Arbitrage Theory Under Portfolio Constraints

Zhi Li

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Abstract

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In this dissertation, we adopt the viability approach to mathematical finance developed in the book of KARATZAS AND KARDARAS (2020) [1], and extend it to settings where portfolio choice is constrained.

We introduce in Chapter 2 the notions of supermartingale numéraire, supermartingale deflator, and viability. After that, we characterize all supermartingale deflators under conic constraints on portfolio choice. Most importantly, we prove a fundamental theorem for equity market structure and arbitrage theory under such conic constraints, to the effect that the existence of the supermartingale numéraire is equivalent to market viability. Further, and always under the assumption of viability, we establish some additional optimality properties of the supermartingale numéraire. In the end of Chapter 2, we pose and solve a problem of robust maximization of asymptotic growth, under some realistic assumptions.

In Chapter 3, we state and prove the Optional Decomposition Theorem under conic constraints. Using this version of the Optional Decomposition Theorem, we deal with the problem, of superhedging contingent claims.

In Chapter 4, we consider yet another portfolio optimization problem. Under simultaneous conic constraints on portfolio choice, and drawdown constraints on their generated wealth, we try to maximize the long-term growth rate from investment. Application of the AZÉMA-YOR transform allows us to show that the optimal portfolio for this optimization problem is a simple

path transformation of a supermartingale numéraire portfolio. Some asymptotic properties of this portfolio are also discussed in Chapter 4.

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Dedication

To my wife Boyuan Zhang

Chapter 1: The Market

1.1 Probabilistic Setup

1.1.1 *Stochastic Structure*

Throughout this dissertation, we shall work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The symbol "$\mathbb{E}^{\mathbb{P}}[f]$" will be used for the LEBESGUE integral, or expectation with respect to the probability measure $\mathbb{P}$, of an $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$, provided that either $f^+ := \max(f, 0)$, the postive part of f, or $f^- := \max(-f, 0)$, the negative part of f, has finite $\mathbb{P}$-expectation.

The filtration $\mathbb{F}$ will denote the flow of relevant information. To be specific, we shall write the filtration $\mathbb{F}$ as $\mathcal{F}(\cdot) \equiv (\mathcal{F}(t))_{t\in\mathbb{R}_+}$, i.e., a family of sub-$\sigma$-algebras of $\mathcal{F}$ which is nondecreasing in the sense that $\mathcal{F}(s) \subseteq \mathcal{F}(t) \subseteq \mathcal{F}$ holds for $0 \le s \le t < \infty$. We only impose the right-continuity assumption on this filtration

$$\mathcal{F}(t) = \mathcal{F}(t+) := \bigcap_{s>t} \mathcal{F}(s), \ \forall\, t \in \mathbb{R}_+.$$

To emphasize, we do not assume that this filtration is augmented by sets of measure zero. We set

$$\mathcal{F}(\infty) := \bigvee_{t\in\mathbb{R}_+} \mathcal{F}(t)$$

and, for concreteness only, take $\mathcal{F}(0) = \{\emptyset, \Omega\}$, mod. $\mathbb{P}$. Moreover, we use the notation $\mathcal{T}$ to represent the collection of stopping times of $\mathcal{F}(\cdot)$.

On such a filtered probability space we shall consider random variables, that is, $(\mathcal{F}/\mathcal{B}(\mathbb{S}))$-measurable mappings $h : \Omega \to \mathbb{S}$ with values in some metric space $\mathbb{S}$ equipped with the σ-algebra $\mathcal{B}(\mathbb{S})$ of its BOREL-measurable mappings $X : \Omega\times\mathbb{R}_+ \to \mathbb{S}$. Unless otherwise explicitly mentioned, identities, inequalities and comparisons involving random variables are interpreted as being valid for $\mathbb{P}$-almost every (a.e.) argument $\omega \in \Omega$.

1.1.2 *Predictability*

Let us recall the definition of adapted processes.

Definition 1.1. The stochastic process X is *adapted* to the filtration $\mathcal{F}(\cdot)$ if, for each $t \geq 0$, $X(t)$ is an $\mathcal{F}(t)$-measurable random variable.

In this paper, all stochastic processes X will be adapted to the underlying filtration $\mathcal{F}(\cdot)$. In fact, processes which model investment decisions will be assumed to satisfy the even more stringent requirement of predictability, that is, to be measurable with respect to the predictable σ-algebra, whose definition we now recall.

Definition 1.2. (**Predictability**). An $\mathbb{R}^n$−valued stochastic process of the form $\sum_{k=1}^m h_j 1_{(t_{j-1},t_j]}$, where $m \in \mathbb{N}$, $0 = t_0 < \cdots < t_m$, and $h_j : \Omega \to \mathbb{R}^n$ is $\mathcal{F}(t_{j-1})$-measurable for all $j \in \{1, \cdots, m\}$, will be called *simple predictable*.

We shall denote by $\mathcal{P}$ the smallest σ-algebra on the product space $\Omega \times \mathbb{R}_+$ with respect to which all simple predictable processes are measurable, and call it the *predictable σ-algebra*. We shall say that a given $\mathbb{R}^n$-valued process is *predictable*, if it is measurable with respect to this σ-algebra $\mathcal{P}$. Given any BOREL subset E of R^n for some integer $n \in \mathbb{N}$, we use $\mathcal{P}(E)$ to denote the class of all E-valued, predictable processes.

Intuitively, predictability means that all decisions should be based only on past and present information.

1.1.3 *Continuous Semimartingales*

A real-valued process X is called *continuous semimartingale*, if it can be written as the sum $X = V + N$ of an adapted continuous process V whose paths have finite variation on compact time intervals, and of a continuous local martingale N.

We shall fix throughout an integer $n \in \mathbb{N}$ and consider a continuous, $\mathbb{R}^n$-valued continuous semimartingale $R \equiv (R_i)_{i\in\{1,\cdots,n\}}$, with $R_i(0) = 0$ and component processes

$$R_i = A_i + M_i, \quad \forall \ i \in \{1, \cdots, n\}. \tag{1.1}$$

Here, the real-valued component processes A_i with $A_i(0) = 0$ are adapted, continuous, and have finite variation on compact time intervals; whereas the real-valued components M_i are continuous local martingales with $M_i(0) = 0$, for all $i \in \{1, \cdots, n\}$.

Next, we introduce the continuous, nondecreasing scalar process

$$O := \sum_{i=1}^{n} \left(\int_0^{\cdot} |\mathrm{d}A_i(t)| + \langle M_i, M_i \rangle \right), \tag{1.2}$$

with $\int_0^T |\mathrm{d}A_i(t)|$ denoting the total variation of A_i on the interval $[0, T]$ for $T \in \mathbb{R}_+$. This scalar process O plays the role of an *"operational clock"* for the vector semimartingale R.

A straightforward application of the RADON-NIKODÝM Theorem implies the existence of predictable processes

$$\alpha \equiv (\alpha_i)_{i\in\{1,\cdots,n\}} \qquad \text{and} \qquad c \equiv (c_{ij})_{1\le i,j\le n}, \tag{1.3}$$

vector-valued and matrix-valued, respectively, such that

$$A = \int_0^{\cdot} \alpha(t)\mathrm{d}O(t), \qquad \text{and} \qquad C \equiv \langle M, M \rangle = \int_0^{\cdot} c(t)\mathrm{d}O(t). \tag{1.4}$$

We use the notation $C \equiv (C_{ij})_{1\le i,j\le n}$ for the nonnegative-definite, matrix-valued process of covariations

$$C_{ij} := \langle M_i, M_j \rangle = \langle R_i, R_j \rangle, \qquad 1 \le i, j \le n. \tag{1.5}$$

The components of the vector-valued process $\alpha \equiv (\alpha_i)_{i\in\{1,\cdots,n\}}$ represent the *local mean rates of return* of individual stocks in the financial market; whereas the entries of the matrix-valued process $c \equiv (c_{ij})_{1\le i,j\le n}$ stand for the *local covariation rates* of these stocks. All these rates are measured with respect to the operational clock O of (1.2).

1.1.4 *Vector Stochastic Integration*

For a given integer $n \in \mathbb{N}$, we consider a continuous vector-valued semimartinagle $R \equiv (R_i)_{i\in\{1,\cdots,n\}}$, with $R_i(0) = 0$ as in Subsection (1.3).

Definition 1.3. (Vector integrands) We denote by $\mathcal{I}(R)$ the class of predictable vector processes

$$\pi \equiv (\pi_i)_{i\in\{1,\cdots,n\}},$$

which are integrable with respect to the vector semimartingale R. Furthermore, for a given BOREL set $E \subseteq \mathbb{R}^n$, we denote the collection of all E-valued processes in $\mathcal{I}(R)$ by

$$\mathcal{I}(R;E) := \mathcal{I}(R) \cap \mathcal{P}(E).$$

The collection $\mathcal{I}(R)$ has a very convenient operational characterization. A predictable process $\pi \equiv (\pi_i)_{i\in\{1,\cdots,n\}}$ belongs to $\mathcal{I}(R)$, if and only if

$$\int_0^T \Big(\mid \pi'(t)\alpha(t) \mid + \pi'(t)c(t)\pi(t)\Big)\mathrm{d}O(t) < \infty, \qquad \forall\ T \in \mathbb{R}_+. \tag{1.6}$$

For instance, this condition is automatically satisfied by any predictable process π which is *locally bounded*; i.e., for which there exist a localizing sequence $(\tau_m)_{m\in\mathbb{N}}$ of stopping times that increase $\mathbb{P}$-a.e. to infinity, and a sequence of nonnegative real constants $(k_m)_{m\in\mathbb{N}}$ such that

$$\max_{i\in\{1,\cdots,n\}} \sup_{t\in[0,\tau_m]} \mid \pi_i(t) \mid \leq k_m, \qquad \forall\ m \in \mathbb{N}.$$

In particular, $\mathcal{I}(R;E) = \mathcal{P}(E)$ holds for a bounded set $E \in \mathcal{B}(\mathbb{R}^n)$.

Thanks to the discussion in KARATZAS AND KARDARAS (2020) [1], subsection 1.1.3., condition (1.6) ensures that every vector process $\pi \in \mathcal{I}(R)$ is both:

- LEBESGUE-integrable with respect to the vector-valued process of finite variation $A = (A_i)_{i\in\{1,\cdots,n\}}$, and the resulting integral process has finite variation. In fact, the first variation process associated with the integral $\int_0^{\cdot} \pi'(t)\mathrm{d}A(t)$ of π with respect to A, equals

$$\int_0^{\cdot} \mid \pi'(t)dA(t) \mid \equiv \int_0^{\cdot} \mid \pi'(t)\alpha(t) \mid dO(t);$$

  and

- ITÔ-integrable with respect to the vector-valued local martingale $M = (M_i)_{i\in\{1,\cdots,n\}}$, and the resulting integral is a continuous local martingale. In fact, the quadratic variation process associated to the stochastic integral $\int_0^{\cdot} \pi'(t)dM(t)$ of π with respect to M, equals

$$\int_0^{\cdot} \pi'(t)dC(t)\pi(t) \equiv \int_0^{\cdot} \pi'(t)c(t)\pi(t)dO(t).$$

These considerations show that the condition (1.6) is necessary and sufficient for the R-integrability of a given $\pi \in \mathcal{P}(\mathbb{R}^n)$. We denote then by

$$\int_0^{\cdot} \sum_{i=1}^{n} \pi_i(t)dR_i(t) \equiv \int_0^{\cdot} \pi'(t)dR(t) = \int_0^{\cdot} \pi'(t)dA(t) + \int_0^{\cdot} \pi'(t)dM(t) \tag{1.7}$$

the stochastic integral of the predictable, vector-valued process π, with respect to the vector-valued semimartingale R.

Remark 1.4. The so-called vector stochastic integral in (1.7) is more general than its componentwise version $\sum_{i=1}^{n} \int_0^{\cdot} \pi_i(t)dR_i(t)$, which would require $\pi_i \in \mathcal{I}(R_i)$ to hold for all $i \in \{1,\cdots,n\}$. Consider for example the case $n = 2$ with $R_2 = -R_1$; then, any predictable process (π_1, π_2) with $\pi_1 = \pi_2$ is R-integrable (with the corresponding integral being equal to zero), regardless of whether $\pi_i \in \mathcal{I}(R_i)$ holds for $i \in \{1, 2\}$. The next definition generalizes this example.

Definition 1.5. (**Null integrands**). Suppose that $\zeta \in \mathcal{P}$ is R-null, by which we mean that $\zeta'\alpha = 0$ and $c\zeta = 0$ hold in the $(\mathbb{P} \otimes O)$-a.e. sense. Therefore, $\zeta \in \mathcal{I}(R)$ and $\int_0^{\cdot} \zeta'(t)\mathrm{d}R(t) \equiv 0$ hold. We call such ζ a *null integrand*.

1.1.5 *Stochastic Exponential and Logarithm*

Let Z be a continuous semimartingale with $Z(0) = 0$. We use $\mathcal{E}(Z)$ to denote the *stochastic exponential* of Z. This process $\mathcal{E}(Z)$ is the unique solution of the linear stochastic integral equation

$$\mathcal{E}(Z) = 1 + \int_0^{\cdot} \mathcal{E}(Z)(t)\mathrm{d}Z(t);$$

whereas straightforward calculation gives

$$\mathcal{E}(Z) = \exp\left(Z - \frac{1}{2}\langle Z, Z\rangle\right).$$

Here $\langle Z\rangle := \langle Z, Z\rangle$ is the quadratic variation process of Z.

The operator $\mathcal{E}(\cdot)$ has an inverse, which we denote by $\mathcal{L}(\cdot)$ and call the *stochastic logarithm*. For any strictly positive continuous semimartingale Y with $Y(0) = 1$, its stochastic logarithm is given by

$$\mathcal{L}(Y) = \int_0^{\cdot} \frac{\mathrm{d}Y(t)}{Y(t)},$$

and we have $Y = \mathcal{E}(\mathcal{L}(Y))$.

Remark 1.6. For a general scalar semimartingale Z with $Z(0) = 0$ and rcll paths (which are right-continuous on $[0, \infty)$ and possess finite left-limits on $(0, \infty)$), and with the notation and assumption

$$\Delta Z(t) := Z(t) - Z(t-) > -1 \quad \text{for } t \in (0, \infty),$$

the stochastic exponential becomes

$$\mathcal{E}(Z) = \exp\Big(Z - \frac{1}{2}\langle Z^c, Z^c\rangle\Big) \cdot \prod_{t\leq\cdot} \big(1 + \Delta Z(t)\big)\exp\big(-\Delta Z(t)\big)$$

and is the unique solution of the linear stochastic integral equation

$$\mathcal{E}(Z) = 1 + \int_0^{\cdot} \mathcal{E}(Z)(t-)dZ(t).$$

For any two scalar semimartingales Z and $\widetilde{Z}$ with these properties, with the notation Z^c for the continuous part of Z, and with the notation

$$\langle Z, \widetilde{Z} \rangle := \langle Z^c, \widetilde{Z}^c \rangle + \sum_{t \leq \cdot} \Delta Z(t) \Delta \widetilde{Z}(t),$$

we have the YOR formula (e.g., JACOD AND SHIRYAEV (2003) [2]):

$$\mathcal{E}(Z)\mathcal{E}(\widetilde{Z}) = \mathcal{E}(Z + \widetilde{Z} + \langle Z, \widetilde{Z} \rangle). \tag{1.8}$$

For additional properties of stochastic exponentials and logarithms, the reader can refer to KARATZAS AND KARDARAS (2020) [1], Exercise 1.8 and Exercise 1.9.

1.2 Assets and Investment

Now we are ready to introduce the model for a financial market based on the probabilistic setting in Section 1.1. Throughout this work, we will study in its context investment strategies and their properties.

1.2.1 *Assets*

We shall consider throughout this dissertation a vector of continuous semimartingales $R \equiv (R_i)_{i \in \{1, \cdots, n\}}$ with $n \in \mathbb{N}$ and $R_i(0) = 0$ for $i \in \{1, \cdots, n\}$, whose component processes represent the cumulative returns of n stocks. We write the semimartingale decomposition

$$R_i = A_i + M_i, \quad i \in \{1, \cdots, n\} \tag{1.9}$$

for $R \equiv (R_i)_{i\in\{1,\cdots,n\}}$ as in Subsection 1.1.3. In this financial market the stock prices $S \equiv (S_i)_{i\in\{1,\cdots,n\}}$ are then $S_i = S_i(0)\mathcal{E}(R_i)$, $i \in \{1, \cdots, n\}$ in the notation of Subsection 1.1.5 for some given real numbers $S_1(0) > 0, \cdots, S_n(0) > 0$, and their dynamics are given formally by

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mathrm{d}R_i(t), \quad t \in \mathbb{R}_+, \quad i \in \{1, \cdots, n\}. \tag{1.10}$$

Together with the specified initial values $S_i(0) > 0$, $i \in \{1, \cdots, n\}$, the dynamics of (1.10) completely determine the continuous, positive semimartingales $S \equiv (S_i)_{i\in\{1,\cdots,n\}}$. Depending on the chosen initial value $S_i(0)$, the process S_i may model the price per share of company i, or may alternatively model the company's total capitalization. This is understood as the product of the actual share price, times the number of shares currently outstanding.

In any case, given $S_i(0) > 0$, the dynamics of (1.10) are expressed equivalently in logarithmic scale as

$$\log S_i = \log S_i(0) + \Gamma_i + M_i, \qquad i \in \{1, \cdots, n\}. \tag{1.11}$$

Here, with $C_{ii} = \langle R_i, R_i\rangle = \langle \log S_i, \log S_i\rangle$ as in (1.5), we have set

$$\Gamma_i := A_i - \frac{1}{2}C_{ii}, \quad i \in \{1, \cdots, n\} \tag{1.12}$$

for the stocks'*cumulative growth* processes.

In terms of the rate processes in (1.3), (1.4), we have now

$$\Gamma_i = \int_0^{\cdot} \gamma_i(t)\mathrm{d}O(t), \quad \text{where} \quad \gamma_i := \alpha_i - \frac{1}{2}c_{ii}, \quad i \in \{1, \cdots, n\}.$$

The processes $\gamma_1, \cdots, \gamma_n$ represent *local growth rates* for the various stocks; we have seen already, that the processes $\alpha_1, \cdots, \alpha_n$ stand for the local *rates of return* of these stocks; whereas the processes $c_{11}, \cdots, c_{nn}$ represent the stocks' local *quadratic variation rates*. All these rates are expressed with respect to the operational clock O of (1.2).

1.2.2 *Investment by Shares versus Proportional Investment*

In the market described above, we place now an investor with initial capital $x \in \mathbb{R}_+$. For a given predictable, vector-valued process

$$\vartheta \equiv (\vartheta_i)_{i \in \{1, \cdots, n\}} \in \mathcal{I}(S),$$

we shall think of the component $\vartheta_i(t)$ of the random vector $\vartheta(t) = (\vartheta_i(t))_{i \in \{1, \cdots, n\}}$ as representing the number of shares held by this investor in the asset $i \in \{1, \cdots, n\}$ at time $t \in \mathbb{R}_+$. With this understanding, the integrand process $\vartheta \in \mathcal{I}(S)$ acquires the significance of *investment strategy*, and its vector stochastic integral

$$X(\cdot; x, \vartheta) := x + \int_0^{\cdot} \vartheta'(t) \mathrm{d}S(t) \equiv x + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t) \mathrm{d}S_i(t) \tag{1.13}$$

becomes the *wealth process* generated by $\vartheta = \{\vartheta(t), 0 \leq t < \infty\}$.

The wealth process X in (1.13) can be understood in the following way. The investor holds $\vartheta_i(t)$ shares of stock i at time $t \in \mathbb{R}_+$. The quantity $\vartheta_i(t) S_i(t)$ is the dollar amount of cash invested in stock $i \in \{1, \cdots, n\}$. In general, the total wealth invested in stocks, namely $\sum_{i=1}^{n} \vartheta_i(t) S_i(t)$, is different from the actual wealth level $X(t; x, \vartheta)$ at time $t \in \mathbb{R}_+$. The discrepancy

$$X(t; x, \vartheta) - \sum_{i=1}^{n} \vartheta_i(t) S_i(t), \quad t \in \mathbb{R}_+ \tag{1.14}$$

is invested in a money market account. When the quantity (1.14) is positive, the investor lends this amount to the money-market; when the amount is negative, its absolute value represents the amount of cash borrowed from the money market. We assume implicitly that the interest rate involved in this situation is zero: the bank pays zero interest to depositors, and charges zero interest to borrowers.

Even in cases where interest rates are insignificant with respect to rates of return of stocks, it is really helpful to be able to account for a non-trivial money market in financial modelling. This can be achieved at no cost to the theory developed here—as long as one thinks of stock returns as being in excess of interest, and of stock processes as being discounted by the value of the money market account.

1.2.3 *Admissibility*

In practice, credit constraints preclude the implementation of any investment strategy imaginable. The notion introduced below reflect this reality.

Definition 1.7. For given initial capital $x \in \mathbb{R}_+$ and investment strategy $\vartheta \equiv (\vartheta_1, \cdots, \vartheta_n) \in \mathcal{I}(S)$, the wealth $X(\cdot; x, \vartheta)$ will be called *admissible*, if solvency $X(\cdot; x, \vartheta) \geq 0$ holds.

Restricting attention to nonnegative wealth processes automatically enforces a mechanism for avoiding debt, a possibility that has to be taken seriously into consideration in economic theory and in financial modelling. From a mathematical viewpoint, the credit constraints implicit in this admissibility requirement ensure that so-called *doubling strategies*, as the ones in the following example from Example 2.3, KARATZAS AND SHREVE (1998) [3], are not allowed.

Example 1.8. (**Doubling Strategy**). Let $n = 1$ and $R \equiv R_1$ be a standard Brownian motion. Restricting attention to the time interval $[0, 1)$, we define a predictable process η and a continuous martingale N via

$$\eta(t) := \frac{1}{S(t)\sqrt{1-t}}, \quad t \in [0, 1); \qquad N := \int_0^{\cdot} \eta(t) \mathrm{d}S(t) = \int_0^{\cdot} \frac{\mathrm{d}R(t)}{\sqrt{1-t}}.$$

Note that $\langle N, N \rangle(t) = -\log(1-t)$ holds for $t \in [0, 1)$, and thus the process W defined via $W(u) := N(1 - \exp(-u)), u \in \mathbb{R}_+$ is standard scalar Brownian motion in its own filtration, by P.LÉVY's charaterization (KARATZAS AND SHREVE (1991) [4], Theorem 3.3.16). It then follows that the stopping times

$$\tau^m := \inf\{t \in [0, 1] \mid N(t) \geq m\}, \quad m \in \mathbb{N},$$

are well defined and $[0,1)$-valued. For every $m \in \mathbb{N}$, consider the strategy $\vartheta^m := \eta 1_{(0,\eta^m]} \in \mathcal{I}(S)$, and note that $X(1;1,\vartheta^m) = 1 + m$ holds for all $m \in \mathcal{N}$. In other words, starting with one currency unit at time $t = 0$, arbitrarily large levels of wealth can be reached at time $t = 1$.

All wealth processes in this example are local martingales; and many of them, for instance the local martingale $X(\cdot;1,\vartheta^m) \equiv 1 + N(\cdot \wedge \tau^m)$, are unbounded form below, signifying the existence of an “infinitely deep credit line". For suppose the local martingale $X(\cdot;1,\vartheta^m)$ were bounded from below by a real constant; then this local martingale would be a supermartingale, leading to the absurd conclusion

$$1 + m = \mathbb{E}^{\mathbb{P}}\left[X(1;1,\vartheta^m)\right] \leq X(0;1,\vartheta^m) = 1.$$

In other words, in the absence of credit constraints, the use of appropriate strategies allows one to attain any target level of wealth imaginable, at any given, fixed time $T \in (0,\infty)$.

The situation is very different when credit constraints are imposed, in recognition of the hard reality that “eventually one runs out of other people’s money". This is the case, for instance, with the admissibility constraint of Definition 1.7, which requires solvency at all times. Then every admissible wealth process $X(\cdot;x,\vartheta)$ is a nonnegative local martingale, thus a supermartingale; and the Optional Sampling Theorem implies $\mathbb{E}^{\mathbb{P}}[X(\tau;x,\vartheta)] \leq x$ for all $\mathbb{P}$-a.e. finite stopping times $\tau \in \mathcal{T}$, drastically limiting the possibilities for gain.

1.2.4 *Finaceability*

In this section we consider one additional behavior of investors: the possibility of withdrawing capital. Withdrawals of capital happen externally and internally. External reasons include a payment stream representing a liability that the investor faces. Internal reasons include consumption. Both of these cases will be important in the sequel.

Definition 1.9. (**Cumulative withdrawal of capital**). Cumulative withdrawal of capital is modelled through elements in $\mathcal{H}$, defined as the set of all nondecreasing, adapted and right-continuous processes $H = \big(H(t)\big)_{0 \leq t < \infty}$ with $H(0) = 0$. Here, $H(t)$ stands for the cumulative (or aggregate) capital withdrawn up to and including time $t \in \mathbb{R}_+$; actual withdrawals in each infinitesimal interval $(t, t + \mathrm{d}t]$ are represented as $\mathrm{d}H(t)$.

Definition 1.10. (**Wealth processes with withdrawals**). Consider an investor with initial capital $x \in \mathbb{R}_+$. For a given predictable, vector-valued investment strategy $\vartheta = (\vartheta_i, i \in \{1, \cdots, n\}) \in \mathcal{I}(S)$, and a given cumulative withdrawal process $H \in \mathcal{H}$, we define

$$X(\cdot; x, \vartheta, H) := x + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t) \mathrm{d}S_i(t) - H \tag{1.15}$$

as the wealth process generated by the investment strategy ϑ and the cumulative withdrawal stream H.

Capital withdrawals are allowed for as long as the investor remains solvent. We state this requirement below.

Definition 1.11. (**Financeability**). For $x \in \mathbb{R}_+, \vartheta \in \mathcal{I}(S)$ and $F \in \mathcal{H}$, the wealth process $X \equiv X(\cdot; x, \vartheta, F)$ is said to *finance* a given cumulative withdrawal stream $H \in \mathcal{H}$, if $X \geq H$ holds. In this case, the process $H \in \mathcal{H}$ will be called *financeable* from the initial capital $x \in \mathbb{R}_+$.

We denote by $\mathcal{H}(x)$ the subset of $\mathcal{H}$ consisting of cumulative capital withdrawal streams financeable from initial capital $x \in \mathbb{R}_+$; formally,

$$\mathcal{H}(x) := \{H \in \mathcal{H} \mid \exists \vartheta \in \mathcal{I}(S), F \in \mathcal{H}\,, \text{ s.t. } X(\cdot; x, \vartheta, F) \geq H\}. \tag{1.16}$$

Some remarks are in order. Suppose that for some given $x \in \mathbb{R}_+$, $\vartheta \in \mathcal{I}(S)$ and $F \in \mathcal{H}$, the wealth process $X \equiv X(\cdot; x, \vartheta, F)$ finances some $H \in \mathcal{H}$. Then X is admissible, as $X \geq H \geq 0$; furthermore the process

$$X(\cdot; x, \vartheta) \equiv X(\cdot; x, \vartheta, 0) \geq X(\cdot; x, \vartheta, F) \geq H \tag{1.17}$$

also finances H. It follows that a given process $H \in \mathcal{H}$ of cumulative withdrawals is financeable from initial capital $x \in \mathbb{R}_+$, if and only if there exists an investment strategy $\vartheta \in \mathcal{I}(S)$ with $X(\cdot; x, \vartheta) \geq H$; therefore,

$$\mathcal{H}(x) = \{H \in \mathcal{H} \mid \exists\, \vartheta \in \mathcal{I}(S), \text{ s.t. } X(\cdot; x, \vartheta, H) \geq 0\}. \tag{1.18}$$

The following observations are immediate from the Definition 1.11 of $\mathcal{H}(x)$ and from its characterization (1.18), for $x \in \mathbb{R}_+$:

- $H \equiv 0 \in \mathcal{H}(0)$.
- For every $x \in \mathbb{R}_+$, the set $\mathcal{H}(x)$ is convex.
- For every $x \in \mathcal{R}_+$, $H \in \mathcal{H}(x)$, and $\overline{H} \in \mathcal{H}$ with $\overline{H} \leq H$, we have $\overline{H} \in \mathcal{H}(x)$.
- For every $0 \leq x_1 \leq x_2 < \infty$, we have $\mathcal{H}(x_1) \subseteq \mathcal{H}(x_2)$.
- For every $x \in (0, \infty)$, we have $\mathcal{H}(x) = x\mathcal{H}(1)$.

Remark 1.12. We introduce now the collection of cumulative capital withdrawal streams in $\mathcal{H}$ which can be financed starting from any (no matter how minuscule) positive initial capital:

$$\mathcal{H}(0+) := \bigcap_{x>0} \mathcal{H}(x) \subseteq \mathcal{H}.$$

Clearly $\{0\} \subseteq \mathcal{H}(0) \subseteq \mathcal{H}(0+) \subseteq \mathcal{H}(x)$ hold, for all $x \in (0, \infty)$.

1.2.5 *Proportional Investment*

We discuss now investment decisions from the point a view of an investor who is not so much interested in how many shares of any given asset he buys or sells at any given time. Rather, this investor keeps track of the proportions of this wealth which get invested in the individual assets. Such a point of view leads naturally to the notion of *portfolio* in Definition 1.15.

Definition 1.13. (**Numéraires**). A wealth process $X \equiv X(\cdot; 1, \vartheta)$ as in (1.13) will be called *numéraire*, if it remains strictly positive at all times: $X > 0$. The collection of all numéraires will be denoted by $\mathcal{X}$. Furthermore, for a given BOREL set $E \subseteq \mathbb{R}^n$, we define $\mathcal{X}(E)$ as the collection of all numéraires generated by all possible portfolios in $\mathcal{I}(R; E)$.

Numéraires are admissible wealth processes but satisfy a more stringent, strict positivity (as opposed to mere non-negativity) constraint. There is a simple normalization here: we require that the wealth process X start with 1 dollar at time $t = 0$. This requirement helps us compare the performance of wealth processes at later times.

By definition, there exists for any given numéraire $X \in \mathcal{X}$ a predictable, vector-valued process $\vartheta \in \mathcal{I}(S)$ with

$$X = 1 + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t) dS_i(t) \equiv X(\cdot; 1, \vartheta) > 0. \tag{1.19}$$

Remark 1.14. (From investment by amounts to proportional investment). Suppose that for a given $x \in (0, \infty)$ and $\vartheta \in \mathcal{I}(S)$, the process $X \equiv X(\cdot\,; x, \vartheta)$ as in (1.13) is strictly positive: $X > 0$. We define then a new predictable, vector-valued process $\pi \equiv (\pi_1, \cdots, \pi_n)$ via

$$\pi_i := S_i\, \vartheta_i / X, \quad \text{for} \quad i \in \{1, \cdots, n\}. \tag{1.20}$$

Because $\vartheta_i(t)$ represents the units of investment (or numbers of shares) held in stock i at time t, the quantity $\pi_i(t)$ gives the fraction (or proportion) of current wealth invested in stock i, and $\sum_{i=1}^{n} \pi_i(t)$ the proportion invested in the entire stock market, at time t. The remaining proportion

$$\pi_0 := 1 - \sum_{i=1}^{n} \pi_i \tag{1.21}$$

of wealth is placed in the money market.

It is straightforward to check that the process π defined from ϑ as in (1.20), is R-integrable. Thus, we may then cast the wealth process of (1.13) as

$$X = x + \int_0^{\cdot} X(t) \sum_{i=1}^{n} \pi_i(t) \mathrm{d}R_i(t) = x + \int_0^{\cdot} X(t) \pi'(t) \mathrm{d}R(t) = x\, \mathcal{E}\Big(\int_0^{\cdot} \sum_{i=1}^{n} \pi_i(t) \mathrm{d}R_i(t) \Big), \tag{1.22}$$

in terms of the resulting process $\pi \in \mathcal{I}(R)$, on account of (1.10).

Definition 1.15. (**Portfolios**). We shall use the term *portfolio* to refer to any predictable, vector-valued process $\pi \in \mathcal{I}(R)$. Such a portfolio generates the corresponding continuous and strictly positive semimartingale wealth process X_π as in (1.22).

Definition 1.16. (**Return and respective Rates**). Given any two portfolios $\pi \in \mathcal{I}(R)$ and $\rho \in \mathcal{I}(R)$, and using the notation of equation (1.4), we define

$$R_\pi := \int_0^\cdot \pi'(t)\mathrm{d}R(t) = \int_0^\cdot \sum_{i=1}^n \pi_i(t)\mathrm{d}R_i(t), \tag{1.23}$$

$$c_{\pi\rho} := \pi' c \rho = \sum_{i=1}^n \sum_{j=1}^n \pi_i c_{ij} \pi_j, \qquad C_{\pi\rho} := \langle R_\pi, R_\rho \rangle = \int_0^\cdot c_{\pi\rho}(t)\mathrm{d}O(t). \tag{1.24}$$

Remark 1.17. (From proportional investment to investment by amounts). Conversely, starting with any predictable, vector-valued process $\pi \in \mathcal{I}(R)$, one may define the vector stochastic integral R_π as in (1.23), and in terms of it the strictly positive continuous semimartingale

$$X_\pi := x\,\mathcal{E}(R_\pi) = x\,\mathcal{E}\Big(\int_0^\cdot \sum_{i=1}^n \pi_i(t)\mathrm{d}R_i(t)\Big), \quad \pi \in \mathcal{I}(R). \tag{1.25}$$

This stochastic exponential process satisfies the dynamics

$$\mathrm{d}X_\pi(t) = X_\pi(t)\mathrm{d}R_\pi(t).$$

Then, with the predictable, vector-valued process $\vartheta \equiv (\vartheta_i)_{i\in\{1,\cdots,n\}} \in \mathcal{I}(S)$ defined via

$$\vartheta_i := \frac{X_\pi\,\pi_i}{S_i} \quad \text{for } i \in \{1, \cdots, n\},$$

it follows that the equation (1.13) holds with X_π replacing X.

The definition (1.25) for $X_\pi = \mathcal{E}(R_\pi)$ and the notation of (1.23) express the log-wealth process generated by a portfolio $\pi \in \mathcal{I}(R)$ as

$$\log X_\pi = R_\pi - \frac{1}{2}C_{\pi\pi} = \int_0^\cdot \pi'(t)\mathrm{d}A(t) - \frac{1}{2}C_{\pi\pi} + \int_0^\cdot \pi'(t)\mathrm{d}M(t). \tag{1.26}$$

The finite first variation part of the above decomposition is called the *cumulative growth* of the portfolio $\pi \in \mathcal{I}(R)$, and is denoted

$$\Gamma_\pi := A_\pi - \frac{1}{2}C_{\pi\pi}, \quad \text{where} \quad A_\pi := \int_0^\cdot \pi'(t)\mathrm{d}A(t). \tag{1.27}$$

The quantities in (1.27) have their “local versions" as below

$$\alpha_\pi := \pi'\alpha, \quad \gamma_\pi := \pi'\alpha - \frac{1}{2}\pi' c\pi = \alpha_\pi - \frac{1}{2}c_{\pi\pi}. \tag{1.28}$$

Here α_π is the *rate of return* of the portfolio $\pi \in \mathcal{I}(R)$, and γ_π is the *growth rate* of this portfolio. These rates are connected to their cumulative versions in (1.27) via

$$A_\pi = \int_0^\cdot \alpha_\pi(t)\mathrm{d}O(t), \qquad \Gamma_\pi = \int_0^\cdot \gamma_\pi(t)\mathrm{d}O(t).$$

Example 1.18. (**Constant-proportion portfolio**). Fix a vector $p = (p_i)_{i\in\{1,\cdots,n\}} \in \mathbb{R}^n$, and let $\pi \equiv p \in \mathcal{I}(R)$ be the portfolio that keeps a fixed proportion p_i of wealth in each stock $i \in \{1, \cdots, n\}$ at all times. In this case, it is clear that $R_p = p'R = \sum_{i=1}^n p_i R_i$.

An important special case is when $p = \mathbf{0}$, meaning $p_i \equiv 0$ for all $i \in \{1, \cdots, n\}$. This portfolio never touches the stock market, keeping all wealth in the money market. Since wealth is denominated in terms of the money market, it is clear that $X_{\mathbf{0}} \equiv 1$.

When p coincides with one of the unit vectors e^i, $i \in \{1, \cdots, n\}$ of $\mathbb{R}^n$, whose coordinates are all zero except for coordinate i which is 1, the entire wealth is invested in stock i at all times. In this case, $X_{e^i} = S_i/S_i(0)$ holds.

Another special case is the equal-weighted portfolio that assigns constant and equal weights to all stocks at all times: $p_i = 1/n$, for all $i \in \{1, \cdots, n\}$. This portfolio never touches the money market.

1.2.6 *Portfolio Constraints*

The introduction of numéraires and proportional investment (i.e., portfolios) facilitates the description of certain common constraints on investment choice. We provide readers some examples of contrained portfolios. For more detailed introduction, readers can refer to subsection 1.3.2 of KARATZAS AND KARDARAS (2020) [1], or KARATZAS AND FERNHOLZ (2009) [5].

Recalling Definition 1.3, we see that prohibition of selling stocks short for a portfolio π, requires $\pi \in \mathcal{I}(R; \mathbb{R}_+^n)$. If borrowing from the money market is further prohibited for π, the constraint $\pi_0 \geq 0$ in the notation of (1.21) should also hold. The requirement then becomes $\pi \in \mathcal{I}(R; \Delta_+^n)$, where the n-dimensional simplex Δ_+^n is defined as

$$\Delta_+^n := \left\{(x_1, \cdots, x_n) \in \mathbb{R}_+^n \;\middle|\; \sum_{i=1}^n x_i \leq 1\right\}. \tag{1.29}$$

Such portfolios, which never sell any stock short and never borrow from the money market, will be called *long-only portfolios*. We note that Δ_+^n-valued predictable processes are bounded and, in particular, R-integrable: $\mathcal{I}(R; \Delta_+^n) = \mathcal{P}(\Delta_+^n)$.

A portfolio that never touches (i.e., neither borrows from, nor lends to) the money market, will be called *stock portfolio*. Clearly, π being a stock portfolio is equivalent to $\pi \in \mathcal{I}(R; \Delta^{n-1})$, where

$$\Delta^{n-1} := \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i = 1\right\} \tag{1.30}$$

is an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$, the lateral face of the unit simplex Δ_+^n in (1.29).

If, additionally, no short-selling of stocks is allowed, the portfolio π will be called *long-only stock portfolio*. This requirement is cast as $\pi \in \mathcal{I}(R; \Delta_+^{n-1}) = \mathcal{P}(\Delta_+^{n-1})$, where

$$\Delta_+^{n-1} := \Delta^{n-1} \bigcap \mathbb{R}_+^n = \left\{(x_1, \cdots, x_n) \in \mathbb{R}_+^n \;\middle|\; \sum_{i=1}^n x_i = 1\right\}. \tag{1.31}$$

Chapter 2: Numéraires and Market Viability

In this chapter, we introduce the notion of *supermartingale numéraire portfolio* $\nu \in \mathcal{I}(R;K)$, among portfolios constrained to take values in a closed, convex set K containing the origin. Such a portfolio has the property that, when measured in terms of the wealth X_ν generated by it, the performance X_π of any other portfolio $\pi \in \mathcal{I}(R;K)$ "can only lag" in the sense that the ratio X_π / X_ν is a supermartingale.

We shall introduce the concept of *market viability* in the second half of this chapter. Furthermore, when K is a closed convex cone, we build up the fundamental Theorem 2.31. This theorem says that a supermartingale numéraire portfolio $\nu \in \mathcal{I}(R;K)$ exists if and only if the market is viable, a concept we shall introduce in Definition 2.20 to formalize the absence of arbitrage opportunities.

We shall also provide some optimal properties for supermartingale numéraire portfolios in the end of this chapter.

2.1 Supermartingale Numéraires

Let us first fix a nonempty closed convex set K of $\mathbb{R}^n$, containing the origin. From now on we only consider portfolios π taking values in this set K, namely, $\pi \in \mathcal{I}(R;K)$.

In this Section, we first introduce the important notion of "supermartingale numéraire". In Proposition 2.8, we give a characterization of a supermartingale numéraire under closed, convex contraints on portfolio choice. Moreover, we point out significant results in Proposition 2.4 and Theorem 2.10, as special cases of Proposition 2.8.

This chapter will focus heavily on market structure and investments. Let us recall the quantities $R_i = A_i + M_i$ of (1.1) which represent the continuous semimartingales of asset returns, and the quantities $C_{ij} = \langle R_i, R_j \rangle$ of (1.5) which represent their covariations.

Definition 2.1. For a given portfolio $\nu \in \mathcal{I}(R; K)$, the wealth process X_ν will be called *supermartingale numéraire*, if for every portfolio $\pi \in \mathcal{I}(R; K)$ the ratio $X_\pi^\nu = X_\pi / X_\nu$ is a supermartingale.

In this case, the process $\nu \in \mathcal{I}(R; K)$ will be called *supermartingale numéraire portfolio*.

The following result shows that, whenever supermartingale numéraires exist, they are unique. This justifies using the definite article for them in the remainder of the text. The corresponding portfolios can only be unique up to portfolios that are R-null, as in Definition 1.5.

Lemma 2.2. *Let Z be a continuous, strictly positive semimartingale. If both Z and $1/Z$ are supermartingales, then Z is constant.*

Proof. Suppose that $Z = Z(0) + M - A$ is a strictly positive, continuous supermartingale in its DOOB-MEYER decomposition, with M a continuous local martingale and A a nondecreasing adapted process and $M(0) = A(0) = 0$. Then the reciprocal

$$\frac{1}{Z} = 1 - \int_0^\cdot \frac{\mathrm{d}Z(t)}{Z^2(t)} + \int_0^\cdot \frac{\mathrm{d}\langle Z\rangle(t)}{Z^3(t)} = 1 - \int_0^\cdot \frac{\mathrm{d}M(t)}{Z^2(t)} + \int_0^\cdot \frac{\mathrm{d}(A + \langle M\rangle)(t)}{Z^3(t)}$$

is a semimartingale. Requiring this semimartingale to be a supermartingale forces, on the strength of the DOOB-MEYER decomposition theorem, the process $A + \langle M \rangle$ has to be non-increasing. But this amounts to $A \equiv 0 \equiv M$, that is, $Z \equiv Z(0)$. □

Remark 2.3. Suppose there exist two supermartingale numéraire portfolios $\nu_1, \nu_2 \in \mathcal{I}(R; K)$. Then X_{ν_1}/X_{ν_2} and X_{ν_2}/X_{ν_1} are both supermartingales by Definition 2.1. Lemma 2.2 indicates that $X_{\nu_1}/X_{\nu_2} \equiv c \in \mathbb{R}_+$. The fact that $X_{\nu_1}(0) = X_{\nu_2}(0) = 1$ indicates that $X_{\nu_1} \equiv X_{\nu_2}$.

2.1.1 *Special case: $K \equiv \mathbb{R}^n$*

When $K \equiv \mathbb{R}^n$, there is no constraint on portfolio choice. In this special case, the following result from Proposition 2.4 in KARATZAS AND KARDARAS (2020) [1], provides a structural condition that characterizes the supermartingale numéraire property in terms of market characteristics. We state it here without proof.

Proposition 2.4. *For $\nu \in \mathcal{I}(R)$, they following statements are equivalent:*

(i) *ν is a supermartingale numéraire portfolio.*

(ii) *The equality*

$$A_i = C_{i\nu} \tag{2.1}$$

holds for all $i \in \{1, \cdots, n\}$.

We shall present a similar result with a convex constraint in Proposition 2.8, and readers can compare that result with Proposition 2.4.

A supermartingale numéraire need not exist. When $K \equiv \mathbb{R}^n$, the following example illustrates this failure in a very simple setting.

Example 2.5. (**Non-existence of supermartingale numéraire**) Consider a standard Brownian motion W on the underlying filtered probability space, and a one-stock model ($n = 1$) with returns process $R_1 =\mid W \mid$.

We write this in its ITÔ-TANAKA decomposition $R_1 = A_1 + M_1$, where A_1 is the local time process of W at the origin, and the process $M_1 \equiv \int_0^{\cdot} \mathrm{sgn}\big(W(t)\big) dW(t)$ a standard Brownian motion. Now A_1 is singular with respect to the LEBESGUE clock $\mathbb{R}_+ \ni t \mapsto C_{11}(t) = \langle M_1, M_1 \rangle(t) = t$, so the requirement (ii) in Proposition 2.4, namely $A_1 = C_{1\nu}$, cannot hold for any $\nu \in \mathcal{I}(R)$. No supermartingale numéraire portfolio exists in this setting.

Remark 2.6. Later in the next subsection, we shall provide another example of a constrained market, in which a supermartingale numéraire portfolio fails to exist.

2.1.2 *Supermartingale Numéraires with Convex Constraints*

Proposition 2.4 fails under general convex constraint set K on portfolio choice. In this subsection, we discuss conditions for supermartingale numéraires under convex constraints on portfolio choice. For emphasis, recall that we require the origin $\mathbf{0}$ to belong to the constraint set K, and K to be closed and convex.

We consider a portfolio $\rho \in \mathcal{I}(R;K)$, which we regard as some sort of "baseline". We define the relative wealth process

$$X_\pi^\rho := \frac{X_\pi}{X_\rho}, \quad \pi \in \mathcal{I}(R;K), \quad \rho \in \mathcal{I}(R;K) \tag{2.2}$$

and remind ourselves that $\pi_0 := 1 - \sum_{i=1}^n \pi_i$. We have the following result.

Proposition 2.7. *For any two portfolios $\pi \in \mathcal{I}(R;K)$ and $\rho \in \mathcal{I}(R;K)$, the relative wealth process of (2.2) admits the representation*

$$X_\pi^\rho = \mathcal{E}(R_\pi^\rho), \quad R_\pi^\rho := R_{\pi-\rho} - C_{\pi-\rho,\rho} = \int_0^\cdot \sum_{i=0}^n \pi_i(t) dR_i^\rho(t). \tag{2.3}$$

Proof. According to the definition of R_π^ρ, we obtain

$$\begin{aligned}\int_0^\cdot \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) &= \int_0^\cdot \left(\sum_{i=1}^n \pi_i(t)\right) dR_0^\rho(t) + \int_0^\cdot \sum_{i=1}^n \pi_i(t) d\big(R_i(t) - C_{i\rho}(t)\big) \\ &= \int_0^\cdot \big(1 - \pi_0(t)\big) dR_0^\rho(t) + (R_\pi - C_{\pi\rho}),\end{aligned}$$

which gives

$$\begin{aligned}\int_0^\cdot \sum_{i=0}^n \pi_i(t) dR_i^\rho(t) &= \int_0^\cdot \pi_0(t) dR_0^\rho(t) + \int_0^\cdot \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) \\ &= R_0^\rho + R_\pi - C_{\pi\rho} = R_{\pi-\rho} - C_{\pi-\rho,\rho}.\end{aligned}$$

Thus, in order to prove (2.3), we only need to show $X_\pi^\rho = \mathcal{E}(R_{\pi-\rho} - C_{\pi-\rho,\rho})$. Basic stochastic calculus completes the proof, by observing

$$\begin{aligned}\log \mathcal{E}(R_{\pi-\rho} - C_{\pi-\rho,\rho}) &= R_{\pi-\rho} - C_{\pi-\rho,\rho} - \frac{1}{2}C_{\pi-\rho,\pi-\rho}\\ &= (R_\pi - \frac{1}{2}C_{\pi\pi}) - (R_\rho - \frac{1}{2}C_{\rho\rho})\\ &= \log X_\pi - \log X_\rho = \log X_\pi^\rho.\end{aligned}$$

□

Supermartingale numéraire under closed-convex-constraints on portfolio choice. We consider now portfolio processes $\pi \in \mathcal{I}(R; K)$ that take values in a given closed, convex subset K of $\mathbb{R}^n$ containing the origin. For any two such portfolios π, ν in this class, we have

$$d\left(\frac{X_\pi(t)}{X_\nu(t)}\right) = \left(\frac{X_\pi(t)}{X_\nu(t)}\right) \cdot \sum_{i=0}^{n} \pi_i(t) \mathrm{d}R_i^\nu(t) \tag{2.4}$$

from Proposition 2.7, where

$$R_i^\nu = R_0^\nu + (R_i - C_{i\nu}), \quad i = 1, \cdots, n \quad \text{and} \quad R_0^\nu = C_{\nu\nu} - R_\nu.$$

Thus, the summation in (2.4) is

$$\begin{aligned}&\left(1 - \sum_{i=1}^{n} \pi_i(t)\right)\mathrm{d}R_0^\nu(t) + \sum_{i=1}^{n} \pi_i(t)\big(\mathrm{d}R_i^\nu(t)\big)\\ =\ & \mathrm{d}R_0^\nu(t) + \sum_{i=1}^{n} \pi_i(t)\big(\mathrm{d}R_i^\nu(t) - \mathrm{d}R_0^\nu(t)\big)\\ =\ & \mathrm{d}C_{\nu\nu}(t) - \mathrm{d}A_\nu(t) - \mathrm{d}M_\nu(t) + \sum_{i=1}^{n} \pi_i(t)\big(\mathrm{d}A_i(t) + \mathrm{d}M_i(t) - \mathrm{d}C_{i\nu}(t)\big)\\ =\ & \mathrm{d}M_{\pi-\nu}(t) + \mathrm{d}A_{\pi-\nu}(t) - \mathrm{d}C_{\pi-\nu,\nu}(t). \end{aligned}\tag{2.5}$$

It develops from (2.5) that, for the portfolio ν to have the supermartingale numéraire property among portfolios that take values in K, it is necessary and sufficient that the finite variation process

$$A_{\pi-\nu} - C_{\pi-\nu,\nu} \quad \text{be nonincreasing for every } \pi \in \mathcal{I}(R;K). \tag{2.6}$$

In terms of rates, this condition can be expressed equivalently as

$$(\pi - \nu)'(\alpha - c\nu) \leq 0, \quad \forall\, \pi \in \mathcal{I}(R;K) \tag{2.7}$$

where the inequality in (2.7) is to be understood as holding for $(\mathbb{P} \otimes \mathbb{O})$-a.e. $(\omega, t) \in \Omega \times [0, \infty)$.

Consider now the local growth rate

$$g(\pi) = \pi'\alpha - (1/2)\pi' c\pi, \quad \pi \in \mathcal{I}(R;K)$$

corresponding to any such portfolio, and try to maximize it. If there exists a portfolio $\rho \in \mathcal{I}(R;K)$ that does this job, then the function

$$[0,1] \ni \epsilon \longrightarrow g\big(\epsilon\pi + (1-\epsilon)\rho\big)$$

attains its maximum at $\epsilon = 0$; and then we have the necessary (first-order) condition for optimality

$$\frac{\partial}{\partial \epsilon} g\big(\epsilon\pi + (1-\epsilon)\rho\big)\bigg|_{\epsilon=0} \leq 0. \tag{2.8}$$

We are using here the convexity of the set K in a crucial manner. Now, elementary computation gives

$$\frac{\partial}{\partial \epsilon} g\big(\epsilon\pi + (1-\epsilon)\rho\big) = (\pi - \rho)'\big[\alpha - c\rho - \epsilon(\pi - \rho)\big]$$

and thus, the condition (2.8) amounts to

$$(\pi - \rho)'(\alpha - c\rho) \leq 0, \quad \forall\, \pi \in \mathcal{I}(R;K). \tag{2.9}$$

We note that this is precisely the condition (2.7), that characterizes the supermartingale numéraire property.

Conversely, suppose that for some portfolio $\rho \in \mathcal{I}(R;K)$ the property (2.9) is satisfied. Then

$$g(\pi) - g(\rho) = (\pi - \rho)'(\alpha - c\rho) - (1/2)(\pi - \rho)'c(\pi - \rho) \leq 0$$

holds for every portfolio $\pi \in \mathcal{I}(R;K)$.

We conclude from this discussion the following result.

Proposition 2.8. *For a given closed, convex subset K of $\mathbb{R}^n$ containing the origin, the following conditions on a given portfolio $\nu \in \mathcal{I}(R;K)$ are equivalent:*

(i) ν has the supermartingale numéraire property among portfolios in $\mathcal{I}(R;K)$;

(ii) ν maximizes the local growth rate among portfolios in $\mathcal{I}(R;K)$;

(iii) $(\pi - \nu)'(\alpha - c\nu) \leq 0$ holds for every $\pi \in \mathcal{I}(R;K)$.

Example 2.9. Let us consider a standard Brownian motion W on the underlying filtered probability space, and a two-stock model ($n = 2$) with returns process $R_1 = 2t + W(t)$ and $R_2 = 3t - W(t)$. Furthermore, we consider the market constraint $K = \{(x, y) \mid x \geq 0,\ y \geq 0\}$.

We write down the local mean rates of return and local covariation rates as below:

$$\begin{aligned} \alpha_1 &\equiv 2, \\ \alpha_2 &\equiv 3, \\ c_{11} = c_{22} &\equiv 1, \\ c_{12} = c_{21} &\equiv -1. \end{aligned}$$

Therefore, we can express the local growth rate explicitly

$$g(x, y) = (2x + 3y) - \frac{1}{2}(x - y)^2.$$

Suppose we have a supermartingale numéraire portfolio $\nu = (\nu_1, \nu_2) \in \mathcal{I}(R; K)$. We construct another portfolio $\rho = (\nu_1 + \nu_2, \nu_1 + \nu_2)$, and compute

$$g(\rho) = 5\nu_1 + 5\nu_2 \geq 2\nu_1 + 3\nu_2 - \frac{1}{2}(\nu_1 - \nu_2)^2 = g(\nu).$$

The inequality holds as equality if and only if $\nu_1 \equiv \nu_2 \equiv 0$. However, $\nu \equiv \mathbf{0}$ is apparently not a supermartingale numéraire portfolio in $\mathcal{I}(R; K)$. Actually, we take another portfolio $\pi \equiv (1, 1) \in \mathcal{I}(R; K)$, and compute

$$g(1, 1) = 5 > g(0, 0).$$

Therefore, in this case, a supermartinale numéraire portfolio fails to exist.

2.1.3 *The Conic Case*

Suppose now that the closed, convex set K is a cone. Then for a given portfolio $\nu \in \mathcal{I}(R; K)$ to have the supermartingale numéraire property among all portfolios in $\mathcal{I}(R; K)$, the condition (2.7) must hold for $\pi = \ell\nu$, for every $\ell \in (0, \infty)$. This means $(\ell - 1)\nu'(\alpha - c\nu) \leq 0$ for every $\ell \in (0, \infty)$, which implies

$$\nu'(\alpha - c\nu) = 0, \quad \text{or equivalently} \;\; A_\nu = C_{\nu\nu}\,. \tag{2.10}$$

Thus, the condition (2.7) takes the more explicit form

$$\pi'(\alpha - c\nu) \leq 0 = \nu'(\alpha - c\nu), \quad \forall\, \pi \in \mathcal{I}(R; K), \tag{2.11}$$

or equivalently

$$A_\pi - C_{\pi\nu} \leq 0 \equiv A_\nu - C_{\nu\nu}, \quad \forall\, \pi \in \mathcal{I}(R; K). \tag{2.12}$$

The discussion above is important enough to be spelled out explicitly.

Theorem 2.10. *Suppose that, for a given portfolio $\nu \in \mathcal{I}(R; K)$, the generated wealth process X_ν has the supermartingale numéraire property among all numéraires in $\mathcal{X}(K)$. Then the portfolio ν*

must satisfy

$$A_\nu = C_{\nu\nu}. \tag{2.13}$$

Compared to the equality (2.1), the equality (2.13) cannot be used to characterize supermartingale numéraire; it is only a necessary condition. Below is a simple example, illustrating its lack of sufficiency.

Example 2.11. Let us consider a standard two-dimensional Brownian motion $W = (W_1, W_2)$ on the underlying filtered probability space, and a two-stock model ($n = 2$) with return processes $R_1(t) = 3t + W_1(t)$ and $R_2(t) = -2t + W_2(t)$. Furthermore, we consider market constraint $K = \{(x, y) \mid x \geq 0, y \leq 0, y \geq -2x/3\}$.

According to Proposition 2.8 condition (ii), we can easily find out the supermartingale numéraire portfolio $\nu(t) = (3, -2)$ for $t \in \mathbb{R}_+$. However, the portfolio $\pi \equiv (0, 0)$ also satisfies the equality (2.13) trivially. Thanks to the uniqueness of supermartingale numéraire from Lemma 2.2, Remark 2.3 and the fact $X_\nu \neq X_\pi$, we see that π is not a supermartingale numéraire portfolio.

2.2 Local finiteness of growth

Assumption 2.12. For the remainder of Chapter 2, we assume further that the closed convex set K is a cone.

We first introduce the linear-subspace-valued process of null investments $\mathfrak{N}$.

Definition 2.13. The linear-subspace-valued process of null investments $\mathfrak{N}$ is the set if vectors (depending on (ω, t)) for which nothing happens if one invests in them, namely

$$\mathfrak{N}(\omega, t) := \left\{\xi \in \mathbb{R}^d \;\middle|\; \xi' c(\omega, t) = 0 \;\text{ and }\; \xi' \alpha(\omega, t) = 0\right\}. \tag{2.14}$$

We have $X_{\pi_1} = X_{\pi_2}$ if and only if $\pi_2(\omega, t) - \pi_1(\omega, t) \in \mathfrak{N}(\omega, t)$ for $(\mathbb{P} \otimes \mathcal{O})$-a.e; then portfolio π_1 and π_2 are considered identical.

From now on, we assume that $\mathfrak{N}(\omega, t) \subseteq K$ holds for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. This requirement says that we are giving investors at least the freedom to do nothing: if an investment is to lead to absolutely no profit or loss, one should be free to do it. A special case is that K contains the origin in $\mathbb{R}^n$.

Definition 2.14. (**Arbitrage opportunities**). Let us first define a set-valued process

$$\mathcal{J}(\omega, t) := \left\{\xi \in \mathbb{R}^n \mid \xi' c(\omega, t) = 0,\ \xi' \alpha(\omega, t) > 0\right\} \tag{2.15}$$

for $(\omega, t) \in \Omega \times [0, \infty)$. Intuitively, the linear subspace $\mathcal{J}(\omega, t)$ denotes all possible "arbitrage opportunities" for a local rate of return $\alpha(\omega, t)$ and a local rate of covariation $c(\omega, t)$.

Let us define the $[0, \infty]$-valued process

$$g_K(\omega, t) := \sup_{p \in K} \left(p' \alpha(\omega, t) - \frac{1}{2} p' c(\omega, t) p\right) \tag{2.16}$$

for $(\omega, t) \in \Omega \times [0, \infty)$. We can take a countable dense subset $\widetilde{K}$ of the cone K, and $g_K \equiv g_{\widetilde{K}}$. Thus g_K is predictable. Recalling the growth rate process of (1.28) in Chapter 1, it follows that this process g_K is the *pointwise maximal growth rate* achievable in the market.

Remark 2.15. Now we build a bridge between the set-valued process $\mathcal{J}$ and the $[0, \infty]$-valued process g_K. For any $(\omega, t) \in \Omega \times [0, \infty)$, according to the quadratic optimality of equation (2.16), the condition

$$g_K(\omega, t) = \infty$$

is equivalent to

$$\mathcal{J}(\omega, t) \cap K \neq \emptyset.$$

In other words, the relation

$$\left\{g_K = \infty\right\} = \left\{\mathcal{J} \cap K \neq \emptyset\right\} \tag{2.17}$$

holds. When $g_K(\omega, t) < \infty$, the quadratic optimization problem

$$\sup_{p \in K \cap \mathfrak{N}^{\perp}} \Big(p'\alpha(\omega, t) - \frac{1}{2} p'c(\omega, t)p \Big) \tag{2.18}$$

has a unique optimizer, so we set

$$\nu := \arg \max_{p \in K \cap \mathfrak{N}^{\perp}} \big(p'\alpha - \frac{1}{2} p'cp \big) \mathbb{1}_{\{g_K < \infty\}}.$$

Thanks to Proposition 2.8 (ii) and equality (2.9), we can use the fact that

$$\nu'\alpha = \nu'c\nu \qquad \text{on the set } \{g_K < \infty\}$$

to rewrite equation (2.16) as

$$g_K = \frac{1}{2}(\nu'c\nu)\mathbb{1}_{\{g_K < \infty\}} + \infty \mathbb{1}_{\{g_K = \infty\}}. \tag{2.19}$$

We define also the *aggregate maximal growth*

$$G := \int_0^{\cdot} g_K(t) \mathrm{d}O(t), \tag{2.20}$$

an adapted and nondecreasing process. If $\{g_K = \infty\}$ has zero $(\mathbb{P} \otimes O)$-measure, then the local integrability of ν with respect to R, is equivalent to

$$\int_0^T \nu'(t)c(t)\nu(t)\mathrm{d}O(t) = G(T) < \infty, \quad \forall\ T > 0. \tag{2.21}$$

On the other hand, if $G(T) < \infty$ holds for every $T \in (0, \infty)$, then $\{g_K = \infty\}$ has zero $(\mathcal{P} \otimes O)$-measure and the requirement (2.21) holds, namely, $\nu \in \mathcal{I}(R; K)$.

The last requirement of (2.21) posits that the maximal aggregate growth achievable in the market over any given finite time interval, is finite. The resulting concept is important enough to merit an appellation.

Definition 2.16. The market has *locally finite growth*, if the process G of (2.20) satisfies the condition $G(T) < \infty$ for all $T \in \mathbb{R}_+$, as in (2.21).

The following lemma is extremely useful. It appears as Exercise 2.16 in KARATZAS AND KARDARAS (2020) [1], where the convex set K is assumed to be $\mathbb{R}^n$. In Lemma 2.17, we prove the same result for a general closed convex cone K containing the origin.

Lemma 2.17. *With X_ν the supermartingale numéraire, we have $C_{\nu\nu} = 2G$ as well as the representations*

$$\frac{1}{X_\nu} = \mathcal{E}(-M_\nu), \quad \log X_\nu = G + \sqrt{2}W(G)$$

for a suitable standard Brownian motion W. In particular, $X_\nu(\infty) = \infty$ holds on the event $\{G(\infty) = \infty\}$.

Proof. From Theorem 2.10, if a supermartingale numéraire portfolio ν exists, then ν satisfies $A_\nu = C_{\nu\nu}$. Thus $2G = 2A_\nu - C_{\nu\nu} = C_{\nu\nu}$.

With the help of stochastic exponentials, we have

$$\frac{1}{X_\nu} = \exp\left(\frac{1}{2}C_{\nu\nu} - R_\nu\right) = \exp\left(\frac{1}{2}C_{\nu\nu} - A_\nu - M_\nu\right) = \mathcal{E}(-M_\nu).$$

Therefore, $1/X_\nu$ is a local martingale. We deduce $\log X_\nu = M_\nu + (1/2)\langle M_\nu, M_\nu\rangle$, and note

$$\langle M_\nu, M_\nu\rangle = C_{\nu\nu} = \int_0^\cdot \nu'(t)c(t)\nu(t)\mathrm{d}O(t) = 2G,$$

so the result follows from the DAMBIS-DUBINS-SCHWARZ theorem (KARATZAS AND SHREVE (1991) [4], Theorem 3.4.6). This gives

$$\log X_\nu(T) = G(T)\left(1 + \sqrt{2}\frac{W(G(T))}{G(T)}\right), \quad 0 \le T < \infty,$$

thus also $\log X_\nu(\infty) = \infty$ on the event $\{G(\infty) = \infty\}$, thanks to the Strong Law of Large Numbers for the Brownian Motion W. □

2.3 Market Viability

We explain in the present Section, how the existence of the supermartingale numéraire makes the entire market *viable* by proscribing certain kinds of *arbitrage*. We develop these crucial notions, and the equally important ones of *superhedging* and *deflator*. We then state and prove the fundamental Theorem 2.31, a cornerstone of the theory developed here.

2.3.1 *Superhedging*

We recall the collection $\mathcal{H}$ consisting of all nondecreasing, adapted and right-continuous processes J with $J(0) = 0$; as well as the collection $\mathcal{H}(x)$ that consists of all processes in $\mathcal{H}$ which are financeable starting from a given initial capital $x \in \mathbb{R}_+$. Clearly $\mathcal{H}(0) \subseteq \mathcal{H}(x) \subseteq \mathcal{H}(y)$ holds for $0 \leq x \leq y < \infty$; and $\mathcal{H}(0+) \triangleq \bigcap_{x \in (0,\infty)} \mathcal{H}(x)$ is the collection of processes in $\mathcal{H}$ that can be financed starting with arbitrary small positive initial capital.

Definition 2.18. (**Superhedging capital**) For any given cumulative capital withdrawal stream $J \in \mathcal{H}$, we introduce the quantity

$$\begin{aligned} x(J) &:= \inf\{x \in \mathbb{R}_+ \big| J \in \mathcal{H}(x)\} \\ &= \inf\{x \in \mathbb{R}_+ \big| \exists \pi \in \mathcal{I}(R;K), F \in \mathcal{H}, s.t. X(\cdot; x, \pi, F) \geq J\} \\ &= \inf\{x \in \mathbb{R}_+ \big| \exists \pi \in \mathcal{I}(R;K), s.t. X(\cdot; x, \pi) \geq J\} \end{aligned} \tag{2.22}$$

and call it the *superhedging capital* associated with the stream $J \in \mathcal{H}$. The standard convention, that the infimum of an empty set is equal to infinity, is in force here.

Corollary 2.19. *The functional $x : \mathcal{H} \to [0, \infty]$ of (2.22) has the following properties:*

- ***Monotonicity**: for J_1 and J_2 in $\mathcal{H}$, $J_1 \leq J_2 \to x(J_1) \leq x(J_2)$.*

- ***Positive homogeneity***: *$x(aJ) = ax(J)$, for $a \in (0, \infty)$ and $J \in \mathcal{K}$.*

- ***Subadditivity***: *for J_1 nad J_2 in $\mathcal{H}$, $x(J_1 + J_2) \le x(J_1) + x(J_2)$.*

2.3.2 *Viability*

A minimal condition which ensures that a market is functioning efficiently, is the absence of so-called *arbitrage opportunities*. Loosely speaking, an arbitrage is created by investing in a way that completely offsets any downside risk, while ensuring that profit will be made, at least under certain probable scenarios.

Finding the exact mathematical formulation for absence of arbitrage, in a way that permits a coherent theory to be developed, is not a straightforward task; and there is no general agreement on this issue. For our purposes, the next definition strikes the perfect balance between financial intuition and practical implementation on the one hand, and mathematical clarity and efficiency on the other.

Definition 2.20. (**Viability**). The market is called *viable* if, when $x(J) = 0$ holds for some cumulative withdrawal stream $J \in \mathcal{H}$ as in (2.22), we have $J \equiv 0$.

Under notation established at the beginning of Subsection 2.3.1, we see that market viability is equivalent to the requirement

$$\mathcal{H}(0+) = \{0\}. \tag{2.23}$$

When market viability fails, there exist *non-zero* cumulative withdrawal streams $J \in \mathcal{H}$ and, for every $m \in \mathbb{N}$, portfolios $\pi_m \in \mathcal{I}(R; K)$ such that

$$X\left(\cdot; \frac{1}{m}, \pi_m, J\right) = X\left(\cdot; \frac{1}{m}, \pi_m\right) - J \ge 0. \tag{2.24}$$

Thus, the non-zero stream J is then financed “essentially for free".

Definition 2.21. (**Gratuities; Arbitrages**). We call *gratuity* (or "free lunch"), a cumulative withdrawal stream $J \in \mathcal{H}$ for which a sequence of portfolios $\Theta = \{\pi_m\}_{m\in\mathbb{N}}$ exists with $X(\cdot; \frac{1}{m}, \pi_m) - J \ge$

0. This sequence Θ is called *arbitrage*.

The next proposition was established in Karatzas and Kardaras (2020) [1], Exercise 2.22, for $K = \mathbb{R}^n$ (no constraints). We prove a new version here under the Assumption 2.12, which offers an alternative characterization of market viability. This result, Proposition 2.22 below, states that, if there exist gratuities, i.e., non-zero cumulative streams that can be financed essentially for free, then these streams can be selected to be of the "contingent claim" form $h1_{[T,\infty)}$.

Proposition 2.22. *The market fails to be viable if, and only if, there exist a $T \in \mathbb{R}_+$ and a nonnegative $\mathcal{F}(T)$-measurable random variable h with $\mathbb{P}(h > 0) > 0$ such that, for every $m \in \mathbb{N}$, there exists a wealth process $X^m \in \mathcal{X}(K)$ with $X^m(T) \geq mh$.*

Proof. " $\Longleftarrow$ " : Suppose there exists a real number $T > 0$ and an $\mathcal{F}(T)$-measurable random variable $h : \Omega \to [0, \infty)$ with $\mathbb{P}(h > 0) > 0$ and the following property: for every integer $m \in \mathbb{N}$, there exists an investment strategy $\pi_m \in \mathcal{I}(R; K)$ with $X(\cdot; 1, \pi_m) > 0$ and $X(T; 1, \pi_m) \geq mh$.

We can define then the non-zero process $J \in \mathcal{H}$ via $J := h1_{[T,\infty)}$, and observe that the inequality $X(t; 1/m, \pi_m) \geq J(t)$ holds for $0 \leq t \leq T$. Then

$$\widetilde{\pi}_m = \pi_m \cdot \mathbb{1}_{[0,T]}$$

generates wealth $X(\cdot; 1/m, \widetilde{\pi}_m, J) \geq 0$, and viability fails also in the sense of Definition 2.21.

" $\Longrightarrow$ " : Suppose now that viability fails. Therefore there exists a non-zero cumulative withdrawal stream $J \in \mathcal{H}$ such that, for every integer $m \in \mathbb{N}$, we can find an investment strategy π_m with $X(\cdot; 1/m, \pi_m, J) = X(\cdot; 1/m, \pi_m) - J \geq 0$, thus $X(\cdot; 1, \pi_m) \geq mJ$. Without loss of generality, we can select π_m so as to satisfy $X(\cdot; 1, \pi_m) > 0$ as well.

It is clear that $\mathbb{P}\big(J(T) > 0\big) > 0$ holds for some real number $T > 0$; for otherwise we would have $\mathbb{P}\big(J(T) > 0\big) = 0$ for all $T \in (0, \infty)$, thus also $\mathbb{P}\big(J(\infty) > 0\big) = 0$, contradicting the assumption of non-zero cumulative withdrawal stream $J \in \mathcal{H}$. For this real number $T > 0$ we consider the $\mathcal{F}(T)$-measurable random variable $h := J(T)$; this takes values in $[0, \infty)$ and satisfies $\mathbb{P}(h > 0) > 0$, as well as $X(T;1,\pi_m) \geq mh$ for every $m \in \mathbb{N}$. It follows that viability fails then also in the sense of the present proposition. $\square$

The result that follows gives a seemingly weaker formulation of market viability, which turns out to be equivalent to Proposition 2.22. This new formulation is often very convenient to work with. We adapt methods from Lemma A.12 in KARATZAS AND KARDARAS (2020) [1] combined with assumption 2.12 in the proof of Proposition 2.23.

Proposition 2.23. *(**Boundedness in Probability**). The market is viable, if and only if we have*

$$\lim_{m\to\infty} \sup_{X\in\mathcal{X}(K)} \mathbb{P}(X(T) > m) = 0, \quad \forall\, T \in \mathbb{R}_+. \tag{2.25}$$

Proof. First we assume that the market is not viable. According to Proposition 2.22, we can find a real number $T > 0$ and a nonnegative $\mathcal{F}(T)$-measurable random variable h with $\mathbb{P}(h > 0) > 0$. Moreover, for every $m \in \mathbb{N}$, there exists $X^m \in \mathcal{X}(K)$ such that $X^m(T) \geq mh$. Pick $\epsilon > 0$ sufficiently small, so that $\mathbb{P}(h > \epsilon) > \epsilon$ holds. Then $\{X_\pi(T) \mid \pi \in \mathcal{I}(R;K)\}$ fails to be bounded in probability, because

$$\liminf_{m\to\infty} \mathbb{P}(X^m(T) > \epsilon m) \geq \liminf_{m\to\infty} \mathbb{P}(X^m(T) \geq mh, h > \epsilon) \geq \epsilon.$$

Conversely, we assume for some $T > 0$, the set $\mathbb{K} := \{X_\pi(T) \mid \pi \in \mathcal{I}(R;K)\}$ is not bounded in probability. This means that we can find a number $\epsilon > 0$ and a sequence $(\pi_m)_{m\in\mathbb{N}} \subseteq \mathcal{I}(R;K)$ that satisfy $\mathbb{P}(X_{\pi_m}(T) > m2^m) > \epsilon$ for all $m \in \mathbb{N}$. Consider the event

$$D := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{X_{\pi_k}(T) > k2^k\} \in \mathcal{F}(T),$$

and note that $\mathbb{P}(D) \geq \epsilon$. For every $m \in \mathbb{N}$, we have the set inclusion $D \subseteq \bigcup_{k=m+1}^{\infty} \{X_{\pi_k}(T) > k2^k\}$, so there is a sufficiently large integer $z_m > m$ such that the set

$$D_m := D \bigcap \big(\bigcup_{k=m+1}^{z_m} \{X_{\pi_k}(T) > k2^k\} \big) \in \mathcal{F}(T)$$

satisfies $\mathbb{P}(D \setminus D_m) \leq \mathbb{P}(D)/2^{m+1}$. Now, the countable intersection

$$E := \bigcap_{m=1}^{\infty} D_m \in \mathcal{F}(T)$$

is a subset of D, and we have

$$\mathbb{P}(D \setminus E) = \mathbb{P}\big(\bigcup_{m=1}^{\infty} (D \setminus D_m) \big) \leq \sum_{m=1}^{\infty} \mathbb{P}(D \setminus D_m) \leq \sum_{m=1}^{\infty} \frac{\mathbb{P}(D)}{2^{m+1}} = \frac{\mathbb{P}(D)}{2},$$

thus also $\mathbb{P}(E) \geq \mathbb{P}(D)/2$, which implies $\mathbb{P}(E) \geq \epsilon/2 > 0$. Let us define

$$\rho_m := \sum_{k=m+1}^{z_m} 2^{-(k-m)} \pi_k, \quad m \in \mathbb{N} \tag{2.26}$$

and note that $X_{\rho_m} \in \mathbb{K}$, because K is a convex cone. Furthermore, we have for every $m \in \mathbb{N}$ the set-inclusions $E \subseteq D_m \subseteq \{X_{\rho_m}(T) > m\}$, from which $X_{\rho_m}(T) \geq m 1_E$ follows. Therefore we have a nonnegative $\mathcal{F}(T)$-measurable random variable 1_E with $\mathbb{P}(E) > 0$ such that, for every $m \in \mathbb{N}$, we have $\rho_m \in \mathcal{I}(R; K)$ and $X_{\rho_m}(T) \geq m 1_E$. In the light of Proposition 2.22, we note that the market is not viable. $\square$

Remark 2.24. From the construction of $\{\rho_m\}_{m \in \mathbb{N}}$ in (2.26), we can obtain the same conclusion when K is a general closed convex set containing the origin, not just a cone. Namely, Proposition 2.23 remains valid when K is a closed convex set containing the origin, and in this case condition (2.25) is again equivalent to market viability.

2.4 Deflators

Associated with the numéraires of Definition 2.1, and in a strong sense dual to them, is the collection of strictly positive supermartingale deflators. We introduce them right below.

Definition 2.25. (**Supermartingale Deflator**). A strictly positive, adapted process Y with right continuous, left-limited paths will be called *supermartingale deflator*, if it satisfies the requirements $Y(0) = 1, Y > 0$, and the product YX_π is a supermartingale for every portfolio $\pi \in \mathcal{I}(R;K)$ with values in the nonempty, closed, convex cone K. We denote by $\mathcal{Y}$ the collection of all supermartingale deflators.

Since we can select $\pi \equiv 0$ in $\mathcal{I}(R;K)$, thus $X_\pi \equiv 1$, every supermartingale deflator $Y \in \mathcal{Y}$ is in particular a supermartingale. If X_ν is the supermartingale numéraire, then clearly $Y_0 = 1/X_\nu$ is a deflator since X_π/X_ν is a supermartingale for every $\pi \in \mathcal{I}(R;K)$. For future developments, we consider also the subset

$$\mathcal{Y}_{loc} := \{Y \in \mathcal{Y} \mid Y \text{ is a local martingale}\} \tag{2.27}$$

of $\mathcal{Y}$. As we shall see in Theorem 2.31, this set in (2.27) is nonempty when the supermartingale numéraire ν exists.

We close this Section with a very important result, describing the exact structure of all elements in the collection $\mathcal{Y}$ of Definition 2.25. This result, Theorem 2.27, characterizes the collection of supermartingale deflators. The reader can compare our result with KARATZAS AND KARDARAS (2020) [1], Exercise 2.29, which deals with the special case $K \equiv \mathbb{R}^n$.

Definition 2.26. We denote by $\mathcal{M}_{loc}^{\perp}(M)$ the collection consisting of all scalar local martingales L with rcll (right continuous with left limits) paths and $L(0) = 0$, which satisfy the orthogonality property

$$\langle L, M_i \rangle = 0, \quad i \in \{1, \cdots, n\};$$

i.e., for which all products $LM_1, \cdots, LM_n$ are local martingales.

Theorem 2.27. (The structure of supermartingale deflators). *We assume the existence of the supermartingale numéraire X_ν.*

(i) With the above notation, every process of the form

$$Y = \frac{1}{X_\nu}\,\mathcal{E}(L + M_\rho) \cdot \exp\left(\widetilde{A}\right) \tag{2.28}$$

belongs to the collection $\mathcal{Y}$ of supermartingale deflators, provided

$$\widetilde{A} \text{ is adapted and of finite variation on compact intervals,} \tag{2.29}$$

$$L \in \mathcal{M}^{\perp}_{loc}(M) \quad \text{and} \quad \Delta L > -1, \tag{2.30}$$

$$\rho \in \mathcal{I}(R), \qquad \pi'\big(\alpha - c(\nu - \rho)\big) \le 0 \quad \text{for every} \quad \pi \in \mathcal{I}(R;K), \tag{2.31}$$

$$\widetilde{A} - C_{\nu\rho} \quad \text{is nonincreasing}. \tag{2.32}$$

(ii) In fact, $\mathcal{Y}$ consists of exactly the processes Y which admit the above representation (2.28) with the properties (2.29)–(2.32).

Proof. (i) Suppose that Y is a supermartingale deflator, according to Definition 2.25. Then

$$N := Y X_\nu$$

is a strictly positive supermartingale with $N(0) = 1$; we denote by

$$\widetilde{N} := \mathcal{L}(N) = \int_0^{\cdot} \frac{\mathrm{d}N(t)}{N(t)}$$

its stochastic logarithm, so that $N = \mathcal{E}(\widetilde{N})$. In particular, $\widetilde{N}$ is a supermartingale with $\widetilde{N}(0) = 0$, and we write its DOOB-MEYER decomposition as $\widetilde{N} = \widetilde{M} + \widetilde{A}$; here $\widetilde{M}$ is a local martingale and $\widetilde{A}$ a nonincreasing process with rcll paths and $\widetilde{M}(0) = 0$, $\widetilde{A}(0) = 0$.

We recall from Proposition 2.7 that for every portfolio $\pi \in \mathcal{I}(R;K)$ we have the relative performance computation

$$\frac{X_\pi}{X_\nu} = \mathcal{E}(M_{\pi-\nu} + A_{\pi-\nu} - C_{\pi-\nu,\nu}). \tag{2.33}$$

We also know $A_\nu = C_{\nu\nu}$ from (2.10), so taking $\pi \equiv 0$ in (2.33) leads to

$$\frac{1}{X_\nu} = \mathcal{E}(-M_\nu). \tag{2.34}$$

The representation (2.28) above gives

$$YX_\pi = \frac{X_\pi}{X_\nu} \cdot N = \frac{X_\pi}{X_\nu} \cdot \mathcal{E}(\widetilde{N}) = \mathcal{E}(M_{\pi-\nu} + A_{\pi-\nu} - C_{\pi-\nu,\nu}) \cdot \mathcal{E}(\widetilde{M} + \widetilde{A}),$$

as well as

$$YX_\pi = \mathcal{E}(M_{\pi-\nu} + \widetilde{M} + A_{\pi-\nu} - C_{\pi-\nu,\nu} + \widetilde{A} + \langle M_{\pi-\nu}, \widetilde{M}\rangle) \tag{2.35}$$

on account of the YOR formula in equation (1.8).

We recall at this point the KUNITA-WATANABE decomposition (KARATZAS AND SHREVE (1991) [4], Proposition 3.4.14),

$$\widetilde{M} = M_\rho + L, \ \text{ for some } \rho \in \mathcal{I}(R) \text{ and } L \in \mathcal{M}_{loc}^{\perp}(M),$$

and obtain

$$\begin{aligned} YX_\pi &= \mathcal{E}\big(M_{\pi-\nu} + M_\rho + L + A_{\pi-\nu} - C_{\pi-\nu,\nu} + \widetilde{A} + \langle M_{\pi-\nu}, M_\rho\rangle\big) \\ &= \mathcal{E}(M_{\pi-\nu+\rho} + L + A_{\pi-\nu} + C_{\pi-\nu,\rho-\nu} + \widetilde{A}) \\ &= \mathcal{E}(M_{\pi-\nu+\rho} + L + A_\pi + C_{\pi,\rho-\nu} - C_{\rho\nu} + \widetilde{A}). \end{aligned} \tag{2.36}$$

We have used for the second equality the property $\langle M_{\pi-\nu}, M_\rho \rangle = C_{\pi-\nu,\rho}$ as well as the fact, given by equation (2.10), that any supermartingale numéraire portfolio ν satisfies $A_\nu = C_{\nu\nu}$. Now, we can rewrite (2.36) as

$$Y X_\pi = \mathcal{E}(M_{\pi-\nu+\rho} + L + V^\pi) \tag{2.37}$$

with the notation

$$V^\pi := A_\pi + C_{\pi,\rho-\nu} - C_{\rho\nu} + \widetilde{A}. \tag{2.38}$$

Since Y is a supermartingale deflator, the product in (2.37) has to be a supermartingale for every portfolio $\pi \in \mathcal{I}(R;K)$; and this amounts to the requirement that the process of finite variation V^π in (2.38) be nonincreasing, for every portfolio $\pi \in \mathcal{I}(R;K)$.

Reading (2.38) with $\pi \equiv 0$ leads to the requirement that $\widetilde{A} - C_{\rho\nu}$ be nonincreasing. But the process

$$V^\pi = \big(\widetilde{A} - C_{\rho\nu}\big) + \int_0^\cdot \pi'\big(\alpha - c(\nu - \rho)\big)\, \mathrm{d}O$$

has to be nonincreasing for every portfolio $\pi \in \mathcal{I}(R)$ that takes values in the set K, and this set K is a cone (which means that π can be scaled by an arbitrarily large constant $m > 0$ without exiting the set). We conclude that the inequality

$$\pi'(\alpha - c(\nu - \rho)) \leq 0$$

must hold $(\mathbb{P} \otimes O)$-a.e., for every portfolio $\pi \in \mathcal{I}(R;K)$.

Finally, we read (2.37), (2.38) with $\pi \equiv \nu$ and deduce

$$Y X_\nu = \mathcal{E}(M_\rho + L + \widetilde{A}) = \mathcal{E}(M_\rho)\mathcal{E}(L) \cdot \exp(\widetilde{A}),$$

as claimed in (2.28).

(ii) Suppose, conversely, that we have a process

$$Y = \frac{1}{X_\nu}\,\mathcal{E}(L)\mathcal{E}(M_\rho)\cdot\exp(\widetilde{A})$$

as in (2.28), with the properties (2.29)–(2.32). Then for every $\pi \in \mathcal{I}(R;K)$, the representation (2.33) and the YOR formula lead to

$$\begin{aligned}
YX_\pi &= \frac{X_\pi}{X_\nu}\,\mathcal{E}(L)\,\mathcal{E}(M_\rho)\cdot\exp(\widetilde{A})\\
&= \mathcal{E}(M_{\pi-\nu}+M_\rho+L)\cdot\exp(A_{\pi-\nu}-C_{\pi-\nu,\,\nu-\rho}+\widetilde{A})\\
&= \mathcal{E}(M_{\pi-\nu}+M_\rho+L)\cdot\exp\big(A_\pi-C_{\pi,\nu-\rho}+(C_{\nu\nu}-A_\nu)+\widetilde{A}-C_{\nu\rho}\big)\\
&= \mathcal{E}(M_{\pi-\nu}+M_\rho+L)\cdot\exp\left(\int_0^\cdot \pi'\big(\alpha-c(\nu-\rho)\big)dO+\widetilde{A}-C_{\nu\rho}\right).
\end{aligned}$$

We have used here again the identity $A_\nu = C_{\nu\nu}$ from (2.10) in previous section. The conditions

$$\pi'\big(\alpha-c(\nu-\rho)\big)\le 0 \quad \text{for every} \quad \pi\in\mathcal{I}(R;K),$$

$$\widetilde{A}-C_{\nu\rho} \quad \text{is nonincreasing,}$$

indicate that $\int_0^\cdot \pi'\big(\alpha-c(\nu-\rho)\big)\mathrm{d}O+\widetilde{A}-C_{\nu\rho}$ is nonincreasing. Thus YX_π is a supermartingale. As $\pi\in\mathcal{I}(R;K)$ is chosen arbitrarily, it follows that Y is then a supermartingale deflator. □

Remark 2.28. In the context of Theorem 2.27, the requirement that $\widetilde{A}-C_{\nu\rho}$ be nonincreasing, implies that $\widetilde{A}$ is nonincreasing.

Indeed, suppose that $\widetilde{A}-C_{\nu\rho}$ is nonincreasing. According to condition (2.31), which requires that

$$\pi'\big(\alpha-c(\nu-\rho)\big)\le 0 \quad \text{should hold for every } \pi\in\mathcal{I}(R;K),$$

taking $\pi=\nu$ in the above inequality implies that

$$\nu'\big(\alpha-c(\nu-\rho)\big)=\nu'(\alpha-c\nu)+\nu c\rho\le 0.$$

Together with equation (2.10), according to which $\nu'(\alpha - c\nu) = 0$, we deduce

$$\nu' c\rho \leq 0, \quad \text{or equivalently that } C_{\nu\rho} \text{ is nonincreasing.}$$

Therefore, the process $\widetilde{A} = (\widetilde{A} - C_{\nu\rho}) + C_{\nu\rho}$ is nonincreasing.

Remark 2.29. Let us recall that Theorem 2.27 only holds for closed, convex cones containing the origin. For a general closed, convex set K containing the origin, conditions (2.31) and (2.32) take the form of the nonincrease of

$$\widetilde{A} + \int_0^{\cdot} (\pi - \nu)'\big(\alpha - c(\nu - \rho)\big)\, \mathrm{d}O, \quad \text{for every portfolio } \pi \in \mathcal{I}(R; K).$$

We cannot simplify this condition further.

The next Lemma is a forerunner for two critical results in this work. It furnishes a crucial step, as well as very important motivation, for the fundamental Theorem 2.31 of the present Chapter. It gives also the “easy halves" for the central result in the next Chapter, namely, the Optional Decomposition Theorem 3.5.

Lemma 2.30. *Assume that the collection $\mathcal{Y}_{loc}$ in (2.27), consisting of supermartingale deflators which are local martingales themselves, is nonempty. Fix a cumulative capital withdrawal stream $J \in \mathcal{H}$, and assume that it can be financed with initial capital $x \in \mathbb{R}_+$; i.e., for some investment strategy $\pi \in \mathcal{I}(R; K)$ and cumulative consumption stream $F \in \mathcal{H}$, we have*

$$X \equiv X(\cdot; x, \pi, F) = x + \int_0^{\cdot} \sum_{i=1}^{n} \frac{\pi_i X}{S_i}(t)\mathrm{d}S_i(t) - F \geq J.$$

Then the superhedging capital of (2.22) satisfies

$$x(J) \geq \sup_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}}\bigg[\int_0^{\infty} Y(t)\mathrm{d}J(t)\bigg]. \tag{2.39}$$

Proof. With

$$\overline{X} := X + F = X(\cdot; x, \pi) \geq F + J$$

and

$$Z := X - J = \overline{X} - (F + J) \geq 0,$$

integration-by-parts gives

$$YZ = Y\overline{X} - Y(F + J) = Y\overline{X} - \int_0^{\cdot} Y(t)(\mathrm{d}F(t) + \mathrm{d}J(t)) - \int_0^{\cdot} (F(t-) + J(t-))\mathrm{d}Y(t)$$

for every $Y \in \mathcal{Y}_{loc}$. Thus,

$$U := YZ + \int_0^{\cdot} Y(t)(\mathrm{d}F(t) + \mathrm{d}J(t)) = Y\overline{X} - \int_0^{\cdot} (F(t-) + J(t-))\mathrm{d}Y(t).$$

Since $Y \in \mathcal{Y}_{loc}$, the process $Y\overline{X}$ is a supermartingale. Also, the requirement $Y \in \mathcal{Y}_{loc}$ ensures that the integral $\int_0^{\cdot} (F(t-) + J(t-))\mathrm{d}Y(t)$ is a local martingale. Therefore, the process U is a local supermartingale. Moreover, $YZ \geq 0$ and $\int_0^{\cdot} Y(t)(\mathrm{d}F(t) + \mathrm{d}J(t)) \geq 0$ indicate that U is non-negative. In conclusion, U is a supermartingale. In a similar spirit, the non-negative process

$$Q := YZ + \int_0^{\cdot} Y(t)\mathrm{d}J(t) = U - \int_0^{\cdot} Y(t)\mathrm{d}F(t)$$

is also a local supermartingale, thus a true supermartingale, and

$$\begin{aligned} x = Q(0) &\geq \mathbb{E}^{\mathbb{P}} Q(\infty) \\ &\geq \sup_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}} \left[\int_0^{\infty} Y(t)\mathrm{d}J(t) \right]. \end{aligned}$$

$\square$

2.5 A Fundamental Theorem for Equity Market Structure and Arbitrage Theory

The next result is very important for equity market structure and arbitrage theory. As a reminder, we shall work under Assumption 2.12. We recall the definition of $\mathcal{J}$ from (2.15) and equation (2.17).

Theorem 2.31. *For the market with conic constraint K, the following statements are equivalent:*

(1) The market is viable.

(2) The supermartingale numeraire X_ν exists.

(3) The market has locally finite growth; namely, the aggregate maximal growth process satisfies $G(T) < \infty$, for all $T > 0$.

(4) The set $\{\mathcal{J} \cap K \neq \emptyset\}$ has zero $(\mathbb{P} \otimes \mathcal{O})$-measure and

$$\nu := \mathbb{1}_{\{\mathcal{J} \cap K = \emptyset\}} \cdot \arg \max_{\pi \in K \cap \mathfrak{N}^\perp} \left(\pi' \alpha - \frac{1}{2} \pi' c \pi \right) \tag{2.40}$$

is integrable with respect to R.

(5) The Boundedness in Probability condition (2.25) holds.

(6) $\mathcal{Y}_{loc} \neq \emptyset$.

Proof. We first prove the equivalence relations among (1) to (5), and deal with (6) in the end. The equivalence (1)$\Longleftrightarrow$(5) is dealt with in Proposition 2.23. The implication (3)$\Longrightarrow$(2) follows from the discussion in Remark 2.15. The equivalence between (3) and (4) is also given in Remark 2.15. It remains to prove the implications (2)$\Longrightarrow$(1) and (1)$\Longrightarrow$(4).

For the implication (2)$\Longrightarrow$(1), assume (2) holds, namely, that we have a portfolio $\nu \in \mathcal{I}(R; K)$ with the supermartingale numéraire property. According to Definition 2.25, the reciprocal of X_ν, namely

$$Y := \frac{1}{X_\nu}$$

is a strictly positive supermartingale deflator; in particular, $\mathcal{Y}_{loc} \neq \emptyset$ (i.e., (2) $\Longrightarrow$ (6)). Now we pick a cumulative capital withdrawal stream $J \in \mathcal{H}$ such that $x(J) = 0$. From Lemma 2.30 and the inequality (2.39), we have

$$0 = x(J) \geq \sup_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty Y(t)\mathrm{d}J(t) \right] \geq 0,$$

which implies that $\mathbb{E}^{\mathbb{P}}\left(\int_0^\infty Y(t)\mathrm{d}J(t) \right) = 0$ holds $\mathbb{P}$-a.e. for every $Y \in \mathcal{Y}_{loc}$. Since such a Y is strictly positive, it follows that $J(\infty) = 0$ also holds $\mathbb{P}$-a.e., and this leads to $J \equiv 0$. The viability of the market follows.

Next, we tackle the implication (1)$\Longrightarrow$(4). Let us assume that the market fails to have locally finite growth; in other words, we need to consider the following two cases:

(A) the set $\{\mathcal{J} \cap K \neq \emptyset\}$ fails to be $(\mathbb{P} \times \mathcal{O})$-null; or

(B) the set $\{\mathcal{J} \cap K \neq \emptyset\}$ is $(\mathbb{P} \times \mathcal{O})$-null, but ν, defined in equation (2.40), is not R-integrable.

For the rest of the argument, we will show that, in each case, the market is not viable. This will establish the implication (1)$\Longrightarrow$(4).

• *Case (A)*: From the argument in Subsection 2.5.1, the set $\{\mathcal{J} \cap K \neq \emptyset\}$ is predictable. Moreover, we can find a portfolio $\pi \in \mathcal{I}(R; K)$, such that

$$J := \int_0^\cdot \pi'(t)dR(t) = \int_0^\cdot \pi'(t)dA(t), \tag{2.41}$$

where $\mathbb{P}[J(\infty) > 0] > 0$. We define $\vartheta \equiv (\vartheta_i)_{i=1,\cdots,n}$ with components $\vartheta_i = \pi_i/S_i$ for $i \in \{1, \cdots, n\}$, and obtain

$$X(\cdot; 0, \vartheta) \equiv \int_0^\cdot \vartheta'(t)dS(t) = \int_0^\cdot \pi'(t)dR(t) = J. \tag{2.42}$$

By definition, we have $x(J) = 0$, and this violates market viability.

• *Case (B)*: Here, we assume that the event $E = \{\mathcal{J} \cap K \neq \emptyset\}$ is $(\mathbb{P} \otimes O)$-null. That means,

$$G(T) = \frac{1}{2} \int_0^T \nu'(t) c(t) \nu(t) \mathrm{d}O(t) \quad \text{holds for all } T \in \mathbb{R}_+. \tag{2.43}$$

Furthermore, we assume

$$\mathbb{P}(G(T) = \infty) > 0 \quad \text{holds for some } T \in (0, \infty). \tag{2.44}$$

In light of Proposition 2.23, it is sufficient to show that the set $\{X_\pi(T) \mid \pi \in \mathcal{I}(R; K)\}$ fails to be bounded in probability. The proof is given in subsection 2.5.2.

For statement (6), we have already argued the implication (2) $\Longrightarrow$ (6). On the other hand, if statement (6) holds, we can select an arbitrary supermartingale deflator $Y \in \mathcal{Y}_{loc}$ and, following the same argument as in the implication (2) $\Longrightarrow$ (1) and Lemma 2.30, deduce that $J \equiv 0$ holds whenever $x(J) = 0$. Therefore, the statement (1) holds. In conclusion, we complete the proof. □

2.5.1 *The set $\{\mathcal{J} \cap K \neq \emptyset\}$ is predictable*

When constructing the portfolio ν in (2.40), we need to make sure ν is predictable by doing appropriate “measurable selection" from the set $\{\mathcal{J} \cap K \neq \emptyset\}$. For this, we will show that the set $\{\mathcal{J} \cap K \neq \emptyset\}$ defined in (2.15) and (2.17), is predictable. We shall first prove that $\mathcal{J} = \left\{\xi \in \mathbb{R}^n \mid \xi' c = 0 \text{ and } \xi' \alpha > 0\right\}$ is a predictable set-valued process.

Definition 2.32. For a fixed $a > 0$, we define $\mathcal{J}^a$ to be the set-valued process such that

$$\mathcal{J}^a(\omega, t) = \left\{\xi \in \mathbb{R}^n \;\middle|\; \xi' c(\omega, t) = 0 \;\text{ and }\; \xi' \alpha(\omega, t) \geq 1/a\right\} \tag{2.45}$$

for $(\omega, t) \in \Omega \times [0, \infty)$.

The following lemma sets forth properties of these sets that we shall find useful.

Lemma 2.33. *With the previous definition we have:*

(i) *$\mathcal{J}^a$ is increasing in $a > 0$; we have $\mathcal{J}^a \subseteq \mathcal{J}$ and $\mathcal{J} = \bigcup_{a>0} \mathcal{J}^a$. In particular, $\mathcal{J} \cap K \neq \emptyset$ if and only if $\mathcal{J}^a \cap K \neq \emptyset$ for all large enough $a > 0$.*

(ii) *For all $a > 0$, $\mathcal{J}^a$ takes values in the collection of closed, convex subsets of $\mathbb{R}^n$.*

Proof. In the proof, we first fix (ω, t). The fact that $\mathcal{J}^a$ is increasing for $a > 0$ is trivial. For $a_1 > a_2 > 0, \xi \in \mathcal{J}^{a_2}$ implies that

$$\xi' c = 0 \quad \text{and} \quad \xi' \alpha > \frac{1}{a_2}. \tag{2.46}$$

Therefore we have

$$\xi' c = 0 \quad \text{and} \quad \xi' \alpha > \frac{1}{a_1}, \tag{2.47}$$

which implies that $\xi \in \mathcal{J}^{a_1}$.

On the other hand, for every $\xi \in \mathcal{J}$, we observe that

$$\xi' c = 0 \quad \text{and} \quad \xi' \alpha = \epsilon > \frac{\epsilon}{2} > 0 \tag{2.48}$$

Therefore $\xi \in \mathcal{J}^{\frac{2}{\epsilon}}$. We can now conclude that $\mathcal{J} = \bigcup_{a>0} \mathcal{J}^a$.

For part (ii), we first show that $\mathcal{J}^a$ is closed. Since $\mathcal{J}^a = \{\xi \in \mathbb{R}^n \mid \xi' c = 0, \text{ and } \xi' \alpha > 1/a\}$, the continuity of maps $\xi \mapsto \xi' c$ and $\xi \mapsto \xi' \alpha$ implies $\mathcal{J}^a$ is closed.

For convexity of $\mathcal{J}^a$, let us consider $\xi_1, \xi_2 \in \mathcal{J}^a$ and $\lambda \in (0, 1)$. For $\xi := \lambda \xi_1 + (1 - \lambda) \xi_2$, we have

$$\xi' c = \lambda \xi_1' c + (1 - \lambda) \xi_2' c = 0, \tag{2.49}$$

$$\xi' \alpha = \lambda \xi_1' \alpha + (1 - \lambda) \xi_2' \alpha \geq \frac{1}{a}, \tag{2.50}$$

which imply that $\xi \in \mathcal{J}^a$. Together with the construction of ξ, we get the convexity of $\mathcal{J}^a$. $\square$

Remark 2.34. In view of $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}^n$ and Lemma 2.53, in order to prove predictability of $\mathcal{J}$, we only have to prove predictability of $\mathcal{J}^n$.

Let $z_1(p) := p'c$ and $z_2(p) := p'\alpha$. By Definition 2.54 in the Appendix (Section 2.8) of this Chapter, z_1 and z_2 are Carathéodory functions. Now we write down $\mathcal{J}^n$ explicitly:

$$\mathcal{J}^n = \left\{ p \in \mathbb{R}^d \,\middle|\, z_1(p) = 0, z_2(p) \geq \frac{1}{n} \right\} = \left\{ p \in \mathbb{R}^d \,\middle|\, z_1(p) = 0 \right\} \bigcap \left\{ p \in \mathbb{R}^d \,\middle|\, z_2(p) \geq \frac{1}{n} \right\}.$$

According to Lemma 2.55, $\mathcal{J}^n$ is the intersection of two closed and predictable set. Thus, with help of Lemma 2.53, $\mathcal{J}^n$ is also predictable.

We define $K^m := \{\xi \in \mathbb{R}^d \,|\, \xi \in K \textit{ and } \|\xi\| \leq m\}$. Apparently K^m is compact. According to Remark 2.52, $\{\mathcal{J} \cap K^m \neq \emptyset\}$ is predictable. Since $\{\mathcal{J}^n \cap K \neq \emptyset\} = \bigcup_{m=1}^{\infty} \{\mathcal{J}^n \cap K^m\}$, Lemma 2.53 implies that $\{\mathcal{J}^n \cap K \neq \emptyset\}$ is also predictable.

Furthermore $\{\mathcal{J} \cap K \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \{\mathcal{J}^n \cap K \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \{\mathcal{J}^n \cap K^m \neq \emptyset\}$ is also predictable.

Remark 2.35. Let us suppose that $\{\mathcal{J} \cap K \neq \emptyset\}$ is not $(\mathbb{P} \times \mathcal{O})$-null. we define

$$U_n := \{p \in \mathbb{R}^d \,|\, \|p\| \leq n \text{ and } p \in \mathcal{J}^n\}.$$

By definition, we have $\mathcal{J} = \bigcup_{n\in\mathbb{N}} U_n$. There exists $n \in \mathbb{N}$ such that the convex, closed and predictable set-valued process

$$\mathcal{B}^n := U_n \cap K \tag{2.51}$$

satisfies $(\mathbb{P} \times \mathcal{O})\big(\{\mathcal{B}^n \neq \emptyset\}\big) > 0$.

From Theorem 2.56, there exists a predictable process π such that $\pi(\omega, t) \in \mathcal{B}^n(\omega, t)$ when $\mathcal{B}^n(\omega, t) \neq \emptyset$, and $\pi(\omega, t) = 0$ if $\mathcal{B}^n(\omega, t) = \emptyset$. This π is bounded, so $\pi \in \mathcal{I}(R; K)$.

We define $J := \int_0^{\cdot} \pi'(t) dR(t) = \int_0^{\cdot} \pi'(t) dA(t)$. The fact that $U_n \cap K \neq \emptyset$ is not $(\mathbb{P} \otimes \mathcal{O})$-null implies that $\mathbb{P}[J(\infty) > 0] > 0$.

We define $\vartheta = (\vartheta_i)_{i=\{1,\cdots,n\}}$ with components $\vartheta_i = \frac{\pi_i}{S_i}$ for $i \in \{1, \cdots, n\}$. We have

$$X(\cdot; 0, \vartheta) \equiv \int_0^{\cdot} \vartheta'(t) \mathrm{d}S(t) = \int_0^{\cdot} \pi'(t) \mathrm{d}R(t) = J.$$

By definition, we have $x(J) = 0$, and this violates market viability.

2.5.2 *Case (B) in Theorem 2.31*

For Case (B) in Theorem 2.31, we will show that by closely following a numéraire portfolio which is not R-integrable up to time T, one can make arbitrarily large gains with fixed, positive probability.

Since the set $\{\mathcal{J} \cap K \neq \emptyset\}$ is $(\mathbb{P}\otimes O)$-null, we consider the portfolio ν defined in equation (2.40). For each $m \in \mathbb{N}$, we set $\nu^m := \nu \mathbb{1}_{\{\|\nu\| \le m\}} \in \mathcal{I}(R; K)$. Direct computation implies that

$$\begin{aligned}\log X_{\nu^m} &= A_{\nu^m} - (1/2)C_{\nu^m\nu^m} + \int_0^{\cdot} (\nu^m)'(t) dM(t) \\ &= \frac{1}{2}\int_0^{\cdot} \mathbb{1}_{\{\|\nu(t)\|\le m\}}\nu'(t)c(t)\nu(t)dO(t) + \int_0^{\cdot} \mathbb{1}_{\{\|\nu(t)\|\le m\}}\nu'(t)dM(t).\end{aligned}$$

Next, we note that the process

$$2G^m := \int_0^{\cdot} \mathbb{1}_{\{\|\nu(t)\|\le m\}}\nu'(t)c(t)\nu(t)dO(t)$$

is the quadratic variation of the local martingale $\int_0^{\cdot} \mathbb{1}_{\{\|\nu(t)\|\le m\}}\nu'(t)\mathrm{d}M(t)$. Therefore, the DAMBIS-DUBINS-SCHWARZ representation, combined with the scaling property of Brownian motion, imply that for every $m \in \mathbb{N}$ there exists a Brownian motion W^m, on a possibly enlarged filtered probability space, such that

$$\log X_{\nu^m} = G^m + \sqrt{2}W^m(G^m).$$

The sequence $\{G^m(T)\}_{m\in\mathbb{N}}$ is nondecreasing and converges to

$$\frac{1}{2}\int_0^T \nu'(t)c(t)\nu(t)dO(t) = G(T)$$

so the strong law of large numbers for Brownian motion gives

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{W^m(G^m(T))}{G^m(T)} \le -\frac{1}{2\sqrt{2}},\ G(T) = \infty\right] = 0.$$

Since $\log X_{\nu^m} = G^m + \sqrt{2}W^m(G^m)$ holds for all $m \in \mathbb{N}$, we obtain

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{\log X_{\nu^m}(T)}{G^m(T)} \le \frac{1}{2},\ G(T) = \infty\right] = 0.$$

Therefore, under the condition $\mathbb{P}[G(T) = \infty] > 0$, the collection of random variables $\{X_{\nu^m}(T) \big| \ m \in \mathbb{N}\} \subseteq \{X(T) \big| \ X \in \mathcal{X}\}$ fails to be bounded in probability, once again violating market viability.

2.6 Optimality Properties of the Supermartingale Numéraire

Let us recall that Assumption 2.12 remains in force through in this section. The existence of a supermartingale numéraire portfolio guarantees market viability. Moreover, such a portfolio has some additional remarkable properties:

- Asymptotic growth-optimality;
- Relative log-optimality.

We will discuss these properties separately in this Section.

2.6.1 *Asymptotic Growth Optimality*

In this subsection, we will discuss asymptotic growth optimality. In a later section, we will use Proposition 2.39 to prove the main Theorem 2.45 for robust maximization of asymptotic growth. We develop an auxiliary result first.

Lemma 2.36. *Suppose ν is a supermartingale numéraire portfolio among portfolios in $\mathcal{I}(R;K)$, and consider an arbitrary portfolio $\pi \in \mathcal{I}(R;K)$. Then, according to notion (1.27), the difference* *of two cumulative growth processes, denoted by* $\Gamma_\nu^\pi = \Gamma_\nu - \Gamma_\pi$*, satisfies*

$$\left\{ \lim_{T\to\infty} \frac{X_\pi(T)}{X_\nu(T)} \in (0,\infty) \right\} = \{\Gamma_\nu^\pi(\infty) < \infty\},$$

$$\lim_{T\to\infty} \frac{1}{\Gamma_\nu^\pi(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) = -1 \quad on \quad \{\Gamma_\nu^\pi(\infty) = \infty\},$$

where the limits are understood in the $\mathbb{P}$*-a.e. sense.*

Remark 2.37. The quantity Γ_ν^π is nondecreasing. The reader can refer to Proposition 2.8 in Chapter 2 or Proposition 2.41 in KARATZAS AND KARDARAS (2020) [1] for detailed argument.

Proof. Recall the representation for the logarithm of the relative wealth $X_\pi^\nu = X_\pi/X_\nu$, with the familiar notation

$$\Gamma_\pi = A_\pi - (1/2)C_{\pi\pi} = \int_0^\cdot \big(\pi'\alpha - (1/2)\pi' c\pi\big) dO = \int_0^\cdot g(\pi) dO$$

we obtain

$$\log\left(\frac{X_\pi}{X_\nu}\right) = M_{\pi-\nu} - (\Gamma_\nu - \Gamma_\pi)$$

By use of inequality (2.11), the fact that X_ν is the supermartingale numéraire implies that

$$\pi'(\alpha - c\nu) \le 0 = \nu'(\alpha - c\nu), \quad \forall\, \pi \in \mathcal{I}(R;K)$$

Therefore we have

$$c_{\pi-\nu,\pi-\nu} = (\pi-\nu)' c(\pi-\nu) \le c_{\pi\pi} - 2\alpha_\pi + c_{\nu\nu} = 2\gamma_\nu - 2\gamma_\pi.$$

These considerations lead to

$$\langle \log X_\pi^\nu, \log X_\pi^\nu \rangle = \langle M_\pi^\nu, M_\pi^\nu \rangle = \int_0^\cdot c_{\pi-\nu,\pi-\nu} \mathrm{d}O(t) = C_{\pi-\nu,\pi-\nu} \le 2\Gamma_\nu^\pi.$$

With help of the DAMBIS-DUBINS-SCHWARZ representation and the scaling property of Brownian motion, the logarithm of the relative wealth has a new representation

$$\log X_\pi^\nu = \log(X_\pi/X_\nu) = -\Gamma_\nu^\pi + \sqrt{2} W\big((1/2)C_{\pi-\nu,\pi-\nu}\big)$$

Thanks to Strong Law of Large Numbers for Brownian motion, we get

$$\frac{W\big((1/2)C_{\pi-\nu,\pi-\nu}\big)}{\Gamma_\nu^\pi} = \frac{W\big((1/2)C_{\pi-\nu,\pi-\nu}\big)}{(1/2)C_{\pi-\nu,\pi-\nu}} \frac{(1/2)C_{\pi-\nu,\pi-\nu}}{\Gamma_\nu^\pi}$$

When $\Gamma_\nu^\pi(\infty) = \infty$, in either case $C_{\pi-\nu,\pi-\nu} < \infty$ or $C_{\pi-\nu,\pi-\nu} = \infty$, we have

$$\lim_{T\to\infty} \frac{W\big((1/2)C_{\pi-\nu,\pi-\nu}(T)\big)}{\Gamma_\nu^\pi(T)} = 0$$

□

Remark 2.38. Readers can refer to KARATZAS AND KARDARAS (2020), Proposition 2.45, which is a special case of Lemma 2.36 where $K \equiv \mathbb{R}^n$.

Proposition 2.39. *(**Asymptotic growth-optimality**). Suppose a supermartingale numéraire portfolio ν exists among $\mathcal{I}(R;K)$, and the aggregate maximal growth process satisfies $G(\infty) = \infty$. Then for any portfolio $\pi \in \mathcal{I}(R;K)$, we have*

$$\limsup_{T\to\infty} \left(\frac{\log X_\pi(T)}{G(T)}\right) \le 1 = \lim_{T\to\infty} \left(\frac{\log X_\nu(T)}{G(T)}\right)$$

Proof. We recall from Lemma 2.17 the representation $\log X_\nu = G + \sqrt{2}W(G)$ for the supermartingale numéraire, for a suitable Brownian motion W. The equality follows then from the Strong Law of Large Numbers for Brownian motion, namely,

$$\lim_{T\to\infty} \frac{W\big(G(T)\big)}{G(T)} = 0; \qquad \text{thus} \qquad \lim_{T\to\infty} \frac{\log X_\nu(T)}{G(T)} = 1.$$

With help of Lemma 2.36 and the assumption $G(\infty) = \infty$, we have

$$\lim_{T\to\infty} \frac{1}{G(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) = 0$$

on the event $\{\Gamma_\nu^\pi(\infty) < \infty\}$. On the event $\{\Gamma_\nu^\pi(\infty) = \infty\}$, Lemma 2.36 indicates that

$$\frac{1}{G(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) = \frac{\Gamma_\nu^\pi(T)}{G(T)} \cdot \frac{1}{\Gamma_\nu^\pi(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right),$$

where $\frac{1}{\Gamma_\nu^\pi(T)} \log \frac{X_\pi(T)}{X_\nu(T)}$ tends to -1 as $T \to \infty$. Moreover, $\frac{\Gamma_\nu^\pi(T)}{G(T)}$ is nonnegative. Thus,

$$\limsup_{T\to\infty} \frac{1}{G(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) \leq 0 \qquad \text{as well as} \qquad \limsup_{T\to\infty} \left(\frac{\log X_\pi(T)}{G(T)}\right) \leq 1$$

hold on the event $\{\Gamma_\nu^\pi(\infty) = \infty\}$. □

2.6.2 *Relative Log-Optimality*

In this Subsection, we first state the notion of relative log-optimality. Later, Proposition 2.43 will show that supermartingale numéraire portfolios are also relative log-optimal; and vice-versa.

Definition 2.40. A process $\rho \in \mathcal{I}(R; K)$ is called *relatively log-optimal portfolio*, if for all portfolio $\pi \in \mathcal{I}(R; K)$ and all stopping times $\tau \in \mathcal{T}$ we have

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{X_\pi(\tau)}{X_\rho(\tau)}\right)^+\right] < \infty \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{X_\pi(\tau)}{X_\rho(\tau)}\right)\right] \leq 0 \tag{2.52}$$

Before showing that supermartingale numérarie portfolios are indeed relative log-optimal portfolios, we state the following Lemma, which plays a key role in the Proposition 2.43.

Lemma 2.41. *Let N be a continuous, strictly positive supermartingale with $N(0) = 1$ and $N = M - A$, where M is a local martingale and A is an adapted, continuous non-dreasing process.*

Then for any stopping time $\tau \in \mathcal{T}$, we have $\mathbb{E}^{\mathbb{P}}[(\log N(\tau))^+] \leq 1$, and

$$\mathbb{E}^{\mathbb{P}}[\log N(\tau)] = -\mathbb{E}^{\mathbb{P}}\left[\int_0^{\tau} \frac{\mathrm{d}A(t)}{N(t)}\right] - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\big[\langle \log N\rangle(\tau)\big]. \tag{2.53}$$

Proof. ITO's Lemma implies that

$$\log N = -B + L, \quad \text{where} \quad B := \frac{1}{2}\langle \log N\rangle \quad \text{and} \quad L := \int_0^{\cdot} \frac{\mathrm{d}N(t)}{N(t)}.$$

We can further decompose the process L into

$$L = E - D \quad \text{where} \quad E := \int_0^{\cdot} \frac{\mathrm{d}M(t)}{N(t)} \quad \text{and} \quad D := \int_0^{\cdot} \frac{\mathrm{d}A(t)}{N(t)}.$$

Here, E is a continuous local martingale with quadratic variation $\langle E\rangle = 2B$. For a fixed stopping time $\tau \in \mathcal{T}$, the inequality

$$\mathbb{E}^{\mathbb{P}}[N(\tau)] \leq 1$$

holds by the Optional Sampling Theorem. Moreover, we have

$$\mathbb{E}^{\mathbb{P}}\big[(\log N(\tau))^+\big] = \int_0^{\infty} \mathbb{P}(N(\tau) > e^t)\mathrm{d}t \leq \mathbb{E}^{\mathbb{P}}[N(\tau)], \tag{2.54}$$

which indicates that $\mathbb{E}^{\mathbb{P}}\big[(\log N(\tau))^+\big] \leq 1$.

We only need to show equation (2.53). We need, therefore, to consider two cases: either

(i) $\mathbb{E}^{\mathbb{P}}[B(\tau)] < \infty$; or

(ii) $\mathbb{E}^{\mathbb{P}}[B(\tau)] = \infty$.

- For case (i), the process $E(\cdot \wedge \tau)$ is actually a square-integrable martingale; in this case we have $\mathbb{E}^{\mathbb{P}}[E(\tau)] = E(0) = 0$, and obtain

$$\mathbb{E}^{\mathbb{P}}[\log N(\tau)] = -\mathbb{E}^{\mathbb{P}}[D(\tau)] - \mathbb{E}^{\mathbb{P}}[B(\tau)] = -\mathbb{E}^{\mathbb{P}}\left[\int_0^{\tau} \frac{\mathrm{d}A(t)}{N(t)}\right] - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\big[\langle \log N\rangle(\tau)\big].$$

- In case (ii), we will show that $\mathbb{E}^{\mathbb{P}}[B(\tau)] = \infty$ implies $\mathbb{E}^{\mathbb{P}}[\log N(\tau)] = -\infty$. We first construct a nonincreasing sequence of functions $(f_k)_{k\in\mathbb{N}}$ as below:

$$f_k(x) = \begin{cases} -(kx)^2/2 + 2kx - (\log k + 3/2), & \text{for } x \in (0, 1/k). \\ \log x, & \text{for } x \in [1/k, \infty). \end{cases}$$

The special structure of $f_k(x)$ makes it increasing and twice continuously differentiable. Furthermore, we have

$$\log(x) \le f_k(x), \quad f_k'(x) \le 1/x, \quad f_k''(x) = -(\min\{k, 1/x\})^2, \quad \forall x \in (0, \infty).$$

We can also verify that each f_k is concave. Thanks to JENSEN's inequality and non-decreasing property of f_k, the process $f_k(N)$ is a bounded from below local supermartingale, thus a supermartingale.

Following ITÔ's Lemma, we can rewrite each $f_k(N)$ as $f_k(N) = E_k - D_k - B_k$, where

$$\begin{aligned} E_k &= \int_0^\cdot f_k'(N(t))\mathrm{d}M(t), \qquad D_k = \int_0^\cdot f_k'(N(t))\mathrm{d}A(t), \\ B_k &= \int_0^\cdot f_k''(N(t))\mathrm{d}\langle N\rangle(t) = \int_0^\cdot \min(\{1, kN(t)\})^2\mathrm{d}B(t). \end{aligned}$$

The bound $f_k'(x) \le 1/x$ indicates that $\langle L_k\rangle = \int_0^\cdot \left[f_k'(N)\right]^2\mathrm{d}N \le \int_0^\cdot \frac{\mathrm{d}N}{N^2} = 2B$. Next, we define two sequences of stopping times

$$\tau_1^m := \inf\{t \in \mathbb{R}_+ \mid B(t) \ge m\}, \quad \tau_2^m := \inf\left\{t \in \mathbb{R}_+ \middle| \int_0^t \frac{\mathrm{d}A(s)}{N(s)} \ge m\right\}$$

for $m \in \mathbb{N}$. We further define the stopping time $\tau^m := \min(\tau_1^m, \tau_2^m)$, $m \in \mathbb{N}$. The stopped local martingale $E_k(\cdot \wedge \tau^m)$ is actually a square-integrable martingale and

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}}[B_k(\tau^m)] < \infty, \qquad \mathbb{E}^{\mathbb{P}}[D_k(\tau^m)] < \infty, \\ \text{thus also} \qquad & \mathbb{E}^{\mathbb{P}}[f_k(N(\tau \wedge \tau^m))] = -\mathbb{E}^{\mathbb{P}}[D_k(\tau \wedge \tau^m)] - \mathbb{E}^{\mathbb{P}}[B_k(\tau \wedge \tau^m)]. \end{aligned}$$

Since the function f_k is bounded from below, FATOU's Lemma and the Monotone Convergence Theorem give

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[f_k(N(\tau))] &\leq \liminf_{m\to\infty} \mathbb{E}^{\mathbb{P}}[f_k(N(\tau \wedge \tau^m))] \\ &= -\limsup_{m\to\infty} \Big(\mathbb{E}^{\mathbb{P}}[D_k(\tau \wedge \tau^m)] + \mathbb{E}^{\mathbb{P}}[B_k(\tau \wedge \tau^m)] \Big) \\ &\leq -\limsup_{m\to\infty} \mathbb{E}^{\mathbb{P}}[B_k(\tau \wedge \tau^m)] = -\mathbb{E}^{\mathbb{P}}[B_k(\tau)]. \end{aligned}$$

The fact that $\log(x) \leq f_k(x)$ indicates that $\mathbb{E}^{\mathbb{P}}[\log N(\tau)] \leq -\mathbb{E}^{\mathbb{P}}[B_k(\tau)]$ for all $k \in \mathbb{N}$. Moreover, the sequence $(B_k(\tau))_{k\in\mathbb{N}}$ is nondecreasing and converges to $B(\tau)$. Thanks to the Monotone Convergence Theorem, we have

$$\mathbb{E}^{\mathbb{P}}[\log N(\tau)] \leq -\mathbb{E}^{\mathbb{P}}[B(\tau)] = -\infty,$$

which complete the proof for case (ii).

□

Remark 2.42. From the DOOB-MEYER Decomposition of the supermartingale $\log N$, we can see that $-\int_0^{\cdot} \frac{\mathrm{d}A(t)}{N(t)} - \frac{1}{2}\langle \log N \rangle$ is actually the adapted, continuous and non-increasing part of $\log N$. Let us write the dynamics

$$\mathrm{d}\left(\frac{X_\pi(t)}{X_\nu(t)}\right) = \left(\frac{X_\pi(t)}{X_\nu(t)}\right)\big[\mathrm{d}M_{\pi-\nu}(t) + \mathrm{d}A_{\pi-\nu}(t) - \mathrm{d}C_{\pi-\nu,\nu}(t)\big] \tag{2.55}$$

in the logarithmic form

$$\mathrm{d}\log\left(\frac{X_\pi(t)}{X_\nu(t)}\right) = \mathrm{d}M_{\pi-\nu}(t) + \mathrm{d}A_{\pi-\nu}(t) - \left(\mathrm{d}C_{\pi-\nu,\nu}(t) + \frac{1}{2}\mathrm{d}C_{\pi-\nu,\pi-\nu}(t)\right). \tag{2.56}$$

Now $C_{\pi-\nu,\nu} + \frac{1}{2}C_{\pi-\nu,\pi-\nu} = \frac{1}{2}C_{\pi\pi} - \frac{1}{2}C_{\nu\nu}$, so with the familiar notation

$$\Gamma_\pi = A_\pi - \frac{1}{2}C_{\pi\pi} = \int_0^\cdot (\pi'\alpha - \frac{1}{2}\pi' c\pi)\mathrm{d}O = \int_0^\cdot g(\pi)\mathrm{d}O \tag{2.57}$$

we obtain

$$\log\left(\frac{X_\pi}{X_\nu}\right) = M_{\pi-\nu} - (\Gamma_\nu - \Gamma_\pi) \tag{2.58}$$

According to Lemma 2.41 and Remark 2.37 for the nondecreasing property of $\Gamma_\nu - \Gamma_\pi$, we have

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{X_\pi(\tau)}{X_\nu \tau}\right)\right] = -\mathbb{E}^{\mathbb{P}}[\Gamma_\nu - \Gamma_\pi] \leq 0 \tag{2.59}$$

for all $\pi \in \mathcal{I}(R;K)$ and $\tau \in \mathcal{T}$. Therefore, a supermartingale numéraire portfolio ν is also relative-optimal. Moreover, the following important Proposition holds.

Proposition 2.43. *A portfolio is relatively log-optimal, if and only if it is a supermartingale numéraire portfolio.*

Proof. (i) Suppose that $\nu \in \mathcal{I}(R;K)$ is a supermartingale numéraire portfolio. By Lemma 2.41 and Remark 2.42, ν is relatively log-optimal.

(ii) For the converse implication, we shall assume throughout the rest of this proof that $\rho \in \mathcal{I}(R;K)$ is a relatively log-optimal portfolio. Without loss of generality, we take $\rho \in K \cap \mathfrak{N}^\perp$.

Remind the definition of $\mathcal{J}$ from (2.15) and equation (2.17), let us suppose that $\{\mathcal{J} \cap K \neq \emptyset\}$ is not $(\mathbb{P} \times O)$-null. According to the proof of Theorem 2.31 for case (A) and Remark 2.35, let $\phi \in \mathcal{I}(R;K)$ be a portfolio such that $\phi \in \mathcal{B}^n$ if $\mathcal{B}^n \neq \emptyset$ and $\phi = 0$ if $\mathcal{B}^n = \emptyset$, where $\mathcal{B}^n$ is defined in (2.51). Straightforward computations based on equation (2.5) show

$$\log X^{\rho}_{\rho+\phi} = \log X_{\rho+\phi} - \log X_{\rho} = \int_0^{\cdot} \phi(t) \mathrm{d}O(t).$$

We have

$$\mathbb{E}^{\mathbb{P}}\left[\left(\frac{X_{\rho+\phi}(\tau)}{X_{\rho}(\tau)}\right)\right] > 0$$

for some stopping time $\tau \in \mathcal{T}$, which contradicts the relatively log-optimality of ρ.

Now, let is recall from Theorem 2.31 and set

$$\nu := \mathbb{1}_{\{\mathcal{J}\cap K=\emptyset\}} \cdot \arg\max_{\pi\in K\cap\mathfrak{N}^{\perp}} \left(\pi'\alpha - \frac{1}{2}\pi' c\pi\right).$$

At this time we don't know whether ν is R-integrable, so we define

$$\rho^m := \rho\mathbb{1}_{\{\rho=\nu\}} + \rho\mathbb{1}_{\{\rho\neq\nu\}}\mathbb{1}_{\{\|\nu\|>m\}} + \nu\mathbb{1}_{\{\rho\neq\nu\}}\mathbb{1}_{\{\|\nu\|\leq m\}}, \quad m \in \mathbb{N},$$

and notice that $\rho^m \in \mathcal{I}(R;K)$ holds for all $m \in \mathbb{N}$. According to the representation (2.5), we obtain that

$$\frac{X_{\rho}}{X_{\rho^m}} \text{ is a local supermartingale,}$$

thanks to the fact that $(\rho - \rho^m)'(\alpha - c\rho_m) \leq 0$. Therefore, the ratio X_rho/X_{ρ^m} is also a supermartingale, as it is positive. Suppose that $\mathbb{P}(X_{\rho}(T) \neq X_{\rho^m}(T)) > 0$ holds for some $T \in \mathbb{R}_+$, the Optional Sampling Theorem and JENSEN's inequality would imply

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{X_{\rho}(T)}{X_{\rho^m}(T)}\right)\right] < \log\mathbb{E}^{\mathbb{P}}\left[\left(\frac{X_{\rho}(T)}{X_{\rho^m}(T)}\right)\right] \leq 0,$$

contradicting the relative log-optimality of ρ.

The property $X_{\rho} = X_{\rho^m}$ follows from this discussion, and from the continuity of the paths of X_{ρ}/X_{ρ^m}. The portfolio $\rho - \rho^m$ is R-null by Definition 1.5. The fact that $\rho - \rho^m \in \mathfrak{N}^{\perp}$ indicates that $\rho = \rho^m$. In conclusion, we have the set

$$\{\rho \neq \nu, \|\nu\| \leq m\}$$

has zero $(\mathbb{P} \otimes O)$-measure and this is correct for all $m \in \mathbb{N}$. Therefore $\{\rho \neq \nu\}$ is $(\mathbb{P} \otimes O)$ null. It shows that $\nu = \rho$ is R-integrable and ρ is a supermartingale numéraire portfolio as well. $\square$

Remark 2.44. In Proposition 2.43, market viability is not assumed. Thus, we can reach another important conclusion:

The existence of a relatively log-optimal portfolio can also be added to the list of equivalences of Theorem 2.31.

2.7 Robust maximization of asymptotic growth

For some given $n \in \mathbb{N}$, we let Ω be the space of continuous function $\omega : \mathbb{R} \to \mathbb{R}^n$, and denote by W the coordinate mapping process $W(\omega, t) = \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}_+$. The filtration $\mathcal{F}(\cdot)$ is the right-continuous augmentation of the one generated by W, and we set $\mathcal{F} = \mathcal{F}(\infty)$. We let $\mathbb{Q}$ denote WIENER measure on $(\Omega, \mathcal{F})$.

Consider now the collection Π of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$, which are absolutely continuous with respect to WIENER measure $\mathbb{Q}$ on $\mathcal{F}(T)$, for every $T \in \mathbb{R}_+$. In particular, GIRSANOV's theorem implies that, for each $\mathbb{P} \in \Pi$, there exists a predictable, n-dimensional process $\lambda^{\mathbb{P}} \equiv (\lambda^{\mathbb{P}}(t) : t \in \mathbb{R}_+)$, such that

$$\int_0^T \|\lambda^{\mathbb{P}}(t)\|^2 dt < \infty, \quad \mathbb{P}-\text{a.e., for every } T \in \mathbb{R}_+; \tag{2.60}$$

and that $W^{\mathbb{P}} := W - \int_0^{\cdot} \lambda^{\mathbb{P}}(t)dt$ is standard, n-dimensional Brownian motion on $\big(\Omega, \mathcal{F}(\cdot), \mathbb{P}\big)$.

We fix also a predictable process $\sigma \equiv (\sigma(t) : t \in \mathbb{R}_+)$ taking values in the space of $(n \times n)$-dimensional matrices, and assume that σ is $(\mathbb{Q} \times \text{Leb})$-a.e. nonsingular. Furthermore, for the predictable process $c := \sigma\sigma'$ taking values in $(n \times n)$-dimensional, nonnegative-definite matrices, we assume that $\int_0^T \sum_{i=1}^n c_{ii}(t)dt < \infty$ holds $\mathbb{Q}$-a.e., for every $T \in \mathbb{R}_+$.

2.7.1 *Background Model, and Problem Statement*

With these ingredients, we introduce a financial market whose stock price processes $(S_i; i \in \{1, \cdots, n\})$ are given via

$$S_i = S_i(0) \cdot \exp\Big(\int_0^{\cdot} \sum_{j=1}^n \sigma_{ij}(t) dW_j(t) - \frac{1}{2} \int_0^{\cdot} c_{ii}(t)dt \Big), \quad i \in \{1, \cdots, n\}$$

with $S_i(0) > 0$ for $i \in \{1, \cdots, n\}$. Note that each S_i, $i \in \{1, \cdots, n\}$ is a continuous semimartingale under $\mathbb{Q}$, and consequently also a semimartingale under each $\mathbb{P} \in \Pi$. In fact, under any given $\mathbb{P} \in \Pi$, we have

$$S_i = S_i(0) \cdot \mathcal{E}\Big(\int_0^{\cdot} \sum_{j=1}^n \sigma_{ij}(t) \big(\lambda_j^{\mathbb{P}}(t)dt + dW_j^{\mathbb{P}}(t) \big) \Big), \quad i \in \{1, \cdots, n\},$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential. Thus, $\lambda^{\mathbb{P}}$ can be interpreted as a "market price of risk" under $\mathbb{P} \in \Pi$. Let now $R \equiv (R_i; i \in \{1, \cdots, n\})$ denote the vector of corresponding returns

$$R_i := \int_0^{\cdot} \sum_{j=1}^n \sigma_{ij}(t) dW_j(t) = \int_0^{\cdot} \sum_{j=1}^n \sigma_{ij}(t) \big(\lambda_j^{\mathbb{P}}(t)dt + dW_j^{\mathbb{P}}(t) \big)$$

for $i \in \{1, \cdots, n\}$; these processes are well-defined under all probability measures $\mathbb{P} \in \Pi$. The interpretation of thses processes "return" rests on the property

$$\frac{dS_i(t)}{S_i(t)} = dR_i(t) = \sum_{j=1}^n \sigma_{ij}(t) \big(\lambda_j^{\mathbb{P}}(t)dt + dW_j^{\mathbb{P}}(t) \big), \quad i = 1, \cdots, n.$$

Under each $\mathbb{P} \in \Pi$, this market has mean return rate vector $\alpha^{\mathbb{P}} = \sigma \lambda^{\mathbb{P}}$ and covariation rate matrix $c = \sigma \sigma'$.

We set $\mathcal{I}(R)$ to be the class of all predictable processes that are R-integrable under $\mathbb{Q}$, which also makes them R-integrable under every $\mathbb{P} \in \Pi$. Moreover, we consider a closed, convex cone

$K \subseteq \mathbb{R}^n$. For any portfolio $\pi \in \mathcal{I}(R;K)$ and probability measure $\mathbb{P} \in \Pi$, we define the asymptotic growth of π under $\mathbb{P} \in \Pi$ as

$$\operatorname{ag}(\pi;\mathbb{P}) := \sup\left\{g \in \mathbb{R} \,\middle|\, \liminf_{T\to\infty}\left(\frac{1}{T}\log X_\pi(T)\right) \geq g, \mathbb{P}\text{-a.e.}\right\}$$

Note that this quantity depends in general on the probability measure $\mathbb{P} \in \Pi$, since probability measures in Π may be singular with respect to each other.

We fix now a nonempty, convex, closed and bounded set $\Lambda \subseteq \mathbb{R}^n$, and define Π_Λ as the class of all probability measures $\mathbb{P} \in \Pi$ such that $\lambda^{\mathbb{P}} \in \Lambda$, $(\mathbb{P} \times \text{Leb})$-a.e. Note that Π_Λ is certainly non-empty; for example, any probability measure $\mathbb{P} \in \Pi$ under which W is a Brownian motion with constant drift $\lambda \in \Lambda$ belongs to Π_Λ. The problem we wish to discuss, is the computation of the robust maximization of the asymptotic growth rate

$$\operatorname{ag}^*_\Lambda = \sup_{\pi\in\mathcal{I}(R;K)} \inf_{\mathbb{P}\in\Pi_\Lambda} \operatorname{ag}(\pi;\mathbb{P}), \tag{2.61}$$

and the identification of a portfolio $\pi^*_\Lambda \in \mathcal{I}(R;K)$ that attains the supremum.

2.7.2 *The Main Result*

In the proof of Theorem 2.45, we shall need the notion of *Carathéodory function* from KARATZAS AND KARDARAS (2007) [6]. We will discuss Carathéodory functions in the appendix in this Chapter. The reader can find the Definition 2.54 for Carathéodory functions on measurable random sets.

We define $\widetilde{\Omega}$ to be $\Omega \times [0,\infty)$ and define a mapping $\phi : \widetilde{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\phi(\widetilde{\omega}, x, y) = x'\sigma(\widetilde{\omega})y - (1/2)x'\sigma(\widetilde{\omega})\sigma'(\widetilde{\omega})x. \tag{2.62}$$

Furthermore, we define

$$h(\widetilde{\omega}, x) := \inf_{y\in\Lambda} \phi(\widetilde{\omega}, x, y). \tag{2.63}$$

With this notation we can state the main theorem and give its proof.

Theorem 2.45. *If the function $h(\widetilde{\omega}, \cdot)$ as in (2.62), (2.63) has a unique maximizer for each $\widetilde{\omega} \in \widetilde{\Omega}$, then there exists a portfolio $\pi^*_\Lambda \in \mathcal{I}(R; K)$ that attains the supremum in (2.61).*

Proof. Thanks to Lemma 2.47, it follows that $h(\widetilde{\omega}, x) = \inf_{y\in\Lambda} \phi(\widetilde{\omega}, x, y)$ is well-defined by (2.63) and continuous, for every $\widetilde{\omega} \in \widetilde{\Omega}$. Therefore, h is a Carathéodory function on K. Since $h(\widetilde{\omega}, x)$ is assumed to have a unique maximizer for each $\widetilde{\omega} \in \widetilde{\Omega}$, Theorem 2.48 implies that

$$\sup_{x\in K} h(\widetilde{\omega}, x) = h(\widetilde{\omega}, x_*(\widetilde{\omega}))$$

where x_* is predictable. Moreover, for every pair $(\widetilde{\omega}, x_*(\widetilde{\omega}))$, there exists an element $y_*(\widetilde{\omega}) \in \Lambda$ such that

$$h(\widetilde{\omega}, x_*(\widetilde{\omega})) = \phi(\widetilde{\omega}), x_*(\widetilde{\omega}), y_*(\widetilde{\omega}))$$

where y_* is also predictable.

Let $\pi^*_\Lambda = x_*$ and $\lambda^{\mathbb{P}^*_\Lambda} = y_*$. SION's Minimax Theorem from SION (1958) [7] implies that

$$\sup_{x\in K} \inf_{y\in\Lambda} \phi(\widetilde{\omega}, x, y) = \inf_{y\in\Lambda} \sup_{x\in K} \phi(\widetilde{\omega}, x, y) = \phi(\widetilde{\omega}, x_*(\widetilde{\omega}), y_*(\widetilde{\omega}));$$

i.e., for every $\widetilde{\omega} = (\omega, t) \in \widetilde{\Omega} = \Omega \times [0, \infty)$, the pair $(x_*(\widetilde{\omega}), y_*(\widetilde{\omega}))$ is a saddle point for $\phi(\widetilde{\omega}, \cdot, \cdot)$ in (2.62). It follows that $\phi(\widetilde{\omega}, x_*(\widetilde{\omega}), y_*(\widetilde{\omega})) \geq \phi(\widetilde{\omega}, x, y_*(\widetilde{\omega}))$, $\forall x \in K$. Namely, the portfolio π^*_Λ has the supermartingale numéraire property under P^*_Λ among portfolios in $\mathcal{I}(R; K)$. Therefore we have the equation

$$A_{\pi^*_\Lambda} = C_{\pi^*_\Lambda \pi^*_\Lambda}.$$

It is important to note that $\pi^*_\Lambda \in \mathcal{I}(R; K)$, namely, π^*_Λ is R-integrable. In fact, the R-integrability of $X_{\pi^*_\Lambda}$ is equivalent to $\int_0^T (\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda dt < \infty$, $\forall T \in \mathbb{R}_+$. Since the boundedness of the set Λ implies that $\lambda^{\mathbb{P}^*_\Lambda}$ is bounded, we also have that $\sigma'\pi^*_\Lambda$ is bounded, which completes the argument that $\pi^*_\Lambda \in \mathcal{I}(R; K)$.

Under any probability measure $\mathbb{P} \in \Pi_\Lambda$, it holds a.e. that

$$\log X_{\pi^*_\Lambda} = \int_0^\cdot \Big((\pi^*_\Lambda)'\sigma\lambda^{\mathbb{P}} - (1/2)(\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda\Big)dt + \int_0^\cdot (\pi^*_\Lambda)'\sigma dW^{\mathbb{P}}.$$

Thanks to

$$h(\widetilde{\omega}, x_*(\widetilde{\omega})) = \inf_{y\in\Lambda} \phi(\widetilde{\omega}, x_*(\widetilde{\omega}), y) = \phi(\widetilde{\omega}, x_*(\widetilde{\omega}), y_*(\widetilde{\omega})),$$

we obtain

$$\frac{1}{T}\log X_{\pi^*_\Lambda}(T) \geq \frac{1}{T}\int_0^T \Big((\pi^*_\Lambda)'\sigma\lambda^{\mathbb{P}^*_\Lambda} - (1/2)(\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda\Big)dt + \frac{1}{T}\int_0^T (\pi^*_\Lambda)'\sigma dW^{\mathbb{P}}.$$

The boundedness of $(\pi^*_\Lambda)'\sigma$ implies

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T (\pi^*_\Lambda)'\sigma dW^{\mathbb{P}^*_\Lambda} &= \lim_{T\to\infty}\left(\frac{\int_0^T (\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda dt}{T}\right)\cdot\left(\frac{\int_0^T (\pi^*_\Lambda)'\sigma dW^{\mathbb{P}^*_\Lambda}}{\int_0^T (\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda dt}\right)\\ &= 0\end{aligned}$$

thanks to the Strong Law of Large Number for the Brownian motion, the DAMBIS-DUBINS-SCHWARZ representation, and the boundedness of $\sigma'\pi^*_\Lambda$. Therefore,

$$ag(\pi^*_\Lambda;\mathbb{P}) \geq \liminf_{T\to\infty}\frac{1}{T}\int_0^T \Big((\pi^*_\Lambda)'\sigma\lambda^{\mathbb{P}^*_\Lambda} - (1/2)(\pi^*_\Lambda)'\sigma\sigma'\pi^*_\Lambda\Big)\mathrm{d}t \tag{2.64}$$

holds for all $\mathbb{P} \in \Pi_\Lambda$.

On the other hand, (2.61) gives

$$ag^*_\Lambda \leq \sup_{\pi\in\mathcal{I}(R;K)} ag(\pi;\mathbb{P}^*_\Lambda).$$

Under the measure $\mathbb{P}^*_\Lambda$, the portfolio π^*_Λ has the supermartingale numéraire property. With the help of Lemma 2.39, straightforward computations give

$$\sup_{\pi \in \mathcal{I}(R;K)} ag(\pi; \mathbb{P}^*_\Lambda) \leq ag(\pi^*_\Lambda, \mathbb{P}^*_\Lambda) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T \Big((\pi^*_\Lambda)' \sigma \lambda^{\mathbb{P}^*_\Lambda} - (1/2)(\pi^*_\Lambda)' \sigma \sigma' \pi^*_\Lambda \Big) dt.$$

Together with the inequality (2.64), this shows that the portfolio $\pi^*_\lambda \in \mathcal{I}(R;K)$ attains the robust maximal asymptotic growth rate in (2.61). □

2.7.3 *Examples*

Next we discuss some examples. We first define

$$\ell^*_\Lambda := \arg\min_{\ell \in \Lambda} \|\ell\|^2.$$

If we have $\ell^*_\Lambda \in \sigma' K$ for all $\tilde{\omega} \in \tilde{\Omega}$, then Theorem 2.45 can be further simplified as follows.

Theorem 2.46. *With the above notation, it holds that*

$$ag^*_\Lambda = \frac{1}{2} \inf_{\ell \in \Lambda} \|\ell\|^2$$

Furthermore, the portfolio $\pi^*_\Lambda := (\sigma')^{-1} \ell^*_\Lambda \in \mathcal{I}(R;K)$ is such that

$$ag(\pi^*_\Lambda, \mathbb{P}) \geq ag^*_\Lambda, \quad \textit{for all} \quad \mathbb{P} \in \Pi_\Lambda.$$

Proof. With compactness of the set Λ, the vector $\ell^*_\Lambda \in \Lambda$ is well defined. Let $\mathbb{P}^*_\Lambda \in \Pi_\Lambda$ denote the probability measure that makes W Brownian motion with drift ℓ^*_Λ. Under this measure $\mathbb{P}^*_\Lambda$, the portfolio π^*_Λ has the supermartingale numéraire property. By Proposition 2.39, we obtain

$$ag^*_\Lambda \leq \sup_{\pi \in \mathcal{I}(R;K)} ag(\pi, \mathbb{P}^*_\Lambda) \leq ag(\pi^*_\Lambda; \mathbb{P}^*_\Lambda) = \frac{1}{2} \|\ell^*_\Lambda\|^2 = \frac{1}{2} \inf_{\ell \in \Lambda} \|\ell\|^2. \tag{2.65}$$

On the other hand, under any $\mathbb{P} \in \Pi_\Lambda$, it holds a.e. that

$$\log X_{\pi^*_\Lambda} = \int_0^\cdot \Big((\ell^*_\Lambda)' \lambda^{\mathbb{P}}(t) - \frac{1}{2} \|\ell^*_\Lambda\|^2 \Big) \mathrm{d}t + (\ell^*_\Lambda)' W^{\mathbb{P}}.$$

Since $\|\ell^*_\Lambda\|^2 \le (\ell^*_\Lambda)' \lambda^{\mathbb{P}}$ holds ($\mathbb{P} \times$ Leb)-a.e., in view of the definition of ℓ^*_Λ and the fact that Λ is convex, we obtain

$$\frac{1}{T} \log X_{\pi^*_\Lambda}(T) \ge \frac{1}{2} \|\ell^*_\Lambda\|^2 + \frac{1}{T} (\ell^*_\Lambda)' W^{\mathbb{P}}(T), \quad T > 0.$$

Therefore, it follows that

$$ag(\pi^*_\Lambda; \mathbb{P}) \ge \frac{1}{2} \|\ell^*_\Lambda\|^2, \quad \text{for all} \quad \mathbb{P} \in \Pi_\Lambda;$$

and combined with inequality (2.65), this completes the proof. □

2.8 Appendix: Measurable Random Subsets and Measurable Selection Results

We collect in this subsection powerful results about measurable selection, that we need throughout the dissertation. The readers can refer to KARATZAS AND KARDARAS (2007) [6], Appendix A for a detailed review. All the proofs are skipped, and we refer to Chapter 17 of ALIPRANTIS AND BORDER [8] for proofs of more general cases.

2.8.1 *Topological Results and Minimax Theorem*

In this subsection, we provide every topological results needed for the proof of Theorem 2.45. We also state a specific version of Minimax Theorem by SION (1958) [7].

Lemma 2.47. *If X, Y are topological spaces and Y is compact, then for any continuous $f : X \times Y \to \mathbb{R}$, the function $g(x) := \inf_{y \in Y} f(x, y)$ is well-defined and continuous.*

Proof. The whole argument proceeds in three parts. That $g(x) > -\infty$ follows from the fact that for any fixed $x \in X$, $f(x, \cdot) : Y \to \mathbb{R}$ is a continuous function defined on a compact space, and hence bounded (in fact the infimum is attained). Then observing that the sets $(-\infty, a)$ and (b, ∞) form a sub-base for the topology of $\mathbb{R}$, it suffices to check that $g^{-1}\big((-\infty, a)\big)$ and $g^{-1}\big((b, \infty)\big)$ are open.

Let proj_X be the canonical projection $\mathrm{proj}_X : X \times Y \to X$, which we recall is continuous and open. It is easy to see that $g^{-1}\big((-\infty, a)\big) = \mathrm{proj}_X \circ f^{-1}\big((-\infty, a)\big)$, so the continuity of f implies that this set is open. (Note that this part of the argument does not depend on the compactness of Y. In fact, a minor modification of the present proof shows that for any given family of upper semicontinuous functions $\{f_c\}_C$, the pointwise infimum $\inf_{c \in C} f_c$ is also upper semicontinuous, a fact that is very useful in convex analysis.)

It is in this last part, showing that $g^{-1}\big((b, \infty)\big)$ is open, that compactness is crucially used. Observe that

$$g(x) > b \Rightarrow f(x, y) > b, \quad \forall\, y \in Y.$$

In other words

$$g(x) > b \Rightarrow (x, y) \in f^{-1}\big((b, \infty)\big), \quad \forall\, y \in Y.$$

The continuity of f implies that the set $f^{-1}\big((b, \infty)\big)$ is open. This in particular implies that for each fixed $x \in g^{-1}\big((b, \infty)\big)$, and every $y \in Y$, there exists a “box" neighborhood

$$U_{x,y} \times V_{x,y}$$

contained in $f^{-1}\big((b, \infty)\big)$. Now using the compactness of Y, a finite collection (x, y_i), $i = 1, \cdots, k$ of all these boxes cover $\{x\} \times Y$. In particular we have

$$\Big(\{x\} \times Y\Big) \subset \Big(\bigcap_{i=1}^{k} U_{x,y_i}\Big) \times Y \subset f^{-1}\big((b, \infty)\big),$$

and hence the set

$$g^{-1}\big((b, \infty)\big) = \bigcup_{x \in g^{-1}\big((b,\infty)\big)} \bigcap_{i=1}^{k(x)} U_{x,y_i}$$

is open. The proof is complete. $\square$

The next result is very useful as measurable selection argument in stochastic optimization problems. It can be found in KARATZAS AND KARDARAS (2007) [6], Theorem A.5.

Theorem 2.48. *Suppose $f : \widetilde{\Omega} \times \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is a Carathéodory function (see Definition 2.54) on a closed and convex set $K \subseteq \mathbb{R}^n$. For the optimization problem $f_*(\widetilde{\omega}) = \sup_{z \in K} f(\widetilde{\omega}, z)$, we have:*

(i) The value function f_* is $\mathcal{P}$-measurable.

(ii) Suppose that $f_*(\widetilde{\omega})$ is finite for all $\tilde{\omega}$, and that there exists a unique $z_*(\tilde{\omega}) \in K$ satisfying $f(\tilde{\omega}, z_*(\tilde{\omega})) = f_*(\tilde{\omega})$. Then the mapping $\tilde{\omega} \mapsto z_*(\tilde{\omega})$ is $\mathcal{P}$-measurable.

Remark 2.49. Theorem 2.48 is a special case of Theorem 2.56. In Theorem 2.56, we generalize the optimization problem $f_*(\widetilde{\omega}) = \sup_{z \in K} f(\widetilde{\omega}, z)$ to Carathéodory functions on closed and convex, measurable nonempty random sets.

In the proof of Theorem 2.45 in Section 2.7, we use the following version of the Minimax Theorem, cf. SION (1958) [7], Theorem 3.4.

Theorem 2.50. *(SION's Minimax Theorem) Let X be a compact convex subset of a linear topological space, and Y a convex subset of a linear topological space. If f is a real-valued function on $X \times Y$ with the properties*

(i) $f(x, \cdot)$ upper semicontinuous and quasi-concave on Y, $\forall\, x \in X$,

(ii) $f(\cdot, y)$ lower semicontinuous and quasi-convex on X, $\forall\, y \in Y$,

then

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

2.8.2 *Measurable Random Subsets*

A random subset of $\mathbb{R}^d$ is just a random variable taking values in $2^{\mathbb{R}^d}$, the powerset (collection of all subsets) of $\mathbb{R}^d$. Thus, a random subset of $\mathbb{R}^d$ is a function $\mathfrak{A} : \tilde{\Omega} \to 2^{\mathbb{R}^d}$. A random subset $\mathfrak{A}$ if $\mathbb{R}^d$ will be called closed (or convex) if the set $\mathfrak{A}(\tilde{\omega})$ is closed (or convex) for every $\tilde{\omega} \in \tilde{\Omega}$.

Measurability requirements on random subsets are given by placing some measurable structure on the space $2^{\mathbb{R}^d}$, which we endow with the smallest σ-algebra that makes the mapping $2^{\mathbb{R}^d} \ni A \mapsto \mathrm{dist}(z, A) \in \mathbb{R}_+ \cup \{+\infty\}$ measurable for all $z \in \mathbb{R}^d$ (by definition, $\mathrm{dist}(z, \emptyset) = +\infty$). The following equivalent formulations are sometimes useful.

Proposition 2.51. *The constructed σ-algebra on $2^{\mathbb{R}^d}$ is also the smallest σ-algebra that makes the collection $\{2^{\mathbb{R}^d} \in A \mapsto \mathbb{1}_{\{A \cap K \neq \emptyset\}}\}$ measurable, for every compact set $K \subseteq \mathbb{R}^d$ of functions.*

Remark 2.52. A random subset $\mathfrak{A}$ of $\mathbb{R}^d$ is measurable if for any compact set $K \subseteq \mathbb{R}^d$, the set $\{\mathfrak{A} \cap K \neq \emptyset\} := \{\tilde{\omega} \in \tilde{\Omega} \mid \mathfrak{A}(\tilde{\omega}) \cap K \neq \emptyset\}$ is $\mathcal{P}$-measurable.

We now deal with unions and intersections of random subsets of $\mathbb{R}^d$.

Lemma 2.53. *Suppose that $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a sequence of measurable random subset of $\mathbb{R}^d$. Then the union $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ is also measurable. Furthermore, if each random subset $\mathfrak{A}_n$ is closed, then the intersection $\bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$ is measurable.*

Definition 2.54. (**Carathéodory function**). For a measurable closed random subset $\mathfrak{A}$ of $\mathbb{R}^d$, a mapping f of $\tilde{\Omega} \times \mathbb{R}^d$ into another topological space will be called *Carathéodory function* on $\mathfrak{A}$, if it is measurable (w.r.t the product σ-algebra on $\tilde{\Omega} \times \mathbb{R}^d$), and if $z \mapsto f(\tilde{\omega}, z)$ is continuous on $\mathfrak{A}(\tilde{\omega})$, for each $\tilde{\omega} \in \tilde{\Omega}$.

Lemma 2.55. *Let E be any topological space, $F \subseteq E$ a closed subset, and $\mathfrak{A}$ a closed and convex random subset of $\mathbb{R}^d$. If $f : \tilde{\Omega} \times \mathbb{R}^d \to E$ is a Carathéodory function on $\mathfrak{A}$, then $\mathbb{C} := \{z \in \mathfrak{A} \mid f(\cdot, z) \in F\}$ is a closed and measurable.*

The last result focuses on the measurability of the "argument" process in stochastic optimization problems.

Theorem 2.56. *Suppose that $\mathfrak{A}$ is a closed and convex, measurable, non-empty random subset of $\mathbb{R}^d$, and $f : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is a Carathéodory function on $\mathfrak{A}$. For the optimization problem $f_*(\tilde{\omega}) = \sup_{z \in \mathfrak{A}} f(\tilde{\omega}, z)$, we have:*

(i) The value function f_* is $\mathcal{P}$-measurable;

(ii) Suppose that $f_*(\tilde{\omega})$ is finite for all $\tilde{\omega}$, and that there exists a unique $z_*(\tilde{\omega}) \in \mathfrak{A}(\tilde{\omega})$ satisfying $f(\tilde{\omega}, z_*(\tilde{\omega})) = f_*(\tilde{\omega})$. Then $\tilde{\omega} \mapsto z_*(\tilde{\omega})$ is $\mathcal{P}$-measurable.

In particular, if $\mathfrak{A}$ is a closed and convex, measurable, non-empty random subset of $\mathbb{R}^d$, we can find a $\mathcal{P}$-measurable $h : \tilde{\Omega} \to \mathbb{R}^d$ with $h(\tilde{\omega}) \in \mathfrak{A}(\tilde{\omega})$ for all $\tilde{\omega} \in \tilde{\Omega}$.

Chapter 3: Optional Decomposition Theorem

In this chapter, we discuss a very important result in stochastic analysis and mathematical finance, the Optional Decomposition Theorem. We shall use this powerful result to establish the *superhedging duality* later in Section 3.2.

In one of its most "classical" forms, following KRAMKOV (1996) [9], the Optional Decomposition Theorem can be stated as follows. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t; t > 0)$, let X be an $\mathbb{R}^n$-valued locally bounded semimartingale. We denote by $\mathcal{P}$ the collection of probability measures $\mathbb{Q}$ which are equivalent to $\mathbb{P}$, and under which X is local martingale. If $\mathcal{P} \neq \emptyset$, then a given nonnegative process V is a supermartingale under all probabilities in $\mathcal{P}$ if, and only if, it admits the "optional" decomposition

$$V = V(0) + \int_0^\cdot \vartheta(t)\mathrm{d}X(t) - H, \tag{3.1}$$

where $\vartheta \in \mathcal{I}(X)$ and H is a nondeacreasing right-continuous adapted (in particular, optional) process with $H(0) = 0$.

In the following sections, we investigate the general structure of such optional decomposition under conic constraints.

The Optional Decomposition Theorem is well studied in KRAMKOV (1996) [9], FÖLLMER AND KRAMKOV (1997) [10] and FÖLLMER AND KABANOV (1997) [11]. However, all these papers treat the Optional Decomposition Theorem with "high end" tools:

- (i) difficult functional-analytic results;
- (ii) deep knowledge of the general theory of processes.

We provide here an elementary proof for the Optional Decomposition Theorem with conic constraints on the integrand $V(t)$ in (3.1). The content of this chapter is consistent with the previous two chapters. Moreover, the proof for the Optional Decomposition Theorem 3.5 will benefit from the characterization of supermartingale deflators in Chapter 2, Section 2.4. Also, we will deploy convex duality results from CVITANIĆ AND KARATZAS (1992) [12].

3.1 Optional Decomposition Theorem

We recall now the closed, convex cone K, where portfolios are constrained to take values, and introduce the support function

$$\mathbb{R}^n \ni x \longmapsto \delta(x) \equiv \delta(x \mid K) := \sup_{y \in K}(-y'x) \in \mathbb{R} \cup \{+\infty\}$$

of the cone $-K$. This is a closed, positively homogeneous, proper convex function on $\mathbb{R}^n$ (ROCKAFELLAR (1970) [13], page 114). It is finite on its *effective domain*

$$\begin{aligned} \widetilde{K} &:= \{x \in \mathbb{R}^n \,:\, \delta(x \mid K) < \infty\} \\ &= \{x \in \mathbb{R}^n \,:\, \exists \beta \in \mathbb{R} \text{ s.t.} - \pi'x \leq \beta, \forall\, \pi \in K\}, \end{aligned}$$

a convex cone called the *barrier cone* of $-K$. As we assume K itself to be a convex cone containing the origin, we can rewrite $\widetilde{K}$ as

$$\widetilde{K} = \{x \in \mathbb{R}^n \,:\, -\pi'x \leq 0, \,\forall\, \pi \in K\}.$$

Straightforward calculation indicates that

$$\delta(x \mid K) \equiv 0 \quad \text{on} \quad \widetilde{K}.$$

Thus, $\delta(\cdot \mid K)$ is an “indicator function"; it indicates with the value 0 on $\widetilde{K}$, and with the value $+\infty$ on $(\widetilde{K})^c$. The next Lemma characterizes the structure of $\mathcal{Y}_{loc}$ defined in (2.27).

Lemma 3.1. *Suppose a supermartingale numéraire portfolio ν exists in $\mathcal{I}(R;K)$. The set $\mathcal{Y}_{loc}$ of (2.27) consists of exactly the processes Y which admit the representation*

$$Y = \mathcal{E}(M_{\rho-\nu} + L), \tag{3.2}$$

where $L \in \mathcal{M}^{\perp}_{loc}(M)$ and $\rho \in \mathcal{I}(R)$ satisfies for every $\pi \in \mathcal{I}(R;K)$ the condition

$$\pi'(\omega,t)\big(\alpha - c(\nu-\rho)\big)(\omega,t) \leq 0, \quad \mathbb{P}\otimes\mathcal{O}\text{-a.e.}. \tag{3.3}$$

Proof. Thanks to Theorem 2.27, in particular equation (2.28), we can rewrite any element Y in $\mathcal{Y}$ as

$$Y = \frac{1}{X_\nu}\,\mathcal{E}(L)\mathcal{E}(M_\rho)\cdot\exp(\widetilde{A}) \tag{3.4}$$

as in (2.28), with $L \in \mathcal{M}^{\perp}_{loc}(M)$ and $\Delta L > -1$ and the conditions of (2.31), (2.32). From (2.34), $1/X_\nu$ can be written as a stochastic exponential

$$\frac{1}{X_\nu} = \mathcal{E}(-M_\nu),$$

which simplifies (3.4) as

$$Y = \mathcal{E}(M_{\rho-\nu} + L + \widetilde{A} - C_{\nu\rho}).$$

Further to request that Y be a local martingale, amounts to imposing $\widetilde{A} - C_{\nu\rho} = 0$. Therefore $Y = \mathcal{E}(M_{\rho-\nu} + L)$, which completes the proof. $\square$

Remark 3.2. Whenever the supermartingale numéraire X_ν exists, the set $\mathcal{Y}_{loc}$ is non-empty. Indeed, by taking $\rho \equiv 0$, the condition

$$\pi'\big(\alpha - c(\nu-\rho)\big) = \pi'(\alpha - c\nu) \leq 0$$

automatically holds, thanks to the inequality (2.11) in Chapter 2. Moreover, we take $L \equiv 0$ and

(3.2) becomes

$$Y = \mathcal{E}(M_{-\nu}) = \mathcal{E}(-M_\nu) = \frac{1}{X_\nu},$$

on account of (2.34). This argument shows $1/X_\nu \in \mathcal{Y}_{loc}$, which means that the supermartingale deflator $1/X_\nu$ is a local martingale.

Corollary 3.3. *For every supermartingale deflator $Y \in \mathcal{Y}$, initial capital $x \in \mathbb{R}_+$, portfolio $\rho \in \mathcal{I}(R; K)$, right continuous nondecreasing and adapted process J, and resulting process*

$$X := x\,\mathcal{E}\left(\int_0^\cdot \sum_{i=1}^n \rho_i(t) \mathrm{d}R_i(t) - J \right), \tag{3.5}$$

the product YX is a supermartingale.

Proof. The notation $X_\rho = \mathcal{E}\left(\int_0^\cdot \sum_{i=1}^n \rho_i(t) \mathrm{d}R_i(t) \right)$ leads to

$$X = xX_\rho \exp(-J), \tag{3.6}$$

so the product YX has the representation

$$YX = xYX_\rho \exp(-J). \tag{3.7}$$

From the defintion of $\mathcal{Y}$ in Definition 2.25, the fact that $Y \in \mathcal{Y}$ implies that YX_ρ is a supermartingale, with DOOB-MEYER decomposition

$$YX_\rho = M - A \tag{3.8}$$

for some local martingale M and nondecreasing process A. From (3.6)-(3.8) and some simple stochastic calculus, we get

$$
\begin{aligned}
YX &= x(M-A)\exp(-J) \\
&= x\Big(\int_0^{\cdot} \exp\big(-J(t-)\big)\mathrm{d}M(t) - \int_0^{\cdot} \exp\big(-J(t-)\big)\mathrm{d}A(t) \\
&\quad - \int_0^{\cdot} \big(M(t)-A(t)\big)\exp\big(-J(t-)\big)\mathrm{d}J(t)\Big).
\end{aligned}
$$

The integral $\int_0^{\cdot} \big(M(t) - A(t)\big)\exp\big(-J(t-)\big)\mathrm{d}J(t)$ is nondecreasing on account of the fact that $M - A = YX_\rho$ is strictly positive, and of the nondecrease of J. In addition, we note that the integral $\int_0^{\cdot} \exp\big(-J(t-)\big)\mathrm{d}A(t)$ is nondecreasing, and the process $\int_0^{\cdot} \exp\big(-J(t-)\big)\mathrm{d}M(t)$ is a local martingale.

In conclusion, YX is a supermartingale. □

Remark 3.4. The process $X > 0$ given in (3.5) satisfies the following dynamics

$$
\frac{\mathrm{d}X(t)}{X(t)} = \sum_{i=1}^{n} \rho_i(t)\frac{\mathrm{d}S_i(t)}{S_i(t)} - \mathrm{d}J(t), \quad \text{with} \quad X(0) = x.
$$

We can further simplfy the above equation as

$$
\begin{aligned}
\mathrm{d}X(t) &= \sum_{i=1}^{n} \vartheta_i(t)\mathrm{d}S_i(t) - X(t)\mathrm{d}J(t) \\
&= \sum_{i=1}^{n} \vartheta_i(t)\mathrm{d}S_i(t) - \mathrm{d}H(t), \quad \text{where } \vartheta_i \triangleq \frac{\rho_i X}{S_i}
\end{aligned}
$$

and $H \triangleq \int_0^{\cdot} X(t)\mathrm{d}J(t) \in \mathcal{H}$. Therefore, by Definition (1.10), we just verified that this $X > 0$ is indeed a “wealth process with withdrawals", to wit, $X = X(\cdot\,; x, \vartheta, H)$.

Theorem 3.5. ***Optional Decomposition***. *Fix a closed, convex cone K containing the origin, where portfolios are constrained to take values; assume that the supermartingale numéraire X_ν exists, and that the matrix-valued covariation process c of (1.3) is nonsingular.*

For a strictly positive, adapted process X with rcll paths and $X(0) = x \in \mathbb{R}_+$, the following statements are then equivalent:

(a) The process YX is a supermartingale, for every local martingale deflator $Y \in \mathcal{Y}_{loc}$.

(b) There exist a portfolio $\pi^* = (\pi_1^*, \cdots, \pi_n^*) \in \mathcal{I}(R; K)$ and a right-continuous, nondecreasing and adapted process J, such that

$$X = x\, \mathcal{E}\bigg(\int_0^{\cdot} \sum_{i=1}^{n} \pi_i^*(t) \mathrm{d}R_i(t) - J \bigg). \tag{3.9}$$

Remark 3.6. Exactly as in Remark 3.4, we can define the investment strategy $\vartheta^* \triangleq \pi^* X/S \in \mathcal{I}(S)$ and the process $H = \int_0^{\cdot} X(t)\mathrm{d}J(t) \in \mathcal{H}$, so that X in (3.9) satisfies

$$\mathrm{d}X(t) = \sum_{i=1}^{n} \vartheta_i^*(t)\mathrm{d}S_i(t) - X(t)\mathrm{d}J(t) = \sum_{i=1}^{n} \vartheta_i^*(t)\mathrm{d}S_i(t) - \mathrm{d}H(t) \tag{3.10}$$

and is indeed a wealth process of the form $X \equiv X(\cdot\,; x, \vartheta^*, H)$. In the proof below, we will benefit from the representation (3.10).

Proof. (**First part of the proof for Theorem 3.5**) For the first part of the proof, we assume that:

all local martingales on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(\cdot), \mathbb{P})$ have continuous paths.

In the second part of the proof, we will deal with general filtrations.

Thanks to Theorem 2.31, the existence of the supermartingale numéraire X_ν indicates that $\mathcal{Y}_{loc}$ is not empty. The implication $(b) \implies (a)$ is covered by Corollary 3.3, so we focus on the reverse implication $(a) \implies (b)$.

We assume that statement (a) holds. For the rest of the proof, we recall that ν is a supermartingale numéraire portfolio. In light of the representation (3.2) for supermartingale martingale deflators which are local martingale themselves, given any $L \in \mathcal{M}_{loc}^{\perp}(M)$ and $\rho \in \mathcal{I}(R)$ satisfying the condition (3.3), the strictly positive process

$$\mathcal{E}(L + M_{\rho-\nu})$$

is an element of $\mathcal{Y}_{loc}$. Let us define the process

$$\widetilde{\rho} := \alpha - c(\nu - \rho) \tag{3.11}$$

for every $\rho \in \mathcal{I}(R)$ which satisfies the condition (3.3). With the help of this $\widetilde{\rho}$, we can restate the condition (3.3) as

$$\pi'(\omega, t)\widetilde{\rho}(\omega, t) \leq 0, \quad \mathbb{P} \otimes O\text{-a.e.} \tag{3.12}$$

for every $\pi \in \mathcal{I}(R; K)$. According to the definition of $\widetilde{K}$, this gives $-\widetilde{\rho} \in \mathcal{I}(R; \widetilde{K})$. Conversely, for any $-\widetilde{\rho} \in \mathcal{I}(R; \widetilde{K})$, condition (3.12) holds, and

$$\rho = c^{-1}(\widetilde{\rho} - \alpha) + \nu \tag{3.13}$$

satisfies the condition (3.3) and $\rho \in \mathcal{I}(R)$.

A special case: We know that $-\widetilde{\rho} \equiv 0 \in \mathcal{I}(R; \widetilde{K})$, thus the corresponding $\rho = \nu - c^{-1}\alpha \in \mathcal{I}(R)$ satisfies the condition (3.3).

Now we define

$$Y_\rho \triangleq \mathcal{E}(M_{\rho-\nu}). \tag{3.14}$$

According to statement (a), we have that XY_ρ is a supermartingale. A straightforward application of the KUNITA-WATANABE decomposition (KARATZAS AND SHREVE (1991) [4], Proposition 3.4.14) leads to

$$XY_\rho = x + M_{\psi_\rho} + L_\rho - J_\rho, \quad M_{\psi_\rho} = \int_0^\cdot \psi'_\rho(t)\mathrm{d}M(t). \tag{3.15}$$

Here $\psi_\rho \in \mathcal{I}(R)$, $L_\rho \in \mathcal{M}^\perp_{loc}(M)$, and J_ρ is a nondecreasing, right-continuous and adapted process with $J_\rho(0) = 0$, i.e., $J_\rho \in \mathcal{K}$.

We first claim that $L_\rho \equiv 0$ in equation (3.15). To see this, the crucial observation is that $\mathcal{E}(M_{\rho-\nu} + mL_\rho) \in \mathcal{Y}_{loc}$ holds for every $m \in \mathbb{N}$. As a consequence, the product $X \cdot \mathcal{E}(M_{\rho-\nu} + mL_\rho)$ is a supermartingale for every $m \in \mathbb{N}$. Since $\langle \mathcal{E}(mL_\rho), M_i \rangle = 0$ holds for $i = 1, \cdots, n$, it follows that $\langle \mathcal{E}(mL_\rho), M_{\psi_\rho}\rangle = 0$, meaning that $\mathcal{E}(mL_\rho)M_{\psi_\rho}$ is a local martingale. These facts, combined with equation (3.14) and (3.15), imply that the process

$$\begin{aligned} \mathcal{E}(mL_\rho)(L_\rho - J_\rho) &= \mathcal{E}(mL_\rho)\big[XY_\rho - (x + M_{\psi_\rho})\big] \\ &= \mathcal{E}(mL_\rho)\cdot X\mathcal{E}(M_{\rho-\nu}) - \mathcal{E}(mL_\rho)(x + M_{\psi_\rho}) \end{aligned}$$

is a supermartingale, for every $m \in \mathbb{N}$. Once again, integration-by-parts gives

$$\begin{aligned} \mathcal{E}(mL_\rho)(L_\rho - J_\rho) &= \int_0^\cdot (L_\rho - J_\rho)(t-)\mathrm{d}\mathcal{E}(mL_\rho)(t) + \int_0^\cdot \mathcal{E}(mL_\rho)(t-)\mathrm{d}L_\rho(t) \\ &\quad + \int_0^\cdot \mathcal{E}(mL_\rho)(t-)\mathrm{d}(\langle mL_\rho, L_\rho\rangle - J_\rho)(t), \end{aligned}$$

from which we deduce that $m\langle L_\rho\rangle - J_\rho$ is nonincreasing for every $m \in \mathbb{N}$. But this is possible only when $\langle L_\rho\rangle \equiv 0$; since $L_\rho(0) = 0$, we obtain $L_\rho \equiv 0$.

As a result of the above discussion, the decomposition (3.15) simplifies to

$$XY_\rho = x + \int_0^\cdot \psi'_\rho \mathrm{d}M - J_\rho. \tag{3.16}$$

Thanks to the notation (3.14) and Itô's rule, we can easily obtain

$$\frac{1}{Y_\rho} = \mathcal{E}(M_{\nu-\rho})\cdot\exp(C_{\rho-\nu,\rho-\nu}).$$

In order to simplify the calculations that follow, we introduce the following processes

$$Q_\rho := x + \int_0^\cdot \psi'_\rho \mathrm{d}M - J_\rho \equiv XY_\rho, \qquad Z_\rho := \mathcal{E}(M_{\nu-\rho})\cdot\exp(C_{\rho-\nu,\rho-\nu}) \equiv \frac{1}{Y_\rho} \tag{3.17}$$

for every portfolio $\rho \in \mathcal{I}(R)$ satisfying condition (3.3). Straightforward calculation indicates that

$$\begin{aligned} \mathrm{d}Q_\rho &= \psi_\rho' \,\mathrm{d}M - \mathrm{d}J_\rho, \\ \mathrm{d}Z_\rho &= Z_\rho\big((\nu-\rho)'\,\mathrm{d}M + \mathrm{d}C_{\rho-\nu,\rho-\nu}\big), \\ \langle \mathrm{d}Q_\rho, \mathrm{d}Z_\rho\rangle &= Z_\rho\,\mathrm{d}C_{\nu-\rho,\psi_\rho} \end{aligned} \tag{3.18}$$

and another application of the product rule to $X = (XY_\rho)(\frac{1}{Y_\rho}) = Q_\rho Z_\rho$ gives

$$\mathrm{d}X = X\left(\left(\nu-\rho+\frac{\psi_\rho}{Q_\rho}\right)'\mathrm{d}M + \mathrm{d}C_{\nu-\rho,\nu-\rho+\frac{\psi_\rho}{Q_\rho}} - \frac{\mathrm{d}J_\rho}{Q_\rho}\right). \tag{3.19}$$

For any other portfolio $\mu \in \mathcal{I}(R)$ satisfying condition (3.3), we have

$$Q_\mu = XY_\mu = X\cdot\mathcal{E}(M_{\mu-\nu}) = Q_\rho\cdot\mathcal{E}(M_{\mu-\rho})\exp(C_{\rho-\nu,\rho-\mu})$$

on account of the definition of Y_μ in (3.14) and Q_μ in (3.17), and straightforward stochastic calculus gives

$$\mathrm{d}Q_\mu = Q_\mu\left(\left(\mu-\rho+\frac{\psi_\rho}{Q_\rho}\right)'\mathrm{d}M + \left(\nu-\rho+\frac{\psi_\rho}{Q_\rho}\right)'c(\mu-\rho)\,\mathrm{d}O - \frac{\mathrm{d}J_\rho}{Q_\rho}\right). \tag{3.20}$$

Comparing the decomposition (3.20) with

$$\mathrm{d}Q_\mu = \psi_\mu'\,\mathrm{d}M - \mathrm{d}J_\mu \tag{3.21}$$

from (3.18), we conclude that

$$\frac{\psi_\rho}{Q_\rho} - \rho = \frac{\psi_\mu}{Q_\mu} - \mu, \quad \mathbb{P}\otimes O\text{-a.e.}$$

and hence that this expression does not depend on $\mu \in \mathcal{I}(R)$. Therefore, we can define a vector-valued process

$$\pi := \frac{\psi_\mu}{Q_\mu} - \mu \tag{3.22}$$

which does not depend on the choice of $\mu \in \mathcal{I}(R)$. Similarly, we conclude from equations (3.20), (3.21) and (3.22), that

$$(\pi+\nu)'c\rho\,\mathrm{d}O + \frac{\mathrm{d}J_\rho}{Q_\rho} = (\pi+\nu)'c\mu\,\mathrm{d}O + \frac{\mathrm{d}J_\mu}{Q_\mu}$$

and hence this expression does not depend on the choice of $\mu \in \mathcal{I}(R)$ either. We add the term $(\pi+\nu)'(\alpha - c\nu)\mathrm{d}O$, which also does not depend on μ, and define for every $\mu \in \mathcal{I}(R)$ satisfying condition (3.3) the process

$$\begin{aligned} \mathrm{d}U &:= (\pi+\nu)'c\mu\,\mathrm{d}O + \frac{\mathrm{d}J_\mu}{Q_\mu} + (\pi+\nu)'(\alpha - c\nu)\mathrm{d}O \\ &= (\pi+\nu)'\big(\alpha - c(\nu-\mu)\big)\mathrm{d}O + \frac{\mathrm{d}J_\mu}{Q_\mu} \end{aligned} \tag{3.23}$$

for every $\mu \in \mathcal{I}(R)$. Reading the equation (3.23) with $\mu = \mu_* \triangleq \nu - c^{-1}\alpha \in \mathcal{I}(R)$, we obtain

$$U = \int_0^{\cdot} \frac{\mathrm{d}J_{\mu_*}(t)}{Q_{\mu_*}(t)},$$

and hence U is an increasing, adapted, rcll process with $U(0) = 0$, i.e., $U \in \mathcal{H}$.

Next, we claim that

$$(\pi+\nu)'(\omega,t)\widetilde{\rho}(\omega,t) \le 0, \quad \mathbb{P}\otimes O\text{-a.e.} \tag{3.24}$$

holds for every $-\widetilde{\rho} \in \mathcal{I}(R;\widetilde{K})$. With the help of the one-to-one correspondence between ρ and $\widetilde{\rho}$ in (3.11) and (3.13), the above statement is equivalent to

$$(\pi+\nu)'(\omega,t)\big(\alpha - c(\nu-\rho)\big)(\omega,t) \le 0 \quad \mathbb{P}\otimes O\text{-a.e.} \tag{3.25}$$

for every $\rho \in \mathcal{I}(R)$ satisfying condition (3.3). Then the arguments of CVITANIĆ AND KARATZAS (1992) [12], Theorem 9.1, lead to

$$(\pi+\nu)(\omega,t) \in K, \quad \mathbb{P}\otimes O\text{-a.e.} \tag{3.26}$$

The result (3.26) provide us with a candidate portfolio $\pi + \nu$ taking values in K, and later we will verify this candidate portfolio is R-integrable as well.

In order to verify the claim (3.24), notice that from (3.23) we obtain

$$\int_0^t \frac{\mathrm{d}J_\rho}{Q_\rho} = U(t) - \int_0^t (\pi + \nu)'\big(\alpha - c(\nu - \rho)\big)\mathrm{d}O.$$

We now fix $\rho \in \mathcal{I}(R)$ satisfying condition (3.3) and define the set

$$F_s := \{\omega \in \Omega\,;\ (\pi + \nu)'(\omega, s)\widetilde{\rho}(\omega, s) \geq 0\}$$

for every $s \in [0, t]$. Here $\widetilde{\rho}$ is defined as (3.11). Moreover let us define a sequence of processes

$$\widetilde{\rho}_n(s) := \big(\widetilde{\rho}(s)\mathbb{1}_{F_s^c} + n\widetilde{\rho}(s)\mathbb{1}_{F_s}\big)\big(1 + \|\widetilde{\rho}(s)\|\big)^{-1}, \quad n \in \mathbb{N}$$

where $\mathbb{1}_F$ is the indicator function for set F. We also define the corresponding process

$$\rho_n := c^{-1}(\widetilde{\rho}_n - \alpha) + \nu$$

via equation (3.13). Then $-\widetilde{\rho}_n \in \mathcal{I}(R; \widetilde{K})$, and the corresponding portfolio $\rho_n \in \mathcal{I}(R)$ satisfies the condition (3.3). Assuming that (3.24) does not hold, we get for n large enough,

$$\begin{aligned} & \mathbb{E}\bigg[\int_0^t \frac{\mathrm{d}J_{\rho_n}(s)}{Q_{\rho_n}(s)}\bigg] \\ = \ & \mathbb{E}\big[U(t)\big] - \mathbb{E}\bigg[\int_0^t \big(1 + \|\widetilde{\rho}(s)\|\big)^{-1}(\pi + \nu)'(s)\widetilde{\rho}(s) \cdot \mathbb{1}_{F_s^c}\,\mathrm{d}O(s)\bigg] \\ & -n\,\mathbb{E}\bigg[\int_0^t \big(1 + \|\widetilde{\rho}(s)\|\big)^{-1}(\pi + \nu)'(s)\widetilde{\rho}(s) \cdot \mathbb{1}_{F_s}\mathrm{d}O(s)\bigg] < 0, \end{aligned}$$

a contradiction.

Now we can put together equations (3.19) and (3.22) to deduce

$$\mathrm{d}X = X\Bigg(\big(\pi+\nu\big)'\mathrm{d}M - \Big(\frac{\mathrm{d}J_\rho}{Q_\rho} - \mathrm{d}C_{\nu-\rho,\nu+\pi}\Big)\Bigg). \tag{3.27}$$

If we take $\widetilde{\rho} \equiv 0$, then we have the corresponding $\rho = \nu - c^{-1}\alpha$ by equation (3.13). Straightforward calculation gives

$$\mathrm{d}C_{c^{-1}\alpha,\nu+\pi} = (\nu+\pi)'cc^{-1}\alpha\mathrm{d}O = (\nu+\pi)'\alpha\mathrm{d}O = (\nu+\pi)'\mathrm{d}A.$$

Thus we can simplify the representation (3.27) as

$$\begin{aligned} \mathrm{d}X &= X\Bigg((\pi+\nu)'\mathrm{d}M - \big(\frac{\mathrm{d}J_{\nu-c^{-1}\alpha}}{Q_{\nu-c^{-1}\alpha}} - \mathrm{d}C_{c^{-1}\alpha,\nu+\pi}\big)\Bigg) \\ &= X\Bigg((\pi+\nu)'\mathrm{d}R - \frac{\mathrm{d}J_{\nu-c^{-1}\alpha}}{Q_{\nu-c^{-1}\alpha}}\Bigg). \end{aligned} \tag{3.28}$$

According to the equation (3.22), we have

$$\pi = \frac{\psi_\rho}{Q_\rho} - \rho \tag{3.29}$$

where $\rho \in \mathcal{I}(R)$ satisfying condition (3.3). Thanks to inequality (2.11) in Chapter 2, to the effect that

$$\pi'(\alpha - c\nu) \leq 0 \ \text{ for every } \ \pi \in \mathcal{I}(R;K),$$

we have a special case of $\rho_0 \equiv 0$ satisfying condition (3.3). By the definition of Y_ρ in (3.14), we obtain

$$Y_{\rho_0} = \mathcal{E}(M_{\rho_0-\nu}) = \mathcal{E}(M_{-\nu}) = \mathcal{E}(-M_\nu) = 1/X_\nu$$

where $\rho_0 \equiv 0$ and X_ν is the supermartingale numéraire, thus

$$Q_{\rho_0} = XY_{\rho_0} = X/X_\nu.$$

We can rewrite then, always with $\rho_0 \equiv 0$, the process π in equation (3.29) as

$$\pi = \frac{X_\nu \psi}{X}, \quad \text{where} \quad \psi \in \mathcal{I}(R). \tag{3.30}$$

With help of equation (3.30), we can rewrite equation (3.28) as

$$\mathrm{d}X = (X_\nu \psi + X\nu)\mathrm{d}R - \big(X/Q_{\nu - c^{-1}\alpha}\big)\mathrm{d}J_{\nu - c^{-1}\alpha}.$$

Therefore, we define the investment strategy $\vartheta \equiv (\vartheta_i)_{i \in 1, \cdots, n}$ as

$$\vartheta_i := \frac{X_\nu}{S_i}\psi_i + \frac{X}{S_i}\nu_i$$

and $\vartheta \in \mathcal{I}(S)$. Since $X > 0$, we can define a new predictable, vector-valued process $\pi^* \equiv (\pi_1^*, \cdots, \pi_n^*)$ via

$$\pi_i^* := \frac{S_i \vartheta_i}{X} = \pi + \nu, \quad \text{for} \quad i \in 1, \cdots, n. \tag{3.31}$$

It is straightforward to check that the process π^* defined from ϑ as in (3.31), is R-integrable. Moreover, by (3.26) we can see that the portfolio π^* takes values in the convex cone K. This completes the proof.

(**Second part of the proof for Theorem 3.5: arbitrary filtrations case**). For arbitrary filtrations, we still focus on the implication $(a) \Longrightarrow (b)$, as the reverse implication $(b) \Longrightarrow (a)$ is a direct consequence of Corollary 3.3.

In the general case here, every local martingale $L \in \mathcal{M}_{loc}^{\perp}(M)$ can be decomposed uniquely as the sum

$$L = L^c + L^d,$$

where $L^c \in \mathcal{M}_{loc}^{\perp}(M)$ is a continuous local martingale, and L^d is a *purely discontinuous* local martingale with $L^d(0) = 0$, meaning that $\langle L^d, V \rangle = 0$ holds for every local martingale V with continuous paths.

Thanks to Lemma 3.1, notation (3.14) and assumption (a) in Theorem 3.5, $Y_\rho = \mathcal{E}(M_{\rho-\nu})$ is an element in $\mathcal{Y}_{loc}$ for $\rho \in \mathcal{I}(R)$ satisfying condition (3.3). Thus, the product XY_ρ is a supermartingale, and a joint DOOB-MEYER and KUNITA-WATANABE decompostion gives

$$XY_\rho = x + M_{\psi_\rho} + L_\rho - J_\rho \tag{3.32}$$

as in equation (3.15). Here $\psi_\rho \in \mathcal{I}(R)$, $L_\rho \in \mathcal{M}_{loc}^{\perp}(M)$ has continuous paths, and the last term J_ρ is a local submartingale with rcll paths with $J_\rho(0) = 0$ and $\langle J_\rho, V\rangle = 0$ for every local martingale V with continuous paths.

For the rest of the proof, we will argue that J_ρ is indeed a nondecreasing process, like it was in equation (3.15). To see this, we rewrite the decomposition of XY_ρ in (3.32) as

$$J_\rho = x + L_\rho + M_{\psi_\rho} - XY_\rho.$$

The continuity of $L_\rho + M_{\psi_\rho}$ and the nonnegativity of XY_ρ indicate that J_ρ is locally bounded from above. We take an arbitrary purely discontinuous local martingale N with $N(0) = 0$ and $\Delta N > -1$. Since N is purely discontinuous, the process $(L_\rho + M_{\psi_\rho})\mathcal{E}(N)$ is a local martingale. Moreover, the characterization of $\mathcal{Y}_{loc}$ from Lemma 3.1 indicates that the product $Y_\rho\mathcal{E}(N) = \mathcal{E}(M_{\rho-\nu} + N)$ is an element in $\mathcal{Y}_{loc}$. Thus, by assumption (1) in Theorem 3.5, the process $XY_\rho\mathcal{E}(N)$ is a supermartingale. As a result,

$$\mathcal{E}(N)J_\rho = \mathcal{E}(N)(x + L_\rho + M_{\psi_\rho} - XY_\rho)$$

is a local submartingale. By simple calculation based on the product rule, we get that

$$J_\rho + \langle J_\rho, N\rangle = \int_0^\cdot \frac{\mathrm{d}\big(\mathcal{E}(N)J_\rho\big)(t)}{\mathcal{E}(N)(t-)} - \int_0^\cdot J_\rho(t-)\mathrm{d}N(t)$$

is also a local submartingale.

Therefore, we obtain the following properties for the process J_ρ:

- J_ρ is a local submartingale with $J_\rho(0) = 0$ and locally bounded from above;
- $\langle J_\rho, V \rangle = 0$ holds for every continuous local martingale V;
- the sum $J_\rho + \langle J_\rho, N \rangle$ is a local martingale for every purely discontinuous local martingale N with $\Delta N > -1$.

In this situation, we can apply Lemma 2.1 in Karatzas and Kardaras (2015) [14] to obtain that L_ρ is in fact a nondecreasing process. Therefore, we get exactly the same form for XY_ρ as in equation (3.16). The rest proof of the implication $(a) \implies (b)$ in Theorem 3.5 for arbitrary filtrations is exactly the same as that for continuous filtrations. □

The following result implies that the decomposition (3.9) is essentially unique.

Proposition 3.7. *(**Uniqueness of wealth process decompositions**). Suppose that the supermartingale numéraire X_ν exists, and let*

$$X \equiv X(\cdot\,;\, x, \vartheta, H) = x + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t)\mathrm{d}S_i(t) - H$$

be a wealth process (not necessarily admissible) with $x \in \mathbb{R}$, $\vartheta \in \mathcal{I}(S)$ and $H \in \mathcal{H}$. If there exists another decomposition of X as

$$X = X(\cdot\,;\, \widetilde{x}, \widetilde{\vartheta}, \widetilde{H}) = \widetilde{x} + \int_0^{\cdot} \sum_{i=1}^{n} \widetilde{\vartheta}_i(t)\mathrm{d}S_i(t) - \widetilde{H},$$

with $\widetilde{x} \in \mathbb{R}$, $\widetilde{\vartheta} \in \mathcal{I}(S)$ and $\widetilde{H} \in \mathcal{H}$, then necessarily $x = \widetilde{x}$, $H = \widetilde{H}$, from which we can also ascertain that $\vartheta = \widetilde{\vartheta}$ has to hold $(\mathbb{P} \otimes O)$-a.e.

Remark 3.8. If X is strictly positive as it is given in Theorem 3.5, we can get more results for the uniqueness of wealth process decomposition. By analogy with Remark 3.6 we can define portfolios

$$\pi_i := \frac{\vartheta_i S_i}{X} \quad \text{and} \quad \widetilde{\pi}_i := \frac{\widetilde{\vartheta}_i S_i}{X} \quad \text{for } i \in \{1, \cdots, n\}.$$

Next we define two right continuous, nondecreasing and adapted processes

$$J := \int_0^\cdot \big(1/X(t)\big)\mathrm{d}H(t) \quad \text{and} \quad \widetilde{J} := \int_0^\cdot \big(1/X(t)\big)\mathrm{d}\widetilde{H}(t).$$

We have

$$x\,\mathcal{E}\bigg(\int_0^\cdot \sum_{i=1}^n \pi_i(t)\mathrm{d}R_i(t) - J\bigg) = X = \widetilde{x}\,\mathcal{E}\bigg(\int_0^\cdot \sum_{i=1}^n \widetilde{\pi}_i(t)\mathrm{d}R_i(t) - \widetilde{J}\bigg).$$

According to Proposition 3.7, $x = \widetilde{x}$, $J = \widetilde{J}$ and $\pi = \widetilde{\pi}$ holds $(\mathbb{P} \otimes O)$-a.e. Further by Theorem 3.5, $\pi = \widetilde{\pi} \in \mathcal{I}(R;K)$.

Proof. Since $X(0) = x$, the claim $x = \widetilde{x}$ follows immediately. We introduce the difference $D_1 := \widetilde{H} - H$, which actually coincides with the stochastic integral of the process $\vartheta - \widetilde{\vartheta}$ with respect to S; in particular, D_1 is continuous. Since the market is viable, the local martingale numéraire X_ν exists. Set $Y := 1/X_\nu$, and observe that $Y \in \mathcal{Y}_{loc}$ and that YD_1 is a continuous supermartingale. Using integration-by-parts, and noting that D_1 has continuous paths of finite variation on compact time intervals, we deduce that the process

$$\int_0^\cdot Y(t)\mathrm{d}D_1(t) = YD_1 - \int_0^\cdot D_1(t)\mathrm{d}Y(t)$$

is also a supermartingale. Therefore we can have the DOOB-MEYER decomposition of the left hand side integral as

$$\int_0^\cdot Y(t)\mathrm{d}D_1(t) = M + A \tag{3.33}$$

where M is a local martingale and A a nonincreasing process with rcll paths and $M(0) = 0$, $A(0) = 0$. Since the left-hand side of equation (3.33) has finite variation on compact time intervals, we deduce that $M \equiv 0$ and $\int_0^\cdot Y(t)\mathrm{d}D_1(t)$ is noninceasing. Furthermore, the fact that Y is strictly positive implies that D_1 is nonincreasing.

We define $D_2 := H - \widetilde{H}$ and by following the same logic, we conclude that D_2 is nonincreasing.

Together with the fact that

$$D_1 = -D_2 \quad \text{and} \quad D_1(0) = D_2(0) = 0,$$

we conclude that $D_1 = D_2 \equiv 0$ holds, as claimed. $\square$

Remark 3.9. Stochastic controllability results, similar to the Optional Decomposition Theorem and obtained via martingale methods, can be traced as far back as DAVIS AND VARAIYA (1973) [15]. Theorem 3.5 offers a rather elementary proof of the Optional Decomposition Theorem with conic constraints. In FÖLLMER AND KRAMKOV (1997) [10], it is assumed that the collection Q of equivalent local martingale measures is nonempty. Instead of this assumption, we use the more "localized" assumption that the class $\mathcal{Y}_{loc}$ defined in (2.27) in non-empty. This assumption $\mathcal{Y}_{loc} \neq \emptyset$ is more general. Moreover, the proof of the Optional Decomposition Theorem in this Section avoids a significant level of sophistication: no difficult functional-analytic results are required.

In a more general setting, e.g., merely with K non-empty and convex, the Optional Decomposition Theorem need not hold. FÖLLMER AND KRAMKOV (1997) [10] consider general convex sets with extra assumptions on semimartingale topology, which is very abstract. Instead of assumptions on semimartingale topology, we construct our results upon nonsingular assumption for covariation process c. Example 1 in CZICHOWSKY, WESTRAY AND ZHENG (2010) [16] shows that there exist non-empty, convex sets K, for which the Optional Decomposition Theorem fails. Example 2 in CZICHOWSKY, WESTRAY AND ZHENG (2010) [16] shows that when K is a convex cone but the covariation process c is non-singular, the Optional Decomposition Theorem can fail as well.

3.2 Financeability

In this section, we pose and answer the following question: *How much initial capital is required, in order to finance a given capital withdrawal stream J via trading in an equity market via portfolios constrained to take values in the given closed, convex cone $K \subseteq \mathcal{R}^n$ that contains the origin?*

Actually we have encountered this question in Section 2. Recalling the definition of superhedging capital $x(J)$ in Definition 2.18, we now try to compute this quantity $x(J)$. From Lemma 2.30, we know that a lower bound for $x(J)$ is the supremum of the expected "deflated" total value $\int_0^\infty Y(t)\mathrm{d}J(t)$ of J, taken over all deflators $Y \in \mathcal{Y}_{loc}$.

We will show in Theorem 3.14 that, under the assumption of market viability—which is equivalent to the requirement $\mathcal{Y}_{loc} \neq \emptyset$ on account of Theorem 2.31—the following equation holds:

$$x(J) = \sup_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty Y(t)\mathrm{d}J(t)\right]. \tag{3.34}$$

3.2.1 *Regularization results*

The key tool used in the proof of Theorem 3.14 is the construction of supermartingales with nice path properties, starting from processes defined in a dense subset of $\mathbb{R}_+$. For the rest of the discussion, let $\mathbb{D}$ be a dense, countable subset of $\mathbb{R}_+$ containing the origin, and define the sequence of random variables

$$Z(t) := \operatorname*{ess\,sup}_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\mathrm{d}J(u)\,\middle|\,\mathcal{F}(t)\right], \quad t \in \mathbb{D}. \tag{3.35}$$

Furthermore, we define

$$\zeta(t;Y) := \mathbb{E}^{\mathbb{P}}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\mathrm{d}J(u)\,\middle|\,\mathcal{F}(t)\right], \quad t \in \mathbb{D}, \quad Y \in \mathcal{Y}_{loc}, \tag{3.36}$$

so that the random variable of (3.35) can be cast more concisely in the form $Z(t) = \operatorname{ess\,sup}_{Y \in \mathcal{Y}_{loc}} \zeta(t;Y)$ for $t \in \mathbb{R}_+$.

Lemma 3.10. ***(Regularization)*** *Let $\mathbb{D}$ be a countable, dense subset of $\mathbb{R}_+$, and $\big(Z(t)\big)_{t\in\mathbb{D}}$ a non-negative $\mathbb{D}$-indexed supermartingale. Then there exists a process $X \equiv \big(X(t)\big)_{t\in\mathbb{D}}$ with the following properties:*

- *X is a supermartingale, with $\mathbb{P}$-a.e. path rcll.*

- *For any fixed $s \in \mathbb{R}_+$, $X(s) = \lim_{\mathbb{D}\ni t\downarrow s} Z(t)$ holds $\mathbb{P}$-a.e.*

- *$X(t) \leq Z(t)$ holds $\mathbb{P}$-a.e., for every $t \in \mathbb{D}$.*

The Proof of this regularization result can be found in MEYER (1966) [17], Theorem T4 of Chapter VI, or in KARATZAS AND SHREVE (1991) [4], Proposition 1.3.14.

Next, we state a concatenation property for elements in $\mathcal{Y}_{loc}$. It is a direct application of supermartingale deflator properties of elements in $\mathcal{Y}_{loc}$ and KARATZAS AND KARDARAS (2020) [1], Exercise 2.26.

Lemma 3.11. *Let Y_1, Y_2 and Y_3 be elements of $\mathcal{Y}_{loc}$, τ be a stopping time of $\mathcal{F}(\cdot)$, and $B \in \mathcal{F}(\tau)$. The process Y defined via the concatenation*

$$f(x) = \begin{cases} Y_1(\omega, t), & \text{for} \quad 0 \leq t \leq \tau(\omega); \\ \big(Y_1(\omega,\tau(\omega))/Y_2(\omega,\tau(\omega))\big)Y_2(\omega,t), & \text{for} \quad t > \tau(\omega), \omega \in B; \\ \big(Y_1(\omega,\tau(\omega))/Y_3(\omega,\tau(\omega))\big)Y_3(\omega,t), & \text{for} \quad t > \tau(\omega), \omega \in \Omega \setminus B. \end{cases}$$

is a strictly positive supermartingale deflator and $Y \in \mathcal{Y}_{loc}$.

In order better to utilize the property of essential supremum in notation (3.35), we prove the next auxilliary lemma.

Lemma 3.12. *Fix $t \in \mathbb{D}$, as well as $Y \in \mathcal{Y}_{loc}$. There exists then a sequence of strictly positive supermartingale deflators $(Y_m)_{m\in\mathbb{N}}$ in $\mathcal{Y}_{loc}$, such that*

(i) *$Y_m(s) = Y(s)$ holds for $s \in [0,t]$;*

(ii) *$\big(\zeta(t;Y_m)\big)_{m\in\mathbb{N}}$ is a nondecreasing sequence;*

(iii) *and*

$$Z(t) = \lim_{m\to\infty} \zeta(t;Y_m) = \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}}\left[\int_t^\infty \frac{Y_m(u)}{Y(t)} \mathrm{d}J(u) \,\middle|\, \mathcal{F}(t)\right]. \tag{3.37}$$

Proof. By the properties of the essential supremum (Proposition VI.1.1 in NEVEU(1975) [18]), we can find out a sequence $(Y'_m)_{m\in\mathbb{N}} \subset \mathcal{Y}_{loc}$ and

$$Z(t) = \sup_{m\in\mathbb{N}} \zeta(t; Y'_m) = \sup_{m\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}\left[\int_t^\infty \frac{Y'_m(u)}{Y'_m(t)} \mathrm{d}J(u) \,\middle|\, \mathcal{F}(t) \right].$$

We further construct a sequence of processes $(Y''_m)_{m\in\mathbb{N}}$ defined as following

$$Y''_m := Y\mathbb{1}_{[0,t]} + \left(\frac{Y(t)}{Y'_m(t)}\right) Y'_m \mathbb{1}_{(t,\infty)}. \tag{3.38}$$

Simple observation indicates that $Y''_m(s) = Y(s)$ holds for $s \in [0,t]$ and $Y''_m(u)/Y''_m(t) = Y'_m(u)/Y'_m(t)$ holds for $u \in (t,\infty)$. As a result, $\zeta(t; Y''_m) = \zeta(t; Y'_m)$.

Upon the definition of Y and $(Y''_m)_{m\in\mathbb{N}}$, we now construct another sequence $(Y_m)_{m\in\mathbb{N}}$ of processes inductively. We set $Y_1 := Y''_1$ and

$$Y_{m+1}(u) := \begin{cases} Y(s), & \text{for} \quad s \in [0,t], \\ Y_m(u)\mathbb{1}_{\{\zeta(t;Y''_{m+1}<\zeta(t;Y_m))\}} + Y''_{m+1}(u)\mathbb{1}_{\{\zeta(t;Y_m)\le\zeta(t;Y''_{m+1})\}}, & \text{for} \quad u \in (t,\infty). \end{cases}$$

The Lemma 3.11 implies that $Y_m \in \mathcal{Y}_{loc}$. Moreover, by definition of Y_{m+1}, we have

$$\zeta(t; Y_{m+1}) = \zeta(t; Y_m) \vee \zeta(t; Y''_{m+1})$$

for all $m \in \mathbb{N}$. Therefore, we can deduce

$$\zeta(t; Y_m) = \bigvee_{k=1}^{m} \zeta(t; Y''_m).$$

The structure of $\zeta(t; Y_m)$ indicates that $\big(\zeta(t; Y_m)\big)_{m\in\mathbb{N}}$ is nodecreasing and the equation (3.37) holds. $\square$

By applying Lemma 3.12, we can replace essential supremum in (3.35) by an increasing limit, namely, the quantity on the right side of equation (3.37): a very significant reduction.

Lemma 3.13. *For the given cumulative withdrawal stream $J \in \mathcal{H}$, let the $\mathbb{D}$-indexed process Z be defined via (3.35). Then, for every strictly positive supermartingale deflator $Y \in \mathcal{Y}_{loc}$, the product $Y(Z+J)$ is a supermartingale when indexed by times in $\mathbb{D}$.*

Proof. For any given $Y \in \mathcal{Y}_{loc}$, integration-by-parts shows that the process

$$YJ - \int_0^{\cdot} Y(t)\mathrm{d}J(t) = \int_0^{\cdot} J(t-)\mathrm{d}Y(t), \tag{3.39}$$

indexed by times in $\mathbb{D}$, is a local martingale. We first prove that the process

$$Q := YZ + \int_0^{\cdot} Y(t)\mathrm{d}J(t) = Y(Z+J) - \int_0^{\cdot} J(t-)\mathrm{d}Y(t),$$

is a supermartingale sampled at times in $\mathbb{D}$. Upon this assertion, $Y(Z+J) = Q + \int_0^{\cdot} J(t-)\mathrm{d}Y(t)$ is seen to be a local supermartingale, thus a true supermartingale as it is non-negative (we note that all three processes YZ, YJ and $\int_0^{\cdot} Y(t)\mathrm{d}J(t)$ are nonnegative).

In order to show that the process Q in the above is a supermartingale, we take arbitrary s and t in $\mathbb{D}$ with $s < t$ and let $(Y_m)_{m\in\mathbb{N}}$ be a sequence as in Lemma 3.12. Then it follows from (3.35) and (3.36) that

$$Z(s) \geq \mathbb{E}^{\mathbb{P}}\left[\int_{(s,t]} \frac{Y(u)}{Y(s)}\mathrm{d}J(u) + \frac{Y(t)}{Y(s)}\zeta(t;Y_m)\,\middle|\,\mathcal{F}(s)\right], \quad m \in \mathbb{N}.$$

Using the representation (3.37) and the monotone convergence theorem on the right-hand-side of the above inequality, we obtain

$$Z(s) \geq \mathbb{E}^{\mathbb{P}}\left[\int_{(s,t]} \frac{Y(u)}{Y(s)}\mathrm{d}J(u) + \frac{Y(t)}{Y(s)}Z(t)\,\middle|\,\mathcal{F}(s)\right]. \tag{3.40}$$

Multiplying now by $Y(s)$, then adding the term $\int_{(0,s]} Y(u)\mathrm{d}J(u)$ to both sides of the resulting inequality, we obtain

$$Y(s)Z(s) + \int_0^s Y(u)\mathrm{d}J(u) \geq \mathbb{E}^{\mathbb{P}}\left[Y(t)Z(t) + \int_0^t Y(u)\mathrm{d}J(u)\,\middle|\,\mathcal{F}(s)\right],$$

which concludes the argument. $\square$

Theorem 3.14. *We fix a closed convex cone K in $\mathbb{R}^n$ containing the origin, and consider only portfolios taking value in K. Suppose that the resulting market is viable, and recall the collection $\mathcal{Y}_{loc}$ from (2.27). Then, the superhedging capital*

$$x(J) = \inf\{x \in \mathbb{R}_+ \,|\, J \in \mathcal{H}(x)\}$$

of (2.22), for any given capital withdrawal stream $J \in \mathcal{H}$, is given as

$$x(J) = \sup_{Y \in \mathcal{Y}_{loc}} \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty Y(t)\mathrm{d}J(t)\right]. \tag{3.41}$$

Proof. Thanks to Theorem 2.31, the assumption of viability indicates that a supermartingale numéraire portfolio $\nu \in \mathcal{I}(R; K)$ exists. Therefore, the set $\mathcal{Y}_{loc}$ in (2.27) is nonempty, as Lemma 2.17 indicates $1/X_\nu \in \mathcal{Y}_{loc}$. Thus, we fix some element $Y_0 \in \mathcal{Y}_{loc}$ of this set. We fix also a cumulative withdrawal stream $J \in \mathcal{H}$ with $z(J) < \infty$, and recall the $\mathbb{D}$-indexed process Z defined in terms of this J via equation (3.35).

From equation (3.39), we can find out that the process $Y_0 J$ is a local submartingale; we note also that this process is locally integrable, and that the random variable $\int_0^\infty Y_0(t)\mathrm{d}J(t)$ is $\mathbb{P}$-integrable, because we are assuming $z(J) < \infty$. Furthermore, Lemma 3.13 implies that

$$Y_0 Z = Y_0(Z + J) - Y_0 J$$

is a nonnegative local supermartingale, thus a true supermartingale, indexed by times in the countable dense set $\mathbb{D}$. Since Y_0 is strictly positive, we invoke the regularization Lemma 3.10 and obtain a nonnegative process $\big(\widetilde{Z}(t)\big)_{t \in \mathbb{R}_+}$ which has rcll paths, satisfies the second and third requirements in the statement of Lemma 3.10, and is such that $Y_0 \widetilde{Z}$ is a supermartingale. In particular, it follows that $\widetilde{Z}(0) \leq Z(0) = z(J)$.

By Lemma 3.13, the quantity $Y(Z + J)$ is a $\mathbb{D}$-indexed supermartingale for every give $Y \in$

$\mathcal{Y}_{loc}$. Moreover, Lemma 3.10 indicates that $Y(\widetilde{Z}+J)$ is a supermartingale indexed by $t \in \mathbb{R}_+$ for every $Y \in \mathcal{Y}_{loc}$. For an arbitiriry number $\epsilon \in \mathbb{R}_+$, the Optional Decomposition Theorem 3.5 and strict positivity of $\epsilon + \widetilde{Z} + J$ guarantee the existence of an investment strategy $\vartheta_J \in \mathcal{I}(S)$ and the corresponding portfolio $\pi_J \in \mathcal{I}(R;K)$ with a cumulative capital withdrawal stream $F \in \mathcal{H}$, such that $\epsilon + \widetilde{Z} + J = X(\cdot; \epsilon + \widetilde{Z}(0), \vartheta_J, F)$, thus

$$\epsilon + z(J) + \int_0^{\cdot} \vartheta_J'(t)\mathrm{d}S(t) \geq \epsilon + \widetilde{Z}(0) + \int_0^{\cdot} \vartheta_J'(t)\mathrm{d}S(t) - F = \epsilon + \widetilde{Z} + J > J \tag{3.42}$$

on account of $\widetilde{Z}(0) \leq Z(0) = z(J)$; in particular,

$$X(\cdot; \epsilon + z(J), \vartheta_J, J) \geq \epsilon + \widetilde{Z}(0) + \int_0^{\cdot} \vartheta_J'(t)\mathrm{d}S(t) - J = \epsilon + \widetilde{Z} + F > F \geq 0. \tag{3.43}$$

This leads to the inequalities $x(J) \leq \epsilon + \widetilde{Z}(0) \leq \epsilon + Z(0) = \epsilon + z(J)$ for every $\epsilon \in \mathbb{R}_+$. Therefore, the inequality $x(J) \leq z(J)$ holds for the quantity of (2.22), which further implies

$$x(J) = \widetilde{Z}(0) = Z(0) = z(J), \tag{3.44}$$

since the inequality $z(J) \leq x(J)$ has already been established in Lemma 2.30. The proof of Theorem 3.14 is complete. $\square$

Chapter 4: Portfolios subject to conic and drawdown constraints

In this chapter, we will focus on topics related to portfolio optimization under both conic and drawdown constrains.

In the rest of this chapter, we fix a closed convex cone K of $\mathbb{R}^n$ containing the origin, and only consider portfolios π taking values in this set K, namely, $\pi \in \mathcal{I}(R; K)$. Moreover, we place *drawdown constraints* on the corresponding numéraires X_π, which simply prohibit X_π from dropping below a certain percentage of its running maximum.

Subject to these two constraints, we try to find portfolios which maximize the long-term growth rate. In Proposition 2.39, we have pointed out that the supermartingale numéraire X_ν maximizes the long-term growth rate among all numéraires X_π with $\pi \in \mathcal{I}(R; K)$. In Proposition 4.14, we will show that a simple path transformation of a supermartingale numéraire portfolio ν generates a wealth process and maximizes the long-term growth rate, under the extra drawdown constraint. We construct this portfolio in Section 4.2.

This is the first time constraints on portfolios and drawdown constraints on numéraires, are considered simultaneously. We utilize a classical path-transformation due to AZÉMA AND YOR (1979) [19]. Actually, in the pioneering work of CVITANIĆ AND KARATZAS (1994) [20], a special case of this transform was already used in the context of drawdown constraints. The “mutual fund" idea of drawdown constrained portfolio was first given in KARDARAS, OBLOJ AND PLATEN (2017) [21].

4.1 Drawdown constraint

We place limitations on the peak-to-through fluctuations of wealth, that investors can endure. This is the so-called "drawdown constraint".

Definition 4.1. X is a continuous semimartingale. We define the process of its running maximum

$$X^* := \sup_{t \in [0,\cdot]} X(t). \tag{4.1}$$

According to this definition, X^* is continuous and nondecreasing.

Definition 4.2. If X is a numéraire in the class $\mathcal{X}(K)$ of Definition 1.13, the process $X^* - X$ is called the *(absolute) drawdown* of X. Whereas the $[0, 1)-$valued process

$$\frac{X^* - X}{X^*} \tag{4.2}$$

is called the *relative drawdown* of X.

For a given number $\delta \in (0, 1)$, we define the sub-collection of "drawdown-constrained numéraires" as

$$^{\delta}\mathcal{X} := \{X \in \mathcal{X} \mid (X^* - X)/X^* < \delta\}. \tag{4.3}$$

That is to say, $^{\delta}\mathcal{X}$ consists of all the numéraires $X \in \mathcal{X}$ whose relative drawdown $(X^* - X)/X^*$ never equals or exceeds the fixed fraction δ. Easily we have $^{1}\mathcal{X} = \mathcal{X}$. Moreover, if in addition we impose constraints on portfolio choice, we can define

$$^{\delta}\mathcal{X}(K) := \{X \in \mathcal{X}(K) \mid (X^* - X)/X^* < \delta\}. \tag{4.4}$$

In order to understand numéraires in $^{\delta}\mathcal{X}$, we have to introduce an important tool—the AZÉMA-YOR transform. This transformation provides a one-to-one correspondence between numéraires in $\mathcal{X}$ and numéraires in the class $^{\delta}\mathcal{X}$, for $\delta \in (0, 1]$. We adapt methods in KARATZAS AND KARDARAS (2020) [1], Lemma 4.1 to show some useful properties of AZÉMA-YOR transform.

Lemma 4.3. *Let X be any strictly positive and continuous semimartingale with $X(0) = 1$, and $F : [1, \infty) \to [1, \infty)$ a continuously differentiable and nondecreasing function with $F(1) = 1$.*

We define the AZÉMA-YOR *transform of X by F, as the process*

$$Z := F(X^*) - F'(X^*)(X^* - X) = F(X^*)\left[1 - \left(\frac{F'(X^*)X^*}{F(X^*)}\right)\frac{X^* - X}{X^*}\right]. \tag{4.5}$$

Then the integral representation

$$Z = 1 + \int_0^{\cdot} F'\big(X^*(t)\big)dX(t) \tag{4.6}$$

holds, as does $Z^* = F(X^*)$.

Proof. We assume first that F is twice continuously differentiable. In this case, the process $F'(X^*)$ is of finite variation on compact intervals, and we have

$$\int_0^{\cdot} \big(X^*(t) - X(t)\big)dF'\big(X^*(t)\big) = \int_0^{\cdot} \big(X^*(t) - X(t)\big)F''\big(X^*(t)\big)dX^*(t) = 0.$$

Here we use the consequence that $\int_0^\infty 1_{\{X(t)<X^*(t)\}}dX^*(t) = 0$, $\mathbb{P}$-a.e and the running maximum process X^* is flat off the set $\{X = X^*\}$. Thanks to the previous observation and the product rule, we can get

$$\begin{aligned} F'(X^*)(X^* - X) &= \int_0^{\cdot} F'\big(X^*(t)\big)d\big(X^*(t) - X(t)\big) \\ &= F(X^*) - 1 - \int_0^{\cdot} F'\big(X^*(t)\big)dX(t), \end{aligned}$$

which proves (4.6) when F is twice continuously differentiable.

For the case where F is only once continuously differentiable, we find a sequence $(F_m; m \in \mathbb{N})$ of twice continuously differentiable, nondecreasing functions with $F_m(1) = 1$, such that $(F_m'; m \in \mathbb{N})$ converges to F' uniformly on compact intervals of $[1, \infty)$. Then, the validity of (4.6) follows by approximation.

According to the fact that $F' \geq 0$ and $X \leq X^*$, it is easy to see from (4.5) that $Z \leq F(X^*)$ holds. Moreover by this fact, we can get $Z^* \leq F(X^*)$, since $F(X^*)$ is nondecreasing and X^* is also nondecreasing. On the other hand, for any $t \in \mathbb{R}_+$, let us define the $[0,t]-$valued random time

$$\psi(t) := \sup\{s \in [0,t] \mid X(s) = X^*(s)\},$$

and note that $X^*(t) = X^*\big(\psi(t)\big) = X\big(\psi(t)\big)$ holds; it then follows that

$$Z^*(t) \geq Z^*\big(\psi(t)\big) \geq Z\big(\psi(t)\big) = F\Big(X^*\big(\psi(t)\big)\Big) = F\big(X^*(t)\big), \text{ for } t \in \mathbb{R}_+.$$

Here the next-to-last equality follows from (4.5) and $X^*\big(\psi(t)\big) = X\big(\psi(t)\big)$, while the last equality is a consequence of $X^*(t) = X^*\big(\psi(t)\big)$. We obtain therefore the comparison $F(X^*) \leq Z^*$ which, combined with the already-established inequality $Z^* \leq F(X^*)$, gives the equality $Z^* = F(X^*)$. $\square$

Remark 4.4. Suppose now in addition, that the condition

$$\frac{xF'(x)}{F(x)} < 1 \tag{4.7}$$

holds, where F is a function as in Lemma 4.3. For a strictly positive and continuous semimartingale X with $X(0) = 1$, it follows from the above conclusion that

$$\frac{Z^* - Z}{Z^*} = \frac{F'(X^*)X^*}{F(X^*)}\frac{X^* - X}{X^*} < \frac{X^* - X}{X^*}.$$

There is an important one-to-one correspondence between numéraires in $\mathcal{X}(K)$ and wealth processes in the class ${}^{\delta}\mathcal{X}(K)$, for any $\delta \in (0,1]$. This bijection can be derived explicitly in terms of the AZÉMA-YOR transform of Lemma 4.3. We first show the bijection between $\mathcal{X}$ and ${}^{\delta}\mathcal{X}$. By using the "mutual fund" idea related to drawdown constrains, we can further establish a bijection between $\mathcal{X}(K)$ and ${}^{\delta}\mathcal{X}(K)$.

(i) First, we show how processes in $\mathcal{X}$ generate processes in ${}^{\delta}\mathcal{X}$. For any given numéraire $X \in \mathcal{X}$

and number $\delta \in (0,1]$, we define a process ${}^\delta X$ via

$$ {}^\delta X := (1-\delta)(X^*)^\delta + \delta X (X^*)^{\delta-1} = (X^*)^\delta \left(1 - \delta \frac{X^* - X}{X^*}\right). \tag{4.8} $$

Actually ${}^\delta X = (X^*)^\delta - \delta (X^*)^{\delta-1}(X^* - X)$ is the AZÉMA-YOR transform of X by taking $F(x) = x^\delta$ in Lemma 4.3. A straightforward application of Lemma 4.3 gives

$$ {}^\delta X = 1 + \int_0^\cdot \delta \big(X^*(t)\big)^{\delta-1} dX(t), \tag{4.9} $$

which implies that ${}^\delta X \in \mathcal{X}$. Moreover we have

$$ {}^\delta X^* = (X^*)^\delta \quad \text{and} \quad \frac{{}^\delta X^* - {}^\delta X}{{}^\delta X^*} = \delta \, \frac{X^* - X}{X^*} < \delta \tag{4.10} $$

on account of (4.8). By definition, ${}^\delta X \in {}^\delta \mathcal{X}$.

(ii) Conversely, for any process Z in the collection ${}^\delta \mathcal{X}$ of (4.3), we will find a numéraire $X \in \mathcal{X}$ such that ${}^\delta X = Z$. By the above discussion, it is necessary to find a $X \in \mathcal{X}$ that

$$ Z^* = (X^*)^\delta \quad \text{and} \quad \frac{Z^* - Z}{Z^*} = 1 - \frac{{}^\delta X}{(X^*)^\delta} = \delta \, \frac{X^* - X}{X^*} $$

hold. In other words, the process X must be

$$ X = X^* \, \frac{X}{X^*} = (Z^*)^{1/\delta} \left(1 - \frac{1}{\delta} \frac{Z^* - Z}{Z^*}\right) = (Z^*)^{1/\delta} - \frac{1}{\delta} (Z^*)^{(1-\delta)/\delta} (Z^* - Z), \tag{4.11} $$

the AZÉMA-YOR transform of Z by the function $[1, \infty) \ni x \to F(x) = x^{1/\delta}$. We define then X as above in terms of Z, and note that Lemma 4.3 gives

$$ X = 1 + \int_0^\cdot \frac{1}{\delta} \big(Z^*(t)\big)^{(1-\delta)/\delta} dZ(t), \quad X^* = (Z^*)^{1/\delta}; \tag{4.12} $$

this implies that $X \in \mathcal{X}$. Finally, it is checked with the help of (4.8) that $Z = {}^\delta X$ holds.

The above discussion can be concluded as follows.

Proposition 4.5. *For any $\delta \in (0,1]$ and with the notation of (4.5), the class of drawdown-constrained numéraires in (4.3) can be expressed as*

$$^{\delta}\mathcal{X} = \{^{\delta}X \mid X \in \mathcal{X}\}.$$

4.1.1 *Non-constant relative drawdown constraint*

More elaborate drawdown constraints can be accommodated as well, allowing the constraint on the relative drawdown to be a function of the running maximum. The following proposition generalizes Proposition 4.5 in this direction.

Proposition 4.6. *Let $d : [1,\infty) \to (0,1]$ be a continuous function, and define*

$$^{d}\mathcal{X} := \{X \in \mathcal{X} \mid (X^* - X)/X^* < d(X^*)\}.$$

Introduce the strictly increasing function $F_d : [1,\infty) \to [1,\infty)$ implicitly, via its strictly increasing inverse

$$F_d^{-1}(z) = \exp\left(\int_1^z \frac{dy}{yd(y)}\right), \quad z \in [1,\infty).$$

(i) For all $x \in [1,\infty)$, we have

$$\frac{xF_d'(x)}{F_d(x)} = d\big(F_d(x)\big), \quad F_d(x) \leq x.$$

(ii) For $X \in \mathcal{X}$, define $^{d}X := F_d(X^*) - F_d'(X^*)(X^* - X)$. We have that $^{d}X^* = F_d(X^*)$ holds, as does

$$\frac{^{d}X^* - {^{d}X}}{^{d}X^*} = d(^{d}X^*)\frac{X^* - X}{X^*}.$$

Therefore, $^{d}X \in {^{d}\mathcal{X}}$.

(iii) Let $Z \in {^{d}\mathcal{X}}(K)$, and set $X := F_d^{-1}(Z^*) - (F_d^{-1})'(Z^*)(Z^* - Z)$. Thus we have $X \in \mathcal{X}$ and

${}^{d}X = Z.$

(iv) As a conclusion we have ${}^{d}\mathcal{X} = \{{}^{d}X \mid X \in \mathcal{X}\}$.

Proof. For part (1), direct computation indicates that

$$F_d'(x) = \frac{1}{(F_d^{-1})'\big(F_d(x)\big)}.$$

By definition of $F_d^{-1}(x)$, we get

$$(F_d^{-1})'(x) = \frac{1}{xd(x)} \cdot F_d^{-1}(x).$$

Thus, we can obtain

$$(F_d^{-1})'\big(F_d(x)\big) = \frac{x}{F_d(x)d\big(F_d(x)\big)},$$

which directly induce the equation

$$\frac{xF_d'}{F_d(x)} = d\big(F_d(x)\big). \tag{4.13}$$

Part (2) is a straightforward application of Lemma 4.3. If we take $F(x) = F_d(x)$ in Lemma 4.3, we have that ${}^{d}X^* = F_d(X^*)$ holds. Moreover, we have

$$\frac{{}^{d}X^* - {}^{\delta}X}{{}^{d}X^*} = \frac{F_d'(X^*)(X^* - X)}{F_d(X^*)}.$$

With help of part (1), we can further get

$$\frac{F_d'(X^*)(X^* - X)}{F_d(X^*)} = d\big(F_d(X^*)\big)\frac{X^* - X}{X^*} = d({}^{d}X^*)\frac{X^* - X}{X^*},$$

which finishes the proof for part (2).

For part (3), thanks to Lemma 4.3, we can derive the integral form of X

$$\begin{aligned} X &= 1 + \int_0^{\cdot} (F_d^{-1})'\big((Z^*(t)\big)\mathrm{d}Z \\ &= 1 + \int_0^{\cdot} F_d^{-1}(Z^*) \cdot \frac{1}{Z^* d(Z^*)}\mathrm{d}Z, \end{aligned}$$

which indicates that $X \in \mathcal{X}$. In order to proof ${}^dX = Z$, we only need to show that ${}^dX^* = Z^*$ and $\frac{{}^dX^* - {}^\delta X}{{}^dX^*} = \frac{Z^* - Z}{Z^*}$.

The fact that ${}^dX^* = Z^*$ is directly from Lemma 4.3 by taking $F = F_d^{-1}$.

On the one hand, $\frac{{}^dX^* - {}^dX}{{}^dX^*} = d({}^dX^*)\frac{X^* - X}{X^*}$ by part (2). On the other hand,

$$\frac{Z^* - Z}{Z^*} = \frac{(F_d^{-1})'(Z^*)(Z^* - Z)}{F_d^{-1}(Z^*)} = d(Z^*)\frac{X^* - X}{X^*}.$$

Thus we have

$$\frac{{}^dX^* - {}^dX}{{}^dX^*} = d({}^dX^*)\frac{X^* - X}{X^*} = d(Z^*)\frac{X^* - X}{X^*} = \frac{Z^* - Z}{Z^*}$$

which completes the proof. □

4.2 Portfolio decomposition via mutual funds

Portfolio decomposition via mutual funds was introduced in Kardaras, Obloj and Platen (2017) [21]. Subsection 4.1.3. in Karatzas and Kardaras (2020) [1] also discusses this idea in detail. In this section, we combine the idea of “mutual funds" with conic constraints on portfolio choice.

Let us fix a number $\delta \in (0, 1]$, as well as a given portfolio $\pi \in \mathcal{I}(R)$ with corresponding numéraire $X_\pi \in \mathcal{X}$. From Proposition 4.5, and with the notation of (4.3) and (4.8), we know that ${}^\delta X_\pi$ is a numéraire in the subclass ${}^\delta\mathcal{X}$ of $\mathcal{X}$. In particular, there exists a portfolio ${}^\delta\pi \in \mathcal{I}(R)$ that finances this numéraire ${}^\delta X_\pi$, i.e., such that ${}^\delta X_\pi = X_{{}^\delta\pi}$ holds.

Now we show that the "financing" portfolio ${}^\delta\pi \in \mathcal{I}(R)$ can actually be described explicitly. In particular, we show that the portfolio ${}^\delta\pi$ which finances the numéraire ${}^\delta X_\pi$ is in fact a convex combination of π, and of the "cash" portfolio $\rho \equiv 0$ which invests only in the money market.

To verify the above claim, we introduce the ratio

$$D_\pi := \frac{X_\pi^* - X_\pi}{X_\pi^*}, \quad \pi \in \mathcal{I}(R), \tag{4.14}$$

the relative drawdown process of the numéraire $X_\pi \in \mathcal{X}$ associated with the given portfolio $\pi \in \mathcal{I}(R)$. For later use, we write the corresponding expression in (4.8) as

$$^{\delta}X_\pi = (X_\pi^*)^\delta (1 - \delta D_\pi). \tag{4.15}$$

Now, the equation (4.6) gives

$$\int_0^{\cdot} \frac{\mathrm{d}\, ^{\delta}X_\pi(t)}{^{\delta}X_\pi(t)} = \int_0^{\cdot} \frac{\delta\big(X_\pi^*(t)\big)^{\delta-1}\mathrm{d}X_\pi(t)}{^{\delta}X_\pi(t)} = \int_0^{\cdot} \frac{\delta\big(X_\pi^*(t)\big)^{\delta-1}X_\pi(t)}{^{\delta}X_\pi(t)} \frac{\mathrm{d}X_\pi(t)}{X_\pi(t)},$$

and one sees immediately from this computation and (4.8), (4.14) that

$$^{\delta}\pi = \left(\frac{\delta(X_\pi^*)^{\delta-1}X_\pi}{^{\delta}X_\pi}\right)\pi = \left(\frac{1 - D_\pi}{1/\delta - D_\pi}\right)\pi. \tag{4.16}$$

The above expression carries a substantial message: the process $^{\delta}X_\pi$ of (4.8) is financed by investing at all times a proportion

$$\frac{1 - D_\pi}{1/\delta - D_\pi} = \frac{\delta - D_{^{\delta}\pi}}{1 - D_{^{\delta}\pi}} \in \big(0, \delta\big]$$

of current wealth in the fund represented by X_π, and the remaining proportion

$$\frac{1 - \delta}{1 - \delta D_\pi} = \frac{1 - \delta}{1 - D_{^{\delta}\pi}} \in \big[1 - \delta, 1\big)$$

in the money market. We have used here a consequence of (4.10), the identity

$$D_{^{\delta}\pi} = \delta D_\pi. \tag{4.17}$$

In other words: the drawdown-constrained numéraire $X_{{}^\delta\pi} = {}^\delta X_\pi$ can be thought of as a mutual fund, created from X_π and from the money market $X_0 \equiv 1$. The fraction of π that gets invested in X_π depends only on the constant $\delta \in (0, 1]$ and on the relative drawdown D_π of X_π in (4.14); it is in fact a decreasing function of D_π.

This observation is of considerable practical importance: the drawdown-constrained portfolio ${}^\delta\pi$ associated with π can be built using quantities readily observable in the market. In fact, because of the special structure of this portfolio, investing according to ${}^\delta\pi$ does not require access to the entire market–only to the money market, and to the fund represented by X_π.

Suppose now we have a numéraire $X \in \mathcal{X}(K)$ such that $X = X_\pi$ with $\pi \in K$. With the help of the "mutual fund" idea related to drawdown constraints, especially the form (4.16), we can easily show that

$$ {}^\delta\pi = \frac{1 - D_\pi}{1/\delta - D_\pi}\pi \in K $$

by the scalable property of cone K. Conversely, for a given numéraire $Z \in {}^\delta\mathcal{X}(K)$, part (ii) of Remark 4.4 has shown that Z is the AZÉMA-YOR transform of some $X \in \mathcal{X}$, namely, $Z = {}^\delta X = X_{{}^\delta\pi}$ with ${}^\delta\pi \in \mathcal{I}(R; K)$. From equation 4.16, we can show that $X = X_\pi$ with $\pi \in K$. The following result summarizes the above discussion.

Proposition 4.7. *For any $\delta \in (0, 1]$ and with the notation of (4.8), the class of drawdown-constrained numéraires in (4.4) can be expressed as*

$$ {}^\delta\mathcal{X}(K) = \{{}^\delta X \mid X \in \mathcal{X}(K)\}. $$

4.3 Asymptotic behavior of porfolios subject to conic and drawdown constraints

The following requirement will be in force for the remaining part of this section.

Assumption 4.8. There exists a supermartingale numéraire X_ν in $\mathcal{X}(K)$, and $G(\infty) = \infty$ holds $\mathbb{P}$-a.e. for the aggregate maximal growth process of (2.20).

Remark 4.9. According to Theorem 2.31, the existence of the supermartingale numŕaire is equivalent to market viability.

According to Assumption 4.8, we have $\lim_{t\to\infty} X_\nu(t) = \infty$, thus also $X_\nu^*(\infty) = \infty$. This is a consequence of Lemma 2.17.

In Chapter 2, we have many useful results related to the supermartingale numéraire X_ν. We have seen that the supermartingale numéraire X_ν has several optimality properties in $\mathcal{X}(K)$. What about the drawdown-constrained case? Moreover, we consider the AZÉMA-YOR transform ${}^\delta X_\nu$ of this supermartingale numéraire. Does ${}^\delta X_\nu$ inherit some of the optimality properties among numéraires in ${}^\delta \mathcal{X}(K)$? The answer is affirmative under some conditions, and negative under others.

We first consider a simple situation, where we only have the conic constraint K and impose no drawdown constraint. For any stopping time T and portfolio $\pi \in \mathcal{I}(R;K)$, $X_\pi^\nu \equiv X_\pi/X_\nu$ is a supermartingale because X_ν is the supermartingale numéraire. Therefore we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{X_\pi}{X_\nu}(T)\right] \leq 1. \tag{4.18}$$

When $T = \infty$, this inequality still holds. The supermartingale convergence theorem implies the existence of the limit

$$\frac{X_\pi}{X_\nu}(\infty) = \lim_{t\to\infty} \frac{X_\pi(t)}{X_\nu(t)}, \quad \pi \in I(R;K). \tag{4.19}$$

What about ${}^\delta X_\pi/{}^\delta X_\nu$? Do we also have $\mathbb{E}^{\mathbb{P}}\left[\frac{{}^\delta X_\pi}{{}^\delta X_\nu}(T)\right] \leq 1$? However, this analogue of inequality (4.18) does not hold for $\delta \in (0,1)$ for some stopping times T. The corresponding inequality remain valid for stopping times T at which the process ${}^\delta X_\nu$ attains its maximum. We will now discuss this topic in this dissertation. A special case for this kind of stopping time is $T = \infty$. We refer readers to KARDARAS, OBLOJ AND PLATEN (2017) [21], Theorem 3.8 for detailed discussion.

The next proposition shows that the supermartingale numéraire portfolio is risky as an investment strategy. Interestingly enough, since $\mathbb{R}^n$ is a special case of close convex cone containing the origin, Proposition 4.10 is true in the unconstrained case, which is also discussed in detail in KARATZAS AND KARDARAS (2020) [1], Proposition 4.6. In this sense, even though we impose a conic constraint K on portfolio selection, there is no way to avoid the oscillating phenomenon described in Proposition 4.10 from the supermartingale numéraire. In the proof of Proposition 4.10, we will see that the oscillating phenomenon is derived from the optimality of the supermartingale numéraire.

Proposition 4.10. *Under Assumption 4.8, we have that*

$$0 = \liminf_{t\to\infty} D_\nu(t) < \limsup_{t\to\infty} D_\nu(t) = 1, \quad \mathbb{P}\text{-}a.e. \tag{4.20}$$

Proof. First we define a sequence of stopping times

$$\tau_m := \inf\{t \in \mathbb{R}_+ \mid \log X_\nu(t) = m\}, \quad m \in \mathbb{N}.$$

Thanks to Lemma 2.17, Assumption 4.8 implies that $\mathbb{P}[\tau_m < \infty] = 1$ holds for all $m \in \mathbb{N}$, as well as $\mathbb{P}[\lim_{m\to\infty} \tau_m = \infty] = 1$. These facts give $\liminf_{t\to\infty} D_\nu(t) = 0$, $\mathbb{P}$-a.e., since $D_\nu(\tau_m) = 0$ holds for all $m \in \mathbb{N}$.

Next we define another sequence of stopping times

$$\sigma_m := \inf\{t \in (\tau_{m-1}, \infty) \mid D_\nu(t) = 1 - 1/m\}, \quad m \in \mathbb{N},$$

quite clearly nondecreasing. We assert that

$$\mathbb{P}[\sigma_m \le \tau_m \mid \mathcal{F}(\tau_{m-1})] \ge 1/m \quad \text{holds for all} \quad m \in \mathbb{N}. \tag{4.21}$$

Under this assertion and the fact that $\{\sigma_m < \tau_m\} \in \mathcal{F}(\tau_m)$ for all $m \in \mathbb{N}$, as well as

$$\sum_{m\in\mathbb{N}} \mathbb{P}[\sigma_m \le \tau_m \mid \mathcal{F}(\tau_{m-1})] = \infty, \tag{4.22}$$

the LÉVY extension of BOREL-CANTELLI lemma (D.WILLIAMS (1991) [22], Page 124) implies

$$\mathbb{P}[\tau_{m-1} \le \sigma_m \le \tau_m < \infty, \text{ for infinitely many } m \in \mathbb{N}] = 1. \tag{4.23}$$

Since $D_\nu(\sigma_m) = 1 - 1/m$ holds on $\{\sigma_m < \infty\}$ for all $m \in \mathbb{N}$, we obtain

$$\limsup_{t\to\infty} D_\nu(t) = 1,$$

which will complete the proof.

Now we go back to prove the assertion (4.21). Since $\int_0^\infty 1_{\{X_\nu(t)<X_\nu^*(t)\}} dX_\nu^*(t) = 0$ holds $\mathbb{P}$-a.e., Itô lemma gives

$$\frac{X_\nu^*(u)}{X_\nu(u)} = 1 + \int_{\tau_{m-1}}^u X_\nu^*(t) d\Big(\frac{1}{X_\nu(t)}\Big) + \log \frac{X_\nu^*(u)}{X_\nu^*(\tau_{m-1})}, \quad m \in \mathbb{N}. \tag{4.24}$$

With help of Lemma 2.17, we know that $1/X_\nu$ is a local martingale. The definition of stopping times τ_m and σ_m indicates that both processes X_ν^*/X_ν and $\log(X_\nu^*/X_\nu^*(\tau_{m-1})$ are bounded on the stochastic interval $[\tau_{m-1}, \sigma_m \wedge \tau_m]$. These facts give the following localization argument

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\frac{X_\nu^*(\sigma_m \wedge \tau_m)}{X_\nu(\sigma_m \wedge \tau_m)} \,\middle|\, \mathcal{F}(\tau_{m-1})\right] &= 1 + \mathbb{E}^{\mathbb{P}}\left[\log \frac{X_\nu^*(\sigma_m \wedge \tau_m)}{X_\nu^*(\tau_{m-1})} \,\middle|\, \mathcal{F}(\tau_{m-1})\right] \\
&\geq 1 + \mathbb{P}\left[\tau_m < \sigma_m \mid \mathcal{F}(\tau_{m-1})\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\frac{X_\nu^*(\sigma_m \wedge \tau_m)}{X_\nu(\sigma_m \wedge \tau_m)} \,\middle|\, \mathcal{F}(\tau_{m-1})\right] &= \mathbb{P}\left[\tau_m < \sigma_m \mid \mathcal{F}(\tau_{m-1})\right] \\
&\quad + m\mathbb{P}\left[\sigma_m \leq \tau_m \mid \mathcal{F}(\tau_{m-1})\right].
\end{aligned}$$

Combining the last two expressions, the assertion (4.21) is verified and the proof is complete. □

With Proposition 4.10 and the relationship between D_{δ_ν} and D_ν, we have a similar oscillatory behavior for the "drawdown-constrained supermartingale numéraire", as the following corollary demonstrates.

Corollary 4.11. *Under Assumption 4.8, we have for every* $\delta \in (0,1]$,

$$0 = \liminf_{t\to\infty} D_{^\delta\nu}(t) < \limsup_{t\to\infty} D_{^\delta\nu}(t) = \delta, \quad \mathbb{P}\text{-}a.e. \tag{4.25}$$

The next example from KARATZAS AND KARDARAS (2020) [1], Example 4.7, illustrates that the ratio $X/{}^\delta X_\nu$ need not be a supermartingale for an arbitrary drawdown-constrained numéraire $X \in {}^\delta\mathcal{X}(K)$. In other word, ${}^\delta X_\nu$ is not the supermartingale numéraire in ${}^\delta\mathcal{X}(K)$.

Example 4.12. For a fixed $\delta \in (0,1)$, we define the stopping time

$$T := \inf\left\{t \in (0,\infty) \;\middle|\; D_\nu(t) = \frac{\delta}{2}\right\}.$$

According to Corollary 4.11, we have $\mathbb{P}(T < \infty) = 1$. Moreover, we set $\eta := \nu 1_{[0,T]} \in \mathcal{I}(R;K)$. Obviously we have $D_\eta \leq \frac{\delta}{2} < \delta$, which implies $X_\eta \in {}^\delta\mathcal{X}(K)$. According to the fact that X_ν is the supermartingale numéraire in $\mathcal{X}(K)$, we can get

$$\mathbb{E}^{\mathbb{P}}\left[\frac{{}^\delta X_\nu}{X_\eta}(T)\right] \leq 1.$$

We also have $\mathbb{P}[{}^\delta X_\nu(T) = X_\eta(T)] < 1$. According to JENSEN's inequality, we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{X_\eta}{{}^\delta X_\nu}(T)\right] > 1.$$

This inequality just implies that the ratio $\frac{X_\eta}{{}^\delta X_\nu}$ is not a supermartingale. Otherwise the optional sampling theorem will provide $\mathbb{E}^{\mathbb{P}}\left[\frac{X_\eta}{{}^\delta X_\nu}(T)\right] \leq 1$.

From Example 4.12, we can see that ${}^\delta X_\nu$ does not hold the supermartingale property in ${}^\delta\mathcal{X}(K)$. However, with the help of the AZÉMA-YOR transform, we can still characterize the limiting behaviour of the ratio ${}^\delta X_\pi/{}^\delta X_\nu$. The next proposition provides both existence and value of the limiting relative ratio. Moreover, Proposition 4.13 generalizes the result in KARATZAS AND KARDARAS (2020) [1], Proposition 4.8, which covers the special case $K \equiv \mathbb{R}^n$.

Proposition 4.13. *Under Assumption 4.8, when $t \to \infty$, both ratios*

$$\frac{{}^{\delta}X_{\pi}}{{}^{\delta}X_{\nu}}(t) \quad and \quad \frac{{}^{\delta}X^{*}_{\pi}}{{}^{\delta}X^{*}_{\nu}}(t)$$

converge $\mathbb{P}$-a.e. Moreover, for any given number $\delta \in (0, 1]$ and portfolio $\pi \in \mathcal{I}(R; K)$, we have

$$\lim_{t\to\infty}\left(\frac{{}^{\delta}X_{\pi}}{{}^{\delta}X_{\nu}}(t)\right) = \lim_{t\to\infty}\left(\frac{{}^{\delta}X^{*}_{\pi}}{{}^{\delta}X^{*}_{\nu}}(t)\right) = \left(\lim_{t\to\infty}\left(\frac{X_{\pi}}{X_{\nu}}(t)\right)\right)^{\delta}. \tag{4.26}$$

Proof. We start with simple case where $\delta = 1$. In this case equation (4.26) becomes

$$\lim_{t\to\infty}\left(\frac{X^{*}_{\pi}(t)}{X^{*}_{\nu}(t)}\right) = \lim_{t\to\infty}\left(\frac{X_{\pi}(t)}{X_{\nu}(t)}\right). \tag{4.27}$$

For any $t \in \mathbb{R}_{+}$, we define the $[0, t]$-valued random time

$$\phi_{\nu}(t) := \sup\{s \in [0, t] \mid X_{\nu}(s) = X^{*}_{\nu}(s)\}.$$

We can observe an important fact that

$$X^{*}_{\nu}(t) = X_{\nu}\big(\phi_{\nu}(t)\big) = X^{*}_{\nu}\big(\phi_{\nu}(t)\big).$$

Furthermore, Assumption 4.8 implies that $\lim_{t\to\infty}\phi_{\nu}(t) = \infty$, $\mathbb{P}$-a.e. Based on these fact, we have

$$\liminf_{t\to\infty}\left(\frac{X^{*}_{\pi}(t)}{X^{*}_{\nu}(t)}\right) = \liminf_{t\to\infty}\left(\frac{X^{*}_{\pi}(t)}{X^{*}_{\nu}\big(\phi_{\nu}(t)\big)}\right) \geq \liminf_{t\to\infty}\left(\frac{X_{\pi}\big(\phi_{\nu}(t)\big)}{X_{\nu}\big(\phi_{\nu}(t)\big)}\right) = \lim_{t\to\infty}\left(\frac{X_{\pi}(t)}{X_{\nu}(t)}\right). \tag{4.28}$$

Next we define another $[0, t]$-valued random time

$$\phi_{\pi}(t) := \sup\{s \in [0, t] \mid X_{\pi}(s) = X^{*}_{\pi}(s)\}.$$

We can observe important facts that

$$X_\pi^*(t) = X_\pi\big(\phi_\pi(t)\big) = X_\pi^*\big(\phi_\pi(t)\big)$$

$$\Big\{ \lim_{t\to\infty} \phi_\pi(t) < \infty \Big\} \subseteq \Big\{ \sup_{t\in\mathbb{R}_+} X_\pi(t) < \infty \Big\} \subseteq \Big\{ \lim_{t\to\infty} \left(\frac{X_\pi(t)}{X_\nu(t)} \right) = 0 \Big\}.$$

Based on these fact,

$$\lim_{t\to\infty} \left(\frac{X_\pi^*(t)}{X_\nu^*(t)} \right) = 0 = \lim_{t\to\infty} \left(\frac{X_\pi(t)}{X_\nu(t)} \right) \text{ holds on } \Big\{ \lim_{t\to\infty} \phi_\pi(t) < \infty \Big\}. \tag{4.29}$$

In the event $\big\{ \lim_{t\to\infty} \phi_\pi(t) = \infty \big\}$, we have

$$\begin{aligned} \limsup_{t\to\infty} \left(\frac{X_\pi^*(t)}{X_\nu^*(t)} \right) &= \limsup_{t\to\infty} \left(\frac{X_\pi^*\big(\phi_\pi(t)\big)}{X_\nu^*(t)} \right) \\ &\leq \limsup_{t\to\infty} \left(\frac{X_\pi\big(\phi_\pi(t)\big)}{X_\nu\big(\phi_\pi(t)\big)} \right) \\ &= \lim_{t\to\infty} \left(\frac{X_\pi(t)}{X_\nu(t)} \right), \text{ on } \Big\{ \lim_{t\to\infty} \phi_\pi(t) = \infty \Big\}. \end{aligned} \tag{4.30}$$

The equation (4.27) now follows from (4.28), (4.29) and (4.30).

Now we deal with the general case of $\delta \in (0, 1)$. Thanks to (4.10), the second equality follows readily from (4.27):

$$\lim_{t\to\infty} \left(\frac{{}^\delta X_\pi^*(t)}{{}^\delta X_\nu^*(t)} \right) = \lim_{t\to\infty} \left(\frac{X_\pi^*(t)}{X_\nu^*(t)} \right)^\delta = \left(\lim_{t\to\infty} \left(\frac{X_\pi^*(t)}{X_\nu^*(t)} \right) \right)^\delta = \left(\lim_{t\to\infty} \left(\frac{X_\pi(t)}{X_\nu(t)} \right) \right)^\delta.$$

We only need to show the first equility in (4.26).

On the strength of (1.13), it follows that

$$\frac{{}^\delta X_\pi}{{}^\delta X_\nu} = \left(\frac{X_\pi^*}{X_\nu^*} \right)^\delta \frac{1 - \delta D_\pi}{1 - \delta D_\nu} = \left(\frac{{}^\delta X_\pi^*}{{}^\delta X_\nu^*} \right) \frac{1 - \delta D_\pi}{1 - \delta D_\nu}. \tag{4.31}$$

According to the fact that $\frac{1-\delta D_\pi}{1-\delta D_\nu} \leq 1/(1-\delta)$, it follows that

$$\lim_{t\to\infty}\left(\frac{{}^\delta X_\pi(t)}{{}^\delta X_\nu(t)}\right) = 0$$

holds on the event $\left\{\lim_{t\to\infty}\left(\frac{{}^\delta X^*_\pi(t)}{{}^\delta X^*_\nu(t)}\right) = 0\right\} = \left\{\lim_{t\to\infty}\left(\frac{X_\pi(t)}{X_\nu(t)}\right) = 0\right\}$; and on $\left\{\lim_{t\to\infty}\left(\frac{X_\pi(t)}{X_\nu(t)}\right) > 0\right\}$, we have

$$\begin{aligned}
\lim_{t\to\infty}\left(\frac{D_\pi(t)}{D_\nu(t)}\right) &= \lim_{t\to\infty}\frac{X^*_\pi - X_\pi}{X^*_\nu - X_\nu}\cdot\frac{X^*_\nu}{X^*_\pi}\\
&= \lim_{t\to\infty}\frac{X^*_\pi - X_\pi}{X^*_\nu - X_\nu}\cdot\lim_{t\to\infty}\frac{X^*_\nu}{X^*_\pi}\\
&= \lim_{t\to\infty}\frac{X^*_\pi}{X^*_\nu}\lim_{t\to\infty}\frac{X^*_\nu}{X^*_\pi} = 1
\end{aligned}$$

on account of the already-established equation (4.27). Furthermore, we have

$$\left|\frac{1-\delta D_\pi}{1-\delta D_\nu} - 1\right| = \delta\left|\frac{D_\pi - D_\nu}{1-\delta D_\nu}\right| \leq \frac{\delta}{1-\delta}\left|\frac{D_\pi}{D_\nu} - 1\right|.$$

Together with (4.31), it follows that

$$\lim_{t\to\infty}\left({}^\delta X_\pi(t)/{}^\delta X_\nu(t)\right) = \lim_{t\to\infty}\left({}^\delta X^*_\pi(t)/{}^\delta X^*_\nu(t)\right)$$

also holds on the event $\left\{\lim_{t\to\infty}\left(\frac{X_\pi(t)}{X_\nu(t)}\right) > 0\right\}$. □

Applying Proposition 4.13, for any $\delta \in (0,1], \pi \in I(R;K)$, it follows that

$$\frac{{}^\delta X_\pi}{{}^\delta X_\nu}(\infty) := \lim_{t\to\infty}\left(\frac{{}^\delta X_\pi}{{}^\delta X_\nu}(t)\right) = \left(\frac{X_\pi}{X_\nu}(\infty)\right)^\delta. \tag{4.32}$$

Therefore, thanks to Jensen's inequality, we have that

$$\mathbb{E}^{\mathbb{P}}\left[\frac{{}^\delta X_\pi}{{}^\delta X_\nu}(\infty)\right] = \mathbb{E}^{\mathbb{P}}\left[\left(\frac{X_\pi}{X_\nu}(\infty)\right)^\delta\right] \leq \left(\mathbb{E}^{\mathbb{P}}\left[\frac{X_\pi}{X_\nu}(\infty)\right]\right)^\delta \leq 1.$$

Before we move towards more asymptotic results for ${}^{\delta}X_{\nu}$, we recall the asymptotic growth optimality of the supermartingale numéraire in Proposition 2.39 in Chapter 2. According to Proposition 4.13, we have the following result, an analogue of Proposition 2.39. It states that ${}^{\delta}X_{\nu}$ inherits the long-run growth optimality from X_{ν}. The reader can compare Proposition 4.14 with Exercise 4.9 in KARATZAS AND KARDARAS (2020) [1], which covers the special case $K \equiv \mathbb{R}^n$.

Proposition 4.14. *Under Assumption 4.8, and for any $\delta \in (0, 1]$ and $\pi \in I(R; K)$, we have*

$$\limsup_{T\to\infty} \Big(\frac{\log {}^{\delta}X_{\pi}(T)}{G(T)}\Big) \le \delta = \lim_{T\to\infty} \Big(\frac{\log {}^{\delta}X_{\nu}(T)}{G(T)}\Big),$$

where X_{ν} is the supermartingale numeraire.

Proof. We recall from Lemma 2.17 the representation $\log X_{\nu} = G + \sqrt{2}W(G)$ for a suitable Brownian motion W. The equality follows then from the strong law of large numbers for Brownian motion; namely,

$$\lim_{T\to\infty} \frac{W\big(G(T)\big)}{G(T)} = 0,$$

Thus

$$\begin{aligned}
\lim_{T\to\infty} \Big(\frac{\log {}^{\delta}X_{\nu}(T)}{G(T)}\Big) &= \lim_{T\to\infty} \frac{\log {}^{\delta}X_{\nu}(T)}{\log X_{\nu}(T)} \frac{\log X_{\nu}(T)}{G(T)} \\
&= \lim_{T\to\infty} \frac{\log {}^{\delta}X_{\nu}(T)}{\log X_{\nu}(T)} \lim_{T\to\infty} \frac{\log X_{\nu}(T)}{G(T)} \\
&= \delta \frac{\log X_{\nu}(T)}{G(T)} = \delta.
\end{aligned}$$

With help of Lemma 2.36 and the assumption $G(\infty) = \infty$, we have

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{G(T)} \log \frac{{}^{\delta}X_{\pi}(T)}{{}^{\delta}X_{\nu}(T)} &= \lim_{T\to\infty} \frac{1}{G(T)} \log \Big(\frac{X_{\pi}(T)}{X_{\nu}(T)}\Big) \cdot \frac{\log\big({}^{\delta}X_{\pi}(T)/{}^{\delta}X_{\nu}(T)\big)}{\log\big(X_{\pi}(T)/X_{\nu}(T)\big)} \\
&= \lim_{T\to\infty} \frac{\delta}{G(T)} \log \Big(\frac{X_{\pi}(T)}{X_{\nu}(T)}\Big)
\end{aligned}$$

When $\Gamma_\nu^\pi(\infty) < \infty$, Lemma 2.36 indicates that

$$\lim_{T\to\infty} \frac{\delta}{G(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) = 0$$

with the notation

$$\Gamma_\nu^\pi = \Gamma_\nu - \Gamma_\pi = G - \Gamma_\pi \geq 0.$$

Thus, on the event $\{\Gamma_\nu^\pi(\infty) < \infty\}$ we have

$$\lim_{T\to\infty} \frac{{}^\delta X_\pi(T)}{G(T)} = \delta$$

On the other hand, on the event $\{\Gamma_\nu^\pi(\infty) = \infty\}$ we write

$$\frac{\delta}{G(T)} \log \frac{X_\pi(T)}{X_\nu(T)} = \frac{\Gamma_\nu^\pi(T)}{G(T)} \cdot \frac{\delta}{\Gamma_\nu^\pi(T)} \log \frac{X_\pi(T)}{X_\nu(T)}.$$

Lemma 2.36 indicates that $\frac{\delta}{\Gamma_\nu^\pi(T)} \log \frac{X_\pi(T)}{X_\nu(T)}$ tends to -1 as $T \to \infty$. Moreover, $\frac{\Gamma_\nu^\pi(T)}{G(T)}$ is nonnegative. Thus,

$$\limsup_{T\to\infty} \frac{\delta}{G(T)} \log\left(\frac{X_\pi(T)}{X_\nu(T)}\right) \leq 0 \quad \text{as well as} \quad \limsup_{T\to\infty} \frac{{}^\delta X_\pi(T)}{G(T)} \leq \delta,$$

hold on the event $\{\Gamma_\nu^\pi(\infty) = \infty\}$. □

Proposition 4.14 points out the trade-off involved, between trying to limit drawdowns on the one hand, and sacrificing long-term growth on the other. Suppose we restrict the relative drawdown by a fraction δ. Then the best we can achieve in terms of long-run growth rate, is the same fraction δ. The more protection we seek against drawdown, the bigger a sacrifice of long-term growth this involves.

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