Columbia University

Department of Economics
Discussion Paper Series

Profitability of Product Bundling

Revision of Discussion Paper No.: 1011-02

Yongmin Chen
Michael H. Riordan

Discussion Paper No.: 1112-05

Department of Economics
Columbia University
New York, NY 10027

March 2012

Profitability of Product Bundling *

Yongmin Chen[†] and Michael H. Riordan[‡]

Revised February 18, 2012

Abstract. Using copulas to model the stochastic dependence of values, this paper establishes new general conditions for the profitability of product bundling. A multiproduct monopolist generally achieves higher profit from mixed bundling than from separate selling if consumer values for two products are negatively dependent, independent, or have limited positive dependence. With more than two goods, the same conditions are sufficient for an optimal monopoly selling scheme to include a bundle of at least two products. The profitability of monopoly bundling also extends to situations where a multiproduct firm competes with a single-product rival.

Keywords: product bundling, mixed bundling, preference dependence, copula.

[†]University of Colorado, Boulder; yongmin.chen@colorado.edu
[‡]Columbia University; mhr21@columbia.edu

*For helpful comments, we thank three referees, Hanming Fang, participants of the 2010 Workshop on Law and Economics at Hitotsubashi University, the 2010 Shanghai Microeconomics Workshop (SHUFE), the 2010 Summer Conference on Industrial Organization (UBC, Vancouver), and seminars at Columbia University, University of Rochester, University of Virginia, and Yonsei University.

1. INTRODUCTION

When is product bundling more profitable than separate selling? The question has long intrigued economists. Stigler (1963) showed with a simple example that bundling can be profitable even without demand complementarity or scope economies. Adams and Yellen (1976) expanded on this view, showing, mostly with examples, that mixed bundling can be a profitable way to segment markets. Schmalensee (1984) studied the profitability of bundling when consumer values for two goods have a bivariate normal distribution, and found for the symmetric case sufficient conditions on the marginal distribution for pure bundling to dominate separate selling for any degree of correlation short of perfect positive correlation. Fang and Norman (2006) provide more general conditions on the distribution of values for the independence case such that pure bundling is more profitable than separate selling. Working with an arbitrary bivariate distribution having a continuous density function, Long (1984) found mixed bundling to be strictly more profitable than separate selling when consumer values are negatively dependent or independent. McAfee, McMillan, and Whinston (1989) relaxed the assumption of a continuous density function to develop a general sufficient condition for the profitability of mixed bundling, albeit one that apparently is difficult to interpret in terms of dependence relations beyond saying bundling is optimal in a broader range of cases than just independence. Chu, Leslie, and Sorensen (2011) showed with numerical analyses that bundling is profitable in an array of special cases, including some featuring limited positive and negative dependence.

Although considerable attention has been directed at how the correlation of values for products matters for the profitability of bundling, the issue remains generally unclear, not only for positively dependent distributions, but also for negatively dependent distributions lacking a continuous density. We revisit the profitability of bundling with a new approach that uses a copula to represent the distribution of consumer values for two products sold by a multiproduct monopolist. A copula is a function that couples marginal distributions of random variables to form a joint distribution,[1] making it straightforward to vary dependence while holding marginal distributions constant. A standard means to describe the dependence of random variables in modern statistics, copulas recently have found useful applications in economics, including, for instance, the modeling of financial time series (Patton, 2008), of product differentiation (Chen and Riordan, 2008, 2010), and of intertemporal dependence of consumer values (Chen and Pearcy, 2010). As we show in this paper, the copula approach also has important advantages for understanding the profitability of

[1] According to Sklar's Theorem any joint distribution can be constructed this way (Nelsen, 2006).

product bundling. It enables new and stronger analytical results that are invariant to the marginal distributions of consumer values and that cover a broader range of dependence conditions under weaker technical assumptions.

Our basic analytical result is a general sufficient condition for the strict profitability of mixed bundling, analogous to the one in McAfee, McMillan, and Whinston (1989). The nice property of our condition is that it is stated solely in terms of the copula. As such, the condition is invariant to the functional form of marginal distributions of values, which is a major advance. The condition immediately implies higher profit under mixed bundling than under separate selling if values for the two products are independently distributed, and, in addition, allows us to develop clear analytical results under conditions of negative and positive dependence. Furthermore, the condition is a powerful tool for evaluating the profitability of bundling directly for classes of joint distributions (analytically or numerically). While it might be straightforward to verify numerically the sufficient condition in McAfee, McMillan, and Whinston (1989) for a particular parametric joint distribution function, our condition, which computes on the unit square an indicator function determined entirely by the copula, has the advantage of applying to entire families of joint distribution functions formed by varying the marginal distributions and by considering copula families with sufficient dependence properties. This generally is much easier than individually checking the infinitely many joint distributions generated from a copula by varying the marginal distributions, or checking the range of joint distributions generated by varying the copula within a family for given marginals. Moreover, Sklar's Theorem says that any joint distribution has an associated copula, and it usually is straightforward with a change of variables to construct the associated copula from primitive joint and marginal distributions (Nelsen, 2006).

We first apply our general condition to the case where consumer values for the two products are negatively dependent. Extending Long (1984), we demonstrate that mixed bundling generally is more profitable than separate selling when values for the two products are negatively dependent, without assuming a continuous joint density. Intuitively, starting from the optimal prices under separating selling, consider adding a bundle to the firm's offerings with a small discount relative to the sum of its individual prices. This reduces the firm's profit from consumers who were already purchasing both goods under separate selling, but increases the firm's profit from consumers who switch from purchasing only one good to purchasing both goods. Under negative dependence, a high value for one product is more likely to be associated with low values for the other. This suggests that, while not too many consumers purchase both products under separate selling, there may be a relatively large number of consumers on the margin, purchasing one product but not the other under separate selling, who are persuaded to purchase the bundle for a small discount, so that the second effect above is more likely to dominate. Our analysis shows this is indeed the case under negative dependence. The result and its limitations are also illustrated with a parameterized family of copulas.

We then consider the case of positive dependence, for which much less is known about the profitability of bundling under general distributions. We show that mixed bundling achieves higher profit than separate selling if positive dependence is not too great, by deriving a bound on the degree of positive dependence that is allowed. The bound is independent of the copula, and it thus applies to arbitrary joint distributions. By disentangling the dependence relationship from the marginal distributions, the copula approach enables us to evaluate the profit effects of bundling from a new perspective, and here it results in a new general profitability condition even though the net effect is not as clear-cut as under negative dependence. We also reach a stronger conclusion for a number of notable copula families, for which mixed bundling is always more profitable than separate selling for the entire range of positive dependence, short of perfect dependence.[2]

We further extend the two-product monopoly model in two directions. First, we consider a multiproduct monopolist selling any number of goods. If consumer values for at least two of the goods are negatively dependent, independent, or have limited positive dependence, then some form of bundling, for instance selling two of the goods in a bundle while also offering all goods on a standalone basis, is more profitable than separate selling. We also consider situations in which a multiproduct firm competes against a single-product rival, with the multiproduct firm producing two distinct products, and the single-product competitor producing a differentiated version of one of them. Under similar dependence conditions as for a multiproduct monopoly, the multiproduct firm optimally chooses bundling in equilibrium, regardless of the dependence relationship between the two differentiated versions of the product that both firms produce.

Product bundling is a familiar marketing practice in modern economies. Mixed bundling, for example, is used by cable companies to offer Internet and television services, by McDonald's to offer burgers and fries, and by tour companies to offer sight-seeing products. The recent empirical industrial organization literature has examined bundling of cable TV channels (Crawford and Yurukoglu 2012), theater tickets (Chu, Leslie and Sorensen 2011), and video-rental distribution (Ho, Ho, and Mortimer 2011). By demonstrating the profitability of bundling in settings substantially more general than previously shown, this paper helps

[2] In the limiting case of perfect dependence, mixed bundling has the same profit as separate selling.

further understand the popularity of such practices.

The rest of the paper is organized as follows. Section 2 sets up a basic model of a monopoly producing two goods, and introduces the copula approach to representing the joint distribution of consumer values for the goods that determines demand. Section 3 establishes a key lemma that provides a general sufficient condition for the profitability of monopoly bundling. The lemma focuses squarely on the properties of the copula, and is employed in subsequent sections in various ways. Sections 4 and 5 study respectively cases where consumer values for the two products are negatively dependent and positively dependent. Section 6 extends the results to a multiproduct monopoly selling any number of goods, and Section 7 to markets where a multiproduct firm competes against a differentiated single-product rival. Section 8 concludes with directions for further research.

2. BASIC MONOPOLY MODEL

Our model of product bundling by a monopolist hews closely to the basic framework of Stigler (1963), Adams and Yellen (1976), Schmalensee (1984), Long (1984) and McAfee, McMillan, and Whinston (1989). There are two goods, X and Y. The size of consumer population is normalized to 1. Each consumer demands at most one unit of each good, and her consumption of one does not affect her demand for the other. A consumer's value for X is u and for Y is v, with marginal distributions $F(u)$ and $G(v)$ on respective supports $[\underline{u}, \bar{u}]$ and $[\underline{v}, \bar{v}]$, with corresponding density functions $f(u) > 0$ and $g(v) > 0$ on $(\underline{u}, \bar{u})$ and $(\underline{v}, \bar{v})$ for $-\infty \leq \underline{u} < \bar{u} \leq \infty$ and $-\infty \leq \underline{v} < \bar{v} \leq \infty$. The value of the outside option is normalized to zero. The constant marginal costs for X and Y are m_X and m_Y, respectively. The value of two goods together is $u + v$, with marginal cost $m_X + m_Y$; thus this framework rules out product complementarity or economies of scale as explanations for bundling.[3] Resale is not possible, and the firm cannot prevent consumers from purchasing both X and Y separately.[4]

A benchmark for evaluating the profitability of bundling is the profit from simple monopoly pricing when the two goods are sold separately. A consumer can be represented by a point $(x, y) \in I^2$ with values $u(x) = F^{-1}(x)$ and $v(y) = G^{-1}(y)$ for the two goods. If X and Y are sold separately at prices p and q, consumers will purchase X if $u(x) \geq p$, or equivalently $x \geq F(p)$, and will purchase Y if $y \geq G(q)$. Therefore, monopoly prices for X

[3] While we follow this maintained assumption in much of the literature to facilitate the comparisons, there are also many situations where goods are substitutes or complements. See, for example, Lewbel (1985) and Armstrong (2010) for analyses of bundling complements and substitutes.

[4] McAfee, McMillan, and Whinston (1989) also study the "monitoring case" with no resale, for which the firm can prevent consumers from purchasing both goods separately, and for which they conclude that bundling generally is profitable.

and Y under separate selling respectively satisfy $p^s \in \arg\max_p \{(p - m_X)[1 - F(p)]\}$ and $q^s \in \arg\max_q \{(q - m_Y)[1 - G(q)]\}$. Following McAfee, McMillan, and Whinston (1989), we assume interior solutions under separate selling:[5]

Assumption 1 p^s and q^s satisfy the first-order conditions

$$1 - F(p^s) - (p^s - m_X) f(p^s) = 0 \tag{1}$$

$$1 - G(q^s) - (q^s - m_Y) g(q^s) = 0 \tag{2}$$

with $0 < F(p^s) < 0$ and $0 < G(q^s) < 1$.

For any given (F, G) and (m_x, m_y), p^s and q^s are given, as are $x^s \equiv F(p^s)$ and $y^s \equiv G(q^s)$. For given (m_x, m_y), we shall call any (F, G) for which Assumption 1 holds *admissible*.[6]

Our point of departure from the previous literature on bundling is to use a copula to describe the population of consumers.[7] Interpreting $(x, y) \in I^2$ as a consumer type, the population of consumers is described by a copula $C(x, y)$. A copula is a bivariate uniform distribution that "couples" arbitrary marginal distributions to form a new joint distribution. By Sklar's Theorem, it is without loss of generality to represent the joint distribution of consumer values for the two products by a copula and the marginal distributions (Nelsen, 2006).[8] Standard uniform margins for x and y imply $C(x, 1) = x$ and $C(1, y) = y$. A copula additionally satisfies $C(x, 0) = 0 = C(0, y)$. Let $x = F(u)$ and $y = G(v)$, and denote the copula associated with the joint distribution of (u, v) by $C(x, y)$. Then the joint distribution of (u, v) is $C(F(u), G(v))$. The partial derivatives, $C_1(x, y) \equiv \partial C(x, y)/\partial x$ and $C_2(x, y) \equiv \partial C(x, y)/\partial y$, exist almost everywhere. Furthermore, $C_1(x, y)$ is the conditional distribution of y given x, and $C_2(x, y)$ is the conditional distribution of x given y.[9]

Under bundling, X and Y are sold individually at prices p and q, respectively, and the XY bundle is sold at price $r \leq p + q$. Pure bundling is a degenerate case in which p and q are high enough to choke off the standalone sales, while mixed bundling admits both bundled

[5] If $\underline{u}$ and $\underline{v}$ are not too much above the marginal costs, then p^s and q^s will be interior values satisfying (1) and (2) below.

[6] The marginal costs can be normalized to zero without loss of generality. With this normalization, u and v are interpreted as consumer values net of marginal costs, and p, q, and r are interpreted as markups.

[7] An exception is Chu, Leslie, and Sorensen (2011) who use a Gaussian copula to model limited correlation for different marginal distribution functions.

[8] Sklar's Theorm holds also for more than two values, where any joint distribution can be represented by marginal distributions and a multivariate copula (Nelsen, 2006). We will use this to generalize our results to $n \geq 2$ products and to bundling under competition.

[9] In the past twenty years or so, it has become standard in statistics to use copulas to describe the dependence of random variables. The analytical utility of copulas is that dependence can be varied while holding constant the marginal distributions of the random variables. An excellent survey of these developments is found in Nelsen (2006).

and separate sales. Consumers are willing to purchase the bundle if $u(x) + v(y) - r \geq \max\{0, u(x) - p, v(y) - q\}$, X alone if $u(x) - p \geq 0$ and $v(y) \leq r - p$, and Y alone if $v(y) - q \geq 0$ and $u(x) \leq r - q$. Consequently, demands for each good and the bundle at an interior solution are, respectively: [10]

$$
\begin{aligned}
Q_X(p,r) &\equiv G(r-p) - C(F(p), G(r-p)); && (3)\\
Q_Y(q,r) &\equiv F(r-q) - C(F(r-q), G(q)); && (4)\\
Q_{XY}(p,q,r) &\equiv \int_{F(r-q)}^{F(p)} [1 - C_1(x, G(r-u(x))]\, dx + [1-F(p)] - Q_X(p,r). && (5)
\end{aligned}
$$

Therefore, the profit function under mixed bundling is

$$\pi(p,q,r) = (p - m_X)\, Q_X(p,r) + (q - m_Y)\, Q_Y(q,r) + (r - m_X - m_Y)\, Q_{XY}(p,q,r)\,. \quad (6)$$

The multiproduct monopolist chooses (p,q,r) to maximize profit subject to $r \leq p+q$. Bundling has higher profit than separate selling if $r < p+q$ at the solution. If $r = p+q$, then profit is the same as under separate selling.

3. PRELIMINARY RESULTS

Our approach to finding a sufficient condition for the profitability of bundling is similar to Long (1984) and McAfee, McMillan, and Whinston (1989). Starting at monopoly pricing under separate selling, the analysis asks whether it is (strictly) profitable to discount the bundle by a small amount.[11] Thus consider the profit function

$$\psi(\varepsilon) \equiv \pi(p^s, q^s, p^s + q^s - \varepsilon)$$

for $\varepsilon \geq 0$. If $\psi(\varepsilon) > \psi(0)$ for some small positive ε, then some form of bundling must be profitable compared to separate selling. If $\pi(p,q,r)$ is differentiable at $(p^s, q^s, p^s + q^s)$, then $\psi'(0) > 0$ is a sufficient condition.

The following lemma is proved by relating the sign of $\psi'(0)$ to the properties of the copula. Indeed, $\psi'(0)$ has the same sign as

[10] Allowance for corner solutions, in which demand for any of the three options is zero, is a straightforword extension of these formulas. Riemann integrability of $C_1(x, G(r-u(x))$ in the formula for Q_{XY} requires continuity almost everywhere, as in the cases of positive or negative dependence examined below.

[11] McAfee, McMillan, and Whinston (1989) also considered raising one of the standalone prices by a small amount, which yields an equivalent condition for profitability. Long (1984) proved his result by interpreting mixed bundling as a two-part tariff and deriving conditions under which it is profitable to raise the fixed fee above zero. This is equivalent to raising the standalone prices *and* the bundle price all by ε, which also yields an equivalent condition.

$$\Delta(x,y) \equiv (1-x)\left[1-C_1(x,y)\right] + (1-y)\left[1-C_2(x,y)\right] - \bar{C}(x,y), \tag{7}$$

where

$$\bar{C}(x,y) \equiv 1 - x - y + C(x,y).$$

$\bar{C}(x,y)$ is the joint survival function for two standard uniform random variables whose joint distribution is $C(x,y)$, i.e., the probability that a consumer's values for X and Y are above $u(x)$ and $v(y)$ respectively (Nelsen, 2006). The lemma provides a general sufficient indicator for the profitability of product bundling in terms of the dependence of consumer values as summarized by the copula.

Lemma 1 *(a) For any given admissible (F,G), bundling is strictly more profitable than separate selling if $\Delta(x^s,y^s) > 0$.*[12] *(b) If $\Delta(x,y) > 0$ for (almost) all $(x,y) \in$ int I^2, then bundling is strictly more profitable than separate selling for (almost) all admissible (F,G).*

Proof. (a) Mixed bundling is strictly more profitable than separate selling if $\psi'(0) > 0$. We have

$$\begin{aligned}\psi'(0) &= -\left(p^s - m_X\right)\frac{\partial Q_X\left(p^s, p^s+q^s\right)}{\partial r} - \left(q^s - m_Y\right)\frac{\partial Q_Y\left(q^s, p^s+q^s\right)}{\partial r} \\ &\quad - \left(p^s + q^s - m_X - m_Y\right)\frac{\partial Q_{XY}\left(p^s, q^s, p^s+q^s\right)}{\partial r} - Q_{XY}\left(p^s, q^s, p^s+q^s\right).\end{aligned}$$

From (3), (4), and (5), simple differentiation and substitution yield:

$$\begin{aligned}\psi'(0) &= \left(p^s - m_x\right) f\left(p^s\right)\left[1 - C_1\left(F\left(p^s\right), G\left(q^s\right)\right)\right] + \left(q^s - m_y\right) g\left(p^s\right)\left[1 - C_2\left(F\left(p^s\right), G\left(q^s\right)\right)\right] \\ &\quad - \left[1 - F\left(p^s\right) - G\left(q^s\right) + C\left(F\left(p^s\right), G\left(q^s\right)\right)\right].\end{aligned}$$

Using first order conditions (1) and (2), and substituting $x^s = F(p^s)$ and $y^s = G(q^s)$, we obtain $\psi'(0) = \Delta(x^s, y^s)$.

(b) $\Delta(x,y)$ exists almost everywhere on int I^2. Therefore, if $\Delta(x,y) > 0$ almost everywhere, then bundling necessarily is profitable for almost all admissible (F,G). Furthermore, if $\Delta(x,y) > 0$ everywhere on int I^2, then bundling must be profitable for all admissible (F,G). ■

Lemma 1(a) is analogous to McAfee, McMillan, and Whinston (1989)'s general sufficient condition for profitable bundling for a given joint distribution function, but uses the copula

[12] $\Delta(x^s,y^s) > 0$ implicitly requires $C_1(x,y)$ and $C_2(x,y)$ to exist at (x^s,y^s), which is almost surely satisfied since $C(x,y)$ is differentiable almoste everywhere (Nelsen, 2006). McAfee, McMillan and Whinston (1989) take a step further, implicitly assuming that $C_1(x,y)$ (or $C_2(x,y)$) exists for all x (or for all y).

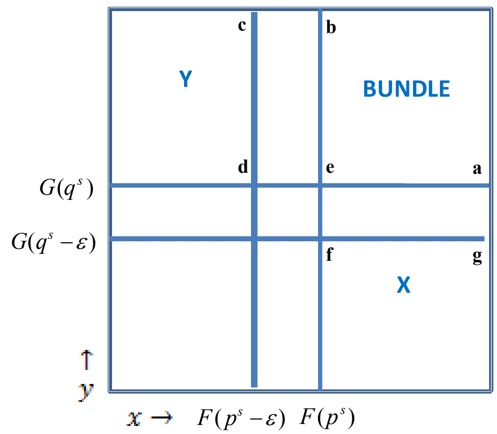

FIG. 1. Monopoly Mixed Bundling with a Small Discount for the Bundle

to describe consumer preferences while also relaxing technical conditions. Some intuition is gained from Fig. 1, which maps McAfee, McMillan, and Whinston (1989)'s Figure III to the consumer type space (I^2). The condition $\Delta(x^s, y^s) > 0$ weighs two effects of a vanishingly small ε discount of the XY bundle relative to separate pricing. The negative first-order effect of an ε discount is to lower the price to those consumers purchasing both products under separate pricing, corresponding to area *aeb* in the figure and to the joint survival term $-\bar{C}(x, y)$ in the definition of $\Delta(x, y)$. The positive effect is to cause some consumers purchasing a single product under separate pricing to purchase the bundle instead, corresponding to area *bcde* and *aefg* and to the remaining terms of $\Delta(x, y)$. The lemma states a general condition for the positive effect to outweigh the negative effect as ε goes to zero.

Lemma 1(b) provides an elegant, powerful, and useful general condition for the profitability of bundling. Assuming an interior solution to the monopoly separate-pricing problem, it states a sufficient condition for profitable bundling only in terms of the copula. The condition dispenses with a joint density function, does not depend on marginal costs, and is a sufficient condition for the profitability of bundling for all admissible marginal distributions rather than just for a given joint probability distribution. Furthermore, the condition can be verified numerically for a given copula by evaluating an indicator function on the unit square.

Lemma 1 immediately establishes the McAfee, McMillan, and Whinston (1989) result that product bundling is optimal under independence.

Proposition 1 (Independence) *For any admissible* (F, G) , *bundling is strictly more prof-*

itable than separate selling if $C(x,y)$ is the independence copula.
Proof. *$C(x,y) = xy$ implies $\bar{C}(x,y) = (1-x)(1-y)$ and*

$$\Delta(x,y) \equiv (1-x)(1-y) > 0$$

for all $(x,y) \in$ int I^2 ■

There is a clear intuition for this result with reference to Fig. 1. The gains from area *bcde* (or *aefg*) exactly offset the losses from area *aeb* Therefore, the gains from area *aefg* (or *bcde*) are pure profit.

It is striking that the profitability condition doesn't depend on marginal costs m_x and m_y except through admissibility, even though a standard intuition going back to Adams and Yellen (1976) is that, by reducing the amount of available surplus that can be captured by the bundle, higher marginal costs make bundling less profitable. This departure from the standard intuition is due to the fact that the profitability condition is based on a local perturbation around the separate pricing solution. Starting at optimal separate prices, which include a markup on cost, the demand-side condition of Lemma 1(b) implies that the firm profitably can capture additional margin by attracting enough consumers to purchase both goods instead of one with a slight discount for the bundle. It also is worth noting that even when the local perturbation argument is sufficient to demonstrate the profitability of bundling, marginal costs still matter for determining *by how much* optimal bundling is more profitable than separate selling, which requires a consideration of global price changes. The following example illustrates.

Example 1 *The Fairlie-Gumbel-Morgenstern (FGM) copula family specifies*

$$C(x,y;\theta) = xy\left[1 + \theta(1-x)(1-y)\right]$$

for $\theta \in [-1,1]$. We can use the sufficient condition in Lemma 1 to directly evaluate the profitability of bundling for joint distributions formed by these copulas. For all $(x,y) \in$ Int I^2, it is easy to establish with simple algebra that

$$\Delta(x,y) = (1-x)(1-y)(3xy\theta - y\theta - x\theta + 1) > 0. \tag{8}$$

Therefore, from Lemma 1(b), bundling is profitable for all admissible (F,G) for all members of the FGM copula family. Furthermore, if, for instance, the marginal distributions are uniform with

$$F(u) = G(u) = \frac{u-4}{5} \quad \text{for } 4 \le u \le 9$$

and $m_X = m_Y = m$, then numerical analysis, which considers global changes in prices, reveals that the profit advantage of bundling over separate selling decreases with both cost (m) and the degree of dependence (θ), while the impacts of bundling on consumer and social welfare, which can be either positive or negative, do not (always) vary monotonically in cost and dependence. The details for this FGM-Uniform case is contained in Appendix A.

4. NEGATIVE DEPENDENCE

We show that a multiproduct monopolist generally achieves higher profit from bundling than from separate selling under negative dependence. Stigler (1963) and Adams and Yellen (1976) found by various examples that bundling can be more profitable than separate selling when values for products are negatively dependent. While the intuition from these studies suggests that bundling generally is profitable under negative dependence, the precise conditions for such a conclusion are subtle and remain unsettled. Long (1984) showed that bundling is profitable if the distribution of consumer values for the two goods has a continuous density and is negatively dependent in a particular way, while, without assuming a continuous density, McAfee, McMillan, and Whinston (1989) did not reach the same strong conclusion.[13]

Long (1984) derived the profitability of bundling under negative dependence by interpreting bundling as a two-part pricing scheme and analyzing demand elasticities. The copula approach provides an alternative statement and extension of Long (1984)'s result. The particular dependence condition identified by Long (1984) is the following:

$$\Pr\{v > q|u > p\} \text{ is nonincreasing in } p \text{ and } \Pr\{u > p|v > q\} \text{ is nonincreasing in } q.$$

In the language of modern statistics, u is *right tail decreasing* in v, and v is *right tail decreasing* in u. Furthermore, if u and v are continuous random variables with a copula $C(x, y)$, then these two properties are equivalent respectively to the two properties of the copula in the following definition (Nelsen, 2006).[14]

[13] McAfee, McMillan, and Whinston (1989) (pp. 379-380) argued informally that their sufficient condition for profitable bundling (Proposition 1) is satisfied if the monopoly price for good Y conditional on knowing the consumer reservation value for good X is decreasing in the value of good X, but concluded that this "cannot be tied solely to the correlation of reservation values."

[14] The equivalence can be seen as follows. The expression $\Pr\{v > q|u > p\} = \frac{\Pr(u>p,v>q)}{\Pr(u>p)}$ is equivalent to $\frac{1-x-y+C(x,y)}{1-x}$ with an appropriate definition of variables. Furthermore, the first expression is nonincreasing in p if the second expression is in nonincreasing in x, which in turn is equivalent to $C_1(x, y) \leq \frac{y-C(x,y)}{1-x}$. See Nelsen (2006) for more details.

Definition 1 *$C(x,y)$ is right tail decreasing at $(x,y) \in I^2$ if*

$$C_1(x,y) \leq \frac{y - C(x,y)}{1-x}, \text{ and } C_2(x,y) \leq \frac{x - C(x,y)}{1-y}. \tag{9}$$

Long (1984)'s argument for the profitability of bundling under negative dependence relies on the condition that the fraction of consumers who buy both products being strictly decreasing in p and in q (at $p = F^{-1}(x^s), q = G^{-1}(y^s)$), which requires $\bar{C}(x^s, y^s) > 0$. Under right tail decreasing, $\bar{C}(x^s, y^s) > 0$ in turn implies:

$$C_i(x^s, y^s) < 1 \text{ for at least one } i, i = 1, 2. \tag{10}$$

Thus Long (1984) implicitly assumes (10).[15] The following proposition extends the negative dependence result in Long (1984), without assuming a continuous density of consumer values.

Proposition 2 (Negative Dependence) *For any admissible (F, G), bundling is strictly more profitable than separate selling if $C(x,y)$ is right tail decreasing and (10) holds at (x^s, y^s).*

Proof. Under right tail decreasing:

$$\begin{aligned}
\Delta(x^s, y^s) &\equiv (1-x^s)\left[1 - C_1(x^s, y^s)\right] + (1-y^s)\left[1 - C_2(x^s, y^s)\right] - \bar{C}(x^s, y^s) \\
&= 1 - C(x^s, y^s) - (1-x^s) C_1(x^s, y^s) - (1-y^s) C_2(x^s, y^s) \\
&\geq 1 - C(x^s, y^s) - (1-x^s)\frac{y^s - C(x^s, y^s)}{(1-x^s)} - (1-y^s)\frac{x^s - C(x^s, y^s)}{(1-y^s)} \\
&= 1 - y^s - x^s + C(x^s, y^s) \equiv \bar{C}(x^s, y^s) \geq 0.
\end{aligned}$$

If $\bar{C}(x^s, y^s) > 0$, then $\Delta(x^s, y^s) > 0$. If $\bar{C}(x^s, y^s) = 0$, then

$$\Delta(x^s, y^s) = (1-x^s)\left[1 - C_1(x^s, y^s)\right] + (1-y^s)\left[1 - C_2(x^s, y^s)\right] > 0$$

by condition (10). In either case the result is immediate from Lemma 1(a). ■

From Lemma 1 (b) and Proposition 2, we immediately have the following generic condition on the profitability of bundling under negative dependence, which takes into account the fact that $C_i(x,y)$ might fail to exist on $(x,y) \in I^2$ only on a set of zero measure.

[15] To see that $\bar{C}(x^s, y^s) > 0$ implies (10), suppose to the contrary that $\bar{C}(x^s, y^s) > 0$ but $C_1(x^s, y^s) = 1$. Then, by negative right tail dependence, $1 = C_1(x^s, y^s) \leq \frac{y^s - C(x^s, y^s)}{1-x^s}$, or $1 - x^s - y^s + C(x^s, y^s) = \bar{C}(x^s, y^s) \leq 0$, which is a contradiction.

Corollary 1 *If $C(x,y)$ is right tail decreasing and (10) holds for (almost) all $(x,y) \in$ int I^2, then bundling is strictly more profitable than separate selling for (almost) all admissible (F,G).*

As at (x^s, y^s), if $C(x,y)$ is right tail decreasing at any $(x,y) \in \text{int } I^2$, then condition (10) is satisfied if $\bar{C}(x,y) > 0$. The property $\bar{C}(x,y) > 0$ for $(x,y) \in \text{int } I^2$ means that some consumers purchase both goods for any interior solution of the independent pricing problem, and is satisfied if $C(x,y)$ has positive support as $x \to 1$ and $y \to 1$ from below. While this clearly holds if $C(x,y)$ has full support on I^2, it is useful to have a more general statement of the negative tail dependence result, because many standard copula families do not have full support (Nelsen, 2006).

To illustrate our results under negative dependence and gain additional insights, consider the following example:

Example 2 *Let*

$$C(x,y;\alpha) = \alpha \max\{x+y-1, 0\} + (1-\alpha)xy, \text{ where } \alpha \in [0,1].$$

This defines a family of copulas parameterized by α, mixing two familiar copulas corresponding to perfect negative dependence and independence.[16] *The entire family of copulas, $C(x,y;\alpha)$, lacks continuous densities except when $\alpha = 0$. For all $\alpha \in [0,1)$:*

$$C_1(x,y;\alpha) = \begin{cases} \alpha + (1-\alpha)y < 1 & if \quad x+y-1 > 0 \\ (1-\alpha)y < 1 & if \quad x+y-1 < 0 \end{cases};$$

$$C_2(x,y;\alpha) = \begin{cases} \alpha + (1-\alpha)x < 1 & if \quad x+y-1 > 0 \\ (1-\alpha)x < 1 & if \quad x+y-1 < 0 \end{cases},$$

and both $C_1(x,y;\alpha) \le \frac{y - C(x,y)}{1-x}$ and $C_2(x,y;\alpha) \le \frac{x-C(x,y)}{1-y}$ are satisfied when $x^s + y^s - 1 \neq 0$ because

$$\frac{y - C(x,y;\alpha)}{1-x} = \begin{cases} = \alpha + (1-\alpha)y & if \quad x+y-1 \ge 0 \\ = \left[1 + \frac{x\alpha}{1-x}\right] y & if \quad x+y-1 < 0 \end{cases},$$

$$\frac{x - C(x,y;\alpha)}{1-y} = \begin{cases} = \alpha + (1-\alpha)x & if \quad x+y-1 \ge 0 \\ = \left[1 + \frac{\alpha y}{1-y}\right] x & if \quad x+y-1 < 0 \end{cases}.$$

[16] A useful result in the theory of copulas is that a convex linear combination of two copulas is a copula (Nelsen, 2006).

Therefore, for $\alpha \in [0,1)$, $C(x,y;\alpha)$ is right tail decreasing and (10) holds for almost all $(x,y) \in int\ I^2$. It follows from Corollary 1 that bundling is profitable for almost all admissible (F,G). Only for (F,G) with $x^s + y^s - 1 = 0$ does Proposition 2 (or Corollary 1) not determine the profitability of bundling.

Assume for instance that $F(t) = G(t) = t$ on $[0,1]$ and $m_x, m_y \in [0,1)$. Then, under separate selling, the firm's optimal prices are

$$p^s(m_x) = \frac{1+m_x}{2} = x^s \in [\frac{1}{2}, 1); \qquad q^s(m_y) = \frac{1+m_y}{2} = y^s \in [\frac{1}{2}, 1),$$

and, for all $m_x, m_y \in (0,1)$, (F,G) is admissible and $x + y - 1 \neq 0$. Therefore, from Proposition 2, bundling is profitable for all joint distributions of consumer values corresponding to $C(x,y;\alpha)$ with $\alpha \in [0,1)$ and with $F(t) = G(t) = t$ for $m_x, m_y \in [0,1)$.

Now, for additional simplicity, assume that $m_x = m_y \equiv m$. When $\alpha = 1$, $C(x,y;1)$ corresponds to the "Hotelling case" of perfect negative dependence:[17]

$$C(x,y;1) = \max\{x+y-1, 0\},$$

where $C_1(x,y) = 1 = C_2(x,y)$ for all $x+y-1 > 0$. Then condition (10) does not hold. Under separate selling, the maximum profit from the two products is $\pi^S = \frac{1}{2}(1-m)^2$. Since all consumers are willing to pay at most $r = 1$ for the bundle, it cannot be profitable to sell the bundle to willing consumers when $m \geq 1/2$. For $m < 1/2$, optimal mixed bundling, with $r^* = 1$ for the bundle and $p^* = q^* = 1 - \frac{m}{2}$ for each standalone good, is strictly more profit than separate selling; i.e. profit is $\pi^B = \frac{1}{2}[(1-m)^2 + (1-2m)] > \pi^S$.

Several interesting points emerge from Example 1. First, as demonstrated in the Hotelling case, the profitability of bundling under negative dependence applies to a larger set of distributions than the set satisfying the conditions of Proposition 2 and its corollary. Second, if the sufficient conditions of Proposition 2 fail, as is in the Hotelling case, bundling may not be profitable if marginal costs are sufficiently high, confirming the standard intuition that higher marginal costs tend to make bundling less profitable by reducing (or in this case eliminating) the surplus that can be captured by the bundle. Third, the intuition that more negative dependence makes bundling more profitable, suggested by Stigler (1963), Adams and Yellen (1976), and Long (1984), is not true generally. In the example, with $F(t) = G(t) = t$ and $m_x = m_y \in [1/2, 1)$, bundling always is strictly more profitable than

[17] As in the Hotelling model of product differentiation, consumer preferences are represented as a uniform distribution of locations on the unit line. The Hotelling case for the bundling model is similar to the negative dependence examples of Stigler (1963) and Adams and Yellen (1976).

separate selling except for $\alpha = 1$, the case of perfect negative dependence.[18] Therefore, given the continuity of the profit function for the symmetric case, profits must decline with the degree of negative dependence for α and m sufficiently high. Fourth, as in Stigler(1963), bundling reduces consumer welfare in the Hotelling case when $m_x = m_y < 1/2$; consumers paying $r^* = 1$ for the bundle are reduced to zero surplus,while other consumers pay a higher price for a standalone good.

We conclude this section by discussing a somewhat stronger negative dependence property. Value u (v) is *stochastically decreasing* in v (u) if the conditional distribution of v (u) is nondecreasing in u (v). These stochastic monotonicity conditions are equivalent to the conditions of the following definition (Nelsen, 2006):

Definition 2 *$C(x, y)$ is negatively stochastic dependent at $(x, y) \in I^2$ if $C_1(x, y)$ is nondecreasing in x and $C_2(x, y)$ is nondecreasing in y.*

From Theorem 5.2.12 of Nelsen (2006), negative stochastic dependence for $(x, y) \in I^2$ implies negative right tail dependence. Proposition 1 then immediately implies:

Corollary 2 *If $C(x, y)$ is negatively stochastic dependent and (10) holds for (almost) all $(x, y) \in$ int I^2, then bundling is strictly more profitable than separate selling for (almost) all admissible (F, G).*

Strict negative dependence (i.e. $C(x, y)$ strictly convex in x and in y) implies an even stronger conclusion about the profitability of bundling. Armstrong (2010) considers the case of independent firms selling X and Y separately, and shows that, starting from separate monopoly prices, with strict negative stochastic dependence, at least one of the two firms has an incentive to offer a discount to consumers buying the other product. Indeed, in our setting, the firm selling Y has an incentive to offer a small $\varepsilon > 0$ discount to consumers buying X if

$$\begin{aligned} & \Delta(x, y) - (1 - x^s)[1 - C_1(x^s, y^s)] \\ = \; & (1 - y^s)[1 - C_2(x^s, y^s)] - \bar{C}(x^s, y^s) \\ = \; & C(x^s, 1) - C(x^s, y^s) - (1 - y^s) C_2(x^s, y^s) > 0, \end{aligned} \tag{11}$$

[18]It might appear counterintuitive that bundling is generally profitable under negative dependence but has an exception under perfect negative dependence. To gain intuition, imagine some sequence of distributions, with $\bar{C}(x, y) > 0$ for $(x, y) \in$ int I^2, that has the Hotelling case as its limit. For any member of this sequence, there are always consumers who are willing to pay up to $r = 2$ for a bundle, which enables a mixed bundle to be profitable even for $m \in [1/2, 1)$. In contrast, in the Hotelling case, all consumers are at most willing to pay $r = 1$ for the bundle, and hence bundling cannot be profitable if $m \geq 1/2$.

which follows from the strict convexity of $C(x,y)$ in y. The graphical interpretation in Fig. 1 helps explain this result: condition (11) states that *bcde* alone exceeds area *aeb*. In other words, with strict negative stochastic dependence, the gain from increased sales of only one of the two products alone outweighs the cost of the discount on the bundle.

5. POSITIVE DEPENDENCE

Less is known about positive dependence. Since bundling is strictly more profitable for the independence copula $C(x,y) = xy$ and any admissible marginal distributions, the same must be true for copulas that are "close" to the independence copula, as observed by McAfee, McMillan, and Whinston (1989). Schmalensee (1984) showed for the symmetric bivariate normal case that bundling is always profitable if demand under separate selling is sufficiently strong (i.e. in our framework if $x^s = y^s$ is sufficiently small) except in the case of perfect positive correlation, for which bundling never has an advantage over separate selling.[19] Beyond Schmalensee (1984)'s bivariate normal results, it remains an open question whether bundling dominates separate selling for any degree of positive dependence short of perfect.

Our main result here is that bundling is profitable if positive stochastic dependence is not too great. The result puts a bound on the degree of positive stochastic dependence that assures profitable bundling. Value u is *stochastically increasing* in v if $\Pr\{u > p|v\}$ is nondecreasing in v; similarly, v is *stochastically increasing* in u if $\Pr\{v > q|u\}$ is nondecreasing in u. These properties are equivalent to the following definition (Nelsen, 2006):

Definition 3 *$C(x,y)$ is positively stochastic dependent at $(x,y) \in I^2$ if $C_1(x,y)$ is nonincreasing in x and $C_2(x,y)$ is nonincreasing in y.*

Given this definition, it is natural to measure the degree of positive stochastic dependence by how negative are $C_{11}(x,y) \equiv \frac{\partial^2 C(x,y)}{\partial x^2}$ and $C_{22}(x,y) \equiv \frac{\partial^2 C(x,y)}{\partial x^2}$, because these second derivatives determine the degree of concavity of $C(x,y)$ in x and in y.[20] Furthermore, if $C(x,y)$ is positively stochastic dependent on the interior of I^2, then these second derivatives exist almost everywhere.

[19] With perfect positive dependence, any feasible mixed bundling scheme is equivalent to a separate selling scheme. This follows from the fact that with perfect positive dependence mixed bundling can have positive standalone sales of only one of the two goods, implying that mixed bundling is equivalent to separately selling one good at p and the other at $r-p$. Consequently, mixed bundling and separate selling have the same outcomes.

[20] For many parameterized copula families, C_{ii} decreases in a parameter that indexes the range of positive dependence. This is true, for example, for the FGM copula family in Example 1, and is also true for the Clayton and Frank copula families discussed later.

Positive stochastic dependence for $(x, y) \in I^2$ implies positive right-tail dependence, and both in turn imply positive quadrant dependence (Nelsen, 2006):

Definition 4 *$C(x, y)$ is positively quadrant dependent at $(x, y) \in I^2$ if $C(x, y) \geq xy$.*

Positive quadrant dependence is used in the proof of the following proposition. The result says that bundling is profitable under positive dependence if $C_{ii}(x, y)$ are not too negative on the boundaries of the set of consumer types purchasing both goods under separate pricing.

Proposition 3 (Positive Dependence) *For any given admissible (F, G), define the constant*

$$\delta^s \equiv \frac{2(1-x^s)(1-y^s)}{(1-x^s)^2 + (1-y^s)^2} > 0. \tag{12}$$

If $C(x, y)$ is positively quadrant dependent at (x^s, y^s), and

$$\min\{C_{11}(x, y^s), C_{22}(x^s, y)) | x \geq x^s, y \geq y^s\} > -\delta^s, \tag{13}$$

then bundling is strictly more profitable than separate selling.

Proof. Since

$$\begin{aligned}
& (1-x)[1 - C_1(x, y)] - \bar{C}(x, y) \\
= \; & y - C(x, y) - (1-x)C_1(x, y) = C(1, y) - C(x, y) - (1-x)C_1(x, y) \\
= \; & \int_x^1 [C_1(z, y) - C_1(x, y)]dz = \int_x^1 (1-z)C_{11}(z, y)\,dz,
\end{aligned}$$

and, similarly,

$$(1-y)[1 - C_2(x, y)] - \bar{C}(x, y) = \int_y^1 (1-z)C_{22}(x, z)\,dz,$$

we have, for any given admissible (F, G),

$$\Delta(x^s, y^s) = \bar{C}(x^s, y^s) + \int_{x^s}^1 (1-x)C_{11}(x, y^s)\,dx + \int_{y^s}^1 (1-y)C_{22}(x^s, y)\,dy.$$

Furthermore, positive quadrant dependence implies $\bar{C}(x^s, y^s) \geq (1-x^s)(1-y^s)$, and

$\min\{C_{11}, C_{22}\} > -\delta^s$ further implies

$$\begin{aligned}
\Delta(x^s, y^s) &> (1-x^s)(1-y^s) - \int_{x^s}^{1}(1-x)\delta^s dx - \int_{y^s}^{1}(1-y)\delta^s dy \qquad (14)\\
&= (1-x^s)(1-y^s) - \left[\frac{(1-x^s)^2}{2} + \frac{(1-y^s)^2}{2}\right]\delta^s = 0.
\end{aligned}$$

■

Note that the bound δ^s is independent of $C(x, y)$ and reaches a maximum of 1 when $x^s = y^s$. Thus, for all admissible marginal distributions, there is some range of positive dependence for which bundling is profitable. This range is larger when market shares under separate pricing are closer together. The result goes substantially beyond McAfee, McMillan, and Whinston (1989)'s observation that bundling is profitable in the neighborhood of independence.

To gain more intuition about condition (13), recall from Figure 1 that, starting from separating selling at (x^s, y^s), adding a bundle with a slightly lower combined price has the positive effect of turning some consumers from purchasing only one good to purchasing the bundle (areas bcde and aefg), but has the negative effect of lower revenues from the consumers previously purchasing both goods (area aeb). Recall also that for independence or negative dependence the gains from one group of consumers switching from a particular good to the bundle at least offsets the losses from discounting the bundle, so the gains from the other group who switch is pure profit. This no longer holds under positive dependence; the profits from both groups of consumers switching to the bundle must be applied to offset the loss from the discount for bundling to be profitable under positive dependence. The bounds on the degree of positive stochastic dependence assure that the gains from each group remain sufficiently large relative to the loss from discounting for a positive total net effect.

The range of positive dependence allowed by (13) is substantial. For instance, for any admissible marginal distribution with $F(p^s) = G(q^s)$ or with $F(p^s) \leq 5/6$ and $G(q^s) \leq 5/6$, all joint distributions formed by FGM family copulas fall below the bound given in (13).[21] The FGM copula family, however, exhibits only a limited range of negative and positive dependence, and $\theta = 1$ does not correspond to perfect positive dependence. For distributions with higher degrees of positive dependence short of perfect dependence, it is often easy to determine the profitability of bundling by directly checking Lemma 1(b)'s

[21] From Example 1, by directly applying Lemma 1(b), bundling is in fact profitable for all FGM family copulas for all admissible marginal distributions.

condition. The examples below illustrate this for several notable copula families.

Example 3 *The following copula family mixes the perfectly dependent copula and the independent copula:*

$$C(x,y) = \alpha \min\{x,y\} + (1-\alpha)\, xy \text{ for } \alpha \in [0,1). \tag{15}$$

This copula family corresponds to situations where consumer values for the two goods have a common component (a common shock) and an independent component. As the copula family in Example 2, this copula family does not have a continuous density function. But when $x \neq y$, we can use the indicator function from Lemma 1 to easily determine the profitability of bundling. Since for $x < y$, $C(x,y) = \alpha x + (1-\alpha)\, xy$, $C_1(x,y) = \alpha + (1-\alpha)\, y$, and $C_2(x,y) = (1-\alpha)\, x$, simple algebraic analysis reveals that

$$\begin{aligned} \Delta(x,y) &= (1-x)\left[1 - C_1(x,y)\right] + (1-y)\left[1 - C_2(x,y)\right] - \bar{C}(x,y) \\ &= (1-x)(1-y)(1-\alpha) > 0. \end{aligned}$$

And similarly $\Delta(x,y) > 0$ for $x > y$. Therefore, short of perfectly positive dependence ($\alpha = 1$), bundling is profitable for the entire copula family defined in (15) for almost any admissible marginal distribution.

Example 4 *The Frank copula family is given by*

$$C(x,y;\theta) = -\frac{1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta x} - 1\right)\left(e^{-\theta y} - 1\right)}{\left(e^{-\theta} - 1\right)}\right) \text{ for } \theta \in (-\infty, \infty)/0.$$

Frank copulas exhibit positive (negative) stochastic dependence if $\theta > (<) 0$ and $\theta = \infty(-\infty)$ corresponds to perfect dependence.[22] *Numerical analysis shows that $\Delta(x,y) > 0$ for $(x,y) \in \operatorname{int} I^2$. Therefore, from Lemma 1(b), bundling apparently is profitable for all Frank copulas for any admissible marginal distributions.*

Example 5 *The Clayton copula family specifies*

$$C(x,y;\theta) = \max\left\{\left[x^{-\theta} + y^{-\theta} - 1\right]^{-1/\theta}, 0\right\} \text{ for } \theta \in [-1, \infty)/0.$$

[22] The Frank family and the Clayton family in the next example are the only "comprehensive" copula families, among the important one-parameter Archimedean copulas listed in Table 4.1 of Nelsen (2006), that also allow independence and the full range of negative dependence, but these cases are covered by Propositions 1 and 2.

Like the Frank family, Clayton copulas exhibit positive (negative) stochastic dependence if $\theta > (<) 0$ and $\theta = \infty$ corresponds to perfect dependence. Numerical analysis verifies $\Delta(x, y) > 0$ for $(x, y) \in$ int I^2, and bundling apparently is profitable for any admissible marginal distributions.

In all three examples above, bundling is profitable for the entire range of positive dependence, except at the limit of perfect dependence. So far, we have not found a counterexample to profitable bundling under positive dependence short of the extreme case of perfect dependence.

6. ANY NUMBER OF GOODS

The profitability conditions for bundling by a monopolist with two goods can be extended to any number of products. To proceed, we generalize the model in Section 2 to any $n \geq 2$ products: X_1, ..., X_n.[23] Let u_i denote the consumer value for X_i, $F_i(u_i)$ the marginal distribution of values, m_i the marginal cost, and p_i^s the single product monopoly price. As with $n = 2$, p_i^s is assumed to be an interior solution of the profit maximization problem, and satisfies

$$1 - F_i(p_i^s) - (p_i^s - m_i) f_i(p_i^s) = 0.$$

Let $\tilde{C}(x_1, ... x_n)$ denote the multivariate copula describing joint distribution of $x_i = F_i(u_i)$ for $i = 1, ... n$. By Sklar's Theorem, the joint distribution of consumer values for the n goods is therefore $\tilde{C}(F_1(u_1), ..., F_n(u_n))$. Assume that the value of two goods X_i and X_j together is $u_i + u_j$, with constant marginal cost $m_i + m_j$, and the values and marginal costs are similarly obtained for l goods together, $l \leq n$. Again, this framework rules out product complementarity or economies of scale as explanations for bundling.

Our previous results on the profitability of bundling for the two-good monopolist extend readily to the n good case. Consider the profitability of selling a two-good bundle $\{X_1, X_2\}$ together with individually-priced goods $X_1, ..., X_n$. Suppose the prices for goods $X_3, ... X_n$ are set at $p_i = p_i^s$, $i = 3, ..., n$, so the profits from the sale of these $(n-2)$ goods is by hypothesis the same as from separate selling. It then suffices to show that profit from goods X_1 and X_2 will be higher under the proposed bundling than under separate selling. Notice that the joint distribution of consumer values for X_1 and X_2 can be represented

[23] In reality, a firm sometimes sells multiple groups of products, and goods within each product group could be substitutes such that a consumer may purchase only one of them. For ease of exposition, we do not explicitly model this situation, but we can accommodate this possibility by allowing the interpretation of X_i, if appropriate, as any (symmetric) good from product group i, $i = 1, ..., n$, where goods within group i are substitutes.

by $C(F_1(u_1), F_2(u_2))$, where $C(x,y) \equiv \tilde{C}(x,y,1,...,1)$ is a bivariate copula. Therefore, Lemma 1 applies, and under the conditions of Propositions 1-3, profits from X_1 and X_2 are higher under mixed bundling than under separate selling. Hence:

Corollary 3 *For a multiproduct monopolist selling $n \geq 2$ products, if consumer values for at least two goods are negatively dependent, independent, or have limited positive dependence, then some form of bundling will have strictly higher profits than separate selling.*

Optimal mixed bundling with n goods is a complex issue. With n goods, mixed bundling involves up to $2^n - 1$ distinct prices, which may not be practical when n is large. Thus it would be desirable to consider whether a subset of relatively simple forms of mixed bundling will strictly dominate separate selling, a perspective adopted by Chu, Leslie, and Sorensen (2011). Corollary 3 identifies a simple instance of mixed bundling that is profitable under the same conditions as in Propositions 1-3. Furthermore, when n is larger, it seems more likely that at least two of the goods will possess the desired dependence properties, and hence more likely that some form of mixed bundling will increase profit.

One may also ask, for instance, whether another form of mixed bundling, which contains a bundle of n goods in addition to n individually-priced goods, will strictly dominate separate selling. A difficulty one encounters in trying to extend the sufficient conditions to this case is that dependence notions for dimensions higher than 2 are less well developed and less suited for our analysis. Nevertheless, there is similar trade off as in the case of $n = 2$: Starting from the separate-pricing solutions, adding the n-goods bundle with a slightly lower combined price reduces profit from consumers already purchasing all n goods, but boosts profit by increasing the number of consumers purchasing all n goods; the former effect is likely more pronounced if values for different goods possess some type of positive dependence, whereas the latter effect is likely more pronounced under negative dependence.

7. PARTIAL COMPETITION

The profitability of bundling under multiproduct monopoly also extends to markets where a multiproduct firm competes against a single-product firm. We focus on the case where the multiproduct firm, A, offers two products X and Y_A, whereas a single-product firm, B, offers a symmetrically differentiated version of product Y, Y_B. The two firms compete by simultaneously choosing prices, where for firm A the prices can either be those under separate selling or those under mixed bundling. We assume a pure strategy equilibrium exists for this model of price competition, and consider whether in equilibrium the multiproduct firm finds higher profits from bundling than from separate selling.

A consumer's value for X is $u(x)$, and for Y_i is $v(y_i)$ with $(x, y_A, y_B) \in I^3$. Therefore, the marginal distribution of consumer values for X is $F(u)$, and the symmetric distribution for each variety of product Y is $G(v)$, with corresponding density functions $f(u) > 0$ and $g(v) > 0$ on their respective supports. The copula $C(x, y_A, y_B)$, with y_A and y_B exchangeable, describes the population of consumers. Adopting stochastic monotonicity dependence concepts, and assuming differentiability, we say that values for X and Y are positively dependent, independent, or negatively dependent when respectively $C_{11}(x, y_A, y_B) \leq 0$, $C_{11}(x, y_A, y_B) = 0$ or $C_{11}(x, y_A, y_B) \geq 0$ for almost all $(x, y_A, y_B) \in I^3$.

Under mixed bundling, let p denote the standalone price of X, q_i the standalone price of Y_i for $i \in \{A, B\}$, and $r \leq p + q_A$ the price of Firm A's bundle. Separate pricing is equivalent to $r = p + q_A$, in which case the demands for X and Y_A are respectively

$$\begin{aligned} Q_X^s(p) &= 1 - F(p), \\ Q_{Y_A}^s(q_A, q_B) &= 1 - C(1, G(q_A), G(q_B)) - \textstyle\int_{G(q_B)}^1 C_3(1, G(q_A + v(y) - q_B), y)\, dy. \end{aligned}$$

The marginal costs for products X and Y are m_X and m_Y, and interior equilibrium prices p^s and q^s in the two product markets satisfy

$$1 - F(p^s) = (p^s - m_X) f(p^s) \tag{16}$$

and

$$\begin{aligned} &\frac{1}{2}\left[1 - C(1, G(q^s), G(q^s))\right] \\ = \; &(q^s - m_Y)\left[C_2(1, G(q^s), G(q^s))\, g(q^s) + \textstyle\int_{G(q^s)}^1 C_{23}(1, y, y)\, g(v(y)) dy\right] \end{aligned} \tag{17}$$

respectively. See Chen and Riordan (2010) for details on the symmetric separate-pricing equilibrium in the Y market.

To evaluate the demand for the products offered by Firm A, consider a type (x, y_A, y_B) consumer who is willing to purchase good X at price p, i.e. $x \geq F(p)$. This consumer also has the opportunity to acquire good Y_A as part of the bundle by paying an incremental price $r - p$, or to purchase Y_B at price q_B. The consumer's choice in the Y market depends on (y_A, y_B) as illustrated in the unit square of Fig. 2. Consumers in region XY_A purchase the bundle, those in in region XY_B separately purchase X and Y_B, and those in region X only purchase good X. Therefore, consumers making standalone purchases of X are those in the union of regions XY_B and X. Denote the aggregate consumer demand functions for goods X, Y_A, Y_B, and bundle XY_A by Q_X, Q_{Y_A}, Q_{Y_B}, and Q_{XY_A}, respectively. The details

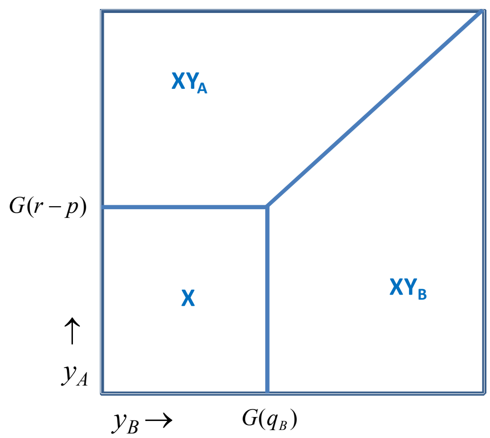

FIG. 2. Demand in Market Y for Consumers Willing to Buy X

about consumers' choices and the demand functions are contained in Appendix B.

With no economies of scope, the profit of Firm A is

$$\begin{aligned}\pi_A(p, q_A, q_B, r) &= (p - m_X) Q_X(p, q_B, r) + (q_A - m_Y) Q_{Y_A}(q_A, q_B, r) \\ &\quad + (r - m_X - m_Y) Q_{XY_A}(p, q_A, q_B, r).\end{aligned}$$

We now extend the profitability of bundling under multiproduct monopoly to this partial competition model, by establishing that in equilibrium the multiproduct firm will optimally choose bundling if values for X and Y are negatively dependent, independent, or have sufficiently limited positive dependence.

Proposition 4 *Let*

$$\bar{\delta} \equiv (q^s - m_Y) g(q^s) [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] > 0. \tag{18}$$

In equilibrium, bundling is strictly more profitable for the multiproduct firm than separate selling if $C_{11} > -4\bar{\delta}$.

Proof. See Appendix B. ■

Thus a multiproduct firm facing a single product competitor optimally chooses bundling in equilibrium for a range of dependence conditions, similarly to a multiproduct monopolist. The main contrast with Proposition 2 is that Proposition 4 employs a stronger negative dependence property, and the main contrast with Proposition 3 is that the bound on the degree of positive dependence in Proposition 3 depends on the copula.[24] A strength of Proposition 4 is that it allows any dependence relation between Y_A and Y_B.

We have confined our analysis of bundling under competition to situations where a multi-product firm competes with a single-product rival. The profitability of bundling is relatively simple in this case, because only the multiproduct firm can choose to bundle its products. In markets where the competition is between multiproduct firms, the issue of bundling is more complex, since the profitability of bundling by one firm may depend on whether or not the other firm bundles. The issue also is more complex because there are dependence relations both between values for different products and between values for products by different firms. We leave it for future research to address the issue of equilibrium product bundling by competing multiproduct firms under general preference dependence conditions.[25]

8. CONCLUSION

Mixed bundling is by construction weakly more profitable than separate selling. The question is when it is strictly more profitable. Our analysis advances the literature by establishing more general sufficient conditions for the strict profitability of bundling, and has found an indicator function that is robust to wide variations in marginal distributions. These results show that a multiproduct firm achieves higher profit from mixed bundling than from separate selling if consumer values for two products are negatively dependent, independent, or have limited positive dependence. When the firm sells more than two products, bundling leads to higher profit under similar conditions. Furthermore, results on the profitability of monopoly bundling extend to markets where a multiproduct firm competes against a single-product rival.

The profitability issue still is not completely settled, as we have found neither useful necessary conditions for the profitability of product bundling, nor a counterexample under positive dependence outside the limiting case of perfect dependence. There also are several other worthy directions for research. For instance, while monopoly bundling often increases the firm's profit, its effects on consumer and social welfare are less clear, as Appendix A illustrates for a special case. Stigler (1963) and some of our examples show that

[24] With (minor) additional restrictions, it's possible to find a lower bound on C_{11} that is a fixed number. For instance, if $Q^s_{Y_A}$, $g(\cdot)$, and $C_{123}(\cdot)$ are all bound above zero, then $\bar{\delta}$ is bound above zero, and the lower bound on C_{11} in Proposition 4 can be stated as $C_{11} \geq -\bar{\delta}$ for some fixed $\bar{\delta} > 0$.

[25] As McAfee, McMillan, and Whinston (1989) observed, under competition between multiproduct firms, if consumer values for all goods are independently distributed, then a firm will find it optimal to engage in mixed bundling if the other firm does not, so that it cannot be an equilibrium for all firms to choose separate selling.

consumers may be worse off with bundling, but it is unclear how robust is this possibility. It would be desirable to find more general conditions for the evaluation of the consumer and welfare effects of monopoly bundling. It would also be interesting to further study the incentives for and effects of bundling under competition. For instance, according to the existing literature, whereas bundling can be an effective entry barrier, it sometimes may also be entry-accommodating by creating (or increasing) product differentiation. It would be desirable to develop an understanding of when bundling forecloses competition and when it softens competition in a more general framework of preference dependence.[26]

APPENDIX A: NUMERICAL ANALYSIS OF THE FGM-UNIFORM CASE

Consider

$$\begin{aligned} F(u) &= G(u) = \frac{u-4}{5}; \\ C(x,y) &= xy + \theta xy(1-x)(1-y); \\ m_X &= m_Y = m, \end{aligned}$$

where the marginal distribution has uniform support for $u \in [4,9]$, and members of the FGM copula family are indexed by the parameter θ, ranging between -1 and 1. The copula parameter indicates the degree of dependence, with $\theta < 0$ indicating negative dependence, $\theta = 0$ independence, and $\theta > 0$ positive dependence (Nelsen, 2006). Below, let Π and W denote (total) profit and (aggregate) consumer welfare respectively under optimal bundling, and $\Delta\Pi$ and ΔW denote increases relative to separate selling. Recall from Example 1 that $\Delta\Pi$ is always positive in this case.

The following numerical analysis considers the consequences of bundling for four levels of marginal cost and for five different dependence conditions. Taken together, the four tables show that the profit advantage of bundling over separate selling decreases with both cost and the degree of dependence. Bundling can either increase or decrease consumer and social welfare; and the effects on consumer welfare (ΔW) and on social welfare ($\Delta\Pi + \Delta W$) may not be monotonic in marginal cost or in the degree of dependence.

[26] The foreclosure theory of bundling was first formalized in Whinston (1990). Other contributions on the foreclosure effects of bundling include Carlton and Waldman (2002), Choi and Stefanadis (2001), and Nalebuff (2004). Our result here shows that even without the foreclosure motive, a multiproduct firm can often profit from bundling its products in competing with a single-product rival. With competition between multiproduct firms, firms may also choose to offer (different) bundles in order to create endogenous product differentiation (Carbajo, De Meza, and Seidman 1990; Chen 1997).

(1) For $m = 0$, the (separate selling) monopoly price is $p^s = 4.5$ for each good, (total) profit is $\Pi^s = 8.1$, and (aggregate) consumer welfare is $W^s = 4.05$. Table A1 shows that pure bundling is optimal, with $p^* \geq 9$ deterring any standalone sales. Furthermore, $r^* > 2p^s$ implies that all consumers necessarily are worse off with optimal bundling. Both the bundle price and profit decrease and consumer welfare increases with θ. The effect of bundling on social welfare ($\Delta\Pi + \Delta W$) is non-monotonic; social welfare is higher under bundling when $\theta = -1$ or $\theta = 1$ but lower for intermediate values including independence.

	p^*	r^*	π^*	$\Delta\Pi$	W^*	ΔW	$\Delta\Pi + \Delta W$
$\theta = -1$	≥ 9	10.564	9.7361	1.6361	2.4877	−1.5623	0.0738
$\theta = -.5$	≥ 9	10.412	9.4636	1.3636	2.6552	−1.3948	−0.0312
$\theta = 0$	≥ 9	10.210	9.2127	1.1127	2.8620	−1.188	−0.0753
$\theta = .5$	≥ 9	9.9556	8.9925	0.8925	3.1102	−0.9398	−0.0473
$\theta = 1$	≥ 9	9.6752	8.8114	0.7114	3.3777	−0.6723	0.0391

Table A1: $m = 0$

(2) When $m = 2$, the monopoly price is $p^s = 5.5$, profit is $\Pi^s = 4.9$, and consumer welfare is $W^s = 2.45$. Table A2 shows that pure bundling is optimal, with $r^* > 2p^s = 11$ when $\theta = -1$ or $\theta = -0.5$, in which cases all consumers are worse of with bundling. Both the bundle price and profit decrease with θ, whereas consumer welfare increases with θ. Consumer welfare is higher under bundling for $\theta = 1$, and social welfare is higher under bundling in all cases.

	p^*	r^*	π^*	$\Delta\Pi$	W^*	ΔW	$\Delta\Pi + \Delta W$
$\theta = -1$	≥ 9	11.100	6.1468	1.2468	2.0079	−0.4421	0.8047
$\theta = -.5$	≥ 9	11.042	5.9421	1.0421	2.1020	−0.348	0.6941
$\theta = 0$	≥ 9	10.961	5.7404	0.8404	2.2121	−0.2379	0.6025
$\theta = .5$	≥ 9	10.846	5.5434	0.6434	2.3458	−0.1042	0.5392
$\theta = 1$	≥ 9	10.680	5.3547	0.4547	2.5152	0.0652	0.5199

Table A2: $m = 2$

(3) When $m = 4$, the monopoly price is $p^s = 6.5$, profit is $\Pi^s = 2.5$, and consumer welfare is $W^s = 1.25$. Table A3 shows mixed bundling is optimal, with $p^* > p^s$, and $r^* < 2p^s = 13$. Furthermore, as θ increases, the standalone price and profit decrease, and the bundle price increases, whereas consumer welfare first increases then decreases. Consumer welfare is lower under bundling for $\theta = 1$, but higher in other cases. Bundling increases social welfare in all cases.

	p^*	r^*	Π^*	$\Delta\Pi$	W^*	ΔW	$\Delta\Pi + \Delta W$
$\theta = -1$	7.4999	12.072	2.9174	0.4174	1.2636	0.0136	0.431
$\theta = -.5$	7.4311	12.170	2.8286	0.3286	1.278	0.028	0.3566
$\theta = 0$	7.3333	12.310	2.746	0.246	1.2749	0.0249	0.2709
$\theta = .5$	7.2111	12.496	2.6741	0.1741	1.2530	0.003	0.1771
$\theta = 1$	7.0917	12.700	2.6171	0.1171	1.2209	-0.0291	0.088

Table A3:. $m = 4$

(4) When $m = 6$, monopoly price is $p^s = 7.5$, profit is $\Pi^s = 0.9$, and consumer welfare is $W^s = 0.45$. Table 4A shows that mixed bundling is optimal, with $p^* > p^s$, and $r^* < 2p^s = 15$. As θ increases, the standalone price and bundle price increase, profit decreases, and consumer welfare varies non-monotonically. Consumer welfare is always lower and social welfare higher under bundling.

	p^*	r^*	Π^*	$\Delta\Pi$	W^*	ΔW	$\Delta\Pi + \Delta W$
$\theta = -1$	7.682	14.226	0.95609	0.05609	0.44433	-0.00567	0.05042
$\theta = -.5$	7.6846	14.381	0.94548	0.04548	0.44677	-0.00323	0.04225
$\theta = 0$	7.6916	14.52	0.93861	0.03861	0.44478	-0.00522	0.03339
$\theta = .5$	7.7046	14.636	0.93751	0.03751	0.42157	-0.02843	0.00908
$\theta = 1$	7.7244	14.730	0.93241	0.03241	0.43518	-0.01482	0.01759

Table A4:. $m = 6$

APPENDIX B: DETAILS UNDER PARTIAL COMPETITION

Using the notation of Section 7, under mixed bundling, consumers will purchase the bundle if

$$\begin{aligned}
u(x) + v(y_A) - r &\geq \max\{0, v(y_B) - q_B\}, \\
u(x) + v(y_A) - r &\geq u(x) - p + \max\{0, v(y_B) - q_B\}, \\
u(x) + v(y_A) - r &\geq v(y_A) - q_A,
\end{aligned}$$

or, equivalently,

$$\begin{aligned} y_A &\geq G\left(r-u(x)+\max\left\{0,v(y_B)-q_B\right\}\right), \\ y_A &\geq G\left(r-p+\max\left\{0,v(y_B)-q_B\right\}\right), \\ x &\geq F\left(r-q_A\right). \end{aligned}$$

Consumers will purchase X as a standalone product, rather than as part of the bundle, if

$$\begin{aligned} x &\geq F\left(p\right), \\ y_A &< G\left(r-p+\max\left\{0,v(y_B)-q_B\right\}\right); \end{aligned}$$

and Y_A as a standalone product if

$$\begin{aligned} x &< F\left(r-q_A\right), \\ y_A &\geq G\left(q_A+\max\left\{0,v(y_B)-q_B\right\}\right). \end{aligned}$$

The following demand function aggregates the consumers purchasing only X:

$$\begin{aligned} & Q_X\left(p,q_B,r\right) \\ =\; & C\left(1,G\left(r-p\right),G\left(q_B\right)\right)-C\left(F\left(p\right),G\left(r-p\right),G\left(q_B\right)\right) \\ & +\int_{G(q_B)}^{1}\left[C_3\left(1,G\left(r-p+v(y)-q_B\right),y\right)-C_3\left(F\left(p\right),G\left(r-p+v(y)-q_B\right),y\right)\right]dy. \end{aligned}$$

Similarly, the standalone demand for Y_A is

$$\begin{aligned} & Q_{Y_A}\left(q_A,q_B,r\right) \\ =\; & C\left(F\left(r-q_A\right),1,G\left(q_B\right)\right)-C\left(F\left(r-q_A\right),G\left(q_A\right),G\left(q_B\right)\right) \\ & +\int_{G(q_B)}^{1}\left[C_3\left(F\left(r-q_A\right),1,y\right)-C_3\left(F\left(r-q_A\right),G\left(q_A+v(y)-q_B\right),y\right)\right]dy, \end{aligned}$$

and the demand for the bundle is

$$\begin{aligned} & Q_{XY_A}\left(p,q_A,q_B,r\right) \\ =\; & 1-F\left(p\right)-Q_X\left(p,q_B,r\right) \\ & +\int_{F(r-q_A)}^{F(p)}\left[C_1\left(x,1,G\left(q_B\right)\right)-C_1\left(x,G\left(r-u(x)\right),G\left(q_B\right)\right)\right]dx \\ & +\int_{F(r-q_A)}^{F(p)}\int_{G(q_B)}^{1}\left[C_{13}\left(x,1,y\right)-C_{13}\left(x,G\left(r-u(x)+v(y)-q_B\right),y\right)\right]dydx. \end{aligned}$$

The demand for Y_B is analogous to the demand for Y_A.

Proof of Proposition 4

We show that, under the conditions specified in the Proposition, at any equilibrium firm, A must choose bundling rather than separate selling; i.e., in equilibrium bundling must have higher profit than separate selling for firm A. This would be true if separate selling cannot be part of any equilibrium.

Let (p^s, q^s) denote the prices of X and Y products in a separate-pricing equilibrium. Then

$$\bar{\psi}(\varepsilon) \equiv \pi_A\left(p^s + \epsilon, q^s, q^s, p^s + q^s\right)$$

is Firm A's profit from increasing the standalone price of X, while holding constant the standalone prices of the Y product and the price of the bundle at $r = p^s + q^s$. Separate selling cannot be part of any equilibrium if $\bar{\psi}'(0) > 0$. Noticing that $\partial Q_{Y_A}(q_A, q_B, r)/\partial p = 0$, we have

$$\begin{aligned}
\bar{\psi}'(0) &= \frac{\partial \pi_A\left(p^s, q^s, q^s, p^s + q^s\right)}{\partial p} \\
&= Q_X\left(p^s, q^s, p^s + q^s\right) + \left(p^s - m_X\right)\frac{\partial Q_X\left(p^*, q^*, p^* + q^*\right)}{\partial p} \\
&\quad + \left(p^s + q^s - m_X - m_Y\right)\frac{\partial Q_{XY_A}\left(p^*, q^*, q^*, p^* + q^*\right)}{\partial p}
\end{aligned}$$

with

$$Q_X\left(p^s, q^s, p^s + q^s\right) \equiv \frac{1}{2}\left[1 - F\left(p^s\right)\right] + \frac{1}{2}\left[C\left(1, G\left(q^s\right), G\left(q^s\right)\right) - C\left(F\left(p^s\right), G\left(q^s\right), G\left(q^s\right)\right)\right],$$

$$\begin{aligned}
\frac{\partial Q_X\left(p^s, q^s, p^s + q^s\right)}{\partial p} \equiv\ & -\left[C_2\left(1, G\left(q^s\right), G\left(q^s\right)\right) - C_2\left(F\left(p^s\right), G\left(q^s\right), G\left(q^s\right)\right)\right] g\left(q^s\right) \\
& - C_1\left(F\left(p^s\right), G\left(q^s\right), G\left(q^s\right)\right) f\left(p^s\right) \\
& - \textstyle\int_{G(q^s)}^{1}\left[C_{23}\left(1, y, y\right) - C_{23}\left(F\left(p^s\right), y, y\right)\right] g\left(v\left(y\right)\right) dy \\
& - \textstyle\int_{G(q^s)}^{1} C_{13}\left(F\left(p^s\right), y, y\right) dy f\left(p^s\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Q_{XY_A}\left(p^s, q^s, p^s + q^s\right)}{\partial p} \equiv\ & -f\left(p^s\right) - \frac{\partial Q_X\left(p^s, q^s, p^s + q^s\right)}{\partial p} \\
& + \left[C_1\left(F\left(p^s\right), 1, G\left(q^s\right)\right) - C_1\left(F\left(p^s\right), G\left(q^s\right), G\left(q^s\right)\right)\right] f\left(p^s\right) \\
& + \textstyle\int_{G(q^s)}^{1}\left[C_{13}\left(F\left(p^s\right), 1, y\right) - C_{13}\left(F\left(p^s\right), y, y\right)\right] dy f\left(p^s\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -f(p^s) + [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] g(q^s) \\
& + \textstyle\int_{G(q^s)}^1 [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)] g(v(y)) dy + C_1(F(p), 1, 1) f(p).
\end{aligned}
$$

Therefore,

$$\bar{\psi}'(0) = Q_X(p^s, q^s, p^s + q^s)$$

$$+ (p^s - m_X) \left\{ \begin{array}{c} -f(p^s) + [C_1(F(p^s), 1, G(q^s)) - C_1(F(p^s), G(q^s), G(q^s))] f(p^s) \\ + \int_{G(q^s)}^1 [C_{13}(F(p), 1, y_2) - C_{13}(F(p), y_2, y_2)] f(p) dy_2 \end{array} \right\}$$

$$+ (q^s - m_Y) \left\{ \begin{array}{c} -f(p^s) + [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] g(q^s) + \\ \int_{G(q^s)}^1 [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)] g(v(y)) dy + C_1(F(p), 1, 1) f(p) \end{array} \right\}$$

Substituting $C_1(F(p^s), 1, 1) = 1$ and $C_{13}(1, y_2, y_2) = \frac{1}{2} \frac{dC_1(1, y_2, y_2)}{dy_2}$, and simplifying, we have

$$
\begin{aligned}
\bar{\psi}'(0) = {} & Q_X(p^s, q^s, p^s + q^s) \\
& + (p^s - m_X) \left\{ \begin{array}{c} [C_1(F(p^s), 1, G(q^s)) - C_1(F(p^s), G(q^s), G(q^s))] f(p^s) \\ -C_1(F(p), 1, G(q^s)) f(p^s) - \int_{G(q^s)}^1 \frac{1}{2} dC_1(F(p), y_2, y_2) f(p) \end{array} \right\} \\
& + (q^s - m_Y) \left\{ \begin{array}{c} [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] g(q^s) \\ + \int_{G(q^s)}^1 [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)] g(v(y)) dy \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & Q_X(p^s, q^s, p^s + q^s) + (p^s - m_X) \left\{ \begin{array}{c} -C_1(F(p^s), G(q^s), G(q^s)) f(p^s) - \\ \frac{1}{2} [C_1(F(p^s), 1, 1) - C_1(F(p^s), G(q^s), G(q^s))] f(p^s) \end{array} \right\} \\
& + (q^s - m_Y) \left\{ \begin{array}{c} [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] g(q^s) \\ + \int_{G(q^s)}^1 [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)] g(v(y)) dy \end{array} \right\}.
\end{aligned}
$$

Substituting for $Q_X(p^s, q^s, p^s + q^s)$, using $(p^s - m_X) f(p^s) = 1 - F(p^s)$, and simplifying

$$
\begin{aligned}
\bar{\psi}'(0) \quad = \quad & \frac{1}{2} \int_{F(p^s)}^1 [C_1(x, G(q^s), G(q^s)) - C_1(F(p^s), G(q^s), G(q^s))] dx \\
& + (q^s - m_Y) \left\{ \begin{array}{c} [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] g(q^s) \\ + \int_{G(q^s)}^1 [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)] g(v(y)) dy \end{array} \right\} \\
& \geq \frac{1}{2} \int_{F(p^s)}^1 \int_{F(p^s)}^x C_{11}(z, G(q^s), G(q^s)) dz dx + \bar{\delta},
\end{aligned}
$$

since $\int_{G(q^s)}^{1}\left[C_{23}\left(1,y,y\right)-C_{23}\left(F\left(p^s\right),y,y\right)\right]g\left(v\left(y\right)\right)dy\geq 0$, where

$$\bar{\delta}=\left(q^s-m_Y\right)g\left(q^s\right)\left[C_2\left(1,G\left(q^s\right),G\left(q^s\right)\right)-C_2\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]>0.$$

Thus, $\bar{\psi}'\left(0\right)>0$ if values for X and Y are negatively dependent ($C_{11}\geq 0$) or independent ($C_{11}=0$). Now, suppose that values for X and Y are positively dependent but $C_{11}>-4\bar{\delta}$. Then

$$\begin{aligned}\bar{\psi}'\left(0\right) &> -\frac{1}{2}\int_{F(q^s)}^{1}\int_{F(q^s)}^{x}4\bar{\delta}dzdx+\bar{\delta}\\ &= -\frac{1}{4}\left(1-F\left(q^s\right)\right)^2 4\bar{\delta}+\bar{\delta}>-\bar{\delta}+\bar{\delta}=0.\end{aligned}$$

Q.E.D.

REFERENCES

[1] Adams, William J. and Janet L. Yellen (1976), "Commodity Bundling and the Burden of Monopoly," *Quarterly Journal of Economics*, 90: 475-498.

[2] Armstrong, Mark (2010), " "Bundling Revisited: Substitute Products and Inter-firm Discounts," MPRA Working Paper 26782.

[3] Carbajo, Jose, David De Meza, and Daniel J. Seidman (1990), "A Strategic Motivation for Commodity Bundling," *Journal of Industrial Economics*, 38: 283-298.

[4] Carlton, D. and M. Waldman (2002), "The Strategic Use of Tying to Preserve and Create Market Power in Evolving Industries", *RAND Journal of Economics*, 32: 194-220.

[5] Chen, Y. (1997), "Equilibrium Product Bundling", *Journal of Business*, 70: 85-103.

[6] Chen, Y. and M. H. Riordan (2008), "Price-Increasing Competition", *RAND Journal of Economics,* 39: 1042-1058.

[7] Chen, Y. and M. H. Riordan (2010), "Preferences, Prices and Performance in Multiproduct Industries", working paper.

[8] Chen, Y. and J. Pearcy (2010), "Dynamic Pricing: When to Entice Brand Switching and When to Reward Consumer Loyalty", *RAND Journal of Economics*, 41: 674-685.

[9] Choi, J. and C. Stefanadis (2001), "Tying, Investment, and the Dynamic Leverage Theory", *RAND Journal of Economics,* 32: 52-71.

[10] Chu, Sean, Phillip Leslie, and Alan Sorensen (2011), "Bundle-Size Pricing as an Approximation to Mixed Bundling," *American Economic Review*, 101: 263–303.

[11] Crawford, G. and A. Yurukoglu (2012), ""The Welfare Effects of Bundling in Multichannel Television Markets," *American Economic Review*, forthcoming.

[12] Fang, Hanming and Peter Norman (2006), "To Bundle or Not to Bundle," *RAND Journal of Economics*, 37, 946-963.

[13] Ho, C., J. Ho and J. Mortimer (2011), "The Use of Full-line Forcing Contracts in the Video Rental Industry," *American Economic Review*, forthcoming.

[14] Lewbel, Arthur (1985), "Bundling of Substitutes or Complements," *International Journal of Industrial Organization*, 3: 101-107.

[15] Long, John B. Jr. (1984), "Comments on 'Gaussian Demand and Commodity Bundling," *Journal of Business*, 57: S235-S246.

[16] McAfee, R. Preston, John McMillan, and Michael D. Whinston (1989), "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *Quarterly Journal of Economics*, 104: 371-383.

[17] Nalebuff, Barry (2004), "Bundling as an Entry Barrier," *Quarterly Journal of Economics*, 119: 159-187.

[18] Nelsen, Roger B. (2006), *An Introduction to Copulas*, 2^{nd} edition, Springer-Verlag.

[19] Patton, A.J. (2008), "Copula-Based Models for Financial Time Series." Oxford Financial Center Economics Series, no. 2008fe21, Oxford University.

[20] Schmalensee, Richard (1984), "Gaussian Demand and Commodity Bundling,"*Journal of Business*, 57: S211-S230.

[21] Stigler, George (1963), "A Note on Block Booking," U.S. Supreme *Court Review*, 1963, 152-175.

[22] Whinston, Michael D. (1990), "Tying, Foreclosure, and Exclusion," *American Economic Review*, 80: 837-859.