Some canonical metrics on Kähler orbifolds

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Abstract

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This thesis examines orbifold versions of three results concerning the existence of canonical metrics in the Kähler setting. The first of these is Yau's solution to Calabi's conjecture, which demonstrates the existence of a Kähler metric with prescribed Ricci form on a compact Kähler manifold. The second is a variant of Yau's solution in a certain non-compact setting, namely, the setting in which the Kähler manifold is assumed to be asymptotic to a cone. The final result is one due to Uhlenbeck and Yau which asserts the existence of Kähler-Einstein metrics on stable vector bundles over compact Kähler manifolds.

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Chapter 1

Introduction

An orbifold is a generalization of a manifold whereby each local chart is equipped with the additional structure of a smooth action of a finite group in such a way that the underlying topological space is locally homeomorphic to the orbit spaces of these actions. Because an orbit space may admit singular points, it is possible to use orbifolds to model objects with specific types of singularities.

Standard notions from complex geometry extend to orbifolds. Some notions extend in an obvious manner: a tensor, for example, is just given locally by an *invariant* tensor on each chart (with some compatibility conditions). However, other notions require more care: a vector bundle associates to each chart an *equivariant* bundle, but with this convention, even though the total space of a vector bundle enjoys a natural projection map to the underlying orbifold, this map will in general fail to be a topological vector bundle in the usual sense.

With the tools from complex geometry available, it therefore makes sense to study the existence of canonical metrics in those situations that have been studied in the literature. While the problem of finding such metrics has been a fruitful direction of research within the field of differential geometry itself, recent developments in the setting of Kähler geometry have unearthed interesting connections to algebraic geometry, specifically to notions of stability motivated by geometric invariant theory. This rich interplay between differential geometry and algebraic geometry suggests there remains a deep trove of outstanding results to be uncovered and understood.

In addition, these interactions with algebraic geometry suggests that generalizations of existence results to such singular objects as orbifolds merit study and, indeed, have been previously considered in the literature (e.g in [56]) not just for the extra degree of generality, but also because constructions for orbifolds can be helpful to similar constructions for manifolds (see, for example, [35] and the gluing construction of [55]). Moreover, in the program of mirror symmetry (resp. string theory), it is known that orbifolds can arise as the mirrors of manifolds (resp. as singular geometries), and hence the study of their complex geometry merits at least as much concern as that of manifolds.

It turns out, perhaps unsurprisingly, that several seminal results concerning the existence of canonical metrics extend to the setting of orbifolds, and this thesis reviews three such extensions, as mentioned in the abstract. Each of these extensions is discussed in a separate paper by the author (see [34, 33, 32]).

One goal of this thesis is to highlight the novel components of these extensions, if there are any. Particular focus is given to those moments where approaches detour from those of the manifold setting. Also indicated, whenever possible, are ways in which approaches from the literature can be consolidated or simplified or—even—improved.

Due to the competing intention of completeness, there is a fear that such novel components could be lost or overlooked among pages of lengthy arguments and details. Because of this, an attempt is made to place a discussion of original ideas either at the beginning of each chapter or in separate "Remark" environments.

One small improvement worth emphasizing here is a new precision with respect to the decay rate concerning solutions to a Monge-Ampère equation corresponding to Yau's Theorem on asymptotically conical manifolds (orbifolds), as discussed in Remark 3.2.

Besides the three kinds of canonical metrics surveyed in this document, there exist many others—such as constant scalar curvature Kähler (cscK) metrics or, more generally, extremal metrics—which leave open other possible directions of study.

In particular, Ross-Thomas [56] have obtained an orbifold variant of a theorem due to Donaldson [27] relating K-semistability to the existence of a cscK metrics. However, their results apply only to those orbifolds with cyclic singluarities, and more work is required to extend their results to arbitrary singularities. In addition, their work suggests further connections to notions of stability and moduli from geometric invariant theory, which could be explored.

Because the landscape of algebraic geometry is so near to many of these results concerning canonical metrics, it is tempting to write in the language of stacks, especially within the orbifold setting that follows. However, in this document—and in those papers by the author which this document examines—there is an attempt to remain faithful to the language of complex geometry and write in a manner that is to be accessible to those working in this area. Nevertheless, it may be a fruitful endeavor in the future to examine relevant results using the language of stacks so as to cement the bridges to algebraic geometry even further.

Chapter 2

Ricci-flat metrics on compact Kähler orbifolds

Yau's solution [65] to Calabi's conjecture [15] states that if R is any $(1,1)$-form representing the first Chern class of a compact Kähler manifold (X, ω), then there is a unique Kähler metric in the cohomology class of ω whose Ricci form is R. If particular, if the first Chern class of X vanishes, then X admits a unique Ricci-flat Kähler metric.

Yau's proof relies upon the fact that to solve Calabi's problem it is sufficient to find a smooth solution φ to the following complex Monge-Ampère equation

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ is a positive form} \end{cases}, \tag{2.1}$$

where n is the complex dimension of X and F is a fixed smooth positive function (corresponding to the desired prescribed Ricci form R). In [34], we offer a self-contained exposition to show that Yau's solution extends to the setting of compact orbifolds.

Theorem 2.1. *Let* $(\mathcal{X}, \omega)$ *be a compact Kähler orbifold and* F *a smooth function on* $\mathcal{X}$ *with average value zero. Then equation* (2.1) *admits a smooth solution* φ*, unique up to additive constant.*

The approach we use in [34] is the classical continuity method, involving a priori estimates to solutions of a family of equations indexed by a parameter t (whose limit as $t \to 1$ is equation (2.1)). Because estimates and tools from the manifold setting extend with little difficulty, there is little novelty to our approach. It was only necessary to develop an appropriate framework of elliptic differential operators and their regularity within the setting of orbifolds, but since these regularity results are essentially local ones, they can be easily adapted to the orbifold setting using arguments involving charts.

It is easily shown that Theorem 2.1 implies—or, more precisely, is equivalent to—the following, which is the analogous extension of the original Calabi conjecture [15] to the setting of orbifolds.

Theorem 2.2. *Let* $(\mathcal{X}, \omega)$ *be a compact Kähler effective orbifold, and let* R *be a* $(1,1)$*-form representing the cohomology class* $2\pi c_1(\mathcal{X}) \in H^2(\mathcal{X}, \mathbb{R})$*. Then there is a unique Kähler form* ω' *on* $\mathcal{X}$ *such that*

(i) ω' *and* ω *represent the same cohomology class and*

(ii) the Ricci form of ω' *is* R*.*

In particular, if $c_1(\mathcal{X}) = 0$ as a cohomology class in $H^2(\mathcal{X}, \mathbb{R})$, then there is a unique Ricci-flat Kähler form on $\mathcal{X}$ (c.f. [17, Theorem 1.3]).

If one instead solves a slightly modified Monge-Ampère equation from the one above, namely,

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F+\varphi}\omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ is a positive form} \end{cases} \tag{2.2}$$

on a compact Kähler orbifold whose first Chern class $c_1(\mathcal{X})$ is negative with Kähler form ω representing $-2\pi c_1(\mathcal{X})$, then one obtains a Kähler-Einstein metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfying $\mathrm{Ric}(\omega_\varphi) = -\omega_\varphi$. Concisely, we have the following additional result concerning the existence of Kähler-Einstein metrics, which was considered by Aubin in the setting of manifolds [3] but which was also considered by Yau in the same setting as a special case of his more general results [65].

Theorem 2.3. *If $\mathcal{X}$ is a compact Kähler effective orbifold satisfying $c_1(\mathcal{X}) < 0$, then there is a Kähler metric $\omega \in -2\pi c_1(\mathcal{X})$ on $\mathcal{X}$ satisfying $\mathrm{Ric}(\omega) = -\omega$.*

One small subtlety in the orbifold setting is that the first Chern class may no longer be an integral class; instead it is a rational class. However, once this is understood, one obtains Ricci-flat metrics on classical examples of Calabi-Yau orbifolds that arise in mirror symmetry, such as hypersurfaces in toric varieties arising from reflexive polyhedra as in Batyrev's mirror construction (see Example 2.34 and more generally Example 2.35). Moreover, by applying Theorem 2.2 to the orbifold obtained as the rth root of an effective Cartier divisor D in a smooth X (see Example 2.36), one obtains Ricci-flat metrics with cone angle $1/r$ along D, thereby encompassing the results of [13] at least for cone angles of the form $\beta = 1/r$ (but also for more general types of cone angles by considering collections of divisors).

2.1 Orbifold preliminaries

The goal of this section is to review the notion of a Kähler orbifold and to review some differential geometric tools and concepts associated to these objects. There is some competing terminology and notation in the literature, so an additional goal of this section is to fix the terminology and notation we will use throughout.

2.1.1 Smooth orbifolds

Let X be a topological space. A real n-dimensional smooth orbifold chart for X consists of a triple (U, G, π) where U is an open connected subset of $\mathbb{R}^n$, G is a finite group of smooth automorphisms of U, and $\pi : U \to X$ is a continuous map which is invariant under the action of G and which induces a homeomorphism of U/G onto an open subset V of X, called the support of the chart (U, G, π). An embedding of a chart $(U_\alpha, G_\alpha, \pi_\alpha)$ into another $(U_\beta, G_\beta, \pi_\beta)$ consists of a smooth embedding $\lambda : U_\alpha \to U_\beta$ such that $\pi_\beta \circ \lambda = \pi_\alpha$. An orbifold atlas for X is a family $\mathcal{U}$ of charts whose supports cover X and which are compatible with one another in the sense that whenever x is a point in the intersection of the supports of two charts $(U_\alpha, G_\beta, \pi_\alpha)$ and $(U_\beta, G_\beta, \pi_\beta)$, then there is a third chart $(U_\gamma, G_\gamma, \pi_\gamma)$ whose support contains x and which enjoys embeddings into both of the charts $(U_\alpha, G_\alpha, \pi_\alpha)$ and $(U_\beta, G_\beta, \pi_\beta)$. An atlas $\mathcal{U}'$ is said to refine another atlas $\mathcal{U}$ if each chart of $\mathcal{U}'$ enjoys an embedding into some chart of $\mathcal{U}$. Two atlases are called equivalent if they share a common refinement.

By an orbifold $\mathcal{X}$ of real dimension n we mean a paracompact Hausdorff space X equipped with an equivalence class of n-dimensional orbifold atlases. We call X the underlying space of the orbifold $\mathcal{X}$. In particular, because X is paracompact, we may assume that X is covered by a locally finite collection V_α of supports with corresponding charts $(U_\alpha, G_\alpha, \pi_\alpha)$, which constitute an atlas for $\mathcal{X}$.

By a smooth map from an orbifold $\mathcal{X}$ into another $\mathcal{X}'$ we mean a map $f : X \to X'$ of their underlying topological spaces satisfying the property that for each $x \in X$, there is a chart (U, G, π) for $\mathcal{X}$ and a chart (U', G', π') for $\mathcal{X}'$ whose supports contain x and $f(x)$ respectively and there is a smooth map $f_{x,U,U'} : U \to U'$ covering the restriction of f to the supports.

Example 2.4. As an important example, suppose G is a compact Lie group acting smoothly, effectively, and almost freely (meaning with finite stabilizers) on a smooth manifold M. Then one can form an effective orbifold $[M/G]$ called the effective quotient orbifold in the following way. The underlying space is the topological quotient M/G, which is paracompact and Hausdorff. Because smooth actions are locally smooth, for a point $x \in M$ with isotropy subgroup G_x, there is a chart U_x (diffeomorphic to $\mathbb{R}^n$) containing x that is G_x-invariant. The orbifold charts are then the triples (U_x, G_x, π) where $\pi : U_x \to U_x/G_x$ is the projection map.

Example 2.5. In particular, as a special case of the previous example, we can form weighted projective space. Let $S^{2n+1} = \{z = (z_0, \dots, z_n) : |z|^2 = 1\} \subset \mathbb{C}^{n+1}$. For coprime integers $a_0, \dots, a_n$, let S^1 act on S^{2n+1} by the rule

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{a_0} z_0, \dots, \lambda^{a_n} z_n).$$

Then the quotient enjoys the structure of an effective quotient orbifold by the above construction, and we denote the orbifold by $\mathbb{CP}[a_0, \dots, a_n]$.

For a smooth manifold F, by a fiber bundle $\mathcal{E}$ over $\mathcal{X}$ with fiber F we mean we are given the data of an atlas of charts $(U_\alpha, G_\alpha, \pi_\alpha)$ for $\mathcal{X}$ together with a fiber bundle E_α with fiber F over each U_α which is equipped with an action of G_α in such a way that the projection of E_α onto U_α is G_α-equivariant. Moreover, to each embedding

$\lambda : U_\alpha \to U_\beta$ of charts, there corresponds a bundle isomorphism $\lambda_* : E_\alpha \to \lambda^* E_\beta$ that is G_α-equivariant. In addition, the bundle isomorphisms are compatible with one another in the sense that $(\lambda' \circ \lambda)_* = (\lambda^* \lambda'_*) \circ \lambda_*$ for a pair of composable embeddings λ, λ'.

For a fiber bundle $\mathcal{E}$ over $\mathcal{X}$, by refining the atlas enough, we may assume that each E_α is isomorphic to $U_\alpha \times F$. In this way, the bundle isomorphisms λ_* correspond to bundle automorphisms of $U_\alpha \times F$.

A fiber bundle $\mathcal{E}$ over $\mathcal{X}$ determines an orbifold, which we denote by $\mathcal{E}$, and a smooth map of orbifolds $p : \mathcal{E} \to \mathcal{X}$. Refining the atlas enough so that each E_α is isomorphic to $U_\alpha \times F$, the underlying space of $\mathcal{E}$ is given by the quotient space

$$E = \left(\bigsqcup_\alpha U_\alpha \times F \right) / \sim$$

where $\sim$ is the equivalence relation determined by the bundle isomorphisms λ_* from the embeddings λ. Orbifold charts can be found by taking the Cartesian product of U_α with a chart for F and then considering the natural image of this product in E. The map of orbifolds $p : \mathcal{E} \to \mathcal{X}$ is described in charts by considering the projection onto the factor U_α. It is important to remark that the underlying space E of a fiber bundle is not necessarily a fiber bundle over the underlying space of the base X in the usual sense of topology.

The notion of a real vector bundle $\mathcal{E}$ of rank r over $\mathcal{X}$ is defined similarly to that of a fiber bundle. In particular, by refining the atlas enough, we may assume that each E_α is isomorphic to $U_\alpha \times \mathbb{R}^r$. In this way, the bundle isomorphisms λ_* correspond to bundle automorphisms of $U_\alpha \times \mathbb{R}^r$ and hence to the data of smooth transition maps $g_\lambda : U_\alpha \to \mathrm{GL}(r, \mathbb{R})$, which satisfy the cocycle condition $g_{\lambda' \circ \lambda}(x) = g_{\lambda'}(\lambda(x)) g_\lambda(x)$.

Example 2.6. The tangent bundle $T\mathcal{X}$ of an orbifold $\mathcal{X}$ of real dimension n is a real vector bundle of rank n defined in the following manner. For a chart $(U_\alpha, G_\alpha, \pi_\alpha)$, the group G_α acts on the tangent bundle TU_α in the following manner. If $g \in G_\alpha$, then g determines a diffeomorphism $g : U_\alpha \to U_\alpha$, and we set $g \cdot (x, \xi) = (g \cdot x, dg_x \xi)$ for $\xi \in T_x U_\alpha$. In this way, the projection map $TU_\alpha \to U_\alpha$ is G_α-equivariant. Each embedding of charts $\lambda : U_\alpha \to U_\beta$ determines a transition function $g_\lambda(x)$ which corresponds to the derivatives $d\lambda_x : T_x U_\alpha \to T_{\lambda(x)} U_\beta$.

The notions of the direct products, tensor products, wedge products, and duals of vector bundles can be defined in terms of transition functions and corresponding representations in the usual manner. For example, if G_α acts on $U_\alpha \times \mathbb{R}^r$ and $g_\lambda : U_\alpha \to GL(r, \mathbb{R})$ is a transition function for $\mathcal{E}$, then G_α acts on $U_\alpha \times (\mathbb{R}^r)^*$ by the dual action and a transition function for $\mathcal{E}^*$ is $g^*_\lambda(x) = (g_\lambda(x)^{-1})^T$.

A smooth section of a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ consists of a collection of G_α-equivariant smooth sections s_α of the bundles E_α over U_α which are compatible with one another in the sense that $\lambda_* \circ s_\alpha = s_\beta \circ \lambda$ whenever $\lambda : U_\alpha \to U_\beta$ is an embedding.

In this way, it makes sense to speak of smooth functions, vector fields, tensors, and differential forms on $\mathcal{X}$. In particular, note that a smooth function is a section of the trivial bundle of rank 1 over $\mathcal{X}$, and hence the equivariance condition implies that a smooth function f is given by a collection f_α of G_α-*invariant* functions on the charts U_α (since the action of G_α on $\mathbb{R}$ or $\mathbb{C}$ is trivial). More generally, one can ascertain that a tensor T over $\mathcal{X}$ corresponds to a collection T_α of G_α-invariant tensors over the charts U_α (with some compatibility conditions). In particular, by a Riemannian metric on $\mathcal{X}$, we mean a positive-definite symmetric $(0, 2)$-tensor in the usual way.

The support of a tensor T is then taken to be the subset of X determined by

$$\mathrm{supp}(T) = \bigcup_{\alpha} \pi_\alpha(\mathrm{supp}(T_\alpha)).$$

We say that an orbifold $\mathcal{X}$ is orientable if all of the smooth automorphisms and embeddings of the charts in an atlas are orientation-preserving. From this point forward, we will assume that $\mathcal{X}$ is orientable and equipped with an orientation.

The integral of a differential n-form ω on $\mathcal{X}$ may defined as follows. If ω is compactly supported with support contained in the support V_α of a chart $(U_\alpha, G_\alpha, \pi_\alpha)$, then we define

$$\int_{\mathcal{X}} \omega = \frac{1}{|G_\alpha|} \int_{U_\alpha} \omega_\alpha.$$

More generally, for an arbitrary n-form, one chooses a partition of unity φ_α subordinate to a locally finite collection V_α of supports and then sets

$$\int_{\mathcal{X}} \omega = \sum_{\alpha} \int_{\mathcal{X}} \varphi_\alpha \omega.$$

The wedge product of forms is a local operation that extends in the usual way, and the naturality of the de Rham differential d ensures that it extends in the obvious way as well. We let $A(\mathcal{X};\mathbb{R})$ denote the algebra of differential forms over $\mathcal{X}$ with graded pieces $A^k(\mathcal{X};\mathbb{R})$. More generally, a section of $\Lambda^k T^*\mathcal{X} \otimes \mathcal{E}$ is called an $\mathcal{E}$-valued k-form, and we let $A^k(\mathcal{E};\mathbb{R})$ denote the space of $\mathcal{E}$-valued k-forms. From this point forward, we will drop the mention of the scalar field (which is $\mathbb{R}$ for a real vector bundle and $\mathbb{C}$ for a complex one) when the choice of scalars is clear, so we will simply write $A(\mathcal{E})$ and $A^k(\mathcal{E})$ with hope that no confusion will arise.

Stokes' theorem extends to compact orbifolds (without boundary) in a natural way. Indeed let ω be any $(n-1)$-form on a compact orbifold $\mathcal{X}$. Choose a partition of unity φ_α subordinate to the supports U_α of the orbifold charts. If we let ω_α denote a G_α-invariant n-form on U_α representing $\varphi_\alpha\omega$, then we find by definition that

$$\int_{\mathcal{X}} d(\varphi_\alpha\omega) = \frac{1}{|G_\alpha|}\int_{U_\alpha} d(\omega_\alpha) = 0$$

where the latter integral vanishes by the ordinary Stokes' theorem, as ω_α has support which is compact and contained within U_α. It now follows that

$$\begin{aligned}\int_{\mathcal{X}} d\omega &= \int_{\mathcal{X}} d\left(\sum_\alpha \varphi_\alpha\omega\right)\\ &= \sum_\alpha \int_{\mathcal{X}} d(\varphi_\alpha\omega)\\ &= 0,\end{aligned}$$

where the interchanging of the sum and the integral sign is valid because, for example, we may suppose that the number of charts is finite as $\mathcal{X}$ is compact. We summarize below.

Lemma 2.7 (Stokes' theorem)**.** *Let ω be an $(n-1)$-form on a compact orbifold $\mathcal{X}$. Then*

$$\int_{\mathcal{X}} d\omega = 0.$$

We say in the usual manner that a connection on $\mathcal{E}$ over $\mathcal{X}$ consists of a linear map $D : A^0(\mathcal{E}) \to A^1(\mathcal{E})$ which satisfies Leibniz rule in the sense that

$$D(fs) = df \otimes s + fDs$$

for functions f on $\mathcal{X}$ and sections s of $\mathcal{E}$. A connection D determines a prolongation

$D : A^k(\mathcal{E}) \to A^{k+1}(\mathcal{E})$ in the usual way by forcing Leibniz rule

$$D(\psi \wedge \xi) = d\psi \otimes \xi + (-1)^k \psi \wedge D\xi$$

for $\psi \in A^k(\mathcal{X})$ and $\xi \in A^0(\mathcal{E})$. The square of a connection $D \circ D : A^0(\mathcal{E}) \to A^2(\mathcal{E})$ is $A^0(\mathcal{X})$-linear and hence corresponds to an $\mathrm{End}(\mathcal{E})$-valued 2-form, which is denoted by F_D and is called the curvature of the connection D. Moreover, any connection D on $\mathcal{E}$ determines one on duals and powers of $\mathcal{E}$ in the standard way by demanding that the connection be compatible with Leibniz rule and contraction (of $\mathcal{E}$ with $\mathcal{E}^*$). In particular, if D is a connection on the tangent bundle $\mathcal{E} = T\mathcal{X}$, then for a Riemannian metric g on $\mathcal{X}$, one finds that

$$d(g(V, W)) = (Dg)(V, W) + g(DV, W) + g(V, DW)$$

for vector fields V, W. A connection D on $T\mathcal{X}$ is said to be compatible with the metric g if $Dg = 0$. For a given metric g, there is a unique symmetric connection on $T\mathcal{X}$ compatible with it, which we call the Levi-Civita connection and which we will denote by ∇.

In addition, a Riemannian metric g provides an identification of the space of vector fields with the space of 1-forms, and so for any two 1-forms η and ζ, the metric g determines a smooth function $g(\eta, \zeta)$. More generally, for a pair of tensors S, T of the same type, the metric determines a smooth function $g(S, T)$. In particular, we write $|S|_g^2$ to denote the smooth function $|S|_g = g(S, S)$.

Finally a Riemannian metric g also determines a unique volume form vol_g compatible with the orientation. In this way, we obtain Sobolev spaces $L^p_k(\mathcal{X})$ by using the usual Sobolev norms

$$\|f\|_{L^p_k} = \left(\sum_{j=0}^{k} \int_{\mathcal{X}} |\nabla^k f|_g^p \mathrm{vol}_g \right)^{1/p}.$$

To reduce the size of subscripts, we often write $\|f\|_p$ to mean the L^p-norm (or L^p_0-norm) of f.

2.1.2 Kähler orbifolds

An orbifold of real dimension $2n$ is called complex (of complex dimension n) if the atlas can be taken to be holomorphic. In particular, this means that each U_α is a subset of $\mathbb{C}^n$, the group G_α acts by biholomorphisms, and the embeddings $\lambda : U_\alpha \to U_\beta$ are holomorphic embeddings.

For a complex orbifold, we have a well-defined complex structure J, that is, a mapping of vector fields to vector fields satisfying $J^2 = -\mathrm{id}$, which can be described locally in the usual way by requiring

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}$$
$$J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}$$

for a choice of holomorphic coordinates $z^j = x^j + \sqrt{-1}y^j$. The complexification of the space of vector fields decomposes into the eigenspaces for J corresponding to $\pm\sqrt{-1}$. Dually the complexification of the space of 1-forms decomposes and a 1-form corresponding to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$) is called a $(1,0)$-form (resp. $(0,1)$-form). Taking higher exterior powers one obtains the notion of a (p,q)-form on

$\mathcal{X}$, that is, we obtain a splitting of the space of complex-valued k-forms

$$A^k(\mathcal{X}) = \bigoplus_{p+q=k} A^{p,q}(\mathcal{X}).$$

The extension of the de Rham operator to the complexified spaces decomposes as $d = \partial + \bar{\partial}$ where ∂ is an operator taking (p,q)-forms to $(p+1,q)$ forms and $\bar{\partial}$ an operator taking (p,q)-forms to $(p,q+1)$-forms. The relation $d^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$.

A real $(1,1)$-form η is called positive (resp. nonnegative) if the corresponding symmetric tensor defined by $(V,W) \mapsto \eta(V,JW)$ is positive (resp. nonnegative) definite for vector fields V,W. Locally this means that if η admits an expression as $\eta = \sqrt{-1}\eta_{j\bar{k}} dz^j \wedge d\bar{z}^k$ in some chart, then the matrix $\eta_{j\bar{k}}$ of smooth functions is positive definite. A real (p,p) form is called positive (resp. nonnegative) if it is the sum of products of positive (resp. nonnegative) real $(1,1)$-forms. The integral of a nonnegative (n,n)-form is nonnegative.

A Riemannian metric g on $\mathcal{X}$ is called hermitian if J is an orthogonal transformation with respect to g. A hermitian metric g gives rise to a real $(1,1)$-form ω defined by $\omega(JV,W) = g(V,W)$. One says that a hermitian metric is Kähler if the corresponding $(1,1)$-form is d-closed. Conversely, we say a real $(1,1)$-form ω is compatible with J if the equality $\omega(JV,JW) = \omega(V,W)$ holds for each pair of vector fields V,W. It is easily shown that the data of a Kähler metric is equivalent to the data of a J-compatible positive d-closed real $(1,1)$-form. By a Kähler orbifold $(\mathcal{X},\omega)$ we mean an orbifold together with a choice of J-compatible positive d-closed real $(1,1)$-form ω.

Many properties of Kähler manifolds, including Kodaira's $\partial\bar{\partial}$-lemma, extend to the setting of Kähler orbifolds (see, for example, [5]).

Lemma 2.8 ($\partial\bar{\partial}$-lemma)**.** *Let $(\mathcal{X}, \omega)$ be a compact Kähler orbifold. If η and η' are two real $(1,1)$-forms in the same cohomology class, then there is a function $f : \mathcal{X} \to \mathbb{R}$ such that $\eta' = \eta + \sqrt{-1}\partial\bar{\partial} f$.*

A Kähler metric g admits a local expression in the charts as

$$g_\alpha = (g_\alpha)_{j\bar{k}} dz_\alpha^j \otimes d\bar{z}_\alpha^k$$

where $(g_\alpha)_{j\bar{k}} = g_\alpha(\partial/\partial z_\alpha^j, \partial/\partial \bar{z}_\alpha^k)$ for local coordinates (z_α^j) on the chart U_α. The corresponding Kähler form ω admits local expression

$$\omega_\alpha = \sqrt{-1}(g_\alpha)_{j\bar{k}} dz_\alpha^j \wedge d\bar{z}_\alpha^k.$$

The Ricci form $\mathrm{Ric}(\omega)$ corresponding to a Kähler form ω is the $(1,1)$-form with local expression

$$\mathrm{Ric}(\omega)_\alpha = -\sqrt{-1}\partial\bar{\partial} \log \det((g_\alpha)_{j\bar{k}}).$$

It can be shown that the cohomology class of $\mathrm{Ric}(\omega)$ does not depend on the particular Kähler metric ω, and thus defines an invariant of the orbifold. The first Chern class $c_1(\mathcal{X})$ can be taken to be the real cohomology class determined by the form $\frac{1}{2\pi}\mathrm{Ric}(\omega)$ for some choice of Kähler form ω. We say that $c_1(\mathcal{X})$ is positive written $c_1(\mathcal{X}) > 0$ (resp. negative written $c_1(\mathcal{X}) < 0$) if $c_1(\mathcal{X})$ is represented by a positive (resp. negative) $(1,1)$-form.

For a Kähler metric g on a complex orbifold, the operator $\bar{\partial} : A^{p,q}(\mathcal{X}) \to A^{p,q+1}(\mathcal{X})$ admits an adjoint, and there is there is a corresponding Laplacian $\Delta = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$. Here we are using the convention (as in [39]) that the Laplacian is a non-negative operator. Acting on the space $A^0(\mathcal{X})$ of functions, we will see that the Laplacian Δ a second-order uniformly elliptic operator (see Section 2.2). In particular, acting on the space $A^0(\mathcal{X})$ of smooth functions, the operator Δ admits a local description as

$$\Delta\varphi = -g^{j\bar{k}}\partial_j\partial_{\bar{k}}\varphi$$

for a choice of holomorphic coordinates in some chart.

In addition, for a Kähler metric g, there is a contraction operator $\Lambda : A^{1,1}(\mathcal{X}) \to A^0(\mathcal{X})$, which associates to every $(1,1)$-form η a smooth function $\Lambda\eta$ satisfying

$$n \cdot \eta \wedge \omega^{n-1} = \Lambda\eta \cdot \omega^n,$$

where n is the complex dimension of $\mathcal{X}$. The function $\Lambda\eta$ measures the component of η along ω. In particular, if η admits a local description as $\eta = \sqrt{-1}\eta_{j\bar{k}}dz^j \wedge d\bar{z}^k$, then the smooth function $\Lambda\eta$ admits a description as

$$\Lambda\eta = g^{j\bar{k}}\eta_{j\bar{k}}$$

where $g^{j\bar{k}}$ is the inverse of $g_{j\bar{k}}$ and $\omega = \sqrt{-1}g_{j\bar{k}}dz^j \wedge d\bar{z}^k$. The operator Λ is related to the Laplacian Δ by the relation

$$\Lambda(\sqrt{-1}\bar{\partial}\partial\varphi) = \Delta\varphi$$

for a smooth function φ. (This relation can be viewed in light of the Kähler identity $\partial^* = i\Lambda\bar{\partial}$ (c.f. [39]).)

For a smooth $\mathbb{C}$-valued function φ on a Kähler orbifold $(\mathcal{X}, \omega)$, we have

$$|d\varphi|_g^2 = |\partial\varphi|_g^2 + |\bar{\partial}\varphi|_g^2.$$

Moreover, if φ is $\mathbb{R}$-valued, then $|\partial\varphi|_g^2 = |\bar{\partial}\varphi|_g^2$, and hence

$$|\partial\varphi|_g^2 = \frac{1}{2}|d\varphi|_g^2.$$

With this convention, it follows that for a smooth $\mathbb{R}$-valued function φ we have

$$n\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{n-1} = |\partial\varphi|_g^2 \cdot \omega^n.$$

2.2 Elliptic operators on orbifolds

The goal of this section is to study some properties and estimates associated to linear elliptic second-order differential operators on a compact orbifold, to construct a Green's function of the complex Laplacian on a compact Kähler orbifold, and to establish some useful inequalities of Poincaré and Sobolev type.

For a bounded domain Ω in $\mathbb{R}^n$, a natural number $k \in \mathbb{N}$, and a number $\alpha \in (0,1)$, recall the $C^{k,\alpha}$-norm of a function f on Ω can be defined by

$$\|f\|_{C^{k,\alpha}(\Omega)} = \sup_{|\ell|\leqslant k} |\partial^\ell f| + \sup_{|\ell|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\ell f(x) - \partial^\ell f(y)|}{|x-y|^\alpha}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ is a multi-index and

$$\partial^\ell = \frac{\partial}{\partial x^{\ell_1}} \cdots \frac{\partial}{\partial x^{\ell_n}}.$$

The space $C^{k,\alpha}(\Omega)$ is then the space of functions on Ω whose $C^{k,\alpha}$-norm is finite.

A uniformly elliptic operator L of second-order with smooth coefficients on Ω

admits an expression of the form

$$L(f) = a^{ij}\partial_i\partial_j f + b^m\partial_m f + cf \tag{2.3}$$

where a^{ij}, b^m, c are smooth functions and where we are using the Einstein summation convention. The uniform ellipticity implies that there are constants $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2$$

for each point $x \in \Omega$ and each vector $\xi \in \mathbb{R}^n$. The following a priori estimates for such operators are well-known (see, for example, [36]).

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $\Omega' \subset\subset \Omega$ be a relatively compact subset, and let $\alpha \in (0,1)$ and $k \in \mathbb{N}$. Then there is a constant C such that*

$$\|f\|_{C^{k+2,\alpha}(\Omega')} \leqslant C(\|Lf\|_{C^{k,\alpha}(\Omega)} + \|f\|_{C^0(\Omega)}).$$

Moreover, the constant C only depends on k, α, the domains Ω and Ω', the $C^{k,\alpha}$-norms of the coefficients of L, and the constants of ellipticity λ, Λ. Finally, it is enough only to assume that $f \in C^2(\Omega)$ so that Lf makes sense, and then it follows that $f \in C^{k+2,\alpha}(\Omega')$ whenever Lf and the coefficients of L are in $C^{k,\alpha}(\Omega)$.

These estimates can be extended to a compact orbifold $\mathcal{X}$ as follows. The Hölder spaces can be defined locally in orbifold charts: Covering $\mathcal{X}$ with orbifold charts, any tensor T has a local expression as a bona fide tensor T_β in these charts, and the $C^{k,\alpha}$-norm of T can be defined to be the supremum of the $C^{k,\alpha}$-norms of these T_β. In particular, if $\mathcal{X}$ is compact, we may achieve that there are finitely many orbifold charts, and the supremum can be taken to be the maximum. A linear second-order differential operator on an orbifold $\mathcal{X}$ is an operator which admits an expression as (2.3) in each orbifold chart.

Theorem 2.10. *Let L be an elliptic second-order linear differential operator with smooth coefficients on a compact Riemannian orbifold $(\mathcal{X}, g)$, and fix $\alpha \in (0,1)$ and $k \in \mathbb{N}$. Then there is a constant C such that*

$$\|f\|_{C^{k+2,\alpha}(\mathcal{X})} \leqslant C(\|Lf\|_{C^{k,\alpha}(\mathcal{X})} + \|f\|_{C^0(\mathcal{X})}).$$

Moreover, the constant C only depends on k, α, the orbifold $\mathcal{X}$, the $C^{k,\alpha}$-norms of the coefficients of L, and the constants of ellipticity λ, Λ. Finally, it is enough to assume only that $f \in C^2(\mathcal{X})$, and then it follows that $f \in C^{k+2,\alpha}(\mathcal{X})$ whenever Lf and the coefficients of L are in $C^{k,\alpha}(\mathcal{X})$.

Proof. As $\mathcal{X}$ is compact, there is a finite collection of orbifold charts $(U_\beta, G_\beta, \pi_\beta)$ whose supports V_β cover X. We may select relatively compact domains $U'_\beta \subset U_\beta$ invariant under the G_β action satisfying the hypotheses of Theorem 2.9, such that the images $\pi_\beta(U'_\beta)$ still cover X. We have a finite list of constants C_β from Theorem 2.9 applied to each pair (U_β, U'_β), and taking the maximum gives a large constant C. Then an application of the previous theorem gives

$$\begin{aligned}
\|f\|_{C^{k+2,\alpha}(\mathcal{X})} &= \max_\beta \|f_\beta\|_{C^{k+2,\alpha}(U'_\beta)} \\
&\leqslant \max_\beta C_\beta(\|h\|_{C^{k,\alpha}(U_\beta)} + \|f\|_{C^0(U_\beta)}) \\
&\leqslant C(\|h\|_{C^{k,\alpha}(\mathcal{X})} + \|f\|_{C^0(\mathcal{X})}),
\end{aligned}$$

as desired. $\square$

Moreover, the mapping properties of such operators on compact orbifolds are well-understood, as demonstrated below. We require the following standard result in functional analysis which can be found, for example, in [50, Proposition 3.9.7].

Lemma 2.11. *Let E_1 and E_2 be Banach spaces, and let $A_1, A_2 : E_1 \to E_2$ be bounded linear operators. Suppose that*

(i) A_1 is injective and the range of A_1 is closed;

(ii) A_2 is a compact operator.

Then the range of $A_1 + A_2$ is closed in E_2.

Lemma 2.12. *Let L be an elliptic second-oder operator with smooth coefficients on a compact orbifold $\mathcal{X}$. Then the range of $L : C^{k+2,\alpha}(\mathcal{X}) \to C^{k,\alpha}(\mathcal{X})$ is closed.*

Proof. Define two Banach spaces E_1 and E_2 by

$$\begin{aligned} E_1 &= C^{k+2,\alpha}(\mathcal{X}) \\ E_2 &= C^{k,\alpha}(\mathcal{X}) \oplus C^{k,\alpha}(\mathcal{X}). \end{aligned}$$

Define two bounded linear operators $A_1, A_2 : E_1 \to E_2$ by

$$\begin{aligned} A_1(f) &= (Lf, f) \\ A_2(f) &= (0, -f). \end{aligned}$$

The operator A_1 is injective because the second component is. Moreover, the range of A_1 is closed by the uniform bound

$$\begin{aligned} \|f\|_{C^{k+2,\alpha}(\mathcal{X})} &\leqslant C(\|Lf\|_{C^{k,\alpha}(\mathcal{X})} + \|f\|_{C^0(\mathcal{X})}) \\ &\leqslant C(\|Lf\|_{C^{k,\alpha}(\mathcal{X})} + \|f\|_{C^{k,\alpha}(\mathcal{X})}) = C\left\|A_1(f)\right\|_{E_2}. \end{aligned}$$

Finally A_2 is compact by the Arzela-Ascoli theorem applied to the inclusion of $C^{k+2,\alpha}(\mathcal{X})$ into $C^{k,\alpha}(\mathcal{X})$. It follows from the previous lemma that $A_1 + A_2$ has closed range. But the range of $A_1 + A_2$ is identified with $\mathrm{Ran}(L) \oplus \{0\}$, and hence L has closed range. $\square$

Theorem 2.13. *Let L be an elliptic second-oder operator with smooth coefficients on a compact Riemannian orbifold $(\mathcal{X}, g)$. Then L is an isomorphism of Banach spaces*

$$L : (\ker L)^{\perp} \cap C^{k+2,\alpha}(\mathcal{X}) \to (\ker L^{*})^{\perp} \cap C^{k,\alpha}(\mathcal{X}),$$

where L^{*} denotes the adjoint of L with respect to the indicated Banach norms.

Proof. The restriction of the domain to $(\ker L)^{\perp}$ ensures that L is injective and the restriction of the codomain to $(\ker L^{*})^{\perp} = \overline{\mathrm{Ran}(L)}$ ensures that L is surjective by the previous lemma. Hence L is a bounded invertible linear operator between Banach spaces. We conclude that the inverse of L is bounded as well from [44, §23 Theorem 2], and therefore L is an isomorphism of Banach spaces. $\square$

Additionally, using Sobolev spaces, one can obtain not only a Green's function of the complex Laplacian, but also inequalities Poincaré and Sobolev stype, as follows.

The following version of Rellich's lemma on orbifolds can be found in [21].

Theorem 2.14 (Rellich)**.** *For a compact Riemannian orbifold $(\mathcal{X}, g)$, the inclusion $L^2_{k+1}(\mathcal{X}) \to L^2_k(\mathcal{X})$ is compact.*

If L is a linear elliptic differential operator of second order, then L defines a map $L : L^p_{k+2}(\mathcal{X}) \to L^p_k(\mathcal{X})$, and similarly to the case of Hölder norms, the mapping properties of L with respect to these Sobolev norms are well-understood.

Theorem 2.15. *Let L be an elliptic operator of second order on a compact Riemannian orbifold $(\mathcal{X}, g)$, and fix a number $p \geqslant 1$ and a number $k \in \mathbb{N}$. Then there is a constant C such that for all smooth functions f we have*

$$\|f\|_{L^p_{k+2}} \leqslant C(\|Lf\|_{L^p_k} + \|f\|_{L^p_k}).$$

Moreover the constant C depends only on $\mathcal{X}, g, L, k, p$. Finally, L induces an isomorphism of Banach spaces

$$L : (\ker L)^{\perp} \cap L^p_{k+2}(\mathcal{X}) \to (\ker L^*)^{\perp} \cap L^p_k(\mathcal{X}).$$

Remark 2.16. We do not outline a complete proof of this theorem here, but a proof would proceed analogusly to that of the manifold setting. It is interesting also to note that just as in the case of Hölder norms, there are local versions of these inequalities: For any relatively compact $\Omega' \subset\subset \Omega$, we have an inequality of the form

$$\|f\|_{L^p_{k+2}(\Omega')} \leqslant C(\|Lf\|_{L^p_k(\Omega)} + \|f\|_{L^p_k(\Omega)}).$$

Example 2.17. In particular, the complex Laplacian Δ on a compact Kähler orbifold $(\mathcal{X}, \omega)$ is an elliptic operator of second order. (Actually it is the negative $-\Delta$ which is elliptic in the strictly positive local sense that we have defined, but it follows immediately that the same a priori estimates hold for Δ as well.) For a point $x \in X$, consider the real-valued function δ_x defined on smooth functions φ by the rule

$$\delta_x(\varphi) = \varphi(x) - \bar{\varphi}$$

where $\bar{\varphi}$ denotes the average value of φ. Then we may regard δ_x as an element of the Hilbert space $L^2(\mathcal{X})$ $(= L^2_0(\mathcal{X}))$ via the L^2-inner product (c.f. [21]). Since δ_x vanishes on smooth functions with average value zero, it follows from Theorem 2.15 that there is an element G_x of $L^2_2(\mathcal{X})$ such that $\Delta G_x = \delta_x$. Such a function G_x is called a Green's function associated to Δ and satisfies

$$\int_{\mathcal{X}} G_x \Delta\varphi \cdot \omega^n = \varphi(x) - \bar{\varphi}.$$

In addition, reasoning in [4, Chapter 4, Section 2] can be applied to show that G_x is bounded from below.

Theorem 2.18 (Poincaré inequality)**.** *Let $(\mathcal{X}, g)$ be a compact Kähler orbifold. There is a constant C depending only on $\mathcal{X}$ and g such that if φ is a smooth function with average value zero, then*

$$\|\varphi\|_2 \leqslant C \|\partial\varphi\|_2 ,$$

where, again, the notation $\|\varphi\|_2$ means $\|\varphi\|_{L^2}$.

Remark 2.19. With this above notation, for a compact Kähler orbifold $(\mathcal{X}, \omega)$ we find that the following coincide $\|df\|_2 = \|\nabla f\|_2 = \sqrt{2}\,\|\partial f\|_2$.

Proof of Theorem 2.18. For a function $\varphi \in L^2_1(\mathcal{X})$ satisfying $\|\varphi\|_2 \neq 0$, let $R(\varphi)$ denote the Rayleigh quotient

$$R(\varphi) = \frac{\|\partial\varphi\|_2}{\|\varphi\|_2}.$$

Let E denote the subspace of $L^2_1(\mathcal{X})$ consisting of all functions with average value zero, and let λ denote the infimum

$$\lambda = \inf_{0 \neq \varphi \in E} R(\varphi).$$

It suffices to show that λ is nonzero, which we show. Suppose not. Let φ_j denote a sequence of elements in E satisfying $\lim_{j\to\infty} R(\varphi_j) = 0$. By scaling by $\|\varphi_j\|_2^{-1}$, we may assume that each φ_j satisfies $\|\varphi_j\|_2 = 1$. It follows that the sequence φ_j is uniformly bounded in $L_1^2(\mathcal{X})$. Rellich's lemma implies that, by passing to a subsequence, we can assume that the φ_j converge in $L^2(\mathcal{X})$ to a function $\varphi \in L^2(\mathcal{X})$. This function must satisfy $\|\varphi\|_2 = 1$. Moreover, for any smooth $(1,0)$-form ψ, if ∂^* denotes the adjoint of ∂ with respect to the L^2-inner product induced by the Kähler metric g, then

$$\begin{aligned}
|\langle \varphi, \partial^* \psi \rangle_{L^2}| &= \lim_{j\to\infty} |\langle \varphi_j, \partial^* \psi \rangle_{L^2}| \\
&= \lim_{j\to\infty} |\langle \partial\varphi_j, \psi \rangle_{L^2}| \\
&\leqslant \lim_{j\to\infty} \|\partial\varphi_j\|_2 \|\psi\|_2 = 0
\end{aligned}$$

so that $\partial\varphi = 0$ in the weak sense. The ellipticity of ∂ implies that φ is actually smooth, and therefore a constant, with average value zero, and hence equal to zero. This contradicts the above deduction that $\|\varphi\|_2 = 1$. □

The following Sobolev inequality on bounded domains is well-known [30].

Lemma 2.20 (Local Sobolev inequality). *Let Ω be a bounded domain in $\mathbb{R}^n$. For a number $p \geqslant 1$, let q denote the Sobolev conjugate satisfying $1/p + 1/q = 1/n$. Then there is a constant C depending on Ω and p such that for any smooth function f with compact support in Ω, we have*

$$\|f\|_q^2 \leqslant C(\|f\|_p^2 + \|\nabla f\|_p^2).$$

In particular, if $\Omega \subset \mathbb{C}^n$ and $p = 2$, then $q = 2n/(n-1)$ and

$$\|f\|_{\frac{2n}{n-1}}^2 \leqslant C(\|f\|_2^2 + \|\partial f\|_2^2).$$

It follows that there is a similar type of inequality on compact Kähler orbifolds.

Theorem 2.21 (Sobolev inequality)**.** *Let $(\mathcal{X}, g)$ be a compact Kähler orbifold of complex dimension n. There is a constant C depending only on $\mathcal{X}$ and g such that if f is a smooth function then*

$$\|f\|^2_{\frac{2n}{n-1}} \leqslant C(\|f\|_2^2 + \|\partial f\|_2^2).$$

Proof. Let φ_α be a partition of unity subordinate to the supports V_α of a finite collection of orbifold charts U_α in an atlas. The smooth function $\varphi_\alpha f$ is compactly supported in the support U_α of a chart U_α. The local Sobolev inequality in this chart U_α implies the existence of a constant C_α such that

$$\|\varphi_\alpha f\|^2_{\frac{2n}{n-1}} \leqslant C_\alpha(\|\varphi_\alpha f\|_2^2 + \|\partial(\varphi_\alpha f)\|_2^2).$$

The triangle inequality implies that

$$|\partial(\varphi_\alpha f)| \leqslant |(\partial\varphi_\alpha)||f| + \varphi_\alpha|\partial f| \leqslant \left(\sup_X |\partial\varphi_\alpha|\right)|f| + |\partial f|.$$

Hence with the previous observation, we find an estimate of the form

$$\|\varphi_\alpha f\|^2_{\frac{2n}{n-1}} \leqslant C_\alpha(\|f\|_2^2 + \|\partial f\|_2^2).$$

Whence if N denotes the number of charts and $C = N \cdot \sup_\alpha C_\alpha$, then

$$\|f\|^2_{\frac{2n}{n-1}} \leqslant \sum_\alpha \|\varphi_\alpha f\|^2_{\frac{2n}{n-1}} \leqslant C(\|f\|_2^2 + \|\partial f\|_2^2),$$

as desired. $\square$

2.3 Yau's theorem on orbifolds

The goal of this section is to outline a proof of Theorem 2.1. Before doing so, let us first demonstrate how Theorem 2.1 implies Theorem 2.2.

Lemma 2.22. *Theorem 2.1 implies Theorem 2.2.*

Proof. Because R and $\mathrm{Ric}(\omega)$ represent the same cohomology class, the $\partial\bar{\partial}$-lemma gives a smooth function $\tilde{F}$ on $\mathcal{X}$ such that

$$R = -\sqrt{-1}\partial\bar{\partial}\tilde{F} + \mathrm{Ric}(\omega).$$

If we let C_1, C_2 denote the positive quantities

$$C_1 = \int_{\mathcal{X}} e^{\tilde{F}}\omega^n \qquad C_2 = \int_{\mathcal{X}} \omega^n$$

and we let F denote the smooth function $F = \tilde{F} + \log(C_2/C_1)$, then F satisfies the integration condition

$$\int_{\mathcal{X}} e^F\omega^n \int_{\mathcal{X}} \omega^n.$$

By Theorem 2.1, there is a smooth function φ satisfying (2.1). The Ricci form of $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ then satisfies

$$\mathrm{Ric}(\omega') = -\sqrt{-1}\partial\bar{\partial}\log(e^F\omega^n) = -\sqrt{-1}\partial\bar{\partial}F + \mathrm{Ric}(\omega) = R.$$

Hence $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a solution to the Calabi conjecture.

Suppose that ω'' is another solution. There is a φ'' such that $\omega'' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi''$. By assumption, the Ricci form of ω'' satisfies

$$\mathrm{Ric}(\omega'') = R = -\sqrt{-1}\partial\bar{\partial}F + \mathrm{Ric}(\omega),$$

or equivalently,

$$-\sqrt{-1}\partial\bar{\partial}(\log(\omega + \sqrt{-1}\partial\bar{\partial}\varphi'')^n) = -\sqrt{-1}\partial\bar{\partial}F - \sqrt{-1}\partial\bar{\partial}\log(\omega^n).$$

Rearranging gives

$$\sqrt{-1}\partial\bar{\partial}\left(\log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi')^n}{\omega^n} - F\right) = 0.$$

Integration by parts shows that there is a constant C such that

$$\log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi'')^n}{\omega^n} - F = C,$$

which implies that

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi'')^n = e^{F+C}\omega^n.$$

The fact that $\int_{\mathcal{X}} e^F\omega^n = \int_{\mathcal{X}} \omega^n$ implies that

$$e^C\int_{\mathcal{X}} e^F\omega^n = \int_{\mathcal{X}} e^{F+C}\omega^n = \int_{\mathcal{X}} \omega^n = \int_{\mathcal{X}} e^F\omega^n.$$

Hence $C = 0$ and we conclude that φ'' is a solution to Theorem 2.1. Thus φ'' and φ' differ by a constant, and so $\omega'' = \omega'$. This shows how the uniqueness in Theorem 2.2 follows from that of Theorem 2.1. $\square$

Let us now move on to a proof of Theorem 2.1. First we deal with uniqueness.

Proposition 2.23. *If φ, φ' are two smooth solutions to* (2.1), *then φ and φ' differ by a constant.*

Proof. Write $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. Then with this notation, we have

$$0 = \omega_\varphi^n - \omega_{\varphi'}^n = \sqrt{-1}\partial\bar{\partial}(\varphi - \varphi') \wedge T$$

where

$$T = \sum_{k=0}^{n-1} \omega_\varphi^k \wedge \omega_{\varphi'}^{n-1-k}$$

is a positive, closed $(n-1, n-1)$ form. Upon integrating by parts, we find

$$0 = \int_{\mathcal{X}} (\varphi - \varphi')(\omega_\varphi^n - \omega_{\varphi'}^n) = -\int_{\mathcal{X}} \sqrt{-1}\partial(\varphi - \varphi') \wedge \bar{\partial}(\varphi - \varphi') \wedge T.$$

The positivity of T implies that the integral is nonnegative, and hence we must have $\partial(\varphi - \varphi') = 0$. Thus $\varphi - \varphi'$ is constant. $\square$

With the tools established in the previous sections, we can formulate a proof of Theorem 2.1 by following exactly the structure of a proof in the smooth setting. In particular, one approach is the following well-known continuity method. For completeness, we outline this approach now.

The idea is to introduce a family of equations

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{tF}\omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ is a Kähler form} \end{cases} \tag{$*_t$}$$

indexed by a parameter $t \in [0, 1]$. The equation $(*_0)$ admits the trivial solution $\varphi \equiv 0$. Thus, if we can show that the set of such $t \in [0, 1]$ for which $(*_t)$ admits a smooth solution is both open and closed, it will follow that we can solve $(*_1)$. For this endeavor, it suffices to prove the following.

Proposition 2.24. *Fix an $\alpha \in (0, 1)$.*

(i) If $(*_t)$ *admits a smooth solution for some* $t < 1$*, then for all sufficiently small* $\epsilon > 0$*, the equation* $(*_{t+\epsilon})$ *admits a smooth solution as well.*

(ii) There is a constant $C > 0$ *depending only on* $\mathcal{X}, \omega, F$*, and* α *such that if* φ *with average value zero satisfies* $(*_t)$ *for some* $t \in [0, 1]$*, then*

- $\|\varphi\|_{C^{3,\alpha}(\mathcal{X})} \leqslant C$ *and*
- $(g_{j\bar{k}} + \partial_j \partial_{\bar{k}} \varphi) > C^{-1}(g_{j\bar{k}})$*, where* $g_{j\bar{k}}$ *are the components of* ω *in local coordinates of any chart and the inequality means that the difference of matrices is positive definite.*

Indeed Proposition 2.24 is sufficient because we can obtain a solution to $(*_1)$ using the following lemma.

Lemma 2.25. *Assume Proposition 2.24. Then if* s *is a number in* $(0, 1]$ *such that we can solve* $(*_t)$ *for all* $t < s$*, then we can solve* $(*_s)$*.*

Proof. Let $t_i \in (0, 1]$ be a sequence of numbers approaching s from below. By assumption, this gives rise to a sequence of functions φ_i satisfying

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_i)^n = e^{t_i F}\omega^n.$$

Proposition 2.24 together with the Arzela-Ascoli theorem implies that after passing to a subsequence, we may assume that φ_i converges in $C^{3,\alpha'}(\mathcal{X})$ to a function φ for some $\alpha' < \alpha$. This convergence is strong enough that we find

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{sF}\omega^n.$$

Moreover, Proposition 2.24 gives that the forms $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_i$ are bounded below by a fixed positive form $C^{-1}\omega$, so that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive form.

It remains to show that φ is smooth. In local coordinates, we find that φ satisfies

$$\log\det(g_{b\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) - \log\det(g_{j\bar{k}}) - sF = 0.$$

Differentiating the equation with respect to the variable z^ℓ we have

$$(g_\varphi)^{j\bar{k}}\partial_j\partial_{\bar{k}}(\partial_\ell\varphi) = s\partial_\ell F + \partial_\ell \log\det(g_{j\bar{k}}) - (g_\varphi)^{j\bar{k}}\partial_\ell g_{j\bar{k}}$$

where $(g_\varphi)^{j\bar{k}}$ is the inverse of the matrix $(g_\varphi)_{j\bar{k}} = g_{j\bar{k}}+\partial_j\partial_{\bar{k}}\varphi$. We think of this equation as a linear elliptic second-order equation $L(\partial_\ell\varphi) = h$ for the function $\partial_\ell\varphi \in C^{2,\alpha'}(\mathcal{X})$. Because the function h and the coefficients of L belong to $C^{1,\alpha'}$, we conclude from Theorem 2.10 that $\partial_\ell\varphi$ belongs to $C^{3,\alpha'}$. Because ℓ was arbitrary, it follows that φ belongs to $C^{4,\alpha'}$. Repeating this argument we obtain that $\varphi \in C^{5,\alpha'}$ and by induction, that φ is actually smooth. This technique of considering the corresponding linear equation to obtain better regularity of solutions is called bootstrapping. □

Let us now prove the first part of Proposition 2.24.

Proof of Proposition 2.24 (i). Let B_1 denote the Banach manifold consisting of those $\varphi \in C^{3,\alpha}(\mathcal{X})$ with average value zero and such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive form. Let B_2 denote the Banach space consisting of those $\varphi \in C^{1,\alpha}(\mathcal{X})$ with average value zero. Define a mapping

$$\begin{aligned} G : B_1 \times [0,1] &\longrightarrow B_2 \\ (\varphi, s) &\longmapsto \log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} - sF. \end{aligned}$$

By assumption, we are given a smooth function φ_t such that $G(\varphi_t, t) = 0$ and $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ is a Kähler form. The partial derivative of G in the direction of φ at the point (φ_t, t) is given by

$$DG_{(\varphi_t,t)}(\psi, 0) = \frac{n\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_t^{n-1}}{\omega_t^n} = -\Delta_t \psi,$$

where $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ and Δ_t denotes the Laplacian with respect to ω_t. Denote this partial derivative by the operator $L(\psi) = -\Delta_t \psi$.

The operator L has trivial kernel. Indeed suppose ψ satisfies $L(\psi) = 0$. Then integration by parts shows that

$$0 = \int_{\mathcal{X}} \psi \Delta_t \psi \omega_t^n = n \int_{\mathcal{X}} \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega_t^{n-1}$$

and the positivity of ω_t^{n-1} implies that $\partial\psi = 0$. We conclude that ψ is a constant, with average value zero, and hence equal to zero.

Moreover two integrations by parts show that the operator L is self-adjoint, and hence L^* has trivial kernel as well. It follows from Theorem 2.13 that L is an isomorphism

$$L : C_0^{3,\alpha}(\mathcal{X}) \to C_0^{1,\alpha}(\mathcal{X})$$

where $C_0^{k,\alpha}(\mathcal{X})$ denotes the subspace of functions in $C^{k,\alpha}(\mathcal{X})$ with average value zero. The implicit function theorem asserts that for s sufficiently close to t, there are functions φ_s in $C_0^{3,\alpha}(\mathcal{X})$ satisfying $G(\varphi_s, s) = 0$. Because $\varphi + \sqrt{-1}\partial\bar{\partial}\varphi_t$ is a positive form, for s close enough to t, we can ensure that each $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_s$ is a positive form as well. Moreover, bootstrapping arguments similar to those described earlier show that φ_s is actually smooth. $\square$

2.4 A uniform $C^{3,\alpha}$-estimate

This section is devoted to proving Proposition 2.24 (ii). There are many expositions of this statement in the smooth setting (see [65, 61, 10, 60]). Essentially any of these arguments can be modified to the orbifold setting, provided the necessary ingredients can be modified to the orbifold setting. We will outline a streamlined version of one of the arguments, which can be found in [60], and we direct the reader to this resource for more details of the reasoning to follow. We also direct the reader's attention to the survey paper [52], which is a survey of some of the recent developments in the theory of complex Monge-Ampère equations.

First we obtain a C^0-estimate using a method of Moser iteration. An argument in the smooth setting can be found in [60], which follows an exposition due to [10]. For completeness, we outline the argument below, demonstrating how the tools of the previous sections (Green's function, Poincare inequality, Sobolev inequality) are used.

Lemma 2.26 (C^0-estimate)**.** *There is a constant C depending on $\mathcal{X}, \omega$, and F such that if φ is a solution to $(*_t)$ with average value zero, then*

$$\|\varphi\|_{C^0(\mathcal{X})} \leqslant C.$$

Proof. Without loss of generality we may assume that ω is rescaled so that $\mathcal{X}$ has volume 1. In addition, to eliminate some minus signs in what follows, by replacing φ with $-\varphi$, we may assume that the Kähler form has description $\omega - \sqrt{-1}\partial\bar{\partial}\varphi$. For such φ with average value zero, it suffices to give a bound

$$\sup_{\mathcal{X}} \varphi - \inf_{\mathcal{X}} \varphi \leqslant C.$$

Thus, shifting φ by a constant, to prove the claim, it suffices to show that for solutions with $\inf \varphi = 1$, we have a bound

$$\sup_{\mathcal{X}} \varphi \leqslant C,$$

which is what we will show.

We first show that we have a uniform bound $\|\varphi\|_1 \leqslant C$ on the L^1-norm of solutions φ. The form $\omega - \sqrt{-1}\partial\bar{\partial}\varphi$ is positive so that after taking the trace with respect to ω we have

$$n + \Delta\varphi > 0.$$

where Δ is the Laplacian with respect to ω. Let x be a point where φ achieves its minimum, and let G_x be a Green's function with respect to the Laplacian Δ. We may assume that G_x is integrable and, shifting by a constant, that G_x is nonnegative. Then

$$\begin{aligned} \varphi(x) = \int_{\mathcal{X}} \varphi \cdot \omega^n + \int_{\mathcal{X}} G_x \Delta\varphi \cdot \omega^n &\geqslant \int_{\mathcal{X}} \varphi \cdot \omega^n - n \int_{\mathcal{X}} G_x \cdot \omega^n \\ &\geqslant \int_{\mathcal{X}} \varphi \cdot \omega^n - C. \end{aligned}$$

It follows that we have a uniform estimate $\|\varphi\|_1 \leqslant C$ as desired.

We next show that we have a uniform estimate $\|\varphi\|_2 \leqslant C$. If we write $\omega_\varphi = \omega - \sqrt{-1}\partial\bar{\partial}\varphi$, then we compute that

$$\int_{\mathcal{X}} \varphi(\omega_\varphi^n - \omega^n) = \int_{\mathcal{X}} \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge T$$

where T is the positive $(n-1, n-1)$ form

$$T = \sum_{k=0}^{n-1} \omega^k \wedge \omega_\varphi^{n-1-k}.$$

It follows that

$$\int_{\mathcal{X}} \varphi(\omega_\varphi^n - \omega^n) \geqslant \int_{\mathcal{X}} \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{n-1}$$

From the observation that $\omega_\varphi^n - \omega^n = (e^{tF} - 1)\omega^n$ together with the estimate of the previous paragraph, we find that

$$\|\partial\varphi\|_2^2 \leqslant C.$$

The Poincaré inequality (Theorem 2.18) then implies that

$$\int_{\mathcal{X}} (\varphi - \|\varphi\|_1)^2 \omega^n \leqslant C \|\partial\varphi\|_2^2 \leqslant C.$$

Hence our bound from the previous paragraph implies a bound $\|\varphi\|_2 \leqslant C$.

Finally it is routine to use a technique called Moser iteration to establish a uniform bound $\sup_{\mathcal{X}} \varphi \leqslant C \|\varphi\|_2$, which will complete the proof. For $p \geqslant 2$, we have

$$\int_{\mathcal{X}} \varphi^{p-1}(\omega_\varphi^n - \omega^n) = \frac{4(p-1)}{p^2} \int_{\mathcal{X}} \sqrt{-1}\partial\varphi^{p/2} \wedge \bar{\partial}\varphi^{p/2} \wedge T,$$

which implies

$$\int_{\mathcal{X}} \varphi^{p-1}(\omega_\varphi^n - \omega^n) \geqslant \frac{4(p-1)}{p^2} \int_{\mathcal{X}} \sqrt{-1}\partial\varphi^{p/2} \wedge \bar{\partial}\varphi^{p/2} \wedge \omega^{n-1}.$$

We deduce that

$$\left\|\partial\varphi^{p/2}\right\|_2^2 \leqslant Cp \|\varphi\|_{p-1}^{p-1}$$

for some constant C independent of p. The Sobolev inequality (Theorem 2.21) applied to $\varphi^{p/2}$ together with this estimate gives that

$$\begin{aligned}\|\varphi\|^p_{\frac{np}{n-1}} = \left\|\varphi^{p/2}\right\|^2_{\frac{2n}{n-1}} &\leqslant C\left(\left\|\varphi^{p/2}\right\|^2_2 + \left\|\partial\varphi^{p/2}\right\|^2_2\right) \\ &\leqslant C\left(\|\varphi\|^p_p + Cp\,\|\varphi\|^{p-1}_{p-1}\right) \\ &\leqslant Cp\,\|\varphi\|^p_p .\end{aligned}$$

If we write $p_k = (n/(n-1))^k p$, then we find

$$\begin{aligned}\|\varphi\|_{p_k} &\leqslant (Cp_{k-1})^{1/(p_k-1)}\,\|\varphi\|_{p_k-1} \\ &\leqslant \|\varphi\|_p \prod_{i=0}^{k-1}(Cp_i)^{1/p_i} \\ &\leqslant \|\varphi\|_p \prod_{i=0}^{\infty}(Cp_i)^{1/p_i}.\end{aligned}$$

If we set $p = 2$ and let $k \to \infty$, then we find that

$$\sup_{\mathcal{X}} \varphi \leqslant C\,\|\varphi\|_2\,,$$

and the estimate from the previous paragraph on $\|\varphi\|_2$ gives the desired bound. $\square$

The following lemma can be proved by a local calculation, which uses the Cauchy-Schwarz inequality twice and which can be found, for example, in [60, Lemma 3.7] (but we require a minus sign because our convention of the Laplacian is the negative of the one appearing in that book).

Lemma 2.27. *There is a constant C depending on $\mathcal{X}$ and ω such that if φ is a* *solution to* $(*_t)$ *with average value zero, then*

$$-\hat{\Delta}\log\Lambda\omega_\varphi \geqslant -C\hat{\Lambda}\omega - \frac{g^{j\bar{k}}\hat{R}_{j\bar{k}}}{\mathrm{tr}_\omega\omega_\varphi}$$

where $\hat{\Delta}$ *is the Laplacian with respect to* ω_φ*,* $\hat{\Lambda}$ *is the trace with respect to* ω_φ*, and* $\hat{R}_{j\bar{k}}$ *is the Ricci curvature of* ω_φ*.*

A C^2-estimate then follows directly from this lemma together with the C^0-estimate, again by a local computation which uses only rudimentary tools such as the Cauchy-Schwarz inequality and which can be found again in [60, Lemma 3.8].

Lemma 2.28 (C^2-estimate)**.** *There is a constant* C *depending on* $\mathcal{X}, \omega, F$ *such that a solution* φ *of* $(*_t)$ *with average value zero satisfies*

$$C^{-1}(g_{j\bar{k}}) < (g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) < C(g_{j\bar{k}}).$$

Let S denote the tensor given by the difference of Levi-Civita connections $S = \hat{\nabla} - \nabla$, where $\hat{\nabla}$ is the connection corresponding to ω_φ and ∇ is the one corresponding to ω. Note that S depends on the third derivatives of φ. So if $|S|$ denotes the norm of S with respect to the metric ω_φ, the fact that the metric $g_{j\bar{k}}$ is uniformly equivalent to the metric $g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$ implies that a bound on $|S|$ gives a C^3-bound on φ.

Lemma 2.29 (C^3-estimate)**.** *There is a constant* C *depending on* $\mathcal{X}, \omega, F$ *such that if* φ *is a solution to* $(*_t)$ *with average value zero, then* $|S| \leqslant C$*, where* $|S|$ *is the norm of* S *computed with respect to the metric* ω_φ*.*

Proof. Again local computations and rudimental local identities from complex geometry can be used first to obtain estimates of the form

$$-\hat{\Delta}|S|^2 \geqslant -C|S|^2 - C$$

and

$$-\hat{\Delta}\Lambda\omega_\varphi \geqslant -C + \epsilon|S|^2$$

where C denotes a large constant and ϵ a small one. We direct the reader again to [60, Lemma 3.9 and Lemma 3.10] for local proofs of these estimates, remarking that the convention of the Laplacian in [60] is the negative of our own so we require a minus sign on the left-hand side of these estimates.

It follows that we may choose a large constant A such that

$$-\hat{\Delta}(|S|^2 + A\Lambda\omega_\varphi) \geqslant |S|^2 - C$$

Suppose that $|S|^2 + A\Lambda\omega_\varphi$ achieves its maximum at $x \in X$. Then in a local chart around x we have a local inequality of the form $-\hat{\Delta}(|S|^2 + A\Lambda\omega_\varphi)(x) \leqslant 0$ and hence also

$$0 \geqslant |S|^2(x) - C$$

so that $|S|^2(x) \leqslant C$. At any other point $y \in X$, this bound together with the C^2-estimate imply any estimate of the form

$$|S|^2(y) \leqslant (|S|^2 + A\Lambda\omega_\varphi)(y) \leqslant (|S|^2 + A\Lambda\omega_\varphi)(x) \leqslant C.$$

This is what we wanted. □

We are now able to complete a proof of Proposition 2.24.

Proof of Proposition 2.24 (ii). Lemma 2.28 shows that the metric ω_φ is uniformly equivalent to the metric ω. Lemma 2.29 implies that we have a uniform bound of the form $\|\varphi\|_{C^3(\mathcal{X})} \leqslant C$, from which it follows that we have a uniform bound of the form $\|\varphi\|_{C^{2,\alpha}(\mathcal{X})} \leqslant C$. Bootstrapping arguments together with Theorem 2.10 and Lemma

2.26 imply that we actually have a uniform bound $\|\varphi\|_{C^{3,\alpha}(\mathcal{X})} \leqslant C$, as desired. $\square$

2.5 Kähler-Einstein metrics for orbifolds with negative first Chern class

The goal of this section is to outline briefly a proof of Theorem 2.3. One way to proceed is by showing that the Kähler-Einstein condition is equivalent to a Monge-Ampère equation that is only slightly different than (2.1), and then use the same technique of the continuity method to solve this slightly modified equation.

Indeed, let ω be any Kähler metric in $-2\pi c_1(\mathcal{X})$. The Ricci form $\mathrm{Ric}(\omega)$ belongs to the class $2\pi c_1(\mathcal{X})$, so by the $\partial\bar{\partial}$-lemma there is a smooth function F such that

$$\mathrm{Ric}(\omega) = -\omega + \sqrt{-1}\partial\bar{\partial}F.$$

Any other Kähler metric can be written as $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for a smooth function φ on $\mathcal{X}$. The Ricci form of ω_φ satisfies

$$\mathrm{Ric}(\omega_\varphi) = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\log\frac{\omega_\varphi^n}{\omega^n}$$

and so the Kähler-Einstein condition $\mathrm{Ric}(\omega_\varphi) = -\omega_\varphi$ reduces to

$$-\sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\partial\bar{\partial}F - \sqrt{-1}\partial\bar{\partial}\log\frac{\omega_\varphi^n}{\omega^n}.$$

For this equation to be true, it suffices to solve the Monge-Ampère equation (2.2).

To solve equation (2.2), one can use a continuity method as in the case of (2.1).

Indeed, one can introduce the family of equations

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{tF+\varphi}\omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ is a Kähler form} \end{cases} \tag{$**_t$}$$

indexed by a parameter $t \in [0,1]$ and show that the set of such t for which $(**_t)$ admits a smooth solution is both open and closed. The openness follows from the implicit function theorem as in the proof of Proposition 2.24 (i), with the only modification being that the linearized operator at t is given by

$$\psi \mapsto -\Delta_t\psi - \psi.$$

The closedness follows from appropriate C^0-, C^2-, and C^3-estimates for solutions of $(**_t)$, which can be obtained from only very slight modifications of the arguments for the corresponding estimates for solutions of $(*_t)$. Moreover, the case of the C^0-estimate is even easier for solutions of $(**_t)$, as one can argue using the maximum principle (see [60, Lemma 3.6] for a proof in the nonsingular case).

2.6 Examples of Calabi-Yau orbifolds

Theorem 2.2 produces Ricci-flat Kähler metrics on orbifolds with $c_1(\mathcal{X}) = 0$ as a real cohomology class in $H^2(\mathcal{X}, \mathbb{R})$. In this section, we give examples of such orbifolds, which we call Calabi-Yau orbifolds.

Previously we had defined the first Chern class of a Kähler orbifold $(\mathcal{X}, \omega)$ using the Ricci form $\mathrm{Ric}(\omega)$. Alternatively, the first Chern class can be defined as a real cohomology class using connections and Chern-Weil theory as usual. In particular, the square of a unitary connection ∇ on $\mathcal{X}$ corresponds to a mapping F_∇ taking

$(1,0)$-tensors to $(1,2)$-tensors which is linear over the ring of smooth complex-valued functions on $\mathcal{X}$. The curvature F_∇ determines a characteristic polynomial in t

$$\det\left(\mathrm{id} - \frac{1}{2\pi\sqrt{-1}}F_\nabla t\right) = \sum_{k=0}^{n} f_k(\mathcal{X})t^k$$

where $f_k(\mathcal{X})$ corresponds to a *real* $2k$-form on $\mathcal{X}$ whose complex dimension we have denoted by n. The cohomology class determined by $f_k(\mathcal{X})$ independent of the choice of unitary connection and defines a real cohomology class $c_k(\mathcal{X}) \in H^{2k}(\mathcal{X}, \mathbb{R})$ called the kth Chern class.

There is also a way to define the Chern classes as integral cohomology classes $c_k(\mathcal{X}) \in H^{2k}(\mathcal{X}, \mathbb{Z})$ (see [2]). (Here, it is important to remark that the cohomology group $H^k(\mathcal{X}, \mathbb{Z})$ is not the same as the integral cohomology group of the underlying topological space $H^k(X, \mathbb{Z})$.) For our purposes, it is enough to know that the integral first Chern class $c_1(\mathcal{X})$ vanishes in $H^2(\mathcal{X}, \mathbb{Z})$ if and only if the sheaf $K_{\mathcal{X}}$ of germs of n-forms is isomorphic to the trivial invertible sheaf $\mathcal{O}_{\mathcal{X}}$ of germs of holomorphic functions. If $c_1(\mathcal{X})$ vanishes as an integral cohomology class, then it must also vanish as a real cohomology class. However, the converse is not true in general, as we will see in Example 2.30 below.

In the special case that $\dim_{\mathbb{C}} \mathcal{X} = 1$, the condition that $c_1(\mathcal{X})$ vanish as a real cohomology class is equivalent to the condition that the degree

$$\deg(K_{\mathcal{X}}) = -\langle c_1(\mathcal{X}), [\mathcal{X}]\rangle = -\int_{\mathcal{X}} c_1(\mathcal{X})$$

of the sheaf $K_{\mathcal{X}}$ vanishes, where $[\mathcal{X}] \in H_2(\mathcal{X}, \mathbb{R})$ denotes the fundamental class determined by a choice of orientation on $\mathcal{X}$. Let us use this observation first to classify all Calabi-Yau orbifold Riemann surfaces, which we call elliptic orbifolds.

Example 2.30. Let $\mathcal{X}$ be a connected closed orbifold Riemann surface with finitely many stacky points $p_1, \ldots, p_n$ in the underlying space X . (We assume that $\mathcal{X}$ contains at least one stacky point, or equivalently, $n > 0$.) For each i, let $m_i > 1$ denote the size of the stabilizer group of p_i. We may assume that we have ordered the points so that $m_1 \geqslant \cdots \geqslant m_n$. The goal of this example is to show that the statement $c_1(\mathcal{X}) = 0$ as a real cohomology class can be stated in terms of the data $n, m_1, \ldots, m_n$, and will hence give a finite list of possibilities.

Let $\mathcal{Y}$ denote the complex manifold whose complex structure is determined by the atlas represented by the open subsets $U \subset \mathbb{C}$ from the charts of $\mathcal{X}$ together with the transition functions determined by the embeddings of these charts. Then $\mathcal{Y}$ is a connected closed smooth Riemann surface of genus g. In particular, $\mathcal{Y}$ is an effective orbifold in which every group in every orbifold chart is trivial. Let $\pi : \mathcal{X} \to \mathcal{Y}$ denote the corresponding canonical smooth map of effective orbifolds, which we study presently.

Away from the stacky points, the map π is an isomorphism of effective orbifolds. More precisely, denote by q_i the point in $\mathcal{Y}$ corresponding to p_i in X. Let U denote the subset $U = X \setminus \{p_1, \ldots, p_n\}$ together with the orbifold structure determined by $\mathcal{X}$, and let V denote the complex submanifold of $\mathcal{Y}$ given by $V = Y \setminus \{q_1, \ldots, q_n\}$ where Y is the underlying space of $\mathcal{Y}$. Then the restriction

$$\pi|_U : U \to V$$

is an isomorphism of effective orbifolds.

Near the point p_i, however, the map π can be described as follows. If w is a local coordinate on $\mathcal{Y}$ near q_i and if z is a local coordinate on $\mathcal{X}$ near p_i, then $\pi^* w = z^{m_i}$. It follows that if $\mathcal{O}_{\mathcal{Y}}(q_i)$ denotes the locally free sheaf corresponding to the divisor q_i in

$\mathcal{Y}$, then upon pulling back to $\mathcal{X}$ we obtain $\pi^*(\mathcal{O}_{\mathcal{Y}}(q_i)) = \mathcal{O}_{\mathcal{X}}(m_i p_i)$. Since the degree of the line bundle $\mathcal{O}_{\mathcal{Y}}(q_i)$ is 1 and pulling back is compatible with taking degree, we find that

$$\deg(\mathcal{O}_{\mathcal{X}}(m_i p_i)) = \deg(\pi^*(\mathcal{O}_{\mathcal{Y}}(q_i))) = \deg(\mathcal{O}_{\mathcal{Y}}(q_i)) = 1.$$

We conclude that we must have

$$\deg(\mathcal{O}_{\mathcal{X}}(p_i)) = \frac{1}{m_i}.$$

If $K_{\mathcal{X}}$ denotes the coherent sheaf of germs of one-forms on $\mathcal{X}$, then we claim that

$$\pi^* K_{\mathcal{Y}} = K_{\mathcal{X}} \left(\sum_{i=1}^{n} (1 - m_i) p_i \right). \tag{2.4}$$

Indeed, away from the stacky points, the map π is an isomorphism, so we have

$$\pi^* K_{\mathcal{Y}}|_U = K_{\mathcal{X}}|_U = K_{\mathcal{X}} \left(\sum_{i=1}^{n} (1 - m_i) p_i \right) \Bigg|_U$$

where the last equality follows because $p_i \notin U$. On the other hand, near the point p_i, the projection map π satisfies

$$\pi^*(dw) = d(z^{m_i}) = m_i z^{m_i - 1} dz.$$

If U_i is a neighborhood of p_i satisfying $U_i \cap \{p_1, \ldots, p_n\} = p_i$, then the previous equality shows that

$$\pi^* K_{\mathcal{Y}}|_{U_i} = K_{\mathcal{X}} \left(\sum_{i=1}^{n} (1 - m_i) p_i \right) \Bigg|_{U_i}.$$

The claim now follows.

Taking the degree of (2.4), we find that

$$2g-2=\deg(K_{\mathcal{X}})+\sum_{i=1}^{n}\frac{1-m_i}{m_i}=\deg(K_{\mathcal{X}})-n+\sum_{i=1}^{n}\frac{1}{m_i}.$$

And hence the degree of the canonical sheaf $K_{\mathcal{X}}$ satisfies

$$\deg(K_{\mathcal{X}})=2g-2+n-\sum_{i=1}^{n}\frac{1}{m_i}.$$

Therefore, the condition $c_1(\mathcal{X})=0$ as a real cohomology class is equivalent to the equation

$$2-2g=\sum_{i=1}^{n}\left(1-\frac{1}{m_i}\right). \tag{2.5}$$

However, this condition is not enough to ensure that the first Chern class vanishes as an *integral* cohomology class, and in fact, even with this condition, the first Chern class is *never* zero in $H^2(\mathcal{X},\mathbb{Z})$ because the integral first Chern class restricts to a generator $c_1(\mathcal{X})|_{p_i}\in H^2(p_i,\mathbb{Z})\simeq\mathbb{Z}/m_i\mathbb{Z}$ of the integral cohomology group of each stacky point. Nevertheless, if m is the least common multiple of $m_1,\ldots,m_n$, then $K_{\mathcal{X}}^{\otimes m}$ is trivial, so that the multiple $mc_1(\mathcal{X})$ *is* zero as an integral cohomology class.

Now let us determine the possibilities for $n,m_1,\ldots,m_n$. Each term on the right hand side of (2.5) is at least $1/2$ and less than 1, so we find that

$$\frac{n}{2}\leqslant 2-2g<n.$$

Since $n\geqslant 1$, we conclude that $g=0$, and thus there are two cases for n: either $n=3$ or $n=4$.

- Suppose that $n = 4$. Then equation (2.5) implies that we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = 2.$$

  There is only one possibility

  (i) $m_1 = m_2 = m_3 = m_4 = 2$.

- Suppose that $n = 3$. Then equation (2.5) implies that we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1.$$

  We also have that $m_1 \geqslant m_2 \geqslant m_3 \geqslant 2$. There are three possibilities

  (ii) $m_1 = m_2 = 4, m_3 = 2$;

  (iii) $m_1 = 6, m_2 = 3, m_3 = 2$;

  (iv) $m_1 = m_2 = m_3 = 3$.

Each of these cases (i) through (iv) can be realized explicitly as a quotient of an elliptic curve. Indeed for a complex number τ in the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, let E_τ denote the smooth elliptic curve

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

The flat Kähler metric on $\mathbb{C}$ descends to a flat Kähler metric on E_τ. If Γ is an finite group acting holomorphically and isometrically on E_τ, then the flat Kähler metric on E_τ descends to a flat Kähler metric (hence Ricci flat) metric on the global quotient orbifold $[E_\tau/\Gamma]$.

(i) If τ is any element of the upper half plane, then the group $\Gamma = \{\pm 1\}$ acts on $\mathbb{C}$ in such a way that the action descends to the quotient E_τ. If $[0]$ denotes the point in E_τ corresponding to $0 \in \mathbb{C}$, then $[0]$ is fixed by every element of Γ. The same is also true of the points $[1/2], [\tau/2]$, and $[(1+\tau)/2]$. It follows that the orbifold $[E_\tau/\Gamma]$ is an elliptic orbifold. Such an orbifold is called a "pillowcase" in the literature.

(ii) Suppose in particular that $\tau = \sqrt{-1}$. The group $\Gamma = \{\pm 1, \pm\tau\} \simeq \mathbb{Z}_4$ acts on $\mathbb{C}$ in such a way that the action descends to one on E_τ. The points $[0]$ and $[(1+\sqrt{-1})/2]$ are fixed by every element of Γ, and hence have a stabilizer group of order 4. The point $[1/2]$ is fixed by the subgroup of size two consisting of $\{1, -1\}$. It follows that the orbifold $[E_\tau/\Gamma]$ corresponds to the elliptic orbifold $\mathbb{P}^1_{4,4,2}$.

(iii) Suppose that $\tau = e^{\pi\sqrt{-1}/3}$. The rotations generated by -1 and $e^{2\pi\sqrt{-1}/3}$ act in a well-defined way on E_τ so that the group $\Gamma = \mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$ acts on E_τ. The points $[0], [(\tau+1)/3]$, and $[(\tau+1)/2]$ have stabilizers of orders $6, 3$, and 2 respectively. It follows that $[E_\tau/\Gamma]$ corresponds to the elliptic orbifold $\mathbb{P}^1_{6,3,2}$.

(iv) Suppose again that $\tau = e^{\pi\sqrt{-1}/3}$. The group $\mathbb{Z}_3$ acts on $\mathbb{C}$ by rotations generated by $\tau^2 = e^{2\pi\sqrt{-1}/3}$. This action descends to a $\mathbb{Z}_3$-action on E_τ. The points $[0], [(1+\tau)/3]$, and $[2(1+\tau)/3]$ are distinct points in the quotient with stabilizers of order 3. It follows that $[E_\tau/\Gamma']$ corresponds to the elliptic orbifold $\mathbb{P}^1_{3,3,3}$.

This completes our discussion of elliptic orbifolds.

Remark 2.31. The slightly different but related problem of the Ricci flow and its convergence is studied on Riemann surfaces with marked points in [53]. There, the convergence of the flow is studied in relation to a notion of stability for the underlying orbifold itself, and the flow is shown to convergence in all three stable, semi-stable, and unstable cases.

For examples of higher dimensional Calabi-Yau orbifolds, one can consider complete intersections in weighted projective space. In this case, one can construct examples where the first Chern class vanishes as an *integral* cohomology class.

Example 2.32. Recall weighted projective space $\mathbb{CP}[a_0, \ldots, a_n]$ of Example 2.5.

It is useful to view $\mathbb{CP}[q_0, \ldots, q_n]$ as a toric variety. Let N be a lattice spanned by vectors $u_0, \ldots, u_n$ satisfying the relation $q_0u_0 + \cdots + q_nu_n = 0$, and let Σ be the fan of all cones generated by proper subsets of $\{u_0, \ldots, u_n\}$. Then $\mathbb{CP}[q_0, \ldots, q_n]$ is the toric variety X_Σ corresponding to the fan Σ. Let D_i denote the torus-invariant divisor in $\mathbb{CP}[q_0, \ldots, q_n]$ corresponding to the one-dimensional cone spanned by u_i. According to [25, Exercise 4.1.5] the class group $\mathrm{Cl}(X_\Sigma)$ can be identified with $\mathbb{Z}$ in such a way that the divisor $\sum_i a_iD_i$ determines the element $\sum_i a_iq_i$ of $\mathbb{Z} \simeq \mathrm{Cl}(X_\Sigma)$. The divisor $-D_0 - \cdots - D_n$ corresponds to the dualizing sheaf of X_Σ which in turn corresponds to the element $-q_0 - \cdots - q_n \in \mathbb{Z} \simeq \mathrm{Cl}(X_\Sigma)$.

Let $\mathbb{C}[x_0, \ldots, x_n]$ denote the polynomial ring corresponding to $\mathbb{CP}[q_0, \ldots, q_n]$, where each x_i has degree q_i. A polynomial F in $\mathbb{C}[x_0, \ldots, x_n]$ has degree d if each monomial x^α appearing in F satisfies $\alpha \cdot (q_0, \ldots, q_n) = d$. Accordingly, a polynonial F of degree d corresponds to a global section of the sheaf corresponding to $d \in \mathbb{Z} \simeq \mathrm{Cl}(X_\Sigma)$, so that the hypersurface $\{F = 0\}$ in $\mathbb{CP}[q_0, \ldots, q_n]$ has normal sheaf corresponding to $d \in \mathbb{Z} \simeq \mathrm{Cl}(X_\Sigma)$.

Let $F_1, \ldots, F_s$ be homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degrees $d_1, \ldots, d_s$ respectively. Then the subset of weighted projective space given by $Y = \{F_1 = \cdots = F_s = 0\}$ is a complete intersection subvariety. If Y has at most quotient singularities, then Y is a complex orbifold. Remarks in the previous paragraph imply that the top power of the normal sheaf of Y is the sheaf corresponding to $d_1+\cdots+d_s \in \mathbb{Z} \simeq \mathrm{Cl}(X_\Sigma)$. Because Y and X are Cohen-Macaulay, adjunction still holds, which gives that

$$K_Y = K_X(d_1 + \cdots + d_s).$$

We find that K_Y is trivial if and only if

$$d_1 + \cdots + d_s = q_0 + \cdots + q_n$$

which is equivalent to $c_1(Y) = 0$ as an integral cohomology class.

Example 2.33. We follow a construction found in [22] to give finite quotients of Calabi-Yau hypersurfaces in weighted projective spaces, which allows us to to realize the elliptic orbifolds (ii) through (iv) as finite quotients of cubic curves in $\mathbb{CP}^2$ (c.f. [45]). A polynomial F in $(n+1)$ variables is called quasi-homogeneous if there is a n-tuple of weights $(c_0, \ldots, c_n)$ such that for any scalar λ we have

$$F(\lambda^{c_0} x_0, \ldots, \lambda^{c_n} x_n) = \lambda F(x_1, \ldots, x_n).$$

We assume that F is non-degenerate in the sense that F defines an isolated singularity at the origin, and we also assume that F is of Calabi-Yau type meaning $\sum_i c_i = 1$. Then the equation $F = 0$ defines a Calabi-Yau hypersurface X_F in the weighted projective space $\mathbb{CP}[q_0, \ldots, q_n]$ where $q_i = c_i/d$ for some common denominator d. (For example, if $F = x^2y + y^3 + xz^2$, then F is quasihomogeneous with weights $(1/3, 1/3, 1/3)$, and hence F defines a Calabi-Yau hypersurface in $\mathbb{CP}^2 = \mathbb{CP}[1,1,1]$.) Let $G_{\max}$ denote the diagonal symmetry group

$$G_{\max} = \{\mathrm{Diag}(\lambda_0, \ldots, \lambda_N) : F(\lambda_0 x_0, \ldots, \lambda_n x_0) = F(x_0, \ldots, x_n)\}.$$

This group contains the element $J = \mathrm{Diag}(e^{2\pi i c_0}, \ldots, e^{2\pi i c_n})$ which acts trivially on X_F. A subgroup G satisfying $J \in G \subset G_{\max}$ acts on X_F with kernel $\langle J \rangle$. Hence the group $\tilde{G} = G/\langle J \rangle$ acts faithfully on X_F, and one obtains an orbifold global quotient $\mathcal{X} = [X_F/\tilde{G}]$. This orbifold global quotient has $c_1(\mathcal{X}) = 0$ as a real cohomology class. In particular, the canonical sheaf $K_{\mathcal{X}}$ is not necessarily trivial, but some power of it is, so that some multiple of $c_1(\mathcal{X})$ vanishes as an integral cohomology class.

One can realize the elliptic orbifolds from Example 2.30 as such quotients of cubic curves in $\mathbb{CP}^2$ (c.f. [45]):

(ii) Consider the curve $X_F \subset \mathbb{CP}^2$ determined by the cubic polynomial $F = x^2y + y^3 + xz^2$. The group $\tilde{G} = \mathbb{Z}_4$ acts on X_F via

$$\xi \cdot [x, y, z] = [\xi^2 x, y, \xi z]$$

so that the quotient orbifold $[X_F/\tilde{G}]$ is the elliptic orbifold $\mathbb{P}_{4,4,2}$. Indeed the points whose stabilizers have orders $4, 4$ and 2 are the images of the points $[1, 0, 0]$, $[0, 0, 1]$, and $[1, \sqrt{-1}, 0]$.

(iii) Consider the curve $X_F \subset \mathbb{CP}^2$ determined by the cubic polynomial $F = x^3 + y^3 + xz^2$. The group $\tilde{G} = \mathbb{Z}_6$ acts on X_F via

$$\xi \cdot [x, y, z] = [\xi^4 x, y, \xi z]$$

so that the quotient orbifold is the elliptic orbifold $\mathbb{P}_{6,3,2}$. Indeed, the three points whose stabilizers have orders 6, 3, and 2 are the images of the points $[0, 0, 1], [1, 0, \sqrt{-1}]$, and $[x, -1, 0]$ where x is a third root of unity.

(iv) Consider the curve $X_F \subset \mathbb{CP}^2$ determined by the cubic polynomial $F = x^3 +$

$y^3 + z^3$. The group $\tilde{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$ acts on X_F via

$$(\xi_1, \xi_2) \cdot [x, y, z] = [x, \xi_1 y, \xi_2 z]$$

so that the quotient orbifold $[X_F/\tilde{G}]$ is the elliptic orbifold $\mathbb{P}_{3,3,3}$. Indeed the points with stabilizers of size three are the images of the points $[-1, 0, z]$, $[0, -1, z]$ and $[-1, y, 0]$ where y, z denote third roots of unity.

In the general case, to each finite quotient Calabi-Yau orbifold $[X_F/\tilde{G}]$, there is an associated Berglund-Hübsch-Krawitz mirror $[X_F^T/\tilde{G}^T]$, which is another Calabi-Yau orbifold that is dual to the other one in the sense that there are symmetric isomorphisms at the level of certain cohomological groups, namely Chen-Ruan orbifold cohomology. It is beyond the scope of this work to describe the mirror here, but we direct the reader to [9] for a detailed description. (Also see [9] for the proposal of this "classical mirror symmetry conjecture" and [22] for the proof.)

Example 2.34. One can consider Calabi-Yau hypersurfaces in toric varieties defined by polyhedra, and in the case that the defining polyhedron is reflexive, there is a so-called Batyrev mirror, which is another Calabi-Yau hypersurface dual to the original hypersurface in the sense of mirror symmetry from mathematical physics [6]. Provided these hypersurfaces are orbifolds, Theorem 2.2 implies that these hypersurfaces and their mirrors admit Ricci-flat metrics. One exposition of this material can be found in [24], which we will follow closely.

More precisely, let N be a lattice, and M the corresponding dual lattice. A full-dimensional integral polytope Δ in M is reflexive if

(i) 0 belongs to the interior of Δ

(ii) there are vectors $v_F \in N$ associated to each codimension-1 face F of Δ such that

$$\Delta = \{m \in M_{\mathbb{R}} : \langle m, v_F \rangle \geqslant -1 \text{ for each } F\}.$$

Such a polytope determines a toric variety X_Δ which is Gorenstein and Fano. The anticanonical sheaf corresponds to the divisor $\sum_\rho D_\rho$ as ρ ranges over the one-dimensional cones in the normal fan Σ_Δ. By adjunction, the zero locus of a generic section of the anti-canonical sheaf determines a hypersurface $\bar{V}$ with trivial canonical sheaf so that $c_1(\bar{V})$ is zero as an integral cohomology class. Let $\mathcal{F}(\Delta)$ denote this family of Calabi-Yau hypersurfaces.

This family $\mathcal{F}(\Delta)$ of Calabi-Yau hypersurfaces is dual to another family of Calabi-Yau hypersurfaces in the following sense. The polar dual of Δ is given by

$$\Delta^\circ = \{n \in N_{\mathbb{R}} : \langle m, n \rangle \geqslant -1 \text{ for all } m \in \Delta\}$$

and is reflexive if and only if Δ is. It follows that Δ° also determines a Gorenstein toric Fano variety X_{Δ°. In this way, one obtains a dual family $\mathcal{F}(\Delta^\circ)$ of Calabi-Yau hypersurfaces determined by generic sections of the anticanonical sheaf of X_{Δ°. The involution taking $\mathcal{F}(\Delta)$ to $\mathcal{F}(\Delta^\circ)$ can be shown to satisfy the properties of the mirror duality in physics [6] in some precise sense involving Hodge numbers.

However, in general, X_Δ may be too singular to study its Hodge theory directly, so one considers a maximal projective simplicial resolution of the normal fan of Δ to obtain a crepant partial resolution $X \to X_\Delta$ by an orbifold X. A general anti-canonical hypersurface V of X is then a Calabi-Yau orbifold which is a proper transform of a corresponding general anticanonical hypersurface $\bar{V}$ of X_Δ, and Batyrev calls V a maximal projective crepant partial desingularization of $\bar{V}$, or a MPCP-desingularization for short. In particular, V is an orbifold, and so it makes sense to consider its Hodge numbers. The precise statement for mirror symmetry then concerns the Hodge numbers of these MPCP-desingularizations V (and their duals V°, which are obtained by applying the same process to the dual polytope Δ°).

Example 2.35. One can generalize the previous example to complete intersection subvarieties of toric varieties and their mirrors [7].

Let X_Δ be a Gorenstein Fano toric variety determined by a reflexive full dimensional integral polytope Δ, and let Σ_Δ denote the corresponding normal fan. In some sense this family of examples is more general than the purpose of this chapter because a general X_Δ may be more singular than an orbifold, but at least X_Δ is an orbifold when the corresponding fan is simplicial.

Let $E = \{e_1, \ldots, e_r\}$ denote the set of vertices of Δ. A representation $E = E_1 \cup \cdots \cup E_s$ as the disjoint union of subsets $E_1, \ldots, E_s$ is called a nef-partition if there are integral convex Σ_Δ-piecewise linear functions $\varphi_1, \ldots, \varphi_s$ on $M_\mathbb{R}$ satisfying

$$\varphi_i(e_j) = \begin{cases} 1 & e_j \in E_i \\ 0 & \text{otherwise} \end{cases}.$$

Such a partition induces a representation of the anticanonical divisor as the sum of s Cartier divisors $\sum_{i=1}^s D_i$ which are nef. A choice of an s-tuple of generic sections of the sheaves corresponding to these divisors gives rise to a complete intersection subvariety V of X_Δ which has trivial canonical sheaf and hence $c_1(V)$ is zero as an integral cohomology class.

There is a duality on such complete intersections which can be described as follows. Let $E = E_1 \cup \cdots \cup E_s$ be a nef partition of the vertices of Δ. For each i, let Δ_i' denote the convex polyhedron

$$\Delta'_i = \{n \in N_{\mathbb{R}} : \langle m, n\rangle \geqslant -\varphi_i(x)\}.$$

Then it can be shown that the lattice polyhedron Δ' defined by

$$\Delta' = \mathrm{Conv}(\Delta'_1 \cup \cdots \cup \Delta'_s).$$

is reflexive [11]. Let E' denote the collection of vertices of Δ'. For each i, if E'_i denotes the collection of vertices of Δ'_i, then $E' = E'_1 \cup \cdots \cup E'_s$ is a nef-partition of E'. In this way, the set of reflexive polyhedra together with nef-partitions enjoys a natural involution

$$(\Delta; E_1, \ldots, E_s) \mapsto (\Delta'; E'_1, \ldots, E'_s).$$

It is shown in [8] that this involution gives rise to mirror symmetric families of Calabi-Yau complete intersections in Gorenstein Fano toric varieties, and the precise statement of this mirror symmetry concerns string-theoretic Hodge numbers.

Example 2.36. Let X be a smooth n-dimensional Kähler manifold and let D be an effective Cartier divisor in X. One can form an orbifold $\mathcal{X} = \sqrt[r]{D/X}$ called the rth root of D in X enjoying a map $\pi : \mathcal{X} \to X$ which is an isomorphism away from D, and moreover such that X is the underlying space of $\mathcal{X}$. Locally, if U is a chart for X intersecting D, then upon restricting U to a polydisc and choosing coordinates such that $U \cap D = \{z_1 = 0\}$, the mapping π admits a local description by the assignment $(z_1, z_2, \ldots, z_n) \mapsto (z_1^r, z_2, \ldots, z_n)$ on U. The group $G = \mathbb{Z}_r$ of rth roots of unity acts on U via the action on the first coordinate to give an orbifold chart for $\mathcal{X}$. (In generaly, the rth root $\sqrt[r]{D/X}$ does *not* admit a description as a global quotient orbifold $[\tilde{X}/\tilde{G}]$, but there are cases in which it is.)

Computations similar to example 2.30 show that the condition $c_1(\mathcal{X})$ vanish as a real cohomology class is equivalent to the condition

$$c_1(X) = \left(1 - \frac{1}{r}\right) c_1(D).$$

In such a case, $\mathcal{X}$ admits a Ricci-flat metric by the main result of this chapter. Such a metric corresponds to a metric on X with edge singularities along D of cone angle $\beta = 1/r$ using the terminology of [13]. (In addition, the condition that $c_1(K_{\mathcal{X}})$ vanish is equivalent to the cohomological assumption of [13].) More generally, it is possible to use the root construction to obtain rational values for β by considering collections of divisors. In this way, the main results of this chapter recover the main results of [13], at least for cone angles β that are rational.

We would like to remark that it is also possible to construct $\sqrt[r]{D/X}$ even when D is singluar. Say that a defining locus for D in some affine chart $U = \mathrm{Spec}(\mathbb{C}[z_1, \ldots, z_n])$ is a polynomial f in the variables $z_1, \ldots, z_n$. If I denotes the ideal of the ring $R = \mathbb{C}[z_1, \ldots, z_n, t]$ generated by $t^r - f$, then the quotient R/I can be identified with a chart $\tilde{U}$ for $\sqrt[r]{D/X}$. There is a natural algebra morphism $\mathbb{C}[z_1, \ldots, z_n] \to R$, which describes a morphism of varieties $\tilde{U} \to U$. In addition, the group $G = \mathbb{Z}_r$ acts on the chart $\tilde{U}$ by the action of G on the coordinate t in such a way that the G-invariant part of $\tilde{U}$ corresponds to the ring $(R/I)^G \simeq \mathbb{C}[z_1, \ldots, z_n, t^r]/\langle t^r - f\rangle$, which may be identified with the ring for U.

There is even a more general construction which constructions a root "stack," and we direct the reader to [14, 1] for these general constructions.

Chapter 3

Ricci-flat metrics on asymptotically conical Kähler orbifolds

It is known that Yau's solution to Calabi's conjecture extends in a certain sense to the setting of non-compact manifolds which are asymptotically conical (AC) [23]. Here, a Kähler manifold (X, g) is called AC if away from some compact set it is diffeomorphic to a Kähler cone $C(\Sigma) = \Sigma \times \mathbb{R}_{>0}$ in such a way that the difference between the two Kähler structures decays rapidly (with some weight λ_g).

In this case, the complex Monge-Ampère equation (2.1) admits solutions, but these solutions depend on the prescribed decay rate of the function F appearing on the right-hand side. In particular, it is natural to require that F decay at least as fast as ρ^β for some weight $\beta < 0$, where ρ is a radius function defined on the underlying AC manifold, and we denote this by $F \in C^\infty_\beta(X)$. The linearization of (2.1) then involves the Laplacian $\Delta : C^\infty_{\beta+2}(X) \to C^\infty_\beta(X)$ corresponding to a Kähler metric, and the Fredholm index of this Laplacian is a monotonic function defined for almost every weight β, with only jump discontinuities at a set $\mathcal{P} \subset \mathbb{R}$ of exceptional weights (where the Fredholm index is not defined). In this way, one obtains different types of existence and uniqueness results for different intervals of decay rates (see [23]).

In [33], we show that these existence results extend to the setting of orbifolds mutatis mutandis.

Theorem 3.1. *Let $(\mathcal{X}, g)$ be a Kähler orbifold of complex dimension n which is asymptotically conical of order λ_g. Given a smooth function $F \in C^\infty_\beta(\mathcal{X})$, consider the equation* (2.1) *to be solved for a smooth function φ on $\mathcal{X}$.*

(i) If $\beta \in (\max\{-4n, \beta_1^-, \lambda_g - 2n\}, -2n)$ where β_1^- is the exceptional weight corresponding to the smallest nonzero eigenvalue of the Laplacian of Σ, then there is a unique solution $\varphi \in \mathbb{R}\rho^{2-2n} \oplus C^\infty_{\beta+2}(\mathcal{X})$.

(ii) If $\beta \in (-2n, -2)$, then there is a unique solution $\varphi \in C^\infty_{\beta+2}(\mathcal{X})$.

(iii) If $\beta \in (-2, 0)$ and $\beta + 2 \notin \mathcal{P}$, then there is a solution $\varphi \in C^\infty_{\beta+2}(\mathcal{X})$.

Remark 3.2. A small improvement in our statement of Theorem 3.1 from the existence theorem of [23] is the precision offered in the statement of the interval for case (i), namely the left-hand endpoint of the interval, which depends on a study of the existence of solutions to the Dirichlet problem for the Laplacian on AC orbifolds and which should be compared to the case of asymptotically locally Euclidean (ALE) manifolds as in [42]. In fact, the number $\beta' := \max\{-4n, \beta_1^-, \lambda_g - 2n\}$ arises in the following manner.

Suppose β is a weight satisfying $\beta' < \beta < -2n$, that is, suppose we are in case (i) of Theorem 3.1. Then because F decays with rate β, it also decays with rate $-2n+\epsilon$, and we may obtain a solution φ to (2.1) from case (ii), but this solution is only known to decay with rate $C_{-2n+\epsilon+2}$. The assumption $\beta' < \beta$ guarantees that we can improve the decay rate of φ in the following four steps.

(a) By expanding (2.1) as

$$-(\Delta\varphi)\omega^n = (1-e^F)\omega^n + \sum_{j=2}^{n}\binom{n}{j}(i\partial\bar{\partial}\varphi)^j \wedge \omega^{n-j},$$

we immediately obtain that $\Delta\varphi$ decays with rate $\max\{\beta, 2(-2n+\epsilon)\}$, and thus the assumption $-4n \leqslant \beta' < \beta$ implies that $\Delta\varphi = O(\rho^{\beta})$.

(b) A weighted version of the Poincaré inequality that holds on AC manifolds [40] (and therefore also on AC orbifolds), allows one demonstrate that for $\beta < -n-1$ the Poisson equation $\Delta u = f$ for $f \in C^{\infty}_{\beta}$ admits a unique solution u belonging to a certain weighted Sobolev space $L^2_{1,1-n}$ (and in fact, such a solution must be smooth by standard regularity arguments).

(c) Moreover, following [49], it is possible to show that assuming $\beta_1^- \leqslant \beta' < \beta < -2n$, the Laplacian $\Delta : C^{k+2,\alpha}_{\beta+2} \to C^{k,\alpha}_{\beta}$ has a range given by all $f \in C^{k,\alpha}_{\beta}$ with zero integral over $\mathcal{X}$, and hence for such an f, we obtain an estimate on solutions u to $\Delta u = f$ the form $\|u\|_{C^{k+2,\alpha}_{\beta+2}} \leqslant C\,\|f\|_{C^{k,\alpha}_{\beta}}$.

(d) However, if f does not have zero integral, then it is possible to show that the assumption $\lambda_g - 2n \leqslant \beta' < \beta < -2n$ implies that a solution u to $\Delta u = f$ must have a contribution at the exceptional weight $-2n$ in the sense that for such a u there is a $v \in C_{\beta+2}$ and a number A such that $u = A\rho^{2-2n} + v$.

(e) Combining (c) and (d) with (a) gives part (i) of Theorem 3.1.

Remark 3.3. It would be interesting to answer the question of whether this number $\max\{-4n, \beta_1^-, \lambda_g - 2n\}$ is as *sharp* as possible, but we do not study this question any further here.

Remark 3.4. Another difference from Conlon-Hein [23] is the inclusion of a detailed outline of a proof of cases (i) and (ii) using a continuity method involving estimates on weighted Hölder spaces, which is a modification of the approach of Joyce [42] (which deals only with the case when the link Σ is S^{2n-1}). (A detailed account of how statement (iii) follows from (ii) is found already in [23].) To fully extend Joyce's claims, the improvement of the previous remark was necessary. In addition, by incorporating modern techniques into Joyce's approach, we are able in [33] to streamline the argument somewhat and avoid some potentially irksome technical lemmas. It would be interesting to investigate whether even more modern approaches, such as an approach involving the ABP estimate, would be applicable in this non-compact setting and would streamline the arguments even further.

Using Theorem 3.1, one can guarantee existence of a one-parameter family of Ricci-flat metrics within each Kähler class satisfying a mild decay condition on an AC orbifold with trivial canonical bundle.

Theorem 3.5. *Let $\mathcal{X}$ be a complex orbifold of complex dimension $n > 2$ with trivial canonical bundle. Let Ω be a nowhere vanishing holomorphic volume form on $\mathcal{X}$. Let Σ be Sasaki-Einstein with associated Calabi-Yau cone (C, g_0, J_0, Ω_0) and radius function r. Suppose that there is a constant $\lambda_\Omega < 0$, a compact subset $K \subset X$, and a diffeomorphism $\Phi : (1, \infty) \times \Sigma \to X \setminus K$ such that*

$$\Phi^*\Omega - \Omega_0 = O(r^{\lambda_\Omega}).$$

Let $\mathfrak{k} \in H^2(\mathcal{X})$ be an almost compactly supported Kähler class of rate $\lambda_{\mathfrak{k}} < 0$ (see Definition 3.41). Denote by λ the maximum of $\lambda_{\mathfrak{k}}$ and λ_Ω, and assume that $\lambda + 2 \notin \mathcal{P} \cap (0, 2)$. Then for all $c > 0$, there is an asymptotically conical Calabi-Yau metric

g_c on $\mathcal{X}$ whose associated Kähler form ω_c lies in $\mathfrak{k}$ and satisfies

$$\Phi^*\omega_c - c\omega_0 = O(r^{\max\{-2n,\lambda\}}).$$

Moreover, if $\lambda < -2n$, then there is an $\epsilon > 0$ such that

$$\Phi^*\omega_c - c\omega_0 = \mathrm{const}\sqrt{-1}\partial\bar{\partial}r^{2-2n} + O(r^{-2n-1-\epsilon}).$$

In particular, one obtains Ricci-flat metrics on orbifold crepant partial resolutions of Calabi-Yau cones, as stated precisely in the following corollary.

Corollary 3.6. *Let $(C(\Sigma), g_0, J_0, \Omega_0)$ be a Calabi-Yau cone of complex dimension $n > 2$, let $p : C \to \Sigma$ denote the radial projection, and let V be the normal affine variety associated to C. Let $\pi : \mathcal{X} \to V$ be a crepant partial resolution by an orbifold $\mathcal{X}$, and let $\mathfrak{k} \in H^2(\mathcal{X})$ be a class that contains positive $(1,1)$-forms. Then for each $c > 0$, there is a complete Calabi-Yau metric g_c on $\mathcal{X}$ such that $\omega_c \in \mathfrak{k}$ and*

$$\omega_c - \pi^* c\omega_0 = O(r^{-2+\delta})$$

for sufficiently small δ. If $\mathfrak{k} \in H^2_c(\mathcal{X})$, then we actually have

$$\omega_c - \pi^*(c\omega_0) = \mathrm{const}\sqrt{-1}\partial\bar{\partial}r^{2-2n} + O(r^{-2n-1-\epsilon})$$

for some $\epsilon > 0$.

This particular result encompasses many well-known examples of Ricci-flat metrics in the non-compact setting, such as Calabi's Ansatz [16] (Example 3.50), Kronheimer's hyper-Kähler metrics on ADE resolutions [46] (Example 3.51), and the small resolution of a conifold considered by Candelas and Xenia [18] (Example 3.52), along with—and this is the main point of Corollary 3.46—all partial resolutions related to these.

3.1 Preliminaries

Definition 3.7. Let (Σ, g_Σ) be a compact Riemannian manifold.

(i) The *Riemannian cone* $C = C(\Sigma)$ over Σ is defined to be the manifold $\mathbb{R}_+ \times \Sigma$ with the metric

$$g_0 = dr^2 + r^2 g_\Sigma$$

where r is a local coordinate on $\mathbb{R}_+ = (0, \infty)$.

(ii) A tensor T on the cone C is said to decay with rate $\lambda \in \mathbb{R}$, written $T = O(r^\lambda)$, if $|\nabla_0^k T|_{g_0} = O(r^{\lambda - k})$ for each $k \in \mathbb{N}$, where ∇_0 denotes the Levi-Civita connection corresponding to g_0. In particular, if T is a tensor on Σ, we may regard T as a tensor on the cone with decay rate $\lambda = -2$.

(iii) We say that the cone (C, g_0) is Kähler if g_0 is Kähler and C is equipped with a choice of g_0-parallel complex structure J_0. In such a case, there is a Kähler form $\omega_0(U, V) = g_0(J_0 U, V)$ with local expression $\omega_0 = (i/2)\partial\bar{\partial} r^2$.

Definition 3.8. Let $(\mathcal{X}, g, J)$ be a complete Kähler orbifold with underlying space X, and let (C, g_0, J_0) be a Kähler cone.

(i) We say that $\mathcal{X}$ is asymptotically conical of rate λ_g with tangent cone C if there is a diffeomorphism $\Phi : C \setminus K \to X \setminus K'$, with K, K' compact, such that $\Phi^* g - g_0 = O(r^{\lambda_g})$ and also $\Phi^* J - J_0 = O(r^{\lambda_g})$. (In particular, this means that all of the orbifold points of $\mathcal{X}$ are contained in K'.)

(ii) A radius function on an asymptotically conical $\mathcal{X}$ is a smooth function ρ on $\mathcal{X}$ with codomain $[1,\infty)$ satisfying $\Phi^*\rho = r$ away from K'.

A radius function ρ on an asymptotically conical $\mathcal{X}$ allows us to define spaces of functions $C^k_\beta(\mathcal{X})$ in the following manner. For an integer $k \geqslant 0$ and a weight $\beta \in \mathbb{R}$, let $C^k_\beta(\mathcal{X})$ be the space of continuous functions f with k continuous derivatives such that the norm

$$\|f\|_{C^k_\beta(\mathcal{X})} = \sum_{j=0}^{k} \sup_{\mathcal{X}} |\rho^{j-\beta}\nabla^j f|_g$$

is finite. Let $C^\infty_\beta(\mathcal{X})$ denote the intersection of all of the $C^k_\beta(\mathcal{X})$ for $k \geqslant 0$.

We may also define weighted Hölder spaces $C^{k,\alpha}_\beta(\mathcal{X})$. The metric g allows us to define the distance $d(x,y)$ between two points x,y in the underlying space X as the infimum of the lengths of all possible continuous *admissible* curves connecting them (see [12, Theorem 38] the notion of admissible curve and the notion of distance). In this way, we obtain the notion of a ball $B_r(x)$ of radius r centered about x. The Levi-Civita connection for g also allows us to say when a path is a geodesic, and hence we can say that a subset Y of the underlying space is strongly convex if any two points are joined by a unique minimal geodesic entirely contained within Y. The convexity radius $r(x)$ at x is defined to be the largest possible radius R such that $B_r(x)$ is strongly convex for all $0 < r < R$. The convexity radius $r(g)$ of g is the infimum over all $r(x)$. Taken over the compact set K', the infimum will be positive by compactness, and over the rest of the orbifold, the infimum will be positive by the asymptotically conical assumption (and the compactness of Σ). For a tensor T on $\mathcal{X}$, we may then define the seminorm

$$[T]_{C^{0,\alpha}_\beta(\mathcal{X})} = \sup_{\substack{x\neq y\in X \\ d(x,y)<r(g)}} \left[\min(\rho(x),\rho(y))^{-\beta}\frac{|T(x)-T(y)|}{d(x,y)^\alpha}\right]$$

where the distance $|T(x) - T(y)|$ is defined via parallel transport along the minimal geodesic from x to y. Then define the weighted Hölder space $C^{k,\alpha}_{\beta}(\mathcal{X})$ to be the space of functions $f \in C^k_\beta(\mathcal{X})$ for which the norm

$$\|f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} = \|f\|_{C^k_\beta(\mathcal{X})} + [\nabla^k f]_{C^{0,\alpha}_{\beta-k-\alpha}}$$

is finite.

For a pair of weights $\beta' < \beta$ and a pair of constants $\alpha' > \alpha$, the inclusion

$$C^{k,\alpha'}_{\beta'}(\mathcal{X}) \hookrightarrow C^{k,\alpha}_{\beta}(\mathcal{X}) \tag{3.1}$$

is continuous. In fact, analogous to the Arzela-Ascoli Theorem for compact manifolds/orbfiolds, this inclusion is compact, as stated below.

Theorem 3.9. *For a pair of weights $\beta' < \beta$ and a pair of constants $\alpha' > \alpha$, the inclusion* (3.1) *is compact.*

This theorem is proved for weighted Hölder spaces on complete non-compact manifolds in [19, Lemme 3]. The arguments presented there can be applied to the setting of orbifolds with only minor notational adjustments.

3.2 Fredholm index of the Laplacian

For an asymptotically conical $(\mathcal{X}, g)$ of complex dimension n, the Laplacian Δ induced from g defines a linear map

$$\Delta : C^{k+2,\alpha}_{\beta+2}(\mathcal{X}) \to C^{k,\alpha}_{\beta}(\mathcal{X}) \tag{3.2}$$

for *any* weight β. In the setting of compact orbifolds, the operator Δ is elliptic, and there are corresponding a prior estimates on solutions to equations involving Δ in corresponding (unweighted) Hölder spaces as descussed in Section 2.2. The same is true in the setting of conical orbifolds in weighted Hölder spaces:

Theorem 3.10 (Weighted Schauder estimates on conical orbifolds)**.** *Let $(\mathcal{X}, g)$ be an asymptotically conical Riemannian orbifold, and let $k, \ell \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there is a constant C such that for each $f \in C^{k+2,\alpha}_{\beta+2}(\mathcal{X})$, we have*

$$\|f\|_{C^{k+2,\alpha}_{\beta+2}(\mathcal{X})} \leqslant C(\|\Delta f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} + \|f\|_{C^0_{\beta+2}(\mathcal{X})}).$$

To prove this theorem, we require first the result for manifolds, which is discussed in greater detail, for example, in Marshall [49].

Theorem 3.11 (Weighted Schauder estimates on conical manifolds)**.** *Let $(C(\Sigma), g_\Sigma)$ be a Riemannian cone, and let $\alpha \in (0,1)$, $k \in \mathbb{N}$, and $\beta \in \mathbb{R}$. Then there is a constant C such that for each $f \in C^{k+2,\alpha}_{\beta+2}(C(\Sigma))$, we have*

$$\|f\|_{C^{k+2,\alpha}_{\beta+2}(C(\Sigma))} \leqslant C(\|\Delta f\|_{C^{k,\alpha}_{\beta}(C(\Sigma))} + \|f\|_{C^0_{\beta+2}(C(\Sigma))}).$$

Proof of Theorem 3.10. Cover the compact set K' by finitely many orbifold charts $(U_\gamma, \pi_\gamma, G_\gamma)$. We may select G_γ-invariant relatively compact subsets $U'_\gamma \subset U_\gamma$ satisfying $d(U'_\gamma, \partial U_\gamma) > 0$ and whose supports still cover K'. Because the collection of supports $\{U'_\gamma\}$ covers K', the norm $\|f\|_{C^{k+2,\alpha}_{\beta+2}(\mathcal{X})}$ is equivalent to the norm defined by

$$\|f\|_{C^{k+2,\alpha}_{\beta+2}(X\setminus K')} + \sum_{\gamma} \|f\|_{C^{k+2,\alpha}(U'_\gamma)} .$$

(Notice the absence of the weights in the norms over the subsets U'_γ.) The ordinary

Schauder estimates in $\mathbb{R}^n$ imply that for each γ, there is a constant C_γ such that

$$\|f\|_{C^{k+2,\alpha}(U'_\gamma)} \leqslant C_\gamma(\|\Delta f\|_{C^{k,\alpha}(U_\gamma)} + \|f\|_{C^0(U_\gamma)}).$$

Moreover, the weighted Schauder estimates on conical manifolds together with the fact that $(\mathcal{X}, g)$ is asymptotically conical imply there is a constant C' such that

$$\|f\|_{C^{k+2,\alpha}_{\beta+2}(X\setminus K')} \leqslant C'(\|\Delta f\|_{C^{k,\alpha}_{\beta}(X\setminus K')} + \|f\|_{C^0_{\beta+2}(X\setminus K')}).$$

If C denotes the maximum of C' and the numbers C_γ, then the result follows. $\square$

In the compact case, the Arzela-Ascoli Theorem—together with (unweighted) Schauder estimates—implies that Δ is Fredholm. However, this implication fails to be true in the asympotically conical case, because, for example, $\mathcal{X}$ is not compact and hence neither is the embedding dealt with by Arzela-Ascoli. Nevertheless, for almost all weights β, the map (3.2) is still Fredholm, as stated below.

Theorem 3.12. *For an asymptotically conical Kähler orbifold $\mathcal{X}$ of complex dimension n with Kähler cone $C(\Sigma)$, define the set of exceptional weights*

$$\mathcal{P} = \left\{ -\frac{m-2}{2} \pm \sqrt{\frac{(m-2)^2}{4} + \mu} : \mu \geqslant 0 \textit{ is an eigenvalue of } \Delta_\Sigma \right\}.$$

where $m = 2n = \dim_{\mathbb{R}} C(\Sigma)$. Then the operator (3.2) *is Fredholm if $\beta + 2 \notin \mathcal{P}$.*

Remark 3.13. In general, the set $\mathcal{P}$ is disjoint from the interval $(-2n+2, 0)$ and symmetric about the point $1 - n = (2-m)/2$.

To prove this result, we require a result from [47] which states that an elliptic operator on the *full cylinder* is an isomorphism away from the exceptional weights. The full cylinder is the product $\mathrm{Cyl}(\Sigma) = \mathbb{R} \times \Sigma$ where we use the coordinate t on $\mathbb{R}$

and t is related to r by the rule $e^t = r$. If Δ is the Laplacian on the cone, then we may regard Δ as a differential operator on the cylinder as well, and the composition $P = e^{2t} \circ \Delta$ can be regarded as an elliptic differential operator

$$P : C^{k+2,\alpha}_{\beta+2}(\mathrm{Cyl}(\Sigma)) \to C^{k,\alpha}_{\beta+2}(\mathrm{Cyl}(\Sigma)) \tag{3.3}$$

which is translation invariant, where the weighted Hölder spaces on the cylinder are exactly those on the cone with the change of variables $r = e^t$. Away from the exceptional weights, it is known that this operator is actually an isomorphism.

Lemma 3.14. *[47, 37] The operator* (3.3) *is an isomorphism provided $\beta + 2 \notin \mathcal{P}$.*

Proof of Theorem 3.12. The argument is essentially the same as that in [47, Section 2]: one splits the estimates into those near the apex of the cone and those near infinity. Since we have weighted conical estimates on an orbifold by Theorem 3.10, we apply these near the apex, and then combine these estimates with estimates near infinity from 3.14 to obtain the desired result.

More precisely, let R be a number so large that $K' \subset \rho^{-1}([1, R])$, and set $X_1 = \rho^{-1}([1, R])$. Let φ_1 be a smooth cutoff function compactly supported on X_1 such that $\varphi_1 \equiv 1$ on K'. Set $\varphi_2 = 1 - \varphi_1$. The Schauder estimates (Theorem 3.10) give that

$$\|\varphi_1 f\|_{C^{k+2,\alpha}_{\beta+2}(\mathcal{X})} \leqslant C(\|\Delta(\varphi_1 f)\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} + \|\varphi_1 f\|_{C^{0}_{\beta+2}(\mathcal{X})}).$$

The composition $P = \rho^2 \circ \Delta$ is a translation invariant differential operator of order 2 which is elliptic. For β satisfying $\beta + 2 \notin \mathcal{P}$, view $\varphi_2 f$ as a function on the full cylinder $\mathrm{Cyl}(\Sigma)$, and then apply Lemma 3.14 to the composition $P = \rho^2 \circ \Delta$ to obtain an estimate of the form

$$\|\varphi_2 f\|_{C^{k+2,\alpha}_{\beta+2}(\mathcal{X})} \leqslant C \left\|\rho^2 \Delta(\varphi_2 f)\right\|_{C^{k,\alpha}_{\beta+2}(\mathcal{X})} = \|\Delta(\varphi_2 f)\|_{C^{k,\alpha}_{\beta}(\mathcal{X})}.$$

Combining these two inequalities, we find that for β satisfying $\beta + 2 \notin \mathcal{P}$, we have

$$\begin{aligned} \|f\|_{C^{k+2,\alpha}_{\beta+2}(\mathcal{X})} \leqslant C \Big(&\|\varphi_1 \Delta f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} + \|[\varphi_1, \Delta] f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} \\ &+ \|\varphi_2 \Delta f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} + \|[\varphi_2, \Delta] f\|_{C^{k,\alpha}_{\beta}(\mathcal{X})} + \|\varphi_1 f\|_{C^{0}_{\beta+2}(\mathcal{X})} \Big) \end{aligned} \tag{3.4}$$

where $[\varphi_i, \Delta] = \varphi_i \Delta - \Delta \varphi_i$.

For β satisfying $\beta + 2 \notin \mathcal{P}$, define two maps

$$A_1, A_2 : C^{k+2,\alpha}_{\beta+2}(\mathcal{X}) \longrightarrow C^{k,\alpha}_{\beta}(\mathcal{X}) \oplus C^{k,\alpha}_{\beta}(\mathcal{X}) \oplus C^{k,\alpha}_{\beta}(\mathcal{X}) \oplus C^{0}_{\beta+2}(\mathcal{X})$$

by the assignments

$$\begin{aligned} A_1(f) &= (\Delta f, -[\varphi_1, \Delta] f, -[\varphi_2, \Delta] f, -\varphi_1 f) \\ A_2(f) &= (0, [\varphi_1, \Delta] f, [\varphi_2, \Delta] f, \varphi_1 f). \end{aligned}$$

Inequality (3.4) asserts that A_1 has trivial kernel and closed image. If we knew that A_2 were a compact mapping, then we could apply Lemma 2.11 to conclude that $\Delta = A_1 + A_2$ has finite dimensional kernel and closed image. Therefore, it suffices to prove that A_2 is compact, which we do presently.

We give only an argument to show that $[\varphi_1, \Delta]$ is a compact mapping

$$[\varphi_1, \Delta] : C^{k+2,\alpha}_{\beta+2}(\mathcal{X}) \to C^{k,\alpha}_{\beta}(\mathcal{X}),$$

claiming that arguments for $[\varphi_2, \Delta]$ and φ_1 are similar. Direct computation shows that

$$[\Delta, \varphi_1]f = (\Delta\varphi_1)f - \langle \nabla\varphi_1, \nabla f \rangle.$$

Because φ_1 is supported only on X_1, we conclude that $[\Delta, \varphi_1]f$ is supported only on X_1 as well. It follows that if f_j is a sequence bounded in $C^{k+2,\alpha}_{\beta+2}(\mathcal{X})$, then $[\Delta, \varphi_1]f_j$ is a sequence bounded in $C^{k+1,\alpha}(X_1)$. The Arzela-Ascoli theorem applied to the compact X_1 ensures that there is a subsequence of $[\Delta, \varphi_1]f_j$ which converges in $C^{k,\alpha}(X_1)$ and hence also in $C^{k,\alpha}_{\beta}(\mathcal{X})$.

Finally the arguments in [47, Section 2] shows that for $\beta + 2 \notin \mathcal{P}$, the map (3.2) has finite co-dimensional range as well, with only minor suitable adaptations to the orbifold setting. □

For a weight $\beta < -2$, the kernel of (3.2) is trivial, so that the Fredholm index of (3.2) is nonpositive for non-exceptional weights $\beta < -2$.

Lemma 3.15. *For $\beta < -2$ satisfying $\beta + 2 \notin \mathcal{P}$, the map* (3.2) *is injective.*

Proof. Let $f \in C^{k+2,\alpha}_{\beta+2}(\mathcal{X})$ be such that $\Delta f = 0$. For $R > 1$, let T_R denote the compact subset of X given by $T_R = \rho^{-1}([1, R])$. The restriction of f to T_R is harmonic and so achieves its maximum on the boundary, which we denote by S_R. In particular, the function f belongs to $C^0_{\beta+2}(\mathcal{X})$ so that there is a constant C such that

$$\max_{T_R} f = \sup_{S_R} f < CR^{\beta+2}.$$

Taking $R \to \infty$ implies that f must be identically zero. □

Lemma 3.16. *For $\beta > -2n$ satisfying $\beta + 2 \notin \mathcal{P}$, the map* (3.2) *is surjective.*

Proof. The formal L^2-adjoint Δ^* is a mapping satisfying

$$\int_{\mathcal{X}} (\Delta f)h \, dV_g = \int_{\mathcal{X}} f(\Delta^* h) \, dV_g$$

for compactly supported smooth functions f, h on $\mathcal{X}$. Because the volume of $\mathcal{X}$ behaves as $O(\rho^{2n})$, we see that this identity extends to functions $f \in C^{\infty}_{\beta+2}(\mathcal{X})$ and $h \in C^{\infty}_{-\beta-2n-\epsilon}(\mathcal{X})$ for any $\epsilon > 0$. Two integrations by parts show that we may take $\Delta^* = \Delta$. We also find that the image of Δ is contained in the subspace perpendicular to the kernel of

$$\Delta : C^{k+2,\alpha}_{-\beta-2n-\epsilon}(\mathcal{X}) \to C^{k,\alpha}_{-\beta-2n-2-\epsilon}(\mathcal{X}) \tag{3.5}$$

for each $\epsilon > 0$. It therefore follows that the image of (3.2) is contained in the subspace perpedicular to the kernel of

$$\Delta : C^{k+2,\alpha}_{-\beta-2n}(\mathcal{X}) \to C^{k,\alpha}_{-\beta-2n-2}(\mathcal{X}). \tag{3.6}$$

The results of [49, Theorem 6.10] extended to the setting of orbifolds imply that the image of (3.2) is actually equal to the subspace perpendicular to the kernel of (3.6). For $\beta > -2n$ satisfying $\beta + 2 \notin \mathcal{P}$, the previous lemma implies that the kernel of (3.6) is trivial, so that (3.2) is surjective. □

Corollary 3.17. *For β satisfying $-2n < \beta < -2$, the map* (3.2) *is an isomorphism.*

3.3 Case (ii) of Theorem 3.1

In this section, we solve equation (2.1) in the case (ii) (where $-2n < \beta < -2$) by the method of continuity as in the previous chapter. (In particular, throughout the section, we will assume that $-2n < \beta < -2$.) For a parameter $t \in [0,1]$, again consider the one-parameter family of equations

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{tF}\omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \end{cases} . \tag{$*_t$}$$

The equation $(*_0)$ admits the trivial solution $\varphi = 0$. The desired equation we want to solve is equation $(*_1)$. To solve this equation, just as in the previous chapter, it suffices to prove the following proposition.

Proposition 3.18.

(i) If $(*_t)$ admits a smooth solution belonging to $C^\infty_{\beta+2}(\mathcal{X})$ for some $t < 1$, then for all sufficiently small $\epsilon > 0$, the equation $(*_{t+\epsilon})$ admits a smooth solution belonging to $C^\infty_{\beta+2}(\mathcal{X})$ as well.

(ii) There is a constant $C > 0$ depending only on $\mathcal{X}, \omega, F$, and α such that if $\varphi \in C^\infty_\gamma(\mathcal{X})$ satisfies $(*_t)$ for some $t \in [0,1]$ and some weight γ satisfying $\beta + 2 \leqslant \gamma < 0$, then

- *$\|\varphi\|_{C^{3,\alpha}_{\beta+2}(\mathcal{X})} \leqslant C$ and*
- *$(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) > C^{-1}(g_{j\bar{k}})$, where $g_{j\bar{k}}$ are the components of ω in local coordinates of any chart and the inequality means that the difference of matrices is positive definite.*

Indeed, Proposition 3.18 is sufficient because we can obtain a solution to $(*_1)$ belonging to $C^\infty_{\beta+2}(\mathcal{X})$ using the following lemma.

Lemma 3.19. *Assume Proposition 3.18. Then if s is a number in $(0,1]$ such that we can solve $(*_t)$ for all $t < s$, then we can solve $(*_s)$ for a smooth function φ belonging to $C^\infty_{\beta+2}(\mathcal{X})$.*

Proof. Let $t_i \in (0,1)$ be a sequence of numbers approaching s from below. By assumption this gives rise to a sequence of smooth functions φ_i satisfying

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_i)^n = e^{t_i F}\omega^n.$$

Proposition 3.18(i) and Theorem 3.9 imply that after passing to a subsequence, we may assume that the φ_i converge in $C^{3,\alpha'}_{\beta'+2}$ to a function φ for some $\alpha' > \alpha$ and $\beta' < \beta$. This convergence is strong enough that φ satisfies the limiting equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{sF}\omega^n.$$

Proposition 3.18(ii) gives that the forms $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_i$ are bounded below by a fixed positive form $C^{-1}\omega$, and hence $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive form. If we knew that φ were smooth, then we could apply Proposition 3.18(i) to obtain that φ belongs to $C^{\infty}_{\beta+2}(\mathcal{X})$. It thus remains to show that φ is smooth.

We show φ is smooth by a standard local bootstrapping argument. In local coordinates, we find that φ satisfies

$$\log\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) - \log\det(g_{j\bar{k}}) - sF = 0.$$

Differentiating the equation with respect to the variable z^ℓ we have

$$(g_\varphi)^{j\bar{k}}\partial_j\partial_{\bar{k}}(\partial_\ell\varphi) = s\partial_\ell F + \partial_\ell \log\det(g_{j\bar{k}}) - (g_\varphi)^{j\bar{k}}\partial_\ell g_{j\bar{k}}$$

where $(g_\varphi)^{j\bar{k}}$ is the inverse of the matrix $(g_\varphi)_{j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$. We think of this equation as a linear elliptic second-order equation $\Delta_\varphi(\partial_\ell\varphi) = h$ for the function $\partial_\ell\varphi \in C^{2,\alpha'}(\mathcal{X})$. Because the function h belongs to $C^{1,\alpha'}$, we conclude from ordinary (unweighted) Schauder estimates that $\partial_\ell \varphi$ belongs to $C^{3,\alpha'}$. Because ℓ was arbitrary, it follows that φ belongs to $C^{4,\alpha'}$. Repeating this argument we obtain that $\varphi \in C^{5,\alpha'}$ and by induction, that φ is actually smooth. $\square$

Let us now prove the first part of Proposition 3.18.

Proof of Proposition 3.18 (i). Let B_1 denote the Banach manifold consisting of those $\varphi \in C^{3,\alpha}_{\beta+2}(\mathcal{X})$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive form. Let B_2 denote the Banach space $B_2 = C^{1,\alpha}_{\beta}(\mathcal{X})$. Define a mapping

$$\begin{aligned} G : B_1 \times [0,1] &\longrightarrow B_2 \\ (\varphi, s) &\longmapsto \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} - sF. \end{aligned}$$

By assumption, we are given a smooth function φ_t belonging to $C^{\infty}_{\beta+2}(\mathcal{X})$ such that $G(\varphi_t, t) = 0$ and $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler form. The partial derivative of G in the direction of φ at the point (φ_t, t) is given by

$$DG_{(\varphi_t,t)}(\psi, 0) = \frac{n\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_t^{n-1}}{\omega_t^n} = -\Delta_t \psi,$$

where $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ and Δ_t denotes the Laplacian with respect to ω_t. Corollary 3.17 gives that Δ_t is an isomorphism

$$\Delta_t : C^{3,\alpha}_{\beta+2}(\mathcal{X}) \to C^{1,\alpha}_{\beta}(\mathcal{X}).$$

The implicit function theorem implies that for s sufficiently close to t, there are functions φ_s in $C^{3,\alpha}_{\beta+2}(\mathcal{X})$ satisfying $(G(\varphi_s), s) = 0$. Because $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ is a positive form, for s close enough to t, we can ensure that each $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_s$ is a positive form as well. Moreover, bootstrapping arguments similar to those described earlier show that φ_s is actually smooth. $\square$

It remains to prove Proposition 3.18 (ii). We do this in the next section.

3.4 A priori estimates

This section is devoted to proving Proposition 3.18 (ii). In particular, we are still assuming that $-2n < \beta < -2$. While the proof is analogous to the compact setting (see Section 2.4), there are a few main differences:

(i) Stokes' Theorem cannot be applied directly since our orbifold is not compact.

(ii) An a priori L^2-bound is replaced by an a priori L^{p_0}-bound for some $p_0 > 1$ (see Lemma 3.22).

(iii) Our bootstrapping arguments (which use weighted Schauder estimates) require a weighted $C^0_{\beta+2}$-estimate on solutions φ.

(iv) Finally, the desired result is a *weighted* $C^{k,\alpha}_{\beta+2}$-estimate, which we show follows from an unweighted $C^{k,\alpha}$-estimate (see Proposition 3.33) as is obtained in the compact setting.

Our methods in this section follow closely those of Joyce in [42] with modifications motivated our approach in the previous chapter, which was influenced strongly by the presentation of [60].

Throughout, let us fix an additional weight γ satisfying $\beta + 2 \leqslant \gamma$. We will be assuming that our solution φ belongs to the weighted Hölder space $C^\infty_\gamma(\mathcal{X})$. Our goal is to obtain a priori estimates on φ. The approach is very similar to that of the continuity method of the first Chapter, but one must obtain additional "weighted" versions of certain a priori estimates.

3.4.1 A C^0-estimate

Lemma 3.20. *For $p > (2-2n)/\gamma$, any smooth solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ to $(*_t)$ satisfies*

$$\int_{\mathcal{X}} \left|\partial|\varphi|^{p/2}\right|_g^2 dV_g \leqslant \frac{np^2}{4(p-1)} \int_{\mathcal{X}} (1-e^{F_t})\varphi|\varphi|^{p-2} dV_g.$$

Proof. For sufficiently large R, if $T_R = \{x \in X : \rho(x) \leqslant R\}$, then by Stokes' Theorem, we have

$$\begin{aligned}
&\sqrt{-1}\int_{T_R} d\left[\varphi|\varphi|^{p-2}\bar{\partial}\varphi \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega_\varphi + \cdots + \omega_\varphi^{n-1})\right] \\
&= \sqrt{-1}\int_{\partial T_R} \varphi|\varphi|^{p-2}\bar{\partial}\varphi \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega_\varphi + \cdots + \omega_\varphi^{n-1}).
\end{aligned}$$

Since $\varphi \in C^\infty_\gamma(\mathcal{X})$, on the boundary ∂S_R, we have that $\varphi = O(R^\gamma), d^c\varphi = O(R^{\gamma-1})$, and $\omega, \omega_\varphi = O(1)$. We also have that $\mathrm{vol}(\partial S_R) = O(R^{2n-1})$. It follows that the right-hand side of the equality is $O(R^{p\gamma+2n-2})$, where, by assumption on p, we have $p\gamma + 2n - 2 < 0$. Taking the limit as $R \to \infty$ and taking the $(1,0)$-part gives that

$$\sqrt{-1}\int_{\mathcal{X}} \partial\left[\varphi|\varphi|^{p-2}\bar{\partial}\varphi \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega_\varphi + \cdots + \omega_\varphi^{n-1})\right] = 0.$$

Expanding the integrand gives the equation

$$\int_{\mathcal{X}} \varphi|\varphi|^{p-2}(1-e^{F_t})\omega^n = (p-1)\int_{\mathcal{X}} |\varphi|^{p-2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi\wedge(\omega^{n-1}+\cdots+\omega_\varphi^{n-1}).$$

Each term on the right is positive so that

$$\int_{\mathcal{X}} \varphi|\varphi|^{p-2}(1-e^{F_t})\omega^n \geqslant (p-1)\int_{\mathcal{X}} |\varphi|^{p-2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi\wedge\omega^{n-1}.$$

But the right-hand side is

$$(p-1)\int_{\mathcal{X}} |\varphi|^{p-2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi\wedge\omega^{n-1} = \frac{4(p-1)}{np^2}\int_{\mathcal{X}} \left|\partial|\varphi|^{p/2}\right|_g^2\omega^n,$$

as we desire. □

There is a type of Sobolev inequality on AC manifolds, which carries over to the orbifold setting without any modification. Joyce [42], who deals with the ALE case, notes that a Sobolev inequality can be deduced from statements concerning elliptic operators in weighted Sobolev spaces, more details of which can be found in [51, Section 13]. Tian-Yau [62, Section 3] provide a Sobolev inequality in several special cases, and general proofs are given by van Coevering [**?**, Section 2.2] and Hein [40, Theorem 1.2].

Lemma 3.21 (Sobolev inequality)**.** *For $n \geqslant 2$, let $\tau = \frac{n}{n-1}$. There is a constant $C > 0$ depending on $(\mathcal{X}, g)$ such that if φ belongs to $L^2_1(\mathcal{X})$, then*

$$\|\varphi\|_{L^{2\tau}} \leqslant C\,\|\partial\varphi\|_{L^2}.$$

With the previous two results, one can obtain a uniform L^{p_0}-estimate:

Lemma 3.22 (An L^{p_0}-estimate)**.** *There are constants $C > 0$ and p_0 larger than $\max\{(2-2n)/\gamma, 3n/2\}$ such that any solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ to $(*_t)$ satisfies*

$$\|\varphi\|_{L^{p_0}} \leqslant C.$$

Proof. Choose p satisfying $p > 1$ and $p > (2-2n)/\gamma$. Let q and r be the numbers $q = np/(p+n-1)$ and $r = \tau p/(p-1)$ where $\tau = n/(n-1)$, and note that q and r

satisfy $1/q + 1/r = 1$. Using Lemma 3.21, we have an estimate of the form

$$\|\varphi\|^p_{L^{\tau p}} \leqslant C \left\|\partial|\varphi|^{p/2}\right\|^2_{L^2}.$$

Then we apply the result of the Lemma 3.20 and Hölder's inequality to obtain that

$$\begin{aligned}\|\varphi\|^p_{L^{\tau p}} &\leqslant \frac{Cnp^2}{4(p-1)}\int_{\mathcal{X}}(1-e^{tF})\varphi|\varphi|^{p-2}dV_g\\ &\leqslant \frac{Cnp^2}{4(p-1)}\left\|1-e^{tF}\right\|_{L^q}\left\||\varphi|^{p-1}\right\|_{L^r}.\end{aligned}$$

But we note from the definition of $r = \tau p/(p-1)$ that

$$\left\||\varphi|^{p-1}\right\|_{L^r} = \|\varphi\|^{p-1}_{L^{r(p-1)}} = \|\varphi\|^{p-1}_{L^{\tau p}}.$$

We conclude that for C sufficiently large, we have

$$\|\varphi\|_{L^{\tau p}} \leqslant Cp\left\|1-e^{tF}\right\|_{L^q}.$$

The condition that $p > (2-2n)/\gamma$ implies that $q\beta < -2n$, so that $\left\|1-e^{tF}\right\|_{L^q}$ exists, and can be bounded by a constant depending on $\mathcal{X}, \omega$, and F. Choosing $p_0 = \tau p$ completes the proof of the claim. □

Lemma 3.23. *Setting $\tau = n/(n-1)$ and with p_0 as in Lemma 3.22, for each integer $k \geqslant 0$, let $p_k = \tau^k p_0$. Then there are constants $A, B > 0$ such that any solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ to $(*_t)$ satisfies*

$$\|\varphi\|_{L^{p_k}} \leqslant A(Bp_k)^{-n/p_k}.$$

Proof. The sequence of norms $\left\|1-e^{tF}\right\|_{L^{p_k}}$ converges (to $\left\|1-e^{tF}\right\|_{C^0}$) and is hence bounded, meaning there is a constant D depending only on $\mathcal{X}, \omega, F$ such that

$$\left\|1 - e^{tF}\right\|_{L^{p_k}} \leqslant D$$

for each k. If C denotes the constant from Lemma 3.21, let $B > 1$ be a constant satisfying

$$\sqrt[3]{B} \geqslant CDn2^n\tau^{n-1}.$$

Then let $A > 1$ denote a constant satisfying

$$A \geqslant (Bp_0)^{n/p_0} \left\|\varphi\right\|_{L^{p_0}}.$$

With these choices for A and B, we prove the claim by induction on the letter k. For $k = 0$, the claim is true by the definitions of A and B.

Now suppose the result has been proved for all integers less than or equal to k, and we prove the result for $k + 1$. If $r = p_k/(p_k - 1)$, then $1/p_k + 1/r = 1$. Using Lemma 3.21, we have an estimate of the form

$$\left\|\varphi\right\|_{L^{p_{k+1}}}^{p_k} = \left\|\varphi\right\|_{L^{\tau p_k}}^{p_k} \leqslant C \left\|\partial|\varphi|^{p_k/2}\right\|_{L^2}^2.$$

We then apply Lemma 3.20 and Hölder's inequality to find that

$$\begin{aligned}
\left\|\varphi\right\|_{L^{p_{k+1}}}^{p_k} &\leqslant \frac{Cnp_k^2}{4(p_k-1)} \left\|1 - e^{tF}\right\|_{L^{p_k}} \left\||\varphi|^{p_k-1}\right\|_{L^r} \\
&\leqslant \frac{CDnp_k^2}{4(p_k-1)} \left\||\varphi|^{p_k-1}\right\|_{L^r}
\end{aligned}$$

by the definition of D. Since $p_k > 1$, we have $p_k^2/(4(p_k-1)) \leqslant p_k$ so that

$$\|\varphi\|_{L^{p_{k+1}}}^{p_k} \leqslant CDnp_k \left\||\varphi|^{p_k-1}\right\|_{L^r}.$$

But

$$\left\||\varphi|^{p_k-1}\right\|_{L^r} = \|\varphi\|_{L^{r(p_k-1)}}^{p_k-1} = \|\varphi\|_{L^{p_k}}^{p_k-1}$$

implies that

$$\|\varphi\|_{L^{p_{k+1}}}^{p_k} \leqslant CDnp_k \|\varphi\|_{L^{p_k}}^{p_k-1}.$$

We apply the inductive hypothesis to the right-hand side to find that

$$\|\varphi\|_{L^{p_{k+1}}}^{p_k} \leqslant A^{p_k-1}CDnp^{n/p_k}B^{n/p_k-1}(Bp_k)^{1-n}.$$

The quantity A is larger than 1, the inequality $p_k^{1/p_k} < 2$ is valid for any positive number $p_k > 1$, and we are assuming that $p_0 > 3n/2$ (so that $B^{n/p_k-1} < B^{-1/3}$), so that we may obtain

$$\|\varphi\|_{L^{p_{k+1}}}^{p_k} \leqslant A^{p_k}CDn2^nB^{-1/3}(Bp_k)^{1-n}.$$

By our assumption on B, we have that $CDn2^nB^{-1/3} \leqslant \tau^{1-n}$, and we conclude that

$$\|\varphi\|_{L^{p_{k+1}}}^{p_k} \leqslant A^{p_k}(B\tau p_k)^{1-n} = (A(Bp_{k+1})^{-n/p_{k+1}})^{p_k}.$$

This completes the inductive step and the proof. $\square$

Proposition 3.24 (A C^0-estimate)**.** *There is a constant C such that any solution $\varphi \in C_\gamma^\infty(\mathcal{X})$ to $(*_t)$ satisfies $\|\varphi\|_{C^0(\mathcal{X})} \leqslant C$.*

Proof. One uses the previous lemma to find that

$$\|\varphi\|_{C^0} \leqslant \lim_{k\to\infty} A(Bp_k)^{-n/p_k} = A$$

as desired. $\square$

3.4.2 A C^3-estimate

Just as in the compact setting, local calculations together with the C^0-estimate then imply the following C^2-estimate (see [65, 60]).

Proposition 3.25 (A C^2-estimate)**.** *There is a constant C depending on $\mathcal{X}, \omega, F$ such that a solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ of $(*_t)$ satisfies*

$$C^{-1}(g_{j\bar{k}}) < (g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) < C(g_{j\bar{k}})$$

where $<$ means that the difference of matrices is positive definite and where ω has local expression $\omega = \sqrt{-1}g_{j\bar{k}}dz^j \wedge d\bar{z}^k$.

There is one significant difference from a proof in the compact setting, however, in that some care has to be given to the application of the maximum principle, since our manifold is non-compact. For a proof in the compact setting, one uses that the function $\Lambda_\omega\omega_\varphi - A\varphi$ achieves a maximum, for some large constant A. In this non-compact setting, this may no longer be the case. However, we do know that because φ belongs to the space C^∞_γ for $\gamma < -2$, the function $\Lambda_\omega\omega_\varphi - A\varphi$ tends to n as ρ tends to infinity, and so if the function has no maximum, then at least we know that it is bounded from above by a uniform constant independent of φ, and this is actually enough to complete the proof.

Let S denote the tensor given by the difference of Levi-Civita connections $S = \hat{\nabla} - \nabla$, where $\hat{\nabla}$ is the connection corresponding to ω_φ and ∇ is the one corresponding to ω. Note that S depends on the second and third derivatives of φ. So if $|S|$ denotes the norm of S with respect to the metric ω_φ, the fact that the metric $g_{j\bar{k}}$ is uniformly equivalent to the metric $g_{j\bar{k}} + \partial_j \partial_{\bar{k}} \varphi$ implies that a bound on $|S|$ gives a C^3-bound on φ. Such a bound follows just as in compact case (see Section 2.4).

Proposition 3.26 (A C^3-estimate)**.** *There is a constant C depending on $\mathcal{X}, \omega, F$ such that if $\varphi \in C^\infty_\gamma(\mathcal{X})$ is a solution to $(*_t)$, then $|S| \leqslant C$, where $|S|$ is the norm of S computed with respect to the metric ω_φ.*

Again there is a difference from the compact setting in that we may not apply a maximum principle directly, but instead, we can use that the function φ decays rapidly to achieve the desired result. In particular, in the proof in the compact case, one considers a point where the function $|S|^2 + A\Lambda_\omega \omega_\varphi$ achieves its maximum. In this non-compact setting, this function may not achieve a maximum, but at least we know that it decays to a uniform constant as $\rho \to \infty$, and as we remarked above, knowing this is enough to complete the proof.

Once a C^3-estimate is known, ordinary bootstrapping arguments together with Schauder estimates and the C^0-estimate then imply the following (see Section 2.4).

Corollary 3.27 (A $C^{k,\alpha}$-estimate)**.** *Let k be a nonnegative integer and $\alpha \in (0,1)$. There is a constant C depending on $\mathcal{X}, \omega, F$ such that if $\varphi \in C^\infty_\gamma(\mathcal{X})$ is a solution to $(*_t)$, then $\|\varphi\|_{C^{k,\alpha}(\mathcal{X})} \leqslant C$.*

3.4.3 A weighted C^0-estimate

We now prove a weighted C^0-estimate. Our goal is specifically to prove a $C^0_{\beta+2}$-estimate. However, to do so, we must first prove a C^0_γ-estimate for a weight γ satisfying

$\beta+2<\gamma$ *and* $-1<\gamma$. A stronger estimate for the weight $\beta+2$ will eventually follow (see Proposition 3.36).

To prove the C^0_γ-estimate, we proceed in a way analogous to the unweighted C^0-estimate presented above, with minor adaptations to deal with the weight γ.

The arguments in [42, Proposition 8.6.7] show directly (in the setting of asymptotically locally Euclidean manifolds) that the following is true.

Lemma 3.28. *There is a constant $C>0$ such that for $p>(2-2n)/\gamma$ and $q\geqslant 0$ satisfying $p\gamma+q<2-2n$, any solution $\varphi\in C^\infty_\gamma(\mathcal{X})$ to $(*_t)$ satisfies*

$$\left\|\partial(|\varphi|^{p/2}\rho^{q/2})\right\|^2_{L^2}\leqslant\frac{np^2}{4(p-1)}\int_{\mathcal{X}}(1-e^{tF})\varphi|\varphi|^{p-2}\rho^q dV \\ +C\frac{q(p+q)}{4(p-1)}\int_{\mathcal{X}}|\varphi|^p\rho^{q-2}dV.$$

In addition, the arguments of Proposition 8.6.8 of [42] prove also the following weighted analogue of the Sobolev inequality. It is convenient to introduce the weighted Sobolev norm

$$\|f\|_{L^q_{k,\beta}}=\left(\sum_{j=0}^k\int_{\mathcal{X}}|\rho^{j-\beta}\nabla^j f|^q\rho^{-2n}\,dV\right)^{1/q}.$$

Lemma 3.29. *Let γ be a weight satisfying $\beta+2<\gamma$ and $-1<\gamma$. There is a constant $C>0$ such that if $p\geqslant 2$ satisfies $p\geqslant(2-2n)/\gamma$ then any solution $\varphi\in C^\infty_\gamma(\mathcal{X})$ of $(*_t)$ satisfies*

$$\|\varphi\|^p_{L^{\tau p}_{0,\gamma}}\leqslant Cp(\|\varphi\|^{p-1}_{L^{p-1}_{0,\gamma}}+\|\varphi\|^p_{L^p_{0,\gamma}}).$$

We may now obtain a uniform weighted L^1-estimate.

Lemma 3.30. *Let γ be a weight satisfying $\beta+2<\gamma$ and $-1<\gamma$. There is a constant $C>0$ such that if $\varphi\in C^\infty_\gamma$ is a solution to $(*_t)$ then $\|\varphi\|_{L^1_{0,\gamma}}\leqslant C$.*

Proof. Let p_0 be chosen from Lemma 3.22. Because $-1<\gamma$, we may also ensure that

p_0 satisfies

$$p_0 < \frac{-2n}{\gamma}.$$

Define r, s by the relations $1/p_0 + 1/r = 1$ and $s = -r(\gamma + 2n)$. Then by Hölder's inequality, we find that

$$\|\varphi\|_{L^1_{0,\gamma}} = \int_{\mathcal{X}} |\varphi|\rho^{-\gamma-2n} dV \leqslant \|\varphi\|_{L^{p_0}} \left[\int_{\mathcal{X}} \rho^s dV\right]^{1/r}.$$

The choice of p_0 satisfying $p_0\gamma > -2n$ ensures that $s < -2n$ so that the integral $\int_{\mathcal{X}} \rho^s dV$ exists. The result now follows from Lemma 3.22. □

By techniques similar to those used in Lemma 3.23, one can use the previous lemmas to prove the following.

Lemma 3.31. *Let γ be a weight satisfying $\beta+2 < \gamma$ and $-1 < \gamma$. With $\tau = n/(n-1)$, for each integer $k \geqslant 0$, let $p_k = \tau^k$. Then there are constants $A, B > 0$ such that any solution $\varphi \in C^\infty_\gamma$ to $(*_t)$ satisfies*

$$\|\varphi\|_{L^{p_k}_{0,\gamma}} \leqslant A(Bp_k)^{-n/p_k}.$$

A C^0_γ-estimate now follows immediately.

Proposition 3.32 (A C^0_γ-estimate)**.** *Let γ be a weight satisfying $\beta + 2 < \gamma$ and $-1 < \gamma$. There is a constant C such that any solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ to $(*_t)$ satisfies $\|\varphi\|_{C^0_\gamma} \leqslant C$.*

3.4.4 A weighted C^3-estimate

The techniques of Theorem 8.6.11 from [42] (which include a priori estimates of elliptic operators on domains in $\mathbb{C}^n$) can be used to show that a C^0_γ-estimate implies a C^3_γ-estimate.

Proposition 3.33. *If $\gamma \geqslant \beta + 2$ is a weight such that we have an estimate of the form $\|\varphi\|_{C^0_\gamma} \leqslant C$, then we also have an estimate of the form $\|\varphi\|_{C^3_\gamma} \leqslant C$, and hence by weighted bootstrapping arguments involving the weighted Schauder estimates of Theorem 3.10, we also have estimates of the form $\|\varphi\|_{C^{k,\alpha}_\gamma} \leqslant C$ for each k, α.*

Moreover, the next proposition asserts that, as soon as we have a C^0_γ-estimate, we may decrease the weight γ to obtain an even stronger weighted estimate, so that we may continue until we obtain a $C^0_{\beta+2}$-estimate. We first require a lemma, the proof of which can be found in [42, Lemma 8.7.1] and involves choosing holomorphic coordinates and higher order estimates on φ.

Lemma 3.34. *For a solution $\varphi \in C^\infty_\gamma(\mathcal{X})$ of $(*_t)$, we have an estimate of the form*

$$|\Delta\varphi + e^{tF} - 1| \leqslant C|\sqrt{-1}\partial\bar{\partial}\varphi|^2$$

where Δ denotes the Laplacian and Levi-Civita connection of either ω or ω_φ, since the corresponding metrics are equivalent by Proposition 3.25.

Lemma 3.35. *Let $\gamma \geqslant \beta + 2$ be a weight such that we have an estimate of the form $\|\varphi\|_{C^0_\gamma} \leqslant C$. If $\gamma' = \max\{2\gamma - 2, \beta + 2\}$, then we also have an estimate of the form $\|\varphi\|_{C^0_{\gamma'}} \leqslant C$.*

Proof. The idea is to use the previous lemma and the fact that the Laplacian (3.2) is an isomorphism. In particular, since we are assuming we have an estimate of the form $\|\varphi\|_{C^0_\gamma} \leqslant C$, Proposition 3.33 shows that we actually have an estimate of the form $\|\varphi\|_{C^{5,\alpha}_\gamma} \leqslant C$. From this, we conclude that $\sqrt{-1}\partial\bar{\partial}\varphi \in C^{3,\alpha}_{\gamma-2}$, and thus that

$|\sqrt{-1}\partial\bar{\partial}\varphi|^2 \in C^{3,\alpha}_{2(\gamma-2)}$. Lemma 3.34 establishes an estimate of the form

$$\Delta\varphi = O(\rho^{2\gamma-4}) + O(\rho^{\beta}).$$

Corollary 3.17 then gives the desired estimate. $\square$

Proposition 3.36 (A $C^0_{\beta+2}$-estimate)**.** *There is a constant C such that any solution $\varphi \in C^\infty_\gamma$ to $(*_t)$ satisfies $\|\varphi\|_{C^0_{\beta+2}} \leqslant C$.*

Proof. Let γ be a weight satisfying $\beta + 2 < \gamma$ and $-1 < \gamma$. Then Proposition 3.32 gives an estimate of the form $\|\varphi\|_{C^0_\gamma} \leqslant C$. Define the sequence of weights $\gamma_0, \gamma_1, \dots$ by the rule $\gamma_0 = \gamma$ and $\gamma_{i+1} = \max\{2\gamma_i - 2, \beta + 2\}$. Then for all sufficiently large i, we have $\gamma_i = \beta + 2$, and the previous lemma therefore gives an estimate of the form $\|\varphi\|_{C^0_{\beta+2}} \leqslant C$. $\square$

3.4.5 Proof of Proposition 3.18(ii)

Proposition 3.36 gives an estimate of the form $\|\varphi\|_{C^0_{\beta+2}} \leqslant C$. Proposition 3.33 then implies that we have an estimate of the form $\|\varphi\|_{C^{k,\alpha}_{\beta+2}} \leqslant C$ for each k, α. Moreover, the metrics determined by ω and $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ are equivalent by Proposition 3.25.

3.5 Cases (i) and (iii) of Theorem 3.1

It remains to discuss Theorem 3.1 in the cases (i) and (iii), that is, if the right-hand side decays fast and slowly respectively. We require some preliminary results.

Lemma 3.37. *Suppose β satisfies $\beta < -n - 1$ and $\beta + 2 \notin \mathcal{P}$. For any $f \in C^\infty_\beta(\mathcal{X})$, there is a unique $u \in C^\infty(\mathcal{X}) \cap L^2_{1,1-n}(\mathcal{X})$ such that $\Delta u = f$, where $L^2_{1,1-n}(\mathcal{X})$ denotes the weighted Sobolev space given by the completion of the space of compactly supported* *smooth functions with respect to the weighted Sobolev norm*

$$\|v\|^2_{L^2_{1,1-n}} = \int_{\mathcal{X}} \left(|v|^2\rho^{-2} + |\nabla v|^2\right)\, dV.$$

Proof. Define a functional

$$E(v) = \int_{\mathcal{X}} \left(\frac{1}{2}|\nabla v|^2 + fv\right)\, dV$$

for all functions v in $L^2_{1,1-n}(\mathcal{X})$.

We first claim that there are constants $\delta, A > 0$ such that

$$E(v) \geqslant \delta \|v\|^2_{L^2_{1,1-n}} - A. \tag{3.7}$$

Indeed, using Hölder's inequality, we can bound $E(v)$ from below by

$$E(v) \geqslant \frac{1}{2}\|\nabla v\|^2_{L^2} - \left\|\rho^{-1}v\right\|_{L^2} \|\rho f\|_{L^2},$$

where the L^2-norm of ρf is finite because $\beta < -n-1$. A geometric mean type of inequality implies that for each $\epsilon > 0$ we have

$$E(v) \geqslant \frac{1}{2}\|\nabla v\|^2_{L^2} - \frac{\epsilon}{2}\left\|\rho^{-1}v\right\|^2_{L^2} - \frac{1}{2\epsilon}\|\rho f\|^2_{L^2}.$$

An orbifold version of Theorem 1.2(ii) from [40] (with $\alpha = 1$ and $\beta = 2n$) gives a weighted Poincaré inequality on $\mathcal{X}$ of the form

$$\left\|\rho^{-1}v\right\|^2_{L^2} = \|v\|^2_{L^2_{0,1-n}} \leqslant C\|\nabla v\|^2_{L^2}. \tag{3.8}$$

It follows that for ϵ sufficiently small, we have

$$E(v) \geqslant \frac{1}{4} \|\nabla v\|_{L^2}^2 - \frac{1}{2\epsilon} \|\rho f\|_{L^2}^2 .$$

We use (3.8) again to find that

$$E(v) \geqslant \frac{1}{8} \|\nabla v\|_{L^2}^2 + \frac{1}{8C} \left\|\rho^{-1} v\right\|_{L^2}^2 - \frac{1}{2\epsilon} \|\rho f\|_{L^2}^2 ,$$

and hence, for C large enough, we have

$$E(v) \geqslant \frac{1}{8C} \|v\|_{L^2_{1,1-n}}^2 - \frac{1}{2\epsilon} \|\rho f\|_{L^2}^2 .$$

By the calculus of variations, the functional E has a unique critical point $u \in L^2_{1,1-n}$ which achieves an absolute minimum of E, and moreover u is a weak solution to the equation $\Delta u = f$. It then follows from standard (local) elliptic regularity arguments that u is actually smooth (since f is). $\square$

Lemma 3.38. *(c.f. [42, Proposition 8.3.4]) For an asymptotically conical orbifold $(\mathcal{X}, g)$ with radius function ρ, we have $\Delta(\rho^{2-2n}) \in C^\infty_{\lambda_g - 2n}(\mathcal{X})$ and*

$$\int_{\mathcal{X}} \Delta(\rho^{2-2n})\, dV = (2n-2)\mathrm{Vol}(\Sigma)$$

where $\mathrm{Vol}(\Sigma)$ *is the volume of the compact manifold* Σ*.*

Proof. For the statement about the decay rate of $\Delta(\rho^{2-2n})$, we know that $\Delta(r^{2-2n}) = 0$ on the cone $C(\Sigma)$. It follows from the definition of the radius function and the fact that $\mathcal{X}$ is asymptotically conical that $\Delta(\rho^{2-2n})$ belongs to $C^\infty_{\lambda_g - 2n}(\mathcal{X})$.

Let S_R be the subset of X given by $S_R = \{x \in X : \rho(x) \leqslant R\}$. Then Stokes'

Theorem gives that

$$\int_{S_R} \Delta(\rho^{2-2n})\, dV = \int_{\partial S_R} [\nabla(\rho^{2-2n}) \cdot \mathbf{n}]\, dV.$$

For R large enough, the quantity $\nabla(\rho^{2-2n}) \cdot \mathbf{n}$ is approximated by $(2n-2)R^{1-2n}$ and $\mathrm{vol}(\partial S_R)$ is approximated by $R^{2n-1}\mathrm{Vol}(\Sigma)$. Letting R tend to ∞ gives the desired result. $\square$

Let μ_1 be the smallest nonzero eigenvalue of Δ_Σ, and let $\beta_1^\pm$ be the exceptional weights corresponding to this eigenvalue in the sense that

$$\beta_1^\pm = -\frac{2n-2}{2} \pm \sqrt{\frac{(2n-2)^2}{4} + \mu_1}. \tag{3.9}$$

Lemma 3.39. *Suppose β satisfies $\beta_1^- < \beta < -2n$, and let f belong to $C^\infty_\beta(\mathcal{X})$.*

(a) If $\int_{\mathcal{X}} f\, dV = 0$, then the unique solution $u \in C^\infty(\mathcal{X}) \cap L^2_{1,1-n}(\mathcal{X})$ to $\Delta u = f$ belongs to the space $C^\infty_{\beta+2}(\mathcal{X})$.

(b) If $\int_{\mathcal{X}} f\, dV \neq 0$ and β satisfies $\lambda_g - 2n < \beta$, then the unique solution $u \in C^\infty(\mathcal{X}) \cap L^2_{1,1-n}(\mathcal{X})$ to $\Delta u = f$ can be written as $u = A\rho^{2-2n} + v$ for a unique number A and a unique function $v \in C^\infty_{\beta+2}(\mathcal{X})$. Moreover, the number A can be computed explicitly as

$$A = \frac{1}{(2n-2)\mathrm{Vol}(\Sigma)} \int_{\mathcal{X}} f\, dV.$$

Proof. For part (a), the proof of Lemma 3.16 states that the range of

$$\Delta : C^{k+2,\alpha}_{\beta+2}(\mathcal{X}) \to C^{k,\alpha}_\beta(\mathcal{X}) \tag{3.10}$$

is the orthogonal complement of the kernel of

$$\Delta : C^{k+2,\alpha}_{-\beta-2n}(\mathcal{X}) \to C^{k,\alpha}_{-\beta-2n-2}(\mathcal{X}). \tag{3.11}$$

Our assumption on β guarantees that $-\beta - 2n$ belongs to the interval $(0, \beta_1^+)$. In this interval, the kernel of (3.11) is the one-dimensional subspace spanned by the constant 1 function. It follows that the range of (3.10) is the subspace W of all $f \in C^{k,\alpha}_{\beta}(\mathcal{X})$ satisfying $\int_{\mathcal{X}} f \, dV = 0$. The restriction

$$\Delta : C^{k,\alpha}_{\beta+2}(\mathcal{X}) \to W$$

is an isomorphism, and hence there is an estimate of the form

$$\|u\|_{C^{k+2,\alpha}_{\beta+2}} \leqslant C \, \|f\|_{C^{k,\alpha}_{\beta}} \qquad \text{for } f = \Delta u \text{ satisfying } \int_{\mathcal{X}} f \, dV = 0.$$

Claim (a) now follows.

For part (b), the integral $\int_{\mathcal{X}} \Delta\rho^{2-2n} \, dV$ is finite and equal to $(2n-2)\mathrm{Vol}(\Sigma)$ by the previous lemma. Because β satisfies $\beta < -2n$, the integral $\int_{\mathcal{X}} f \, dV$ is also finite. Let A denote the constant

$$A = \frac{\int_{\mathcal{X}} f \, dV}{\int_{\mathcal{X}} \Delta\rho^{2-2n} \, dV} = \frac{1}{(2n-2)\mathrm{vol}(\Sigma)} \int_{\mathcal{X}} f \, dV.$$

Since β satisfies $\lambda_g - 2n < \beta$, the function $f - \Delta(A\rho^{2-2n})$ belongs to $C^{\infty}_{\beta}(\mathcal{X})$ and has zero integral over $\mathcal{X}$. By part (i), there is a unique $v \in C^{\infty}_{\beta+2}(\mathcal{X})$ such that $\Delta v = f - \Delta(A\rho^{2-2n})$. Upon rearranging, we find that the proof is complete. $\square$

3.5.1 Case (i)

Let us now discuss the case (i) of Theorem 3.1. In this case, we are assuming that β satisfies $\max\{-4n, \beta_1^-, \lambda_g - 2n\} < \beta < -2n$. The idea is that there is an inclusion $C^\infty_\beta(\mathcal{X}) \hookrightarrow C^\infty_{\beta'}(\mathcal{X})$ for β' satisfying $\beta < -2n < \beta'$ so that we can use the existence from case (ii), noting, however, that the solution we obtain may not belong to the desired space of functions. Nevertheless, because the solution satisfies a Monge-Ampère equation, we will be able to use Lemma 3.39(b) to conclude that the solution belongs to the space we want.

More precisely, for a small number $\epsilon > 0$, there is an inclusion $C^\infty_\beta(\mathcal{X}) \hookrightarrow C^\infty_{-2n+\epsilon}(\mathcal{X})$. It therefore follows from case (ii), that there is a unique solution φ to (2.1) satisfying $\varphi \in C^\infty_{2-2n+\epsilon}(\mathcal{X})$. By expanding the Monge-Ampère equation (2.1) and using the relation

$$n\sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega^{n-1} = -(\Delta\varphi)\omega^n,$$

we find that

$$-(\Delta\varphi)\omega^n = (1 - e^F)\omega^n + \sum_{j=2}^{n} \binom{n}{j} (\sqrt{-1}\partial\bar{\partial}\varphi)^j \wedge \omega^{n-j}.$$

The term $(1 - e^F)\omega^n$ belongs to $O(\rho^\beta)$ by assumption on F. All of the terms in the summation belong to $O(\rho^{-4n+2\epsilon})$ because $j \geqslant 2$. It follows that $\Delta\varphi$ belongs to $O(\rho^\beta)$, and hence by Lemma 3.39(b) we find that $\varphi \in \mathbb{R}\rho^{2-2n} \oplus C^\infty_{\beta+2}(\mathcal{X})$ as desired.

3.5.2 Case (iii)

We finish by discussing the case (iii) of Theorem 3.1. In this case, we are assuming that β satisfies $-2 < \beta < 0$ and $\beta + 2 \notin \mathcal{P}$. The idea is to reduce again to the case (ii), using the following lemma.

Lemma 3.40. *Suppose that β satisfies $-2 < \beta < 0$ and $\beta + 2 \notin \mathcal{P}$. If F belongs to $C^\infty_\beta(\mathcal{X})$, then there is a function $\varphi_1 \in C^\infty_{\beta+2}(\mathcal{X})$ satisfying*

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_1)^n = e^{F-F_1} \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi_1 > 0 \end{cases} \tag{3.12}$$

for some $F_1 \in C^\infty_{2\beta}(\mathcal{X})$.

Proof. Identify $X \setminus K'$ with $(1, \infty) \times \Sigma$. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function satisfying

$$\eta(t) = \begin{cases} 0 & r \leqslant 1 \\ 1 & r \geqslant 2 \end{cases}.$$

For $R \geqslant 1$, let η_R be the composition $\eta_R(r) = \eta(r/R)$. Since $\Delta : C^\infty_{\beta+2}(\mathcal{X}) \to C^\infty_\beta(\mathcal{X})$ is surjective by Lemma 3.16, there is a function $\hat{\varphi}_1 \in C^\infty_{\beta+2}(\mathcal{X})$ such that $\Delta\hat{\varphi}_1 = -F$ on $\mathcal{X}$. We claim that $\varphi_1 = \eta_R\hat{\varphi}_1$ has the desired properties for R sufficiently large.

We first claim that the form $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_1$ is a positive form. Indeed, we compute that

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\varphi_1 = \eta_R\sqrt{-1}\partial\bar{\partial}\hat{\varphi}_1 + \sqrt{-1}\frac{\eta'}{R}(\partial\hat{\varphi}_1 \wedge \bar{\partial}r + \partial r \wedge \bar{\partial}\hat{\varphi}_1) \\ + \sqrt{-1}\hat{\varphi}_1\left(\frac{\eta'}{R}\partial\bar{\partial}r + \frac{\eta''}{R^2}\partial r \wedge \bar{\partial}r\right). \end{aligned}$$

Because $\hat{\varphi}_1 \in C^\infty_{\beta+2}(\mathcal{X})$ and η_R is supported only for $r > R$, we find that the length of the first term $\eta_R\sqrt{-1}\partial\bar{\partial}\hat{\varphi}_1$ is $O(R^\beta)$. The derivatives of η are only supported for $r \in [R, 2R]$ so that the lengths of the other terms are also $O(R^\beta)$. We conclude that $\sup|\sqrt{-1}\partial\bar{\partial}\varphi_1| \to 0$ as $R \to \infty$, and so we can ensure that the form $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_1$ is positive for R large enough.

Now for the complex Monge-Ampère equation, we note that for $r > 2R$, we have $\varphi_1 = \hat{\varphi}_1$ and hence for such r we have

$$\begin{aligned}(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_1)^n &= (1 - \Delta\hat{\varphi}_1)\,\omega^n + \sum_{j\geqslant 2}\binom{n}{j}(\sqrt{-1}\partial\bar{\partial}\hat{\varphi}_1)^j \wedge \omega^{n-j} \\ &= (1 + F + O(r^{2\beta}))\omega^n.\end{aligned}$$

If we set $F_1 = F - \log(1 + F + O(r^{2\beta}))$, then we have (by the Taylor series for log) that $F_1 = O(r^{2\beta})$. The result follows. □

Indeed, we may now finish the proof of case (iii) in the following manner. By the previous lemma, we obtain a $\varphi_1 \in C^\infty_{\beta+2}(\mathcal{X})$ and an $F_1 \in C^\infty_{2\beta}(\mathcal{X})$ satisfying (3.12). In particular, the form $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_1$ is a Kähler form. If it happens that $2\beta < -2$, then we can use case (ii) to obtain a solution $\varphi_2 \in C^\infty_{2\beta+2}(\mathcal{X})$ to the equation $(\omega_1 + \sqrt{-1}\partial\bar{\partial}\varphi_2)^n = e^{F_1}\omega_1^n$, and setting $\varphi = \varphi_1 + \varphi_2$, we find that φ belongs to $C^\infty_\beta(\mathcal{X})$ and that φ solves (2.1). If 2β is not smaller than -2, then we can use the previous argument to find $\varphi_2 \in C^\infty_{2\beta+2}(\mathcal{X})$ such that $(\omega_1 + \sqrt{-1}\partial\bar{\partial}\varphi_2)^n = e^{F_1 - F_2}\omega_1^n$ for some $F_2 \in C^\infty_{4\beta}(\mathcal{X})$. If $4\beta < -2$, then we can use the previous argument to solve the desired equation (2.1). If not, then we can proceed iteratively to solve (2.1) in a finite number of steps.

3.6 Calabi-Yau metrics

This section is devoted to proving Theorem 3.5, which states the existence of Ricci-flat metrics in certain Kähler classes which decay rapidly in the following precise sense.

Definition 3.41. Let $\mathcal{X}$ be a compact orbifold and $K \subset X$ a compact subset of the underlying space, and let $C = \mathbb{R}_+ \times \Sigma$ be a cone with cone metric g_0. Suppose that there is a diffeomorphism $\Phi : (1, \infty) \times \Sigma \to X \setminus K$. A class $\mathfrak{k}$ in $H^2(\mathcal{X})$ is called almost compactly supported of rate $\lambda_{\mathfrak{k}} < 0$ if the class can be represented by a Kähler form ω on $\mathcal{X}$ such that there is a compact set $K' \supset K$ and a real smooth 1-form η on $X \setminus K'$ such that the difference $\omega - d\eta$ decays with rate $\lambda_{\mathfrak{k}}$.

Our proof of the theorem follows very closely the proof presented in [**?**, Theorem 2.4] and relies upon the following lemmas. The first lemma can be found in [23], and the proof given there still holds in the orbifold setting because the arguments are given outside of the compact subset K' in which our orbifold is isomorphic to a cone.

Lemma 3.42. *With the hypotheses of Theorem 3.5, we have* $\Phi^* J - J_0 = O(r^{\lambda_\Omega})$ *(in the sense of Definition 3.7).*

The second lemma can also be found in [23], and the exact same proof involving cut-off functions extends to the orbifold case with almost no adjustments.

Lemma 3.43. *With the hypotheses of Theorem 3.5, for each* $\alpha > 0$*, there is a smooth plurisubharmonic function* h_α *on* $\mathcal{X}$ *which is strictly plurisubharmonic (this means in particular that* $\sqrt{-1}\partial\bar{\partial}h_\alpha$ *is a positive form) and whose pullback to* $(1, \infty) \times \Sigma$ *agrees with* $r^{2\alpha}$ *outside of a compact subset* $K_\alpha \subset X$*.*

The final lemma is a version of the $\partial\bar{\partial}$-lemma that holds outside of a compact subset. Again, this lemma follows from the manifold case simply because away from the orbifold points our orbifold is isomorphic to a manifold so that the results [23, Proposition A.2(ii), Corollary A.3(ii)] still hold.

Lemma 3.44. *Let* $\mathcal{X}$ *be an AC Kähler orbifold with trivial canonical bundle. If* $n = \dim_{\mathbb{C}} \mathcal{X} > 2$ *and if* α *is an exact real* $(1,1)$*-form on* $X \setminus K$ *for some compact* K *containing all of the orbifold points, then there is a compact* $K' \supset K$ *and a smooth function* u *on* $X \setminus K'$ *such that* $\alpha = \sqrt{-1}\partial\bar{\partial}u$ *on* $X \setminus K'$*.*

We are now in a position to prove Theorem 3.5. The proof of Theorem 2.4 from [23] applies almost directly, but we sketch the arguments here for completeness.

Proof of Theorem 3.5. We identify $X \setminus K$ with $(1, \infty) \times \Sigma$ via Φ, and we let ourselves work with increasingly large compact subsets K if necessary. By assumption, there is a smooth 1-form η on $X \setminus K$ such that the difference $\omega - d\eta$ decays with rate $\lambda_{\mathfrak{t}}$. By Lemma 3.44, there is a smooth function u on $X \setminus K$ such that $d\eta = -\sqrt{-1}\partial\bar{\partial}u$. By Lemma 3.43, for each $\alpha > 0$, there is a smooth plurisubharmonic function h_α on $\mathcal{X}$ which is strictly plurisubharmonic and whose pullback to $(1, \infty) \times \Sigma$ agrees with $r^{2\alpha}$ outside of some compact subset $K_\alpha \subset X$.

Fix some $\alpha \in (0, 1)$. Ensure that the compact set K contains K_α and K_1. Let R be a number so large that $K \subset \{r \leqslant R\}$. Fix a cutoff function ψ on $\mathcal{X}$ satisfying

$$\psi(x) = \begin{cases} 0 & \rho(x) < 2R \\ 1 & \rho(x) > 3R. \end{cases}$$

For a constant $S > 2$, let ψ_S denote the rescaled cutoff function satisfying by

$$\psi_S(x) = \begin{cases} 0 & \rho(x) < 2RS \\ 1 & \rho(x) > 3RS. \end{cases}$$

For a constant $c > 0$ and a constant C, let $\hat{\omega}$ be the form

$$\hat{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}(\psi u) + C\sqrt{-1}\partial\bar{\partial}((1 - \psi_S)h_\alpha) + c\sqrt{-1}\partial\bar{\partial}h_1.$$

In [23], it is shown that for suitable choices of S, c, and C, the form $\hat{\omega}$ is a Kähler form on $\mathcal{X}$ in such a way that $(\mathcal{X}, \hat{\omega})$ is asymptotically conical of rate $\lambda < 0$. The

Kähler form $\hat{\omega}$ has global Ricci potential given by

$$\hat{f} = \log\left(\frac{i^{n^2}\Omega \wedge \bar{\Omega}}{(\hat{\omega}/c)^n}\right)$$

belonging to the space $C^\infty_\lambda(\mathcal{X})$. We would like to use Theorem 3.1 to solve the equation

$$(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^n = e^{\hat{f}}\hat{\omega}^n$$

for $\hat{\varphi}$, and we would obtain a Ricci-flat metric. Let us consider two cases for λ: either $\lambda < -2n$ or $-2n < \lambda < 0$.

(i) If $\lambda < -2n$, then by considering $\lambda' \geqslant \lambda$ in the interval $(\max\{-4n, \beta_1^-, \lambda_g - 2n\}, -2n)$ of case (i) of Theorem 3.1 and the inclusion $C^\infty_\lambda(\mathcal{X}) \hookrightarrow C^\infty_{\lambda'}(\mathcal{X})$, we may view $\hat{f}$ as having decay rate λ', and therefore use case (i) of Theorem 3.1 to obtain a solution $\hat{\varphi} \in \mathbb{R}\rho^{2-2n} \oplus C^\infty_{\beta+2}(\mathcal{X})$ whose corresponding Kähler form decays with rate $-2n = \max\{\lambda, -2n\}$.

(ii) If $-2n < \lambda < 0$, then we may use either case (ii) or (iii) to find a solution $\hat{\varphi} \in C^\infty_{\lambda+2}(\mathcal{X})$ whose corresponding Kähler form decays with rate $\lambda = \max\{\lambda, -2n\}$.

Remark 3.45. In [23, Remark 2.10], it is shown using the Lichnerowicz-Obata Theorem that if $\mathrm{Ric}(g_0) \geqslant 0$, then $\mathcal{P} \cap (0, 2) = \mathcal{P} \cap [1, 2)$ and moreover that the exceptional weights in the interval $(1, 2)$ are associated with the growth rates of plurisubharmonic functions on the cone C. This remark justifies the slight difference in the statement of Theorem 3.5 from that of [23, Theorem 2.4].

Corollary 3.46. *Let $(C(\Sigma), g_0, J_0, \Omega_0)$ be a Calabi-Yau cone of complex dimension $n > 2$, let $p : C \to \Sigma$ denote the radial projection, and let V be the normal affine variety associated to C. Let $\pi : \mathcal{X} \to V$ be a crepant partial resolution by an orbifold*

$\mathcal{X}$, and let $\mathfrak{k} \in H^2(\mathcal{X})$ be a class that contains positive $(1,1)$-forms. Then for each $c > 0$, there is a complete Calabi-Yau metric g_c on $\mathcal{X}$ such that $\omega_c \in \mathfrak{k}$ and

$$\omega_c - \pi^* c\omega_0 = O(r^{-2+\delta}) \tag{3.13}$$

for sufficiently small δ. If $\mathfrak{k} \in H^2_c(\mathcal{X})$, then we have

$$\omega_c - \pi^*(c\omega_0) = \mathrm{const}\sqrt{-1}\partial\bar{\partial} r^{2-2n} + O(r^{-2n-1-\epsilon})$$

for some $\epsilon > 0$.

Proof. We first claim that we have an exact sequence of the form

$$0 \to H^2_c(\mathcal{X}, \mathbb{R}) \to H^2(\mathcal{X}, \mathbb{R}) \to H^{1,1}_{\mathrm{pr,b}}(\Sigma).$$

Indeed, if $X_1 \subset \mathcal{X}$ denotes the suborbifold $X_1 = \{x \in X : \rho(x) \leqslant 1\}$, then we may view Σ as the boundary of X_1. Considering the pair (X_1, Σ), we have a long exact sequence in cohomology of the form

$$\cdots \to H^{k-1}(\Sigma, \mathbb{R}) \to H^k(X_1, \Sigma, \mathbb{R}) \to H^k(X_1, \mathbb{R}) \to H^k(\Sigma, \mathbb{R}) \to \cdots$$

Using the identifications $H^k(X_1, \mathbb{R}) \simeq H^k(\mathcal{X}, \mathbb{R})$ and $H^k(X_1, \Sigma, \mathbb{R}) \simeq H^k_c(\mathcal{X}, \mathbb{R})$, we obtain a long exact sequence, a portion of which is

$$\cdots \to H^1(\Sigma, \mathbb{R}) \to H^2_c(\mathcal{X}, \mathbb{R}) \to H^2(\mathcal{X}, \mathbb{R}) \to H^2(\Sigma, \mathbb{R}) \to \cdots$$

In [64], it is shown that $H^1(\Sigma, \mathbb{R}) = 0$ (because, for example, we may choose a metric on Σ with positive Ricci curvature). Moreover, the Bochner formula (see [37, Lemma 5.3]) gives that $H^2(\Sigma)$ can be identified with $H^{1,1}_{\mathrm{pr,b}}(\Sigma)$, the primitive basic $(1,1)$-cohomology group associated with the Sasaki structure on Σ. The claim now follows.

Let ω be a closed positive $(1,1)$-form in the class $\mathfrak{k}$. From exact sequence of the previous paragraph, there is a compact subset $K \subset X$ such that away from K, we have

$$\omega = d\eta + p^*\xi$$

for some real 1-form η and some primitive basic $(1,1)$-form ξ on Σ. Noting that $p^*\xi = O(r^{-2})$ shows that we can take $\lambda_{\mathfrak{k}} = -2 + \delta$ (and the fact that Ω agrees with Ω_0 outside of a compact set implies that $\lambda_\Omega = -\infty$). Theorem 3.5 now gives the result. □

Remark 3.47. The arguments in [37] (see Proof of Theorem 5.1) can actually be used to give a stronger version of Corollary 3.46, whereby the relation (3.13) is replaced by the relation

$$\omega_c - \pi^* c\omega_0 = p^*\xi + O(r^{-4}) \tag{3.14}$$

where ξ is the primitive basic harmonic $(1,1)$-form on Σ that represents the restriction of κ to Σ. If $\xi = 0$, or equivalently, if $\mathfrak{k} \in H^2_c(\mathcal{X})$, then we have

$$\omega_c - \pi^*(c\omega_0) = \mathrm{const}\sqrt{-1}\partial\bar{\partial} r^{2-2n} + O(r^{-2n-1-\epsilon})$$

for some $\epsilon > 0$.

Remark 3.48. Moreover, the same arguments and method of proof in [37] (see Proof of Theorem 5.1) can be used to deal with the surface case ($n = 2$) of Corollary 3.46. In particular, in this case, the Kähler class must belong to $H^2_c(\mathcal{X})$.

3.7 Examples

We consider examples of crepant partial resolutions of Calabi-Yau cones, to which one can apply the results of Corollary 3.46 (and Remark 3.48 for the case $n = 2$) to obtain Ricci-flat Kähler metrics.

Our first example is that of the canonical bundle over projective space $\mathbb{CP}^{n-1}$, which is actually a manifold, and which is covered by the results in [23]. However, we find it useful to review this particular example, as it contains a construction that will be repeated in further examples.

Example 3.49. Projective space $\mathbb{CP}^{n-1}$ equipped with the Fubini-Study metric is a Kähler-Einstein Fano manifold of dimension $n-1$ (with Kähler-Einstein constant n). The tautological line bundle $\mathcal{O}(-1)$ is a Hermitian-Einstein vector bundle over $\mathbb{CP}^{n-1}$ when equipped with the hermitian metric h induced by viewing $\mathcal{O}(-1)$ as a subbundle of the trivial vector bundle of rank n. Let t denote the smooth nonnegative function on the total space L of $\mathcal{O}(-1)$ defined by

$$t(\eta) = h_x(\eta, \eta) = |\eta|^2$$

for η a vector in the fiber of L_x over $x \in \mathbb{CP}^{n-1}$. Let $\Sigma \subset L$ denote the corresponding S^1-bundle given by $\Sigma = t^{-1}(1)$. Then Σ may be identified with the sphere S^{2n-1}, viewed as an S^1-bundle over $\mathbb{CP}^{n-1}$ by considering the inclusion $S^{2n-1} \hookrightarrow \mathbb{C}^n \setminus 0$ followed by the projection onto $\mathbb{CP}^{n-1}$. The group $\mathbb{Z}_n$ of nth roots of unity acts freely on Σ via the diagonal action of $\mathbb{Z}_n$ on $S^{2n-1} \subset \mathbb{C}^n \setminus 0$. The variety V associated to the cone $C(\Sigma/\mathbb{Z}_n) = C(S^{2n-1}/\mathbb{Z}_n)$ may be identified with the quotient variety $\mathbb{C}^n/\mathbb{Z}_n$, which carries a global holomorphic volume form from that of $\mathbb{C}^n$. There is a crepant resolution

$$\pi : K_{\mathbb{CP}^{n-1}} \to \mathbb{C}^n/\mathbb{Z}_n,$$

which contracts the zero section of $K_{\mathbb{CP}^{n-1}}$ to the singular point of $\mathbb{C}^n/\mathbb{Z}_n$. Calabi [16] lifts the Kähler metric on $\mathbb{CP}^{n-1}$ to a Sasaki-Einstein metric on the S^1-bundle $\Sigma \simeq S^{2n-1}$, so that the cone $C(\Sigma/\mathbb{Z}_n) \simeq C(S^{2n-1}/\mathbb{Z}_n)$ is a Calabi-Yau cone.

Corollary 3.46 now abstractly proves the existence of a one-parameter family of AC Calabi-Yau metrics on $K_{\mathbb{CP}^{n-1}}$ in each Kähler class $\mathfrak{k}$ that contains positive $(1,1)$-forms. In particular, by solving an ODE, Calabi [16] explicitly constructs a family of Ricci-flat Kähler metrics on the total space of the canonical bundle $K_{\mathbb{CP}^{n-1}} = \mathcal{O}(-n)$, and the classes represented by these metrics are compactly supported.

The next example we consider is a genearlization of the previous example in the sense that we consider the canonical bundle over *any* Kähler-Einstein Fano manifold. Again this example is actually a manifold and is covered by the previous results from [23].

Example 3.50. Let (M, g) be a Kähler-Einstein Fano manifold of dimension $n-1$ with Kähler-Einstein constant k_0. Let L denote the total space of a maximal root of the canonical bundle (meaning that if ι is the largest integer that divides K_M in $\mathrm{Pic}(M)$, then $L^\iota = K_M$). The function $(\det g)^{-1}$ describes a hermitian metric on K_M with Hermitian-Einstein constant k_0. The corresponding metric on L described by $h = (\det g)^{-1/\iota}$ is also Hermitian-Einstein with constant $\ell = k_0/\iota$. Let t denote the smooth nonnegative function on the total space of L determined by h, and let Σ denote the corresponding S^1-bundle over M given by $\Sigma = t^{-1}(1) \subset L$. The fiberwise action of $\mathbb{Z}_\iota$ on Σ is free. The total space of the canonical bundle K_M is a smooth crepant resolution of the variety associated to the cone $C(\Sigma/\mathbb{Z}_\iota)$, which enjoys a global holomorphic volume form. By lifting the metric on M, Calabi [16] constructs a Sasaki-Einstein metric on Σ, and in this way, the cone $C(\Sigma/\mathbb{Z}_\iota)$ enjoys the structure of a Calabi-Yau cone.

Corollary 3.46 now abstractly proves the existence of a one-parameter family of AC Calabi-Yau metrics on K_M in each Kähler class $\mathfrak{k}$ that contains positive $(1,1)$-forms. By solving an ODE, Calabi [16] explicitly constructs a family of Ricci-flat Kähler metrics on the total space of the canonical bundle K_M, and the classes represented by these metrics are compactly supported.

Example 3.51. Let us fix our attention to the variety $X = \mathbb{C}^2/\Gamma$ for a finite subgroup Γ of $SU(2)$. The ADE classification gives a one-to-one correspondence between finite subgroups of $SU(2)$ and simply laced Dynkin diagrams of the form A_n for $n \geqslant 1$, D_n for $n \geqslant 4$, E_6, E_7 and E_8. Moreover, if $\pi : \tilde{X} \to X$ denotes the minimal resolution of $X = \mathbb{C}^2/\Gamma$, then the Dynkin diagram is the dual graph of the exceptional set of the resolution, which is a union of #Vert copies of $\mathbb{P}^1$, where #Vert is the number of vertices in the Dynkin diagram corresponding to Γ. Using this correspondence, Kronheimer [46] constructs ALE hyper-Kähler metrics on the minimal resolution $\tilde{X}$. In this case, any Kähler class is compactly supported, so Corollary 3.46 implies in addition that any intermediate crepant partial resolution $\underline{X}$ factoring π

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[dr, "\pi"'] & \underline{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

admits a $b_2(\underline{X})$-parameter family of AC Calabi-Yau metrics as well. The second Betti number of $\underline{X}$ satisfies $b_2(\underline{X}) = \dim H^2_c(\underline{X})$, and moreover, the inequality

$$H^2_c(\underline{X}) = b_2(\underline{X}) \leqslant b_2(\tilde{X}) = \#\mathrm{Vert}$$

always holds.

Example 3.52. Candelas and de la Ossa [18] construct explicitly a metric that behaves with $\xi \neq 0$ in (3.14), as discussed in [23]. Let L denote the total space of two copies $\mathcal{O}(-1)^{\oplus 2}$ of the tautological line bundle over $\mathbb{CP}^1$, and let h denote the hermitian metric obtained as the product of the metrics induced on each factor separately. If t denotes the corresponding nonnegative smooth function on the total space L, then the subset $\Sigma = t^{-1}(1) \subset L$ is an S^3-bundle over $\mathbb{CP}^1$. Moreover, L is a subset of the product space $\mathbb{CP}^1 \times (\mathbb{C}^2)^2$, and the projection of Σ onto the factor $(\mathbb{C}^2)^2 = \mathbb{C}^4$ shows that Σ is also an S^2-bundle over S^3. Any such bundle is known to be trivial by Steenrod's classification [59]. The variety V associated to the cone $C(\Sigma) \simeq C(S^3 \times S^2)$ may be identified with the affine variety $V = \{z_1^2 + z_2^2 + z_3^2 + z_3^2 = 0\} \subset \mathbb{C}^4$ considered by [18]. There is a crepant resolution $\pi : L \to V$ which contracts the zero section of L to the singular point of V. There is a Sasaki-Einstein metric on $S^3 \times S^2$ so that $C(S^3 \times S^2)$ becomes a Calabi-Yau cone.

Corollary 3.46 now abstractly proves the existence of a one-parameter family of AC Calabi-Yau metrics on L in each Kähler class $\mathfrak{k}$ that contains positive $(1,1)$-forms. We note that such Kähler classes are not compactly supported because in fact, if E denotes the zero section, which is isomorphic to $\mathbb{CP}^1$, then $H^2_c(L) \simeq H_{2n-2}(E) = 0$. Moreover, since $b_2(L) = 1$, there is at most a one-parameter family of such Kähler classes that contain positive $(1,1)$-forms. In [18], an explicit one-parameter family of AC Kähler metrics on L is constructed.

Example 3.53. More generally, if Γ is a finite subgroup of $SU(2)$ acting freely on $S^3 \subset \mathbb{C}^2$, then we obtain a corresponding action of Γ on the sphere bundle Σ, and hence also on the total space L of two copies $\mathcal{O}(-1)^{\oplus 2}$ of the tautological line bundle over $\mathbb{CP}^1$. The global quotient orbifold $[L/\Gamma]$ is a crepant partial resolution of the variety associated to the Calabi-Yau cone $C((S^3/\Gamma)\times S^2)$, so Corollary 3.46 abstractly proves the existence of a family of AC Calabi-Yau metrics on $[L/\Gamma]$ in each Kähler class that contains $(1,1)$-forms. Moreover, the orbifold $[L/\Gamma]$ may be resolved fully to obtain a smooth resolution $\tilde{Y}$, and in analogy with Example 3.51, each intermediate partial resolution $\underline{Y}$ admits a $b_2(\underline{Y})$-parameter family of AC Calabi-Yau metrics. In this case, each $\underline{Y}$ is a $\mathbb{CP}^1$-fibration of a partial resolution $\underline{X}$ of the variety $X = \mathbb{C}^2/\Gamma$ from Example 3.51, and the second Betti number of $\underline{Y}$ satisfies

$$b_2(\underline{Y}) = 1 + \dim H^2_c(\underline{Y}) = 1 + b_2(\underline{X}).$$

Chapter 4

Hermitian-Einstein metrics on stable vector bundles over compact Kähler orbifolds

A seminal result due to Uhlenbeck-Yau [63] states that a stable vector bundle over a compact Kähler manifold admits a unique Hermitian-Einstein metric. This result was also proved for surfaces by Donaldson [26], and in that paper, he studied a corresponding variational problem to introduce a functional M_K on the space of Hermitian metrics whose critical points are the desired Hermitian-Einstein ones. This functional was studied in a slightly more general setting by Simpson [57], who related the properness of this functional (in a certain sense) to the stability of the bundle in order to provide another approach to proving the result of Uhlenbeck-Yau.

In [32], we show it is possible to extend these results to the setting of orbifolds to obtain the following.

Theorem 4.1. *Let $\mathcal{E}$ be an indecomposable holomorphic vector bundle over a compact Kähler orbifold $(\mathcal{X}, \omega)$. The following statements are equivalent.*

(i) The bundle $\mathcal{E}$ is stable.

(ii) For each metric K on $\mathcal{E}$, the Donaldson functional M_K is proper (in the sense of Definition 4.21).

(iii) There is a Hermitian-Einstein metric on $\mathcal{E}$.

This result arguably requires more care when passing to the orbifold setting than the results in the previous sections. In particular, the pullback and pushforward constructions for vector bundles and sheaves respectively require care. Nevertheless, in [32], we only require these pullback and pushforward constructions along very specific types of maps—namely, projection maps corresponding to fiber bundles—and we show that for such maps, these constructions still remain valid, thereby allowing us to mimic the manifold approach, as demonstrated with more detail in the following remark.

Remark 4.2. The purpose of this remark is to outline how a certain regularity argument necessary in our proof of the implication (i) $\implies$ (ii) of Theorem 4.1 follows from similar regularity statements from the setting of manifolds.

Assuming (ii) does not hold for some metric K, one can follow an approach by [57] to construct a weakly holomorphic subbundle for $\mathcal{E}$ that is destabilizing. Here a weakly holomorphic subbundle means an L_1^2 section Π of $\mathrm{End}(\mathcal{E})$ which satisfies $\Pi = \Pi^2 = \Pi^*$ (where the adjoint is computed with respect to K) and $(I_{\mathcal{E}} - \Pi)\bar{\partial}\Pi = 0$. An argument by Uhlenbeck-Yau [63] shows that these conditions on Π ensure that it may be extended to a rational section (that is, a holomorphic section outside of a subvariety of codimension at least 2). In such a case, we explain in [32] that Π may be regarded as a rational section of the fiber bundle $\mathrm{Gr}(s, \mathcal{E})$ over $\mathcal{X}$ of s-planes in $\mathcal{E}$, where s denotes the trace of Π.

The rational section Π of the bundle $\mathrm{Gr}(s, \mathcal{E})$ then determines a coherent sheaf (which is a subsheaf of $\mathcal{E}$) over $\mathcal{X}$ in the following manner. In [32], we show that it is possible to pull back the bundle $\mathcal{E}$ along the projection $p : \mathrm{Gr}(s, \mathcal{E}) \to \mathcal{X}$ to obtain a vector bundle $p^*\mathcal{E}$ of rank r over $\mathrm{Gr}(s, \mathcal{E})$, which contains a universal subbundle $\mathcal{S}$ of rank s via the ordinary incidence correspondence. The coherent sheaf $\mathcal{S}$ restricts to one on the closure $\mathcal{Y}$ of the image of the rational section in the total space $\mathrm{Gr}(s, \mathcal{X})$. Pushing forward this sheaf to $\mathcal{X}$, one obtains a sheaf over $\mathcal{X}$ which is coherent by Grauert's direct image theorem.

The reader may also be interested in Eyssidieux and Sala [31], who provide stacky analogues of the Uhlenbeck-Yau theorem and some of its variants while also studying applications to ALE spaces.

4.1 Preliminaries

By an analytic subvariety $\mathcal{V}$ of $\mathcal{X}$ we mean we are given the data of an atlas of charts $(U_\alpha, G_\alpha, \pi_\alpha)$ for $\mathcal{X}$ and for each α there corresponds a subvariety V_α of U_α satisfying $G_\alpha \cdot V_\alpha = V_\alpha$. Moreover, these V_α are required to agree with one another with respect to the embeddings $\lambda : U_\alpha \to U_\beta$. An analytic subvariety $\mathcal{V}$ determines a subset V of the underlying space X in a natural way.

For an analytic subvariety $\mathcal{V}$ of $\mathcal{X}$, the orbifold structure on X induces an orbifold structure on the complement $X \setminus V$, and we denote the resulting orbifold by $\mathcal{X} \setminus \mathcal{V}$.

A Hermitian metric on a complex vector bundle $\mathcal{E}$ consists of a collection of hermitian metrics H_α on bundles E_α over U_α which are invariant under the action of G_α on U_α and which are compatible with the embeddings in the sense that for each embedding $\lambda : U_\alpha \to U_\beta$, the pullback metric $\lambda^* H_\beta$ agrees with H_α. A Hermitian metric H can be regarded as a section of the bundle $\mathcal{E}^* \otimes \overline{\mathcal{E}}^*$. A Hermitian metric is said to be compatible with a connection D if $DH = 0$.

A complex vector bundle $\mathcal{E}$ of rank k over a complex $\mathcal{X}$ of dimension n is called holomorphic if the transition functions $g_\lambda : U_\alpha \to GL(k, \mathbb{C})$ can be taken to be holomorphic. In such a case, the orbifold $\mathcal{E}$ enjoys the structure of a complex orbifold (of dimension $n+k$) in such a way that the map of orbifolds $p : \mathcal{E} \to \mathcal{X}$ is holomorphic. A holomorphic structure on $\mathcal{E}$ determines a raising operator $\bar{\partial} : A^0(\mathcal{E}) \to A^{0,1}(\mathcal{E})$ by the usual local definition, and $\bar{\partial}$ satisfies the property that if s is a holomorphic section of $\mathcal{E}$, then $\bar{\partial}s = 0$. We say that a connection D on $\mathcal{E}$ is compatible with the holomorphic structure if $D'' = \bar{\partial}$, where D'' denotes the composition of D with the projection of $A^1(\mathcal{E})$ onto $A^{0,1}(\mathcal{E})$.

Example 4.3. The complexified tangent bundle $T\mathcal{X}$ of a complex orbifold $\mathcal{X}$ is a holomorphic vector bundle in the same way that it is for manifolds.

Just as in the manifold setting, if $\mathcal{E}$ is a holomorphic vector bundle, then a Hermitian metric H on $\mathcal{E}$ determines a unique Chern connection, denoted d_H, which is compatible with H and which is compatible with the holomorphic structure. The curvature F_H of d_H is an $\mathrm{End}(\mathcal{E})$-valued $(1,1)$-form.

If two metrics H, K on $\mathcal{E}$ satisfy

$$\langle \xi, \eta \rangle_H = \langle h\xi, \eta \rangle_K$$

for a positive endomorphism h of $\mathcal{E}$, we write $H = Kh$. One can show that in such a case, the endomorphism h is self-adjoint with respect to K (and also H). In addition, the curvatures F_H and F_K are related by

$$F_H = F_K + \bar{\partial}(H^{-1}\partial_K H),$$

where we are using the notation ∂_K to denote the $(1,0)$-component of the Chern connection d_K.

Lemma 4.4. *If two metrics H, K satisfy $H = Ke^s$ for an endomorphism s that is self-adjoint with respect to K, then*

(i) *the adjoint of $\partial_K s$ is $\bar{\partial} s$.*

(ii) $\Delta_{\partial_K} s = i\Lambda(F_H - F_K)$

(iii) $\Delta_{\bar{\partial}} s = i\Lambda(F_H - F_K) - i\Lambda F_K s$

(iv) $\|\partial_K s\|^2_{L^2_K} = \langle i\Lambda(F_H - F_K), s\rangle_{L^2_K}$

(v) $\Delta|s|^2_K = \langle 2i\Lambda(F_H - F_K), s\rangle - \langle i\Lambda F_K s, s\rangle - |d_K s|^2$

Proof. For (i), upon differentiating the relation $\langle s\eta, \xi\rangle = \langle \eta, s\xi\rangle$, we find

$$\langle (\partial_K s)\eta + s(\partial_K \eta), \xi\rangle + \langle s\eta, \bar{\partial}\xi\rangle = \langle \partial_K \eta, s\xi\rangle + \langle \eta, (\bar{\partial} s)\xi + s\bar{\partial}\xi\rangle.$$

Because s is self-adjoint we are left with

$$\langle (\partial_K s)\eta, \xi\rangle = \langle \eta, (\bar{\partial} s)\xi\rangle.$$

For (ii), the curvatures are related by

$$F_H = F_K + \bar{\partial}(H^{-1}\partial_K H) = F_K + \bar{\partial}(\partial_K s).$$

The Kähler identities (see [39]) extend to identities on bundle-valued forms to imply the relation

$$\partial_K^* = i\Lambda\bar{\partial},$$

which gives that

$$i\Lambda F_H = i\Lambda F_K + \partial_K^* \partial_K s.$$

We conclude that

$$\Delta_{\partial_K} s = i\Lambda(F_H - F_K),$$

as claimed.

For (iii), we recall that F_K is given by $F_K = \partial_K \bar{\partial} + \bar{\partial}\partial_K$ so that the curvatures are related by

$$F_H = F_K + (F_K - \partial_K \bar{\partial})s.$$

We then use the Kähler identity $\bar{\partial}^* = -i\Lambda\partial_K$ to obtain that

$$i\Lambda F_H = i\Lambda F_K + i\Lambda F_K s + \Delta_{\bar{\partial}} s.$$

Rearranging gives (iii).

For (iv), multiplying the equality of (ii) on the right by s, then taking the trace, and then integrating gives

$$\begin{aligned}\int_{\mathcal{X}} |\partial_K s|_K^2 \frac{\omega^n}{n!} &= \int_{\mathcal{X}} \langle \partial_K^* \partial_K s, s\rangle_K \frac{\omega^n}{n!} \\ &= \int_{\mathcal{X}} \langle i\Lambda(F_H - F_K), s\rangle_K \frac{\omega^n}{n!},\end{aligned}$$

as desired.

For (v), we have the identity

$$\Delta |s|_K^2 = \frac{1}{2}\Delta_d |s|_K^2 = \langle \Delta_K s, s\rangle - |d_K s|^2$$

regardless of whether s is self-adjoint. (Here Δ_d denotes the de Rham Laplacian.)

Because $\Delta_K = \Delta_{\partial_K} + \Delta_{\bar{\partial}}$, we obtain from parts (ii) and (iii) that

$$\Delta|s|_K^2 = \langle 2i\Lambda(F_H - F_K), s\rangle - \langle i\Lambda F_K s, s\rangle - |d_k s|^2$$

as desired. □

4.1.1 Stable bundles and sheaves

Chern-Weil theory may be used as usual to define Chern classes (or more generally characteristic classes) of vector bundles. We will discuss characteristic classes in greater detail in Section 4.1.2, but for now let us at least note that the first Chern class $c_1(\mathcal{E})$ can be defined as the cohomology class represented by the $(1,1)$-form

$$\frac{i}{2\pi}\mathrm{Tr}(F_H)$$

for any choice of Hermitian metric H on $\mathcal{E}$. The degree of $\mathcal{E}$ is then the integral of $\frac{i}{2\pi}\Lambda\mathrm{Tr}(F_H)$ over $\mathcal{X}$

$$\deg(\mathcal{E}) = \frac{i}{2\pi}\int_{\mathcal{X}} \Lambda\mathrm{Tr}(F_H)\cdot \mathrm{vol} = \int_{\mathcal{X}} c_1(\mathcal{E})\wedge\frac{\omega^{n-1}}{(n-1)!},$$

and the slope of $\mathcal{E}$ is the ratio

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Given an action $\sigma : G\times U \to U$ of a finite group G on $U \subset \mathbb{C}^n$ by biholomorphisms, a G-equivariant sheaf over U consists of the data of a sheaf $\mathcal{F}$ of $\mathcal{O}_U$-modules together with an isomorphism of sheaves of $\mathcal{O}_{G\times U}$-modules

$$\rho : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$$

which satisfies the cocycle relation

$$p_{23}^*\rho \circ (1_G \times \sigma)^*\rho = (m \times 1_U)^*\rho$$

where m denotes multiplication $m : G \times G \to G$ and $p_{23} : G \times G \times U \to G \times U$ is the projection onto the second two factors.

By a sheaf $\mathcal{F}$ over $\mathcal{X}$ we mean we are given the data of an atlas $(U_\alpha, G_\alpha, \pi_\alpha)$ of orbifold charts together with a G_α-equivariant sheaf $\mathcal{F}_\alpha$ over each U_α. For each embedding $\lambda : U_\alpha \to U_\beta$ there also corresponds a sheaf isomorphism $\tau_\lambda : \mathcal{F}_\alpha \to \lambda^*\mathcal{F}_\beta$. Moreover these isomorphisms are compatible with one another in the sense that whenever $\lambda : U_\alpha \to U_\beta$ and $\lambda' : U_\beta \to U_\gamma$ are a pair of composable embeddings, then $\tau_{\lambda'\circ\lambda} = \lambda^*\tau'_\lambda \circ \tau_\lambda$.

The notion of a sheaf $\mathcal{F}$ over an analytic subvariety $\mathcal{V}$ of $\mathcal{X}$ is defined similarly. In particular, if $\mathcal{V}$ is given locally by subvarieties V_α of charts $(U_\alpha, G_\alpha, \pi_\alpha)$, then a sheaf assigns to each V_α a G_α-equivariant sheaf $\mathcal{F}_\alpha$, and moreover to each embedding of charts, there corresponds a sheaf isomorphism as above (and these isomorphisms are compatible with one another). It is important to note that a sheaf $\mathcal{F}$ over $\mathcal{X}$ is not the same thing as a sheaf over the underlying topological space X.

Example 4.5. A complex orbifold $\mathcal{X}$ enjoys a structure sheaf $\mathcal{O}_\mathcal{X}$ of holomorphic $\mathbb{C}$-valued functions, and more generally, any analytic subvariety $\mathcal{V}$ of $\mathcal{X}$ determines a structure sheaf $\mathcal{O}_\mathcal{V}$.

A sheaf $\mathcal{F}$ is called coherent (resp. torsion-free) if each F_α is. If $\mathcal{F}$ is coherent and torsion-free of rank r, then one can define the determinant line bundle associated to $\mathcal{F}$ to be

$$\det(\mathcal{F}) = (\Lambda^r \mathcal{F})^{**}.$$

It follows that for a coherent torsion-free sheaf $\mathcal{F}$ we have a well-defined notion of degree

$$\deg(\mathcal{F}) = \deg(\det(\mathcal{F}))$$

and slope

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}.$$

In addition, one can show that a torsion-free coherent sheaf is locally free outside of a subset of codimension at least two. A proof of this can be found for example in [43]. The argument given there extends to the setting of orbifolds because one can apply the argument to each G_α-equivarariant sheaf over each chart.

Lemma 4.6. *If $\mathcal{F}$ is a torsion-free coherent sheaf, then there is a subvariety $\mathcal{V}$ of codimension at least 2 in X such that the restriction of $\mathcal{F}$ to $\mathcal{X} \setminus \mathcal{V}$ is locally free.*

The notion of slope allows one to introduce the usual notion of (slope) stability in the standard way.

Definition 4.7. One says that a coherent torsion-free sheaf $\mathcal{F}$ is semi-stable if for each proper coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$, we have the inequality $\mu(\mathcal{F}') \leqslant \mu(\mathcal{F})$. If moreover the strict inequality $\mu(\mathcal{F}') < \mu(\mathcal{F})$ holds for each proper coherent subset $\mathcal{F}'$ satisfying $0 < \operatorname{rank}(\mathcal{F}') < \operatorname{rank}(\mathcal{F})$, then we say that $\mathcal{F}$ is stable. In addition, a holomorphic vector bundle $\mathcal{E}$ is called (semi)-stable if its corresponding sheaf of sections is.

4.1.2 The heat flow and Donaldson functional

Let us fix from this point forward a holomorphic vector bundle $\mathcal{E}$ of rank r over a Kähler orbifold $(\mathcal{X}, \omega)$. There will be no loss of generality in assuming in addition that $\mathcal{E}$ is indecomposable.

Definition 4.8. A Hermitian metric H on $\mathcal{E}$ is called Hermitian-Einstein if there is a constant λ such that

$$\Lambda F_H = \lambda \cdot I_{\mathcal{E}} \in A^0(\mathrm{End}(\mathcal{E})) \tag{4.1}$$

where $I_{\mathcal{E}}$ denotes the identity automorphism on $\mathcal{E}$.

Remark 4.9. The constant $\lambda = \lambda(\mathcal{X}, \omega, \mathcal{E})$ can be determined by the Kähler class $[\omega]$ and the slope of $\mathcal{E}$. In particular, taking the trace of both sides of (4.1) and integrating over $\mathcal{X}$ gives

$$\deg(\mathcal{E}) = \frac{\lambda i}{2\pi} \mathrm{rank}(\mathcal{E}) \mathrm{vol}(\mathcal{X})$$

so that

$$\lambda = \frac{-2\pi i \cdot \mu(E)}{\mathrm{vol}(\mathcal{X})}.$$

Definition 4.10. By a heat flow with initial data H_0 we mean a flow of metrics H_t satisfying the differential equation

$$\dot{H}_t = -\frac{i}{2} H_t(\Lambda F_t - \lambda \cdot I_{\mathcal{E}}). \tag{4.2}$$

In particular, note that stable points of this flow must be Hermitian-Einstein metrics. Donaldson studied this flow in [26], and some of the results from that paper can be summarized in the following theorem.

Theorem 4.11. *For any initial metric H_0 on $\mathcal{E}$, the heat flow* (4.2) *has a unique smooth solution defined for* $0 \leqslant t < \infty$.

This theorem is valid for orbifolds for a few reasons. First, the short-time existence is a local argument involving a linearization of the flow, which can be studied in a local orbifold chart with no changes from the manifold setting. The long-time existence involves estimates to solutions of the flow, which remain valid in the orbifold setting since in particular the estimates are valid on each chart in some open cover, whereby computations agree with those in the manifold case. Some of the intermediary results Donaldson obtained in order to establish long-time existence included the following two propositions.

Proposition 4.12. *For an initial metric $H_0 = K$ on $\mathcal{E}$, the function*

$$\sup_{\mathcal{X}} |\Lambda F_t - \lambda I_{\mathcal{E}}|_K^2$$

is decreasing along the heat flow.

Proposition 4.13. *For an initial metric $H_0 = K$ on $\mathcal{E}$, let H_t be a one-parameter family of metrics for $0 \leqslant t < T$. Assume that H_t converges in C^0-norm (with respect to K) to some continuous metric as $t \to T$ and also that we have a uniform bound on* $\sup_{\mathcal{X}} |\Lambda F_t|_K^2$. *Then we also have a uniform L_2^p-bound on H_t for each $p < \infty$ (where the norm is computed with respect to K). Moreover, this result is still true when we allow $T = \infty$.*

As a result of this former proposition, we have the following corollary.

Corollary 4.14. *Let H_t be a solution to the heat flow with initial condition $H_0 = K$. If H_t is uniformly bounded with respect to the C^0-norm, then H_t is also uniformly* *bounded with respect to the L_1^2-norm, where here all norms are computed with respect to the initial metric K.*

Proof. Proposition 4.12 implies that $|\Lambda F_t|_K$ is uniformly bounded with respect to the C^0-norm. In addition, we are assuming a C^0-bound on H_t. Thus the right-hand side of equality (iv) of Lemma 4.4 is bounded uniformly by a constant independent of t. The result then follows. □

Another ingredient found in Donaldson [26] is the following result concerning the C^0-norm of solutions to the heat flow. For two metrics H_1, H_2, let $\sigma(H_1, H_2)$ denote the number

$$\sigma(H_1, H_2) = \mathrm{Tr}(H_1^{-1} H_2) + \mathrm{Tr}(H_2^{-1} H_1) - 2r,$$

where r is the rank of $\mathcal{E}$. Then the assignment σ does not quite define a metric, but we do have in fact that a sequence H_i converges to H in C^0 if and only if $\sup_{\mathcal{X}} \sigma(H_i, H) \to 0$. (In fact, the space of hermitian metrics is the set of sections of a fiber bundle, which, on each fiber admits a natural distance function d coming from the description of the fiber as a homogeneous space $GL(r, \mathbb{C})/U(r)$, and Donaldson [26] asserts that the function σ compares uniformly with d, in the sense that $\sigma \leqslant f(d)$ and $d \leqslant F(\sigma)$ for monotone f, F.) Donaldson [26] then proves the following by a direct calculation.

Proposition 4.15. *If H_t, K_t are two solutions to the heat flow and $\sigma = \sigma(H_t, K_t)$, then*

$$\left(\frac{\partial}{\partial t} + \Delta\right) \sigma \leqslant 0.$$

With this, it is possible to obtain the following result which relates the L^2-convergence of solutions to the heat flow with C^0-convergence.

Corollary 4.16. *For a real number $\tau > 0$ and a solution H_t to the heat flow, let $\sigma^\tau = \sigma(H_t, H_{t+\tau})$. Then for each $t' > 0$, we have*

$$\sup_{\mathcal{X}} \sigma^\tau(t + t') \leqslant c(t') \int_{\mathcal{X}} \sigma^\tau(t) \frac{\omega^n}{n!},$$

where $c(t')$ denotes the (finite) supremum of the heat kernel at time t'. In particular, if the sequence H_t coverges in L^2 as $t \to \infty$, then it also converges in C^0, where here the norms are to be computed with respect to the fixed initial metric $K = H_0$.

Proof. The inequality follows immediately from the previous proposition together with a type of Green's formula involving the heat kernel (see [29]). For the second part about convergence, suppose that H_t is Cauchy in L^2. Let $\epsilon > 0$ be given. Because H_t is Cauchy in L^2 and because the function σ compares uniformly with the norm afforded by K, there is a time $T > 0$ such that

$$\int_{\mathcal{X}} \sigma^\tau(t) \frac{\omega^n}{n!} < c(1)^{-1} \epsilon$$

for each $t > T$ and each $\tau > 0$. We therefore find that

$$\sup_{\mathcal{X}} \sigma^\tau(t + 1) \leqslant c(1) \int_{\mathcal{X}} \sigma^\tau(t) \frac{\omega^n}{n!} < \epsilon$$

for each $t > T$ and each $\tau > 0$. This means precisely that

$$\sup_{\mathcal{X}} \sigma(H_{t+1}, H_{t+1+\tau}) < \epsilon$$

for each $t > T$ and each $\tau > 0$. We conclude that the H_t are uniformly Cauchy with respect to the C^0-norm determined by K. $\square$

Also in [26], Donaldson considered a corresponding variational approach and introduced a functional whose critical points correspond to stable points of the heat flow. The introduction of such a functional requires the notion of secondary characteristic classes, which we review now.

A p-multilinear function φ on $\mathfrak{gl}(r,\mathbb{C})$ is called invariant if it is invariant under the (diagonal) adjoint action of $GL(r,\mathbb{C})$ on p copies of $\mathfrak{gl}(r,\mathbb{C})$. Such a function assigns to any metric H on $\mathcal{E}$ a (p,p)-form

$$\varphi(F_H) := \varphi(F_H,\dots,F_H) \in A^{p,p}(\mathcal{X}).$$

The cohomology class represented by the (p,p)-form $\varphi(F_H)$ is independent of the choice of metric H and is called the first characteristic class associated to φ. In addition the $\partial\bar{\partial}$-lemma implies that the difference between two such forms is $\partial\bar{\partial}$-exact. In fact, the following more precise statement is true (see [26]).

Proposition 4.17. *If H, K are two metrics on $\mathcal{E}$, then for each φ there is an invariant*

$$R_\varphi(H,K) \in A^{p-1,p-1}(\mathcal{X})/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}),$$

called the secondary characteristic class associated to φ, satisfying the following three properties.

(i) We have $R_\varphi(K,K) = 0$ and for any third metric J, we have

$$R_\varphi(H,K) = R_\varphi(H,J) + R_\varphi(J,K).$$

(ii) If H_t is a smooth family of metrics, then

$$\frac{d}{dt}R_\varphi(H_t,K) = -i\varphi(F_{H_t}; H_t^{-1}\dot{H}),$$

where $\varphi(F_{H_t}; H_t^{-1}\dot{H})$ denotes the sum

$$\varphi(F_{H_t}; H_t^{-1}\dot{H}) = \sum_{k=1}^{p} \varphi(F_{H_t}, \ldots, \overbrace{H_t^{-1}\dot{H}}^{k}, \ldots, F_{H_t}).$$

(iii) We have

$$i\bar{\partial}\partial R_\varphi(H,K) = \varphi(F_H) - \varphi(F_K) \in A^{p,p}(X).$$

Example 4.18. For our purposes, we only need two such invariants R_φ. The first R_1 is associated to the trace $\varphi_1(A) = \mathrm{Tr}A$, and the second R_2 is associated to the Killing form $\varphi_2(A,B) = -\mathrm{Tr}(AB)$.

This means in particular that given a path H_t of metrics with $H_0 = K$ and $H_1 = H$, we may set

$$R_1(H,K) = -i\int_0^1 \mathrm{Tr}(H_t^{-1}\dot{H})\, dt$$
$$R_2(H,K) = 2i\int_0^1 \mathrm{Tr}(H_t^{-1}\dot{H}F_t)\, dt.$$

The particular integrals may depend on the choice of path, but, modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$, they do not.

Definition 4.19. With these two invariants, Donaldson introduced a functional for surfaces, whose extension to arbitrary dimensions can be described as

$$M(H,K) = \int_{\mathcal{X}} (R_2 + 2\lambda R_1\omega) \wedge \frac{\omega^{n-1}}{(n-1)!}. \tag{4.3}$$

For a fixed metric K, we can consider the functional $M(-,K)$ on the space of metrics, and it turns out that the critical points of this functional (if they exist) are Hermitian-Einstein metrics.

Proposition 4.20. *For a fixed metric K, let M_K denote the functional on the space of metrics described by $M_K(H) = M(H, K)$.*

(i) If H_t is any smooth path of metrics, then the variation of $M_K(H_t)$ along H_t is given by

$$\frac{\partial}{\partial t} M_K(H_t) = 2i \int_{\mathcal{X}} \mathrm{Tr}(H_t^{-1}\dot{H}_t(F_t - \lambda\omega I_{\mathcal{E}})) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

(ii) If H is a critical point of M_K, then H is a Hermitian-Einstein metric.

(iii) In particular, if H_t is a solution to the heat flow (4.2), then

$$\frac{\partial}{\partial t} M_K(H_t) = -\left\| \Lambda F_t - \lambda I_{\mathcal{E}} \right\|^2_{L^2_{H_t}},$$

meaning that M_K is a non-increasing function of t along the heat flow.

Proof. For part (i), we compute using the definitions of the invariants R_1 and R_2 as follows

$$\begin{aligned} \frac{d}{dt} M_K(H_t) &= \int_{\mathcal{X}} \left(2i\mathrm{Tr}(H_t^{-1}\dot{H}F_t) - 2i\lambda \mathrm{Tr}(H_t^{-1}\dot{H})\omega \right) \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &= 2i \int_{\mathcal{X}} \mathrm{Tr}(H_t^{-1}\dot{H}(F_t - \lambda\omega I_{\mathcal{E}})) \wedge \frac{\omega^{n-1}}{(n-1)!} \end{aligned}$$

Part (ii) is then immediate from the computation in part (i). Part (iii) then follows from using the flow (4.2) and the fact that $\Lambda F_t - \lambda I_{\mathcal{E}}$ is skew-adjoint with respect to H_t. □

Part (iii) of the previous proposition says that M_K is non-increasing along the heat flow, and we will later show that the functional M_K is convex in a certain sense (see Proposition 4.32). In general, however, M_K may not be bounded from below. Because the critical points of M_K are the desired metrics, it would be useful to understand exactly when M_K admits such critical points. Motivated by Proposition 5.3 of [57], we introduce the following notion of properness for M_K.

Definition 4.21. We say that M_K is proper if there are positive constants C_1, C_2 such that for the solution $H_t = Ke^{s_t}$ to the heat flow with initial condition $s_0 = 0$, we have

$$\sup_{\mathcal{X}} |s_t|_K \leqslant C_1 + C_2 M_K(Ke^{s_t})$$

for each $t \geqslant 0$.

Corollary 4.22. *If M_K is proper and H_t is a solution to the heat flow with initial condition $H_0 = K$, then the following statements are true.*

(i) $M_K(H_t)$ is bounded from below (by $-C_1/C_2$).

(ii) $\|H_t\|_{C^0_K}$ is bounded from above.

(iii) $\Lambda F_{H_t} \to \lambda I_{\mathcal{E}}$ in L^2_K as $t \to \infty$.

Proof. Part (i) is obvious. Part (ii) follows from the fact that $t \mapsto M_K(H_t)$ is decreasing along the heat flow and $M_K(H_0) = M_K(K) = 0$. For part (iii), because $M_K(H_t)$ is bounded from below and non-increasing, we know that

$$\lim_{t\to\infty} \frac{\partial}{\partial t} M_K(H_t) = 0$$

and by the previous proposition we conclude that

$$\lim_{t\to\infty} \|\Lambda F_t - \lambda I_{\mathcal{E}}\|^2_{L^2_{H_t}} = 0.$$

Because H_t is uniformly bounded in C^0, the L^2-norm with respect to H_t is equivalent to the L^2-norm with respect to K in a uniform manner, meaning that there is a constant C independent of t such that

$$\|\Lambda F_t - \lambda I_{\mathcal{E}}\|^2_{L^2_K} \leqslant C \|\Lambda F_t - \lambda I_{\mathcal{E}}\|^2_{L^2_{H_t}} .$$

Taking the limit of both sides gives the required convergence. $\square$

4.2 Proof of the main result

Let us recall that our objective is to prove the following.

Theorem 4.23. *Assume $\mathcal{E}$ is indecomposable. Then the following are equivalent.*

(i) The bundle $\mathcal{E}$ is stable.

(ii) For each metric K on $\mathcal{E}$, the Donaldson functional M_K is proper in the sense of Definition 4.21.

(iii) There is a Hermitian-Einstein metric on $\mathcal{E}$.

Let us immediately deal with the implication (iii) $\implies$ (i). A proof can be found, for example, in [48], but we outline a proof now for the sake of completeness.

Proposition 4.24. *Assume $\mathcal{E}$ is indecomposable. If there is a Hermitian-Einstein metric H on $\mathcal{E}$, then $\mathcal{E}$ is stable.*

Proof. Let $\mathcal{E}'$ be a proper coherent subsheaf of $\mathcal{E}$ of rank r' with torsion-free quotient $\mathcal{E}/\mathcal{E}'$. Lemma 4.6 implies that $\mathcal{E}'$ is locally free outside of a subvariety of codimension at least 2. From this point forward, we work away from this subvariety so that for example the metric H restricts to a metric on $\mathcal{E}'$ with corresponding curvature denoted F'. It is standard (see [39, Chapter 1, Section 5]) to show that the difference of curvatures

$$F|_{\mathcal{E}'} - F'$$

is a semi-positive $\mathrm{End}(\mathcal{E}')$-valued $(1,1)$-form and moreover vanishes if and only if the orthogonal complement of $\mathcal{E}'$ is holomorphic. Here we are using the convention as in [39] that semi-positive implies in particular that

$$\frac{i}{2\pi} \cdot \mathrm{Tr}_{\mathcal{E}'}(F) - \frac{i}{2\pi} \mathrm{Tr}_{\mathcal{E}'} F'$$

is a positive $(1,1)$-form. Integrating over $\mathcal{X}$ we obtain the inequality

$$\frac{i}{2\pi} \int_{\mathcal{X}} \mathrm{Tr}_{\mathcal{E}'} \Lambda F \cdot \mathrm{vol} \geqslant \deg(\mathcal{E}') \tag{4.4}$$

which is valid because we are working outside of a subset of codimension at least two. Now the Hermitian-Einstein condition guarantees that

$$\mathrm{Tr}_{\mathcal{E}}(\Lambda F) = \mathrm{Tr}_{\mathcal{E}}(\lambda I_{\mathcal{E}}) = r \cdot \lambda$$

and hence also that

$$\mathrm{Tr}_{\mathcal{E}'} \Lambda F = r' \cdot \lambda = \frac{r'}{r} \mathrm{Tr}_{\mathcal{E}} \Lambda F.$$

Using these we find that (4.4) is equivalent to

$$\frac{r'}{r} \deg(\mathcal{E}) \geqslant \deg(\mathcal{E}').$$

But the inequality is actually strict because equality would mean that the complement of $\mathcal{E}'$ is holomorphic, which is a contradiction to the assumption that $\mathcal{E}$ is indecomposable. We conclude that $\mathcal{E}$ is stable. □

A proof of the implication (ii) $\implies$ (iii) for manifolds can be found in [57], and for this, one uses the heat flow of Definition 4.10. Indeed the argument roughly proceeds as follows. The assumption (ii) guarantees the functional M_K has a unique critical point belonging to a certain Sobolev space. The heat flow approaches this critical point H_∞ and certain estimates involving this flow and the functional M_K allow one to obtain enough regularity on this critical point to ascertain that H_∞ corresponds to a bona fide smooth metric. Since stable points of the heat flow are Hermitian-Einstein, we find that H_∞ is. Let us now be more precise.

Proposition 4.25. *Assume $\mathcal{E}$ is indecomposable. If for each fixed metric K on $\mathcal{E}$, the Donaldson functional M_K is proper in the sense of Definition 4.21, then there is a Hermitian-Einstein metric on $\mathcal{E}$.*

Proof. Let us denote by H_t a solution to the heat flow with initial condition $H_0 = K$. Corollary 4.22 applies so in particular we have a uniform C^0-bound on H_t. (Here all norms will be computed with respect to the fixed initial metric K.) We have a uniform C^0-bound on ΛF_t by Proposition 4.12. It follows from Corollary 4.14 that we have a uniform L^2_1-bound on H_t. A compactness theorem (Theorem 2.14) now guarantees the existence of a sequence of times $t_i \to \infty$ and a limit $H_\infty \in L^2_1$ such that the sequence H_{t_i} converges in L^2 to H_∞. Corollary 4.16 implies actually that H_{t_i} converges to H_∞ in C^0-norm. Proposition 4.13 now gives that H_{t_i} is in fact uniformly bounded in L^p_2 for each $p < \infty$. It follows that the weak limit F_∞ exists in L^p for each $p < \infty$ and moreover that the weak equation $\Lambda F_\infty = \lambda \cdot I_{\mathcal{E}}$ holds by Corollary 4.22 (iii). Elliptic regularity (Section 2.2) now implies that H_∞ is in fact smooth. $\square$

The remaining implication is (i) $\implies$ (ii), and this is proved for manifolds in [57] with the help of a regularity statement concerning weakly holomorphic subbundles from [63], another proof of which can be found in [54]. The precise notion of a weakly holomorphic subbundle that we will use is the following.

Definition 4.26. By a weakly holomorphic subbundle of $\mathcal{E}$ (with respect to a fixed metric K) we mean an L_1^2 section Π of $\mathrm{End}(\mathcal{E})$ which satisfies $\Pi = \Pi^* = \Pi^2$ (where the adjoint is computed with respect to K) and $(I_{\mathcal{E}} - \Pi)\bar{\partial}\Pi = 0$.

A weakly holomorphic subbundle determines a degree via Chern-Weil theory, which may be computed as

$$\deg(\Pi) = \frac{i}{2\pi}\int_{\mathcal{X}} \mathrm{Tr}(\Pi\Lambda F_K)\frac{\omega^n}{n!} - \frac{1}{2\pi}\int_{\mathcal{X}} |\bar{\partial}\Pi|_K^2 \frac{\omega^n}{n!} \tag{4.5}$$

(compare to [57, Lemma 3.2] or [63, Proposition 4.2]). It therefore makes sense to say when a weakly holomorphic subbundle is destabilizing for $\mathcal{E}$.

Remark 4.27. Let us verify equation (4.5) for the case of a smooth subbundle $\mathcal{S}$ of $\mathcal{E}$. Let Π denote the projection endomorphism of $\mathcal{E}$ corresponding to $\mathcal{S}$. If we write $D_{\mathcal{E}}$ for the Chern connection on $\mathcal{E}$ determined by the metric K, then there is a connection $D_{\mathcal{S}}$ on $\mathcal{S}$ described by the composition $\Pi \circ D_{\mathcal{E}}$. The difference $A = D_{\mathcal{E}}|_{\mathcal{S}} - D_{\mathcal{S}}$ may be considered as a map from $A^0(\mathcal{S})$ to $A^1(\mathcal{S}^\perp)$. In fact, A is a map to $A^{1,0}(\mathcal{S}^\perp)$ (see [39]) and corresponds to the composition $\Pi^\perp \circ \partial_{\mathcal{E}}\Pi$ (see [63, Proposition 4.2]), where here $\partial_{\mathcal{E}}$ denotes the $(1,0)$-component of $D_{\mathcal{E}}$. In [39], the curvature of the subbundle is related to the curvature of the ambient bundle by $F_{\mathcal{S}} = \Pi \circ F_{\mathcal{E}} - A \wedge A^*$. Taking the trace and integrating over $\mathcal{X}$ we find that

$$\begin{aligned}
\deg(\mathcal{S}) &= \frac{i}{2\pi}\int_{\mathcal{X}} \mathrm{Tr}(F_{\mathcal{S}}) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
&= \frac{i}{2\pi}\int_{\mathcal{X}} \mathrm{Tr}(\Pi F_{\mathcal{E}})\frac{\omega^{n-1}}{(n-1)!} - \frac{1}{2\pi}\int_{\mathcal{X}} \mathrm{Tr}(iA \wedge A^*) \wedge \frac{\omega^{n-1}}{(n-1)!} \\
&= \frac{i}{2\pi}\int_{\mathcal{X}} \mathrm{Tr}(\Pi\Lambda F_{\mathcal{E}})\frac{\omega^n}{n!} - \frac{1}{2\pi}\int_{\mathcal{X}} |\Pi^\perp \circ \partial_{\mathcal{E}}\Pi|_K^2 \frac{\omega^n}{n!}.
\end{aligned}$$

Because $\Pi \circ \partial_{\mathcal{E}}\Pi = 0$, we conclude that in fact $\Pi^{\perp} \circ \partial_{\mathcal{E}}\Pi = \partial_{\mathcal{E}}\Pi$. The fact that Π is self-adjoint also implies that $|\partial_{\mathcal{E}}\Pi| = |\bar{\partial}\Pi|$ and the formula (4.5) is verified.

With these notions, the implication (i) $\implies$ (ii) then follows immediately from the following two lemmas.

Lemma 4.28. *Suppose M_K is not proper. Then there is a weakly holomorphic subbundle of $\mathcal{E}$ which is destabilizing for $\mathcal{E}$.*

Lemma 4.29. *Let Π be a weakly holomorphic subbundle of $\mathcal{E}$. Then there is a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ and an analytic subvariety $\mathcal{V}$ of codimension at least two in $\mathcal{X}$ such that*

(i) The map Π is smooth away from $\mathcal{V}$ and there we have $\Pi = \Pi^* = \Pi^2$ and $(I_{\mathcal{E}} - \Pi) \circ \bar{\partial}\Pi = 0$.

(ii) Outside of $\mathcal{V}$ the subsheaf $\mathcal{F}$ agrees with the image of Π and is a holomorphic subbundle of $\mathcal{E}|_{\mathcal{X}\setminus\mathcal{V}}$.

Lemma 4.28 is proved for manifolds in Proposition 5.3 of [57], and exactly the same method of proof applies in our setting. We reserve the final section following this one for a discussion of this method. The basic idea is the following. Assuming M_K is not proper, we can obtain a sequence s_k of sections of $\mathrm{End}(\mathcal{E})$ with larger and larger norms. An appropriate normed sequence u_k then tends to a weak limit u_∞ in L^2_1, whose eigenvalues are constant almost everywhere. The eigenspaces of u_∞ then give rise to a filtration of $\mathcal{E}$ by L^2_1-subbundles, for which, it is possible to show that one must be destabilizing.

Assuming Lemma 4.28 then, for now, it remains only to discuss Lemma 4.29. A version of this result can be found in the original paper by Uhlenbeck-Yau [63], and we aim to explain how it extends to the orbifold setting.

For the holomorphic vector bundle $\mathcal{E}$ of rank r over $\mathcal{X}$, it is possible to form the Grassmannian bundle $\mathrm{Gr}(s, \mathcal{E})$ of s-planes in $\mathcal{E}$, which is a holomorphic fiber bundle over $\mathcal{X}$ with fiber $\mathrm{Gr}(s, r)$. Locally if E_α is a G_α-equivariant vector bundle of rank r over a chart U_α, then the Grassmannian bundle $\mathrm{Gr}(s, \mathcal{E})$ associates to U_α the fiber bundle $\mathrm{Gr}(s, E_\alpha)$ of s-planes in E_α, where the fiber over a point $x \in U_\alpha$ is the Grassmannian $\mathrm{Gr}(s, (E_\alpha)_x)$ of s-planes in the fiber $(E_\alpha)_x$. The bundle $\mathrm{Gr}(s, \mathcal{E}_\alpha)$ enjoys an induced action of G_α on it coming from the action of G_α on E_α, and the induced action is such that the natural projection onto U_α is G_α-equivariant. For an embedding $\lambda : U_\alpha \to U_\beta$, the bundle isomorphism $E_\alpha \to \lambda^* E_\beta$ induces a bundle isomorphism $\mathrm{Gr}(s, E_\alpha) \to \lambda^* \mathrm{Gr}(s, E_\beta) \simeq \mathrm{Gr}(s, \lambda^* E_\beta)$.

A holomorphic subbundle $\mathcal{E}'$ of $\mathcal{E}$ of rank s determines a section of the fiber bundle $\mathrm{Gr}(s, \mathcal{E})$. In addition, any section of the bundle $\mathrm{Gr}(s, \mathcal{E})$ determines a holomorphic subbundle $\mathcal{E}'$ of $\mathcal{E}$ which corresponds to the image of the section in $\mathrm{Gr}(s, \mathcal{E})$.

If $p : \mathrm{Gr}(s, \mathcal{E}) \to \mathcal{X}$ denotes the projection map, then there is a way of pulling back the vector bundle $\mathcal{E}$ along p to obtain a vector bundle $p^*\mathcal{E}$ of rank r over $\mathrm{Gr}(s, \mathcal{E})$, which is described as follows. There is an atlas of charts $(U_\alpha, G_\alpha, \pi_\alpha)$ for $\mathcal{X}$ such that $(U_\alpha \times \mathrm{Gr}(s, r), G_\alpha, \pi'_\alpha)$ is an atlas of charts for $\mathrm{Gr}(s, \mathcal{E})$, where π'_α denotes the natural map from $U_\alpha \times \mathrm{Gr}(s, r)$ to its image in $\mathrm{Gr}(s, \mathcal{E})$. To each embedding of charts $\lambda : U_\alpha \to U_\beta$, there corresponds an embedding of charts $\lambda' : U_\alpha \times \mathrm{Gr}(s, r) \to U_\beta \times \mathrm{Gr}(s, r)$ such that the diagram

$$\begin{tikzcd} U_\alpha \times \mathrm{Gr}(s, r) \arrow[r, "\lambda'"] \arrow[d, "\mathrm{pr}_1^\alpha"'] & U_\beta \times \mathrm{Gr}(s, r) \arrow[d, "\mathrm{pr}_1^\beta"] \\ U_\alpha \arrow[r, "\lambda"'] & U_\beta, \end{tikzcd} \tag{4.6}$$

commutes, where pr_1^α denotes projection onto the first factor. The vector bundle

$\mathcal{E}$ associates to each U_α a G_α-equivariant vector bundle E_α, and we may consider the pullback $p^*E_\alpha = (\mathrm{pr}_1^\alpha)^*E_\alpha$ of E_α along the projection $U_\alpha \times \mathrm{Gr}(s,r)$ onto the first factor. The pullback p^*E_α enjoys an action of G_α in the following manner: if $g \in G_\alpha$ and if ξ is an element of p^*E_α in the fiber over $(x,V) \in U_\alpha \times \mathrm{Gr}(s,r)$, then in fact ξ is an element in the fiber $(E_\alpha)_x$ and $g \cdot \xi$ is an element of the fiber $(E_\alpha)_{g\cdot x} = (p^*E_\alpha)_{g\cdot(x,V)}$. In addition, to each embedding of charts $\lambda' : U_\alpha \times \mathrm{Gr}(s,r) \to U_\beta \times \mathrm{Gr}(s,r)$, there corresponds a bundle isomorphism $p^*E_\alpha \to (\lambda')^*p^*E_\beta$ which is described as $(\mathrm{pr}_1^\alpha)^*\lambda_*$ where $\lambda_* : E_\alpha \to \lambda^*E_\beta$ is the bundle isomorphism induced by λ. This makes sense because there is an isomorphism of bundles $(\lambda')^*p^*E_\beta \simeq (\mathrm{pr}_1^\alpha)^*\lambda^*E_\beta$ by the commutativity of the diagram (4.6).

Remark 4.30. We remark that using the previous construction, it is actually possible to pull back a vector bundle $\mathcal{E}$ over $\mathcal{X}$ along the projection map $\mathcal{E}' \to \mathcal{X}$ of *any* fiber bundle $\mathcal{E}'$ over $\mathcal{X}$. However, it is not immediately clear that this construction of the pullback is readily available for each smooth map of orbifolds $\mathcal{X}' \to \mathcal{X}$. In Section 4.4 of [20], the authors introduce the notion of a "good" smooth map of orbifolds, and using this notion, they show that a vector bundle may be pulled back along such maps. In particular, the projection map $\mathcal{E}' \to \mathcal{X}$ for a fiber bundle is a "good" map, so their construction applies in this situation, as we have just described.

There is a universal subbundle $\mathcal{S}$ of $p^*\mathcal{E}$ of rank s over $\mathrm{Gr}(s,\mathcal{E})$ described as the incidence correspondence in the usual way.

If $\mathcal{Y}$ is an analytic subvariety of $\mathrm{Gr}(s,\mathcal{E})$ and $\mathcal{F}$ is a coherent sheaf over $\mathcal{Y}$, then there is a way of pushing forward the sheaf via the restriction of the projection $p : \mathrm{Gr}(s,\mathcal{E}) \to \mathcal{X}$ to $\mathcal{Y}$ to obtain a coherent sheaf $p_*\mathcal{F}$ on $\mathcal{X}$ as follows. The subvariety $\mathcal{Y}$ associates to each chart $U_\alpha \times \mathrm{Gr}(s,r)$ a G_α-invariant subvariety $V_\alpha \subset U_\alpha \times \mathrm{Gr}(s,r)$ and the sheaf $\mathcal{F}$ associates a G_α-equivariant sheaf $\mathcal{F}_\alpha$ over V_α. We may consider the pushforward $p_*\mathcal{F}_\alpha$ onto U_α using the projection pr_1^α onto the first factor. The resulting sheaf $p_*\mathcal{F}_\alpha$ is G_α-equivariant since pr_1^α is. For each embedding $\lambda : U_\alpha \to U_\beta$ of charts, there is an isomorphism of sheaves $p_*\mathcal{F}_\alpha \to \lambda^* p_* \mathcal{F}_\beta$ described as $(\mathrm{pr}_1^\alpha)_* \tau_{\lambda'}$, where $\tau_{\lambda'}$ denotes the isomorphism of sheaves $\mathcal{F}_\alpha \to \lambda'^* \mathcal{F}_\beta$. This makes sense because there is an isomorphism of sheaves $\lambda^* p_* \mathcal{F}_\beta \simeq (\mathrm{pr}_1^\alpha)_* \lambda'^* \mathcal{F}_\beta$ by the commutativity of the following diagram (compare to (4.6)):

$$\begin{tikzcd}
V_\alpha \arrow[r, "\lambda'|_{V_\alpha}"] \arrow[d] & V_\beta \arrow[d] \\
U_\alpha \times \mathrm{Gr}(s,r) \arrow[r, "\lambda'"] \arrow[d, "\mathrm{pr}_1^\alpha"'] & U_\beta \times \mathrm{Gr}(s,r) \arrow[d, "\mathrm{pr}_1^\beta"] \\
U_\alpha \arrow[r, "\lambda"] & U_\beta.
\end{tikzcd}$$

In addition, the resulting sheaf is coherent by the Grauert direct image theorem [38] because the maps pr_1^α are proper maps between complex spaces (as $\mathcal{Y}$ is compact and properness is preserved under base change).

By a rational map from a holomorphic orbifold $\mathcal{X}$ into another $\mathcal{X}'$ we mean we are given an analytic subvariety $\mathcal{V}$ of codimension at least 2 or more in $\mathcal{X}$ together with a holomorphic map from $\mathcal{X} \setminus \mathcal{V}$ into $\mathcal{X}'$.

We assert that a rational section of $\mathrm{Gr}(s, \mathcal{E})$ over $\mathcal{X}$ determines a coherent subsheaf of $\mathcal{E}$ in the following manner. Let $\mathcal{Y}$ denote the closure of the image of the section in $\mathrm{Gr}(s, \mathcal{E})$. The restriction of the universal bundle $\mathcal{S}$ to $\mathcal{Y}$ is a coherent sheaf of rank s over $\mathcal{Y}$. In addition, as a closed subset of a compact space, $\mathcal{Y}$ is compact itself, and so the projection of $\mathcal{Y}$ onto $\mathcal{X}$ is proper. Pushing forward the restriction $\mathcal{S}|_{\mathcal{Y}}$ of the universal bundle via the projection of $\mathcal{Y}$ onto $\mathcal{X}$, we obtain a sheaf $\mathcal{F}$ over $\mathcal{X}$, which is coherent by our above observations.

Proof of Lemma 4.29. A weakly holomorphic subbundle Π determines a map from a set of full measure in $\mathcal{X}$ to the total space of the bundle $\mathrm{Gr}(s, \mathcal{E})$. Uhlenbeck-Yau demonstrate how the assumptions $\Pi = \Pi^2 = \Pi^* \in L_1^2$ and $(I_{\mathcal{E}} - \Pi)\bar{\partial}\Pi = 0$ imply that Π extends to a rational section of the bundle $\mathrm{Gr}(s, \mathcal{E})$. The previous discussion explains how this rational section furnishes a coherent subsheaf of $\mathcal{E}$ over $\mathcal{X}$. $\square$

4.3 Simpson's method

This section is devoted to proving Lemma 4.28 following an approach from [57]. We first require a somewhat technical estimate relating the C^0-norm to the L^2-norm for solutions to the heat flow. This result is intended to replace Assumption 3 from [57]. It is important to note that in this section all inner products and norms are to be computed with respect to K unless otherwise indicated. In particular, this means that we will use the notation L^p to denote the L^p-norm of a section with respect to the fixed metric K.

Lemma 4.31. *Fix a metric K. Then there are positive constants C_1, C_2 such that the following is true. Let H_t be a solution to the heat flow with initial condition $H_0 = K$ and write $H_t = Ke^{s_t}$ for a path $t \mapsto s_t$ of self-adjoint endomorphisms with initial condition $s_t = 0$. Then for any t we have*

$$\sup_{\mathcal{X}} |s_t|_K \leqslant C_1 + C_2 \left\| |s_t|_K^2 \right\|_{L^2}^{1/2}.$$

Proof. In the course of the proof, we let $C_1, C_2, \ldots$ denote constants that are independent of t but which may vary from step to step. Recall from Lemma 4.4 (v), we know that

$$\Delta |s_t|_K^2 = \langle 2i\Lambda(F_t - F_K), s_t \rangle - \langle i\Lambda F_K s_t, s_t \rangle - |d_K s_t|^2$$

and hence because the last term is a square we find

$$\Delta|s_t|_K^2 \leqslant \langle 2i\Lambda(F_t - F_K), s_t\rangle - \langle i\Lambda F_K s_t, s_t\rangle.$$

Proposition 4.12 (with the Schwarz inequality) implies there are positive constants C_1, C_2 such that

$$\begin{aligned}\Delta|s_t|_K^2 &\leqslant C_1|s_t| + C_2|s_t|^2 \\ &\leqslant C_1 + C_2|s_t|^2.\end{aligned}$$

Let p_t be a point where $|s_t|_K^2$ achieves its maximum. Let $G_{p_t} \in L_2^2$ be Green's function for Δ. Green's formula gives that

$$|s_t|^2(p_t) = \frac{1}{\mathrm{vol}(\mathcal{X})}\int_{\mathcal{X}} |s_t|^2\frac{\omega^n}{n!} + \int_{\mathcal{X}} G_{p_t}\Delta|s_t|^2\frac{\omega^n}{n!}.$$

Because G_{p_t} is bounded from below, we may assume by shifting by a constant that G_{p_t} is positive (c.f. [4]). Moreover, we may assume that G_{p_t} is square integrable. In fact, because $\mathcal{X}$ is compact, there is a constant C (independent of t) such that $\|G_{p_t}\|_{L^2} \leqslant C$. Using the previous paragraph and the Schwarz inequality we find that we have an inequality of the form

$$|s_t|^2(p_t) \leqslant C_1\left\||s_t|^2\right\|_{L^1} + C_2 + C_3\left\||s_t|^2\right\|_{L^2}.$$

The inclusion of L^2 into L^1 implies that

$$|s_t|^2(p_t) \leqslant C_1 + C_2\left\||s_t|^2\right\|_{L^2}.$$

Now, we also know that

$$(\sup_{\mathcal{X}} |s_t|)^2 \leqslant 1 + \sup_{\mathcal{X}} |s_t|^2,$$

and so in conjunction with the previous paragraph we have

$$\sup |s_t| \leqslant C_1 + C_2 \left\| |s_t|^2 \right\|_{L^2}^{1/2},$$

as desired. □

It is also prudent to understand the variation of M_K along a path of the form $t \mapsto Ke^{ts}$ for a fixed endomorphism s of $\mathcal{E}$ that is self-adjoint with respect to K and which satisfies $\int_{\mathcal{X}} \mathrm{Tr}(s)\omega^n = 0$. We will see that the functional M_K is convex along such a path.

Proposition 4.32. *If $H_t = Ke^{ts}$ for an endomorphism s with $\int_{\mathcal{X}} \mathrm{Tr}(s)\omega^n = 0$ which is self-adjoint with respect to K, then*

$$\frac{\partial}{\partial t} M_K(Ke^{ts}) = 2i \int_{\mathcal{X}} \mathrm{Tr}(sF_t) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

and

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = 2 \int_{\mathcal{X}} |\bar{\partial} s|_{H_t}^2 \frac{\omega^n}{n!}.$$

Proof. Note that along this path, we have $\dot{H}_t = H_t s$, and so using Proposition 4.20 (i), we readily verify the formula of the first variation (using that $\int_{\mathcal{X}} \mathrm{Tr}(s)\omega^n = 0$).

Upon taking another derivative, we find

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = 2i \int_{\mathcal{X}} \mathrm{Tr}(s\dot{F}_t) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

But because $H_t = Ke^{ts}$, we have that the curvatures are related by

$$F_t = F_K + \bar{\partial}(H_t^{-1}\partial_K H_t) = F_K + t\bar{\partial}(\partial_K(s))$$

so that $\dot{F}_t = \bar{\partial}\partial_K s$, from which we find

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = 2i \int_{\mathcal{X}} \mathrm{Tr}(s \wedge \bar{\partial}\partial_K s) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

One integration by parts shows that

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = -2i \int_{\mathcal{X}} \mathrm{Tr}(\bar{\partial} s \wedge \partial_K s) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

And another shows that

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = -2i \int_{\mathcal{X}} \mathrm{Tr}(\partial_K \bar{\partial} s \wedge s) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

(Note that the sign is preserved here because $\bar{\partial} s$ is a 1-form.) Then the Kähler identity $\bar{\partial}^* = -i\Lambda\partial$ shows that

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = 2 \int_{\mathcal{X}} \mathrm{Tr}(\bar{\partial}^* \bar{\partial} s \wedge s) \wedge \frac{\omega^n}{n!}.$$

Now using that s is self-adjoint with respect to H_t, we find that this is equal to

$$\frac{\partial^2}{\partial t^2} M_K(Ke^{ts}) = 2 \int_{\mathcal{X}} \langle \bar{\partial}^* \bar{\partial} s, s \rangle_{H_t} \frac{\omega^n}{n!}.$$

This is equivalent to the desired formula. $\square$

This proposition allows one to obtain a slightly different expression for the functional M_K, which can be found for example in [57, 58, 41, 28]. Indeed, let us write $M(t)$ for the value $M(t) = M_K(Ke^{ts})$. Then with this convention, we have from the previous proposition that $M'(0)$ is given by

$$M'(0) = 2i \int_{\mathcal{X}} \mathrm{Tr}(sF_K) \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

The fundamental theorem of Calculus in conjunction with the previous proposition gives

$$M'(t) = 2i \int_{\mathcal{X}} \mathrm{Tr}(sF_K) \wedge \frac{\omega^{n-1}}{(n-1)!} + 2\int_0^t \int_{\mathcal{X}} |\bar{\partial}s|^2_{H_u} \frac{\omega^n}{n!} du.$$

The condition $M(0) = 0$ implies then that $M(1)$ is given by an additional integration

$$M_K(Ke^s) = M(1) = 2i \int_{\mathcal{X}} \mathrm{Tr}(sF_K) \wedge \frac{\omega^{n-1}}{(n-1)!} + 2\int_0^1 \left(\int_0^t \int_{\mathcal{X}} |\bar{\partial}s|^2_{H_u} \frac{\omega^n}{n!} du \right) dt.$$

We now follow [28] to write the second term on the right-hand side with a local expression involving frames.

Let us fix a smooth unitary (with respect to K) frame for $\mathcal{E}$ for which the matrix of s with respect to this frame is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_r$. (The matrix of $\bar{\partial}s$ may not be diagonal because the frame is only *smooth*.) With these conventions, then the integrand of the second equality in the previous proposition becomes

$$\begin{aligned} |\bar{\partial}s|^2_{H_t} &= (\bar{\partial}s)^\beta_\alpha \overline{(\bar{\partial}s)^\rho_\gamma} (H_t)^{\alpha\gamma} (H_t)_{\beta\rho} \\ &= (\bar{\partial}s)^\beta_\alpha \overline{(\bar{\partial}s)^\rho_\gamma} (e^{-t\lambda_\alpha}\delta^{\alpha\gamma})(e^{t\lambda_\beta}\delta_{\beta\rho}) \\ &= \sum_{\alpha,\beta} |(\bar{\partial}s)^\beta_\alpha|^2 e^{(\lambda_\beta - \lambda_\alpha)t}. \end{aligned}$$

Integrating once we obtain

$$\int_0^t |\bar{\partial} s|^2_{H_u} du = \sum_{\alpha,\beta} |(\bar{\partial} s)^\beta_\alpha|^2 \frac{e^{(\lambda_\beta-\lambda_\alpha)t}-1}{\lambda_\beta-\lambda_\alpha}.$$

And integrating again gives

$$\int_0^1 \left(\int_0^t |\bar{\partial} s|^2_{H_u} du\right) dt = \sum_{\alpha,\beta} |(\bar{\partial} s)^\beta_\alpha|^2 \frac{e^{\lambda_\beta-\lambda_\alpha}-(\lambda_\beta-\lambda_\alpha)-1}{(\lambda_\beta-\lambda_\alpha)^2}.$$

What we have shown therefore is that

$$M(Ke^s, K) = 2i\int_{\mathcal{X}} \operatorname{Tr}(s\Lambda F_K)\frac{\omega^n}{n!} + 2\int_{\mathcal{X}} \sum_{\alpha,\beta} |(\bar{\partial} s)^\beta_\alpha|^2 \frac{e^{\lambda_\beta-\lambda_\alpha}-(\lambda_\beta-\lambda_\alpha)-1}{(\lambda_\beta-\lambda_\alpha)^2}\frac{\omega^n}{n!}, \tag{4.7}$$

where the summand is interpreted as $\frac{1}{2}|(\bar{\partial} s)^\beta_\alpha|^2$ if $\alpha=\beta$.

Following [58] and [28, Lemma 24] it is then possible to obtain the following estimate, which we won't really need, but which we collect for completeness.

Corollary 4.33. *For any endomorphism s with $\int_{\mathcal{X}} \operatorname{Tr}(s)\omega^n = 0$ that is self-adjoint with respect to K, we have*

$$\|D_K s\|^2_{L^1} \leqslant 2(\sqrt{2}\,\|s\|_{L^1} + \operatorname{vol}(\mathcal{X}))\left(M_K(Ke^s) - 2i\int_{\mathcal{X}} \operatorname{Tr}(s\Lambda F_K)\frac{\omega^n}{n!}\right).$$

Proof. For any real number u, we have the following inequality

$$\frac{1}{2\sqrt{u^2+1}} \leqslant \frac{e^u-u-1}{u^2},$$

which is verified, for example, in [58]. From this it follows immediately upon setting

$u = \lambda_\beta - \lambda_\alpha$ that

$$\frac{1}{2\sqrt{(\lambda_\beta - \lambda_\alpha)^2 + 1}} \leqslant \frac{e^{\lambda_\beta - \lambda_\alpha} - (\lambda_\beta - \lambda_\alpha) - 1}{(\lambda_\beta - \lambda_\alpha)^2}.$$

The inequality

$$2(\lambda_\beta^2 + \lambda_\alpha^2) = (\lambda_\beta - \lambda_\alpha)^2 + (\lambda_\beta + \lambda_\alpha)^2 \geqslant (\lambda_\beta - \lambda_\alpha)^2$$

implies also that

$$\frac{1}{2\sqrt{2(\lambda_\beta^2 + \lambda_\alpha^2) + 1}} \leqslant \frac{e^{\lambda_\beta - \lambda_\alpha} - (\lambda_\beta - \lambda_\alpha) - 1}{(\lambda_\beta - \lambda_\alpha)^2}.$$

And finally the inequality

$$|s|_K^2 = \sum_\alpha \lambda_\alpha^2 \geqslant \lambda_\beta^2 + \lambda_\alpha^2$$

implies

$$\frac{1}{2\sqrt{2|s|_K^2 + 1}} \leqslant \frac{e^{\lambda_\beta - \lambda_\alpha} - (\lambda_\beta - \lambda_\alpha) - 1}{(\lambda_\beta - \lambda_\alpha)^2}.$$

Now using that $|D_K s|_K^2 = 2|\bar\partial s|_K^2$ (because s is self-adjoint), we find that

$$\frac{|D_K s|_K^2}{4\sqrt{2|s|_K + 1}} = \frac{|\bar\partial s|_K^2}{2\sqrt{2|s|_K + 1}} \leqslant \sum_{\alpha,\beta} |\bar\partial s_\alpha^\beta|^2 \frac{e^{\lambda_\beta - \lambda_\alpha} - (\lambda_\beta - \lambda_\alpha) - 1}{(\lambda_\beta - \lambda_\alpha)^2}.$$

Upon integrating over $\mathcal{X}$ and using formula (4.7), we find

$$\frac{1}{2}\int_{\mathcal{X}} \frac{|D_K s|_K^2}{\sqrt{2|s|_K + 1}} \frac{\omega^n}{n!} \leqslant M_K(Ke^S) - 2i\int_{\mathcal{X}} \mathrm{Tr}(s\Lambda F_K)\frac{\omega^n}{n!}. \tag{4.8}$$

On the other hand, if we write

$$D_K s = \frac{D_K s}{(2|s|_K + 1)^{1/4}}(2|s|_K + 1)^{1/4}$$

and use the Cauchy inequality, we find that

$$\begin{aligned}\left(\int_{\mathcal{X}} |D_K s|_K \frac{\omega^n}{n!}\right)^2 &\leqslant \left(\int_{\mathcal{X}} \frac{|D_K s|_K^2}{\sqrt{2|s|_K + 1}} \frac{\omega^n}{n!}\right)\left(\int_{\mathcal{X}} (2|s|_K^2 + 1)^{1/2} \frac{\omega^n}{n!}\right) \\ &\leqslant \left(\int_{\mathcal{X}} \frac{|D_K s|_K^2}{\sqrt{2|s|_K + 1}} \frac{\omega^n}{n!}\right)\left(\int_{\mathcal{X}} (\sqrt{2}|s|_K + 1) \frac{\omega^n}{n!}\right) \\ &\leqslant \left(\int_{\mathcal{X}} \frac{|D_K s|_K^2}{\sqrt{2|s|_K + 1}} \frac{\omega^n}{n!}\right)\left(\sqrt{2}\,\|s\|_{L^1} + \mathrm{vol}(\mathcal{X})\right),\end{aligned}$$

and then using (4.8) we conclude

$$\|D_K s\|_{L^1}^2 \leqslant 2\left(\sqrt{2}\,\|s\|_{L^1} + \mathrm{vol}(\mathcal{X})\right)\left(M_K(Ke^s) - 2i\int_{\mathcal{X}} \mathrm{Tr}(s\Lambda F_K)\frac{\omega^n}{n!}\right),$$

as desired. □

Notation 4.34. Let us follow [57] to introduce briefly some notation that will allow us to express formula (4.7) in a global manner.

For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, an endomorphism s of $\mathcal{E}$ that is self-adjoint with respect to K, we let $\varphi(s)$ denote the endomorphism described in the following manner. If $\{e_1, \ldots, e_r\}$ is a smooth unitary (with respect to K) frame for E with respect to which s is diagonal with entries $\lambda_1, \ldots, \lambda_r$, then $\varphi(s)$ is the endomorphism with diagonal entries $\varphi(\lambda_1), \ldots, \varphi(\lambda_r)$.

In addition, for a smooth function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of two variables, a self-adjoint endomorphism $s \in \mathrm{End}(\mathcal{E}, K)$, and an endomorphism $A \in \mathrm{End}(\mathcal{E})$, we let $\Phi(s)(A)$ denote the endomorphism of $\mathcal{E}$ described in the following manner. If $\{e_1, \ldots, e_r\}$ is a smooth unitary (with respect to K) frame of E with respect to which s is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_r$ and A has the local expression $A = A^\beta_\alpha e^\alpha \otimes e_\beta$ where e^α is the frame dual to e_β, then the endomorphism $\Phi(s)(A)$ has local expression

$$\Phi(s)(A) = \sum_{\alpha,\beta} \Phi(\lambda_\alpha, \lambda_\beta) A^\beta_\alpha e^\alpha \otimes e_\beta.$$

The construction Φ enables one to express the derivatives of construction φ in the following way.

Lemma 4.35. *Given a $\varphi : \mathbb{R} \to \mathbb{R}$, if we set $d\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to be the difference quotient defined by*

$$d\varphi(u, v) = \frac{\varphi(u) - \varphi(v)}{u - v}$$

for $u \neq v$ and $d\varphi(u, u) = \frac{d}{du}\varphi(u)$ along the diagonal, then we have

$$\bar{\partial}(\varphi(s)) = d\varphi(s)(\bar{\partial} s).$$

In addition, if $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is any smooth function which agrees with $d\varphi$ along the diagonal, then

$$\mathrm{Tr}(\Phi(s)(\bar{\partial} s)) = \mathrm{Tr}(d\varphi(s)(\bar{\partial} s)).$$

Proof. Let e_α be a smooth unitary frame for $\mathcal{E}$ with respect to which a local expression for s is

$$s = \sum_\alpha \lambda_\alpha e^\alpha \otimes e_\alpha$$

for some local smooth functions λ_α. Let us also write

$$\bar{\partial} e_\alpha = \theta^\beta_\alpha e_\beta$$

for some local $(0,1)$-forms θ_α^β. The relation $e^\alpha(e_\beta) = \delta_\beta^\alpha$ implies then that we also have

$$\bar\partial e^\alpha = -\theta_\beta^\alpha e^\beta.$$

It follows that a local expression for $\bar\partial s$ is given by

$$\begin{aligned}\bar\partial s &= (\bar\partial\lambda_\alpha)e^\alpha \otimes e_\alpha + \sum_{\alpha,\beta}(-\lambda_\alpha\theta_\beta^\alpha e^\beta \otimes e_\alpha + \lambda_\alpha\theta_\alpha^\beta e^\alpha \otimes e_\beta)\\ &= (\bar\partial\lambda_\alpha)e^\alpha \otimes e_\alpha + \sum_{\alpha\neq\beta}(\lambda_\alpha - \lambda_\beta)\theta_\alpha^\beta e^\alpha \otimes e_\beta,\end{aligned}$$

which means precisely that the coefficients of $\bar\partial s$ are given by

$$(\bar\partial s)_\alpha^\beta = \begin{cases}(\lambda_\alpha - \lambda_\beta)\theta_\alpha^\beta & \alpha\neq\beta\\ \bar\partial\lambda_\alpha & \alpha=\beta\end{cases}.$$

More generally, a similar computation shows that the coefficients of $\bar\partial(\varphi(s))$ are given by

$$(\bar\partial(\varphi(s))_\alpha^\beta = \begin{cases}(\varphi\circ\lambda_\alpha - \varphi\circ\lambda_\beta)\theta_\alpha^\beta & \alpha\neq\beta\\ \bar\partial(\varphi\circ\lambda_\alpha) & \alpha=\beta\end{cases}.$$

On the other hand, by definition, the endomorphism $d\varphi(s)(\bar\partial s)$ has coefficients

$$\begin{aligned}(d\varphi(s)(\bar\partial s))_\alpha^\beta &= d\varphi(\lambda_\alpha,\lambda_\beta)(\bar\partial s)_\alpha^\beta\\ &= \begin{cases}\frac{\varphi\circ\lambda_\alpha - \varphi\circ\lambda_\beta}{\lambda_\alpha-\lambda_\beta}(\lambda_\alpha - \lambda_\beta)\theta_\alpha^\beta & \alpha\neq\beta\\ \varphi'(\lambda_\alpha)\bar\partial\lambda_\alpha & \alpha=\beta\end{cases}.\end{aligned}$$

Comparing coefficients, we find that the first part of the lemma follows.

For the second part about the trace, suppose that $\Phi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is any smooth function which agrees with $d\varphi$ along the diagonal. Then the trace of $\Phi(s)(\bar\partial s)$ is given by

$$\begin{aligned}\mathrm{Tr}(\Phi(s)(\bar\partial s)) &= \sum_\alpha \Phi(\lambda_\alpha,\lambda_\alpha)(\bar\partial s)^\alpha_\alpha\\ &= \sum_\alpha \varphi'(\lambda_\alpha)\bar\partial\lambda_\alpha\\ &= \mathrm{Tr}(d\varphi(s)(\bar\partial s)),\end{aligned}$$

as desired. $\square$

With these conventions, we see that formula (4.7) is equivalent to

$$M_K(Ke^s) = 2i\int_{\mathcal{X}} \mathrm{Tr}(s\Lambda F_K)\frac{\omega^n}{n!} + 2\int_{\mathcal{X}} \langle \Psi(s)(\bar\partial s), \bar\partial s\rangle_K \frac{\omega^n}{n!}, \tag{4.9}$$

where $\Psi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the function

$$\Psi(u,v) = \frac{e^{v-u}-(v-u)-1}{(v-u)^2},$$

which is extended continuously (and smoothly) along the diagonal by requiring that $\Psi(u,u) = 1/2$.

The construction Φ extends to L^p-spaces of endomorphisms in the following way. Because Φ is smooth, there is a positive constant C depending on Φ such that we have the pointwise estimate

$$|\Phi(s)(A)|_K \leqslant C|s|_K|A|_K$$

for any endomorphism A and self-adjoint endomorphism s. Given any $1\leqslant p<q$, if

r is the number $1/p = 1/q + 1/r$, then Hölder's inequality implies that

$$\||s|_K |A|_K\|_{L^p} \leqslant \|s\|_{L^r} \|A\|_{L^q} .$$

It follows that for $1 \leqslant p < q$, given a self-adjoint endomorphism $s \in L^r(\mathrm{End}(\mathcal{E}))$, the construction $A \mapsto \Phi(s)(A)$ describes a bounded linear operator

$$\Phi(s) : L^q(\mathrm{End}(\mathcal{E})) \to L^p(\mathrm{End}(\mathcal{E}))$$

whose norm satisfies

$$\|\Phi(s)\| \leqslant C \|s\|_{L^r} .$$

In this way, we may think of Φ as a mapping

$$\Phi : L^r(\mathrm{End}(\mathcal{E}, K)) \to \mathrm{Hom}(L^q(\mathrm{End}(\mathcal{E})), L^p(\mathrm{End}(\mathcal{E}))).$$

Moreover, it also follows that if s_k is a sequence that converges in the L^r-norm to s_∞, then the sequence $\Phi(s_k)$ of operators converges in the operator norm to $\Phi(s_\infty)$. We summarize in the following proposition.

Proposition 4.36. *For $1 \leqslant p < q$, the construction Φ describes a continuous mapping*

$$\Phi : L^r(\mathrm{End}(\mathcal{E}, K)) \to \mathrm{Hom}(L^q(\mathrm{End}(\mathcal{E})), L^p(\mathrm{End}(\mathcal{E}))),$$

where r is the number satisfying $1/p = 1/q + 1/r$.

Proof of Lemma 4.28. Assuming the properness condition in Definition 4.21 is violated, we will construct explicitly a weakly holomorphic subbundle that is destabilizing.

For a solution H_t to the heat flow with initial condition $H_0 = K$, let us write $H_t = Ke^{s_t}$ for a path of self-adjoint endomorphisms s_t with initial condition $s_0 = 0$.

We first claim that we have $\int_{\mathcal{X}} \mathrm{Tr}(s_t)\omega^n = 0$ along the path $t \mapsto s_t$. Indeed, the heat equation (4.2) implies that

$$\dot{s}_t = -\frac{i}{2}(\Lambda F_t - \lambda I_{\mathcal{E}}).$$

Upon taking the trace and integrating over $\mathcal{X}$, we find that the right-hand side vanishes, and so the quantity $\int_{\mathcal{X}} \mathrm{Tr}(s_t)\omega^n$ must be constant. The initial condition $s_0 = 0$ implies that this constant must be zero, as desired.

Now assume the properness condition of Definition 4.21 is violated. By Lemma 4.31, we can find s_t contradicting the estimate with $\||s_t|_K^2\|_{L^2}$ arbitrarily large, or else the resulting bound on the C^0-norm would make the estimate of Definition 4.21 hold trivially after adjusting C_1. We thus have a sequence of times t_k and corresponding self-adjoint endomorphisms s_k whose L^2-norms $\||s_k|^2\|_{L^2}$ tend to ∞ and which satisfy

$$\left\||s_k|^2\right\|_{L^2}^{1/2} \geqslant kM_K(Ke^{s_k}). \tag{4.10}$$

Let us define a sequence of normalized endomorphisms $u_k = \ell_k^{-1}s_k$, where ℓ_k is the number

$$\ell_k = \left\||s_k|^2\right\|_{L^2}^{1/2}.$$

Note that the u_k are indeed normalized in the sense that $\||u_k|^2\|_{L^2}^{1/2} = 1$. The uniform estimate of Lemma 4.31 implies that

$$\ell_k \sup_{\mathcal{X}} |u_k| \leqslant C_1 + C_2\ell_k \left\||u_k|^2\right\|_{L^2}^{1/2},$$

and so we obtain a uniform C^0-bound on the sequence u_k.

We now prove the following useful lemma.

Lemma 4.37. *After passing to a subsequence, we may assume that the sequence u_k converges in u_∞ weakly in L^2_1. If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive smooth function satisfying $\Phi(u, v) < (u - v)^{-1}$ whenever $u > v$, then*

$$i \int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K) \frac{\omega^n}{n!} + \int_{\mathcal{X}} \langle \Phi(u_\infty)(\bar{\partial} u_\infty), \bar{\partial} u_\infty \rangle_K \frac{\omega^n}{n!} \leqslant 0,$$

where $\Phi(u_\infty)(\bar{\partial} u_\infty)$ is the endomormorphism of $\mathcal{E}$ constructed as in Notation 4.34.

Proof of Lemma 4.37. Condition (4.10) can be written as

$$2i\ell_k \int_{\mathcal{X}} \mathrm{Tr}(u_k \Lambda F_K) \frac{\omega^n}{n!} + 2\ell_k^2 \int_{\mathcal{X}} \langle \Psi(\ell_k u_k)(\bar{\partial} u_k), \bar{\partial} u_k \rangle_K \frac{\omega^n}{n!} \leqslant \frac{1}{k} \ell_k.$$

As $\ell \to \infty$, the expression

$$\ell \Psi(\ell u, \ell v) = \frac{\ell e^{\ell(v-u)} - \ell^2 (v-u) - \ell}{\ell^2 (v-u)^2}$$

increases monotonically to $(u - v)^{-1}$ for $u > v$ and to ∞ for $u \leqslant v$.

Fix Φ as in the statement of the lemma. Because the construction $\Phi(u_k)$ depends only on the eigenvalues of u_k and these are bounded uniformly in k (by the C^0-bound on the sequence), we may assume that Φ is compactly supported. Then the assumption on Φ guarantees that $\Phi(u, v) < \ell \Psi(\ell u, \ell v)$ for ℓ sufficiently large. It follows from the previous paragraph that for k sufficiently large, we have

$$i \int_{\mathcal{X}} \mathrm{Tr}(u_k \Lambda F_K) \frac{\omega^n}{n!} + \int_{\mathcal{X}} \langle \Phi(u_k)(\bar{\partial} u_k), \bar{\partial} u_k \rangle_K \frac{\omega^n}{n!} \leqslant \frac{1}{2k}. \tag{4.11}$$

The C^0-bound on the sequence u_k implies that the operator norms of $\Phi(u_k)$ are bounded uniformly, and hence we obtain from this inequality a uniform bound on $\left\|\bar{\partial} u_k\right\|_{L^2}$. Therefore we may choose a subsequence so that $u_k \to u_\infty$ weakly in L_1^2 (and strongly in L^2).

Moreover, the fact that we have a uniform C^0-bound on the sequence u_k implies that the sequence u_k converges to u_∞ in L^r for any $r > 2$. Indeed, let us write b for a uniform C^0-bound for the sequence u_k. Then we compute that

$$
\begin{aligned}
\|u_k - u_j\|_{L^r}^r &= \int_{\mathcal{X}} |u_k - u_j|_K^r \frac{\omega^n}{n!} \\
&= \int_{\mathcal{X}} |u_k - u_j|_K^{r-2} |u_k - u_j|_K^2 \frac{\omega^n}{n!} \\
&\leqslant (2b)^{r-2} \int_{\mathcal{X}} |u_k - u_j|_K^2 \frac{\omega^n}{n!} \\
&= (2b)^{r-2} \|u_k - u_j\|_{L^2}^2 .
\end{aligned}
$$

This estimate implies that if the sequence u_k is Cauchy in L^2 then it is also Cauchy in L^r for $r > 2$.

The proof of this lemma would be complete if we knew we could take a limit of the inequality (4.11) as $k \to \infty$. We can do so for the following reasons. Let $\epsilon > 0$ be arbitrary. Notice that

$$
\left\|\Phi^{1/2}(u_k)(\bar{\partial} u_k)\right\|_{L^2}^2 = \int_{\mathcal{X}} \langle \Phi(u_k)(\bar{\partial} u_k), \bar{\partial} u_k \rangle_K \frac{\omega^n}{n!}.
$$

The mapping

$$
u \mapsto i \int_{\mathcal{X}} \mathrm{Tr}(u \Lambda F_K) \frac{\omega^n}{n!}
$$

is continuous for $u \in L^2$, so the inequality (4.11) implies that for k sufficiently large we have

$$
i \int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K) \frac{\omega^n}{n!} + \left\|\Phi^{1/2}(u_k)(\bar{\partial} u_k)\right\|_{L^2}^2 \leqslant \epsilon.
$$

This estimate implies in particular that the sequence of numbers $\left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^2}^2$ is bounded uniformly. Let p be a number satisfying $1 < p < 2$, and let r be the positive number such that $1/p = 1/2 + 1/r$. The inequality $1 < p$ implies that $r > 2$. By the previous paragraph, because $r > 2$, the sequence u_k converges in L^r. It follows from Proposition 4.36 that the sequence of operators $\Phi^{1/2}(u_k)$ converges to $\Phi^{1/2}(u_\infty)$ in the space $\mathrm{Hom}(L^2(\mathrm{End}(\mathcal{E})), L^p(\mathrm{End}(\mathcal{E}))$. The sequence $\bar{\partial}u_k$ is bounded in L^2, so we may appeal to Proposition 4.36 to find that $\Phi^{1/2}(u_k)(\bar{\partial}u_j) \to \Phi^{1/2}(u_\infty)(\bar{\partial}u_j)$ for fixed j as $k \to \infty$ in L^p. This means that for k sufficiently large we have

$$\left\|\Phi^{1/2}(u_\infty)(\bar{\partial}u_j)\right\|_{L^p}^2 \leqslant \left\|\Phi^{1/2}(u_k)(\bar{\partial}u_j)\right\|_{L^p}^2 + \epsilon,$$

where this estimate is independent of j because the sequence $\bar{\partial}u_j$ is bounded uniformly in L^2. In addition, the sequence $\Phi^{1/2}(u_\infty)(\bar{\partial}u_j)$ converges to $\Phi^{1/2}(u_\infty)(\bar{\partial}u_\infty)$ weakly in L^p, so by the lower semicontinuity of the norm, we find that for j, k sufficiently large, we have

$$\begin{aligned}\left\|\Phi^{1/2}(u_\infty)(\bar{\partial}u_\infty)\right\|_{L^p}^2 &\leqslant \left\|\Phi^{1/2}(u_\infty)(\bar{\partial}u_j)\right\|_{L^p}^p + \epsilon \\ &\leqslant \left\|\Phi^{1/p}(u_k)(\bar{\partial}u_j)\right\|_{L^p}^2 + 2\epsilon.\end{aligned}$$

Moreover, we have an estimate of the form

$$\|f\|_{L^p} \leqslant (\mathrm{vol}(\mathcal{X}))^{1/r} \|f\|_{L^2}$$

for $f \in L^2$. As $p \to 2$, we have $r \to \infty$, so by choosing p sufficiently close to 2, we may ensure that $(\mathrm{vol}(\mathcal{X}))^{1/r}$ is sufficiently close to 1. Because the sequence of numbers $\left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^2}^2$ is bounded uniformly, we may now ensure that by taking p close enough to 2 that we have a uniform estimate of the form

$$\left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^p}^2 \leqslant \left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^2}^2 + \epsilon.$$

Collecting all of the above, what we have shown therefore is that for k sufficiently large and for p sufficiently close to 2, we have

$$\begin{aligned}
& i\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!} + \left\|\Phi^{1/2}(u_\infty)(\bar{\partial}u_\infty)\right\|_{L^p}^2 \\
& \leqslant i\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!} + \left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^p}^2 + 2\epsilon \\
& \leqslant i\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!} + \left\|\Phi^{1/2}(u_k)(\bar{\partial}u_k)\right\|_{L^2}^2 + 3\epsilon \\
& \leqslant 4\epsilon.
\end{aligned}$$

If a measurable function satisfies an inequality involving the L^p-norm uniformly for $p < 2$, then it satisfies the same inequality involving the L^2-norm. Because $\epsilon > 0$ was arbitrary, the lemma now follows. □

We also claim the limit u_∞ is nontrivial. Indeed because the sequence u_k converges to u_∞ in L^2, we find that the sequence also converges in L^4 by the ideas in the proof of the lemma. But we have $\|u_k\|_{L^4}^4 = \||u_k|_K^2\|_{L^2}^2 = 1$. We therefore find that $\|u_\infty\|_{L^4} = 1$, and u_∞ is nontrivial.

With the lemma, it is possible to see that the eigenvalues of u_∞ are constant. To demonstrate this, we argue that if $\varphi : \mathbb{R} \to \mathbb{R}$ is any smooth function, then the function $\mathrm{Tr}(\varphi(u_\infty))$ is constant. (Here we are using Notation 4.34.) To prove that this function is constant, we will consider its derivative $\bar{\partial}\mathrm{Tr}(\varphi(u_\infty))$. If $d\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the difference quotient of φ as in Lemma 4.35, then we have $\bar{\partial}\mathrm{Tr}(\varphi(u_\infty)) = \mathrm{Tr}(d\varphi(u_\infty)(\bar{\partial}u_\infty))$. Let N be a large number. Choose $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which agrees with $d\varphi$ along the diagonal in the sense that $\Phi(u, u) = d\varphi(u, u) = \varphi'(u)$ and also ensure Φ satisfies

$$N\Phi^2(u, v) < (u - v)^{-1}$$

for $u < v$. Then by Lemma 4.35, we have

$$\bar{\partial}\mathrm{Tr}(\varphi(u_\infty)) = \mathrm{Tr}(d\varphi(u_\infty)(\bar{\partial}u_\infty)) = \mathrm{Tr}(\Phi(u_\infty)(\bar{\partial}u_\infty)),$$

and using Lemma 4.37, we find that

$$i\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!} + N\int_{\mathcal{X}} \langle \Phi^2(u_\infty)(\bar{\partial}u_\infty), \bar{\partial}u_\infty\rangle_K \frac{\omega^n}{n!} \leqslant 0,$$

that is,

$$\int_{\mathcal{X}} |\Phi(u_\infty)(\bar{\partial}u_\infty)|_K^2 \frac{\omega^n}{n!} \leqslant -\frac{i}{N}\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!}.$$

The Schwarz inequality implies that

$$\mathrm{Tr}(\Phi(u_\infty)(\bar{\partial}u_\infty)) = \langle \Phi(u_\infty)(\bar{\partial}u_\infty), I_{\mathcal{E}}\rangle_K \leqslant r^2|\Phi(u_\infty)(\bar{\partial}u_\infty)|_K^2,$$

from which we obtain

$$\left\|\bar{\partial}\mathrm{Tr}(\varphi(u_\infty))\right\|_{L^1} = \left\|\mathrm{Tr}(\Phi(u_\infty)(\bar{\partial}u_\infty))\right\|_{L^1} \leqslant -\frac{r^2 i}{N}\int_{\mathcal{X}} \mathrm{Tr}(u_\infty \Lambda F_K)\frac{\omega^n}{n!}.$$

We conclude that

$$\left\|\bar{\partial}\mathrm{Tr}(\varphi(u_\infty))\right\|_{L^1} \leqslant \frac{C}{N}.$$

The fact that N was arbitrary implies that $\bar{\partial}\mathrm{Tr}(\varphi(u_\infty)) = 0$. Because the function $\mathrm{Tr}(\varphi(u_\infty))$ is real, we conclude that it must be a constant, as desired.

If $\nu_1 \leqslant \cdots \leqslant \nu_r$ denote the eigenvalues of u_∞ (which are constant almost everywhere), then we claim that not all ν_α are equal. Indeed because each s_k satisfies $\int_{\mathcal{X}} \mathrm{Tr}(s_k)\omega^n = 0$, we find also that $\int_{\mathcal{X}} \mathrm{Tr}(u_k)\omega^n = 0$, and hence we have $\int_{\mathcal{X}} \mathrm{Tr}(u_\infty)\omega^n = 0$ as well. But u_∞ is nontrivial, and so at least one eigenvalue must be nonzero.

It follows that the eigenspaces of u_∞ give rise to a nontrivial flag of L^2_1-subbundles of $\mathcal{E}$ which we denote by

$$0 \subset \pi_1 \subset \cdots \subset \pi_r = I_{\mathcal{E}},$$

where π_α denotes projection onto the sum of the first α eigenspaces of u_∞. Note that by construction the π_α are self-adjoint with respect to K and satisfy $\pi_\alpha^2 = \pi_\alpha$.

We claim that each π_α represents a weakly holomorphic subbundle of $\mathcal{E}$ in the sense of Definition 4.26. For this, it remains only to check that $(I_{\mathcal{E}} - \pi_\alpha)\bar{\partial}\pi_\alpha = 0$. We will use Notation 4.34 to write π_α as $\pi_\alpha = p_\alpha(u_\infty)$ where $p_\alpha : \mathbb{R} \to \mathbb{R}$ is a smooth real-valued function satisfying

$$p_\alpha(\nu_\beta) = \begin{cases} 1 & \beta \leqslant \alpha \\ 0 & \beta > \alpha \end{cases},$$

from which it follows that $\bar{\partial}\pi_\alpha = dp_\alpha(u_\infty)(\bar{\partial}u_\infty)$ by Lemma 4.35.

If we set $\Phi_\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to be

$$\Phi_\alpha(u, v) = (1 - p_\alpha)(v)dp_\alpha(u, v)$$

where 1 denotes the constant 1 function, then then we claim that

$$(I_{\mathcal{E}} - \pi_\alpha)\bar{\partial}\pi_\alpha = \Phi_\alpha(u_\infty)(\bar{\partial}u_\infty). \tag{4.12}$$

Indeed let e_β be a unitary basis for $\mathcal{E}$ with respect to which a local expression for u_∞ is

$$u_\infty = \sum_\beta \nu_\beta e^\beta \otimes e_\beta.$$

Reasoning in Lemma 4.35 shows that

$$(\bar{\partial} u_\infty)^\gamma_\beta = \begin{cases} (\nu_\beta - \nu_\gamma)\theta^\gamma_\beta & \beta \neq \gamma \\ 0 & \beta = \gamma \end{cases}$$

where θ^γ_β is the matrix of $\bar{\partial}$. We then compute that the coefficients of $\Phi_\alpha(u_\infty)(\bar{\partial} u_\infty)$ are given by

$$\begin{aligned}(\Phi_\alpha(u_\infty)(\bar{\partial} u_\infty))^\gamma_\beta &= \begin{cases} (1 - p_\alpha)(\nu_\gamma) dp_\alpha(\nu_\beta, \nu_\gamma)(\nu_\beta - \nu_\gamma)\theta^\gamma_\beta & \beta \neq \gamma \\ 0 & \beta = \gamma \end{cases} \\ &= \begin{cases} (1 - p_\alpha(\nu_\gamma))(p_\alpha(\nu_\beta) - p_\alpha(\nu_\gamma))\theta^\gamma_\beta & \beta \neq \gamma \\ 0 & \beta = \gamma \end{cases} \\ &= \begin{cases} (p_\alpha(\nu_\beta) - p_\alpha(\nu_\gamma))\theta^\gamma_\beta & \beta \neq \gamma, \gamma > \alpha \\ 0 & \beta = \gamma \text{ or } \gamma \leqslant \alpha \end{cases}.\end{aligned}$$

On the other hand, the previous paragraph implies that the coefficients of $\bar{\partial}\pi_\alpha$ are given by

$$(\bar{\partial}\pi_\alpha)^\mu_\beta = \begin{cases} (p_\alpha(\nu_\beta) - p_\alpha(\nu_\mu))\theta^\mu_\beta & \beta \neq \mu \\ 0 & \beta = \mu \end{cases}$$

and also the coefficients of $I_{\mathcal{E}} - \pi_\alpha$ are given by

$$(I_{\mathcal{E}} - \pi_\alpha)^\gamma_\mu = \begin{cases} \delta^\gamma_\mu & \gamma > \alpha \\ 0 & \gamma \leqslant \alpha \end{cases}.$$

The composition $(I_{\mathcal{E}} - \pi_\alpha)\bar{\partial}\pi_\alpha$ therefore has coefficients

$$\begin{aligned}((I_{\mathcal{E}} - \pi_\alpha)\bar{\partial}\pi_\alpha)^\gamma_\beta &= (I_{\mathcal{E}} - \pi_\alpha)^\gamma_\mu(\bar{\partial}\pi_\alpha)^\mu_\beta \\ &= \begin{cases} (\bar{\partial}\pi_\alpha)^\gamma_\beta & \gamma > \alpha \\ 0 & \gamma \leqslant \alpha \end{cases}.\end{aligned}$$

Comparing with the coefficients for $\Phi_\alpha(u_\infty)(\bar{\partial}u_\infty)$ we find the relation (4.12) is indeed true.

We next claim that for $\nu_\gamma > \nu_\beta$, we have $\Phi_\alpha(\nu_\gamma, \nu_\beta) = 0$. There are two possibilities for ν_β: either $\nu_\beta \leqslant \nu_\alpha$ or $\nu_\beta > \nu_\alpha$. If $\nu_\beta \leqslant \nu_\alpha$, then $p_\alpha(\nu_\beta) = 1$ and so $\Phi_\alpha(\nu_\gamma, \nu_\beta) = 0$ by definition. On the other hand, if $\nu_\beta > \nu_\alpha$, then for $\nu_\gamma > \nu_\beta \geqslant \nu_\alpha$, we have that each $p_\alpha(\nu_\gamma) = p_\alpha(\nu_\beta) = 0$, and so the difference quotient $dp(\nu_\gamma, \nu_\beta)$ vanishes. The claim now follows.

Because the eigenvalues of u_∞ are constant almost everywhere, the construction $\Phi_\alpha(u_\infty)$ depends only on the values of $\Phi_\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on the pairs of eigenvalues (ν_β, ν_γ). So by replacing Φ_α with Φ^N_α satisfying $\Phi^N_\alpha(\nu_\gamma, \nu_\beta) = \Phi_\alpha(\nu_\gamma, \nu_\beta)$ and

$$N(\Phi^N_\alpha)^2(u, v) < (u - v)^{-1} \quad \text{for } u > v,$$

then we find that still we have

$$(I_{\mathcal{E}} - \pi_\alpha)\bar{\partial}\pi_\alpha = \Phi_\alpha^N(u_\infty)(\bar{\partial}u_\infty),$$

but now we have guaranteed in addition that $\left\|\Phi_\alpha^N(u_\infty)(\bar{\partial}u_\infty)\right\|_{L^2}^2 \leqslant C/N$ by following the line of reasoning from earlier in the argument. Because N is arbitrary, we can conclude that $\Phi_\alpha^N(u_\infty)(\bar{\partial}u_\infty) = 0$. This therefore completes the proof that π_α is a weakly holomorphic subbundle.

We finally show that at least one of the π_α for $\alpha < r$ is destabilizing. In a telescoping manner we may write

$$u_\infty = \nu_r I_{\mathcal{E}} - \sum_{\alpha=1}^{r-1}(\nu_{\alpha+1} - \nu_\alpha)\pi_\alpha.$$

Then, according to Definition 4.26, the following sum of degrees is given by

$$\begin{aligned}
W &= \nu_r \deg(E) - \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha)\deg(\pi_\alpha) \\
&= \nu_r \frac{i}{2\pi}\int_{\mathcal{X}} \operatorname{Tr}(\Lambda F_K) - \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha)\left(\frac{i}{2\pi}\int_{\mathcal{X}} \operatorname{Tr}(\pi_\alpha \Lambda F_K) - \frac{1}{2\pi}\int_{\mathcal{X}} |\bar{\partial}\pi_\alpha|_K^2\right) \\
&= \frac{i}{2\pi}\int_{\mathcal{X}} \operatorname{Tr}(u_\infty \Lambda F_K) + \frac{1}{2\pi}\int_{\mathcal{X}} \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha)|\bar{\partial}\pi_\alpha|_K^2.
\end{aligned}$$

Because $\bar{\partial}\pi_\alpha = p_\alpha(u_\infty)(\bar{\partial}u_\infty)$, we obtain

$$W = \frac{i}{2\pi}\int_{\mathcal{X}} \operatorname{Tr}(u_\infty \Lambda F_K) + \frac{1}{2\pi}\int_{\mathcal{X}} \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha)\langle (dp_\alpha)^2(u_\infty)(\bar{\partial}u_\infty), \bar{\partial}u_\infty\rangle_K.$$

For fixed $\nu_\beta > \nu_\gamma$, if ν_α satisfies $\nu_\beta > \nu_\alpha \geqslant \nu_\gamma$, then $dp_\alpha(\nu_\beta, \nu_\gamma)^2 = (\nu_\beta - \nu_\gamma)^{-2}$, and vanishes otherwise. It follows that for $\nu_\beta > \nu_\gamma$, the (telescoping) sum satisfies

$$\sum_\alpha (\nu_{\alpha+1} - \nu_\alpha)(dp_\alpha)^2(\nu_\gamma, \nu_\beta) = \frac{\nu_\beta - \nu_\gamma}{(\nu_\beta - \nu_\gamma)^2} = \frac{1}{\nu_\beta - \nu_\gamma}.$$

Lemma 4.37 implies that $W \leqslant 0$, which means that

$$\nu_r \deg(\mathcal{E}) \leqslant \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha) \deg(\pi_\alpha). \tag{4.13}$$

On the other hand, the trace of u_∞ is zero, which means that

$$\nu_r \mathrm{rk}(\mathcal{E}) = \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha) \mathrm{Tr}(\pi_\alpha).$$

If each $\deg(\pi_\alpha)$ satisfied $\deg(\pi_\alpha) < \mathrm{Tr}(\pi_\alpha)(\deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}))$, then we would have

$$\sum_\alpha (\nu_{\alpha+1} - \nu_\alpha) \deg(\pi_\alpha) < \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \sum_\alpha (\nu_{\alpha+1} - \nu_\alpha) \mathrm{Tr}(\pi_\alpha) = \nu_r \deg(\mathcal{E}),$$

which contradicts (4.13). It follows that at least one π_α has $\mu(\pi_\alpha) \geqslant \mu(\mathcal{E})$.

This completes the proof of Lemma 4.28. $\square$

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