Some Applications of Nonlocal Models to Smoothed Particle Hydrodynamics-like Methods

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Abstract

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Smoothed Particle Hydrodynamics (SPH) is a meshless numerical method which has long been put into practice for scientific and engineering applications. It arises as a numerical discretization of convolution-like integral operators that approximate local differential operators. There have been many studies on the SPH with an emphasis on its role as a numerical scheme for partial differential equations while little attention is paid to the underlying continuum nonlocal models that lie intermediate between the two. The main goal of this thesis is to provide mathematical understanding of the SPH-like meshless methods by means of ongoing developments in studies of nonlocal models with a finite range of nonlocal interactions. It is timely for such a work to be initiated with growing interests in the nonlocal models.

The thesis touches on numerical, theoretical and modeling aspects of the nonlocal integro-differential equations pertaining to the SPH-like schemes. As illustrative examples of each aspect it presents robust SPH-like schemes for advection-convection equations, discusses the stabilities of nonsymmetric nonlocal gradient operators, and proposes a new formulation of nonlocal Dirichlet-like type boundary conditions.

Table of Contents

Acknowledgements iv

Chapter 1: Introduction 1

1.1 From nonlocal models to asymptotically compatible SPH-like schemes 6

1.2 Loss of radial symmetry for variational stability of nonlocal gradient operators . . . 7

1.3 Linear extrapolation by nonlocal gradient for nonlocal Dirichlet boundary conditions 8

Chapter 2: Asymptotically compatible particle discretizations for linear advection problems 9

2.1 Model I: biased interaction kernel 11

2.1.1 Continuum formulation of Model I 11

2.1.2 Numerical Discretization of Model I 12

2.1.3 Convergence of the numerical scheme to Model I and its local limit 13

2.1.4 Moving mesh particle-like approximations 17

2.2 Model II: Vanishing nonlocal viscosity 20

2.2.1 Continuum formulation of Model II 20

2.2.2 Numerical Discretization of Model II 22

2.2.3 Convergence of the numerical scheme to Model II and its local limit 23

2.2.4 A moving particle approximation 28

2.3 Numerical experiments 31

2.3.1 Tests with Model I . . . 31
2.3.2 Tests with Model II . . . 32
2.3.3 Comparisons and discussions . . . 34
2.4 Discussion . . . 35
Chapter 3: Nonlocal gradient operators with hemispherical nonlocal interaction neighborhoods . . . 38
3.1 Nonlocal nonsymmetric gradient operators . . . 39
3.1.1 Consistency for linear functions . . . 40
3.1.2 Adjoint operator . . . 41
3.1.3 Representation in Fourier space . . . 41
3.1.4 Spectral estimates . . . 43
3.1.5 Orientation dependence . . . 46
3.2 Applications of the nonlocal gradient operators . . . 48
3.2.1 Nonlocal Stokes Equation . . . 48
3.2.2 Nonlocal Helmholtz decomposition . . . 56
3.2.3 Nonlocal correspondence models of isotropic linear elasticity . . . 59
3.3 Another look at the nonlocal gradient operators . . . 66
3.4 Discussion . . . 71
Chapter 4: Second order accurate nonlocal Dirichlet boundary conditions . . . 74
4.1 Well-posedness of nonlocal formulation . . . 76
4.2 Asymptotic localization as $\delta \to 0$ . . . 82
4.2.1 A two dimensional case study . . . 94

4.3 Discussion 97

Conclusion 99

References 101

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Chapter 1: Introduction

Smoothed particle hydrodynamics (SPH) is a meshless numerical method developed independently by Lucy [70], and Gingold and Monaghan [50] in 1977. It was originally designed for astrophysical problems, but its simplicity and efficiency have led to its popularity as a numerical technique for a wider range of problems involving complex flows. As suggested by its very name the method is based on mollification of a function f against some smooth kernel w_δ

$$f(x) \approx f_\delta(x) := \int f(y) w_\delta(x-y)\,dy$$

which in turn is used to yield an integral approximations of the derivatives of f such as the gradient

$$\nabla f(x) \approx \nabla f_\delta(x) = \int f(y) \nabla w_\delta(x-y)\,dy.$$

The kernel is often assumed to be nonnegative, radially symmetric and compactly supported in $B_\delta(0)$ with its L^1 norm normalized to unity. We note that in the absence of boundaries these assumptions on the kernel imply the mollification and the gradient operator is commutative, that is, $\nabla f_\delta = (\nabla f)_\delta$. The integral approximations are then discretized using numerical quadratures

$$\nabla f_\delta(x_i) \approx \sum_{j \in N_i^\delta} f(x_j) \nabla_i w_\delta(x_i - x_j) \frac{m_j}{\rho_j}$$

where $x_i, N_i^\delta, m_i, \rho_i$ denote the position, δ-ball neighborhood, mass and density of the ith particle, respectively. In many instances the quadrature weights m_i/r_i are changing with time as the particle densities themselves are subject to the distribution of the particle masses. The meshless feature is evident from the neighborhood N_i^δ, which does not need information about the connectivity of the quadrature points x_i.

Along with the growing popularity of the SPH has come its various developments ranging from addressing its shortfalls such as tensile instability [99] to the development of the variant of SPH and other related numerical methods such as the Finite Pointset Method [63] and Voronoi-SPH method [5]. Despite the significant amount of existing and ongoing research on the SPH, there has been a limited amount of rigorous mathematical study of the SPH such as its convergence proof under realistic conditions. This is in part due to the fact that the SPH has been applied to those settings for which the classical mesh based numerical methods are not well suited. We are interested in analyzing mathematical underpinnings of the SPH, or more generally SPH-like methods that rely on integro-differential operators with a finite range of interactions. Such operators form a particular subclass of generic nonlocal operators, which in the case of linear operators amount to integral operators against some kernel function k by the Schwartz kernel Theorem [29]. The scope of the thesis is mainly concerned with linear nonlocal operators with a finite range of nonlocal interactions, not only for the sake of simplicity but also in the spirit of laying some foundational studies towards a clearer mathematical understanding of the SPH-like methods.

While nonlocal models in general have been increasingly adopted in many scientific fields [3, 4, 8, 12, 13, 14, 15, 17, 20, 21, 47, 49, 53, 60, 68, 89, 95, 106, 107, 109, 48, 120], those integro-differential models with a finite range of interactions have arisen as an alternative to conventional partial differential based models; the peridynamics models proposed by Silling [95] is one such example to study material defects and singularities. In addition they appear in the context of non-local convection and diffusion models [25, 32, 103] as well as in the description of a Markovian jump process [24]. The computational aspects of the nonlocal models can also be found in existing literature such as [108, 38] that are focused on robust and effective numerical discretizations of the models, the essence of which is encapsulated in the notion of asymptotic compatibility [105]. More on the theoretical ends is the development of nonlocal vector calculus [30, 1], which gives a rise to the nonlocal variational framework. A series of rigorous functional analytic studies were carried out as in the work [8] on nonlocal characterizations of Sobolev spaces, which inspired the subsequent works [78, 77] on localization of nonlocal gradients of scalar-valued and vector-valued functions. We refer to [27] for a more comprehensive overview of the various aspects in studies of the nonlocal models. We also note that the nonlocal models have begun to be harnessed in the emerging technologies of automated vehicles [59] and in conjunction with machine learning [88]. Of interest to us is to channel aforementioned research efforts on nonlocal modeling into mathematical studies of the SPH-like methods in order to address some of the issues of the methods.

There are numerous outstanding questions to be answered in the context of the SPH-like methods and one important issue is concerned with the convergence of the methods. It was proved in [7] that the SPH schemes are convergent to the scalar nonlinear conservation laws provided that the ratio of the particle spacing to the smoothing length of the SPH kernel to vanish to zero. This requirement has been numerically validated in a later work [121] and it is quite typical in the analysis of other particle-like methods. Such a requirement contrasts with the commonly adopted practice of having the smoothing length δ proportional to particle spacing h to allow efficient evaluation of neighboring particle interactions. An important, yet challenging task is thus to restore, in the language of nonlocal modeling framework, the asymptotic compatibility (AC) of the SPH schemes; convergence to the nonlocal models with a fixed smoothing length as the spatial discretization gets refined plus the convergence to the local PDE as the smoothing length of the kernel and the spatial discretization parameter go to zero simultaneously in an arbitrary manner. The AC property was originally motivated for the robust discretization of peridynamics models and nonlocal diffusions, yet it is also essential for the SPH-like schemes since the number of surrounding interacting particles may not be able to keep increasing in practical applications. Within the SPH community, there has been made a progress in relaxing the dependence of consistency of the SPH on the ratio of h to δ via the introduction of symmetric tensors that renormalize the discrete SPH approximations of the first [86] and second spatial derivatives [44]. However a complete relaxation of the condition on the ratio of $\frac{h}{\delta}$ has not yet been achieved, lacking the full force of AC property per se.

Another issue of the SPH-like methods is concerned with their formulations of nonlocal diffusion operators. There have been claims in literature that the SPH approximations of diffusion are observed "to be sensitive to particle disorder" [80]. Thus an alternative integral formulation by Brookshaw [11] is commonly used, namely

$$(u_{xx})_i \approx 2 \int \frac{u(y) - u(x)}{x - y} \partial_x w_\delta(x - y) dy$$

which can be seen as a combination of first order finite differences and SPH approximations of derivatives. The Brookshaw's formulation has a striking similarity with the nonlocal diffusion operators in the setting of bond-based models of peridynamics [95]. The existing body of literature on the latter provides a solid foundation to the former, but conversely the emergence of a generalization of the bond-based models motivates us to revisit the issue of SPH formulations of diffusion. In particular we draw a parallel to the correspondence models of peridynamics [98] by considering the composition of the SPH divergence and the SPH gradient operators as a way to formulate the SPH diffusion. We point out that the authors in [112] promoted the composition approach by demonstrating its accuracies. However they noted some numerical instabilities in their numerical examples, which reminds us of the recent works [36, 96] on the instabilities observed in the numerical simulations of correspondence models. After all the essence of the issue is to delineate, both on the continuum and the discrete levels, the properties of nonlocal gradient operators similar to those studied in [78, 77] that are characterized with a finite range of nonlocal interaction. Our interest in studying such nonlocal gradient operators is fitting to broader and overarching mathematical investigations of generic nonlocal gradient operators which have been applied to various contexts including a nonlocal chemotactic model [53] and nonlocal image analysis [49] among many others; more on a theoretical front one can also find some work such as [73] on a related notion of fractional gradient and divergence.

While SPH was originally devised for astrophysical problems in the free space, its increasing applications to a wider range of problems entailed the necessity to enforce boundary conditions in the presence of physical boundaries. There evidently is a deficiency of particles near the boundaries due the truncation of the nonlocal interaction neighborhoods, hence only partially available nonlocal information is gathered to compute the quantities of interest; consequently accuracy of the computed solutions can be compromised. It has been an important yet challenging task to address various types of boundary conditions in SPH, a “sore point“ [92] in SPH, despite the progresses already made and ongoing research efforts towards that end [81, 9, 114, 64, 45, 93, 79]. We pay a special attention to the solid wall boundary conditions in SPH, for which some general techniques are based on fictitious particles [18, 69] that populate the δ-collar regions outside the domain, or boundary particles [81] that are placed along the boundary surfaces to exert repulsive forces. Examples of the solid boundary conditions include no or free slip, and impermeability conditions. In order to facilitate our study we first turn to the notion of volumetric constraints [31] which has been proposed as nonlocal analogues of local boundary conditions. This formalized notion is in unison with the fictitious particles approaches in SPH as it is concerned with imposing a constraint on the δ-collar surrounding the domain. In this thesis we are interested in volumetric constraints for local Dirichlet boundary conditions that would lead to second order convergence rates of nonlocal solutions to the local ones. The second order rates are expected to be optimal since the SPH approximations of local differential operators are second order accurate in the bulk domain away from the boundaries. Our interest in Dirichlet boundary conditions is in parallel with the previous works on the Neumann cases [100, 116]. To our best understanding popular approaches for the Dirichlet conditions are by means of linear extrapolations of the given boundary data on the surface of codimension one onto the $\delta-$ collar volume. In SPH the method of Morris et al [82] is “the unoffical standard" [55] that uses finite difference approximations of the derivatives to extrapolate the values of the fictitious particles. We are interested in proposing an alternative method to the method of Morris et al which has two shortcomings as mentioned in [57]. One such drawback is potentially expensive calculations of distances between particles and the domain boundaries whereas the other is the need to limit the relative magnitudes of the computed distances for numerical stabilities.

In this thesis we discuss the three specific questions pertaining to the aforementioned issues in SPH. In chapter 2 we propose asymptotically compatible SPH-like schemes for one dimensional linear advection problems. The results therein are reproduced from the work [65]. In chapter 3 we study coercivity properties of nonlocal gradient operators with half-spherical interaction neighborhoods as well as some applications of those operators. The results therein are reproduced from the work [66]. In chapter 4 we present our second order accurate formulations of nonlocal Dirichlet boundary conditions for nonlocal linear diffusion problems. Finally in chapter 5 we provide conclusions with future research directions. Before we present the details of our results let us provide a summary of each later chapter as well as the connections between them.

1.1 From nonlocal models to asymptotically compatible SPH-like schemes

In chapter 2 we present nonlocal models of linear advection which we discretize to obtain particle methods that are asymptotically compatible. We consider the whole real line for our spatial domain which simplifies our analysis, but we do take into account the variable coefficients case to lay the foundational work for future developments. We put forth two nonlocal models, the first one based on upwinding nonlocal kernels and the second involving the nonlocal diffusion terms, both of which are meant to ensure the numerical stabilities in their discretizations. We note that our choice of upwinding kernel which is biased towards one sided information is in resemblance with that of hemispherical nonlocal interaction neighborhoods for nonlocal gradient operators in chapter 3. In both cases there is a unifying notion that a loss of radial symmetries plays an important role in stabilities. So far as the our second model is concerned we can draw a connection to our study in chapter 4. The addition of nonlocal diffusion to nonlocal advection is an illustration of nonlocal-nonlocal coupling, another instance of which is our use of nonlocal gradient operators to enforce the boundary conditions for nonlocal diffusion problems. When it comes to discretizations of our nonlocal models, both models are discretized using quadrature-based finite differences, but the distinction between them is that the schemes for the first model are based on the first moment of the nonlocal kernel while those for the other are conceived based on the renormalized SPH. We recall that the asymptotic compatibilities which we prove for our schemes involve two asymptotic regions, one where the nonlocality δ is fixed with the vanishing particle spacing and the other where both δ and h tend to zero. To our best understanding the latter regime has been the primary interest in the existing literature, with which we comply, but our analysis of the former regime is so as to place our nonlocal continuum models on the same mathematical footing as conventional PDE based models.

1.2 Loss of radial symmetry for variational stability of nonlocal gradient operators

In chapter 3 we analyze nonlocal gradient operators which are representative examples of nonlocal differential operators that constitute the backbones of nonlocal models. The variational setting for our investigation is the nonlocal Dirichlet energies wherein the energy densities are quadratic in the nonlocal gradients. We depart from the existing studies which clarifies the link between the coercivity of the Dirichlet energies and the interaction strengths of radially symmetric kernels that constitute nonlocal gradient operators in the form of integral operators. Instead we adopt a different perspective and focus on nonlocal gradient operators with a non-spherical interaction neighborhood. In fact our geometric consideration of using non-radially symmetric neighborhood can also be seen in our formulation of nonlocal gradient operators used in chapter 4. We show that the truncation of the spherical interaction neighborhood to a half sphere helps making nonlocal gradient operators well-defined and the associated nonlocal Dirichlet energies coercive. As opposed to the case with full spherical neighborhoods, our results are possible without any extra assumption on the strengths of the kernels near the origin. We then present some applications of the nonlocal gradient operators with non-spherical interaction neighborhoods. These include nonlocal linear models in mechanics such as nonlocal isotropic linear elasticity and nonlocal Stokes equations, and a nonlocal extension of the Helmholtz decomposition. We show in particular that these well-posed nonlocal models recover their local counterparts when the latter are mathematically valid, which can be seen as the limiting case of the second regime of asymptotic compatibility considered in chapter 1, corresponding to the case $\delta \to 0$ and $h = 0$.

1.3 Linear extrapolation by nonlocal gradient for nonlocal Dirichlet boundary conditions

In chapter 4 we introduce a continuum formulation of nonlocal linear diffusion model subject to Dirichlet-type boundary conditions. The essence of our approach is to linearly extrapolate boundary data onto outer δ-layer surrounding the domain, thereby resolving the deficiency of nonlocal interaction information near the boundary. We use nonlocal gradient operators for the extrapolation instead of the first order finite difference approximations of the local derivative which is a widespread practice in literature. Our specialized choice of the constant kernel in the nonlocal gradient allows us to enforce the consistency of the operator without explicit calculations of the distances to the boundary, which is regarded as a computational disadvantage of the finite difference based approach. We show the well-posedness of our nonlocal formulation and study the localization of the nonlocal solution to the local one as the nonlocality parameter vanishes. We prove the second order rate of convergence which is the optimal order attainable in the free space. The results in this chapter are complementary to our studies in chapter 2 and 3, where physical boundaries are not present or assumed to periodic, respectively. The nonlocal gradient operators in this chapter involve the normalization factors that are similar in spirit to the one used in Model I of chapter 2. Our nonlocal continuum formulation involves the interplay of nonlocal gradient and diffusion operators that are not related by the variational principle, which departs from our discussion of the nonlocal models of mechanics in chapter 3.

Chapter 2: Asymptotically compatible particle discretizations for linear advection problems

This chapter discusses asymptotically compatible (AC) SPH-like particle discretizations of the following model equation

$$\begin{aligned} u_t(x,t) + \tfrac{\partial}{\partial x}(c(x)u(x,t)) &= 0, \qquad x \in \mathbb{R}, t > 0, \\ u(x,0) &= u_0(x)\,. \end{aligned} \tag{2.1}$$

The AC property is aimed at making the discretization more robust to the changes in modeling and discretization parameters, which bears particular importance to particle methods like SPH which in general exhibit heterogeneous distributions of moving particles. So far as our choice of the model equation is concerned we note that advection phenomena often play an essential role in fluid mechanics and they amount to the simplest prototypical equations in the study of hyperbolic conservation laws; the added simplicity due to one dimensional setting allows us to present our ideas without too many technical complications. The study of advection presents itself as an applicable setting of earlier works that connected SPH like approximations with nonlocal relaxations of conventional local conservation laws given by partial differential equations. In [26], this connection is mainly provided for a scalar diffusion model, while in [37] the focus is on a steady state linear Stokes system in multiple space dimension. The latter offers a rigorous analysis of a nonlocal relaxation and asymptotically compatible discretization to the Stokes equation.

We point out that a nonlocal model of general advection and diffusion process can take on a form:

$$u_t(t,x) + \int_{\mathbb{R}} (\alpha'_\delta(x,x')u(t,x') - \alpha_\delta(x',x)u(t,x))dx' = 0$$

for some nonlocal interaction kernels α_δ and α'_δ. Upon suitable choices of the kernels, we can view the classical advection equation as the local limit of the above nonlocal model. In this chapter, we present two different constructions that can be localized to the same equation (2.1). The first is the model proposed in [103], which is in fact a nonlocal convection diffusion model. Their model constructs the convection term via the kernel which is not symmetric but biased towards the upwinding direction and this leads to stability of the model in the sense of satisfying the maximum principle. In our first nonlocal model we will adopt the similar idea by encoding the upwinding in the kernel, but we will take a simpler approach by using kernels that take into account neighboring particles either to the left or the right of the particle depending only on the upwinding direction of that particle. Despite the simplification we show that a stable numerical scheme is obtained when we recast the model as a pairwise interaction model of [32] by introducing the first moment of the kernel. On the other hand, our second continuum nonlocal model is a bona fide SPH kernel approximation with an addition of nonlocal viscosity. We consider the viscosity in the framework of the nonlocal vector calculus [30], which in fact has already been studied in the context of peridynamics [104]. We adopt the idea of vanishing viscosity that vanishes in the limit of $\delta \to 0$ where δ is the smoothing length of the SPH kernel. The consistency of a corresponding numerical scheme is ensured by encapsulating the idea of renormalized SPH of Vila [111].

The main contribution of this chapter is that our result is among the first, as far as we are aware, to propose and provide mathematical justifications of particle methods with the full force of the AC property. Our work can be seen as a extension of the existing AC numerical schemes developed in the setting of nonlocal models discretized on stationary uniform grids. We note that non-uniform distributions of particles have implications for our study in chapter 4 on nonlocal Dirichlet boundary conditions. Near physical boundaries the nonlocal interaction neighborhoods are truncated, which could make it challenging to distribute particles uniformly in the first place. In order to achieve the AC property in that setting we may follow the similar steps as we have done in this chapter by starting first with a nonlocal continuum model. We take this very first step in chapter 4 of proposing our nonlocal model which we show is an accurate approximation of the local one.

2.1 Model I: biased interaction kernel

Our first approach is to look for a nonlocal model in which stabilization effect is provided by the kernel that has a built-in bias in deciding which neighboring particles to interact with. Such approaches in the context of nonlocal modeling have been studied previously, see for example, [32] for one dimensional nonlinear nonlocal conservation laws and [103] multidimensional linear convection diffusion equations.

2.1.1 Continuum formulation of Model I

In more specific terms, let us define the kernel

$$w_c^\delta(x,y) = \begin{cases} 1_{y<x}\eta^\delta(y-x) & \text{if } c(x) > 0, \\ 1_{y>x}\eta^\delta(y-x) & \text{otherwise,} \end{cases}$$

where $\eta^\delta(s) = \frac{1}{\delta^2}\eta(\frac{s}{\delta})$ is a scaled kernel for some odd function η. The function η is assumed to satisfy $\eta(z) \geq 0$ on $z \geq 0$ and to be supported on $(-1,1)$. Note the use of the subscript c to elucidate the dependence of the kernel on the velocity field c. We then propose the following nonlocal model (referred as the Model I from here on)

$$\begin{aligned} u_t(x,t) + \int_{\mathbb{R}} (c(y)u(y,t) - c(x)u(x,t))w_c^\delta(x,y)dy &= 0\,, \\ u(x,0) &= u_0(x)\,. \end{aligned} \tag{2.2}$$

In order to ensure (at least formal) consistency of the Model I with the local PDE, we need to impose a suitable normalization condition on w_c^δ, and to this end fix x and assume without loss of generality $c(x) > 0$. Then we consider the nonlocal operator L^δ defined as:

$$L^\delta(u)(x,t) = \int_0^\delta (c(x)u(x,t) - c(x-y)u(x-y,t))\eta^\delta(y)dy \tag{2.3}$$

so that we can rewrite eq. (2.2) as

$$u_t + L^\delta(u) = 0\,.$$

Under the assumption of smooth u and c, direct calculation based on Taylor expansion shows that we need the following condition on the kernel

$$\int_0^\delta y\eta^\delta(y)dy = 1 \tag{2.4}$$

which we assume from here on.

2.1.2 Numerical Discretization of Model I

We are interested in discretization of eq. (2.3) that is asymptotically compatible, so we follow the approach in [32]. Assuming $c(x) > 0$, we first rewrite the operator $L^\delta(u)(x,t)$:

$$L^\delta(u)(x,t) = \int_0^\delta \frac{c(x)u(x,t) - c(x-y)u(x-y,t)}{y} y\eta^\delta(y)dy$$

from which we see that the operator L^δ represents a continuum of one-sided finite differences. In comparison with the pairwise interaction nonlocal model of [32], our Model I allows the particle at x_j to interact only with the neighboring paticles to its left. We are motivated to introduce this bias in our model based on the simple fact that the use of the upwinding flux in discretization of the advection PDE with a constant velocity brings about numerical stability by default (that is, without explicit consideration of how much numerical viscosity should be added). An extension of that simple idea is what we choose to study in the setting of nonlocal advection with variable speed.

Moving particles at each fixed time instant constitutes a grid that is in general irregular, so let us first consider the case of general stationary grid, denoted by $\{x_j\}_{j\in\mathbb{Z}}$. We propose the following discretization:

$$L_h^\delta(u)(x_j) = \sum_{k=1}^{L(j)} (c_j u_j - c_{j-k}u_{j-k})W_{k,j}^L$$

where

$$L(j) = \max\{\max\{l : 0 \le l, |x_j - x_{j-l}| \le \delta\}, 1\}$$

and

$$W_{k,j}^L = \frac{1}{x_j - x_{j-k}} \int_{x_j - x_{j-k+1}}^{x_j - x_{j-k}} y\eta^\delta(y)dy + \frac{\mathbf{1}_{k=L(j)}}{x_j - x_{j-k}} \int_{x_j - x_{j-k}}^{\delta} y\eta^\delta(y)dy.$$

For simplicity we adopt the forward Euler time stepping which then yields the following fully explicit scheme

$$u_j^{n+1} = u_j^n - \Delta t \sum_{k=1}^{L(j)} (c_j u_j^n - c_{j-k} u_{j-k}^n) W_{k,j}^L. \tag{2.5}$$

Remark 1. *If the spacing of the grid points is fixed but $\delta \to 0$, then the scheme eq.* (2.5) *reduces to a local finite difference scheme*

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{x_j - x_{j-1}} (c_j u_j^n - c_{j-1} u_{j-1}^n).$$

Remark 2. *By construction $W_{k,j}^L$ are nonnegative with a weighted sum satisfying the identity*

$$\sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = 1.$$

2.1.3 Convergence of the numerical scheme to Model I and its local limit

We are interested in the convergence of numerical solutions obtained by the numerical scheme eq. (2.5) in two cases:

$$\begin{cases} \bullet\ \bar{h} \to 0 \text{ with } \delta > 0 \text{ fixed,} \\ \bullet\ \bar{h}, \delta \to 0 \text{ arbitrarily.} \end{cases} \tag{2.6}$$

where $\bar{h} = \sup_i |x_{i+1} - x_i|$. Since the scheme is linear, it suffices to establish consistency and stability. At the expense of assuming smooth solutions and data, we can show the convergence in the L^∞ norm.

Convergence to the nonlocal Model I Let us first present a consistency estimate.

Lemma 1. *(Consistency) Suppose c, u, c', u_x are bounded. If c' and u_x are Lipscthiz, then*

$$\sup_j |L^\delta(u)(x_j,t) - L_h^\delta(u)(x_j,t)| \leq C\bar{h}$$

for some constant C independent of δ.

Proof. Let us define

$$G(s) = \frac{c(x)u(x,t) - c(x-s,t)u(x-s,t)}{s}$$

and note that direct calculation shows the uniform boundedness of $G'(s)$. Then applying the argument of Lemma 5.1 in [104] yields the result.

□

The next is concerned with the stability.

Lemma 2. *(L^∞-Stability) Assume c, c' are uniformly bounded. Provided that the following CFL condition holds*

$$\frac{\Delta t \|c\|_{L^\infty(\mathbb{R})}}{\underline{h}} \leq 1,$$

where $\underline{h} = \inf_i |x_{i+1} - x_i|$ then

$$\|u^{n+1}\|_{L^\infty(\mathbb{R})} \leq \|u^n\|_{L^\infty(\mathbb{R})}(1 + \Delta t\|c'\|_{L^\infty(\mathbb{R})})\ \forall n \geq 0.$$

Proof. We can rewrite the scheme as

$$u_j^{n+1} = (1 - \Delta t c_j \sum_{k}^{L(j)} W_{k,j}^L)u_j^n + \Delta t \sum_{k=1}^{L(j)} c_j W_{k,j}^L u_{j-k}^n + \Delta t \sum_{k=1}^{L(j)} (c_{j-k} - c_j) W_{k,j}^L u_{j-k}^n$$

Then since

$$0 \leq \Delta t \sum_{k=1}^{L(j)} c_j W_{k,j}^L \leq \frac{\Delta t\|c\|_{L^\infty(\mathbb{R})}}{\underline{h}} \sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = \frac{\Delta t\|c\|_{L^\infty(\mathbb{R})}}{\underline{h}}$$

and

$$\sum_{k=1}^{L(j)} \left|(c_{j-k} - c_j) W_{k,j}^L\right| \leq \|c'\|_{L^\infty(\mathbb{R})} \sum_{k=1}^{L(j)} (x_j - x_{j-k}) W_{k,j}^L = \|c'\|_{L^\infty(\mathbb{R})},$$

the result follows. □

We note that one can immediately observe that the CFL condition in lemma 2 is in fact that of local PDE. Moreover, if $c(\cdot)$ = constant we indeed have the maximum principle $\|u^n\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$.

Asymptotic convergence to the local advection equation eq. (2.1) We note that the CFL condition in lemma 2 is independent of δ. Hence if we choose the time steps Δt to satisfy that CFL condition, the scheme eq. (2.5) is Lax-Richtmyer stable in L^∞ uniformly in δ. It only remains then to show consistency to the local PDE.

Lemma 3. *(Consistency) If $u, u_x, u_{xx}, c, c', c''$ are bounded, then*

$$\sup_j \left|L_h^\delta(u)(x_j, t) - (c(x)u(x,t))_x \,|_{x=x_j}\right| = O(\max(\bar{h}, \delta)).$$

Proof. Using Taylor series expansion, we have

$$c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t) = (c(x)u(x,t))_x\Big|_{x=x_j}(x_j - x_{j-k}) + O(|x_j - x_{j-k}|^2).$$

Now for $|x_j - x_{j-1}| > \delta$,

$$\begin{aligned}
&\sum_{k=1}^{L(j)} (c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t)) W_{k,j}^L \\
&\quad = (c(x)u(x,t))_x \,|_{x=x_j} + O(|x_j - x_{j-1}|) \int_0^\delta y\eta^\delta(y)dy \\
&\quad = (c(x)u(x,t))_x \,|_{x=x_j} + O(|x_j - x_{j-1}|),
\end{aligned}$$

whereas for $|x_j - x_{j-1}| \le \delta$,

$$\begin{aligned}
&\sum_{k=1}^{L(j)} (c(x_j)u(x_j,t) - c(x_{j-k})u(x_{j-k},t))W_{k,j}^L \\
&\quad = (c(x)u(x,t))_x \mid_{x=x_j} + \sum_{k=1}^{L(j)} O(|x_j - x_{j-k}|)(x_j - x_{j+k})W_{k,j}^L \\
&\quad = (c(x)u(x,t))_x \mid_{x=x_j} + O(\delta) \sum_{k=1}^{L(j)} (x_j - x_{j+k})W_{k,j}^L = (c(x)u(x))_x \mid_{x=x_j} + O(\delta).
\end{aligned}$$

This leads to the conclusion in the lemma. □

In summary, we state:

Theorem 1. *Let $T > 0$ be a fixed terminal time. Assume c and c' are bounded. Let U_j^n denote the numerical solution given by the scheme eq.* (2.5) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition of lemma 2. Let $u(x,t), u^\delta(x,t)$ denote the solutions of eq.* (2.1) *and eq.* (2.2)*, respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. *Suppose c' is Lipschitz in $\mathbb{R}$. If u_x^δ is Lipschitz in $\mathbb{R}$ uniformly in $t \in [0,T]$ and $u^\delta, u_x^\delta, u_{tt}^\delta \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u^\delta(x_j,T) - U_j^N| \le C_\delta(\bar{h} + \Delta t)$$

for some $C_\delta > 0$ depending on $\|\cdot\|_{L^\infty(\mathbb{R}\times[0,T])}$ norms of u^δ and u_{tt}^δ, and Lipschitz constant of u_x^δ.

2. *If $c'' \in L^\infty(\mathbb{R})$ and $u, u_x, u_{xx}, u_{tt} \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u(x_j,T) - U_j^N| = O(\max(\bar{h},\delta)) + O(\Delta t).$$

Proof. The proof follows from the Lax Equivalence Theorem. □

2.1.4 Moving mesh particle-like approximations

With our convergence results in the case of general non-uniform stationary particles, we can easily establish the corresponding results for moving particles. To this end, we first introduce moving coordinates $x(t)$ subject to some prescribed velocity field $b(x)$, which is not necessarily the underlying velocity field $c(x)$ in the similar spirit as in the Arbitrary Euler-Lagrangian Methods [54],

$$\dot{x}(t) = b(x(t)), \qquad x(0) = x_0. \tag{2.7}$$

The local PDE can then be rewritten in the Lagrangian frame

$$\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + (c'(x(t)) - b'(x(t)))u(x(t),t)) = 0 \tag{2.8}$$

so that our non-local integral relaxation of the the flux term yields:

$$\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + \int_{x(t)-\delta}^{x(t)+\delta} ((c(y) - b(y))u(y,t) - \dots \\ (c(x(t)) - b(x(t)))u(x(t),t))w^{\delta}_{c-b}(x(t),y)dy = 0. \tag{2.9}$$

Note the dependence of the kernel on the *relative* velocity field $\hat{c} = c - b$. With $\hat{c}_{j,n} = \hat{c}(x_j(n\Delta t)) > 0$, we then obtain the following discretization:

$$u_j^{n+1} = u_j^n - \Delta t(b'_{j,n}u_j^n + \sum_{k=1}^{L(j)}(\hat{c}_{j,n}u_j^n - \hat{c}_{j-k,n}u_{j-k}^n)W^L_{k,j}). \tag{2.10}$$

We assume for simplicity that the initial distribution of particles is uniform with a spacing of size h. We also assume the velocity field b is smooth with bounded first derivative, which implies that there exist constants C_1 and C_2 depending only on the terminal time T such that

$$C_1(T)h \leq |x_{j+1} - x_j| \leq C_2(T)h$$

for all $j \in \mathbb{Z}$ and $t \in [0, T]$ [72]. Then it is a straightforward extension of the results in section 2.1.3 to establish:

Theorem 2. *Let $T > 0$ be a fixed terminal time. Assume c, c', b, b' are uniformly bounded. Let U_j^n denote the numerical solution given by the scheme eq.* (2.10) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition*

$$\frac{\Delta t \|c - b\|_{L^\infty(\mathbb{R})}}{C_1(T)h} \leq 1.$$

Let $u(x,t), u^\delta(x,t)$ denote the solutions of eq. (2.8) *and eq.* (2.9)*, respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. *Suppose c' and b' are Lipschitz in $\mathbb{R}$. If u_x^δ is Lipschitz in $\mathbb{R}$ uniformly in $t \in [0, T]$ and $u^\delta, u_x^\delta, u_{tt}^\delta \in L^\infty(\mathbb{R} \times [0, T])$, then*

$$\sup_j |u^\delta(x_j, T) - U_j^N| \leq C_\delta(\bar{h} + \Delta t)$$

for some $C_\delta > 0$ depending on $\| \cdot \|_{L^\infty(\mathbb{R}\times[0,T])}$ norms of u^δ and u_{tt}^δ, and Lipschitz constant of u_x^δ.

2. *If $c'', b'' \in L^\infty(\mathbb{R})$ and $u, u_x, u_{xx}, u_{tt} \in L^\infty(\mathbb{R} \times [0, T])$, then*

$$\sup_j |u(x_j, T) - U_j^N| = O(\max(\bar{h}, \delta)) + O(\Delta t).$$

Proof. The arguments are analogous to the case of stationary non-uniform grid in the section 2.1.3. □

The appearance of the local derivative term in eq. (2.9) suggests that one may consider its nonlocal relaxation to obtain the following fully nonlocal equation

$$\frac{d}{dt}u(x(t),t) + \left(\int_{x(t)-\delta}^{x(t)+\delta} (b(y) - b(x(t)))w_{\hat{c}}^{\delta}(x(t),y)dy\right) u(x(t),t) + \dots$$
$$\int_{x(t)-\delta}^{x(t)+\delta} ((c(y) - b(y))u(y,t) - (c(x(t)) - b(x(t)))u(x(t),t))w_{\hat{c}}^{\delta}(x(t),y)dy = 0 \qquad (2.11)$$

This formulation fits well with our nonlocal framework when the velocity field $b(x)$ is chosen to be equal to $c(x)$ in which case eq. (2.9) reduces to a local PDE. We can see that the use of the kernel $w_{\hat{c}}^{\delta}$ in our nonlocal relaxation of b' is a convenient choice as it yields, in the case of $\hat{c}_{j,n} > 0$, the following simple numerical discretization

$$u_j^{n+1} = u_j^n - \Delta t \sum_{k=1}^{L(j)} ((c_{j,n} - b_{j-k,n})u_j^n - \hat{c}_{j-k,n}u_{j-k}^n)W_{k,j}^L. \qquad (2.12)$$

The convergence of the above discretization can be shown:

Theorem 3. *Suppose the same assumptions are made as in theorem 2. Then with u and u^{δ} now denoting the solutions of eq.* (2.8) *and eq.* (2.11)*, respectively, the same conclusions as in theorem 2 hold.*

Proof. The proof is exactly analogous to that of theorem 2. □

We make some observations concerning the two schemes in eq. (2.10) and eq. (2.12). It is clear that the computational complexity of the latter involves $O(N)$ more operations than the former where N denotes the number of particles. In terms of memory storage, however, the former requires additional $O(N)$ storage for the terms $(b')_j^n$ than the latter which only needs the values of b_j that have already been computed for the nonlocalized flux. Meanwhile we note that computations of the weights $W_{k,j}^L$ for both schemes can be significantly simplified by choosing a singular kernel such as $\eta^{\delta}(z) = \frac{1_{(-\delta,\delta)}}{\delta z}$.

2.2 Model II: Vanishing nonlocal viscosity

It is a well-known practice in the study of conservation laws to seek solutions of an inviscid problem by taking the limit solutions of the corresponding viscous problem as the viscosity effect vanishes. In the particular case of advection equation, one way to bring in the idea of vanishing viscosity is through advection-diffusion equations where the diffusive terms are introduced through nonlocal operators such as a fractional power of the Laplacian [2, 22, 32]. In this section we present a nonlocal continuum advection-diffusion model which will in turn give a rise to another AC particle method for the local advection equation eq. (2.1).

2.2.1 Continuum formulation of Model II

In more specific terms, we propose the following nonlocal advection-diffusion model (referred to as Model II from here on):

$$\begin{aligned} u_t(x,t) + \int_{\mathbb{R}} c\left(\frac{x+y}{2}\right)(u(x,t)+u(y,t))w^\delta(y-x)dy - \dots \\ \delta\mu \int_{\mathbb{R}} (u(y,t)-u(x,t))\frac{w^\delta(y-x)}{y-x}dy = 0, \\ u(x,0) = u_0(x) \end{aligned} \tag{2.13}$$

where $\delta > 0$ is a nonlocal horizon and $\mu > 0$ is a positive coefficient depending only on the velocity field c. We assume that $w^\delta(z) = \frac{1}{\delta^2} w(\frac{z}{\delta})$ where w is an anti-symmetric integrable kernel that is supported on $(-1,1)$ and satisfies $w^\delta(z) \geq 0$ on $z \geq 0$. In the language of nonlocal vector calculus [30], the model can be rewritten as:

$$u_t + \mathcal{A}_\delta(u) - \delta\mu\mathcal{D}_\delta(u) = 0$$

where $\mathcal{A}_\delta$ corresponds to a nonlocal divergence operator with the vector two-point function $\mathbf{v}(x,y) = c(\frac{x+y}{2})u(x,t)$ and the antisymmetric two point function $\alpha = \frac{y-x}{|y-x|}|w^\delta(y-x)|$ whereas $\mathcal{D}_\delta$ corresponds to a nonlocal diffusion operator with the symmetric kernel $\frac{w^\delta(y-x)}{y-x}$. As the focus of this chapter is on initial Cauchy value problems, discussions on the nonlocal boundary conditions, or constrained values [23], and modifications to the nonlocal diffusion operators near the boundary [26], if any, are not presented here.

Following discussions in [30, 24, 103], it is immediate to see that with a fixed $\delta > 0$ the model is a nonlocal analogue of the modified equation

$$u_t(x,t) + (c(x)u(x,t))_x - \frac{\delta\mu}{2}u_{xx}(x,t) = 0$$

$$\int_{-\delta}^{\delta} hw^\delta(h)dh = 1 \tag{2.14}$$

which we assume from here on. Note that such assumptions together with the integrability of w implies that

$$\|w^\delta\|_{L^1(\mathbb{R})} = O\left(\frac{1}{\delta}\right). \tag{2.15}$$

A connection to the SPH can be made by setting $w^\delta(z) = -\partial_z\left(\frac{1}{\delta}\rho(\frac{z}{\delta})\right)$ where ρ is a radially symmetric, nonnegative, differentiable function that decreases with increasing radial distances and is compactly supported on $(-1, 1)$ with $\|\rho\|_{L^1(\mathbb{R})} = 1$.

The presence of nonlocal diffusion term in eq. (2.13) suggests that a simple minded discretization of the model may not need additional stabilization and that such discretization, with vanishing δ, is expected to converge to the advection equation. This is indeed the case as we establish the convergence results of our discretization of the Model II. We assume $\|c\|_{L^\infty(\mathbb{R})} < \infty$ and we choose to take

$$\mu \geq \|c\|_{L^\infty(\mathbb{R})}.$$

This leads to sufficient damping to the nonlocal model to ensure desirable physical features related to stability. For example, for constant velocity c, the above condition implies a maximum principle for the resulting nonlocal convection-diffusion model.

2.2.2 Numerical Discretization of Model II

Similar to our discretization of the Model I, we present our numerical discretization of the Model II wherein the kernels of the nonlocal integral operators are integrated analytically. What is distinct in this case, however, is that we adopt the idea presented in the renormalized SPH by Vila [111] instead of considering the first moment of the kernel. Before presenting our discretization on a set of moving particles, we first consider a set of stationary particles $\{x_i\}_{i\in\mathbb{Z}}$ and propose the following discretization:

$$\begin{aligned}\mathcal{A}_\delta^h(u(x_j,t)) &= \frac{1}{N_{j,\delta}}\sum_{k=-l_j}^{r_j} c\left(\frac{x_j+x_{j+k}}{2}\right)(u(x_j,t)+u(x_{j+k},t))W_{j,j+k}\\ \mathcal{D}_\delta^h(u(x_j,t)) &= \frac{1}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}\frac{(u(x_{j+k},t)-u(x_j,t))}{x_{j+k}-x_j}W_{j,j+k}\end{aligned}$$

where $l_j = \max\{\max\{l\in\mathbb{Z}^+ \mid x_j - x_{j-l}\le\delta\},1\}$,$r_j = \max\{\max\{r\in\mathbb{Z}^+ \mid x_{j+r}-x_j\le\delta\},1\}$,

$$W_{j,j+k} = \int_{x_{j+k-1}}^{x_{j+k}} w^\delta(y-x_j)dy + 1_{k=r_j}\int_{x_{j+k}}^{x_j+\delta} w^\delta(y-x_j)dy$$

for $k>0$ (analogously for $W_{j,j+k}$ with $k<0$) and the renormalization factor

$$N_{j,\delta} = \sum_{k=-l_j}^{r_j}(x_{j+k}-x_j)W_{j,j+k}.$$

Let us then summarize the properties of $N_{j,\delta}$ that will be used in our analysis to follow.

Lemma 4. *There exists a constant C such that*

$$0\le N_{j,\delta}-1\le C\frac{\bar{h}}{\delta}.$$

Moreover

$$\frac{1}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}W_{j,j+k}=0,\qquad \frac{1}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}(x_{j+k}-x_j)W_{j,j+k}=1 \tag{2.16}$$

and

$$\left| \frac{\sum_{k=-l_j}^{r_j} (x_{j+k} - x_j)^2 W_{j,j+k}}{N_{j,\delta}} \right| = O(\max(\bar{h}, \delta)). \tag{2.17}$$

Proof. Assume without loss of generality $x_j = 0$ and note

$$N_{j,\delta} - 1 = \sum_{k=1}^{r_j} \int_{x_{j+k-1}}^{x_{j+k} + 1_{k=r_j}(\delta - x_{j+k})} (y\big|_{y=x_{j+k}} - y) w^\delta(y) dy + \dots$$
$$\sum_{k=-l_j}^{-1} \int_{x_{j+k} + 1_{k=l_j}(-\delta - x_{j+k})}^{x_{j+k+1}} (y\big|_{y=x_{j+k}} - y) w^\delta(y) dy.$$

Then the first claim follows from nonnegativity of the integrands and (2.15). eq. (2.16) is immediate from the definition of $N_{j,\delta}$. eq. (2.17) is a consequence of nonnegativity of each $(x_{j+k} - x_j) W_{j,j+k}$ in $N_{j,\delta}$. □

We adopt the forward Euler time stepping, thereby obtaining the following explicit scheme:

$$u_j^{n+1} = u_j^n - \Delta t (\mathcal{A}_\delta^h(u_j^n) - \mu \max\{\delta, x_j - x_{j-l_j}, x_{j+r_j} - x_j\} \mathcal{D}_\delta^h(u_j^n)). \tag{2.18}$$

As will be seen in the proof of lemma 6, approximating δ with $\max\{\delta, x_j - x_{j-l_j}, x_{j+r_j} - x_j\}$ in the viscosity term ensures that sufficient numerical viscosity is present when $\delta < x_j - x_{j-l_j}$ or $\delta < x_{j+r_j} - x_j$.

2.2.3 Convergence of the numerical scheme to Model II and its local limit

Again, we show that the numerical solutions obtained by the scheme eq. (2.18) are convergent in L^∞ norm in the following two regimes outlined in eq. (2.6). Since the scheme eq. (2.18) is linear, we establish consistency and stability.

Convergence to the nonlocal Model II The consistency is stated first below.

Lemma 5. *If $u, u_x, u_{xx}, c, c', c''$ are uniformly bounded, then*

$$\sup_j |\mathcal{A}_\delta^h(u(x_j,t)) - \mathcal{A}_\delta(u(x_j,t))| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right).$$

If u_{xxx} is uniformly bounded, then

$$\sup_j |\mathcal{D}_\delta^h(u(x_j,t)) - \mathcal{D}_\delta(u(x_j,t))| = O(\bar{h}) + O\left(\frac{\bar{h}^3}{\delta^2}\right).$$

Proof. For convenience of notation, let us denote $f(y,t) = c\left(\frac{x_j+y}{2}\right)(u(x_j,t) + u(y,t))$ and apply Taylor series expansion to write

$$f(y,t) = f(x_j,t) + (y - x_j) f_y(x_j,t) + R_1(y,t)$$

where $R_1(y,t) = \int_{x_j}^{y} (y-s) f_{yy}(s,t) ds$. Then the anti-symmetry of w^δ together with the moment condition eq. (2.14) yields

$$\mathcal{A}_\delta(u(x_j,t)) = f_y(x_j,t) + \int_{x_j-\delta}^{x_j+\delta} R_1(y,t) w^\delta(y - x_j) dy.$$

On the other hand, we can apply Taylor series expansion and the identities eq. (2.16) to obtain

$$\mathcal{A}_\delta^h(u(x_j,t)) = f_y(x_j,t) + \frac{1}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} R_1(x_{j+k},t) W_{j,j+k}.$$

We then have

$$
\begin{aligned}
&|\mathcal{A}_\delta(u(x_j,t)) - \mathcal{A}_\delta^h(u(x_j,t))| \\
&= \frac{1}{N_{j,\delta}} \left| \sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - N_{j,\delta} \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy \right| \\
&\leq \left| \sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy \right| + \dots \\
&\quad O\left(\frac{\bar{h}}{\delta}\right) \left| \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy \right|
\end{aligned}
$$

where the last inequality is due to $N_{j,\delta} \geq 1$ and $N_{j,\delta} = 1 + O\left(\frac{\bar{h}}{\delta}\right)$ in lemma 4. But $|R_1(y)| \leq C\delta^2$ together with eq. (2.15) implies

$$
\begin{aligned}
&|\mathcal{A}_\delta(u(x_j,t)) - \mathcal{A}_\delta^h(u(x_j,t))| \\
&\leq \left| \sum_{k=-l_j}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy \right| + O(\bar{h}).
\end{aligned}
$$

Then the first claim follows from the estimate

$$
\begin{aligned}
&\left|\sum_{k=1}^{r_j} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j}^{x_j+\delta} R_1(y,t)w^\delta(y-x_j)dy\right| \\
&= \sum_{k=1}^{r_j}\left|\int_{x_{j+k-1}}^{x_{j+k}+1_{k=r_j}(x_j+\delta-x_{j+k})} (R_1(y,t)\big|_{y=x_{j+k}} - R_1(y,t))w^\delta(y-x_j)dy\right| \\
&\leq \sum_{k=1}^{r_j}\int_{x_{j+k-1}}^{x_{j+k}+1_{k=r_j}(x_j+\delta-x_{j+k})}\left(\underbrace{\left|\int_{x_j}^{y}(x_{j+k}-y)f_{yy}(s,t)ds\right|}_{O(\delta\bar{h})}+\dots\right. \\
&\left.\underbrace{\left|\int_{y}^{x_{j+k}}(x_{j+k}-s)f_{yy}(s,t)ds\right|}_{O(\bar{h}^2)}\right)|w^\delta(y-x_j)|dy \leq O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right)
\end{aligned}
$$

where eq. (2.15) is used in the last inequality, and the analogous estimate

$$
\begin{aligned}
&\left|\sum_{k=-l_j}^{-1} R_1(x_{j+k},t)W_{j,j+k} - \int_{x_j-\delta}^{x_j} R_1(y,t)w^\delta(y-x_j)dy\right| \\
&\leq \sum_{k=-l_j}^{-1}\left|\int_{x_{j+k}+1_{k=-l_j}(x_j-\delta-x_{j+k})}^{x_{j+k+1}} (R_1(y,t)\big|_{y=x_{j+k}} - R_1(y,t))w^\delta(y-x_j)dy\right| \\
&\leq O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right).
\end{aligned}
$$

The second estimate follows analogously from Taylor series expansion of $u(y,t)$ up to third order in y. □

So far as the stability of the scheme eq. (2.18) is concerned, our choice of non-local viscosity term ensures sufficient numerical viscosity to show the following L^∞ stability result:

Lemma 6. *Assume c, c' are bounded. If*

$$
\frac{2\Delta t\mu}{\underline{h}} \leq 1,
$$

then $\|u^{n+1}\|_{L^\infty(\mathbb{R})} \le (1+\Delta t\|c'\|_{L^\infty(\mathbb{R})})\|u^n\|_{L^\infty(\mathbb{R})}, \quad \forall n \ge 0.$

Proof. The scheme can be rewritten as

$$
\begin{aligned}
&u_j^{n+1}\\
&=\left(1-\frac{\Delta t}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}\left(c_{\frac{j,j+k}{2}}+\frac{\max\{\delta, x_j-x_{j-l_j}, x_{j+r_j}-x_j\}\mu}{x_{j+k}-x_j}\right)W_{j,j+k}\right)u_j^n+\\
&\quad\left(-\frac{\Delta t}{N_{j,\delta}}\sum_{k=-l_j}^{r_j}\left(c_{\frac{j,j+k}{2}}-\frac{\max\{\delta, x_j-x_{j-l_j}, x_{j+r_j}-x_j\}\mu}{x_{j+k}-x_j}\right)W_{j,j+k}\right)u_{j+k}^n.
\end{aligned}
$$

Then since

$$
\frac{\sum_{k=-l_j}^{r_j}|W_{j,j+k}|}{N_{j,j+k}} \le \frac{1}{\underline{h}}\frac{\sum_{k=-l_j}^{r_j}(x_{j+k}-x_j)W_{j,j+k}}{N_{j,j+k}} = \frac{1}{\underline{h}}
$$

and

$$
\begin{aligned}
\left|\frac{\sum_{k=-l_j}^{r_j} c_{\frac{j,j+k}{2}}W_{j,j+k}}{N_{j,k}}\right| &= \left|\frac{\sum_{k=-l_j}^{r_j}(c_{\frac{j,j+k}{2}}-c_j)W_{j,j+k}}{N_{j,k}}\right|\\
&\le \frac{\frac{1}{2}\|c'\|_{L^\infty(\mathbb{R})}\sum_{k=-l_j}^{r_j}(x_{j+k}-x_j)W_{j,j+k}}{N_{j,k}} = \frac{\|c'\|_{L^\infty(\mathbb{R})}}{2},
\end{aligned}
$$

the claim follows from the positivity of the coefficients of u_j^n and u_{j+k}^n. □

Convergence to the local advection equation eq. (2.1) We have the CFL stability condition in lemma 6 that is independent of δ, so it only remains to show consistency:

Lemma 7. *If* $u, u_x, u_{xx}, u_{xxx}, c, c', c''$ *are uniformly bounded, then*

$$
\sup_j |\mathcal{A}_\delta^h(u(x_j,t)) - (c(x)u(x,t^n))_x\big|_{x=x_j}| = O(\max(\bar{h},\delta)),
$$

$$
\sup_j |\max\{\delta, x_j-x_{j-l_j}, x_{j+r_j}-x_j\}\mathcal{D}_\delta^h(u(x_j,t))| = O(\max(\bar{h},\delta)).
$$

Proof. The results follow from Taylor series expansions, and the identities eq. (2.16) and the estimate eq. (2.17) in lemma 4. □

In summary, we deduce

Theorem 4. *Let $T > 0$ be a fixed terminal time. Assume c, c', c'' are bounded. Let U_j^n denote the numerical solution given by the scheme eq.* (2.18) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition of lemma 6. Let $u(x,t), u^\delta(x,t)$ denote the solutions of eq.* (2.13) *and eq.* (2.1)*, respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$. Then,*

1. *if $u^\delta, u^\delta_x, u^\delta_{xx}, u^\delta_{xxx}, u^\delta_{tt} \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u^\delta(x_j, T) - U_j^N| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right) + O\left(\frac{\bar{h}^3}{\delta^2}\right) + O(\Delta t)$$

2. *if $u, u_x, u_{xx}, u_{xxx}, u_{tt} \in L^\infty(\mathbb{R} \times [0,T])$, then*

$$\sup_j |u(x_j, T) - U_j^N| = O(\max(\bar{h}, \delta)) + O(\Delta t).$$

Proof. The proof follows from the Lax Equivalence Theorem. □

2.2.4 A moving particle approximation

It is now straightforward to consider the case of moving particles that advect according to the velocity field $b(x)$. If we recall the PDE in Lagrangian frame eq. (2.8), we see that the corresponding nonlocal Model II is given by

$$\frac{d}{dt}u(x(t),t) + b'(x(t))u(x(t),t) + \dots$$

$$\int_{\mathbb{R}} (c-b)\left(\frac{x(t)+y}{2}\right)(u(x(t),t) + u(y,t))w^\delta(y - x(t))dy - \dots \tag{2.19}$$

$$\mu \int_{\mathbb{R}} (u(y,t) - u(x(t),t))\frac{w^\delta(y - x(t))}{y - x(t)}dy = 0 \tag{2.20}$$

which in turn leads to the following scheme:

$$u_j^{n+1} = u_j^{n+1} - \Delta t(b'_{j,n}u_j^n + \mathcal{A}^h_{\delta,b(\cdot)}(u_j^n) - \dots$$
$$\mu \max\{\delta, (x_{j,n} - x_{j-l_j,n}), (x_{j+r_j,n} - x_{j,n})\}\mathcal{D}^h_\delta(u_j^n)) \tag{2.21}$$

where $\mathcal{A}^h_{\delta,b(\cdot)}$ is the modification of $\mathcal{A}^h_\delta$ obtained by replacing c with $c - b$, $N_{j,\delta}$ within $\mathcal{A}^h_{\delta,b(\cdot)}$ and $\mathcal{D}^h_\delta$ is evaluated using $\{x_{i,n}\}_{i\in\mathbb{Z}} = \{x_i(\Delta tn)\}_{i\in\mathbb{Z}}$, and

$$\mu \geq \|c - b\|_{L^\infty(\mathbb{R})}.$$

As in the case of Model I, we take initial distribution of particles to be uniform with a particle spacing of size h, and velocity b to be smooth such that

$$C_1(T)h \leq |x_{j+1} - x_j| \leq C_2(T)h$$

for all $j \in \mathbb{Z}$ and $t \in [0, T]$. Then one can show:

Theorem 5. *Let $T > 0$ be a fixed terminal time. Assume c, c', c'', b, b', b'' are bounded. Let U_j^n denote the numerical solution given by the scheme eq.* (2.21) *at $x = x_j$ and $t = n\Delta t$, $n \in \mathbb{N}$, where Δt is chosen to satisfy the CFL condition*

$$\frac{2\Delta t\mu}{C_1(T)h} \leq 1.$$

Let $u(x,t)$, $u^\delta(x,t)$ denote the solutions of eq. (2.8) *and eq.* (2.19)*, respectively, and $N \in \mathbb{N}$ so that $N\Delta t = T$.*

1. If $u^\delta, u^\delta_x, u^\delta_{xx}, u^\delta_{xxx}, u^\delta_{tt} \in L^\infty(\mathbb{R}\times[0,T])$, then

$$\sup_j |u^\delta(x_j, T) - U_j^N| = O(\bar{h}) + O\left(\frac{\bar{h}^2}{\delta}\right) + O\left(\frac{\bar{h}^3}{\delta^2}\right) + O(\Delta t)$$

2. *If* $u, u_x, u_{xx}, u_{xxx}, u_{tt} \in L^\infty(\mathbb{R} \times [0, T])$*, then*

$$\sup_j |u(x_j, T) - U_j^N| = O(\max(\bar{h}, \delta))) + O(\Delta t).$$

Proof. The arguments are analogous to the case of stationary particles in the section 2.2.3. □

Following our approach in the case of Model I, we consider the fully nonlocalized equation

$$\begin{aligned} \frac{d}{dt}u(x(t), t) + 2 \left(\int_{\mathbb{R}} b\left(\frac{x(t) + y}{2}\right) w^\delta(y - x(t)) dy \right) u(x(t), t) + \dots \\ \int_{\mathbb{R}} (c - b)\left(\frac{x(t) + y}{2}\right) (u(x(t), t) + u(y, t)) w^\delta(y - x(t)) dy - \dots \\ \delta\mu \int_{\mathbb{R}} (u(y, t) - u(x(t), t)) \frac{w^\delta(y - x(t))}{y - x(t)} dy = 0 \end{aligned} \tag{2.22}$$

which in turn leads to the following scheme:

$$\begin{aligned} u_j^{n+1} = u_j^{n+1} - \Delta t(\mathcal{B}_\delta^h(u_j^n) + \mathcal{A}_{\delta, b(\cdot)}^h(u_j^n) - \dots \\ \mu \max\{\delta, (x_{j,n} - x_{j-l_j,n}), (x_{j+r_j,n} - x_{j,n})\} \mathcal{D}_\delta^h(u_j^n)) \end{aligned} \tag{2.23}$$

where

$$\mathcal{B}_\delta^h(u_j^n) = \left(\frac{2}{N_{j,\delta}} \sum_{k=-l_j}^{r_j} b\left(\frac{x_{j,n} + x_{j+k,n}}{2}\right) W_{j,j+k} \right) u_j^n.$$

Then it can be shown:

Theorem 6. *Suppose the same assumptions are made as in theorem 5. Then with* u *and* u^δ *now denoting the solutions of eq.* (2.8) *and eq.* (2.22)*, respectively, the same conclusions as in theorem 5 hold.*

Proof. The proof is exactly analogous to that of theorem 5. □

2.3 Numerical experiments

We present some numerical tests to demonstrate the convergence of our particle schemes. We take the interval $[0, 2\pi]$ as our computational domain and impose periodic boundary conditions. The underlying velocity field is prescribed as $c(x) = \frac{1}{3}\cos(2x)$ and the particle velocity is taken to be $b(x) = \sin(x)$ so that the trajectories of the particles are given analytically by

$$x(t) = 2\cot^{-1}(\exp(c - t)) \quad \text{where} \quad c = \log\left(\frac{\cot(x(0))}{2}\right).$$

The terminal time is chosen to be $T = 1$ and the initial data is $u_0(x) = \cos(x)$. By the method of manufactured solutions we take the exact solution to be $u(x,t) = \cos((x - t))$ and solve the corresponding inhomogeneous problems. The non-local integrals in the resulting inhomgeneous terms are evaluated using MATLAB built-in integration routine. In all our numerical experiments, numerical solution errors are measured in L^∞ norm in accordance with our theoretical analysis. The initial distribution of the particles is chosen to be uniform with the discretization parameter $h = \frac{2\pi}{N}$, where N denotes the number of particles. In order to satisfy the CFL conditions, we pick $\Delta t = \frac{h}{4}$ and $\Delta t = \frac{h}{8}$ for all our numerical experiments on Model I and II, respectively when validating each model individually. On the other hand, we choose the smaller time step size, i.e. $\Delta t = \frac{h}{8}$ for both models when comparing them side by side. Our particular choices of time step sizes are immaterial to demonstrate the theoretically predicted convergence rates of the schemes so far as the step sizes are chosen to meet the CFL conditions. All the convergence results are presented with respect to the spatial parameters δ and h only, which has been the focus of our theoretical analysis thus far presented.

2.3.1 Tests with Model I

We take the singular kernel $\eta^\delta(z) = \frac{1_{(-\delta,\delta)}}{\delta z}$ and compute the numerical solutions using both schemes eq. (2.10) and eq. (2.12) (referred to as the Scheme 1 and the Scheme 2 in this subsection for short).

Example 1: Fixed horizon We fix the horizon length $\delta = 0.2$ and study the convergence of the numerical solutions to the exact solution as $h \to 0$. In particular, we take the mesh refinement path $h = \frac{2\pi}{160}, \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}$. For both Schemes 1 and 2, the convergence rates are expected to be first order in h which is experimentally validated in Figure 2.1.

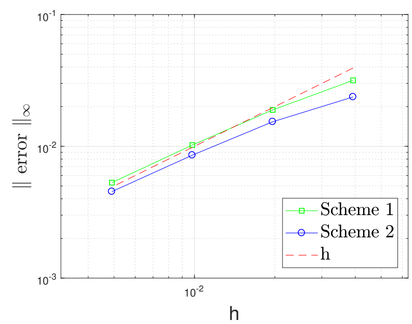


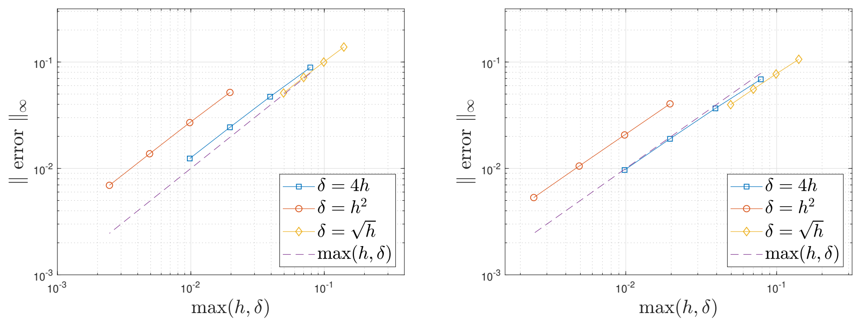
Figure 2.1: Convergence of the numerical solution errors in the L^∞ norm

Example 2: Asymptotic compatibility We consider the three parameter refinement paths (i) $(\delta, h) = (4h, h)$, (ii) $(\delta, h) = (\sqrt{h}, h)$ and (iii) $(\delta, h) = (h^2, h)$ to illustrate the convergence in the cases $\delta = O(h), h = o(\delta)$ and $\delta = o(h)$, respectively. We refine $h \to 0$ along $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$. Our theoretical analysis predicts the convergence rates to be the first order in $\max\{h, \delta\} = \delta, \delta$ and h along the refinement paths (i),(ii) and (iii), respectively, which agrees with our simulation results in Figure 2.2.

2.3.2 Tests with Model II

We take $w^\delta(z) = -\frac{d}{dz}\left(\frac{1}{\delta}\rho(\frac{z}{\delta})\right)$ where $\rho(z)$ is the classical B-spline kernel [7] defined as

$$\rho(z) = \frac{2}{3}\begin{cases} 1 - \frac{3}{2}z^2 + \frac{3}{4}z^3 & \text{if } 0 \le |z| \le 1 \\ \frac{1}{4}(2-z)^3 & \text{if } 1 \le |z| \le 2 \\ 0 & \text{otherwise} \end{cases}$$



(a) Scheme 1 (b) Scheme 2

Figure 2.2: Asymptotic convergence of the numerical solution errors in the L^∞ norm. Each colored solid line should be viewed one at a time since the horizontal scales $\max(h, \delta)$ are different for the different parameter refinement paths. The dotted lines in purple are the comparison lines only for the case (i). The comparison lines for the other cases are not shown.

and compute the numerical solutions using both schemes eq. (2.21) and eq. (2.23) (referred to as the Scheme 1 and the Scheme 2 in this subsection for short). We take $\mu = \|c - b\|_{L^\infty(\mathbb{R})}$. Note the support of w^δ is in fact $(-2\delta, 2\delta)$, which we will take into account in our numerical experiments in this subsection by letting $\bar{\delta} = 2\delta$.

Example 1: Fixed horizon We fix the horizon $\bar{\delta} = 0.2$ and take the refinement path $h = \frac{2\pi}{160}, \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}$ as in the Model I. For both Schemes 1 and 2, the convergence rates are expected to be first order in h, which we can observe in Figure 2.3.

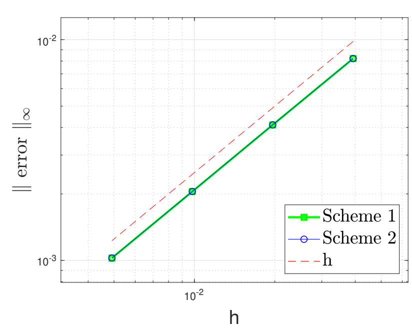
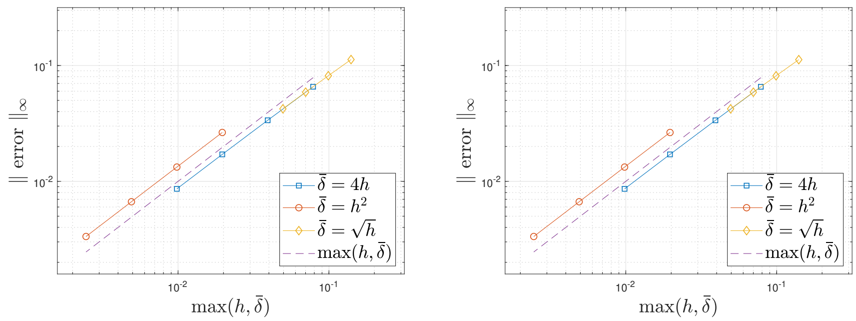


Figure 2.3: Convergence of the numerical solution errors in the L^∞ norm

Example 2: Asymptotic compatibility We consider the three parameter refinement paths (i) $(\bar{\delta}, h) = (4h, h)$, (ii) $(\bar{\delta}, h) = (\sqrt{h}, h)$ and (iii) $(\bar{\delta}, h) = (h^2, h)$ along $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$.

Figure 2.4 verifies our theoretical prediction of the first order convergence rate in $\max\{h, \bar{\delta}\} = \bar{\delta}, \bar{\delta}$ and h along the refinement paths (i),(ii),(iii), respectively.



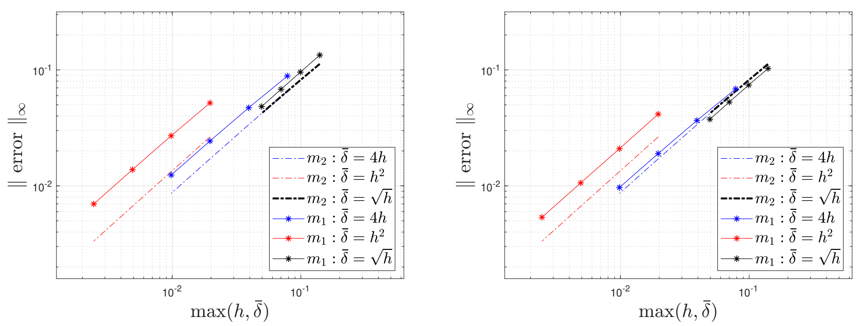
(a) Scheme 1 (b) Scheme 2

Figure 2.4: Asymptotic convergence of the numerical solution errors in the L^∞ norm. Each colored solid line should be viewed one at a time since the horizontal scales $\max(h, \delta)$ are different for the different parameter refinement paths. The dotted lines in purple are the comparison lines only for the case (i). The comparison lines for the other cases are not shown.

2.3.3 Comparisons and discussions

Beyond validation of our theoretical analysis, it is of practical interest to conduct additional *empirical* investigation of our numerical schemes. After all it remains to answer important and practical questions such as which of our particle methods one should use. We focus in particular on comparing the accuracy of the particle methods for the Model I and the Model II, thereby providing some insights into gauging practical effectiveness of the methods. We choose the kernel $w^\delta(z)$ to be $-2\frac{d}{dz}\left(\frac{1}{\delta}\rho(\frac{z}{\delta})\right)$ for the Model I where ρ is the B-spline kernel. Note that this choice of kernel satisfies the moment condition eq. (2.4). We then perform our numerical experiments for the Model I with $h = \frac{2\pi}{320}, \frac{2\pi}{640}, \frac{2\pi}{1280}, \frac{2\pi}{2560}$ along (i) $(\bar{\delta}, h) = (4h, h)$, (ii) $(\bar{\delta}, h) = (\sqrt{h}, h)$ and (iii) $(\bar{\delta}, h) = (h^2, h)$. The numerical results thus obtained are compared with the corresponding results for the Model II obtained in the previous subsection.

It can be seen from Figure 2.5 that the particle methods for the Model II produce numerical solutions that are more accurate than those obtained by the methods for the Model I except in the regime $\delta = \sqrt{h}$ for the fully nonlocalized schemes. One heuristic explanation of the better accuracy of the former might be that their truncation errors contain approximation of the vanishing second moment of the anti-symmetric kernel whereas those for the latter methods involve the *non-vanishing* second moment of the upwinding kernel. On the other hand, the poorer accuracy of the scheme eq. (2.23) relative to the scheme eq. (2.12) when $\delta \gg h$ could be attributed to the pronounced influence of the nonlocal viscosity term in the Model II which is taken to be proportional to δ. This in turn alludes to the importance of non-local continuum models in designing effective numerical methods for the corresponding local continuum models.



(a) Schemes eq. (2.10) and eq. (2.21) (b) Schemes eq. (2.12) and eq. (2.23)

Figure 2.5: Comparison of numerical solution errors in L^∞ norm: m_1 and m_2 denote the Model I and II, respectively.

2.4 Discussion

This chapter presents two nonlocal advection models and their asymptotically compatible particle discretizations. In short both models are equipped with stabilizers on the continuum level: one via use of biased kernels and the other via nonlocal diffusion. This is in sharp contrast to the classical SPH in which a simple minded discretization of the continuum model is unstable unless numerical/artificial viscosity is introduced at the discretization level. We reiterate that these built-in stability features are reminiscent of the corresponding ideas well known in the setting of numerical methods for PDE, namely upwinding and vanishing artificial viscosity.

At the discrete level we can also delineate some differences between the Model I and the Model II. On one hand, our particle methods for the Model I achieve consistency with the local PDE by ensuring that the discrete flux between each pair of particles is captured at the correct spatial scale. On the other hand, our particle discretization of the Model II assures that the sum of all pairwise non-local fluxes between a particle and all its neighbors is normalized to the desired local flux value. More on the practical side, we see that the particle methods for the Model I admit singular kernels as well as integrable kernels, hence possibly offering a greater flexibility in numerical simulations than the methods for the Model II which rely on integrable kernels. The latter methods however are simpler in the sense that they do not require any information of the upwinding directions, which in the case of the former methods is necessary for each moving particle at each instance of time.

Despite the robustness of our particle methods, we point out that they are all limited in terms of their accuracy being only first order accurate. Moreover our particle methods fail to be conservative schemes, hence possibly limited in their applicability to nonlinear problems. On a related note, we remark that the results in this chapter are limited to scalar one dimensional linear problems, though nonlocal formulations of the Stokes systems in multi-dimensional spaces have also been considered recently [37] and there are natural connections with earlier studies on numerical discretizations of other nonlocal models such as nonlocal diffusion and peridynamics [23, 105]. Indeed the practicality of SPH-like particle methods makes it imperative to extend our current work to the simplest multidimensional setting of two dimensional linear problems, for which some speculations can be offered. As far as the stability is concerned, one may conceive a two dimensional generalization of the one dimensional upwinding kernel so that the domain of nonlocal interactions is limited to a semi-disc depending on the direction of the velocity field. However the generic difficulty of two dimensional interpolation on nonuniformly distributed data [113] poses a significant challenge to determination of suitable quadrature weights to ensure the AC property. The special case of no neighboring particle within the horizon could potentially be treated with a rather simple idea of using the closest particle and seeking appropriate interpolation schemes on just two data points. The more likely situation where there are some neighboring particles present is yet to be further investigated, possibly in the similar spirits as in the recent works [108, 35]. In addition, issues related to model nonlinearity and effects of physical boundary are also important topics to be further explored.

We have conveyed in this chapter the importance of continuum nonlocal models as a bridge between continuum local PDE models and their numerical approximations. This is the rationale that underlies our studies in chapters 3 and 4. After all the nonlocal models are based on nonlocal integro-differential operators, hence it is of great importance to better understand their mathematical properties which can be translated into the discrete analogues. One such example is the coercivity of nonlocal gradient operators, which we can guarantee by breaking the radial symmetries of the operators in chapter 3. In the meantime our analysis of the continuum nonlocal models in chapter 4 provides us with the assurance that no loss of order of accuracy is incurred at least on the continuum nonlocal level due to the presence of boundaries, which in turn can offer us a possible route to design accurate numerical discretizations.

Chapter 3: Nonlocal gradient operators with hemispherical nonlocal interaction neighborhoods

As we have alluded in chapter 1 what motivates our study in this chapter is the stability of nonlocal systems associated with nonlocal gradient operators as discussed in the context of correspondence theory of peridynamics [36] and as utilized in the setting of fluid dynamics [37]. Du and Tian [36, 37] have established the stability provided that nonlocal interaction kernels are radially symmetric and have suitably strong singularity of fractional type at the center of the nonlocal interaction neighborhood. We extend their analysis by proving that the use of nonspherical interaction neighborhood (or non-radially defined nonlocal interaction kernel) allows the assumptions on singular interaction at the center to be removed when establishing the coercivity of the nonlocal Dirichlet energies. The idea of breaking the radial symmetry in the nonlocal interaction contributes to the second motivation of our work in connection to the study of such nonlocal operators for nonlocal convections [103] and nonlocal in time dynamic processes [40]. The former is important to preserve the upwinding feature of the physical transport process while the latter reflects the time-irreversibility.

Existing studies in the literature have demonstrated successful applications of nonlocal operators with non-radially symmetric kernels to nonlocal modeling. In [40], the well-posedness and localization of nonlocal in time parabolic equation is established for a wide class of kernels when the support of the kernels for nonlocal-in-time derivative operators are truncated to yield one-sided backward in time nonlocal derivative operators. In study of compact embeddings of nonlocal function spaces of L^p vector fields [33], it is shown that the monotonicity assumption of the kernels for the nonlocal diffusion operators can be relaxed if they are non-radially symmetric and remain non-negative on a conic region. In [46], the unique solvability is examined for nonlocal diffusion operators with kernels that may not have any symmetry at all. Our main contribution is to establish the coercivity result on the correspondence nonlocal Dirichlet energies provided that, for nonlocal gradient operators used, one starts with radially symmetric kernels and breaks the symmetry by truncating the support of the kernels via multiplication by characteristic functions associated with half-spheres.

The main contribution of this chapter is that it brings theoretical underpinnings of the nonlocal gradient operators to a closer alignment with the SPH settings where the nonlocal kernels are almost always taken to be nonsingular. It can serve as a starting point to address the numerical instabilities of SPH discretizations of the composition of nonlocal divergence and gradient operators, just as we have revisited the long standing issue of asymptotic compatibility using the continuum nonlocal models in chapter 2. Another contribution is the systematic emergence of our nonlocal systems such as the nonlocal Stokes model which distinguish from the scalar settings that are dominant in the rest of this thesis. The role of the nonlocal gradient operators as a building block can also potentially lead to modeling capabilities of nonlocal approximations of nonlinearity.

3.1 Nonlocal nonsymmetric gradient operators

A general form of the nonlocal gradient operator $\mathcal{G}_\delta$ studied in earlier works [27, 30, 34, 77] is given by

$$\mathcal{G}_\delta u(x) = 2\int_{\mathbb{R}^d} \underline{\omega}_\delta(y-x)(y-x)\otimes\frac{u(y)-u(x)}{|y-x|}dy \tag{3.1}$$

for a suitably defined function $u = u(x) : \mathbb{R}^d \to \mathbb{R}^1$ or $\mathbb{R}^d$, and a nonlocal interaction kernel $\underline{\omega}_\delta = \underline{\omega}_\delta(z)$. A common case for the latter is given by a radially symmetric kernel. In this work, we define a nonlocal gradient operator $\mathcal{G}_\delta^{\vec{n}}$ acting on u as

$$\mathcal{G}_\delta^{\vec{n}} u(x) = 2\int_{\mathbb{R}^d} \chi_{\vec{n}}(y-x)w_\delta(|y-x|)(y-x)\otimes\frac{u(y)-u(x)}{|y-x|}dy \tag{3.2}$$

where $\chi_{\vec{n}}(z)$ denotes the characteristic function of the half-space $\mathcal{H}_{\vec{n}} = \{\vec{z} \in \mathbb{R}^d : \vec{z}\cdot\vec{n} \geq 0\}$ parameterized by the unit vector $\vec{n}$. Here, the scalar-valued nonnegative function w_δ is a radially symmetric nonlocal kernel that measures the strength of the nonlocal interaction. We consider in particular a scaled kernel of the form $w_\delta(|x|) = \frac{1}{\delta^{d+1}} w(\frac{|x|}{\delta})$ where w is nonnegative and compactly supported on the unit ball with a bounded first moment

$$\int_{\mathbb{R}^d} w(|x|)|x|dx = \int_{|x|\leq 1} w(|x|)|x|dx = d. \tag{3.3}$$

Corresponding to (3.1), we have $\underline{\omega}_\delta(z) = \chi_{\vec{n}}(z) w_\delta(|z|)$, which is no longer radially symmetric. The support of w_δ is given by the ball of radius δ, which is called a nonlocal horizon or smoothing length depending on different contexts [95, 37]. Effectively, the nonlocal interaction neighborhood (the domain of integration) in (3.2) is given by the support of $\underline{\omega}_\delta$ inside a half sphere defined by $\vec{n}$ with radius δ. The operators $\mathcal{G}_\delta^{\vec{n}}$ are d-dimensional versions of the following one dimensional one-sided nonlocal gradient operators [34, 37, 40]

$$\mathcal{G}_\delta^{\pm} u(x) = \pm 2 \int_0^\delta \omega_\delta(s)(u(x \pm s)) - u(x))ds. \tag{3.4}$$

Nonlocal derivative operators with a nonradial interaction kernel or a nonspherical neighborhood have also been used to study spatially inhomogeneous nonlocal convection in multidimensional cases [103]. However, systematic studies remain limited. We thus present some further analysis in this section.

3.1.1 Consistency for linear functions

Let us first note that the operators $\mathcal{G}_\delta^{\vec{n}}$ coincide with their local counterparts for linear functions.

Lemma 8. *For every unit vector $\vec{n}$ in $\mathbb{R}^d$ and every affine function $u : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ where $\tilde{d} = 1$ or d,*

$$\mathcal{G}_\delta^{\vec{n}}(u)(x) = \nabla u(x) \quad \forall x \in \mathbb{R}^d.$$

Proof. We assume without loss of generality $\tilde{d} = d$ and $x = 0$. Let us write $u(x) = Ax + b$ for some $d \times d$ real-valued matrix A and $b \in \mathbb{R}^d$. If R is a rotation matrix that aligns $\vec{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$

with $\vec{n}$, then we have

$$\begin{aligned}\mathcal{G}_\delta^{\vec{n}}(u)(0) &= 2\int_{\mathcal{H}_{\vec{n}}} w_\delta(|y|) y \otimes \frac{Ay}{|y|} dy = 2\int_{\mathcal{H}_{\vec{e}_1}} w_\delta(|z|) Rz \otimes \frac{ARz}{|z|} dz \\ &= R\left(2\int_{\mathcal{H}_{\vec{e}_1}} w_\delta(|z|) z \otimes \frac{z}{|z|} dz\right) R^T A^T = R I_d R^T A^T = A^T = \nabla u(0)\end{aligned}$$

where T, I_d denote the transpose operator and the $d \times d$ identity matrix, respectively, and the moment condition (3.3) is used in the third to the last equality. □

3.1.2 Adjoint operator

In analogy with the local setting, the definition of the operator $\mathcal{G}_\delta^{\vec{n}}$ leads to the consideration of its adjoint nonlocal divergence operator $\mathcal{D}_\delta^{\vec{n}}$. That is, we define the operator $\mathcal{D}^{\vec{n}}$ by

$$\int_{\mathbb{R}^d} u(x) \cdot \mathcal{G}_\delta^{\vec{n}} v(x) dx = -\int_{\mathbb{R}^d} \mathcal{D}_\delta^{\vec{n}} u(x) v(x) dx \tag{3.5}$$

for suitable functions u and v such that both sides of the equality make sense. In explicit terms, the operator $\mathcal{D}_\delta^{\vec{n}}$ takes the form

$$\mathcal{D}_\delta^{\vec{n}} u(x) = 2\int_{\mathbb{R}^d} \chi_{\vec{n}}(s) \frac{w^\delta(|s|)}{|s|} s \cdot (u(x) - u(x-s)) ds.$$

Note that in an analogy to the local diffusion operator $\Delta = \text{div grad}$, we may also define the associated nonlocal diffusion operator $\mathcal{L}_\delta^{\vec{n}} = \mathcal{D}_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}}$ where $\circ$ denotes the composition. More discussions on $\mathcal{L}_\delta^{\vec{n}}$ are given later.

3.1.3 Representation in Fourier space

For simplicity and definiteness, we assume from here on the two dimensional setting $d = 2$ unless otherwise noted. Moreover we focus only on the action of the nonlocal operators on functions that are periodic on the domain $\Omega = (-\pi, \pi)^2$. We can then exploit the periodicity to examine the Fourier symbols of the nonlocal operators introduced thus far. In particular, for any periodic function $u(x)$ on Ω, we let $\widehat{u} = \widehat{u}(\xi)$ denote its Fourier coefficients, hence the Fourier expansion

$$u(x) = \sum_{\xi \in \mathbb{Z}^2} \widehat{u}(\xi) e^{i\xi \cdot x}.$$

The loss of symmetry in the integration domain in the definitions of the nonlocal operators, in contrast to the symmetric case studied in [37], is manifested in terms of the real parts of the Fourier symbols.

Lemma 9. *For locally integrable, periodic u, v, w where $u, w : \Omega \to \mathbb{R}^1$ or $\mathbb{R}^2$ and $v : \Omega \to \mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$, the Fourier symbols of the operators $\mathcal{G}_\delta^{\vec{n}}, \mathcal{D}_\delta^{\vec{n}}, \mathcal{L}_\delta^{\vec{n}}$ are given by*

$$\begin{aligned}
\widehat{\mathcal{G}_\delta^{\vec{n}} u}(\xi) &= \lambda_\delta^{\vec{n}}(\xi)(\widehat{u}(\xi))^T \\
\widehat{\mathcal{D}_\delta^{\vec{n}} v}(\xi) &= -\overline{\lambda_\delta^{\vec{n}}(\xi)}^T \widehat{v}(\xi) = \lambda_\delta^{-\vec{n}}(\xi)^T \widehat{v}(\xi) \\
\widehat{\mathcal{L}_\delta^{\vec{n}} w}(\xi) &= -|\lambda_\delta^{\vec{n}}(\xi)|^2 \widehat{w}(\xi)
\end{aligned}$$

where $\xi \in \mathbb{Z}^2$ and

$$\begin{aligned}
\Re(\lambda_\delta^{\vec{n}}(\xi)) &= 2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}(\cos(\xi \cdot s) - 1)ds \\
\Im(\lambda_\delta^{\vec{n}}(\xi)) &= 2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|} \sin(\xi \cdot s)ds.
\end{aligned}$$

The above results are immediate from the definitions of the operators. It is natural to compare $\lambda_\delta^{\vec{n}}(\xi)$ to the Fourier symbol of the local gradient operator given by $i\xi$ to relate the imaginary part of $\lambda_\delta^{\vec{n}}(\xi)$ to its local counterpart. The former is shown to be a scalar multiple of the latter independently of $\vec{n}$ due to some symmetry of the integrand on half-spheres.

Lemma 10. *For each $\vec{n}$ and $\xi \in \mathbb{Z}^2 \backslash \{0\}$, the Fourier symbol $\lambda_\delta^{\vec{n}}(\xi)$ in Lemma 9 can be expressed as*

$$\lambda_\delta^{\vec{n}}(\xi) = i\Lambda_\delta(|\xi|)\frac{\xi}{|\xi|} + \Re(\lambda_\delta^{\vec{n}}(\xi)) \tag{3.6}$$

where

$$\Lambda_\delta(|\xi|) = 4\int_{s_1\geq 0, s_2\geq 0} \frac{w_\delta(|s|)s_1}{|s|}\sin(|\xi|s_1)ds. \tag{3.7}$$

On the other hand, $\Re(\lambda_\delta^{\vec{n}}(\xi))$ *is a scalar multiple of* $\frac{\xi}{|\xi|}$ *if and only if* $\vec{n}$ *is a scalar multiple of* ξ.

Proof. We observe that

$$\Im(\lambda_\delta^{\vec{n}}(\xi)) = 2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}\sin(\xi\cdot s)ds = \int_{B_\delta(0)} \frac{w_\delta(|s|)s}{|s|}\sin(\xi\cdot s)ds$$

where the equality is due to the odd symmetry of $s\sin(\xi\cdot s)$. Then the first claim follows from Lemma 3 in [37]. Next we consider the real part of $\lambda_\delta^{\vec{n}}(\xi)$. If we let $\xi^\perp$ denote a vector orthogonal to ξ, then it follows that

$$|\xi^\perp\cdot\Re(\lambda_\delta^{\vec{n}}(\xi)| = 2\int_{\mathcal{I}_{\vec{n},\xi}} \frac{w_\delta(|s|)|\xi^\perp\cdot s|}{|s|}(1-\cos(\xi\cdot s))ds$$

where $\mathcal{I}_{\vec{n},\xi} = \{s\in\mathcal{H}_{\vec{n}} : s - 2(s\cdot\xi)\xi\in\mathcal{H}_{\vec{n}}\}$, thus the second claim holds since $|\mathcal{I}_{\vec{n},\xi}| = 0$ precisely when $\vec{n}$ is a scalar multiple of ξ. □

3.1.4 Spectral estimates

We now present a key theorem concerning the spectral property of the nonlocal gradient operator $\mathcal{G}_\delta^{\vec{n}}$. The theorem implies, in particular the strong coercivity, uniformly in δ and $\vec{n}$, of the Dirichlet energies given by $\|\mathcal{G}_\delta^{\vec{n}}(\cdot)\|_2^2$ with respect to the norm $\|\cdot\|_2 + \|\mathcal{G}_\delta^{\vec{n}}(\cdot)\|_2$.

Theorem 7. *There exists a positive constant* C *independent of* $\vec{n}$, ξ *and* δ *(as* $\delta\to 0$*) such that*

$$C \leq |\lambda_\delta^{\vec{n}}(\xi)| \leq 2\sqrt{2}|\xi|, \quad \forall\xi\in\mathbb{Z}^2/\{0\}.$$

Proof. Let $k = \delta|\xi|$, we show the bound on $|\lambda_\delta^{\vec{n}}(\xi)|$ using two separate estimates on the imaginary and real parts of $\lambda_\delta^{\vec{n}}(\xi)$ respectively depending on $k < 1$ or $k \geq 1$.

For $k < 1$, we estimate the imaginary part. Using Lemma 10, we have

$$\Lambda_\delta(\xi) = 4\int_{s_1\geq 0, s_2\geq 0} \frac{w_\delta(|s|)s_1}{|s|}\sin(|\xi|s_1)ds = 4\int_{r=0}^{\delta}\int_{\theta=0}^{\frac{\pi}{2}} w_\delta(r)r\cos(\theta)\sin(|\xi|r\cos(\theta))drd\theta$$
$$= \frac{4}{\delta}\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r)r\cos(\theta)\sin(|\xi|\delta r\cos(\theta))drd\theta\,.$$

By the inequality $\sin(x) \geq x - x^3/6$ for $0 \leq x \leq 1$, we obtain

$$\Lambda_\delta(\xi) \geq \frac{4k}{\delta}\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r)r^2\cos^2(\theta)drd\theta - \frac{4k^3}{6\delta}\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r)r^4\cos^4(\theta)drd\theta$$
$$\geq \frac{Ck}{\delta} = C|\xi|,$$

for a constant $C > 0$, independent of ξ, δ and $\vec{n}$.

Next, for $1 \geq k$, we consider $\Re(\lambda_\delta^{\vec{n}}(\xi))$. Note that

$$|\Re(\lambda_\delta^{\vec{n}}(\xi))| \geq |\vec{n}\cdot\Re(\lambda_\delta^{\vec{n}}(\xi))| = 2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)}{|s|}(n_1s_1 + n_2s_2)(1-\cos(\xi\cdot s))ds$$
$$\geq 2\cos\left(\frac{\pi}{4}\right)\int_{H_{\vec{n},\frac{\pi}{4}}} w_\delta(|s|)(1-\cos(\xi\cdot s))ds$$

where $\mathcal{H}_{\vec{n},\frac{\pi}{4}}$ denotes the set of those points in $\mathcal{H}_{\vec{n}}$ that have angles with $\vec{n}$ between $-\pi/4$ and $\pi/4$. In terms of polar coordinates, we can then write

$$|\Re(\lambda_\delta^{\vec{n}}(\xi))| \geq C\int_{r=0}^{\delta}\int_{\theta_1}^{\theta_1+\frac{\pi}{2}} w_\delta(r)(1-\cos(|\xi|r\cos(\psi_\theta)))rdrd\theta$$
$$= \frac{C}{\delta}\underbrace{\int_{r=0}^{1}\int_{\theta_1}^{\theta_1+\frac{\pi}{2}} w(r)(1-\cos(kr|\cos(\psi_\theta)|))rdrd\theta}_{J_\delta(\xi)}$$

for some θ_1 depending on $\vec{n}$. Here $0 \leq \psi_\theta \leq \pi$ denotes the angle between the vector ξ and the vector with the polar coordinates (r,θ).

We now introduce a possible cut-off of w at the origin to get an absolutely integrable kernel ϕ, that is, ϕ is a radial function such that

$$0 \leq \phi(|x|) \leq w(|x|) \text{ and } 0 < I := \int_{\mathbb{R}^2} \phi(|x|)dx < \infty.$$

We then discuss the different cases separately. First, let us consider the case with $1 \leq k \leq \lambda$ where λ is to be specified.

If we let

$$A_\xi = \left(\theta_1, \theta_1 + \frac{\pi}{2}\right) - \left\{\theta \in (0, 2\pi) : \left|\psi_\theta - \frac{\pi}{2}\right| \leq \frac{\pi}{8}\right\},$$

then

$$\frac{\cos\left(\frac{3\pi}{8}\right)}{2\lambda} \leq kr|\cos(\psi_\theta)| \leq 1 \quad \text{for } (r,\theta) \in \left(\frac{1}{2\lambda}, \frac{1}{\lambda}\right) \times A_\xi$$

so that

$$\begin{aligned} J_\delta(\xi) &\geq \frac{1}{\delta} \int_{r=\frac{1}{2\lambda}}^{\frac{1}{\lambda}} \int_{A_\xi} w(r)r(1 - \cos(kr|\cos(\psi_\theta)|))drd\theta \\ &\geq \frac{1}{\delta}\left(1 - \cos\left(\frac{\cos\left(\frac{3\pi}{8}\right)}{2\lambda}\right)\right) \int_{r=\frac{1}{2\lambda}}^{\frac{1}{\lambda}} \int_{A_\xi} w(r)rdrd\theta \geq \frac{C}{\delta} \end{aligned}$$

where the last inequality is due to the non-degeneracy of $|A_\xi| \geq \frac{\pi}{4}$ uniformly in ξ.

Next, we consider the case $\lambda < k$. Then with the same A_ξ defined in the case (a)

$$\begin{aligned} J_\delta(\xi) &\geq \frac{1}{\delta} \int_{A_\xi} \int_{r=0}^{1} w(r)r\left(1 - \cos\left(kr|\cos(\psi_\theta)|\right)\right) drd\theta \\ &\geq \frac{1}{\delta} \int_{A_\xi} \left(\frac{I}{2\pi} - \int_{r=0}^{1} \phi(r)r\cos(kr|\cos(\psi_\theta)|)dr\right) d\theta. \end{aligned}$$

By the Riemann-Lebesgue lemma, there exists a constant $c > 0$ such that for $j > c$

$$\left|\int_{r=0}^{1} \phi(r)r\cos(jr)\,dr\right| < \frac{I}{4\pi}.$$

Then, since $\cos(\frac{3\pi}{8}) \leq |\cos(\psi_\theta)|$ for $\theta \in A_\xi$ we set $\lambda = c/\cos(\frac{3\pi}{8})$ to obtain that for some $\tilde{C}$,

$$J_\delta(\xi) \geq \frac{\tilde{C}}{\delta}.$$

In summary, we have $|\lambda_\delta^{\vec{n}}(\xi)| \geq \min\{C_1, \frac{C_2}{\delta}\}$ for positive constants C_1 and C_2.

In order to prove the uniform upper bound on $|\lambda_\delta^{\vec{n}}(\xi)|$ we observe

$$|\Re(\lambda_\delta(\xi))| = \left|2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}(\cos(\xi \cdot s) - 1)ds\right| \leq 2\int_{\mathcal{H}_{\vec{n}}} w_\delta(|s|)|\xi \cdot s|ds \leq 2|\xi|$$

and

$$|\Im(\lambda_\delta(\xi))| = \left|\int_{B_\delta(0)} \frac{w_\delta(|s|)s}{|s|}\sin(\xi \cdot s)ds\right| \leq \int_{B_\delta(0)} w_\delta(|s|)|\xi||s|ds = 2|\xi|$$

following the derivation in the proof of Lemma 10. □

3.1.5 Orientation dependence

Before we turn to applications of the nonlocal operators, we discuss an important issue that is naturally concerned with the introduction of the parameter $\vec{n}$ in our formulation of the operators. We recall that the $\vec{n}$ itself belongs to the non-trivial ambient space $S^{d-1} = \{\vec{n} \in \mathbb{R}^d : \|\vec{n}\|_2 = 1\}$ for $d \geq 2$ when the one dimensional half-spaces $(0, \infty)$ and $(-\infty, 0)$ are generalized to the higher dimensional analogues $\mathcal{H}_{\vec{n}}$. A specific choice of $\vec{n}$ leads to orientation dependence. It is a legitimate question to ask if such dependence is necessary (while maintaining a coercive Dirichret integral). One can also ask how to pick $\vec{n}$, in case that it is needed, in practice. Since the coercivity result of Theorem 7 is true for all $\vec{n}$, one possible approach is to eliminate the dependence on $\vec{n}$ by defining a new energy functional in terms of the average over $\vec{n} \in S^{d-1}$. Indeed, if we remain in the two dimensional domain $\Omega = (-\pi, \pi)^2$ as in Section 1.4 and take for concreteness a periodic scalar valued $u : \Omega \to \mathbb{R}$, we may consider the averaged nonlocal Dirichlet integral

$$\mathcal{E}_\delta^{avg}(u) = \frac{1}{2\pi}\int_{S^1}\int_\Omega |\mathcal{G}_\delta^{\vec{n}} u(x)|^2 dx dS.$$

The coercivity of $\mathcal{E}_\delta^{avg}$ is immediate from that of each Dirichlet integral associated with $\mathcal{G}_\delta^{\vec{n}}$. One can view $\mathcal{E}_\delta^{avg}$ as a stabilized symmetric nonlocal Dirichlet integral

$$\mathcal{E}_\delta^{sym}(u) = \int_\Omega |\frac{1}{2}(\mathcal{G}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}})u(x)|^2 dx$$

since

$$\mathcal{E}_\delta^{avg}(u) = \mathcal{E}_\delta^{sym}(u) + \sum_{\xi\in\mathbb{Z}^2/\{0\}} \frac{2}{\pi}\int_{B_\delta(0)}\int_{B_\delta(0)} \frac{w_\delta(|a|)w_\delta(|b|)a\cdot b}{|a||b|} \arcsin\left(\frac{a\cdot b}{|a||b|}\right)(\cos(\xi\cdot a)-1)(\cos(\xi\cdot b)-1)dadb.$$

Unfortunately, we are unable to express $\mathcal{E}_\delta^{avg}(u)$ as a Dirichlet integral of a nonlocal gradient operator. However, based on the above calculation and using a crude estimate $\pi x^2 \geq x\arcsin(x)$ we may alternatively consider a simpler looking, yet still coercive functional

$$\mathcal{E}_\delta^{\star,\vec{k}}(u) = \int_\Omega |\mathcal{G}_\delta^{\star,\vec{k}} u(x)|^2 dx$$

where for any vector $\vec{k} \in \mathbb{R}^d$, $\mathcal{G}_\delta^{\star,\vec{k}} u(x)$ is a modified nonlocal gradient operator given by

$$\mathcal{G}_\delta^{\star,\vec{k}} u(x) = \int_{\mathbb{R}^d} w_\delta(|y-x|)\frac{y-x}{|y-x|}(u(y)-u(x))dy + \left(\int_{\mathbb{R}^d} w_\delta(|y-x|)(u(y)-u(x))dy\right)\vec{k}. \tag{3.8}$$

We remark that the adjoint operator of $\mathcal{G}_\delta^{\star,\vec{k}}$ can be defined in a similar fashion as in section 1.2. This may prove to be useful in nonlocal modeling. Note that $\mathcal{G}_\delta^{\star,\vec{k}}$ can be seen as a special form of more general nonlocal gradient operators studied in [77]. It is also related to the more standard nonlocal gradient operator with a spherical interaction neighborhood corresponding to the form of $\mathcal{G}_\delta^{\star,\vec{k}}$ with $\vec{k} = \vec{0}$ (the zero vector). According to [36, 37], the coercivity of the Dirichret integral corresponding to this case, i.e., $\vec{k} = \vec{0}$, depends on the choices of the kernel w_δ. For nonzero $\vec{k}$, we can have the coercivity of the Dirichret integral for $\mathcal{G}_\delta^{\star,\vec{k}}$ due to the second term in (3.8). This orientation dependent term is $O(\delta)$, due to the moment condition in (3.3), similar in spirit to how stability is attained in our recent work on the deterministic particle methods [65].

3.2 Applications of the nonlocal gradient operators

We now illustrate how the coercivity of the nonlocal Dirichlet energies can be utilized in several applications. These include the application of our nonlocal operators as building blocks for an alternative formulation of the nonlocal Stokes equation studied in [37]. Another application is to establish a nonlocal version of the Helmholtz decomposition. In addition, the operators are used to construct well-defined models of nonlocal isotropic linear elasticity that converge to the classical counterpart for any Poisson's ratio.

We use the nonsymmetric operator $\mathcal{G}_\delta^{\vec{n}}$ with a unit vector $\vec{n}$ defined in (3.2) for illustration, though similar discussions can be made for $\mathcal{G}_\delta^{\star,\vec{k}}$ with a constant nonzero vector $\vec{k} \neq \vec{0}$ defined in (3.8). To avoid further technical complications, we assume from here on the kernels adopted in our nonlocal operators are positive almost everywhere. Moreover, as in Section 1 we present our results in the two dimensional setting on the domain $\Omega = (-\pi, \pi)^2$ unless specified otherwise.

3.2.1 Nonlocal Stokes Equation

We first consider the steady nonlocal stokes equation

$$\begin{aligned} -\mathcal{L}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}} &= \mathbf{f} \text{ in } \Omega \\ -\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= 0 \text{ in } \Omega \end{aligned} \tag{3.9}$$

where $\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}}, \mathbf{f}$ are periodic functions on Ω and assumed to have zero means. One of the motivation for studying nonlocal Stokes model is to better understand methods like the smoothed particle hydronamics (SPH) [50, 70], see [37] for more references and discussions.

The nonlocal Stokes equation is obtained by applications of the nonlocal operators $\mathcal{L}_\delta^{\vec{n}}, \mathcal{G}_\delta^{\vec{n}}, \mathcal{D}_\delta^{\vec{n}}$

to their local counterparts in the classical Stokes equation

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \text{ in } \Omega \\ -\nabla \cdot \mathbf{u} &= 0, \text{ in } \Omega. \end{aligned} \tag{3.10}$$

In the Fourier space the system (3.9) can be written as

$$A_\delta^{\vec{n}}(\xi) \begin{bmatrix} \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi) \\ \widehat{p_\delta^{\vec{n}}}(\xi) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}}(\xi) \\ 0 \end{bmatrix} \tag{3.11}$$

where

$$A_\delta^{\vec{n}} = \begin{bmatrix} |\lambda_\delta^{\vec{n}}(\xi)|^2 I_2 & \lambda_\delta^{\vec{n}}(\xi) \\ \overline{\lambda_\delta^{\vec{n}}(\xi)}^T & 0 \end{bmatrix}.$$

The non-degeneracy of $\lambda_\delta^{\vec{n}}$ assured by Theorem 7 yields the well-posedness of (3.11).

Lemma 11. *The system (3.11) has a unique solution given by*

$$\begin{bmatrix} \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi) \\ \widehat{p_\delta^{\vec{n}}}(\xi) \end{bmatrix} = (A_\delta^{\vec{n}})^{-1} \begin{bmatrix} \widehat{\mathbf{f}}(\xi) \\ 0 \end{bmatrix} \tag{3.12}$$

where

$$(A_\delta^{\vec{n}})^{-1} = \begin{bmatrix} \frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\left(I - \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\right) & \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \\ \frac{\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2} & -1 \end{bmatrix}.$$

Moreover, there exists a constant $C > 0$ independent of f and δ, as $\delta \to 0$, such that

$$\|\mathbf{u}_\delta^{\vec{n}}\|_{[S_\delta^{\vec{n}}(\Omega)]^2} + \|p_\delta^{\vec{n}}\|_{L^2(\Omega)} \le C\|\mathbf{f}\|_{[(S_\delta^{\vec{n}}(\Omega))^\star]^2} \tag{3.13}$$

where $S_\delta^{\vec{n}}(\Omega) = \{h \in L^2(\Omega) : h$ is periodic and $\mathcal{G}_\delta^{\vec{n}} h \in [L^2(\Omega)]^2\}$ is the energy space equipped with the energy norm $\|h\|_{S_\delta^{\vec{n}}(\Omega)} = \left(\|\mathcal{G}_\delta^{\vec{n}} h\|_{[L^2(\Omega)]^2}\right)^{1/2}$ for $h \in S_\delta^{\vec{n}}(\Omega)$, and $(S_\delta^{\vec{n}}(\Omega))^\star$ is its dual space with respect to the standard L^2 duality pairing.

Let us examine the limit of the nonlocal solutions $(u_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ as $\delta \to 0$. As can be expected by Lemma 10, it is worthy noting that in the case studied in this work, the nonlocal velocity is only approximately local divergence free, which is in contrast with the nonlocal Stokes equation in [37] that gives equivalent local and nonlocal divergence free vector fields.

Proposition 1. *Let $(\mathbf{u}, p)$ and $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ denote the solutions of (3.9) and (3.10) respectively. Then there exists a constant $C > 0$ independent of f and δ, as $\delta \to 0$, such that*

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{[L^2(\Omega)]^2} + \|p_\delta^{\vec{n}} - p\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}_\delta^{\vec{n}}\|_{L^2(\Omega)} \le C\delta\|\mathbf{f}\|_{[L^2(\Omega)]^2}. \tag{3.14}$$

Proof. We can see from (3.12)

$$\begin{aligned}
|\widehat{\mathbf{u}}(\xi) - \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi)| &\le \left(\left| \frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{1}{|\xi|^2} \right| + \left| \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^4} - \frac{i\xi(-i\xi)^T}{|\xi|^4} \right| \right) |\widehat{\mathbf{f}}(\xi)| \\
&\le 2\left(\frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|} + \frac{1}{|\xi|} \right) \left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\widehat{\mathbf{f}}(\xi)| \\
&\le C\left(\left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| \right) |\widehat{\mathbf{f}}(\xi)|
\end{aligned}$$

where the last inequality is due to Theorem 7. Then since

$$|\widehat{p}(\xi) - \widehat{p_\delta^{\vec{n}}}(\xi)| \le \left| \frac{-\overline{\lambda_\delta^{\vec{n}}(\xi)}}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\hat{\mathbf{f}}(\xi)| = \left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\widehat{\mathbf{f}}(\xi)|,$$

it is sufficient to prove

$$\left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| \le C\delta \qquad (\dagger)$$

for some C independent of δ and ξ. To this end let us explicitly write $\lambda_\delta^{\vec{n}}(\xi) = a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)$ to obtain

$$\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right| = \left|\frac{a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right|$$
$$\leq \underbrace{\left|\frac{ib_\delta(\xi)}{|ib_\delta(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right|}_{I_1} + \underbrace{\left|\frac{ib_\delta(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2} - \frac{ib_\delta(\xi)}{|ib_\delta(\xi)|^2}\right|}_{I_2} + \underbrace{\left|\frac{a_\delta^{\vec{n}}(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2}\right|}_{I_3}$$

and consider cases depending on the values of $k = \delta|\xi|$ as in the proof of Theorem 7.

1. $k < 1$. We first consider I_1. Using Lemma 10 we have

$$I_1 \leq \left|\frac{1}{\Lambda_\delta(\xi)} - \frac{1}{|\xi|}\right| = \delta\left|\frac{1}{4\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r)r\cos(\theta)\sin(kr\cos(\theta))drd\theta} - \frac{1}{k}\right|.$$

Since

$$x - \frac{x^3}{6} \leq \sin(x) \leq x \text{ for } 0 \leq x \leq 1,$$

we obtain

$$2\pi k - \frac{\pi k^3}{4} \leq 4\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r)r\cos(\theta)\sin(kr\cos(\theta))drd\theta \leq 2\pi k,$$

which implies

$$\frac{1}{\delta}\left|\frac{1}{\Lambda_\delta(\xi)} - \frac{1}{|\xi|}\right| \leq \frac{1}{2\pi}\left(\frac{1}{k - \frac{k^3}{8}} - \frac{1}{k}\right) = \frac{1}{2\pi}\frac{k}{8 - k^2} \leq \frac{k}{2\pi}.$$

Hence $I_1 \leq \frac{\delta k}{2\pi} \leq \frac{\delta}{2\pi}$.

Next, for I_2, since $|\cos(x) - 1| \leq \frac{x^2}{2}$, it is clear that

$$|a_\delta^{\vec{n}}(\xi)| = \left|2\int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}(\cos(\xi\cdot s) - 1)ds\right| \leq C\delta|\xi|^2.$$

where we have used the moment condition (3.3).

On the other hand we can see from the proof of Theorem 7 that

$$|b_\delta(\xi)| \geq C|\xi|\,.$$

Hence it follows that

$$I_2 \leq \frac{|a_\delta^{\vec{n}}(\xi)|^2}{(|a_\delta^{\vec{n}}(\xi)|^2 + |b_\delta(\xi)|^2)|b_\delta(\xi)|} \leq \frac{C\delta^2|\xi|^4}{|\xi|^3} \leq C\delta\,.$$

As for I_3, similar calculation as in the case of I_2 shows $I_3 \leq C\delta$.

2. $k \geq 1$. We observe from the proof of Theorem 7 that $|\lambda_\delta^{\vec{n}}(\xi)| \geq \frac{C}{\delta}$, hence

$$\frac{1}{\delta}\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{\xi}{|\xi|^2}\right| \leq \frac{1}{\delta|\lambda_\delta^{\vec{n}}(\xi)|} + \frac{1}{\delta|\xi|} \leq C.$$

Lastly the local divergence of u_δ can be estimated as

$$\begin{aligned}|\widehat{\nabla\cdot\mathbf{u}_\delta^{\vec{n}}}(\xi)| &\leq \frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left(\frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right| + \left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\right|\right)|\widehat{\mathbf{f}}(\xi)| \\ &\leq \frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left(\frac{2|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|} + 1\right)\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right||\widehat{\mathbf{f}}(\xi)| \leq C\delta|\widehat{\mathbf{f}}(\xi)|\end{aligned}$$

where the last inequality is due to the estimate (†). □

The study on the nonlocal Stokes equation can be extended to time-dependent case. Let us consider

$$\begin{aligned}(\mathbf{u}_\delta^{\vec{n}})_t - \mathcal{L}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}}p_\delta^{\vec{n}} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\ -\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= 0 \text{ in } (0,T)\times\Omega \\ \mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{u}_0 \text{ in } \Omega\end{aligned} \tag{3.15}$$

along with its counterpart local equation

$$
\begin{aligned}
\mathbf{u}_t - \Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } (0,T)\times\Omega \\
-\nabla\cdot\mathbf{u} &= 0 \text{ in } (0,T)\times\Omega \\
\mathbf{u}|_{t=0} &= \mathbf{u_0} \text{ in } \Omega
\end{aligned}
\tag{3.16}
$$

where all the local and nonlocal field variables as well as the data are assumed to be periodic on Ω with zero means. We then have the analogous results as in the steady case.

Proposition 2. *Assume* $\mathbf{f} \in L^2(0,T : [(S_\delta^{\vec{n}}(\Omega))^\star]^2)$ *and* $\mathbf{u}_0 \in [L^2(\Omega)]^2$ *with* $-\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_0 = 0$ *in* Ω *. Then the nonlocal Stokes equation (3.15) has a unique solution* $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ *where* $\mathbf{u}_\delta^{\vec{n}} \in L^2(0,T : [S_\delta^{\vec{n}}(\Omega)]^2) \cap C(0,T : [L^2(\Omega)]^2)$*,* $(\mathbf{u}_\delta^{\vec{n}})_t \in L^2(0,T : [(S_\delta^{\vec{n}}(\Omega))^\star]^2)$ *and* $p_\delta^{\vec{n}} \in L^2(0,T : L^2(\Omega))$*.*

Proof. Let us write $P_\delta^{\vec{n}}$ to denote the nonlocal Leray operator which in the Fourier space is given by

$$
\widehat{P_\delta^{\vec{n}}}(\xi) := I - \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}.
$$

One can check that $P_\delta^{\vec{n}}\mathbf{u} = \mathbf{u}$ for $\mathcal{D}_\delta^{\vec{n}}\mathbf{u} = 0$, $P_\delta^{\vec{n}}$ commutes with $\mathcal{L}_\delta^{\vec{n}}$, and $P_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}} = 0$, hence the nonlocal system (3.15) is equivalent to

$$
\begin{aligned}
(\mathbf{u}_\delta^{\vec{n}})_t - \mathcal{L}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= P_\delta^{\vec{n}}\mathbf{f} \text{ in } (0,T)\times\Omega \\
\mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}} &= \mathbf{f} - P_\delta^{\vec{n}}\mathbf{f} \text{ in } (0,T)\times\Omega \\
\mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{u}_0 \text{ in } \Omega
\end{aligned}
$$

of which the unique solutions are given by Duhamel's principle

$$
\widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi,t) = \widehat{\mathbf{u}}_0(\xi)\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2 t) + \int_0^t \exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2(t-s))\widehat{P_\delta^{\vec{n}}}(\xi)\widehat{\mathbf{f}}(\xi,s)ds
$$

and

$$\widehat{p_\delta^{\vec{n}}}(\xi,t) = \frac{\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}(I - \widehat{P_\delta^{\vec{n}}}(\xi))\widehat{\mathbf{f}}(\xi,t).$$

We may then apply the standard energy arguments to show $\mathbf{u}_\delta^{\vec{n}}$, $(\mathbf{u}_\delta^{\vec{n}})_t$ and $p_\delta^{\vec{n}}$ belong to the appropriate spaces. In order to show the continuity of $\mathbf{u}_\delta$, we first deduce from Theorem 7 that $S_\delta^{\vec{n}}(\Omega) \subset L^2(\Omega) \subset (S_\delta^{\vec{n}}(\Omega))^\star$ is a Hilbert triple and then apply the classical interpolation result [102]. □

To conclude our discussion of the nonlocal Stokes equation we prove that the nonlocal solutions of (3.15) converge to the corresponding local ones as the nonlocal parameter δ vanishes. Given a locally divergence free initial velocity u_0, however, we need to exercise care in prescribing the initial velocity for the nonlocal Stokes equations since Lemma 10 shows that $\mathbf{u}_0$ is in general not nonlocally divergence free.

Proposition 3. *Suppose $\mathbf{f} \in L^2(0,T : [L^2(\Omega)]^2)$ and $u_0 \in L^2(\Omega)$ with $-\nabla \cdot \mathbf{u}_0 = 0$ in Ω. Assume $\mathbf{u}_{0,\delta}^{\vec{n}} \to \mathbf{u}_0$ in $[L^2(\Omega)]^2$ as $\delta \to 0$ with $-\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_{0,\delta}^{\vec{n}} = 0$. Then the unique solution $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ of (3.15) where $\mathbf{u}_0$ is replaced by $\mathbf{u}_{0,\delta}^{\vec{n}}$ converges to the unique solution $(\mathbf{u}, p)$ of (3.16) in $L^2(0,T : [L^2(\Omega)]^2)$ as $\delta \to 0$.*

Proof. Let us define the local Leray projector P_0 in the Fourier space

$$\widehat{P_0}(\xi) = I - \frac{\xi\xi^T}{|\xi|^2}.$$

The local solutions u and p are then given by

$$\widehat{\mathbf{u}}(\xi,t) = \widehat{\mathbf{u}_0}(\xi)\exp(-|\xi|^2 t) + \int_0^t \exp(-|\xi|^2(t-s))\widehat{P_0}(\xi)\widehat{\mathbf{f}}(\xi,s)ds$$

and

$$\widehat{p}(\xi,t) = \frac{-i\xi^T}{|\xi|^2}(I - \widehat{P_0}(\xi))\widehat{\mathbf{f}}(\xi,t).$$

We first consider

$$\widehat{\mathbf{u}^{\vec{n}}_{\delta}}(\xi,t) - \widehat{\mathbf{u}}(\xi,t) = \widehat{\mathbf{u}^{\vec{n}}_{0,\delta}}(\xi)\exp(-|\lambda^{\vec{n}}_{\delta}(\xi)|^2 t) - \widehat{\mathbf{u}_0}(\xi)\exp(-|\xi|^2 t)$$
$$+\int_0^t \left(\exp(-|\lambda^{\vec{n}}_{\delta}(\xi)|^2(t-s))\widehat{P^{\vec{n}}_{\delta}}(\xi) - \exp(-|\xi|^2(t-s))P_0(\xi)\right)\widehat{\mathbf{f}}(\xi,s)ds$$

where $\widehat{P^{\vec{n}}_{\delta}}(\xi)$ is the nonlocal Leray operator used in the proof of Proposition 2, hence

$$\int_0^T \|\mathbf{u}^{\vec{n}}_{\delta} - \mathbf{u}\|^2_{[L^2(\Omega)]^2} dt \leq C_T \int_0^T \left(\sum_{\xi\in\mathbb{Z}^2,\xi\neq 0} |\widehat{\mathbf{u}_0}(\xi)|^2 \left|\exp(-|\lambda^{\vec{n}}_{\delta}(\xi)|^2 t) - \exp(-|\xi|^2 t)\right|^2 + \right.$$
$$\left. |\widehat{\mathbf{u}^{\vec{n}}_{0,\delta}}(\xi) - \widehat{\mathbf{u}_0}(\xi)|^2|\exp(-|\lambda^{\vec{n}}_{\delta}(\xi)|^2 t)|^2 + \int_0^t \left|\left(\exp(-|\lambda^{\vec{n}}_{\delta}(\xi)|^2(t-s))\widehat{P^{\vec{n}}_{\delta}}(\xi) - \exp(-|\xi|^2(t-s))\widehat{P_0}(\xi)\right)\widehat{\mathbf{f}}(\xi,s)\right|^2 ds \right)$$

where C_T is a constant depending only on T. We observe that since $\mathbf{u}^{\vec{n}}_{0,\delta} \to \mathbf{u}_0$ in $[L^2(\Omega)]^2$ as $\delta \to 0$ the integrand in the parentheses can be bounded, uniformly in δ (as $\delta \to 0$) and t, by some constant C depending only on $\mathbf{u}_0$ and $\mathbf{f}$. One can easily verify

$$|\lambda^{\vec{n}}_{\delta}(\xi) - i\xi| \to 0, \text{ hence } \left|\widehat{P^{\vec{n}}_{\delta}}(\xi) - \widehat{P_0}(\xi)\right| \leq \left|\widehat{P^{\vec{n}}_{\delta}}(\xi) - \widehat{P_0}(\xi)\right|_F \to 0 \text{ as } \delta \to 0 \text{ for each } \xi \in \mathbb{Z}^2, \xi \neq 0$$

so that it follows the dominated convergence theorem yields $\mathbf{u}^{\vec{n}}_{\delta} \to \mathbf{u}$ in $L^2(0,T:[L^2(\Omega)]^2)$ as $\delta \to 0$.

We apply the similar argument to the expression

$$\int_0^T \|p^{\vec{n}}_{\delta} - p\|^2_{L^2(\Omega)} dt = \int_0^T \sum_{\xi\in\mathbb{Z}^2,\xi\neq 0} \left(\frac{\overline{\lambda^{\vec{n}}_{\delta}(\xi)}^T}{|\lambda^{\vec{n}}_{\delta}(\xi)|^2}(I - \widehat{P_{\delta}}(\xi))\widehat{\mathbf{f}}(\xi,t) + \frac{i\xi^T}{|\xi|^2}(I - \widehat{P_0}(\xi))\widehat{\mathbf{f}}(\xi,t) \right)^2 dt$$

to conclude $p^{\vec{n}}_{\delta} \to p$ in $L^2(0,T:L^2(\Omega))$ as $\delta \to 0$.

Lastly we remark that nonlocally divergence free initial velocity $\mathbf{u}^{\vec{n}}_{0,\delta}$ as in the assumption of the theorem can be explicitly constructed by taking $\widehat{\mathbf{u}^{\vec{n}}_{0,\delta}}(\xi) = \widehat{P^{\vec{n}}_{\delta}}(\xi)\widehat{\mathbf{u}_0}(\xi)$. □

We note that one can also get the order of convergence as in the time-independent case. The details are omitted.

3.2.2 Nonlocal Helmholtz decomposition

The nonlocal Leray operators introduced in the proof of Proposition 2 clearly imply a nonlocal version of the classical Helmholtz decomposition theorem which warrants a more detailed discussion.

We begin with the following two dimensional result.

Theorem 8. *If* $\mathbf{u} : \Omega \to \mathbb{R}^2$ *is square integrable and periodic with zero mean, then there exist unique periodic, zero mean scalar potentials* $p_\delta^{\vec{n}}, q_\delta^{\vec{n}} \in S_\delta^{\vec{n}}(\Omega)$ *such that*

$$\mathbf{u}(x) = \mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}}(x) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathcal{G}_\delta^{-\vec{n}} q_\delta^{\vec{n}}(x)$$

with $\mathcal{D}_\delta^{\vec{n}} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathcal{G}_\delta^{-\vec{n}} q_\delta^{\vec{n}} \right)(x) = 0$. *In addition we have the estimate*

$$\|p_\delta^{\vec{n}}\|_{S_\delta^{\vec{n}}(\Omega)} + \|q_\delta^{\vec{n}}\|_{S_\delta^{\vec{n}}(\Omega)} \leq C \|\mathbf{u}\|_{[L^2(\Omega)]^2}$$

for some constant C *independent of* $\vec{n}$ *and also of* δ *as* $\delta \to 0$. *Here* $S_\delta^{\vec{n}}(\Omega)$ *is the energy space as in Lemma 11.*

Proof. In the Fourier space the unique solutions are given by

$$\widehat{p_\delta^{\vec{n}}}(\xi) = -\frac{\lambda_\delta^{-\vec{n}}(\xi)^T \widehat{\mathbf{u}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \quad \text{and} \quad \widehat{q_\delta^{\vec{n}}}(\xi) = -\frac{\lambda_\delta^{\vec{n}}(\xi)^T}{|\lambda_\delta^{-\vec{n}}(\xi)|^2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(I + \frac{\lambda_\delta^{\vec{n}}(\xi) \lambda_\delta^{-\vec{n}}(\xi)^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \right) \widehat{\mathbf{u}}(\xi)$$

and the rest of the claim is due to $\lambda_\delta^{-\vec{n}}(\xi) = -\overline{\lambda_\delta^{\vec{n}}(\xi)}$ and Theorem 7. □

In three dimensions we introduce nonlocal curl operators

$$\mathcal{C}_\delta^{\vec{n}} v(x) = 2\int_{\mathbb{R}^3} \chi_{\vec{n}}(y-x)\underline{\omega}_\delta(y-x)(y-x)\times\frac{v(y)-v(x)}{|y-x|}dy,$$

based on which we deduce the following.

Theorem 9. *If* $\mathbf{u} : \widetilde{\Omega} = (-\pi,\pi)^3 \to \mathbb{R}^3$ *is square integrable and periodic with zero mean, then there exist unique periodic, zero mean scalar and vector potentials* $p_\delta^{\vec{n}} \in S_\delta^{\vec{n}}(\widetilde{\Omega})$ *and* $\mathbf{v}_\delta^{\vec{n}} \in [S_\delta^{\vec{n}}(\widetilde{\Omega})]^3$*, respectively, such that*

$$\mathbf{u}(x) = \mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}}(x) + \mathcal{C}_\delta^{-\vec{n}} \mathbf{v}_\delta^{\vec{n}}(x)$$

with the nonlocal Gauge condition $\mathcal{D}_\delta^{-\vec{n}}\mathbf{v}_\delta^{\vec{n}}(x) = 0$*. Moreover* $(\mathcal{C}_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}})p_\delta^{\vec{n}}(x)$ *and* $(\mathcal{D}_\delta^{\vec{n}} \circ \mathcal{C}_\delta^{-\vec{n}})\mathbf{v}_\delta^{\vec{n}}(x)$ *vanish along with the analogous estimate as in Theorem 8 with* $\mathbf{v}_\delta^{\vec{n}}$ *in place of* $p_\delta^{\vec{n}}$*.*

Proof. One can verify that a nonlocal vector identity

$$(\mathcal{C}_\delta^{-\vec{n}} \circ \mathcal{C}_\delta^{\vec{n}})\mathbf{f}(x) = (\mathcal{G}_\delta^{\vec{n}} \circ \mathcal{D}_\delta^{\vec{n}})\mathbf{f}(x) - \mathcal{L}_\delta^{\vec{n}}\mathbf{f}(x)$$

holds for any periodic $\mathbf{f} : (-\pi,\pi)^3 \to \mathbb{R}^3$. Solving for $\mathbf{f}$ in $-\mathcal{L}_\delta^{\vec{n}}\mathbf{f} = \mathbf{u}$ then yields the Fourier representations of the unique solutions

$$\widehat{p}_\delta^{\vec{n}}(\xi) = -\frac{\lambda_\delta^{-\vec{n}}(\xi)^T\widehat{\mathbf{u}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \quad \text{and} \quad \widehat{\mathbf{v}}_\delta^{\vec{n}}(\xi) = \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\times\widehat{\mathbf{u}}(\xi).$$

We refer to the proof of Theorem 8 for the rest of the claim. □

Let us point out that the proof of Theorem 9 reveals the well-posedness of a nonlocal version of the classical first order div-curl elliptic system as stated below.

Theorem 10. *Given periodic, zero mean data* $f \in L^2((-\pi,\pi)^3)$ *and* $\mathbf{g} \in [L^2((-\pi,\pi)^3)]^3$ *with* $\mathcal{D}_\delta^{-\vec{n}}\mathbf{g} = 0$*, there exist a unique periodic, zero mean vector field* $\mathbf{u} \in [S_\delta^{\vec{n}}((-\pi,\pi)^3)]^3$ *satisfying the* *following nonlocal div-curl system*

$$\begin{aligned} \mathcal{D}_\delta^{\vec{n}}\mathbf{u} &= f \textit{ in } (-\pi,\pi)^3 \\ C_\delta^{\vec{n}}\mathbf{u} &= \mathbf{g} \textit{ in } (-\pi,\pi)^3 \end{aligned} \tag{3.17}$$

and the estimate

$$\|\mathbf{u}\|_{[S_\delta^{\vec{n}}((-\pi,\pi)^3)]^3} \leq C\left(\|f\|_{L^2((-\pi,\pi)^3)} + \|\mathbf{g}\|_{[L^2((-\pi,\pi)^3)]^3}\right) \tag{3.18}$$

for some positive constant C independent of $\mathbf{u}$, $\vec{n}$ and δ (as $\delta \to 0$).

The estimate (3.18) is a nonlocal version of the second Friedrichs inequality [62, 94], which can be easily shown with the help of Theorem 7. We omit the details. Let us remark that the well-posed nonlocal div-curl system can play an important role in the applications of SPH-like numerical methods for fluid flows. If we introduce the notion of nonlocal vorticity as the nonlocal curl of velocity, the well-posedness of the nonlocal div-curl system could facilitate studies of nonlocal vorticity formulation on the continuum level en route to the ultimate goals of developing their effective numerical discretizations. The recovered nonlocal velocity from the nonlocal vorticity could in turn be used as the underlying velocity field for nonlocal models of convective flows, which would depart from our practice of adopting the local underlying velocity field in Chapter 2.

We note that classical Helmholtz decomposition have wide applications of mechanics and electromagnetics. For nonlocal and fractional versions, one can also check [101]. The further study of nonlocal div-curl systems, including nonlocal versions of the div-curl lemma [83, 19], may also be of interests and will be left for future works.

3.2.3 Nonlocal correspondence models of isotropic linear elasticity

We study the elastic potential energy given by

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) = \frac{1}{2}\lambda\|\mathcal{D}_\delta^{\vec{n}}\mathbf{u}(\mathbf{x})\|_2^2 + \mu\|e_\delta^{\vec{n}}(\mathbf{u}(\mathbf{x}))\|_2^2 \tag{3.19}$$

where λ, μ are Lamé coefficients and $e_\delta^{\vec{n}}(\mathbf{u})$ is the nonlocal strain tensor

$$e_\delta^{\vec{n}}(\mathbf{u}) = \frac{\mathcal{G}_\delta^{\vec{n}}\mathbf{u} + (\mathcal{G}_\delta^{\vec{n}}\mathbf{u})^T}{2}$$

for a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$. This can be viewed as a so-called nonlocal correspondence model where the local stress tensors in the local energy density are replaced by the nonlocal counterparts [36, 96]. We assume $\mu > 0$ and $\lambda + 2\mu > 0$ for which it is well known that the corresponding local elastic energy is variationally stable over the energy space

$$V_0 = \{\mathbf{u} \in \mathbf{H}^1(\Omega) | \mathbf{u} \text{ is periodic with zero mean}\}, \qquad \|\cdot\|_{V_0} = \|\cdot\|_{\mathbf{H}^1(\Omega)},$$

where $\mathbf{H}^1(\Omega)$ denotes the standard $\mathbf{R}^2$-valued Sobolev space. The energy $\mathcal{E}_\delta^{\vec{n}}$ can be seen as a linear elastic potential energy of isotropic materials in the peridynamics correspondence theory [95, 96]. As in the one dimensional case with radially symmetric interaction kernels [36], let us first define the nonlocal energy space $V_\delta^{\vec{n}}$ to be the closure of C^∞, zero mean, periodic $\mathbb{R}^2$-valued functions on Ω with respect to the norm

$$\|\mathbf{u}\|_{V_\delta^{\vec{n}}} = (\|\mathbf{u}\|_2^2 + \mathcal{E}_\delta^{\vec{n}}(\mathbf{u}))^{1/2}.$$

A precise statement of the variational stability of $\mathcal{E}_\delta^{\vec{n}}$ is then given by

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C\|\mathbf{u}\|_{V_\delta^{\vec{n}}}^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}} \tag{3.20}$$

for some constant $C > 0$ independent of $\vec{n}$ and δ, as $\delta \to 0$. In order to establish the stability let us introduce the nonlocal Naviér operator $M_\delta^{\vec{n}}$ defined as

$$M_\delta^{\vec{n}}(\mathbf{u}) = -\mu \mathcal{L}_\delta^{\vec{n}} \mathbf{u} - (\lambda + \mu) \mathcal{G}_\delta^{\vec{n}} (\mathcal{D}_\delta^{\vec{n}} \mathbf{u}) \tag{3.21}$$

so that for $\mathbf{u} = (u_1, u_2) \in V_\delta^{\vec{n}}$

$$(M_\delta^{\vec{n}}(\mathbf{u}), \mathbf{u})_2 = \lambda \int_\Omega |\mathcal{D}_\delta^{\vec{n}} \mathbf{u}(\mathbf{x})|^2 d\mathbf{x} + \mu \int_\Omega \left(|\mathcal{D}_\delta^{\vec{n}} \mathbf{u}(\mathbf{x})|^2 + \sum_{i=1,2} |\mathcal{G}_\delta^{\vec{n}} u_i(\mathbf{x})|^2 \right) d\mathbf{x} = 2\mathcal{E}_\delta^{\vec{n}}(\mathbf{u})$$

due to (3.5). We observe from Lemma 9 $\mathcal{D}_\delta^{\vec{n}} \mathbf{u} = \mathrm{Tr}(e_\delta^{-\vec{n}}(\mathbf{u}))$ is not in general equal to $\mathrm{Tr}(e_\delta^{\vec{n}}(\mathbf{u}))$ where Tr denotes the trace operator, which is different from the local case.

Similar to the studies on the classical Korn's inequality and the nonlocal versions in [61, 74, 84], we have a nonlocal version for the nonlocal Naviér system as follows.

Lemma 12. *(Nonlocal Korn's inequality). There exists a constant $C > 0$ independent of $\vec{n}$ and δ such that*

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C \|\mathcal{G}_\delta^{\vec{n}} \mathbf{u}(\mathbf{x})\|_2^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}}.$$

Proof. We can see

$$\begin{aligned}\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) = \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2, \xi \neq (0,0)} M_\delta^{\vec{n}}(\xi) \widehat{\mathbf{u}}(\xi) \cdot \widehat{\mathbf{u}}(\xi) &= \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2, \xi \neq (0,0)} \mu |\lambda_\delta^{\vec{n}}(\xi)|^2 |\widehat{\mathbf{u}}(\xi)|^2 + (\mu + \lambda) |\lambda_\delta^{\vec{n}}(\xi) \cdot \widehat{\mathbf{u}}(\xi)|^2 \\ &\geq \frac{1}{2} \min(\mu, \lambda + 2\mu) \sum_{\xi \in \mathbb{Z}^2, \xi \neq (0,0)} |\lambda_\delta^{\vec{n}}(\xi)|^2 |\widehat{\mathbf{u}}(\xi)|^2\end{aligned}$$

which proves the claim. □

We point out that by setting $\lambda = 0$, one can recover the periodic versions of nonlocal Korn's inequality studied in [74, 84].

Now using Theorem 7, which serves like a nonlocal Poincare inequality [27, 77], we readily have the following coercivity result and the well-posedness.

Lemma 13. *There exists a constant $C > 0$ independent of $\vec{n}$ and δ as $\delta \to 0$ such that*

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C\|\mathbf{u}\|_2^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}}.$$

Moreover, the problem

$$M_\delta^{\vec{n}}(\mathbf{u}) = \mathbf{f} \tag{3.22}$$

is well-posed over $V_\delta^{\vec{n}}$ where $\mathbf{f} \in \mathbf{L^2}(\Omega)$ is periodic with zero mean.

We note that the same remains valid for $\mathbf{f}$ belonging to the dual space $(V_\delta^{\vec{n}})^*$. Indeed, given the equivalence of $\mathcal{E}_\delta^{\vec{n}}(\cdot)$ with $\|\cdot\|_{V_\delta^{\vec{n}}}$ due to Lemma 13, an explicit characterization of the dual space with the L^2 duality pairing can also be obtained as done in [120]. Not only is the nonlocal solution $\mathbf{u}_\delta^{\vec{n}} \in V_\delta^{\vec{n}}$ to (3.22) important in its own right as a unique minimizer of $\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) - (\mathbf{f}, \mathbf{u})_2$ but it can also be shown to recover the corresponding local solution to the local Naviér equation, when the latter is well-posed. To this end let us first present the following embedding result.

Lemma 14. *There exists a constant C independent of δ and $\vec{n}$ such that*

$$\|\mathbf{u}\|_{V_\delta^{\vec{n}}} \leq C\|\mathbf{u}\|_{V_0} \quad \forall \mathbf{u} \in V_0.$$

Proof. As can be observed in the proof of Lemma 12, we have

$$M_\delta^{\vec{n}}(\xi)\widehat{\mathbf{u}}(\xi) \cdot \widehat{\mathbf{u}}(\xi) \leq \max(\mu, \lambda + 2\mu)|\lambda_\delta^{\vec{n}}(\xi)|^2|\widehat{\mathbf{u}}(\xi)|^2$$

to which applying Theorem 7 proves the claim. □

Not only is it in $\|\cdot\|_2$ but also in the nonlocal norm $\|\cdot\|_{V_\delta^{\vec{n}}}$ that we have the convergence of the nonlocal solution $\mathbf{u}_\delta$ to its local counterpart.

Proposition 4. *Assume $\mathbf{f} \in \mathbf{L^2}(\Omega)$ is periodic with zero mean. Let $\mathbf{u}_\delta^{\vec{n}}$ denote the solution of the nonlocal Naviér equation (3.22). Then there exists a constant C independent of δ and $\vec{n}$ as $\delta \to 0$*

such that

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_2 \leq C\delta\|\mathbf{f}\|_2$$

where $\mathbf{u}$ *is the solution to the local Naviér equation*

$$M_0(\mathbf{u}) := -\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\nabla\cdot\mathbf{u} = \mathbf{f} \quad \text{in } \Omega.$$

If we further assume that $\mathbf{f}$ *belongs to the fractional Sobolev space* $\mathbf{H}^{\frac{1}{2}}(\Omega)$*, then we have*

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{V_\delta^{\vec{n}}} \leq C\delta\|\mathbf{f}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}.$$

Proof. Let us first write

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_2^2 \leq \sum_{\xi\in\mathbb{Z}^2, \xi\neq(0,0)} \left|(\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1}\right|^2 \left|\widehat{\mathbf{f}}(\xi)\right|^2.$$

Based on the explicit expressions for the matrices $M_\delta^{\vec{n}}(\xi)$ and $M_0(\xi)$, we can apply Theorem 7 and the following estimate shown in the proof of Proposition 1

$$\left|\frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|} - \frac{1}{|\xi|}\right| \leq C\delta \qquad (\star)$$

to verify that

$$|(\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1}|_F \leq C\delta$$

where $|\cdot|_F$ denotes the Frobenius norm, from which the convergence in L^2 follows. We next consider

$$
\begin{aligned}
\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{V_\delta^{\vec{n}}}^2 &= \sum_{\xi\in\mathbb{Z}^2, \xi\neq(0,0)} (I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1})\widehat{\mathbf{f}}(\xi) \cdot ((\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1})\widehat{\mathbf{f}}(\xi) \\
&\leq C\delta \sum_{\xi\in\mathbb{Z}^2, \xi\neq(0,0)} \left| I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1}) \right| \left| \widehat{\mathbf{f}}(\xi) \right|^2 .
\end{aligned}
$$

If we denote $\xi = (\xi_1, \xi_2)$ and $\lambda_\delta^{\vec{n}}(\xi) = ([\lambda_\delta^{\vec{n}}(\xi)]_1, [\lambda_\delta^{\vec{n}}(\xi)]_2)$, Theorem 7 and the estimate ($\star$) from above allow us to obtain

$$
\begin{aligned}
&\left| I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1} \right| \\
&\leq \left| I - \frac{1}{\mu(\lambda+2\mu)|\xi|^4} \begin{bmatrix} (\lambda+2\mu)|[\lambda_\delta^{\vec{n}}(\xi)]_1|^2 + \mu|[\lambda_\delta^{\vec{n}}(\xi)]_2|^2 & (\lambda+\mu)[\lambda_\delta^{\vec{n}}(\xi)]_1\overline{[\lambda_\delta^{\vec{n}}(\xi)]_2} \\ (\lambda+\mu)\overline{[\lambda_\delta^{\vec{n}}(\xi)]_1}[\lambda_\delta^{\vec{n}}(\xi)]_2 & \mu|[\lambda_\delta^{\vec{n}}(\xi)]_1|^2 + (\lambda+2\mu)|[\lambda_\delta^{\vec{n}}(\xi)]_2|^2 \end{bmatrix} \right. \\
&\qquad \left. \cdot \begin{bmatrix} \mu\xi_1^2 + (\lambda+2\mu)\xi_2^2 & -(\lambda+\mu)\xi_1\xi_2 \\ -(\lambda+\mu)\xi_1\xi_2 & \mu\xi_2^2 + (\lambda+2\mu)\xi_1^2 \end{bmatrix} \right|_F \\
&\leq C\delta|\xi|,
\end{aligned}
$$

which proves the convergence in $V_\delta^{\vec{n}}$. □

As in the previous section on the nonlocal Stokes equation, we now consider the time dependent nonlocal Naviér equation

$$
\begin{aligned}
(\mathbf{u}_\delta^{\vec{n}})_{tt} + M_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\
\mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{g} \text{ in } \Omega \\
(\mathbf{u}_\delta^{\vec{n}})_t|_{t=0} &= \mathbf{h} \text{ in } \Omega
\end{aligned} \tag{3.23}
$$

in juxtaposition with the local Naviér equation

$$
\begin{aligned}
\mathbf{u}_{tt} + M_0\mathbf{u} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\
\mathbf{u}|_{t=0} &= \mathbf{g} \text{ in } \Omega \\
\mathbf{u}_t|_{t=0} &= \mathbf{h} \text{ in } \Omega
\end{aligned} \tag{3.24}
$$

where all the local and nonlocal field variables as well as the data are assumed to be periodic on Ω with zero means. Since the Hermitian matrix $P_\delta^{\vec{n}}$ is positive definite, we can apply the similar argument as in [43] to establish the following well-posedness result for which we omit the details.

Proposition 5. *Suppose* $\mathbf{g} \in V_\delta^{\vec{n}}, \mathbf{h} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^2(0,T : L^2(\Omega))$. *Then there exists a unique solution* $\mathbf{u}_\delta^{\vec{n}}$ *to (3.23) such that*

$$
\mathbf{u}_\delta^{\vec{n}} \in C(0,T : V_\delta^{\vec{n}}), \quad (\mathbf{u}_\delta^{\vec{n}})_t \in L^2(0,T : L^2(\Omega)).
$$

For completeness we consider convergence of the time dependent nonlocal solution $\mathbf{u}_\delta^{\vec{n}}$ to the corresponding local solution as $\delta \to 0$. As in the steady case this can be readily established using the explicit Fourier representations of both nonlocal and local solutions.

Proposition 6. *Suppose* $\mathbf{g} \in V_0(\Omega), \mathbf{h} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^2(0,T : L^2(\Omega))$. *Let* $\mathbf{u}_\delta^{\vec{n}}$ *and* $\mathbf{u}$ *denote the solutions of nonlocal and local Naviér equations, respectively, with the same initial displacement field* $\mathbf{g}$ *and velocity field* $\mathbf{h}$. *Then we have* $\mathbf{u}_\delta^{\vec{n}} \to \mathbf{u}$ *in* $L^2(0,T : V_\delta^{\vec{n}}(\Omega)) \bigcap H^1(0,T : L^2(\Omega))$.

Proof. We use the explicit Fourier representation of the solutions as given in the proof of Theorem

2.25 in [43] to write

$$\widehat{\mathbf{u}_\delta^{\vec{n}}}(t,\xi) - \widehat{\mathbf{u}}(t,\xi) = \underbrace{\left(\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)}_{\widehat{\mathbf{u}}_{c1}} + \underbrace{\left(\frac{\sin\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right)}{\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)}} - \frac{\sin\left(\sqrt{\widehat{M}_0(\xi)t}\right)}{\sqrt{\widehat{M}_0(\xi)}}\right)\widehat{\mathbf{h}}(\xi)}_{\widehat{\mathbf{u}}_{c2}}$$

$$+ \underbrace{\int_0^t \left(\frac{\sin\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)s}\right)}{\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)}} - \frac{\sin\left(\sqrt{\widehat{M}_0(\xi)s}\right)}{\sqrt{\widehat{M}_0(\xi)}}\right)\widehat{\mathbf{f}}(t-s,\xi)ds}_{\widehat{\mathbf{u}}_{c3}}.$$

We cannot argue as in [43] since the operators $M_\delta^{\vec{n}}$ and M_0 do not commute, hence not simultaneously diagonalizable. Instead we argue directly as follows. We have

$$\begin{aligned}
&\widehat{M}_\delta^{\vec{n}}(\xi)\left(\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\cdot\left(\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\\
&\le C\widehat{M}_0(\xi)\left(\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\cdot\left(\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\\
&\le C|\widehat{M}_0(\xi)||\widehat{\mathbf{g}}(\xi)|^2
\end{aligned}$$

where the second inequality is due to Lemma 14, hence $\|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}} \le C\|\mathbf{g}\|^2_{V_0}$ uniformly in $t \in [0,T]$, which in turn implies

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}}dt \le CT\|\mathbf{g}\|^2_{V_0}.$$

Let us now note that for each fixed $\xi \in \mathbb{Z}^2, \xi \neq (0,0)$ and $t \in (0,T)$ we get

$$\left|\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right| \le \left|\cos\left(\sqrt{\widehat{M}_\delta^{\vec{n}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M}_0(\xi)t}\right)\right|_F \to 0 \quad \text{as } \delta \to 0$$

since it can be easily checked that $|\lambda_\delta^{\vec{n}}(\xi) - i\xi| \to 0$, hence $\left|\widehat{M}_\delta^{\vec{n}}(\xi) - \widehat{M}_0(\xi)\right|_F \to 0$, as $\delta \to 0$. The dominated convergence theorem then implies

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}} dt \to 0 \quad \text{as } \delta \to 0.$$

Similarly the rest of the claims can be proved and we omit the details. □

3.3 Another look at the nonlocal gradient operators

We are motivated to revisit the nonlocal operators upon juxtaposing our application results in Section 2 with the popular practice in nonlocal modelling of taking into account a full spherical neighborhood. In more specific terms we consider the queston of whether there is a connection between the nonlocal diffusion operator based on the non-symmetric formulations of nonlocal divergence and gradient and other existing formulations of nonlocal diffusion (Laplacian) operator. We begin by recalling from [36] that symmetric gradient operator is closely related to a bond-based, in the language of peridynamics, nonlocal diffusion operator. We show that the similar conclusion can be drawn in our formulation with some choices of kernels.

To illustrate this point it is sufficient to consider the one dimensional setting wherein the nonlocal Dirichlet integrals are

$$\mathcal{E}_\delta^\pm(u) = \int_{-\pi}^{\pi} |\mathcal{G}_\delta^\pm(u)(x)|^2 dx$$

for which we have the following.

Lemma 15. *For a smoothly defined periodic $u : (-\pi, \pi) \to \mathbb{R}$, if $w_\delta(|x|)$ is integrable then*

$$\mathcal{E}_\delta^+(u) = \mathcal{E}_\delta^-(u) = 2\int_{-\pi}^{\pi}\int_0^\delta \rho_\delta(a) \left|\frac{u(x+a) - u(x)}{a}\right|^2 da dx$$

where $\rho_\delta(a) = \rho_\delta(|a|)$ is a radial (even) function given by

$$\rho_\delta(a) = 2a^2 \int_0^\delta w_\delta(b)\,(w_\delta(a) - w_\delta\,(a+b))\,db = -a^2(\mathcal{G}_\delta^+ w_\delta)(a), \quad \forall a \in (0,\delta) \tag{3.25}$$

supported on $(-\delta,\delta)$ and satisfies $\|\rho_\delta\|_{L^1} = 1$. Moreover if ω is non-increasing in $(0,1)$, then ρ_δ is non-negative.

Proof. Lemma 10 shows $|\lambda_\delta^{\vec{n}}(\xi)| = |\lambda_\delta^{-\vec{n}}(\xi)|$, thus it is immediate to see $\mathcal{E}_\delta^+(u) = \mathcal{E}_\delta^-(u)$. We consider only $\mathcal{E}_\delta^+(u)$ and follow the argument in [36] to write $D_\delta = [-\delta,0]^2\bigcup[0,\delta]^2$ so that

$$\mathcal{E}_\delta^+(u) = 2\underbrace{\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)(u(x+t)-u(x))^2 dsdtdx}_{I_1} + \dots$$
$$\underbrace{-\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)((u(x+s)-u(x+t))^2 dsdtdx}_{I_2}.$$

Let us first observe

$$I_1 = \int_{-\pi}^{\pi}\int_{-\delta}^{\delta} k_\delta(|a|)\left|\frac{u(x+a)-u(x)}{a}\right|^2 dadx$$

where

$$k_\delta(|a|) = 2w_\delta(|a|)|a|^2\int_0^\delta w_\delta(|b|)db.$$

Next we consider I_2 and rewrite it as

$$\begin{aligned} I_2 &= -\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)((u(x+s-t)-u(x))^2 dsdtdy \\ &= -\int_{-\pi}^{\pi}\int_{\hat{D}_\delta} \frac{a^2}{2} w_\delta\left(\left|\frac{a+b}{2}\right|\right) w_\delta\left(\left|\frac{b-a}{2}\right|\right)\frac{((u(x+a)-u(x))^2}{a^2} dadbdx \end{aligned}$$

where we have used the periodicity of u in the first equality and the change of variables $a = s-t$ and $b = s+t$ in the second with the corresponding change of integration domain from D_δ to $\hat{D}_\delta$. That is we have

$$I_2 = \int_{-\pi}^{\pi}\int_{-\delta}^{\delta} h_\delta(|a|)\frac{(u(y+a)-u(y))^2}{a^2} dadx$$

where

$$h_\delta(|a|) = -\frac{a^2}{2}\left(\int_{|a|}^{-|a|+2\delta} + \int_{|a|-2\delta}^{-|a|}\right) w_\delta(|\frac{a+b}{2}|)w_\delta(|\frac{b-a}{2}|)db = -a^2\int_{|a|}^{-|a|+2\delta} w_\delta(|\frac{a+b}{2}|)w_\delta(|\frac{b-a}{2}|)db.$$

We note the almost everywhere finiteness of $h_\delta(|a|)$ since

$$\int_{-a}^{a} h_\delta(|a|)da = 1 - \left(\int_{-\delta}^{\delta} s^2 w_\delta(|s|)ds\right)\left(\int_{-\delta}^{\delta} w_\delta(|t|)dt\right)$$

which can be easily verified. Then the desired claim follows from setting $\rho_\delta = k_\delta + h_\delta$ and observing that for $0 < a < \delta$,

$$h_\delta(|a|) = -2a^2 \int_0^{\delta-a} w_\delta(|z+a|)w_\delta(|z|)dz = -2a^2 \int_0^{\delta} w_\delta(|z+a|)w_\delta(|z|)dz$$

where the last equality holds since w_δ is supported on $(-\delta, \delta)$, and analogously for $-\delta < a < 0$.

Finally it is clear from (3.25) that ρ_δ is non-negative for non-increasing ω_δ. □

We note that Lemma 15 remains valid for any periodic function u with well-defined $\mathcal{E}_\delta^\pm(u)$. Naturally, one may be interested in extending the result of Lemma 15 to more general kernels w. Let us consider first a special example

$$w(|x|) = \begin{cases} \frac{C_\beta}{|x|^\beta}, & |x| \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

for $1 \leq \beta < 2$, and $C_\beta > 0$ is chosen to satisfy the moment condition (3.3). Then the result of Lemma 15 also holds. Indeed, let us fix $0 < a < \delta$ without loss of generality. We then have

$$\frac{\rho_\delta(a)}{2a^2C_\beta^2} = \int_0^{\delta-a} \frac{1}{z^\beta}\left(\frac{1}{a^\beta} - \frac{1}{(a+z)^\beta}\right)dz + \int_{\delta-a}^{\delta} \frac{1}{z^\beta}\frac{1}{a^\beta}dz \leq \int_0^{\delta-a} \frac{1}{z^\beta}\left(\frac{\beta\delta^{\beta-1}z}{a^\beta(a+z)^\beta}\right)dz + \int_{\delta-a}^{\delta} \frac{1}{z^\beta}\frac{1}{a^\beta}dz < \infty.$$

Moreover if we define

$$\rho_\delta^\epsilon(a) = \chi_{(\epsilon,\infty)}(|a|)2a^2 \int_\epsilon^\delta w_\delta(b)\left(w_\delta(a) - w_\delta\left(a + \frac{a}{|a|}b\right)\right)db$$

which are nonnegative, monotonically increasing in ϵ, pointwise convergent approximations of ρ_δ

as $\epsilon \to 0$, then a direct calculation shows

$$\|\rho_\delta\|_{L^1(\mathbb{R})} = \lim_{\epsilon \to 0} \int_{-\delta}^{\delta} \rho_\delta^\epsilon(a) da = 1$$

where the first equality is due to the Monotone Convergence Theorem.

We can extend the definition of ρ_δ to a more broader class of non-integrable kernels w_δ that include the above fractional ones as a special case. To this end let us assume that w_δ is non-increasing and consider

$$w_\delta^\epsilon(|x|) = \begin{cases} w_\delta(|x|), & |x| > \epsilon \\ \inf_{|y| \le \epsilon} w_\delta(|y|), & \text{otherwise} \end{cases}. \tag{3.26}$$

Note that the modified nonnegative and radial kernel w_δ^ϵ is integrable and also nonincreasing in $(0, \delta)$. Then

$$\rho_\delta^\epsilon(a) = 2a^2 \int_0^\delta w_\delta^\epsilon(b) \left(w_\delta^\epsilon(a) - w_\delta^\epsilon\left(a + \frac{a}{|a|} b\right)\right) db \tag{3.27}$$

is non-negative, monotonically increasing in ϵ and satisfies $\lim_{\epsilon \to 0} \|\tilde{\rho}_\delta^\epsilon\|_{L^1(\mathbb{R})} = 1$. Hence we may define

$$\rho_\delta(a) = \lim_{\epsilon \to 0} \rho_\delta^\epsilon(a) \tag{3.28}$$

which satisfies $\|\rho_\delta\|_{L^1(\mathbb{R})} = 1$ due to the Monotone Convergence Theorem. We thus get the following more general result.

Lemma 16. *For a smoothly defined periodic function $u : (-\pi, \pi) \to \mathbb{R}$, if $w_\delta(|x|)$ is non-increasing with bounded first moment then*

$$\mathcal{E}_\delta^+(u) = \mathcal{E}_\delta^-(u) = 2 \int_{-\pi}^{\pi} \int_0^\delta \rho_\delta(a) \left| \frac{u(x+a) - u(x)}{a} \right|^2 da dx$$

where $\rho_\delta(a) = \rho_\delta(|a|)$ is a radial (even) function defined by (3.26)-(3.27)-(3.28).

Again, the Lemma 16 remains valid for any periodic function u with well-defined $\mathcal{E}_\delta^\pm(u)$. We remark that the non-negativity of ρ_δ for non-increasing w_δ is in a clear contrast with the case of symmetric nonlocal gradient operators wherein the corresponding kernel is always sign changing. On the other hand it is easy to see that ρ_δ is not always non-negative and may change sign as in the case of

$$\rho_\delta(x) = \pi x^2 \sin(\pi|x|) + \frac{\pi^2 x^2}{4}\left((|x|-1)\cos(\pi x) - \frac{\sin(\pi|x|)}{\pi}\right)$$

which amounts to $w_\delta(x) = \frac{\pi}{2}\sin(\pi|x|)$ with $\delta = 1$.

As a corollary of Lemmas 15 and 16, we have the equivalence of the corresponding nonlocal diffusion operator based on the nonsymmetric gradient and divergence

$$\mathcal{L}_\delta^+ u(x) = (\mathcal{D}_\delta^+ \circ \mathcal{G}_\delta^+)u(x) = 4\int_0^\delta \int_0^\delta w_\delta(|s|)w_\delta(|t|)(u(x+s) - u(x) - u(x+s-t) + u(x-t))dsdt$$

and the conventional bond based nonlocal diffusion operator

$$\mathcal{L}_\delta u(x) = 2\int_{-\delta}^\delta k_\delta(|a|)(u(x+a) - u(x))da$$

upon setting $k_\delta(|a|) = \frac{\rho_\delta(|a|)}{a^2}$. At the same time, under the same assumptions on u and w_δ as in Lemma 15, we can further relate the operator $\mathcal{L}_\delta^+$ to another nonlocal diffusion operator in a similar form, namely a doubly nonlocal Laplace operator $\mathcal{L}_{\delta,\epsilon}^{double}$ proposed in [91]

$$\mathcal{L}_{\delta,\epsilon}^{double} u(x) = \int_{-\delta}^\delta \int_{-\epsilon}^\epsilon \gamma_\delta(y)\eta_\epsilon(r)(u(x+y+r) - u(x) - u(x+r) + u(x+y))dydr.$$

Here γ_δ and η_ϵ are assumed to be radial, non-negative and compactly supported on $(-\delta,\delta)$ and $(-\epsilon,\epsilon)$, respectively. Further assumptions on the moments of the kernels are made, namely the normalized second moment of γ_δ and integrability of η_ϵ with unit mass. For clarity of comparison, we first observe

$$\mathcal{L}_{\delta,\epsilon}^{double} u(x) = (\mathcal{L}_\delta \circ \mathcal{A}_\epsilon) u(x) = (\mathcal{A}_\epsilon \circ \mathcal{L}_\delta) u(x)$$

if we let $k_\delta = \gamma_\delta$ in $\mathcal{L}_\delta$ and define the averaging operator $\mathcal{A}_\epsilon$ by

$$\mathcal{A}_\epsilon u(x) = \frac{1}{2}\int_{-\epsilon}^{\epsilon} \eta_\epsilon(|z|)(u(x+z) + u(x))dz = \frac{1}{2}u(x) + \frac{1}{2}\int_{-\epsilon}^{\epsilon} \eta_\epsilon(|z|)u(x+z)dz.$$

Evidently the operator $\mathcal{A}_\epsilon$ provides a simple averaging, and does not fundamentally alter the spectral properties of $\mathcal{L}_{\delta,\epsilon}^{double}$. Thus, while involving an extra kernel, it does not change the modeling capability overall.

3.4 Discussion

This chapter provides a study of nonlocal gradient operators $\mathcal{G}_\delta^{\vec{n}}$ wherein the support of a positive kernel is prescribed to be any half-sphere parameterized by a unit vector $\vec{n}$. This can be seen as extensions of the one-sided nonlocal derivative operators for scalar functions of a single variable. It is interesting to observe that our nonlocal gradient operators with nonspherical interaction neighborhood can be effectively applied to model inherently symmetric phenomena as illustrated in our study of nonlocal Naviér equation of isotropic linear elasticity and nonlocal Stokes models of incompressible viscous flows. We also remark that by removing any singular growth assumption on the kernels, our nonsymmetric gradient operators are well suited to numerical quadrature based discretizations.

We have demonstrated that the symmetry of the nonlocal interaction neighborhood is not essential for nonlocal modeling and the related mathematical theory. While the analysis is focused on the half-sphere case, one may study further extensions that may involve only sectors of the sphere such as those used [103] for nonlocal convection and in the studies of [6, 46, 33] on nonlocal variational problems. The analytical results in this work are largely based on the Fourier analysis which is limited to problems defined over periodic cells. On one hand it will be interesting to consider the analogue on more general domains with more general boundary conditions or nonlocal constraints, which could be facilitated by further developments of nonlocal vector calculus leading to the results like nonlocal vector identities that are tailored to our nonlocal operators, particularly so for Neumann and Robin conditions [23, 30, 27, 78, 77]. Without the Fourier analysis mathematical investigation of the associated nonlocal energy spaces will need to be carried out using different techniques and more sophisticated analytical tools, such as those in similar spirits to the works of [76, 78, 77, 8]. On the other hand our analysis on a periodic setting does have a direct impact on the scientific communities working on numerical methods that are based on nonlocal integro differential operators. For example, one such method is the SPH mentioned earlier, which has indeed been directly applied for numerical simulations on periodic domains [122, 85]. Among a myriad of choices of kernels our current work selects for researchers a particular family of kernels that are theoretically verified to be effective. Our study also imparts to the researchers the message that the underlying nonlocal continuum formulations are bona fide mathematical objects deserving to be scrutinized in view of the local setting wherein successful developments of numerical methods have been built upon rigorous theoretical understandings on the continuum level. Meanwhile we point out that the well-posedness of our nonlocal Stokes and Naviér equations in the periodic setting is naturally linked to the consideration of their Fourier spectral discretizations and related numerical issues such as the asymptotic compatibility [104] as in [37]. Further numerical analysis of other discretizations are also important for applications and will be left to future works.

We have demonstrated in this chapter the significance of choice of nonlocal interactions kernels since the nonlocal hemispherical interaction neighborhoods are after all encoded in terms of the characteristic functions. Indeed suitable evaluations of nonlocal interactions is a challenge that is pervasive in nonlocal modeling in general. With hindsight we point out that in chapter 2 we have chosen the same kernel w_δ for the nonlocal operators $\mathcal{A}_\delta$ and $\mathcal{D}_\delta$ in our Model II, which simplified our analysis there. On the other hand let us recall that the kernel choice for $\mathcal{D}_\delta^{\vec{n}}$ in this chapter is simply determined by that for $\mathcal{G}_\delta^{\vec{n}}$ due to the mirroring of the adjoint relationships in the local case. However a more complicated situation involving two nonlocal kernels may arise and we illustrate in the next chapter that nonlocal continuum formulation can help answer the question of what conditions to be satisfied by one kernel given that the other kernel has already been prescribed.

Chapter 4: Second order accurate nonlocal Dirichlet boundary conditions

This chapter presents an approach to impose nonlocal analogues of local Dirichlet boundary conditions for nonlocal integro-differential equations with a finite range of interactions. As mentioned in chapter 1 our goal is to ensure that the solutions of our nonlocal continuum formulation converge to the local solutions at the quadratic convergence rate in the limit of $\delta \to 0$. The particular setting in which which we present our approach is the following nonlocal linear diffusion models

$$\mathcal{L}_\delta u(x) := \int_{B_\delta(x)} (u(x) - u(y)) w_\delta(x, y) dy = f(x), \quad x \in \Omega$$

on a domain Ω where $w_\delta(x, y)$ is a nonlocal interaction kernel. Those appear in the bond-based peridynamic models [76, 97] with connections to the SPH setting [37, 26]. On one hand the simplicity of the model allows us to present our findings without much technical difficulties. On the other hand, despite a large number of theoretical and numerical studies on the linear nonlocal diffusion models [52, 108, 75, 39, 117, 90], an outstanding issue in the existing body of literature is that of enforcing the suitable volume constraints [23] which are nonlocal equivalents to the local boundary conditions. In the presence of physical boundaries an intrinsic challenge is to prescribe the nonlocal interactions in the δ-layer outside the domain, which is in fact a distinct feature of generic nonlocal models with a finite δ range of nonlocal interactions.

To better contextualize our study let us recall that the method of Morris et al, one of the most popular strategies to enforce Dirichlet boundary conditions, has two drawbacks [58]; the first is the computation of distances to the boundary which could be costly, and the second is the necessity to cap the relative magnitudes of the computed distances for numerical stabilities. The paper [114] discusses an improvement of the method of Morris et al, addressing the complication of non-unique normal distances of a point to the boundary, yet the two drawbacks of the original method remain unresolved in their approach. The work of [56] proposes an efficient way to approximate the ratios of distances needed for extrapolation by Morris et al. However their method, an estimation after all, can potentially deteriorate the accuracy of the generated numerical solutions. Apart from the discrete settings of the SPH the authors in [42] prove the second order convergence of the nonlocal solution to the local one when the linear extrapolation is based on the exact derivatives of the latter, which however may not be known a priori. We should note the recent work [119] in the context of peridynamic modeling presents an automated algorithm that promotes the so called mirror-based fictions nodes methods, an equivalent to the SPH ghost particle method [18]. Despite its design to avoid the numerical stability issue of the method of Morris, the mirroring approach has been shown in [71] to yield less accurate approximates of the nonlocal diffusion operator.

Our main contribution is a theoretical proof of the localization, at the second order rate, of our nonlocal continuum formulation which is similar to that of Morris. Our approach, however, departs from the existing ones in that nonlocal gradient operators, instead of the finite differences, are used to extrapolate the boundary data to the volumetric data. With a suitable choice of kernels in the nonlocal operators we illustrate that we can circumvent the explicit calculations of the distances to the boundaries. The nonlocal interaction neighborhoods of nonlocal gradient operators extend into the interior of the domain, which we exploit to implicitly compute the appropriate normalization factors to enforce consistency of the nonlocal operators with the local ones. Moreover the computed normalization factors, for a fixed nonlocal horizon, are nonvanishing, which can potentially amount to a relaxed consideration of numerical stability. After all the rationale behind our work is to enforce the coupling of the nonlocal diffusion operator in the bulk domain with another nonlocal operator to handle the boundary. In order to clearly demonstrate our idea we focus on one dimensional setting just as the analogous result in one dimensional setting for Neumann boundary conditions [100] is established first before its extension to two dimensions in a subsequent work [116].

Our approach of using the nonlocal gradient operators in place of the finite differences is closely linked to the view that nonlocal operators can be seen as a weighted average of discrete finite difference operators. This perspective is in fact what we rely on in Chapter 2 to propose our continuum nonlocal models, apart from the interpolation-based interpretations of the SPH-like methods. As then expected our analysis in this chapter comes down to analyzing the nonlocal continuum equations with the truncation errors as the source terms. On a related note let us emphasize that the quadratic rate of convergence we show is with respect to the uniform, instead of the L^2 norm as adopted in the localization results of the nonlocal mechanics models in Chapter 3.

4.1 Well-posedness of nonlocal formulation

For the rest of the paper we consider the one dimensional domain $\Omega = (0, 1)$ unless otherwise noted. We seek a nonlocal relaxation of the classical local PDE with homogeneous boundary conditions

$$\begin{cases} -\Delta u(x) = f(x), \quad x \in \Omega \\ u(0) = u(1) = 0 \end{cases} \tag{4.1}$$

in the form of

$$\begin{cases} \widetilde{\mathcal{L}}_\delta u(x) := (\mathcal{L}_\delta - \mathcal{M}_\delta) u(x) = f(x), \quad x \in \Omega \\ u(x) = 0, \quad x \in (-\delta, 0) \cup (1, 1+\delta) \end{cases} . \tag{4.2}$$

Here M_δ is a perturbation of $\mathcal{L}_\delta$

$$\mathcal{M}_\delta u(x) = \int_{B_\delta(x)\backslash\Omega} \big(u(x) + \mathcal{G}_\delta u(x)(y-x)\big)\, w_\delta(x,y) dy$$

where $\mathcal{G}_\delta(u)(x)$ is a nonlocal gradient operator given by

$$\mathcal{G}_\delta u(x) = \begin{cases} \frac{1}{\int_{B_\delta(x)\cap\Omega} \operatorname{dist}(y,\partial\Omega)\rho(x,y)dy} \begin{cases} \int_{B_\delta(x)\cap\Omega} u(y)\rho(x,y)dy, & x \in (0,\delta) \\ \int_{B_\delta(x)\cap\Omega} (-u(y))\rho(x,y)dy, & x \in (1-\delta,1) \end{cases} \\ 0, \quad \text{otherwise} \end{cases}$$

for some nonlocal interaction kernel ρ, which we want to choose in order to dispense with calculating the distances $\operatorname{dist}(y,\partial\Omega)$ to the boundaries. Indeed there is such a choice, namely $\rho \equiv 1$, which we utilize for the rest of the paper. Considering without loss of generality $x \in (0,\delta)$, we note that the convexity of $B_\delta(x) \cap \Omega$ gives

$$\int_{B_\delta(x)\cap\Omega} \operatorname{dist}(y,\partial\Omega)dy = \int_0^{x+\delta} |y-0|dy = \int_0^{x+\delta} |y-x-\delta|dy = \frac{(x+\delta)^2}{2}.$$

Next we specify the assumptions on the kernel $w_\delta(x,y)$ of the nonlocal diffusion operator $\mathcal{L}_\delta$. We assume $w_\delta(x,y) = \frac{1}{\delta^3} w\left(\frac{|x-y|}{\delta}\right)$ where

$$\begin{cases} w \text{ is continuous, nonincreasing and positive on (0,1) outside which it} \\ \text{vanishes, and satisfies } \int_{\mathbb{R}} w(|z|)|z|^2 dz = 2. \end{cases} \tag{A1}$$

We can then rewrite the equation eq. (4.2) as the following equivalent integral equation

$$\widetilde{\mathcal{N}}_\delta u = f \quad \text{in } \Omega \tag{4.3}$$

where

$$\widetilde{\mathcal{N}}_\delta u(x) := a_\delta(x)u(x) - \int_\Omega u(y)\left(w_\delta(x,y) - b_\delta(x)\chi_{[0,\delta]}(|y-x|)\right)dy.$$

Here $\chi_{[\cdot]}$ denotes the characteristic function,

$$a_\delta(x) = \int_\Omega w_\delta(x,y)dy$$

and

$$b_\delta(x) = \begin{cases} \dfrac{2}{(x+\delta)^2} \displaystyle\int_{B_\delta(x)\backslash\Omega} (x-y) w_\delta(x,y)dy, & x \in (0,\delta) \\ \dfrac{2}{(1-x+\delta)^2} \displaystyle\int_{B_\delta(x)\backslash\Omega} (y-x) w_\delta(x,y)dy, & x \in (1-\delta, 1) \\ 0 \text{ otherwise} \end{cases}.$$

We note the kernel in the second term of $\widetilde{\mathcal{N}_\delta}$ is in general sign changing and translation-variant. This poses a challenge to adapt the technique of [75] because the kernel considered therein is translation invariant. Instead we will resort to the idea of comparison matrix used in [67, 87] and introduce analogously the comparison operator

$$\widetilde{\mathcal{P}}_\delta(u)(x) := a_\delta(x)u(x) - \int_0^1 u(y)\widetilde{w}_\delta(x,y)dy$$

where $\widetilde{w}_\delta(x,y) = |w_\delta(x,y) - b_\delta(x)\chi 1_{(0,\delta)}(|y-x|)|$. We in turn present the comparison problem

$$\widetilde{\mathcal{P}}_\delta v_\delta = f \quad \text{in } \Omega \tag{4.4}$$

the solvability of which will lead to that of the original problem eq. (4.3). In the meantime we it is not difficult to deduce from the assumptions on w that both $\widetilde{\mathcal{N}_\delta}$ and $\widetilde{\mathcal{P}}_\delta$ are bounded operators on $L^2(\Omega)$, which then leads us to consider $f \in L^2(\Omega)$. With the aforementioned preparations we can now state the well-posedness of eq. (4.4).

Proposition 7. *Suppose that in addition to* (A1) *the kernel w_δ satisfies*

$$a_\delta(x) \geq \int_\Omega \tilde{w}_\delta(x,y)dy, \quad x \in \Omega \tag{A2}$$

Then $\widetilde{\mathcal{P}}_\delta$ is invertible with

$$\|(\widetilde{\mathcal{P}}_\delta)^{-1}\|_2 \leq C$$

for some $C(w, \delta) > 0$.

Proof. We first argue $\tilde{\mathcal{P}}_\delta$ satisfies the Fredholm alternative. Since $a_\delta(x) > 0$ we have that $a_\delta(x)I$ is an invertible operator. Meanwhile the continuity of w_δ implies $\int_\Omega u(y)\tilde{w}_\delta(x, y)dy$ is a compact operator.

Next we show the kernel of $\tilde{P}_\delta$ is trivial, which then would imply that $\tilde{P}_\delta$ is invertible. To this end let $v \in L^2(\Omega)$ such that $\tilde{\mathcal{P}}_\delta v \equiv 0$. That is,

$$v(x) = \frac{1}{a_\delta(x)} \int_\Omega v(y)\tilde{w}_\delta(x, y)dy.$$

We see v is continuous on $\overline{\Omega}$ due to the continuities of a_δ and $\tilde{w}_\delta(x, y)$. Then we apply the assumption (A2) to obtain that v is a constant function. To rule out nonzero constants it is sufficient to show that the inequality in (A2) is strict in some subdomain of Ω. Indeed for $x \in (0, \delta)$ the non-increasing assumption on w gives

$$w_\delta(x, y) \geq b_\delta(x)$$

when $|y - x| \geq x$, so that

$$a_\delta(x) - \int_0^1 \tilde{w}_\delta(x, y)dy \geq xb_\delta(x) > 0 \quad x \in \left(\frac{\delta}{2}, \frac{3\delta}{4}\right)$$

where the last inequality is due to the positivity of w_δ.

Finally we conclude the proof by invoking the bounded inverse theorem. □

Remark 3. *Since w is assumed to be a scaled kernel and Ω is an interval, the condition* (A2) *holds for all $\delta \in (0, \delta_0)$ if it is satisfied for some δ_0. In particular the constant function as well as the piecewise linear function used in [41] satisfy* (A2). *An analogues of the condition in the discrete* *setting would amount to diagonal dominance.*

Next result we present is concerned with a comparison principle of the comparison problem eq. (4.4). One can almost expect the result to hold from the nonnegativity of the kernel $\widetilde{w}_\delta$. We will utilize the comparison principle in proving the main result of the section, theorem 11 on the well-posedness of the original problem eq. (4.3).

Lemma 17. *Under the same assumptions in Proposition 7, $\widetilde{\mathcal{P}}_\delta\phi \geq 0$ implies $\phi \geq 0$ for $\phi \in L^2(\Omega)$*

Proof. A proof is essentially an adaption of the argument in [51] in the finite dimensional setting, but we include it here for completeness. Let us first write

$$u = \phi^+ - \phi^-$$

into positive and negative parts. Then since $\tilde{w}_\delta$ is nonnegative we have

$$(\widetilde{\mathcal{P}}_\delta\phi^+, \phi^-)_{L^2} = \underbrace{\int_0^1 a_\delta(x)\phi^+(x)\phi^-(x)dx}_{=0} - \int_0^1 \underbrace{\left(\int_0^1 \phi^+(y)\widetilde{w}_\delta(x,y)dy\right)}_{\geq 0}\phi^-(x)dx \leq 0$$

so that if $\phi^- \neq 0$

$$0 \leq (\widetilde{\mathcal{P}}_\delta\phi, \phi^-) = (\widetilde{\mathcal{P}}_\delta\phi^+, \phi^-) - (\widetilde{\mathcal{P}}_\delta\phi^-, \phi^-) < 0$$

which is a contradiction. Here the last inequality is due to the fact that $\widetilde{P}_\delta$ has a trivial kernel. □

Theorem 11. *Under the assumptions* (A1) *and* (A2)*, the same conclusions as in Proposition 7 hold for the problem eq.* (4.3) *in place of eq.* (4.4).

Proof. By linearity it is sufficient consider the case $f \geq 0$. Since $a_\delta(x) > 0$ we can write

$$\widetilde{\mathcal{N}}_\delta = I - \widehat{N}_\delta$$

$$\widetilde{\mathcal{P}}_\delta = I - \widehat{P}_\delta$$

where

$$\widehat{N}_\delta\phi(x) = \int_\Omega \phi(y)\frac{w_\delta(x,y) - b_\delta(x)\chi_{[0,\delta]}(|y-x|)}{a_\delta(x)}dy$$

$$\widehat{P}_\delta\phi(x) = \int_\Omega \phi(y)\frac{|w_\delta(x,y) - b_\delta(x)\chi_{(0,\delta)}(|y-x|)|}{a_\delta(x)}dy.$$

By construction we observe

$$|\widehat{N}_\delta^n f(x)| := |\underbrace{\widehat{N}_\delta \circ \cdots \circ \widehat{N}_\delta}_{\text{applied n times}} f(x)| \leq \widehat{P}_\delta^n f(x), n \in \mathbb{N}, x \in \Omega.$$

On the other hand we have

$$(\widetilde{P}_\delta)^{-1} f \geq \sum_{j=0}^{n} \widehat{P}_\delta^n f, n \in \mathbb{N}$$

since we can apply lemma 17 to

$$\widetilde{P}_\delta\left((\widetilde{P}_\delta)^{-1} f - \sum_{j=0}^{n} \widehat{P}_\delta^j f\right) = \widehat{P}_\delta^{n+1} f \geq 0$$

Consequently we obtain

$$|\sum_{j=0}^{\infty} \widehat{N}_\delta^j f(x)| \leq (\widetilde{P}_\delta)^{-1} f(x), x \in \Omega.$$

which proves the existence of $\widetilde{\mathcal{N}}_\delta^{-1} : L^2(\Omega) \to L^2(\Omega)$. As in the proof of Proposition 7 the proof is complete due to the bounded inverse theorem. □

Before we discuss the local limit of our formulation let us briefly mention our nonlocal treatment of the case where the local PDE (4.1) is subject to the inhomogeneous Dirichlet boundary conditions $u(0) = a$ and $u(1) = b$. In that case we may take the linear function ϕ which satisfies those boundary conditions, and solve eq. (4.3) with f replaced by $f - \widetilde{\mathcal{N}}_\delta\phi$. Alternatively we may modify the definition of $\mathcal{G}_\delta u(x)$ for $x \in (0,\delta) \cup (1-\delta, 1)$ into

$$\frac{1}{\int_{B_\delta(x)\cap\Omega} \text{dist}(y,\partial\Omega)\rho(x,y)dy} \begin{cases} \int_{B_\delta(x)\cap\Omega}(u(y)-a)\rho(x,y)dy, & x\in(0,\delta) \\ \int_{B_\delta(x)\cap\Omega}(b-u(y))\rho(x,y)dy, & x\in(1-\delta,1) \end{cases}$$

without affecting the validity of our analysis.

4.2 Asymptotic localization as $\delta \to 0$

It is important to analyze consistency between our nonlocal formulation and the corresponding local one as the former is conceived as a relaxation of the latter. We are interested in studying the asymptotic behavior of the nonlocal solution as the nonlocality vanishes, namely its convergence to the local one when the local formulation is mathematically valid. To this end we turn to the variational framework and prove a sharper version of the stability result than theorem 11 wherein the constant C is independent of δ. As in the previous section we will first work with the comparison problem by introducing the quadratic form

$$\begin{aligned} Q_\delta(u) := (\widetilde{\mathcal{P}}_\delta u, u) = \frac{1}{2}\int_0^1\int_0^1 (u(y)-u(x))^2 \tilde{w}^s_\delta(x,y)dxdy + \\ \int_0^1 u(x)^2\left(\int_0^1 (w_\delta(x,y) - \tilde{w}^s_\delta(x,y))dy\right)dx. \end{aligned}$$

One can find in literature the related results when nonlocal interaction kernels are radially symmetric. We would like to make use of those results in our setting and as the first step we show that the symmetric kernel $\widetilde{w}^s_\delta(x,y)$ can be bounded from below by some radially symmetric one.

Lemma 18. *Suppose w satisfies* (A1).*Then there exists a radial, monotone, non-negative kernel ρ that is compactly supported on* $(0,1)$ *and strictly positive on* $(0,\sigma)$ *for some* $0<\sigma$*, satisfying*

$$\rho_\delta(|x-y|) := \frac{1}{\delta^3}\rho_\delta\left(\frac{|x-y|}{\delta}\right) \le \widetilde{w}^s_\delta(x,y) \quad \forall x,y\in\Omega$$

and

$$0 < \int_{\mathbb{R}} \rho_\delta(z)|z|^2 < \infty.$$

Proof. We consider two cases

1. w is constant, i.e. $w_\delta(|x|) = \frac{3}{\delta_3}\chi_{[0,\delta]}(|x|)$. In this case

$$\widetilde{w}_\delta(x,y) = \chi_{[0,\delta]}(|x-y|) \times \begin{cases} \frac{6x}{\delta^3(x+\delta)} \text{ if } x \in (0,\delta) \cup (1-\delta,1) \\ \frac{3}{\delta^3} \text{ otherwise} \end{cases}$$

so that

$$\widetilde{w}^s_\delta(x,y) = \frac{3}{\delta^3}\chi_{[0,\delta]}(|x-y|) \times \begin{cases} \left(\frac{x}{x+\delta} + \frac{y}{y+\delta}\right) \text{ if } x,y \in (0,\delta) \cup (1-\delta,1) \\ \left(\frac{x}{x+\delta} + \frac{1}{2}\right) \text{ if } x \in (0,\delta) \cup (1-\delta,1), y \in (\delta, 1-\delta) \\ \left(\frac{y}{y+\delta} + \frac{1}{2}\right) \text{ if } x \in (\delta,1-\delta), y \in (0,\delta) \cup (1-\delta,1) \\ 1 \text{ if } x,y \in (\delta,1-\delta) \end{cases}$$

We set

$$\rho(|x-y|) = \chi_{[0,1]}(|x-y|)\frac{3}{2}\frac{|x-y|}{|x-y|+1}.$$

2. Otherwise we let

$$\rho(|x|) = \frac{1}{2}\max\left(w(|x|) - 2\int_0^1 yw(|y|)dy, 0\right)$$

which is supported on $(0,\sigma)$ for some $0 < \sigma < 1$ due to (A1).

In both cases one can verify $\rho_\delta(|x-y|) := \frac{1}{\delta^3}\rho_\delta\left(\frac{|x-y|}{\delta}\right) \leq \frac{1}{2}w_\delta(x,y)$, where ρ satisfies the desired properties, completing the proof. □

Next result we present is concerned with a sharper version of the inequality in (A2). The nonlocal variational framework which we would like to rely on is pertinent to the cased of positive definite nonlocal operators, which we have not yet established for the operator $\widetilde{\mathcal{P}}_\delta$. The following result will help us stride towards that direction.

Lemma 19. *Suppose that w satisfies* (A1) *and*

$$|\{y \in \Omega : w_\delta(x,y) - b_\delta(x)\chi_{[0,\delta]}(|y-x|) \geq 0\}| \geq |\{y \in \Omega : w_\delta(x,y) - b_\delta(x)\chi_{[0,\delta]}(|y-x|) \leq 0\}|, \quad x \in \Omega. \tag{A2s}$$

Then

$$a_\delta(x) - \int_\Omega \tilde{w}_\delta(x,y)dy \geq C\delta b_\delta(x), \quad x \in \Omega$$

for some $C > 0$ independent of δ.

Proof. We consider $x \in (0,\delta)$ and recall $w_\delta(|x|) \geq b_\delta(x)$ due to (A1). Then we have

$$\begin{aligned}\widetilde{\mathcal{P}}_\delta(1) = a_\delta(x) - \int_\Omega \tilde{w}_\delta(x,y)dy &= \int_0^1 \left(w_\delta(x,y) - |w_\delta(x,y) - b_\delta(x)\chi_{(0,\delta)}(|y-x|)|\right) dy \\ &= b_\delta(x)(x + (2r(x)-1)\delta) + 2\int_{x+r(x)\delta}^{x+\delta} w_\delta(x,y)dy\end{aligned}$$

where

$$r(x) = \sup_{z\in[0,1]} \{w_\delta(x, x+z\delta) - b_\delta(x) \geq 0\}.$$

Since $b_\delta(x)$ is decreasing on $(0,\delta)$ it follows from (A1) and (A2s) that $r(x)$ is increasing on $(0,\delta)$ with $r(0) \geq \frac{1}{2}$ and $r(\delta) = 1$. Let us consider the cases.

- If there exists $\lambda \in (0,1)$ such that $\frac{1}{2} < r(\lambda) < 1$, then we have

$$\widetilde{\mathcal{P}}_\delta(1) \geq \begin{cases} 2\int_{x+r(\lambda)\delta}^{x+\delta} w_\delta(x,x+y)dy = \frac{2}{\delta^2}\int_{r(\lambda)}^1 w(y)dy, x \in (0,\lambda\delta) \\ b_\delta(x)(2r(\lambda)-1)\delta, x \in (\lambda\delta,\delta) \end{cases}$$

- Otherwise

$$r(x) = \begin{cases} \frac{1}{2}, & 0 < x < t \\ 1, & t < x < 1 \end{cases}$$

for some $t \in (0,1)$, so that

$$\widetilde{\mathcal{P}}_\delta(1) \geq \begin{cases} 2\int_{x+\frac{\delta}{2}}^{x+\delta} w_\delta(x, x+y)dy = \frac{2}{\delta^2}\int_{\frac{1}{2}}^{1} w(s)ds, x \in (0, t\delta) \\ b_\delta(x)\delta \quad \text{ if } x \in (t\delta, \delta) \end{cases}$$

The assumption (A1) implies $\int_s^1 w(y)dy = C_1(s) > 0$ for any $s \in (0,1)$ while we have $\delta b_\delta(0) = \frac{C_2}{\delta^2} \geq \delta b_\delta(x)$, hence proving the desired claim. □

We now present the uniform Poincare's inequality which amounts to variational stabilities of the nonlocal solutions.

Proposition 8. *Suppose w_δ satisfies* (A1)*,* (A2s) *and*

$$a_\delta(x) \geq \int_\Omega \tilde{w}_\delta(y, x)dy \quad \forall x \in \Omega. \tag{A3}$$

Then there exists $\delta_0 > 0$ and $C(w)$ such that

$$\|(\widetilde{\mathcal{N}}_\delta)^{-1}\|_2 \leq C$$

for $\delta \in (0, \delta_0)$.

Proof. We observe

$$|(\widetilde{N}_\delta)^{-1} f(x)| = |(\widetilde{N}_\delta)^{-1} f^+(x) - (\widetilde{N}_\delta)^{-1} f^-(x)| \leq 2(\widetilde{P}_\delta)^{-1}|f(x)|, \quad x \in \Omega$$

from the proof of theorem 11. Thus it is sufficient to prove

$$\|(\widetilde{\mathcal{P}}_\delta)^{-1}\|_2 \leq C$$

for some constant $C > 0$ uniformly in δ. We have

$$Q_\delta(u) \geq \frac{1}{2}\int_\Omega \int_\Omega (u(y) - u(x))^2 \rho_\delta(|x-y|) dx dy + \frac{\tilde{C}}{2}\int_\Omega u^2(x) \int_{B_\delta(x)\setminus\Omega} \frac{|y-x|}{\delta} w_\delta(|y-x|) dy dx$$

due to lemma 18, (A3) and lemma 19. If we now let

$$k_\delta(|x-y|) = \min\left\{\rho_\delta(|x-y|), \frac{|y-x|}{\delta} w_\delta(|y-x|)\right\}$$

which has a nonzero finite second moment, then we have

$$CQ_\delta(u) \geq \frac{1}{2}\int_\Omega \int_\Omega (u(y) - u(x))^2 k_\delta(|x-y|) dx dy + \int_\Omega u^2(x) \int_{B_\delta(x)\setminus\Omega} k_\delta(|x-y|) dy dx$$

Now we deduce from (A1) and the monotonity of ρ_δ that k_δ satisfies the hypothesis of Theorem 1.3 in [90] so that we conclude

$$Q_\delta(u) \geq C\|u\|_2^2.$$

□

Remark 4. *The assumption* (A3) *would amount to column diagonal dominance in a discrete setting. It is to be checked on* 2δ*-inner layer of* Ω*, that is,* $(0, 2\delta) \cup (1 - 2\delta, 1)$*, as opposed to the* δ *inner layer which is the case for* (A2)*. One can verify that it holds true for the constant as well as linear kernels, both of which also satisfy* (A2s)*.*

Another ingredient that is often used to study the local limit is a compactness result such as Lemma 5.2 in [76] which is applicable to uniformly (in δ) bounded nonlocal energies. However we are not aware of such result when the kernel is neither radially symmetric nor nonnegative. Instead we take a detour by considering the following truncation error analysis.

Lemma 20. *For $\phi \in C^4(\bar{\Omega})$ with $\phi(1) = \phi(0) = 0$, we have*

$$T_\delta(\phi)(x) := (-\Delta + \widetilde{\mathcal{N}}_\delta)\phi(x) = \begin{cases} O(\delta^2) \text{ for } x \in (\delta, 1-\delta) \\ O(1) \text{ otherwise} \end{cases}.$$

More specifically there exists a constant $M > 0$ independent of δ such that

$$|T_\delta(\phi)(x)| \le M\delta^3 b_\delta(x) + O(\delta^2).$$

Proof. The results are straightforward by a Taylor series expansion and (A1). □

Now we establish the convergence in L^2 norm of the solutions to the nonlocal problems as the nonlocality parameter vanishes.

Theorem 12. *Under the same assumptions as in Proposition 8, if u_δ solves (4.3) and u_0 solves (4.1), then*

$$\lim_{\delta\to 0} \|u_\delta - u_0\|_2 = 0$$

Proof. Without loss of generality we consider a sequence of $\delta_n \to 0$ and denote the corresponding nonlocal solution $u_{\delta_n} = (\widetilde{\mathcal{N}}_{\delta_n})^{-1} f$ by u_n. Let $\{f_m\}_{m=1}^{\infty}$ be a sequence of smooth functions on Ω such that $\lim_{m\to\infty} \|f_m - f\|_2 = 0$. Now fix $\epsilon > 0$.

Let $u_{m,0}$ solve eq. (4.3) with f_m in place of f, and let $u_{m,n} = (\widetilde{\mathcal{N}}_{\delta_n})^{-1} f_m$. Then we obtain

$$\begin{aligned}
&\|u_0 - u_n\|_2 \\
&\le \|u_0 - u_{m,0}\|_2 + \|u_{m,0} - u_{m,n}\|_2 + \|u_{m,n} - u_n\|_2 \\
&\le \|u_0 - u_{m,0}\|_2 + \|(\widetilde{\mathcal{N}}_{\delta_n})^{-1}\|_2 \left(\|T_{\delta_n} u_{m,0}\|_2 + \|f_m - f\|_2\right) \\
&\le \|u_0 - u_{m,0}\|_2 + C \left(\|T_{\delta_n} u_{m,0}\|_2 + \|f_m - f\|_2\right)
\end{aligned}$$

where the last inequality is due to Proposition 8. Now a very crude estimate from lemma 20 shows

$$\|T_{\delta_n} u_{m,0}\|_2 \le C_1(m)(\delta_n)^{1/2}.$$

Hence if we pick a particular $\hat{m}$ so that

$$\|u_0 - u_{\hat{m},0}\|_2 + C\|f_{\hat{m}} - f\|_2 < \frac{2\epsilon}{3}$$

then we have

$$C\|T_{\delta_n} u_{m,0}\|_2 \leq C_1(m)(\delta_n)^{1/2} < \frac{\epsilon}{3}$$

for all n sufficiently large, proving the claim. $\square$

We finally turn to the order of convergence rate when the local solution is smooth. We would like to apply the barrier function technique as done in [100]. At our disposal is the comparison principle lemma 17 for the comparison operator $\widetilde{P}_\delta$ so our strategy is to consider a suitable comparison problem. In our explicit construction of a barrier function we exploit the fact that the conclusion of lemma 19 provides a lower bound for $\widetilde{P}_\delta 1$ in terms of b_δ which is involved in the upper bounds on the truncation error in lemma 20.

Proposition 9. *Suppose w satisfies* (A1) *and* (A2s). *Assume further f is regular enough such that the local solution u to* (4.1) *is smooth. If we denote by u_δ the nonlocal solution to* (4.3)*, then there exists $\delta_0 > 0$ and $C > 0$ such that*

$$\|u_\delta - u\|_\infty \leq C\delta^2$$

for $\delta \in (0, \delta_0)$

Proof. We consider

$$\widetilde{\mathcal{P}}_\delta v_\delta(x) = |T_\delta u_0(x)|$$

so that we have $|e_\delta| \leq 2v_\delta$ as in the proof of Proposition 8.

Let $\phi(x) = C + \psi(x)$ where $C > 0$ is a constant to be specified, and ψ is the solution to the problem

$$-\Delta\psi = 1, \qquad \text{in } \Omega$$

$$\psi = 0, \qquad \text{on } \partial\Omega$$

We have then

$$\begin{aligned}
&\widetilde{P}_\delta\phi(x) \geq C(a_\delta(x) - \|\tilde{w}_\delta(x,\cdot)\|_{L^1(\Omega)}) - C_1\left|\int_\Omega (x-y)\widetilde{w}_\delta(x,y)dy\right| \\
&-\psi''(x)\int_\Omega \frac{(y-x)^2}{2}\widetilde{w}_\delta(x,y)dy \\
&\geq CC_2\delta b_\delta(x) - C_1\left|\int_\Omega (x-y)\widetilde{w}_\delta(x,y)dy\right| + \int_\Omega \frac{(y-x)^2}{2}\widetilde{w}_\delta(x,y)dy \\
&\geq CC_2\delta b_\delta(x) - C_1\left(\left|\int_\Omega (x-y)w_\delta(x,y)dy\right| + b_\delta(x)\underbrace{\left|\int_\Omega (x-y)\chi_{(0,\delta)}(|y-x|)dy\right|}_{O(\delta^2)}\right) \\
&\int_\Omega \frac{|y-x|^2}{2}w_\delta(x,y)dy - b_\delta(x)\underbrace{\left(\int_\Omega \frac{|y-x|^2}{2}\chi_{(0,\delta)}(|y-x|)dy\right)}_{O(\delta^3)}
\end{aligned}$$

for some constants $C_1, C_2 > 0$. Here the first, second and third inequalities hold due to the non-negativity of ψ, lemma 19, and the property $\widetilde{w}(x,y) = \widetilde{w}(x,z)$ for $|y-x| = |z-x|$, respectively. Thus for sufficiently large $\tilde{C} > 0$ we can write

$$\widetilde{P}_\delta\phi(x) \geq M\delta b_\delta(x) - C_1\left|\int_\Omega (x-y)w_\delta(x,y)dy\right| + \int_\Omega \frac{|y-x|^2}{2}w_\delta(x,y)dy$$

But then since $\left|\int_\Omega (x-y)w_\delta(x,y)dy\right| \geq \delta^2 b_\delta(x)$ there exists $\delta_0 > 0$ and $C > 0$ such that

$$\widetilde{P}_\delta\phi(x) \geq M\delta b_\delta(x) + \int_\Omega \frac{|y-x|^2}{2}w_\delta(x,y)dy \geq M\delta b_\delta(x) + C$$

for all $\delta < \delta_0$, where the assumption (A1) is used in the last inequality. Therefore we obtain

$$|v_\delta(x)| \leq \left(\sup_{x\in\Omega}\phi(x)\right)\left(\sup_{x\in\Omega}\frac{|T_\delta u_0(x)|}{\widetilde{\mathcal{P}}_\delta\phi(x)}\right) \leq C_1 \sup_{x\in\Omega}\frac{M_1\delta^3 b_\delta(x) + O(\delta^2)}{M\delta b_\delta(x) + C} \leq C\delta^2$$

where the first and second inequalities are due to lemma 17 and lemma 20, respectively, completing the proof. □

Remark 5. *In comparison with theorem 12 we have removed the assumption* (A2s) *at the expense of a stronger regularity assumption on f.*

Based on our choice of the barrier function in the proof of Proposition 9 we can in fact revisit the existing methods of enforcing nonlocal Dirichlet boundary conditions to prove their convergence rates as $\delta \to 0$. The two approaches that we focus on are the constant extension methods [71] and the methods of Morris et al [82]. Notwithstanding their numerically demonstrated performances, we are not aware, to our best understanding, of existing theoretical justification for the convergence rates of their associated nonlocal solutions.

We first turn to the constant extension methods wherein the nonlocal volumetric constraints are prescribed via constant extension of the local boundary values, that is, $u(x) = 0$ for $x \in (-\delta, 0) \cup (1, 1+\delta)$. Hence the nonlocal continuum formulation, in the case of homogeneous boundary conditions as before, is given by

$$\begin{cases} \mathcal{L}_\delta u(x) = f(x), \quad x \in \Omega \\ u(x) = 0, \quad x \in (-\delta, 0) \cup (1, 1+\delta) \end{cases} . \tag{4.5}$$

One can find a multitude of thorough mathematical investigations on this formulation; see, for example, [28] and the references cited therein. It is straightforward to check that the truncation errors of the formulation, in the sense of lemma 20, are bounded by $M\delta^2 b_\delta(x)$ for some M independent of δ on $(-\delta, 0) \cup (1, 1+\delta)$. With the same choice of the barrier function as before we can prove

Proposition 10. *Suppose w satisfies* (A1). *Assume further f is regular enough such that the local solution u to* (4.1) *is smooth. If we denote by u_δ the nonlocal solution to* (4.5)*, then there exists $\delta_0 > 0$ and $C > 0$ such that*

$$\|u_\delta - u\|_\infty \leq C\delta$$

for $\delta \in (0, \delta_0)$.

Next we analyze the methods of Morris et al which is based on extrapolation via finite difference approximations of local gradients. The underlying continuum nonlocal formulation amounts to

$$\begin{cases} \widehat{\mathcal{L}}_\delta u(x) := (\mathcal{L}_\delta + \mathcal{F}_\delta)u(x) = f(x), \quad x \in \Omega \\ u(x) = 0, \quad x \in (-\delta, 0) \cup (1, 1+\delta) \end{cases} \tag{4.6}$$

where

$$\begin{aligned} &\mathcal{F}_\delta(u)(x) \\ &= \begin{cases} \frac{u(x)}{|x-p(x)|} \int_{B_\delta(x)\setminus\Omega} \left|(p(x)-y)\cdot \frac{x-p(x)}{|x-p(x)|}\right| w_\delta(x,y)dy, \quad x \in (0,\delta)\cup(1-\delta,1) \\ 0, \quad \text{otherwise} \end{cases} . \end{aligned}$$

Here $p(x)$ denotes the closest point to x on $\partial\Omega$. As in our formulation we can rewrite the formulation into

$$\widehat{\mathcal{N}}_\delta u = f \quad \text{ in } \Omega \tag{4.7}$$

where $\widehat{\mathcal{N}}_\delta$ is given by

$$\begin{aligned} \widehat{\mathcal{N}}_\delta u(x) := \left(\int_{B_\delta(x)\setminus\Omega} w_\delta(|x-y|)dy\right) u(x) + \mathcal{F}_\delta(u)(x) \\ + \int_\Omega (u(x)-u(y)) w_\delta(|x-y|)dy. \end{aligned}$$

We then define the associated energy space $V_\delta = \{u \in L^2(\Omega) : (\widehat{\mathcal{N}}_\delta u, u) < \infty\}$ for which we have the following characterization.

Lemma 21. *Assume w satisfies* (A1). *Then the problem* (4.7) *is well-posed over the space V_δ, a weighted L^2 space with the weight η_δ*

$$\eta_\delta(x) = 1 + \int_{B_\delta(x)\setminus\Omega} \frac{|(p(x)-y)\cdot(x-p(x))|}{|x-p(x)|^2} w_\delta(x,y)dy.$$

Moreover $H_0^1(\Omega)$ is a proper subset of V_δ.

Proof. Since the operator $\int_\Omega (u(x) - u(y)) w_\delta(|x-y|) dy$ is a bounded operator on $L^2(\Omega)$ we see that V_δ is the weighted L^2 space which is complete, hence the well-posedness follows due to the Riesz representation theorem. For the second claim we consider

$$\int_0^\delta \frac{u^2(x)}{x} \left(\int_{x-\delta}^0 |y| w_\delta(x,y) dy \right) dx$$
$$\leq \left(\int_{\mathbb{R}} |z| w_\delta(|z|) dz \right) \left(\int_0^\delta \frac{u^2(x)}{x^2} dx \right) \leq \left(\int_{\mathbb{R}} |z| w_\delta(|z|) dz \right) \left(\int_0^\delta (u'(x))^2 \right)$$

for $u \in H_0^1(0,1)$, where the last inequality is due to the Hardy's inequality [10]. Finally we can take $\phi(x) = \sqrt{x} - x$ to show the inclusion is proper, completing the proof. □

The inclusion of the Sobolev space $H_0^1(\Omega)$ in the nonlocal space V_δ clarifies the connection that the integral equation (4.7) is a suitable nonlocal candidate to approximate the local differential equation (4.1). At the same time we point out V_δ is strictly included in the solution space $L^2(\Omega)$ of our formulation (4.3) since a nonzero constant function does not belong to V_δ; the two may indeed be viewed as exploiting distinct degrees of nonlocal relaxations. A simple calculation shows that the same truncation error estimates hold for $\widehat{\mathcal{N}}_\delta$ as in lemma 20, and the same barrier function as before can be used to show the following.

Proposition 11. *Suppose w satisfies* (A1). *Assume further f is regular enough such that the local solution u to* (4.1) *is smooth. If we denote by u_δ the nonlocal solution to* (4.7)*, then there exists $\delta_0 > 0$ and $C > 0$ such that*

$$\|u_\delta - u\|_\infty \leq C\delta^2$$

for $\delta \in (0, \delta_0)$.

In comparison with our approach the two existing methods admit a wider range of nonlocal interaction kernels due to their more relaxed assumptions on the kernels. The simplicity of the constant extension method is its attractive feature which comes with the price of being only first order accurate. As opposed to the method of Morris et al, our formulation is based on the nonlocal operator that is not self-adjoint. Our approach however is closely related to the recent work [118] by Shi and Zhang which utilizes yet another non self-adjoint nonlocal operator in order to obtain the second order accurate approximation. In contrast with our choice of w_δ they adopt more regularly scaled kernel $W_\delta(x,y) = C_\delta W\left(\frac{|x-y|^2}{4\delta^2}\right)$, where W is radial, non-negative, smooth and compactly supported on $(-2\delta, 2\delta)$, and C_δ is a normalization constant. Their formulation can be expressed in the current $1D$ setting as

$$\begin{aligned}&\int_\Omega (u(x)-u(y))\frac{W_\delta(|x-y|)}{\delta^2}dy - \widehat{\mathcal{G}}_\delta u(x)\overline{W}_\delta(|x-p(x)|)\\&= \int_\Omega (f(y)+f(p(x))|x-p(x)|)\overline{W}_\delta(|x-y|)dy\end{aligned} \tag{4.8}$$

where $\overline{W}_\delta(|x-y|) = C_\delta \overline{W}(\frac{|x-y|}{4\delta^2})$, $\overline{W}(r) = \int_r^1 W(z)dz$, and

$$\widehat{\mathcal{G}}_\delta u(x) = \begin{cases} \frac{-2}{4\delta^2\overline{\overline{W}}_\delta(0)}\begin{cases}\int_0^{2\delta}\left(u(y)\overline{W}_\delta(|y|)+\delta^2 f(y)\overline{\overline{W}}_\delta(|y|)\right)dy, & x\in(0,2\delta)\\ \int_{1-2\delta}^1\left(u(y)\overline{W}_\delta(|1-y|)+\delta^2 f(y)\overline{\overline{W}}_\delta(|1-y|)\right)dy, & x\in(1-2\delta,1)\end{cases}\\ 0, \quad \text{otherwise}\end{cases}$$

for $\overline{\overline{W}}_\delta(|x-y|) = C_\delta\overline{\overline{W}}(\frac{|x-y|}{4\delta^2})$, $\overline{\overline{W}}(r) = \int_r^1 \overline{W}(z)dz$. We note that the operators $\widehat{\mathcal{G}}_\delta u(x)$ approximate the outward normal local derivatives with $O(\delta^2)$ error and result in $O(\delta)$ truncation error of their formulation in $(0,2\delta)\cup(1-2\delta,1)$, which is one order higher than our formulation. Our nonlocal gradient operator $\mathcal{G}_\delta u(x)$ is simpler than $\widehat{\mathcal{G}}_\delta u(x)$ with no dependence on the source term $f(x)$, but nevertheless our formulation is still able to achieve the second order accuracy. Another aspect of the simplicity of our approach is that no modification of the source term f is needed in our nonlocal equation (4.3) as opposed to the Zhang and Shi's formulation (4.8).

Notwithstanding the lack of self adjointness in our approach we illustrate in the sequel a scenario in two dimensions where the method of Morris et al may lose its optimal rate of convergence in the presence of a relatively simple circular boundary, for which we propose ours as a viable alternative. We remark that Shi and Zhang's formulation is originally given in a two dimensional setting, but as is the case in $1D$ their formulation is more complicated than what we present below involving the terms concerning the curvature of the boundary as well as the boundary surface (curve) integrals.

4.2.1 A two dimensional case study

In this subsection we present an extension of our formulation to a two dimensional setting focusing on its quadratic convergence rate. We promote our extension as a more accurate approach than the method of Morris et al which is expected to provide the sub-optimal convergence rate in our chosen two dimensional domain. Specifically let us consider as in [16] the punctured periodic domain $\Omega = [-2,2]^2 \backslash U$ where $U = \{\vec{x} = (x_1, x_2) \in \mathbb{R}^2 : |\vec{x} \leq 1\}$. We assume a sufficiently smooth f such that there exists a smooth u which is periodic on $\partial\Omega \backslash \partial U$ and satisfies

$$\begin{cases} -\Delta u = f, & x \in \Omega \\ u \equiv 0, & x \in \partial U \end{cases} \tag{4.9}$$

We choose for simplicity the constant kernel $w_\delta(\vec{x}, \vec{y}) = \frac{16}{\delta^4}\chi_{(0,\delta)}(|\vec{x}-\vec{y}|)$ which satisfies the normalization condition $\int_{\mathbb{R}^2} w_\delta(|\vec{x}|)|\vec{x}|^2 d\vec{x} = 4$. For concreteness we fix $\vec{x}^\star = (1+\epsilon, 0)$ where $0 < \epsilon < \delta$. We first recall the nonlocal operator in our $1D$ formulation which can be rewritten as

$$\int_0^1 (u(x) - u(y)) w_\delta(x,y) dy + \left(\int_0^1 \min(|z|, |z-1|) \chi_{(0,\delta)}(|x-z|) dz \right)^{-1} \cdot \left(\int_0^1 u(z) \chi_{(0,\delta)}(|x-z|) dz \right) \left| \int_0^1 (y-x) w_\delta(x,y) dy \right|.$$

We benchmark this in our current two dimensional setting by first defining the direction vector

$$n_\delta(\vec{x}^\star) = \frac{\frac{16}{\delta^4}\int_\Omega (\vec{y} - \vec{x}^\star) \chi_{(0,\delta)}(|\vec{y} - \vec{x}^\star|) d\vec{y}}{\left| \frac{16}{\delta^4}\int_\Omega (\vec{y} - \vec{x}^\star) \chi_{(0,\delta)}(|\vec{y} - \vec{x}^\star|) d\vec{y} \right|} = (1, 0).$$

Next we compute a nonlocal directional derivative in the direction of $n_\delta(\vec{x}^\star)$

$$\mathcal{G}_\delta^{n_\delta(\vec{x}^\star)}u(\vec{x}^\star) = \left(\int_\Omega |q_{n_\delta(\vec{x}^\star)}(\vec{y}) - \vec{y}|\chi_{\mathcal{I}_\delta(\vec{x}^\star)}(\vec{y})d\vec{y}\right)^{-1} \int_\Omega u(\vec{y})\chi_{\mathcal{I}_\delta(\vec{x}^\star)}(\vec{y})d\vec{y}$$

where $q_{n_\delta(\vec{x}^\star)}(\vec{y})$ is the projection of $\vec{y}$ onto $\partial\Omega$ along $n_\delta(\vec{x}^\star)$ and $\mathcal{I}_\delta(\vec{x}^\star) = B_\delta(\vec{x}^\star) \cup \{\vec{y} \in \mathbb{R}^2 : \vec{y} \cdot n_\delta(\vec{x}^\star) = y_1 \leq 0\}$. Now the geometry of $\mathcal{I}_\delta(\vec{x}^\star) \cap \Omega$ allows us to bypass the calculations of $q_{n_\delta(\vec{x}^\star)}(\vec{y})$ by using the distance along $n_\delta(\vec{x}^\star)$ to the segment $\{\vec{y} \in \partial B_\delta(\vec{x}^\star) \cap \Omega : (\vec{y}-\vec{x}^\star)\cdot\vec{n}_\delta(\vec{x}^\star) \geq 0\}$

$$\int_\Omega |q_{n_\delta(\vec{x}^\star)}(\vec{y}) - \vec{y}|\chi_{\mathcal{I}_\delta(\vec{x}^\star)}(\vec{y})d\vec{y} = \int_\Omega (1 + \epsilon + \sqrt{\delta^2 - y_2^2} - y_1)\chi_{\mathcal{I}_\delta(\vec{x}^\star)}(\vec{y})dy_1dy_2.$$

Finally our nonlocal formulation at $\vec{x} = \vec{x}^\star$ is given by

$$\int_\Omega (u(\vec{x}^\star) - u(\vec{y}))\frac{16}{\delta^4}d\vec{y} + \mathcal{G}_\delta^{n_\delta(\vec{x}^\star)}u(\vec{x}^\star)\int_\Omega (\vec{y} - \vec{x}) \cdot n_\delta(\vec{x}^\star)\frac{16}{\delta^4}d\vec{y} = f(\vec{x}^\star)$$

which has the $O(1)$ truncation error by construction. As in our analysis of the $1D$ case we can rewrite our formulation into the integral equation

$$a_\delta(\vec{x})u(\vec{x}) - \int_\Omega u(\vec{y})\left(\frac{16}{\delta^4} - \widehat{b}_\delta(\vec{x})\chi_{\mathcal{I}_\delta(\vec{x})}(\vec{y})\right)d\vec{y} = f(\vec{x}) \tag{4.10}$$

where

$$\widehat{b}_\delta(\vec{x}) = \left(\int_\Omega |q_{n_\delta(\vec{x})}(\vec{y}) - \vec{y}|\chi_{\mathcal{I}_\delta(\vec{x})}(\vec{y})d\vec{y}\right)^{-1}\left(\int_\Omega (\vec{y} - \vec{x}) \cdot n_\delta(\vec{x})\frac{16}{\delta^4}d\vec{y}\right)$$

if $\vec{x}$ belongs to the δ-layer $\{\vec{z} \in \Omega : \text{dist}(\vec{z}, \partial U) < \delta\}$ whereas $\mathcal{L}_\delta u(\vec{x}) = f$ otherwise. Then we have the following quadratic convergence result.

Proposition 12. *The nonlocal equation eq.* (4.10) *is well-posed over* $L^2(\Omega)$. *Moreover its unique solution* u_δ *converges in* L^∞ *to the local solution* u_0 *of eq.* (4.9) *at the rate of* $O(\delta^2)$ *as* $\delta \to 0$.

Proof. The similar line of analysis as in $1D$ can be adopted, hence we only sketch the key steps without detailed calculations. The first is to check that the comparison operator satisfies

$$a_\delta(\vec{x}) - \int_\Omega \left| \frac{16}{\delta^4} - \widehat{b}_\delta(\vec{x}) \chi_{I_\delta(\vec{x})}(\vec{y}) \right| d\vec{y} \geq C\delta \int_\Omega (\vec{y} - \vec{x}) \cdot n_\delta(\vec{x}) \frac{16}{\delta^4} d\vec{y}$$

for some $C > 0$ in parallel with lemma 19. The second is to apply the Fredholm argument as in theorem 11 for the well-posedness of the comparison problem, hence the original nonlocal equation (4.10). Lastly we use the comparison principle together with the barrier function $\tilde{C} + \phi$ for sufficiently large $\tilde{C} > 0$, where ϕ solves eq. (4.9) with $f \equiv 1$. □

Returning to the method of Morris et al let us note $p(\vec{x}^\star) = (1, 0)$ and the leading term in the truncation error $T_\delta(\vec{x}^\star)$ is given by

$$\frac{16}{\delta^4} \partial_1 u(1 + \epsilon, 0) \underbrace{\left(\int_{B_\delta(\vec{x}^\star) \cap \Omega} (y_1 - 1 - \epsilon) dy_1 dy_2 + \int_{B_\delta(\vec{x}^\star) - \Omega} (y_1 - 1) dy_1 dy_2 \right)}_{O(\delta^3)} = O\left(\frac{1}{\delta}\right).$$

It should be noted that this estimate is sharp since the term in the parenthesis is for instance, $\approx$ $2.9749e{-}04, 3.7654e{-}05, 4.7393e{-}06, 5.9457e{-}07$ when $\epsilon = \frac{\epsilon}{3}$ and $\delta = 0.1, 0, 05, 0.025, 0.00125$, respectively. Consequently there is a loss of opitmal second order convergence which can be seen from the equation for the error $\widehat{\mathcal{N}}_\delta e_\delta = T_\delta$ at $\vec{x} = \vec{x}^\star$, that is

$$\begin{aligned} &\frac{\delta^4}{16} \underbrace{\left(\int_{B_\delta(\vec{x}^\star) \cap \Omega} d\vec{y} + \frac{1}{\epsilon^2} \int_{B_\delta(\vec{x}^\star) \setminus \Omega} \left| (p(\vec{x}^\star) - \vec{y}) \cdot (\vec{x}^\star - p(\vec{x}^\star)) \right| d\vec{y} \right)}_{O\left(\frac{1}{\delta^2}\right)} e_\delta(\vec{x}^\star) \\ &+ \frac{C}{\delta^4} \int_{\Omega \cap B_\delta(\vec{x}^\star)} e_\delta(\vec{y}) d\vec{y} = T_\delta(\vec{x}^\star) \end{aligned}$$

This shows that if $e_\delta = O(\delta^2)$ then the hand side would be $O(\delta)$, which contradicts the right hand side being $O(\frac{1}{\delta})$. Our analysis in section 4.2 suggests the action of $\widetilde{\mathcal{N}}_\delta^{-1}$ lifts the order of the truncation error by $O(\delta^2)$ in the δ-layer, hence we can instead expect to get only $O(\delta)$ error in the uniform norm.

4.3 Discussion

This chapter proposes a formulation of enforcing Dirichlet type boundary conditions in the context of nonlocal linear diffusion problems. Our formulation rests on applying nonlocal gradient operators for linear extrapolation and we have specified the conditions on nonlocal interactions kernels which ensures the well-posedness of the nonlocal problem. Of perhaps more interests to scientific communities at large is the second order quadratic convergence rate at which the solution to the nonlocal problem converges the local counterpart uniformly in Ω. We point out that our convergence analysis supplements the previous work [71] which focuses on the consistency between the nonlocal and local operators as opposed to the solutions of their continuum formulations.

Our treatment of the boundary conditions for the nonlocal diffusion brings us to the question of whether a similar idea could be applied to the boundary conditions of some nonlocal convection models such as our upwinding model in chapter 2. Possibly more careful constructions of nonlocal gradients would be necessary in order to take into account the characteristics of the local convection in relation to the boundary data. The interplay between the kernels of the nonlocal operators is likely to be a crucial issue that warrants thorough investigations, for which it may prove useful to generalize our discussion in chapter 3 of nonlocal gradient operators with non-spherical interaction neighborhoods. In the meantime it is also of importance to investigate higher order extrapolations which involve higher order nonlocal integro-differential operators than nonlocal gradients.

While the focus of this chapter is to study the nonlocal models on the continuum level, its practical implications motivate us to carry out numerical studies in our future studies. At the same time we are working on extension of our current work to more general two dimensional domains for which precise assumptions on their geometries need to be carefully taken into account. We also plan to investigate our treatment for Dirichlet type boundary conditions in the context of a broader range of nonlocal problems such as those with nonlinear effects.

Lastly let us revisit the view, mentioned in the beginning of this chapter, of nonlocal operators as an averaged superposition of discrete difference operators. The quadratic convergence rate of our model, despite the $O(1)$ truncation near the inner δ-layer, resonates with the analogous results in the finite difference setting [115]. We may also understand the stability of our nonlocal operator, despite the sign changing nature of the nonlocal kernel, is attributed to the effect of averaging dominantly positive (hence stable) and the remaining non-positive parts of the kernel.

Conclusion

In this thesis we have tackled three specific mathematical issues that arise in the context of studying SPH-like particle methods. Within the framework of nonlocal continuum models with a finite range of interactions we have developed in chapter 2 asymptotically compatible particle methods for linear advection problems, analyzed in chapter 3 the coercivity of nonlocal gradient operators with a half-spherical nonlocal interaction neighborhoods, and proposed in chapter 4 second order accurate nonlocal approximations of Dirichlet boundary conditions. We point out that the mathematical developments of generic, or non-SPH, nonlocal models can not only foster but at the same time benefit from the specialized research efforts made towards the improvement of SPH. For instance the notion of asymptotic compatibility provides a criterion to assess robustness of the SPH whereas the SPH standard way of enforcing no slip boundary conditions help designing an approach to handle nonlocal Dirichlet boundary conditions for generic nonlocal linear diffusion models. It should also be noted that this thesis echoes the motto put forward in [27] of nonlocal thinking and local acting. For example, the upwinding nonlocal model is motivated by the upwinding finite difference schemes whereas the analysis of the nonlocal Dirichlet energies is carried out by means of Fourier analysis as would be done in the local setting. While an important premise of this thesis is the role of nonlocal models as a bridging tool to study SPH-like methods, it remains an interesting question to investigate whether nonlocal models of fluid can stand alone as bona-fide physical models that can complement the conventional local ones as in the relation of peridynamics modeling to classical theories of elasticity.

This thesis is just a starting point for mathematical studies of the SPH-like methods via nonlocal models. Besides the future work left for each of the specific problems considered in the thesis, there are many outstanding mathematical questions to be answered which can help resolve the challenges for those particle methods. The examples include nonlocal continuum formulations for incompressibility constraints, turbulence modeling, multiphase flows, vorticity and coupling of nonlocal SPH formulations with local PDEs [110], to name a few. At the same time it may prove fruitful to extend some of the existing studies of nonlocal models such as [34] on nonlocal gradient recovery which can have broader implications beyond the context of SPH-like schemes. Lastly we remark that what has been rather elusive in this thesis are the practical computational aspects of the SPH-like methods such as discussion on fast solvers and scalable algorithms. Given the real industrial and engineering applications of those methods we would like to pursue their numerical studies on the implementation side along with the indispensable theoretical investigations of consistency, stability and convergence.

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