based on Equations (2.2.1) and (2.2.4). The gray dotted line is the upper bound of ATPP that the two-stage LASSO can reach. Notice that even for LASSO, there is an upper bound which it cannot exceed. .......................................................... 12

3.1 The constant coefficient of the second order term in (3.2.2). We set $G = M$. As the signal strength $M$ increases, the optimal choice of $q$ shifts towards 1. .......................................................... 22

3.2 Absolute relative error of first-order and second-order approximations of AMSE under large noise scenario. In these four figures, $p_B = (1 - \epsilon)\delta_0 + \epsilon\delta_1$, $\delta = 0.4$, $\epsilon = 0.2$. .......................................................... 24

3.3 The constant $c_q$ in Theorem 3.3.1 part (ii). The maximum is achieved at $q = 2$. .......................................................... 24

3.4 Absolute relative error of first-order and second-order approximations of AMSE under large sample scenario. In these four figures, $p_B = (1 - \epsilon)\delta_0 + \epsilon\delta_1$, $\epsilon = 0.5$, $\sigma = 1$. .......................................................... 28
3.5 Comparison of AFDP-ATPP curve between LASSO and two-stage debiased LASSO. Here we pick the setting $\delta = 0.8$, $\epsilon = 0.3$, $\sigma \in \{0.5, 0.22, 0.15\}$, $p_G = \delta_1$. For the two-stage debiased LASSO, we use optimal tuning $\lambda_1^*$ in the first stage. The gray dotted line is the upper bound for the two-stage LASSO without debiasing can reach. . . . . 33

3.6 Comparison of AFDP-ATPP curve between SIS and the two-stage debiased LASSO. Here we pick the setting $\delta = 0.8$, $\epsilon = 0.3$, $\sigma \in \{0.5, 0.22, 0.15\}$, $p_G = \delta_1$. For the two-stage debiased LASSO, we use optimal tuning $\lambda_1^*$ in the first stage. The gray dotted line is the upper bound that the two-stage LASSO without debiasing can reach. . . . . 34

3.7 Top row: AFDP-ATPP curve under the setting $\delta = 0.8$, $\epsilon = 0.2$, $\sigma \in \{1.5, 3, 5\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and ridge. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\delta = 2$, $\epsilon = 0.4$, $\sigma \in \{2, 4, 8\}$. . . 39

3.8 Top row: AFDP-ATPP curve under the setting $\delta = 0.6$, $\epsilon = 0.4$, $\sigma \in \{0.25, 0.75, 2\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and ridge. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\delta = 0.9$, $\epsilon = 0.4$, $\sigma \in \{1.2, 1.5, 1.9\}$. 40

3.9 Top row: AFDP-ATPP curve under the setting $\epsilon = 0.1$, $\sigma = 0.4$, $\delta \in \{2, 3, 4\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and LASSO. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\epsilon = 0.3$, $\sigma = 0.4$, $\delta \in \{2, 3, 4\}$. . 42
3.10 Top row: AFDP-ATPP curve under the setting \( \epsilon = 0.4, \sigma = 0.22, \delta \in \{0.7, 0.8, 1.2\} \). Second row: Y-axis is the difference of AFDP between the other bridge estimators and LASSO. One standard deviation of the difference is added. ................................. 43

3.11 Top row: AFDP-ATPP curve under the setting \( b_\epsilon = 4/\sqrt{\epsilon}, \sigma = 3, \delta = 0.8, \epsilon \in \{0.25, 0.0625, 0.04\} \). Second row: AFDP-ATPP curve under the setting \( b_\epsilon = 4/\sqrt{\epsilon}, \sigma = 5, \delta = 0.8, \epsilon \in \{0.25, 0.0625, 0.04\} \). One standard deviation is added. ................................. 44

3.12 LASSO vs. two-stage LASSO. Here \( \delta = 0.8, \epsilon = 0.2, M = 8, \sigma \in \{1, 3, 5\} \). The outperformance of two-stage LASSO is the most significant when the noise level is low. When noise gets higher, the gap becomes smaller and smaller. ................................. 44

3.13 Large/small noise scenario under correlated design. ................................. 46

3.14 Large/small noise scenario under i.i.d. non-Gaussian design. We set \( \delta = 0.9, \epsilon = 0.4, M = 8, \sigma \in \{0.8, 1, 2\} \). The degrees of freedom of the t-distribution is \( \nu = 3 \). ................................. 47

3.15 Nearly black object with correlated design. We fix \( \delta = 0.8, \sigma = 3 \) and \( b_\epsilon = 4/\sqrt{\epsilon}, \epsilon \in \{0.25, 0.0625, 0.04\} \). The correlation \( \rho \) is set to 0.5 and 0.9 in the two rows. ................................. 48

3.16 Nearly black object with i.i.d. non-Gaussian design. We fix \( \delta = 0.8, \sigma = 3 \) and \( b_\epsilon = 4/\sqrt{\epsilon}, \epsilon \in \{0.25, 0.0625, 0.04\} \). The degrees of freedom for the t-distribution design is \( \nu = 3 \). ................................. 48

3.17 LASSO vs. two-stage LASSO under general designs. Here \( \delta = 0.8, \epsilon = 0.2, M = 8, \sigma \in \{1, 3, 5\} \). The first two rows are for \( \rho = 0.5, 0.9 \) in correlated design. The last row is for \( \nu = 3 \) in i.i.d. non-Gaussian design. 49
4.1 Comparing the MSE of SLOPE estimator and the expected MSE $\delta(\sigma^2 - \sigma_w^2)$ via state evolution equation (4.1.4), (4.1.5). The SLOPE weights $\{\lambda\}$ is equally spaced within $[0.01, 1]$. Other model parameters are $p = 1000$; The components of $x$ are iid samples from $5 \ast$ Bernoulli(prob = 0.3); The components of $z$ are iid samples from $\mathcal{N}(0, \sigma^2_w)$. 60

5.1 MSE of SLOPE, LASSO and Ridge estimators. SLOPE: BH and SLOPE: unif denote the SLOPE estimators with weights $\lambda_i = \Phi^{-1}(1 - \frac{iq}{2p})/\Phi^{-1}(1 - \frac{q}{2p})$ with $q = 0.5$ and $\lambda_i = 1 - 0.99(i - 1)/p$, respectively. Other model parameters are $\delta = 0.9$, $\epsilon = 0.5$; The nonzero components of $x$ are iid samples from Uniform$[0, 5]$; $\sigma_w \in [0, 5]$. 105

5.2 MSE of SLOPE, LASSO and Ridge estimators, when the system is above phase transition for both SLOPE and LASSO. A case of $\delta < 1$ ($\epsilon = 0.2$) is presented in the upper panel, while one for $\delta > 1$ ($\epsilon = 0.5$) is in the lower panel. The other parameters are the same as in Figure 5.1. 106

5.3 MSE of SLOPE, LASSO and Ridge estimators, when there are tied non-zero elements in the signal. SLOPE:max2 denotes the SLOPE estimator with weights $\lambda_1 = \lambda_2 = 1$ and $\lambda_i = 0$ for $i \geq 3$. SLOPE:unif is the same as in Figure 5.1. We set $\delta = 0.9$, $\epsilon = 0.7$. The non-zero components of $x$ all equal to 5. 107
Acknowledgments

First and foremost, I want to sincerely thank my Ph.D. advisor Prof. Arian Maleki. He has given me tremendous support and guidance in the past six years. His patience and instructions helped me get through the hard-time in this memorable journey. His insightful vision, decent work ethics and meticulous attitude on research also shaped my research values and helped me to stand at where I am now. The advice I have received from him on research and career path is invaluable. I could not have done this much without him. It is my great honor to be his Ph.D student.

I would also like to thank Prof. Ming Yuan, Prof. John Wright, Prof. Cindy Rush and Prof. Yang Feng for serving on my defense committee and providing helpful feedback. In addition, I want to acknowledge all the faculty and staff in the Department of Statistics at Columbia University. It is their consistent endeavors that establish such a great Ph.D. program. I am particularly thankful to Dood Kalicharan and Anthony Cruz for their consistent assistance.

Further I would like to thank all my collaborators and fellow friends at Columbia. In particular, I want to express my gratitude to a great collaborator and friend: Haolei Weng. His solid techniques and deep insights on high dimensional Statistics always make it a joyful experience to work with him. I would also like to mention a close friend, Yuanjun Gao. Being a warm and considerate person, he helped me through a lot of hard time in my life. I want to thank Wenda Zhou, a smart, passionate and hard-working peer who I collaborated with, Jin Hyung Lee and Timothy Jones, two friends with whom I shared a lot of memory. There are more people I want to express
my gratitude, to name a few: Miao Zhang, Rishabh Dudeja, Phyllis Wan, Lisha Qiu, Lydia Hsu, Chengliang Tang, Morgane Austern, Leo Neoffurt, Gabriel Gazzo, Adji Dieng, Yixin Wang, Haihao Lu, Vahab Mirrokni, Xuan Yang, Yilong Zhang, Yuting Ma, Swupnil Sahai, Xiaoxia Han. It was these people who built up my precious PhD memory.

    Last but definitely not the least, I want to thank my parents, Yaling Li and Junbo Wang, for their unconditional support and encouragement. I owe so much to them for their love. This thesis is for them.
Chapter 1

Introduction

1.1 Objective and Organization

Linear model, as a basic element of statistics, is still an important topic due to its simplicity and interpretability. Consider the linear model $y = Ax + w$ with $y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times p}$. We are interested in studying the properties of the unknown coefficient $x$ given observed $y$ and $A$. Depending on the application we may pursue one of the following two goals:

1. Estimation: Here, the goal is to obtain a good estimator $\hat{x}$ which approximates the true coefficient vector $x$.

2. Variable selection: Here, we assume that only $k$ out of $p$ components of $x$ are nonzero and the goal is to identify those nonzero components.

These two problems have been studied extensively in the field of high-dimensional statistics. However, a majority of previous researches focused on asymptotic frameworks where either $n$ grows much faster than $p$ or the sparsity level $k$ is much smaller than $p$. Although guaranteed quality of the obtained estimator or variable selection procedure, such as consistency, can be achieved under these settings, many of the real world problems do not necessarily follow these asymptotic frameworks. Hence, it is
CHAPTER 1. INTRODUCTION

of great interests to study the estimators or variable selection techniques in the other
asymptotic regimes, where the variable selection consistency or exact recovery of \( \mathbf{x} \)
is not possible. This is one of the main objectives of this thesis.

Towards this goal, we focus on an asymptotic framework where \( \frac{n}{p} \to \delta > 0 \) and
\( \frac{k}{p} \to \epsilon \in (0, 1] \). We note that under this framework, consistency of both estimation
and variable selection are not achievable [Wai09b, DT05]. Despite this fact, we will
show in this thesis that we can still provide accurate comparison among different
estimators and variable selection techniques.

Regarding variable selection, we study the performance of a class of two stage
variable selection procedures, where an estimator of \( \mathbf{x} \) is proposed in the first stage
and further thresholded in the second stage, in Chapter 2. This class of variable
selection techniques covers many of the popular variable selection techniques that are
used in practice, such as LASSO. We study two natural questions that arise here: (i).
How are such two stage methods compared with one stage methods, such as LASSO
[Tib96]? (ii). What is the optimal variable selection scheme in the class of two stage
techniques we study in the thesis? By further limiting our scope of the estimator in
the first stage to the class of Bridge regression estimators, defined as [KF00]:

\[
\hat{\mathbf{x}} := \arg\min_{\mathbf{x}} \frac{1}{2} \| \mathbf{y} - A\mathbf{x} \|_2^2 + \gamma \| \mathbf{x} \|_q^q,
\]

we are able to answer the above questions accurately. We first show that the two-stage
approach with the optimally tuned LASSO in the first stage outperforms the LASSO.
Further in regards to what Bridge estimator should be used in the first stage, we prove
that those estimators with smaller mean-square error (MSE) in the first stage, after
thresholding, provide better variable selection performances.

This leads to our discussion in Chapter 3 where we compare the MSE of different
Bridge estimators. By using the approximate message passing framework [DMM09,
BM11, BM12, BLM+15, WMZ18], we exactly characterize the limiting MSE as \( p \to \infty \).
This analysis enables us to provide a precise comparison of different Bridge
estimator. It turns out that different Bridge estimators may outperform the others
in different model regimes.

In recent years, a new member of penalties has received a lot of attention: the sorted $L_1$ norm (SLOPE) \cite{BvdBS15}, defined as:

$$\hat{x} = \arg\min_x \frac{1}{2} \|y - Ax\|_2^2 + \gamma \sum_{i=1}^P \lambda_i |x|_{(i)}.$$ 

The original motivation for proposing SLOPE was to control the false discovery rate (FDR) in linear model \cite{BBSC13,BvdBS15}. However, later works \cite{SC16,BLT18} justified its minimax optimality in estimation, without the knowledge of the underlying sparsity of $x$. As a comparison, LASSO can only achieve the same optimality by absorbing the knowledge of the unknown sparsity into its tuning parameter $\gamma$ (either with oracle or with data-driven sparsity estimation). However there are a few missing parts in these studies. First, it is not clear that if both are given good choices of tuning, which would perform better. This is particularly important given that in practice, data dependent algorithms are used for tuning the parameters of LASSO and SLOPE. These motivate our study of SLOPE in Chapters 4 and 5. The nonseparability of the SLOPE norm brings difficulties in obtaining the limiting MSE. Hence, in Chapter 4, we prove finite-sample concentration inequalities for the MSE of SLOPE. The concentrated “mean” can be further characterized by a system of two equations with explicit form.

In Chapter 5, we make use of our results in Chapter 4, combined with our results on Bridge estimator in Chapter 3, to compare the performance of SLOPE with Bridge estimators.

### 1.2 Asymptotic framework

In this section, we review the asymptotic framework under which our studies are performed. We start with the definition of a converging sequence adopted from \cite{BM12}.
Definition 1.2.1. The sequence of instances \( \{ \mathbf{x}(p), w(p), A(p) \}_{p \in \mathbb{N}} \), indexed by \( p \), is said to be a standard converging sequence if

(a) \( n = n(p) \) such that \( \frac{n}{p} \to \delta \in (0, \infty) \).

(b) The empirical distribution of the entries of \( \mathbf{x}(p) \) converges weakly to a probability measure \( p_x \) on \( \mathbb{R} \). Our discussion in Chapters 2 and 3 require \( p_x \) to have finite second moment. Further, \( \frac{1}{p} \sum_{i=1}^{p} x_i(p)^2 \) converges to the second moment of \( p_B \); and \( \frac{1}{p} \sum_{i=1}^{p} I(x_i(p) = 0) \to p_B(\{0\}) \). Chapters 4 and 5 require \( p_x \) to be bounded from above.

(c) The empirical distribution of the entries of \( w(p) \) converges weakly to a zero-mean distribution with variance \( \sigma^2 \). Furthermore, \( \frac{1}{n} \sum_{i=1}^{n} w_i(p)^2 \to \sigma^2 \).

(d) \( A_{ij}(p) \overset{i.i.d.}{\sim} N(0, \frac{1}{n}) \).

The asymptotic scaling \( n/p \to \delta \) specified in Condition (a) was proposed by Huber in 1973 [H+73], and has become one of the most popular asymptotic settings especially for studying problems with moderately large dimensions [EK+10, EKBB+13, DM16, SCC17, DW+18, SC18]. Regarding Condition (b), suppose the entries of \( \mathbf{x}(p) \) form a stationary ergodic sequence with marginal distribution determined by some probability measure \( p_B \). According to Birkhoff’s ergodic theorem, it is clear that Condition (b) will hold almost surely. Thus Condition (b) can be considered as a weaker notion of this Bayesian set-up. Similar interpretation works for Condition (c). Regarding Condition (d), many related works assume it as well [DMM09, BM12, BvdBS+15]. Moreover, we would like to point out that there are a lot of empirical and a few theoretical studies revealing the universal behavior of i.i.d. Gaussian design matrices over a wider class of distributions. See [BLM+15] and the references therein. Hence, the Gaussianity of the design does not play a critical role in our final results. The numerical studies presented in Section 3.7.7 confirm this claim. The independence assumption of the design entries is critical for our analysis. Given that our analyses...
for i.i.d. matrices are already complicated, and the obtained results are highly non-trivial (as will be seen in Section 3.2, 3.3, 3.4), we leave the study of general design matrices for a future research. However, the numerical studies performed in Section 3.7.7 imply that the main conclusions of our paper are valid even when the design matrix is correlated.

In the rest of the paper, we assume the vector of regression coefficients $\mathbf{x}$ is sparse. More specifically, we assume $p_x = (1 - \epsilon)\delta_0 + \epsilon \mathbf{p}_G$, where $\delta_0$ denotes a point mass at 0 and $\mathbf{p}_G$ is a probability measure without any point mass at 0. Accordingly, the mixture proportion $\epsilon$ represents the sparsity level of $\mathbf{x}(p)$ in the converging sequence. $Z$ represents a standard normal random variable, while $\mathbf{g}, \mathbf{h}$ are reserved for standard Gaussian vectors with dimensions specified in their contexts. Regarding notation, we use bold uppercase letters for matrices and bold lowercase letters for vectors. Unbold letters are for scalar variables. Subscripts like $i$ attached to a vector are used to denote its $i$th component.
Chapter 2

Bridge regression based two stage variable selection procedure

In this chapter, we discuss the variable selection problem for linear models under the high dimensional asymptotics defined in Section 1.2. Consider the linear regression model

\[ y = Ax + w, \]

with \( y \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times p} \), \( x \in \mathbb{R}^p \) and \( w \in \mathbb{R}^n \). Suppose only a few elements of \( x \) are nonzero. The problem of variable selection is to find these nonzero locations of \( x \). Motivated by the concerns about the instability and high computational cost of classical variable selection techniques, such as best subset selection and stepwise selection, Tibshirani proposed LASSO [Tib96] to perform parameter estimation and variable selection simultaneously. The LASSO estimate is given by

\[ \hat{x}(1, \lambda) := \arg \min_x \frac{1}{2} \| y - Ax \|_2^2 + \lambda \| x \|_1, \quad (2.0.1) \]

where \( \lambda \) is the tuning parameter, and \( \| \cdot \|_1 \) is the \( \ell_1 \) norm. The regularization term \( \| x \|_1 \) stabilizes the variable selection process while the convexity of (2.0.1) reduces the computational cost.

Compared to LASSO, other convex regularizers, such as \( \| x \|_2^2 \), introduce larger
penalties to the large components of \( \mathbf{x} \). Hence, their estimates might be more stable than LASSO. Even though the solutions of many of these regularizers are not sparse (and thus not automatically perform variable selection), we may threshold their estimates to select variables. This observation leads us to the following questions: can such two-stage methods with other regularizers outperform LASSO in variable selection? If so, which regularizer should be used in the first stage? We would like to address these questions in this Chapter.

In particular, we study the performances of the two-stage variable selection (TVS) techniques mentioned above, with the first stage based on the class of bridge estimators [FF93]:

\[
\hat{\mathbf{x}}(q, \lambda) := \arg\min_{\mathbf{x}} \frac{1}{2} \| \mathbf{y} - \mathbf{A}\mathbf{x} \|^2_2 + \lambda \| \mathbf{x} \|^q_q, \tag{2.0.2}
\]

where \( \| \mathbf{x} \|^q_q = \sum_i |x_i|^q \) with \( q \geq 1 \). We address the following question: Which value of \( q \) offers the best variable selection performance? Does LASSO outperform the two-stage methods based on other bridge estimators?

### 2.1 A two stage variable selection (TVS) procedure

The two stage variable selection technique we consider takes \( \hat{\mathbf{x}}(q, \lambda) \) and returns the sparse estimate \( \bar{\mathbf{x}}(q, \lambda, s) \) defined as follows:

\[
\bar{\mathbf{x}}(q, \lambda, s) = \eta_0(\hat{\mathbf{x}}(q, \lambda); s^2/2), \tag{2.1.1}
\]

where \( \eta_0(u; \chi) = u \mathbb{I}_{\{|u| \geq \sqrt{2\chi}\}} \) denotes the hard threshold function and it operates on a vector in a component-wise manner. The nonzero elements of \( \bar{\mathbf{x}}(q, \lambda, s) \) are used as selected variables.

In the rest of this chapter, we give a thorough investigation of such TVS techniques under the asymptotic setting \( n/p \to \delta \in (0, \infty) \).
CHAPTER 2. BRIDGE REGRESSION BASED TWO STAGE VARIABLE SELECTION PROCEDURE

Under our asymptotic framework, we are able to obtain a sharp characterization of the variable selection “error” (we will clarify our definition of this error in Section 1.2). The asymptotically exact expressions we derive for the error open a new way for comparing the aforementioned variable selection techniques accurately.

It turns out that the variable selection performance of TVS is closely connected with the estimation quality of the bridge estimator in the first stage; a bridge estimator with a smaller asymptotic mean square error (AMSE) in the first stage offers a better variable selection performance in the TVS. This novel observation enables us to connect and translate the study of TVS to the comparison of the estimation accuracy of different bridge estimators.

Since later on, we will discuss TVS at \( \hat{x} \) with optimally tuned \( \lambda \), we clarify the following notation here: The asymptotic mean square error (AMSE) of the bridge estimator \( \hat{x}(q, \lambda) \) is defined as the almost sure limit

\[
\text{AMSE}(q, \lambda) \triangleq \lim_{p \to \infty} \frac{1}{p} \| \hat{x}(q, \lambda) - x \|_2^2.
\]

According to [BM11, WMZ18], AMSE\( (q, \lambda) \) is well defined for \( q \in [1, \infty) \) and \( \lambda > 0 \). The optimal tuning \( \lambda_q^* \) is defined as

\[
\lambda_q^* \triangleq \arg \min_{\lambda > 0} \text{AMSE}(q, \lambda).
\]

2.2 Variable selection performance of TVS

In order to further study the performance of different TVS, in this section we provide exact formula that describes the behavior of the TVS defined in (2.1.1). Since under our asymptotic setting the exact recovery of the non-zero locations of \( x \) is impossible [Wai09b, RG13], we expect to observe both false positives and false negatives. Hence, for a given sparse estimator \( \hat{x} \), we follow [SBC15] and measure its variable selection performance by the false discovery proportion (FDP) and true positive proportion
CHAPTER 2. BRIDGE REGRESSION BASED TWO STAGE VARIABLE SELECTION PROCEDURE

(TPP), defined as:

\[
FDP(\hat{x}) = \frac{\# \{ i : \hat{x}_i \neq 0, x_i = 0 \}}{\# \{ i : \hat{x}_i \neq 0 \}}, \quad TPP(\hat{x}) = \frac{\# \{ i : \hat{x}_i \neq 0, x_i \neq 0 \}}{\# \{ i : x_i \neq 0 \}}.
\]

In particular, our study will focus on the asymptotic version of FDP and TPP for the LASSO estimate \(\hat{x}(1, \lambda)\) and thresholded estimators \(\bar{x}(q, \lambda, s)\). We define (the limits are in almost surely senses)

\[
AFDP(1, \lambda) = \lim_{p \to \infty} FDP(\hat{x}(1, \lambda)), \quad AFDP(q, \lambda, s) = \lim_{p \to \infty} FDP(\bar{x}(q, \lambda, s)).
\]

Similar definitions are used for ATPP(1, \lambda) and ATPP(q, \lambda, s). The following result adapted from \[BvdBSC13\] characterizes the AFDP and ATPP for LASSO.

**Lemma 2.2.1.** For any given \(\lambda > 0\), almost surely

\[
AFDP(1, \lambda) = \frac{(1 - \epsilon)\mathbb{P}(|Z| > \alpha)}{(1 - \epsilon)\mathbb{P}(|Z| > \alpha) + \epsilon \mathbb{P}(|G + \tau Z| > \alpha \tau)}, \\
ATPP(1, \lambda) = \mathbb{P}(|G + \tau Z| > \alpha \tau),
\]

(2.2.1)

where \((\alpha, \tau)\) is the unique solution to the following equations with \(q = 1:\)

\[
\tau^2 = \sigma^2 + \frac{1}{\delta} \mathbb{E}(\eta_q(B + \tau Z; \alpha \tau^{2-q}) - B)^2, \quad (2.2.2)\\n\lambda = \alpha \tau^{2-q} \left(1 - \frac{1}{\delta} \mathbb{E}\eta'_q(B + \tau Z; \alpha \tau^{2-q})\right), \quad (2.2.3)
\]

with \(\eta_q(\cdot; \cdot)\) being the proximal operator defined as

\[
\eta_q(u; \chi) = \arg \min_z \frac{1}{2}(u - z)^2 + \chi |z|^q,
\]

and \(\eta'_q(\cdot; \cdot)\) being the derivative of \(\eta_q\) with respect to its first argument.

The formulas in this lemma have been derived in terms of convergence in probability in \[BvdBSC13\]. The extension to almost sure convergence is straightforward and is hence skipped. See Appendix C.1 of \[WWM20\] for more information. One of the main goals of this paper is to compare the performance of two-stage variable selection techniques with LASSO. In the next lemma, we derive the AFDP and ATPP of the thresholded estimate \(\bar{x}(q, \lambda, s)\).
CHAPTER 2. BRIDGE REGRESSION BASED TWO STAGE VARIABLE SELECTION PROCEDURE

Lemma 2.2.2. For any given $q \in [1, \infty), \lambda > 0, s > 0$, almost surely

$$\text{AFDP}(q, \lambda, s) = \frac{(1 - \epsilon)\mathbb{P}(\eta_q(|Z|; \alpha) > \frac{s}{\tau})}{(1 - \epsilon)\mathbb{P}(\eta_q(|Z|; \alpha) > \frac{s}{\tau}) + \epsilon\mathbb{P}(|\eta_q(G + \tau Z; \alpha \tau^{2-q})| > s)},$$

$$\text{ATPP}(q, \lambda, s) = \mathbb{P}(|\eta_q(G + \tau Z; \alpha \tau^{2-q})| > s), \hspace{1cm} (2.2.4)$$

where $(\alpha, \tau)$ is the unique solution of (2.2.2) and (2.2.3).

The proof of this lemma is presented in Appendix B.1. With Lemma 2.2.1 and 2.2.2, we are able to compare the performance of LASSO and the performance of TVS with different Bridge estimator in the first stage.

2.3 Comparison between LASSO and LASSO based TVS

The main objective of this chapter is to compare the performance of the TVS techniques under the asymptotic setting of Section 1.2. A natural way for performing this comparison is to set ATPP to a fixed value $\zeta \in [0, 1]$ for different variable selection schemes and then compare their AFDPs.

The first challenge we face in such a comparison is that the TVS may have many different ways for setting ATPP to $\zeta$. If $q > 1$, Lemma 2.2.2 shows that for every given value of the regularization parameter $\lambda$, we can set $s$ (the threshold parameter) in a way that it returns the right level of ATPP. Which of these parameter choices should be used when we compare a TVS with another variable selection technique, such as LASSO? Despite the fact that different choices of $(\lambda, s)$ achieve the same ATPP level $\zeta$, they may result in different values of AFDP. Thus for a fair comparison we pick the one that minimizes AFDP. The next theorem explains how this optimal pair can be found.

Theorem 2.3.1. Consider $q \in (1, \infty)$. Given an ATPP level $\zeta \in [0, 1]$, for every value of $\lambda > 0$ there exists $s = s(\lambda, \zeta)$ such that $\text{ATPP}(q, \lambda, s) = \zeta$. Furthermore, the value of $\lambda$ that minimizes $\text{AFDP}(q, \lambda, s(\lambda, \zeta))$ also minimizes $\text{AMSE}(q, \lambda)$. 
The proof of this theorem can be found in Appendix B.3. Before discussing the implications of this theorem, we state a similar result for LASSO.

**Theorem 2.3.2.** For any $\zeta \in [0, \text{ATPP}(1, \lambda^*_1)]$, there exists at least one $\lambda$ such that $\text{ATPP}(1, \lambda) = \zeta$. Further there exists a unique $s = s_\zeta$ such that $\text{ATPP}(1, \lambda^*_1, s) = \zeta$. There may also exist other $(\lambda, s)$ s.t. $\text{ATPP}(1, \lambda, s) = \zeta$. Among all these estimators, the one that offers the minimal AFDP is $\tilde{x}(1, \lambda^*_1, s_\zeta)$, i.e., the two-stage LASSO with the optimal tuning value $\lambda = \lambda^*_1$.

The proof of this theorem can be found in Appendix B.4. There are a couple of points we would like to emphasize here:

(i) Consider a TVS technique. According to Theorems 2.3.1 and 2.3.2, for $q \in (1, \infty)$, the optimal choice of $\lambda$ does not depend on the ATPP level $\zeta$ we are interested in. Even for $q = 1$, the optimal choice of $\lambda$ is independent of $\zeta$ in a large range of ATPPs. It is the optimal tuning $\lambda^*_q$ for AMSE.

(ii) An implication of Theorem 2.3.2 is that, for a wide range of $\zeta$, a second thresholding step helps with the variable selection of LASSO. Figure 2.1 compares the AFDP-ATPP curve of LASSO with that of the two-stage LASSO. As is clear in this figure, when SNR is higher, the gap between the performance of two-stage LASSO and LASSO becomes larger. We should emphasize that the ATPP level of the two-stage LASSO (with optimal tuning) can not exceed that of $\hat{x}(1, \lambda^*_1)$. We discuss debiasing to resolve this issue in Section 3.6.

(iii) Theorems 2.3.1 and 2.3.2 do not explain how $\lambda^*_q$ can be estimated in practice. This issue will be discussed in Section 3.7. But in a nutshell, any approach that optimizes $\lambda$ for minimizing the out-of-sample prediction error works well.

**Remark 2.3.1.** Theorems 2.3.1 and 2.3.2 prove that the optimal way to use two-stage variable selection is to set $\lambda = \lambda^*_q$ for the regularization parameter in the first stage.
Figure 2.1: Comparison of AFDP-ATPP curve between LASSO and two-stage LASSO. Here we pick the setting $\delta = 0.8$, $\epsilon = 0.3$, $\sigma \in \{0.5, 0.22, 0.15\}$, $p_G = \delta_1$. For two-stage LASSO, we use optimal tuning $\lambda^*_1$ in the first stage. All the curves are calculated based on Equations (2.2.1) and (2.2.4). The gray dotted line is the upper bound of ATPP that the two-stage LASSO can reach. Notice that even for LASSO, there is an upper bound which it cannot exceed.

It is important to point out that $\lambda^*_q$ minimizes $\text{AMSE}(q, \lambda)$ and thus is the optimal tuning for parameter estimation. Therefore, the optimal tuning of the regularization parameter in bridge regression is the same for estimation and variable selection.

In the rest of the chapter, we will use the notation $s^*_q(\zeta)$ for the value of threshold that satisfies $\text{ATPP}(q, \lambda^*_q, s^*_q(\zeta)) = \zeta$.

The two theorems we presented in the last section pave our way in addressing the question we raised in the beginning of this Chapter, i.e., finding the best bridge estimator based TVS technique. Consider $q_1, q_2 \in [1, \infty)$. We would like to compare $\text{AFDP}(q_1, \lambda^*_{q_1}, s^*_{q_1}(\zeta))$ and $\text{AFDP}(q_2, \lambda^*_{q_2}, s^*_{q_2}(\zeta))$. The following corollary of Theorems 2.3.1 and 2.3.2 shows the equivalence of the variable selection and estimation performance of bridge estimators.

**Corollary 2.3.1.** Let $q_1, q_2 \geq 1$. If $\text{AMSE}(q_1, \lambda^*_{q_1}) < \text{AMSE}(q_2, \lambda^*_{q_2})$, then for every $\zeta \in [0, 1]$

$$\text{AFDP}(q_1, \lambda^*_{q_1}, s^*_{q_1}(\zeta)) \leq \text{AFDP}(q_2, \lambda^*_{q_2}, s^*_{q_2}(\zeta)).$$
CHAPTER 2. BRIDGE REGRESSION BASED TWO STAGE VARIABLE SELECTION PROCEDURE

The proof of this result is presented in Appendix B.5. According to Corollary 2.3.1, in order to see which two-stage method is better, we can compare their AMSE under optimal tuning $\lambda_q^*$. Such AMSE is given by (see Theorem A.2.1 and Lemma A.2.1 in the appendix)

$$\text{AMSE}(q, \lambda_q^*) = \mathbb{E} \left( \eta_q(B + \tau_*Z; \alpha_\ast \tau_*^{2-q}) - B \right)^2,$$

where $\tau_*$ and $\alpha_\ast$ satisfy (2.2.2) and (2.2.3) with $\lambda = \lambda_q^*$.

2.4 Related Work

The literature on variable selection is very rich. Hence, the related works we choose to discuss can only be illustrative rather than exhaustive.

Traditional methods of variable selection include best subset selection and stepwise procedures. Best subset selection suffers from high computational complexity and high variance. The greedy nature of stepwise procedures reduces the computational complexity, but limits the number of models that are checked by such procedures. See [Mil02] for a comprehensive treatment of classical subset selection. To overcome these limitations, [Tib96] proposed the LASSO that aims to perform variable selection and parameter estimation simultaneously. Both the variable selection and estimation performance of LASSO have been studied extensively in the past decade. It has been justified in the works of [MB06, ZY06, ZH08] that a type of “irrepresentable condition” is almost sufficient and necessary to guarantee sign consistency for the LASSO. Later [Wai09b] established sharp conditions under which LASSO can perform a consistent variable selection. One implication of [Wai09b] that is relevant to our paper is that, consistent variable selection is impossible under the linear asymptotic regime\(^1\) that we

\(^1\)Throughout the paper, the linear asymptotic is referred to the asymptotic setting with (a) and (b) in Definition 2.1 satisfied. Typically in this case, we have $n$, $p$ and the number of nonzero coefficients $k$ go to infinity proportionally.
consider in this paper. This result is consistent with that of [SBC15] and our paper. Hence, we should expect that both the true positive proportion (TPP) and false discovery proportion (FDP) play a major role in our analyses and comparisons. It is worth mentioning that the rate of convergence for variable selection under Hamming loss has been studied in a sequel of works [GJWY12, KJF14, JZZ14, BNS+18].

Since LASSO requires strong conditions for variable selection consistency, several authors have considered a few variants, such as adaptive LASSO [Zou06] and thresholded LASSO [MY09]. Thresholded LASSO is an instance of two-stage variable selection schemes we study in this paper. [MY09] proved that thresholded LASSO offers a variable selection consistency under weaker conditions than the irrepresentable condition required by LASSO. As we will see later, even the thresholded LASSO does not obtain variable selection consistency under the asymptotic framework of this paper. However, we will show that it outperforms the LASSO in variable selection. Other authors have also studied two-step or even multi-step variable selection schemes in the hope of weakening the required conditions [Zho09, Z+09, LC14, WFQ17]. Note that none of these methods provide consistent variable selection under the linear asymptotic setting we consider in this paper. Study and comparison of these other schemes under our asymptotic setting is an interesting open problem for future research.

A more delicate study of the LASSO estimator and more generally the bridge estimators is necessary for an accurate analysis of two-stage methods under the linear asymptotic regime. Our analysis relies on the recent results in the study of bridge estimators [DMM09, DMM11, BM11, BM12, WMZ18, MAYB13, SBC15]. These papers use the platform offered by approximate message passing (AMP) to characterize sharp asymptotic properties. In particular, the most relevant work to our paper is [SBC15] which studies the solution path of LASSO through the trade-off diagram of the asymptotic FDP and TPP. The present paper makes further steps in the analysis of bridge estimator based two-stage methods under various interesting signal-to-noise ratio settings that have not been considered in [SBC15].
Another line of two-stage methods is the idea of screening [FL08, WR09, JJ+12, CF12]. For instance, in [FL08] a preliminary estimate of the $j$th regression coefficient is obtained by regressing $y$ on only the $j$th predictor. Then a hard threshold function is applied to all the estimates to infer the location of the non-zero coefficients. As we will discuss in Section 3.6.2, this approach is a special form of our TVS with a debiasing performed in the first stage, and hence our variable selection technique under appropriate tuning outperforms Sure Independence Screening of [FL08]. Compared to Sure Independence Screening, the work of [WR09] uses more complicated estimators in the first stage, which is more aligned to our approach. However, [WR09] requires data splitting. While this data splitting achieves certain theoretical improvement, in practice (especially in high-dimensions) this may degrade the performance of a variable selection technique. In this paper, we avoid data splitting. We should also mention that two-stage or multi-stage methods (that have a thresholding step) are also popular for estimation purposes. See for instance [YLR14]. Due to limited space, the current paper will be focused on variable selection and not discuss the estimation performance of TVS. However, an accurate analysis of multi-stage estimation techniques is an interesting problem to study.

Finally, there exists one stream of research with emphasis on the derivation of sufficient and necessary conditions for variable selection consistency under different types of restrictions on the model parameters [FRG09, Wai09a, ASZ10, WWR10, Rad11, DI17, NT18]. These works typically assume that all the entries of the design matrix $A$ and error vector $w$ are independent zero-mean Gaussian, with which they are able to obtain accurate information theoretical thresholds and phase transition for exact support recovery of the coefficients $x$. We refer to [NT18] for a detailed discussion of such results. As will be shown shortly in Section 1.2, we make the same assumption on the design $A$, but allow much weaker conditions on the error term $w$. More importantly, we push the analysis one step further by analyzing a class of TVS when exact recovery is impossible information theoretically.
Chapter 3

Estimation Accuracy of Bridge regression

3.1 Introduction

Results proved in Chapter 2 imply that in order to compare TVS with different optimally tuned Bridge estimators in the first stage, we can simply compare the AMSE of these Bridge estimators. This connects the variable selection problem with the estimation problem. Moreover, a good understanding of the estimation properties of different Bridge estimator is of its own interests.

Note that in the calculation of $\text{AMSE}(q, \lambda_\mathrm{q}^*)$, the values of $\alpha_*$ and $\tau_*$ are required and can only be calculated through the fixed point equations (2.2.2) and (2.2.3). Therefore, we have no access to an explicit formula for $\text{AMSE}(q, \lambda_\mathrm{q}^*)$. Furthermore, due to the nature of different $\ell_q$ regularizers, each bridge estimator has its own strength under different model settings. Many factors including $\delta$, $\sigma$ and $p_B$ may affect our comparison. This poses an extra challenge to completely evaluate and compare AMSE for different values of $q$. To address these issues, we focus on a few regimes that researchers have found useful in applications, and develop techniques to obtain explicit and accurate expressions for $\text{AMSE}(q, \lambda_\mathrm{q}^*)$. These sharp results enable
an accurate comparison among different TVS methods in each setting. The regimes we will consider are the following:

(i) Nearly black objects or rare signals: In this regime, \( \epsilon \) is assumed to be small. In other words, there are very few non-zero coefficients that need to be detected. This model is called nearly black objects [DJHS92] or rare signals [DJ15]. Intuitively speaking, it is also equivalent to the models considered in many other papers in which the sparsity level is assumed to be much smaller than the number of features. See for instance, [MB06, ZY06, ZH08] and the references therein. We will allow the signal strength to vary with respect to \( \epsilon \). It turns out that the rate of signal strength affects the choice of optimal bridge estimator.

(ii) Low SNR: In this model, \( \sigma \) is considered to be large. This assumption is accurate in many social and medical studies. For more information, the reader may refer to [HTT17]. To explain the effect of SNR on the best choice of \( q \), we will also mention a result for high SNR. Such assumption is also standard in the engineering applications, where the quality of measurements is carefully controlled. The analysis that is performed under the low noise setting is often called phase transition analysis, noise sensitivity analysis, or nearly exact recovery. See for instance [OH16, DT05, DMM11].

(iii) Large sample regime: In this regime the per-feature sample size \( \delta \) is large. This regime, as will be seen later, is closely related to the classical asymptotic regime \( n/p \rightarrow \infty \), and is appropriate for traditional applied statistical problems. See for instance [KF00] for the asymptotic analysis of bridge estimators.

For the first two, new phenomena are discovered: the Ridge estimator is optimal among all the bridge estimators in large noise settings; in the setting of rare signals, LASSO achieves the best performance when the signal strength exceeds a certain level. However for signals below that level, other bridge estimators may outperform LASSO. In the large sample scenario, we connect our analyses with the fruits of the
classical low-dimensional asymptotic studies. We will provide new comparison results not available in classical asymptotic analyses of bridge estimators.

In summary, our studies reveal the intricate impact of the combination of SNR and sparsity level on the estimation of the coefficients. We present our contributions more formally in the subsequent sections of this chapter.

3.2 Analysis of AMSE for nearly black objects

As discussed in the preceding section, the formulas of AMSE are implicit and depend on \( \delta, \sigma \) and \( p_B \) in a complicated way. The goal of this section is to obtain explicit and accurate expressions for \( \text{AMSE}(q, \lambda_q^*) \) when \( \epsilon \) is small (i.e. the signal is very sparse).

As discussed in e.g. [DJHS92] for the case of orthogonal design, a major challenge is that the strength of the signal affects the performance of each estimator. Hence, in our analysis we let the strength of the signal vary with \( \epsilon \). This generalization requires an extra notation we introduce here. Let \( G \) denote the random variable with probability measure \( p_G \), which determines the values of the non-zero entries of \( x \). Define

\[
b_{\epsilon} = \sqrt{\mathbb{E}G^2}, \quad \tilde{G} = G/b_{\epsilon}.
\]

Under this parameterization, \( \mathbb{E}\tilde{G}^2 = 1 \) and \( b_{\epsilon} \) represents the (average) magnitude of each non-zero coefficient. We refer to \( b_{\epsilon} \) as the signal strength and will allow it to change with the sparsity level \( \epsilon \). Our first theorem characterizes the behavior of bridge estimators for \( q > 1 \) and small values of \( \epsilon \).

**Theorem 3.2.1.** Suppose that \( b_{\epsilon} \to \infty \) and \( b_{\epsilon} = O(1/\sqrt{\epsilon}) \). For \( q > 1 \), we have

- If \( b_{\epsilon} = \omega(\epsilon^{-\frac{1}{2-q}}) \), then

\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{2-q}} b_{\epsilon}^{2(q-1)} \text{AMSE}(q, \lambda_q^*) = q(q-1)^{\frac{1}{q-1}} \sigma^2 \left[ \mathbb{E}|Z|^{\frac{2}{q-1}} \right]^{\frac{q-1}{q}} \left[ \mathbb{E}|\tilde{G}|^{2q-2} \right]^{\frac{1}{q}}.
\]

\(^1\)O notation used here is the standard big-O notation. We will also use other standard asymptotic notations. If the reader is not familiar with these notation, he/she may refer to Appendix A.1.
• If \( b_\epsilon = o(\epsilon^{-\frac{q}{2}}) \), then \( \lim_{\epsilon \to 0} \epsilon^{-1} b_\epsilon^{-2} \text{AMSE}(q, \lambda_q^*) = 1 \).

• If \( \lim_{\epsilon \to 0} b_\epsilon \epsilon^{-\frac{q}{2}} = c_\epsilon \in (0, \infty) \), then

\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{2}} b_\epsilon^{-\frac{2(q-1)}{q}} \text{AMSE}(q, \lambda_q^*) = \min_C h(C),
\]

where \( h : \mathbb{R}^+ \to \mathbb{R} \) and \( h(C) \triangleq (Cq)^{-\frac{2}{q-1}} \sigma^2 \mathbb{E}[Z|^2] + \mathbb{E}\left( \eta_q(c_\epsilon \tilde{G}; C\sigma^{2-q}) - c_\epsilon \tilde{G} \right)^2 \).

Furthermore, the minimizer of \( h(C) \) is finite.

We note that when \( q > 2 \), \( b_\epsilon = o(\epsilon^{-\frac{q}{2}}) \) always holds, hence only the second item applies. When \( q = 2 \), only the second and the third items apply.

This theorem is proved in Appendix C.2. Before we interpret this result, we characterize \( \text{AMSE}(1, \lambda_1^*) \) in Theorem 3.2.2.

**Theorem 3.2.2.** Suppose that \( b_\epsilon \to \infty \) and \( b_\epsilon = O(1/\sqrt{\epsilon}) \). We have

• If \( b_\epsilon = \omega(\sqrt{\log \epsilon^{-1}}) \), then \( \lim_{\epsilon \to 0} \frac{\epsilon}{\log \epsilon^{-1}} \frac{\text{AMSE}(1, \lambda_1^*)}{\sigma^2} = 2 \sigma^2 \).

• If \( b_\epsilon = o(\sqrt{\log \epsilon^{-1}}) \), then \( \lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^*)}{\sigma^2} = 1 \);

• If \( \frac{b_\epsilon}{\sqrt{2 \log \epsilon^{-1}}} \to c \in (0, \infty) \), then \( \lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^*)}{\sigma^2} = \mathbb{E}(\eta_1(c \tilde{G}; \sigma) - c \tilde{G})^2 \).

This theorem will be proved in Appendix C.3. There are a few points that we should emphasize about Theorems 3.2.1 and 3.2.2.

**Remark 3.2.1.** First let us discuss the assumptions of these two theorems. It is straightforward to show that with \( b_\epsilon = \omega(1/\sqrt{\epsilon}) \), the SNR per measurement goes to infinity. Such scenarios seem uncommon in applications, and for the sake of brevity we have only considered \( b_\epsilon = O(1/\sqrt{\epsilon}) \). Otherwise, the techniques we developed can be applied to higher SNR as well. Furthermore, we postpone the discussion about the case \( b_\epsilon = O(1) \) to Theorem 3.2.3.\(^2\)

\(^2\)For the definitions of the asymptotic notations such as \( \Omega \) refer to Appendix A.1.
Remark 3.2.2. The work of [DJHS92] has studied the problem of estimating an extremely sparse signal under the orthogonal design. The main goal of [DJHS92] is to obtain the minimax risk for the class of $\epsilon$-sparse signals (similar to our model) without any constraint on the signals’ power. They have shown that the approximately least favorable distribution has a point mass at $\Theta(\sqrt{\log(\epsilon^{-1})})$, and that LASSO achieves the minimax risk. Note that there are two major differences between Theorem 3.2.2 and the work of [DJHS92]: (i) our result is for non-orthogonal design, and (ii) we are not concerned with the minimax performance. In fact, we fix the power of the signal and obtain the asymptotic mean square error. This platform enables us to observe several delicate phenomena that are not observed in minimax settings. For instance, as is clear from Theorem 3.2.2, the rate of AMSE($1, \lambda^*_1$) undergoes a transition at the signal strength level $\Theta(\sqrt{\log(\epsilon^{-1})})$. As we will discuss later, below this threshold, LASSO is not necessarily optimal. However, since the risk of the Bayes estimator and LASSO is maximized for $b_\epsilon = \Theta(\sqrt{\log(\epsilon^{-1})})$, this important information is missed in minimax analysis.

Remark 3.2.3. Compared to other bridge estimators, the performance of LASSO is much less sensitive to the strength of the signal: AMSE($1, \lambda^*_1$) $\sim \epsilon \log \epsilon^{-1}$ as long as $b_\epsilon = \Omega(\sqrt{\log \epsilon^{-1}})$, while the order of AMSE($q, \lambda^*_q$) continuously changes as $b_\epsilon$ varies.

Theorems 3.2.1 and 3.2.2 can be used for comparing different bridge estimators, as clarified in our next corollary.

Corollary 3.2.1. Suppose that $b_\epsilon = \epsilon^{-\gamma}$ for $\gamma \in (0, 1/2]$. We have

- If $q > 2\gamma + 1$, then AMSE($q, \lambda^*_q$) $\sim \epsilon^{1-2\gamma}$.
- If $1 < q \leq 2\gamma + 1$, then AMSE($q, \lambda^*_q$) $\sim \epsilon^{\frac{1-2\gamma(q-1)}{q}}$.
- If $q = 1$, then AMSE($q, \lambda^*_q$) $\sim \epsilon \log(\epsilon^{-1})$.

The above result implies that in a wide range of signal strength, $q = 1$ offers the smallest AMSE when the value of $\epsilon$ is very small. Consequently, according to
Corollary 2.3.1, the two-stage LASSO provides the best variable selection performance. One can further confirm that the same conclusion continues to hold as long as \( b_\epsilon = \omega(\sqrt{\log \epsilon^{-1}}) \).

So far, we have seen that if the signal is reasonably strong, i.e. \( b_\epsilon = \omega(\sqrt{\log \epsilon^{-1}}) \), then two-stage LASSO outperforms all the other variable selection techniques. However, once \( b_\epsilon = O(\sqrt{\log \epsilon^{-1}}) \), we can see that \( \text{AMSE}(q, \lambda_q^*) \sim \epsilon b_\epsilon^2 \) for all \( q \geq 1 \). Hence, in order to provide a fair comparison, one should perform finer analyses and obtain a more accurate expression for AMSE. Our next result shows how this can be done.

**Theorem 3.2.3.** Consider \( b_\epsilon = 1 \) and hence \( \tilde{G} = G \). Assume \( G \) is bounded from above. Then we have

\[
\text{AMSE}(1, \lambda_1^*) = \epsilon \mathbb{E}G^2 + o(\epsilon^k), \quad \forall k \in \mathbb{N};
\]

\[
\text{AMSE}(q, \lambda_q^*) = \epsilon \mathbb{E}G^2 - \epsilon^2 \frac{\mathbb{E}^2 \left( \frac{G}{\sigma} + Z \right) \frac{1}{\tau^2} \text{sgn} \left( \frac{G}{\sigma} + Z \right) G}{\mathbb{E}|Z| \frac{2}{\tau^2}} + o(\epsilon^2),
\]

for \( q > 1 \) and where \( \text{sgn} (\cdot) \) denotes the sign of a random variable.

The proof of this theorem is presented in Appendix C.4. The first interesting observation about this theorem is that, the first dominant term of AMSE is the same for all bridge estimators. The second dominant term, on the other hand, is much smaller for \( q = 1 \) compared to the other values of \( q \). Hence, LASSO is *suboptimal* in this setting. Accordingly, two-stage LASSO is outperformed by other TVS methods. However, as is clear from Theorem 3.2.3, we should not expect the bridge estimator with \( q > 1 \) to outperform LASSO by a large margin when \( \epsilon \) is too small. In fact, the second dominant term is proportional to \( \epsilon^2 \) (for \( q > 1 \)), while the first dominant term is proportional to \( \epsilon \). Hence, the second dominant term is expected to become important for moderately small values of \( \epsilon \). In such cases, we expect \( q > 1 \) to offer more significant improvements. Regarding the optimal choice of \( q \), it is determined
Figure 3.1: The constant coefficient of the second order term in (3.2.2). We set $G = M$. As the signal strength $M$ increases, the optimal choice of $q$ shifts towards 1.

by the constant of the second order term in (3.2.2). As is shown in Figure 3.1, while the optimal value of $q$ is case-dependent, it gets closer to 1 as the signal strength increases. This observation is consistent with the message delivered by Theorems 3.2.1 and 3.2.2.

### 3.3 Analysis of AMSE in large noise scenario

This section aims to obtain explicit formulas for the optimal AMSE of bridge estimators in low SNR. This regime is particularly important, since in many social and medical studies, variable selection plays a key role and the SNR is low. The following theorem summarizes the main result of this section.

**Theorem 3.3.1.** As $\sigma \to \infty$, we have the following expansions of $\operatorname{AMSE}(q, \lambda_q^*)$:

(i) For $q = 1$, when $G$ has a sub-Gaussian tail, we have

$$\operatorname{AMSE}(1, \lambda_q^*) = \epsilon \mathbb{E}|G|^2 + o(e^{-C^2\sigma^2}),$$

where $C$ can be any positive number smaller than $C_0$, and $C_0 > 0$ is a constant only depending on $\epsilon$ and $G$. The explicit definition of $C_0$ can be found in the proof.
(ii) For $1 < q \leq 2$, if all the moments of $G$ are finite, then

\[
\text{AMSE}(q, \lambda_q^*) = \epsilon \mathbb{E}|G|^2 - \frac{\epsilon^2 (\mathbb{E}|G|^2)^2 c_q}{\sigma^2} + o(\sigma^{-2}),
\]

with $c_q = \frac{(\mathbb{E}|Z|^{\frac{2-q}{q-1}})^2}{(q-1)^2 \mathbb{E}|Z|^{\frac{2}{q-1}}}$. 

(iii) For $q > 2$, if $G$ has sub-Gaussian tail, then (3.3.2) holds.

We present our proofs in Appendix C.5. Figure 3.2 compares the accuracy of the first-order approximation and second-order approximation for moderate values of $\sigma$. As is clear, for $q \in (1, \infty)$, the second-order approximation provides an accurate approximation of $\text{AMSE}(q, \lambda_q^*)$ for a wide range of $\sigma$. Moreover, the first-order approximation for $\text{AMSE}(1, \lambda_1^*)$ is already accurate as can be justified by its exponentially small second order term in (3.3.1).

According to this theorem, we can conclude that for sufficiently large $\sigma$, two-stage method with any $q > 1$ can outperform the two-stage LASSO. This is because while the first dominant term is the same for all the bridge estimators with $q \in [1, \infty)$, the second order term for LASSO is exponentially smaller (in magnitude) than that of the other estimators. More interestingly, the following lemma shows that in fact $q = 2$ leads to the smallest $\text{AMSE}$ in the large noise regime.

**Lemma 3.3.1.** The maximum of $c_q$, defined in Theorem 3.3.1, is achieved at $q = 2$.

See Figure 3.3 for the plot of $c_q$.

**Proof.** A simple integration by part yields:

\[
\mathbb{E}|Z|^{\frac{2-q}{q-1}} = 2(q - 1) \int_0^{\infty} z^{\frac{q}{q-1}} \phi(z)dz = (q - 1) \mathbb{E}|Z|^{\frac{2}{q-1}}
\]

We can then apply Hölders's inequality to obtain

\[
c_q = \frac{(\mathbb{E}|Z|^{\frac{2}{q-1}})^2}{\mathbb{E}|Z|^{\frac{2}{q-1}}} \leq \frac{\mathbb{E}|Z|^{2} \mathbb{E}Z^2}{\mathbb{E}|Z|^{\frac{2}{q-1}}} = 1 = c_2.
\]

$\blacksquare$
Figure 3.2: Absolute relative error of first-order and second-order approximations of AMSE under large noise scenario. In these four figures, $p_B = (1 - \epsilon)\delta_0 + \epsilon\delta_1$, $\delta = 0.4$, $\epsilon = 0.2$.

Figure 3.3: The constant $c_q$ in Theorem 3.3.1 part (ii). The maximum is achieved at $q = 2$. 
Therefore, while the AMSE of all bridge estimators share the same first dominant
term, Ridge offers the largest second dominant term (in magnitude), and hence the
lowest AMSE. If we combine this result with Corollary 2.3.1, we conclude that in low
SNR regime, two-stage Ridge obtains the best variable selection performance among
TVS schemes with their first stage picked from the class of bridge estimators.

A comparison of this result with that for the high SNR derived in [WMZ18]
clarifies the impact of SNR on the best choice of \( q \).

**Theorem 3.3.2.** Assume \( \epsilon \in (0, 1) \). As \( \sigma \to 0 \), we have the following expansions of
AMSE\((q, \lambda^*_q)\) in terms of \( \sigma \).

(i) For \( q = 1 \), if \( \mathbb{P}(|G| \geq \mu) = 1 \) for some \( \mu > 0 \), \( \delta > M_1(\epsilon) \), and \( \mathbb{E}|G|^2 < \infty \), then
\[
\text{AMSE}(1, \lambda^*_1) = \frac{\delta M_1(\epsilon)}{\delta - M_1(\epsilon)} \sigma^2 + o\left(e^{\frac{(M_1(\epsilon) - \delta)\mu^2}{2\sigma^2}}\right), \tag{3.3.3}
\]
where \( M_1(\epsilon) = \min(1 - \epsilon)\mathbb{E} \eta_1^2(Z; \chi) + \epsilon(1 + \chi^2) \), and \( \tilde{\mu} \) can be any positive
number smaller than \( \mu \).

(ii) For \( 1 < q < 2 \), if \( \mathbb{P}(|G| \leq x) = O(x) \) (as \( x \to 0 \)), \( \delta > 1 \), and \( \mathbb{E}|G|^2 < \infty \) then
\[
\text{AMSE}(q, \lambda^*_q) = \frac{\sigma^2}{1 - 1/\delta} - \sigma^2 \delta^{q+1}(1 - \epsilon)^2(\mathbb{E}|Z|^q)^2 + o(\sigma^{2q}). \tag{3.3.4}
\]

(iii) For \( q = 2 \), if \( \delta > 1 \) and \( \mathbb{E}|G|^2 < \infty \), we have
\[
\text{AMSE}(2, \lambda^*_2) = \frac{\sigma^2}{1 - 1/\delta} - \sigma^4 \frac{\delta^3}{(\delta - 1)^3 \epsilon \mathbb{E}|G|^2} + o(\sigma^4). \tag{3.3.5}
\]

(iv) For \( q > 2 \), if \( \delta > 1 \) and \( \mathbb{E}|G|^{2q-2} < \infty \), then
\[
\text{AMSE}(q, \lambda^*_q) = \frac{\sigma^2}{1 - 1/\delta} - \sigma^4 \frac{\delta^3 \epsilon (q - 1)^2(\mathbb{E}|G|^{q-2})^2}{(\delta - 1)^3 \mathbb{E}|G|^{2q-2} + o(\sigma^4)}. \tag{3.3.6}
\]

The results for \( q \in [1, 2] \) are taken from [WMZ18]. The proof for the case \( q > 2 \)
can be found in Appendix I of [WWM20]. It is straightforward to see that \( M_1(\epsilon) \) is
an increasing function of \( \epsilon \in [0, 1] \) and \( M_1(1) = 1 \). This implies that AMSE\((1, \lambda^*_1)\)
is the smallest among all AMSE\((q, \lambda^*_q)\) with \(q \in [1, \infty)\). As is clear, the first order terms in the expansion of AMSE\((q, \lambda^*_q)\) are the same for all \(q \in (1, \infty)\). However, the second dominant term shows that the smaller values of \(q\) are preferable (note the strict monotonicity only occurs in the range \((1, 2]\)).

Combining the above results with Corollary 2.3.1 implies that in the high SNR setting, two-stage LASSO offers the best variable selection performance. We should also emphasize that as depicted in Figure 2.1, in this regime two-stage LASSO offers a much better variable selection performance than LASSO.

**Remark 3.3.1.** Theorems 3.3.1 and 3.3.2 together give a full and sharp evaluation of the noise-sensitivity of bridge estimators. Among all the bridge estimators with \(q \in [1, \infty)\), LASSO and Ridge are optimal for parameter estimation and variable selection, in the low and large noise settings respectively. This result delivers an intriguing message: sparsity inducing regularization is not necessarily preferable even in sparse models. Such phenomenon might be well explained by the bias-variance tradeoff: variance is the major factor in very noisy settings, thus a regularization that produces more stable estimator is preferred, when the noise is large.

### 3.4 Analysis of AMSE in large sample scenario

Our analysis in this section is concerned with the large \(\delta\) regime. Since \(n/p \to \delta\) in our asymptotic setting, large \(\delta\) means large sample size (relative to the dimension \(p\)). Intuitively speaking, this is similar to the classical asymptotic setting where \(n \to \infty\) and \(p\) is fixed (specially if we assume the fixed number \(p\) is large). We will later connect the results we derive in the large \(\delta\) regime to those obtained in classical asymptotic regime, and provide new insights.

In our original set-up, the elements of the design matrix are \(A_{ij} \overset{i.i.d.}{\sim} N(0, \frac{1}{n})\). This means the SNR \(\text{var}(\sum_j A_{ij} x_j)/\text{var}(w_i) \to \frac{E|B|^2}{\sigma^2} = \delta\) as \(n \to \infty\). Therefore, if we let \(\delta \to \infty\), the SNR will decrease to zero, which is not consistent with the classical
asymptotics in which the SNR is assumed to be fixed. To resolve this discrepancy we scale the noise term by $\sqrt{\delta}$ and use the model:

$$y = Ax + \frac{1}{\sqrt{\delta}} w,$$  \hspace{1cm} (3.4.1)

where \{x, w, A\} is the converging sequence in Definition 1.2.1. Under this model we compare the AMSE of different bridge estimators. The next theorem summarizes the main result.

**Theorem 3.4.1.** Consider the model in (3.4.1) and $\epsilon \in (0, 1)$. As $\delta \to \infty$, we have

(i) For $q = 1$, if $\mathbb{P}(|G| \geq \mu) = 1$ for some $\mu > 0$ and $\mathbb{E}|G|^2 < \infty$, then

$$\text{AMSE}(1, \lambda^*_1) = \frac{M_1(\epsilon)\sigma^2}{\delta} + o(\delta^{-1}),$$  \hspace{1cm} (3.4.2)

where $M_1(\epsilon)$ has the same definition as in Theorem 3.3.2 (i).

(ii) For $1 < q < 2$, if $\mathbb{P}(|G| \leq x) = O(x)$ (as $x \to 0$) and $\mathbb{E}|G|^2 < \infty$, then

$$\text{AMSE}(q, \lambda^*_q) = \frac{\sigma^2}{\delta} - \frac{\sigma^{2q} (1 - \epsilon)^2 (\mathbb{E}|Z|^q)^2}{\epsilon \mathbb{E}|G|^{2q-2}} + o(\delta^{-q})$$  \hspace{1cm} (3.4.3)

(iii) For $q = 2$, if $\mathbb{E}|G|^2 < \infty$, then we have

$$\text{AMSE}(2, \lambda^*_2) = \frac{\sigma^2}{\delta} + \frac{\sigma^2}{\delta^2} \left[ 1 - \frac{\sigma^2}{\epsilon \mathbb{E}G^2} \right] + o(\delta^{-2})$$  \hspace{1cm} (3.4.4)

(iv) For $q > 2$, if $\mathbb{E}|G|^{2q-2} < \infty$, then

$$\text{AMSE}(q, \lambda^*_q) = \frac{\sigma^2}{\delta} + \frac{\sigma^2}{\delta^2} \left[ 1 - \frac{\epsilon(q - 1)^2 \sigma^2 (\mathbb{E}|G|^{q-2})^2}{\mathbb{E}|G|^{2q-2}} \right] + o(\delta^{-2}).$$  \hspace{1cm} (3.4.5)

The proof of Theorem 3.4.1 can be found in Appendix C.6. Figure 3.4 compares the accuracy of the first and second order expansions in large range of $\delta$. As is clear from this figure, the second-order term often offers an accurate approximation over a wide range of $\delta$. 
Chapter 3. Estimation Accuracy of Bridge Regression

Figure 3.4: Absolute relative error of first-order and second-order approximations of AMSE under large sample scenario. In these four figures, $p_B = (1 - \epsilon)\delta_0 + \epsilon\delta_1$, $\epsilon = 0.5$, $\sigma = 1$.

**Remark 3.4.1.** As mentioned in Section 3.3, $M_1(\epsilon)$ is an increasing function of $\epsilon \in [0, 1]$ and $M_1(1) = 1$. This implies that AMSE(1, $\lambda_1^*$) is the smallest among all AMSE(q, $\lambda_q^*$) with q $\in$ [1, $\infty$). Therefore, in this regime LASSO gives the smallest estimation error and thus two-stage LASSO offers the best variable selection performance.

**Remark 3.4.2.** The AMSE(q, $\lambda_q^*$) with q > 1 share the same first dominant term, but have different second order terms. Furthermore, for q $\in$ (1, 2], the smaller q is, the better its performance will be. Such monotonicity does not hold beyond q = 2.
3.5 Connection between large sample scenario with results in classical settings

We now connect our results in this large $\delta$ regime to those obtained in classical asymptotic setting. The classical asymptotics ($p$ fixed) of bridge estimators for all the values of $q \in [0, \infty)$ is studied in [KF00]. We explain LASSO first. According to [KF00], if $\frac{\lambda}{\sqrt{n}} \to \lambda_0 \geq 0$ and $\frac{1}{n} A^\top A \to C$, then

$$\sqrt{n}(\hat{x} - x) \xrightarrow{d} \arg\min_u V(u),$$

(3.5.1)

where $V(u) = -2u^\top W + u^\top Cu + \lambda_0 \sum_{j=1}^p [u_j \text{sgn}(x_j) \mathbb{1}_{\{x_j \neq 0\}} + |u_j| \mathbb{1}_{\{x_j = 0\}}]$ with $W \sim \mathcal{N}(0, \sigma^2 C)$. We will do the following calculations to explore the connections. Since $A_{ij} \sim \mathcal{N}(0, 1/n)$ in our paper, we first make the following changes to LASSO to make our set-up consistent with that of [KF00]:

$$\frac{1}{2} \| y - Ax \|_2^2 + \lambda \| x \|_1 = \frac{1}{2} \left( \| y - \sqrt{n} A \frac{x}{\sqrt{n}} \|_2^2 + 2\sqrt{n} \lambda \| \frac{x}{\sqrt{n}} \|_1 \right).$$

We thus have $C = \frac{1}{n}(\sqrt{n} A)^\top (\sqrt{n} A) \to I$ and $\lambda_0 = 2\lambda$. Now suppose the result (3.5.1) works for $\hat{x}(1, \lambda)$. Then we have

$$\hat{x}(1, \lambda) - x \xrightarrow{d} \arg\min_u V(u),$$

(3.5.2)

where $V(u) = -2u^\top W + u^\top u + 2\lambda \sum_{j=1}^p [u_j \text{sgn}(x_j) \mathbb{1}_{\{x_j \neq 0\}} + |u_j| \mathbb{1}_{\{x_j = 0\}}]$ with $W \sim \mathcal{N}(0, \frac{\sigma^2}{\delta} I)$. It is straightforward to see that the optimal choice of $u$ in (3.5.2) has the following form:

$$\hat{u}_j = \begin{cases} W_j - \lambda \text{sgn}(x_j) & \text{when } x_j \neq 0 \\ W_j - \lambda s(\hat{u}_j) & \text{when } x_j = 0 \end{cases}$$

where $s(u_j) = \text{sgn}(u_j)$ when $u_j \neq 0$ and $|s(u_j)| \leq 1$ when $u_j = 0$. Furthermore, for the case of $x_j = 0$, $\hat{u}_j = 0$ is equivalent to $|W_j| \leq \lambda$ and $\text{sgn}(W_j) = \text{sgn}(\hat{u}_j)$ when $\hat{u}_j \neq 0$. Based on this result, we do the following heuristic calculation to connect our
results with those of [KF00]:

\[
\frac{1}{p} \left\| \hat{x}(1, \lambda) - x \right\|_2^2 \\
\approx \frac{1}{p} \mathbb{E} \left[ \sum_{j:x_j \neq 0} \left( W_j^2 - 2\lambda \text{sgn}(x_j)W_j + \lambda^2 \right) + \sum_{j:x_j=0, \hat{u}_j \neq 0} \left( W_j^2 - 2\lambda W_j \text{sgn}(\hat{u}_j) + \lambda^2 \right) \right] \\
\approx \frac{1}{p} \left[ \sum_{j:x_j \neq 0} \left( \frac{\sigma^2}{\delta} + \lambda^2 \right) + \sum_{j:x_j=0} \mathbb{E} \eta_1^2(W_j; \lambda) \right] = \frac{k}{p} \left( \frac{\sigma^2}{\delta} + \lambda^2 \right) + \frac{p - k}{p} \mathbb{E} \eta_1^2(W_j; \lambda) \\
= \frac{\sigma^2}{\delta} \left[ \frac{p - k}{p} \mathbb{E} \eta_1^2(Z; \sqrt{\delta \lambda/\sigma}) + \frac{k}{p} (1 + (\sqrt{\delta \lambda/\sigma})^2) \right],
\]

where \( k \) is the number of non-zero elements of \( x \) and \( Z \sim N(0,1) \). Note that in our asymptotic setting \( k/p \to \epsilon \) and we consider the optimal tuning \( \lambda_1^* \). Therefore following the above calculations we obtain

\[
\min_{\lambda} \frac{1}{p} \left\| \hat{x}(1, \lambda) - x \right\|_2^2 \approx \frac{\sigma^2}{\delta} \min_{\chi} (1 - \epsilon) \mathbb{E} \eta_1^2(Z; \chi) + \epsilon (1 + \lambda^2) = \frac{M_1(\epsilon)\sigma^2}{\delta}.
\]

This is consistent with (3.4.2) in our asymptotic analysis. We can do similar calculations to show that the asymptotic analysis of [KF00] leads to the first order expansion of AMSE in Theorem 3.4.1 for the case \( q > 1 \).

Based on this heuristic argument, we may conclude that the information provided by the classical asymptotic analysis is reflected in the first order term of \( \text{AMSE}(q, \lambda^*_q) \). Moreover, our large sample analysis is able to derive the second dominant term for \( q > 1 \). This term enables us to compare the performance of different values of \( q > 1 \) more accurately (note they all have the same first order term). Such comparisons cannot be performed in [KF00].

### 3.6 Debiasing

#### 3.6.1 Implications of debiasing for LASSO

As is clear from Theorem 2.3.2, since LASSO produces a sparse solution, it is not possible for a LASSO based two-stage method to achieve ATPP values beyond what
is already reached by the first stage. This problem can be resolved by debiasing. In this approach, instead of thresholding the LASSO estimate (or in general a bridge estimate), we threshold its debiased version. Below we will add a dagger $\dagger$ to aforementioned notations to denote their corresponding debiased version. Recall $\hat{x}(q, \lambda)$ denotes the solution of bridge regression for any $q \geq 1$. Define the debiased estimates as

(i) For $q = 1$,

$$\hat{x}^\dagger(1, \lambda) \triangleq \hat{x}(1, \lambda) + A^\top \frac{y - A\hat{x}(1, \lambda)}{1 - \|\hat{x}(1, \lambda)\|_0/n},$$

where $\| \cdot \|_0$ counts the number of non-zero elements in a vector.

(ii) For $q > 1$,

$$\hat{x}^\dagger(q, \lambda) \triangleq \hat{x}(q, \lambda) + A^\top \frac{y - A\hat{x}(q, \lambda)}{1 - f(\hat{x}(q, \lambda), \hat{\gamma}_0)/n},$$

where $f(v, w) = \sum_{i=1}^{p} \frac{1}{1 + w(q-1)|v_i|^q}$ and $\gamma = \hat{\gamma}_0$ is the unique solution of the following equation:

$$\frac{\lambda}{\gamma} = 1 - \frac{1}{n} f(\hat{x}(q, \lambda), \hat{\gamma}).$$

We have the following theorem to confirm the validity of the debiasing estimator $\hat{x}^\dagger(q, \lambda)$.

**Theorem 3.6.1.** For any given $q \in [1, \infty)$, with probability one, the empirical distribution of the components of $\hat{x}^\dagger(q, \lambda) - x$ converges weakly to $N(0, \tau^2)$, where $\tau$ is the solution of $(2.2.2)$ and $(2.2.3)$.

See Appendix C.7 for the proof. In order to perform variable selection, one may apply the hard thresholding function to these debiased estimates, i.e.,

$$\bar{x}^\dagger(q, \lambda, s) = \eta_0(\hat{x}^\dagger(q, \lambda); s^2/2) = \hat{x}^\dagger(q, \lambda)\mathbb{1}_{\{|\hat{x}^\dagger(q, \lambda)| \geq s\}}.$$

We use the notations ATPP$^\dagger(q, \lambda, s)$ and AFDP$^\dagger(q, \lambda, s)$ to denote the ATPP and AFDP of $\bar{x}^\dagger(q, \lambda, s)$ respectively. In the case of LASSO, note that unlike $\hat{x}(1, \lambda)$
the debiased estimator $\hat{x}^\dagger(1, \lambda)$ is dense. Hence we expect the two-stage variable selection estimate $\tilde{x}^\dagger(1, \lambda, s)$ to be able to reach any value of ATPP between $[0, 1]$.

The following theorem confirms this claim.

**Theorem 3.6.2.** Given the ATPP level $\zeta \in [0, 1]$, for every value of $\lambda > 0$, there exists $s(\lambda, \zeta)$ such that $\text{ATPP}^\dagger(1, \lambda, s(\lambda, \zeta)) = \zeta$. Furthermore, whenever $\tilde{x}^\dagger(1, \lambda, s)$ and $\bar{x}(1, \lambda, \bar{s})$ reach the same level of ATPP, they have the same AFDP. The value of $\lambda$ that minimizes $\text{AFDP}^\dagger(1, \lambda, s(\lambda, \zeta))$ also minimizes $\text{AMSE}(1, \lambda)$.

As expected since the solution of bridge regression for $q > 1$ is dense, the debiasing step does not help variable selection for $q > 1$. Our next theorem confirms this claim.

**Theorem 3.6.3.** Consider $q > 1$. Given the ATPP level $\zeta \in [0, 1]$, for every value of $\lambda > 0$, there exists $s(\lambda, \zeta)$ such that $\text{ATPP}^\dagger(q, \lambda, s(\lambda, \zeta)) = \zeta$. Furthermore, whenever $\tilde{x}^\dagger(q, \lambda, s)$ and $\bar{x}(q, \lambda, \bar{s})$ reach the same level of ATPP, they have the same AFDP. Also, the value of $\lambda$ that minimizes $\text{AFDP}^\dagger(q, \lambda, s(\lambda, \zeta))$ also minimizes $\text{AMSE}(q, \lambda)$. As a result, the optimal value of $\text{AFDP}^\dagger(q, \lambda, s(\lambda, \zeta))$ is the same as $\text{AFDP}(q, \lambda^*, s^*_q(\zeta))$.

For the proof of Theorems 3.6.2 and 3.6.3, please refer to Appendix C.7.

**Remark 3.6.1.** Comparing Theorem 3.6.2 with Theorem 2.3.2, we see that replacing LASSO in the first stage with the debiased version enables to achieve wider range of ATPP level. On the other hand, given the value of $\lambda$, if $\tilde{x}^\dagger(1, \lambda, s)$ and $\bar{x}(1, \lambda, \bar{s})$ reach the same level of ATPP, their AFDP are equal as well. Therefore, the debiasing for LASSO expands the range of AFDP-ATPP curve without changing the original one. Figure 3.5 compares the variable selection performance of LASSO with that of the two-stage scheme having the debiased LASSO estimate in the first stage. Compare this figure with Figure 2.1 to see the difference between the two-stage LASSO and two-stage debiased LASSO.
Figure 3.5: Comparison of AFDP-ATPP curve between LASSO and two-stage debiased LASSO. Here we pick the setting \( \delta = 0.8, \epsilon = 0.3, \sigma \in \{0.5, 0.22, 0.15\}, p_G = \delta_1 \). For the two-stage debiased LASSO, we use optimal tuning \( \lambda^*_1 \) in the first stage. The gray dotted line is the upper bound for the two-stage LASSO without debiasing can reach.

Remark 3.6.2. The debiasing does not present any extra gain to the two-stage variable selection technique based on bridge estimators with \( q > 1 \). In other words, debiasing does not change the AFDP-ATPP curve for \( q > 1 \).

3.6.2 Debiasing and Sure Independence Screening

Sure Independence Screening (SIS) is a variable selection scheme proposed for ultra-high dimensional settings [FL08]. Our asymptotic setting is not considered an ultra-high dimensional asymptotic. We are also aware that SIS is typically used for screening out irrelevant variables and other variable selection methods, such as LASSO, will be applied afterwards. Nevertheless, we present a connection and comparison between our two-stage methods and SIS in the linear asymptotic regime. Such comparisons shed more light on the performance of SIS. It is straightforward to confirm that Sure Independence Screening is equivalent to

\[
\hat{x}^\dagger(q, \infty, s) = \eta_0(\hat{x}^\dagger(q, \infty); s^2/2) = \eta_0(A^Ty; s^2/2).
\]
Figure 3.6: Comparison of AFDP-ATPP curve between SIS and the two-stage debiased LASSO. Here we pick the setting $\delta = 0.8$, $\epsilon = 0.3$, $\sigma \in \{0.5, 0.22, 0.15\}$, $p_G = \delta_1$. For the two-stage debiased LASSO, we use optimal tuning $\lambda_1^*$ in the first stage. The gray dotted line is the upper bound that the two-stage LASSO without debiasing can reach.

Therefore, the main difference between the approach we propose in this paper and SIS, is that SIS sets $\lambda$ to $\infty$, while we select the value of $\lambda$ that minimizes AMSE.\(^3\) This simple difference may give a major boost to the variable selection performance. The following lemma confirms this claim.

**Lemma 3.6.1.** Consider $q \geq 1$. Given any ATPP level $\zeta \in [0, 1]$, let $\text{AFDP}_{\text{sis}}(\zeta)$ and $\text{AFDP}^\dagger(q, \lambda_q^*, s(\lambda_q^*, \zeta))$ denote the asymptotic FDP of SIS and two-stage debiased bridge estimator respectively, when their ATPP is equal to $\zeta$. Then, $\text{AFDP}^\dagger(q, \lambda_q^*, s(\lambda_q^*, \zeta)) \leq \text{AFDP}_{\text{sis}}(\zeta)$.

Refer to Appendix C.7 for the proof. Note that when the noise $\sigma$ is large, we expect the optimally tuned $\lambda$ to be large, and hence the performance of SIS gets closer to the TVS. However, as $\sigma$ decreases, the gain obtained from using a better estimator in the first stage improves. Figure 3.6 compares the performance of SIS and TVS under different noise settings.

\(^3\)Our approach is more aligned with the approach proposed in [WR09]. However, [WR09] uses data splitting to select $\lambda$. 
3.7 Numerical experiments

3.7.1 Objective and Simulation Set-up

This section aims to investigate the finite sample performances of various two-stage variable selection estimators under the three different regimes analyzed in Section 3.1. In particular, we will study to what extent our theory works for more realistic situations, where model parameters $\sigma$, $\epsilon$, $\delta$ are of moderate magnitudes or the iid-Gaussian design assumption is violated. For brevity, we will use bridge estimator to refer to the corresponding two-stage method whenever it does not cause any confusion. More specifically, in all the figures, $\ell_q$ will be used to denote the TVS that uses the bridge estimator with $q$ in the first stage, and $\ell_1$-db denotes the two-stage debiased LASSO. The performances of different methods will be compared via the AFDP-ATPP curves.\footnote{Since the simulations are in finite samples, the curve we calculate is actually FDP-TPP instead of the asymptotic version. With a little abuse of notation, we will call it AFDP-ATPP curve throughout the section.}

The organization of this section is as follows. In Sections 3.7.2 - 3.7.6, we focus on experiments under iid-Gaussian design as assumed in our theories. In Section 3.7.7, we present numerical results for non-i.i.d. or non-Gaussian designs to evaluate the accuracy of our results, when i.i.d. Gaussian assumption on $A$ is violated.

We adopt the following settings for iid-Gaussian design. The settings for general design are described in Section 3.7.7.

1. Number of variables is fixed at $p = 5000$. Sample size $n = p\delta$ is then decided by $\delta$.

2. Given the values of $\delta$, $\epsilon$, $\sigma$, we sample $A \in \mathbb{R}^{n \times p}$ with $A_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{n})$. We pick the probability measure $p_G$ as a point mass at $M$ where $M$ will be specified in each scenario. We generate $x \in \mathbb{R}^p$ with $x_i \overset{i.i.d.}{\sim} p_B = (1-\epsilon)\delta_0 + \epsilon p_G$, and $w \in \mathbb{R}^n$. 

3. Since the simulations are in finite samples, the curve we calculate is actually FDP-TPP instead of the asymptotic version. With a little abuse of notation, we will call it AFDP-ATPP curve throughout the section.
with \( w_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \) or \( \mathcal{N}(0, \sigma^2/\delta) \).\(^5\) Construct \( y \) according to \( y = Ax + w \).

3. For each data set \((y, A)\), AFDP-ATPP curves will be generated for different variable selection methods. In each setting of parameters, 80 samples are drawn and the average AFDP-ATPP curves are calculated. The associated one standard deviation confidence interval will be presented.

We compute bridge estimators via coordinate descent algorithm, with the proximal operator \( \eta_q(x; \tau) \) calculated through a properly implemented Newton’s method.

We discuss how to pick optimal tuning under iid-Gaussian design in Section 3.7.2. Section 3.7.3 presents the large/small noise scenario. Section 3.7.4 is devoted to the large sample regime. Section 3.7.5 covers the nearly black object scenario. In Section 3.7.6, we compare the performance of LASSO and two-stage LASSO to shed more lights on our two-stage methods.

### 3.7.2 Estimating the optimal tuning \( \lambda^*_q \)

For two-stage variable selection procedures, it is critical to have a good estimator in the first step. One challenge here is to search for the optimal tuning that minimizes AMSE of \( \hat{x}(q, \lambda) \). According to the result of Theorem A.2.1 and the definition of AMSE in (2.1), it is straightforward to see that \( \tau^2 = \sigma^2 + \frac{1}{\delta} \text{AMSE} \). Hence, one can minimize \( \tau^2 \) to achieve the same optimal tuning. Motivated by [MMB18], we can obtain a consistent estimator of \( \tau^2 \):

\[
q = 1: \quad \hat{\tau}^2 = \frac{\|y - A\hat{x}(1, \lambda)\|_2^2}{n(1 - \|\hat{x}(1, \lambda)\|_0/n)^2}, \quad q > 1: \quad \hat{\tau}^2 = \frac{\|y - A\hat{x}(1, \lambda)\|_2^2}{n(1 - f(\hat{x}(q, \lambda), \hat{\gamma}_\lambda)/n)^2},
\]

where \( f(\cdot, \cdot), \hat{\gamma}_\lambda \) are the same as the ones in (3.6.1) and (3.6.2). The consistency \( \hat{\tau} \to \tau \) can be easily seen from the proof of Theorem 3.6.1. We thus do not repeat it. As a result, we approximate \( \lambda^*_q \) by searching for the \( \lambda \) that minimizes \( \hat{\tau}^2 \). Notice

\(^5\)The setting \( w_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2/\delta) \) will be used in the large sample scenario, since we have scaled the error term by \( \sqrt{\delta} \) in our asymptotic analysis in Section 3.4.
that this problem has been studied for LASSO in [MMB18] and a generalization is straightforward for other bridge estimators. We use the following grid search strategy:

- Initialization: An initial search region \([a, b]\), a window size \(\Delta\) and a grid size \(m\).
- Searching: A grid with size \(m\) is built over \([a, b]\), upon which we search in descending order for \(\lambda\) that minimizes \(\hat{\tau}^2\) with warm initialization.
  - If the minimal point \(\hat{\lambda} \in (a, b)\), stop searching and return \(\hat{\lambda}\).
  - If \(\hat{\lambda} = a \) or \(b\), update the search region with \([\frac{a}{10}, a]\) or \([b, b + \Delta]\) and do the next round of searching.
- Stability: If the optimal \(\hat{\lambda}\) obtained from two consecutive search regions are smaller than a threshold \(\epsilon_0\), we stop and return the previous optimal \(\hat{\lambda}\); If the number of non-zero locations of a LASSO estimator is larger than \(n\) (which may happen numerically for very small tuning), we set its \(\hat{\tau}^2\) to \(\infty\).

For our experiments, we pick the initial \([a, b] = [0.1, \frac{1}{2} \| A^\top y \|_\infty], \Delta = \frac{1}{2} \| A^\top y \|_\infty\) and \(m = 15\).

### 3.7.3 From large noise to small noise

Theorems 3.3.1 and 3.3.2 showed that in low and high SNR situations, ridge and LASSO offer the best performances respectively. These results are obtained for limiting cases \(\sigma \to \infty\) and \(\sigma \to 0\). In this section, we run a few simulations to clarify the scope of applicability of our analysis. Toward this goal, we fix the probability measure \(p_G = \delta_M\) with \(M = 8\) and run TVS for \(q \in \{1, 1.2, 2, 4\}\) and debiased LASSO\(^6\) under four settings:

\(^6\)We include the results for two-stage debiased LASSO in Sections 3.7.3 - 3.7.5 to validate the effect of debiasing stated in Theorem 3.6.2 and Remark 3.6.1.
1. $\delta = 0.8$, $\epsilon = 0.2$: The results are shown in Figure 3.7. Here we pick $\sigma \in \{1.5, 3, 5\}$. As expected from our theoretical results, for small values of noise LASSO offers the best performance. As we increase the noise, eventually ridge outperforms LASSO and the other bridge estimators. Note that under this setting, the outperformance occurs at a high noise level so that all estimators have large errors. In this example, we make $1 > \delta > M_1(\epsilon)$. Refer to Theorem 3.3.2 for the importance of this condition.

2. $\delta = 2$, $\epsilon = 0.4$: The results are included in Figure 3.7. Here we pick $\sigma \in \{2, 4, 8\}$. Similar phenomena are observed. However for all choices of $\sigma$, the AFDP-ATPP curves of different methods are quite close to each other.

3. $\delta = 0.6$, $\epsilon = 0.4$: Figure 3.8 contains the results for this part. Here we have $\sigma \in \{0.25, 0.75, 2\}$. An important feature of this simulation is that $\delta < M_1(\epsilon)$, which does not satisfy the condition of Theorem 3.3.2. It is interesting to observe that in this case, ridge outperforms LASSO even for small values of the noise. We thus see that the superiority of LASSO in small noise characterized by Theorem 3.3.2 may not hold when the conditions of the theorem are violated. In fact, Theorem 3.3.2 is restricted to the regime below the phase transition (i.e., when the signal can be fully recovered without noise). However, in the current setting, the optimal AMSE for $q = 1, 1.2, 2, 4$ at $\sigma = 0$ are 14.9, 12.2, 10.2, 11.6, respectively.

4. $\delta = 0.9$, $\epsilon = 0.4$: The results are shown in Figure 3.8. Here we have $\sigma \in \{1.2, 1.5, 1.9\}$. This group of figures provide us with examples where ridge based TVS outperforms the other two-stage methods, and at the same time reaches a quite satisfactory AFDP-ATPP trade-off. For instance, when $\sigma = 1.5$ and AFDP $\approx 0.2$, for ridge we have ATPP $\approx 0.85$ while that for LASSO is around 0.75. Note that here $M_1(\epsilon) < \delta < 1$. 


Figure 3.7: Top row: AFDP-ATPP curve under the setting $\delta = 0.8, \epsilon = 0.2, \sigma \in \{1.5, 3, 5\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and ridge. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\delta = 2, \epsilon = 0.4, \sigma \in \{2, 4, 8\}$. 
Figure 3.8: Top row: AFDP-ATPP curve under the setting $\delta = 0.6$, $\epsilon = 0.4$, $\sigma \in \{0.25, 0.75, 2\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and ridge. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\delta = 0.9$, $\epsilon = 0.4$, $\sigma \in \{1.2, 1.5, 1.9\}$. 


3.7.4 Large sample regime

We will validate the results in Theorem 3.4.1, which are obtained under the limiting case \( \delta \to \infty \). We fix the probability measure \( p_G = \delta M \) with \( M = 1 \) and consider the following settings for \( q \in \{1, 1.5, 2, 4\} \) and debiased LASSO:

1. \( \epsilon = 0.1, \sigma = 0.4 \): The results for this setting are shown in Figure 3.9. We vary \( \delta \in \{2, 3, 4\} \). As is clear, LASSO starts to outperform the others even when \( \delta = 2 \). As \( \delta \) increases, LASSO remains the best, but all the methods are becoming better and the AFDP-ATPP curves get closer to each other.

2. \( \epsilon = 0.3, \sigma = 0.4 \): The results can be found in Figure 3.9. Again \( \delta \in \{2, 3, 4\} \). Similar phenomena are observed. Compared to the previous setting, a larger \( \epsilon \) leads to a higher SNR and all the methods have improved performances.

3. \( \epsilon = 0.4, \sigma = 0.22 \): The results are shown in Figure 3.10. We set \( \delta \in \{0.7, 0.8, 1.2\} \). When \( \delta \) is 0.7 or 0.8, ridge significantly outperforms the others. As \( \delta \) is increased to 1.2, LASSO starts to lead the performances.

3.7.5 Nearly black object

In this section, we verify our theoretical results which are presented in Section 3.2 for the nearly black object setting. Recall \( b_\epsilon = \sqrt{\mathbb{E}G^2} \) and \( \tilde{G} = G/b_\epsilon \). We consider the following setting: \( \delta = 0.8, \sigma \in \{3, 5\}, b_\epsilon = 4/\sqrt{\epsilon}, \tilde{G} = 1, \epsilon \in \{0.25, 0.0625, 0.04\} \). The simulation results are displayed in Figure 3.11. We observe that under both noise levels \( \sigma = 3, 5 \), LASSO is suboptimal at sparsity level \( \epsilon = 0.25 \). As \( \epsilon \) decreases, LASSO becomes better. When \( \epsilon \) is reduced to 0.04, LASSO outperforms the other bridge estimators by a large margin. Note that in this simulation, the signal strength \( b_\epsilon \) scales with \( \epsilon \) at the rate \( \epsilon^{-1/2} \). This is the regime where LASSO is proved to be optimal in Section 3.2.
Figure 3.9: Top row: AFDP-ATPP curve under the setting $\epsilon = 0.1, \sigma = 0.4, \delta \in \{2, 3, 4\}$. Second row: Y-axis is the difference of AFDP between the other bridge estimators and LASSO. One standard deviation of the difference is added. Third and fourth rows: the same type of plots as in the first two rows, under the setting $\epsilon = 0.3, \sigma = 0.4, \delta \in \{2, 3, 4\}$. 
**3.7.6 LASSO vs. two-stage LASSO**

In Theorem 2.3.2 we proved that two-stage LASSO with its first stage optimally tuned outperforms LASSO on variable selection. We now provide a brief simulation to verify this result. We choose $p_G = \delta M$ with $M = 8$ and set $\delta = 0.8, \epsilon = 0.2, \sigma \in \{1, 3, 5\}$. As shown in Figure 3.12, two-stage LASSO improves over LASSO. When the noise is small ($\sigma = 1$), the improvement is the most significant. As the noise level increases, the difference between the two approaches becomes smaller. When the noise is large ($\sigma = 5$), both have large errors.

**3.7.7 General design**

In this section, we extend our simulations to general design matrices. Given that our theoretical results in Section 3.2, 3.3, 3.4 are derived under the i.i.d. Gaussian assumption on $A$, the aim of this section is to numerically study the validity scope
\[ \sigma = 3, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 8 \]  

\[ \sigma = 3, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 16 \]  

\[ \sigma = 3, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 20 \]  

\[ \sigma = 5, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 8 \]  

\[ \sigma = 5, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 16 \]  

\[ \sigma = 5, \epsilon \in \{0.25, 0.0625, 0.04\}, b_\epsilon = 20 \]  

\[ \sigma \in \{1, 3, 5\} \]  

\[ \sigma \in \{1, 3, 5\} \]  

\[ \sigma \in \{1, 3, 5\} \]  

Figure 3.11: Top row: AFDP-ATPP curve under the setting \( b_\epsilon = 4/\sqrt{\epsilon}, \sigma = 3, \delta = 0.8, \epsilon \in \{0.25, 0.0625, 0.04\} \). Second row: AFDP-ATPP curve under the setting \( b_\epsilon = 4/\sqrt{\epsilon}, \sigma = 5, \delta = 0.8, \epsilon \in \{0.25, 0.0625, 0.04\} \). One standard deviation is added.

Figure 3.12: LASSO vs. two-stage LASSO. Here \( \delta = 0.8, \epsilon = 0.2, M = 8, \sigma \in \{1, 3, 5\} \). The outperformance of two-stage LASSO is the most significant when the noise level is low. When noise gets higher, the gap becomes smaller and smaller.
of our main conclusions when such an assumption does not hold. In particular, we consider the following correlated designs and i.i.d. non-Gaussian designs:

- **Correlated design**: We consider the model \( y = A \Sigma^{\frac{1}{2}} x + w \), where \( A_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{n}) \) and \( \Sigma \) is a Toeplitz matrix with \( \Sigma_{ij} = \rho^{|i-j|} \). Here \( \rho \in (0, 1) \) controls the correlation strength.

- **i.i.d. non-Gaussian design**: We generate \( A \) with i.i.d. components \( A_{ij} \sim \sqrt{\frac{\nu - 2}{n \nu}} t_{\nu} \) where \( t_{\nu} \) is the t-distribution with degrees of freedom \( \nu \). The scaling \( \sqrt{\frac{\nu - 2}{n \nu}} \) ensures \( \text{var}(A_{ij}) = \frac{1}{n} \) as in the i.i.d Gaussian case.

Throughout this section, we choose \( p = 2500, p_G = \delta M, n = \delta p, w_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \).

**Large/small noise** We set \( M = 8, \delta = 0.9, \epsilon = 0.4 \). For correlated design, we vary \( \rho \in \{0.1, 0.5, 0.9\} \) to allow for different levels of correlations among the predictors. Figure 3.13 shows the simulation results. There are a few important observations:

(i) For a given \( \rho \in \{0.1, 0.5, 0.9\} \), the comparison of bridge estimators under different noise levels is similar to what we observe for i.i.d. Gaussian designs: LASSO performs best in low noise case, and ridge becomes optimal when the noise is large.

(ii) Given the noise level \( \sigma = 0.8 \), as the design correlation \( \rho \) varies in \( \{0.1, 0.5, 0.9\} \), it is interesting to observe that, LASSO outperforms the other estimators when the correlation is not high \( (\rho = 0.1, 0.5) \), while ridge becomes the optimal one when the correlation is increased to 0.9. Similar phenomenon happens at the noise level \( \sigma = 1 \). It seems that in terms of variable selection performance comparison of TVS, adding dependency among the predictors is like increasing the noise level in the system. We leave a theoretical analysis of the impact of correlation on our results as an interesting future research.
Regarding i.i.d. non-Gaussian design, we choose the t-distribution $t_\nu$ with $\nu = 3$. Note that among all the t-distributions $\{t_\nu, \nu \in \mathbb{N}\}$ with finite variance, $t_3$ has the heaviest tail. The results are shown in Figure 3.14. We again observe the comparison predicted by our theory: LASSO outperforms the other bridge estimators when the noise level is low ($\sigma = 0.8$), and ridge performs best as the noise level increases to $\sigma = 2$. 

Figure 3.13: Large/small noise scenario under correlated design.
Figure 3.14: Large/small noise scenario under i.i.d. non-Gaussian design. We set \( \delta = 0.9, \epsilon = 0.4, M = 8, \sigma \in \{0.8, 1, 2\} \). The degrees of freedom of the t-distribution is \( \nu = 3 \).

**Nearly black object** For nearly black objects, we consider \( \delta = 0.8, \sigma = 3, b_\epsilon = \frac{4}{\sqrt{\epsilon}}, \check{G} = 1, \epsilon \in \{0.25, 0.0625, 0.04\} \). We construct the design matrix in the following ways:

(i) Set a correlated Gaussian design with correlation levels \( \rho = 0.5, 0.9 \).

(ii) Set an i.i.d. non-Gaussian design with \( t_3 \).

Figures 3.15 and 3.16 contain the results for the correlated design and i.i.d. non-Gaussian design, respectively. We can see that as the model becomes sparser, LASSO starts to outperform other choices of bridge estimator and eventually becomes optimal. This is consistent with the main conclusion we have proved for the i.i.d. Gaussian designs.

**LASSO vs two-stage LASSO** We compare LASSO and two-stage LASSO under more general designs. As in Section 3.7.6 for i.i.d. Gaussian design, we set \( \delta = 0.8, \epsilon = 0.2, M = 8 \) and \( \sigma = 1, 3, 5 \). For correlated designs, we pick \( \rho = 0.5, 0.9 \). For i.i.d. non-Gaussian design, we choose \( \nu = 3 \). As is seen in Figure 3.17, the same phenomenon observed in i.i.d. Gaussian design also occurs under general designs:
Figure 3.15: Nearly black object with correlated design. We fix $\delta = 0.8$, $\sigma = 3$ and $b_\epsilon = 4/\sqrt{\epsilon}, \epsilon \in \{0.25, 0.0625, 0.04\}$. The correlation $\rho$ is set to 0.5 and 0.9 in the two rows.

Figure 3.16: Nearly black object with i.i.d. non-Gaussian design. We fix $\delta = 0.8$, $\sigma = 3$ and $b_\epsilon = 4/\sqrt{\epsilon}, \epsilon \in \{0.25, 0.0625, 0.04\}$. The degrees of freedom for the t-distribution design is $\nu = 3$. 
two-stage LASSO outperforms LASSO by a large margin when the noise is small, and the outperformance becomes marginal in large noise.

Figure 3.17: LASSO vs. two-stage LASSO under general designs. Here \(\delta = 0.8, \epsilon = 0.2, M = 8, \sigma \in \{1, 3, 5\}\). The first two rows are for \(\rho = 0.5, 0.9\) in correlated design. The last row is for \(\nu = 3\) in i.i.d. non-Gaussian design.
3.8 Tuning parameter selection for a two-stage variable selection scheme

Two-stage variable selection techniques discussed in this paper have two tuning parameters: the regularization parameter \( \lambda \) in the first stage and the threshold \( s \) from the second stage. Furthermore, given that TVS using different bridge estimators offer the best performance in different regimes, we may see \( q \) as another tuning parameter. How can these parameters be optimally tuned in practice? As proved in Chapter 2, the TVS with an estimator of smaller AMSE in the first stage provides a better variable selection. Hence, the parameter \( \lambda \) can be set by minimizing the estimated risk of the bridge estimator. Similarly, one can estimate the risk for different values of \( q \) and choose the one that offers the smallest estimated risk. Section 3.7.2 has showed how this can be done.

It remains to determine the parameter \( s \). As presented in our results, the threshold \( s \) controls the trade-off between AFDP and ATPP. By increasing \( s \) we decrease the number of false discoveries, but at the same time, we decrease the number of correct discoveries. Therefore, the choice of \( s \) depends on the accepted level of false discoveries (or similar quantities). For instance, one can control the false discovery rate by combining the two-stage approach with the knockoff framework [BC+15]. Specifically, if we would like to control FDP at a rate of \( \rho \in (0, 1) \), we can go through the following procedure.

1. Construct the knockoff features \( \tilde{A} \in \mathbb{R}^{n \times p} \) as stated in [BC+15];

2. Run bridge regression on the joint design \([A, \tilde{A}]\) and obtain the corresponding estimator\( \begin{bmatrix} \hat{x} \\ \tilde{x} \end{bmatrix} \). Let \( W_j = \max( |\hat{x}_j|, |\tilde{x}_j| ) \text{sign}( |\hat{x}_j| - |\tilde{x}_j| ) \), \( j = 1, 2, \ldots, p \). Define the threshold \( s \) as \( s = \min \left\{ t > 0 : \frac{1 + \# \{ j : W_j \leq -t \} }{\# \{ j : W_j \geq t \} \vee 1} \leq \rho \right\} \).

3. Select all the predictors with \( \{ j : W_j \geq s \} \).
The above procedure only works for $n \geq p$. We may adapt the new knockoff approach in [CFJL18] when $n < p$.

### 3.9 Conclusion

We studied two-stage variable selection schemes for linear models under the high-dimensional asymptotic setting, where the number of observations $n$ grows at the same rate as the number of predictors $p$. Our TVS has a bridge estimator in the first stage and a simple threshold function in the second stage. For such schemes, we proved that for a fixed ATPP, in order to obtain the smallest AFDP one should pick an estimator that minimizes the asymptotic mean square error in the first stage of TVS. This connection between parameter estimation and variable selection further led us to a thorough investigation of the AMSE under different regimes including rare and weak signals, small/large noise, and large sample. Our analyses revealed several interesting phenomena and provided new insights into variable selection. For instance, the variable selection of LASSO can be improved by debiasing and thresholding; a TVS with ridge in its first stage outperforms TVS with other bridge estimators for large values of noise; the optimality of two-stage LASSO among two-stage bridge estimators holds for very sparse signals until the signal strength is below some threshold. We conducted extensive numerical experiments to support our theoretical findings and validate the scope of our main conclusions for general design matrices.
Chapter 4

Concentration of SLOPE MSE

The study presented in the last chapter was based on the accurate characterization of the asymptotic mean square error. In many situations, we not only care about the limiting behavior, but also the finite sample results. This is due to either the interest in bounding the convergence speed, or due to the fact that the sequence of target variable does not necessarily converge or the limit is hard to characterize. In high dimensional settings, this is often achieved through concentration inequalities.

In this chapter, we study the MSE of SLOPE [BvdBS15]:

$$\hat{x} = \arg\min_x \frac{1}{2} \|y - Ax\|_2^2 + \gamma \|x\|_\lambda$$  \hspace{1cm} (4.0.1)

where given a sequence of weights $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$, the sorted $L_1$ penalty $\|x\|_\lambda$ is defined as

$$\|x\|_\lambda := \sum_{i=1}^p \lambda_i |x|(i).$$

While we would like to study the asymptotic behavior of SLOPE as $n, p \to \infty, \frac{n}{p} \to \delta$, we adopt a specific angle.

The initiative for proposing SLOPE is to control the false discovery rate (FDR). [BvdBS15] has shown that SLOPE can control the FDR under the orthogonal design. In general, experiments confirm that SLOPE can to some extent control the FDR. [SC+16, BLT+18] further prove that SLOPE achieves minimax optimal estimation.
rate $2\sigma^2_w k \log \frac{p}{k}$ without any prior knowledge on the sparsity level $k$. As a comparison, LASSO can only achieve the minimax optimal rate given a sparsity-dependent tuning parameter. Note that LASSO is also minimax optimal with a data-driven tuning [BLT+18].

There are a few reasons we are interested in studying SLOPE:

- As discussed before, SLOPE has a few nice variable selection and estimation properties. However, it is not clear how this algorithm is compared with other standard techniques, such as bridge estimators.

- The sorted $L_1$ norm is nonseparable, which is different from many existing methods. Moreover, the way it imposes weights on the ordered signal components is less intuitive since it introduces larger bias in the larger (“more significant”) signal components.

- Previous work has mainly focused on the extreme sparse setting where the sparsity level $\frac{k}{p} \to 0$. Hence, the performance of SLOPE is not well understood when the signal is not extremely sparse.

In this chapter, we will show that the MSE of SLOPE concentrates around a constant characterized by a system of equations similar to the ones we derived for the AMSE of Bridge estimator in (2.2.2) and (2.2.3). This explicit form of the concentrated quantity enables us to do further study of the finite sample property of SLOPE, which will be the content of our next chapter.

### 4.1 Concentration inequalities of SLOPE MSE

In this section, we present our main results. We will show a concentration inequality for the MSE of SLOPE estimator. In Chapter 5, we perform the noise sensitivity analysis of SLOPE, and provide a detailed comparison with the standard bridge
estimators. Before delving into the details, we first clarify the setup of our study. In this work, we consider the following linear model:

\[ y = Ax + w, \quad (4.1.1) \]

where \( A \in \mathbb{R}^{n \times p} \) is the design matrix, \( y \in \mathbb{R}^n \) is the response vector, \( x \in \mathbb{R}^p \) is the unknown \( k \)-sparse coefficient vector that we want to estimate, and \( w \in \mathbb{R}^n \) denotes the noise. We study the family of SLOPE estimators given by

\[ \hat{x}(\gamma) \in \arg\min_x \frac{1}{2} \|y - Ax\|_2^2 + \gamma \|x\|_\lambda, \quad (4.1.2) \]

where \( \gamma > 0 \) is a regularization parameter. Note that for notational simplicity, we have suppressed \( \lambda \) in \( \hat{x}(\gamma) \). Given a weight vector \( \lambda \in \mathbb{R}^p \), (4.1.2) defines a SLOPE estimator. We observe that setting \( \lambda_1 = \cdots = \lambda_p = 1 \) in (4.1.2) yields the LASSO estimator. In our analysis of the SLOPE estimators, we make the following assumptions. Once we mention all the assumptions, we will provide a detailed discussion of why each assumption has been made.

**Assumption 4.1.1** (Linear scaling). \( \|x\|_0 = k > 0 \). Furthermore, there exist \( \kappa_1, \kappa_2, \kappa_3 > 0 \), such that

\[ \kappa_1 \leq \frac{n}{p} = \delta < \kappa_2, \]
\[ \kappa_3 \leq \frac{k}{p} = \epsilon < \delta. \]

**Assumption 4.1.2** (IID Gaussian design). \( A_{ij} \text{ i.i.d. } \sim \mathcal{N}(0, \frac{1}{n}) \).

**Assumption 4.1.3** (Noise distribution). \( w_i \)’s are i.i.d. sub-Gaussian with \( \mathbb{E}w_i = 0 \), \( \text{Var}(w_i) = \sigma_w^2 \). Furthermore, there exists \( \kappa_4 > 0 \), such that \( \|w_i\|_{\psi_2} \leq \sigma_w \kappa_4. \)

\[ \text{The sub-Gaussian norm of a random variable } Z \text{ is defined as } \]
\[ \|Z\|_{\psi_2} = \inf\{s > 0 : \mathbb{E}(e^{s^2/2}) - 1 \leq 1\}. \]

---

<table>
<thead>
<tr>
<th>1The sub-Gaussian norm of a random variable ( Z ) is defined as</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ |Z|_{\psi_2} = \inf{s &gt; 0 : \mathbb{E}(e^{s^2/2}) - 1 \leq 1}. ]</td>
</tr>
</tbody>
</table>
Assumption 4.1.4 (Bounded signal). There exist $\kappa_5, \kappa_6 > 0$ such that
\[
\kappa_5 \leq \frac{\|x\|_2}{\sqrt{p}} \leq \|x\|_\infty \leq \kappa_6.
\]

Assumption 4.1.5 (Reasonable weights). $\lambda_1 \leq 1$, $\frac{\|\lambda\|_2^2}{p} \geq \kappa_7$, for some $\kappa_7 > 0$.

Before we proceed to our main results, let us discuss these assumptions. Assumption 4.1.1 specifies the high-dimensional regime that our analysis will focus on. As we discussed in the last section, the usefulness of this regime has led many researchers to adopt this framework [DMM09, BM12, WMZ18, Sto09, Sto13, TOH15, TAH18, DTL17, LBEK18, WZL+18, WZM+20, SCC17]. The condition $\epsilon < \delta$ is very mild, as it merely requires the sample size to be larger than the number of non-zero elements of the signal. This is the information-theoretic limit for the exact recovery of a sparse signal from noiseless undersampled linear measurements [WV10].

Assumption 4.1.2 is also a standard assumption that has been made in the linear asymptotic studies we cited above. While this assumption is admittedly restrictive, it has allowed a careful analysis of many estimators/algorithms and provided an accurate prediction of phenomena that are observed in high-dimensional settings, such as phase transitions. Furthermore, extensive simulation results reported elsewhere (see e.g. [WWM20, MMB18]) have confirmed that the conclusions drawn for iid matrices hold for much broader classes of matrices. We will also report simulation results in Section 5.4 that show our main conclusions regarding SLOPE continue to hold for dependent and non-Gaussian designs.

Assumption 4.1.3 is another standard assumption in high-dimensional asymptotics. This assumption can possibly be weakened at the expense of obtaining slower concentration. However, to keep the discussion as simple as possible we consider sub-Gaussian noises.

In Assumption 4.1.4, the normalized $\ell_2$ norm square of the signal $x$ is assumed to be of order one. This together with Assumptions 4.1.1, 4.1.2, and 4.1.3 guarantees that the signal-to-noise ratio in each observation remains bounded. To clarify why one would like to keep the signal-to-noise ratio of order one, let us consider the well-
studied LASSO problem. If the signal-to-noise ratio in each observation goes to $\infty$, then as $n,p \to \infty$ it is known that the estimation error of LASSO converges to zero above its phase transition\(^2\) (hence the problem is very similar to the noiseless setting). Furthermore, if the signal-to-noise ratio goes to zero, then no estimator can provide an accurate estimation of the signal under the linear asymptotic regime [WV10]. Hence, this assumption ensures that the signal-to-noise ratio is fixed and the estimation problem does not have a trivial estimation error.

Assumption 4.1.4 also imposes the constraint that the elements of $\mathbf{x}$ are uniformly bounded. It might be possible to remove this condition and work out the explicit dependence of the results on $\|\mathbf{x}\|_\infty$. However, for notational simplicity, we stick to the boundedness assumption.

Assumption 4.1.5 imposes some constraints on the weights so that neither the loss function nor the penalty term in (4.1.2) dominate. The upper bound in Assumption 4.1.5 can be assumed without loss of generality due to the existence of the tuning parameter $\gamma$. Under these assumptions, we have the following theorem:

**Theorem 4.1.1.** Assume $\sigma_w, \gamma > 0$. Let $\mathbf{h} \sim \mathcal{N}(0, I_p)$. Under Assumptions 4.1.1-4.1.4, we have

\[
\mathbb{P}
\left( \frac{1}{\sqrt{p}} \|\hat{\mathbf{x}}(\gamma) - \mathbf{x}\|_2 - \sqrt{\mathbb{E}\|\eta(\mathbf{x} + \sigma^* \mathbf{h}; \sigma^* \chi^*) - \mathbf{x}\|_2^2} > t \right) \leq \frac{A_1(\gamma, \sigma_w)}{t^4} e^{-c a(\gamma, \sigma_w) p t^4},
\]

(4.1.3)

for $t \leq A_2(\sigma_w, \gamma)$. Here, $c$ is an absolute constant, and $a, A_1, A_2$ are functions that will be specified below under different scenarios. The unknown parameter $(\sigma^*, \chi^*)$ in (4.1.3) can be obtained from the following two equations:

\[
(\sigma^*)^2 = \sigma_w^2 + \frac{1}{\delta p} \mathbb{E}\|\eta(\mathbf{x} + \sigma^* \mathbf{h}; \sigma^* \chi^*) - \mathbf{x}\|^2,
\]

(4.1.4)

\[
\gamma = \sigma^* \chi^* \left( 1 - \frac{1}{\delta \sigma^* p} \mathbb{E}\langle \eta(\mathbf{x} + \sigma^* \mathbf{h}; \sigma^* \chi^*), \mathbf{h} \rangle \right).
\]

(4.1.5)

\(^2\)“Above (below) phase transition” refers to the success (failure) regime for exact recovery.
These two equations will be referred to as state evolution throughout the paper. Below we consider three different scenarios and explain how $a, A_1, A_2$ are set in each case. The importance of these three cases in our paper will be clarified right after the theorem. Define the quantity

$$M_\lambda(\chi^*):=\lim_{\sigma\to 0}\frac{1}{p}\mathbb{E}\|\eta(x/\sigma+h;\chi^*)-x/\sigma\|^2_2.$$  \hspace{1cm} (4.1.6)

(i) If $M_\lambda(\chi^*)<\delta$, $x$ has no tied nonzero components, and $\sigma_w<\frac{\|x\|_2}{\sqrt{2p}}\sqrt{\frac{2\delta-M_\lambda(\chi^*)}{\delta M_\lambda(\chi^*)}}$, then we have

$$a=\frac{\gamma^2}{\sigma_w C_M^8}\exp\left(-2\left(1+\frac{1}{\delta}+C^2 C_M^2\right)\right),$$

$$A_1=\frac{\sigma^4_w}{\gamma^2}\frac{(C_M^8+1/\delta^2)(1+1/\delta+C^2 C_M^2)^8}{\exp\left(-6\left(1+\frac{1}{\delta}+C^2 C_M^2\right)\right)},$$

$$A_2=\sqrt{\sigma_w}\cdot \exp\left(-\frac{3}{2}\left(1+\frac{1}{\delta}+C^2 C_M^2\right)\right)\frac{\delta^{4/3}}{\sqrt{C_\epsilon\delta+1}},$$

where $C_M=\sqrt{\frac{\delta}{\delta-M_\lambda(\chi^*)}}$ and $C_\epsilon=\sqrt{\frac{\delta-\epsilon}{\epsilon}}$.

(ii) For $\gamma$ picked in a way such that $M_\lambda(\chi^*)>\delta$ and $\mathbb{E}\|\eta(x+\sigma^* h;\sigma^* x^*)-x\|^2_2<\(1-\Delta\)\|x\|^2_2$, where $\Delta>0$ is a constant, we have

$$a=\sigma^4_w, \quad A_1=\frac{\text{Poly}(\sigma_w)}{\sigma^4_w}, \quad A_2=\Theta\left(\frac{1}{\sigma_w}\right).$$

(iii) Let $\theta=\|\eta(h;\frac{2\delta+1}{\sigma_w}\gamma)\|_2$ and $C_\delta=\frac{\delta+1}{\delta}$. If $\frac{\gamma}{\sigma_w}\geq\frac{1}{\|x\|^2_2/p}\sqrt{0\vee \log \frac{16\delta+8}{\theta}}$ and

$$\sigma_w\geq\frac{\sqrt{2(\delta+1)\|x\|_2^2}}{\delta \sqrt{p}},$$

then we have

$$a=\frac{\theta^2}{C_\delta^3\sigma_{\xi}^4}, \quad A_1=\frac{\text{Poly}(C_\delta,\sigma_w,\gamma)}{\theta^2}, \quad A_2=\left(\frac{\sigma_w}{\gamma+\delta\sigma_w}\right)^{\frac{3}{2}}\left(\frac{\theta}{C_\delta}\right)^{\frac{1}{2}}\sigma_w.$$ 

The proof of Theorem 4.1.1 is presented in Section 4.2. We make several important remarks below to interpret and discuss the results of Theorem 4.1.1.

---

3State evolution is a term that is used for these two equations in the message passing literature [DMM09].
Remark 4.1.1. Theorem 4.1.1 shows that the MSE of a given SLOPE estimator concentrates tightly around \( \frac{1}{p}E\|\eta(x + \sigma^* h; \sigma^* \chi^*) - x\|^2 \), which equals to \( \delta((\sigma^*)^2 - \sigma^2_w) \) from (4.1.4). Given all the model and SLOPE parameters, we can compute the preceding quantity from the state evolution equations (4.1.4) and (4.1.5). Such a quantity is expected to be an accurate prediction for MSE. This is empirically verified in Figure 4.1.

Remark 4.1.2. Given that the SLOPE estimation problem involves several important parameters, such as the noise level \( \sigma_w \) and the tuning parameter \( \gamma \), we should expect these quantities to play a role in the concentration of the mean square error. Hence, obtaining a single concentration inequality that exhibits the accurate dependence on all the parameters seems to be remarkably challenging. As described in Theorem 4.1.1, to overcome this difficulty, we have chosen to derive concentration results under three different scenarios. We now discuss the result of each scenario below.

(1) Scenarios (i) and (ii) are concerned with the concentration in the low noise regime. Scenario (i) considers the case in which the sample size (per dimension), \( \delta \), is above the threshold \( M_\lambda(\chi^*) \). Note that in this case, if we choose \( \gamma = \Theta(\sigma_w) \), it is clear that the probability bound becomes smaller as the noise level decreases (ignoring the polynomial term), which captures qualitatively the correct effect of \( \sigma_w \). As will be seen in the proof of Theorem 5.2.1, the condition \( \gamma = \Theta(\sigma_w) \) holds for the optimal tuning of the parameter \( \gamma \). The assumption that \( x \) has no tied nonzero components is crucial for the comparison of different SLOPE estimators. We will discuss this assumption in more details in Section 5.2. Finally, note that the condition \( \delta > M_\lambda(\chi^*) \) ensures that SLOPE is “performing above its phase transition”, i.e., as the noise level \( \sigma_w \to 0 \), the MSE \( \frac{1}{p}E\|\eta(x + \sigma^* h; \sigma^* \chi^*) - x\|^2 \) goes to zero as well. For studying the important features of the phase transition, the reader may refer to [WMZ18].

(2) Scenario (ii) characterizes the behavior when \( \delta \) is below the threshold \( M_\lambda(\chi^*) \).
The additional condition $E\|\eta(x + \sigma^* h; \sigma^* \chi^*) - x\|_2^2 < (1 - \Delta)\|x\|_2^2$ is mild. This is because the term $E\|\eta(x + \sigma^* h; \sigma^* \chi^*) - x\|_2^2$ converges to $\|x\|_2^2$ as $\gamma \to \infty$, and an appropriately chosen $\gamma$ will make that term smaller. Unfortunately, the probability bound we derived in this scenario becomes degenerate as $\sigma_w$ approaches zero, hence does not reveal the accurate expression of the noise level in the concentration inequality. Nevertheless, the concentration inequality is still valid in terms of the dimension or sample size, given all the other parameters. Moreover, as will be clear in Chapter 5, this scenario is not of particular interest for our low noise sensitivity analysis.

(3) Scenario (iii) shows the concentration result in the large noise regime. The requirement on the tuning $\gamma \geq \frac{1}{\|\lambda\|_{\ell_1}^2} \sqrt{0 \lor \log \frac{16\delta^2 \cdot \sigma_w}{\delta}}$ is reasonable in this setting, because it is desirable to set a large value of the tuning to reduce the variance of the SLOPE estimate, when the noise level is high. In particular, as we will discuss in Chapter 5, the condition is satisfied by the optimal tuning. Note that as the system has larger noise ($\sigma_w$ increases), the concentration is expected to become worse. Our probability bound (ignoring the polynomial term) is consistent with such intuition.

Remark 4.1.3. [HL19] has showed that as $n \to \infty$, the MSE of a given SLOPE estimator converges to the limit of $\frac{1}{p}E\|\eta(x + \sigma^* h; \sigma^* \chi^*) - x\|_2^2$ for specialized weight sequence $\{\lambda_i\}$. Using Borel-Cantelli lemma, such asymptotic result is directly obtained from the concentration inequality (4.1.3). Moreover, setting $\lambda_1 = \cdots = \lambda_p = 1$ recovers the asymptotic result of LASSO [DMM11, BM12].

Remark 4.1.4. The non-separability of the sorted $\ell_1$ norm in SLOPE and the complicated form of the equations (4.1.4) (4.1.5) bring substantial difficulty to derive the concentration inequality. Hence we do not claim our results to be the optimal ones. For example, there might exist a sharper result for LASSO due to its amenable structure. Moreover, it is possible to directly analyze the first order derivative of the ob-
Figure 4.1: Comparing the MSE of SLOPE estimator and the expected MSE $\delta(\sigma^2 - \sigma_w^2)$ via state evolution equation (4.1.4), (4.1.5). The SLOPE weights $\{\lambda\}$ is equally spaced within $[0.01, 1]$. Other model parameters are $p = 1000$; The components of $\mathbf{x}$ are iid samples from $5 \ast \text{Bernoulli}(\text{prob} = 0.3)$; The components of $\mathbf{z}$ are iid samples from $\mathcal{N}(0, \sigma_w^2)$.

In Figure 4.1, we present simulation results on the finite sample concentration of the SLOPE MSE.

4.2 Sketch of the proof

In this section, we sketch out the proof idea. To obtain the concentration of the MSE we use the convex Gaussian minimax theorem (CGMT) approach that was first developed in [TOH15]. Recall $\hat{x} = \arg\min_x \frac{1}{2}\|y - Ax\|_2^2 + \gamma\|x\|_\lambda$. Denote
\[ \hat{z} = \frac{\hat{x} - z}{\sqrt{p}},\ m_n = \frac{1}{p} \mathbb{E} \| x + \sigma^* h; \sigma^* \chi^* - x \|_2^2, \] where \((\sigma^*, \chi^*)\) is specified in Theorem 4.1.1. We aim to show \(\| \hat{z} \|_2^2\) concentrates around \(m_n\). First, it is straightforward to confirm that

\[ \hat{z} = \arg\min_z \frac{1}{2} \| \sqrt{p} A z - w \|_2^2 + \gamma \| \sqrt{p} z + x \| : = \arg\min_z F_n(z). \]

As we will show in Lemma 4.3.2, there exists a finite constant \(c_1\) such that
\[ \| H^c \|_2^2 \leq c_1^2. \] Define the sets
\[ S_w = \{ z : \| z \|_2 \leq 2c_1 \}, \quad H^c = \{ z : \| z \|_2 - \sqrt{m_n} \geq \epsilon \}. \]

We will choose \(\epsilon\) small to ensure \(H^c \subseteq S_w\). If we are able to prove that

\[ \min_{z \in S_w} F_n(z) < \min_{z \in S_w \cap H^c} F_n(z), \]

then \(\| \hat{z} \|_2 - \sqrt{m_n} \leq \epsilon\). To see why this is true, it is clear that (4.2.1) implies \( \hat{z} \in H^c \cup S_w^c\). Suppose \( \hat{z} \in S_w^c\), and denote \( z^* = \arg\min_z F_n(z) \). Since \( z^* \in H^c \subseteq S_w \) and \( \hat{w} \in S_w^c\), there exists a constant \(\lambda \in (0, 1)\) such that \(\lambda z^* + (1 - \lambda) \hat{z} \in S_w \cap H^c\).

By the convexity of \(F_n(z)\), it holds that

\[ \min_{z \in S_w \cap H^c} F_n(z) \leq F_n(\lambda z^* + (1 - \lambda) \hat{z}) \leq \lambda F_n(z^*) + (1 - \lambda) F_n(\hat{z}) \leq \min_{z \in S_w} F_n(z). \]

This is a contradiction. Hence, \( \hat{z} \in H^c \). Based on the proceeding arguments, it is sufficient to obtain \( \min_{z \in S_w} F_n(z) < \min_{z \in S_w \cap H^c} F_n(z) \) w.h.p. Towards this goal, in Section 4.3.1, we will define a function \( \hat{\Lambda}(\alpha, \beta, T_h) \) and use it to establish a tight “upper bound” for \( \min_{z \in S_w} F_n(z) \) and a “lower bound” for \( \min_{z \in S_w \cap H^c} F_n(z) \) in the following way:

\[ \frac{1}{p} \min_{z \in S_w} F_n(z) \leq_p \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h} \hat{\Lambda}(\alpha, \beta, T_h), \]

\[ \frac{1}{p} \min_{z \in S_w \cap H^c} F_n(z) \geq_p \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h} \hat{\Lambda}(\alpha, \beta, T_h). \]

An accurate explanation of \( \leq_p \) and \( \geq_p \) is presented in Lemma 4.3.1 and Theorem D.2.1. For now one may treat them as normal \( \leq \) and \( \geq \). As a result, as long as we
can prove

\[
\min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h \leq c_3} \hat{\Lambda}(\alpha, \beta, T_h) < \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h \leq c_3} \Lambda(\alpha, \beta, T_h), \quad \text{w.h.p.}
\]

(4.2.2)

our goal is achieved. To obtain this result, we show a uniform concentration between \(\hat{\Lambda}\) and its mean, denoted as \(\Lambda(\alpha, \beta, T_h)\), in Section 4.3.3. By using the nice form of \(\Lambda(\alpha, \beta, T_h)\), we will show \(\sqrt{m_n}\) is the minimum of \(\max_{0 \leq \beta \leq c_2} \max_{0 \leq T_h \leq c_3} \Lambda(\alpha, \beta, T_h)\) and hence

\[
\min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h \leq c_3} \Lambda(\alpha, \beta, T_h) < \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq c_2} \max_{0 < T_h \leq c_3} \Lambda(\alpha, \beta, T_h), \quad \text{w.h.p.}
\]

This further transfers back to \(\hat{\Lambda}\) through the concentration result, and completes our proof for Theorem 4.1.1.

### 4.3 Proof of Theorem 4.1.1

#### 4.3.1 The upper and lower bounds involving \(\hat{\Lambda}\)

Recall the notations \(\mathbf{y} = A\mathbf{x} + \mathbf{w}\) and \(\mathbf{g} \sim \mathcal{N}(0, \mathbf{I}_n)\). Define the function \(\hat{\Lambda}\) in the following way:

\[
\hat{\Lambda}(\alpha, \beta, T_h) = \left\{ \begin{array}{ll}
\frac{\sqrt{n}}{p} \|\mathbf{w}\|_2^{2\beta} - \frac{n\beta^2}{2p} + \frac{2}{p} \|\mathbf{x}\|_{\Lambda}, & \text{if } \alpha = 0, \\
-\frac{n}{2p} \beta^2 + \frac{\|\sqrt{n}\mathbf{g} - \sqrt{n}\mathbf{w}\|_2^2}{p} \beta - \frac{\alpha T_h}{2} + \frac{h^\top \mathbf{x}}{p} \beta + \frac{T_h}{2\alpha p} (\|\mathbf{x}\|_2^2 - \|\eta(h + \frac{\alpha^2}{T_h^2} \mathbf{h}; \frac{\alpha^2}{T_h^2})\|_2^2), & \text{otherwise}. 
\end{array} \right.
\]

The role of this quantity in our analysis was described in the last section. The following lemma relates \(F_n\) with \(\hat{\Lambda}\).

**Lemma 4.3.1.** We have the following inequality holds (recall the definition of the
symbol \( \leq_p \) and \( \geq_p \) after Theorem D.2.1),

\[
\frac{1}{p} \min_{z \in S_w} F_n(z) \leq_p \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq 2c_2} \max_{0 < T_h \leq c_3} \hat{\Lambda}(\alpha, \beta, T_h),
\]

\[
\frac{1}{p} \min_{z \in S_w \cap H_t} F_n(z) \geq_p \min_{0 \leq \alpha \leq 2c_1} \max_{0 \leq \beta \leq 2c_2 \leq H_t \leq c_3} \hat{\Lambda}(\alpha, \beta, T_h).
\]

**Proof.** We prove these two bounds separately.

**The upper bound:** Using the identity \( \frac{1}{2} \| b \|^2 = \max_u \sqrt{n} u^\top b - \frac{n}{2} \| u \|^2 \) with \( u \in \mathbb{R}^n \), we obtain

\[
\min_{z \in S_w} F_n(z) = \min_{z \in S_w} \max_u \sqrt{n} u^\top \tilde{A}z - \sqrt{n} u^\top w - \frac{n}{2} \| u \|^2 + \gamma \sqrt{p} z + x \|_\lambda.
\]

The maximum is attained at \( u = \frac{\sqrt{p} A z - w}{\sqrt{n}} \). Since for any given \( z \in S_w \), \( \| z \|_2 \leq 2c_1 \), there exists a constant \( c_2 \) such that \( \| \sqrt{p} A z - w \|_2 \leq c_2 \sqrt{n} \) w.h.p.\(^4\) Denoting \( S_w = \{ u : \| u \|_2 \leq c_2 \} \), we then have with high probability

\[
\min_{z \in S_w} F_n(z) = \min_{z \in S_w} \max_u \sqrt{n} u^\top \tilde{A}z - \sqrt{n} u^\top w - \frac{n}{2} \| u \|^2 + \gamma \sqrt{p} z + x \|_\lambda,
\]

\[
= \min_{z \in S_w} \max_u \sqrt{n} u^\top \tilde{A}z - \sqrt{n} u^\top w - \frac{n}{2} \| u \|^2 + s^\top (\sqrt{p} z + x), \quad (4.3.1)
\]

where \( \tilde{A} = \sqrt{n} A \) has independent standard normal entries, \( D_\gamma \) is the dual SLOPE norm ball with radius \( \gamma \) defined in (D.0.2). Note that the second equality is due to the fact that \( D_\gamma \) is the dual norm ball of \( \| \cdot \|_\lambda \). According to the CGMT (Part (ii) in Theorem D.2.1), the expression in (4.3.1) is closely related to

\[
\max_{u \in S_u} \max_{s \in D_\gamma} \min_{z \in S_w} \sqrt{p} \| g^\top u + \sqrt{p} u^\top h^\top z - \sqrt{n} u^\top w - \frac{n}{2} \| u \|^2 + s^\top (\sqrt{p} z + x).
\]

\[
:= f(z, u, s)
\]

---

\(^4\)\( \| \sqrt{p} A z - w \|_2 \leq \sqrt{p} \lambda_{\max}(A) \| z \|_2 + \| w \|_2 \leq 2c_1 \sqrt{p} \lambda_{\max}(A) + \| w \|_2 \). We note that \( \lambda_{\max}(A) \leq \sqrt{n}(1 + \min(\frac{p}{n}, \frac{n}{p})) + \epsilon \) w.h.p. for some small enough \( \epsilon > 0 \); \( \| w \|_2 \leq \sqrt{n} \sigma \) with high probability. This provides us with an upper bound.
Specifically, whenever \(\max_{u \in S_u} \max_{s \in D_s} \min_{z \in S_w} \hat{f}(z, u, s) \leq c\) with probability at least \(1 - \Delta\), we will have \(\min_{z \in S_w} \max_{u \in S_u} \max_{s \in D_s} \hat{f}(z, u, s) \leq c\) with probability at least \(1 - 2\Delta\). We now simplify \(\max_{u \in S_u} \max_{s \in D_s} \min_{z \in S_w} \hat{f}(z, u, s)\).

\[
\max \max \min \hat{f}(z, u, s) = \max \max \min \hat{f}(z, u, s)
\]

\[
= \max \min \max \sqrt{\rho \alpha g^\top u} - \sqrt{\beta} \|u\|_2 + s \|\alpha - \sqrt{\rho} u^\top w - \frac{n}{2} \|u\|_2^2 + s^\top x
\]

\[
\leq \max \min \max \sqrt{\rho \alpha g^\top u} - \sqrt{\beta} \|u\|_2 + s \|\alpha - \sqrt{\rho} u^\top w - \frac{n}{2} \|u\|_2^2 + s^\top x
\]

\[
= \max \min \max \sqrt{\rho \alpha g^\top u} - \sqrt{\beta} \|u\|_2 + s \|\alpha - \sqrt{\rho} u^\top w - \frac{n}{2} \|u\|_2^2 + s^\top x
\]

where the two inequalities above follow from the weak duality, and the third equality holds w.h.p. for a positive constant \(c_3\) because the maximum is obtained at \(T_h = \|\beta h + s\|_2/\sqrt{\beta}\) and \(\|s\|_2 \leq \gamma \|\alpha\|_2\) for \(s \in D_s\).

The next step is to simplify \(\hat{f}(\alpha, \beta, T_h)\). It is clear that \(\hat{f}(0, \beta, T_h) = \sqrt{\rho \|w\|_2 \beta - \frac{n}{2} \beta^2 + \gamma \|x\|_\lambda}\). When \(\alpha \neq 0, T_h > 0\), we have

\[
\hat{f}(\alpha, \beta, T_h)
\]

\[
= -\frac{n}{2} \beta^2 + \|\sqrt{\rho \alpha g} - \sqrt{\rho} w\|_2 - \frac{\alpha T_h}{2} + \frac{T_h \|x\|_2^2 - 2h^\top x \alpha}{2 \alpha} +
\]

\[
= -\frac{\alpha}{2 T_h} \|s + \beta h - \frac{T_h x}{\alpha}\|_2^2
\]

\[
= -\frac{n}{2} \beta^2 + \|\sqrt{\rho \alpha g} - \sqrt{\rho} w\|_2 - \frac{\alpha T_h}{2} + \frac{T_h \|x\|_2^2 - 2h^\top x \alpha}{2 \alpha}
\]

\[
= -\frac{\alpha}{2 T_h} \left( \|T_h x/\alpha - \beta h; \gamma\|_2^2 - \frac{T_h \|x\|_2^2 - 2h^\top x \alpha}{2 \alpha}\right)
\]

\[
= -\frac{n}{2} \beta^2 + \|\sqrt{\rho \alpha g} - \sqrt{\rho} w\|_2 - \frac{\alpha T_h}{2} - \frac{h^\top x}{2} \beta
\]

\[
+ \frac{T_h}{2 \alpha} \left( \|x\|_2^2 - \|x - \frac{\alpha \beta h}{T_h}; \frac{\alpha \gamma}{T_h}\|_2^2 \right).
\]

The last two equalities are due to Lemma D.1.1 and Lemma D.1.2 (i), respectively.
The lower bound: Similar to (4.3.1) we have with high probability

$$\min_{z \in S_w \cap H} F_n(z) = \min_{z \in S_w \cap H} \max_{u \in S_u} \max_{s \in D} f(z, u, s)$$

From the CGMT (Part (i) in Theorem D.2.1), we would like to find a lower bound

$$\min_{z \in S_w \cap H} \max_{u \in S_u} \max_{s \in D} \tilde{f}(z, u, s) = \min_{z \in S_w \cap H} \max_{0 \leq \beta \leq c_2} \max_{\|u\|_2 = \beta} \sqrt{p} h^\top z \beta - \frac{n}{2} \beta^2 + \|\sqrt{p}\| \beta^2 + s^\top (\sqrt{p} z + x)$$

The rest of the proof is the same as the one for the upper bound.

### 4.3.2 Solution analysis of $\Lambda$}

The bounds we obtained in Section 4.3.1 are in the min-max form of the function $\hat{\Lambda}(\alpha, \beta, T_h)$. To simplify the bounds further, we will connect $\hat{\Lambda}$ with its expectation $\Lambda$. In this section, we analyze the properties of the saddle point of $\Lambda$. Then in Section 4.3.3, we study the uniform concentration of $\hat{\Lambda}(\alpha, \beta, T_h)$ around its mean $\Lambda(\alpha, \beta, T_h)$.

Let $\delta = \frac{n}{p}$ and define

$$\Lambda(\alpha, \beta, T_h) =$$

$$\begin{cases} 
\sqrt{\alpha^2 \delta + \delta^2 \sigma_w^2 \beta - \frac{\alpha T_h}{2} - \frac{\delta}{2} \beta^2 + \frac{T_h}{2} \|x\|^2_{E} \|\eta(x + \frac{\alpha T_h}{p} \frac{\alpha^2}{T_h})\|^2_{\frac{2}{p}}}, & \text{if } \alpha > 0, \beta > 0, T_h > 0, \\
\delta \sigma_w \beta - \frac{\delta}{2} \beta^2 + \frac{2\|x\|_2}{p}, & \text{if } \alpha = 0, \beta \geq 0, T_h > 0, \\
-\frac{\alpha T_h}{2} + \frac{T_h}{2} \|x\|^2_{E} \|\eta(x + \frac{\alpha T_h}{p} \frac{\alpha^2}{T_h})\|^2_{\frac{2}{p}}}, & \text{if } \alpha > 0, \beta = 0, T_h \geq 0, \\
-\infty, & \text{if } \alpha \geq 0, \beta > 0, T_h = 0, \\
0, & \text{if } \alpha = \beta = T_h = 0.
\end{cases}$$

These extended definitions make $\Lambda$ closed and provide closeness for the level sets.
Lemma 4.3.2. Consider the min-max problem,
\[
\min_{\alpha \geq 0} \max_{\beta \geq 0} \max_{T_h \geq 0} \Lambda(\alpha, \beta, T_h).
\]

If \( \sigma_w \geq 0, \gamma > 0 \), then the following results hold:

(i) \( \Lambda(\alpha, \beta, T_h) \) is convex in \( \alpha \) and jointly concave in \( (\beta, T_h) \);

(ii) The set of saddle points is non-empty and compact.

(iii) Let \((\alpha^*, \beta^*, T_h^*)\) be a saddle point of the system. Then we have \( \alpha^*, \beta^*, T_h^* > 0 \).

(iv) Any saddle point \((\alpha^*, \beta^*, T_h^*)\) satisfies the following system of equations:
\[
\begin{align*}
(\alpha^*)^2 &= \frac{1}{p} \mathbb{E}[\|\eta(x + \frac{\alpha^* \beta^*}{T_h} h; \frac{\alpha^* \gamma}{T_h}) - x\|_2^2], \\
&= \frac{1}{p} \mathbb{E} \langle h, \eta(x + \frac{\alpha^* \beta^*}{T_h} h; \frac{\alpha^* \gamma}{T_h}) \rangle = \sqrt{(\alpha^*)^2 \delta + \delta^2 \sigma_w^2} - \delta \beta^*, \quad (4.3.3) \\
\frac{\alpha^* \beta^*}{T_h} &= \frac{\sqrt{(\alpha^*)^2 \delta + \delta^2 \sigma_w^2}}{\delta}.
\end{align*}
\]

Moreover, by setting \( \alpha^* = \sqrt{\delta(\sigma^2 - \sigma_w^2)}, \beta^* = \frac{\gamma}{\chi}, T_h^* = \frac{\gamma \sqrt{\delta(\sigma^2 - \sigma_w^2)}}{\sigma \chi} \) the above three equations are simplified to
\[
\begin{align*}
\sigma^2 &= \sigma_w^2 + \frac{1}{\delta p} \mathbb{E}[\|\eta(x + \sigma h; \sigma \chi) - x\|_2^2], \\
\gamma &= \sigma \chi \left(1 - \frac{1}{\delta \sigma p} \mathbb{E}[\langle \eta(x + \sigma h; \sigma \chi), h \rangle]\right). \quad (4.3.4)
\end{align*}
\]

Proof. Part (i): Let \( e_f(x; \tau) := \min_v \frac{1}{2\tau}\|x - v\|^2 + f(v) \) be the Moreau Envelope of the function \( f \). We note that the following identity hold:
\[
e_f(x; \tau) + e_{f^*}(x/\tau; 1/\tau) = \frac{\|x\|^2}{2\tau}. \quad (4.3.5)
\]
where \( f^*(x') = \sup_v \{v^\top x' - f(v)\} \) is the convex conjugate of \( f \). Since we have not found any direct proof of the above identity in the literature, we refer our readers to Lemma D.2.7 for a simple proof of (4.3.5).
Since the convex conjugate of $\| \cdot \|_\lambda$ is $\mathbb{I}_{\mathcal{D}}(\cdot)$, by making use of the above identity and Lemma D.1.5 (ii), we are able to prove the following form of $\Lambda$:

$$\Lambda(\alpha, \beta, T_h)$$

$$= \beta \sqrt{\alpha^2 \delta + \sigma_w^2 \delta^2} - \frac{\alpha T_h}{2} - \frac{\delta \beta^2}{2} + \frac{T_h \| x \|^2}{2p\alpha} - \frac{\gamma}{p} \mathbb{E}_{\mathcal{E} \subseteq \mathcal{D}_1} \left( \frac{T_h}{\alpha \gamma} x + \frac{h}{\gamma}; \frac{T_h}{\alpha \gamma} \right)$$

$$(4.3.6)$$

$$= \beta \sqrt{\alpha^2 \delta + \sigma_w^2 \delta^2} - \frac{\alpha T_h}{2} - \frac{\delta \beta^2}{2} + \frac{\gamma}{p} \mathbb{E} \min_{\| s \|_{\lambda^*} \leq 1} \left\{ \langle x, \frac{\beta}{\gamma} h - s \rangle + \frac{\alpha \gamma}{2 T_h} \| \frac{\beta}{\gamma} h - s \|_2^2 \right\}$$

$$(4.3.7)$$

Since $e_f(x; \tau)$ is jointly convex in both of its two arguments, (4.3.6) implies the joint concavity of $\Lambda$ in $(\beta, T_h)$. Furthermore, (4.3.7) implies that $\Lambda$ is convex in $\alpha$.

**Part (ii):** We aim to apply the Saddle Point Theorem (Theorem D.2.4). The rest of the proof is to verify the required conditions. According to part (i), the function $\Lambda(\alpha, \beta, T_h)$ is convex in $\alpha$ over $\mathbb{R}^+$ for each $(\beta, T_h) \in \mathbb{R}^+ \times \mathbb{R}^{++}$, and jointly concave in $(\beta, T_h)$ over $\mathbb{R}^+ \times \mathbb{R}^{++}$ for each $\alpha \in \mathbb{R}^+$. It is also clear that the closeness of $\Lambda(\alpha, \beta, T_h)$ will be satisfied if $\Lambda(\alpha, \beta, T_h)$ is continuous at $\alpha = 0$ for each given $(\beta, T_h)$. This is true because of the following decomposition:

$$\frac{\| x \|^2}{p} - \left\| \eta \left( x + \frac{\alpha \beta h}{T_h}; \frac{\alpha \gamma}{T_h} \right) \right\|_{L_2}^2$$

$$= \left\| \eta \left( x + \frac{\alpha \beta h}{T_h}; \frac{\alpha \gamma}{T_h} \right) - x - \frac{\alpha \beta h}{T_h} \right\|_{L_2}^2 - \frac{\alpha^2 \beta^2}{2} + \frac{2 \alpha \gamma \mathbb{E} \| \eta(\frac{x + \frac{\alpha \beta h}{T_h}}{\alpha \gamma}; \frac{\alpha \gamma}{T_h}) \|_\lambda}{p T_h} = o\left( \frac{\sigma_w^2}{T_h^2} \right).$$

Note that we have used Lemma D.1.5 (iv) (setting $\gamma_2 = 0$ therein). Finally, we need to find $(\bar{\alpha}, \bar{\beta}, \bar{T}_h) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{++}, \bar{c} \in \mathbb{R}$ such that the following two sets are nonempty and compact:

$$\mathcal{H}_1 = \{ \alpha \geq 0 : \Lambda(\alpha, \bar{\beta}, \bar{T}_h) \leq \bar{c} \}, \quad \mathcal{H}_2 = \{ \beta \geq 0, T_h > 0 : \Lambda(\bar{\alpha}, \beta, T_h) \geq \bar{c} \}.$$

Since $\gamma > 0$, we are able to choose $\bar{T}_h > 0$ small enough so that $\frac{1}{p} \mathbb{E} \| \eta(\frac{h}{\sqrt{\gamma}}; \frac{T_h}{\gamma}) \|_2^2 < \frac{1}{3}$ and $\bar{\beta} = \frac{\bar{T}_h}{\sqrt{\gamma}}$. Then we have for $\alpha > 0$

$$\Lambda(\alpha, \bar{\beta}, \bar{T}_h) = \alpha \bar{T}_h \left( \sqrt{1 + \frac{\delta \sigma_w^2}{\alpha^2} - \frac{1}{2} \frac{T_h}{2p\alpha} + \frac{\| x \|^2}{2p\alpha^2} - \frac{\mathbb{E} \| \eta(\frac{x}{\alpha}; \frac{h}{\beta \gamma}; \frac{T_h}{\gamma}) \|_2^2}{2p} } \right).$$

$$(4.3.8)$$
Also, by Lemma D.1.5 Part (i), we conclude that
\[
\left\| \eta \left( \frac{x}{\alpha} + \frac{h}{\sqrt{\delta}} \right) \right\|_{L_2}^2 \\
\leq 2 \left( \eta \left( \frac{x}{\alpha} + \frac{h}{\sqrt{\delta}} \right) - \eta \left( \frac{h}{\sqrt{\delta}} \right) \right)_{L_2}^2 + 2 \left\| \eta \left( \frac{h}{\sqrt{\delta}} \right) \right\|_{L_2}^2 \leq \frac{2\|x\|_{L_2}^2}{p\alpha^2} + \frac{2}{3} \quad (4.3.9)
\]

Combining (4.3.8) and (4.3.9) we know that \( \lim_{\alpha \to \infty} \Lambda(\alpha, \bar{\beta}, T_h) = +\infty \). Hence, we can choose \( \bar{\alpha} > 0 \) and \( \bar{c} < \infty \) such that

\[
\bar{c} = \Lambda(\bar{\alpha}, \bar{\beta}, T_h) > 1. \quad (4.3.10)
\]

Under our choice of \( (\bar{\alpha}, \bar{\beta}, T_h) \) and \( \bar{c} \), clearly \( \mathcal{H}_1, \mathcal{H}_2 \) are nonempty. To obtain the compactness of \( \mathcal{H}_1 \), it is sufficient to show \( \mathcal{H}_1 \) is bounded because \( \Lambda(\alpha, \bar{\beta}, T_h) \) is continuous in \( \alpha \) over \( \mathbb{R}_+ \). The boundedness is further guaranteed by \( \lim_{\alpha \to \infty} \Lambda(\alpha, \bar{\beta}, T_h) = +\infty \). Regarding \( \mathcal{H}_2 \), we first show it is bounded. If this is not true, there exists a sequence \( \{ (\beta_k, T_{h,k}) \} \subset \mathcal{H}_2 \) and one of the following three cases has to hold: (1) \( \beta_k \to \infty, T_{h,k} \to c_0 < \infty \); (2) \( \beta_k \to \infty, T_{h,k} \to \infty \); (3) \( \beta_k \to c_0 < \infty, T_{h,k} \to \infty \).

Assuming case (1) holds, then
\[
\lim_{k \to \infty} \Lambda(\alpha, \beta_k, T_{h,k}) \leq \frac{c_0\|x\|_{L_2}^2}{2\alpha p} + \lim_{k \to \infty} \left( \sqrt{\alpha^2 \delta + \delta^2 \sigma_w^2 \beta_k - \frac{\delta \beta_k^2}{2}} \right) = -\infty,
\]

contradicting \( \inf_k \Lambda(\alpha, \beta_k, T_{h,k}) \geq \bar{c} \). For the other two cases, the same contradiction can be drawn based on (4.3.8) and the Cauchy-Schwarz inequality \( \|u\|_\lambda \leq \|\lambda\|_2 \|u\|_2 \).

Now given that \( \mathcal{H}_2 \) is bounded and \( \Lambda(\alpha, \beta, T_h) \) is continuous in \( (\beta, T_h) \), if \( \mathcal{H}_2 \) is not compact, there must exist a sequence \( \{ (\beta_k, T_{h,k}) \} \subset \mathcal{H}_2 \), such that \( T_{h,k} \to 0 \) as \( k \to \infty \).

In this case, if \( \beta_k \to c_0 > 0 \) then
\[
\lim_{k \to \infty} \Lambda(\alpha, \beta_k, T_{h,k}) \leq c_0 \sqrt{\bar{\alpha}^2 \delta + \delta^2 \sigma_w^2} - \lim_{k \to \infty} \frac{1}{2\alpha p T_{h,k}} \mathbb{E}\|\eta(T_{h,k}x + \bar{\alpha} \beta_k h; \bar{\alpha} \gamma)\|_2^2 = -\infty.
\]

If \( \beta_k \to 0 \), then
\[
\lim_{k \to \infty} \Lambda(\alpha, \beta_k, T_{h,k}) \leq \lim_{k \to \infty} \left[ \sqrt{\bar{\alpha}^2 \delta + \delta^2 \sigma_w^2 \beta_k - \frac{\delta \beta_k^2}{2}} - \frac{\bar{\alpha} T_{h,k}}{2} + \frac{T_{h,k}\|x\|_{L_2}^2}{2\alpha p} \right] = 0.
\]
Both contradict with the fact that \( \inf_k \Lambda(\bar{\alpha}, \beta_k, T_{h,k}) \geq \bar{c} > 1 \). This completes our proof of part (ii).

**Part (iii):** We proceed by analyzing the first order conditions of \( \Lambda \) w.r.t. \( \alpha, \beta \) and \( T_h \) respectively. We have the following equations:

\[
\frac{\partial \Lambda}{\partial \alpha} = -\frac{T_h}{2} + \frac{\alpha \beta \delta}{\sqrt{\alpha^2 \delta + \delta^2 \sigma_w^2}} - \frac{T_h}{2\alpha^2 p} \mathbb{E} \left\| \eta \left( x + \frac{\alpha \beta}{T_h} h; \frac{\alpha \gamma}{T_h} \right) - x \right\|^2_2, \tag{4.3.11}
\]

\[
\frac{\partial \Lambda}{\partial \beta} = \sqrt{(\alpha)^2 \delta + \delta^2 \sigma_w^2} - \delta \beta - \frac{1}{p} \mathbb{E} \left\langle \eta \left( x + \frac{\alpha \beta}{T_h}; \frac{\alpha \gamma}{T_h} \right), h \right\rangle; \tag{4.3.12}
\]

\[
\frac{\partial \Lambda}{\partial T_h} = -\frac{\alpha}{2} + \frac{1}{2\alpha p} \mathbb{E} \left\| \eta \left( x + \frac{\alpha \beta}{T_h} h; \frac{\alpha \gamma}{T_h} \right) - x \right\|^2_2. \tag{4.3.13}
\]

We first prove \( \alpha^* > 0 \) by contradiction. Suppose \( \alpha^* = 0 \). From (4.3.2), we know that \( \beta^* = \sigma_w \) and \( T_{h^*} > 0 \) and \( \Lambda(0, \sigma_w, T_{h^*}) = \frac{\delta \sigma_w^2}{2} + \gamma \| x \|_2 \). However, based on (4.3.11), we have that

\[
\left. \frac{\partial \Lambda}{\partial \alpha} \right|_{\alpha=0, \beta=\sigma_w, T_h>0} = -\frac{T_h}{2} - \lim_{T_h \to 0} \frac{T_h^2}{\alpha^2 \beta^2 p} \mathbb{E} \left\| \eta \left( x + \frac{\alpha \beta}{T_h} h; \frac{\alpha \gamma}{T_h} \right) - x \right\|^2_2 < 0.
\]

This implies that \( \Lambda(\bar{\alpha}, \sigma_w, T_{h^*}) < \Lambda(0, \sigma_w, T_{h^*}) \) for some small enough \( \bar{\alpha} > 0 \) which contradicts with the fact that \( (\alpha^*, \beta^*, T_{h^*}) \) is a saddle point.

Now for \( \alpha^* > 0 \), we want to prove \( \beta^*, T_{h^*} > 0 \). It is obvious that \( (\beta^*, T_{h^*}) \notin \mathbb{R}_+ \times \{0\} \) where \( \Lambda = -\infty \) since \( \Lambda(\alpha^*, 0, 0) = 0 \). Further for any \( T_h > 0 \), \( \frac{\partial \Lambda}{\partial \beta} \bigg|_{\alpha=\alpha^*, \bar{\beta}=\sigma_w, T_h} = \sqrt{\delta(\alpha^*)^2 + \delta^2 \sigma_w^2} > 0 \) which implies that \( (\beta^*, T_{h^*}) \notin \{0\} \times \mathbb{R}_+ \). Hence if we show \( (\beta^*, T_{h^*}) \neq (0, 0) \), then can claim \( (\beta^*, T_{h^*}) \in \mathbb{R}_+ \times \mathbb{R}_+ \). We note that if we pick \( \beta = \frac{T_h}{\sqrt{\alpha^*}} \), then since \( \gamma > 0 \), we can set \( T_h \) small enough such that \( \mathbb{E} \| \eta(x + \frac{\alpha^* \gamma}{\sqrt{\beta}} h; \frac{\alpha^* \gamma}{T_h}) \|^2_2 < \frac{\| x \|^2_2}{2} \). Hence, we are able to obtain a positive value of \( \Lambda \) for small \( T_h \):

\[
\Lambda \left( \alpha^*, \frac{T_h}{\sqrt{\delta}}, T_h \right) > \left( \sqrt{(\alpha^*)^2 + \delta \sigma_w^2} - \frac{\alpha^*}{2} + \frac{\| x \|^2_2}{4\alpha^* p} \right) T_h - \frac{T_h^2}{2} \geq \frac{\| x \|^2_2}{\sqrt{2p}} T_h - \frac{T_h^2}{2} > 0, \quad \forall T_h < \frac{\sqrt{2} \| x \|^2_2}{\sqrt{p}}.
\]

This indicates that \( (\alpha^*, 0, 0) \) is not the optima when \( \alpha^* > 0 \).
**Part (iv):** For any saddle point \((\alpha^*, \beta^*, T_h^*)\), our results in part (iii) make sure they lie in the interior of the domain. As a result we have \(\frac{\partial \Lambda}{\partial \alpha}(\alpha^*, \beta^*, T_h^*) = 0, \frac{\partial \Lambda}{\partial \beta}(\alpha^*, \beta^*, T_h^*) = 0, \frac{\partial \Lambda}{\partial T}(\alpha^*, \beta^*, T_h^*) = 0\). By further making use of (4.3.11), (4.3.12), (4.3.13), it is straightforward to confirm that the first order equations can be simplified to (4.3.3). The equivalence between the three-equation system (4.3.3) and the two-equation system (4.3.4) can be directly verified.

In order to transfer the concentration of \(\hat{\Lambda}\) around \(\Lambda\) to the concentration of the minimizer \(\hat{\alpha}\) around \(\alpha^*\) we need bounds on \(\alpha^*\) and the optimizer \((\beta^*(\alpha), T_h^*(\alpha))\) in the maximization step of \(\Lambda\) for each given \(\alpha\). Since \(\sigma\) and \(\chi\), defined in (4.3.4), have better interpretations and are easier to analyze, we set \(\sigma^*(\alpha) = \frac{\alpha \beta^*(\alpha)}{T_h^*(\alpha)}, \chi^*(\alpha) = \frac{\gamma}{\beta^*(\alpha)}\), and consider the bounds on these two quantities instead. Recall that we defined

\[
M_\Lambda(\chi^*) := \lim_{\sigma \to 0} \frac{1}{p} \mathbb{E}\|\eta(\frac{\sigma}{\sigma} + h; \chi^*) - \frac{\sigma}{\sigma}\|^2.
\]

The following lemma summarizes the bounds we have derived for these quantities.

**Lemma 4.3.3.** Below we summarize our bounds for \(\alpha^*\) and our upper bound on \(\chi^*(\alpha)\) in three different cases:

(i) If \(M_\Lambda(\chi^*) < \delta\), then we have

\[
\frac{\frac{2}{p} \mathbb{E}\|\eta(h; \chi^*)\|_2^2 \sigma_w^2}{\delta - \frac{1}{p} \mathbb{E}\|\eta(h; \chi^*)\|_2^2} \leq (\alpha^*)^2 \leq \frac{\delta M_\Lambda(\chi^*) \sigma_w^2}{\delta - M_\Lambda(\chi^*)}.
\]

Furthermore, when \(\sigma_w \leq \sqrt{\frac{2-\chi^*}{\delta M_\Lambda(\chi^*)} \|x\|_2} \sqrt{2p}\), we have

\[
\chi^*(\alpha) \leq 1 + \frac{48}{\|x\|_\infty} \left( \frac{1}{\sqrt{\delta}} + \frac{\delta \gamma \sqrt{2}}{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}} \right).
\]

Finally, in this case we must have \(\gamma \leq \sqrt{\frac{\delta - \epsilon}{\epsilon}} \sqrt{\frac{\delta - M_\Lambda(\chi^*)}{M_\Lambda(\chi^*)}} \|x\|_2 \sigma_w\).

(ii) Suppose that \(M_\Lambda(\chi^*) > \delta\), and \(\mathbb{E}\|\eta(x + \sigma h; \sigma^* \chi^*) - x\|_2^2 \leq (1 - \varepsilon)\|x\|_2^2\) for some \(\varepsilon > 0\). Further pick \(\sigma_0\) such that \(\delta \sigma_0^2 = \frac{1}{p} \mathbb{E}\|\eta(x + \sigma_0 h; \sigma_0 \chi^*) - x\|_2^2\) and set \(b_0 = \frac{\frac{\partial}{\partial \sigma^2}}{\frac{\partial}{\partial \sigma^2} \frac{1}{p} \mathbb{E}\|\eta(x + \sigma h; \sigma^* \chi^*) - x\|_2^2}_{\sigma = \sigma_0}\). If \(\sigma^2 \leq \frac{\delta - b_0}{b_0} (\frac{1}{p} (1 - \varepsilon)\|x\|_2^2 - \delta \sigma_0^2)\), then
we have

\[
\frac{\delta^*}{\sigma^*} + \frac{1}{p} \mathbb{E} \left\| \eta(h; \chi^*) \right\|^2 \leq \left( \frac{\alpha^*}{\sigma^*} \right)^2 \leq \left( \frac{\delta^*}{\sigma^*} + \frac{b_0 \delta^*_w}{\delta - b_0} \right) \land \left\| x \right\|^2_p,
\]

and

\[
\chi^*(\alpha) \leq 1 \lor \frac{16 \left\| x \right\|_\infty}{\varepsilon \sqrt{1 - \varepsilon \left\| x \right\|_2}} \left( \frac{1}{\sqrt{\delta}} + \frac{\delta \gamma \sqrt{2}}{\sqrt{\delta \alpha^2 + \delta^2 \sigma^2_w}} \right).
\]

(iii) If \( \sigma_w > \sqrt{2(\delta + 1) \left\| x \right\|^2} \), and \( \gamma \sigma_w > \frac{1}{\left\| x \right\|_2^{1/p}} \sqrt{0 \lor \log \frac{10 \delta + 8 \delta^2}{\delta^2}} \), then we have

\[
\left\| \eta(h; \chi^*) \right\|^2 \sigma^2_w \leq \left\| \eta(x + \sigma_w h; \sigma_w \chi^*) - x \right\|^2 \leq (\alpha^*)^2 \leq \frac{\delta \left\| \eta(h; \chi^*) \right\|^2 \sigma^2_w + \frac{\delta}{p} \left\| x \right\|^2}{\delta - \left\| \eta(h; \gamma \sigma_w) \right\|^2 \sigma^2_w},
\]

and

\[
\chi^*(\alpha) \leq \frac{2 \delta + 1}{\sqrt{\delta \alpha^2 + \delta^2 \sigma^2_w}}
\]

(iv) Once we obtain an upper bound of \( \chi^*(\alpha) \) in (i) - (iii), denoted by \( U_\chi \), we have the following bounds for \( \sigma^*(\alpha) \).

\[
\frac{\alpha^2}{1 + U^2_\chi} \leq (\sigma^*(\alpha))^2 \leq \frac{\alpha^2}{\left\| \eta(h; U_\chi) \right\|^2 \sigma^2_w}.
\]

In addition, we have a common lower bound on \( \chi^*(\alpha) \) for all cases in (i) - (iii):

\[
\chi^*(\alpha) \geq \frac{\sqrt{\delta \gamma \sigma^* \chi^*}}{\sqrt{\alpha^2 + \delta^2 \sigma^2_w}} := L_\chi.
\]

Proof. First, let us recall the equations \( \sigma^* \) and \( \chi^* \) should satisfy, i.e., (4.3.18) and (4.3.19):

\[
\delta - \frac{\delta \sigma^2}{(\sigma^*)^2} = \frac{1}{p} \mathbb{E} \left\| \eta \left( \frac{x}{\sigma^*} + h; \chi^* \right) - \frac{x}{\sigma^*} \right\|^2,
\]

\[
\frac{\delta \gamma}{\sigma^* \chi^*} = \delta - \frac{1}{p} \mathbb{E} \left\langle \eta \left( \frac{x}{\sigma^*} + h; \chi^* \right), h \right\rangle,
\]

and the equations of \( \sigma^*(\alpha) \) and \( \chi^*(\alpha) \) for a given \( \alpha \) in (4.3.20) and (4.3.21):

\[
\frac{\alpha^2}{\sigma^2} = \frac{1}{p} \mathbb{E} \left\| \eta \left( \frac{x}{\sigma} + h; \chi \right) - \frac{x}{\sigma} \right\|^2,
\]

\[
\frac{\delta \gamma}{\sigma \chi} = \frac{\sqrt{\delta \alpha^2 + \delta^2 \sigma^2_w}}{\sigma} - \frac{1}{p} \mathbb{E} \left\langle \eta \left( \frac{x}{\sigma} + h; \chi \right), h \right\rangle.
\]
These results will be repeatedly used in our proof of this lemma. Now we prove the claims in (i) - (iv).

**Proof of (i).** Recall $M_\lambda(\chi) = \lim_{\sigma \to 0} \frac{1}{p} \mathbb{E} \left\| \eta(\frac{x}{\sigma} + h; \chi) - \frac{x}{\sigma} \right\|_2^2$. Since $\chi^*$ enables the SLOPE to stay above the phase transition, by Lemma D.1.7, we have the following upper bound on $\chi^*$:

$$\delta \geq M_\lambda(\chi^*) \geq \epsilon + \frac{(\chi^*)^2}{p} \sum_{i=1}^k \lambda_i^2, \quad \Rightarrow \quad \chi^* \leq \sqrt{\frac{\delta - \epsilon}{\frac{1}{p} \left\| \lambda_{[1:k]} \right\|_2^2}} \leq \sqrt{\frac{\delta - \epsilon}{\frac{1}{p} \left\| \lambda \right\|_2}}.$$

Now we are ready to bound $\sigma^*$ and $\alpha^*$. We first have

$$(\sigma^*)^2 \leq \frac{\delta \sigma_w^2}{\delta - M_\lambda(\chi^*)}, \quad \Rightarrow \quad (\alpha^*)^2 \leq \frac{\delta M_\lambda(\chi^*) \sigma_w^2}{\delta - M_\lambda(\chi^*)}.$$

In addition, since $\delta - \frac{\delta \sigma_w^2}{(\sigma^*)^2} \geq \frac{1}{p} \mathbb{E} \left\| \eta(h; \chi^*) \right\|_2^2$, we have

$$(\sigma^*)^2 \geq \frac{\delta \sigma_w^2}{\delta - \frac{1}{p} \mathbb{E} \left\| \eta(h; \chi^*) \right\|_2^2}, \quad \Rightarrow \quad (\alpha^*)^2 \geq \frac{\delta \mathbb{E} \left\| \eta(h; \chi^*) \right\|_2^2 \sigma_w^2}{\delta - \frac{1}{p} \mathbb{E} \left\| \eta(h; \chi^*) \right\|_2^2}.$$

We note that arbitrary choices of $\gamma$ may not lead to $\lim_{\sigma \to 0} \frac{1}{p} \mathbb{E} \left\| \eta(\frac{x}{\sigma} + h; \chi) - \frac{x}{\sigma} \right\|_2 < \delta$. Such $\gamma$ satisfies

$$\gamma \leq \chi^* \sigma^* \leq \sqrt{\frac{\delta - \epsilon}{\epsilon}} \sqrt{\frac{\delta}{\delta - M_\lambda(\chi^*)}} \frac{\sqrt{p}}{\left\| \lambda \right\|_2 \sigma_w^*}.$$

Our next aim is to bound $\chi^*(\alpha)$. Toward this goal, we are going to apply Lemma D.1.5 (vi). Let $u = \frac{\left\| \lambda \right\|_2}{p}$. We evaluate the value of $\frac{1}{p} \mathbb{E} \left\| \eta(x + \sigma h; \sigma \chi) \right\|_2^2$ at $\sigma = \frac{c\|x\|_\infty}{\chi}$ for some constant $c > \frac{1}{u}$:

$$\left\| \eta(x + \sigma h; \sigma \chi) \right\|_2^2 \bigg|_{\sigma = \frac{c\|x\|_\infty}{\chi}} \overset{(a)}{\leq} \frac{1}{p} \sum_{i=1}^p \mathbb{E} \left( |x_i + c\|x\|_\infty h_i/\chi| - c\|x\|_\infty u \right)^2 \overset{(b)}{\leq} \frac{c^2\|x\|_\infty^2 \mathbb{E} \left( |h_i| - \chi(cu - 1)/c \right)^2}{\chi^2} \overset{(b)}{\leq} \frac{2c^2\|x\|_\infty^2}{\chi^2} e^{-\frac{\chi^2(cu - 1)^2}{2c^2}},$$

where step (a) is due to Lemma D.1.5 (vi) and step (b) is due to Lemma D.2.8. We may set $c = \frac{2}{u}$. For any given $1 > \epsilon > 0$, if $\chi \geq 1 \vee \frac{1}{u} \sqrt{\log \frac{4\sqrt{2}\|x\|_\infty}{uc\|x\|_2/\sqrt{p}}} \vee 0$, it is not hard
to verify that
\[ \| \eta(x + \sigma h; \sigma \chi) \|_{L_p^2}^2 \leq \frac{\epsilon^2 \| x \|_2^2}{4p}, \]
\[ \Rightarrow \| \eta(x + \sigma h; \sigma \chi) - x \|_{L_p^2}^2 \geq \frac{(1 - \epsilon) \| x \|_2^2}{p}. \]

The above result implies the following: if \( \chi \geq 1 \vee \frac{4}{u} \sqrt{\log \frac{4\sqrt{2} \| x \|_{\infty}}{u \| x \|_2 / \sqrt{p}}} \), then \( \frac{1}{p} \| \eta(x + \sigma h; \sigma \chi) - x \|_2^2 \) as a function of \( \sigma^2 \), becomes greater than \( \frac{(1 - \epsilon) \| x \|_2^2}{4p} \) at \( \sigma = \frac{2 \| x \|_{\infty}}{u \chi} \). Since \( \frac{1}{\sigma^2 p} \| \eta(x + \sigma h; \sigma \chi) - x \|_2^2 \) is decreasing in \( \sigma^2 \), we know that \( \frac{1}{p} \| \eta(x + \sigma h; \sigma \chi) - x \|_2^2 \geq \frac{(1 - \epsilon) \| x \|_2^2 \sigma^2}{4p \| x \|_{\infty}^2} \) for any \( \sigma \leq \frac{2 \| x \|_{\infty}}{u \chi} \).

Since \( \alpha^2 \leq \frac{\delta M_A(\chi^*)^2}{2 - M_A(\chi^*)} \), when \( \sigma_w \leq \frac{\sqrt{\delta M_A(\chi^*)} \sqrt{1 - \epsilon} \| x \|_2}{\sqrt{p}} \), we have \( \alpha \leq \frac{\sqrt{1 - \epsilon} \| x \|_2}{\sqrt{p}} \).

Now according to (4.3.20), this implies that \( \alpha^2 \geq \frac{(1 - \epsilon) \| x \|_2^2 \sigma^2}{4p \| x \|_{\infty}^2} \). Additionally by (4.3.16), we know \( \sigma^*(\alpha) \geq \frac{\alpha}{\sqrt{1 + (\chi^*(\alpha))^2}} \).

On \( \chi^*(\alpha) \geq 1 \), we have that
\[
1 \geq \frac{1}{p} \mathbb{E} \langle \eta(x/\sigma^*(\alpha) + h; \chi^*(\alpha)), h \rangle = \frac{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}}{\sigma^*(\alpha)^2} \frac{\delta \gamma}{\chi^*(\alpha)^2} - \frac{\delta \gamma}{\chi^*(\alpha)}. \]

These leads to
\[
\chi^*(\alpha) \leq 1 \vee \frac{16}{u^2 \varepsilon \| x \|_2 / \sqrt{p}} \vee 2 \| x \|_{\infty} \frac{\alpha}{\sqrt{1 - \varepsilon} \| x \|_2 / \sqrt{p}} \frac{\alpha + \delta \gamma \sqrt{2}}{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}} \leq 1 \vee \frac{16 \| x \|_{\infty}}{\varepsilon \sqrt{1 - \varepsilon} \| x \|_2 / \sqrt{p}} \left( \frac{1}{\sqrt{\delta}} + \frac{\gamma \sqrt{2} \delta}{\sqrt{\alpha^2 + \delta \sigma_w^2}} \right) \]
where in the last step we use the fact that \( \sqrt{2 \log x} \leq x \) for \( x \geq 1 \). Setting \( \varepsilon = \frac{1}{2} \) completes the proof.

**Proof of (ii).** When \( M_A(\chi^*) > \delta \), let \( \sigma_0 \) be the value that satisfies
\[ \delta \sigma_0^2 = \frac{1}{p} \mathbb{E} \| \eta(x + \sigma_0 h; \sigma_0 \chi^*) - x \|_2^2. \]
Let \( b_0 = \frac{\partial}{\partial \sigma^2} \frac{1}{p} \mathbb{E} \| \eta(x + \sigma h; \sigma^* x) - x \|_2^2 \bigg| \sigma = \sigma_0 < \delta \). Then, for the same \( \sigma^* \), we have

\[
\sigma_0^2 + \frac{\delta \sigma_w^2}{\delta - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2} \leq (\sigma^*)^2 \leq \sigma_0^2 + \frac{\delta \sigma_w^2}{\delta - b_0},
\]

and further lower and upper bounds on \( \alpha^* \) as below:

\[
\delta \sigma_0^2 + \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2 \delta \sigma_w^2 \leq (\sigma^*)^2 \leq \delta \sigma_0^2 + \frac{b_0 \delta \sigma_w^2}{\delta - b_0}
\]

In regards to the upper bound of \( \chi^*(\alpha) \), we can slightly adapt the proof of (i) to obtain the following upper bound:

\[
\chi^*(\alpha) \leq 1 \vee \frac{16 \| x \|_\infty}{\varepsilon \sqrt{1 - \varepsilon^2 \| x \|_2^2}} \left( \frac{1}{\sqrt{\delta}} + \frac{\delta \sqrt{2}}{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}} \right).
\]

We note that an implicit assumption on \( \sigma_w \) to enable the above bound is \( \sigma_w^2 \leq \frac{\delta - b_0}{\delta b_0} \left( \frac{\delta - b_0}{\delta b_0} \right) \).

**Proof of (iii).** We note that \((\sigma^*)^2 \) stays above \( \sigma_w^2 + \frac{1}{p} \mathbb{E} \| \eta(x + \sigma_w h; \sigma_w^* x) - x \|_2^2 + \frac{1}{\delta p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2 ((\sigma^*)^2 - \sigma_w^2) \) and \( \sigma_w^2 + \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2 \). These observations lead to the following results:

\[
\frac{\delta \sigma_w^2 - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2 \delta \sigma_w^2 + \frac{1}{p} \mathbb{E} \| \eta(x + \sigma_w h; \sigma_w^* x) - x \|_2^2}{\delta - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2} \leq (\sigma^*)^2 \leq \frac{\delta \sigma_w^2 + \frac{1}{p} \| x \|_2^2}{\delta - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2},
\]

which in turn leads to the following range of \( \alpha \) that contains \( \alpha^* \).

\[
\frac{\delta \mathbb{E} \| \eta(x + \sigma_w h; \sigma_w^* x) - x \|_2^2}{\delta - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2} \leq (\sigma^*)^2 \leq \frac{\delta \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2 \sigma_w^2 + \frac{1}{p} \| x \|_2^2}{\delta - \frac{1}{p} \mathbb{E} \| \eta(h; \sigma^* x) \|_2^2}
\]

The further lower bound for \( \alpha^* \) is obvious due to Lemma D.2.4.

Regarding upper bounds on \( \chi^*(\alpha) \), for any constant \( c > 1 \vee \sqrt{\delta} \), if \( \chi \geq \frac{\delta c}{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}} \), by (4.3.20) and (4.3.21) we have that

\[
\frac{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}}{c \alpha} \| \eta \bigg( \frac{x}{\sigma} + h; \chi \bigg) - \frac{x}{\sigma} \|_2^2 = \frac{\sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}}{c \sigma} \leq \frac{1}{p} \langle \eta \bigg( \frac{x}{\sigma} + h; \chi \bigg), h \rangle \leq \| \eta \bigg( \frac{x}{\sigma} + h; \chi \bigg) - \frac{x}{\sigma} \|_2^2,
\]

which leads to

\[
\alpha^2 \geq \frac{\delta^2}{c^2 - \delta \sigma_w^2}.
\]
When $\delta^2 > \frac{1}{p} \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]$, it is not hard to see that
\[
\frac{\delta^2}{\sigma_w^2} \geq \frac{\mathbb{E}[(\eta(h; \chi^*))^2]^2}{\sigma_w^2 + \frac{1}{p} \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}
\]
when $\sigma_w^2 \geq \frac{(c^2 - \delta) \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}{\delta - \frac{1}{p} \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}$ which forms a contradiction with the range of $\alpha^*$. As a result, we have $\chi \leq \frac{\delta \gamma}{\delta^2 + \delta \sigma_w^2}$ if $c - 1$. If we are able to pick the model parameters such that $\mathbb{E}[\|\eta(h; \chi^*)\|^2_2] < \frac{\delta^2}{4\delta + 2}$, then we may set $c = \sqrt{2\delta + 1}$ which will lead to a condition on $\sigma_w$ as $\sigma_w \geq \frac{\sqrt{2(\delta + 1)} \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}{\delta \sigma_w}$. We next prove our claim.

We argue that when $\sigma_w \geq \frac{\sqrt{\delta + 1} \mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}{\delta \sqrt{\sigma}}$, $\chi > \frac{\gamma}{2\sigma_w}$. With this claim, in order to have $\mathbb{E}[\|\eta(h; \chi^*)\|^2_2] < \frac{\delta^2}{4\delta + 2}$, using Lemma D.1.5 (vi) and Lemma D.2.8, we can simply set $\frac{\gamma}{\sigma_w} \leq \frac{1}{\mathbb{E}[\|\eta(h; \chi^*)\|^2_2]} \sqrt{0 \log \frac{16\delta + 8}{\delta^2}}$.

Let $\sigma_w \geq \frac{\mathbb{E}[\|\eta(h; \chi^*)\|^2_2]}{C \sqrt{\sigma}}$ for now. Using (4.3.18) we obtain
\[
\frac{\delta \sigma_w^2}{\sigma^2} \geq \delta - \|h - \Pi_{D_{\chi^*}}(x/\sigma^* + h)\|^2_2
\]
\[
= \delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)] + \frac{1}{p} \mathbb{E}[x/\sigma^* + h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)]
\]
\[
- \|\Pi_{D_{\chi^*}}(x/\sigma^* + h)\|^2_2 - \frac{1}{p} \mathbb{E}[x/\sigma^*, \Pi_{D_{\chi^*}}(x/\sigma^* + h)]
\]
\[
\geq \delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)] - \frac{1}{p} \mathbb{E}[x/\sigma^*, \Pi_{D_{\chi^*}}(x/\sigma^* + h)]
\]
\[
\geq \delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)] - \frac{\|x\|^2_2}{\sigma^* \sqrt{p}} \sqrt{1 + \frac{\|x\|^2_2}{(\sigma^*)^2 p}}
\]
\[
\geq \delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)] - \frac{\|x\|^2_2}{\sigma^* \sqrt{p}} \sqrt{1 + C_1^2}
\]
where step (a) is by Lemma D.1.5 (i), (b) is by the Cauchy Schwarz inequality. This implies that
\[
\frac{\delta \sigma_w + \sqrt{1 + C_1^2} \mathbb{E}[\|x\|^2_2/\sqrt{p}]}{\sigma^*} \geq \delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)]
\]
Now by (4.3.19), we have
\[
\delta \gamma = \sigma^* \chi^* (\delta - 1 + \frac{1}{p} \mathbb{E}[h, \Pi_{D_{\chi^*}}(x/\sigma^* + h)]) \leq (\delta \sigma_w + \sqrt{1 + C_1^2} \mathbb{E}[\|x\|^2_2/\sqrt{p}]) \chi^*,
\]
which implies that
\[
\chi^* \geq \frac{\delta \gamma}{\delta \sigma_w + \sqrt{1 + C_1^2} \mathbb{E}[\|x\|^2_2/\sqrt{p}]} \geq \frac{\delta}{\delta + C_1 \sqrt{1 + C_1^2} \sigma_w} \gamma.
\]
By setting $C_2^2 = \frac{1}{2}(\sqrt{4\delta^2 + 1} - 1)$, we have $C_1\sqrt{1 + C_2^2} = \delta$, and hence $\chi^* \geq \frac{\gamma}{2\sigma_w}$.

We note that decreasing $C_1$ does not affect the above results. This gives us a cleaner form of $C_1 = \frac{\delta}{\sqrt{\delta^2 + 1}}$. The proof is hence completed.

As the last step, we aim to simplify the lower bound of $\alpha^*$. Let $\bar{D}_\chi = \{v : |v|_{(1)} \leq \chi\|\lambda\|_2^2/p\}$. We have

$$
\|\eta(x + \sigma_w h; \sigma_w \chi^*) - \eta(\sigma_w h; \sigma_w \chi^*)\|_{L_2} \leq \frac{1}{\sqrt{p}}(1 - \mathbb{P}(x + \sigma_w h \in \bar{D}_{\sigma_w \chi^*}, \sigma_w h \in \bar{D}_{\sigma_w \chi^*})).
$$

Proof of (iv). By Lemma 5.1.1 (iii), $\frac{1}{\sigma^2} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|_2^2$ is a decreasing function of $\sigma^2$. Since $\frac{1}{\sigma^2} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|_2^2 \leq 1 + \chi^2$ by Lemma D.1.6 and $\lim_{\sigma \to \infty} \frac{1}{\sigma^2} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|_2^2 = \mathbb{E}\|\eta(h; \chi)\|_2^2$, we know that

$$
\frac{1}{p} \mathbb{E}\|\eta(h; U_\chi)\|_2^2 \leq \frac{\alpha^2}{(\sigma^2(\alpha))^2} \leq 1 + U_{\chi}^2.
$$

This proves (4.3.16). (4.3.17) is a simple result of (4.3.21) and the fact that $\mathbb{E}(\eta(\frac{x}{\sigma} + h; \chi), h) = \mathbb{E}(\eta(\frac{x}{\sigma} + h; \chi) - \eta(\frac{x}{\sigma}; \chi), h) \geq \mathbb{E}\|\eta(\frac{x}{\sigma} + h; \chi) - \eta(\frac{x}{\sigma}; \chi)\|_2^2 \geq 0$. 

\section{4.3.3 Concentration of $\hat{\Lambda}$ on $\Lambda$}

In this section, we justify the concentration of the objective function $\hat{\Lambda}$ around $\Lambda$ under different model settings. Let us start with a few basic concentration results which can proved from standard results, and will serve as the building blocks for our subsequent analyses.

\textbf{Lemma 4.3.4.} Let $c$ and $C$ denote absolute constants. We have the following concentration results:

(i) For any $t \geq 0$, we have

$$
\mathbb{P}\left(\frac{1}{\sqrt{p}}\|\sqrt{\delta}\alpha g - \delta w\|_2 - \sqrt{\delta\alpha^2 + \delta^2\sigma_w^2} > t\right) \leq 6e^{-Ct^2/\alpha^2 + \delta^2\sigma_w^2}.
$$

Furthermore,

$$
\mathbb{P}\left(\sup_{\alpha \leq U_\alpha} \frac{1}{\sqrt{p}}\|\alpha g - \sqrt{\delta}w\|_2 - \sqrt{\delta\alpha^2 + \delta^2\sigma_w^2} > t\right)
\leq C\sqrt{\delta}U_\alpha + (\sqrt{\delta} + 1)t e^{-\frac{c}{U_{\alpha} + \sqrt{\delta}\sigma_w^2}} := P_1(U_\alpha, t).
$$
(ii) We have that
\[
P\left( \frac{1}{\sigma_p^2} |\langle h, x \rangle| > t \right) \leq 2e^{-\frac{ct^2}{|\alpha|^2/\sigma_p^2}} := P_2(t), \quad \forall t \geq 0.
\] (4.3.26)

(iii) If \( \theta = \frac{1}{\sqrt{p}} \mathbb{E}[\eta(x + \sigma h; \sigma \chi)]_2 \), then \( P\left( \frac{1}{\sigma_p^2} \|\eta(x + \sigma h; \sigma \chi)\|_2^2 - \mathbb{E}[\eta(x + \sigma h; \sigma \chi)]_2^2 \right) > t \right) \leq 3e^{-\frac{ct^2}{\sigma_t^2}}.

**Proof. Proof of (i).** First, by applying the Bernstein’s inequality (please refer to Theorem D.2.2) and Lemma D.2.1, we have
\[
P\left( \frac{1}{n} \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2^2 - (\delta \alpha^2 + \delta^2 \sigma_w^2) \right) > t \right)
\]
\[
\leq P\left( \frac{1}{n} \left\| g \right\|_2^2 - 1 > \frac{t}{(\sqrt{\delta \alpha + \delta \sigma_w})^2} \right) + P\left( \left\| g \right\|_2 - \frac{\sigma_w t}{(\sqrt{\delta \alpha + \delta \sigma_w})^2} \right)
\]
\[
+ P\left( \frac{1}{n} \left\| w \right\|_2^2 - \sigma_w^2 > \frac{\sigma_w^2 t}{(\sqrt{\delta \alpha + \delta \sigma_w})^2} \right)
\]
\[
\leq 6e^{-\frac{ct^2}{(\sqrt{\delta \alpha + \delta \sigma_w})^2 + (\sqrt{\delta \alpha + \delta \sigma_w})^2}}.
\]

Since\(^5\) \( |z - c| > \delta \) implies that \( |z^2 - c^2| > c \delta \vee \delta^2 \), we have
\[
P\left( \frac{1}{n} \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2 - \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2} > t \right)
\]
\[
\leq P\left( \frac{1}{n} \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2 - (\delta \alpha^2 + \delta^2 \sigma_w^2) > \left( \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2} t \vee \delta^2 \right) \right)
\]
\[
\leq 6e^{-\frac{ct^2}{\alpha^2 + \delta^2 \sigma_w^2}}.
\]

This completes the first part. To extend the result to bounding the supreme difference, we first note that
\[
\left\| \frac{1}{\sqrt{n}} \left( \sqrt{\delta \alpha} g - \delta w \right) \right\|_2 - \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2} \left\| \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2 - \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2} \right\|
\]
\[
\leq \frac{1}{\sqrt{n}} \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2 - \left\| \sqrt{\delta \alpha} g - \delta w \right\|_2 + \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2} - \sqrt{\delta \alpha^2 + \delta^2 \sigma_w^2}
\]
\[
\leq \sqrt{\delta} \left\| \frac{g}{\sqrt{n}} \right\|_2 \left| \alpha_1 - \alpha_2 \right| + \left| \int_{\alpha_1}^{\alpha_2} \frac{\delta t}{\sqrt{\delta t^2 + \delta^2 \sigma_w^2}} dt \right|
\]
\[
\leq \left( \frac{\|g\|_2}{\sqrt{n}} + 1 \right) \sqrt{\delta} |\alpha_1 - \alpha_2|.
\]

\(^5\)We refer the reader to [Ver18] for the result: \( |z - 1| > \delta \) implies \( |z^2 - 1| > \delta \vee \delta^2 \). A simply adaption gives us what we claim here.
CHAPTER 4. CONCENTRATION OF SLOPE MSE

The Lipschitz constant of the difference is \((2 + s_1)\sqrt{\delta}\) on \(\|g\|_2 \leq \sqrt{n}(1 + s_1)\). Let \(\mathcal{T}\) be an \(\epsilon\)-net on \((0, U_\alpha]\) with \(\epsilon = \frac{t}{(4 + 2s_1)\sqrt{\delta}}\). It is straightforward to confirm that \(|\mathcal{T}| \leq U_\alpha(8 + 4s_1)\sqrt{\delta}t\). Therefore, we have

\[
\mathbb{P}\left(\sup_{0 < \alpha \leq U_\alpha} \left|\frac{1}{\sqrt{n}}\|\sqrt{\delta}\alpha g - \delta w\|_2 - \sqrt{\delta}\alpha^2 + \delta^2\sigma_w^2\right| > t\right) \\
\leq \frac{U_\alpha(8 + 4s_1)\sqrt{\delta}}{t} e^{-\frac{ct}{\sqrt{U_\alpha + \sqrt{\delta}\sigma_w}}} + e^{-ct^2}(U_\alpha + (\sqrt{\delta} + 1)t) e^{-\frac{ct}{\sqrt{U_\alpha + \sqrt{\delta}\sigma_w}}}.
\]

where in the last step we set \(s_1 = \frac{t}{U_\alpha + \sqrt{\delta}\sigma_w}\).

**Proof of (ii).** We note that \(\langle h, x \rangle\) is Lipschitz in \(h\). The result then follows by applying the Gaussian Lipschitz concentration (please refer to Theorem D.2.3);

**Proof of (iii).** In the rest of this proof, to simplify the notations, we use \(\eta\) to denote the proximal operator \(\eta(x + \sigma h; \sigma \chi)\) as a function of \(h\). It is not hard to see that \(\|\|\eta\|_2\|_{\text{Lip}} \leq \sigma\). Therefore, by Theorem D.2.3,

\[
\mathbb{P}(\|\|\eta\|_2 - \mathbb{E}\|\eta\|_2\| > t) \leq 2e^{-\frac{ct^2}{\sigma^2}}.
\]

This further implies that

\[
\mathbb{P}(\|\|\eta\|_2^2 - [\mathbb{E}\|\eta\|_2]^2\| > t) \leq \mathbb{P}(\|\|\eta\|_2 - \mathbb{E}\|\eta\|_2\| > s) + \mathbb{P}(\|\|\eta\|_2 - \mathbb{E}\|\eta\|_2\| > \frac{t}{2\mathbb{E}\|\eta\|_2 + s})
\]

Setting \(s = -\mathbb{E}\|\eta\|_2 + \sqrt{[\mathbb{E}\|\eta\|_2]^2 + t}\), we obtain that

\[
\mathbb{P}(\|\|\eta\|_2^2 - \mathbb{E}\|\eta\|_2^2\| > t) \leq 3e^{-\frac{ct^2}{\sigma^2(\mathbb{E}\|\eta\|_2^2 + t)}}.
\]

This gives us the final result. \(\square\)

Part (iii) in Lemma 4.3.4 further gives us the following supremum bound.

**Lemma 4.3.5.** Suppose that \(\sigma \in [0, U_\sigma]\) and \(\chi \leq U_\chi\). Then, we have the following
concentration:

\[
P\left(\sup_{\sigma,\chi} \frac{1}{\sigma_p} \left| \|\eta(x + \sigma h; \sigma \chi)\|_2^2 - \mathbb{E}\|\eta(x + \sigma h; \sigma \chi)\|_2^2 \right| > t \right)
\leq P_2(t) + C \frac{(1 + x_0^2/p + \frac{t^4}{\|x\|_2^2/p^2} + U_\chi^4 + U_\sigma^4)(U_\sigma^2 + U_\chi^2)}{t^2} e^{-\frac{c t^2}{\|x\|_2^2/p + U_\sigma^2 + U_\chi^2}}.
\]

where \(P_2(t)\) was defined in the statement of Lemma 4.3.4.

\(\textbf{Proof.}\) Consider the function \(S(\sigma, \chi)\) defined as

\[
S(\sigma, \chi) := \frac{1}{\sigma} \left( \|\eta(x + \sigma h; \sigma \chi)\|_2^2 - \mathbb{E}\|\eta(x + \sigma h; \sigma \chi)\|_2^2 \right).
\]

\(S\) is obviously continuous in \((\sigma, \chi)\). In addition, by L'Hopital rule, it is not hard to see that \(\lim_{\sigma \to 0} S(\sigma, \chi) < \infty\). We would like to justify the Lipschitz property of \(S\).

From (D.2.3), we can obtain the following expressions for the partial derivative of \(S\):

\[
\frac{\partial S}{\partial \sigma} = \frac{1}{\sigma^2} \left( \|\eta(x + \sigma h; \sigma \chi)\|_2^2 - \mathbb{E}\|\eta(x + \sigma h; \sigma \chi)\|_2^2 - \|x\|_2^2 \right),
\]

\[
\frac{\partial S}{\partial \chi} = \frac{2}{\sigma \chi} \left( - \langle \Pi_{D\sigma \chi}(x + \sigma h), \eta(x + \sigma h, \sigma \chi) \rangle + \mathbb{E}\langle \Pi_{D\sigma \chi}(x + \sigma h), \eta(x + \sigma h, \sigma \chi) \rangle \right).
\]

Using Lemma D.1.6, we obtain the following upper bounds for \(\frac{\partial S}{\partial \sigma}\) and \(\frac{\partial S}{\partial \chi}\):

\[
\frac{\partial S}{\partial \sigma} \leq \|h\|_2^2 + p + 2 \chi^2 \|\lambda\|_2^2 := \xi_\sigma,
\]

\[
\frac{\partial S}{\partial \chi} \leq 2 \|\lambda\|_2(\|\eta(x + \sigma h; \sigma \chi)\|_2 + \mathbb{E}\|\eta(x + \sigma h; \sigma \chi)\|_2) := \xi_\chi.
\]

Therefore, \(\|S\|_{\text{Lip}} \leq \xi_\sigma + \xi_\chi := p \xi\). On the set \(\{\|h\|_2 \leq \sqrt{p}(1 + s)\}\), we may use the following bound for \(\xi\):

\[
\xi \lesssim 1 + \|x\|_2 / \sqrt{p} + s^2 + U_\chi^2 + U_\sigma^2.
\]

Now for a given \(t > 0\), consider an \(\epsilon\)-net \(T\) of the compact set \(\{\sigma, \tau : 0 \leq \sigma \leq U_\sigma, L_\chi \leq \chi \leq U_\chi\}\) with \(\epsilon = \frac{t}{2c}\). The basic result (see, for example, [Ver18] Corollary
4.2.13) in metric entropy tells us that the $\frac{1}{2\xi}$-covering number in two-dimensional space satisfies

$$|T| \lesssim \frac{\xi^2(U_\sigma^2 + U_\chi^2)}{t^2}.$$ 

$\|S\|_{\text{Lip}} \leq \xi$ guarantees that $S$ over its domain will not be “too far away” from its value on $T$. By the concentration bound we derived in Lemma 4.3.4 (iii), we obtain the following union bound after setting $s = \frac{t}{\|x\|_2/\sqrt{p}}$:

$$\mathbb{P}\left( \sup_{\alpha \leq U_\sigma, \chi \leq U_\chi} S(\sigma, \tau) > pt \right) \leq |T| \sup_{\alpha \leq U_\sigma, \chi \leq U_\chi} \mathbb{P}\left( |S(\sigma, \chi)| > pt \right) + \mathbb{P}(\|h\|_2 \geq \sqrt{p}(1 + s)) \lesssim \frac{\xi^2(U_\sigma^2 + U_\chi^2)}{t^2} e^{-\frac{ct^2}{\|x\|_2^2/p + U_\sigma^2 + U_\chi^4}} + P_2(t)$$

With some algebra, it is not hard to see that $\xi^2(U_\sigma^2 + U_\chi^2) \lesssim C(1 + \|x\|_2^2/p + \frac{t^4}{\|x\|_2^2/p^2} + U_\chi^4 + U_\sigma^4)(U_\sigma^2 + U_\chi^2)$. \hfill \Box

With the above results, we are ready to prove the concentration of $\hat{\Lambda}$ on $\Lambda$.

**Lemma 4.3.6.** Consider the set $K = \{(\alpha, \beta, T_h) : 0 \leq \alpha \leq U_\sigma, L_\chi \leq \frac{\gamma}{\beta} \leq U_\chi, \frac{\alpha \beta}{T_h} \leq U_\sigma\}$, then for any $t \geq 0$, we have that

$$\mathbb{P}\left( \sup_{(\alpha, \beta, T_h) \in K} \frac{1}{\gamma}(\hat{\Lambda}(\alpha, \beta, T_h) - \Lambda(\alpha, \beta, T_h)) > t \right) \leq P_1(U_\sigma, L_\chi t) + P_2(L_\chi t) + P_3(U_\sigma, U_\chi, L_\chi t). \tag{4.3.32}$$

where $P_1$, $P_2$ and $P_3$ are presented in (4.3.25), (4.3.26) and (4.3.27) respectively.

**Proof.** First we know that

$$\frac{1}{\beta}(\hat{\Lambda} - \Lambda) = \underbrace{\|\sqrt{p}\alpha g - \sqrt{n}w\|_2}_{\mathcal{J}_1} - \sqrt{\alpha^2 \delta + \delta^2 \sigma_w^2} - \underbrace{\frac{h^\top x}{p}}_{\mathcal{J}_2} - \underbrace{\frac{T_h}{2\alpha \beta} \left( \|\eta(x + \frac{\alpha \beta}{T_h} h; \frac{\alpha \beta}{T_h} \|_2^2 - \mathbb{E}\|\eta(x + \frac{\alpha \beta}{T_h} h; \frac{\alpha \beta}{T_h})\|_2^2 \right)}_{\mathcal{J}_3}.$$
CHAPTER 4. CONCENTRATION OF SLOPE MSE

This leads to the union bound,
\[
P \left( \max_{k} \frac{1}{\gamma} |\hat{\Lambda} - \Lambda| > t \right)
\leq P \left( \max_{0 < \alpha \leq U_k \alpha} |J_1| > \frac{L_\Lambda t}{3} \right) + P \left( |J_2| > \frac{L_\Lambda t}{3} \right) + P \left( \max_{k} |J_3| > \frac{L_\Lambda t}{3} \right).
\]

The result then follows from Lemma 4.3.4 (i), (ii) and Lemma 4.3.5. \(\square\)

In order to transfer the above concentration results from the objective functions to the saddle points, we need to make use of the Hessian information of the objective function. The rest of this section is devoted to bounding the eigenvalues of the Hessian matrix of \(\Lambda\) w.r.t. \((\alpha, \beta, T_h)\).

**Lemma 4.3.7.** Suppose \(P \left( \sup_{(\alpha, \beta, T_h) \in K} |\hat{\Lambda} - \Lambda| > t \right) \leq p_0\) for some set \(K\). Furthermore, for any given \(\alpha \in [L_\alpha, U_\alpha]\) with some \(L_\alpha < U_\alpha\), let \((\beta^*(\alpha), T^*_h(\alpha)) = \arg \sup_{\beta, T_h} \Lambda(\alpha, \beta, T_h)\), the slice of \((\beta, T_h)\) in \(K\) given \(\alpha\) contains an open ball \(B_t(\alpha) ((\beta^*(\alpha), T^*_h(\alpha)))\) and the smallest eigenvalue of negative Hessian of \(\Lambda\) w.r.t. \((\beta, T_h)\) can be lower bounded by \(\psi_1(\alpha)\), then we have that
\[
P \left( \left| \sup_{\beta, T_h} \hat{\Lambda} - \sup_{\beta, T_h} \Lambda \right| > t \right) \leq p_0, \quad \forall \alpha \in [L_\alpha, U_\alpha], \quad t \leq \frac{r^2(\alpha)\psi_1(\alpha)}{4}.
\]

*Proof.* From now on we only focus on the subset of the probability space where \(\sup_K |\hat{\Lambda} - \Lambda| \leq t\). For any \(\alpha \in [L_\alpha, U_\alpha]\), the condition guarantees that \((\alpha, \beta^*(\alpha), T^*_h(\alpha)) \in K\). Hence, we have
\[
\sup_{\beta, T_h} \Lambda - \sup_{\beta, T_h} \hat{\Lambda} \leq \Lambda(\alpha, \beta^*(\alpha), T^*_h(\alpha)) - \hat{\Lambda}(\alpha, \beta^*(\alpha), T^*_h(\alpha)) \leq t.
\]

Furthermore, the distance between the supremum of \(\hat{\Lambda}\) and the supremum of \(\Lambda\) cannot be larger than \(2\sqrt{\frac{t}{\psi_1(\alpha)}}\). Hence, we have that the supremum of \(\hat{\Lambda}\) also stays in \(K\) when \(t \leq \frac{r^2(\alpha)\psi_1(\alpha)}{4}\). Let \((\beta^*_1(\alpha), T^*_{h1}(\alpha)) = \arg \sup_{\beta, T_h} \hat{\Lambda}(\alpha, \beta, T_h)\). Since \((\alpha, \beta^*_1(\alpha), T^*_{h1}(\alpha)) \in K\), we have
\[
\sup_{\beta, T_h} \hat{\Lambda} - \sup_{\beta, T_h} \Lambda \leq \hat{\Lambda}(\alpha, \beta^*_1(\alpha), T^*_{h1}(\alpha)) - \Lambda(\alpha, \beta^*_1(\alpha), T^*_{h1}(\alpha)) \leq t.
\]
This gives us
\[ \left| \sup_{\beta, T_h} \hat{\Lambda} - \sup_{\beta, T_h} \Lambda \right| \leq t. \]

The convexity easily follows. \[ \Box \]

**Remark 4.3.1.** We emphasize that here the supremum for \( \hat{\Lambda} \) is its global maximum given any \( \alpha \), instead of the supremum within the set \( K \). In addition, both \( \sup_{\beta, T_h} \hat{\Lambda}(\alpha) \) and \( \sup_{\beta, T_h} \Lambda(\alpha) \) are convex.

**Remark 4.3.2.** Suppose for each \( \alpha \), we have \( (\alpha \beta, \frac{\gamma}{\beta}) \in [\frac{L_{\sigma}}{2}, 2U_{\sigma}] \times [\frac{L_{\chi}}{2}, 2U_{\chi}] \) while the optimal pair \( (\alpha \beta^*, \frac{\gamma^*}{\beta^*}) \in [L_{\sigma}, U_{\sigma}] \times [L_{\chi}, U_{\chi}] \). Then, we can pick
\[ r(\alpha) = \frac{\alpha \gamma}{U_{\chi} \max\{2U_{\sigma}, \sqrt{L_{\sigma}^2 + 4\alpha^2}\}}. \]

**Remark 4.3.3.** Regarding \( \psi_1(\alpha) \), we can obtain a bound using the following arguments. Assume \( L_{\sigma} \leq \frac{\alpha \beta}{T_h} \leq U_{\sigma}, L_{\chi} \leq \frac{\gamma}{\beta} \leq U_{\chi} \). Define \( M_{hh} = \frac{1}{p} \mathbb{E} \sum_{I \in \mathcal{P}_0} (\sum_{j \in I} h_j s_j)^2, \)
\( M_{h\lambda} = \frac{1}{p} \mathbb{E} \sum_{I \in \mathcal{P}_0} (\sum_{j \in I} h_j \lambda_j)(\sum_{j \in I} \lambda_j), \)
\( M_{\lambda\lambda} = \frac{1}{p} \mathbb{E} \sum_{I \in \mathcal{P}_0} (\sum_{j \in I} \lambda_j)^2, \)
where \( s_j = \text{sign}(x_j + \frac{\alpha \beta}{T_h} h_j) \). Then with some calculations we can represent the second order derivatives of \( \Lambda \) w.r.t. \((\beta, T_h)\) as
\[ \frac{\partial^2 \Lambda}{\partial \beta^2} = -\delta - \frac{\alpha}{T_h} M_{hh}, \]
\[ \frac{\partial^2 \Lambda}{\partial \beta \partial T_h} = \frac{\alpha \beta}{T_h^2} M_{hh} - \frac{\alpha \gamma}{T_h^2} M_{h\lambda}, \]
\[ \frac{\partial^2 \Lambda}{\partial T_h^2} = -\frac{\sigma^2}{\alpha T_h} (M_{hh} - 2\chi M_{h\lambda} + \chi^2 M_{\lambda\lambda}), \]

Therefore the determinant and the trace of the negative hessian take the following forms:
\[ \text{det} = \delta \frac{\sigma^2}{\alpha T_h} (M_{hh} - 2\chi M_{h\lambda} + \chi^2 M_{\lambda\lambda}) + \frac{\sigma^2 \chi^2}{T_h^2} (M_{hh} M_{\lambda\lambda} - M_{h\lambda}^2), \]
\[ \text{trace} = \delta + \frac{\alpha}{T_h} M_{hh} + \frac{\sigma^2}{\alpha T_h} (M_{hh} - 2\chi M_{h\lambda} + \chi^2 M_{\lambda\lambda}). \]

Furthermore, we note that \( M_{hh} - 2\chi M_{h\lambda} + \chi^2 M_{\lambda\lambda} \) is the derivative of \( \| \eta(x + \sigma h; \sigma \chi) - x \|_2^2 \) w.r.t. \( \sigma^2 \) (please refer to Lemma 5.1.1), and hence the following
relation holds:

\[
M_{hh} - 2\chi M_{h\lambda} + \chi^2 M_{\lambda\lambda} = \frac{1}{p} \mathbb{E} \sum_{I \in \mathcal{P}_0} \frac{(\sum_{j \in I} h_j s_j - \chi \lambda_j)^2}{|I|} \geq \lim_{\sigma \to \infty} \frac{\partial \|\eta(x + \sigma h; \sigma \chi) - x\|_2^2}{\partial (\sigma^2)} = \frac{1}{p} \mathbb{E} \|\eta(h; \chi)\|_2^2.
\]

An upper bound for \(\frac{1}{\lambda_{\min}}\) is \(\frac{\text{trace}}{\det}\). Hence, we have that

\[
\frac{1}{\lambda_{\min}} \leq \frac{1}{\delta} + \frac{\sigma^2}{\alpha^2 \gamma} \|\eta(h; \chi)\|_2^2 \leq \frac{1}{\delta} + \frac{\delta \gamma + L \sigma \chi}{\delta L^2 \sigma \chi} \|\eta(h; U \chi)\|_2^2.
\]

**Lemma 4.3.8.** We have the following bound for \(\frac{\partial^2 (\lambda |_{\beta_*=\beta^*(\alpha), T_h=T_h^*(\alpha)})}{\partial \alpha^2}\):

\[
\frac{1}{\gamma} \frac{\partial^2 \Lambda}{\partial \alpha^2} \geq \frac{\sigma^2 \delta^3}{\chi (\delta \alpha^2 + \delta^2 \sigma^2_w)^2}.
\]

**Proof.** Recall the notation \(\mathcal{P}_0\) we defined after (D.0.3). Using (4.3.11), it is not hard to verify that

\[
\frac{1}{\gamma} \frac{\partial^2 \Lambda}{\partial \alpha^2} = \frac{\sigma^2}{\alpha^2 \gamma} \|\eta(h; \chi)\|_2^2 + \frac{1}{\alpha^2 \sigma^*(\alpha) \chi^*(\alpha) p} \left( \|x\|_2^2 - \mathbb{E} \sum_{I \in \mathcal{P}_0} \frac{(\sum_{k \in I} x_k \cdot \text{sign}(x_k + \sigma h_k))^2}{|I|} \right).
\]

Furthermore, by the chain rule we have

\[
\frac{\partial^2 (\Lambda |_{\beta_*=\beta^*(\alpha), T_h=T_h^*(\alpha)})}{\partial \alpha^2} = \frac{\partial^2 \Lambda}{\partial \alpha^2} - \mathbf{a}^\top \mathbf{B} \mathbf{a},
\]

where \(\mathbf{a}^\top = (\frac{\partial^2 \Lambda}{\partial \alpha \partial \beta}, \frac{\partial^2 \Lambda}{\partial \alpha \partial T_h})\), \(\mathbf{B}\) is the Hessian of \(\Lambda\) w.r.t. \((\beta, T_h)\), and hence \(-\mathbf{B}\) is nonnegative definite.

### 4.3.4 Concentration of MSE

In this section, we are finally ready to prove Theorem 4.1.1. Before delving into the technical part, we first make some connections between the concentration of \(\hat{\alpha}\) on \(\alpha^*\) and the concentration between the MSE on \(\alpha^*\). If we are able to prove
\( \mathbb{P}(|\hat{\alpha} - \alpha^*| > t) \leq \kappa \) (which is what we are going to justify in the rest of this section), then we are able to show (4.2.2) holds with \( \epsilon \) replaced by \( t \) w.h.p. The parameters \( c_1, c_2, c_3 \) in (4.2.2) can be either picked as some large value (for example, we may pick \( c_1 = \alpha^* + 2t \)), or we can simply use the bounds on \( \alpha, \sigma^*(\alpha) \) and \( \chi^*(\alpha) \) we derived in Lemma 4.3.3. Now CGMT theorem finally transfer (4.2.2) to that on \( F_n(\omega) \) in Section 4.2. This will complete our proof.

**Proof of Theorem 4.1.1 (i).** We can first pick the following lower and upper bounds for \( \alpha, \sigma \) and \( \chi \) from Lemma 4.3.3 (i). Let \( C_M = \sqrt{\frac{\delta}{\delta - M\beta}} \), and \( C_\epsilon = \sqrt{\frac{2\epsilon}{\epsilon}} \). Then, we have that

\[
\|\eta(h; \chi^*)\|^2_{L_2} \leq (\alpha^*)^2 \leq C_M^2 M_\lambda(\chi^*)^2 w = \delta(C_M^2 - 1)w^2,
\]

\[
L_\lambda = \frac{\gamma}{C_M w}, \quad U_\lambda \leq 1 + \frac{1}{\sqrt{\delta}} + \frac{\gamma}{\sigma_w} \leq 1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M.
\]

\[
L^2_\sigma = \frac{\alpha^2}{1 + (1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M)^2},
\]

\[
U^2_\sigma = \frac{\alpha^2}{\|\eta(h; 1 + \frac{1}{\sqrt{\delta}} + \sigma_w)\|^2_{L_2}} \leq \frac{\|x\|^2/2p}{\|\eta(h; 1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M\|^2_{L_2}},
\]

where we know that \( \sigma_w \leq \frac{1}{C_M M_\lambda(\chi^*) \sqrt{2p}} = \frac{1}{\sqrt{\delta}(C_M^2 - 1)\sqrt{2p}} \) and \( \gamma \leq C_M \sigma_w \). Let \( \mathcal{K} = \{ (\alpha, \beta, T_h) : L_\alpha \leq \alpha \leq U_\alpha, L_\lambda \leq U_\lambda, L_\sigma \leq \frac{\alpha}{T_h} \leq U_\sigma \} \), we have \( (\alpha, \sigma^*(\alpha), \chi^*(\alpha)) \in \mathcal{K} \).

Using Lemma 4.3.6 and doing some calculations, we could see that \( P_3 \) dominates the result, implying the uniform closeness between \( \hat{\Lambda} \) and \( \Lambda \) as:

\[
\mathbb{P}(\sup_{\mathcal{K}}|\hat{\Lambda} - \Lambda| > t) \lesssim \frac{(C_M^2 \sigma_w^2 + \frac{1}{C_M^2 \sigma_w^2} t^4)(1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M^6)^6}{\|\eta(h; C(1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M)\|^6_{L_2} t^2} e^{-\frac{\epsilon^{\|\eta(h; U)\|^2_{L_2}}}{(C_M^2 \sigma_w^2)(1 + \sqrt{\delta}(M\beta))}}.
\]

Next we would like to apply Corollary 4.3.7. By Remark 4.3.2, we know that we can pick \( r(\alpha) = \frac{\gamma^{\|\eta(h; U)\|^2_{L_2}}}{U_{\chi}} \) and \( \psi_1 = \frac{\delta^{\frac{1}{2}}(\|\eta(h; U)\|^2_{L_2})}{(C_M^2 + 1)U_{\chi}} \) by Remark 4.3.3. Hence, we can conclude that \( \sup_{\mathcal{K}} \beta, T_h \hat{\Lambda} - \sup_{\mathcal{K}} \beta, T_h \Lambda \mid < t \) over \( \alpha \in [0, U_\alpha] \) w.h.p. when \( t \leq \frac{\sqrt{\gamma} C M^{\|\eta(h; 1 + \frac{1}{\sqrt{\delta}} + C_\epsilon C_M)\|^2_{L_2}}}{C_\epsilon (C_M^2 + 1)(1 + \sqrt{\delta}(M\beta))^{3/2}} \) at any given \( \alpha \).
By Lemma 4.3.8, we have the following lower bound for \( \frac{\partial^2 \Lambda}{\partial \alpha^2} \):

\[
\frac{\partial^2 \Lambda}{\partial \alpha^2} \geq \frac{\gamma}{C^2_M (1 + \frac{1}{\sqrt{\delta}} + C_c C_M) \sigma_w} := \mu.
\]

As a result, we have that

\[
\sup_{\beta,T} \Lambda(\alpha) - \sup_{\beta,T} \Lambda(\alpha^*) \geq \frac{\mu}{2} (\alpha - \alpha^*)^2. \tag{4.3.33}
\]

We claim that the minimizer \( \hat{\alpha} \) of \( \max_{\beta,T} \Lambda \hat{\Lambda} \) must be close to the minimizer \( \alpha^* = 0 \) of \( \sup_{\beta,T} \Lambda \). For any \( |\alpha - \alpha^*| > 2\sqrt{\frac{t}{\mu}} \), (4.3.33) implies that \( \left( \sup_{\beta,T} \Lambda(\alpha) - \sup_{\beta,T} \Lambda(0) \right) > 2t \) and hence

\[
\sup_{\beta,T} \hat{\Lambda}(\alpha) \geq \sup_{\beta,T} \Lambda(\alpha) - t > \sup_{\beta,T} \Lambda(\alpha^*) + t \geq \sup_{\beta,T} \hat{\Lambda}(\alpha^*), \quad \Rightarrow \quad |\hat{\alpha} - \alpha^*| \leq 2\sqrt{\frac{t}{\mu}}.
\]

As a result, after reloading the parameters, we have that for any

\[
t \leq \frac{\delta^{1/4} C^2_M \|\eta(h;1 + \frac{1}{\sqrt{\delta}} + C_c C_M)\|_{L^2}^{5/4}}{C^2_M \sqrt{\sigma_w} \|C(1 + \frac{1}{\sqrt{\delta}} + C_c C_M)\|_{L^2}^{1/4}} \sqrt{\sigma_w},
\]

\[
P(|\hat{\alpha} - \alpha^*| > t) \lesssim \left( C^4_M \sigma_w^4 + \frac{C^4_M}{C^2_M (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)} t^4 \right) (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)^8 e^{-\frac{c\|\eta(h;1 + \frac{1}{\sqrt{\delta}} + C_c C_M)\|_{L^2}^2}{C^2_M \sigma_w^2 (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)^2 (1 + \sqrt{\delta} M \Lambda(\alpha^*)^{-2})}}
\]

\[
\lesssim \left( C^8_M \sigma_w^4 t^8 (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)^8 e^{-\frac{c\|\eta(h;1 + \frac{1}{\sqrt{\delta}} + C_c C_M)\|_{L^2}^2}{C^2_M \sigma_w^2 (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)^2}} \right) e^{-\frac{c\|\eta(h;1 + \frac{1}{\sqrt{\delta}} + C_c C_M)\|_{L^2}^2}{C^2_M \sigma_w^2 (1 + \frac{1}{\sqrt{\delta}} + C_c C_M)^2}}.
\]

\[\square\]

For the proof of (ii) and (iii) of Theorem 4.1.1, we basically go through the same process as that of Theorem 4.1.1 (i). Hence below we only point out the differences for each specific scenario.

**Proof of Theorem 4.1.1 (ii).** First by Lemma 4.3.3 (ii) we have:

\[
\alpha^* = \Theta(1), \quad \chi^*(\alpha) = \Theta(1), \quad \sigma^*(\alpha) = \Theta(\alpha).
\]
By Lemma 4.3.6, all the terms are of the same order, implying that

$$\mathbb{P}(\sup_K |\hat{\Lambda} - \Lambda| > t) \lesssim \frac{1 + t^4}{t^2} e^{-\frac{c\sigma^2}{t^{1+\sigma^2}}}.$$  

By Corollary 4.3.7 and Remark 4.3.2, 4.3.3, we can pick \( r(\alpha) = \Theta(1) \) and \( \psi_1 = \Theta(1). \) Hence \( \sup_{\beta,T_h} \hat{\Lambda} - \sup_{\beta,T_h} \Lambda < t \) over \( \alpha \in [L_\alpha, U_\alpha] \) w.h.p. when \( t \leq \Theta(1) \) at any given \( \alpha. \)

By Lemma 4.3.8, we have \( \frac{\partial^2 \Lambda}{\partial \alpha^2} \geq \frac{\gamma \sigma^2 \psi^3}{\chi_0(\alpha)(6\alpha^2 + \delta^2 \sigma^2)} \geq \Theta(\sigma^2_w). \) Hence similar as the previous proof, we have when \( t \leq \Theta\left(\frac{1}{\sigma_w}\right), \)

$$\mathbb{P}(|\hat{\alpha} - \alpha^*| > t) \leq \frac{\text{Poly}(\sigma_w)}{\sigma_w^2 t^4} e^{-\frac{c\sigma^2}{1+\sigma^2}}.$$  

\( \square \)

**Proof of Theorem 4.1.1 (iii).** Let \( C_\delta = \frac{\delta + 1}{\delta} \) and \( \theta = \|\eta(h; \frac{2\delta + 1}{\delta} \gamma / \sigma_w)\|_{L_2}. \) We can first pick the following lower and upper bounds for \( \alpha, \sigma \) and \( \chi \) by using Lemma 4.3.3 (iii):

\[
L_\alpha^2 = \|\eta(h; \frac{2\delta + 1}{\delta} \gamma / \sigma_w)\|^2_{L_2} \sigma_w^2, \quad U_\alpha^2 = \frac{\delta \|\eta(h; \frac{\gamma}{2\sigma_w})\|^2_{L_2} \sigma_w^2 + \|x\|^2_p}{\delta - \|\eta(h; \frac{\gamma}{2\sigma_w})\|^2_{L_2}},
\]

\[
U_\chi = \frac{(2\delta + 1)\gamma}{\sqrt{\delta^2 \sigma_w^2 + \gamma^2}} \leq \frac{2\delta + 1}{\sigma_w},
\]

\[
L_\chi = \frac{\sqrt{\gamma}}{\sqrt{\delta^2 \sigma_w^2 + \gamma^2}} \geq \frac{\sqrt{\gamma}}{\sqrt{\delta^2 \sigma_w^2 + \gamma^2}} \geq \frac{\gamma}{\sqrt{\sigma_w}},
\]

\[
L_\sigma^2 = \frac{\alpha^2}{1 + U_\chi^2}, \quad U_\sigma^2 \leq \frac{\|\eta(h; \frac{\gamma}{2\sigma_w})\|^2_{L_2} \sigma_w^2 + \|x\|^2_p}{\|\eta(h; U_\chi)\|^2_{L_2}} \leq \frac{4 \|\eta(h; \frac{\gamma}{2\sigma_w})\|^2_{L_2} \sigma_w^2 + \|x\|^2_p}{3 \delta + 2},
\]

where we note the condition \( \sigma_w \geq \sqrt{2(\delta + 1)/\delta} \) and \( \|\eta(h; \frac{\gamma}{2\sigma_w})\|^2_{L_2} \leq \frac{\alpha^2}{\delta^2}. \) Again \( P_3 \) dominate the rates and we have that

$$\mathbb{P}(\sup_K |\hat{\Lambda} - \Lambda| > t)$$

$$\lesssim \left( \frac{\sigma^2_w + \frac{t^4}{\sigma^4_w}}{t^2} \right) \left( \frac{\sigma^2_w + \frac{1+1/\delta}{2\delta} \gamma^2}{\sigma_w^2} + \left( \frac{\sigma^2_w + \frac{1+1/\delta}{2\delta} \gamma^2}{\sigma_w^2} + \frac{\gamma^2}{\sigma_w^2} \right) \right)^3 e^{-\frac{c\sigma^2}{(1 + \frac{\sigma^2_w + \frac{1+1/\delta}{2\delta} \gamma^2}{\sigma_w^2})}}.$$
Next we would like to apply Corollary 4.3.7. By Remark 4.3.2, we know that we can pick $r(\alpha) = \frac{\delta}{2\delta_1 + 1} \sigma_w \theta$. Furthermore, according to Remark 4.3.3, we may set $\psi_1 \geq \frac{\sigma_2^4 \theta^3}{\gamma + \delta \sigma_w}$. Hence $|\sup_{\beta,T_h} \hat{\Lambda} - \sup_{\beta,T_h} \Lambda| < t$ over $\alpha \in [L_\alpha, U_\alpha]$ w.h.p. when $t \leq \frac{\sigma_2^4 \theta^5}{\gamma + \delta \sigma_w}$. By Lemma 4.3.8, we have the following lower bound for 

\[
\frac{\partial^2 \Lambda}{\partial \alpha^2} \geq \frac{\gamma \sigma_w^2 \delta^3}{\delta + 1} \frac{\gamma \sigma_w^2 \delta^3}{(\delta - \|\eta(h; \gamma \sigma_w^2 \theta^2)\|^2_{L^2})^{3/2}} \geq \frac{\sigma_2^3 \delta^4 (\delta - \|\eta(h; \gamma \sigma_w^2 \theta^2)\|^2_{L^2})^{3/2}}{(\delta + 1) (\delta^2 \sigma_w^2 + \|x\|_2^2/p)^{3/2}} \geq \frac{\delta^4 (\delta - \|\eta(h; \gamma \sigma_w^2 \theta^2)\|^2_{L^2})^{3/2}}{(\delta + 1) (\delta^2 \sigma_w^2 + \|x\|_2^2/p)^{3/2}} \geq \frac{\delta}{\delta + 1} : = \mu.
\]

As a result, for any $t \leq \frac{\sigma_2^5 \theta^5 / \gamma + \delta \sigma_w)^{3/2}}{C_\delta^2 \sigma_2^2 \theta^5 / \gamma + \delta \sigma_w)^{3/2}}$ we have 

\[
P(\sup_{K} |\hat{\alpha} - \alpha^*| > t) \leq \frac{(C_\delta^2 \sigma_2^8 + \frac{1}{C_\delta^2 \sigma_2^8})(\sigma_2^2 + C_\delta + C_\delta^2 \sigma_2^2)^3}{\theta^2 t^4} e^{-\frac{\sigma_2^4 t^4}{\gamma + \delta \sigma_w + \|x\|_2^2/p}} \leq \frac{\sigma_2^5 \delta^4}{\gamma + \delta \sigma_w + \|x\|_2^2/p} e^{-\frac{\sigma_2^4 t^4}{\gamma + \delta \sigma_w + \|x\|_2^2/p}}.
\]

Based on the results in this chapter, we are able to further study the performance of SLOPE on estimation and compare it with other existing regularizations.
Chapter 5

SLOPE versus bridge estimators

In the last chapter, we proved that the MSE of SLOPE concentrates around a quantity characterized by (4.1.4) and (4.1.5). The goal of this chapter is to use this concentration result to provide an accurate comparison between the SLOPE and the bridge estimators. Define

\[ e_\lambda(\gamma, \sigma) = \frac{1}{p} E \| \eta(\mathbf{x} + \sigma \mathbf{h}; \sigma \mathbf{x}) - \mathbf{x} \|^2, \]  

(5.0.1)

where \((\sigma, \chi)\) is the solution to the state evolution equations (4.1.4) and (4.1.5). According to Theorem 4.1.1, the squared error \(\frac{1}{p} \| \hat{\mathbf{x}}_\lambda(\gamma, \sigma_w) - \mathbf{x} \|^2\) of the SLOPE estimator \(\hat{\mathbf{x}}_\lambda(\gamma, \sigma_w)\) concentrates around \(e_\lambda(\gamma, \sigma_w)\). Hence, the quantity \(e_\lambda(\gamma, \sigma_w)\) measures the finite-sample performance of the estimator \(\hat{\mathbf{x}}_\lambda(\gamma, \sigma_w)\). Based on \(e_\lambda(\gamma, \sigma_w)\), we can evaluate and compare the performance of different SLOPE estimators. However, as is clear from the expressions in (4.1.4)(4.1.5)(5.0.1), the value of \(e_\lambda(\gamma, \sigma_w)\) depends on the signal \(\mathbf{x}\), the noise level \(\sigma_w\), the regularization parameter \(\gamma\), and the sample size \(\delta\) (relative to the dimension \(p\)) in a nonlinear and implicit way. In order to gain useful information about the performance of \(\hat{\mathbf{x}}_\lambda(\gamma, \sigma_w)\), we will focus our study on the impact of the noise level \(\sigma_w\) on \(e_\lambda(\gamma, \sigma_w)\). Specifically, we analyze \(e_\lambda(\gamma, \sigma_w)\) under the low noise and large noise settings in Sections 5.2 and 5.3, respectively. Our delicate noise sensitivity analysis will turn \(e_\lambda(\gamma, \sigma_w)\) into explicit and informative quantities that
provide us interesting insights into the behavior of the family of SLOPE estimators. Towards that goal, we consider the value of $\gamma$ that minimizes $e_\lambda(\gamma, \sigma_w)$,

$$\gamma^*_\lambda = \arg\min_{\gamma > 0} e_\lambda(\gamma, \sigma_w).$$

Thus, $e_\lambda(\gamma^*_\lambda, \sigma_w)$ characterizes the performance of $\hat{x}_\lambda(\gamma^*_\lambda, \sigma_w)$, i.e., the SLOPE estimator under optimal tuning $\gamma = \gamma^*_\lambda$. This is the best performance that each SLOPE estimator can possibly achieve. Our subsequent analyses and results are tailored to estimators with the regularization parameter $\gamma$ optimally tuned. To start with, in Section 5.1 we state some useful results which will be repeatedly used in the proof later.

## 5.1 Properties of the MSE of SLOPE

Recall that $e_\lambda(\gamma^*_\lambda, \sigma_w) = \frac{1}{p} E \| \eta(x + \sigma^* h; \sigma^* \chi^*) - x \|_2^2$, where $\gamma^*_\lambda = \arg\min_{\gamma > 0} e_\lambda(\gamma, \sigma_w)$ and the pair $(\sigma^*, \chi^*)$ is obtained from the equations

$$\begin{align*}
(\sigma^*)^2 &= \sigma_w^2 + \frac{1}{\delta \sigma^* p} E \| \eta(x + \sigma^* h; \sigma^* \chi^*) - x \|_2^2, \\
\gamma^*_\lambda &= \sigma^* \chi^* \left(1 - \frac{1}{\delta \sigma^* \sigma^* p} E \langle \eta(x + \sigma^* h; \sigma^* \chi^*), h \rangle \right).
\end{align*}$$

The main proof for Theorems 5.2.1 and 5.3.1 is to analyze the above state evolution equations as $\sigma_w \to 0$ or $\sigma_w \to \infty$. The quantity $E \| \eta(x + \sigma^* h; \sigma^* \chi^*) - x \|_2^2$ plays a critical role in the analysis. Lemma 5.1.1 below characterizes several important properties of this quantity that will be useful in the later proofs.

**Lemma 5.1.1.** For any fixed $\chi > 0$, define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f(v) = E \| \eta(x + \sqrt{v} h; \sqrt{v} \chi) - x \|_2^2,$$

where $h \sim \mathcal{N}(0, I_p)$. Then $f(v)$ has the following properties:

(i) $f(v)$ is continuous at $v = 0$ and has derivatives of all orders on $(0, +\infty)$. 

(ii) \( f(v) \) is strictly increasing over \([0, +\infty)\).

(iii) \( \frac{f(v)}{v} \) is decreasing over \((0, +\infty)\), and strictly decreasing if \(x \neq 0\).

**Proof. Part (i):** Observe that

\[
f(v) = v\mathbb{E}\|\eta(x/\sqrt{v} + h; \chi) - x/\sqrt{v}\|_2^2, \quad \text{for } v > 0.
\]

To show \( f(v) \) is smooth over \((0, +\infty)\), it is sufficient to show for each \(1 \leq i \leq p\), \( \mathbb{E}\eta_i^2(x/\sqrt{v} + h; \chi) \) and \( \mathbb{E}x_i\eta_i(x/\sqrt{v} + h; \chi) \) are both smooth for \( v \in (0, +\infty) \). We have

\[
\mathbb{E}\eta_i^2(x/\sqrt{v} + h; \chi) = (2\pi)^{-p/2} \int \eta_i^2(h; \chi) e^{-\frac{\|h-x/\sqrt{v}\|_2^2}{2}} dh.
\]

Given that \( \eta_i^2(x/\sqrt{v} + h; \chi) \leq 2\|x\|_2^2/v + 2\|h\|_2^2 \), we can apply the mean value theorem and the Dominated Convergence Theorem (DCT) to conclude the existence of derivatives of all orders for \( \mathbb{E}\eta_i^2(x/\sqrt{v} + h; \chi) \). Similar arguments work for \( \mathbb{E}x_i\eta_i(x/\sqrt{v} + h; \chi) \). We next show the continuity of \( f(v) \) at \( v = 0 \). From Lemma D.1.6 we have

\[
\sup_{v > 0} \mathbb{E}\|\eta(x/\sqrt{v} + h; \chi) - x/\sqrt{v}\|_2^2 \leq p + \chi^2\|\lambda\|_2^2.
\]

Hence \( |f(v)| \leq (p + \chi^2\|\lambda\|_2^2) \cdot |v| \), yielding that \( \lim_{v \to 0} f(v) = 0 \).

**Part (ii):** Recall the notation \( I, \mathcal{P} \) and \( \mathcal{P}_0 \) defined in and after (D.0.3). Let \( r_j \) be the rank of \( |x_j + \sqrt{v}h_j| \) in the sequence \( \{|x_i + \sqrt{v}h_i|\}_{i=1}^p \). Using the form of \( \eta_i \) presented in Lemma D.1.2 Part (iv), combined with DCT and Lemma D.2.3 we can compute the derivative \( f'(v) \),

\[
f'(v) = \frac{1}{v} \mathbb{E} \left( \|\eta(x + \sqrt{v}h; \sqrt{v}\chi)\|_2^2 - 2\langle x, \eta(x + \sqrt{v}h; \sqrt{v}\chi) \rangle \right)
\]

\[
+ \sum_{I \in \mathcal{P}_0} \frac{1}{|I|} \left( \sum_{j \in I} x_j \cdot \text{sign}(x_j + \sqrt{v}h_j) \right)^2
\]

\[
= \mathbb{E} \sum_{I \in \mathcal{P}_0} \frac{1}{|I|} \left( \sum_{j \in I} (h_j \cdot \text{sign}(x_j + \sqrt{v}h_j) - \chi r_j) \right)^2 > 0.
\]
Therefore, \( f'(v) > 0 \) for \( v \in (0, +\infty) \). Also \( f(v) \) is continuous at \( v = 0 \) from Part (i). Thus \( f(v) \) is strictly increasing over \([0, +\infty)\).

**Part (iii):** Utilizing the result from Part (ii), we compute the derivative when \( v > 0 \),

\[
\left( \frac{f(v)}{v} \right)' = \frac{f'(v)v - f(v)}{v^2} = -\frac{1}{v^2} \mathbb{E} \left[ \|x\|^2 - \sum_{I \in \mathcal{P}_0} \frac{1}{|I|} \left( \sum_{j \in I} x_j \cdot \text{sign}(x_j + \sqrt{v} h_j) \right)^2 \right]
\]

\[
= -\frac{1}{v^2} \mathbb{E} \left[ \sum_{I \in \mathcal{P} \setminus \mathcal{P}_0} \sum_{j \in I} x_j^2 + \sum_{I \in \mathcal{P}_0} \left( \sum_{j \in I} x_j^2 \right) - \frac{1}{|I|} \left( \sum_{j \in I} x_j \cdot \text{sign}(x_j + \sqrt{v} h_j) \right)^2 \right]
\]

\[
\leq 0,
\]

where the last inequality is due the arithmetic-mean square-mean inequality. We can further argue that the strict inequality holds in (5.1.3) when \( x \neq 0 \). This is because Lemma D.1.1 implies that \( \eta(x + \sqrt{v} h; \sqrt{v} \chi) = 0 \) if and only if

\[
h \in \mathcal{O}_x \triangleq \left\{ h \in \mathbb{R}^p : \sum_{i=1}^j |x/\sqrt{v} + h|_{(i)} = \chi \sum_{i=1}^j \lambda_i, 1 \leq j \leq p \right\}.
\]

The set \( \mathcal{O}_x \) is convex and has positive Lebesgue measure. We can then continue from (5.1.3) to obtain

\[
\left( \frac{f(v)}{v} \right)' = \frac{f'(v)v - f(v)}{v^2} \leq -\frac{\|x\|^2}{v^2} \cdot \mathbb{P}(h \in \mathcal{O}_x) < 0.
\]

The equations (5.1.1) and (5.1.2) that we aim to analyze seem rather complicated, because the regularization parameter \( \gamma^*_\lambda \) is chosen to be the optimal one instead of an arbitrarily given value. Lemma 5.1.2 shows us that the choice of the optimal tuning simplifies the equations to some extent, and sets the stage for the noise sensitivity analysis.
Lemma 5.1.2. If $\sigma^*$ is the unique solution to the equation

$$\sigma^2 = \sigma_w^2 + \frac{1}{\delta p} \inf_{\chi > 0} \mathbb{E} \| \eta(x + \sigma h; \sigma \chi) - x \|_2^2,$$

then we have

$$e_{\lambda}(\gamma^*, \sigma_w) = \delta((\sigma^*)^2 - \sigma_w^2).$$

Proof. We first prove (5.1.4) has a unique solution. Denote

$$G(\sigma) = \frac{\sigma_w^2}{\sigma^2} + \frac{1}{\delta p} \inf_{\chi > 0} \mathbb{E} \| \eta(x + \sigma h; \chi) - x \|_2^2.$$

Then (5.1.4) is equivalent to $G(\sigma) = 1$. Lemma 5.1.1 Part (iii) shows that $\mathbb{E} \| \eta(x + \sigma h; \chi) - x \|_2^2$ is a decreasing function of $\sigma$ over $(0, \infty)$. As a result, so is $\inf_{\chi > 0} \mathbb{E} \| \eta(x + \sigma h; \chi) - x \|_2^2$. Hence $G(\sigma)$ is a continuous and strictly decreasing function for $\sigma \in (0, \infty)$. Moreover,

$$0 \leq G(\sigma) \leq \frac{\sigma_w^2}{\sigma^2} + \frac{1}{\delta p} \lim_{\chi \to \infty} \mathbb{E} \| \eta(x + \sigma h; \chi) - x \|_2^2 = \frac{\sigma_w^2}{\sigma^2} + \frac{\| x \|_2^2}{\delta p \sigma^2},$$

yielding that $\lim_{\sigma \to \infty} G(\sigma) = 0$. It is also clear that $\lim_{\sigma \to 0} G(\sigma) = +\infty$. Thus, $G(\sigma) = 1$ has a unique solution $\sigma = \sigma^*$. It remains to prove (5.1.5). Consider any given $\gamma > 0$. We have

$$e_{\lambda}(\gamma, \sigma_w) = \frac{1}{p} \mathbb{E} \| \eta(x + \bar{\sigma} h; \bar{\sigma} \bar{\chi}) - x \|_2^2 = \delta(\bar{\sigma}^2 - \sigma_w^2),$$

with $(\bar{\sigma}, \bar{\chi})$ being the solution to (4.1.4) and (4.1.5). Equation (4.1.4) can be rewritten as

$$1 = \frac{\sigma_w^2}{\sigma^2} + \frac{1}{\delta p} \mathbb{E} \| \eta(x + \bar{\sigma} h; \bar{\chi}) - x \|_2^2,$$

with which we obtain

$$G(\sigma^*) = 1 \geq \frac{\sigma_w^2}{\sigma^2} + \frac{1}{\delta p} \inf_{\chi > 0} \mathbb{E} \| \eta(x + \sigma h; \chi) - x \|_2^2 = G(\bar{\sigma}),$$

which implies that $\sigma^* \leq \bar{\sigma}$ due to the monotonicity of $G(\sigma)$. Hence,

$$\delta((\sigma^*)^2 - \sigma_w^2) \leq \delta(\bar{\sigma}^2 - \sigma_w^2) = e_{\lambda}(\gamma, \sigma_w), \quad \forall \gamma > 0.$$
Finally, we need to show the above lower bound is attained by $e_{\lambda}(\gamma^*, \sigma_w)$ for some value $\gamma^*$. Define

$$\chi^* = \arg\min_{\chi > 0} \mathbb{E}\|\eta(\frac{x}{\sigma^*} + h; \chi) - \frac{x}{\sigma^*}\|_2^2.$$ (5.1.6)

Note that $\chi^*$ might not be unique and it can be any minimizer. We then pick the following tuning:

$$\gamma^* = \chi^* \sigma^*(1 - \frac{1}{\delta p} \mathbb{E}[\nabla \cdot \eta(\frac{x + \sigma^* h}{\sigma^* \chi^*})]).$$ (5.1.7)

Based on (5.1.6) and (5.1.7) together with the result $G(\sigma^*) = 1$, it is straightforward to verify that

$$e_{\lambda}(\gamma^*, \sigma_w) = \delta((\sigma^*)^2 - \sigma_w^2).$$

Next we prove Theorem 5.2.1 and Proposition 5.2.2. Therein we need to first characterize the connection between $M_\lambda$ and (5.1.1), of which the proof is delayed to Lemma 5.2.1 after we finish the main proof.

## 5.2 Low noise sensitivity analysis of SLOPE

In this section, we study the low noise sensitivity of SLOPE. We first state our main result.

**Theorem 5.2.1.** Let $k = \|x\|_0$ and suppose $x \in \mathbb{R}^p$ does not have tied non-zero elements. Define

$$M_\lambda = \frac{1}{p} \inf_{\alpha > 0} \left\{ k + \alpha^2 \sum_{i=1}^{k} \lambda_i^2 + \mathbb{E}\|\eta(\tilde{h}; \alpha, \lambda_{[k+1:p]})\|_2^2 \right\},$$ (5.2.1)

where $\tilde{h} \in \mathbb{R}^{p-k} \sim \mathcal{N}(0, I_{p-k})$. Then, we have

(a)

$$\lim_{\sigma_w \to 0} e_{\lambda}(\gamma^*_\lambda, \sigma_w) = \begin{cases} > 0, & \text{if } \delta < M_\lambda, \\ = 0, & \text{if } \delta > M_\lambda. \end{cases}$$
(b) Furthermore,

\[
\lim_{\sigma_w \to 0} \frac{e_\lambda(\gamma^*_\lambda, \sigma_w)}{\sigma^2_w} = \begin{cases} 
\infty, & \text{if } \delta < M_\lambda, \\
\frac{\delta M_\lambda}{\delta - M_\lambda}, & \text{if } \delta > M_\lambda.
\end{cases}
\]

The proof of this theorem can be found below. Several remarks are in order.

**Remark 5.2.1.** The low noise sensitivity analysis is aligned with the concentration results of Scenarios (i) and (ii) in Theorem 4.1.1. As will be shown in Lemma 5.2.1, \( M_\lambda \) defined in (5.2.1) equals to \( M_\lambda(\chi^*) \) in (4.1.6) under optimal tuning \( \gamma = \gamma^*_\lambda \). Thus, the cases \( \delta > M_\lambda \) and \( \delta < M_\lambda \) correspond to Scenarios (i) and (ii), respectively.

**Remark 5.2.2.** Part (a) in Theorem 5.2.1 characterizes the phase transition of SLOPE estimators. Specifically, as the noise vanishes, SLOPE can fully recover the \( k \)-sparse signal \( x \) if and only if \( \delta > M_\lambda \). Thus, \( M_\lambda \) is the sharp threshold of SLOPE for the exact recovery.

**Remark 5.2.3.** Part (b) in Theorem 5.2.1 further reveals the low noise sensitivity of SLOPE. Above phase transition where exact recovery is attainable, the error \( e_\lambda(\gamma^*_\lambda, \sigma_w) \) of all the SLOPE estimators reduces to zero at the same rate of \( \sigma^2_w \). Hence the constant \( \frac{\delta M_\lambda}{\delta - M_\lambda} \) represents the noise sensitivity of each SLOPE estimator. The smaller \( M_\lambda \) is, the smaller the constant is.

The explicit formulas we derived in Theorem 5.2.1 enable us to compare different SLOPE estimators with each other and also with more standard estimators such as bridge regression. According to this theorem, the key quantity that determines the performance of SLOPE is \( M_\lambda \). Hence, in order to find the best SLOPE estimator we should find the sequence \( \lambda \) that minimizes \( M_\lambda \). The following proposition addresses this issue.

**Proposition 5.2.2.** \( M_\lambda \) as a function of \( \lambda \), is minimized when \( \lambda_1 = \cdots = \lambda_p \).
The proof of this proposition can be found in the rest of this section.

According to this proposition, we can conclude that LASSO is optimal among all SLOPE estimators in the low noise scenario. Note that it has been proved that LASSO outperforms all the convex bridge estimators in the low-noise regime [WMZ18], but not necessarily the non-convex bridge estimators [ZMW+17].

**Remark 5.2.4.** We should emphasize that the requirement that the unknown signal \( x \) does not have tied non-zero components is critical for both Theorem 5.2.1 and Proposition 5.2.2. Intuitively speaking, for signal \( x \) with tied non-zero components, given the fact that setting unequal weights \( \{ \lambda_i \} \) can produce estimators having tied non-zero elements (cf. Lemma D.1.2 Part (iv)), a SLOPE estimator (with appropriately chosen weights) makes better use of the signal structure than LASSO does. Hence, the optimality of LASSO will not hold for such signals. We provide some empirical results in Section 5.4 to support this claim.

**Proof of Theorem 5.2.1.** The first part of the proof is to analyze \( \sigma^* \) when \( \sigma_w \to 0 \). Lemma 5.1.2 proves that \( e_{\lambda}(\gamma^*, \sigma_w) = \delta((\sigma^*)^2 - \sigma_w^2) \) with \( \sigma = \sigma^* \) being the solution to the equation

\[
\sigma^2 = \sigma_w^2 + \frac{1}{\delta p} \inf_{\chi > 0} \mathbb{E} \| x + \sigma h; \sigma \chi - x \|^2_2. \tag{5.2.2}
\]

The first part of the proof is to analyze \( \sigma^* \) when \( \sigma_w \to 0 \).

(i) The case \( \delta < M_\lambda \). We prove that in this case \( \lim_{\sigma_w \to 0} e_{\lambda}(\gamma^*, \sigma_w) > 0 \). It is equivalent to show \( \lim_{\sigma_w \to 0} \sigma^* > 0 \). Suppose this is not true. Then from (5.2.2) we obtain

\[
\frac{1}{p} \inf_{\chi > 0} \mathbb{E} \| x/\sigma^* + h; \chi - x/\sigma^* \|^2_2 < \delta.
\]

According to lemma 5.2.1, letting \( \sigma_w \to 0 \) on both sides of the above inequality yields that \( M_\lambda \leq \delta \). This is a contradiction.
(ii) The case $\delta > M_\lambda$. Lemma 5.1.1 Part (iii) together with (5.2.2) gives us that
\[
\frac{(\sigma^*)^2 - \sigma_w^2}{(\sigma^*)^2} = \frac{1}{\delta} \inf_{\lambda > 0} \mathbb{E}\|x/\sigma^* + h; \chi\|_2^2 - \frac{\sigma^2_w}{\sigma^2} \leq \frac{1}{\delta} \lim_{\sigma \to 0} \inf_{\lambda > 0} \mathbb{E}\|x/\sigma + h; \chi\|_2^2 = \frac{M_\lambda}{\delta},
\]
where the last equality is due to Lemma 5.2.1. Hence,
\[
0 \leq (\sigma^*)^2 \leq \frac{\sigma_w^2}{1 - M_\lambda/\delta} \to 0, \quad \text{as } \sigma_w \to 0.
\]
Now given that $\lim_{\sigma_w \to 0} \sigma^* = 0$, letting $\sigma_w \to 0$ on both sides of (5.2.3) delivers
\[
\lim_{\sigma_w \to 0} \frac{(\sigma^*)^2}{\sigma_w^2} = \frac{\delta}{\delta - M_\lambda},
\]
leading to $\lim_{\sigma_w \to 0} \frac{c_\lambda(\gamma^*, \sigma_w)}{\sigma_w^2} = \frac{\delta M_\lambda}{\delta - M_\lambda}$. 

Proof of Proposition 5.2.2. For this part of the proof, we show that the quantity
\[
M_\lambda = \inf_{\alpha > 0} \left\{ k + \alpha^2 \sum_{i=1}^k \lambda_i^2 + \mathbb{E}\|\eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]})\|_2^2 \right\}
:= h(\lambda, \alpha)
\]
is minimized when $\lambda_1 = \cdots = \lambda_p$. Define the set
\[
\mathcal{W}_\lambda = \{ \lambda \in \mathbb{R}^p : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \tilde{\lambda} \geq \lambda_{k+1} \geq \cdots \geq \lambda_p \geq 0 \}.
\]
For any $\lambda \in \mathcal{W}_\lambda$, it is clear that $\sum_{i=1}^k \lambda_i^2 \geq \tilde{\lambda}^2$. Moreover, according to Lemma D.1.1, $\eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]}) = h_{[k+1:p]} - \Pi_{\tilde{\mathcal{D}}_\alpha}(h_{[k+1:p]})$, where $\tilde{\mathcal{D}}_\alpha \subset \mathbb{R}^{p-k}$ is the dual SLOPE norm ball of radius $\alpha$ with the weight sequence $\lambda_{[k+1:p]}$. Clearly, among the choices of $\lambda \in \mathcal{W}_\lambda$, $\tilde{\mathcal{D}}_\alpha$ becomes the largest convex set $\tilde{\mathcal{D}}_\alpha$ when $\lambda_i = \tilde{\lambda}$, $i = k + 1, \ldots, p$, which in turn implies that the residual norm $\|\eta(h_{[k+1:p]}; \alpha; \lambda_{[k+1:p]})\|_2$ is minimized with the same selection. We therefore have shown that
\[
\min_{\lambda \in \mathcal{W}_\lambda} h(\lambda, \alpha) = k + k\alpha^2 \tilde{\lambda}^2 + (p-k)\mathbb{E}\eta_{\tilde{\mathcal{D}}_\alpha}^2(\tilde{z}; \alpha \tilde{\lambda}),
\]
where \( \eta_\ell (z; \alpha \tilde{\lambda}) = \text{sign}(z)(|z| - \alpha \tilde{\lambda})_+ \) is the soft thresholding operator and 
\( z \sim \mathcal{N}(0, 1) \). The equation above holds for any \( \tilde{\lambda} \geq 0 \), we thus can conclude that 
\[
\inf_{\lambda: \lambda_1 \geq \cdots \lambda_p \geq 0} M_\lambda = \inf_{\alpha > 0, \lambda \geq 0} \lambda \in W \| \eta_\ell (z; \alpha \tilde{\lambda}) \|_2 = \inf_{\alpha > 0} \left\{ k + k \alpha^2 \tilde{\lambda}^2 + (p - k) \mathbb{E} \eta_\ell^2 (z; \alpha \tilde{\lambda}) \right\}
\]
which is precisely the \( M_\lambda \) when all the elements of \( \lambda \) are equal. \( \square \)

**Lemma 5.2.1.** Suppose \( x \in \mathbb{R}^p \) does not have non-zero tied components with \( \|x\|_0 = k \). Then, it holds that 
\[
\lim \inf_{v \to 0} \mathbb{E} \| \eta(x/\sqrt{v} + h; \chi) - x/\sqrt{v} \|_2^2 = M_\lambda.
\]

**Proof.** For any given \( v > 0 \), define the optimal value of \( \chi \) as 
\[
\chi(v) = \arg \min_{\chi > 0} \mathbb{E} \| \eta(x/\sqrt{v} + h; \chi) - x/\sqrt{v} \|_2^2.
\]

When there are multiple solutions, we define \( \chi(v) \) as the one with the smallest value. 
We first assume the limit \( \lim_{v \to 0} \chi(v) = \alpha^* \in [0, \infty] \) exists, but will validate this 
assumption later. Recall the definition of the dual-norm ball \( D_\gamma \) of SLOPE norm in (D.0.2). Here, we consider the projection of \( x/\sqrt{v} + h \) on \( D_{\chi(v)} \). Suppose \( \alpha^* = \infty \). 
Since \( \|x/\sqrt{v} + h\|_2 \to \infty \) as \( v \to 0 \), we obtain 
\[
\| \Pi_{D_{\chi(v)}} (x/\sqrt{v} + h) \|_2 \to \infty, \quad \text{as } v \to 0.
\]
Hence, from Lemma D.1.1 we conclude that as \( v \to 0 \), we have 
\[
\| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|_2 = \| \Pi_{D_{\chi(v)}} (x/\sqrt{v} + h) - h \|_2 \to +\infty,
\]
with which Fatou’s lemma yields that 
\[
\lim_{v \to 0} \mathbb{E} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|_2^2 \geq \mathbb{E} \lim_{v \to 0} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|_2^2 = +\infty.
\]
This contradicts with the boundedness due to the definition of \( \chi(v) \):
\[
\mathbb{E} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|_2^2 \leq \mathbb{E} \| \eta(x/\sqrt{v} + h; 0) - x/\sqrt{v} \|_2^2 = p.
\]
Hence \( \alpha^* \in [0, \infty) \) and \( \chi(v) \) is bounded. Lemma D.1.6 gives us that
\[
\| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|^2_2 \leq \chi^2(v) \| \lambda \|^2_2 + \| h \|^2_2.
\]
Thus DCT enables us to obtain
\[
\lim_{v \to 0} \mathbb{E} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|^2_2 = \mathbb{E} \lim_{v \to 0} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|^2_2.
\]
(5.2.5)

To compute the limit on the right-hand side of the above equation, we apply Lemma D.1.7 and obtain that
\[
\lim_{v \to 0} \mathbb{E} \| \eta(x/\sqrt{v} + h; \chi(v)) - x/\sqrt{v} \|^2_2 = k + (\alpha^*)^2 \| \lambda_{[1:k]} \|^2_2 + \mathbb{E} \| \eta(h_{[k+1:p]}; \alpha^*, \lambda_{[k+1:p]}) \|^2_2.
\]
(5.2.6)

Define \( g(\alpha) := k + \alpha^2 \| \lambda_{[1:k]} \|^2_2 + \mathbb{E} \| \eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]}) \|^2_2 \). Since \( \chi(v) \) is defined as the optimal tuning, it has to hold that \( \alpha = \alpha^* \) minimizes \( g(\alpha) \). Finally, we need to prove the existence of \( \lim_{v \to 0} \chi(v) \) that we assumed at the beginning of the proof. We take an arbitrarily convergent sequence \( \{ \chi(v_n) \}_{n=1}^{\infty} \) with \( v_n \to 0 \), as \( n \to \infty \). Denote \( \lim_{n \to \infty} \chi(v_n) = \tilde{\alpha} \). Note that the preceding arguments hold for any such sequence as well. Thus \( \alpha = \tilde{\alpha} \) minimizes \( g(\alpha) \) over \((0, \infty)\). The proof will be completed if we can show \( g(\alpha) \) has a unique minimizer. According to Lemma D.2.2, it is direct to compute
\[
g'(\alpha) = 2\alpha \sum_{i=1}^{k} \lambda_i^2 - \frac{2}{\alpha} \mathbb{E} [\langle \eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]}), h \rangle - \| \eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]}) \|^2_2]
\]
\[= -2\alpha \sum_{i=1}^{k} \lambda_i^2 - 2\mathbb{E} \| \eta(h_{[k+1:p]}; \alpha, \lambda_{[k+1:p]}) \| \lambda_{[k+1:p]},
\]
where in the last equality we applied Lemma D.1.5 (ii). It is not hard to see that \( g'(\alpha) \) is increasing with \( g'(0) = -2\mathbb{E} \| \eta(h_{[k+1:p]}; 0, \lambda_{[k+1:p]}) \| \lambda_{[k+1:p]} \) and \( g'(\infty) = \infty \). Thus \( g(\alpha) \) is strictly convex and has a unique minimizer. \( \square \)
5.3 Large noise sensitivity analysis of SLOPE

In this section, we study the behavior of $e_\lambda(\gamma^{*}_\lambda, \sigma_w)$ as $\sigma_w \rightarrow \infty$. Again we first state our main theorem in this part.

**Theorem 5.3.1.** As $\sigma_w \rightarrow \infty$,

$$e_\lambda(\gamma^{*}_\lambda, \sigma_w) = \frac{1}{p} \|x\|_2^2 + O(\exp(-c\sigma_w^2)),$$

where $c > 0$ is a constant possibly depending on $\kappa_5, \kappa_6,$ and $\kappa_7$ in Assumptions 4.1.4 and 4.1.5.

The proof can be found below. The large noise sensitivity analysis in this theorem is consistent with Scenario (iii) in Theorem 4.1.1. As will be seen in the proof the optimal tuning $\gamma^{*}_\lambda = \Omega(\sigma_w^2)$, thus satisfying the requirement of the tuning in Scenario (iii). To provide a good benchmark to understand and interpret Theorem 5.3.1, let us mention the large noise sensitivity result for bridge regression from [WWM20]. Consider the bridge estimator

$$\hat{x}(\gamma) \in \arg\min_x \frac{1}{2} \|y - Ax\|_2^2 + \gamma \cdot \sum_{i=1}^p |x_i|^q.$$

Let $e_q(\gamma, \sigma_w)$ denote the (asymptotically) exact expression of $\frac{1}{p} \|\hat{x}(\gamma) - x\|_2^2$ and define

$$\gamma^{*}_q = \arg\min_{\gamma > 0} e_q(\gamma, \sigma_w).$$

Thus, $e_q(\gamma^{*}_q, \sigma_w)$ measures the performance of the bridge estimator under optimal tuning. It has been proved [WWM20] that

$$e_q(\gamma^{*}_q, \sigma_w) = \frac{1}{p} \|x\|_2^2 - \frac{c_q \|x\|_2^p}{\sigma_w^{2q-2}} + o(\sigma_w^{-2}), \quad \text{for } q \in (1, \infty).$$

Here, the positive constant $c_q$ only depends on $q$. Combing the results (5.3.1) and (5.3.2), we reach the following conclusions:

1. The SLOPE and bridge estimators share the same first order term $\|x\|_2^2/p$. This is expected because as the noise level goes to infinity, the variance will dominate the estimation error and thus the optimal estimator will eventually converge to zero.
2. The second order term is exponentially small for all SLOPE estimators, while it is negative and polynomially small for all bridge estimators with $q \in (1, \infty)$. Hence, bridge estimators outperform all the SLOPE estimators in the large noise scenario. Moreover, [WWM20] showed that the constant $c_q$ in (5.3.2) attains the maximum at $q = 2$. Therefore, Ridge regression turns out to be the optimal bridge estimator in the large noise scenario. In Section 5.4, we use the Ridge estimator as a representative bridge estimator for numerical studies.

3. Theorem 5.3.1 does not answer which SLOPE estimator is optimal. However, together with the result (5.3.2) it reveals that the family of SLOPE estimators generally do not perform well compared with bridge estimators. We may prefer using bridge regression such as Ridge to estimate the sparse vector $x$ in the large noise scenario.

According to Lemma 5.1.2, the key step is to analyze the equation

$$\sigma^2 = \sigma_w^2 + \frac{1}{\delta p} \inf_{x > 0} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|^2, \quad (5.3.3)$$

when $\sigma_w \to \infty$. Let $\sigma = \sigma^*$ be the solution to the above equation. First observe that $\forall \sigma > 0$,

$$\inf_{x > 0} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|^2 \leq \lim_{\chi \to \infty} \mathbb{E}\|\eta(x + \sigma h; \sigma \chi) - x\|^2 = \|x\|^2. \quad (5.3.4)$$

This result combined with (5.3.3) yields

$$1 \leq \frac{(\sigma^*)^2}{\sigma_w^2} \leq 1 + \frac{\|x\|^2}{\delta p \sigma_w^2},$$

from which letting $\sigma_w \to \infty$ we obtain

$$\lim_{\sigma_w \to \infty} \frac{(\sigma^*)^2}{\sigma_w^2} = 1. \quad (5.3.5)$$

Moreover, adopting the notation from Lemma 5.3.1 we know

$$e_\lambda(\gamma_\lambda, \sigma_w) - \frac{\|x\|^2}{p} = \frac{1}{p} \left[ \mathbb{E}\|\eta(x + \sigma^* h; \sigma^* \chi(\sigma^*))\|^2 - 2\mathbb{E}\langle \eta(x + \sigma^* h; \sigma^* \chi(\sigma^*)), x \rangle \right]$$

$$:= \Delta(\sigma^*).$$
Since $\Delta(\sigma^*) \leq 0$ implied by (5.3.4), it holds that
\[
|\Delta(\sigma^*)| \leq \frac{2}{p} \mathbb{E} \langle \eta(x + \sigma^* h; \sigma^* \chi(\sigma^*)), x \rangle \leq \frac{2\sigma^* \|x\|_2^2}{\sqrt{p}} \|\eta(x/\sigma^* + h; \chi(\sigma^*))\|_{L_2}
\leq \frac{2\sigma^* \|x\|_2}{\sqrt{p}} \left[ \frac{1}{p} \sum_{i=1}^{p} \mathbb{E}(|x_i/\sigma^* + h_i| - \chi(\sigma^*) \|\lambda\|_2^2/p)^2 \right]^{1/2},
\]
where the third inequality is due to Lemma D.1.5 (vi). As we will show in Lemma 5.3.1, $\chi(\sigma^*) = \Omega(\sigma^*)$. This guarantees that as $\sigma^* \to \infty$, we will have $\|x\|_{\sigma^*} \leq \chi(\sigma^*) \|\lambda\|_2^2/2p$.

Using Gaussian tail inequality in Lemma D.2.8, it is hence straightforward to calculate that for each $i = 1, \ldots, p$, as $\sigma^* \to \infty$,
\[
\mathbb{E}(|x_i/\sigma^* + h_i| - \chi(\sigma^*) \|\lambda\|_2^2/p)^2 \leq \mathbb{E}[|h_i| - (\chi(\sigma^*) \|\lambda\|_2^2/p - |x_i|/\sigma^*)]_+^2 \\
\leq O(e^{-\frac{1}{4}\chi^2(\sigma^*) \|\lambda\|_4^4/p^2}).
\]

Based on Lemma 5.3.1, the above result together with (5.3.5) and (5.3.6) completes the proof.

**Lemma 5.3.1.** Suppose $x \neq 0$. Define
\[
\chi(\sigma) = \arg \min_{\chi > 0} \mathbb{E} \|\eta(x/\sigma + h; \chi) - x/\sigma\|_2^2.
\]
It holds that
\[
\chi(\sigma) = \Omega(\sigma), \quad \text{as } \sigma \to \infty.
\]

**Proof.** We first claim that $\chi(\sigma) \to \infty$, as $\sigma \to \infty$. Otherwise, consider a sequence $\sigma_n \to \infty$ such that $\chi(\sigma_n) \to \chi^* \in [0, \infty)$, as $n \to \infty$. Then Dominated Convergence Theorem enables us to compute
\[
\lim_{n \to \infty} \mathbb{E} \|\eta(x/\sigma_n + h; \chi(\sigma_n)) - x/\sigma_n\|_2^2 = \mathbb{E} \|\eta(h; \chi^*)\|_2^2 > 0.
\]

On the other hand, by the definition of $\chi(\sigma_n)$, we obtain
\[
\lim_{n \to \infty} \mathbb{E} \|\eta(x/\sigma_n + h; \chi(\sigma_n)) - x/\sigma_n\|_2^2 \leq \lim_{n \to \infty} \frac{\|x\|_2^2}{\sigma_n^2} = 0.
\]
This is a contradiction. We next analyze the rate of $\chi(\sigma)$. As we have shown in (5.3.4), $\mathbb{E}\|\eta(x/\sigma + h; \chi(\sigma)) - x/\sigma\|_2^2 \leq \frac{1}{\sigma^2}\|x\|_2^2$, it holds that $\forall \sigma > 0$,

$$\mathbb{E}\|\eta(x/\sigma + h; \chi(\sigma))\|_2^2 \leq \frac{2}{\sigma}\mathbb{E}\langle \eta(x/\sigma + h; \chi(\sigma)), x \rangle. \quad (5.3.7)$$

With a change of variables, we can rewrite the terms as

$$\mathbb{E}\|\eta(x/\sigma + h; \chi(\sigma))\|_2^2 = \frac{\chi^{p+2}(\sigma)}{(2\pi)^{p/2}} \int \|\eta(h; 1)\|_2^2 \cdot \exp\left(-\frac{\chi^2(\sigma)}{2}\|h - \frac{x}{\sigma\chi(\sigma)}\|_2^2\right) dh.$$ 

$$\mathbb{E}\langle \eta(x/\sigma + h; \chi(\sigma)), x \rangle = \frac{\chi^{p+1}(\sigma)}{(2\pi)^{p/2}} \int \langle \eta(h; 1), x \rangle \cdot \exp\left(-\frac{\chi^2(\sigma)}{2}\|h - \frac{x}{\sigma\chi(\sigma)}\|_2^2\right) dh.$$ 

By Laplace’s approximation of multi-dimensional integrals [7], we can conclude that

$$\frac{\mathbb{E}\|\eta(x/\sigma + h; \chi(\sigma))\|_2^2}{\mathbb{E}\langle \eta(x/\sigma + h; \chi(\sigma)), x \rangle} \propto \frac{\sigma}{\chi(\sigma)}, \quad \text{as } \sigma \to \infty.$$ 

Therefore, if $\chi(\sigma) = o(\sigma)$, the above result will contradict with (5.3.7).

\section{5.4 Numerical Experiments}

In this section, we present our numerical studies. We pursue the following goals in our simulations:

1. Check the accuracy of our conclusions for finite sample sizes.

2. Show that the main conclusions hold even if some of the assumptions that we made in our theoretical studies, such as the independence or Gaussianity of the elements of $A$, are violated.

3. Show that if the non-zero elements of $x$ are equal, then LASSO might not be the optimal SLOPE estimator in the low noise regime. Hence, the assumption that $x$ has no tied non-zero components in Theorem 5.2.1 and Proposition 5.2.2 is necessary in this sense.

We consider the following simulation setups:
• Design: \( A = \tilde{A} \Sigma^{\frac{1}{2}} \) where the \( \tilde{A}_{ij} \)'s (up to a scaling) are iid \( t \)-distributed with degrees of freedom equal 3 to test the validity of our conclusions when the elements of \( A \) have a heavy-tailed distribution, and iid Gaussian otherwise. The elements \( \tilde{A}_{ij} \) are re-scaled by \( \sqrt{n} \text{std}(\tilde{A}_{ij}) \). Furthermore, in our simulation results we will consider two choices of \( \Sigma \): \( \Sigma_{ij} = \rho^{i-j} \) with (i) \( \rho = 0.8 \), and (ii) \( \rho = 0 \). The first choice will test the validity of our conclusions for the case that the elements of \( A \) are dependent.

• Noise: \( w \sim \mathcal{N}(0, \sigma_w^2 I_p) \). The values for \( \sigma_w \) will be specified in each simulation below.

• Signal: for a given value of \( \epsilon \) and \( p \), we randomly set \((1 - \epsilon)p\) components of \( x \) as 0. For the rest non-zero components, two configurations are considered: (i) iid samples from \( \text{Unif}[0,5] \); (ii) all equal to 5. We use the second case to show that when the non-zero coefficients are equal, then LASSO might not be optimal in the low noise scenario.

• \( p = 500, n = \delta p \). \( \delta \) and \( \epsilon \) will be determined later.

• Once each problem instance is set, we will run our simulations \( m = 20 \) times, and we will report the average MSE and the standard error bars.

• Recall that the comparison results in Section 5.2, 5.3 are valid for optimally-tuned estimators. In our simulations, we use 5-fold cross-validation to find the optimal tuning parameters.

Figure 5.1 shows the MSE of SLOPE, LASSO and Ridge estimators under different types of design matrices. The estimator denoted by SLOPE:BH is the SLOPE estimator that was proposed in [BvdBS+15] and shown to be minimax optimal in [SC+16, BLT+18]. We first discuss the results for iid Gaussian designs in the first plot. We set the parameters \((\delta, \epsilon) = (0.9, 0.5)\) so that the setting is above phase
transitions for LASSO, and below phase transition for the two SLOPE estimators.\(^1\) It is clear that LASSO outperforms the SLOPE estimators when the noise level is low, as predicted by Theorem 5.2.1 and Proposition 5.2.2. Moreover, as the noise level increases above \(\sigma_w = 2\), Ridge starts to have a smaller MSE compared to LASSO and SLOPE. This is consistent with the result from Theorem 5.3.1. These phenomena are also observed in the other three plots where iid Gaussian assumptions are not satisfied on the design matrix. Such empirical results suggest that the main comparison conclusions drawn from Proposition 5.2.2 and Theorems 5.2.1 and 5.3.1 are valid for non-Gaussian and correlated designs too. We leave a precise analysis of such designs as an open avenue for a future research.

In Figure 5.2, we further compare the MSE of LASSO with that of SLOPE in two cases when the system is above phase transition for both SLOPE and LASSO. As is clear from the first column, for iid Gaussian designs, LASSO has a smaller MSE when \(\sigma_w\) is small, which is accurately characterized in Theorem 5.2.1 and Proposition 5.2.2. Again, similar result seems to hold under more general settings, including correlated design, heavy tail design and a combination of the two, as shown in the rest of the graphs.

Finally, we examine the condition that the signal \(x\) does not have tied non-zero components, as required in Theorem 5.2.1 and Proposition 5.2.2. We empirically demonstrate in Figure 5.3 that the condition is necessary for Theorem 5.2.1 and Proposition 5.2.2 to hold. As is clear from the figure, for the signal \(x\) of which the non-zero components are all equal to 5, LASSO is significantly outperformed by the SLOPE estimator (SLOPE:max2) with \(\lambda_1 = \lambda_2 = 1 > 0 = \lambda_3 = \ldots = \lambda_p\) in the low noise scenario. This is because the sorted \(\ell_1\) penalty in SLOPE (with

\(^1\)From Theorem 5.2.1 we know that \(\delta > M_\lambda\) means the corresponding setting is above phase transition. For LASSO, the inequality can be simplified as \(\delta \geq \inf_x 2(1 - \epsilon)((1 + x^2)\Phi(-\chi) - \chi\phi(\chi)) + \epsilon(1 + x^2)\), and analytically verified. For the two SLOPE estimators, since \(M_\lambda\) can not be directly evaluated, we conclude it is below phase transition based on the numerical results in the figure.
Figure 5.1: MSE of SLOPE, LASSO and Ridge estimators. SLOPE:BH and SLOPE:unif denote the SLOPE estimators with weights $\lambda_i = \Phi^{-1}(1 - \frac{iq}{2p})/\Phi^{-1}(1 - \frac{q}{2p})$ with $q = 0.5$ and $\lambda_i = 1 - 0.99(i - 1)/p$, respectively. Other model parameters are $\delta = 0.9$, $\epsilon = 0.5$; The nonzero components of $x$ are iid samples from Uniform[0, 5]; $\sigma_w \in [0, 5]$.

appropriately chosen weights) promotes estimators that have tied non-zero elements, while $\ell_1$ penalty can only promote sparsity. Therefore, SLOPE better exploits the existing structures in the signals. Note that the choice of the penalty weights is critical for SLOPE to take full advantage of the signal structures. For example, the other SLOPE estimator (SLOPE:unif), with the (unordered) weights being uniformly sampled, does not behave as well as SLOPE:max2.
CHAPTER 5. SLOPE VERSUS BRIDGE ESTIMATORS

Figure 5.2: MSE of SLOPE, LASSO and Ridge estimators, when the system is above phase transition for both SLOPE and LASSO. A case of $\delta < 1$ ($\epsilon = 0.2$) is presented in the upper panel, while one for $\delta > 1$ ($\epsilon = 0.5$) is in the lower panel. The other parameters are the same as in Figure 5.1.
Figure 5.3: MSE of SLOPE, LASSO and Ridge estimators, when there are tied non-zero elements in the signal. SLOPE:max2 denotes the SLOPE estimator with weights $\lambda_1 = \lambda_2 = 1$ and $\lambda_i = 0$ for $i \geq 3$. SLOPE:unif is the same as in Figure 5.1. We set $\delta = 0.9$, $\epsilon = 0.7$. The non-zero components of $\mathbf{x}$ all equal to 5.
Chapter 6

Discussions

6.1 Beyond IID Gaussian assumption

One of the main assumptions in our entire study is the IID Gaussian condition on the design matrix $X$. On one hand, this is a fundamental setting under which many algorithms could be accurately studied; On the other hand, without incorporating the correlation structure into the design, it would be difficult to rigorously argue the generality of our results. The next step is to generalize our studies to correlated Gaussian design. In this section, we briefly discuss this assumption.

The existing approaches that enable accurate analyses of penalized regression problem under IID Gaussian design and linear asymptotic framework include approximate message passing (AMP) [DMM09, BM11], convex Gaussian minimax theorem (CGMT) [TOH15, TAH18], and the “double leave-one-out” approach [BBEK+12, EKBB+13]. For the former two approaches, we have not seen straightforward ways to generalize them to handle correlated Gaussian design due to the key roles of independence in the derivation. The “double leave-one-out” approach, however, tries to study the property of these estimators by evaluating the accumulated differences between an estimator and its twisted version (by training on the design matrix either with one predictor or one observation left out). This approach is more flexible in
handling correlations. Despite this flexibility, it is still very challenging to fully characterize the MSE for correlated design problem and we leave the complete answer for future research.

6.2 Nonconvex bridge estimators

In Chapters 2 and 3, our discussion has been focused on the bridge estimators with $q \in [1, \infty)$. When $q$ falls in $[0, 1)$, the corresponding bridge regression becomes a nonconvex problem. Given that certain nonconvex regularizations have been shown to achieve variable selection consistency under weaker conditions than LASSO [LW+17], it is of great interest to analyze the variable selection performance of nonconvex bridge estimators. An early work [HHM+08] has showed that bridge estimators for $q \in (0, 1)$ enjoy an oracle property in the sense of [FL01] under appropriate conditions. However, the asymptotic regime considered in [HHM+08] is fundamentally different from the linear asymptotic in the current paper. A more relevant work is [ZMW+17] which studied the estimation property of bridge regression when $q$ belongs to $[0, 1]$ under a similar asymptotic framework to ours. Nevertheless, the main focus of [ZMW+17] is on the estimators returned by an iterative local algorithm. The analysis of the global minimizer in [ZMW+17] relies on the replica method [RGF09] from statistical physics, which has not been fully rigorous yet. To the best of our knowledge, under the linear asymptotic setting, no existing works have provided a fully rigorous analysis of the global solution from nonconvex regularization in linear regression models. We leave this important and challenging problem as a future research.
6.3 Improving the concentration inequality of the MSE

In Chapter 4, we have shown the concentration inequalities of the SLOPE estimator. However there are a few open questions we have not yet addressed.

1. Improving the rate from $t^4$ to $t^2$. The current rate we have shown comes from a locally subGaussian concentration of the objective function and a second order Taylor expansion around its minimizer. This could be potentially improved by justifying a same-rate concentration inequality directly on the derivative of the objective function. However technical difficulties exist in the more complicated form of the derivative.

2. Extending the results to general settings of the model parameters. The fact that our concentration inequalities rely on different cases of noise level comes from the nonseparability of the SLOPE norm. We expect a cleaner form for simpler regularizers, such as the Bridge estimator where $\|x\|_q^q$.

In general, we would expect such concentration inequality to hold for a wider range of penalized regression problem:

$$\min_x \frac{1}{2} \|y - Ax\|_2^2 + \lambda R(x)$$

where $R(x)$ is a convex regularizer of unknown parameter $x$. As long as the structure of $R$ enables us to obtain rich characterization of its proximal operator, we are hopeful to obtain something meaningful. We leave these for future research.
Appendix A

Preliminaries on Proofs in Chapter 2 and 3

A.1 Some notations

We will use the following notations throughout this supplementary file:

(i) We will use $\partial_i f$ to denote the partial derivative of $f(x, y, \ldots)$ with respect to its $i^{th}$ argument. Also for the ease of organizing the proof, we may use $\partial_y f$ to be the partial derivative of $f$ with respect to its argument $y$, which is equivalent to $\partial_2 f$.

(ii) We will use DCT as a short name for Dominated Convergence Theorem.

(iii) Recall we have $p_B = (1 - \epsilon)\delta_0 + \epsilon p_G$. By symmetry, it can be easily verified that $B$ and $G$ appearing in the subsequent proofs can be equivalently replaced by $|B|$ and $|G|$. Hence without loss of generality, we assume $B$ and $G$ are nonnegative random variables.

(iv) Let $\Phi$ and $\phi$ denote the cumulative distribution function and probability density function of a standard normal random variable respectively. Integration by parts
gives us the standard result on the Gaussian tails expansion: for $k \in \mathbb{N}^+, s > 0$
\begin{equation}
\Phi(-s) = \phi(s) \left[ \sum_{i=0}^{k-1} \frac{(-1)^i(2i - 1)!!}{s^{2i+1}} + (-1)^k (2k - 1)!! \int_s^\infty \frac{\phi(t)}{t^{2k}} dt \right], \tag{A.1.1}
\end{equation}
where $(2i - 1)!! \triangleq 1 \times 3 \times 5 \times \ldots \times (2i - 1)$.

(v) As $a \to 0$ (or $a \to \infty$), $g(a) = O(f(a))$, means that there exists a constant $C$ such that for small enough (or large enough) values of $a$, $g(a) \leq Cf(a)$. Furthermore, $g(a) = o(f(a))$ if and only if $\lim_{a \to 0} \frac{g(a)}{f(a)} = 0$ (or in case of $a \to \infty$, $\lim_{a \to \infty} \frac{g(a)}{f(a)} = 0$).

(vi) As $a \to 0$ (or $a \to \infty$), $g(a) = \Omega(f(a))$, if and only if $f(a) = O(g(a))$. Similarly, $g(a) = \omega(f(a))$ if and only if $f(a) = o(g(a))$. Finally, $f(a) = \Theta(g(a))$, if and only if $f(a) = O(g(a))$ and $g(a) = O(f(a))$.

A.2 State evolution and properties of the proximal operator

**Definition A.2.1** (pseudo-Lipschitz function). A function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is said to be pseudo-Lipschitz, if $\exists L > 0$ s.t., $\forall x, y \in \mathbb{R}^2$, $|\psi(x) - \psi(y)| \leq L(1 + \|x\|_2 + \|y\|_2)\|x-y\|_2$.

The following theorem proved by [BM11] and [WMZ18] will be used in our proof.

**Theorem A.2.1.** ([BM11], [WMZ18]) For a given $q \in [1, \infty)$, let $\hat{x}(q, \lambda)$ be the bridge estimator defined in (2.0.2). Consider a converging sequence $\{\mathbf{x}(p), \mathbf{A}(p), w(p)\}$. Then, for any pseudo-Lipschitz function $\psi : \mathbb{R}^2 \to \mathbb{R}$, almost surely
\begin{equation}
\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \psi(\hat{x}_i(q, \lambda), x_i(p)) = \mathbb{E}\psi(\eta_{q}(B + \tau Z; \alpha \tau^{2-q}), B),
\end{equation}
where $B \sim p_B$ and $Z \sim N(0, 1)$ are two independent random variables; $\alpha$ and $\tau$ are two positive numbers satisfying (2.2.2) and (2.2.3).
For each tuning parameter $\lambda > 0$, [WMZ18] has proved that the solution pair $(\alpha, \tau)$ to the nonlinear equations (2.2.2) and (2.2.3) is unique. We will denote this unique solution pair for the optimal tuning value $\lambda = \lambda_q^*$ by $(\alpha^*_q, \tau^*_q)$. Note that we omit the dependency of these two quantities on $q$, since when they appear in this paper, $q$ is clear from the context.

**Lemma A.2.1.** If $(\alpha^*_q, \tau^*_q)$ are the solutions of (2.2.2) and (2.2.3) for $\lambda = \lambda_q^*$, then $\tau^*_q$ satisfies the following equation:

$$
\tau^2_q = \sigma^2 + \frac{1}{\delta} \min_{\alpha > 0} \mathbb{E}(\eta_q(B + \tau Z; \alpha \tau^{2-q}) - B)^2,
$$

$$
\alpha^*_q = \arg\min_{\alpha > 0} \mathbb{E}(\eta_q(B + \tau Z; \alpha \tau^{2-q}) - B)^2
$$

(A.2.1)

and

$$
\text{AMSE}(q, \lambda_q^*) = \mathbb{E}(\eta_q(B + \tau^*_Z; \alpha^*_q \tau^{2-q}) - B)^2.
$$

This is a simple extension of Lemma 15 in Appendix E of [WMZ18]. Hence we skip the proof. Define

$$
R_q(\alpha, \tau) \triangleq \mathbb{E}(\eta_q(B/\tau + Z; \alpha) - B/\tau)^2,
$$

(A.2.2)

$$
\alpha_q(\tau) \triangleq \arg\min_{\alpha \geq 0} R_q(\alpha, \tau).
$$

(A.2.3)

For the definition (A.2.3), if the minimizer is not unique, we choose the smallest one.

Recall the proximal operator:

$$
\eta_q(u; \chi) = \arg\min_z \frac{1}{2}(u - z)^2 + \chi |z|^q.
$$

Note that $\eta_q(u; \chi)$ does not have an explicit form except for $q = 0, 1, 2$. In the following lemma, we summarize some properties of $\eta_q(u; \chi)$. They will be used to prove our theorems.

**Lemma A.2.2.** For any $q \in (1, \infty)$, we have

(i) $\eta_q(u; \chi) = -\eta_q(-u; \chi)$. 


(ii) \( u = \eta_q(u; \chi) + \chi q|\eta_q(u; \chi)|^{q-1}\text{sgn}(u) \), where \( \text{sgn} \) denotes the sign of a variable.

(iii) \( \eta_q(\alpha u; \alpha^{2-q} \chi) = \alpha \eta_q(u; \chi) \), for \( \alpha > 0 \).

(iv) \( \partial_1 \eta_q(u; \chi) = \frac{1}{1+\chi q(q-1)|\eta_q(u; \chi)|^{q-2}} \).

(v) \( \partial_2 \eta_q(u; \chi) = \frac{-q |\eta_q(u; \chi)|^{q-1}\text{sgn}(u)}{1+\chi q(q-1)|\eta_q(u; \chi)|^{q-2}} \).

(vi) \( 0 \leq \partial_1 \eta_q(u; \chi) \leq 1 \).

(vii) If \( 1 < q < \frac{1}{2} \), then \( \lim_{u \to 0} \frac{|u|}{|\eta_q(u; \chi)|^{q-1}} = \chi q \).

(viii) If \( 1 < q < \frac{1}{2} \), then \( \lim_{u \to \infty} \frac{|u|}{|\eta_q(u; \chi)|} = 1 \).

**Proof.** Please refer to Lemmas 7 and 10 in [WMZ18] for the proof of \( q \in (1, 2] \). The proof for \( q > 2 \) is the same. Hence we do not repeat it.
Appendix B

Proofs of Chapter 2

B.1 Proof of Lemma 2.2.2

Define $FP = \sum_{i=1}^{p} \mathbb{I}(\bar{x}_i(q, \lambda, s) \neq 0, x_i = 0)$, $TP = \sum_{i=1}^{p} \mathbb{I}(\bar{x}_i(q, \lambda, s) \neq 0, x_i \neq 0)$.

First note that according to Theorem A.2.1, almost surely the empirical distribution of $(\hat{x}(q, \lambda), x)$ converges weakly to the distribution of $(\eta(B + \tau Z; \alpha \tau^2-q), B)$. We now choose a sequence $t_m \to 0$ as $m \to 0$ such that $G$ does not have any point mass on that sequence. Then by portmanteau lemma we have almost surely

$$\lim_{m \to \infty} \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \mathbb{I}(\bar{x}_i(q, \lambda, s) \neq 0, |x_i| \leq t_m) = \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \mathbb{I}(|\hat{x}_i(q, \lambda)| > s, |x_i| \leq t_m)$$

$$= \mathbb{P}(|\eta(B + \tau Z; \alpha \tau^2-q)| > s, |B| \leq t_m)$$

$$= (1 - \epsilon) \mathbb{P}(|\eta_\tau(Z; \alpha \tau^2-q)| > s) + \epsilon \mathbb{P}(|\eta_\tau(G + \tau Z; \alpha \tau^2-q)| > s, |G| \leq t_m),$$

which leads to

$$\lim_{m \to \infty} \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \mathbb{I}(\bar{x}_i(q, \lambda, s) \neq 0, |x_i| \leq t_m) = (1 - \epsilon) \mathbb{P}(|\eta_\tau(Z; \alpha \tau^2-q)| > s). \quad (B.1.1)$$
Moreover, it is clear that
\[
\frac{1}{p} \left| \sum_{i=1}^{p} I(\hat{x}_i(q, \lambda, s) \neq 0, |x_i| \leq t_m) - \text{FP} \right| \\
\leq \frac{1}{p} \sum_{i=1}^{p} I(|\hat{x}_i(q, \lambda)| > s) \cdot I(0 < |x_i| \leq t_m) \\
\leq \left[ \frac{1}{p} \sum_{i=1}^{p} I(|\hat{x}_i(q, \lambda)| > s) \right] \cdot \sqrt{\frac{1}{p} \sum_{i=1}^{p} I(0 < |x_i| \leq t_m)} \\
\xrightarrow{a.s.} \left[ \mathbb{P}(|\eta_q(B + \tau Z; \alpha \tau^{2-q})| > s) \right]^{1/2} \cdot \epsilon^{1/2} \left[ \mathbb{P}(0 < |G| \leq t_m) \right]^{1/2} \quad \text{as} \quad p \to \infty.
\]

Hence we obtain almost surely
\[
\lim_{m \to \infty} \lim_{p \to \infty} \frac{1}{p} \left| \sum_{i=1}^{p} I(\hat{x}_i(q, \lambda, s) \neq 0, |x_i| \leq t_m) - \text{FP} \right| = 0.
\]

This combined with (B.1.1) implies that as \( p \to \infty \)
\[
\frac{\text{FP}}{p} \xrightarrow{a.s.} (1 - \epsilon) \mathbb{P}(|\eta_q(\tau Z; \alpha \tau^{2-q})| > s).
\]

We can now conclude that
\[
\text{AFDP}(q, \lambda, s) = \lim_{p \to \infty} \frac{\text{FP}}{p} \\
= \frac{\text{FP}}{p} \sum_{i=1}^{p} I(\hat{x}_i(q, \lambda) > s)/p \\
= \frac{(1 - \epsilon) \mathbb{P}(|\eta_q(\tau Z; \alpha \tau^{2-q})| > s)}{\mathbb{P}(|\eta_q(\tau Z; \alpha \tau^{2-q})| > s)}, \quad \text{a.s.}
\]

The formula of \( \text{AFDP}(q, \lambda, s) \) in Lemma 2.2.2 can then be obtained by Lemma A.2.2 part (iii). Regarding \( \text{ATPP}(q, \lambda, s) \) we have
\[
\text{ATPP}(q, \lambda, s) = \frac{\text{lim}_{p \to \infty} \sum_{i=1}^{p} I(\hat{x}_i(q, \lambda) > s)/p - \text{lim}_{p \to \infty} \text{FP}/p}{\text{lim}_{p \to \infty} \sum_{i=1}^{p} I(x_i \neq 0)/p} \\
\mathbb{P}(|\eta_q(G + \tau Z; \alpha \tau^{2-q})| > s), \quad \text{a.s.}
\]
B.2 Proofs of Theorems 2.3.1, 2.3.2 and Corollary 2.3.1

We present the proofs of Theorems 2.3.1, 2.3.2 and Corollary 2.3.1 in Sections B.3, B.4 and B.5, respectively.

B.3 Proof of Theorem 2.3.1

Proof. According to Lemma 2.2.2, we know

\[ \text{ATPP}(q, \lambda, s) = \mathbb{P}(\eta_q(G + \tau Z; \alpha \tau^{2-q}) > s) \]

where \((\alpha, \tau)\) is the unique solution to (2.2.2) and (2.2.3). From Lemma A.2.2 part (iv), the proximal function \(\eta_q(u; \chi) = 0\) if and only if \(u = 0\) for \(q > 1\). Since \(G + \tau Z \neq 0\) a.s., we have ATPP\((q, \lambda, 0) = 1\). Moreover, it is clear that ATPP\((q, \lambda, +\infty) = 0\), and ATPP\((q, \lambda, s)\) is a continuous and strictly decreasing function of \(s\) over \([0, \infty]\). Hence there exists a unique \(s\) for which ATPP\((q, \lambda, s) = \zeta \in [0, 1]\).

Now consider all possible pairs \((\lambda, s)\) such that \(\text{ATPP}(q, \lambda, s) = \zeta\). Let \((\alpha^*_q, \tau^*_q, s^*_q)\) be the triplet corresponding to the optimal tuning \(\lambda^*_q\) (it minimizes AMSE\((q, \lambda)\)), and \((\alpha, \tau, s)\) be the one that corresponds to any other \(\lambda\). According to Theorem A.2.1, we know AMSE\((q, \lambda) = \delta(\tau^2 - \sigma^2)\). So \(\tau^*_q < \tau\). By the strict monotonicity and symmetry of \(\eta_q\) with respect to its first argument (see Lemma A.2.2 parts (i)(iv)), ATPP\((q, \lambda^*_q, s^*_q) = \text{ATPP}(q, \lambda, s)\) implies that

\[
\mathbb{P}(\eta_q(G/\tau + Z) > s^*_q/\tau^*_q; \alpha^*_q) = \mathbb{P}(\eta_q(G/\tau + Z) > s^*_q/\tau^*_q; \alpha^*_q),
\]

(B.3.1)

where \(\eta^{-1}_q\) is the inverse function of \(\eta_q\). Now we claim AFDP\((q, \lambda^*_q, s^*_q) < \text{AFDP}(q, \lambda, s)\).

Otherwise, from the formula of AFDP in (2.2.4), we will have

\[
\mathbb{P}(\eta_q(|Z|; \alpha_s) > s_s/\tau_s) \geq \mathbb{P}(\eta_q(|Z|; \alpha_s) > s/\tau),
\]

which is equivalent to \(\mathbb{P}(|Z| > \eta^{-1}_q(s_s/\tau_s; \alpha_s)) \geq \mathbb{P}(|Z| > \eta^{-1}_q(s/\tau; \alpha_s))\). This implies \(\eta^{-1}_q(s_s/\tau_s; \alpha_s) \leq \eta^{-1}_q(s/\tau; \alpha_s)\). However, combining this result with \(\tau^*_q < \tau\) and the
fact that $\mathbb{P}(|\mu + Z| > t)$ is an strictly increasing function of $\mu$ over $[0, \infty)$, we must have

\[
\mathbb{P}\left(\left|\frac{G}{\tau_s} + Z\right| > \eta^{-1}_q\left(\frac{s}{\tau_s}; \alpha_s\right)\right) > \mathbb{P}\left(\left|\frac{G}{\tau} + Z\right| > \eta^{-1}_q\left(\frac{s}{\tau}; \alpha\right)\right).
\]

This is in contradiction with (B.3.1). The conclusion follows. \qed

### B.4 Proof of Theorem 2.3.2

According to Lemma 2.2.1,

\[
\text{ATPP}(1, \lambda) = \mathbb{P}(|G + \tau Z| > \alpha \tau) = \mathbb{E}[\Phi(G/\tau - \alpha) + \Phi(-G/\tau - \alpha)].
\]

It has been shown in [BM12] that, $\alpha$ is an increasing and continuous function of $\lambda$, and $\alpha \to \infty$ as $\lambda \to \infty$. Hence, ATPP($1, \lambda$) is continuous in $\lambda$ and $\lim_{\lambda \to \infty} \text{ATPP}(1, \lambda) = \lim_{\alpha \to \infty} \mathbb{P}(|G + \tau Z| > \alpha \tau) = 0$. Now let $(\alpha_*, \tau_*)$ be the solution to (2.2.2) and (2.2.3) when $\lambda = \lambda_1^*$. As we decrease $\lambda$ from $\infty$ to $\lambda_1^*$, ATPP($1, \lambda$) continuously changes from 0 to ATPP($1, \lambda_1^*$). Therefore, for any ATPP level $\zeta \in [0, \text{ATPP}(1, \lambda_1^*)]$, there always exists at least a value of $\lambda \in [\lambda_1^*, \infty]$ such that ATPP($1, \lambda$) = $\zeta$. Regarding the thresholded LASSO $\hat{x}(1, \lambda_1^*, s)$, Lemma 2.2.2 shows that

\[
\text{ATPP}(1, \lambda_1^*, s) = \mathbb{P}(|\eta_1(G + \tau_s Z; \alpha_s \tau_s)| > s).
\]

Note that when $s = 0$ the thresholded LASSO is equal to LASSO and thus ATPP($1, \lambda_1^*, 0$) = ATPP($1, \lambda_1^*$). It is also clear that ATPP($1, \lambda_1^*, s$) is a continuous and strictly decreasing function of $s$ on $[0, \infty]$. As a result, a unique threshold $s_\zeta$ exists s.t. ATPP($1, \lambda_1^*, s_\zeta$) reaches a given level $\zeta \in [0, \text{ATPP}(1, \lambda_1^*)]$. We now compare the AFDP of different estimators that have the same ATPP. Suppose $\hat{x}(1, \lambda)$ and $\hat{x}(1, \lambda_1^*, s)$ reach the same level of ATPP. We have

\[
\mathbb{P}(|\eta_1(G + \tau Z; \alpha \tau)| > 0) = \mathbb{P}(|\eta_1(G + \tau_s Z; \alpha_s \tau_s)| > s),
\]
which is equivalent to
\[ P(|G/\tau + Z| > \alpha) = P(|G/\tau_x + Z| > \alpha_x + s/\tau_x). \] (B.4.1)

Similar to the argument in the proof of Theorem 2.3.1, we have \( \alpha < \alpha_x + s/\tau_x \), since otherwise the left hand side in (B.4.1) will be smaller than the right hand side. Hence, we obtain
\[ P(|Z| > \alpha) > P(|Z| > \alpha_x + s/\tau_x) = P(|\eta_1(Z;\alpha_x)| > s/\tau_x). \]

This implies \( \text{AFDP}(1,\lambda) > \text{AFDP}(1,\lambda_1^*,s) \) based on Lemmas 2.2.1 and 2.2.2. By the same argument, we can show that \( \bar{x}(1,\lambda_1^*,s) \) also has smaller AFDP than \( \bar{x}(1,\lambda,s) \) if \( \lambda \neq \lambda_1^* \).

### B.5 Proof of Corollary 2.3.1

This theorem compares the two-stage estimators \( \bar{x}(q,\lambda^*_q,s) \) for \( q \in [1,\infty) \). Consider \( q_1, q_2 \geq 1 \), and \( \text{AMSE}(q_1,\lambda^*_q) < \text{AMSE}(q_2,\lambda^*_q) \). Let \( (\alpha_{q_i}, \tau_{q_i}) \) be the solution to (2.2.2) and (2.2.3) when \( \lambda = \lambda^*_q \), for \( i = 1, 2 \). Then, according to Theorem A.2.1, \( \tau_{q_1} < \tau_{q_2} \). Suppose \( \text{ATPP}(q_1,q_1,s_1) = \text{ATPP}(q_2,q_2,s_2) \), i.e.,
\[ P(\eta_{q_1}(G + \tau_{q_1}Z;\alpha_{q_1}\tau_{q_1}^{2-q_1}) > s_1) = P(\eta_{q_2}(G + \tau_{q_2}Z;\alpha_{q_2}\tau_{q_2}^{2-q_2}) > s_2). \]

When the ATPP level is 0 or 1, we see \( s_1 \) and \( s_2 \) are either both \( \infty \) or 0. The corresponding AFDP will be the same. We now consider the level of ATPP belong to \((0,1)\). Using arguments similar to the ones presented in the proof of Theorem 2.3.1, we can conclude \( \eta_{q_1}^{-1}(s_1/\tau_{q_1}^*;\alpha_{q_1}^*) > \eta_{q_2}^{-1}(s_2/\tau_{q_2}^*;\alpha_{q_2}^*) \). This gives us
\[ P(\eta_{q_1}(Z;\alpha_{q_1}^*)| > s_1/\tau_{q_1}^*) = P(|Z| > \eta_{q_1}^{-1}(s_1/\tau_{q_1}^*;\alpha_{q_1}^*)) < P(|Z| > \eta_{q_2}^{-1}(s_2/\tau_{q_2}^*;\alpha_{q_2}^*)) = P(|\eta_{q_2}(Z;\alpha_{q_2}^*)| > s_2/\tau_{q_2}^*), \]

implying \( \text{AFDP}(q_1,\lambda_{q_1},s_1) < \text{AFDP}(q_2,\lambda_{q_2},s_2) \).

\(^1\)Note that \( \eta_{q_1}^{-1}(u;\chi) \) is not well defined for \( u = 0 \) and we define it as \( \eta_{q_1}^{-1}(0;\chi) = \chi \).
Appendix C

Proof of Chapter 3

C.1 Proof sketch for Theorem 3.2.1 - 3.4.1

In Appendix C.2 - C.6 we prove Theorem 3.2.1 - 3.4.1. Since the proofs share some similarities, we sketch the proof idea in this section.

The results in Theorem 3.2.1 - 3.4.1 characterize the asymptotic expansion of the optimal AMSE\((q, \lambda_q^*)\) under different scenarios we considered. In Lemma A.2.1, we connect AMSE\((q, \lambda_q^*)\) with \((\alpha_*, \tau_*)\) through the state evolution equations. Hence in order to prove our theorems, we will characterize the behavior of the solution \((\alpha_*, \tau_*)\) of the fixed point equations (2.2.2) and (2.2.3) with \(\lambda = \lambda_q^*\) under different scenarios. This can be achieved by making use of (A.2.1) and its first order condition (notice \(\alpha_*\) minimize the AMSE).

Depending on different scenarios, (A.2.1) may be presented in slightly different ways. Specifically for nearly black object, we replace \(B\) by \(b_0\tilde{B}\) with \(p_{\tilde{B}} = (1 - \epsilon)\delta_0 + \epsilon p_G\); For large sample scenario, we replace \(\sigma^2\) by \(\frac{\sigma^2}{\delta}\).

For \(R_q(\alpha, \tau)\), the following decomposition holds:

\[
R_q(\alpha, \tau) = (1 - \epsilon)\mathbb{E}\eta_q^2(Z; \alpha) + \epsilon\mathbb{E}[\eta_q(G/\tau + Z; \alpha) - G/\tau]^2.
\]

Since both terms are positive, either can be used as a lower bound for \(R_q(\alpha, \tau)\).
For LASSO, the $\ell_1$ norm enables a simple form for $\eta_1$ and hence for (A.2.1) and its first order derivative. We present some useful formula below.

\[
R_1(\alpha, \tau) = (1 - \epsilon)^2 \mathbb{E}_{\eta_1^2(Z; \alpha)} + \epsilon \mathbb{E} \left[ \eta_1(b \tilde{G} + \tau Z; \alpha \tau) - b \tilde{G} - \tau Z \right]^2 - \epsilon \tau^2 \\
\approx F_1 \\
+ 2\epsilon \tau^2 \mathbb{E} \partial_1 \eta_1(b \tilde{G} + \tau Z; \alpha \tau) \\
\triangleq F_2 \\
= 2(1 - \epsilon)[(1 + \alpha^2)\Phi(-\alpha) - \alpha \phi(\alpha)] + \epsilon \mathbb{E}_G \left[ (1 + \alpha^2 - \frac{G^2}{\tau^2})\Phi\left(\frac{G}{\tau} - \alpha\right) + \\
(1 + \alpha^2 - \frac{G^2}{\tau^2})\Phi\left(-\frac{G}{\tau} - \alpha\right) - (\alpha + \frac{G}{\tau})\phi(\alpha - \frac{G}{\tau}) - \\
(\alpha - \frac{G}{\tau})\phi(\alpha + \frac{G}{\tau}) + \frac{G^2}{\tau^2} \right] \\
\triangleq F_3 \\
(C.1.1)
\]

(C.1.2)

Each of the two expansions (C.1.1) and (C.1.2) will be handy in certain case. Note that

\[
F_1 = 2(1 - \epsilon)^\tau^2 \int_\alpha^\infty (z - \alpha)^2 \phi(z) dz = 2(1 - \epsilon)[(1 + \alpha^2)\Phi(-\alpha) - \alpha \phi(\alpha)]. \\
(C.1.3)
\]

We also provide the following expansion for the first order derivative $\partial_\alpha R_1(\alpha, \tau)$.

\[
\partial_\alpha R_1(\alpha, \tau) = 2(1 - \epsilon)[-\phi(\alpha) + \alpha \Phi(-\alpha)] + \epsilon \mathbb{E}_G \left[ \alpha \Phi\left(\frac{|G|}{\tau} - \alpha\right) - \phi\left(\alpha - \frac{|G|}{\tau}\right) \right] \\
+ \epsilon \mathbb{E}_G \left[ \alpha \Phi\left(-\frac{|G|}{\tau} - \alpha\right) - \phi\left(\alpha + \frac{|G|}{\tau}\right) \right] \\
\triangleq F_4 \\
(C.1.4)
\]

### C.2 Proof of Theorem 3.2.1

#### C.2.1 Roadmap of the proof

As we have mentioned in Section C.1, we will characterize the behavior of $(\alpha_*, \tau_*)$ defined through equation (A.2.1). Since we are dealing with the nearly black object model, we replace $B$ by $b \tilde{B}$ with $p \tilde{B} = (1 - \epsilon)\delta_0 + \epsilon p_B$. We first handle $q < 2$ in Section C.2.2 - C.2.5. Then in Section C.2.6 we deal with $q \geq 2$. We will prove in Section C.2.2 that as $\epsilon \to 0$, $\tau_* \to \sigma$. Furthermore, it is straightforward to see that
\( \alpha_\ast \to \infty \) as \( \epsilon \to 0 \). Otherwise, if \( \alpha_\ast \to C \), then

\[
\lim_{\epsilon \to 0} E(\eta_q(b_\epsilon \tilde{B} + \tau_\ast Z; \alpha_\ast \tau_\ast^{-q}) - b_\epsilon \tilde{B})^2 \\
\geq \lim_{\epsilon \to 0} (1 - \epsilon)E\eta_q^2(\tau_\ast Z; \alpha_\ast \tau_\ast^{-q}) \\
= E\eta_q^2(\sigma Z; C\sigma^{-q}) > 0.
\]

However, in Section C.2.2 we will prove that \( \lim_{\epsilon \to 0} E(\eta_q(b_\epsilon \tilde{B} + \tau_\ast Z; \alpha_\ast \tau_\ast^{-q}) - b_\epsilon \tilde{B})^2 \to 0 \). In order to show the optimal AMSE vanishes as \( \epsilon \to 0 \), we need to characterize the rate at which \( \alpha_\ast \to \infty \). This requires an accurate analysis of \( \arg \min_{\alpha} \tilde{R}_q(\alpha, \epsilon, \tau_\ast) \), where

\[
\tilde{R}_q(\alpha, \epsilon, \tau) \triangleq E(\eta_q(b_\epsilon \tilde{B} + \tau Z; \alpha \tau^{-q}) - b_\epsilon \tilde{B}).
\]  

We note the slight differences between \( \tilde{R}_q \) and \( R_q \) in (A.2.2) and \( \text{AMSE}(q, \lambda_\ast^q) = \tilde{R}(\alpha_\ast, \epsilon, \tau_\ast) \). The behavior of \( \alpha_\ast \) depends on the relation between \( b_\epsilon \) and \( \epsilon \) in the following way:

- **Case I** - If \( b_\epsilon = o(\epsilon^{-\frac{q}{2}}) \), then \( \lim_{\epsilon \to 0} \epsilon^{-1} b_\epsilon^{-2} \tilde{R}_q(\alpha_\ast, \epsilon, \tau_\ast) = E|\tilde{G}|^2 \). This claim is proved in Section C.2.3. Note that \( \lim_{\alpha \to \infty} \tilde{R}_q(\alpha, \epsilon, \tau) = \epsilon b_\epsilon^2 E|\tilde{G}|^2 \) too.

- **Case II** - If \( b_\epsilon = \omega(\epsilon^{-\frac{q}{2}}) \), then \( \epsilon^{\frac{q-1}{2q}} b_\epsilon^{\frac{(q-1)^2}{q}} \alpha_\ast = \Theta(1) \). This claim is proved in Section C.2.4. Furthermore, we will show that

\[
\epsilon^{-\frac{1}{7}} b_\epsilon^{-\frac{2(q-1)}{q}} \tilde{R}_q(\alpha_\ast, \epsilon, \tau_\ast) \to q(q - 1)^{\frac{1}{7} - 1}\sigma^{\frac{2}{7}} \left[ E|Z|^{\frac{2}{7}} \right]^{\frac{q-1}{2}} \left[ E|\tilde{G}|^{2q - 2} \right]^{\frac{1}{9}}
\]

- **Case III** - If \( b_\epsilon = \Theta(\epsilon^{-\frac{q}{2}}) \), then still the optimal choice of \( \alpha \) satisfies \( \epsilon^{\frac{q-1}{2q}} b_\epsilon^{\frac{(q-1)^2}{q}} \alpha_\ast = \Theta(1) \). This will be proved in Section C.2.5. After obtaining this result, we will show

\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{7}} b_\epsilon^{-\frac{2(q-1)}{q}} \tilde{R}_q(\alpha_\ast, \epsilon, \tau_\ast) = \min_C h(C),
\]

where \( h(C) \triangleq (Cq)^{-\frac{2}{7}} \sigma^{\frac{2}{7}} E|Z|^{\frac{2}{7}} + E(\eta_q(c_r G; C\sigma^{-q}) - c_r G)^2 \), and \( c_r \triangleq \lim_{\epsilon \to 0} b_\epsilon^{\frac{q-1}{2q}} \).
C.2.2 Proof of $\tau_* \to \sigma$ as $\epsilon \to 0$

We first prove a simple lemma which helps with bounding the optimal $\tau_*^2$.

**Lemma C.2.1.** For any value of $\epsilon > 0$ we have

$$\sigma^2 \leq \tau_*^2 \leq \sigma^2 + \frac{\epsilon b^2}{\delta} \mathbb{E}\tilde{G}^2.$$ 

**Proof.** $\tau_* > \sigma$ is clear from $\tau_*^2 = \sigma^2 + \frac{1}{\delta} \min_{\alpha > 0} \tilde{R}_q(\alpha, \epsilon, \tau_*)$. Furthermore,

$$\tau_*^2 - \sigma^2 = \frac{1}{\delta} \min_{\alpha > 0} \tilde{R}_q(\alpha, \epsilon, \tau_*) \leq \frac{1}{\delta} \lim_{\alpha \to \infty} \tilde{R}_q(\alpha, \epsilon, \tau_*) = \frac{\epsilon b^2}{\delta} \mathbb{E}\tilde{G}^2.$$ 

If $\sqrt{\epsilon} b \to 0$ as $\epsilon \to 0$, by Lemma C.2.1, we have $\tau_* \to \sigma$. So next we focus on the case when $\sqrt{\epsilon} b \to c$, where $c \in (0, \infty)$. In order to prove $\tau_* \to \sigma$ under this case, we prove $\tilde{R}_q(\alpha, \epsilon, \tau_*) \to 0$ for a specific choice of $\alpha$.

**Lemma C.2.2.** If $\sqrt{\epsilon} b \to c$, and $\tilde{\sigma}^2 \triangleq \sigma^2 + \frac{\epsilon b^2}{\delta} \mathbb{E}\tilde{G}^2$, then as $\epsilon \to 0$

$$\sup_{\sigma \leq \tau \leq \tilde{\sigma}} \tilde{R}_q \left( \epsilon \frac{(2-q)(q-1)}{-2q}, \epsilon, \tau \right) \to 0.$$ 

**Proof.** Define $\alpha_0 \triangleq \epsilon \frac{(2-q)(q-1)}{-2q}$. We have

$$\tilde{R}_q(\alpha_0, \epsilon, \tau) \equiv A_1(\epsilon) + A_2(\epsilon) \tag{C.2.2}$$

We first prove that $\overline{\lim}_{\epsilon \to 0} \sup_{\sigma \leq \tau \leq \tilde{\sigma}} A_1(\epsilon) = 0$. Note that

$$\alpha_0^{\frac{2}{q-1}} A_1(\epsilon) \overset{(a)}{=} (1 - \epsilon) \tau^2 q^{-\frac{2}{q-1}} \mathbb{E}(|Z| - |\eta_{q}(Z; \alpha_0)|)^{\frac{2}{q-1}} \tag{C.2.3}$$

Equality (a) is due to Lemma A.2.2 (ii) (iii). Hence,

$$\sup_{\sigma \leq \tau \leq \tilde{\sigma}} \alpha_0^{\frac{2}{q-1}} A_1(\epsilon) \leq (1 - \epsilon) \tilde{\sigma}^2 q^{-\frac{2}{q-1}} \mathbb{E}|Z|^{\frac{2}{q-1}}.$$
Since $\alpha_0 \to \infty$ as $\epsilon \to 0$, this immediately implies that $\lim_{\epsilon \to 0} \sup_{\sigma \leq \tau \leq \tilde{\sigma}} A_1(\epsilon) = 0$. Now we discuss $A_2(\epsilon)$. We have

$$A_2(\epsilon) = \epsilon \mathbb{E}(\eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau^{2-q}) - b_\epsilon \tilde{G} - \tau Z)^2 + \epsilon \tau^2 + 2 \epsilon \mathbb{E}(Z(\eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau^{2-q}) - b_\epsilon \tilde{G} - \tau Z)) \triangleq \epsilon B_1(\epsilon) + \epsilon \tau^2 + 2 \epsilon B_2(\epsilon).$$

We study $B_1(\epsilon)$ and $B_2(\epsilon)$ separately.

$$B_1(\epsilon) \overset{(a)}{=} q^2 \alpha_0^2 \tau^{4-2q} \mathbb{E}|\eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau^{2-q})|^{2q-2},$$

where Equality (a) is due to Lemma A.2.2(ii). We note that the choice of $\alpha_0$ implies $\epsilon \alpha_0^2 b_\epsilon^{2q-2} \to 0$. Hence, as $\epsilon \to 0$

$$\epsilon B_1(\epsilon) \leq \epsilon \alpha_0^2 q^2 \tau^{4-2q} \mathbb{E}|b_\epsilon \tilde{G} + \tau Z|^{2q-2} \to 0 \quad \text{(C.2.4)}$$

It is straightforward to see that $\lim_{\epsilon \to 0} \sup_{\tau \leq \tilde{\sigma}} \epsilon B_1(\epsilon) = 0$. Now let us discuss $B_2(\epsilon)$. By using Stein’s lemma we have

$$B_2(\epsilon) = \tau^2 \mathbb{E}(\partial_1 \eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau^{2-q}) - 1)$$

$$\overset{(a)}{=} \tau^2 \mathbb{E}\left[ \frac{-\alpha_0 \tau^{2-q} q(q-1)|\eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau)|^{q-2}}{1 + \alpha_0 \tau^{2-q} q(q-1)|\eta_q(b_\epsilon \tilde{G} + \tau Z; \alpha_0 \tau^{2-q})|^{q-2}} \right].$$

Equality (a) is due to Lemma A.2.2(v). Hence, $|B_2(\epsilon)| < \tau^2$ and

$$\sup_{\tau \leq \tilde{\sigma}} \epsilon B_2(\epsilon) \to 0. \quad \text{(C.2.5)}$$

Combining (C.2.2), (C.2.3), (C.2.4), and (C.2.5) completes the proof.

Now Lemma C.2.1 implies that $\tau_* \in [\sigma, \tilde{\sigma}]$. By combining this observation with Lemma C.2.2, it is straightforward to conclude that

$$\delta(\tau_*^2 - \sigma^2) = \min_{\alpha > 0} \tilde{R}_q(\alpha, \epsilon, \tau_*) \leq \tilde{R}_q \left( \epsilon \left( \frac{(2-q)(q-1)}{-2q} \right), \epsilon, \tau_* \right) \leq \sup_{\sigma < \tau < \tilde{\sigma}} \tilde{R}_q \left( \epsilon \left( \frac{(2-q)(q-1)}{-2q} \right), \epsilon, \tau \right) \to 0.$$

This finishes our proof of $\tau_* \to \sigma$. 
C.2.3 Case I - $b_\epsilon = o(\epsilon^{\frac{1-q}{2}})$

Since Case I is the simplest case, we start with this one. As discussed in Section C.2.1, $\alpha_* \to \infty$ as $\epsilon \to 0$. In Lemma C.2.3 we use this fact to derive a lower bound for $\tilde{R}_q(\alpha, \epsilon, \tau)$, then use it to obtain a finer information about $\alpha_*$.

**Lemma C.2.3.** If $\alpha \to \infty$ and $\tau \to \sigma > 0$ as $\epsilon \to 0$, then

$$\lim_{\epsilon \to 0} \alpha^{-\frac{2}{q-1}} \tilde{R}_q(\alpha, \epsilon, \tau) \geq \sigma^2 q^{-\frac{2}{q-1}} \mathbb{E}|Z|^{\frac{2}{q-1}}$$

**Proof.** First note that

$$\alpha^{-\frac{2}{q-1}} \tilde{R}_q(\alpha, \epsilon, \tau) \overset{(a)}{=} (1 - \epsilon) \alpha^{\frac{2}{q-1}} \tau^2 \mathbb{E} \eta_q^2(Z; \alpha) \overset{(b)}{=} (1 - \epsilon) \tau^2 q^{-\frac{2}{q-1}} \mathbb{E}|Z| - |\eta_q(Z; \alpha)| \frac{2}{q-1}.$$

where inequality (a) is due to Lemma A.2.2(iii) and inequality (b) is due to Lemma A.2.2(ii). We note that an application of DCT proves that the last term of expectation converges to $\mathbb{E}|Z|^{\frac{2}{q-1}}$ as $\epsilon \to 0$. We should mention that it is straightforward to prove that for every $u$, $\eta_q(u; \alpha) \to 0$ as $\alpha \to \infty$.

The rest of the proof goes as follows: we first use Lemma C.2.3 to prove $b_\epsilon \alpha_*^{-\frac{1}{2-q}} \to 0$. This will further help us to characterize the accurate behavior of $\tilde{R}_q(\alpha_*, \epsilon, \tau_*)$.

**Lemma C.2.4.** We have $\lim_{\alpha \to \infty} \tilde{R}_q(\alpha, \epsilon, \tau) = \epsilon b_\epsilon^2 \mathbb{E}|\tilde{G}|^2$.

The proof of this lemma is straightforward and is hence skipped.

**Lemma C.2.5.** If $b_\epsilon = o(\epsilon^{\frac{1-q}{2}})$, then $b_\epsilon \alpha_*^{-\frac{1}{2-q}} \to 0$ as $\epsilon \to 0$.

**Proof.** We prove by contradiction. Assume the assertion of the lemma is incorrect, i.e. $\frac{\alpha_*}{b_\epsilon^{\frac{1-q}{2}}} = O(1)$. Then,

$$\epsilon^{-1} b_\epsilon^{-2} \tilde{R}_q(\alpha_*, \epsilon, \tau_*) = (\alpha_*^{-1} b_\epsilon^{2-q})^{\frac{2}{q-1}} (b_\epsilon^{\frac{2}{q-1}} \epsilon)^{-1} \alpha_*^{\frac{2}{q-1}} \tilde{R}_q(\alpha_*, \epsilon, \tau_*)$$

According to Lemma C.2.3, since $\alpha_* \to \infty$ and $\tau_* \to \sigma$, we have $\alpha_*^{\frac{2}{q-1}} \tilde{R}_q(\alpha_*, \epsilon, \tau_*) = \Omega(1)$. Furthermore our assumption indicates that $\alpha_*^{-1} b_\epsilon^{2-q} = \Omega(1)$. Finally, due to the
condition of the lemma, we have $b_c^{2/q(1-q)} \epsilon \to 0$. Hence, $\epsilon^{-1} b_c^{-2} \tilde{R}_q(\alpha, \epsilon, \tau\alpha) \to \infty$. Based on Lemma C.2.4, $\lim_{\epsilon \to 0} \tilde{R}_q(\alpha, \epsilon, \tau\alpha)$ is proportional to $eb_c^2$. This forms a contradiction with the optimality of $\alpha\alpha$ and completes the proof.

In the next theorem we use Lemma C.2.5 to characterize $\tilde{R}_q(\alpha, \epsilon, \tau\alpha)$.

**Theorem C.2.1.** If $b_c = o(\epsilon^{1-q})$, then $\lim_{\epsilon \to 0} \frac{\tilde{R}_q(\alpha, \epsilon, \tau\alpha)}{eb_c^2|G|^{2}} \geq 1$.

**Proof.** It is not hard to see that $\frac{\tilde{R}_q(\alpha, \epsilon, \tau\alpha)}{eb_c^2|G|^{2}} \geq E \left[ \eta_q \left( \tilde{G} + \frac{\tau\alpha}{b_c^2} Z; \tau\alpha^{2-q} \frac{\alpha\alpha}{b_c^{-q}} \right) - \tilde{G} \right]^2$. Since $b_c \to \infty$ and according to Lemma C.2.5, $\frac{\alpha\alpha}{b_c^{-q}} \to \infty$, it is straightforward to apply DCT and obtain that $E \left[ \eta_q \left( \tilde{G} + \frac{\tau\alpha}{b_c^2} Z; \tau\alpha^{2-q} \frac{\alpha\alpha}{b_c^{-q}} \right) - \tilde{G} \right]^2 \to E|\tilde{G}|^2$. The conclusion then follows.

A direct corollary of Lemma C.2.4 and Theorem C.2.1 is if $b_c = o(\epsilon^{1-q})$, then $\tilde{R}_q(\alpha, \epsilon, \tau\alpha) \sim eb_c^2E|\tilde{G}|^2$. This completes the first piece of Theorem 3.2.1.

**C.2.4 Case II - $b_c = \omega(\epsilon^{1-q})$**

We first characterize the risk for a specific choice of $\alpha$. This offers an upper bound for $\tilde{R}_q(\alpha, \epsilon, \tau\alpha)$, and will later help us obtain the exact behavior of $\alpha\alpha$.

**Lemma C.2.6.** Suppose that $b_c = \omega(\epsilon^{1-q})$. If $\alpha = C\epsilon^{\frac{1-q}{2}} b_c^{-\frac{(q-1)^2}{q}}$, then,

$$\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{q}} b_c^{-\frac{2(q-1)}{q}} \tilde{R}(\alpha, \epsilon, \tau\alpha) = C^{-2} q^{-\frac{2}{q-1}} \sigma^2 E|Z|^{\frac{2}{q-1}} + C^2 q^2 \sigma^{q-2q} E|\tilde{G}|^{2(q-1)}.$$

**Proof.** We again start our argument with the same decomposition as in (C.2.2). Note that as $\epsilon \to 0$, $\alpha \to \infty$, and as discussed in Section C.2.2, $\tau\alpha \to \sigma$. We further have

$$\epsilon^{-\frac{1}{q}} b_c^{-\frac{2(q-1)}{q}} A_1(\epsilon) = (1 - \epsilon) \epsilon^{-\frac{1}{q}} b_c^{-\frac{2(q-1)}{q}} \tau\alpha^2 E\eta_q^2(Z; \alpha)$$

(C.2.6)

$$\equiv (1 - \epsilon) q^{-\frac{2}{q-1}} \left( \frac{\alpha b_c^{q-1}}{q} q^{-\frac{(q-1)^2}{q}} \right)^{-\frac{2}{q-1}} \tau\alpha^2 E|Z| - |\eta_q(Z; \alpha)|^{\frac{2}{q-1}} \to q^{-\frac{2}{q-1}} C^{-2} \sigma^{q-2q} E|Z|^{\frac{2}{q-1}}.$$

Equality (a) is due to Lemma A.2.2(ii), and the last step is a result of DCT. We should also emphasize that since $\alpha \to \infty$, $|\eta_q(Z; \alpha)| \to 0$. Furthermore, similar to the
steps in the proof of Lemma (C.2.2), we can obtain
\[
A_2(\epsilon) = \epsilon\alpha^2 \tau_{\alpha}^{4-2q} q^2 \mathbb{E}\left| \eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) \right|^{2q-2} + \epsilon \tau_{\alpha}^2 \mathbb{E}\left[ \partial_1 \eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) - 1 \right].
\] (C.2.7)

First we have that
\[
\epsilon^{-\frac{1}{2}} b^{-\frac{2(q-1)}{q}} \mathbb{E}\left| \partial_1 \eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) \right|^{2q-2} = \alpha^2 \epsilon^{-\frac{1}{q}} b^{-\frac{2(q-1)}{q}} \mathbb{E}\left[ \left| \eta_q(b, \alpha^{-\frac{1}{2-q}} \tilde{G} + \alpha^{-\frac{1}{2-q}} \tau_{\alpha} Z; \tau_{\alpha}^{2-q}) \right|^{2q-2} \right] \] (C.2.8)

We note our condition on the growth of \( \alpha \) and the following relation:
\[
b \alpha^{-\frac{1}{2-q}} = C^{-\frac{1}{2-q}} \left[ \epsilon^{-\frac{1}{q}} b^{-\frac{2(q-1)}{q}} \right] \to \infty, \quad \epsilon^{-\frac{1}{2}} b^{-\frac{2(q-1)}{q}} \to 0.
\]

The first relation above implies that
\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{1}{2}} b^{-\frac{2(q-1)}{q}} \mathbb{E}\left| \partial_1 \eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) \right|^{2q-2} = C^2 \sigma^{4-2q} q^2 \mathbb{E}|\tilde{G}|^{2q-2} \] (C.2.9)

Since \( |\partial_1 \eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) - 1| \leq 1 \), we are able to conclude that
\[
\epsilon^{-\frac{1}{2}} b^{-\frac{2(q-1)}{q}} A_2(\epsilon) \to C^2 \sigma^{4-2q} q^2 \mathbb{E}|\tilde{G}|^{2q-2} \] (C.2.10)

where the last step is a simple application of DCT (According to Lemma A.2.2(ii) \( \frac{|\eta_q(u, \chi)|}{|u|} \leq 1 \) for every \( u \) and \( \chi \)), combined with the fact that \( b \to \infty \) and \( \tau_{\alpha} \to \sigma \) as \( \epsilon \to 0 \). Combining (C.2.2), (C.2.6), (C.2.7), and (C.2.10) finishes the proof.

So far, we know that \( \alpha_{\ast} \to \infty \). Our next theorem provides more accurate information about \( \alpha_{\ast} \).

**Theorem C.2.2.** If \( b \to \omega(1) \), then \( b \alpha_{\ast}^{-\frac{1}{2-q}} \to \infty \).

**Proof.** Suppose this is not correct, then \( b \alpha_{\ast}^{-\frac{1}{2-q}} = O(1) \). According to (C.2.7) and (C.2.2) we have
\[
\tilde{R}_q(\alpha_{\ast}, \epsilon, \tau_{\alpha}) \geq \epsilon\mathbb{E}(\eta_q(b, \tilde{G} + \tau_{\alpha} Z; \alpha \tau_{\alpha}^{2-q}) - b \tilde{G})^2 \overset{(a)}{=} \epsilon b^{2q} \mathbb{E}(\eta_q(\tilde{G} + b^{-1} \tau_{\alpha} Z; b^{q-2} \alpha \tau_{\alpha}^{2-q}) - \tilde{G})^2,
\] (C.2.11)
where the last equality is due to Lemma A.2.2(iii). Note that
\[
\frac{\tilde{R}_q(\alpha_*, \epsilon, \tau_*)}{\tilde{R}_q(C\sqrt{\epsilon^{-q} b^{-2} \frac{(q-1)^2}{q}}, \epsilon, \tau_*)} = \frac{\epsilon^{-1} b^{-2} \tilde{R}_q(\alpha_*, \epsilon, \tau_*)}{\epsilon^{-\frac{1}{q}} b^{-\frac{2(1-q)}{q}} \tilde{R}_q(C\sqrt{\epsilon^{-q} b^{-2} \frac{(q-1)^2}{q}}, \epsilon, \tau_*)} \times (\epsilon^{q-1} b^{2})^{\frac{1}{q}}.
\]

According to Lemma C.2.6, \( \epsilon^{-\frac{1}{q}} b^{-\frac{2(1-q)}{q}} \tilde{R}_q(C\sqrt{\epsilon^{-q} b^{-2} \frac{(q-1)^2}{q}}, \epsilon, \tau_*) = \Theta(1) \). By using the DCT in (C.2.11) (combined with the assumption that \( b_{\alpha_*}^{-\frac{1}{q}} = O(1) \)), it is straightforward to confirm that \( \epsilon^{-1} b^{-2} \tilde{R}_q(\alpha_*, \epsilon, \tau_*) = \Omega(1) \). Since, \( (\epsilon^{q-1} b^{2})^{\frac{1}{q}} \to \infty \), we conclude that
\[
\lim_{\epsilon \to 0} \frac{\tilde{R}_q(\alpha_*, \epsilon, \tau_*)}{\tilde{R}_q(C\sqrt{\epsilon^{-q} b^{-2} \frac{(q-1)^2}{q}}, \epsilon, \tau_*)} = \infty.
\]
This is contradicted with the optimality of \( \alpha_* \). Hence, \( b_{\alpha_*}^{-\frac{1}{q}} \to \infty \).

Finally, we are ready to prove the main claim of this section.

**Theorem C.2.3.** Suppose that \( b_* = \omega(\epsilon^{1\frac{q}{q-1}}) \). Then, \( \epsilon^{-\frac{q-1}{q}} b^{-\frac{(q-1)^2}{q}} \alpha_* \to C_* \), where \( C_* = \left[ \frac{\sigma^{2q-2} \mathbb{E}|Z|^{\frac{q}{q-1}}}{(q-1)q^{\frac{q}{q-1}} \mathbb{E}|\tilde{G}|^{2q-2}} \right]^{\frac{q-1}{q}} \). Furthermore,
\[
\epsilon^{-\frac{1}{q}} b^{-\frac{2(q-1)}{q}} \tilde{R}_q(\alpha_*, \epsilon, \tau_*) \to (C_* q)^{\frac{2}{q-1}} \sigma^2 \mathbb{E}|Z|^{\frac{2}{q-1}} + (C_* q)^2 \sigma^4 \mathbb{E}|\tilde{G}|^{2(q-1)}. \tag{C.2.12}
\]

**Proof.** We know that \( \frac{\partial \tilde{R}_q(\alpha_*, \epsilon, \tau_*)}{\partial \alpha_*} = 0 \). Hence,
\[
0 = (1 - \epsilon) \tau_*^q \mathbb{E}[\eta(Z; \alpha_*) \partial_2 \eta(Z; \alpha_*)]
+ \epsilon \tau_*^{2-q} \mathbb{E}\left[ (\eta(Z; \alpha_*) \partial_2 \eta(Z; \alpha_* Z; \alpha_* \tau_*^{2-q})) - b_\epsilon \tilde{G} \right] \partial_2 \eta(Z; \alpha_* Z; \alpha_* \tau_*^{2-q})
\]
\[
\quad (a) = (1 - \epsilon) \tau_*^{q\epsilon} \mathbb{E}\left[ \frac{-q|\eta(Z; \alpha_*)|^q}{1 + \alpha_* q(q-1)|\eta(Z; \alpha_*)|^{q-2}} \right] + \epsilon \tau_*^{3-q} \mathbb{E}[Z \partial_2 \eta(Z; \alpha_* \tau_* Z; \alpha_* \tau_*^{2-q})] + \epsilon \alpha_* \tau_*^{4-2q} q^2 \mathbb{E}\left[ \frac{|\eta(Z; \alpha_* \tau_* Z; \alpha_* \tau_*^{2-q})|^{2q-2}}{1 + \alpha_* \tau_*^{2-q} q(q-1)|\eta(Z; \alpha_* \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-2}} \right]
\]
\[
\quad \Delta = (1 - \epsilon) q \tau_*^{2} H_1 + \epsilon \tau_*^{3-q} H_2 + \epsilon \alpha_* \tau_*^{4-2q} q^2 H_3. \tag{C.2.13}
\]

To obtain (a) we have used Lemma A.2.2 parts (ii) and (v). We now study each term separately. We should mention at this point that the rest of our analyses relies
heavily on Theorem C.2.2. By using Lemma A.2.2(iii) we have
\[
H_1 = \mathbb{E} \left[ \frac{|\eta_\phi(Z; \alpha_*)|^q}{1 + \alpha_* q(q - 1)|\eta_\phi(Z; \alpha_*)|^{q-2}} \right] \\
= \alpha_*^{-1} \mathbb{E} \left[ \frac{|Z| - |\eta_\phi(Z; \alpha_*)|}{q\alpha_*} \left| \frac{2^{\frac{2}{q-1}} |\eta_\phi(\alpha_*^{-\frac{1}{q-2}} Z; 1)|^{q-2}}{1 + q(q - 1)|\eta_\phi(\alpha_*^{-\frac{1}{q-2}} Z; 1)|^{q-2}} \right] .
\]  
(C.2.14)

Since $|\eta_\phi(\alpha_*^{-\frac{1}{q-2}} Z; 1)| \to 0$, we have
\[
\frac{|\eta_\phi(\alpha_*^{-\frac{1}{q-2}} Z; 1)|^{q-2}}{1 + q(q - 1)|\eta_\phi(\alpha_*^{-\frac{1}{q-2}} Z; 1)|^{q-2}} \to \frac{1}{q(q - 1)} .
\]  
(C.2.15)

By combining (C.2.14) and (C.2.15) we have
\[
\alpha_*^{\frac{q+1}{q-1}} H_1 \to (q - 1)^{-1} q^{-\frac{q+1}{q-1}} \mathbb{E}|Z|^{\frac{2}{q-1}} .
\]  
(C.2.16)

Now we focus on $H_3$. Define $D \triangleq 1 + q(q - 1)\tau_*^{2-q}|\eta_\phi(\frac{b_\tau \hat{G}}{\alpha_*^{2-q}}; \frac{\tau_* Z}{\alpha_*^{2-q}})|^{q-2}$.

Then,
\[
H_3 = \mathbb{E} \left[ \frac{|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_*^{2-q})|^{2q-2}}{1 + \alpha_* \tau_*^{2-q} q(q - 1)|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_*^{2-q})|^{q-2}} \right] \\
= \mathbb{E} \left[ \frac{b_\tau^{2q-2} |\eta_\phi(\hat{G} + b_\tau^{-1} \tau_* Z; \alpha_* b_\tau^{q-2} \tau_*^{2-q})|^{2q-2}}{1 + \alpha_* b_\tau^{q-2} \tau_*^{2-q} q(q - 1)|\eta_\phi(\hat{G} + b_\tau^{-1} \tau_* Z; \alpha_* b_\tau^{q-2} \tau_*^{2-q})|^{q-2}} \right] .
\]

According to Theorem C.2.2, $\alpha_* b_\tau^{q-2} \to 0$. This drives the term $\alpha_* \tau_*^{2-q} q(q - 1)|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-2} \to 0$ in the denominator. It is then straightforward to use DCT and show that
\[
\lim_{\epsilon \to 0} b_\tau^{2-2q} H_3 = \mathbb{E} |\hat{G}|^{2q-2} .
\]  
(C.2.17)

Finally, we discuss $H_2$. By using Stein’s lemma and after some algebraic calculations we have
\[
H_2 = -q \mathbb{E} \left[ Z |\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-1} \text{sign}(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q}) \right] \\
= -q(q - 1) \tau_* \mathbb{E} \left[ \frac{|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-2}}{(1 + \alpha_* \tau_*^{2-q} q(q - 1)|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-2})^3} \right] \\
- q^2 (q - 1) \alpha_* \tau_*^{3-q} \mathbb{E} \left[ \frac{|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{2q-4}}{(1 + \alpha_* \tau_*^{2-q} q(q - 1)|\eta_\phi(b_\tau \hat{G} + \tau_* Z; \alpha_* \tau_*^{2-q})|^{q-2})^3} \right] \\
\triangleq -q(q - 1) \tau_* H_4 - q^2 (q - 1) \alpha_* \tau_*^{3-q} H_5 .
\]
Now we bound $H_4$ and $H_5$. Due to exactly the same reason when we analyzing $H_3$, the denominator of $H_4$ and $H_5$ converges to 1. According to Lemma A.2.2(iii), we have

$$b_\epsilon^{2q}H_4 = \mathbb{E}\left[ \frac{|\eta_q(\tilde{G} + b_\epsilon^{-1}\tau_s Z; \alpha_s b_\epsilon^{1-2q}\tau_s^{1-2q})|^{q-2}}{(1 + \alpha_s \tau_s^{2-q}\eta_q(q-1)|\eta_q(b_\epsilon \tilde{G} + \tau_s Z; \alpha_s \tau_s^{2-q})|^{q-2})^3} \right] \to \mathbb{E}|	ilde{G}|^{q-2} \quad (C.2.18)$$

Similarly for $H_5$ we have that

$$b_\epsilon^{4-2q}H_5 = \mathbb{E}\left[ \frac{|\eta_q(\tilde{G} + b_\epsilon^{-1}\tau_s Z; \alpha_s b_\epsilon^{1-2q}\tau_s^{1-2q})|^{2q-4}}{(1 + \alpha_s \tau_s^{2-q}\eta_q(q-1)|\eta_q(b_\epsilon \tilde{G} + \tau_s Z; \alpha_s \tau_s^{2-q})|^{q-2})^3} \right] \to \mathbb{E}|	ilde{G}|^{2q-4} \quad (C.2.19)$$

From (C.2.13) and with some algebra we have

$$1 = \lim_{\epsilon \to 0} \frac{\epsilon\alpha_s^{3-q}H_2 + \epsilon\alpha_s^{4-2q}q^2H_3}{(1 - \epsilon)\tau_s^{2\epsilon}H_1} = \lim_{\epsilon \to 0} \frac{\epsilon\alpha_s^{3-q}\alpha_s^{-1}\tau_s^{2-2q}H_2 + \epsilon\alpha_s^{4-2q}q^2\tau_s^{2-2q}H_3}{(1 - \epsilon)\tau_s^q\alpha_s^{q-1}H_1}$$

$$= \frac{(q-1)q^{2q}}{\sigma^{2q-2}\mathbb{E}|Z|^{\frac{q}{2}}\epsilon^2} \lim_{\epsilon \to 0} \frac{\epsilon\alpha_s^{2q-2}}{\tau_s^{2q-2}} b_\epsilon^{2q-2}$$

where in the last step, we use the fact that $\alpha^{-1}b_\epsilon^{-2q}H_2 \to 0$ which is an implication of (C.2.18) and (C.2.19). We also used (C.2.16) and (C.2.17) to simplify the part involving $H_1$ and $H_3$. Overall, these give us that

$$\lim_{\epsilon \to 0} \epsilon\alpha_s^{2q-2} b_\epsilon^{2q-2} = \frac{\sigma^{2q-2}\mathbb{E}|Z|^{\frac{q}{2}}}{(q-1)q^{2q}} \mathbb{E}|	ilde{G}|^{2q-2} = C_s^{2q-2} \quad (C.2.20)$$

This proves the first claim of our theorem. The behavior of $\tilde{R}(\alpha_s, \epsilon, \tau_s)$ now follows once we combine (C.2.20) with Lemma C.2.6. In order to obtain the final form presented in Theorem 3.2.1, we need to substitute $C_s$ into (C.2.12) and simplify the expression.

\[ \square \]

### C.2.5 Case III - $\frac{b_\epsilon}{\sqrt{\epsilon^{1-q}}} \to c_r$ for $c_r \in (0, \infty)$

Since the proof is very similar to the one we presented in the last section, we only present the sketch of the proof, and do not discuss the details. We only emphasize on the major differences. First note that similar to what we had before

$$\tilde{R}_q(\alpha, \epsilon, \tau_s) = (1 - \epsilon)\tau_s^2\mathbb{E}\eta_q^2(Z; \alpha) + \epsilon\mathbb{E}\left[ \eta_q(b_\epsilon \tilde{G} + \tau_s Z; \alpha \tau_s^{2-q}) - b_\epsilon \tilde{G} \right]^2.$$
We can prove the following claims:

1. It is straightforward to prove that
\[
\lim_{\alpha \to \infty} \tilde{R}_q(\alpha, \epsilon, \tau^*) = \epsilon b_\tau^2 \mathbb{E}(\tilde{G})^2. \tag{C.2.21}
\]

2. We claim that 
\[
\alpha_{\ast}^{-\frac{1}{2-q}} b_\tau = O(1). \tag{C.2.22}
\]

The reasoning for (a) is similar to what we did in (C.2.6). We connect (C.2.21) and (C.2.22) through the optimality of \( \alpha_{\ast} \) to conclude that
\[
\epsilon b_\tau^2 \alpha_{\ast}^{\frac{2}{2-q}} \geq \Theta(1)
\]

Our claim then follows by substituting the relation \( \epsilon \sim b_\tau^{-\frac{2}{2-q}} \) into the above equation.

3. Given the previous case two scenarios can happen, each of which is discussed below:

- **Case I:** \( \alpha_{\ast}^{-\frac{1}{2-q}} b_\tau \to 0 \). Note that in this case, we have
\[
\lim_{\epsilon \to 0} \frac{\tilde{R}_q(\alpha_{\ast}, \epsilon, \tau^*)}{\epsilon b_\tau^2 \mathbb{E}(\tilde{G})^2} \geq \lim_{\epsilon \to 0} \frac{\mathbb{E}(\eta_q(\tilde{G} + b_\tau^{-1} \tau^* Z; \alpha_{\ast} b_\tau^{-2} \varpi^{-q}) - \tilde{G})^2}{\mathbb{E}(\tilde{G})^2} = 1,
\]

where the last step is a result of DCT and the assumption that \( \alpha_{\ast}^{-\frac{1}{2-q}} b_\tau \to 0 \). Note that the lower bound is achievable by \( \alpha = \infty \).

- **Case II:** \( \alpha_{\ast}^{-\frac{1}{2-q}} b_\tau = \Theta(1) \). Under this assumption, we have \( \alpha_{\ast} \epsilon^{\frac{(q-1)(2-q)}{2}} \to C \), where \( C > 0 \) is fixed. We will specify the optimal choice of \( C \) later. Furthermore,
\[
b_\tau \alpha_{\ast}^{-\frac{1}{2-q}} \to c_r C^{-\frac{1}{2-q}}.
\]

We remind the reader that \( c_r \triangleq \lim_{\epsilon \to 0} \frac{b_\tau}{\sqrt{\epsilon^{1-q}}} \in (0, \infty) \). Similar to the proof of Lemma C.2.3 we have
\[
\alpha_{\ast}^{\frac{2}{2-q}} \mathbb{E} \eta_q^2(Z; \alpha_{\ast}) \to q^{-\frac{2}{2-q}} \mathbb{E}|Z|^{\frac{2}{2-q}}. \tag{C.2.23}
\]
Also, with DCT we can show that, as $\epsilon \to 0$

\[
\alpha_*^{-\frac{2}{q\epsilon}} \mathbb{E}(\eta_q(b_\epsilon \tilde{G} + \tau_* Z; \alpha_\tau^{2-q}) - b_\epsilon \tilde{G})^2
\rightarrow \mathbb{E}(\eta_q(c_\tau C^{-\frac{1}{2\epsilon}} G; \sigma^{2-q}) - c_\tau C^{-\frac{1}{2\epsilon}} G)^2. \tag{C.2.24}
\]

Now we can characterize the risk accurately as

\[
\lim_{\epsilon \to 0} \epsilon^{q-2} \tilde{R}_q(\alpha_*, \epsilon, \tau_*) = \lim_{\epsilon \to 0} \epsilon^{q-2} \alpha_*^{-\frac{2}{q\epsilon}} \tau_*^{2-q} \alpha_*^{\frac{2}{q\epsilon}} \mathbb{E}(Z; \alpha_*)
+ \lim_{\epsilon \to 0} \epsilon^{q-1} \alpha_*^{-\frac{2}{q\epsilon}} \alpha_*^{\frac{2}{q\epsilon}} \mathbb{E}(\eta_q(b_\epsilon \tilde{G} + \tau_* Z; \alpha_\tau^{2-q}) - b_\epsilon \tilde{G})^2
\overset{(a)}{=} C^{-\frac{2}{q\epsilon}} \sigma^2 q^{-\frac{2}{q\epsilon}} \mathbb{E}|Z|^\frac{2}{q\epsilon} + \mathbb{E}(\eta_q(c_\tau G; C\sigma^{2-q}) - c_\tau G)^2 =: h(C)
\]

To obtain Equality (a), we have combined (C.2.23), (C.2.24) and the fact that $\epsilon^{q-2} \alpha_*^{-\frac{2}{q\epsilon}} \rightarrow C^{-\frac{2}{q\epsilon}}$, $\epsilon^{q-1} \alpha_*^{\frac{2}{q\epsilon}} \rightarrow C^{\frac{2}{q\epsilon}}$.

To get the optimal choice of $C$, we take the derivative with respect to $C$ and obtain

\[
h'(C) = -\frac{2}{q - 1} C^{-\frac{q+1}{q\epsilon}} \sigma^2 q^{-\frac{2}{q\epsilon}} \mathbb{E}|Z|^\frac{2}{q\epsilon}
+ 2\sigma^{2-q} \mathbb{E}[\eta_q(c_\tau G; C\sigma^{2-q}) - c_\tau G] \partial_2 \eta_q(c_\tau G; C\sigma^{2-q})
\overset{(i)}{=} -\frac{2}{q - 1} C^{-\frac{q+1}{q\epsilon}} \sigma^2 q^{-\frac{2}{q\epsilon}} \mathbb{E}|Z|^\frac{2}{q\epsilon}
+ 2C\sigma^{4-2q} q^{2q} \mathbb{E}\left[\frac{|\eta_q(c_\tau G; C\sigma^{2-q})|^{2q-2}}{1 + C\sigma^{2-q}(q - 1)|\eta_q(c_\tau G; C\sigma^{2-q})|^{q-2}}\right]
\]

To obtain the last equality we have used Lemma A.2.2 parts (i), (ii), and (v). We would like to show that the optimal choice of $C$ is finite. Toward this goal, we characterize the limiting behavior of the ratio of the positive and negative terms in $\frac{dh(C)}{dC}$. First it is straightforward to see that
\[ \lim_{C \to 0} h'(C) = -\infty. \] Further we have

\[
\lim_{C \to \infty} C^{\frac{q}{q-1}} h'(C) = - \lim_{C \to \infty} \frac{2}{q-1} C^{\frac{1}{q-1}} \sigma^2 q^{-\frac{2}{q-1}} \mathbb{E}[Z]^\frac{2}{q-1}
\]

\[
+ 2\sigma^{q-2} q^2 \lim_{C \to \infty} C^{\frac{q}{q-1}} C \mathbb{E} \left[ \left| \eta_q(c_r, G; C \sigma^{2-q}) \right|^{q-2} \left( 1 + C \sigma^{2-q} q(q-1) \left| \eta_q(c_r, G; C \sigma^{2-q}) \right|^{q-2} \right) \right]
\]

\[
= 2\sigma^{q-2} q^2 \lim_{C \to \infty} \mathbb{E} \left[ \left| c_r G \right| - \left| \eta_q(c_r, G; C \sigma^{2-q}) \right| \frac{\eta_q(c_r C^{-\frac{1}{q-1}} G; \sigma^{2-q})^{q-2}}{1 + \sigma^{2-q} q(q-1) \left| \eta_q(c_r C^{-\frac{1}{q-1}} G; \sigma^{2-q}) \right|^{q-2}} \right] \frac{\eta_q(c_r G; C \sigma^{2-q})^{q-2}}{q^{q-2} \mathbb{E} |c_r G|^{q-1}} \cdot \mathbb{E} |c_r G|^{q-1} > 0
\]

where in the last step we used the fact that \( |\eta_q(c_r C^{-\frac{1}{q-1}} G; \sigma^{2-q})| \to 0 \) as \( C \to \infty \). We should finally emphasize that, since \( \lim_{C \to \infty} h(C) \) equals the risk of \( \lim_{\epsilon \to 0} \epsilon^{q-2} \tilde{R}_q(\alpha, \epsilon, \sigma) \) when \( \alpha_*^{-\frac{1}{q-1}} b_\epsilon \to 0 \), we conclude that \( \alpha_* \epsilon^{(q-1)(2-q)} \to C_* \), where \( C_* \) is the minimizer of \( h(C) \).

**C.2.6 \( q \geq 2 \)**

In this part, we prove the rate for \( q \geq 2 \) in the nearly black object model. The proof when \( \sqrt{\epsilon} b_\epsilon = o(1) \) can be simply obtained according to the previous proof for \( q < 2 \). When \( \sqrt{\epsilon} b_\epsilon = \Theta(1) \), a slightly longer argument is involved.

\( \sqrt{\epsilon} b_\epsilon = o(1) \) In this case, we have \( \tau_* \to \sigma \) according to Lemma C.2.1. Using the same argument as the start of Section C.2.1, we know \( \alpha_* \to \infty \). Blessed by the condition \( q \geq 2 \), we know \( b_\epsilon = o(\epsilon^{\frac{2-q}{2}}) \) and \( \alpha_* b_\epsilon^{q-2} \to \infty \). The conclusion of Lemma C.2.4 and Theorem C.2.1 simply follows and we have \( \tilde{R}(\alpha_*, \epsilon, \tau_*) \sim \epsilon b_\epsilon^2 \).

\( \sqrt{\epsilon} b_\epsilon = \Theta(1) \) Assume \( \lim_{\epsilon \to 0} \sqrt{\epsilon} b_\epsilon = c > 0 \). Let \( \lim_{\epsilon \to 0} \alpha_* = \alpha_0 \in [0, \infty] \) (we may focus on one of the convergent subsequences). The limit of the optimal \( \tau_*^2 \) is bounded in the sense that \( \sigma^2 \leq \lim_{\epsilon \to 0} \tau_*^2 \leq \lim_{\epsilon \to 0} \tau_*^2 \leq \sigma^2 + \frac{1}{\delta} c^2 \). Let us consider a convergent subsequence of \( \tau_* \) (since it is bounded). By using the state-evolution equation we
have
\[
\lim_{\epsilon \to 0} \tau_s^2 = \sigma^2 + \frac{1}{\delta} \lim_{\epsilon \to 0} \mathbb{E}[\eta_q(B + \tau_s Z; \alpha_s \tau_s^{2-q} - B)]^2
\]
\[
= \sigma^2 + \frac{1}{\delta} \lim_{\epsilon \to 0} \left[ (1 - \epsilon) \tau_s^2 \mathbb{E}[\eta_q^2(Z; \alpha_s)] + \epsilon b^2 \mathbb{E}[\eta_q(\bar{G} + \tau_s^{-1} Z; \alpha_s b^{-2} \tau_s^{2-q} - \bar{G})]^2 \right]
\]
\[
= \sigma^2 + \frac{1}{\delta} \lim_{\epsilon \to 0} \tau_s^2 \mathbb{E}[\eta_q^2(Z; \alpha_0)] + \frac{c^2}{\delta} \mathbb{E}[\eta_q(\bar{G}; \lim_{\epsilon \to 0} (\alpha_s b^{2-2} \lim_{\epsilon \to 0} \tau_s^{2-q} - \bar{G})]^2]
\]

Under the assumption $\delta < 1$, the right hand side is always larger than the left hand side when $\alpha_0 = 0$. This implies that $\alpha_0 > 0$.

When $q > 2$, we have $\lim_{\epsilon \to 0} \alpha_s b^{2-2} \to \infty$. This leads to the following result for $\tau_s^2$:

\[
\lim_{\epsilon \to 0} \tau_s^2 = \frac{\sigma^2 + \frac{c^2}{\delta}}{1 - \frac{1}{\delta} \mathbb{E}[\eta_q^2(Z; \alpha)]}
\]

The larger $\alpha_0$ is, the smaller $\tau_s^2$ is. Hence we have $\alpha_s \to \infty$ and $\tau_s^2 \to \sigma^2 + \frac{c^2}{\delta}$.

This gives us
\[
\bar{R}_q(\alpha_s, \epsilon, \tau_s) \to c^2, \quad \text{when} \quad q > 2.
\]

When $q = 2$, the above argument becomes invalid. However in this case $\eta_q(u; \chi) = \frac{u}{1+2\chi}$, leading to an explicit form of the optimal $\alpha_s$ and $\tau_s$. A careful calculation exhibits that

\[
\alpha_s = \frac{1}{4} \left( \frac{\sigma^2}{\mathbb{E}B^2} + \frac{1}{\delta} - 1 + \sqrt{\left( \frac{\sigma^2}{\mathbb{E}B^2} + \frac{1}{\delta} - 1 \right)^2 + \frac{4\sigma^2}{\mathbb{E}B^2}} \right)
\]
\[
= \frac{1}{4} \left( \frac{\sigma^2}{c^2} + \frac{1}{\delta} - 1 + \sqrt{\left( \frac{\sigma^2}{c^2} + \frac{1}{\delta} - 1 \right)^2 + \frac{4\sigma^2}{c^2}} \right)
\]

The corresponding limit of MSE can then be explicitly represented as

\[
\bar{R}_2(\alpha_s, \epsilon, \tau_s) = \frac{\delta \sigma^2 + 4\delta \alpha_s^2 c^2}{(1 + 2\alpha_s)^2 \delta - 1}
\]

This completes the proof.
C.3 Proof of Theorem 3.2.2

C.3.1 Roadmap of the proof

The roadmap of the proof is similar to the one presented in Section C.2.1. As we discussed there, the main goal is to characterize the behavior of \((\alpha^*, \tau^*)\) in (A.2.1) for \(q = 1\) with \(B\) replaced by \(b_\tilde{B}\), where \(\tilde{B} = (1 - \epsilon)\delta_0 + \epsilon p_\tilde{G}\).

Similar to the proof in Section C.2.1, we can again prove that as \(\epsilon \to 0\), (i) \(\tau^* \to \sigma\), and (ii) \(\alpha^* \to \infty\). For the sake of brevity we skip the proof of this claim. The rest of this proof is to obtain a more accurate statement about the behavior of \(\alpha^*\) and AMSE(1, \(\lambda_1^*\)). The optimal choice of \(\alpha\) depends on the relation between \(b_\epsilon\) and \(\epsilon\) in the following way:

- **Case I** - \(b_\epsilon = \omega(\sqrt{-\log \epsilon})\). Under this rate, we will prove that \(\lim_{\epsilon \to 0} \frac{\alpha_\epsilon}{\sqrt{-2\log \epsilon}} = 1\). We then use this result to show \(\lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^*)}{\epsilon b_\epsilon^2} = 2\sigma^2\). The proofs are presented in C.3.2.

- **Case II** - \(b_\epsilon = o(\sqrt{-\log \epsilon})\). If \(b_\epsilon = \omega(1)\), then \(\lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^*)}{\epsilon b_\epsilon^2} = \mathbb{E}\tilde{G}^2\). We prove this result in Section C.3.3.

- **Case III** - \(b_\epsilon = \Theta(\sqrt{-\log \epsilon})\). If \(\frac{b_\epsilon}{\sqrt{-2\log \epsilon}} \to c\), then \(\lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^*)}{-2\epsilon \log \epsilon} = \mathbb{E}(\eta_1(c\tilde{G}; \sigma) - c\tilde{G})^2\). This claim is proved in Section C.3.4.

C.3.2 Case I - \(b_\epsilon = \omega(\sqrt{-\log \epsilon})\)

Before we start, we would like to remind our reader of the definition of \(\tilde{R}\) in (C.2.1). We will study the behavior of \(\tilde{R}\) as \(\epsilon \to 0\) to obtain the rate of AMSE(1, \(\lambda_1^*\)). Similar to the procedures in Section C.2.1, we characterize the rate of \(\alpha^*\) in several steps: First we describe the behavior of the AMSE for a specific choice of \(\alpha^*_*\). The suboptimality of this special choice then narrow down the scope of the optimal \(\alpha^*_*\). Finally, this information about \(\alpha^*_*\) enables us to accurately analyze the derivative of the risk with respect to \(\alpha\) and the increasing rate of \(\alpha^*_*\).
Lemma C.3.1. Suppose that \( b_\epsilon = \omega(\sqrt{-\log \epsilon}) \). If \( \alpha = \sqrt{-2 \log \epsilon} \), then

\[
\lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha, \epsilon, \tau_*)}{-\epsilon \log \epsilon} = 2\sigma^2. \tag{C.3.1}
\]

Proof. Recall the expansions of \( R_1(\alpha, \tau) \) in (C.1.1), we have the following expansion for \( \tilde{R}_1(\alpha, \epsilon, \tau_*) \).

\[
\tilde{R}_1(\alpha, \epsilon, \tau_*) = (1 - \epsilon) \tau_*^2 \mathbb{E} \eta_1^2(Z; \alpha) + \epsilon \mathbb{E}[\eta_1(b_\epsilon \tilde{G} + \tau_* Z; \alpha \tau_*) - b_\epsilon \tilde{G} - \tau_* Z]^2
- \epsilon \tau_*^2 + 2\epsilon \tau_*^2 \mathbb{E}[\partial_1 \eta_1(b_\epsilon \tilde{G} + \tau_* Z; \alpha \tau_*)] = \tau_*^2(F_1 + F_2 - F_3 + F_4). \tag{C.3.2}
\]

As what we pointed out in (C.1.3), \( F_1 = 2(1 - \epsilon)[(1 + \alpha_*^2) \Phi(-\alpha_* - \alpha_* \phi(\alpha_*))] \). Since \( \alpha = \sqrt{-2 \log \epsilon} \to \infty \), (A.1.1) implies that

\[
\lim_{\epsilon \to 0} \frac{F_1}{4\phi(\alpha)/\alpha^3} = 1. \tag{C.3.3}
\]

To calculate \( F_2 \) we note that \(|\eta_1(b_\epsilon \tilde{G} + \tau_* Z; 1) - b_\epsilon \tilde{G} - \tau_* Z| \leq 1\) and \( \tau_* \to \sigma \). By using DCT and the fact that \( \frac{b}{\alpha} \to \infty \) we have

\[
\lim_{\epsilon \to 0} \frac{F_2}{\epsilon \alpha^2} = 1. \tag{C.3.4}
\]

It is straightforward to check that \(|\partial_1 \eta_1(b_\epsilon \tilde{G} + \tau_* Z; \alpha \tau_*)| < 1\), these give us that

\[
F_3 = O(\epsilon), \quad F_4 = O(\epsilon). \tag{C.3.5}
\]

By combining (C.3.2), (C.3.3), (C.3.4), and (C.3.5) we obtain (C.3.1). \qed

Our next lemma provides a more refined information about \( \alpha_* \).

Lemma C.3.2. For \( b_\epsilon = \omega(\sqrt{-\log \epsilon}) \) there exists a \( c \in [0, 1] \), such that

\[
\lim_{\epsilon \to 0} \frac{\sqrt{-2 \log \epsilon}}{\alpha_*} = c.
\]

Proof. From (C.3.2) we have

\[
\lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{\tilde{R}_1(\sqrt{-2 \log \epsilon}, \epsilon, \tau_*)} \geq \lim_{\epsilon \to 0} \frac{(1 - \epsilon) \tau_*^2 \mathbb{E} \eta_1^2(Z; \alpha_*)}{\tilde{R}_1(\sqrt{-2 \log \epsilon}, \epsilon, \tau_*)} = \lim_{\epsilon \to 0} \frac{2(1 - \epsilon) \phi(\alpha_*)}{-\alpha_*^2 \epsilon \log \epsilon}, \tag{C.3.6}
\]
where the second inequality is due to (C.3.1) and (C.3.3). Furthermore, if
\[ \lim_{\epsilon \to 0} \frac{\sqrt{-2 \log \epsilon}}{\alpha_*} > 1, \] then
\[ \lim_{\epsilon \to 0} \frac{2(1 - \epsilon)\phi(\alpha_*)}{-\alpha_*^2 \epsilon \log \epsilon} > \lim_{\epsilon \to 0} \frac{(1 - \epsilon)\phi(\alpha_*)}{\sqrt{2\epsilon(-\log \epsilon)^2}} \to \infty. \] (C.3.7)

This is in contradiction with the optimality of \( \alpha_* \).

**Lemma C.3.3.** For \( b_\epsilon = \omega(\sqrt{-\log \epsilon}) \), we have
\[ \lim_{\epsilon \to 0} \frac{b_\epsilon}{\alpha_*} = \infty. \]

**Proof.** We would like to prove this with contradiction. First, suppose that \( \lim_{\epsilon \to 0} \frac{b_\epsilon}{\alpha_*} = 0 \). Under this assumption, we have
\[ \lim_{\epsilon \to 0} \frac{\mathbb{E}[\eta_1(b_\epsilon \tilde{G} + \tau_* Z; \alpha_* \tau_*) - b_\epsilon \tilde{G}]^2}{\mathbb{E}(b_\epsilon^2 \tilde{G}^2)} = \lim_{\epsilon \to 0} \frac{\mathbb{E}[\eta_1(\tilde{G} + \frac{\tau_*}{b_\epsilon} Z; \frac{\alpha_*}{b_\epsilon} \tau_*) - \tilde{G}]^2}{\mathbb{E}(\tilde{G}^2)} = 1, \] (C.3.8)
where to obtain the last equality we have used DCT. Now we have
\[ \lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{\tilde{R}_1(\sqrt{-2 \log \epsilon}, \epsilon, \tau_*)} \geq \lim_{\epsilon \to 0} \frac{\epsilon \mathbb{E}[\eta_1(b_\epsilon \tilde{G} + \tau_* Z; \alpha_* \tau_*) - b_\epsilon \tilde{G}]^2}{\tilde{R}_1(\sqrt{-2 \log \epsilon}, \epsilon, \tau_*)} \overset{(a)}{=} \lim_{\epsilon \to 0} \frac{\epsilon b_\epsilon^2 \mathbb{E}(\tilde{G}^2)}{-\epsilon \log \epsilon} = \infty. \] (C.3.9)
Equality (a) is due to (C.3.8), and the last equality is in contradiction with the optimality of \( \alpha_* \). Similarly, we can show that if \( \lim_{\epsilon \to 0} \frac{b_\epsilon}{\alpha_*} = c \) \( (c < \infty) \), then
\[ \lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{\tilde{R}_1(\sqrt{-2 \log \epsilon}, \epsilon, \tau_*)} = \infty, \] which is again in contradiction with the optimality of \( \alpha_* \). For brevity we skip this proof.

**Theorem C.3.1.** If \( b_\epsilon = \omega(\sqrt{-\log \epsilon}) \), then
\[ \lim_{\epsilon \to 0} \frac{\alpha_*}{\sqrt{-2 \log \epsilon}} = 1. \] (C.3.10)

**Proof.** We analyze the derivative of the risk with respect to \( \alpha \). Recall the form of
\[
\frac{\partial R_1(\alpha, \tau)}{\partial \alpha} \quad \text{in (C.1.4)}, \quad \text{we have}
\]

\[
\frac{1}{\tau_*^2} \frac{\partial R_1^2(\alpha, \tau)}{\partial \alpha} \bigg|_{\alpha = \alpha_*} = 2(1 - \epsilon) \left[ -\phi(\alpha_*) + \alpha_* \Phi(-\alpha_*) \right] := D_1
\]

\[
+ \epsilon \mathbb{E} \left[ \alpha_* \Phi \left( \frac{|b_\tau G|}{\tau_*} - \alpha_* \right) - \phi(\alpha_* - \frac{|b_\tau G|}{\tau_*}) \right] := D_2
\]

\[
+ \epsilon \mathbb{E} \left[ \alpha_* \Phi \left( -\frac{|b_\tau G|}{\tau_*} - \alpha_* \right) - \phi(\alpha_* + \frac{|b_\tau G|}{\tau_*}) \right] := D_3
\]

\[
= (C.3.11)
\]

Since \( \alpha_* \to \infty \), similar calculations as the one presented (C.3.3) lead to the rate for \( D_1 \); On the other hand, according to Lemma C.3.3, \( b_\epsilon / \alpha_* \to \infty \), This gives us the rate for \( D_2 + D_3 \). Overall we have

\[
\lim_{\epsilon \to 0} \frac{D_1}{\phi(\alpha_*)/\alpha_*^2} \to -1, \quad \lim_{\epsilon \to 0} \frac{D_2 + D_3}{\alpha_*} \to 1. \quad \text{(C.3.12)}
\]

Hence, by combining (C.3.11) and (C.3.12), we have

\[
\lim_{\epsilon \to 0} \frac{2\phi(\alpha_*)/\alpha_*^2}{\epsilon \alpha_*} = \lim_{\epsilon \to 0} \frac{-\frac{\epsilon D_2 + \epsilon D_3}{\alpha_*}}{D_1 \frac{\alpha_*^2}{\phi(\alpha_*)}} = 1.
\]

By taking logarithm, \( \lim_{\epsilon \to 0} -\frac{\alpha_*^2}{2} - 3 \log(\alpha) - \log(\epsilon) = 0 \). Since \( \alpha \to \infty \), \( \lim_{\epsilon \to 0} -\frac{1}{2} - \frac{\log \epsilon}{\alpha^2} = 0 \).

Combining Lemma C.3.1 and Theorem C.3.1 proves \( \lim_{\epsilon \to 0} \frac{\text{AMSE}(1, \lambda_1^\epsilon)}{-2\sigma^2 \epsilon \log \epsilon} = 1 \).

### C.3.3 Case II- \( b_\epsilon = o(\sqrt{-\log \epsilon}) \)

**Lemma C.3.4.** If \( b_\epsilon = \omega(1) \) and \( b_\epsilon = o(\sqrt{-\log \epsilon}) \), then there exists \( c \in [0, 1] \), such that

\[
\lim_{\epsilon \to 0} \sqrt{-2 \frac{\log \epsilon}{\alpha_*}} = c, \quad \text{(C.3.13)}
\]

**Proof.** Since \( \lim_{\alpha \to \infty} \tilde{R}_1(\alpha, \epsilon, \tau_*) = b_\epsilon^2 \mathbb{E} \tilde{G}^2 \), we have

\[
\lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{\epsilon b_\epsilon^2} \leq \mathbb{E} \tilde{G}^2. \quad \text{(C.3.14)}
\]
Note that $\tilde{R}_1(\alpha_*, \epsilon, \tau_*) \geq (1 - \epsilon)\tau_*^2 \mathbb{E} \eta_1^2(Z; \alpha_*)$. Hence,

$$\lim_{\epsilon \to 0} \frac{(1 - \epsilon)\tau_*^2 \mathbb{E} \eta_1^2(Z; \alpha_*)}{eb_\epsilon^2} \leq \mathbb{E} \tilde{G}^2.$$  

In (C.1.3) and (C.3.3) we prove that $\lim_{\epsilon \to 0} \frac{(1 - \epsilon)\mathbb{E} \eta_1^2(Z; \alpha_*)}{\phi(\alpha_*)\alpha_*^3} = 1$. Hence,

$$\lim_{\epsilon \to 0} \frac{\tau_*^2 \phi(\alpha_*)\alpha_*^{-3}}{eb_\epsilon^2} = \lim_{\epsilon \to 0} \frac{\sigma^2 \phi(\alpha_*)\alpha_*^{-3}}{eb_\epsilon^2} \leq \mathbb{E}(\tilde{G})^2. \quad (C.3.15)$$

It is straightforward to see that if (C.3.13) does not hold, then (C.3.15) will not be correct either. Hence, our claim is proved. \qed

**Theorem C.3.2.** If $b_\epsilon = \omega(1)$ and $b_\epsilon = o(\sqrt{-\log \epsilon})$, then

$$\lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{eb_\epsilon^2} = \mathbb{E}(\tilde{G})^2.$$  

In other words, its dominant term is the same as that of $\alpha = \infty$.

**Proof.** Note that

$$\tilde{R}_1(\alpha_*, \epsilon, \tau_*) = (1 - \epsilon)\tau_*^2 \mathbb{E} \eta_1^2(Z; \alpha_*) + eb_\epsilon^2 \mathbb{E} \left[ \eta_1 \left( \tilde{G} + b_\epsilon^{-1}\tau_*Z; b_\epsilon^{-1}\alpha_*\tau_* \right) - \tilde{G} \right]^2$$

According to Lemmas C.3.4 we know that $\alpha_*/b_\epsilon \to \infty$ and $\alpha_* \to \infty$. Hence, by using DCT we can prove that

$$\lim_{\epsilon \to 0} \mathbb{E} \left[ \eta_1 \left( \tilde{G} + b_\epsilon^{-1}\tau_*Z; b_\epsilon^{-1}\alpha_*\tau_* \right) - \tilde{G} \right]^2 = \mathbb{E}(\tilde{G})^2.$$  

Hence,

$$\lim_{\epsilon \to 0} \frac{\tilde{R}_1(\alpha_*, \epsilon, \tau_*)}{eb_\epsilon^2} \geq \lim_{\epsilon \to 0} \mathbb{E} \left[ \eta_1 \left( \tilde{G} + b_\epsilon^{-1}\tau_*Z; b_\epsilon^{-1}\alpha_*\tau_* \right) - \tilde{G} \right]^2 = \mathbb{E}(\tilde{G})^2.$$  

On the other hand, this lower bound is achieved for $\alpha = \infty$. Hence, the proof is complete. \qed
C.3.4 Case III- $b_\epsilon = \Theta(\sqrt{-\log \epsilon})$

**Theorem C.3.3.** If $\frac{b_\epsilon}{\sqrt{-2\log \epsilon}} \to c$, then

$$\lim_{\epsilon \to 0} \frac{\hat{R}(\alpha_*, \epsilon, \tau_*)}{-2c \log \epsilon} = \mathbb{E}(\eta_1(c\tilde{G}; \sigma) - c\tilde{G})^2.$$ 

**Proof.** Since the proof is very similar to the proof of Theorem C.3.1, we only present a proof sketch here. It is straightforward to check the following steps:

1. If $\alpha = \sqrt{-2\log \epsilon}$, then $\lim_{\epsilon \to 0} \frac{\hat{R}_1(\alpha_*, \epsilon, \tau_*)}{-2c \log \epsilon} = \mathbb{E}(\eta_1(c\tilde{G}; \sigma) - c\tilde{G})^2$. The proof is similar to the proof of Lemma C.3.1.

2. $\lim_{\epsilon \to 0} \frac{\sqrt{-2\log \epsilon}}{\alpha_*} = \frac{1}{\tilde{c}}$, where $\tilde{c} \in [1, \infty)$. The proof is exactly the same as the proof of Lemma C.3.2. $\tilde{c}$ can reach $\infty$?

3. For notational simplicity, suppose $\alpha = \tilde{c}\sqrt{-2\log \epsilon}$, where $\tilde{c} \geq 1$ (it is straightforward to show that $\lim_{\epsilon \to 0} \frac{\alpha}{\sqrt{-2\log \epsilon}}$ is not infinite. This will be clear from the rest of the proof too.). Recall the expansion of $\hat{R}_1(\alpha, \epsilon, \tau_*) = \tau_*^2(F_1 + F_2 - F_3 + F_4)$ in (C.3.2). It is first straightforward to confirm the following claims.

$$\frac{F_1}{-2c \log \epsilon} \to 0, \quad \frac{F_3}{-2c \log \epsilon} \to 0, \quad \frac{F_4}{-2c \log \epsilon} \to 0.$$ 

Furthermore, it is straightforward to show that

$$\frac{\tau_*^2 F_2}{-2c \log \epsilon} \to \mathbb{E}(\eta_1(c\tilde{G}; \tilde{c}\sigma) - c\tilde{G})^2.$$ 

Note that

$$(\eta_1(c\tilde{G}; \tilde{c}\sigma) - c\tilde{G})^2 = \min(\tilde{c}^2\sigma^2, c^2\tilde{G}^2) \geq \min(\sigma^2, c^2\tilde{G}^2),$$

where the last inequality is due to $\tilde{c} \geq 1$. Hence, for any $\alpha = \tilde{c}\sqrt{-2\log \epsilon}$, we have the following lower bound:

$$\lim_{\epsilon \to 0} \frac{\hat{R}_1(\alpha, \epsilon, \tau_*)}{-2c \log \epsilon} \geq \mathbb{E}\min(\sigma^2, c^2\tilde{G}^2).$$

Note that this lower bound is achieved by $\alpha = \sqrt{-2\log \epsilon}$. This completes the proof.

$\square$
C.4 Proof of Theorem 3.2.3

Before we discuss the proof of our main theorem, we mention a preliminary lemma that will later be used in our proof.

C.4.1 Preliminaries

Lemma C.4.1 (Laplace Approximation). Suppose $G$ is nonnegative and $\text{esssup}(G) = M$. Then for arbitrary nonnegative continuous function $f$ we have $\frac{E(f(G)e^{\alpha G})}{f(M)E(e^{\alpha G})} \to 1$ as $\alpha \to \infty$.

Proof of Lemma C.4.1. Let $G$ denote the distribution of $G$. For any small $\delta > 0$ and large $A$, let $0 < \delta_1 < \delta$, we have

$$\frac{\int_{G > M - \delta} e^{\alpha G} dG}{\int_{G \leq M - \delta} e^{\alpha G} dG} \geq \frac{\int_{G > M - \delta_1} e^{\alpha G} dG}{\int_{G \leq M - \delta} e^{\alpha G} dG} \geq \frac{e^{\alpha(M - \delta_1)} \mathbb{P}(G > M - \delta_1)}{e^{\alpha(M - \delta)} \mathbb{P}(G \leq M - \delta)} \geq \frac{\mathbb{P}(G > M - \delta_1)}{\mathbb{P}(G \leq M - \delta)} e^{\alpha(\delta - \delta_1)} > A$$

for large enough $\alpha$. This implies that $\frac{\int_{G > M - \delta} e^{\alpha G} dG}{E(e^{\alpha G})} \geq \frac{A}{A+1}$ for large $\alpha$. Notice the continuity of $f$, we have $|f(G) - f(M)| < \epsilon$ when $G$ is close enough to $M$ and $|f| \leq C$ on $[0, M]$. Thus we have

$$\frac{f(M) - \epsilon}{f(M)} \frac{A}{A+1} \leq \frac{\int_{G > M - \delta} f(G)e^{\alpha G} dG}{f(M)E(e^{\alpha G})} \leq \frac{E(f(G)e^{\alpha G})}{f(M)E(e^{\alpha G})} \leq \frac{\int_{G > M - \delta} f(G)e^{\alpha G} dG}{f(M) \int_{G > M - \delta} e^{\alpha G} dG} \leq \frac{f(M) + \epsilon}{f(M)} + \frac{C}{Af(M)}$$

This holds for arbitrary $\epsilon, \delta$ and $A$. Our conclusion follows.

C.4.2 $q = 1$: LASSO case

Recall the definition of $(\alpha_*, \tau_*)$ in (A.2.1). As we discussed in Section C.1, the main objective is to characterize the behavior of $\alpha_*$ and $\tau_*$ for large values of $\epsilon$. First, we
prove that \( \alpha_\epsilon \to \infty \) and \( \tau_\epsilon \to \sigma \) as \( \epsilon \to 0 \).

**Lemma C.4.2.** As \( \epsilon \to 0 \), we have \( \alpha_\epsilon \to \infty \) and \( \tau_\epsilon \to \sigma \).

**Proof.** First as \( \epsilon \to 0 \), we can pick the sequence of \( \alpha \to \infty \), noticing that \( \tau^2 = \frac{\delta \alpha^2}{\delta - \mathbb{E}[\eta_1(x/\tau + z; \alpha) - x/\tau]^2} \), the corresponding fixed point solution \( \tau^2 \to \sigma^2 \). Now suppose \( \lim_{\epsilon \to 0} \alpha_\epsilon < \infty \), then we consider a convergent subsequence \( \alpha_\epsilon \to \bar{\alpha} \). If \( \tau_\epsilon \to \infty \), then \( \lim_{\epsilon \to \infty} \tau_\epsilon^2 = \frac{\delta \bar{\alpha}^2}{\delta - \mathbb{E}[\eta_1(Z; \bar{\alpha})]^2} < \infty \) which forms a contradiction. Assume \( \tau_\epsilon \to \bar{\tau} < \infty \) (say we pick a subsequence), then it is not hard to see \( \bar{\tau}^2 = \sigma^2 + \frac{1}{\delta} \mathbb{E}[\eta_1(x + \bar{\tau} Z; \bar{\alpha} \bar{\tau}) - x]^2 > \sigma^2 \). This forms a contradiction with the optimality of \( \alpha_\epsilon \). \( \square \)

The next step is to obtain more accurate information about \( \alpha_\epsilon \) and \( \sigma_\epsilon \). **Lemma C.4.3** paves the way toward this goal.

**Lemma C.4.3.** For any given \( h(G) \) being a positive function over \( [0, +\infty) \) with \( \mathbb{E}h(|G|) < \infty \), there exists a constant \( \xi > 0 \) such that the following results hold for all sufficiently small \( \epsilon \),

\[
\mathbb{E}[h(|G|)\phi(\alpha_\epsilon - |G|/\tau_\epsilon) \cdot 1(\xi \leq |G| \leq \epsilon \sigma_\alpha)] > \mathbb{E}[h(|G|)\phi(\alpha_\epsilon - |G|/\tau_\epsilon) \cdot 1(|G| \leq \xi)].
\]

**Proof.** We present the proof for \( \alpha_\epsilon \). Let \( p(x) \) be the probability function of \( |G| \) and \( g_{\xi, \tau}(\alpha) = \int_0^\xi \int_0^\tau h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx \). For fixed \( \xi, \tau > 0 \), we calculate the derivative of \( g \) with respect to \( \alpha \):

\[
g'_{\xi, \tau}(\alpha) = \frac{\tau \sigma(h(\alpha)p(x)e^{\frac{\alpha x}{\tau}} - \frac{x^2}{2\tau^2} p(x))}{\int_0^\xi h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx} + \int_0^\xi \int_0^\tau h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx \cdot \int_0^\tau \frac{2}{\tau} h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx
\]
\[
- \int_0^\xi h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx \cdot \int_0^\tau \frac{2}{\tau} h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx
\]
\[
\geq \frac{\tau \sigma h(x)p(x)e^{\frac{\alpha x}{\tau}} - \frac{x^2}{2\tau^2} p(x)}{\int_0^\tau h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx} \cdot \frac{2}{\tau} h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx
\]
\[
- \int_0^\tau h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx \cdot \int_0^\tau \frac{2}{\tau} h(x)p(x)e^{\frac{\alpha x}{\tau}} \frac{x^2}{2\tau^2} dx
\]
\[
\geq 0.
\]
Hence \( g_{\xi,\tau}(\alpha) \) is an increasing function of \( \alpha \), for any fixed \( \xi, \tau > 0 \). Now we consider a small neighbor around \( \sigma : \mathcal{I}_\Delta = [\sigma - \Delta, \sigma + \Delta] \), where \( \Delta > 0 \) is small enough so that \( \sigma - \Delta > 0 \). We would like to show there exists a positive constant \( \xi_0 \) s.t. the following holds,

\[
g_{\xi_0,\tau}(1) > 1, \quad \forall \tau \in \mathcal{I}_\Delta. \quad (C.4.1)\]

To show the above, we first notice that \( \forall \tau \in \mathcal{I}_\Delta \)

\[
\int_{\xi}^{t_\sigma} h(x)p(x)e^{\frac{x}{\alpha\tau}} \frac{x^2}{2\tau^2} dx \geq \int_{\xi}^{t_\sigma} h(x)p(x)e^{\frac{x}{\sigma\Delta}} \frac{x^2}{2(\sigma - \Delta)^2} dx \quad (C.4.2)
\]

\[
\int_{0}^{\xi} h(x)p(x)e^{\frac{x}{\alpha\tau}} \frac{x^2}{2\tau^2} dx \leq \int_{0}^{\xi} h(x)p(x)e^{\frac{x}{\sigma\Delta}} \frac{x^2}{2(\sigma + \Delta)^2} dx. \quad (C.4.3)
\]

Moreover, we can easily pick a small constant \( \xi_0 > 0 \) to satisfy

\[
\int_{\xi_0}^{t_\sigma} h(x)p(x)e^{\frac{x}{\alpha\tau}} \frac{x^2}{2(\sigma - \Delta)^2} dx > \int_{0}^{\xi_0} h(x)p(x)e^{\frac{x}{\sigma\Delta}} \frac{x^2}{2(\sigma + \Delta)^2} dx,
\]

which together with \( (C.4.2)(C.4.3) \) proves \( (C.4.1) \). Since \( g_{\xi_0,\tau}(\alpha) \) is monotonically increasing, we have

\[
g_{\xi_0,\tau}(\alpha_*) > 1, \forall \tau \in \mathcal{I}_\Delta.
\]

Since \( \tau^* \to \sigma \), we know when \( \epsilon \) is small enough, \( \tau^* \in \mathcal{I}_\Delta \). This implies that \( g_{\xi_0,\tau^*}(\alpha_*) > 1 \). It is straightforward to use this inequality to derive the result for \( \alpha_* \).

The next Lemma obtains a simple equation between \( \alpha_* \) and \( \tau_* \). This equation will be later used to obtain an accurate characterization of \( \alpha_* \).

**Lemma C.4.4.** Assume there exists a constant \( c > 0 \) such that the tail of \( G \) satisfies

\[
P(|G| \geq a) \leq e^{-c(a^2 - b^2)} \quad \text{for } a > b > 0.
\]

Then there exists a \( 0 < t_* < 1 \) such that for any given \( 1 > t > t_* \), the following holds

\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ \frac{(\alpha_*)^2 |G|}{\alpha_* \tau_* - |G|} \exp \left( \frac{\alpha_* |G|}{\tau_*} - \frac{G^2}{2\tau_*^2} \right) \mathbb{1}(|G| \leq t \alpha_* \tau_*) \right] = 2. \quad (C.4.4)
\]
Proof of Lemma C.4.4. Since, $\alpha_*$ minimizes $\mathbb{E}(\eta_1(B + \tau_\ast Z; \alpha \tau_\ast) - B)^2$, we have
\[
\left. \frac{\partial R_1(\alpha, \tau_\ast)}{\partial \alpha} \right|_{\alpha = \alpha_*} = 0.
\]
By setting (C.1.4) as 0, we obtain that
\[
2(1 - \epsilon) \left( \phi(\alpha_*) - \alpha_\ast \Phi(-\alpha_*) \right) + \epsilon \mathbb{E} \left[ \phi(\alpha_\ast + \frac{|G|}{\tau_\ast}) - \alpha_\ast \Phi(-\frac{|G|}{\tau_\ast}) \right] = \epsilon \mathbb{E} \left[ \alpha_\ast \Phi\left( \frac{|G|}{\tau_\ast} - \alpha_* \right) - \phi(\alpha_* - \frac{|G|}{\tau_\ast}) \right].
\]
(C.4.5)

Since $\alpha_\ast \to \infty$, according to (A.1.1), $G_1 \sim \frac{\phi(\alpha_\ast)}{(\alpha_\ast)^2}$ and
\[
\mathbb{E} \left[ \frac{|G|}{\alpha_\ast \tau_\ast + |G|} \phi(\alpha_\ast + |G|/\tau_\ast) \right] \leq G_3 \leq \mathbb{E} \left[ \frac{|G|\phi(\alpha_\ast + |G|/\tau_\ast)}{\alpha_\ast |G| + |G|} + \frac{\alpha_\ast \phi(\alpha_\ast + |G|/\tau_\ast)}{\alpha_\ast + |G|/\tau_\ast} \right].
\]

Hence,
\[
\frac{|G_3|}{G_1} \leq \mathbb{E} \left[ \frac{\alpha_*^2 |G|}{\alpha_* \tau_\ast + |G|} \exp\left( -\frac{\alpha_* |G| - |G|^2}{2 \tau_\ast^2} \right) \right] + \mathbb{E} \left[ \frac{\alpha_*^3}{(\alpha_\ast + |G|/\tau_\ast)^3} \exp\left( -\frac{\alpha_* |G| - |G|^2}{2 \tau_\ast^2} \right) \right].
\]
(C.4.6)

Moreover, it is straightforward to see that
\[
\frac{(\alpha_\ast)^2 |G|}{\alpha_* \tau_\ast + |G|} \exp\left( -\frac{\alpha_* |G| - |G|^2}{2 \tau_\ast^2} \right) \leq \frac{\alpha_* |G|}{\tau_\ast} \exp\left( -\frac{\alpha_* |G|}{\tau_\ast} \right) \leq e^{-1}
\]

We can then apply DCT to conclude the first term on the right hand side of (C.4.6) goes to zero. Similar arguments work for the second term. Hence $G_3/G_1 \to 0$, as $\epsilon \to 0$. Hence, according to (C.4.5)
\[
\lim_{\epsilon \to 0} \frac{\epsilon \alpha_*^2 G_2}{\phi(\alpha_\ast)} = 2.
\]
(C.4.7)

Next we would like to simplify $G_2$. The idea is to approximate $\Phi(|G|/\tau_\ast - \alpha_\ast)$ by $\frac{1}{\alpha_* - |G|/\tau_\ast} \phi(\alpha_* - |G|/\tau_\ast)$, but since $|G|$ is not necessarily bounded, the approximation may not be accurate. Therefore, we first consider an approximation to a truncated version of $G_3$. More specifically, given a constant $0 < t < 1$, we focus on
\[
T \triangleq \mathbb{E} \left\{ [\alpha_* \Phi(|G|/\tau_\ast - \alpha_\ast) - \phi(\alpha_* - |G|/\tau_\ast)] \cdot \mathbb{1}(|G| \leq t \tau_\ast \alpha_\ast) \right\}.
\]
It is straightforward to confirm that
\[
-\mathbb{E}\left[ \frac{\alpha_s}{(\alpha_s - |G|/\tau_s)^3} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]
\leq T - \mathbb{E}\left[ \frac{|G|}{\alpha_s\tau_s - |G|} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right] \leq 0. \tag{C.4.8}
\]

To show $T$ has the same order as the gaussian tail approximation, we analyze the following ratio:
\[
\mathbb{E}\left[ \frac{\alpha_s}{(\alpha_s - |G|/\tau_s)^3} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]
\leq \frac{1}{\alpha_s\tau_s} \mathbb{E}\left[ |G|\phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]
\leq \frac{\tau_s}{\alpha_s(1-t)^3} \mathbb{E}\left[ |G|\phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]. \tag{C.4.9}
\]

According to Lemma C.4.3 and the fact $\tau_s > \sigma$, there exists $\xi > 0$ such that
\[
\mathbb{E}\left[ \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right] > \mathbb{E}\left[ \phi(\alpha_s - |G|/\tau_s) \cdot 1(\xi \leq |G| \leq t\sigma\alpha_s) \right]
> \frac{1}{2} \mathbb{E}\left[ \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right].
\]

Hence,
\[
\mathbb{E}\left[ |G|\phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right] \geq \mathbb{E}\left[ |G|\phi(\alpha_s - |G|/\tau_s) \cdot 1(\xi \leq |G| \leq t\tau_s\alpha_s) \right]
\geq \xi \mathbb{E}\left[ \phi(\alpha_s - |G|/\tau_s) \cdot 1(\xi \leq |G| \leq t\tau_s\alpha_s) \right] > \frac{\xi}{2} \mathbb{E}\left[ \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right].
\]

The above inequality together with (C.4.9) yields,
\[
\frac{\mathbb{E}\left[ \frac{\alpha_s}{(\alpha_s - |G|/\tau_s)^3} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]}{\mathbb{E}\left[ \frac{|G|}{\alpha_s\tau_s - |G|} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right]} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.
\]

This combined with (C.4.8) gives us,
\[
T \sim \mathbb{E}\left[ \frac{|G|}{\alpha_s\tau_s - |G|} \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| \leq t\tau_s\alpha_s) \right].
\]

Now we turn to analyzing the term
\[
G_2 - T = \mathbb{E}\left\{ -\alpha_s \Phi(|G|/\tau_s - \alpha_s) - \phi(\alpha_s - |G|/\tau_s) \cdot 1(|G| > t\tau_s\alpha_s) \right\}.
\]
We aim to show $G_2 - T$ has smaller order than $T$. Equivalently, we would like to prove
\[
\mathbb{E}\left\{ \left[ \alpha_* \Phi(\frac{|G|}{\tau_*} - \alpha_*) - \phi(\alpha_* - |G|/\tau_*) \right] \cdot 1(|G| > t\tau_* \alpha_*) \right\} = o(1). \tag{C.4.10}
\]

First note that
\[
\mathbb{E}\left\{ \left[ \alpha_* \Phi(\frac{|G|}{\tau_*} - \alpha_*) - \phi(\alpha_* - |G|/\tau_*) \right] \cdot 1(|G| > t\tau_* \alpha_*) \right\} = O(\alpha_* P(|G| > t\tau_* \alpha_*)�)
\]

Furthermore, for any $0 < \tilde{t} < t$, we have
\[
\mathbb{E}\left[ \frac{|G|}{\alpha_* \tau_* - |G|} \phi(\alpha_* - |G|/\tau_*) \cdot 1(|G| \leq t\tau_* \alpha_*) \right] > \mathbb{E}\left[ \frac{|G|}{\alpha_* \tau_* - |G|} \phi(\alpha_* - |G|/\tau_*) \cdot 1(\tilde{t}\tau_* \alpha_* \leq |G| \leq t\tau_* \alpha_*) \right] \\
\geq \mathbb{E}\left[ \frac{\tilde{t}\tau_* \alpha_*}{\tau_* \alpha_* - \tilde{t}\tau_* \alpha_*} \phi(\alpha_* - \tilde{t}\tau_* \alpha_*/\tau_*) \cdot 1(\tilde{t}\tau_* \alpha_* \leq |G| \leq t\tau_* \alpha_*) \right] \\
= \frac{\tilde{t}}{1 - \tilde{t}} \phi((1 - \tilde{t})\alpha_*) P(\tilde{t}\tau_* \alpha_* \leq |G| \leq t\tau_* \alpha_*)�
\]

So (C.4.10) would hold if we can show
\[
\frac{\alpha_* P(|G| > t\tau_* \alpha_*)}{\phi((1 - \tilde{t})\alpha_*) P(|G| > t\tau_* \alpha_*)} = o(1) \tag{C.4.11}
\]

Based on the condition we impose on the tail probability of $G$ in the statement of Lemma C.4.4, (C.4.11) is equivalent to
\[
\frac{\alpha_* P(|G| > t\tau_* \alpha_*)}{\phi((1 - \tilde{t})\alpha_*) P(|G| > t\tau_* \alpha_*)} = o(1).
\]

Using the tail probability condition again, we obtain
\[
\frac{\alpha_* P(|G| > t\tau_* \alpha_*)}{\phi((1 - \tilde{t})\alpha_*) P(|G| > t\tau_* \alpha_*)} = O(\alpha_* e^{\frac{(1 - \tilde{t})^2(\alpha_*)^2}{2}} \cdot e^{-c(t^2 - \tilde{t}^2)(\tau_* \alpha_*)^2}). \tag{C.4.12}
\]

Also note that
\[
c(t^2 - \tilde{t}^2)\sigma^2 - \frac{(1 - \tilde{t})^2}{2} = c(t^2 - 1)\sigma^2 + (1 - \tilde{t}) \left[ c(1 + \tilde{t})\sigma^2 - \frac{1}{2}(1 - \tilde{t}) \right].
\]
Hence we can choose \( t \) and \( \tilde{t} \) close to 1 so that \( c(t^2 - \tilde{t}^2)\sigma^2 - \frac{(1-\tilde{t})^2}{2} > 0 \) (Set \( t = 1 \) and \( \tilde{t} \) close enough to 1, the expression is negative. Conclusion follows by the continuity in \( t \)). Since \( \tau_* \to \sigma \), when \( \alpha_* \) is large enough, \( c(t^2 - \tilde{t}^2)(\tau_*)^2 - \frac{(1-\tilde{t})^2}{2} \) is bounded below away from zero. This implies the term on the right hand side of (C.4.12) is \( o(1) \). Putting the preceding results we derived so far, we have shown that

\[
G_2 \sim \mathbb{E}\left[ \frac{|G|}{\alpha_* \tau_* - |G|} \phi(\alpha_* - |G|/\tau_*) \cdot 1(|G| \leq t\tau_* \alpha_*) \right].
\]

This result together with (C.4.7) completes the proof. \( \square \)

Equation (C.4.4) can potentially enable us to obtain accurate information about \( \alpha_* \). The only remaining difficulty is the existence of \( \tau_* \) in this equation that can depend on \( \epsilon \). Our next lemma proves that (C.4.4) still holds, even if we replace \( \tau_* \) with \( \sigma \). Hence, we obtain a simple equation for \( \alpha_* \).

**Lemma C.4.5.** Under the conditions of Lemma C.4.4, we have

\[
\lim_{\epsilon \to 0} \mathbb{E}\left[ \frac{\alpha_*^2 |G|}{\alpha_* \sigma - |G|} \exp\left( \frac{\alpha_* |G|}{\sigma} - \frac{G^2}{2\sigma^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right] = 2. \tag{C.4.13}
\]

**Proof.** Firstly, it is not hard to confirm that the proof of Lemma C.4.4 works through if we consider the truncation \( 1(|G| \leq t\alpha_* \sigma) \), which leads to the following result

\[
\lim_{\epsilon \to 0} \mathbb{E}\left[ \frac{\alpha_*^2 |G|}{\alpha_* \tau_* - |G|} \exp\left( \frac{\alpha_* |G|}{\tau_*} - \frac{G^2}{2\tau_*^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right] = 2. \tag{C.4.14}
\]

Denoting \( h(z) = \mathbb{E}\left[ \frac{\alpha_*^2 |G|}{\alpha_* z - |G|} \exp\left( \frac{\alpha_* |G|}{z} - \frac{G^2}{2z^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right], \) we have

\[
\epsilon h(\tau_*) - \epsilon h(\sigma) = \epsilon h'(\tilde{\tau})(\tau_* - \sigma), \tag{C.4.15}
\]

where \( \tilde{\tau} \) is between \( \tau_* \) and \( \sigma \). We then calculate the derivative

\[
h'(z) = \mathbb{E}\left[ \left( - \frac{\alpha_*^2 |G|}{(\alpha_* z - |G|)^2} - \frac{\alpha_*^2 |G|^2}{z^3} \right) \exp\left( \frac{\alpha_* |G|}{z} - \frac{G^2}{2z^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right]
\]

Hence we have

\[
|h'(\tilde{\tau})| \leq \left( \frac{1}{\sigma^2(1-t)^2} + \frac{t\alpha_*^2}{\sigma^2} \right) \mathbb{E}\left[ \alpha_* |G| \exp\left( \frac{\alpha_* |G|}{\tilde{\tau}} - \frac{G^2}{2\tilde{\tau}^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right]
\]

\[
\leq \left( \frac{1}{\sigma^2(1-t)^2} + \frac{t\alpha_*^2}{\sigma^2} \right) \mathbb{E}\left[ \alpha_* |G| \exp\left( \frac{\alpha_* |G|}{\sigma} - \frac{G^2}{2\sigma^2} \right) \cdot 1(|G| \leq t\alpha_* \sigma) \right]. \tag{C.4.16}
\]
We have used $\tau_s \geq \hat{\tau} \geq \sigma$ in the above derivation. Moreover, according to (C.4.14), it is easily seen that

$$\Theta(1) = e \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\tau_s} - \frac{G^2}{2\tau_s^2} \right) \cdot 1(|G| \leq t \alpha_s \sigma) \right]$$

$$\geq e \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\tau_s} - \frac{G^2}{2\tau_s^2} \right) \cdot 1(\xi \leq |G| \leq t \alpha_s \sigma) \right]$$

$$\geq e \alpha_s \xi e^{\frac{\alpha_s \xi}{\tau_s^2}} P(\xi \leq |G| \leq t \alpha_s \sigma) = \Theta(e \alpha_s e^{\frac{\alpha_s \xi}{\tau_s^2}}), \quad (C.4.17)$$

where $\xi$ is a small constant such that $P(\xi \leq |G|) > 0$. This tells us that $e \alpha_s^2 \rightarrow 0$. Thus we have

$$0 \leq \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\sigma} - \frac{G^2}{2\tau_s^2} \right) \cdot 1(|G| \leq t \alpha_s \sigma) \right]$$

$$- \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\tau_s} - \frac{G^2}{2\tau_s^2} \right) \cdot 1(|G| \leq t \alpha_s \sigma) \right]$$

$$= e \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\tau_s} - \frac{G^2}{2\tau_s^2} \right) \left( \exp \left( \frac{\alpha_s |G|}{\sigma \tau_s} (\tau_s - \sigma) \right) - 1 \right) \cdot 1(|G| \leq t \alpha_s \sigma) \right]$$

$$\leq e \mathbb{E} \left[ \alpha_s |G| \exp \left( \frac{\alpha_s |G|}{\tau_s} - \frac{G^2}{2\tau_s^2} \right) \cdot 1(\xi \leq |G| \leq t \alpha_s \sigma) \right]$$

$$\approx 2 \left( \exp \left( \frac{t \alpha_s^2}{\tau_s^2} (\tau_s - \sigma) \right) - 1 \right)$$

Note that according to (A.2.1) we have

$$\frac{\delta(\tau_s^2 - \sigma^2)}{\tau_s^2} = \frac{1}{\tau_s^2} \mathbb{E} (\eta_1 (B + \tau_s Z; \alpha_s \tau_s) - B)^2. \quad (C.4.18)$$

By canceling terms in (C.1.2) using (C.1.4)=0, we also have

$$\frac{1}{\tau_s^2} \mathbb{E} (\eta_1 (B + \tau_s Z; \alpha_s \tau_s) - B)^2$$

$$= 2(1 - \epsilon) \Phi(-\alpha_s) + e \mathbb{E}_G \left[ \left( 1 - \frac{G^2}{\tau_s^2} \right) \Phi \left( \frac{G}{\tau_s} - \alpha_s \right) \right.$$

$$\left. + \left( 1 - \frac{G^2}{\tau_s^2} \right) \Phi \left( \frac{-G}{\tau_s} - \alpha_s \right) - \frac{G}{\tau_s} \phi(\alpha_s - \frac{G}{\tau_s}) + \frac{G}{\tau_s} \phi(\alpha_s + \frac{G}{\tau_s}) + \frac{G^2}{\tau_s^2} \right]$$

$$\leq \frac{2 \phi(\alpha_s)}{\alpha_s} + \epsilon O(1). \quad (C.4.19)$$
By combining (C.4.18) and (C.4.19), we have
\[ 2\alpha^2_*(\tau_* - \sigma) \leq 2\phi(\alpha_*)\alpha_* + \epsilon\alpha^2_*O(1) \to 0 \]

This shows
\[ \epsilon\Ex[\alpha_*|G]\exp\left(\frac{\alpha_*|G|}{\sigma} - \frac{G^2}{2\tau^2_*}\right) : 1(|G| \leq t\alpha_*\sigma) = \Theta(1). \quad (C.4.20) \]

By combining (C.4.15), (C.4.16), (C.4.17), (C.4.20) and (C.4.14) we have \( \epsilon h(\tau_*) - \epsilon h(\sigma) = o(1) \), which completes the proof.

Based on Lemma C.4.5, we can build the following explicit convergence of \( \alpha \) w.r.t. \( \epsilon \).

**Lemma C.4.6.** Assume esssup\( (G) = M < \infty \), then we have \( \frac{\alpha_*}{\log \frac{1}{\epsilon}} \to \frac{M}{\sigma} \).

**Proof.** Obviously the condition of Lemma C.4.5 is satisfied when \( G \) is bounded. It is then easy to see that (C.4.13) becomes
\[ \lim_{\epsilon \to 0} \frac{\epsilon\alpha_*}{\sigma} \Ex[G] \exp\left(\frac{\alpha_*|G|}{\sigma} - \frac{G^2}{2\sigma^2}\right) = 2 \]

By Lemma C.4.1, we have
\[ \lim_{\epsilon \to 0} \frac{\epsilon\alpha_*}{\sigma} M e^{-\frac{M^2}{2\sigma^2}} \Ex[\phi(\tau_*)] = 2 \]

Some simple algebra proves \( \frac{\log \epsilon}{\alpha_*} + \frac{M}{\sigma} \to 0 \).

Now we are ready to establish the first part of Theorem 3.2.3. For \( G \) bounded as in Lemma C.4.6, by the first order condition, we have
\[ \lim_{\epsilon \to 0} \frac{\alpha^3_*}{\phi(\alpha_*)} \left( R - \Ex[x^2/\tau^2_*] \right) = \lim_{\epsilon \to 0} 4(1 - \epsilon)(1 + O(\alpha^{-2}_*)) \]
\[ + 2\epsilon\alpha_*\Ex\left[ - \frac{\alpha_*G}{\tau_*} - \frac{\alpha^2_*G}{2\tau^2_*} - \frac{\alpha^2_*G}{2\tau^2_*} \right] = \lim_{\epsilon \to 0} 4 - 2\epsilon\alpha_*\Ex\left( \frac{G}{\tau_*} - \frac{G^2}{2\tau^2_*} \right) = 0 \]

Using the divergent speed of \( \alpha_* \) derived in Lemma C.4.6, we can obtain that
\[ \text{AMSE}_1^* = \Ex[x^2] + \alpha^{-3}_*\phi(\alpha_*)o(1) = \Ex[x^2] + o(\epsilon^{\frac{\sigma^2}{2M^2}\log \frac{1}{\epsilon}}(1 + o(1))) = \Ex[x^2] + o(\epsilon^k), \quad \forall k \in \mathbb{N}. \]
C.4.3 $q > 1$

Again recall (A.2.1) which defines $(\alpha_s, \tau_s)$. Similar to the proof of Lemma C.4.2 we can show $\alpha_s \rightarrow \infty$ and $\tau_s \rightarrow \sigma$. The next step is to get more accurate info about $\alpha_s$.

**Lemma C.4.7.** If all the moments of $B$ are bounded, then

$$
\lim_{\epsilon \to 0} \frac{\alpha_s}{\epsilon^{1-q}} = \frac{1}{q} \left( \frac{\sigma \mathbb{E}|Z|^{\frac{2}{q-1}}}{\mathbb{E}[(G/\sigma + Z)^{\frac{1}{q-1}} \text{sign}(G/\sigma + Z)]} \right)^{q-1}.
$$

**Proof.** From $u - \eta_q(u, \alpha) = \alpha q \text{sgn}(u) |\eta_q(u, \alpha)|^{q-1}$ and $\eta_q \to 0$ as $\alpha \to \infty$, we have $\lim_{\alpha \to \infty} \alpha |\eta_q(u, \alpha)|^{q-1} = \frac{|u|}{q}$. Since $\alpha_s$ is optimal, the first order optimality condition yields

$$
\epsilon \mathbb{E} \left[ \frac{q|\eta_q(G/\tau + Z; \alpha_s)|^q}{1 + \alpha_s q(q-1)|\eta_q(G/\tau + Z; \alpha_s)|^{q-2}} \right] + (1 - \epsilon) \mathbb{E} \left[ \frac{q|\eta_q(Z; \alpha_s)|^q}{1 + \alpha_s q(q-1)|\eta_q(Z; \alpha_s)|^{q-2}} \right] = \epsilon \mathbb{E} \left[ \frac{G q \eta_q(G/\tau + Z; \alpha_s)|^{q-1} \text{sgn}(\frac{G}{\tau} + Z)}{1 + \alpha_s q(q-1)|\eta_q(G/\tau + Z; \alpha_s)|^{q-2}} \right].
$$

Denote the three parts inside $\mathbb{E}(\cdot)$ by $T_1, T_2, T_3$ respectively. Multiply each side by $\alpha_s^{\frac{q+1}{q-1}}$, and note that

$$
0 \leq \alpha_s^{\frac{q+1}{q-1}} T_1 = \frac{q\alpha_s^{\frac{1}{q-1}} |\eta_q(G/\tau + Z; \alpha_s)|^q}{\alpha_s^{\frac{1}{q-1}} + q(q-1)\alpha_s^{\frac{1}{q-1}} |\eta_q(G/\tau + Z; \alpha_s)|^{q-2}} \leq \frac{1}{q-1} \left| \alpha_s^{\frac{1}{q-1}} |\eta_q(G/\tau + Z; \alpha_s)|^2 \right| = \frac{1}{q-1} \left| \frac{G/\tau + Z - \eta_q(G/\tau + Z; \alpha_s)}{q} \right|^{\frac{2}{q-1}} \leq \frac{1}{q-1} \left| \frac{G/\sigma + Z}{q} \right|^{\frac{2}{q-1}} < \infty.
$$

The last step holds if we assume finite moments of all orders for $G$. Similar inequalities hold for $T_2, T_3$. Thus by DCT, we have

$$
\lim_{\epsilon \to 0} \epsilon \alpha_s^{\frac{q+1}{q-1}} \mathbb{E} T_1 = 0,
$$

$$
\lim_{\epsilon \to 0} \alpha_s^{\frac{q+1}{q-1}} \mathbb{E} T_2 = \frac{\mathbb{E}|Z|^\frac{2}{q-1}}{(q-1)q^{\frac{2}{q-1}}},
$$

$$
\lim_{\epsilon \to 0} \alpha_s^{\frac{q}{q-1}} \mathbb{E} T_3 = \frac{\mathbb{E}[G/\sigma + Z]^\frac{1}{q-1} \text{sgn}(G/\sigma + Z)}{\sigma(q-1)q^{\frac{1}{q-1}}},
$$

$$
\mathbb{E}|Z|^{\frac{2}{q-1}} 
$$
and
\[ \lim_{\epsilon \to 0} \epsilon \alpha_s^{\frac{1}{q-1}} = \frac{1}{q^{\frac{2}{q-1}}} \frac{\mathbb{E}|Z|^{\frac{2}{q-1}}}{\mathbb{E}[\frac{G}{\sigma} + Z]^{\frac{1}{q-1}} \sgn(\frac{G}{\sigma} + Z)}. \]

Now we prove the second part of Theorem 3.2.3. From (A.2.1) we have
\[ \mathbb{E}[\eta_q(B/\tau_s + Z; \alpha_s) - B/\tau_s]^2 = \mathbb{E} \eta_q^2(B/\tau_s + Z; \alpha_s) - \frac{2}{\tau_s} \mathbb{E}[B\eta_q(B/\tau_s + Z; \alpha_s)] + \frac{\mathbb{E}B^2}{\tau_s}. \]

Thus we have
\[ \frac{\text{AMSE}(q, \lambda_q^*)}{\sigma^2} - \epsilon \frac{\mathbb{E}G^2}{\tau_s^2} \]
\[ = c\mathbb{E} \eta_q^2(G/\tau_s + Z; \alpha_s) + (1 - \epsilon) \mathbb{E} \eta_q^2(Z; \alpha_s) - \frac{2\epsilon}{\tau_s} \mathbb{E}[G\eta_q(G/\tau_s + Z; \alpha_s)]. \]

Note that by DCT (similar argument as the ones mentioned in Lemma C.4.7), we have
\[ \lim_{\epsilon \to 0} \alpha_s^{\frac{2}{q-1}} \mathbb{E} \eta_q^2(G/\tau_s + Z; \alpha_s) = \lim_{\epsilon \to 0} q^{-\frac{2}{q-1}} \mathbb{E} \left[ \left| \frac{G}{\tau_s} + Z \right| - \left| \eta_q \left( \frac{G}{\tau_s} + Z; \alpha_s \right) \right| \right]^{\frac{2}{q-1}} \]
\[ = q^{-\frac{2}{q-1}} \mathbb{E} \left| \frac{G}{\sigma} + Z \right|^{\frac{2}{q-1}} \]

Similarly,
\[ \lim_{\epsilon \to 0} \alpha_s^{\frac{2}{q-1}} \mathbb{E} \eta_q^2(Z; \alpha_s) = \lim_{\epsilon \to 0} q^{-\frac{2}{q-1}} \mathbb{E}[|Z| - |\eta_q(Z; \alpha_s)|]^{\frac{2}{q-1}} = q^{-\frac{2}{q-1}} \mathbb{E}|Z|^{\frac{2}{q-1}}, \]

and finally,
\[ \alpha_s^{\frac{1}{q-1}} \mathbb{E}[G\eta_q(G/\tau_s + Z; \alpha_s)] = \alpha_s^{\frac{1}{q-1}} \mathbb{E}[G|\eta_q(G/\tau_s + Z; \alpha_s)|\sgn(G/\tau_s + Z)] \]
\[ = q^{-\frac{1}{q-1}} \mathbb{E}[G(|G/\tau_s + Z| - |\eta_q(G/\tau_s + Z; \alpha_s)|)^{\frac{1}{q-1}}\sgn(G/\tau_s + Z)] \]
\[ \rightarrow q^{-\frac{1}{q-1}} \mathbb{E}[G|G/\sigma + Z|^{\frac{1}{q-1}}\sgn(G/\sigma + Z)]. \]
Based on these information, we have
\[
\lim_{\epsilon \to 0} \frac{\text{AMSE}(q, \lambda^*_q) - \epsilon \mathbb{E}G^2}{\epsilon^2 \tau^2} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E} \eta^2_q(G/\tau_0 + Z; \alpha_*) + \frac{1}{\epsilon^2} \mathbb{E} \eta^2_q(Z; \alpha_*) - \frac{2}{\epsilon \tau_*} \mathbb{E} [G \eta_q(G/\tau_0 + Z; \alpha_*)]
\]
\[
= \mathbb{E}^2[G|G/\sigma + Z|^{1/\tau} \text{sgn}(G/\sigma + Z)] - \frac{2}{\sigma^2 \mathbb{E}|Z|^{2/\tau}} \mathbb{E}^2[G|G/\sigma + Z|^{1/\tau} \text{sgn}(G/\sigma + Z)]
\]
\[
= - \frac{\mathbb{E}^2[G|G/\sigma + Z|^{1/\tau} \text{sgn}(G/\sigma + Z)]}{\sigma^2 \mathbb{E}|Z|^{2/\tau}}.
\]

Hence, we have
\[
\frac{\text{AMSE}(q, \lambda^*_q) - \epsilon \mathbb{E}G^2}{\epsilon^2} = - \frac{\mathbb{E}^2[G|G/\sigma + Z|^{1/\tau} \text{sgn}(G/\sigma + Z)]}{\mathbb{E}|Z|^{2/\tau}}.
\]

## C.5 Proof of Theorem 3.3.1

### C.5.1 Preliminaries

Before we start the proof we discuss a useful lemma.

**Lemma C.5.1.** Consider a nonnegative random variable \(X\) with probability distribution \(\mu\) and \(\mathbb{P}(X > 0) = 1\). Let \(\xi > \zeta > 0\) be the points such that \(\mathbb{P}(X \leq \zeta) \leq \frac{1}{4}\) and \(\mathbb{P}(\zeta < X \leq \xi) \geq \frac{1}{4}\). Let \(a, b, c : \mathbb{R}_+ \to \mathbb{R}_+\) be three deterministic positive functions such that \(a(s), c(s) \to \infty\) as \(s \to \infty\). Then there exists a positive constant \(s_0\) depending on \(a, c, X\), such that when \(s > s_0\),
\[
\int_0^{a(s)} e^{b(s)x - \frac{x^2}{a(s)}} d\mu(x) \leq 3 \int_0^{a(s)} e^{b(s)x - \frac{x^2}{c(s)}} d\mu(x).
\]

**Proof.** For large enough \(s\) such that \(a(s) > \xi\),
\[
\int_\zeta^{a(s)} e^{b(s)x - \frac{x^2}{a(s)}} d\mu(x) \geq \int_\zeta^\xi e^{b(s)x - \frac{x^2}{a(s)}} d\mu(x) \geq e^{b(s)\zeta - \frac{\zeta^2}{a(s)}} \mathbb{P}(\zeta < X \leq \xi)
\]
\[
\geq e^{b(s)\zeta - \frac{\zeta^2}{a(s)}} \mathbb{P}(X \leq \zeta) \geq e^{-\frac{\zeta^2}{a(s)}} \int_0^\zeta e^{b(s)x - \frac{x^2}{c(s)}} d\mu(x).
\]

As a result we have the following inequality,
\[
\int_0^{a(s)} e^{b(s)x - \frac{x^2}{c(s)}} d\mu(x) \leq (1 + e^{\frac{\zeta^2}{c(s)}}) \int_\zeta^{a(s)} e^{b(s)x - \frac{x^2}{a(s)}} d\mu(x).
\]
For sufficiently large $s$ such that $e^{\xi^2/(1-s)} < 2$, the conclusion follows.

C.5.2 Roadmap

Recall that we have $(\alpha_\ast, \tau_\ast)$ in (A.2.1). As mentioned in Section C.1, we need to characterize $(\alpha_\ast, \tau_\ast)$ as $\sigma \to \infty$. Accordingly AMSE$(q, \lambda_\ast^q) = \delta(\tau_\ast^2 - \sigma^2)$. It is clear from (A.2.1) that $\tau_\ast \to \infty$ as $\sigma \to \infty$. However, to derive the second order expansion of AMSE$(q, \lambda_\ast^q)$ as $\sigma \to \infty$, we need to obtain the convergence rate of $\tau_\ast$. We will achieve this goal by first characterizing the convergence rate of the term $\min_{\alpha \geq 0} \mathbb{E}(\eta_q(B + \tau_\ast Z; \alpha \tau_\ast^2 - q) - B)^2$ as $\tau_\ast \to \infty$. We then use that result to derive the convergence rate of $\tau_\ast$ based on (A.2.1) and finally calculate AMSE$(q, \lambda_\ast^q)$. Since the proof techniques look different for $q = 1, 1 < q \leq 2, q > 2$, we prove the theorem for these three cases in Sections C.5.3, C.5.4 and C.5.5 respectively.

C.5.3 Proof of Theorem 3.3.1 for $q = 1$

As explained in the roadmap of the proof, the key step is to characterize the convergence rate of $\tau_\ast$. Towards this goal, we first derive the convergence rate of $\alpha_\ast(q)$ as $\tau \to \infty$ in Section C.5.3.1. We then bound the convergence rate of $R_q(\alpha_\ast(q), \tau)$ as $\tau \to \infty$ in Section C.5.3.2. This enables us to study the rate of $\tau_\ast$ when $\sigma \to \infty$, and derive the expansion of AMSE$(q, \lambda_\ast^q)$ as $\sigma \to \infty$ in Section C.5.3.3.

C.5.3.1 Deriving the convergence rate of $\alpha_\ast(q)$ as $\tau \to \infty$ for $q = 1$

We first prove $\alpha_\ast(q) \to \infty$ as $\tau \to \infty$ in the next lemma.

Lemma C.5.2. Recall the definition of $\alpha_\ast(q)$ in (A.2.3). Assume $\mathbb{E}|G|^2 < \infty$. Then, $\alpha_\ast(q) \to \infty$ as $\tau \to \infty$.

Proof. Suppose this is not true, then there exists a sequence $\{\tau_n\}$ such that $\alpha_\ast(\tau_n) \to \alpha_0 < \infty$ and $\tau_n \to \infty$, as $n \to \infty$. Notice that

$$|\eta_q(B/\tau_n + Z; \alpha_\ast(\tau_n))| \leq |B|/\tau_n + Z \leq |B| + Z,$$
for sufficiently large $n$. We can apply DCT to obtain

$$\lim_{n \to \infty} R_q(\alpha_q(\tau_n), \tau_n) = \mathbb{E} \eta_q^2(Z; \alpha_0) > 0.$$  

On the other hand, since $\alpha = \alpha_q(\tau_n)$ minimizes $R_q(\alpha, \tau_n)$

$$\lim_{n \to \infty} R_q(\alpha_q(\tau_n), \tau_n) \leq \lim_{n \to \infty} \lim_{\alpha \to \infty} R_q(\alpha, \tau_n) = 0.$$  

A contradiction arises.

Based on Lemma C.5.2, we can further derive the convergence rate of $\alpha_q(\tau)$.

**Lemma C.5.3.** If $G$ has a sub-Gaussian tail, then

$$\lim_{\tau \to \infty} \frac{\alpha_q(\tau)}{\tau} = C_0,$$

where $C = C_0$ is the unique solution of the following equation:

$$\mathbb{E} \left( e^{CG}(CG - 1) + e^{-CG}(-CG - 1) \right) = \frac{2(1 - \epsilon)}{\epsilon}.$$  

**Proof.** Since $\alpha = \alpha_q(\tau)$ minimizes $R_q(\alpha, \tau)$, we know $\partial_1 R_q(\alpha_q(\tau), \tau) = 0$. To simplify the notation, we will simply write $\alpha$ for $\alpha_q(\tau)$ in the rest of this proof. Rearranging the terms in (C.1.4) gives us

$$\frac{2(1 - \epsilon)}{\epsilon} = \mathbb{E} \frac{\alpha^2}{\phi(\alpha)} \left[ \alpha \Phi(\frac{|G| - \alpha}{\tau}) + \alpha \Phi(\frac{-|G| - \alpha}{\tau}) - \phi(\frac{|G| - \alpha}{\tau}) - \phi(\frac{|G| + \alpha}{\tau}) \right].$$

Fixing $t \in (0, 1)$, we reformulate the above equation in the following way:

$$\frac{2(1 - \epsilon)}{\epsilon} = \mathbb{E}[T(G, \alpha, \tau)I(|G| \leq t\tau\alpha)] + \mathbb{E}[T(G, \alpha, \tau)I(|G| > t\tau\alpha)]. \tag{C.5.1}$$

We now analyze the two terms on the right hand side of the above equation. Since $G$ has a sub-Gaussian tail, there exists a constant $\gamma > 0$ such that $\mathbb{P}(|G| > x) \leq e^{-\gamma x^2}$ for $x$ large. We can then have the following bound,

$$|\mathbb{E}[T(G, \alpha, \tau)I(|G| > t\tau\alpha)]| \leq \frac{\alpha^2}{\phi(\alpha)} (2\alpha + \sqrt{2/\pi}) \mathbb{P}(|G| > t\tau\alpha)$$

$$\leq \alpha^2 (2\sqrt{2\pi} + 2)e^{-(\gamma t^2\tau^2 - \frac{1}{4})\alpha^2} \to 0, \quad \text{as } \tau \to \infty,$$
where we have used the fact that $\alpha \to \infty$ as $\tau \to \infty$ from Lemma C.5.2. This result combined with (C.5.1) implies that as $\tau \to \infty$

$$\mathbb{E}[T(G, \alpha, \tau)\mathbb{I}(|G| \leq t\tau\alpha)] \to \frac{2(1-\epsilon)}{\epsilon}. \quad (C.5.2)$$

Moreover, using the tail approximation of normal distribution in (A.1.1) with $k = 3$, we have for sufficiently large $\tau$,

$$\mathbb{E}[T(G, \alpha, \tau)\mathbb{I}(|G| \leq t\tau\alpha)] \\
\leq \mathbb{E} \left[ \frac{\alpha}{\alpha - |G|/\tau} e^{\frac{\alpha|G|}{\tau} - \frac{\alpha^2}{2\tau^2}} \left( \frac{\alpha|G|}{\tau} - \frac{\alpha^2}{(\alpha - |G|/\tau)^2} + \frac{3\alpha^2}{(\alpha - |G|/\tau)^4} \right) \right]_{U_1(G,\alpha,\tau)} \\
+ \frac{\alpha}{\alpha + |G|/\tau} e^{\frac{-\alpha|G|}{\tau} - \frac{\alpha^2}{2\tau^2}} \left( -\frac{\alpha|G|}{\tau} - \frac{\alpha^2}{(\alpha + |G|/\tau)^2} + \frac{3\alpha^2}{(\alpha + |G|/\tau)^4} \right) \cdot \mathbb{I}(|G| \leq t\tau\alpha).$$

Similarly applying (A.1.1) with $k = 2$ gives us for large $\tau$

$$\mathbb{E}[T(G, \alpha, \tau)\mathbb{I}(|G| \leq t\tau\alpha)] \\
\geq \mathbb{E} \left[ \frac{\alpha}{\alpha - |G|/\tau} e^{\frac{\alpha|G|}{\tau} - \frac{\alpha^2}{2\tau^2}} \left( \frac{\alpha|G|}{\tau} - \frac{\alpha^2}{(\alpha - |G|/\tau)^2} \right) \right]_{L_1(G,\alpha,\tau)} \\
+ \frac{\alpha}{\alpha + |G|/\tau} e^{\frac{-\alpha|G|}{\tau} - \frac{\alpha^2}{2\tau^2}} \left( -\frac{\alpha|G|}{\tau} - \frac{\alpha^2}{(\alpha + |G|/\tau)^2} \right) \cdot \mathbb{I}(|G| \leq t\tau\alpha).$$

We claim based on the two bounds that $\lim_{\tau \to \infty} \frac{\alpha}{\tau} = C_1$ with $0 < C_1 < \infty$. Otherwise:

- If $C_1 = \infty$, there exists a sequence $\alpha_n/\tau_n \to \infty$ and $\tau_n \to \infty$, as $n \to \infty$. Since $|L_2(G, \alpha_n, \tau_n)| \leq e^{-\frac{\alpha_n|G|}{\tau_n}} \left( \frac{\alpha_n|G|}{\tau_n} + 1 \right) \leq 2$, we can apply DCT to obtain

  $$\lim_{n \to \infty} \mathbb{E}(L_2(G, \alpha_n, \tau_n)\mathbb{I}(|G| \leq t\tau_n\alpha_n)) = 0.$$ 

  Furthermore, we choose a positive constant $\zeta > 0$ satisfying the condition in
Lemma C.5.1 for the nonnegative random variable $|G|$. Then

$$\mathbb{E}(L_1(G, \alpha_n, \tau_n)\mathbb{I}(|G| \leq t\tau_n\alpha_n))$$

$$\geq \mathbb{E}\left[ e^{\frac{\alpha_n|G|}{\tau_n}} - \frac{3\alpha_n^2}{2\tau_n^2} \left( \frac{\alpha_n|G|}{\tau_n} - \frac{1}{(1-t)^3} \right) \mathbb{I}(|G| \leq t\tau_n\alpha_n) \right]$$

$$\geq \int_{\zeta < g \leq t\tau_n\alpha_n} e^{\frac{\alpha_n g}{\tau_n}} - \frac{3\alpha_n^2}{2\tau_n^2} \alpha_n g \, dF(g) - \int_{g \leq t\tau_n\alpha_n} \frac{1}{(1-t)^3} e^{\frac{\alpha_n g - \alpha_n^2}{2\tau_n^2}} \, dF(g)$$

$$\geq \left( \frac{\zeta \alpha_n}{\tau_n} - \frac{2}{(1-t)^3} \right) \int_{\zeta < g \leq t\tau_n\alpha_n} e^{\frac{\alpha_n g}{\tau_n}} - \frac{3\alpha_n^2}{2\tau_n^2} \alpha_n g \, dF(g)$$

$$\geq \left( \frac{\zeta \alpha_n}{\tau_n} - \frac{2}{(1-t)^3} \right) \int_{\zeta < g \leq t\tau_n\alpha_n} e^{-\frac{\alpha_n^2}{2\tau_n^2}} \, dF(g) \to \infty,$$

where we have used Lemma C.5.1 in (a). This forms a contradiction.

- If $C_1 = 0$, for large enough $\tau$ we have $\frac{\alpha}{\tau} < 1$ and then on $|G| \leq t\tau\alpha$,

$$|U_1(G, \alpha, \tau) + U_2(G, \alpha, \tau)| \leq \frac{2}{1-t} e^G \left[ G + \frac{1}{(1-t)^2} + \frac{3}{\alpha^2(1-t)^4} \right],$$

which is integrable since $G$ has sub-Gaussian tail. Hence we apply DCT to obtain as $\tau \to \infty$

$$\mathbb{E}[(U_1(G, \alpha, \tau) + U_2(G, \alpha, \tau))\mathbb{I}(|G| \leq t\tau\alpha)] \to -2$$

This forms another contradiction.

Similar to the above arguments, we can conclude that $\lim_{\tau \to \infty} \frac{\alpha}{\tau} = C_2 \in (0, \infty)$. Now that $\frac{\alpha}{\tau} = O(1)$, we can use DCT to obtain

$$\lim_{\tau \to \infty} \mathbb{E}\left[ \frac{\alpha}{\alpha \pm |G|/\tau} e^{\frac{\alpha |G|}{\tau}} - \frac{3\alpha^2}{(\alpha \pm |G|/\tau)^4} \mathbb{I}(|G| \leq t\tau\alpha) \right] = 0.$$

This result combined together with (C.5.2) and the upper and lower bounds on $\mathbb{E}[T(G, \alpha, \tau)\mathbb{I}(|G| \leq t\tau\alpha)]$ enables us to show

$$\lim_{\tau \to \infty} \mathbb{E}[(L_1(G, \alpha, \tau) + L_2(G, \alpha, \tau))\mathbb{I}(|G| \leq t\tau\alpha)] = \frac{2(1-\epsilon)}{\epsilon}.$$

Now consider a convergent sequence $\frac{\alpha_n}{\tau_n} \to C_1 \in (0, \infty)$ and $\tau_n \to \infty$ as $n \to \infty$. On $|G| \leq t\tau_n\alpha_n$ we can bound for large $n$

$$|L_1(G, \alpha_n, \tau_n) + L_2(G, \alpha_n, \tau_n)| \leq \frac{2}{1-t} e^{2C_1G} \left( 2C_1G + \frac{1}{(1-t)^2} \right),$$

where we have used Lemma C.5.1 in (a). This forms a contradiction.
which is again integrable. Thus DCT gives us

\[
\lim_{n \to \infty} E\left[ (L_1(G, \alpha_n, \tau_n) + L_2(G, \alpha_n, \tau_n)) \mathbb{I}(|G| \leq t \tau_n \alpha_n) \right] = E\left[ e^{C_1|G|}(C_1|G| - 1) + e^{-C_1|G|}(-C_1|G| - 1) \right].
\]

For \( C_2 \) the same equation holds. By calculating the derivative we can easily verify

\[ h(c) = e^{c|G|}(c|G| - 1) + e^{-c|G|}(-c|G| - 1), \]

as a function of \( c \) over \((0, \infty)\), is strictly increasing. This determines \( C_1 = C_2 \). Above all we have shown

\[ \frac{\alpha_n(\tau)}{\tau} \to C_0, \quad \text{as } \tau \to \infty, \]

where \( E\left[ e^{C_0G}(C_0G - 1) + e^{-C_0G}(-C_0G - 1) \right] = \frac{2(1-\epsilon)}{\epsilon}. \)

C.5.3.2 Bounding the convergence rate of \( R_1(\alpha_1(\tau), \tau) \) as \( \tau \to \infty \)

We state the main result in the next lemma.

**Lemma C.5.4.** If \( G \) has sub-Gaussian tail, then as \( \tau \to \infty \)

\[ R_1(\alpha_1(\tau), \tau) = \frac{\epsilon \mathbb{E}|G|^2}{\tau^2} + o\left( \frac{\phi(\alpha_1(\tau))}{\alpha_1^3(\tau)} \right). \]

**Proof.** For notational simplicity, we will use \( \alpha \) to denote \( \alpha_1(\tau) \) in the rest of the proof. Rearranging (C.1.2), we can write \( R_1(\alpha, \tau) \) in the following form:

\[
R_1(\alpha, \tau) = 2(1 - \epsilon)[(1 + \alpha^2)\Phi(-\alpha) - \alpha\phi(\alpha)] \\
+ \epsilon \mathbb{E}\left[ (1 + \alpha^2 - G^2/\tau^2)[\Phi(G/\tau - \alpha) + \Phi(-G/\tau - \alpha)] \right]_{S_1(G, \alpha, \tau)} \\
- (G/\tau + \alpha)\phi(\alpha - G/\tau) + (G/\tau - \alpha)\phi(\alpha + G/\tau) + G^2/\tau^2 \right]_{S_2(G, \alpha, \tau)}.
\]
Hence, we have
\[
\lim_{\tau \to \infty} \frac{\alpha^3}{\phi(\alpha)} \left( R_q(\alpha, \tau) - \frac{\epsilon \mathbb{E}|G|^2}{\tau^2} \right) = 2(1 - \epsilon) \lim_{\tau \to \infty} \frac{\alpha^3}{\phi(\alpha)} \left[ (1 + \alpha^2)\Phi(-\alpha) - \alpha \phi(\alpha) \right] \\
+ \epsilon \lim_{\tau \to \infty} \frac{\alpha^3}{\phi(\alpha)} \mathbb{E}[S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)]
\]
\[
\overset{(a)}{=} 4(1 - \epsilon) + \epsilon \lim_{\tau \to \infty} \frac{\alpha^3}{\phi(\alpha)} \mathbb{E}[S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)].
\]

We use the tail expansion (A.1.1) with \( k = 3, 4 \) to obtain (a). Since \(|x\phi(x)| \leq \frac{e^{-1/2}}{\sqrt{2\pi}}\), we have
\[
|S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)| \leq \frac{2e^{-1/2}}{\sqrt{2\pi}} + \frac{4\alpha}{\sqrt{2\pi}} + 2 \left( 1 + \alpha^2 + \frac{G^2}{\tau^2} \right).
\]

Moreover, it is not hard to use the sub-Gaussian condition \( \mathbb{P}(|G| > x) \leq e^{-\gamma x^2} \) to obtain
\[
\mathbb{E}(G^2 \mathbb{I}(|G| > t\tau \alpha)) = \int_0^{t\tau \alpha} 2x \mathbb{P}(G > t\tau \alpha)dx + \int_{t\tau \alpha}^{\infty} 2x \mathbb{P}(G > x)dx \\
\leq (t\tau \alpha)^2 e^{-\gamma t^2 \tau^2 \alpha^2} + \frac{1}{\gamma} e^{-\gamma t^2 \tau^2 \alpha^2},
\]
where \( t \in (0, 1) \) is a constant. Combining the last two bounds we can derive
\[
\frac{\alpha^3}{\phi(\alpha)} \mathbb{E}[(S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau))\mathbb{I}(|G| > t\tau \alpha)] \leq \alpha^3(2e^{-1/2} + 4\alpha + 2\sqrt{2\pi}(1 + \alpha^2))e^{-\gamma t^2 \tau^2 \alpha^2} + \\
\frac{2\sqrt{2\pi} \alpha^3}{\tau^2} (t^2 \tau^2 \alpha^2 + 1/\gamma) e^{-\gamma (t^2 \tau^2 \alpha^2 - \frac{1}{\gamma})} \to 0, \quad \text{as } \tau \to \infty.
\]

On the other hand, we can build an upper bound and lower bound for \(|S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)| \) on \(|G| \leq t\tau \alpha\) with the tail expansion (A.1.1) as we did in the proof of Lemma C.5.3. For both bounds we can argue they converge to the same limit as \( \tau \to \infty \) by using DCT and Lemma C.5.3. Here we give the details of using DCT for
the upper bound. Using (A.1.1) with \(k = 3\) we can obtain the upper bound,

\[
\frac{\alpha^3}{\phi(\alpha)} (S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)) \\
\leq \frac{\alpha^3 \phi(\alpha - G/\tau)}{\phi(\alpha)} \left[ \frac{2G^2/\tau^2 - 2\alpha G/\tau - 1}{(\alpha - G/\tau)^3} + \frac{3(1 + \alpha^2 - G^2/\tau^2)}{(\alpha - G/\tau)^5} \right] + \\
\frac{\alpha^3 \phi(\alpha + G/\tau)}{\phi(\alpha)} \left[ \frac{2G^2/\tau^2 + 2\alpha G/\tau - 1}{(\alpha + G/\tau)^3} + \frac{3(1 + \alpha^2 - G^2/\tau^2)}{(\alpha + G/\tau)^5} \right].
\]

It is straightforward to see that on \(\{|G| \leq t\tau \alpha\}\) for sufficiently large \(\alpha\), there exist three positive constants \(C_1, C_2, C_3\) such that the upper bound can be further bounded by \(\left[ C_1|G|^{1-1} + \frac{C_2|G|^{1+1}}{(1+\epsilon)^1} + \frac{C_3|G|^{1+1}}{(1+\epsilon)^1} \right] e^{C_3|G|}\), which is integrable by the condition that \(G\) has sub-Gaussian tail. Hence we can apply DCT to derive the limit of the upper bound. Similar arguments enable us to calculate the limit of the lower bound. By calculating the limits of the upper and lower bounds we can obtain the following result:

\[
\frac{\alpha^3}{\phi(\alpha)} \mathbb{E}[\{S_1(G, \alpha, \tau) + S_2(G, \alpha, \tau)\} \mathbb{I}(\{\{|G| \leq t\tau \alpha\}\)] \\
\to -2 \mathbb{E} \left(e^{C_0G}(C_0G - 1) + e^{-C_0G}(-C_0G - 1)\right) = -\frac{4(1 - \epsilon)}{\epsilon}.
\]

This completes the proof. \(\square\)

### C.5.3.3 Deriving the expansion of AMSE\((q, \lambda_q^*)\) for \(q = 1\)

We are now in the position to derive the result (3.3.1) in Theorem 3.3.1. As we explained in the roadmap, we know

\[
\text{AMSE}(q, \lambda_q^*) = \tau_*^2 R_q(\alpha_q(\tau_*), \tau_*) = \delta(\tau_*^2 - \sigma^2). \quad (C.5.4)
\]

First note that \(\tau_* \to \infty\) as \(\sigma \to \infty\) since \(\tau_* \geq \sigma\). According to Lemma C.5.4 and (C.5.4), we have

\[
\lim_{\sigma \to \infty} \frac{\sigma^2}{\tau_*^2} = \lim_{\tau_* \to \infty} \frac{\sigma^2}{\tau_*^2} = \lim_{\tau_* \to \infty} \left(1 - \frac{R_q(\alpha_q(\tau_*), \tau_*)}{\delta}\right) = 1. \quad (C.5.5)
\]
Furthermore, Lemma C.5.3 shows that
\[
\lim_{\sigma \to \infty} \frac{\alpha_q(\tau_s)}{\tau_s} = \lim_{\tau_s \to \infty} \frac{\alpha_q(\tau_s)}{\tau_s} = C_0. \tag{C.5.6}
\]
Combining Lemma C.5.4 with (C.5.4), (C.5.5), and (C.5.6) we obtain as \(\sigma \to \infty\),
\[
e^{-\frac{c^2}{2} \tau_s^2} (\text{AMSE}(q, \lambda^*_q) - \epsilon \mathbb{E}|G|^2) = e^{-\frac{c^2}{2} \tau_s^2} (R_q(\alpha_q(\tau_s), \tau_s) - \epsilon \mathbb{E}|G|^2) \tag{C.5.7}
\]
\[
= e^{-\frac{c^2}{2} \tau_s^2} e^{-\frac{\alpha_q^2(\tau_s)}{\tau_s^2}} o(1) = e^{-\frac{c^2}{2} \tau_s^2} e^{-\frac{\alpha_q^2(\tau_s)}{\tau_s^2}} \tau_s^2 (\alpha_q(\tau_s))^{-3} o(1) = o(1).
\]
We have used the fact \(0 < C < C_0\) to get the last equality.

### C.5.4 Proof of Theorem 3.3.1 for \(q \in (1, 2]\)

The basic idea of the proof for \(q \in (1, 2]\) is the same as that for \(q = 1\). We characterize the convergence rate of \(R_q(\alpha_q(\tau), \tau)\) in Section C.5.4.1. We can derive the expansion of \(\text{AMSE}(q, \lambda^*_q)\) in Section C.5.4.2.

#### C.5.4.1 Characterizing the convergence rate of \(R_q(\alpha_q(\tau), \tau)\) as \(\tau \to \infty\) for \(q \in (1, 2]\)

We first derive the convergence rate of \(\alpha_q(\tau)\) as \(\tau \to \infty\).

**Lemma C.5.5.** For \(q \in (1, 2]\), assume \(G\) has finite moments of all order. We have,
\[
\frac{\alpha_q(\tau)}{\tau^{2(q-1)}} \to \left( \frac{q - 1}{\eta_2 \mathbb{E}|Z|^q \mathbb{E} B^2 \mathbb{E}|Z|^q} \right)^{q-1}, \quad \text{as } \tau \to \infty.
\]

**Proof.** First note that Lemma C.5.2 holds for \(q \in (1, 2]\) as well. Hence \(\alpha_q(\tau) \to \infty\) as \(\tau \to \infty\). We aim to characterize its convergence rate. Since \(\eta_2(u; \chi) = \frac{u}{1+2\chi}\), the result can be easily verified for \(q = 2\). We will focus on the case \(q \in (1, 2)\). For notational simplicity, we will use \(\alpha\) to represent \(\alpha_q(\tau)\) in the rest of the proof. By the first order condition of the optimality, we have \(\partial_1 R_q(\alpha, \tau) = 0\), which can be further
written out:

\[
0 = \mathbb{E}[(\eta_q(B/\tau + Z; \alpha) - B/\tau)\partial_2\eta_q(B/\tau + Z; \alpha)]
\]

\[
= \mathbb{E} \left[ \frac{-q|\eta_q(B/\tau + Z; \alpha)|^q}{1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2}} \right]_{H_1}
\]

\[
+ \mathbb{E} \left[ \frac{Bq|\eta_q(B/\tau + Z; \alpha)|^{q-1}\text{sgn}(B/\tau + Z)}{\tau(1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2})} \right]_{H_2}
\]

(C.5.7)

where we have used Lemma A.2.2 part (v). We now analyze the two terms \(H_1\) and \(H_2\) respectively. Regarding \(H_1\) from Lemma A.2.2 part (ii) we have

\[
\frac{\alpha^{\frac{q+1}{2}} q|\eta_q(B/\tau + Z; \alpha)|^q}{1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2}} \leq \frac{\alpha^{\frac{1}{2}}|\eta_q(B/\tau + Z; \alpha)|^2}{q - 1}
\]

\[
= \frac{|B/\tau + Z| - |\eta_q(B/\tau + Z; \alpha)|}{q^{\frac{1}{q}}(q - 1)} \leq (|B| + |Z|)^{\frac{2}{q}} q^{-\frac{2}{q-1}}(q - 1), \text{ for } \tau \geq 1.
\]

Since \(G\) has finite moments of all orders, the upper bound above is integrable. Hence DCT enables us to conclude

\[
\lim_{\tau \to \infty} \alpha^{\frac{q+1}{2}} H_1 = \frac{\mathbb{E}|Z|^{\frac{2}{q-1}}}{q^{\frac{2}{q-1}}(1 - q)}.
\]

(C.5.8)

For the term \(H_2\), according to Lemma A.2.2 parts (ii)(iv) we can obtain

\[
H_2 = \frac{1}{\tau\alpha} \mathbb{E} \left[ \frac{B(B/\tau + Z - \eta_q(B/\tau + Z; \alpha))}{1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2}} \right]_{I_1}
\]

\[
+ \mathbb{E} \left[ \frac{\tau\alpha}{\tau^2\alpha} \frac{B^2}{1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2}} \right]_{I_2}
\]

\[
- \frac{1}{\tau\alpha} \mathbb{E} \left[ \frac{B\eta_q(B/\tau + Z; \alpha)}{1 + \alpha q(q - 1)|\eta_q(B/\tau + Z; \alpha)|^{q-2}} \right]_{I_3}.
\]

By a similar argument and using DCT, it is not hard to see that,

\[
\lim_{\tau \to \infty} \tau^2\alpha^{\frac{2}{q-1}} I_1 = \frac{\mathbb{E}B^2\mathbb{E}|Z|^{\frac{2}{q-1}}}{q^{\frac{2}{q-1}}(q - 1)}, \quad \lim_{\tau \to \infty} \tau\alpha^{\frac{q+1}{2}} I_3 = \frac{\mathbb{E}B\mathbb{E}|Z|^{\frac{2}{q-1}}\text{sgn}(Z)}{q^{\frac{2}{q-1}}(q - 1)} = 0.
\]

(C.5.9)
Regarding the term $I_2$, by using Stein’s lemma and Taylor expansion, we can obtain a sequel of equalities:

\[
I_2 = \frac{\mathbb{E}[B(Z^2 - 1)\eta_q(B/\tau + Z; \alpha)]}{\alpha\tau} = \frac{\mathbb{E}[B(Z^2 - 1)(\eta_q(Z; \alpha) + \partial_1\eta_q(\gamma B/\tau + Z; \alpha)B/\tau)]}{\alpha\tau} = \frac{\mathbb{E}[B^2(Z^2 - 1)\partial_1\eta_q(\gamma B/\tau + Z; \alpha)]}{\alpha\tau^2} = \frac{1}{\alpha\tau^2}\mathbb{E}\left[\frac{B^2(Z^2 - 1)}{1 + \alpha(q - 1)|\eta_q(\gamma B/\tau + Z; \alpha)|^{q-2}}\right]
\]

where the second step is simply due to Lemma A.2.2 part (i); $\gamma \in (0, 1)$ is a random variable depending on $B$ and $Z$. With a similar argument to verify the conditions of DCT we obtain

\[
\lim_{\tau \to \infty} \alpha\frac{q-1}{q-2}\tau^2 I_2 = \frac{(2 - q)\mathbb{E}B^2\mathbb{E}|Z|^{\frac{2-q}{q-1}}}{q\mathbb{E}|Z|^{\frac{2-q}{q-1}}(q - 1)^2}.
\] (C.5.10)

Finally, (C.5.7), (C.5.8), (C.5.9) and (C.5.10) together enable us to have as $\tau \to \infty$,

\[
\frac{\alpha}{\tau^{2(q-1)}} = \left[\frac{\alpha^{q+1}(I_3 - H_1)}{\tau^2\alpha^{q-1}(I_1 + I_2)}\right]^{q-1} \to \left(\frac{q - 1}{q^{\frac{1}{q-1}} \mathbb{E}|Z|^{\frac{2-q}{q-1}}}\right)^{q-1}
\]

We now characterize the convergence rate of $R_q(\alpha_q(\tau), \tau)$.

**Lemma C.5.6.** Suppose $1 < q \leq 2$ and $G$ has finite moments of all orders, then as $\tau \to \infty$,

\[
R_q(\alpha_q(\tau), \tau) = \frac{\epsilon\mathbb{E}|G|^2}{\tau^2} - \frac{\epsilon^2(\mathbb{E}|G|^2\mathbb{E}|Z|^{\frac{2-q}{q-1}})^2}{(q - 1)^2\mathbb{E}|Z|^{\frac{2-q}{q-1}}} \frac{1}{\tau^q} + o(1/\tau^4).
\]

**Proof.** It is straightforward to prove the result for $q = 2$. Now we only consider $1 < q < 2$. We write $\alpha$ for $\alpha_q(\tau)$ in the rest of the proof to simplify the notation. First we have

\[
R_q(\alpha, \tau) = \frac{\epsilon\mathbb{E}|G|^2}{\tau^2} = \mathbb{E}\eta_q^2(B/\tau + Z; \alpha) - 2\mathbb{E}[\eta_q(B/\tau + Z; \alpha)B/\tau] = \mathbb{E}\eta_q^2(B/\tau + Z; \alpha) - 2\mathbb{E}[\eta_q(Z; \alpha) + \partial_1\eta_q(\gamma B/\tau + Z; \alpha)B/\tau] = \mathbb{E}\eta_q^2(B/\tau + Z; \alpha) - 2\mathbb{E}[\partial_1\eta_q(\gamma B/\tau + Z; \alpha)B^2/\tau^2],
\] (C.5.11)
where we have used Taylor expansion in the second step and \( \gamma \in (0, 1) \) is a random variable depending on \( B, Z \). According to Lemma A.2.2 part (ii), for \( \tau \geq 1 \),

\[
\alpha^{2-\tau} \eta_q^2(B/\tau + Z; \alpha) = q^{2-\tau}(|B/\tau + Z| - |\eta_q(B/\tau + Z; \alpha)|)^{2-\tau} \leq q^{2-\tau}(|B| + |Z|)^{2-\tau}.
\]

The upper bound is integrable since \( G \) has finite moments of all orders. Hence we can apply DCT to obtain

\[
\lim_{\tau \to \infty} \alpha^{2-\tau} \eta_q^2(B/\tau + Z; \alpha) = q^{2-\tau} \mathbb{E}|Z|^{2-\tau}.
\]

(C.5.12)

We can follow a similar argument to use DCT to have

\[
\lim_{\tau \to \infty} \alpha^{2-\tau} \mathbb{E} \left[ \partial_1 \eta_q(\gamma B/\tau + Z; \alpha) B^2/\tau^2 \right] = q^{2-\tau} \mathbb{E}|Z|^{2-\tau}.
\]

(C.5.13)

where (a) holds due to Lemma A.2.2 part (iv); we have used Lemma C.5.5 and DCT to obtain (b). Finally, we put the results (C.5.11), (C.5.12), (C.5.13) and Lemma C.5.5 together to derive

\[
\lim_{\tau \to \infty} \tau^4 (R_q(\alpha, \tau) - \epsilon \mathbb{E}|G|^2/\tau^2)
\]

\[
= \lim_{\tau \to \infty} \frac{\tau^4}{\alpha^{2-\tau}} \cdot \left[ \lim_{\tau \to \infty} \alpha^{2-\tau} \mathbb{E} \eta_q^2(B/\tau + Z; \alpha) - 2 \lim_{\tau \to \infty} \alpha^{2-\tau} \mathbb{E} \left( \partial_1 \eta_q(\gamma B/\tau + Z; \alpha) B^2/\tau^2 \right) \right]
\]

\[
= \left( q - 1 \frac{\mathbb{E}|Z|^{2-\tau}}{\mathbb{E} B^2 \mathbb{E}|Z|^{2-\tau}} \right)^{-2} \cdot \left( q^{2-\tau} \mathbb{E}|Z|^{2-\tau} - 2 q^{2-\tau} \mathbb{E}|Z|^{2-\tau} \right) = -\frac{\epsilon^2 \left( \mathbb{E}|G|^2 \mathbb{E}|Z|^{2-\tau} \right)^2}{(q - 1)^2 \mathbb{E}|Z|^{2-\tau}}.
\]

This finishes the proof.

\[\square\]

C.5.4.2 Deriving the expansion of \( \text{AMSE}(q, \lambda_q^*) \) for \( q \in (1, 2] \)

The way we derive the result (3.3.2) of Theorem 3.3.1 is similar to that in Section C.5.3.3. We hence do not repeat all the details. The key step is applying Lemma...
C.5.6 to obtain

\[
\lim_{\sigma \to \infty} \sigma^2(\text{AMSE}(q, \lambda_q^*) - \epsilon \mathbb{E}|G|^2) = \lim_{\tau_\ast \to \infty} \frac{\tau_\ast^4 (R_q(\alpha_q(\tau_\ast), \tau_\ast) - \epsilon \mathbb{E}|G|^2/\tau_\ast^2)}{\tau_\ast^2} = -\epsilon^2 (\mathbb{E}|G|^2)^2 c_q.
\]

C.5.5 Proof of Theorem 3.3.1 for \( q > 2 \)

We aim to prove the same results as presented in Lemmas C.5.5 and C.5.6. However, many of the limits we took when proving for the case \( 1 < q \leq 2 \) become invalid for \( q > 2 \) because DCT may not be applicable. Therefore, here we assume a slightly stronger condition that \( G \) has a sub-Gaussian tail and use a different reasoning to validate the results in Lemmas C.5.5 and C.5.6. Throughout this section, we use \( \alpha \) to denote \( \alpha_q(\tau) \) for simplicity. First note that Lemma C.5.2 holds for \( q > 2 \) as well. Hence we already know \( \alpha \to \infty \) as \( \tau \to \infty \). The following key lemma paves our way for the proof.

**Lemma C.5.7.** Suppose function \( h : \mathbb{R}^2 \to \mathbb{R} \) satisfies \( |h(x, y)| \leq C(|x|^{m_1} + |y|^{m_2}) \) for some \( C > 0 \) and \( 0 \leq m_1, m_2 < \infty \). \( B \) has sub-Gaussian tail. Then the following result holds for any constants \( v \geq 0, \gamma \in [0, 1] \) and \( q > 2 \),

\[
\lim_{\tau \to \infty} \alpha^{v+1} q^{\frac{v}{q-1}} \mathbb{E}\left[\frac{h(B, Z)|\eta_q(B/\tau + Z; \alpha)|^v}{1 + \alpha q(q-1)|\eta_q(\gamma B/\tau + Z; \alpha)|^{q-2}}\right] = \frac{q^{\frac{v}{q-1}}}{q-1} \mathbb{E}[h(B, Z)|Z|^{\frac{v+2-q}{q-1}}], \quad \text{as } \tau \to \infty. \tag{C.5.14}
\]

Moreover, there is a finite constant \( K \) such that for sufficiently large \( \tau \),

\[
\max_{0 \leq \gamma \leq 1} \alpha^{v+1} q^{\frac{v}{q-1}} \mathbb{E}\left[\frac{|h(B, Z)||\eta_q(B/\tau + Z; \alpha)|^v}{1 + \alpha q(q-1)|\eta_q(\gamma B/\tau + Z; \alpha)|^{q-2}}\right] \leq K. \tag{C.5.15}
\]

**Proof.** Define \( A = \{|\eta_q(\gamma B/\tau + Z; \alpha)| \leq \frac{1}{2} |\gamma B/\tau + Z|\} \).

We evaluate the expectation on the set \( A \) and its complement \( A^c \) respectively. Recall we use \( p_B \) to denote the distribution of \( B \). By a change of variable we then
have
\[
\mathbb{E} \left[ \frac{h(B, Z)|\eta_B(B/\tau + Z; \alpha)|^v}{1 + \alpha q(q - 1)|\eta_B(\gamma B/\tau + Z; \alpha)|^{q-2}} \right] = \int \frac{h(x, y - \gamma x/\tau)|\eta_B(y + (1 - \gamma) x/\tau; \alpha)|^v}{1 + \alpha q(q - 1)|\eta_B(y; \alpha)|^{q-2}} \phi(y - \gamma x/\tau) dy dp_B(x).
\]

We have on \( \{|\eta_B(y; \alpha)| \leq \frac{1}{2} |y|\} \) when \( \tau \) is large enough,
\[
\frac{\alpha^{v+1}}{1 + \alpha q(q - 1)|\eta_B(y; \alpha)|^{q-2}} \phi(y - \gamma x/\tau)
\leq \frac{h(x, y - \gamma x/\tau) |\eta_B(y + (1 - \gamma) x/\tau; \alpha)|^v}{q(q - 1)|\eta_B(y; \alpha)|^{q-2}} \phi(y - \gamma x/\tau)
\leq \frac{q^{\frac{2-q}{q-1}} |h(x, y - \gamma x/\tau)| |y + (1 - \gamma) x/\tau|^\frac{v}{q-1}}{q(q - 1)|\eta_B(y; \alpha)|^{\frac{2}{q-2}}} \phi(y/\sqrt{2}) e^{-\frac{1}{2} (y - \sqrt{2})^2 + \frac{2}{\sqrt{2}} x^2}
\leq \frac{2^{\frac{q-2}{q-1}} q^{\frac{2-q}{q-1}} |h(x, y - \gamma x/\tau)| |y + (1 - \gamma) x/\tau|^\frac{v}{q-1}}{q(q - 1)|\eta_B(y; \alpha)|^{\frac{2}{q-2}}} \phi(y/\sqrt{2}) e^{\frac{2}{\sqrt{2}} x^2}
\leq \frac{2^{\frac{q-2}{q-1}} q^{\frac{2-q}{q-1}} (|x|^{m_1} + (|y| + |x|)^{m_2}) |y + |x|)^\frac{v}{q-1}}{q(q - 1)|\eta_B(y; \alpha)|^{\frac{2}{q-2}}} \phi(y/\sqrt{2}) e^{\frac{2}{\sqrt{2}} x^2}.
\]

We have used Lemma A.2.2 part (ii) to obtain (a)(b); (c) is due to the condition \( |\eta_B(y; \alpha)| \leq \frac{1}{2} |y| \); and (d) holds because of the condition on the function \( h(x, y) \). Notice that the numerator of the upper bound is essentially a polynomial in \( |x| \) and \( |y| \). Since \( B \) has sub-Gaussian tail, if we choose \( c_0 \) small enough (when \( \tau \) is sufficiently large), the integrability with respect to \( x \) is guaranteed. The integrability w.r.t. \( y \) is clear since \( (2 - q)/(q - 1) > -1 \). Thus we can apply DCT to obtain
\[
\lim_{\tau \to \infty} \alpha^{v+1} \mathbb{E} \left[ \frac{h(B, Z)|\eta_B(B/\tau + Z; \alpha)|^v}{1 + \alpha q(q - 1)|\eta_B(\gamma B/\tau + Z; \alpha)|^{q-2}} \right] = \int \frac{h(x, y - \gamma x/\tau)|\eta_B(y + (1 - \gamma) x/\tau; \alpha)|^v}{\alpha^{\frac{v+1}{q-1}} + q(q - 1)|\eta_B(y; \alpha)|^{q-2}} \phi(y - \gamma x/\tau) dy dp_B(x)
= \int \frac{q^{\frac{1-v}{q-1}} h(x, y)}{(q - 1)|\eta_B(y; \alpha)|^{\frac{2-v}{q-1}}} \phi(y) dy dp_B(x)
= \frac{q^{\frac{1-v}{q-1}} \mathbb{E}[h(B, Z)] Z^{\frac{2-v}{q-1}}}.\]
We now evaluate the expectation on the event \( A^c \). Note that \( A^c \) implies

\[
|\gamma B/\tau + Z| = \alpha q |\eta_q(\gamma B/\tau + Z; \alpha)|^{q-1} + |\eta_q(\gamma B/\tau + Z; \alpha)| > \frac{\alpha q}{2^{q-1}} |\gamma B/\tau + Z|^{q-1} + \frac{1}{2} |\gamma B/\tau + Z|.
\]

This implies \( |\gamma B/\tau + Z| < 2(\alpha q)^{\frac{1}{q-1}} \) on \( A^c \). Hence we have the following bounds,

\[
\alpha^{\frac{1}{q-1}} \mathbb{E} \left[ \frac{|h(B, Z)| \cdot |\eta_q(B/\tau + Z; \alpha)|^v}{1 + \alpha q(q - 1)|\eta_q(\gamma B/\tau + Z; \alpha)|^{q-2}} \right]_{A^c}
\]

\[
\leq \alpha^{\frac{1}{q-1}} \mathbb{E}(|h(B, Z)| \cdot |\eta_q(B/\tau + Z; \alpha)|^v \mathbb{I}_{A^c})
\]

\[
\leq \alpha^{\frac{1}{q-1}} \int_{|y| < 2(\alpha q)^{\frac{1}{q-1}}} |h(x, y - \frac{\gamma x}{\tau})| \cdot |\alpha^{\frac{1}{q-1}} \eta_q(y + \frac{1 - \gamma}{\tau} x; \alpha)|^v \phi(y) e^{\frac{\gamma y x}{\tau}} dy dp_B(x)
\]

\[
\leq \frac{\alpha^{\frac{1}{q-1}}}{q^{\frac{1}{q-1}}} \int_{|y| < 2(\alpha q)^{\frac{1}{q-1}}} |h(x, y)|^{m_1} + (|y| + |x|)^{m_2} (|y| + |x|)^{\frac{v}{q-1}} \phi(y) e^{2(\alpha q)^{\frac{1}{q-1}} \gamma y x} dy dp_B(x)
\]

\[
\leq c_1 \alpha^{\frac{1}{q-1}} \int_{|y| < 2(\alpha q)^{\frac{1}{q-1}}} \tilde{P}(|x|, |y|) \phi(y) e^{2(\alpha q)^{\frac{1}{q-1}} \gamma y x} dy dp_B(x)
\]

\[
\leq c_2 \alpha^{\frac{1}{q-1}} \int \tilde{P}(|x|) e^\gamma dp_B(x) \leq c_2 \alpha^{\frac{1}{(q-1)(q-2)}} \to \infty \text{ as } \tau \to \infty,
\]

where \((e)\) is due to Lemma A.2.2 part (ii) and condition on \( h(x, y) \); \( P(\cdot, \cdot), \tilde{P}(\cdot) \) are two polynomials; the extra term \( \alpha^{\frac{1}{q-1}} \) in step \((f)\) is derived from the condition \(|y| < 2(\alpha q)^{\frac{1}{q-1}} \). We thus have finished the proof of (C.5.14). Finally, note that the two upper bounds we derived do not depend on \( \gamma \), hence (C.5.15) follows directly. \( \Box \)

We are now ready to prove Theorem 3.3.1 for \( q > 2 \). We will prove the results of Lemmas C.5.5 and C.5.6 for \( q > 2 \). After that the exactly same arguments presented in Section C.5.4.2 will close the proof. Since the basic idea of proving Lemmas C.5.5 and C.5.6 for \( q > 2 \) is the same as for the case \( q \in (1, 2] \), we do not detail out the entire proof and instead highlight the differences. The major difference is that we apply Lemma C.5.7 to make some of the limiting arguments valid in the case \( q > 2 \). Adopting the same notations in Section C.5.4.1, we list the settings in the use of Lemma C.5.7 below

\[
\bullet \text{ Lemma C.5.5 } I_1: \text{ set } h(x, y) = x^2, v = 0, \gamma = 1.
\]
Lemma C.5.5 I₃: set \( h(x, y) = x \text{sgn}(\frac{x}{\tau} + y) \), \( v = 1, \gamma = 1 \). Note that the dependence of \( h(x, y) \) on \( \tau \) does not affect the result.

Lemma C.5.5 I₂: Notice we have
\[
\alpha^{q-1} \tau^2 I_2 = \alpha^{q-1} \tau \mathbb{E} \left[ B(Z^2 - 1) \left( \eta_q(Z; \alpha) + \frac{B}{\tau} \int_0^1 \partial_1 \eta_q(sB/\tau + Z; \alpha) \, ds \right) \right]
= \alpha^{q-1} \int_0^1 \mathbb{E} \left[ B^2(Z^2 - 1) \partial_1 \eta_q(sB/\tau + Z; \alpha) \right] \, ds
= \int_0^1 \alpha^{q-1} \mathbb{E} \left[ \frac{B^2(Z^2 - 1)}{1 + \alpha q(q - 1) \eta_q(sB/\tau + Z; \alpha)} \right] \, ds.
\]
We have switched the integral and expectation in the second step above due to the integrability. Set \( h(x, y) = x^2(y^2 - 1), v = 0, \gamma = s \); then by the bound (C.5.15) in Lemma C.5.7, we can bring the limit \( \tau \to \infty \) inside the above integral to obtain the result of \( \lim_{\tau \to \infty} \alpha^{q-1} \tau^2 I_2 \).

In Lemma C.5.6, we need to rebound the term \( \mathbb{E}[\eta_q(B/\tau + Z; \alpha)B/\tau] \) in (C.5.11).
\[
\alpha^{q-1} \tau^2 \mathbb{E}[\eta_q(B/\tau + Z; \alpha)B/\tau]
= \alpha^{q-1} \tau^2 \mathbb{E} \left[ B \left( \eta_q(Z; \alpha) + \frac{B}{\tau} \int_0^1 \partial_1 \eta_q(sB/\tau + Z; \alpha) \, ds \right) \right]
= \int_0^1 \alpha^{q-1} \mathbb{E} \left[ \frac{B^2}{1 + \alpha q(q - 1) \eta_q(sB/\tau + Z; \alpha)} \right] \, ds
\]
We set \( h(x, y) = x^2, v = 0, \gamma = s \). The rest arguments are similar to the previous one.

### C.6 Proof of Theorems 3.4.1

Since the roadmap of the proof is similar to that of Theorem 3.3.1, we will not repeat it. We suggest the reader study Appendix C.5 before reading this appendix.

We remind the reader that in the large sample regime, we have scaled the noise term. Hence \( \tau_* \) will satisfy
\[
\tau_*^2 = \frac{\sigma^2}{\delta} + \frac{\tau_*^2 R_q(\alpha_q(\tau_*), \tau_*)}{\delta}.
\]
We first derive the convergence rate of $\tau_*$ as $\delta \to \infty$.

**Lemma C.6.1.** For a given $q \in [1, \infty)$, as $\delta \to \infty$,

$$
\tau_*^2 = \frac{\sigma^2}{\delta} + o\left(\frac{1}{\delta}\right).
$$

**Proof.** Since $\alpha = \alpha_q(\tau_*)$ minimizes $R_q(\alpha, \tau_*)$, from (C.6.1) we obtain

$$
\delta(\tau_*^2 - \sigma^2/\delta) \leq R_q(0, \tau_*) = \tau_*^2,
$$

which yields $\tau_*^2 \leq \frac{\sigma^2}{\delta - 1} \to 0$ as $\delta \to \infty$. This completes the proof.

Lemma C.6.1 shows that $\tau_* \to 0$ as $\delta \to \infty$. Hence we need to characterize the convergence rate of $R_q(\alpha_q(\tau), \tau)$ as $\tau \to 0$. The results have been derived in the small noise regime analysis. We collect the results together in the next lemma.

**Lemma C.6.2.** As $\tau \to 0$ we have

1. For $q = 1$, assume $\mathbb{P}(|G| \geq \mu) = 1$ with $\mu$ a positive constant and $\mathbb{E}|G|^2 < \infty$, then

   $$
   R_q(\alpha_q(\tau), \tau) - f(\chi_0) = O(\phi(\mu/\tau - \chi_0)),
   $$

   where $\chi = \chi_0$ is the minimizer of $f(\chi) = (1 - \epsilon)\mathbb{E}\eta_0^2(Z; \chi) + \epsilon(1 + \chi^2)$.

2. For $1 < q < 2$, assume $\mathbb{P}(|G| \leq x) = O(x)$ (as $x \to 0$) and $\mathbb{E}|G|^2 < \infty$,

   $$
   R_q(\alpha_q(\tau), \tau) = 1 - \frac{(1 - \epsilon)^2(\mathbb{E}|Z|^q)^2}{c\mathbb{E}|G|^{2q-2}}\tau^{2q-2} + o(\tau^{2q-2}).
   $$

3. For $q > 2$, assume $\mathbb{E}|G|^{2q-2} < \infty$, then

   $$
   R_q(\alpha_q(\tau), \tau) = 1 - \tau^2 \epsilon(q - 1)^2(\mathbb{E}|G|^{q-2})^2 + o(\tau^2).
   $$

**Proof.** Result (1) is Lemma 5 in [WMZ18]; Result (2) is Lemma 20 in [WMZ18]; Result (3) is Lemma I.2 in [WWM20].
We now use the results in Lemmas C.6.1 and C.6.2 to prove Theorem 3.4.1. We only present the proof for \( q \in (1, 2) \). Similar arguments work for other values of \( q \). By Theorem A.2.1 and (C.6.1),

\[
\delta^q (\text{AMSE}(q, \lambda_q^*) - \sigma^2/\delta) = \delta^q (\tau_q^2 R_q(\alpha_q(\tau_*), \tau_*) - \tau_*^2 + \tau_*^2 R_q(\alpha_q(\tau_*), \tau_*)/\delta) = \delta^q \tau_*^2 (R_q(\alpha_q(\tau_*), \tau_*) - 1) + \delta^{q-1} \tau_*^2 R_q(\alpha_q(\tau_*), \tau_*) \overset{(a)}{\rightarrow} - \sigma^2 q (1 - \epsilon)^2 (\mathbb{E}|Z|^q)^2 / \epsilon \mathbb{E}|G|^{2q-2}, \text{ as } \delta \to \infty.
\]

Step (a) is due to Lemmas C.6.1 and C.6.2 part (2). This finishes the proof.

C.7 Proofs of Theorem 3.6.1, 3.6.2, 3.6.3, Lemma 3.6.1

The proof of Theorems 3.6.1, 3.6.2 (3.6.3) and Lemma 3.6.1 can be found in Sections C.7.1, C.7.2 and C.7.3 respectively.

C.7.1 Proof of Theorem 3.6.1

Since some technical details for \( q = 1 \) and \( q > 1 \) are different, we prove the two cases separately in Sections C.7.1.2 and C.7.1.1 respectively.

C.7.1.1 Proof of Theorem 3.6.1 for \( q = 1 \)

In this section, we apply the approximate message passing (AMP) framework to prove the result for LASSO. We first briefly review the approximate message passing algorithm and state some relevant results that will be later used in the proof. We then describe the main proof steps.

I. Approximate message passing algorithms. [BM12] has utilized AMP theory to characterize the sharp asymptotic risk of LASSO. The authors considered
a sequence of estimates \( \mathbf{x}^t \in \mathbb{R}^p \) generated from an approximate message passing algorithm with the following iterations (initialized at \( \mathbf{x}^0 = 0, z^0 = y \)):

\[
\begin{align*}
\mathbf{x}^{t+1} &= \eta_q(\mathbf{A}^\top z^t + \mathbf{x}^t; \alpha \tau_t^{2-q}), \\
z^t &= y - \mathbf{A}x^t + \frac{1}{\delta}z^{t-1}\langle\partial_1\eta_q(\mathbf{A}^\top z^{t-1} + \mathbf{x}^{t-1}; \alpha \tau_{t-1}^{2-q})\rangle,
\end{align*}
\]

(C.7.1)

where \( \langle v \rangle = \frac{1}{p} \sum_{i=1}^{p} v_i \) denotes the average of a vector’s components; \( \alpha \) is the solution to Equations (2.2.2) and (2.2.3); and \( \tau_t \) satisfies (\( \tau_0^2 = \sigma^2 + \mathbb{E}|B|^2/\delta \)):

\[
\tau_{t+1}^2 = \sigma_w^2 + \frac{1}{\delta} \mathbb{E}[\eta_q(B + \tau_tZ; \alpha \tau_t^{2-q}) - B]^2, \quad t \geq 0.
\]

(C.7.2)

The asymptotics of many quantities in AMP can be sharply characterized. We summarize some results of [BM12] that we will use in our proof.

**Theorem C.7.1** ([BM12]). Let \( \{\mathbf{x}(p), \mathbf{A}(p), w(p)\} \) be a converging sequence, and \( \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a pseudo-Lipschitz function. For \( q = 1 \), almost surely

\[
\begin{align*}
(i) \quad & \lim_{t \to \infty} \lim_{p \to \infty} \frac{1}{p} \|\hat{x}(1, \lambda) - \mathbf{x}^t\|_2^2 = 0, \\
(ii) \quad & \lim_{n \to \infty} \frac{1}{n} \|z^t\|_2^2 = \tau_t^2, \quad \lim_{t \to n \to \infty} \frac{1}{n} \|z^t - z^{t-1}\|_2^2 = 0, \\
(iii) \quad & \lim_{n \to \infty} \frac{1}{p} \|\mathbf{x}^t\|_0 = \mathbb{P}(|B + \tau_tZ| > \alpha \tau_t), \\
(iv) \quad & \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \psi(\mathbf{x}^t_i + (\mathbf{A}^\top z^t)_i, x_i) = \mathbb{E}\psi(B + \tau_tZ, B), \\
(v) \quad & \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \psi(\mathbf{x}^t_i, x_i) = \mathbb{E}\psi(\eta_1(B + \tau_tZ), B),
\end{align*}
\]

where \( \hat{x}(1, \lambda) \) is the LASSO solution and \( \tau_t \) is defined in (C.7.2).
II. Main proof steps. We first have the following bounds:

\[
\frac{1}{p} \| \hat{x}^\dagger(1, \lambda) - x^t - A^\top \frac{y - Ax^t}{1 - \|x^t\|_0/n} \|_2^2 \\
\leq \frac{2}{p} \| \hat{x}(1, \lambda) - x^t \|_2^2 + \frac{8}{p(1 - \mathbb{P}(|B + \tau Z| > \alpha \tau)/\delta)^2} \| A^\top A(\hat{x}(1, \lambda) - x^t) \|_2^2 \\
+ \frac{8}{p} \| A^\top (y - A\hat{x}(1, \lambda)) \|_2^2 \left( \frac{1}{1 - \|\hat{x}(1, \lambda)\|_0/n} - \frac{1}{1 - \mathbb{P}(|B + \tau Z| > \alpha \tau)/\delta} \right)^2 \\
+ \frac{8}{p} \| A^\top (y - Ax^t) \|_2^2 \left( \frac{1}{1 - \|x^t\|_0/n} - \frac{1}{1 - \mathbb{P}(|B + \tau Z| > \alpha \tau)/\delta} \right)^2,
\]

where \((\alpha, \tau)\) is the solution to (2.2.2) and (2.2.3). From Theorem C.7.1 part (i), we know \(\lim_{t \to \infty} \lim_{p \to \infty} Q_1 = 0\), a.s. Since the largest singular value of \(A\) is bounded almost surely [BY93], we can also obtain \(\lim_{t \to \infty} \lim_{p \to \infty} Q_2 = 0\), a.s. Moreover, from Theorem A.2.1 we can easily see the term \(\| A^\top (y - A\hat{x}(1, \lambda)) \|_2^2 / p \leq 2 \| A^\top A(\hat{x}(1, \lambda) - x) \|_2^2 / p + 2 \| A^\top w \|_2^2 / p\) is almost surely bounded. Also we know from [BvdBSC13] that \(\frac{1}{p} \| \hat{x}(1, \lambda) \|_0 = \mathbb{P}(|B + \tau Z| > \alpha \tau), a.s.\) Therefore, we obtain \(\lim_{p \to \infty} Q_3 = 0\), a.s.

Regarding \(Q_4\), it is not hard to see from (C.7.2) that \(\tau_t \to \tau\) as \(t \to \infty\). Then a similar argument as for \(Q_3\) combined with Theorem C.7.1 parts (i)(iii) gives us \(\lim_{t \to \infty} \lim_{p \to \infty} Q_4 = 0\), a.s. Above all we are able to derive almost surely

\[
\lim_{t \to \infty} \lim_{p \to \infty} \frac{1}{p} \| \hat{x}^\dagger(1, \lambda) - x^t - A^\top \frac{y - Ax^t}{1 - \|x^t\|_0/n} \|_2^2 = 0. \tag{C.7.3}
\]

Next from Equation (C.7.1) we have the following,

\[
A^\top z^t - A^\top \frac{y - Ax^t}{1 - \|x^t\|_0/n} = A^\top \frac{\|x^t\|_0/\sqrt{n}(z^t + z^t-1n/p\delta))}{1 - \|x^t\|_0/n}.
\]

Using the result of Theorem C.7.1 part (ii) and \(n/p \to \delta\), we can obtain

\[
\lim_{t \to \infty} \lim_{p \to \infty} \frac{1}{p} \| A^\top z^t - A^\top \frac{y - Ax^t}{1 - \|x^t\|_0/n} \|_2^2 = 0, \tag{C.7.4}\]

The results (C.7.3) and (C.7.4) together imply that

\[
\lim_{t \to \infty} \lim_{p \to \infty} \frac{1}{p} \| \hat{x}^\dagger(1, \lambda) - x^t - A^\top z^t \|_2^2 = 0, \tag{C.7.5}\]

Next from Equation (C.7.1) we have the following,
According to Theorem C.7.1 part (iv), for any bounded Lipschitz function $L(x) : \mathbb{R} \rightarrow \mathbb{R}$, 
$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{i=1}^{p} L(\mathbf{x}_i^t + (\mathbf{A}^\top \mathbf{z})_i - x_i) = \mathbb{E}L(\tau Z).$$
Putting the last two results together, it is not hard to confirm 
$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{i=1}^{p} L(\hat{\mathbf{x}}_i^\top(1, \lambda) - x_i) = \mathbb{E}L(\tau Z).$$

Hence, the empirical distribution of $\hat{\mathbf{x}}_i^\top(1, \lambda) - \mathbf{x}$ converges to the distribution of $\tau Z$.

### C.7.1.2 Proof of Theorem 3.6.1 for $q > 1$

The proof idea for $q > 1$ is the same as for $q = 1$. However since the debiased estimator for $q > 1$ takes a different form, we need take care of some subtle details.

Recall the definition of $f(v, w), \hat{\gamma}_\lambda$ in (3.6.1). We first obtain the bound:

$$\frac{1}{p} \left\| \frac{\hat{\mathbf{x}}_i^\top(q, \lambda) - \mathbf{x}_i^\top - \mathbf{A}^\top \frac{y - \mathbf{A} \mathbf{x}_i^t}{1 - f(\mathbf{x}_i^t, \alpha \tau_i^{2-q})/n} \right\|^2$$

$$\leq \frac{2}{p} \left\| \hat{\mathbf{x}}_i(q, \lambda) - \mathbf{x}_i^\top \right\|^2 + \frac{8}{p(1 - f(\eta_q(B + \tau Z; \alpha \tau_2^{2-q}), \alpha \tau_2^{2-q})/\delta)^2} \left\| \mathbf{A}^\top \mathbf{A}(\hat{\mathbf{x}}_i(q, \lambda) - \mathbf{x}_i^\top) \right\|^2$$

$$+ \frac{8}{p} \left\| \mathbf{A}^\top(y - \mathbf{A} \hat{\mathbf{x}}_i(q, \lambda)) \right\|^2 \left[ \frac{1}{1 - \frac{1}{n} f(\hat{\mathbf{x}}_i(q, \lambda), \hat{\gamma}_\lambda)} - \frac{1}{1 - \frac{1}{n} f(\eta_q(B + \tau Z; \alpha \tau_2^{2-q}), \alpha \tau_2^{2-q})} \right]$$

As in the proof of $q = 1$, we show that $Q_i (i = 1, 2, 3, 4)$ vanishes asymptotically. For that purpose we first note that Theorem C.7.1 (except part (iii)) holds for $q > 1$ as well. Hence the same argument for $q = 1$ gives us $Q_1, Q_2 \xrightarrow{a.s.} 0$. Regarding $Q_3$, by the facts that the empirical distribution of $\hat{\mathbf{x}}_i(q, \lambda)$ converges weakly to the distribution of $\eta_q(B + \tau Z; \alpha \tau_2^{2-q})$ and $\frac{1}{1 + \alpha \tau_2^{2-q} \tau_2^{q-1} |x|^2}$ is a bounded continuous function of $x$, we have 

$$\lim_{p \rightarrow \infty} \frac{1}{p} f(\hat{\mathbf{x}}_i(q, \lambda), \alpha \tau_2^{2-q}) = f(\eta_q(B + \tau Z; \alpha \tau_2^{2-q}), \alpha \tau_2^{2-q}), \text{ a.s.}$$
Moreover, according to Lemma C.7.1 we obtain as $p \to \infty$,
\[
\frac{1}{p} |f(\hat{x}(q, \lambda), \alpha \tau^{2-q}) - f(\hat{x}(q, \lambda), \hat{\gamma}_\lambda)| \leq \alpha^{-1} t^{-2} |\hat{\gamma}_\lambda - \alpha \tau^{2-q}| \overset{a.s.}{\to} 0.
\]

The last two results together lead to $Q_3 \overset{a.s.}{\to} 0$. For $Q_4$, it is not hard to apply Theorem C.7.1 part (v) and the fact $\tau_t \to \tau$ to show
\[
\lim_{t \to \infty} \lim_{p \to \infty} \frac{1}{p} f(x^t, \alpha \tau^{2-q}) = f(\eta_q(B + \tau Z; \alpha \tau^{2-q}), \alpha \tau^{2-q}), \quad a.s.
\]
which implies $Q_4 \overset{a.s.}{\to} 0$. The rest of the proof is almost the same as the one for $q = 1$. We hence do not repeat the arguments.

**Lemma C.7.1.** For $\hat{\gamma}_\lambda$ defined in (3.6.2), as $p \to \infty$
\[
\hat{\gamma}_\lambda \overset{a.s.}{\to} \alpha \tau^{2-q}.
\]

**Proof.** Denote $a(\gamma) = \delta(1 - \frac{\lambda}{\gamma}), \hat{b}(\gamma) = \text{Ave} \left[ \frac{1}{1+\gamma q(q-1)|x(q,\lambda)|^{q-2}} \right]$, $b(\gamma) = \mathbb{E} \left[ \frac{1}{1+\gamma q(q-1)|\eta_q(B+\tau Z, \alpha \tau^{2-q})|} \right]$, and $\gamma \lambda = \alpha \tau^{2-q}$. Clearly from (3.6.2) and (2.2.3), $\hat{\gamma}_\lambda$ is the unique solution of $a(\gamma) = \hat{b}(\gamma)$, and $\gamma \lambda$ is the unique solution of $a(\gamma) = b(\gamma)$.

As a simple corollary of Theorem A.2.1, almost surely the empirical distribution of $\hat{x}(q, \lambda)$ converges weakly to the distribution of $\eta_q(B + \tau Z; \alpha \tau^{2-q})$. As a result, for $h(x) = \frac{1}{1+\gamma q(q-1)|x|^{q-2}}$ which is bounded and continuous on $\mathbb{R}$, we have almost surely
\[
\hat{b}(\gamma) \to b(\gamma), \quad \text{as} \ p \to \infty.
\]

The above convergence is pointwise in $\gamma$. In fact we can obtain a stronger result. That is, there is a $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$, such that for any $\omega \in \Omega_0$, $\hat{b}(\gamma, \omega) \to b(\gamma)$ for all $\gamma \geq 0$. (we can first construct $\Omega_0$ for $\gamma \in \mathbb{Q}$, then extend to $\mathbb{R}_+$ by continuity and monotonicity of $a(\gamma), \hat{b}(\gamma)$ and $b(\gamma)$.)

Now for any $\omega \in \Omega_0$, for any $\varepsilon > 0$, consider the neighborhood $[\gamma_\lambda - \varepsilon, \gamma_\lambda + \varepsilon]$. Let $\eta_\varepsilon = \min \{ b(\gamma_\lambda - \varepsilon) - a(\gamma_\lambda - \varepsilon), a(\gamma_\lambda + \varepsilon) - b(\gamma_\lambda + \varepsilon) \}$. Monotonicity of $a(\gamma), b(\gamma)$ and uniqueness of the solution $\gamma_\lambda$ guarantee $\eta_\varepsilon > 0$. At $\gamma_\lambda - \varepsilon$ and $\gamma_\lambda + \varepsilon$, we know as $p \to \infty$,
\[
\hat{b}(\gamma_\lambda - \varepsilon, \omega) \to b(\gamma_\lambda - \varepsilon), \quad \hat{b}(\gamma_\lambda + \varepsilon, \omega) \to b(\gamma_\lambda + \varepsilon).
\]
Thus there exists $N_\epsilon(\omega)$, for any $p > N_\epsilon(\omega)$,

$$|\hat{b}(\gamma_\lambda - \epsilon, \omega) - b(\gamma_\lambda - \epsilon)| < \frac{\eta_\epsilon}{2}, \quad |\hat{b}(\gamma_\lambda + \epsilon, \omega) - b(\gamma_\lambda + \epsilon)| < \frac{\eta_\epsilon}{2}.$$  

By noticing the distance between $a(\gamma)$ and $b(\gamma)$ on the two end-points, we have

$$|\hat{b}(\gamma_\lambda - \epsilon, \omega) - a(\gamma_\lambda - \epsilon)| > \frac{\eta_\epsilon}{2} \quad \text{and} \quad |a(\gamma_\lambda + \epsilon) - \hat{b}(\gamma_\lambda + \epsilon, \omega)| > \frac{\eta_\epsilon}{2}. \quad \text{The monotonicity of}\ \hat{b}(\gamma, \omega) \\text{determines that}\ \hat{\gamma}_\lambda(\omega) \in (\gamma_\lambda - \epsilon, \gamma_\lambda + \epsilon), \ \text{i.e.,}\ \left|\hat{\gamma}_\lambda(\omega) - \gamma_\lambda\right| < \epsilon. \ \text{As a conclusion, we have}\ \hat{\gamma}_\lambda \xrightarrow{a.s.} \gamma_\lambda. \quad \Box$$

### C.7.2 Proof of Theorems 3.6.2 and 3.6.3

We only prove the case $q = 1$. The proof for $q > 1$ is similar. In the proof of Theorem 3.6.1, we have showed

$$\lim_{t \to \infty} \lim_{p \to \infty} \|\hat{x}_t^\dagger(1, \lambda) - (x_t' + A^T z_t')\|_2 = 0, \quad a.s.$$  

Combining this result with Theorem C.7.1 part (iv), we know for any bounded Lipschitz function $\psi : \mathbb{R}^2 \to \mathbb{R}$

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^p \psi(\hat{x}_i^\dagger(1, \lambda), x_i) = \mathbb{E}\psi(B + \tau Z, B).$$  

Hence the empirical distribution of $(\hat{x}_i^\dagger(1, \lambda), x)$ converges weakly to the distribution of $(B + \tau Z, B)$. We then follow the same calculations as the proof of Lemma 2.2.2 and obtain

$$\text{AFDP}^\dagger(1, \lambda, s) = \frac{(1 - \epsilon)\mathbb{P}(|\tau|Z > s)}{(1 - \epsilon)\mathbb{P}(|\tau|Z > s) + \epsilon\mathbb{P}(|G + \tau Z| > s)},$$

$$\text{ATPP}^\dagger(1, \lambda, s) = \mathbb{P}(|G + \tau Z| > s) \quad (C.7.5)$$

Notice that $\tau > 0$ and $G + \tau Z$ is continuous. Thus as we vary $s$, $\text{ATPP}^\dagger(1, \lambda, s)$ can reach all values in $[0, 1]$. Furthermore, by comparing the formula above with that in Lemma 2.2.2 (when $q = 1$), it is clear that $\text{ATPP}^\dagger(1, \lambda, s) = \text{ATPP}(1, \lambda, \tilde{s})$ will imply $\text{AFDP}^\dagger(1, \lambda, s) = \text{AFDP}(1, \lambda, \tilde{s})$. The proof for the second part is similar to that of Theorem 2.3.1, and is skipped.
C.7.3 Proof of Lemma 3.6.1

SIS thresholds $A^\top y$, which is also the initialization of AMP in (C.7.1). By setting $t = 0$ in Theorem C.7.1 (iv), we obtain

$$\lim_{p \to \infty} \sum_{i=1}^{p} \psi((A^\top y)_i, x_i) = \mathbb{E}\psi(B + \tau_0 Z, B)$$

(C.7.6)

where $\tau_0^2 = \sigma^2 + \frac{E x^2}{\delta} > 0$. This implies that almost surely the empirical distribution of $\{(A^\top y)_i, x_i\}_{i=1}^{p}$ converges weakly to $(B + \tau_0 Z, B)$. Following the same argument as in the proof of Lemma 2.2.2, we have

$$AFDP_{sis}(s) = \frac{(1 - \epsilon) \mathbb{P}(\tau_0 |Z| > s)}{(1 - \epsilon) \mathbb{P}(\tau_0 |Z| > s) + \epsilon \mathbb{P}(|G + \tau_0 Z| > s)}$$

$$ATPP_{sis}(s) = \mathbb{P}(|G + \tau_0 Z| > s)$$

On the other hand, based on Equation (C.7.5), we obtain

$$AFDP^\dagger(q, \lambda_q^*, s) = \frac{(1 - \epsilon) \mathbb{P}(\tau_* |Z| > s)}{(1 - \epsilon) \mathbb{P}(\tau_* |Z| > s) + \epsilon \mathbb{P}(|G + \tau_* Z| > s)}$$

$$ATPP^\dagger(q, \lambda_q^*, s) = \mathbb{P}(|G + \tau_* Z| > s)$$

Note that

$$\tau_*^2 \leq \sigma^2 + \frac{1}{\delta} \lim_{\alpha \to \infty} \mathbb{E}(\eta_q(B + \tau_* Z; \alpha \tau_*^{2-q}) - B)^2 = \tau_0^2.$$ 

With the same argument as the proof of Theorem 2.3.1, we have the conclusion follow.
Appendix D

Proof of Chapter 4

In this section, we present the proofs of Proposition 5.2.2, and Theorems 4.1.1, 5.2.1, and 5.3.1. The proof of Theorem 4.1.1 is presented in Chapter 4. Proofs of Proposition 5.2.2 and Theorems 5.2.1 and 5.3.1 are then given in Chapter 5. Some basic properties of the SLOPE proximal operator $\eta$ that are frequently used in the main proofs are provided in Section D.1. Lastly, Section D.2 collects some other lemmas used in the proofs.

Before proceeding, we introduce some notations that will be used in the proofs. Recall

$$\eta(u; \gamma, \lambda) = \arg\min_x \frac{1}{2} \|u - x\|_2^2 + \gamma \|x\|_\lambda.$$  \hspace{1cm} (D.0.1)

When the value of $\lambda$ is clear from the context, we suppress $\lambda$ and simply use $\eta(u; \gamma)$ to denote the proximal operator. We also denote by $D_\gamma$ the dual SLOPE norm ball with radius $\gamma$:

$$D_\gamma := \{v : \|v\|_{\lambda^*} \leq \gamma\}$$

$$= \left\{ v : \sum_{i=1}^j |v|_{(i)} \leq \gamma \sum_{i=1}^j \lambda_i, \ 1 \leq j \leq p \right\}.$$  \hspace{1cm} (D.0.2)

with $\|\cdot\|_{\lambda^*}$ the dual norm of $\|\cdot\|_\lambda$. The characterization of $D_\gamma$ in (D.0.2) is proved in Lemma D.1.1. Furthermore, in Lemma D.1.2 we will show that the sorted components
of \{|\eta_i|\} are piecewise constant. Hence, for each \(i = 1, \ldots, p\), we define

\[ \mathcal{I}_i = \{1 \leq j \leq p : |\eta_j(u; \gamma, \lambda)| = |\eta_i(u; \gamma, \lambda)|\}. \quad \text{(D.0.3)} \]

This induces a partition \(\mathcal{P}\) of \([p]\), defined as

\[ \mathcal{P} = \{\mathcal{I}_i, 1 \leq i \leq p\}. \]

We note that \(\mathcal{P}\) only keeps the unique values of \(\{\mathcal{I}_i\}\). Further we define \(\mathcal{P}_0\) as a subset of \(\mathcal{P}\):

\[ \mathcal{P}_0 = \{\mathcal{I} \in \mathcal{P} : \eta_i \neq 0 \text{ for } i \in \mathcal{I}\}. \]

It is important to note that \(\{\mathcal{I}_i\}_i, \mathcal{P}\) and \(\mathcal{P}_0\) all depend on \(u, \gamma\) and \(\lambda\). Since this dependency is often clear from the context, we typically suppress this dependency in the notations.

Finally, given a set \(C \subset \mathbb{R}^p\) and a point \(u \in \mathbb{R}^p\), we use \(\Pi_C(u)\) to denote the projection of \(u\) on \(C\), and \(\mathbb{I}_C(u)\) to denote the function with value 0 when \(u \in C\) and \(\infty\) otherwise. We also reserve the notation \(h \sim \mathcal{N}(0, I_p)\) and \(g \sim \mathcal{N}(0, I_n)\).

## D.1 Basic properties of the proximal operator of SLOPE norm

In this section, we list some properties of \(\eta\). We would like to remind our readers of the definitions in (D.0.1), (D.0.2), and (D.0.3). The first property is a dual characterization of the primal definition of \(\eta\) in (D.0.1).

**Lemma D.1.1.** The primal convex problem (D.0.1) has the dual form

\[ v^* \in \arg\min_{v \in D_\gamma} \|u - v\|_2^2, \quad \text{(D.1.1)} \]

where \(D_\gamma\) is defined in (D.0.2). Furthermore, strong duality holds, and the primal and dual solution pair \((\eta(u; \gamma), v^*)\) is unique and satisfies

\[ v^* = u - \eta(u; \gamma). \]
Proof. First of all, it is clear that (D.0.1) is strictly convex and \( \eta(u; \gamma) \) is unique. The optimization (D.1.1) can be considered as projecting the point \( u \in \mathbb{R}^p \) onto the closed convex set \( D_\gamma \), thus a unique solution \( v^* \) exists. Now we connect the primal form and the dual form using the classical Fenchel duality framework. Let \( z = u - x \). By substituting in \( z \) and adding a Lagrangian multiplier \( v \) for the constraint \( z = u - x \), we obtain the following equivalent form of (D.0.1):

\[
\max_v \min_{x,z} \frac{1}{2} \|z\|_2^2 + \gamma \|x\|_\lambda - \langle v, z - u + x \rangle
\]

\[
= \max_v \min_{x,z} \left\{ \frac{1}{2} \|z\|_2^2 - \langle v, z \rangle \right\} + \gamma \|x\|_\lambda - \langle v, x \rangle + \langle v, u \rangle.
\]

The optimal \( z^* = v \). Regarding minimizing over \( x \), we have\(^1\) \( \min_x \gamma \|x\|_\lambda - \langle v, x \rangle = -\|x\|_\lambda \max_x \{\langle v, x \rangle / \|x\|_\lambda \} - \gamma = \mathbb{I}_{\mathcal{D}_\gamma}(v) \). Now the above Lagragian form reduces to

\[
-\frac{1}{2} \|u\|_2^2 + \min_{v \in \mathcal{D}_\gamma} \frac{1}{2} \|u - v\|_2^2,
\]

which naturally leads to the optimal solution

\[
v^* = \Pi_{\mathcal{D}_\gamma}(u),
\]

The strong duality holds in this case, implying that

\[
v^* = z^* = u - x^* = u - \eta(u; \gamma).
\]

The last piece of the proof deals with the characterization of \( \mathcal{D}_\gamma \) in (D.0.2). We will use the relation \( \|v\|_{\lambda^*} = \max_{\|a\|_\lambda \leq 1} \langle a, v \rangle \), Without loss of generality, we assume \( v_1 \geq \ldots \geq v_p \geq 0 \) (otherwise we permute the order and swap the signs of the components of \( a \) accordingly). It is not hard to see that the optimization problem can be rewritten as:

\[
\max_{a} \langle a, v \rangle, \quad \text{subject to} \quad \|a\|_\lambda \leq 1, \quad a_1 \geq \ldots \geq a_p \geq 0.
\]

\(^1\)Here we use the fact that \( \|v\|_* = \max_{\|u\| \leq 1} \langle u, v \rangle \) for any norm \( \| \cdot \| \) and its dual norm \( \| \cdot \|_* \) in a Hilbert space.
It is equivalent to re-parameterize $a$ using a vector $b$ with $a_i = \sum_{j=1}^{p} b_j$ and $b_j \geq 0$. Transforming the above constraints as Lagrange multipliers and optimizing over $b$, we get the following dual problem:

$$\min_{\theta, \theta_i} \theta, \quad \text{subject to} \quad \sum_{i=1}^{j} v_i - \theta \sum_{i=1}^{j} \lambda_i + \theta_j \leq 0, \quad \theta_j \geq 0, \quad \forall 1 \leq j \leq p, \quad \theta \geq 0.$$

Obviously given $\{\theta_j\}$,

$$\hat{\theta} = \max_j \left\{ \frac{\sum_{i=1}^{j} v_i + \theta_j}{\sum_{i=1}^{j} \lambda_i} \right\}.$$

To further minimize over $\theta_j$, obviously we should set $\theta_j = 0$ for all $j$ and the optimal value, $\|v\|_{\lambda^*}$, equals:

$$\|v\|_{\lambda^*} = \max_j \left\{ \frac{\sum_{i=1}^{j} v_i}{\sum_{i=1}^{j} \lambda_i} \right\}.$$

As a corollary of this result, we may characterize $D_{\gamma}$ as

$$D_{\gamma} = \{v : \|v\|_{\lambda^*} \leq \gamma\} = \{v : \sum_{i=1}^{j} v_i \leq \gamma \sum_{i=1}^{j} \lambda_i, \quad \forall 1 \leq j \leq p\}.$$

The primal form (D.0.1) and the dual form (D.1.1) enable us to obtain several useful properties of $\eta(u; \gamma)$. We select some of them to present here. We first analyze the primal form (D.0.1) to derive some properties of $\eta(u; \gamma)$.

**Lemma D.1.2.** Consider any given $u \in \mathbb{R}^p$ with $u_1 \geq u_2 \geq \cdots \geq u_p \geq 0$. The following results hold:

(i) $\eta(tu; t\gamma) = t\eta(u, \gamma)$ for $t \geq 0$.

(ii) $\eta_1(u; \gamma) \geq \eta_2(u; \gamma) \geq \cdots \geq \eta_p(u; \gamma) \geq 0$.

(iii) $u_i \geq \eta_i(u; \gamma), 1 \leq i \leq p$.

(iv) $\eta_j(u; \gamma) = \frac{\sum_{i \in \mathcal{I}_j} (u_i - \gamma \lambda_i)_+}{|\mathcal{I}_j|}$, where $\mathcal{I}_j$ is defined in (D.0.3).
Proof. Part (i) is because: \( \eta(tu; t\gamma) = \arg\min_x \frac{1}{2}\|x - tu\|^2 + t\gamma\|x\| \leq \arg\min_x \frac{1}{2}\|x/t - u\|^2 + \gamma\|x/t\| \). Part (ii) is taken from Proposition 2.2 in [BvdBS+15]. For Part (iii), note that \( u - \eta(u; \gamma) = \Pi_{D_\gamma}(u) \) with \( D_\gamma \) being symmetric around \( 0 \), we have \( u_i \geq u_i - \eta_i(u; \gamma) \). This implies (iii). We now prove Part (iv). First consider \( \eta_j(u; \gamma) > 0 \). Denote \( I_{j}^{\min} = \min\{i : i \in I_j\}, I_{j}^{\max} = \max\{i : i \in I_j\} \) and \( \eta_i(u; \gamma) = a > 0, i \in I_j \). There exists a sufficiently small \( \delta > 0 \), such that (we adopt the notation \( \eta_0(u; \gamma) = +\infty, \eta_{p+1}(u; \gamma) = 0) \\
\eta_{I_{j}^{\max}+1}(u; \gamma) < a - \delta < a + \delta < \eta_{I_{j}^{\min}-1}(u; \gamma). \\
 Define a vector \( z(b) \in \mathbb{R}^p : z_i(b) = \eta_i(u; \gamma) \) for \( 1 \leq i \leq I_j^{\min} - 1, I_j^{\max} + 1 \leq i \leq p \), and \( z_i(b) = b \) for other \( i \)'s. Since \( \eta(u; \gamma) \) is the minimizer of (D.0.1) we know \\
\frac{1}{2}\|u - \eta(u; \gamma)\|^2 + \gamma \langle \lambda, \eta(u; \gamma) \rangle \leq \frac{1}{2}\|u - z(b)\|^2 + \gamma \langle \lambda, z(b) \rangle, \\
holds for any \( b \in [a - \delta, a + \delta] \). Due to the choice of \( z(b) \), we can further simplify the above inequality to obtain \\
\frac{1}{2}\sum_{i=I_j^{\min}}^{I_j^{\max}} (u_i - a)^2 + \gamma \sum_{i=I_j^{\min}}^{I_j^{\max}} \lambda_i a \leq \frac{1}{2}\sum_{i=I_j^{\min}}^{I_j^{\max}} (u_i - b)^2 + \gamma \sum_{i=I_j^{\min}}^{I_j^{\max}} \lambda_i b : = G(b), \\
where \( b \in [a - \delta, a + \delta] \). Hence, \( a \) is a local minimas of the quadratic function \( G(\cdot) \). Therefore \\
\frac{dG(b)}{db} \bigg|_{b=a} = \sum_{i=I_j^{\min}}^{I_j^{\max}} (a - u_i + \gamma \lambda_i), \ \ \Rightarrow \ \ a = \frac{1}{I_j^{\max} - I_j^{\min} + 1} \sum_{i=I_j^{\min}}^{I_j^{\max}} u_i - \gamma \lambda_i. \\
Regarding \( \eta_j(u; \gamma) = 0 \), we can use the same arguments to conclude that \( 0 \) is local minima of \( G(\cdot) \) in \([0, \delta] \). So \( \frac{dG(b)}{db} \bigg|_{b=0} \geq 0 \) leads to the result. \( \square \) \\
The next two lemmas study the differentiability of \( \eta(u; \gamma) \) that are useful in the later proof. According to Lemma D.1.5 (i), \( \eta(u; \gamma) \) is Lipschitz continuous, hence differentiable almost everywhere (with respect to \( u \)). In fact, from Lemma D.1.2 (iv), it seems possible to calculate the derivatives of \( \eta(u; \gamma) \) outside a set of Lebesgue
measure zero. Towards that goal, we slightly extend the notation of the partition \( P \) of \([p]\) to \( P(u, \gamma) \) to mark the dependency of the partition on \( u \) and \( \gamma \). \( P_0 \) and \( I \) are extended in a similar fashion.

**Lemma D.1.3.** Given any \( \gamma > 0 \), there exists a Lebesgue measure zero set \( \mathcal{L}_\gamma \subset \mathbb{R}^p \) such that for each \( u \in \mathcal{L}_\gamma \),

(i) There exists a sufficiently small ball \( B_\varepsilon(u) = \{ \tilde{u} : \| \tilde{u} - u \|_2 \leq \varepsilon \} \) such that the partition \( P(\tilde{u}; \gamma) \) remains the same over \( B_\varepsilon(u) \).

(ii) \( \eta(\cdot; \gamma) \) is differentiable at \( u \).

(iii) \( \forall u, v \in \mathbb{R}^p, \sum_{i=1}^{p} u_i \langle \nabla \eta_i(u; \gamma), v \rangle = \sum_{I \in P_0} \frac{1}{|I|} (\sum_{i \in I} u_i)(\sum_{i \in I} v_i) \).

**Proof.** Part (i), since the dual SLOPE norm ball is a polygon (with many faces), the orthogonal space of each face cut the entire space into many small regions, where the projection within each region is differentiable. Obviously the union of the boundaries of these regions is of measure 0. Let \( S \) be the union of these boundaries. Then \( S^c \subset \mathbb{R}^p \) is an open set, within which the projection is differentiable. This further implies the differentiability of \( \eta(u; \gamma) \) in \( u \) in \( S^c \).

Part (ii) is a simple result of Part (i) and Lemma D.1.2 (iv). For Part (iii), according to Part (i) and Lemma D.1.2 (iv), it is clear that

\[
\nabla \eta_j(u; \gamma) = 0, \quad \text{if} \quad \eta_j(u; \gamma) = 0,
\]

\[
[\nabla \eta_j(u; \gamma)]_i = \begin{cases} 
\frac{1}{|I_j|}, & i \in I_j \\
0, & i \notin I_j
\end{cases}, \quad \text{if} \quad \eta_j(u; \gamma) > 0.
\]

The identity in Part (iii) can then be directly verified based on the above results. \( \square \)

**Lemma D.1.4.** Given almost any \( u \in \mathbb{R}^p \), there exists a Lebesgue measure zero set \( \mathcal{L}_u \subset \mathbb{R}^{++} \) such that for each \( \gamma \in \mathcal{L}_u^c \),

(i) The partition \( P(u; \tilde{\gamma}) \) remains the same for all \( \tilde{\gamma} \in [\gamma - \epsilon, \gamma + \epsilon] \) with \( \epsilon \) sufficiently small.
(ii) $\eta(u; \cdot)$ is differentiable at $\gamma$. Assuming $u_1 \geq u_2 \geq \cdots \geq u_p \geq 0$, for all $1 \leq j \leq p$,

$$\frac{\partial \eta_j(u; \gamma)}{\partial \gamma} = \begin{cases} 0 & \text{if } \eta_j(u; \gamma) = 0 \\
\sum_{i \in I_j} \lambda_i |I_j| & \text{otherwise} \end{cases}$$

**Proof.** Part (i): Without loss of generality, we consider $u_1 > u_2 > \cdots > u_p > 0$ and $\tilde{\gamma} = \gamma + \Delta$. Choosing $\Delta$ small enough gives that

$$\eta_1(u; \gamma) \geq \eta_2(u; \gamma) \geq \cdots \geq \eta_p(u; \gamma) = 0 = \cdots = \eta_p(u; \tilde{\gamma}),$$

$$\eta_1(u; \tilde{\gamma}) \geq \eta_2(u; \tilde{\gamma}) \geq \cdots \geq \eta_p(u; \tilde{\gamma}) > 0 = \cdots = \eta_p(u; \tilde{\gamma}),$$

where $k$ and $\tilde{k}$ are the number of zero components that $\eta(u; \gamma)$ and $\eta(u; \tilde{\gamma})$ have, respectively. The key inequality is

$$\| \eta(u; \tilde{\gamma}) - \eta(u; \gamma) \|_2^2 \leq \sum_{j=p-k+1}^p |\eta(u; \tilde{\gamma}) - \eta(u; \gamma)|^2,$$

where (a) is by Lemma D.1.2 (i) and (b) is due to Lemma D.1.5 (i). Then (D.1.2) enables us to choose $\Delta$ small enough so that $\tilde{k} \leq k$. For the rest of the proof, we have

(1) We first show $\tilde{k} = k$, which is equivalent to

$$\sum_{j=p-k+1}^p |\eta(u; \tilde{\gamma})|^2 = 0,$$

when $\Delta$ is small. Suppose this is not true. Then there exist $\Delta_n \to 0$ and $p-k+1 \leq j_{\Delta_n} \leq p$ such that $\eta_{\Delta_n}(u; \tilde{\gamma}) \neq 0$. Lemma D.1.2 Part (iv) gives that

$$\eta_{\Delta_n}(u; \tilde{\gamma}) = \sum_{i \in I_{j_{\Delta_n}}} (u_i - \tilde{\gamma}_i \lambda_i),$$

and the inequality (D.1.2) implies that

$$\lim_{\Delta \to 0} \sum_{j=p-k+1}^p |\eta(u; \tilde{\gamma})|^2 = 0.$$
These result combined with the fact that \( \lim_{n \to \infty} \frac{\sum_{i \in I_j \Delta_n (u; \tilde{\gamma})} (u_i - \gamma \lambda_i)}{|I_j \Delta_n (u; \tilde{\gamma})|} = 0 \) yield

\[
\lim_{n \to \infty} \sum_{i \in I_j \Delta_n (u; \tilde{\gamma})} (u_i - \gamma \lambda_i) = 0. \tag{D.1.3}
\]

Consider the set \( L_1 = \{ \gamma \in \mathbb{R}^+: \sum_{i \in K} (u_i - \gamma \lambda_i) = 0 \text{ for some } K \subseteq \{1, 2, \ldots, p\} \} \).

Since \( u_i > 0 \) for all \( 1 \leq i \leq p \), \( L_1 \) has finite elements thus of Lebesgue measure zero. Hence, as long as \( \gamma \in L_1^c \), (D.1.3) is impossible to hold.

(2) We next show \( I_j (u; \gamma) = I_j (u; \tilde{\gamma}) \) for \( 1 \leq j \leq p - k \), where these sets are defined in (D.0.3). Lemma D.1.2 Part (iv) and the inequality (D.1.2) together imply that for each \( 1 \leq j \leq p - k \),

\[
\lim_{\Delta \to 0} \left| \frac{\sum_{i \in I_j (u; \gamma)} (u_i - \gamma \lambda_i)}{|I_j (u; \gamma)|} - \frac{\sum_{i \in I_j (u; \tilde{\gamma})} (u_i - \gamma \lambda_i)}{|I_j (u; \tilde{\gamma})|} \right| = 0. \tag{D.1.4}
\]

Now define the vector \( h^\Delta \in \mathbb{R}^p \) so that for each \( 1 \leq i \leq p \),

\[
h^\Delta_i = \begin{cases} 
0 & \text{if } i \notin I_j (u; \gamma), \text{ and } i \notin I_j (u; \tilde{\gamma}), \\
\frac{1}{|I_j (u; \gamma)|}, & \text{if } i \in I_j (u; \gamma) \text{ and } i \notin I_j (u; \tilde{\gamma}), \\
\frac{-1}{|I_j (u; \gamma)|}, & \text{if } i \notin I_j (u; \gamma) \text{ and } i \in I_j (u; \tilde{\gamma}), \\
\frac{1}{|I_j (u; \gamma)|} - \frac{1}{|I_j (u; \tilde{\gamma})|}, & \text{otherwise}.
\end{cases}
\]

Then, (D.1.4) can be rewritten as \( \lim_{\Delta \to 0} \langle h^\Delta, u - \gamma \lambda \rangle = 0 \). Consider the set \( L_2 = \{ \gamma \in \mathbb{R}^+: \langle h^\Delta, u - \gamma \lambda \rangle = 0 \text{ for some } h^\Delta \neq 0 \} \). We know such set has finite elements as long as \( u \) does not belong to the Lebesgue measure zero set \( \{ u : \langle h^\Delta, u \rangle = 0 \text{ for some } h^\Delta \neq 0 \} \). Moreover, since the set \( \{ h^\Delta \in \mathbb{R}^p : \Delta \text{ is small} \} \) is finite, it holds that \( \min_{h^\Delta \neq 0} \langle h^\Delta, u - \gamma \lambda \rangle > 0 \) for small \( \Delta \) when \( \gamma \in L_2^c \). This combined with (D.1.4) implies that \( h^\Delta = 0 \) when \( \Delta \) is small enough.

Part (ii): It is a simple result of Part (i) and Lemma D.1.2 (iv).

Next we list some properties which are relevant to the Lipschitz continuity, convexity and norm bounds of \( \eta \).
Lemma D.1.5. For any $u \in \mathbb{R}^p$, the proximal operator $\eta(u; \gamma)$ satisfies,

(i) $\|\eta(u_1; \gamma) - \eta(u_2; \gamma)\|_2^2 \leq \langle u_1 - u_2, \eta(u_1; \gamma) - \eta(u_2; \gamma) \rangle \leq \|u_1 - u_2\|_2^2$;

(ii) $\frac{1}{2}\|u - \eta(u; \gamma)\|_2^2 + \gamma\|\eta(u; \gamma)\|_\lambda = \frac{1}{2}(\|u\|_2^2 - \|\eta(u; \gamma)\|_2^2)$;

(iii) $\|\eta(u; \gamma)\|_2^2$ is convex in $u$ and non-increasing in $\gamma$.

(iv) $\|\eta(u; \gamma_1) - \eta(u; \gamma_2)\|_2 \leq \|\lambda\|_2 |\gamma_1 - \gamma_2|$.

(v) $\|\eta(u; \gamma)\|_\lambda \geq \frac{\|\lambda\|_1}{p}\|\eta(u; \gamma)\|_1$.

(vi) $\|\eta(u; \gamma, \lambda)\|_2^2 \leq \|\eta(u; \gamma, \frac{\|\lambda\|_2}{p}1)\|_2^2$. The right hand side is the $\ell_2$ norm square of a $\ell_1$ proximal operator.

Proof. To prove (i), we know that $\Pi_{D,\gamma}(u_1) = u_1 - \eta(u_1; \gamma)$ and $\Pi_{D,\gamma}(u_2) = u_2 - \eta(u_2; \gamma)$. The property of projection onto a convexity body implies that

$\langle u_1 - \Pi_{D,\gamma}(u_1), \Pi_{D,\gamma}(u_2) - \Pi_{D,\gamma}(u_1) \rangle \leq 0, \quad \langle u_2 - \Pi_{D,\gamma}(u_2), \Pi_{D,\gamma}(u_1) - \Pi_{D,\gamma}(u_2) \rangle \leq 0$.

Adding the two inequalities above up gives the first inequality of (i). The second one is by a simple use of Cauchy-Schwarz inequality. Part (ii) is the strong duality property.

For Part (iii), the equation in Part (ii) is equivalent to

$$\max_x \langle u, x \rangle - \frac{1}{2}\|x\|_2^2 - \gamma\|x\|_\lambda = \frac{1}{2}\|\eta(u; \gamma)\|_2^2.$$  

The term on the left-hand side is the maximum of a series of linear functions in $u$, hence convex. The monotonicity in $\gamma$ is obvious.

Part (iv): We first prove the inequality holds for $u$ that satisfies Lemma D.1.4. In this case we know there are finite number of discontinuity points of $\eta$ w.r.t. $\gamma$. Hence
for all such $u$,

$$
\| \eta(u; \gamma + \Delta) - \eta(u; \gamma) \|_2 = \left\| \Delta \int_0^1 \frac{\partial \eta(u; \gamma + t\Delta)}{\partial \gamma} dt \right\|_2 \\
\leq |\Delta| \int_0^1 \left\| \frac{\partial \eta(u; \gamma + t\Delta)}{\partial \gamma} \right\|_2 dt = (a) |\Delta| \int_0^1 \sqrt{\sum_{I \in \mathcal{P}_0} \frac{1}{|I|} \left(\sum_{i \in I} \lambda_i \right)^2} dt \\
\leq |\Delta| \int_0^1 \sqrt{\sum_{z \in \mathcal{P}_0} \sum_{i \in z} \lambda_i^2} dt \leq |\Delta| \| \lambda \|_2,
$$

where $(a)$ is due to Lemma D.1.4 (ii). For other $u$’s, since they all belong to a Lebesgue measure zero set, there exists a sequence $u_m \to u$ and $u_m$ satisfies Part (iv). Hence Part (iv) holds for other $u$’s as well due to the continuity of $\eta(\cdot; \gamma)$.

**Part (v):** Denote the uniform permutation $T: [p] \mapsto [p]$. It is clear that

$$
\| \eta(u; \gamma) \|_2 = \sum_{i=1}^p |\eta(u; \gamma)|_{(i)} \lambda_i \geq E_T \sum_{i=1}^p |\eta(u; \gamma)|_{(i)} \lambda_T(i) = \sum_{i=1}^p |\eta(u; \gamma)|_{(i)} E_T \lambda_T(i).
$$

The proof is completed by verifying that $E_T \lambda_T(i) = \frac{\sum_{i=1}^p \lambda_i}{p}$ for all $1 \leq i \leq p$.

For Part (vi), first we realize that it is equivalent to showing $D_\gamma \supset \{ v : |v|_{(i)} \leq \frac{\gamma \| \lambda \|_2^2}{p} \}$. According to the structure of $D_\gamma$ in (D.0.2), the above set relation can be further translated as

$$
\frac{\gamma \| \lambda \|_2^2}{p} \leq \min_{1 \leq s \leq p} \frac{\gamma}{2} \sum_{i=1}^s \lambda_i = \frac{\gamma}{2} \sum_{i=1}^p \lambda_i = \frac{\gamma}{2} \sum_{i=1}^p \lambda_i.
$$

This is directly verified by Lemma D.2.5, the proof is hence completed.

The last lemma in this section characterizes the diameter of the dual norm ball $D_1$.

**Lemma D.1.6.** We have the following results for the unit dual norm ball $D_1$:

$$
\max\{ \| z \|_2 : z \in \partial D_1 \} = \| \lambda \|_2.
$$

This implies that $\mathbb{E} \| \eta(x + h; \chi) - x \|_2^2 \leq p + \chi^2 \| \lambda \|_2^2$.

**Proof.** First it is not hard to see that $z = \lambda \in D_1$ and $\| z \|_2 = \| \lambda \|_2$. Now we show this is the largest possible value that can be reached. First we note that $\lambda$ is the only point in $D_1$ that meet all the constraints in (D.0.2) up to arbitrary signs on each component. Now for any other $z \in D_1$, let $j_1 = \min\{ k : \sum_{i=1}^k |z|_{(i)} < \sum_{i=1}^k \lambda_k \}$.
Then we have $|z|_{(i)} = \lambda_i$ for any $i < j_1$ and $|z|_{(j_1)} < \lambda_j$. Now let $j_2 = \min\{k : k > j_1, \sum_{i=1}^{k} |z|_{(i)} = \lambda_i\}$. If $j_2$ is not defined (the set that defines $j_2$ is empty), then we can safely increase $|z|_{(j_1)}$ without violating the constraints and also make $\|z\|_2$ larger. Otherwise, we can pick a pair of new values for $(|z|_{(j_1)}, |z|_{(j_2)}) = (|z|_{(j_1)} + \Delta, |z|_{(j_2)} - \Delta)$ for some small enough $\Delta > 0$ without violating any of the constraints. We note that this new pick will increase $\|z\|_2^2$ since $(|z|_{(j_1)} + \Delta)^2 + (|z|_{(j_2)} - \Delta)^2 > |z|_{(j_1)}^2 + |z|_{(j_2)}^2$. Hence as long as one of the $\sum_{i=1}^{k} |z|_{(i)} < \sum_{i=1}^{k} \lambda_i$ for some $1 \leq k \leq p$, we can vary $z$ to increase $\|z\|_2^2$. This completes the proof.

To justify the rest of the conclusions, we note that

$$E\|\eta(x + h; \chi) - x\|_2^2 = E\|h - \Pi_{\chi}(x + h)\|_2^2 \leq p + E\|\Pi_{\chi}(x + h)\|_2^2 \leq p + \chi^2 \|\lambda\|_2^2,$$

where we used the fact that $E\langle h, \Pi_{\chi}(x + h)\rangle = p - E\langle h, \eta(x + h; \chi) - \eta(x; \chi)\rangle \geq 0$ due to Lemma D.1.5 (i).

**Lemma D.1.7.** We have the following characterization of the limiting quantity:

$$\lim_{\sigma \to 0} \|\eta(x/\sigma + h; \chi) - x/\sigma\|_2^2 = \frac{k}{p} + \frac{\chi^2}{p} \|\lambda[1:k]\|_2^2 + \|\eta(h[k+1:p]; \chi, \lambda[k+1:p])\|_2^2.$$

**Proof.** If we assume the nonzero components of $x$ are different from each other. Without loss of generality, suppose $|x_1| > \ldots > |x_k| > x_{k+1} = \ldots = x_p = 0$. Then as $\sigma \to 0$, the gap between any two consecutive terms of $\{|x_i/\sigma + h_i|\}_{i=1}^k$ converges to infinity. As a result, the proximal operator on this part becomes componentwise soft-thresholding. On the other hand, the rest $p-k$ components interact with $\lambda[k+1:p]$ to form a proximal operator independently from the first $k$ components. This leads to the following observation:

$$\lim_{\sigma \to 0} \eta_{i}(\frac{x}{\sigma} + h; \chi, \lambda) - \frac{x_i}{\sigma} = (h_i - \chi \lambda_i) \text{sign}(x_i), \quad 1 \leq i \leq k, \quad (D.1.5)$$

$$\lim_{\sigma \to 0} \eta_{[k+1:p]}(\frac{x}{\sigma} + h; \chi, \lambda) - \frac{x[k+1:p]}{\sigma} = \eta(h[k+1:p]; \chi, \lambda[k+1:p]). \quad (D.1.6)$$

It is important to note that here $h_i$ are not ordered according to $x_i$ and hence $h[k+1:p] \sim \mathcal{N}(0, I_{p-k})$ and is independent from $h_i$ for $i \leq k$. 

---

**APPENDIX D. PROOF OF CHAPTER 4**

186
This indicates the identity below and our goal naturally follows.

\[
\lim_{\sigma \to 0} \|\eta(x/\sigma + h; \chi) - x/\sigma\|_2^2 = \|h_{[1:k]}\|_2^2 + \chi^2 \|\lambda_{[1:k]}\|_2^2 + \|\eta(h_{[k+1:p]}; \chi, \lambda_{[k+1:p]})\|_2^2.
\]

This leads to the final goal we would like to prove.

**Lemma D.1.8.** For two sequences of weights \(\lambda_1\) and \(\lambda_2\), if \(\lambda_{1,i} \geq \lambda_{2,i}\) \(\forall 1 \leq i \leq p\), then \(|\eta(u; \chi, \lambda_1)| \leq |\eta(u; \chi, \lambda_2)|\) for any \(1 \leq i \leq p\).

**Proof.** Without loss of generality, we assume \(u_1 \geq \ldots \geq u_p \geq 0\). Let \(a_{1,i} = u_i - \chi \lambda_{1,i}\), \(a_{2,i} = u_i - \chi \lambda_{2,i}\). Then we have \(a_{1,i} \leq a_{2,i}\) for any \(1 \leq i \leq p\). The next two steps for executing SLOPE is to run a isotonic regression on each sequence \(a_1\) and \(a_2\) and threshold each one at 0 respectively. Suppose \(b_j\) is the isotonic regressor of \(a_j\) for \(j = 1, 2\), i.e.,

\[
b_j = \arg \min_b \|a_j - b\|_2^2, \quad \text{subject to} \quad b_1 \geq \ldots \geq b_p.
\]

As long as we can show \(b_{1,i} \leq b_{2,i}\) for all \(1 \leq i \leq p\) the proof is completed. We prove this claim by contradiction. Suppose this is not the case, then we define two new sequence \(\tilde{b}_1\) and \(\tilde{b}_2\) as

\[
\tilde{b}_{j,i} = \begin{cases} 
    b_{j,i}, & \text{if } b_{1,i} \leq b_{2,i}, \\
    \frac{1}{2}(b_{1,i} + b_{2,i}), & \text{if } b_{1,i} > b_{2,i}
\end{cases}, \quad \text{for } i \in \{1, 2\}.
\]

Obviously both \(\tilde{b}_1\) and \(\tilde{b}_2\) are nonincreasing sequence. In addition, for any index \(i\) where \(b_{1,i} > b_{2,i}\), it is easy to verify \((a_{1,i} - \tilde{b}_{1,i})^2 + (a_{2,i} - \tilde{b}_{2,i})^2 < (a_{1,i} - b_{1,i})^2 + (a_{2,i} - b_{2,i})^2\), which is equivalent to \(\frac{1}{2}(b_{1,i} - b_{2,i})^2 + (a_{2,i} - a_{1,i})(b_{1,i} - b_{2,i}) > 0\). Overall, this indicates that

\[
\|a_1 - \tilde{b}_1\|_2^2 + \|a_2 - \tilde{b}_2\|_2^2 < \|a_1 - b_1\|_2^2 + \|a_2 - b_2\|_2^2
\]

hence at least \(\|a_j - \tilde{b}_j\|_2 < \|a_j - b_j\|_2\) for one of \(j \in \{1, 2\}\). This contradicts the property of the regressor. \(\square\)
D.2 Reference materials

In this section, we summarize a few results which have been proved in previous works and are used in our paper.

D.2.1 Convex Gaussian Min-max Theorem (CGMT)

The Convex Gaussian Min-max Theorem (CGMT) provides a powerful tool to reduce the analysis of SLOPE estimator under iid Gaussian design. Denote

$$\Phi(G) := \min_{z \in S_z} \max_{u \in S_u, v \in S_v} u^\top G z + \psi(z, u, v),$$

$$\phi(g, h) := \min_{z \in S_z} \max_{u \in S_u, v \in S_v} \|z\|_2 g^\top u + \|u\|_2 h^\top z + \psi(z, u, v),$$

$$\tilde{\phi}(g, h) := \max_{u \in S_u, v \in S_v} \min_{z \in S_z} \|z\|_2 g^\top u + \|u\|_2 h^\top z + \psi(z, u, v),$$

where $G \in \mathbb{R}^{n \times p}$, $h \in \mathbb{R}^p$, $g \in \mathbb{R}^n$ have independent standard normal entries.

**Theorem D.2.1.** (CGMT). Suppose $S_z, S_u, S_v$ are all compact sets, and $\psi(z, u, v)$ is continuous on $S_z \times S_u \times S_v$, then the following results hold:

(i) For all $c \in \mathbb{R}$,

$$\mathbb{P}(\Phi(G) \leq c) \leq 2\mathbb{P}(\phi(g, h) \leq c).$$

(ii) Further assume that $S_z, S_u, S_v$ are convex sets, and $\psi(z, u, v)$ is convex on $S_z$ and concave on $S_u \times S_v$. Then for all $c \in \mathbb{R}$,

$$\mathbb{P}(\Phi(G) \geq c) \leq 2\mathbb{P}(\tilde{\phi}(g, h) \geq c).$$

The above results are essentially taken from Theorem 3 in [TOH15]. The minor difference is that the current version involves an extra vector $v$, and $\tilde{\phi}(g, h)$ appears in Part (ii) instead of $\phi(g, h)$. By a rather straightforward inspection of the proof in [TOH15], these changes continue to hold.

In addition, when case (i) in Theorem D.2.1 occurs, we say $\Phi(G) \geq_p \phi(g, h)$; Similarly when case (ii) happens, we say $\Phi(G) \leq_p \tilde{\phi}(g, h)$. 
D.2.2 Basic results in concentration inequalities

In this section we list some basic concentration inequalities that are used in this paper. Again in these theorems, $C,c$ are used to denote absolute constants.

**Theorem D.2.2** (Bernstein’s inequality). Let $x_1, \ldots, x_n$ be independent, mean zero, sub-exponential random variables. Then for every $t \geq 0$, we have

$$P\left(\left|\sum_{i=1}^{n} x_i\right| \geq t\right) \leq 2 \exp \left[-c_B \min \left(\frac{t^2}{\sum_{i=1}^{n} \|x_i\|_{\psi_1}^2}, \frac{t}{\max_i \|x_i\|_{\psi_1}}\right)\right],$$

where $c_B > 0$ is an absolute constant, and $\|\cdot\|_{\psi_1}$ is the sub-exponential norm defined as $\|x\|_{\psi_1} = \inf\{t > 0 : \mathbb{E}e^{\|x\|/t} \leq 2\}$.

Please refer to Theorem 2.8.1 in [Ver18] for a proof.

**Lemma D.2.1.** For two sub-Gaussian random variable $X$ and $Y$, we have $\|XY\|_{\psi_1} \leq \|X\|_{\psi_2} \|Y\|_{\psi_2}$.

Please refer to Lemma 2.7.7 in [Ver18] for a proof.

**Theorem D.2.3** (Gaussian Lipschitz). Consider a random vector $X \sim \mathcal{N}(0, I_p)$ and a Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$, then

$$P(|f(X) - \mathbb{E}f(X)| > t) \leq 2e^{-\frac{ct^2}{\|f\|_{\text{Lip}}^2}}, \quad \forall t \geq 0.$$

Please refer to Theorem 5.2.2 in [Ver18] for a proof.

D.2.3 Other results

We list the rest of the results we used in the main proof here.

**Theorem D.2.4** (Saddle Point Theorem). Let $X$ and $Z$ be two nonempty convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; and $\phi : X \times Z \to \mathbb{R}$ be a function such that $\phi(\cdot, z)$ is convex and closed over $X$ for each $z \in Z$, and $-\phi(x, \cdot)$ is convex and closed over $Z$ for each $x \in X$. If for some $\bar{x} \in X, \bar{z} \in Z, \bar{c} \in \mathbb{R}$, the levels sets

$$\{x \in X : \phi(x, \bar{z}) \leq \bar{c}\}, \quad \{z \in Z : \phi(\bar{x}, z) \geq \bar{c}\},$$
are nonempty and compact, then the set of saddle points of $\phi$ is nonempty and compact.

The above theorem is Proposition 5.5.7 in [Ber09].

For the proof of the following lemmas, we recall our readers of the notations defined in (D.0.1), (D.0.2) and (D.0.3).

**Lemma D.2.2.** Let $f(a, b) = \|\eta(x + ah; b)\|_2^2$, then at those differentiable points of $f$, we have the following equations for the partial derivatives of $f$

\[
\frac{\partial f}{\partial a} = 2\langle \eta(x + ah; b), h \rangle, \quad (D.2.1)
\]
\[
\frac{\partial f}{\partial b} = \frac{2}{b}||\eta(x + ah; b)||_2^2 - \frac{2}{b}\langle \eta(x + ah; b), x + ah \rangle. \quad (D.2.2)
\]

**Proof.** (D.2.1) is a simple application of the chain rule. Regarding (D.2.2), since $f(a, b) = b^2\|\eta(x/b + ah/b; 1)\|_2^2$, we have that

\[
\frac{\partial f}{\partial b} = 2b\|\eta(x/b + ah/b; 1)\|_2^2 + 2b^2\langle \eta(x/b + ah/b; 1), -x/b^2 - ah/b^2 \rangle,
\]
\[
= \frac{2}{b}||\eta(x + ah; b)||_2^2 - \frac{2}{b}\langle \eta(x + ah; b), x + ah \rangle.
\]

**Lemma D.2.3.** Let $G(a) = \|\eta(x + ach; ab) - x\|_2^2$, then we have

\[
G'(a) = \frac{2}{a}\left(\|\eta(x + ach; ab)\|_2^2 - 2\langle \eta(x + ach; ab), x \rangle + \sum_{\sum_{k \in \mathcal{I}} x_k \cdot \text{sign}(x_k + ach_k)}^{\mathcal{I}} \frac{\|\sum_{k \in \mathcal{I}} x_k \cdot \text{sign}(x_k + ach_k)\|_2}{|\mathcal{I}|}\right).
\]

**Proof.** Since we only care about the derivative, we first ignore the constant term and rewrite $G(a) = \|\eta(x + ach; ab)\|_2^2 - 2\langle \eta(x + ach; ab), x \rangle$. By Lemma D.2.2, it is not hard to verify that

\[
\frac{d}{da} \|\eta(x + ach; ab)\|_2^2 = \frac{2}{a}\|\eta(x + ach; ab)\|_2^2 - \frac{2}{a}\langle \eta(x + ach; ab), x \rangle. \quad (D.2.3)
\]
Now for the second term, we have that
\[
\frac{d}{da}(\eta(x + ah; ab), x) = \frac{d}{da}(\eta(x/a + ch; b), x)
\]
\[
= \frac{1}{a} \langle \eta(x + ah; ab), x \rangle + a \sum_{k=1}^{p} x_k \langle \nabla \eta_k \left( \frac{x}{a} + ch; b \right), -x/a^2 \rangle
\]
\[
= \frac{1}{a} \langle \eta(x + ah; ab), x \rangle - \frac{1}{a} \sum_{I \in P_0} \frac{1}{|I|} \left( \sum_{k \in I} x_k \cdot \text{sign}(x_k + ach_k) \right)^2.
\]

The proof is completed by combining the above two parts.

\[\square\]

**Lemma D.2.4.** Let \( f(v) = \|\eta(x + \sqrt{v}h; \sqrt{v} \chi) - x\|_2^2 \). Then, we have the following upper bound for \( f(v) \):
\[
f(v) \leq \|\eta(h; \chi)\|_2^2 v + \|x\|_2^2 := l(v).
\] (D.2.4)

In addition, \( l(v) \) is the asymptote of \( f(v) \) as \( v \to \infty \).

**Proof.** Let \( g(a) = \|\eta(ax + h; \chi)\|_2^2 \). We can express \( f(v) - \|\eta(h; \chi)\|_2^2 v \) as the following:
\[
f(v) - \|\eta(h; \chi)\|_2^2 v = \left( \|\eta(x + \sqrt{v}h; \chi)\|_2^2 - \frac{2}{\sqrt{v}} \langle \eta(x + \sqrt{v}h; \chi), x \rangle - \|\eta(h; \chi)\|_2^2 \right) + \|x\|_2^2
\]
\[
= - \frac{g(0) - g\left( \frac{1}{\sqrt{v}} \right) + \frac{1}{\sqrt{v}} g'\left( \frac{1}{\sqrt{v}} \right)}{1/v} + \|x\|_2^2 = \|x\|_2^2 - \frac{1}{2} g''\left( \frac{1}{\sqrt{v}} \right),
\]
where \( \tilde{v} > v \) is some value for Taylor expansion.

By inspecting the proof of Lemma D.2.2 and Lemma D.2.3, it is not hard to see that
\[
g''(a) = 2 \sum_{I \in P_0} \frac{1}{|I|} \left( \sum_{k \in I} x_k \cdot \text{sign}(ax_k + h_k) \right)^2.
\]
The conclusion then follows.

\[\square\]

**Lemma D.2.5.** For a sequence of decreasing weights \( \lambda = (\lambda_1, \ldots, \lambda_p) \), if \( \lambda_1 \leq 1 \), then for any \( 0 \leq \lambda < \frac{\|\lambda\|_2^2}{\sqrt{p}} \), we have that
\[
\frac{\# \{ \lambda_i > \lambda \}}{p} \geq 1 - \frac{1}{1 - \lambda^2}.
\]
In addition, the following lower bound for \( \frac{\|\lambda\|_1}{p} \) holds.

\[
\frac{\|\lambda\|_1}{p} \geq \frac{\|\lambda\|_1^2}{p^2}.
\]

Proof. For any \( \lambda < \frac{\|\lambda\|_2}{\sqrt{p}} \), we have that

\[
\frac{\|\lambda\|_2^2}{p} = \sum_{i, \lambda_i \leq \lambda} \lambda_i^2 + \sum_{\lambda_i > \lambda} \lambda_i^2 \leq \lambda^2 \#\{\lambda_i \leq \lambda\} + \#\{\lambda_i > \lambda\},
\]

\[
\Rightarrow \frac{\#\{\lambda_i \leq \lambda\}}{p} \leq \frac{1 - \frac{\|\lambda\|_2^2}{p}}{1 - \lambda^2}.
\] (D.2.5)

This completes the first part of the result. To show the second part of the lemma, we need to solve the following optimization problem:

\[
\min_{\lambda} \sum_{i=1}^{p} \lambda_i, \quad \text{subject to} \quad 0 \leq \lambda_p \leq \ldots \leq \lambda_1 = 1, \quad \|\lambda\|_2^2 = \kappa p,
\]

where we introduce a fixed value of \( \kappa \) to make the optimization notationally valid.

We argue that the optimal sequence of \( \lambda \) must satisfy \( \lambda_i \in \{0, 1\} \) for all but at most one \( i \in \{1, \ldots, p\} \). In other words, \( \lambda = (1, \ldots, 1, c, 0, \ldots, 0) \) with some \( c \in [0, 1] \). To show this, we note that for any \( (\lambda_i, \lambda_j) \) with \( i > j \), if \( \lambda_i > \lambda_j \), then given a fixed squared sum of the two \( S = \lambda_i^2 + \lambda_j^2, \lambda_i + \sqrt{S - \lambda_i^2} \) is a decreasing function in \( \lambda_i \) hence the larger \( \lambda_i \) (equivalently the smaller \( \lambda_j \)) is, the larger \( \lambda_i + \lambda_j \) we would obtain. As a result, the optimal choice for \( (\lambda_i, \lambda_j) \) is either \( (\sqrt{S}, 0) \) when \( S \leq 1 \) or \( (1, \sqrt{S - 1}) \) when \( S > 1 \). We call this procedure “extremize”.

Now we optimize the entire sequence \( \lambda \). For any given initial sequence \( 1 \geq \lambda_1 \geq \ldots \geq \lambda_p \geq 0 \), we set \( i_L = \min\{i : \lambda_i < 1\}, \ i_R = \max\{i : \lambda_i > 0\} \). Then we extremize \( (\lambda_{i_L}, \lambda_{i_R}) \) such that \( \lambda_{i_L} = 1 \) or \( \lambda_{i_R} = 0 \). Next we update the value of \( (i_L, i_R) \) and continue this extremization procedure. The mechanism of “extremize” guarantees that \( i_R - i_L \) decreases by at least 1 after each round of operations hence this procedure stops in at most \( p \) steps. The resulting sequence of \( \lambda \) will have the largest \( \|\lambda\|_1 \) with \( \|\lambda\|_2^2 = \kappa p \) and takes the aformentioned optimal form.
For the optimal sequence $\lambda = (1, \ldots, 1, c, 0, \ldots, 0)$, we have $\#\{\lambda_i = 1\} = \|\lambda\|_2^2$ and $c = \sqrt{\|\lambda\|_2^2 - \|\lambda\|_2^2}$. This implies that $\|\lambda\|_1 = \|\lambda\|_2^2 + c \geq \|\lambda\|_2^2 + c^2 = \|\lambda\|_2^2$.

This proof is hence completed.

\[ \tag*{\text{Lemma D.2.6.}} \]

We have that $\#\{|x_i| \geq \|x\|_\infty \sqrt{2^p} \} \geq \|x\|_2^2 \|x\|_\infty^2 \tau$. The result then follows.

\[ \tag*{\text{Lemma D.2.7.}} \]

Recall the definition of Moreau envelope: $e_f(x; \tau) = \inf_u \frac{1}{2\tau} \|x - u\|_2^2 + f(u)$, we have the following identity hold:

$e_f(x; \tau) + e_{f^*}(x/\tau; 1/\tau) = \|x\|_2^2/2\tau$.

\[ \tag*{\text{Proof.}} \]

We verify the equation by definition:

$e_{f^*}(x/\tau; 1/\tau) = \inf_u \frac{\tau}{2} \|x/\tau - u\|_2^2 + f^*(u)$

$= \inf_u \frac{\tau}{2} \|x/\tau - u\|_2^2 + \sup_v \{v, u\} - f(v)$

$\overset{(a)}{=} \sup_v -f(v) + \inf_u \frac{\tau}{2} \|x/\tau - u\|_2^2 + \langle v, u \rangle$

$= \sup_v -f(v) + \frac{\|x\|_2^2}{2\tau} - \frac{\|x - x\|_2^2}{2\tau} + \inf_u \frac{\|v - x\|_2^2}{2\tau} + \langle v - x, u \rangle + \frac{\tau}{2} \|u\|_2^2$

$= \frac{\|x\|_2^2}{2\tau} - e_f(x, \tau),$

where step (a) is due to the fact that the target function is convex-concave in $(u, v)$, and the saddle point could be attained in a compact set (we may limit our scope on the set where $\|u\|_2 \leq \|x\|_2/\tau$ and $\|v\|_2 \leq \|x\|_2$). The proof is then completed due to common Minimax theorem.
Lemma D.2.8. Let $Z \sim \mathcal{N}(0, 1)$. We have the following inequalities about $Z$ hold:

(i) $\frac{1}{2} E(|Z| - x)^2 = (1 + x^2) \Phi(-x) - x \phi(x) \geq \frac{\phi(x)}{2} \Phi(-x) \geq \frac{\phi(\sqrt{2}x)}{2}$.

(ii) $\frac{\phi(x)}{1 + x} \leq \Phi(-x) \leq \frac{2\phi(x)}{1 + x}$.

(iii) $\phi(x) - x \Phi(-x) \geq \frac{1}{2} \phi(\sqrt{2}x), \frac{\phi(x)}{x} \geq \phi(\sqrt{2}x)$.

We note that (ii) serves as an upper bound for $\frac{1}{2} E(|Z| - x)^2$ since $(1 + x^2) \Phi(-x) - x \phi(x) \leq \Phi(-x)$.


[CFJL18] Emmanuel Candès, Yingying Fan, Lucas Janson, and Jinchi Lv. Panning for gold: model-xknockoffs for high dimensional controlled variable


[DTL17] Oussama Dhifallah, Christos Thrampoulidis, and Yue M Lu. Phase retrieval via linear programming: Fundamental limits and algorithmic


