Geometric Pullback Formula For Unitary Shimura Varieties

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Abstract

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In this thesis we study Kudla's special cycles of codimension r on a unitary Shimura variety $\mathrm{Sh}(\mathrm{U}(n-1,1))$ together with an embedding of a Shimura subvariety $\mathrm{Sh}(\mathrm{U}(m-1,1))$. We prove that when $r = n - m$, for certain cuspidal automorphic representations π of the quasi-split unitary group $\mathrm{U}(r,r)$ and certain cusp forms $f \in \pi$, the geometric volume of the pullback of the arithmetic theta lift of f equals the special value of the standard L-function of π at $s = \frac{m-r+1}{2}$. As ingredients of the proof, we also give an exposition of Kudla's geometric Siegel-Weil formula and Yuan-Zhang-Zhang's pullback formula in the setting of unitary Shimura varieties, as well as Qin's integral representation result for L-functions of quasi-split unitary groups.

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Dedication

To my parents

Nguyễn Văn Đệ

and

Trần Thị Ngân

and my sister

Nguyễn Thị Thanh Vân

Chapter 1: Introduction

1.1 Background and motivation

Number Theory is among the areas of contemporary Mathematics that is often able to build surprising and fascinating bridges with other fields. In particular, connections between arithmetic geometry, the study of rational solutions to polynomial equations, and automorphic forms, the study of complex analytic functions with nice transformational properties, are prevalent today and are an actively ongoing area of research. Some of the more famous open problems in the field to name a few include the celebrated Birch and Swinnerton-Dyer conjecture relating the algebraic and the analytic rank of an elliptic curve, and the Beilinson-Bloch conjecture relating the order of vanishing of the motivic L-function of a variety to the rank of one of its Chow groups. One of the first results in this direction dates back to 1986 when Gross and Zagier proved in [GZ86] and [GKZ87] that the central derivative of the L-function of an elliptic curve over an imaginary quadratic field is related to the Neron-Tate height of a Heegner point on that curve.

How does the special point from their formula generalize to higher-dimensional varieties? Kudla proposes in that case a definition of a more general cycle $Z(T, \varphi^\infty)_K$ of codimension r – see Definition 2.4 – on a Shimura variety of orthogonal type associated to a quadratic space of signature $(n, 2)$. It generalizes the Heegner points from above: in fact, the latter can be recovered as a special case of the former when $n = r = 1$ by [Kud04, Proposition A.I.1].

If we now switch from the orthogonal case to the unitary case of a Shimura variety X_K associated to a Hermitian space of signature $(n-1, 1)$, then the Neron-Tate height pairing is replaced by the Beilinson-Bloch one on the homologically trivial part of the Chow group of

X_K and [LL21] prove an analogue of the Gross-Zagier formula above known as the *arithmetic inner product formula*. When n is even and $r = \frac{n}{2}$ is the middle codimension, it connects again the central L-derivative of an automorphic representation π of the quasi-split unitary group $\mathrm{U}(r, r)$ to the Beilinson-Bloch height pairing of special cycles in the π-nearly isotypic component of $\mathrm{CH}^r(X_K)^0_{\mathbb{C}}$ produced by means of *arithmetic theta lifting* – see Definition 4.2.

1.2 Statement of the main result

In this thesis we prove an identity in a similar spirit to Gross-Zagier and the arithmetic inner product formula, but for a special value of the L-function at a non-central point rather than for its central derivative. Given a cuspidal automorphic representation π of the quasi-split unitary group $G = \mathrm{U}(r, r)$ with prescribed archimedean components, we relate the geometric volume of the pullback of the arithmetic theta lift $\Theta_{\varphi^\infty}(f)$ of particular cusp forms $f \in \pi$ to a special value of the standard L-function of π. The inspiration for such a connection comes from [KY13] where the authors work out an arithmetic pullback formula in the case of a modular curve. We prove a geometric version of that statement for unitary Shimura varieties of any dimension.

To that end we consider the following setup. Let E/F be a CM-field with F a totally real field of degree $d > 1$ over $\mathbb{Q}$. Let V be a Hermitian space over E/F with unitary group $H = \mathrm{U}(V)$ over F whose signature is $(n-1, 1)$ at a distinguished archimedean place and positive-definite at the rest. Then V is automatically anisotropic and for a neat choice of a compact open subgroup $K \in H(\mathbb{A}_F^\infty)$ the unitary Shimura variety X_K associated to V has a canonical model over its reflex field E. It is projective of dimension $n - 1$.

The aforementioned Kudla's special cycles $Z(T, \varphi^\infty)_K$ with $T \in \mathrm{Herm}_r(F)$ and $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ live on this variety and we would like to study all at once by forming a natural generating series from them. For this purpose, we turn to the Weil representation for pair of unitary groups – see Section 2.2.1 – and fix correspondingly a splitting character χ :

$\mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ such that

$$\chi|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}^n$$

where $\varepsilon_{E/F} : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is the quadratic character attached to E/F by class field theory. We also associate to this character χ an integer sequence $(l_v^\chi)_{v\in\mathcal{V}_{F,\infty}}$ at the archimedean places of F as in Equation 2.5. In particular, when n is even we can take χ to be the trivial character and therefore $l_v^\chi = 0$ for all $v \in \mathcal{V}_{F,\infty}$.

Given a K-invariant Bruhat-Schwartz function $\varphi^\infty \in S(V(\mathbb{A}_F^\infty)^r)$, we can then define a formal generating series $Z_{\varphi^\infty}(g)$ of the codimension r special cycles $Z(T,\varphi^\infty)_K$ on X_K – see Definition 2.19 – for all values of $g \in G(\mathbb{A}_F)$ where

$$G = \mathrm{U}(r,r) = \left\{ g \in GL_{2r}(E) \mid {}^t\overline{g} J_r g = J_r \right\}$$

is the quasi-split unitary group over F preserving the skew-Hermitian form

$$J_r = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}.$$

We will assume that it is absolutely convergent, or equivalently modular by [Liu11a, Theorem 3.5]. This generating series is valued in $\mathrm{CH}^r(X_K)_\mathbb{C}$ and in order to obtain a scalar-valued function we can apply a particular linear functional

$$\mathrm{vol} : \mathrm{CH}^r(X_K(\mathbb{C}))_\mathbb{C} \to \mathbb{C}$$

(see Definition 2.6) which computes the volume of a cycle by integration against a suitable volume form given by a power of a section of the Hodge line bundle on X_K.

Let $i : V_0 \hookrightarrow V$ be an embedding of a subspace of signature $(m-1,1)$ at the distinguished archimedean place and we suppose that the inner product on its orthogonal complement $V_0^\perp$ is represented by the matrix $\mathcal{T} \in \mathrm{Herm}_r(F)$. There is a corresponding embedding of Shimura varieties $i : X_{K_0} \hookrightarrow X_K$ and we will be interested in the image under the pullback map on the level of Chow groups

$$i^* : \mathrm{CH}^r(X_K)_{\mathbb{C}} \to \mathrm{CH}^r(X_{K_0})_{\mathbb{C}}$$

of the arithmetic theta lift

$$\Theta_{\varphi^\infty}(f) = \int_{G(F)\backslash G(\mathbb{A}_F)} \overline{f(g)} Z_{\varphi^\infty}(g)\, dg \in \mathrm{CH}^r(X_K)_{\mathbb{C}}$$

(see Definition 4.2) of certain cusp forms f belonging to a cuspidal automorphic representation $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$.

In particular, we prove the following

Theorem 1.1. *Assume the following conditions:*

① $F \neq \mathbb{Q}$ *so that the unitary Shimura variety* X_K *is compact;*

② $n \geq 3$ *and* $1 \leq m \leq n-1$ *and* $1 \leq r \leq m-1$*;*

③ *the generating series* $Z_{\varphi^\infty}(g)$ *of special cycles of codimension* r *on* X_K *is absolutely convergent;*

④ $r = n - m$.

Let $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$ *be a cuspidal automorphic representation of the group* G *such that for all* $v \in \mathcal{V}_{F,\infty}$ *its archimedean component* π_v *is a holomorphic discrete series of weight* $\left(\frac{n+l_v^\chi}{2}, \frac{n-l_v^\chi}{2}\right)$ *(see Definition 4.1) coming from [Liu11a, Theorem 3.5].*

Let S *be a finite subset of the places* $\mathcal{V}_F$ *of* F *that contains (see Definition 3.1):*

- *all archimedean places of* F *and all finite ones that are ramified in* E*;*
- *all ramified places of the representation* π*;*
- *all finite places* v *at which the character* $\chi_v \stackrel{\text{def}}{=} \chi\big|_{E_v^\times}$ *is ramified;*

- *all finite places v at which the local Hermitian space $V_0^{\perp}(F_v)$ is non-split.*

If $f \in \pi$ is a cusp form that is K_v-fixed for all places $v \notin S$, which is possible by our choice of S, then the identity

$$\frac{\mathrm{vol}\, i^*(\Theta_{\varphi^\infty}(f))}{\mathrm{vol}\, X_{K_0}} = I_S(s_0, \varphi, \overline{f}) \frac{L^S\left(s_0 + \frac{1}{2}, \pi\right)}{D^S(s_0)} \tag{1.1}$$

holds at the special value $s_0 = \frac{m-r}{2} = \frac{2m-n}{2}$.

Here the Bruhat-Schwartz test function

$$\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{S}(V_0(\mathbb{A}_F)^r) \otimes \mathcal{S}(V_0^{\perp}(\mathbb{A}_F)^r) = \mathcal{S}(V(\mathbb{A}_F)^r)$$

is chosen so that:

① $\varphi_1 = \varphi_{1,\infty} \otimes \bigotimes_{v \in \mathcal{V}_F^\infty} \varphi_{1,v}$ *and* $\varphi_2 = \varphi_{2,\infty} \otimes \bigotimes_{v \in \mathcal{V}_F^\infty} \varphi_{2,v}$ *are factorizable and K-invariant;*

② $\varphi_{1,v} = \mathbb{1}_{V_0(\mathcal{O}_v)^r}$ *and* $\varphi_{2,v} = \mathbb{1}_{V_0^{\perp}(\mathcal{O}_v)^r}$ *are the characteristic functions of maximal $\mathcal{O}_v$-lattices in $V_0(F_v)^r$ and $V_0^{\perp}(F_v)^r$ respectively for all $v \notin S$;*

③ $\varphi_{1,\infty}$ *is the archimedean component on $V_0(\mathbb{A}_{F,\infty})^r$ from the geometric Siegel-Weil formula in Theorem 2.13 and $\varphi_{2,\infty}$ is the standard Gaussian function on $V_0^{\perp}(\mathbb{A}_{F,\infty})^r$.*

Moreover $D^S(s_0) = \prod_{v \notin S} D_v(s_0)$ and $I_S(s, \varphi, \overline{f}) = \prod_{v \in S} I_v(s, \varphi_v, \overline{f})$ are products of local factors at the good and bad places respectively given in Theorem 3.2 by

$$D_v(s) = \begin{cases} \prod_{i=1}^{2r} L_v(2s+i, \varepsilon_{E/F}^i) \prod_{i=r+1}^{2r} L_v(2s+i, \varepsilon_{E/F}^i) & , \ v \text{ inert in } E \\ \prod_{i=1}^{2r} \zeta_v(2s+i) \prod_{i=r+1}^{2r} \zeta_v(2s+i) & , \ v \text{ split in } E \end{cases}$$

and

$$I_v(s, \varphi_v, \overline{f}) = \int_{K_v} \int_{M_v} \overline{f_{\mathcal{T}}(m(a_v)k_v)} \Phi_{\varphi_{1,v}}(k_v, s) \times (\chi_v^2 |\cdot|_E^s)(\det a_v) \times (\omega_\chi(k_v)\varphi_{2,v})(a_v)\, d^\times a_v\, dk_v.$$

Example 1.2. In order to get a better understanding of the theorem, here are some edge cases of triplets of numbers satisfying the conditions of the theorem:

① $(r,m,n) = (1,n-1,n)$ – the case of a divisor X_{K_0} on X_K. The arithmetic theta lift $\Theta_{\varphi^\infty}(f)$ is a sum of divisors on X_K indexed by $T \in F_{\geq 0}$ and $i^*(\Theta_{\varphi^\infty}(f)) \in \mathrm{CH}^2(X_K)_{\mathbb{C}}$ is the intersection of $\Theta_{\varphi^\infty}(f)$ with X_{K_0}. The analytic side on the other hand involves the L-value at $s = \frac{n-1}{2}$ of a cuspidal automorphic representation π of the quasi-split unitary group $\mathrm{U}(1,1)$ with archimedean components π_v being holomorphic discrete series of weight $\left(\frac{n+l_v^X}{2}, \frac{n-l_v^X}{2}\right)$ for all $v \in \mathcal{V}_{F,\infty}$;

② $(r,m,n) = \left(\frac{n-1}{2}, \frac{n+1}{2}, n\right)$ and n is odd – the case of a mid-dimensional cycle X_{K_0} on X_K. The arithmetic theta lift $\Theta_{\varphi^\infty}(f)$ is a formal sum of mid-dimensional cycles on X_K indexed by $T \in \mathrm{Herm}_{\frac{n-1}{2}}(F)^+$ and $i^*(\Theta_{\varphi^\infty}(f)) \in \mathrm{CH}_0(X_K)_{\mathbb{C}}$ is the 0-dimensional intersection of $\Theta_{\varphi^\infty}(f)$ with X_{K_0}, so its volume is simply its degree as a 0-cycle. The analytic side on the other hand involves the L-value at $s = 1$ of a cuspidal automorphic representation π of the quasi-split unitary group $\mathrm{U}\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$ with archimedean components π_v being holomorphic discrete series of weight $\left(\frac{n+l_v^X}{2}, \frac{n-l_v^X}{2}\right)$ for all $v \in \mathcal{V}_{F,\infty}$.

Remark 1.3. It is desirable to show that for all values of s_0 in the theorem, the data of the Bruhat-Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$ and the cusp form $f \in \pi$ at the finite places in S can be chosen depending on s_0 so that $I_S(s_0, \varphi, \overline{f}) \neq 0$ and consequently both sides of the identity are nonzero. We do not address that question in this thesis, but see [PSR88, Theorem 2.1].

Remark 1.4. Using the Rankin-Selberg integral representation of L-functions for symplectic groups Sp_{2r} by [PSR88], we should be able to prove an analogous theorem for the pullback of special cycles between orthogonal Shimura varieties. We also expect an arithmetic pullback formula when the geometric Chow group $\mathrm{CH}^r(X_K)$ on the left-hand side is replaced by the arithmetic Chow group $\widehat{\mathrm{CH}}^r(\mathcal{X}_K)$ of an integral model $\mathcal{X}_K$ over O_E, and the special L-value on the right-hand side is replaced by a special L-derivative following the general philosophy of the Kudla program. However, this is more difficult compared to the case considered in our thesis and would be interesting to study further.

1.3 Structure of the thesis

Chapter 2 deals with the geometric side of Theorem 1.1. We recall the classical Siegel-Weil formula (Theorem 2.13) and show a geometric version of it in Theorem 2.7 that relates the volumes of the special cycles of codimension r on the Shimura variety X_K to an Eisenstein series as in [Kud04, Theorem 4.1], but here in the unitary case. Then we consider an embedding of unitary Shimura varieties $\mathrm{Sh}(\mathrm{U}(m-1,1)) \hookrightarrow \mathrm{Sh}(\mathrm{U}(n-1,1))$ for $m < n$ coming from an embedding of a Hermitian subspace, and prove in Theorem 2.21 an equivalent in the unitary case of Yuan-Zhang-Zhang's pullback formula for the special cycles [YZZ09, Proposition 3.1].

Chapter 3 deals with the analytic side of Theorem 1.1. We give an exposition of Qin's paper [Qin07] on the integral representation of the standard L-function of a cuspidal automorphic representation π of the quasi-split unitary group $\mathrm{U}(r,r)$ over F. In particular, we record its main result [Qin07, Theorem 6.3] in Theorem 3.2.

Finally, Chapter 4 combines the results from the previous chapters to derive an expression for the geometric volume of the pullback of the arithmetic theta lift of particular cusp forms $f \in \pi$ and we relate it to the special value of the standard L-function of π at $s = \frac{m-r+1}{2}$ as in Theorem 1.1.

1.4 Notations and conventions

We introduce the following notations and conventions used commonly throughout the thesis:

- For a general number field L we denote by $\mathcal{V}_L, \mathcal{V}_L^\infty, \mathcal{V}_{L,\infty}$ the set of all places of L, the set of all finite ones, and the set of all archimedean ones respectively;
- Similarly $\mathbb{A}_L, \mathbb{A}_L^\infty, \mathbb{A}_{L,\infty}$ denote the ring of full adèles of L, the ring of finite adèles, and the ring of infinite adèles respectively;
- $|\cdot|_L = \prod_{v \in \mathcal{V}_L} |\cdot|_{L_v}$ is the adèlic norm on $\mathbb{A}_L$;
- F is a totally real number field of degree $d > 1$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}$;
- E is a totally imaginary CM-extension of F of degree $2d$ over $\mathbb{Q}$;
- V is a Hermitian space over E/F with unitary group $H = \mathrm{U}(V)$ over F whose inner product $\langle -, - \rangle$ has signature $(n-1, 1)$ at a distinguished archimedean place $\iota : F \hookrightarrow \mathbb{R}$ and is positive-definite at the rest;
- $\{X_K\}_K$ is the inverse system of unitary Shimura varieties over E associated to the datum of H;
- $\mathrm{CH}^r(X_K)$ and $\mathrm{CH}^r(X_K(\mathbb{C}))$ are the Chow groups with integer coefficients of codimension r cycles on the Shimura variety X_K over E and over $\mathbb{C}$ respectively, whereas the subscript $\mathbb{C}$ on them denotes the same groups with complex coefficients instead;
- $V(R)^r$ is the space $V^r \otimes_F R$ over $E \otimes_F R$ for any $r \in \mathbb{N}$ and any local complete F-algebra R;
- $\mathcal{S}(V(R)^r)$ is the space of Bruhat-Schwartz functions on $V(R)^r$;
- $T : V(F)^r \longrightarrow \mathrm{Herm}_r(F)$ is the moment map given by

$$T(\mathbf{x}) = \frac{1}{2}\langle x_i, x_j \rangle_{1 \le i,j \le r} \ \forall\ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix};$$

- $\varphi_\infty^0(\mathbf{x}) \stackrel{\text{def}}{=} e^{-2\pi \operatorname{Tr} T(\mathbf{x})}$ is the Gaussian function on V^r at the archimedean places;

- W is a split skew-Hermitian space over E/F of dimension $2r$ with quasi-split unitary group $G = \mathrm{U}(W) = \mathrm{U}(r,r)$ over F preserving the form

$$J_r = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix};$$

- $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$ is the additive character given by $\psi_\mathbb{Q} \circ \operatorname{Tr}_{F/\mathbb{Q}}$ where

$$\psi_{\mathbb{Q},p}(x) = e^{-2\pi i \{x_p\}}, \ \psi_{\mathbb{Q},\infty}(x) = e^{2\pi i x_\infty};$$

- $\varepsilon_{E/F} : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$ is the quadratic character associated to the extension E/F by class field theory;

- $\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ is a fixed splitting character satisfying the condition

$$\chi|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}^n,$$

  to which we associate an integer sequence $(l_v^\chi)_{v \in \mathcal{V}_{F,\infty}}$ at the archimedean places of F such that

$$\chi_v(z) = z^{l_v^\chi}$$

  for all $v \in \mathcal{V}_{F,\infty}$ and all $z \in E_v^{\times,1} \cong \mathbb{C}^{\times,1}$;

- $I(\chi, s)$ is the degenerate principal series representation $\operatorname{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \left(\chi|\cdot|_E^{s+\frac{r}{2}}\right)$ of $G(\mathbb{A}_F)$;

- $V_0 \subset V$ is a subspace of V with inner product represented by the matrix $\mathcal{T} \in \operatorname{Herm}_r(F)$ whose signature is $(m-1,1)$ at the distinguished archimedean place ι and is positive-definite at the rest;

- $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$ is a cuspidal automorphic representation of the group G with contragradient $\pi^\vee$ such that the archimedean components π_v are holomorphic discrete series of weight $\left(\frac{n+l_v^\chi}{2}, \frac{n-l_v^\chi}{2}\right)$ for all $v \in \mathcal{V}_{F,\infty}$;

- For a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ the function

$$f_T(g) \stackrel{\text{def}}{=} \int\limits_{\mathrm{Herm}_r(F)\backslash\,\mathrm{Herm}_r(\mathbb{A}_F)} f(n(X)g)\psi(\mathrm{Tr}\,TX)\,dX$$

  denotes the T-th Fourier coefficient of the cusp form $f \in \pi$.

In addition, for any place $v \in \mathcal{V}_F$ the subscript v on a global object will denote its adèlic component at the place v, e.g. K_v is the standard maximal compact subgroup $G(\mathcal{O}_v)$ of $G(F_v)$.

Chapter 2: Geometric side – Special cycles on unitary Shimura varieties

2.1 General setup

By analogy with the treatment of orthogonal Shimura varieties described in [Kud04], in the unitary case we consider a totally imaginary CM-extension E/F with F a totally real field of degree $d > 1$ over $\mathbb{Q}$. Let V be a Hermitian space over E/F whose inner product is non-degenerate of signature $(n-1, 1)$ at a distinguished archimedean place $\iota : F \hookrightarrow \mathbb{R}$ and $(n, 0)$ at the rest. Let $H = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V)$ be the Weil restriction to $\mathbb{Q}$ of the unitary group of V and D be the open connected subset of $\mathbb{P}(V_\iota(\mathbb{R}))$ consisting of negative-definite $\mathbb{C}$-lines in $V_\iota(\mathbb{R})$ with complex structure endowed by means of a complex embedding of E over ι. Then H is a reductive group over $\mathbb{Q}$ with associated Hermitian symmetric domain D. From a geometric point of view $D = \left\{z \in \mathbb{C}^{n-1} \mid |z| < 1\right\}$ is the open unit ball of dimension $n-1$ and can be canonically identified with the $H(\mathbb{R})$-conjugacy class of the Hodge map

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow H_\mathbb{R} \cong \mathrm{U}(n-1,1)_\mathbb{R} \times \mathrm{U}(n,0)_\mathbb{R}^{d-1}$$
$$h(z) = \left(\begin{pmatrix} \mathbb{1}_{n-1} & \\ & \overline{z}/z \end{pmatrix}, \mathbb{1}_n, \ldots, \mathbb{1}_n \right).$$

The pair (H, D) then forms the data of a Shimura variety X with canonical model over its reflex field E such that for any compact open subgroup $K \subset H(\mathbb{A}_\mathbb{Q}^\infty)$ its complex uniformization is given by

$$X_K(\mathbb{C}) \cong H(\mathbb{Q})\backslash D \times H(\mathbb{A}_{\mathbb{Q}}^{\infty})/K.$$

After fixing a finite coset decomposition

$$H(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \bigsqcup_j H(\mathbb{Q})h_j K,$$

we can rewrite this in classical language as

$$X_K(\mathbb{C}) \cong \bigsqcup_j \Gamma_j\backslash D$$

where $\Gamma_j = H(\mathbb{Q}) \cap h_j K h_j^{-1}$. The Shimura variety X_K is quasi-projective of dimension $n-1$ and smooth once we fix a neat choice of the subgroup K. By the Borel-Harish-Chandra theorem it is projective if and only if the space V is anisotropic, which is automatically true by our assumption that $F \neq \mathbb{Q}$. We will assume this from now on to avoid complications arising from compactification.

In general, X_K may have several connected components which are only defined over some extension E_K/E. If we let $T = H/H^{\mathrm{der}}$ be the maximal abelian quotient of H, then these components are in bijective correspondence with the double cosets

$$T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathbb{Q}}^{\infty})/\det(K)$$

and the Galois group $\mathrm{Gal}(E_K/E)$ permutes them transitively so that

$$X_K(\mathbb{C}) \cong \bigsqcup_{\sigma\in\mathrm{Gal}(E_K/E)} Y_K^{\sigma}$$

where $Y_K = \Gamma_1\backslash D$ is a fixed connected component.

The Shimura variety X_K has a rich source of natural cycles defined in the following way. Let $1 \leq r \leq n-1$ be an integer and $\mathbf{x} \in V(F)^r$ be an r-tuple of vectors in V with E-linear span $\underline{\mathbf{x}}$. Denoting by

$$\begin{aligned} r_{\mathbf{x}} &= \dim \underline{\mathbf{x}} \\ H_{\mathbf{x}} &= \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(\underline{\mathbf{x}}^{\perp}) = \mathrm{Stab}_H \mathbf{x} \\ D_{\mathbf{x}} &= \left\{ z \in D \mid z \subseteq \underline{\mathbf{x}}^{\perp} \right\}, \end{aligned} \tag{2.1}$$

we arrive at following

Definition 2.1. For an r-tuple of vectors $\mathbf{x} \in V(F)^r$ the pair $(H_{\mathbf{x}}, D_{\mathbf{x}})$ forms a Shimura subvariety giving rise to a *Kudla cycle*

$$Z(\underline{\mathbf{x}})_K \stackrel{\mathrm{def}}{=} \mathrm{Sh}(H_{\mathbf{x}}, D_{\mathbf{x}})_K \in \mathrm{CH}^{r_{\mathbf{x}}}(X_K).$$

This cycle is rational over E of codimension $r_{\mathbf{x}}$ and if the moment matrix

$$T(\mathbf{x}) = \frac{1}{2} \langle x_i, x_j \rangle_{1 \leq i,j \leq r}$$

of inner products of components of $\mathbf{x}$ is not positive-definite or if $\mathrm{rk}\, T(\mathbf{x}) < r_{\mathbf{x}}$, then we formally set $Z(\underline{\mathbf{x}})_K = \emptyset$.

Definition 2.2. For an r-tuple of vectors $\mathbf{x} \in V(F)^r$ and an element $h \in H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ the *Hecke-translated Kudla cycle* is given by

$$\begin{aligned} Z(\underline{\mathbf{x}}, h)_K \stackrel{\mathrm{def}}{=} H_{\mathbf{x}}(\mathbb{Q}) \backslash D_{\mathbf{x}} \times H_{\mathbf{x}}(\mathbb{A}_{\mathbb{Q}}^{\infty}) / K_{\mathbf{x}}^h &\in \mathrm{CH}^{r_{\mathbf{x}}}(X_K) \\ H_{\mathbf{x}}(\mathbb{Q})(z, g) K_{\mathbf{x}}^h &\longmapsto H(\mathbb{Q})(z, gh) K \in X_K \end{aligned}$$

where $K_{\mathbf{x}}^h = H_{\mathbf{x}}(\mathbb{A}_{\mathbb{Q}}^{\infty}) \cap hKh^{-1}$. Note that this cycle is again rational over E.

Consider the restriction $\mathcal{L}_D$ to D of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V_{\iota}(\mathbb{R}))$ whose fiber over a point $z \in D$ is the line $z \in V_\iota(\mathbb{R})$. The action of $H(\mathbb{R})$ on D lifts naturally to an action on $\mathcal{L}_D$ and therefore $\mathcal{L}_D$ descends to a holomorphic line bundle $\mathcal{L}_K$ on the Shimura variety X_K:

$$\mathcal{L}_K(\mathbb{C}) \cong H(\mathbb{Q})\backslash \mathcal{L}_D \times H(\mathbb{A}_{\mathbb{Q}}^\infty)/K \longrightarrow X_K.$$

This line bundle is algebraic with canonical model over E and on each connected component $\Gamma_j\backslash D$ has the form $\Gamma_j\backslash\mathcal{L}_D$. We can equip $\mathcal{L}_D$ with the Hermitian metric

$$h_{\mathcal{L}_D}(v_1, v_2) = -\langle v_1, v_2\rangle,$$

whose invariance under the action of $H(\mathbb{R})$ means that it descends to a Hermitian metric on $\mathcal{L}_K$ as well.

For a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ let

$$\Omega_T \stackrel{\mathrm{def}}{=} \{\mathbf{x} \in V^r \mid T(\mathbf{x}) = T\}.$$

If $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ is a K-invariant Bruhat-Schwartz function for the left regular representation of $H(\mathbb{A}_{\mathbb{Q}}^\infty)$ on $V(\mathbb{A}_F^\infty)^r$, then after fixing an element $\mathbf{x} \in \Omega_T(E)$, assuming that the last set is non-empty, we can write

$$\Omega_T(\mathbb{A}_E^\infty) \cap \operatorname{supp}\varphi^\infty = \bigsqcup_j K h_j^{-1}\mathbf{x}$$

with $h_j \in H(\mathbb{A}_{\mathbb{Q}}^\infty)$. Note that the disjoint union is finite since the K-orbits give an open cover of the compact set on the left-hand side.

Definition 2.3. For a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ and a K-invariant Schwartz function $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ we define the *weighted Kudla cycle*

$$Z(\underline{T}, \varphi^\infty)_K \stackrel{\mathrm{def}}{=} \sum_j \varphi^\infty(h_j^{-1}\mathbf{x}) Z(\underline{\mathbf{x}}, h_j)_K \in \mathrm{CH}^{\mathrm{rk}\,T}(X_K)_{\mathbb{C}}$$

as a weighted sum of the Hecke-translated Kudla cycles $Z(\underline{\mathbf{x}}, h_j)_K$ with $T(\mathbf{x}) = T$.

Unlike the latter, this cycle behaves well under pullback. Namely if $K' \subset K$ is another compact open subgroup and $\mathrm{pr} : X_{K'} \to X_K$ denotes the natural étale covering map, then

$$\mathrm{pr}^* Z(\underline{T}, \varphi^\infty)_K = Z(\underline{T}, \varphi^\infty)_{K'}$$

and we get a well-defined element of the direct limit $\mathrm{CH}^r(X)_\mathbb{C} \overset{\mathrm{def}}{=} \varinjlim_K \mathrm{CH}^r(X_K)_\mathbb{C}$, which justifies suppressing the subscript K from the notation.

The weighted Kudla cycle has codimension equal to the rank of T. In order to remove the dependence on it, we can take an intersection product inside the Chow group of X_K with a suitable power of the first Chern class of $\mathcal{L}_K^\vee$. This motivates the following

Definition 2.4. For a weighted Kudla cycle $Z(\underline{T}, \varphi^\infty)_K$ the corresponding *normalized Kudla cycle* is the cycle

$$Z(T, \varphi^\infty)_K \overset{\mathrm{def}}{=} Z(\underline{T}, \varphi^\infty)_K \cdot c_1(\mathcal{L}_K^\vee)^{r - \mathrm{rk}\, T} \in \mathrm{CH}^r(X_K)_\mathbb{C}.$$

It has exact codimension r in X_K.

2.2 Kudla's generating series of special cycles

Instead of studying each special cycle individually, we would like to find a way to treat all systematically. The natural idea is to form a generating series from them and the correct domain of definition in this case turns out to be the Hermitian symmetric domain of the quasi-split unitary group $\mathrm{U}(r, r)$ over F preserving the skew-Hermitian form

$$J_r = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}.$$

This Hermitian symmetric domain

$$\mathscr{H}_r = \left\{ \mathbf{z} = (\mathbf{z}_v)_v = (\mathbf{x}_v + \mathbf{y}_v i)_v \;\middle|\; \mathbf{x}_v, \mathbf{y}_v \in \mathrm{Herm}_r(\mathbb{R}), \mathbf{y}_v > 0 \;\forall\; v \in \mathcal{V}_{F,\infty} \right\}$$

is known as the *Hermitian upper half-plane of genus* r.

Definition 2.5. For a K-invariant Schwartz function $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ the *generating series of codimension* r *special cycles on* X_K is the formal sum

$$Z_{\varphi^\infty}(\mathbf{z}) \overset{\mathrm{def}}{=} \sum_{T \in \mathrm{Herm}_r(F)} Z(T, \varphi^\infty)_K q^T$$

for all $\mathbf{z} \in \mathscr{H}_r$ where

$$q^T \overset{\mathrm{def}}{=} \exp\left(2\pi i \sum_{v \in \mathcal{V}_{F,\infty}} \mathrm{Tr}\, T_v \mathbf{z}_v \right).$$

We will assume from now on that for every linear functional $l : \mathrm{CH}^r(X_K)_{\mathbb{C}} \to \mathbb{C}$ the resulting series $l(Z_{\varphi^\infty}(\mathbf{z}))$ is absolutely convergent, which by [Liu11a, Theorem 3.5] also implies that it is modular on $\mathscr{H}_r$. In particular, in order to obtain a scalar-valued function we can make use of the degree map

$$\deg : \mathrm{CH}^{n-1}(X_K(\mathbb{C}))_{\mathbb{C}} \longrightarrow \mathbb{C}$$

sending the class of a point to 1. More generally,

Definition 2.6. We define the *volume linear functional* as the map

$$\mathrm{vol} : \mathrm{CH}^r(X_K(\mathbb{C}))_{\mathbb{C}} \to \mathbb{C}$$

sending the class of a cycle $Z \in \mathrm{CH}^r(X_K(\mathbb{C}))_{\mathbb{C}}$ to

$$\mathrm{vol}\, Z \overset{\mathrm{def}}{=} \deg Z \cdot c_1(\mathcal{L}_K^\vee)^{n-1-r}.$$

In particular, for a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ and a K-invariant Schwartz function $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ the volume of the normalized Kudla cycle $Z(T,\varphi^\infty)_K$ is given by

$$\operatorname{vol} Z(T,\varphi^\infty)_K = \deg Z(\underline{T},\varphi^\infty)_K \cdot c_1(\mathcal{L}_K^\vee)^{n-1-\mathrm{rk}\,T}.$$

Note that taking the intersection product with $c_1(\mathcal{L}_K^\vee)$ determines a Lefschetz operator $L : \mathrm{CH}^r(X_K(\mathbb{C}))_\mathbb{C} \to \mathrm{CH}^{r+1}(X_K(\mathbb{C}))_\mathbb{C}$ and there is a similar corresponding operator on cohomology $L : H^{2r}_{dR}(X_K) \to H^{2r+2}_{dR}(X_K)$ given by wedge product with the Chern form of $\mathcal{L}_K^\vee$

$$\Omega_{\mathcal{L}_K^\vee} = \frac{i}{2\pi}\partial\overline{\partial}\log\|s\|^2_{h_{\mathcal{L}_K}}$$

where s is any non-vanishing holomorphic section of the metrized line bundle $\mathcal{L}_K$. These operators commute with the cycle class map $\mathrm{cl} : \mathrm{CH}^r(X_K(\mathbb{C}))_\mathbb{C} \to H^{2r}_{dR}(X_K)$ so that the diagram

$$\begin{array}{ccc} \mathrm{CH}^r(X_K(\mathbb{C}))_\mathbb{C} & \xrightarrow{c_1(\mathcal{L}_K^\vee)} & \mathrm{CH}^{r+1}(X_K(\mathbb{C}))_\mathbb{C} \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^{2r}_{dR}(X_K) & \xrightarrow{\Omega_{\mathcal{L}_K^\vee}} & H^{2r+2}_{dR}(X_K) \end{array}$$

is commutative. It follows therefore that we can compute the volume of a special cycle as

$$\operatorname{vol} Z(T,\varphi^\infty)_K = \int_{Z(T,\varphi^\infty)_K} \Omega_{\mathcal{L}_K^\vee}^{n-1-\mathrm{rk}\,T}. \tag{2.2}$$

The main result of this section is the geometric Siegel-Weil formula which asserts that the volume of the generating series $Z_{\varphi^\infty}(\mathbf{z})$ is a modular form on $\mathscr{H}_r$.

Theorem 2.7 (Geometric Siegel-Weil formula). *Let $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$ be the Bruhat-Schwartz function whose non-archimedean component is φ^∞ and whose archimedean one is given by*

$$\bigotimes_{v\neq\iota} \varphi^0_\infty \otimes \left(\varphi^r_{KM} \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee}\Big|_{\mathbb{1}_{n-1,n-1}}\right) \in \mathcal{S}(V(\mathbb{A}_{F,\infty})^r)$$

such that:

① φ^r_{KM} *is the Kudla-Milson form from Subsection 2.2.2 at the distinguished place* ι, *and the wedge product is evaluated on a properly oriented vector* $\mathbb{1}_{n-1,n-1} \in \bigwedge^{n-1,n-1} \mathfrak{p}$ *with* $\mathfrak{p}$ *the* -1 *eigenspace in the Cartan decomposition of* $\mathrm{Lie}(H)$;

② $\varphi^0_\infty(\mathbf{x}) = e^{-2\pi \operatorname{Tr} T(\mathbf{x})_v}$ *is the Gaussian function on* $V(F_v)^r$ *at all other archimedean places* v *of* F.

Then for all $\mathbf{z} \in \mathscr{H}_r$ *and all codimensions* $1 \leq r \leq n-1$ *the identity*

$$\operatorname{vol} Z_{\varphi^\infty}(\mathbf{z}) = \operatorname{vol}(X_K)\mathcal{E}(\mathbf{z}, s_0, \varphi) \tag{2.3}$$

holds at the special value $s_0 = \frac{n-r}{2}$ *where*

$$\mathcal{E}(\mathbf{z}, s, \varphi) = \prod_{v \in \mathcal{V}_{F,\infty}} (\det \mathbf{y}_v)^{-\frac{n}{2}} \sum_{\gamma \in P(F) \backslash G(F)} \Phi_\varphi(\gamma \mathbf{z}, s)$$

is a classical Eisenstein series of genus r *and weight* n.

Remark 2.8. This result is well-known in the case of orthogonal Shimura varieties by [Kud03] and generalizes to other classical groups as well. However, we did not find a proof in the literature for the unitary case of particular interest to us and therefore provide it here for completeness.

2.2.1 The classical Siegel-Weil formula

The first step towards the proof of Theorem 2.7 will be to relate both sides to a familiar identity, namely the Siegel-Weil formula. We recall the setup for a pair of unitary groups from [Liu11a] and [Liu11b]. Let W be the standard split skew-Hermitian space over E/F of dimension $2r$ with unitary group $G = \mathrm{U}(W) = \mathrm{U}(r, r)$. We fix a corresponding ordered basis $\{e_i\}_{i=1}^{2r}$ such that its inner product $\langle \cdot, \cdot \rangle$ satisfies

$$\left\langle e_i, e_j \right\rangle_{1 \leq i,j \leq 2r} = J_r.$$

The subspaces $X = \operatorname{span}\{e_i\}_{i=1}^{r}$ and $Y = \operatorname{span}\{e_i\}_{i=r+1}^{2r}$ are then maximal isotropic and form a complete polarization $W = X \bigoplus Y$. The *Siegel parabolic subgroup* $P \subset G$ stabilizing the flag $0 \subset Y \subset W$ has a Levi decomposition $P = MN$ with

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\overline{a}^{-1} \end{pmatrix} \middle| \, a \in \mathrm{GL}_r(E) \right\}$$

$$N = \left\{ n(b) = \begin{pmatrix} \mathbb{1}_r & b \\ 0 & \mathbb{1}_r \end{pmatrix} \middle| \, b \in \mathrm{Herm}_r(F) \right\}$$

and together with J_r it generates G.

Consider now the adèlic points of these groups. Given a Hecke character $\chi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$, we can extend it to a character on $P(\mathbb{A}_F)$ that is trivial on $N(\mathbb{A}_F)$ by the formula

$$\chi(m(a)n(b)) = \chi(\det a).$$

Definition 2.9. For any complex number $s \in \mathbb{C}$ the *degenerate principal series representation of* $G(\mathbb{A}_F)$

$$I(\chi, s) \stackrel{\text{def}}{=} \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \left(\chi |\cdot|_E^{s+\frac{r}{2}} \right)$$

is the non-normalized smooth parabolic induction from $P(\mathbb{A}_F)$ of the character $\chi|\cdot|_E^{s+\frac{r}{2}}$. More concretely, if K is the maximal compact subgroup

$$\prod_{v \in \mathcal{V}_F^\infty} G(\mathcal{O}_v) \prod_{v \in \mathcal{V}_{F,\infty}} \mathrm{U}(r)_\mathbb{R} \times \mathrm{U}(r)_\mathbb{R}$$

of $G(\mathbb{A}_F)$, then this representation consists of the smooth K-finite functions $\Phi : G(\mathbb{A}_F) \to \mathbb{C}$ satisfying

$$\Phi(m(a)n(b)g) = \chi(\det a) |\det a|_E^{s+\frac{r}{2}} \Phi(g)$$

for all $m(a) \in M(\mathbb{A}_F), n(b) \in N(\mathbb{A}_F), g \in G(\mathbb{A}_F)$.

By the Iwasawa decomposition $G = MNK$, a section $\Phi \in I(\chi, s)$ is uniquely determined by its values on K. We call therefore such a section *standard* if these values are independent of the choice of s, and *unramified* if $\Phi(-, s)\big|_K = 1$.

Definition 2.10. For a standard section $\Phi(-, s) \in I(\chi, s)$ we define the *Siegel Eisenstein series* as

$$E(g, s, \Phi) \stackrel{\text{def}}{=} \sum_{\gamma \in P(F)\backslash G(F)} \Phi(\gamma g, s).$$

It is absolutely convergent for $\mathrm{Re}(s) > \frac{r}{2}$ and has a meromorphic continuation to the entire complex plane which is holomorphic at $s = 0$.

Let now V be a Hermitian space over E/F of dimension n with inner product $(\cdot, \cdot)$ and unitary group $H = \mathrm{U}(V)$. The tensor product $\mathbb{W} = \mathrm{Res}_{E/F} W \otimes_E V$ admits the natural alternating form $\mathrm{Tr}_{E/F} \overline{\langle \cdot, \cdot \rangle} \otimes (\cdot, \cdot)$ making it into a symplectic space of dimension $4rn$ over F, and the groups G and H form a dual reductive pair of type I inside $\mathrm{Sp}(\mathbb{W})$. We fix now:

- the standard additive character

$$\psi = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}} : \mathbb{A}_F/F \to \mathbb{C}^\times \tag{2.4}$$

  where $\psi_{\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ is given by $\psi_{\mathbb{Q},p}(x) = e^{-2\pi i \{x_p\}},\ \psi_{\mathbb{Q},\infty}(x) = e^{2\pi i x_\infty}$;

- a splitting character $\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ such that

$$\chi\big|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}^n \tag{2.5}$$

  where $\varepsilon_{E/F} : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$ is the quadratic character attached to the extension E/F by class field theory. We associate to this character χ a sequence of integers $(l_v^\chi)_{v \in \mathcal{V}_{F,\infty}}$ at the archimedean places of F such that

$$\chi_v(z) = z^{l_v^\chi}$$

for all $v \in \mathcal{V}_{F,\infty}$ and all $z \in E_v^{\times,1} \cong \mathbb{C}^{\times,1}$ (see [Liu11a, Section 3A]). In particular n and l_v^χ have the same parity.

If $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F) \to \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F)$ is the metaplectic double cover, then with respect our choice of χ there is a splitting

$$\begin{array}{ccc} & & \mathrm{Mp}(\mathbb{W})(\mathbb{A}_F) \\ & \overset{\widetilde{\iota}_\chi}{\nearrow} & \downarrow \\ G(\mathbb{A}_F) \times H(\mathbb{A}_F) & \xrightarrow{\iota} & \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F) \end{array}$$

and the Weil representation ω_χ of $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F)$ pulls back to a representation of $G(\mathbb{A}_F) \times H(\mathbb{A}_F)$ with a Schrödinger model on the Bruhat-Schwartz space $\mathcal{S}(V(\mathbb{A}_F)^r)$. Its action is given explicitly by the formulas

- $\omega_\chi(m(a))\varphi(\mathbf{x}) = \chi(\det a)|\det a|_E^{\frac{n}{2}}\varphi(\mathbf{x}a)$ $\qquad m(a) \in M(\mathbb{A}_F)$;
- $\omega_\chi(n(b))\varphi(\mathbf{x}) = \psi(\mathrm{Tr}\, bT(\mathbf{x}))\varphi(\mathbf{x})$ $\qquad n(b) \in N(\mathbb{A}_F)$;
- $\omega_\chi(J_r)\varphi(\mathbf{x}) = c_V^r\widehat{\varphi}(\mathbf{x})$ $\qquad J_r = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$;
- $\omega_\chi(h)\varphi(\mathbf{x}) = \varphi(h^{-1}\mathbf{x})$ $\qquad h \in H(\mathbb{A}_F)$.

Here c_V is the Weil constant of the quadratic space V and

$$\widehat{\varphi}(\mathbf{x}) \overset{\mathrm{def}}{=} \int_{V(\mathbb{A}_F)^r} \varphi(\mathbf{y})\psi((\mathbf{x},\mathbf{y}))\, d\mathbf{y}$$

is the Fourier transform of $\varphi(\mathbf{x})$ with respect to ψ and its associated self-dual measure $d\mathbf{y}$.

Definition 2.11. For a Bruhat-Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$ we define the *theta function associated to* φ as

$$\theta(g,h,\varphi) \overset{\mathrm{def}}{=} \sum_{\mathbf{x}\in V(F)^r} (\omega_\chi(g,h)\varphi)(\mathbf{x}) = \sum_{\mathbf{x}\in V(F)^r} (\omega_\chi(g)\varphi)(h^{-1}\mathbf{x}).$$

It is absolutely convergent and slowly increasing as a function on $G(F)\backslash G(\mathbb{A}_F) \times H(F)\backslash H(\mathbb{A}_F)$.

Definition 2.12. Given a Bruhat-Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$, we can associate to it a canonical Siegel-Weil section $\Phi_\varphi(-, s)$ of the degenerate principal series representation by the formula

$$\Phi_\varphi(g, s) \stackrel{\text{def}}{=} (\omega_\chi(g)\varphi)(\mathbf{0})|\det a|_E^{s-\frac{n-r}{2}} \in I(\chi, s).$$

Then by abuse of notation we define the Eisenstein series

$$E(g, s, \varphi) \stackrel{\text{def}}{=} \sum_{\gamma \in P(F)\backslash G(F)} \Phi_\varphi(\gamma g, s).$$

Finally we set

$$I(g, \varphi) \stackrel{\text{def}}{=} \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(g, h, \varphi)\, dh$$

to be the theta lift of the constant function 1 on $H(\mathbb{A}_F)$ with respect to a normalized measure dh. This theta lift is related to a special value of a Siegel Eisenstein series by the following result [Liu11a, Theorem 2.1]

Theorem 2.13 (Classical Siegel-Weil formula). *Let $s_0 = \frac{n-r}{2}$.*

① *If $n > 2r$, then $E(g, s_0, \varphi)$ is absolutely convergent and*

$$E(g, s_0, \varphi) = I(g, \varphi)$$

② *If $r < n \leq 2r$ and V is anisotropic, then $E(g, s, \varphi)$ is holomorphic at $s = s_0$ and*

$$E(g, s_0, \varphi) = I(g, \varphi)$$

③ *If $r = n$ and V is anisotropic, then $E(g, s, \varphi)$ is holomorphic at $s = s_0 = 0$ and*

$$E(g, 0, \varphi) = 2I(g, \varphi).$$

2.2.2 The Kudla-Milson form

In order to make use of the Siegel-Weil formula, we need to pick an appropriate Bruhat-Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$. Its archimedean component at the distinguished place ι will be a special *Kudla-Milson form* constructed in [KM86] whose properties we outline below. To that end, we switch the notation from the previous section to the archimedean case and let now V_ι be a Hermitian space over $\mathbb{C}$ of signature $(n-1, 1)$ and W_ι be a skew-Hermitian space over $\mathbb{C}$ with quasi-split unitary group $G_\iota = \mathrm{U}(r, r)_\mathbb{R}$. The unitary group $H_\iota = \mathrm{U}(V_\iota)$ has associated symmetric space $D_\iota = H_\iota / K_{H_\iota}$ with $K_{H_\iota} = \mathrm{U}(n-1)_\mathbb{R} \times \mathrm{U}(1)_\mathbb{R}$ a maximal compact subgroup. If we denote by $\Omega^{r,r}(D_\iota)$ the space of smooth differential forms on D_ι of type (r, r), then for any $1 \leq r \leq n-1$ the Kudla-Milson form φ^r_{KM} is an element of $[\mathcal{S}(V_\iota(\mathbb{R})^r) \otimes \Omega^{r,r}(D_\iota)]^{H_\iota(\mathbb{R})}$ satisfying the following properties [Kud97, Theorem 7.1]:

① it is closed, i.e.

$$d\varphi^r_{KM} = 0$$

for the standard differential $d = \partial + \overline{\partial}$;

② it is $H_\iota(\mathbb{R})$-invariant, i.e.

$$\varphi^r_{KM}(h\mathbf{x}, h\mathbf{z}) = \varphi^r_{KM}(\mathbf{x}, \mathbf{z})$$

for all $h \in H_\iota(\mathbb{R})$;

③ it is an eigenfunction of K_{G_ι} for the Weil representation of the pair (G_ι, H_ι) on $\mathcal{S}(V_\iota(\mathbb{R})^r)$, i.e.

$$\omega_\chi(k)\varphi^r_{KM} = \det(k)^n \varphi^r_{KM}$$

for all

$$k \in K_{G_\iota} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a + bi \in \mathrm{U}(r)_{\mathbb{R}} \right\};$$

④ it is compatible with wedge product in the sense that

$$\varphi_{KM}^{r_1} \wedge \varphi_{KM}^{r_2} = \varphi_{KM}^{r_1+r_2};$$

⑤ $\varphi_{KM}^{r}(\mathbf{0}) = \Omega^{r}_{\mathcal{L}_K^\vee}$.

Consider in addition a positive-definite Hermitian space V_+ over $\mathbb{C}$ of dimension n. Let

$$\varphi_+(\mathbf{x}) \overset{\text{def}}{=} e^{-2\pi \operatorname{Tr} T(\mathbf{x})} \in \mathcal{S}(V_+(\mathbb{R})^r)$$

be the Gaussian function and ω_+ be the Weil representation of the pair $(\mathrm{U}(V_+), \mathrm{U}(W))$ on $\mathcal{S}(V_+(\mathbb{R})^r)$. For a Hermitian matrix $T \in \mathrm{Herm}_r(\mathbb{R})$ and $\mathbf{x} \in V_+(\mathbb{R})^r$ with $T(\mathbf{x}) = T$ there is then a holomorphic Whittaker function

$$W_T(g) \overset{\text{def}}{=} \omega_+(g)\varphi_+(\mathbf{x}) \ \forall \ g \in G_\iota(\mathbb{R}).$$

Given an Iwasawa decomposition

$$g = \begin{pmatrix} \mathbb{1}_r & b \\ 0 & \mathbb{1}_r \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^t\overline{a}^{-1} \end{pmatrix} k,$$

we can unwind the Weil action explicitly to compute that

$$W_T(g) = |\det(a)|^{\frac{n}{2}} q^T \det(k)^n \tag{2.6}$$

where $q^T = e^{2\pi i \operatorname{Tr} T\mathbf{z}}$ with $\mathbf{z} = b + a{}^t\overline{a}i$. More generally,

Definition 2.14. For a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ the *holomorphic Whittaker function*

$$W_T(g) \stackrel{\text{def}}{=} \prod_{v \in \mathcal{V}_{F,\infty}} W_{T_v}(g_v) \ \forall \ g \in G(\mathbb{A}_{F,\infty})$$

is the product of the local ones over the archimedean places of F.

Returning to our global Hermitian space V over E/F, recall that for any $\mathbf{x} \in V(F)^r$ we have defined $r_{\mathbf{x}}, H_{\mathbf{x}}, D_{\mathbf{x}}$ as in Equation 2.1. Let now $\Gamma_{\mathbf{x}} \subset H_{\mathbf{x}}$ be a discrete, torsion-free, co-compact subgroup. There is a fibration

$$\mathrm{pr} : \Gamma_{\mathbf{x}} \backslash D \to \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x}}$$

obtained from the natural projection map $\mathrm{pr} : D \to D_{\mathbf{x}}$, and its Thom class admits the following nice characterization in terms of the Kudla-Milson form [KM87, Theorem 4.1]

Proposition 2.15 (Thom property). *For all $\mathbf{x} \in V(F)^r$, all $g \in G(\mathbb{A}_{F,\infty})$ and all closed and bounded forms $\eta \in \Omega^{n-1-r,n-1-r}(\Gamma_{\mathbf{x}} \backslash D)$ we have*

$$\int_{\Gamma_{\mathbf{x}} \backslash D} (\omega_\chi(g)\varphi_{KM}^r)(\mathbf{x}) \wedge \eta = W_{T(\mathbf{x})}(g) \int_{\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x}}} \Omega_{\mathcal{L}_K^\vee}^{r-r_{\mathbf{x}}} \wedge \mathrm{pr}_* \eta.$$

2.2.3 Proof of the geometric Siegel-Weil formula

We will now try to relate the right-hand side of Proposition 2.15 to the volume of the generating series of the special cycles, and the left-hand side to a theta integral, hence by extension to an Eisenstein series as we already know from the classical Siegel-Weil formula. For this purpose, we choose a Bruhat-Schwartz form

$$\widehat{\varphi} = \varphi^\infty \otimes \widehat{\varphi}_\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r) \otimes [\mathcal{S}(V(\mathbb{A}_{F,\infty})^r) \otimes \Omega^{r,r}(D)]^{H(\mathbb{A}_{F,\infty})}$$

so that its non-archimedean component φ^∞ matches the Bruhat-Schwartz function in the generating series $Z_{\varphi^\infty}(\mathbf{z})$ and the archimedean part satisfies

$$\widehat{\varphi}_\infty = \bigotimes_{v\neq\iota} \varphi^0_\infty \otimes \varphi^r_{KM}.$$

Here $\varphi^r_{KM} \in [\mathcal{S}(V(F_\iota)^r) \otimes \Omega^{r,r}(D)]^{H(F_\iota)}$ is the Kudla-Milson form at the distinguished place ι and

$$\varphi^0_\infty(\mathbf{x}) = e^{-2\pi \operatorname{Tr} T(\mathbf{x})}$$

is the Gaussian function of weight n at all other archimedean places.

For any $g \in G(\mathbb{A}_F)$ and $h \in H(\mathbb{A}_F^\infty)$ we can form the theta series

$$\widehat{\theta}(g,h,\widehat{\varphi}) = \sum_{\mathbf{x}\in V(F)^r} (\omega_\chi(g)\widehat{\varphi})(h^{-1}\mathbf{x})$$

and will view it naturally as a closed (r,r)-form $\widehat{\theta}(g,\widehat{\varphi})$ on X_K. Indeed, after fixing a finite choice of coset representatives

$$H(\mathbb{A}_F^\infty) = \bigsqcup_j H(F)h_jK$$

and setting $\Gamma_j = H(F) \cap h_jKh_j^{-1}$, there is a canonical isomorphism

$$\Omega^{r,r}(X_K) \cong [\Omega^{r,r}(D) \otimes C^\infty(H(\mathbb{A}_F^\infty))]^{H(F)\times K} \cong \prod_j \Omega^{r,r}(D)^{\Gamma_j}$$

where the last map is given by evaluation at h_j.

After grouping the terms in the theta series by their moment matrix $T(\mathbf{x})$, we arrive at the Fourier expansion

$$\widehat{\theta}(g,h,\widehat{\varphi}) = \sum_{T\in\operatorname{Herm}_r(F)} \sum_{\mathbf{x}\in\Omega_T(E)} (\omega_\chi(g)\widehat{\varphi})(h^{-1}\mathbf{x}) \overset{\text{def}}{=} \sum_{T\in\operatorname{Herm}_r(F)} \widehat{\theta}_T(g,h,\widehat{\varphi}). \tag{2.7}$$

The crucial property of these Fourier coefficients is that their de Rham cohomology class equals the product of a Whittaker function with a normalized Kudla cycle. More precisely,

Proposition 2.16. *For all $g \in G(\mathbb{A}_{F,\infty})$ and all $T \in \mathrm{Herm}_r(F)$ the identity*

$$\widehat{\theta}_T(g,\widehat{\varphi}) = W_T(g) Z(T,\varphi^\infty)_K$$

holds in $H^{2r}_{dR}(X_K)$.

Proof. If $\eta \in \Omega^{n-1-r,n-1-r}(X_K)$ is a closed and bounded form, then its canonical integration pairing $\langle -,- \rangle : \Omega^{r,r}(X_K) \otimes \Omega^{n-1-r,n-1-r}(X_K) \to \mathbb{C}$ with the left-hand side satisfies

$$\begin{aligned}
\left\langle \widehat{\theta}_T(g,\widehat{\varphi}),\eta \right\rangle &\overset{\text{def}}{=} \int_{X_K} \widehat{\theta}_T(g,\widehat{\varphi}) \wedge \eta \\
&= \sum_j \int_{\Gamma_j \backslash D} \widehat{\theta}_T(g,h_j,\widehat{\varphi}) \wedge \eta(h_j) \\
&= \sum_j \int_{\Gamma_j \backslash D} \sum_{\mathbf{x} \in \Omega_T(E)} (\omega_\chi(g)\widehat{\varphi})(h_j^{-1}\mathbf{x}) \wedge \eta(h_j) \\
&= \sum_j \sum_{\mathbf{x} \in \Gamma_j \backslash \Omega_T(E)} \int_{\Gamma_{j,\mathbf{x}} \backslash D} (\omega_\chi(g)\widehat{\varphi})(h_j^{-1}\mathbf{x}) \wedge \eta(h_j) \\
&= \sum_j \sum_{\mathbf{x} \in \Gamma_j \backslash \Omega_T(E)} \varphi^\infty(h_j^{-1}\mathbf{x}) \int_{\Gamma_{j,\mathbf{x}} \backslash D} (\omega_\chi(g)\widehat{\varphi}_\infty)(\mathbf{x}) \wedge \eta(h_j) \\
[\text{Proposition 2.15}] &= W_T(g) \sum_j \sum_{\mathbf{x} \in \Gamma_j \backslash \Omega_T(E)} \varphi^\infty(h_j^{-1}\mathbf{x}) \int_{\Gamma_{j,\mathbf{x}} \backslash D_{\mathbf{x}}} \Omega^{r-r_{\mathbf{x}}}_{\mathcal{L}_K^\vee} \wedge \eta(h_j) \\
&= W_T(g) \langle Z(T,\varphi^\infty)_K, \eta \rangle.
\end{aligned}$$

Here $\Gamma_{j,\mathbf{x}}$ denotes the stabilizer of $\mathbf{x}$ in Γ_j and the last equality comes from the decomposition formula [Kud97, Proposition 5.4] for the special cycle into classical cycles. The proposition now follows by the non-degeneracy of the pairing $\langle -,- \rangle$. ∎

Remark 2.17. The only non-zero Fourier coefficients $\widehat{\theta}_T(g,\widehat{\varphi})$ are in fact those for which the matrix T is positive semi-definite.

Corollary 2.18. *By setting $\eta = \Omega^{n-1-r}_{\mathcal{L}_K^\vee}$ in the proposition above we obtain the identity*

$$\int_{X_K} \widehat{\theta}_T(g, \widehat{\varphi}) \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee} = W_T(g) \int_{Z(T,\varphi^\infty)_K} \Omega^{n-1-\mathrm{rk}\,T}_{\mathcal{L}_K^\vee} = W_T(g) \operatorname{vol} Z(T, \varphi^\infty)_K$$

for all $g \in G(\mathbb{A}_{F,\infty})$ and all $T \in \mathrm{Herm}_r(F)$.

Consider now the real Lie group $H(F_\iota)$ at the distinguished archimedean place of F. If $\mathfrak{h}$ and $\mathfrak{k}$ denote the complexifications of the Lie algebras of $H(F_\iota) \cong \mathrm{U}(n-1,1)_{\mathbb{R}}$ and its maximal compact subgroup $\mathrm{U}(n-1)_{\mathbb{R}} \times \mathrm{U}(1)_{\mathbb{R}}$ respectively, then recall that there is a Cartan decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$$

corresponding to the ± 1 eigenspaces of a Cartan involution. It induces an identification

$$\begin{aligned}
\Omega^{n-1,n-1}(X_K) &\cong [\Omega^{n-1,n-1}(D) \otimes C^\infty(H(\mathbb{A}_F^\infty))]^{H(F)\times K} \\
&\cong [C^\infty(H(\mathbb{A}_{F,\infty})/K_\infty) \otimes \bigwedge^{n-1,n-1} \mathfrak{p}^* \otimes C^\infty(H(\mathbb{A}_F^\infty))]^{H(F)\times K} \\
&\cong [C^\infty(H(F)\backslash H(\mathbb{A}_F))]^{K_\infty K}
\end{aligned} \tag{2.8}$$

where the last isomorphism is given by evaluation on a properly oriented element $\mathbb{1}_{n-1,n-1} \in \bigwedge^{n-1,n-1} \mathfrak{p}$. We can normalize the measure on the right-hand side so that for any smooth function f on X_K, the image $\widetilde{f}$ of the top-degree differential form $f\Omega^{n-1}_{\mathcal{L}_K^\vee}$ under this isomorphism satisfies

$$\int_{X_K} f\Omega^{n-1}_{\mathcal{L}_K^\vee} = \mathrm{vol}(X_K) \int_{H(F)\backslash H(\mathbb{A}_F)} \widetilde{f}(h)\, dh. \tag{2.9}$$

In particular, if $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^r)$ is the Bruhat-Schwartz function

$$\varphi \overset{\mathrm{def}}{=} \widehat{\varphi} \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee}\big|_{\mathbb{1}_{n-1,n-1}},$$

then for the theta form $\widehat{\theta}_T(g,h,\widehat{\varphi})$ from Equation 2.7 viewed as a closed (r,r)-form on X_K we have the correspondence

$$\widehat{\theta}_T(g,h,\widehat{\varphi}) \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee} \longleftrightarrow \widehat{\theta}_T(g,h,\widehat{\varphi}) \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee}\Big|_{\mathbb{1}_{n-1,n-1}} = \theta_T(g,h,\varphi)$$

where the last term is the theta function

$$\theta_T(g,h,\varphi) = \sum_{\mathbf{x}\in V(F)^r} (\omega_\chi(g)\varphi)(h^{-1}\mathbf{x})$$

associated to φ from Definition 2.11. The normalization in Equation 2.9 tells us therefore that

$$\int_{X_K} \widehat{\theta}_T(g,\widehat{\varphi}) \wedge \Omega^{n-1-r}_{\mathcal{L}_K^\vee} = \mathrm{vol}(X_K) \int_{H(F)\backslash H(\mathbb{A}_F)} \theta_T(g,h,\varphi).$$

The left-hand side is now reminiscent of Corollary 2.18 and the right-hand side of the classical Siegel-Weil formula from Theorem 2.13, so combining the two results we get

$$\mathrm{vol}(X_K)E_T(g,s_0,\varphi) = \mathrm{vol}(X_K) \int_{H(F)\backslash H(\mathbb{A}_F)} \theta_T(g,h,\varphi)\,dh = W_T(g)\,\mathrm{vol}\,Z(T,\varphi^\infty)_K \;\forall\; g\in G(\mathbb{A}_{F,\infty})$$

at the special value $s_0 = \frac{n-r}{2}$. When $g = \mathbf{z} = \mathbf{x}+\mathbf{y}i \in \mathscr{H}_r$, the value of the Whittaker function is given explicitly by the formula

$$W_T(\mathbf{z}) = q^T \prod_{v\in\mathcal{V}_{F,\infty}} (\det \mathbf{y}_v)^{\frac{n}{2}}$$

coming from Definition 2.14 and Equation 2.6. Therefore the geometric Siegel-Weil formula follows by setting the classical Eisenstein series

$$\mathcal{E}(\mathbf{z},s_0,\varphi) \stackrel{\mathrm{def}}{=} E(\mathbf{z},s_0,\varphi) \prod_{v\in\mathcal{V}_{F,\infty}} (\det \mathbf{y}_v)^{-\frac{n}{2}}.$$

2.3 Yuan-Zhang-Zhang's geometric pullback formula

We showed in the beginning of this section how to attach a generating series of special cycles to a unitary Shimura variety. Given now two such varieties and a morphism between them, we would like to study what happens with their respective generating series under pullback along this morphism. In order to do this in greater generality, we first describe the generating series as a function on the adèlic points.

An E-vector subspace $V' \in V(\mathbb{A}_F^\infty)$ is called *admissible* if the inner product on V' takes E-rational values and is positive-definite. For an element $\mathbf{x} \in K\backslash V(\mathbb{A}_F^\infty)^r$ we denote by $\underline{\mathbf{x}}$ the subspace of $V(\mathbb{A}_F^\infty)$ generated by the E-linear span of its components. Then on X_K we define the Kudla special cycle

$$Z(\mathbf{x})_K \stackrel{\text{def}}{=} \begin{cases} Z(\underline{\mathbf{x}})_K \cdot c_1(\mathcal{L}_K^\vee)^{r-r_{\mathbf{x}}} & , \ \underline{\mathbf{x}} \text{ is admissible} \\ \emptyset & , \text{ otherwise.} \end{cases}$$

Definition 2.19. For a K-invariant Bruhat-Schwartz function $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ the *adèlic lift of the generating series of codimension r special cycles on X_K* is the function on $G(\mathbb{A}_F)$ given by

$$Z_{\varphi^\infty}(g) \stackrel{\text{def}}{=} \sum_{T \in \mathrm{Herm}_r(F)} \left(Z(T, \omega_\chi(g^\infty)\varphi^\infty)_K \prod_{v \in \mathcal{V}_{F,\infty}} \omega_\chi(g_v)\varphi_\infty^0(T_v) \right).$$

It has a Fourier-Whittaker expansion

$$Z_{\varphi^\infty}(g) = \sum_{\mathbf{x} \in K\backslash V(\mathbb{A}_F^\infty)^r} \left((\omega_\chi(g^\infty)\varphi^\infty)(\mathbf{x}) Z(\mathbf{x})_K \prod_{v \in \mathcal{V}_{F,\infty}} W_{T(\mathbf{x})_v}(g_v) \right)$$

where the Whittaker functions are well-defined since only those terms for which $T(\mathbf{x}) \in \mathrm{Herm}_r(F)$ contribute to the sum.

In view of this definition, we can perform adèlic lift on both sides of Theorem 2.7 to obtain the following

Corollary 2.20 (Adèlic geometric Siegel-Weil formula). *Under the same assumptions as in Theorem 2.7, for any $g \in G(\mathbb{A}_F)$ the identity*

$$\operatorname{vol} Z_{\varphi^\infty}(g) = \operatorname{vol}(X_K) E(g, s_0, \varphi)$$

holds at the special value $s_0 = \frac{n-r}{2}$.

Consider now a subspace $V_0 \subset V$ of dimension m with orthogonal complement $V_0^\perp$ whose signature is $(m-1, 1)$ at the distinguished archimedean place ι and is positive-definite at the rest. Analogously to Equation 2.1, we denote

$$H_0 = \mathrm{U}(V_0) = \operatorname{Stab}_H V_0^\perp$$
$$D_0 = \{z \in D \mid z \subseteq V_0\}$$
$$K_0 = K \cap H_0(\mathbb{A}_F^\infty).$$

The natural inclusion $i : H_0 \hookrightarrow H$ induces an embedding of unitary Shimura varieties

$$i : X_{K_0} = \mathrm{Sh}(H_0, D_0)_{K_0} \hookrightarrow X_K = \mathrm{Sh}(H, D)_K$$

with $\dim X_{K_0} = m - 1$ and we let

$$i^* : \mathrm{CH}^r(X_K)_\mathbb{C} \longrightarrow \mathrm{CH}^r(X_{K_0})_\mathbb{C}$$

be the pullback map on their Chow groups along this morphism.

The Bruhat-Schwartz space of V^r admits a decomposition

$$\mathcal{S}(V(\mathbb{A}_F)^r) = \mathcal{S}(V_0(\mathbb{A}_F)^r) \otimes \mathcal{S}(V_0^\perp(\mathbb{A}_F)^r),$$

and for $\varphi^\infty \in \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$ we can pullback the normalized Kudla cycle $Z(T, \varphi^\infty)_K$ on X_K along

i to get a cycle on X_{K_0}. By applying this construction to the entire generating series $Z_{\varphi^\infty}(g)$, we obtain the following identity [YZZ09, Proposition 3.1]

Theorem 2.21. *Let the Bruhat-Schwartz function*

$$\varphi^\infty = \varphi_1^\infty \otimes \varphi_2^\infty \in \mathcal{S}(V_0(\mathbb{A}_F^\infty)^r) \otimes \mathcal{S}(V_0^\perp(\mathbb{A}_F^\infty)^r) = \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$$

be such that both φ_1^∞ and φ_2^∞ are K-invariant. Then for all $g \in G(\mathbb{A}_F)$ and all codimensions $1 \le r \le m-1$ the identity

$$i^*(Z_{\varphi^\infty}(g)) = Z_{\varphi_1^\infty}(g)\theta(g, \varphi_2)$$

holds in the group $\mathrm{CH}^r(X_{K_0})_{\mathbb{C}}$. Here

$$\theta(g, \varphi_2) = \sum_{\mathbf{x} \in V_0^\perp(F)^r} (\omega_\chi(g)\varphi_2)(\mathbf{x})$$

is the theta function on $V_0^\perp$ from Definition 2.11 with archimedean component

$$\varphi_{2,\infty}(\mathbf{x}) = \prod_{v \in \mathcal{V}_{F,\infty}} \varphi_\infty^0(T(\mathbf{x})_v)$$

given by the Gaussian function.

Proof. If $\mathbf{x} \in V(\mathbb{A}_F^\infty)^r$, then by an analogue of [YZZ09, Proposition 2.6] the intersection components of $Z(\mathbf{x})_K$ and $Z(V_0^\perp)_K$ are indexed by admissible classes in $K\backslash KV_0^\perp + K\mathbf{x}$. For such a class $(V_0^\perp, k\mathbf{x})$ the projection $\mathbf{z}$ of $k\mathbf{x}$ to $V_0^\perp(\mathbb{A}_F^\infty)^r$ must lie in $V_0^\perp(F)^r$ by the definition of admissibility, so $\mathbf{y} \overset{\mathrm{def}}{=} k\mathbf{x} - \mathbf{z} \in V_0(\mathbb{A}_F^\infty)^r$. Conversely, given $\mathbf{y} + \mathbf{z} \in V_0(\mathbb{A}_F^\infty)^r \oplus V_0^\perp(F)^r$ the class $(V_0^\perp, \mathbf{y} + \mathbf{z})$ is admissible. Therefore in the Chow group of X_{K_0} the pullback of $Z(\mathbf{x})_K$ admits a decomposition

$$i^*(Z(\mathbf{x})_K) = \sum_{(\mathbf{y},\mathbf{z})} Z(\mathbf{y})_{K_0}$$

where the sum is over all admissible $\mathbf{y} \in K_0 \backslash V_0(\mathbb{A}_F^\infty)^r$ and all $\mathbf{z} \in V_0^\perp(F)^r$. It follows that

$$\begin{aligned} i^*(Z_{\varphi^\infty}(g)) &= \sum_{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^r} (\omega_\chi(g^\infty)\varphi^\infty)(\mathbf{x}) i^*(Z(\mathbf{x})_K) \prod_{v \in \mathcal{V}_{F,\infty}} W_{T(\mathbf{x})_v}(g_v) \\ &= \sum_{\mathbf{y} \in K_0 \backslash V_0(\mathbb{A}_F^\infty)^r} (\omega_\chi(g^\infty)\varphi_1^\infty)(\mathbf{y}) Z(\mathbf{y})_{K_0} \prod_{v \in \mathcal{V}_{F,\infty}} W_{T(\mathbf{y})_v}(g_v) \\ &\times \sum_{\mathbf{z} \in V_0^\perp(F)^r} (\omega_\chi(g^\infty)\varphi_2^\infty)(\mathbf{z}) \prod_{v \in \mathcal{V}_{F,\infty}} W_{T(\mathbf{z})_v}(g_v) \\ &= Z_{\varphi_1^\infty}(g)\theta(g, \varphi_2) \end{aligned}$$

as claimed. ■

Chapter 3: Analytic side – Integral representation of L-functions for quasi-split unitary groups

3.1 Statement of the integral representation

Let E/F be a CM-extension with F a totally real field of degree d over $\mathbb{Q}$ and $G = \mathrm{U}(r,r)$ be the quasi-split unitary group over F preserving the skew-Hermitian form

$$J_r = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}.$$

If π is a cuspidal automorphic representation of G, we will show in this chapter that its standard L-function possesses a Rankin-Selberg integral representation against a suitable theta series akin to the classical Shimura integral described in [Shi73] and [Shi75]. For this purpose, let first V' be a Hermitian space over E/F of dimension r with inner product represented by the matrix $\mathcal{T} \in \mathrm{Herm}_r(F)$.

Definition 3.1. We fix a finite subset $S \subset \mathcal{V}_F$ of the places of F that contains:

- all archimedean places of F and all finite ones that are ramified in E;
- all ramified places of the representation $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$;
- all finite places v at which the character $\chi_v \overset{\text{def}}{=} \chi\big|_{E_v^\times}$ is ramified;
- all finite places v at which the local Hermitian space $V'(F_v)$ is non-split.

With this definition, we now have the following identity [Qin07, Theorem 6.3]

Theorem 3.2 (Integral representation of the L-function). *Let:*

① $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$ *be a cuspidal automorphic representation of the group* G*;*

② $f \in \pi = \bigotimes'_{v \in \mathcal{V}_F} \pi_v$ *be a cusp form that is* K_v*-fixed for all* $v \notin S$*;*

③ $\Phi(g,s) = \prod_{v \in \mathcal{V}_F} \Phi_v(g,s) \in \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi|\cdot|_E^{s+\frac{r}{2}})$ *be a section of the degenerate principal series representation of* G *such that for all* $v \notin S$

$$\Phi_v(-,s)\big|_{K_v} = 1;$$

④ $\varphi = \prod_{v \in \mathcal{V}_F} \varphi_v \in \mathcal{S}(V'(\mathbb{A}_F)^r)$ *be a Bruhat-Schwartz function such that for all* $v \notin S$

$$\varphi_v = \mathbb{1}_{V'(O_v)^r}.$$

Then the Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma \in P(F)\backslash G(F)} \Phi(\gamma g, s)$$

and the theta series

$$\theta(g,\varphi) = \sum_{\mathbf{x} \in V'(F)^r} (\omega_\chi(g)\varphi)(\mathbf{x})$$

satisfy the identity

$$\int_{G(F)\backslash G(\mathbb{A}_F)} f(g)E(g,s,\Phi)\theta(g,\varphi)\,dg = I_S(s,\varphi,\Phi,f)\frac{L^S\left(s+\frac{1}{2},\pi\right)}{D^S(s)}$$

for all values of $s \in \mathbb{C}$*. Here* $D^S(s) = \prod_{v \notin S} D_v(s) = \prod_{v \notin S} d_v(s) j_v(s)$ *(see Theorem 3.4 and Lemma*

3.7) with

$$D_v(s) = \begin{cases} \prod_{i=1}^{2r} L_v(2s+i, \varepsilon^i_{E/F}) \prod_{i=r+1}^{2r} L_v(2s+i, \varepsilon^i_{E/F}) & , v \text{ inert in } E \\ \prod_{i=1}^{2r} \zeta_v(2s+i) \prod_{i=r+1}^{2r} \zeta_v(2s+i) & , v \text{ split in } E \end{cases}$$

and $I_S(s, \varphi, \Phi, f) = \prod_{v \in S} I_v(s, \varphi_v, \Phi_v, f)$ *with*

$$I_v(s, \varphi_v, \Phi_v, f) = \int_{K_v} \int_{M_v} f_{\mathcal{T}}(m(a_v)k_v)\Phi_v(k_v, s)(\chi_v^2 |\cdot|_E^s)(\det a_v) \times (\omega_\chi(k_v)\varphi_v)(a_v)\, d^\times a_v\, dk_v$$

where $f_{\mathcal{T}}$ *denotes the* $\mathcal{T}$*-th Fourier coefficient of* f*.*

3.2 The doubling method

The classical Rankin-Selberg integral constructed by Jacquet, Piatetski-Shapiro and Shalika is well-known to represent the L-function of cuspidal automorphic representations of $\mathrm{GL}_n \times \mathrm{GL}_m$ and, as a special case of that, GL_n. For other reductive groups whose representations are not necessarily generic, e.g. Sp_{2n}, the doubling method of Piatetski-Shapiro and Rallis provides a similar way of obtaining integral representations of their automorphic L-functions. We shall use in particular the results by [Qin07] and [Liu11a] on the doubling method for quasi-split unitary groups and review them here for completeness.

Let us fix an unramified non-archimedean place v of F and set $E_v = E \otimes_F F_v$. Then E_v is either:

- a quadratic extension of F_v with involution ρ the non-trivial element of $\mathrm{Gal}(E_v/F_v)$ if v is inert in E;
- a direct sum of two copies of F_v with involution ρ exchanging the two factors if v is

split in E.

In either case, consider a skew-Hermitian space W over E_v/F_v with unitary group $G(F_v) = \mathrm{U}(W)$ and choose an ordered basis $\{e_i\}_{i=1}^{2r}$ such that its skew-Hermitian form $\langle\cdot,\cdot\rangle$ satisfies

$$\left\langle e_i, e_j \right\rangle_{1\le i,j\le 2r} = J_r.$$

We denote by $-W$ the skew-Hermitian space whose skew-Hermitian form is $-\langle\cdot,\cdot\rangle$ and fix an ordered basis $\left\{e_i^-\right\}_{i=1}^{2r}$ such that

$$-\left\langle e_i^-, e_j^- \right\rangle_{1\le i,j\le 2r} = -J_r.$$

Let $W^2 = W \oplus -W$ be their direct sum endowed with the skew-Hermitian form $\langle\cdot,\cdot\rangle \oplus -\langle\cdot,\cdot\rangle$ and whose unitary group we denote by $G^2(F_v) = \mathrm{U}(W^2)$. Then under the ordered basis

$$\{e_1,\ldots,e_r; e_1^-,\ldots,e_r^-; e_{r+1},\ldots,e_{2r}; e_{r+1}^-,\ldots,e_{2r}^-\}$$

of W^2 there is a closed embedding $i : G(F_v)\times G(F_v) \hookrightarrow G^2(F_v)$ given explicitly by $i(g_1,g_2) = i_0(g_1, g_2^\vee)$ where

$$g^\vee = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix} g \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}^{-1}$$

and

$$i_0\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

For the complete polarization $W^2 = X \oplus Y$ with

$$X = \{(w,-w) \mid w \in W\},\ Y = \{(w,w) \mid w \in W\}$$

we denote by $Q(F_v)$ the maximal parabolic subgroup of $G^2(F_v)$ preserving Y. A general element of $Q(F_v)$ can be written as $m(a)n(b)$ with

$$m(a) \in M(F_v) = \left\{ \begin{pmatrix} a & \\ & {}^t a^{-\rho} \end{pmatrix} \middle| \, a \in \mathrm{GL}_{2r}(E_v) \right\}$$

a Levi subgroup of $Q(F_v)$ and

$$n(b) \in N(F_v) = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \middle| \, b \in \mathrm{Herm}_{2r}(F_v) \right\}$$

a unipotent radical of $Q(F_v)$.

Let $\gamma_v : F_v^\times \to \mathbb{C}^\times$ be an unramified character. We consider the space $I(\gamma_v, s)$ of functions $f : G^2(F_v) \to \mathbb{C}$ satisfying

$$f(m(a)n(b)g) = \gamma(\det a)|\det a|_{F_v}^{s+\frac{4r+1}{2}} f(g)$$

for all $m(a)n(b) \in Q(F_v)$ and $g \in G^2(F_v)$. It has a natural right multiplication action by the group $G^2(F_v)$ and if $I(\gamma_v, s)^{K_v^2}$ is the subspace fixed by the maximal compact subgroup $K_v^2 \stackrel{\text{def}}{=} G^2(\mathcal{O}_v)$, then by Frobenius reciprocity for the unramified character γ_v we have

$$\dim I(\gamma_v, s)^{K_v^2} = 1.$$

The unique therefore K_v^2-invariant function f_0 with value 1 on K_v^2 turns out to be also $K_v \stackrel{\text{def}}{=} G(\mathcal{O}_v)$ bi-invariant. More precisely, [Li92, Lemma 3.2]

Lemma 3.3. *For all $k_1, k_2 \in K_v$ and all $g_v \in G(F_v)$ the identity*

$$f_0(k_1 g k_2, 1) = f_0(g, 1)$$

holds under the embedding $(g,1) \in G(F_v) \times G(F_v) \overset{i}{\hookrightarrow} G^2(F_v)$.

3.3 L-functions for quasi-split unitary groups

Let π_v be an irreducible unramified representation of $G(F_v)$ and $\pi_v^\vee$ be its contragradient representation realized on the space of complex conjugates of functions in π_v. If

$$\left\langle f, f^\vee \right\rangle \overset{\text{def}}{=} \int_{G(F_v)} f(g) f^\vee(g)\, dg$$

is the canonical pairing between them with respect to a Haar measure on $G(F_v)$, then we can define a matrix coefficient of π_v by the formula

$$\omega_{\pi_v}(g, f, f^\vee) \overset{\text{def}}{=} \left\langle \pi_v(g) f, f^\vee \right\rangle \; \forall\, g \in G(F_v).$$

Moreover, if f and $f^\vee$ are K_v-fixed vectors of π_v and $\pi_v^\vee$ respectively with $\langle f, f^\vee \rangle = 1$, then

$$\omega_{\pi_v}(g) \overset{\text{def}}{=} \omega_{\pi_v}(g, f, f^\vee)$$

is the zonal spherical function of π_v with respect to K_v.

The Langlands dual group of $G(F_v)$ is

$$^{L}G(F_v) = \begin{cases} \mathrm{GL}_{2r}(\mathbb{C}) \rtimes \mathrm{Gal}(E_v/F_v) & , \; v \text{ inert in } E \\ \mathrm{GL}_{2r}(\mathbb{C}) & , \; v \text{ split in } E, \end{cases}$$

and in the former case the action of $\mathrm{Gal}(E_v/F_v)$ on $\mathrm{GL}_{2r}(\mathbb{C})$ is given by

$$g^\rho = A_{2r}\, {}^t g^{-1} A_{2r}^{-1} \; \forall\, g \in \mathrm{GL}_{2r}(\mathbb{C})$$

where

$$A_{2r} = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}.$$

Since π_v is an irreducible unramified representation of $G(F_v)$, by the Satake isomorphism it determines a unique semisimple conjugacy class τ in ${}^L G(F_v)$ whose representative can be taken of the form

$$\tau = \begin{cases} \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} & , \ v \text{ inert in } E \\ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{2r} \end{pmatrix} & , \ v \text{ split in } E \end{cases}$$

with $a_i \in \mathbb{C}^\times$ for all $1 \leq i \leq 2r$ by [Bor79, Section 6.9].

Let now μ be the natural action of $\mathrm{GL}_{2r}(\mathbb{C})$ on $\mathbb{C}^{2r}$ and σ be the induced representation

$$\sigma = \begin{cases} \mathrm{Ind}_{\mathrm{GL}_{2r}(\mathbb{C})}^{{}^L G} \mu & , \ v \text{ inert in } E \\ \mathrm{Ind}_{\mathrm{GL}_{2r}(\mathbb{C})}^{\mathrm{GL}_{2r}(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}} \mu & , \ v \text{ split in } E. \end{cases}$$

We can associate to it a local L-factor $L(s,\pi_v)$ – see [Qin07, Section 4] – given by

$$L(s,\pi_v) \overset{\text{def}}{=} \begin{cases} \det(1-\sigma(\tau)q^{-s})^{-1} = \prod_{i=1}^{r}\left((1-a_iq^{-2s})(1-a_i^{-1}q^{-2s})\right)^{-1} & , \ v \text{ inert in } E \\ \det(1-\sigma(\tau)q^{-s})^{-1} = \prod_{i=1}^{2r}\left((1-a_iq^{-s})(1-a_i^{-1}q^{-s})\right)^{-1} & , \ v \text{ split in } E \end{cases}$$

where q is the cardinality of the residue field of F_v. The relationship between the functions f_0, ω_{π_v} and the local L-factor above is explained by the following result [Li92, Theorem 3.1]

Theorem 3.4. *For all $s \in \mathbb{C}$ we have*

$$\int_{G(F_v)} f_0(g,1)\omega_{\pi_v}(g)\,dg = \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)}$$

where

$$d_v(s) = \begin{cases} \prod_{i=1}^{2r} L_v(2s+i,\varepsilon_{E/F}^i) & , \ v \text{ inert in } E \\ \prod_{i=1}^{2r} \zeta_v(2s+i) & , \ v \text{ split in } E. \end{cases}$$

Here $\zeta_v(s)$ is the local zeta function of F_v and $L_v(s,\varepsilon_{E/F}^i)$ is the local Hecke L-function for the character $\varepsilon_{E/F}^i : F_v^\times \to \mathbb{C}^\times$.

An explicit formula for $f_0(g,1)$ can be given as follows. For $g \in G(F_v)$ let

$$\delta(g) = \begin{cases} \operatorname{diag}(\omega_E^{l_1},\ldots,\omega_E^{l_r}),\ l_1 \geq \ldots \geq l_r \geq 0 & , \ v \text{ inert in } E \\ \operatorname{diag}(\omega^{l_1},\ldots,\omega^{l_{2r}}),\ l_1 \geq \ldots \geq l_{2r} & , \ v \text{ split in } E \end{cases}$$

where ω, respectively ω_E, is a uniformizer of O_{F_v}, respectively O_{E_v}, and the exponents l_i are chosen so that g belongs to the double coset $K_v m(\delta(g))K_v$ in the inert case, respectively to

$K_v\delta(g)K_v$ in the split case. Now the K_v bi-invariance of $f_0(g,1)$ by Lemma 3.3 implies that

$$f_0(g,1) = \begin{cases} f_0(m(\delta(g)),1) & , v \text{ inert in } E \\ f_0(\delta(g),1) & , v \text{ split in } E, \end{cases}$$

and if we define the function

$$\Delta(g) = \begin{cases} |\det \delta(g)|_{E_v}^{-1} & , v \text{ inert in } E \\ |\det \delta(g)|_{F_v}^{-1} & , v \text{ split in } E, \end{cases}$$

then analogously to [Li92, p.197] one can show that

$$f_0(g,1) = \Delta^{-(s+r)}(g).$$

In such case, Theorem 3.4 becomes equivalent to the following

Corollary 3.5. *For all $s \in \mathbb{C}$ we have*

$$\int_{G(F_v)} \Delta^{-(s+r)}(g)\omega_{\pi_v}(g)\, dg = \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)}.$$

3.3.1 Local computation

In order to compute the integral on the left-hand side of Corollary 3.5, we will need some information about the Fourier coefficients of the function $\Delta(g)$. Since we will only be working locally at the fixed unramified non-archimedean place v of F, we drop the subscript v from the notation in the rest of this subsection.

Let ψ be therefore an additive character of the now local field F and (π, V_π) be an unramified irreducible admissible representation of $G(F)$. If $T \in \mathrm{Herm}_r(F)$ is a Hermitian matrix in the inert case, respectively $T \in \mathrm{Mat}_{r\times r}(F)$ in the split case, we will be interested in linear functionals $l_T : V_\pi \to \mathbb{C}$ satisfying

$$l_T\left(\pi\begin{pmatrix}\mathbb{1}_r & X\\ 0 & \mathbb{1}_r\end{pmatrix}f\right) = \overline{\psi(\operatorname{Tr} XT)}l_T(f)$$

for all $f \in V_\pi$ and all $X \in \operatorname{Herm}_r(F)$ in the inert case, respectively all $X \in \operatorname{Mat}_{r\times r}(F)$ in the split case.

For example, if $\pi = \bigotimes'_{v\in\mathcal{V}_F} \pi_v$ is a restricted product of cuspidal automorphic representations π_v of $G(F_v)$, then we know that for all but finitely many places v the local components π_v are unramified. If $T \in \operatorname{Herm}_r(F_v)$ in the inert case, respectively $T \in \operatorname{Mat}_{r\times r}(F_v)$ in the split case, then the linear functional on the space $\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$ of cuspidal automorphic forms defined by

$$l_T(f) = \int f\left(\begin{pmatrix}\mathbb{1}_r & X_v\\ 0 & \mathbb{1}_r\end{pmatrix}\right)\psi(\operatorname{Tr} X_v T_v)\, dX_v$$

satisfies the condition above.

The reason why the integral on the left-hand side of Corollary 3.5 is connected to Fourier coefficients is explained by the following

Lemma 3.6. *Let ψ be an unramified additive character of F and T be a square matrix such that $T \in \operatorname{Herm}_r(F)$ in the inert case, respectively $T \in \operatorname{Mat}_{r\times r}(F)$ in the split case. Let (π, V_π) be an unramified irreducible admissible representation of $G(F)$ and $f \in V_\pi^K$ be a vector fixed under the maximal compact subgroup $K = G(\mathcal{O}_F)$ of $G(F)$. Then for all linear functionals l_T as above and all $s \in \mathbb{C}$ we have*

$$\int_{G(F)} \Delta^{-(s+r)}(g) l_T(\pi(g)f)\, dg = l_T(f)\frac{L\left(s+\frac{1}{2},\pi\right)}{d(s)}.$$

Proof. When $\operatorname{Re}(s)$ is sufficiently large, the left-hand side converges absolutely and we have

$$\begin{aligned}\int_{G(F)} \Delta^{-(s+r)}(g) l_T(\pi(g)f)\,dg &\overset{\text{[Lemma 3.3]}}{=} \int_{G(F)} \int_K \Delta^{-(s+r)}(kg) l_T(\pi(g)f)\,dk\,dg \\ &\overset{g\mapsto k^{-1}g}{=} \int_{G(F)} \Delta^{-(s+r)}(g) \int_K l_T(\pi(kg)f)\,dk\,dg.\end{aligned} \tag{3.1}$$

We claim that the inner integral equals $l_T(f)\omega_\pi(g)$. Indeed, since it is a K bi-invariant matrix coefficient of π, it follows that

$$\int_K l_T(\pi(kg)f)\,dk = \lambda\omega_\pi(g)$$

for some constant λ and by plugging in $g = 1$ we find that $\lambda = l_T(f)$. In such case

$$\begin{aligned}\text{Equation 3.1} &= l_T(f) \int_{G(F)} \Delta^{-(s+r)}(g)\omega_\pi(g)\,dg \\ &= l_T(f)\frac{L\left(s+\frac{1}{2},\pi\right)}{d(s)}\end{aligned}$$

by Corollary 3.5. ∎

We use now the Iwasawa decomposition $G(F) = M(F)\times N(F)\times K$ to rewrite the left-hand side of Lemma 3.6 as

$$\int_{M(F)\times N(F)\times K} \Delta^{-(s+r)}(m(a)n(b)k) \times l_T(\pi(m(a)n(b)k)f) \times \delta_P^{-1}(m(a))\,dk\,dn(b)\,dm(a). \tag{3.2}$$

Here δ_P is the modular function of $P(F)$ given by $\delta_P(m(a)) = |\det a|_E^r$ in the inert case, respectively $\delta_P(m(a,b)) = |\det ab|_F^r$ in the split case where

$$m(a,b) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b^{-1} \end{pmatrix} \,\middle|\, a,b \in \mathrm{GL}_r(F) \right\}.$$

The right invariance of $\Delta(g)$ and $f(g)$ under K makes the integral over K trivial and we can simplify

$$\text{Equation 3.2} = \int\limits_{M(F)\times N(F)} \Delta^{-(s+r)}(m(a)n(b))\times\overline{\psi(\operatorname{Tr} bT)}\times l_T(\pi(m(a))f)\times\delta_P^{-1}(m(a))\,dn(b)\,dm(a). \tag{3.3}$$

By setting

$$J(s,a) \stackrel{\text{def}}{=} \int\limits_{N(F)} \Delta^{-(s+r)}(m(a)n(b))\overline{\psi(\operatorname{Tr} bT)}\,dn(b)$$

as the integral over the unipotent radical, we can rewrite

$$\text{Equation 3.3} = \int\limits_{M(F)} J(s,a)l_T(\pi(m(a))f)\delta_P^{-1}(m(a))\,dm(a).$$

The convergence and meromorphic continuation to the complex plane of the function $J(s,a)$ are well-studied in [Shi83, Proposition 3.3] and help us establish the following explicit formula for it [Qin07, Lemma 5.3]

Lemma 3.7. *Let ψ be an unramified character of F and T be a square matrix such that $T \in \mathrm{GL}_r(\mathcal{O}_E)\cap\mathrm{Herm}_r(F)$ in the inert case, respectively $T \in GL_r(\mathcal{O}_F)$ in the split case. Then we have*

$$J(s,a) = \begin{cases} |\det a|_E^{s+r}\, j(s) & ,\ a \in M(\mathcal{O}_F) \\ 0 & ,\ \textit{otherwise.} \end{cases}$$

Here

$$j(s) = \begin{cases} \displaystyle\int\limits_{\mathrm{Herm}_r(F)} \Delta^{-(s+r)}(n(b))\overline{\psi(\operatorname{Tr} bT)}\,db = \prod_{i=r+1}^{2r} L(2s+i,\varepsilon_{E/F}^i) & ,\ v \textit{ inert in } E \\ \displaystyle\int\limits_{\mathrm{Mat}_{r\times r}(F)} \Delta^{-(s+r)}(n(b))\overline{\psi(\operatorname{Tr} bT)}\,db = \prod_{i=r+1}^{2r} \zeta(2s+i) & ,\ v \textit{ split in } E. \end{cases}$$

Using the last lemma we arrive at the following

Corollary 3.8. *Let $f \in \pi$ be a K-fixed vector and T be a square matrix such that $T \in \mathrm{GL}_r(\mathcal{O}_E) \cap \mathrm{Herm}_r(F)$ in the inert case, respectively $T \in \mathrm{GL}_r(\mathcal{O}_F)$ in the split case. Then for all values of $s \in \mathbb{C}$ we have*

$$l_T(f)\frac{L\left(s+\frac{1}{2},\pi\right)}{d(s)j(s)} = \int\limits_{M(\mathcal{O}_F)} l_T(\pi(m(a))f)|\det a|_E^s \, dm(a).$$

Proof. By Lemma 3.6 and Lemma 3.7 we conclude that

$$\begin{aligned} l_T(f)\frac{L\left(s+\frac{1}{2},\pi\right)}{d(s)} &= \int\limits_{G(F)} \Delta^{-(s+r)}(g) l_T(\pi(g)f)\, dg \\ &= \int\limits_{M(F)} J(s,a) l_T(\pi(m(a))f)\delta_P^{-1}(m(a))\, dm(a) \\ &= j(s) \int\limits_{M(\mathcal{O}_F)} l_T(\pi(m(a))f)|\det a|_E^s\, dm(a). \end{aligned}$$

■

3.3.2 Global computation

We turn back now our attention to the global case where E/F is a CM-extension of number fields. Let $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$ be a cuspidal automorphic representation of the group G and $f \in \pi$ be a cusp form belonging to it.

Definition 3.9. For any $T \in \mathrm{Herm}_r(F)$ the *T-th Fourier coefficient of f* is the function on $G(\mathbb{A}_F)$ given by

$$f_T(g) \overset{\mathrm{def}}{=} \int\limits_{\mathrm{Herm}_r(F)\backslash \mathrm{Herm}_r(\mathbb{A}_F)} f(n(X)g)\psi(\mathrm{Tr}\, TX)\, dX \;\forall\; g \in G(\mathbb{A}_F).$$

If $T_1, T_2 \in \mathrm{Herm}_r(F)$ are such that $T_1 = {}^t a^\rho T_2 a$ for some $a \in \mathrm{GL}_r(E)$, then we have

$$f_{T_1}(g) = f_{T_2}(m(a)g) \ \forall\ g \in G(\mathbb{A}_F).$$

Consider a non-degenerate Hermitian space V' over E/F of dimension r with unitary group $H' = \mathrm{U}(V')$. By our discussion in Subsection 2.2.1, the group $G \times H'$ possesses a Weil representation ω_χ on the Bruhat-Schwartz space $\mathcal{S}(V'(\mathbb{A}_F)^r)$ associated to our fixed additive character ψ and multiplicative character χ. Recall that by Definition 2.10 and Definition 2.11, to a standard section $\Phi(g,s)$ of the degenerate principal series representation $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi|\cdot|_E^{s+\frac{r}{2}})$ and a Bruhat-Schwartz function $\varphi \in \mathcal{S}(V'(\mathbb{A}_F)^r)$ we associated an Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma \in P(F)\backslash G(F)} \Phi(\gamma g, s)$$

and a theta series

$$\theta(g,\varphi) = \sum_{\mathbf{x} \in V'(F)^r} (\omega_\chi(g)\varphi)(\mathbf{x}).$$

Definition 3.10. We define the expression

$$I(s,\varphi,\Phi,f) \stackrel{\mathrm{def}}{=} \int_{G(F)\backslash G(\mathbb{A}_F)} f(g)E(g,s,\Phi)\theta(g,\varphi)\, dg$$

as our Rankin-Selberg integral of interest.

This integral first appeared in the seminal paper [Shi73] where it represented the L-function of a newform of weight 2, and later in [Shi75] where it represented the symmetric square L-function. Even though $\theta(g,\varphi)$ is slowly increasing and $E(g,s,\Phi)$ is of moderate growth, the cusp form f is rapidly decreasing and therefore the integral is convergent for the values of s where the Eisenstein series is holomorphic. Expanding the definition and unfolding the integral above gives

$$I(s,\varphi,\Phi,f) = \int\limits_{G(F)\backslash G(\mathbb{A}_F)} f(g) \sum_{\gamma\in P(F)\backslash G(F)} \Phi(\gamma g,s) \sum_{\mathbf{x}\in V'(F)^r} (\omega_\chi(g)\varphi)(\mathbf{x})\,dg =$$
$$= \int\limits_{P(F)\backslash G(\mathbb{A}_F)} f(g)\Phi(g,s) \sum_{\mathbf{x}\in V'(F)^r} (\omega_\chi(g)\varphi)(\mathbf{x})\,dg =$$
$$= \int\limits_K \int\limits_{P(F)\backslash P(\mathbb{A}_F)} f(pk)\Phi(pk,s) \sum_{\mathbf{x}\in V'(F)^r} (\omega_\chi(pk)\varphi)(\mathbf{x})\,dp\,dk.$$

Now under the Iwasawa decomposition $g = pk = m(a)n(b)k$, a (left) Haar measure on the parabolic subgroup P satisfies $dp = |\det a|_E^{-r}\,d^\times a\,db$. Moreover, by definition

$$\Phi(pk,s) = \chi(\det a)|\det a|_E^{s+\frac{r}{2}}\Phi(k,s)$$

and after unwinding the action of the Weil representation ω_χ we get

$$I(s,\varphi,\Phi,f) = \int\limits_K \int\limits_{M(F)\backslash M(\mathbb{A}_F)} \int\limits_{N(F)\backslash N(\mathbb{A}_F)} f(m(a)n(b)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a)$$
$$\times \sum_{\mathbf{x}\in V'(F)^r} \psi(\operatorname{Tr} bT(\mathbf{x})) \times (\omega_\chi(k)\varphi)(\mathbf{x}a)\,db\,d^\times a\,dk.$$

The integral over the unipotent radical can be recognized as a Fourier coefficient, whereas the last sum can be split into an outer sum over images of the moment map $T : V'(F)^r \to \operatorname{Herm}_r(F)$

$$T(\mathbf{x}) = \frac{1}{2}\langle x_i, x_j\rangle_{1\le i,j\le r},$$

and an inner sum over their corresponding preimages. Note that $N(F) = \operatorname{Herm}_r(F)$ carries an adjoint action of $M(F) = \mathrm{GL}_r(E)$: for $m(a) \in M(F)$ and $n(b) \in N(F)$ it is given by the formula

$$a \cdot b \overset{\text{def}}{=} ab\,{}^t a^\rho$$

so that $n(a \cdot b) = m(a)n(b)m(a)^{-1}$. We can therefore further split the first sum over the images of T into an outer sum over a fixed set of coset representatives of $M(F)\backslash \operatorname{Im}(T)$ and an inner sum over their corresponding stabilizers, in other words

$$\begin{aligned} I(s,\varphi,\Phi,f) &= \int_K \int_{M(F)\backslash M(\mathbb{A}_F)} \sum_{T' \in \operatorname{Im}(T)} \sum_{\mathbf{x} \in T^{-1}(T')} f_{T'}(m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \\ &\quad \times (\omega_\chi(k)\varphi)(\mathbf{x}a)\, d^\times a\, dk \\ &= \sum_{T' \in M(F)\backslash \operatorname{Im}(T)} \int_K \int_{M(F)\backslash M(\mathbb{A}_F)} \sum_{m(a') \in M_{T'}(F)\backslash M(F)} \sum_{\mathbf{x} \in T^{-1}(T')} f_{T'}(m(a')m(a)k)\Phi(k,s) \\ &\quad \times (\chi^2|\cdot|_E^s)(\det a) \times (\omega_\chi(k)\varphi)(\mathbf{x}a'a)\, d^\times a\, dk \end{aligned}$$

where $M_{T'}$ denotes the stabilizer of T' under the action of M. By setting

$$\begin{aligned} I_{T'}(s) &\overset{\text{def}}{=} \int_K \int_{M(F)\backslash M(\mathbb{A}_F)} \sum_{m(a') \in M_{T'}(F)\backslash M(F)} \sum_{\mathbf{x} \in T^{-1}(T')} f_{T'}(m(a')m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \\ &\quad \times (\omega_\chi(k)\varphi)(\mathbf{x}a'a)\, d^\times a\, dk \end{aligned}$$

as the inner expression, we can simplify

$$I(s,\varphi,\Phi,f) = \sum_{T' \in M(F)\backslash \operatorname{Im}(T)} I_{T'}(s). \tag{3.4}$$

The advantage of this decomposition is that almost all terms in the last sum vanish: in fact, by [Qin07, Lemma 6.1] we have $I_{T'}(s) = 0$ for all s and all $T' \in M(F)\backslash \operatorname{Im}(T)$ with $\det T' = 0$. In such case, since V' has dimension r the matrix $\mathcal{T}$ representing its non-degenerate inner product belongs to the unique open orbit

$$\{T' \in \operatorname{Im} T \mid \det T' \neq 0\}$$

of $M(F)\backslash \operatorname{Im}(T)$ for which $I_{T'}(s) \neq 0$. Therefore we conclude that

Proposition 3.11. *The Rankin-Selberg integral of interest satisfies*

$$I(s,\varphi,\Phi,f) = \int\limits_K \int\limits_{M(\mathbb{A}_F)} f_{\mathcal{T}}(m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \times (\omega_\chi(k)\varphi)(a)\, d^\times a\, dk.$$

Proof. The stabilizer $M_{\mathcal{T}}$ is isomorphic by definition to the unitary group $H'(F) = \mathrm{U}(\mathcal{T})$, hence after identifying the components of $\mathbf{x} \in T^{-1}(\mathcal{T})$ with the columns of $H'(F)$ Equation 3.4 reduces to

$$\begin{aligned} I(s,\varphi,\Phi,f) &= \int\limits_K \int\limits_{M(F)\backslash M(\mathbb{A}_F)} \sum_{m(a')\in H'(F)\backslash M(F)} f_{\mathcal{T}}(m(a')m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \\ &\quad \times \sum_{\mathbf{x}\in H'(F)} (\omega_\chi(k)\varphi)(\mathbf{x}a'a)\, d^\times a\, dk \\ &= \int\limits_K \int\limits_{M(\mathbb{A}_F)} f_{\mathcal{T}}(m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \times (\omega_\chi(k)\varphi)(a)\, d^\times a\, dk. \end{aligned}$$

■

3.4 Proof of the integral representation

We are now ready to proceed with the proof of Theorem 3.2. Recall that $S \subset \mathcal{V}_F$ is a fixed finite subset of places of F satisfying the conditions in Definition 3.1. For any superset $S' \supseteq S$ we naturally define

$$K_{S'} = \prod_{v\in S'} K_v, \;\; M(F_{S'}) = \prod_{v\in S'} M(F_v)$$

and

$$I_{S'}(s,\varphi,\Phi,f) = \int\limits_{K_{S'}} \int\limits_{M(F_{S'})} f_{\mathcal{T}}(m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \times (\omega_\chi(k)\varphi)(a)\, d^\times a\, dk.$$

With our particular choice of S, we can state the following

Proposition 3.12. *Under the same assumptions as in Theorem 3.2, for any place $v \notin S$ the identity*

$$I_{S\cup v}(s,\varphi,\Phi,f) = \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)j_v(s)} I_S(s,\varphi,\Phi,f)$$

holds for all values of $s \in \mathbb{C}$.

Proof. Let us rewrite first using the definitions

$$\begin{aligned} I_{S\cup v}(s,\varphi,\Phi,f) &= \int\limits_{K_{S\cup v}} \int\limits_{M(F_{S\cup v})} f_{\mathcal{T}}(m(a)k)\Phi(k,s) \times (\chi^2|\cdot|_E^s)(\det a) \times (\omega_\chi(k)\varphi)(a)\, d^\times a\, dk \\ &= \int\limits_{K_S M(F_S)} \int\limits_{K_v M(F_v)} f_{\mathcal{T}}(m(a_S)k_S m(a_v)k_v)\Phi_S(k_S,s)\Phi_v(k_v,s) \times (\chi_S^2|\cdot|_E^s)(\det a_S) \\ &\times (\chi_v^2|\cdot|_E^s)(\det a_v) \times (\omega_\chi(k_S)\varphi_S)(a_S) \times (\omega_\chi(k_v)\varphi_v)(a_v)\, d^\times a_v\, dk_v\, d^\times a_S\, dk_S. \end{aligned}$$

Now we make the following observations that for all $k_v \in K_v$ and $m(a_v) \in M(F_v)$:

- $f_{\mathcal{T}}(m(a_S)k_S m(a_v)k_v) = f_{\mathcal{T}}(m(a_S)k_S m(a_v))$ by K_v-invariance of f;
- $\Phi_v(k_v,s) = 1$ by the choice of Φ;
- $\chi_v^2(\det a_v) = 1$ because by unramifiedness at v we must have $\chi_v(x) = |x|_E^{\lambda_v}\ \forall\, x \in E_v^\times$ for some $\lambda_v \in \mathbb{C}$ if v is inert in E, respectively $\chi_v(x) = |x|_F^{\lambda_v}\ \forall\, x \in F_v^\times$ if v is split in E. On the other hand $\chi\big|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}^n$, so χ_v^2 is trivial on $F_v^\times$ and the only possibility for this is when $\lambda_v = 0 \implies$ in either case χ_v^2 is the trivial character;
- $\omega_\chi(k_v)\varphi_v = \varphi_v$ by the choice of φ_v and unramifiedness of the characters ψ_v and χ_v.

After piecing all of this together we get

$$
\begin{aligned}
I_{S\cup v}(s,\varphi,\Phi,f) &= \int\limits_{K_S M_S}\int\limits_{M_v} f_{\mathcal{T}}(m(a_S)k_S m(a_v))\Phi_S(k_S,s)\times(\chi_S^2|\cdot|_E^s)(\det a_S) \\
&\times |\det a_v|_E^s \times (\omega_\chi(k_S)\varphi_S)(a_S)\times\varphi_v(a_v)\, d^\times a_v\, d^\times a_S\, dk_S \\
&= \int\limits_{K_S M_S} \Phi_S(k_S,s)\times(\chi_S^2|\cdot|_E^s)(\det a_S)\times(\omega_\chi(k_S)\varphi_S)(a_S) \\
&\times \int\limits_{M_v} f_{\mathcal{T}}(m(a_S)k_S m(a_v))\times|\det a_v|_E^s\varphi_v(a_v)\, d^\times a_v\, d^\times a_S\, dk_S.
\end{aligned}
$$

Since $\varphi_v = \mathbb{1}_{V'(\mathcal{O}_v)^r}$ and $M(F_v)\cap V'(\mathcal{O}_v)^r = M(\mathcal{O}_v)$, it follows that

$$
\begin{aligned}
&\int\limits_{M(F_v)} f_{\mathcal{T}}(m(a_S)k_S m(a_v))\times|\det a_v|_E^s\varphi_v(a_v)\, d^\times a_v \\
&= \int\limits_{M(\mathcal{O}_v)} f_{\mathcal{T}}(m(a_S)k_S m(a_v))\times|\det a_v|_E^s\, d^\times a_v \\
[\text{Corollary 3.8}] &= \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)j_v(s)} f_{\mathcal{T}}(m(a_S)k_S);
\end{aligned}
$$

here we are viewing $l_{\mathcal{T}}(f) = f_{\mathcal{T}}(m(a_S)k_S)$ as a linear functional on π_v. Therefore

$$
\begin{aligned}
I_{S\cup v}(s,\varphi,\Phi,f) &= \int\limits_{K_S M_S} \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)j_v(s)} f_{\mathcal{T}}(m(a_S)k_S)\Phi_S(k_S,s) \\
&\times(\chi_S^2|\cdot|_E^s)(\det a_S)\times(\omega_\chi(k_S)\varphi_S)(a_S)\, d^\times a_S\, dk_S \\
&= \frac{L\left(s+\frac{1}{2},\pi_v\right)}{d_v(s)j_v(s)} I_S(s,\varphi,\Phi,f)
\end{aligned}
$$

as claimed. ∎

Let us set

$$
D^S(s) = \prod_{v\notin S} D_v(s) \overset{\text{def}}{=} \prod_{v\notin S} d_v(s)j_v(s)
$$

and define the partial L-function of π as

$$L^S\left(s+\frac{1}{2},\pi\right) \overset{\text{def}}{=} \prod_{v\notin S} L\left(s+\frac{1}{2},\pi_v\right).$$

By adding one place at a time to the set $S' \supseteq S$, Proposition 3.12 above tells us that indeed

$$\int_{G(F)\backslash G(\mathbb{A}_F)} f(g)E(g,s,\Phi)\theta(g,\varphi)\,dg = I_S(s,\varphi,\Phi,f)\frac{L^S\left(s+\frac{1}{2},\pi\right)}{D^S(s)}$$

where

$$\begin{aligned} I_S(s,\varphi,\Phi,f) &= \prod_{v\in S} I_v(s,\varphi_v,\Phi_v,f) \\ &= \prod_{v\in S}\int_{K_v}\int_{M_v} f_{\mathcal{T}}(m(a_v)k_v)\Phi_v(k_v,s)\times(\chi_v^2|\cdot|_E^s)(\det a_v)\times(\omega_\chi(k_v)\varphi_v)(a_v)\,d^\times a_v\,dk_v \end{aligned}$$

is a factorizable factor coming from the bad places in S, which completes the proof of Theorem 3.2. Lastly, note that the partial L-function and the Eisenstein series are meromorphic functions of s, hence so is $I_S(s,\varphi,\Phi,f)$.

Chapter 4: Arithmetic theta lifting and pullback of the special cycles

In this last chapter we will combine the results from the previous chapters to derive a proof of the main result from Theorem 1.1 alluded to in the introduction. We recall first the following definition of archimedean weight from [Liu11a].

Definition 4.1. For a pair of integers (a,b) with $a+b>0$, a cuspidal automorphic representation π of the quasi-split unitary group $G=\mathrm{U}(r,r)$ over F is said to have *weight* (a,b) *at the archimedean place* $v\in\mathcal{V}_{F,\infty}$ if the restriction of π_v to the maximal compact subgroup $\mathrm{U}(r)_{\mathbb{R}}\times\mathrm{U}(r)_{\mathbb{R}}$ contains the character $\det_1^a\boxtimes\det_2^{-b}$ where $\det_i$ denotes the determinant on the i-th factor $\mathrm{U}(r)_{\mathbb{R}}$.

We also define the arithmetic theta lift following the convention in [LL21, Definition 4.8].

Definition 4.2. For a cusp form $f\in\pi$ the *arithmetic theta lift of* f is the codimension r cycle on X_K given by

$$\Theta_{\varphi^\infty}(f)\stackrel{\text{def}}{=}\int\limits_{G(F)\backslash G(\mathbb{A}_F)}\overline{f(g)}Z_{\varphi^\infty}(g)\,dg\in\mathrm{CH}^r(X_K)_{\mathbb{C}},$$

which is well-defined assuming the properness of the Shimura variety X_K and the absolute convergence of the generating series $Z_{\varphi^\infty}(g)$. In other words, it is the adèlic Petersson inner product of $f(g)$ with $Z_{\varphi^\infty}(g)$.

Equipped with these definitions, we are now ready to complete the proof of our main theorem.

Proof of Theorem 1.1. By Corollary 2.20, we know that the identity

$$\operatorname{vol} Z_{\varphi^\infty}(g) = \operatorname{vol}(X_K)E(g, s_1, \varphi)$$

holds at the special value $s_1 = \frac{n-r}{2}$. If we choose now the Bruhat-Schwartz function

$$\varphi^\infty = \varphi_1^\infty \otimes \varphi_2^\infty \in \mathcal{S}(V_0(\mathbb{A}_F^\infty)^r) \otimes \mathcal{S}(V_0^\perp(\mathbb{A}_F^\infty)^r) = \mathcal{S}(V(\mathbb{A}_F^\infty)^r)$$

so that both φ_1^∞ and φ_2^∞ are K-invariant, then by Theorem 2.21 the geometric pullback i^* of the generating series from X_K to X_{K_0} satisfies

$$i^*(Z_{\varphi^\infty}(g)) = Z_{\varphi_1^\infty}(g)\theta(g, \varphi_2)$$

where

$$\theta(g, \varphi_2) = \sum_{\mathbf{x} \in V_0^\perp(F)^r} (\omega_\chi(g)\varphi_2)(\mathbf{x})$$

is the standard theta function on the orthogonal complement of V_0 with archimedean component

$$\varphi_{2,\infty}(\mathbf{x}) = \prod_{v \in \mathcal{V}_{F,\infty}} \varphi_\infty^0(T(\mathbf{x})_v)$$

the Gaussian function. In particular, by applying the volume linear functional on both sides of the identity we get

$$\frac{\operatorname{vol} i^*(Z_{\varphi^\infty}(g))}{\operatorname{vol} X_{K_0}} = E(g, s_0, \varphi_1)\theta(g, \varphi_2) \tag{4.1}$$

now at the new special value $s_0 = \frac{m-r}{2}$ where $\varphi_{1,\infty}$ is the archimedean component from the geometric Siegel-Weil formula 2.7.

Note that $\dim V_0^\perp = n - m = r$, so if $H_0^\perp = \mathrm{U}(V_0^\perp)$ is the unitary group of $V_0^\perp$, then $(G, H_0^\perp)$ is a dual reductive pair satisfying the conditions of the integral representation in Theorem 3.2 with $V' = V_0^\perp$. Hence given a cuspidal automorphic representation $\pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F))$

and a cusp form $f \in \pi$ which is K_v-fixed for all $v \notin S$, we have

$$\int_{G(F)\backslash G(\mathbb{A}_F)} f(g)E(g,s,\Phi)\theta(g,\varphi)\,dg = I_S(s,\varphi,\Phi,f)\frac{L^S\left(s+\frac{1}{2},\pi\right)}{D^S(s)}. \tag{4.2}$$

Recall here that the choice of local components outside of S is such that

$$\Phi_v(-,s)|_{K_v} = 1, \;\; \varphi_v = \mathbb{1}_{V_0^\perp(O_v)^r}.$$

Observe now that with our choice of φ_1 and φ_2 as in Theorem 1.1, the functions Φ_{φ_1} and φ_2 satisfy these conditions and therefore the product of $f(g)$ with the right-hand side of Equation 4.1 matches the integrand on the left-hand side of Equation 4.2 when $s = s_0$. It follows that the left-hand side of Equation 4.2 represents the Petersson inner product of $\overline{f(g)}$ with $\operatorname{vol} i^*(Z_{\varphi^\infty}(g))$, which is nonzero at the archimedean places by their matching weights coming from our assumption on π. In such case, by combining the two equations we get

$$\begin{aligned}
\frac{\operatorname{vol} i^*(\Theta_{\varphi^\infty}(f))}{\operatorname{vol} X_{K_0}} &= \frac{1}{\operatorname{vol} X_{K_0}} \operatorname{vol} i^* \int_{G(F)\backslash G(\mathbb{A}_F)} \overline{f(g)} Z_{\varphi^\infty}(g)\,dg \\
&= \frac{1}{\operatorname{vol} X_{K_0}} \int_{G(F)\backslash G(\mathbb{A}_F)} \overline{f(g)} \operatorname{vol} i^*(Z_{\varphi^\infty}(g))\,dg \\
&= \int_{G(F)\backslash G(\mathbb{A}_F)} \overline{f(g)} E(g,s_0,\varphi_1)\theta(g,\varphi_2)\,dg \\
&= I_S(s_0,\varphi,\overline{f})\frac{L^S\left(s_0+\frac{1}{2},\pi^\vee\right)}{D^S(s_0)}
\end{aligned}$$

since both vol and i^* are linear functionals on $\mathrm{CH}^r(X_K)_{\mathbb{C}}$. Finally, note that $\pi_v^\vee \cong \pi_v$ for all $v \notin S$ since they have the same Satake parameters, hence

$$L^S(s,\pi^\vee) = L^S(s,\pi)$$

and this concludes the proof. ■

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