On the method of auxiliary Monge-Ampère equations in complex geometry

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2025

Abstract

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A priori estimates, especially the L^∞ estimates, are the core to solving partial differential equations (PDE) arising from complex geometry. It has been a longstanding question whether one can find a PDE method to prove a L^∞ estimate of the complex Monge-Ampère equation with a singular right-hand side. Recently, the method of auxiliary Monge-Ampère equations developed by B. Guo, D. H. Phong, and F. Tong has emerged to be a major advance regarding this question, with immense generality. We use this method to prove the L^∞ estimate for complex Monge-Ampère and Hessian questions on a nef cohomology class, and for the $(n-1)$-form equations on Hermitian manifolds. We also prove a sharp uniform modulus of continuity estimate for complex Monge-Ampère equations, together with its geometric application on diameter bounds for Kähler metrics.

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Acknowledgements

I am deeply grateful to my Ph.D. advisor, Professor Duong Hong Phong, for his unwavering guidance and support over the years. Not only is he a wonderful teacher of mathematics, but he has always inspired me personally to be diligent and noble, *labor omnia vincit*.

I am indebted to Professor Simon Brendle for his generous support, and to Bin Guo, Freid Tong, Xi Sisi Shen, and Donovan McFeron for their unreserved guidance and invaluable advice. I have also greatly benefited from many helpful conversations with Shuang Liang, Nikita Klemyatin, and Kevin Smith.

I would like to thank Professor Ngaiming Mok for introducing me to the field of complex geometry. His pursuit of *Wissenschaft* has always been a source of motivation for me. I am also grateful to Professors Zheng Hua and Jianghua Lu, who opened the door to advanced mathematics for me.

I am fortunate to have shared my journey with many friends, whose companionship and encouragement sustained me—especially Ji Zeng, Jiang Qian, and Yi Shang. My thanks also go to teachers and friends outside mathematics for their intellectual influence, notably Daniel Fleming and Hui Qin.

Above all, I thank my family for their enduring love and support. In particular, I remember my late maternal grandmother Xianyu Meng, who nurtured my love of reading from my earliest days. Without them, I would not be where I am today.

Chuwen Wang

Chapter 3 contains material from "On L^∞ estimates for Monge-Ampère and Hessian equations on nef classes" that appears in *Analysis & PDE*, vol. 17, no. 2 (2024), pp. 749–756, joint with B. Guo, D. H. Phong and F. Tong.

Chapter 4 contains material from "On the modulus of continuity of solutions to complex Monge-Ampère equations" which is submitted for publication and available on arXiv, joint with B. Guo, D. H. Phong, and F. Tong.

Chapter 5 contains material from "On uniform estimates for $(n-1)$-form fully nonlinear partial differential equations on compact Hermitian manifolds" which is available on arXiv, joint with N. Klemyatin and S. Liang.

Dedication

To the suffering yet resilient ethnoreligious minorities in Iraq, Syria, and Turkey. May the land of Mesopotamia flourish in peace once more.

Chapter 1: Introduction and Background

1.1 Overview

Solutions to geometric partial differential equations (PDE) yield canonical geometric structures, which is a central topic in differential geometry. The key to solving such equations lies in deriving *a priori* estimates.

In the seminal work of Yau [46], he proved the Calabi conjecture, showing that any compact Kähler manifold X with first Chern class $c_1(X) = 0$ admits a Ricci-flat Kähler metric by solving the following Monge-Ampère equation

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n \tag{1.1}$$

The key to his proof was *a priori* estimates, especially L^∞ estimates. Yau used Moser iteration to prove the L^∞ estimates, which work not only for a smooth right-hand side, but also for $e^F \in L^p$ when $p > n$. The next major advance was by Kołodziej [31], who established the L^∞ estimates when the right-hand side e^F is in some Orlicz space, in particular, L^q for $q > 1$, which has profound geometric implications [35, 14, 12, 38, 41]. His proof relied heavily on pluripotential theory developed by Bedford and Taylor [1, 2]. Later on Demailly and Pali [12] and Eyssidieux, Guedj, and Zeriahi [14] generalized Kołodziej's result to degenerating background metrics, where pluripotential theory continues to be essential. Boucksom, Eyssidieux, Guedj, and Zeriahi also generalized the estimates to big cohomology classes using pluripotential theory [6].

It has been a longstanding question whether one can find partial differential equation methods to prove the above estimates. PDE methods would be generally more flexible and broadly applicable than pluripotential theory, which is tailored to Monge–Ampère equations. Błocki [5] obtained the L^∞ estimates for L^p right hand side with $p > 2$ using Alexandrov-Bakelman-Pucci (ABP) maximum principle. However, his method cannot be extended to degenerating background metrics or the full range of p. On the other hand, Wang, Wang, and Zhou [45] proved the L^∞ estimates for domains in $\mathbb{C}^n$, by deriving a Trudinger-type inequality, but their method cannot be adoptable to compact manifolds.

In their work on constant scalar curvature Kähler metrics, Chen and Cheng [8] used auxiliary Monge-Ampère equations to obtain entropy estimates. Comparing solutions to solutions of auxiliary equations has been a powerful tool since De Giorgi [17], and was used notably in the proof of Hölder continuity of solutions to Monge-Ampère equations by Dinew and Kołodziej [11, 13]. Complex Monge-Ampère equations are in particular suitable as candidates for auxiliary equations, since the existence and smoothness of the solution has been established by Yau [46] in the compact case and by Caffarelli, Kohn, Nirenberg, and Spruck [7] in the case of the Dirichlet problem.

Recently, by combining ideas from [8, 45], Guo, Phong, and Tong [21] proposed a method using auxiliary Monge-Ampère equations to derive L^∞ estimates, which has been proved to have immense generality and applications [21, 22, 25, 24, 23, 20, 19]. In Chapter 2, we will give a basic introduction to this method as in Guo, Phong, and Tong's work [21]. In Chapters 3-5, we will discuss the main results of this paper, the application of the auxiliary Monge-Ampère equation method to the case of nef cohomology classes on Monge-Ampère and Hessian equations (joint work with B. Guo, D.H. Phong, and F. Tong), to the case of modulus of continuity of Monge-Ampère equations (joint work with B. Guo, D.H. Phong, and F. Tong), and to the case of $(n-1)$-form equations on Hermitian manifolds (joint work with N. Klemyatin and S. Liang).

1.2 Preliminaries

1.2.1 Equations on Kähler manifolds

Let (X, ω) be a compact Kähler manifold of complex dimension n without boundary. If φ is a real smooth function on X, we denote $\omega_\varphi = \omega + i\partial\bar\partial\varphi$, the classical complex Monge-Ampère equation takes the form

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n \tag{1.2}$$

where F is a smooth function on X satisfying the integrability condition $\int_X e^F \omega^n = \int_X \omega^n$, and the solution $\varphi \in PSH(X, \omega)$ can be normalized, say $\sup_X \varphi = 0$. With this setup, Yau's theorem [46] tells us the equation admits a unique smooth solution $\varphi \in PSH(X, \omega) \cap C^\infty(X) = \{u \in C^\infty(X) | \omega + i\partial\bar{\partial}u > 0\}$.

We can also consider a family of equations. Let χ be a closed (1,1)-form on X, say semipositive. For $t \in (0, 1]$, denote $\omega_t = \chi + t\omega$ as the degenerating family of background metrics. Then we look for solutions of the following family of Monge-Ampère equations

$$(\omega_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0 \tag{1.3}$$

where F is a smooth function on X with $\int_X e^F \omega^n = \int_X \omega^n$, $c_t = \frac{\int_X \omega_t^n}{\int_X \omega^n}$ is the volume normalizing constant, and φ_t is plurisubharmonic for each t.

The complex Monge-Ampère equations can be generalized to complex Hessian equations. Let $\sigma_j(\lambda)$ denote the j-th elementary symmetric polynomial for $\lambda \in \mathbb{R}^n$, for example, $\sigma_1(\lambda) = \lambda_1 + \lambda_2 + ... + \lambda_n, \sigma_2(\lambda) = \sum_{1 \leq j < l \leq n} \lambda_j \lambda_l$, and $\sigma_n(\lambda) = \lambda_1 \lambda_2 ... \lambda_n$.

Fix an integer k with $1 \leq k \leq n$, the k-th Garding cone Γ_k is a convex cone in $\mathbb{R}^n$ defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, ...\sigma_k(\lambda) > 0\} \tag{1.4}$$

For example, $\Gamma_1 = \{\lambda \in \mathbb{R}^n | \lambda_1 + ... + \lambda_n > 0\}$ is a half-space, and

$$\Gamma_n = \{\lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, ...\sigma_n(\lambda) > 0\} = \{\lambda \in \mathbb{R}^n | \lambda_1 > 0, ...\lambda_n > 0\}$$

is the cone of the first octant.

A C^2 function φ on X is said to be k-subharmonic with respect to the metric ω if

$$(\omega + i\partial\bar{\partial}\varphi)^j \wedge \omega^{n-j}$$

is positive for all j such that $1 \leq j \leq k$. The σ_k equation is of the form

$$(\omega + i\partial\bar{\partial}\varphi)^k \wedge \omega^{n-k} = e^F \omega^n, \quad \sup_X \varphi = 0 \tag{1.5}$$

where F is a smooth function on X satisfying the integrability condition $\int_X \omega^n = \int_X e^F \omega^n$, and $\varphi \in SH_k(X, \omega)$ is k-subharmonic with respect to ω. Dinew and Kołodziej [13] proved the L^∞ estimate for complex Hessian equations. Note that when $k = n$, this is our usual Monge-Ampère equation.

We can generalize the previous two cases to a broad class of equations. Let $h_\varphi : TX \to TX$ be the endomorphism defined by ω_φ relative to ω, i.e. in local coordinates, if we write $\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k, \omega_\varphi = \omega + i\partial\bar{\partial}\varphi = i\tilde{g}_{\bar{k}j} dz^j \wedge d\bar{z}^k$, then $(h_\varphi)^j{}_i = g^{j\bar{k}} \tilde{g}_{\bar{k}i}$, where $(g^{i\bar{j}})$ denotes the inverse of $g_{\bar{k}j}$. We denote by $\lambda[h_\varphi]$ the (unordered) vector of eigenvalues of h_φ. Note that $\lambda[h_\varphi] \in \Gamma_n$ is equivalent to the condition that φ is ω-plurisubharmonic, $(\omega + i\partial\bar{\partial}\varphi)^n \geq 0$, and $\lambda[h_\varphi] \in \Gamma_k$ is equivalent to the condition that φ is k-subharmonic.

Given a smooth function F on X, we can consider the fully nonlinear partial differential equation

$$f(\lambda[h_\varphi]) = e^F, \tag{1.6}$$

with $\sup_X \varphi = 0$ and $\lambda[h_\varphi] \in \Gamma$ on X. F is normalized so that $\int_X e^{nF} \omega^n = \int_X \omega^n$.

Naturally, we need assumptions on f for the equation to admit a solution, or even just the L^∞ estimates. Throughout the paper, we will make the following three assumptions on f,

- $f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}_+$, where Γ is a symmetric cone satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. We also require $f(\lambda)$ to be symmetric in $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma$ and it is homogeneous of degree one, i.e., $f(r\lambda) = rf(\lambda)$ for $r > 0$ and $\lambda \in \Gamma$.

- f is elliptic, in the sense that $\frac{\partial f}{\partial \lambda_j} > 0$ for each $j = 1, \ldots, n$ and $\lambda \in \Gamma$.

- f satisfies the structure condition, i.e., there is a $\gamma > 0$ such that

$$\prod_{j=1}^{n} \frac{\partial f}{\partial \lambda_j} \geq \gamma, \quad \forall \lambda \in \Gamma. \tag{1.7}$$

Note that if we take $f(\lambda[h_\varphi]) = \left(\frac{\omega_\varphi^n}{\omega^n}\right)^{1/n} = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}, \Gamma = \Gamma_n$, this becomes the complex Monge-Ampère equation, and if we take $f(\lambda[h_\varphi]) = \left(\frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n}\right)^{1/k} = \sigma_k(\lambda)^{\frac{1}{k}}, \Gamma = \Gamma_k$, this is the complex Hessian equation. There are also a broad class of equations satisfying these conditions, for example the p-Monge-Ampère equation of Harvey and Lawson [26, 27], with

$$f(\lambda) = \Big(\prod_I \lambda_I\Big)^{\frac{n!}{(n-p)!p!}}$$

where I runs over all distinct multi-indices $1 \leq i_1 < \cdots < i_p \leq n$, $\lambda_I = \lambda_{i_1} + \cdots + \lambda_{i_p}$, and Γ is the cone defined by $\lambda_I > 0$ for all p-indices I.

The p-th Nash entropy of a function F is define to be

$$Ent_p(F) = \|e^F\|_{L^1(\log L)^p} := \int_X e^F |F|^p \omega^n, \quad p \geq 1$$

This is often referred to as the $L^1(\log L)^p$ norm of e^F, although technically this is not a norm but rather describes functions in some Orlicz space. Note that $L^1(\log L)^p \subset L^1(\log L)^q$ for $p \geq q \geq 1$.

In [21], Guo, Phong, and Tong proved the following L^∞ estimates: For solutions to equation 1.6 satisfying the three conditions 1.2.1, if we fix $p > n$, then for any solution $\varphi \in C^2(X)$ there is a constant C depends only on $n, p, \omega, Ent_p(F)$ and γ, so that $\sup_X \varphi \leq C$.

This result generalizes well to the degenerating family case. If we consider the family of equations

$$f(\lambda[h_{t,\varphi}]) = c_t e^{F_t}, \quad \sup_X \varphi_t = 0, \quad \lambda[h_{t,\varphi}] \in \Gamma, \quad t \in (0,1] \tag{1.8}$$

where F_t is normalized $\int_X e^{nF_t}\omega^n = \int_X \omega^n$, the volume $V_t = \int_X \omega_t^n = \int_X (\chi + t\omega)^n$, then if we fix $p > n$, there is a constant C such that $\sup_X \varphi_t \leq C$ for all $t \in (0,1]$, and C depends only on $n, p, \chi, \gamma, \omega$, and the upper bounds for $\frac{c_t}{V_t}$, $Ent_p(F_t) = \frac{1}{V_t}\int_X e^{nF_t}|F_t|^p\omega^n$ and the energy $E_t(\varphi_t) = \frac{c_t^n}{V_t}\int_X(-\varphi_t)e^{F_t}\omega^n$.

In Chapter 2 we will discuss the proof of the Monge-Ampère case as in Guo, Phong, and Tong [21]. The case for the general class of equations will be discussed in Chapter 5 when we deal with $(n-1)$-form equations on Hermitian manifolds.

1.2.2 Nef and big cohomology classes

Let (X, ω) be a compact Kähler manifold of dimension n, χ a closed $(1,1)$ form on X satisfying the normalization $\int_X \omega^n = 1$.

Definition 1.1 *The class of χ is called nef if the class $\{\hat{\omega}_t = \chi + t\omega\}$ is Kähler for any $t > 0$, i.e., if it lies in the closure of the Kähler cone. Equivalently, $[\chi]$ is nef if for all $\varepsilon > 0$, there exists a smooth closed form $\theta_\varepsilon \in [\chi]$ such that $\theta_\varepsilon \geq -\varepsilon\omega$.*

If we can set $\varepsilon = 0$ in the above definition, $[\chi]$ is called semipositive, in which case it can be represented by a semipositive form θ.

We remark that when X is projective and $\{D\}$ is a divisor class, this agrees with the nefness in algebraic geometry. In this section, we will assume the form χ is nef. Let $\nu \in \{0, 1, ..., n\}$ be the numerical dimension, i.e.,

$$\nu = max\{k|[\chi]^k \neq 0 \in H^{k,k}(X, \mathbb{C})\} \tag{1.9}$$

Definition 1.2 *A cohomology class $[\chi]$ is called big if it contains a strictly positive $(1,1)$-current. For a nef class $[\chi]$ this is equivalent to the numerical dimension $\nu = n$.*

1.3 Main results

We will now present the main results of this paper.

1.3.1 Equation on nef classes

In Chapter 3, we will discuss complex Monge-Ampère and Hessian equations, cf. [25]. For a nef class $[\chi]$ on X, consider $\hat{\omega}_t = \chi + t\omega$ for $t \in (0,1]$, $\hat{\omega}_t$ might not be positive, but its class is Kähler . Therefore we consider the degenerate family of complex Monge-Ampère equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0 \tag{1.10}$$

where F is a smooth function satisfying $\int_X e^F \omega^n = \int_X \omega^n$ and $c_t = [\hat{\omega}_t]^n = O(t^{n-\nu})$ a volume normalizing constant. Define the upper envelope $V_t = \sup\{v \mid v \in PSH(X, \hat{\omega}_t),\ v \leq 0\}$, we have the following L^∞ estimates.

Theorem 1.1 *Consider equation (1.10). Fix $p > n$, there is a uniform constant C depending only on n, p, ω, χ, and $\|e^F\|_{L^1(\log L)^p}$, such that for all $t \in (0,1]$, we have*

$$0 \leq -\varphi_t + V_t \leq C$$

This gives a PDE-based proof of an earlier result obtained by Boucksom, Eyssidieux, Guedj, and Zeriahi [6] and Fu, Guo, and Song [16]. A major advantage of a PDE proof is that it can be readily adjusted for a general class of equations. We can use this method to obtain the L^∞ estimates for complex Hessian equations stated below, which is a new result.

Consider the σ_k-equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^k \wedge \omega^{n-k} = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0. \tag{1.11}$$

Define the corresponding envelope

$$\tilde{V}_{t,k} = \sup\{v \mid v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2,\ v \leq 0\}$$

where $v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2$ means that the eigenvalue vector of the linear transformation

$\omega^{-1} \cdot (\hat{\omega}_t + i\partial\bar{\partial}v)$ lies in the Γ_k-cone.

Theorem 1.2 *Let φ_t be the solution to 1.11, then there exists a constant depending on $\|e^{\frac{n}{k}F}\|_{L^1(\log L)^p}$, $\frac{c_t}{[\hat{\omega}_t]^n}$, $p > n$ and $\bar{E}_t$ such that*

$$0 \le -\varphi_t + \tilde{V}_{t,k} \le C.$$

where $\bar{E}_t$ be an upper bound of $E_t(\varphi_t) = \int_X (-\varphi_t + \tilde{V}_{t,k}) e^{nF/k}\omega^n$

1.3.2 Uniform modulus of continuity for Monge-Ampère equations

In Chapter 4, we will discuss the modulus of continuity for Monge-Ampère equations and its geometric application, cf. [24]. Let (X, ω) be a compact Kähler manifold and consider the complex Monge-Ampère equation

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F\omega^n,\ \omega + i\partial\bar{\partial}\varphi > 0,\ \inf_X \varphi = 1 \tag{1.12}$$

where $\int_X e^F\omega^n = \int_X \omega^n$ satisfies the compatibility condition. If the right-hand side is in some Orlicz space, Kołodziej [31] showed that the solution must be in L^∞ and continuous. If we assume further e^F is in L^q for some $q > 1$, Kołodziej and Demailly et al. showed that the solution must be Hölder continuous [33, 11]. When it is not in L^q for $q > 1$, there are examples that the solution fails to be Hölder continuous. However, we are still able to derive a uniform estimate and show that the solution is continuous with order $O(|\log d|^{-\alpha})$.

Theorem 1.3 *Consider equation 1.12. Fix $p > n$. There exists a constant $C > 0$ depending on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ such that the following uniform estimate holds for any $x, y \in X$*

$$|\varphi(x) - \varphi(y)| \le \frac{C}{|\log d(x, y)|^\alpha} \tag{1.13}$$

Here $d(x, y)$ denotes the Riemannian geodesic distance and $\alpha = \min\{\frac{p-n}{n}, \frac{p}{n+1}\}$.

The same method can be applied to derive a PDE-based proof of Hölder continuity when the right-hand side is L^p for $p > 1$, first obtained by Kołodziej and Demailly et al. [33, 11]. We can obtain the same Hölder exponent as in [11]

Theorem 1.4 *Consider equation 1.12. If $e^F \in L^p$ for some $p > 1$, and let $q = \frac{p}{p-1}$ be its Hölder conjugate, then the solution φ is Hölder continuous, i.e.,*

$$|\varphi(x) - \varphi(y)| \leq Cd(x, y)^{\alpha_0} \tag{1.14}$$

for any $x, y \in X$, where $d(x, y)$ is the geodesic distance of the two points x, y in the Riemannian manifold (X, ω) and $\alpha_0 < \frac{2}{1+nq}$.

A trick of Li [34] linked Dini continuity to the diameter bounds for Kähler metrics, where he obtained a diameter bound for solutions with L^p right-hand side, $p > 1$. Using this, we can get the following improved diameter bound with only $L^1(\log L)^p$ right-hand side.

Theorem 1.5 *Let ω_φ be the solution to equation 1.12. Suppose $p > 3n$, in other words, $\alpha > 2$, then there exists a constant $C > 0$ depending only on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ such that*

$$\operatorname{diam}(X, \omega_\varphi) \leq C.$$

1.3.3 The $(n-1)$-form equations on Hermitian manifolds

In Chapter 5, we will discuss the $(n-1)$-form equations on Hermitian manifolds, cf.[30]. So far we have focused on the Kähler case. However, the method of auxiliary Monge-Ampère equations also generalizes to Hermitian manifolds. Guo and Phong [23] derived a local version of the comparison method to obtain L^∞ estimates on Hermitian manifolds, where we need our auxiliary Monge-Ampère equation to be the solution to a Dirichlet problem on a ball. The existence and smoothness in such cases are given by Caffarelli, Kohn, Nirenberg, and Spruck [7].

Let (X, ω) be a compact Hermitian manifold without boundary, ω_h be another Hermitian metric on X. For any C^2 function φ, we set

$$\tilde{\omega} := \omega_h + \frac{1}{n-1}((\Delta_\omega \varphi)\omega - i\partial\bar{\partial}\varphi). \tag{1.15}$$

Here

$$\Delta_\omega \varphi = n\frac{i\partial\bar{\partial}\varphi \wedge \omega^{n-1}}{\omega^n}$$

is the complex Laplacian of φ with respect to ω.

Let $h_\varphi := \omega^{-1}\tilde{\omega}, TX \to TX$ be the relative endomorphism defined by $\tilde{\omega}$ relative to ω. Again denote $\lambda[h_\varphi]$ as the eigenvalues vector of h_φ.

Given a smooth function F on X, a $(n-1)$-form equation is a fully nonlinear partial differential equation of the following type:

$$f(\lambda[h_\varphi]) = e^F, \ \sup_X \varphi = 0, \text{ and } \lambda[h_\varphi] \in \Gamma. \tag{1.16}$$

Theorem 1.6 *Let φ be a C^2 solution on a compact Hermitian manifold (X, ω) of the equation 1.2.1, where f satisfies the three conditions in 1.2.1.*

Fix any $p > n$, we have

$$||\varphi||_{L^\infty(X)} \leq C \tag{1.17}$$

where C is a constant depending only on $\omega, \omega_h, n, p, \gamma$, and $\|e^F\|_{L^1(\log L)^p}$.

Note that our relative endomorphism h_φ is different from 1.6 due to a different form of unknown metric. Therefore the corresponding cone is different too. The condition $\lambda \in \Gamma$ means $\lambda[\omega^{-1} \cdot \tilde{\omega}] \in \Gamma$, instead of $\lambda[\omega^{-1} \cdot (\omega + i\partial\bar{\partial}\varphi)] \in \Gamma$. For example, when $\Gamma = \Gamma_n$, this requires φ being $(n-1)$-plurisubharmonic in the sense of [27, 28], instead of being plurisubharmonic.

Examples of such equations arose in [15, 37, 44], in particular, $(n-1)$-form Monge-Ampère equations arise in the solution to the Gauduchon conjecture. With $f(\lambda[h_\varphi]) = \left(\frac{\tilde{\omega}^n}{\omega^n}\right)^{1/n}$, $\Gamma = \Gamma_n$, the equation is the $(n-1)-$form Monge-Ampère equation considered by Tosatti-Weinkove [42]. They showed when ω is Kähler, the equation is solvable provided F is modified by a suitable additive constant, in which the C^0 estimates is an essential step in their proof. In the Hermitian case, the L^∞ estimates were first obtained by Tosatti-Weinkove [43] using Moser iteration. A second proof was given by Székelyhidi [39] using the ABP maximum principle. The third proof by Guo and Phong [23] using auxiliary Monge-Ampère equations gives a sharp dependence on the right-hand side.

Chapter 2: Method of auxilliary Monge-Ampère equations

In this chapter, we will demonstrate the method of auxiliary Monge-Ampère equations by a basic example. Following Guo, Phong, and Tong [21], we will prove the L^∞ estimate below.

Theorem 2.1 *Let (X,ω) be a compact Kähler manifold, consider the complex Monge-Ampère equation*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F\omega^n, \quad \sup_X \varphi = 0 \tag{2.1}$$

where $F \in C^\infty(X)$ such that $\int_X e^F\omega^n = \int_X \omega^n$. Let $\varphi \in C^2(X)$ be a solution, fix $p > n$, then there exists a constant C only depends on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$, such that

$$\sup_X \varphi \le C \tag{2.2}$$

W remark that this result was first proved by Kolodziej [31] using pluripotential theory. We will first prove a weaker version, allowing the constant C to depend on the energy $E = E(\varphi) = \int_X(-\varphi)e^F\omega^n$, and then show the energy term can be controlled by the entropy. Note that using ABP method, Guo, Phong, and Tong [21] proved that the energy term $E(\varphi) = \int_X(-\varphi)e^F\omega^n$ can be controlled by the entropy $\|e^F\|_{L^1(\log L)^p}$ for the general class of equations satisfying the three conditions in 1.2.1, therefore a uniform constant C will only depends on n, p, ω and $\|e^F\|_{L^1(\log L)^p}$.

The key to the proof is to derive a Trudinger-type inequality. For any $s > 0$, let

$$\Omega_s = \{x \in X | \varphi(x) \le -s\}$$

be the sub-level set of φ, we want to show that

Lemma 2.1 *There are uniform constants* $C = C(n, \omega) > 0$ *and* $\alpha_0 = \alpha_0(n, \omega) > 0$ *such that*

$$\int_{\Omega_s} \exp\Big\{\alpha_0 \Big(\frac{-(\varphi + s)}{A_s^{1/(1+n)}}\Big)^{\frac{n+1}{n}}\Big\}\omega^n \leq C \exp(CE), \tag{2.3}$$

where $A_s = \int_{\Omega_s}(-\varphi - s)e^F\omega^n$ *and* $E = E(\varphi) = \int_X(-\varphi)e^F\omega^n$.

Proof of Lemma 2.1. Consider a sequence of smooth positive function $\tau_k : \mathbb{R} \to \mathbb{R}_+$ such that $\tau_k(x)$ decrease pointwise to $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$, where $\chi_{\mathbb{R}_+}$ is the characteristic function on $\mathbb{R}_+$. Consider the following auxiliary Monge-Ampère equation on X

$$(\omega + i\partial\bar{\partial}\psi_k)^n = \frac{\tau_k(-\varphi + s)}{A_{s,k}} e^F \omega^n, \quad \sup_X \psi_k = 0, \tag{2.4}$$

where $A_{s,k} = \int_X \tau_k(-\varphi + s)e^F\omega^n$. Note that $\lim_{k\to\infty} A_{s,k} = A_s$, and $A_s \leq E(\varphi)$. The auxiliary equation admits a smooth solution ψ_k by Yau's theorem.

Now let us consider a comparison function

$$\Phi = -\varepsilon(-\psi_k + \Lambda)^{n/(n+1)} - (\varphi + s)$$

where the two constants are chosen to be $\varepsilon = (\frac{n+1}{n})^{\frac{n}{n+1}} A_{s,k}^{\frac{1}{n+1}}$ and $\Lambda = \frac{n^{n+1}}{(n+1)^{n+1}}\varepsilon^{n+1} = \frac{n}{n+1}A_{s,k}$. Note that $-\psi_k + \Lambda > 0$ by our normalization of the solution ψ_k. Now Φ is a smooth function on a compact manifold X, therefore its maximum must be achieved at some point $x_0 \in X$. Suppose x_0 is not in the interior of Ω_s°, i.e., $-(\varphi(x_0) + s) \leq 0$, then

$$\sup_X \Phi = \Phi(x_0) \leq -(\varphi(x_0) + s) \leq 0$$

since the first term is negative. On the other hand if x_0 is in the interior Ω_s°, then we can apply maximum principle by calculating the Laplacian $\Delta_\varphi\Phi(x_0)$ with respect to the metric $\omega_\varphi = \omega +$

$i\partial\bar{\partial}\varphi$,

$$\begin{aligned}
0 &\geq \Delta_\varphi \Phi(x_0) \\
&= -\varepsilon\frac{n}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}}\mathrm{tr}_{\omega_\varphi}(-i\partial\bar{\partial}\psi_k) - \mathrm{tr}_{\omega_\varphi}(i\partial\bar{\partial}\varphi) + \frac{n\varepsilon}{(n+1)^2}(-\psi_k+\Lambda)^{-\frac{n+2}{n+1}}\mathrm{tr}_{\omega_\varphi} i\partial(\psi_k)\wedge\bar{\partial}(\psi_k) \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}}\mathrm{tr}_{\omega_\varphi}(\omega_{\psi_k}-\omega) + \mathrm{tr}_{\omega_\varphi}\omega - n \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}} n\Big(\frac{\omega^n_{\psi_k}}{\omega^n_\varphi}\Big)^{1/n} + (1-\frac{n\varepsilon}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}})\mathrm{tr}_{\omega_\varphi}\omega - n \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}}\Big(\tau_k(-\varphi-s)A^{-1}_{s,k}\Big)^{1/n} + (1-\frac{n\varepsilon}{n+1}\Lambda^{-\frac{1}{n+1}})\mathrm{tr}_{\omega_\varphi}\omega - n \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_k+\Lambda)^{-\frac{1}{n+1}}(-\varphi-s)^{1/n}A^{-1/n}_{s,k} - n.
\end{aligned}$$

In the second inequality we drop a positive term as the trace of a positive (1,1)-form, while in the third inequality we used arithmetic-geometric inequality, note that $\mathrm{tr}_{\omega_\varphi}\omega_{\psi_k}$ and $\frac{\omega^n_{\psi_k}}{\omega^n_\varphi}$ are sum and product of the eigenvalues of the relative endomorphism, respectively. In the fourth one we use the Monge-Ampère equation and in the last one we used our choice of the constant Λ, so that $1-\frac{n\varepsilon}{n+1}\Lambda^{-\frac{1}{n+1}}=0$. After rearranging, we have at $x_0\in\Omega_s^\circ$,

$$-(\varphi+s) \leq (\frac{n+1}{n\varepsilon})^n A_{s,k}(-\psi_k+\Lambda)^{\frac{n}{n+1}} = \varepsilon(-\psi_k+\Lambda)^{\frac{n}{n+1}},$$

i.e. $\Phi(x_0)\leq 0$. Note that here we used our choice of constant ε. Combining the two cases we can conclude that $\sup_X \Phi \leq 0$, which means on Ω_s°,

$$(-\varphi-s)^{\frac{n+1}{n}} \leq C_n A^{1/n}_{s,k}(-\psi_k + A_{s,k})$$

where C_n is a constant only depends on n. Now if we take a constant α_0, consider $\alpha_0\frac{(-\varphi-s)^{\frac{n+1}{n}}}{A^{1/n}_{s,k}}$ and integrate the exponential over Ω_s, we have

$$\int_{\Omega_s}\exp(\alpha_0\frac{(-\varphi-s)^{\frac{n+1}{n}}}{A^{1/n}_{s,k}})\omega^n \leq e^{C_n\alpha_0 A_{s,k}}\int_{\Omega_s} e^{-C_n\alpha_0\psi_k}\omega^n \tag{2.5}$$

We shall use the α-invariant estimate of Hörmander and Tian [29, 40]. Recall that for a Kähler manifold (X,ω) and a Kähler class $[\chi]$, there exists a constant $\alpha = \alpha(X,[\chi])$ such that for any $\alpha_0 < \alpha$ and any χ-plurisubharmonic function ψ, we have

$$\int_X e^{-\alpha_0\psi}\omega^n \leq C(\alpha,n,\chi,\omega) \tag{2.6}$$

If we take $\chi = C_n\omega$, $\psi = C_n\psi_k$, then $\chi + i\partial\bar{\partial}\psi > 0$, ψ is χ-plurisubharmonic. Choose $\alpha_0 < \alpha = \alpha(n,\omega)$, we can then combine 2.5 and 2.6, we have

$$\int_{\Omega_s} \exp\left(\alpha_0\frac{(-\varphi-s)^{\frac{n+1}{n}}}{A_{s,k}^{1/n}}\right)\omega^n \leq e^{CA_{s,k}}\int_{\Omega_s} e^{-C_n\alpha_0\psi_k}\omega^n \leq e^{CA_{s,k}}\int_X e^{-C_n\alpha_0\psi_k}\omega^n \leq Ce^{CA_{s,k}}$$

for some constant $C = C(n,\omega)$. Now let $k\to\infty$, we have

$$\int_{\Omega_s} \exp\left(\alpha_0\frac{(-\varphi-s)^{\frac{n+1}{n}}}{A_s^{1/n}}\right)\omega^n \leq Ce^{CE(\varphi)} \tag{2.7}$$

Note that the constant α_0 and C only depend on n and ω, hence the lemma is proved. □

From the lemma to the proof of Theorem 2.1, we shall use generalized Young's inequality and a De Giorgi iteration trick. Let $\eta(x) := (\log(1+x))^p$ be a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ for some fixed $p > n$. Note that η is a strictly increasing function with $\eta(0) = 0$, so the inverse is well-defined, and let $\eta^{-1}(y) = e^{y^{\frac{1}{p}}} - 1$ be its inverse. Set

$$v := \frac{\alpha_0}{2}\left(\frac{-\varphi-s}{A_s^{1/(n+1)}}\right)^{(n+1)/n} \tag{2.8}$$

We use the generalized Young's inequality with respect to η, for any $z \in \Omega_s$,

$$\begin{aligned}
v(z)^p e^{F(z)} &\leq \int_0^{\exp(F(z))} \eta(x)dx + \int_0^{v(z)^p} \eta^{-1}(y)dy \\
&\leq \exp(F_t(z))(\log(1+\exp(F_t(z))))^p + v(z)^p(\exp(v(z))-1) \\
&\leq \exp(F_t(z))(1+|F_t(z)|)^p + v(z)^p \exp(v(z)) \\
&\leq \exp(F(z))(1+|F(z)|)^p + C(p)\exp(2v(z))
\end{aligned}$$

Here we are using a simple integral inequality, $\int_0^a f(x)dx \leq af(a)$ for a positive increasing function f. Now we can integrate both sides over $z \in \Omega_s$,

$$\begin{aligned}
\int_{\Omega_s} v(z)^p e^{F(z)}\omega^n &\leq \int_{\Omega_s} e^F(1+|F(z)|)^p\omega^n + C\int_{\Omega_s} e^{2v(z)}\omega^n \\
&\leq \|e^F\|_{L^1(\log L)^p} + C + C\int_{\Omega_s} \exp(\alpha_0(\frac{-\varphi - s}{A_s^{1/(n+1)}})^{(n+1)/n})\omega^n \\
&\leq \|e^F\|_{L^1(\log L)^p} + C + Ce^{CE}
\end{aligned}$$

for some constant C only depends on n, p, ω. This means

$$\int_{\Omega_s} (-\varphi - s)^{\frac{(n+1)p}{n}} e^{F(z)}\omega^n \leq \frac{2^p}{\alpha_0^p} A_s^{\frac{p}{n}}(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE}) \tag{2.9}$$

Now recall the definition $A_s = \int_{\Omega_s}(-\varphi - s)e^F\omega^n$, by Hölder inequality,

$$\int_{\Omega_s}(-\varphi - s)e^F\omega^n \leq (\int_{\Omega_s}(-\varphi - s)^{\frac{(n+1)p}{n}} e^{F(z)}\omega^n)^{\frac{n}{(n+1)p}}(\int_{\Omega_s} e^F\omega^n)^{\frac{1}{q}} \tag{2.10}$$

where q is the Hölder conjugate of $\frac{(n+1)p}{n}$, $1 - \frac{n}{(n+1)p} = \frac{1}{q}$. Combining 2.9 and 2.10, we have

$$A_s \leq CA_s^{\frac{1}{n+1}}(\int_{\Omega_s} e^F\omega^n)^{\frac{1}{q}}$$

where the constant C depends on $n, \omega, p, \|e^F\|_{L^1(\log L)^p}$ and E. Then

$$A_s \leq C(\int_{\Omega_s} e^F \omega^n)^{\frac{n+1}{qn}} \tag{2.11}$$

Now recall the definition of Ω_s, for any $r > 0$ and $s > 0$, $\Omega_{s+r} = \{x \in X | r \leq -\varphi(x) - s\}$, hence

$$r \int_{\Omega_{r+s}} e^F \omega^n \leq \int_{\Omega_{r+s}} (-\varphi - s) e^F \omega^n \leq \int_{\Omega_s} (-\varphi - s) e^F \omega^n \leq C(\int_{\Omega_s} e^F \omega^n)^{\frac{n+1}{qn}} \tag{2.12}$$

where in the first inequality we used Markov's inequality. Note that $\frac{n+1}{qn} = \frac{(n+1)p-n}{np} = 1 + \frac{p-n}{pn}$, we require $p > n$ therefore $1 < \frac{n+1}{qn} = 1 + \delta_0$. Now we are ready to apply De Giorgi's trick (cf. [31, 21]). Let $\phi(s) = \int_{\Omega_s} e^F \omega^n$, 2.12 tells us

$$r\phi(r + s) \leq C\phi(s)^{1+\delta_0} \tag{2.13}$$

for any $r, s \geq 0$. We need the following lemma.

Lemma 2.2 *Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_{\geq 0}$ *be a decreasing right-continuous function with* $\lim_{x\to\infty} \phi(x) = 0$. *Assuming that* $r\phi(r + s) \leq C\phi(s)^{1+\delta_0}$ *for all* $s > 0$ *and* $r \in [0, 1]$, *then there exists a constant* $S_\infty = S_\infty(\phi, C, \delta_0)$ *that* $\phi(s) = 0$ *for all* $s > S_\infty$.

Proof. Since $\lim_{x\to\infty} \phi(x) = 0$, there exists $s_0 \in \mathbb{R}$ such that $\phi(s_0)^{\delta_0} < \frac{1}{2C}$. Now for $j \in \{1, 2, ...\}$, we define inductively that

$$s_j = \sup\{x \in \mathbb{R} | \phi(x) > \frac{1}{2}\phi(s_{j-1})\}$$

By right-continuity of ϕ, we have $\phi(s_j) \leq \frac{1}{2}\phi(s_{j-1})$. Also, take $r = 1$, by our assumption, $\phi(s_j + 1) \leq C\phi(s_j)^{1+\delta_0} = \phi(s_j)(C\phi(s_j)^{\delta_0}) \leq \phi(s_j)(C\phi(s_0)^{\delta_0}) \leq \frac{1}{2}\phi(s_j)$, so $1 + s_j > s_{j+1}$. Now for any $s \in (s_j, s_{j+1})$, $\frac{1}{2}\phi(s_j) \leq \phi(s) \leq \phi(s_j)$, so take $r = s - s_j$

$$(s - s_j)\phi(s) \leq C\phi(s_j)^{1+\delta_0} \leq 2C\phi(s)\phi(s_j)^{\delta_0}$$

which gives us

$$s_{j+1} - s_j \leq 2C\phi(s_j)^{\delta_0} \leq 2C2^{-j\delta_0}\phi(s_0)^{\delta_0} \leq 2^{-j\delta_0}$$

Hence $S_\infty = s_0 + \sum_j (s_{j+1} - s_j) \leq s_0 + \frac{1}{1-2^{-\delta_0}}$ □

Now back to our main theorem, note that by Markov inequality and definition of Ω_s,

$$\phi(s) = \int_{\Omega_s} e^F \omega^n \leq \frac{1}{s}\int_{\Omega_s} (-\varphi)e^F\omega^n \leq \frac{E}{s} \tag{2.14}$$

Therefore $\lim_{s\to\infty}\phi(s) = 0$ and we can take $s_0 = (2C)^{\frac{1}{\delta_0}}E = (2C)^{\frac{np}{p-n}}E$ for the iteration, so our constant S_∞ only depends on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ and E. Note that $\Omega_\infty = \{\varphi \leq -S_\infty\}$ is an open set with Lebesgue measure 0, it must be empty, which gives a L^∞ estimate for φ.

Now we have proved the L^∞ estimate for φ with dependence on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ and E, the only thing left is to remove the dependence on $E(\varphi) = \int_X (-\varphi)e^F\omega^n$. This comes from Jensen's inequality.

Lemma 2.3 *There is a constant C depends only on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ that*

$$E = \int_X (-\varphi)e^F\omega^n \leq C$$

Proof. Note that φ is ω-plulrisubharmonic. By the α-invariant estimate, there exists $\alpha_0 = \alpha_0(n, \omega) > 0$ that

$$\int_X e^{-\alpha_0\varphi}\omega^n \leq C \tag{2.15}$$

Hence

$$\int_X e^{-F-\alpha_0\varphi}\omega_\varphi^n = \int_X e^{-F-\alpha_0\varphi}e^F\omega^n = \int_X e^{-\alpha_0\varphi}\omega^n \leq C$$

Apply Jensen's inequality, we have

$$\int_X (-F - \alpha_0\varphi)\omega_\varphi^n \leq C$$

which gives

$$\int_X (-\alpha_0 \varphi)\omega_\varphi^n \leq \int_X F e^F \omega^n + C$$

Hence $E = \int_X (-\varphi) e^F \omega^n$ is controlled by the $L^1(\log L)^1$ norm of e^F. □

Chapter 3: Equations on nef classes

In this section we will prove Theorem 1.1 and 1.2, the L^∞ estimate for Monge-Ampère and Hessian equations on nef classes.

Let (X, ω) be a compact Kähler manifold of dimension n, χ a closed $(1,1)$ form on X satisfying normalization $\int_X \omega^n = 1$. Recall a nef class means it lies in the closure of the Kähler cone. In this section we will assume the form χ is nef.

3.1 Complex Monge-Ampère equations

We will first consider the case of Monge-Ampère equations on nef classes. Consider $\hat{\omega}_t = \chi + t\omega$ for $t \in (0,1]$, $\hat{\omega}_t$ might not be positive but its class is Kähler . Therefore we consider the degenerate family of complex Monge-Ampère equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0 \tag{3.1}$$

where F is a smooth function satisfying $\int_X e^F \omega^n = \int_X \omega^n$ and $c_t = [\hat{\omega}_t]^n = O(t^{n-\nu})$ a volume normalizing constant. By Yau's theorem [46] it admits a unique smooth solution. As χ is not semipositive and the usual L^∞ estimate for φ_t does not hold Instead, the envelope V_t of the class $[\hat{\omega}_t]$

$$V_t = \sup\{v \mid v \in PSH(X, \hat{\omega}_t),\ v \le 0\} \tag{3.2}$$

must be addressed.

As in the last chapter, we will derive the L^∞ estimate by proving a Trudinger-type inequality. For any $s > 0$, let $\Omega_s = \{x \in X | \varphi_t(x) - V_t(x) \le -s\}$ be the sub-level set of $\varphi_t - V_t$, we need to show that

Lemma 3.1 *There are constants* $C = C(n, \omega, \chi) > 0$ *and* $\alpha_0 = \alpha_0(n, \omega, \chi) > 0$ *such that*

$$\int_{\Omega_s} \exp\Big\{\alpha_0\Big(\frac{-(\varphi_t - V_t + s)}{A_s^{1/(1+n)}}\Big)^{\frac{n+1}{n}}\Big\}\omega^n \le C\exp(CE_t) \tag{3.3}$$

where $A_s = \int_{\Omega_s}(-\varphi_t + V_t - s)e^F\omega^n$ *and* $E_t = \int_X(-\varphi_t + V_t)e^F\omega^n$ *is the energy.*

In the nef case, the key is to smoothly approximate the envelope V_t, by the following lemma of Berman [3], so that the auxiliary Monge-Ampère equations can be set up whose solution is guaranteed by Yau's theorem.

Lemma 3.2 *Consider the Monge-Ampère equation*

$$(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^n = e^{\beta u_\beta}\omega^n \tag{3.4}$$

and its smooth solution u_β*. If* $\beta \to \infty$*,* u_β *will converge uniformly to the envelope* V_t*.*

Remark. V_t is actually a $C^{1,1}$ function by [9].

Now let us begin the proof of Lemma 3.1. Consider a sequence of smooth positive function $\tau_k(x)$ converging to $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$. For each t we fix a smooth function $u_{\beta,t}$ as in Lemma 3.2. For simplicity, we omit the subscript t and consider the following auxiliary Monge-Ampère equation on X

$$(\hat{\omega}_t + i\partial\bar{\partial}\psi_{t,k})^n = c_t\frac{\tau_k(-\varphi_t + u_\beta + s)}{A_{s,k,\beta}}e^F\omega^n, \quad \sup_X \psi_{t,k} = 0, \tag{3.5}$$

where $A_{s,k,\beta} = \int_X \tau_k(-\varphi_t + u_\beta + s)e^F\omega^n$. From definition of the envelope V_t, we have $\psi_{t,k} < V_t$. And u_β converges to V_t uniformly thanks to Berman's lemma 3.2. Therefore we can assume $\psi_{t,k} < u_\beta + 1$ by taking β large enough.

Define a function

$$\Phi = -\varepsilon(-\psi_{t,k} + u_\beta + 1 + \Lambda)^{n/(n+1)} - (\varphi_t - u_\beta + s),$$

with the constants $\varepsilon^{n+1} = A_{s,k,\beta}n^{-n}(n+1)^n$, and $\Lambda = n^{n+1}(n+1)^{-n-1}\varepsilon^{n+1}$, whose choice will be apparent later. Note that in the first term $-\psi_{t,k} + u_\beta + 1 + \Lambda > 0$ by our choice above. Now Φ is a smooth function on a compact manifold X, therefore its maximum must be achieved at some point $x_0 \in X$. Suppose x_0 is not in the interior of Ω_s°, then

$$\sup_X \Phi = \Phi(x_0) \leq -(\varphi_t - u_\beta + s) \leq -V_t + u_\beta \leq \epsilon_\beta$$

for some $\epsilon_\beta \to 0$ as $\beta \to \infty$. The second inequality follows from the definition of the sub-level set Ω_s, so we will have $\Phi(x) \leq 0$ for all $x \in X$. On the other hand if x_0 is in the interior Ω_s°, then we can calculate the Laplacian $\Delta_t \Phi(x_0)$ with respect to the metric $\omega_t = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$,

$$\begin{aligned}
0 &\geq \Delta_t \Phi(x_0) \\
&= -\varepsilon \frac{n}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \mathrm{tr}_{\omega_t}(-i\partial\bar{\partial}\psi_{t,k} + i\partial\bar{\partial}u_\beta) - \mathrm{tr}_{\omega_t}(i\partial\bar{\partial}\varphi_t - i\partial\bar{\partial}u_\beta) \\
&\quad + \frac{n\varepsilon}{(n+1)^2}(-\psi_{t,k} + u_\beta + 1 + \Lambda)^{-\frac{n+2}{n+1}} \mathrm{tr}_{\omega_t} i\partial(\psi_{t,k} - u_\beta) \wedge \bar{\partial}(\psi_{t,k} - u_\beta) \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \mathrm{tr}_{\omega_t}(\hat{\omega}_{t,\psi_{t,k}} - \hat{\omega}_{t,u_\beta}) + \mathrm{tr}_{\omega_t}\hat{\omega}_{t,u_\beta} - n \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} n\Big(\frac{\hat{\omega}^n_{t,\psi_{t,k}}}{\omega_t^n}\Big)^{1/n} + (1 - \frac{n\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}})\mathrm{tr}_{\omega_t}\hat{\omega}_{t,u_\beta} - n \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \Big(\tau_k(-\varphi_t + u_\beta - s)A^{-1}_{s,k,\beta}\Big)^{1/n} + (1 - \frac{n\varepsilon}{n+1}\Lambda^{-\frac{1}{n+1}})\mathrm{tr}_{\omega_t}\hat{\omega}_{t,u_\beta} - n \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}}(-\varphi_t + u_\beta - s)^{1/n} A^{-1/n}_{s,k,\beta} - n.
\end{aligned}$$

In the second inequality we drop a positive term as the trace of a positive $(1,1)$-form, while in the third inequality we used arithmetic-geometric inequality. In the fourth one we use the Monge-Ampère equation and in the last one we used our choice of the constant Λ, so that $1 - \frac{n\varepsilon}{n+1}\Lambda^{-\frac{1}{n+1}} = 0$. After rearranging, we have at $x_0 \in \Omega_s^\circ$,

$$-(\varphi_t - u_\beta + s) \leq (\frac{n+1}{n\varepsilon})^n A_{s,k,\beta}(-\psi_{t,k} + u_\beta + \Lambda + 1)^{\frac{n}{n+1}} = \varepsilon(-\psi_{t,k} + u_\beta + \Lambda + 1)^{\frac{n}{n+1}},$$

i.e. $\Phi(x_0) \leq 0$. Note that here we used our choice of constant ε. Combining the two cases we can conclude that $\sup_X \Phi \le \epsilon_\beta \to 0$ as $\beta \to \infty$, which means on Ω_s°,

$$(-\varphi_t + u_\beta - s)^{\frac{n+1}{n}} \le C_n A_{s,k,\beta}^{1/n}(-\psi_{t,k} + u_\beta + 1 + A_{s,k,\beta}) + \epsilon_\beta^{(n+1)/n}$$

where C_n is a constant only depends on n. Set $A_{s,k} = \int_X \tau_k(-\varphi_t + V_t + s)e^F\omega^n$, we would have

$$(-\varphi_t + V_t - s)^{\frac{n+1}{n}} \le C_n A_{s,k}^{1/n}(-\psi_{t,k} + V_t + 1 + A_{s,k}),$$

if we let $\beta \to \infty$. By definition $V_t \le 0$, and from the α-invariant estimate [29, 40], there exists an $\alpha_0(n, \omega, \chi)$ such that

$$\int_{\Omega_s} \exp\Big(\alpha_0 \frac{(-\varphi_t + V_t - s)^{\frac{n+1}{n}}}{A_{s,k}^{1/n}}\Big)\omega^n \le \int_{\Omega_s} \exp\Big(\alpha_0 C_n(-\psi_{t,k} + 1 + A_{s,k})\Big)\omega^n \le Ce^{CA_{s,k}}. \tag{3.6}$$

Note that $A_s < E_t$ for any s, thus Lemma 3.1 is proved by letting $k \to \infty$. □

Having established Lemma 3.1, we follow the same argument in Chapter 2 to prove Theorem 1.1.

Let $\eta(x) := (\log(1 + x))^p$ be a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ for some fixed $p > n$. Note that η is a strictly increasing function with $\eta(0) = 0$, so the inverse is well-defined, and let η^{-1} be its inverse. Set

$$v := \frac{\alpha_0}{2}\Big(\frac{-\varphi_t + V_t - s}{A_s^{1/(n+1)}}\Big)^{(n+1)/n} \tag{3.7}$$

then by the generalized Young's inequality with respect to η, for any $z \in \Omega_s$,

$$\begin{aligned} v(z)^p e^{F(z)} &\le \int_0^{\exp(F(z))} \eta(x)dx + \int_0^{v(z)^p} \eta^{-1}(y)dy \\ &\le \exp(F_t(z))(\log(1 + \exp(F_t(z))))^p + \int_0^{\exp(v(z))-1} x\eta'(x)dx \\ &\le \exp(F_t(z))(1 + |F_t(z)|)^p + v(z)^p \exp(v(z)) \\ &\le \exp(F(z))(1 + |F(z)|)^p + C(p)\exp(2v(z)) \end{aligned}$$

Here we are only using a simple integral inequality, $\int_0^a f(x)dx \leq af(a)$ for a positive increasing function f. Now we integrate both sides over $z \in \Omega_s$,

$$\begin{aligned}\int_{\Omega_s} v(z)^p e^{F(z)}\omega^n &\leq \int_{\Omega_s} e^F(1+|F(z)|)^p\omega^n + C\int_{\Omega_s} e^{2v(z)}\omega^n \\ &\leq \|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t},\end{aligned}$$

where C is a constant only depends on p, n, χ and ω. Here we used Lemma 3.1. Now this means

$$\int_{\Omega_s}(-\varphi_t + V_t - s)^{\frac{(n+1)p}{n}} e^{F(z)}\omega^n \leq 2^p\alpha_0^{-p}A_s^{\frac{p}{n}}\left(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}\right). \tag{3.8}$$

from the definition of v. Recall the definition for A_s,

$$\begin{aligned}A_s &= \int_{\Omega_s}(-\varphi_t + V_t - s)e^F\omega^n \\ &\leq \Big(\int_{\Omega_s}(-\varphi_t + V_t - s)^{\frac{(n+1)p}{n}} e^F\omega^n\Big)^{\frac{n}{(n+1)p}} \cdot \Big(\int_{\Omega_s} e^F\omega^n\Big)^{1/q} \\ &\leq A_s^{\frac{1}{n+1}}\Big(2^p\alpha_0^{-p}\big(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}\big)\Big)^{\frac{n}{(n+1)p}} \cdot \Big(\int_{\Omega_s} e^F\omega^n\Big)^{1/q}\end{aligned}$$

Here we used the Hölder inequality with $q = \frac{p(n+1)}{p(n+1)-n} > 1$. This means we have

$$A_s \leq \Big(2^p\alpha_0^{-p}\big(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}\big)\Big)^{1/p} \cdot \Big(\int_{\Omega_s} e^F\omega^n\Big)^{\frac{1+n}{qn}}. \tag{3.9}$$

Now we can apply the De Giorgi type iteration argument. Note that $\frac{1+n}{qn} = \frac{n+1}{n} - \frac{1}{p} = \frac{pn+p-n}{pn} = 1 + \delta_0 > 1$ in the exponent, with $\delta_0 := \frac{p-n}{pn} > 0$. Set

$$B_0 := \Big(2^p\alpha_0^{-p}\big(\|e^F\|_{L^1(\log L)^p} + C + Ce^{CE_t}\big)\Big)^{1/p}. \tag{3.10}$$

inequality 3.9 becomes

$$A_s \leq B_0\Big(\int_{\Omega_s} e^F\omega^n\Big)^{1+\delta_0} \tag{3.11}$$

Now define $\phi(s) := \int_{\Omega_s} e^F \omega^n, \mathbb{R}_+ \to \mathbb{R}_+$. For any $r \in [0,1]$ and $s \geq 0$, by definition of Ω_s, we have $-\varphi_t + V_t - (s+r) \geq 0$ on Ω_{s+r}, i.e., $-\varphi_t + V_t - s \geq r$, therefore $A_s = \int_{\Omega_s} (-\varphi_t + V_t - s) e^F \omega^n \geq r \int_{\Omega_{s+r}} e^F \omega^n = r\phi(s+r)$. Combining this with inequality 3.11 we have

$$r\phi(s+r) \leq B_0 \phi(s)^{1+\delta_0} \tag{3.12}$$

Then by the De Giorgi type lemma 2.2 in Chapter 2, we can conclude that there exists some S_∞ such that $\phi(s) = 0$ for any $s \geq S_\infty$. Similar to Chapter 2, by Markov inequality and definition of Ω_s,

$$\phi(s) = \int_{\Omega_s} e^F \omega^n \leq \frac{1}{s} \int_{\Omega_s} (-\varphi_t + V_t) e^F \omega^n \leq \frac{E_t}{s} \tag{3.13}$$

we can take $s_0 = (2c)^{\frac{1}{\delta_0}} \bar{E}_t$ to start the iteration, where $\bar{E}_t$ is an upper bound of $E_t(\varphi_t) = \int_X (-\varphi_t + V_t) e^F \omega^n$. This implies our S_∞ only depends on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ and $\bar{E}_t$. Here we need to derive a bound of the energy in the family case.

Lemma 3.3 *There is a constant C depends only on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}, \chi$ that*

$$E_t(\varphi_t) = \int_X (-\varphi_t + V_t) e^F \omega^n \leq C$$

The proof is similar to 2.3 using Jensen's inequality. Note that since all Kähler metrics are equivalent on a compact Kähler manifold, without loss of generality we can assume $\chi \leq \omega$, so that the $\hat{\omega}_t$-plurisubharmonic function φ_t is also 2ω-plurisubharmonic, in order to apply the α-invariant estimate.

This finished our proof of Theorem 1.1. □

3.2 Complex Hessian equations

Now we consider the case for complex Hessian equations. With the same notation in the Monge-Ampère case, for $k \in \{0, 1, ...n\}$, consider the σ_k-equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^k \wedge \omega^{n-k} = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0. \tag{3.14}$$

We need parallel concepts from the Monge-Ampère cases. Let $\sigma_j(\lambda)$ denote the the j-th elementary symmetric polynomial for $\lambda \in \mathbb{R}^n$, the Γ_k cone is a convex cone in $\mathbb{R}^n$ defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, ...\sigma_k(\lambda) > 0\} \tag{3.15}$$

Define the envelope corresponding to the Γ_k-cone

$$\tilde{V}_{t,k} = \sup\{v |\ v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2,\ v \leq 0\}$$

where $v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2$ means that the vector of eigenvalues of the linear transformation $\omega^{-1} \cdot (\hat{\omega}_t + i\partial\bar{\partial}v)$ lies in the Γ_k-cone.

Let

$$E_t(\varphi_t) = \int_X (-\varphi_t + \tilde{V}_{t,k}) e^{nF/k} \omega^n$$

be the energy associated to the equation (3.14) as in [21] and let $\bar{E}_t$ be an upper bound of $E_t(\varphi_t)$. Then recall the main statement Theorem 1.2:

Theorem 3.1 *Let φ_t be the solution to 3.14, then there exists a constant depending on $\|e^{\frac{n}{k}F}\|_{L^1(\log L)^p}$, $\frac{c_t}{[\hat{\omega}_t]^n}$, $p > n$ and $\bar{E}_t$ such that*

$$0 \leq -\varphi_t + \tilde{V}_{t,k} \leq C.$$

where $\bar{E}_t$ be an upper bound of $E_t(\varphi_t) = \int_X (-\varphi_t + \tilde{V}_{t,k}) e^{nF/k} \omega^n$

Theorem 3.1 can be derived using a similar argument as in the previous section with suitable modifications for σ_k equations. One novel ingredient is the smooth approximation of $\tilde{V}_{t,k}$, as in Lemma 3.2. One can adapt the method in [3] to derive this required approximation. We present a sketch of the proof here.

Lemma 3.4 *Fix $t \in (0,1]$. There exists a sequence of smooth functions $u_\beta \in SH_k(X,\omega,\hat{\omega}_t)$ converging uniformly to $\tilde{V}_{t,k}$ as $\beta \to \infty$.*

Proof. Let $u_\beta \in SH_k(X,\omega,\hat{\omega}_t)$ be the solution to the σ_k-equations

$$(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^n = c_t e^{\beta u_\beta}\omega^n, \tag{3.16}$$

which admits a unique smooth solution by [13]. We claim that there is a constant $C_t > 0$ such that

$$\sup_X |u_\beta - \tilde{V}_{t,k}| \leq \frac{C_t \log \beta}{\beta},$$

from which the lemma follows.

By the maximum principle, at a maximum point of u_β, $i\partial\bar{\partial}u_\beta \leq 0$, so $\beta u_\beta \leq \log \frac{\hat{\omega}_t^k \wedge \omega^{n-k}}{c_t \omega^n} \leq C_t$, that is $u_\beta - \frac{C_t}{\beta} \leq 0$. By the definition of $\tilde{V}_{t,k}$, it follows that

$$u_\beta - \frac{C_t}{\beta} \leq \tilde{V}_{t,k}. \tag{3.17}$$

On the other hand, we fix a smooth $u \leq 0$ such that $\hat{\omega}_t + i\partial\bar{\partial}u > 0$. Such a u exists because $[\hat{\omega}_t]$ is a Kähler class by assumption. For any $v \in SH_k(X,\omega,\hat{\omega}_t) \cap C^2$ with $v \leq 0$, we consider the barrier function

$$\tilde{u} = \frac{1}{\beta}u + (1 - \frac{1}{\beta})v - \frac{C_t' \log \beta}{\beta}$$

where $C_t' > 0$ is a large constant to be determined. By direct calculation, we have

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \geq \frac{1}{\beta^n}(\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k} \geq e^{\beta\tilde{u}}\omega^n$$

where the last inequality holds if we choose C_t' large enough so that

$$e^{-C_t' \log \beta} \leq \frac{1}{\beta^k} \min_X \frac{(\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Therefore we get

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \geq e^{\beta(\tilde{u}-u_\beta)}(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}.$$

At the maximum point of $\tilde{u} - u_\beta$, $(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \leq (\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}$. This shows that $\tilde{u} - u_\beta \leq 0$ on X. Taking supremum over all such v's in $\tilde{u}$, it follows that

$$(1 - \frac{1}{\beta})\tilde{V}_{t,k} \leq u_\beta + \frac{C_t \log \beta}{\beta}.$$

The lemma follows from this and 3.17. □

Remark. If the right hand side $\|e^F\|_{L^p} < \infty$ for some $p > \frac{n}{k}$, then we can bound both the relative volume $\frac{c_t}{[\hat{\omega}_t]^n}$ and the energy E_t uniformly by $\|e^F\|_{L^p} < \infty$, as in [21]. Both follow from Hölder inequality, if we further assume the class $[\chi]$ to be big.

Chapter 4: Modulus of continuity for Monge-Ampère equations

In this section, we will discuss the modulus of continuity for Monge-Ampère equations and a geometric application for diameter bounds.

4.1 Modulus of continuity estimate

Let (X, ω) be a compact Kähler manifold of complex dimension n. We consider the complex Monge-Ampère equation

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F \omega^n \tag{4.1}$$

Here we let $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0$ and normalize the solution φ so that $\inf_X \varphi = 1$. Again F here is a smooth function satisfying the compatibility condition $\int_X e^F \omega^n = \int_X \omega^n$. Denote the $L^1(\log L)^p$ norm as usual, $\|e^F\|_{L^1(\log L)^p} = \int_X e^F |F|^p \omega^n$. We will derive a uniform continuity estimate for the solution φ when the right hand side e^F lies in some Orlicz space.

Theorem 4.1 *Fix $p > n$. There exists a constant $C > 0$ depending on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ such that the following uniform estimate holds:*

$$|\varphi(x) - \varphi(y)| \leq \frac{C}{|\log d(x, y)|^\alpha} \tag{4.2}$$

for any $x, y \in X$ and here $d(x, y)$ denotes the geodesic distance of the two points x, y in the Riemannian manifold (X, ω), and $\alpha = \min\{\frac{p-n}{n}, \frac{p}{n+1}\}$.

By the example below on Riemann surfaces, the solution φ is not guaranteed to be (uniformly) Hölder continuous if $e^F \notin L^p$ for any $p > 1$. However, Theorem 4.1 tells us the solution is still uniformly continuous with order $O(|\log d|^{-\alpha})$, and the exponent α is sharp.

Also we remark that the estimate 4.2 continues to hold even when e^F is not smooth, as long as $\|e^F\|_{L^1(\log L)^p} < \infty$. This can be seen by taking a smooth approximation of e^F, along with the stability estimates of complex Monge-Ampère equations [32, 22].

Example 4.1 *Let* $\mathbf{D} \subset \mathbb{C} \subset \mathbb{CP}^1$ *be the disk with radius* $1/2$. *Consider the function* $\varphi(z) = (-\log|z|^2)^{-a}$ *for some* $a > 0$ *defined on* $\mathbf{D}$ *where* z *is the standard coordinate on* $\mathbb{C}$. *Then on we have* $\mathbf{D}\backslash\{0\}$

$$i\partial\bar{\partial}\varphi = a(a+1)\frac{idz \wedge d\bar{z}}{|z|^2(-\log|z|^2)^{a+2}} = e^{\tilde{F}}.$$

Note that $e^{\tilde{F}} \in L^1(\log L)^p(\mathbf{D})$ for any $p < a+1$. The exponent α in Theorem 4.1 is $p-1$ in this case. This example shows that the exponent α is sharp. Moreover, φ is not Hölder continuous for any exponent.

Note that here φ is singular. In order to get a smooth example we can consider a regularization of φ, say $\varphi_\epsilon(z) = (-\log(\epsilon + |z|^2))^{-a}$ for $\epsilon \to 0^+$. We can also glue φ or φ_ϵ to the whole space $\mathbb{CP}^1$ to get an example on a compact Kähler manifold.

4.1.1 Demailly's regularization

In order to set up the key proof, we need Demailly's regularization lemma.

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a smoothing kernel which is supported in $[0,1]$ and normalized to satisfy $\int_{\mathbb{R}_+} \rho(t)dt = 1$ and $\rho(t) = \text{const}$ for $t \in [0, 3/4]$. Given a function $u \in L^1(X)$ and $\delta \in (0,1)$, we define its δ-regularization to be

$$\rho_\delta u(z) = \frac{1}{\delta^{2n}} \int_{\zeta \in T_z X} u(\exp_z(\zeta))\rho(\delta^{-1}|\zeta|^2_\omega) dV_\omega(\zeta), \tag{4.3}$$

where $\exp_z : T_z X \to X$ is the exponential map of the Riemannian manifold (X, ω). $\rho_\delta u$ is a suitable weighted average of u over the geodesic ball $B_\omega(z, \delta)$, so if $0 \le u \in PSH(X, \omega)$, by mean value inequality $\rho_\delta u(z)$ control the maximum of u over $B_\omega(z, \delta/2)$. The following lemma is proved in [11, 4].

Lemma 4.1 *Let* u *be an* L^1 *function in* $PSH(X, \omega)$. *Then*

1. *([4]) There exists a constant $K > 0$ depending on the curvature of (X, ω) such that $t \mapsto \rho_t u(z) + Kt^2$ is monotone increasing for any $z \in X$.*

2. *([11]) There exists a constant $C > 0$ depending on only n, ω such that*

$$\int_X |\rho_\delta u - u|\omega^n \leq C\delta^2, \quad \forall \delta \in (0, 1] \tag{4.4}$$

For a given small $c > 0$, we define the Kiselman-Legendre transformation of u as

$$u_{c,\delta}(z) = \inf_{t \in (0,\delta]} \{\rho_t u(z) + Kt^2 - c \log \frac{t}{\delta} - K\delta^2\} \tag{4.5}$$

where $K > 0$ is the constant in Lemma 4.1. By applying Kiselman's minimum principle it can be shown that (see [10, 4]) for $u \in PSH(X, \omega)$

$$\omega + i\partial\bar{\partial}u_{c,\delta} \geq -(Ac + K\delta^2)\omega \tag{4.6}$$

where $-A$ is a lower bound of the bisectional curvature of the fixed Kähler metric ω.

4.1.2 Proof of the main theorem

Let φ be the solution to the equation 4.1, note that we have normalized φ so that $inf_X \varphi = 1$. We call a constant uniform if it only depends on $n, p > n, \omega$ and $\|e^F\|_{L^1(\log L)^p}$. The L^∞ estimates 2.1 tells us there is a uniform constant C_0 such that $1 \leq \varphi \leq C_0$ on X.

We fix a small $\delta > 0$. Let $c = \frac{1}{|\log \delta|^\alpha}$ for some $\alpha = \min(\frac{p-n}{n}, \frac{p}{n+1}) > 0$, and $\Phi_\delta = \varphi_{c,\delta}$ be the Kiselman-Legendre transformation of φ at the level c as in 4.5. From 4.6 we get

$$\omega + i\partial\bar{\partial}\Phi_\delta \geq -(Ac + K\delta^2)\omega \geq -A'c\omega$$

for some $A' = A'(n, \omega) > 0$. Hence the function $\varphi_\delta = \frac{\Phi_\delta}{1+A'c} \leq \Phi_\delta$ belongs to $PSH(X, \omega)$, i.e. $\omega + i\partial\bar{\partial}\varphi_\delta \geq 0$. Note from 4.5 and our normalization of φ that φ_δ is positive.

For any $s \geq 0$ we denote the sub-level set

$$\Omega_s = \{\varphi \leq -2\delta + (1-r)\varphi_\delta - s\}, \tag{4.7}$$

where $r = |\log\delta|^{-\frac{p}{n+1}} > 0$ is a small constant.

Lemma 4.2 *There is a uniform constant C_1 such that*

$$\int_{\Omega_0} e^F \omega^n \leq \frac{C_1}{|\log\delta|^p}.$$

Proof. Note that

$$\Omega_0 = \{2\delta \leq (1-r)\varphi_\delta - \varphi\} \subset \{2\delta \leq \varphi_\delta - \varphi\} \subset \{2\delta \leq \Phi_\delta - \varphi\} \subset \{2\delta \leq \rho_\delta\varphi - \varphi\} =: \Omega$$

The last inclusion comes from the definition of $\Phi_\delta = \rho_{c,\delta}\varphi \leq \rho_\delta\varphi$. Thus it suffices to prove the lemma for the domain Ω.

We will utilize general Young's inequality. Define $v = \log\frac{\rho_\delta\varphi - \varphi}{\delta^{3/2}}$ as a function on Ω. We have $v \geq \log\frac{2}{\delta^{1/2}} > 0$ on Ω. Again define our weight function $\eta(x) := (\log(1+x))^p$, $\mathbb{R}_+ \to \mathbb{R}_+$, and we apply the generalized Young's inequality with this weight. At any point $z \in \Omega$,

$$v^p e^F \leq \int_0^{e^F} \eta(x)dx + \int_0^{v^p} \eta^{-1}(y)dy \leq e^F(1+|F|)^p + v^p e^v.$$

Integrating the above over Ω, we get

$$\int_\Omega v^p e^F \omega^n \leq C + \int_\Omega C|\log\delta|^p \frac{\rho_\delta\varphi - \varphi}{\delta^{3/2}}\omega_0^n \leq C,$$

where in the last inequality we used Demailly's Lemma 4.1. Since on Ω, $v \geq \log 2 + \frac{1}{2}|\delta| \geq \frac{1}{2}|\log\delta|$, we conclude that

$$\frac{1}{2^p}|\log\delta|^p \int_\Omega e^F\omega^n \leq \int_\Omega v^p e^F \omega^n \leq C.$$

Then the lemma follows. □

Again let τ_k a sequence of smooth positive function $\tau_k : \mathbb{R} \to \mathbb{R}_+$ such that $\tau_k(x)$ converge pointwise to $x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$, we set up our auxiliary Monge-Ampère equation as

$$(\omega + i\partial\bar{\partial}\psi_{s,k})^n = \frac{\tau_k(-\varphi + (1-r)\varphi_\delta - 2\delta - s)}{A_{s,k}} e^F \omega^n, \quad \sup_X \psi_{s,k} = 0, \tag{4.8}$$

where $A_{s,k} := \int_X \tau_k(-\varphi + (1-r)\varphi_\delta - 2\delta - s)e^F\omega^n$, whose limit as $k \to \infty$ is

$$A_s := \int_{E_s} (-\varphi + (1-r)\varphi_\delta - 2\delta - s)e^F\omega^n$$

The auxiliary equation admits a unique smooth solution by Yau's theorem, and again we shall compare the solutions using the maximum principle. Let

$$\Psi = -\varepsilon(-\psi_{s,k} + \Lambda)^{\frac{n}{n+1}} + (-\varphi + (1-r)\varphi_\delta - 2\delta - s)$$

where this time we choose the constants $\varepsilon = (\frac{n+1}{n})^{\frac{n}{n+1}} A_{s,k}^{\frac{1}{n+1}}$ and $\Lambda = \frac{n}{n+1}\frac{A_{s,k}}{r^{n+1}}$. Ψ archives its maximum at $x_0 \in X$. If $\mathbf{x}_0 \in X \backslash \Omega_s$, then $\sup_X \Psi < 0$ by the definition of Ω_s. Otherwise $x_0 \in \Omega_s$, and we can take the Laplacian with respect to ω_φ at x_0,

$$\begin{aligned}
0 \;\geq\; & \Delta_\varphi\Psi \geq \frac{n\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}\Delta_\varphi\psi_{s,k} + (1-r)\Delta_\varphi\varphi_\delta - \Delta_\varphi\varphi \\
\geq\; & \frac{n\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}\mathrm{tr}_{\omega_\varphi}\omega_{\psi_{s,k}} + (1-r)\mathrm{tr}_{\omega_\varphi}\omega_{\varphi_\delta} - n + (r - \frac{n\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}})\mathrm{tr}_{\omega_\varphi}\omega \\
\geq\; & \frac{n^2\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}\left(\frac{\omega^n_{\psi_{s,k}}}{\omega^n_\varphi}\right)^{1/n} - n \\
=\; & \frac{n^2\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}\left(\frac{\tau_k(-\varphi + (1-r)\varphi_\delta - 2\delta - s)}{A_{s,k}}\right)^{1/n} - n \\
\geq\; & \frac{n^2\varepsilon}{n+1}(-\psi_{s,k} + \Lambda)^{-\frac{1}{n+1}}\left(\frac{-\varphi + (1-r)\varphi_\delta - 2\delta - s}{A_{s,k}}\right)^{1/n} - n
\end{aligned}$$

where by convention $\omega_u = \omega + i\partial\bar{\partial}u$ for $u \in PSH(X, \omega)$. Again we use our choice of Λ in the third inequality. Similar as in previous chapters, we conclude $\sup_X \Psi \leq 0$, which means on Ω_s,

$$\frac{(-\varphi + (1-r)\varphi_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \leq C(n)(-\psi_{s,k} + \frac{A_{s,k}}{r^{n+1}}).$$

Integrating both sides, by α-invariant estimate, there exists $\alpha_0 > 0$ depends on n and ω so that

$$\begin{aligned} \int_{\Omega_s} \exp\Big(\alpha_0 \frac{(-u + (1-r)u_\delta - 2\delta - s)^{(n+1)/n}}{A_{s,k}^{1/n}}\Big)\omega^n &\leq \int_X \exp(-C(n)\alpha_0\psi_{s,k} + C(n)\alpha_0 \frac{A_{s,k}}{r^{n+1}}) \\ &\leq C\exp(C\frac{A_{s,k}}{r^{n+1}}). \end{aligned} \tag{4.9}$$

By letting $k \to \infty$, we obtain a Trudinger-type inequality

$$\int_{\Omega_s} \exp\Big(\alpha_0 \frac{(-\varphi + (1-r)\varphi_\delta - 2\delta - s)^{(n+1)/n}}{A_s^{1/n}}\Big)\omega^n \leq C\exp(C\frac{A_s}{r^{n+1}}) \tag{4.10}$$

Lemma 4.3 *There is a uniform constant $C > 0$, in particular independent of δ and s, such that*

$$\frac{A_s}{r^{n+1}} \leq C$$

Proof. Note that

$$A_s = \int_{\Omega_s} (-\varphi + (1-r)\varphi_\delta - 2\delta - s)e^F\omega^n \leq C\int_{\Omega_s} e^F\omega^n \leq C\int_{\Omega_0} e^F\omega^n \leq \frac{C}{|\log\delta|^p} = Cr^{n+1}$$

where the first inequality is the L^∞ estimate for φ (and φ_δ), the last inequality is Lemma 4.2 and the last equality comes from our choice of the constant r. □

Combining both Lemma 4.3 and 4.10, we have

$$\int_{\Omega_s} \exp\Big(\alpha_0 \frac{(-\varphi + (1-r)\varphi_\delta - 2\delta - s)^{(n+1)/n}}{A_s^{1/n}}\Big)\omega^n \leq C, \tag{4.11}$$

As in the previous chapters, we can apply the generalized Young's inequality with weight $\eta(x) =$

$(\log(1+x))^p$ to conclude that

$$\int_{\Omega_s} [-\varphi + (1-r)\varphi_\delta - 2\delta - s]^{(n+1)p/n} e^F \omega^n \le C A_s^{p/n},$$

where $C > 0$ is independent of s and δ. By Hölder inequality we have

$$\begin{aligned} A_s &\le \Big(\int_{\Omega_s} [-\varphi + (1-r)\varphi_\delta - 2\delta - s]^{(n+1)p/n} e^F \omega^n\Big)^{\frac{n}{p(n+1)}} \Big(\int_{\Omega_s} e^F \omega^n\Big)^{1/q} \\ &\le C A_s^{\frac{1}{n+1}} \Big(\int_{\Omega_s} e^F \omega^n\Big)^{1/q}, \end{aligned} \tag{4.12}$$

where $q = \frac{p(n+1)}{p(n+1)-n}$ is the Hölder conjugate of $\frac{p(n+1)}{n}$. Thus

$$A_s \le C\Big(\int_{\Omega_s} e^F \omega^n\Big)^{(1+n)/qn} = C\Big(\int_{\Omega_s} e^F \omega^n\Big)^{1+a_0}$$

where $a_0 = \frac{p-n}{pn} > 0$. Now we are ready to apply the De Giorgi trick with some modifications.

Define $\phi(s) = \int_{\Omega_s} e^F \omega^n$, similar as in previous chapters, we have for any $r \ge 0$ and $s \ge 0$

$$r\phi(s+r) \le C_3 \phi(s)^{1+a_0} \tag{4.13}$$

for some uniform constant $C_3 > 0$ independent of $\delta \in (0, 1/2]$. However, we need to be careful here because of the comparison function we choose and the estimates for $\phi(0)$. Fix a $\delta_0 > 0$ small which depends only on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ such that for all $\delta \in (0, \delta_0]$, we would have

$$C_3\phi(0)^{a_0} = C_3\Big(\int_{E_0} e^F \omega^n\Big)^{a_0} \le \frac{C_3 C_1^{a_0}}{|\log\delta|^{pa_0}} < \frac{1}{2}.$$

from Lemma 4.2. Hence $\phi(0)^{a_0} \le C|\log\delta|^{-pa_0}$. With this choice of δ_0, by a iteration argument as in Lemma 2.2, we get the set $\Omega_s = \emptyset$ for all $s > S_\infty$, where

$$S_\infty \le \frac{2C_3}{1-2^{-a_0}}\phi(0)^{a_0} \le \frac{C}{|\log\delta|^{pa_0}}.$$

We thus conclude that

$$\varphi_\delta - \varphi \le 2\delta + r\varphi_\delta + \frac{C}{|\log\delta|^{pa_0}}, \quad \text{on } X.$$

This means

$$\Phi_\delta - \varphi \le 2\delta + r\varphi_\delta + A'c\varphi + \frac{C}{|\log\delta|^{pa_0}}. \tag{4.14}$$

At every point $z \in X$, there is a $t_z \in (0, \delta]$ realizing the infimum of $\Phi_\delta = \varphi_{c,\delta}$ in the definition 4.5 of Φ_δ. From 4.14 it holds that

$$\rho_{t_z}\varphi + Kt_z^2 - \varphi - c\log\frac{t_z}{\delta} - K\delta^2 \le 2\delta + r\varphi_\delta + A'c\varphi + \frac{C}{|\log\delta|^{pa_0}}.$$

Also note that $\rho_{t_z}\varphi + Kt_z^2 - \varphi \ge 0$ by Demailly's Lemma 4.1. So

$$\log\frac{t_z}{\delta} \ge \frac{-K\delta^2 - 2\delta}{c} - \frac{r}{c}\varphi_\delta - A'\varphi - \frac{C}{|\log\delta|^{pa_0}c},$$

Note that we can still choose the small constant c. If we let $c = \frac{1}{|\log\delta|^\alpha}$, where $\alpha = \min(pa_0, \frac{p}{n+1})$, we would have $\frac{r}{c} < C$ and $\frac{1}{|\log\delta|^{pa_0}c} < C$ for some uniform constant C. Therefore by the L^∞ estimate of φ and φ_δ there exists a uniform constant $C > 0$ such that $\log\frac{t_z}{\delta} \ge -C$, which means $t_z \ge \theta\delta$ for some uniform $\theta \in (0,1)$. At $z \in X$, we have

$$\begin{aligned}\rho_{\theta\delta}\varphi + K\theta^2\delta^2 - \varphi &\le \rho_{t_z}\varphi + Kt_z^2 - \varphi \\ &\le K\delta^2 + c\log\frac{t_z}{\delta} + 2\delta + r\varphi_\delta + A'c\varphi + \frac{C}{|\log\delta|^{pa_0}} \le \frac{C}{|\log\delta|^\alpha}.\end{aligned}$$

Note here the first inequality comes from the monotonicity of $t \mapsto \rho_t\varphi(z) + Kt^2$ in Demailly's Lemma 4.1. This tells us for any $z \in X$ and $\delta \in (0, \delta_0]$, $\rho_{\theta\delta}\varphi(z) - \varphi(z) \le \frac{C}{|\log\delta|^\alpha}$; or equivalently for any $\delta \in (0, \theta\delta_0]$

$$\rho_\delta\varphi(z) - \varphi(z) \le \frac{C_4}{|\log\delta|^\alpha}, \tag{4.15}$$

for some uniform constant $C_4 > 0$.

Finally, we need a comparison between the two modulus of continuity barriers. We follow the ideas in [18]. For any $\delta > 0$, we denote

$$\bar{\varphi}_\delta(z) = \max_{x \in \overline{B(z,\delta)}} \varphi(x),$$

where $B(z,\delta)$ denotes the geodesic ball with center z and radius δ in the Riemannian manifold (X,ω). We claim that there exists a uniform constant $C > 0$ such that $\bar{\varphi}_\delta(z) - \varphi(z) \le \frac{C}{|\log \delta|^\alpha}$ for any $z \in X$ and $\delta \in (0, \theta\delta_0/\beta_n]$ for some $\beta_n > 0$ sufficiently large depending only on n and α. This will suffice to prove the theorem. To be more general (but less precise) we can formulate it in the following way.

Lemma 4.4 *Fix $\alpha \in (0,1)$, the following two conditions are equivalent:*

1. *There exists $\delta_1, A > 0$ such that at any point $z \in X$ and for any $\delta \in (0,\delta_1)$,*

$$\varphi_\delta - \varphi < \frac{A}{|\log \delta|^\alpha}$$

2. *There exists $\delta_2, B > 0$ such that at any point $z \in X$ and for any $\delta \in (0,\delta_2)$,*

$$\bar{\varphi}_\delta - \varphi < \frac{B}{|\log \delta|^\alpha}$$

We will proceed to prove the more precise version. From now on we assume $\delta_0 > 0$ is chosen to be smaller than the injectivity radius of (X,ω), so the exponential maps considered are diffeomorphisms on relevant domains.

We denote $\Omega(\delta) = \sup_{z\in X}(\bar{\varphi}_\delta(z) - \varphi(z))$. Take a uniform constant $A > 0$ whose choice will be given later 4.21, which depends only on n, p, ω and $C_4 > 0$, then we claim that $\Omega(\delta) \le \frac{A}{|\log \delta|^\alpha}$ for any $\delta \in (0, \theta\delta_0/\beta_n]$. Suppose not, then there exists some $0 < \delta' < \theta\delta_0/\beta_n$ such that $\Omega(\delta') > \frac{A}{|\log \delta'|^\alpha}$. We define

$$\delta := \inf\{0 < t < \frac{\theta\delta_0}{\beta_n} |\ \Omega(s) \le \frac{A}{|\log s|^\alpha} \text{ for all } s \in [t, \frac{\theta\delta_0}{\beta_n}]\}. \tag{4.16}$$

The existence of δ' implies that $\delta \geq \delta' > 0$. Since φ is continuous and X is compact, there exists $z_0 \in X$ such that $\Omega(\delta) = \bar{\varphi}_\delta(z_0) - \varphi(z_0) = \varphi(w_0) - \varphi(z_0)$ for some $w_0 \in \overline{B(z_0, \delta)}$. From the definition of δ in 4.16, it follows that

$$\Omega(\delta) = \frac{A}{|\log \delta|^\alpha}, \quad \text{and } \Omega(s) \leq \frac{A}{|\log s|^\alpha} \text{ for all } s \in [\delta, \frac{\theta\delta_0}{\beta_n}]. \tag{4.17}$$

We fix a constant $b > 1$ but close to 1. For any $x \in B(z_0, 3b\delta)\backslash B(w_0, b\delta)$, we have $d(x, w_0) \geq b\delta$. Hence by 4.16

$$\varphi(w_0) - \varphi(x) \leq \Omega(d(x, w_0)) \leq \frac{A}{|\log d(x, w_0)|^\alpha} = \frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha}\Omega(\delta) \leq \hat{C}\Omega(\delta), \tag{4.18}$$

where $\hat{C} > 0$ is an upper bound of $\frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha}$ for all $\delta \in (0, \theta\delta_0/\beta_n]$, which is uniform. By taking β_n large enough (depending only on n and p), we can choose $\hat{C}$ arbitrarily close to 1, since $\lim_{\delta\to 0} \frac{|\log \delta|^\alpha}{|\log (6b\delta)|^\alpha} = 1$. Thus we can assume that $\hat{C} < 1 + \nu^n$ for some small constant $\nu < \frac{1}{16}$. Rewrite 4.18 as

$$\varphi(x) \geq \varphi(w_0) - \hat{C}\Omega(\delta), \quad \forall\, x \in B(z_0, 3b\delta)\backslash B(w_0, b\delta). \tag{4.19}$$

By the definition of $\rho_{3b\delta}u$ in 4.3, we have

$$\begin{aligned}
\rho_{3b\delta}\varphi(z_0) &= \frac{1}{(3b\delta)^{2n}} \int_{\zeta \in T_{z_0}X} \varphi(\exp_{z_0}(\zeta))\rho(\frac{|\zeta|^2_\omega}{(3b\delta)^2})dV_\omega(\zeta) \\
&\geq \frac{1}{(3b\delta)^{2n}} \int_F \varphi(\exp_{z_0}(\zeta))\rho(\frac{|\zeta|^2_\omega}{(3b\delta)^2})dV_\omega(\zeta) + (1 - \varepsilon_0)(\varphi(w_0) - \hat{C}\Omega(\delta)),
\end{aligned}$$

where $F \subset T_{z_0}X$ is the inverse image of $B(w_0, b\delta)$ under $\exp_{z_0} : T_{z_0} \to X$, and

$$\varepsilon_0 = \frac{1}{(3b\delta)^{2n}} \int_F \rho(\frac{|\zeta|^2_\omega}{(3b\delta)^2})dV_\omega(\zeta) \in [0, 1].$$

Note that we can choose $\varepsilon_0 \geq \frac{1}{4^{2n}}$. By Gauss' Lemma, $|\zeta|^2_\omega \leq (b + 1)^2\delta^2$ for any $\zeta \in F$. By the choice of the kernel function ρ, we have $\rho(\frac{|\zeta|^2_\omega}{(3b\delta)^2}) = \text{const}$ for such ζ. So

$$\frac{1}{(3b\delta)^{2n}}\int_F \varphi(\exp_{z_0}(\zeta))\rho(\frac{|\zeta|^2_\omega}{(3b\delta)^2})dV_\omega(\zeta) = \varepsilon_0\frac{1}{\mu(B(w_0,b\delta))}\int_{B(w_0,b\delta)}\varphi(z)d\mu(z),$$

where $d\mu = (\exp_{z_0})_* dV_{\omega(z_0)}$ is the pushforward of the "Euclidean measure" in $T_{z_0}X$ to X under the exponential map $\exp_{z_0}$. We observe that $B(w_0, b\delta)$ can be viewed as a domain in the normal coordinates chart at z_0, and under this coordinates system, the measure μ differs from the Euclidean one by $C\delta$ (for some uniform $C = C(\omega)$). Moreover, $\varphi + \varphi_{z_0}$ is pluri-subharmonic for some local potential φ_{z_0} of ω which satisfies $|\phi_{z_0}| \le C\delta$ (e.g. consider $\varphi_{z_0} - \varphi_{z_0}(w_0)$ if necessary). Then by the standard mean-value inequality for subharmonic functions in Euclidean space, we get

$$\varepsilon_0\frac{1}{\mu(B(w_0,b\delta))}\int_{B(w_0,b\delta)}\varphi(z)d\mu(z) \ge \varepsilon_0\varphi(w_0) - C_5\delta,$$

where $C_5 > 0$ is a constant depending only on n, ω. Combining the above we get

$$\begin{aligned}
\rho_{3b\delta}\varphi(z_0) &\ge \varphi(w_0) - C_5\delta - (1-\varepsilon_0)\hat{C}\Omega(\delta) \\
&= \varphi(z_0) - C_5\delta + (1-(1-\varepsilon_0)\hat{C})\Omega(\delta) \\
&\ge \varphi(z_0) - C_5\delta + \nu^n\Omega(\delta),
\end{aligned}$$

where in the last inequality, we use the choices of $\nu^n > \varepsilon_0 \ge 4^{-2n}$ and $\hat{C} \le 1 + \nu^n$. Combined with 4.15, this yields that

$$\frac{C_4}{|\log 3b\delta|^\alpha} + C_5\delta \ge \nu^n\Omega(\delta) = \nu^n\frac{A}{|\log\delta|^\alpha}. \tag{4.20}$$

Then we can derive a contradiction by letting the constant $A > 0$ at the beginning be

$$A = 1 + \Big|\log\frac{\theta\delta_0}{\beta_n}\Big|^\alpha\Omega(\frac{\theta\delta_0}{\beta_n}) + \sup_{\delta\in(0,\theta\delta_0/\beta_n]}\nu^n|\log\delta|^\alpha(\frac{C_4}{|\log 3b\delta|^\alpha} + C_5\delta) \tag{4.21}$$

The proof of Theorem 4.1 is thus complete. □

4.2 Hölder continuity

The method of auxiliary Monge-Ampère equations also gives us a PDE approach to reestablish Hölder continuity of the Monge-Ampère equation with L^p right-hand side for $p > 1$.

Theorem 4.2 *Consider equation 4.1*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F\omega^n, \quad \inf_X \varphi = 1$$

If $e^F \in L^p$ for some $p > 1$, and let $q = \frac{p}{p-1}$ be its Hölder conjugate, then the solution φ is Hölder continuous,

$$|\varphi(x) - \varphi(y)| \leq Cd(x,y)^{\alpha_0} \tag{4.22}$$

for any $x, y \in X$, where $d(x,y)$ is the geodesic distance of the two points x, y in the Riemannian manifold (X, ω) and $\alpha_0 < \frac{2}{1+nq}$.

Remark. Examples in [36, 18] shows the optimal Hölder exponent cannot be better than $\frac{2}{nq}$. Similarly, we call a constant uniform if it only depends on p, n, ω and $\|e^F\|_{L^p}$.

We will explain how the proof follows closely from parallel arguments in the last section, by changing $\frac{1}{|\log d(x,y)|^\alpha}$ into δ^α. Again we can assume F is smooth and derive the estimate 4.22. If e^F is only L^p, we take a smooth approximate and apply the stability estimate [32, 22].

Suppose $e^F \in L^p$ for some $p > 1$ and $q = \frac{p}{p-1}$. Fix $a_0 < \frac{1}{n}$, denote $\alpha_0 = \frac{2a_0}{a_0+q} = \frac{2}{1+\frac{1}{a_0}q} < \frac{2}{1+nq}$. By our normalization and L^∞ estimate, $1 \leq \varphi \leq C$ for some uniform constant C on X. Fix some small $\delta > 0$, and let $c = \delta^{\alpha_0}$, let $\Phi_\delta = \varphi_{c,\delta}$ be the Kiselman-Legendre transformation of φ as in 4.5. Again by 4.6 we get

$$\omega + i\partial\bar{\partial}\Phi_\delta \geq -(Ac + K\delta^2)\omega \geq -A'c\omega$$

for some $A' = A'(n,\omega) > 0$, and the positive function $\varphi_\delta = \frac{\Phi_\delta}{1+A'c} \leq \Phi_\delta$ lies in $PSH(X, \omega)$.

For $s \geq 0$ he sub-level sets Ω_s in 4.7 is now defined to be

$$\Omega_s = \{\varphi \leq -2\delta^{\alpha_0} + (1-r)\varphi_\delta - s\}$$

where we take $r = \delta^{(2-\alpha_0)/(n+1)q}$. Now with these choices, a parallel version of Lemma 4.2 follows from Hölder inequality

Lemma 4.5 *There is a uniform constant C such that*

$$\int_{\Omega_0} e^F \omega^n \leq C\delta^{(2-\alpha_0)/q}$$

Proof. As in Lemma 4.1, there exists a constant $C > 0$ depending on only n, ω such that

$$\int_X |\rho_\delta \varphi - \varphi| \omega^n \leq C\delta^2 \tag{4.23}$$

Similar to Lemma 4.2, we have the inclusion

$$\Omega_0 = \{2\delta^{\alpha_0} \leq (1-r)\varphi_\delta - \varphi\} \subset \{2\delta^{\alpha_0} \leq \rho_\delta\varphi - \varphi\} =: \Omega$$

Now by Markov's inequality

$$\int_{\Omega_0} \omega^n \leq \int_\Omega \omega^n \leq \frac{1}{\delta^{\alpha_0}} \int_X |\rho_\delta\varphi - \varphi|\omega^n \leq C\delta^{2-\alpha_0}$$

Then by Hölder inequality,

$$\int_{\Omega_0} e^F \omega^n \leq (\int_{\Omega_0} e^{pF}\omega^n)^{\frac{1}{p}} (\int_{\Omega_0} \omega^n)^{\frac{1}{q}} \leq C(n, \omega, \|e^F\|_{L^p})\delta^{(2-\alpha_0)/q}$$

□

Setting up our auxiliary Monge-Ampère equation as

$$(\omega + i\partial\bar{\partial}\psi_{s,k})^n = \frac{\tau_k(-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)}{A_{s,k}} e^F \omega^n, \quad \sup_X \psi_{s,k} = 0, \tag{4.24}$$

where $A_{s,k} := \int_X \tau_k(-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)e^F\omega^n$ and $A_s := \int_{E_s}(-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)e^F\omega^n$. Choose our comparison function

$$\Psi = -\varepsilon(-\psi_{s,k} + \Lambda)^{\frac{n}{n+1}} + (-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)$$

with the constants $\varepsilon = (\frac{n+1}{n})^{\frac{n}{n+1}} A_{s,k}^{\frac{1}{n+1}}, \Lambda = \frac{n}{n+1}\frac{A_{s,k}}{r^{n+1}}$, we can use maximum principle in the exact same way, and derive the Trudinger-type inequality

$$\int_{\Omega_s} \exp\Big(\alpha_0 \frac{(-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)^{(n+1)/n}}{A_s^{1/n}}\Big)\omega^n \le C \exp\big(C\frac{A_s}{r^{n+1}}\big) \tag{4.25}$$

Now similar as in Lemma 4.3, we have

Lemma 4.6 *There is a uniform constant $C > 0$ such that*

$$\frac{A_s}{r^{n+1}} \le C$$

Proof. Again we have

$$A_s = \int_{\Omega_s}(-\varphi + (1-r)\varphi_\delta - 2\delta^{\alpha_0} - s)e^F\omega^n \le C\int_{\Omega_s} e^F\omega^n \le C\int_{\Omega_0} e^F\omega^n \le C\delta^{\frac{2-\alpha_0}{q}} = Cr^{n+1}$$

by our choice of r. $\square$

Now define $\phi(s) = \int_{\Omega_s} e^F\omega^n$, we can proceed exactly as before to conclude

$$r\phi(s+r) \le C\phi(s)^{1+a_0} \tag{4.26}$$

for a uniform constant C with any $a_0 < \frac{1}{n}$, because in this case we can take p as large as we like.

The same iteration argument gives that

$$\varphi_\delta - \varphi \leq C\delta^{\alpha_0}$$

Recall that we take $c = \delta^{\alpha_0}$, which will impliy for some uniform $\delta_0 \in (0,1]$ and uniform $\theta \in (0,1)$,

$$\rho_{\theta\delta}\varphi - \varphi \leq C\delta^{\alpha_0}$$

for any $\delta \in (0, \delta_0]$. Finally, the argument can be completed by invoking an equivalence between the Hölder continuity barriers, one can either apply the argument in the last section or directly quote [18]. □

4.3 Geometric application

It is natural to study the geometry of the Kähler metric satisfying a Monge-Ampère equation. In this section, we will show how the uniform modulus of continuity estimate can be used to obtain a diameter bound on Kähler metrics $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi$, by applying a trick from Li [34].

Theorem 4.3 *Let ω_φ be the solution to equation 4.1. Suppose $p > 3n$, in other words, $\alpha > 2$, then there exists a constant $C > 0$ depending only on $n, p, \omega, \|e^F\|_{L^1(\log L)^p}$ such that*

$$\text{diam}(X, \omega_\varphi) \leq C.$$

Remark. Li [34] showed a diameter bound can be obtained if the right-hand side e^F lies in L^p for $p > 1$. Here we generalize his result by only requiring a $L^1(\log L)^p$ right-hand side.

Definition 4.1 *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called* Dini continuous, *if $\int_0^1 \frac{f(r)}{r} dr < \infty$.*

As before, we denote $\Omega(r) = \sup_{d(x,y)\leq r} |\varphi(x) - \varphi(y)|$ to be the modulus of continuity of φ, which is the oscillation of u over geodesic balls of radius r.

Lemma 4.7 *On the Kähler manifold (X, ω), let φ be a smooth and strictly ω-plurisubharmonic function. If $\sqrt{\Omega}$ is Dini continuous, then the diameter of the Kähler manifold (X, ω_φ) is bounded by a constant depending on ω and $\int_0^1 \frac{\sqrt{\Omega(r)}}{r} dr$.*

Proof. Since (X, ω) is compact, we can take a finite open cover $\{U_a\}_{a=1}^N$, where each U_a is a bounded domain in $\mathbb{C}^n$, and without loss of generality we assume each U_a is biholomorphic to the Euclidean ball $B_{\mathbb{C}^n}(0, 2)$ and $\{\frac{1}{2}U_a\}$ also covers X. It is clear that $\omega|_{U_a}$ is equivalent to $\omega_{\mathbb{C}^n}|_{U_a}$. For notational convenience, in the proof of this lemma we write $B_r(z) = B_{\mathbb{C}^n}(z, r)$ and $\omega_E = \omega_{\mathbb{C}^n}$.

We consider the function $\rho(z) = d_{\omega_\varphi}(z, 0)$, which is a Lipschitz function. We fix a cut-off function $\chi : \mathbb{R}_+ \to [0, 1]$ such that $\chi(x) = 1$ for $x \in [0, 1]$ and vanishes on $[2, \infty)$. Following [34], we look at the integral of $|\nabla\rho|^2_\omega$. For any fixed $r < 1$ and any $p \in \frac{1}{2}U_a \cong B_1(0)$, we have

$$\begin{aligned}
\int_{B_r(p)} |\nabla\rho|^2_{\omega_E}\omega_E^n &\leq \int_{B_r(p)} |\nabla\rho|^2_{\omega_\varphi}(\mathrm{tr}_{\omega_E}\omega_\varphi)\omega_E^n = \int_{B_r(p)} (n + \Delta_{\omega_E}\varphi)\omega_E^n \\
&\leq Cr^{2n} + \int_{B_{2r}(p)} \Delta_{\omega_E}\chi\left(\frac{d_E(z, p)}{r}\right) \cdot (\varphi(z) - \varphi(p))\omega_E^n \\
&\leq Cr^{2n} + Cr^{2n-2}\Omega(2r),
\end{aligned}$$

where in the second line we apply the integration by parts. By Poincare inequality, it follows that

$$⨍_{B_r(p)} \left(\rho - \rho_{r,p}\right)^2 \omega_E^n \leq r^2 ⨍_{B_r(p)} |\nabla\rho|^2_{\omega_E}\omega_E^n \leq Cr^2 + C\Omega(r), \tag{4.27}$$

where $⨍_{B_r(p)} f$ denotes the average of f over the ball $B_r(p)$, $\rho_{r,p} = ⨍_{B_r(p)} \rho\omega_E^n$, and in the last inequality we have applied $\Omega(2r) \leq 2\Omega(r)$ which follows from the triangle inequality.

We now follow closely the proof of the classical Morrey's lemma in PDE theory. By Hölder inequality and 4.27

$$|\rho_{r,p} - \rho_{r/2,p}| \leq ⨍_{B_{r/2}(p)} |u(z) - \rho_{r,p}|\omega_E^n \leq Cr + C\sqrt{\Omega(r)}. \tag{4.28}$$

We apply 4.28 with $r = 2^{-j}$ for $j = 1, 2, 3, \cdots$. Then

$$|\rho_{2^{-j},p} - \rho_{2^{-j-1},p}| \le C2^{-j} + C\Omega(2^{-j})^{1/2}. \tag{4.29}$$

Under the assumption that $\sqrt{\Omega(r)}$ is *Dini continuous*, we see that the series $\hat{\rho} = \sum_{j=1}^{\infty}(\rho_{2^{-j},p} - \rho_{2^{-j-1},p})$ converges absolutely, and $|\hat{\rho}|$ is uniformly bounded, since $\sum_j 2^{-j}$ converges and $\sum_j \Omega(2^{-j})^{1/2} \le 2\int_0^1 \frac{\sqrt{\Omega(t)}}{t}dt < \infty$. By Lebesgue differentiation theorem it is clear that

$$d_{\omega_\varphi}(p, 0) = \rho(p) = \lim_{j\to\infty} \rho_{2^{-j},p} = \rho_{1/2,p} - \hat{\rho}. \tag{4.30}$$

To get the desired bound on $d_{\omega_\varphi}(p, 0)$ it suffices to estimate $\rho_{1/2,p}$. To this end, we observe that the inequalities above are uniform for any $p \in B_1(0)$. In particular we can apply 4.28 and 4.29 with $r = 3 \cdot 2^{-j}$ for $j = 1, 2, \cdots$ and $p = 0$ to conclude that

$$d_{\omega_\varphi}(0, 0) = 0 = \rho_{3/2,0} - O(1)$$

where $O(1)$ denotes a uniformly bounded constant. This gives the bound on $\rho_{3/2,0} = ⨍_{B_{3/2}} \rho$. Finally for any $p \in B_1(0)$, we have $B_{1/2}(p) \subset B_{3/2}(0)$ by triangle inequality, hence

$$\rho_{1/2,p} = ⨍_{B_{1/2}(p)} \rho\omega_E^n \le C ⨍_{B_{3/2}(0)} \rho\omega_E^n = C\rho_{3/2,0}$$

is uniformly bounded, as desired. Combined with 4.30, this gives the expected bound on $d_{\omega_\varphi}(p, 0)$ for any $p \in B_1(0)$. Since finitely many of these balls cover (X, ω), we get the diameter bound of (X, ω_φ). The proof of the lemma is complete. □

Proof of Theorem 4.3. Let φ be the solution to 4.1. Suppose $p > 3n$, then

$$\alpha = \min\{\frac{p-n}{n}, \frac{p}{1+n}\} > 1.$$

Theorem 4.1 implies that $|\varphi(x) - \varphi(y)| \le \frac{C}{|\log d(x,y)|^\alpha}$ for any $x, y \in X$ and a uniform constant C.

So we have for the modulus of continuity of u, $\Omega(r) \leq \frac{C}{|\log r|^{\alpha}}$. Then we have

$$\int_0^{1/2} \frac{\sqrt{\Omega(r)}}{r} dr \leq \int_{\log 2}^{\infty} \frac{C}{t^{\frac{\alpha}{2}}} dt < \infty.$$

Now the diameter bound will be implied by Lemma 4.7. □

Example 4.2 *Let (X, ω) be a compact Kähler manifold and $L \to X$ be a holomorphic line bundle over X. Suppose $s \in H^0(X, \mathcal{O}_X(L))$ is a nonzero holomorphic section of $\mathcal{O}_X(L)$. Consider the following complex Monge-Ampère equations (for $\epsilon \in (0, 1]$)*

$$(\omega + i\partial\bar{\partial}\varphi_\epsilon)^n = \frac{C_\epsilon}{(\epsilon + |s|_h^2)(-\log(\epsilon + |s|_h^2))^a} \omega^n,$$

where h is a Hermitian metric on L such that $|s|_h^2 < 1$, $C_\epsilon > 0$ is a normalizing constant so that the equation is solvable, and $a > 0$ is a constant.

Note that the function on the right-hand side belongs to $L^1(\log L)^p$ for any $p < a + 1$. So for any *fixed* $a > n - 1$, Theorem 4.1 implies a uniform estimate on the modulus of continuity of φ_ϵ. Note that this estimate still holds for $\epsilon = 0$, if we interpret the complex Monge-Ampère equation in the sense of Bedford-Taylor.

Now if $a > 3n - 1$, Theorem 4.3 implies a uniform diameter bound of the Kähler metrics $\omega_\epsilon = \omega + i\partial\bar{\partial}\varphi_\epsilon$, which is independent of $\epsilon \in (0, 1]$.

Chapter 5: The $(n-1)$-form equations on Hermitian manifolds

In this chapter, we will prove Theorem 1.6. Recall on a compact Hermitian manifold (X, ω), where ω_h be another Hermitian metric on X. Define $\tilde{\omega} := \omega_h + \frac{1}{n-1}((\Delta_\omega \varphi)\omega - i\partial\bar{\partial}\varphi)$ for any $\varphi \in C^2(X)$, where $\Delta_\omega \varphi = n\frac{i\partial\bar{\partial}\varphi \wedge \omega^{n-1}}{\omega^n}$ is the complex Laplacian. Let $h_\varphi := \omega^{-1}\tilde{\omega}, TX \to TX$ be the relative endomorphism defined by $\tilde{\omega}$ relative to ω. Again denote $\lambda[h_\varphi]$ as the eigenvalues vector of h_φ.

Given a smooth function F on X, a $(n-1)$-form equation is a fully nonlinear partial differential equation of the following type

$$f(\lambda[h_\varphi]) = e^F, \text{ and } \lambda[h_\varphi] \in \Gamma. \tag{5.1}$$

with $\sup_X \varphi = 0$ and $\lambda[h_\varphi] \in \Gamma$ on X.

We will prove an L^∞ estimate for the solution while f satisfies the following three conditions.

- $f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}_+$, where Γ is a symmetric cone satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. We also require $f(\lambda)$ to be symmetric in $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma$ and it is homogeneous of degree one, i.e., $f(r\lambda) = rf(\lambda)$ for $r > 0$ and $\lambda \in \Gamma$.
- f is elliptic, in the sense that $\frac{\partial f}{\partial \lambda_j} > 0$ for each $j = 1, \ldots, n$ and $\lambda \in \Gamma$.
- f satisfies the structure condition, i.e., there is a $\gamma > 0$ such that

$$\prod_{j=1}^{n} \frac{\partial f}{\partial \lambda_j} \geq \gamma, \quad \forall \lambda \in \Gamma. \tag{5.2}$$

Proof of Theorem 1.6. We denote by $G^{i\bar{j}} = \frac{\partial \log f(\lambda[h])}{\partial h_{ij}} = \frac{1}{f}\frac{\partial f(\lambda[h])}{\partial h_{ij}}$ the coefficients of the linearization of the operator $\log f(\lambda[h])$.

It follows from the structure conditions of f that $G^{i\bar{j}}$ is positive definite at h_φ and

$$\det(G^{i\bar{j}}) = \frac{1}{f^n}\det(\frac{\partial f(\lambda[h])}{\partial h_{ij}}) \geq \frac{\gamma}{f(\lambda)^n}. \tag{5.3}$$

Fix any point $z_0 \in X$, by simple linear algebra there is a smooth local frame $\{e^i\}$ for the holomorphic cotangent bundle $T^{*1,0}X$ in a neighborhood of z_0, such that $\omega = \sqrt{-1}\sum_j e^j \wedge e^{\bar{j}}$ and $\tilde{\omega} = \sqrt{-1}\sum_j \lambda_j e^j \wedge e^{\bar{j}}$. In particular, $(h_\varphi)_{\bar{j}i} = \lambda_j \delta_{ij}$ and $G^{i\bar{j}} = \frac{1}{f}\frac{\partial f}{\partial \lambda_j}\delta_{ij}$ in this frame. Therefore, we have in the neighborhood of z_0

$$\mathrm{tr}_G \tilde{\omega} = \sum_{i,j} G^{i\bar{j}} \tilde{g}_{\bar{j}i} = \sum_j \frac{1}{f}\frac{\partial f}{\partial \lambda_j}\lambda_j = 1. \tag{5.4}$$

The last equality follows from Euler's relation and the assumption that $f(\lambda)$ is homogeneous of degree 1. As z_0 is arbitrary, we know that $\mathrm{tr}_G\tilde{\omega} = 1$ holds everywhere on X.

Define the following tensor $\Theta^{i\bar{j}}$ by

$$\Theta^{i\bar{j}} = \frac{1}{n-1}\Big((G^{k\bar{l}} g_{\bar{l}k}) g^{i\bar{j}} - G^{i\bar{j}}\Big).$$

Remark. This tensor is similar to the tensor $\Theta^{i\bar{j}}$ defined in [42, p.17] and [23, p.20], except that we put $G^{i\bar{j}}$ in place of $\tilde{g}^{i\bar{j}}$ in order to deal with more general fully nonlinear equations.

We summarize the key properties of $\Theta^{i\bar{j}}$ in the following lemma.

Lemma 5.1 *The tensor $\Theta^{i\bar{j}}$ then satisfies the following: for $\lambda \in \Gamma$,*

(a) $\Theta^{i\bar{j}}\varphi_{i\bar{j}} = 1 - G^{i\bar{j}}(g_h)_{\bar{j}i}$ *and* $G^{i\bar{j}}(g_h)_{\bar{j}i} \geq 0$*;*

(b) $\Theta^{i\bar{j}}$ *is positive definite, and* $\det(\Theta^{i\bar{j}}) \geq \frac{\gamma}{f(\lambda)^n}$.

Proof. We have

$$\begin{aligned} 1 &= \mathrm{tr}_G \tilde{\omega} \\ &= G^{i\bar{j}}\Big((g_h)_{\bar{j}i} + \frac{1}{n-1}(g^{k\bar{l}}\varphi_{\bar{k}l}g_{\bar{j}i} - \varphi_{\bar{j}i})\Big) \\ &= G^{i\bar{j}}(g_h)_{\bar{j}i} + \frac{1}{n-1}\Big((G^{k\bar{l}}g_{\bar{l}k})g^{i\bar{j}} - G^{\bar{j}i}\Big)\varphi_{\bar{j}i} \\ &= G^{i\bar{j}}(g_h)_{\bar{j}i} + \Theta^{i\bar{j}}\varphi_{\bar{j}i} \end{aligned}$$

where the first equality follows from the equation (5.4), the second equality follows from definition of $\tilde{\omega}$ and the last equality follows from the definition of Θ. Since $G^{i\bar{j}}$ is positive definite, we know $G^{i\bar{j}}(g_h)_{\bar{j}i} \geq 0$. This proves (a).

We can choose holomorphic coordinates at a given point $p \in X$ such that $g_{i\bar{j}}|_p = \delta_{ij}$ and $G_{i\bar{j}}|_p = \mu_i\delta_{ij}$. Note that $\mu_i > 0$ for each $i = 1, \cdots, n$ since $G^{i\bar{j}}$ is positive definite. Then we have

$$\Theta^{i\bar{j}}|_p = \frac{1}{n-1}(\sum_{k\neq i}\mu_k)\delta_{ij}$$

which is clearly positive definite. Moreover,

$$\det(\Theta^{i\bar{j}})|_{x_0} = \frac{1}{(n-1)^n}\prod_{i=1}^{n}(\sum_{k\neq i}\mu_k) \geq \prod_{i=1}^{n}\Big(\prod_{k\neq i}\mu_k\Big)^{1/(n-1)} = \prod_{i=1}^{n}\mu_i = \det(G^{i\bar{j}}) \geq \frac{\gamma}{f(\lambda)^n},$$

where the middle inequality follows from the arithmetic-geometric (AG) inequality and the last inequality follows from the equation (5.3). This proves (b). □

Now we consider a point x_0 where φ attains its minimum, i.e. $\varphi(x_0) = \inf_X \varphi$. We assume $-\varphi(x_0) \geq 2$, otherwise Theorem 1.6 already holds. There exists a local holomorphic coordinate system z centered at x_0 such that

$$\frac{1}{2}i\partial\bar{\partial}|z|^2 \leq \omega \leq 2i\partial\bar{\partial}|z|^2 \text{ in } B(x_0, 2r_0) \tag{5.5}$$

where $B(x_0, 2r_0)$ is the Euclidian ball of radius $2r_0$ defined by this coordinate. Note that any dependence on r_0 can be absorbed into a dependence on X and ω. For simplicity, denote $\Omega := B(x_0, 2r_0)$.

We fix a small constant $\epsilon' > 0$ such that

$$\omega_h \geq \frac{2\epsilon'}{n-1}(\mathrm{tr}_\omega \omega_h)\omega \text{ in } \Omega. \tag{5.6}$$

Let $s_0 = 4\epsilon' r_0^2$. For any $s \in (0, s_0)$, we set

$$u_s(z) := \varphi(z) - \varphi(x_0) + \epsilon'|z|^2 - s, \quad \forall z \in \Omega = B(x_0, 2r_0). \tag{5.7}$$

Define

$$\Omega_s = \{z \in \Omega; u_s(z) < 0\}. \tag{5.8}$$

Note that when $z \in \partial\Omega$, $u_s(z) \geq 4\epsilon' r_0^2 - s > 0$. Therefore the sub-level set of u_s $\Omega_s := \{z|\ u_s(z) < 0\} \cap \Omega$ is relatively compact in Ω, and by definition Ω_s is an open set.

Set

$$A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega^n. \tag{5.9}$$

To make the right-hand side of our auxiliary Monge-Ampère equation smooth, we choose the following sequence of smooth positive functions $\tau_k : \mathbb{R} \to \mathbb{R}_+$ such that

$$\tau_k(x) = x + \frac{1}{k}, \quad \text{when } x \geq 0, \tag{5.10}$$

and

$$\tau_k(x) = \frac{1}{2k}, \quad \text{when } x \leq -\frac{1}{k},$$

and $\tau_k(x)$ lies between $1/2k$ and $1/k$ for $x \in [-1/k, 0]$. Clearly τ_k converge pointwise to $\tau_\infty(x) =$

$x \cdot \chi_{\mathbb{R}_+}(x)$ as $k \to \infty$, where $\chi_{\mathbb{R}_+}$ denotes the characteristic function of $\mathbb{R}_+$.

The auxiliary Monge-Ampère equation we consider is the following

$$(i\partial\bar{\partial}\psi_{s,k})^n = \frac{\tau_k(-u_s)}{A_{s,k}} e^{nF}\omega^n \text{ in } \Omega, \quad \psi_{s,k} = 0 \text{ on } \partial\Omega \tag{5.11}$$

with $i\partial\bar{\partial}\psi_{s,k} \geq 0$, and $A_{s,k}$ is defined by

$$A_{s,k} = \int_\Omega \tau_k(-u_s)e^{nF}\omega^n. \tag{5.12}$$

By Caffarelli-Kohn-Nirenberg-Spruck [7], this Dirichlet problem admits a unique solution $\psi_{s,k}$ which is of class $C^\infty(\bar{\Omega})$, with $\psi_{s,k} \leq 0$. By the definition of $A_{s,k}$, we have $A_{s,k} \to A_s$ as $k \to \infty$, and

$$\int_\Omega (i\partial\bar{\partial}\psi_{s,k})^n = 1. \tag{5.13}$$

Now we are ready to establish the following key comparison lemma

Lemma 5.2 *Let u_s be a C^2 solution of the fully non-linear equation (5.1) and $\psi_{s,k}$ be the solutions of the complex Monge-Ampère equation (5.11) as defined above. Then we have*

$$-u_s \leq \varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} \text{ on } \bar{\Omega} \tag{5.14}$$

where ε is the constant defined by $\varepsilon^{n+1} = A_{s,k}\gamma^{-1}\frac{(n+1)^n}{n^{2n}}$.

Proof. We show that the function

$$\Phi = -\varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} - u_s \tag{5.15}$$

is always ≤ 0 on $\bar{\Omega}$. Let $x_{\max} \in \overline{\Omega}$ be a maximum point of Φ. If $x_{\max} \in \overline{\Omega}\backslash\Omega_s$, clearly $\Phi(x_{\max}) \leq 0$ by the definition of Ω_s and the fact that $\psi_{s,k} < 0$ in Ω. If $x_{\max} \in \Omega_s$, then we have $i\partial\bar{\partial}\Phi(x_{\max}) \leq 0$

by the maximum principle. Since $\Theta^{i\bar{j}}$ is positive definite, we calculate at $x_{\max}$,

$$
\begin{aligned}
0 \;&\geq\; \Theta^{i\bar{j}}\Phi_{\bar{j}i} \\
&= \frac{n\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\Theta^{i\bar{j}}(\psi_{s,k})_{i\bar{j}} + \frac{\varepsilon n}{(n+1)^2}(-\psi_{s,k})^{-\frac{n+2}{n+1}}\Theta^{i\bar{j}}(\psi_{s,k})_i(\psi_{s,k})_{\bar{j}} \\
&\quad -1 + \mathrm{tr}_G\omega_h - \frac{\epsilon'}{n-1}(\mathrm{tr}_G\omega \cdot \mathrm{tr}_\omega\omega_{\mathbb{C}^n} - \mathrm{tr}_G\omega_{\mathbb{C}^n}) \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\Theta^{i\bar{j}}(\psi_{s,k})_{i\bar{j}} - 1 + \mathrm{tr}_G[\omega_h - \frac{2\epsilon'}{n-1}(\mathrm{tr}_\omega\omega_h)\omega] \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}(\det\Theta^{i\bar{j}})^{1/n}[\det(\psi_{s,k})_{i\bar{j}}]^{1/n} - 1 \\
&\geq \frac{n^2\varepsilon}{n+1}(-\psi_{s,k})^{-\frac{1}{n+1}}\frac{\gamma^{1/n}}{f}\frac{(-u_s)^{1/n}e^F}{A_{s,k}^{1/n}} - 1 \\
&\geq \frac{n^2\varepsilon}{n+1}\gamma^{1/n}(-\psi_{s,k})^{-\frac{1}{n+1}}\frac{(-u_s)^{1/n}}{A_{s,k}^{1/n}} - 1.
\end{aligned}
$$

Here the first equality follows from the definition of $\Theta^{i\bar{j}}$ and part (a) of Lemma 5.1

$$\Theta^{i\bar{j}}\varphi_{i\bar{j}} = 1 - \mathrm{tr}_G\omega_h.$$

The third line follows from the choice of ϵ' in (5.6). In the fourth line, we applied the standard arithmetic-geometric inequality. The fifth line follows from part (b) of Lemma 5.1 and the definition of $\psi_{s,k}$. The last line follows from the equation (4.2). By the choice of ε, this implies that $\Psi(x_0) \leq 0$. Hence $\sup_\Omega \Psi \leq 0$. □

Along the same spirit in previous chapters, as long as we establish the comparison between u_s and $\psi_{s,k}$ as in Lemma 5.2, we could derive the L^∞ estimate, without referring to the differential equations satisfied by u_s and $\psi_{s,k}$. We cite the lemma here without repeating the proof.

Lemma 5.3 *Let $0 < s < s_0 = 2r_0^2$. Assume that we have functions u_s, $u_s > 0$ on $\partial\Omega$, and let Ω_s, A_s, and $A_{s,k}$ be the corresponding notions as defined by. Assume that the inequality (5.14) holds, that is,*

$$-u_s \leq C(n,\gamma)\,A_{s,k}^{\frac{1}{n+1}}(-\psi_{s,k})^{\frac{n}{n+1}} \text{ on } \bar{\Omega} \tag{5.16}$$

for some constant $C(n,\gamma)$, where $\psi_{s,k}$ are plurisubharmonic functions on Ω, $\psi_{s,k}=0$ on $\partial\Omega$, and $\int_\Omega (i\partial\bar\partial\psi_{s,k})^n = 1$. Then for any $p>n$, we have

$$-\varphi(x_0) \le C(n,\omega,\gamma,p,\|\varphi\|_{L^1(\Omega,\omega^n)}). \tag{5.17}$$

Now, Theorem 1.6 follows from Lemma 5.2 and Lemma 5.3, except that the a priori L^∞ bound of φ may rely on the L^1 norm of φ. We can remove such dependence as in [23].

Lemma 5.4 *For any $\varphi \in C^2(X)$ such that $\lambda[\omega^{-1}\cdot(\omega+i\partial\bar\partial\varphi)] \in \Gamma_1$ and $\sup_X \varphi = 0$, there exists a uniform constant $C>0$ depending only n,ω such that*

$$\int_X (-\varphi)\omega^n \le C.$$

Remark. We stress again that our definition of relative endomorphism $h_\varphi := \omega^{-1}\tilde\omega$ is different from those in [21, 23] due to a different form of unknown metric (1.15). So the condition $\lambda\in\Gamma$ means $\lambda[\omega^{-1}\cdot\tilde\omega]\in\Gamma$, instead of $\lambda[\omega^{-1}\cdot(\omega+i\partial\bar\partial\varphi)]\in\Gamma$.

Since our assumption for φ is that $\lambda[\omega^{-1}\cdot\tilde\omega]$ lies in $\Gamma\subset\Gamma_1$, i.e. $\mathrm{tr}_\omega\tilde\omega \ge 0$. This implies that $\Delta_\omega\varphi \ge -\mathrm{tr}_\omega\omega_h \ge -C'$ by (1.15), for some positive constant C'. Therefore $\mathrm{tr}_\omega(\omega+i\partial\bar\partial(\frac{n\varphi}{C'}))\ge 0$. In other words, $\lambda[\omega^{-1}\cdot(\omega+i\partial\bar\partial(\frac{n\varphi}{C'}))]$ is in Γ_1. Lemma 5.4 then yields the desired L^1 bound of φ given our normalization $\sup_X\varphi=0$. The proof of Theorem 1.6 is complete. □

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