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MEASURING UNCERTAINTY WITHOUT A NORM

by

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Abstract

Traub, Wasilkowski, and Woźniakowski have shown how uncertainty can be defined and analyzed without a norm or metric. Thier theory is based on two natural and non-restrictive axioms. We show that these axioms induce a family of pseudometrics, and that balls of radius ε are (roughly) the ε-approximations to the solution. In addition, we show that a family of pseudometrics is necessary, even for the problem of computing x such that $|f(x)| \leq \varepsilon$, where f is a real function.

1. Introduction

In two recent monographs ([3],[4]), Traub and his colleagues have studied the optimal solution of problems which are solved approximately, that is, where there is uncertainty in the answer. In [4], uncertainty was measured by a norm. For some problems, this is not an appropriate or natural assumption. Therefore, in [3] it is shown how uncertainty can be introduced via two natural and non-restrictive axioms.

In a private communication, Traub asked about the strength of these axioms. That is, do the axioms generate any interesting structures? In Section 2 of this paper, we show that these axioms induce a family of pseudometrics. Moreover, we show that the balls of radius ε generated by this family of pseudometrics are (roughly speaking) the ε-approximations to the solution.

Is a family of pseudometrics necessary? We give an affirmative answer in Section 3, using the problem of computing x such that $|f(x)| \leq \varepsilon$, where f is a real function.

2. Solution Operators Are Generated by Families of Pseudometrics

We first recall the definition of a solution operator from [3]. Let F and G be sets, and let 2^G denote the power set of G, i.e., the class of all subsets of G. Let $\mathbb{R}^+$ denote the non-negative real numbers. If $S : F \times \mathbb{R}^+ \to 2^G$ is an operator such that

$$(2.1) \qquad \forall f \in F, \qquad S(f,0) \neq \emptyset$$

and

$$(2.2) \qquad \forall f \in F, \ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{R}^+ \text{ with } \varepsilon_1 \leq \varepsilon_2, \ S(f,\varepsilon_1) \subseteq S(f,\varepsilon_2),$$

then S is said to be a <u>solution operator</u>, and $S(f,\varepsilon)$ is said to be the set of ε-<u>approximations</u> to the (exact) <u>solution</u> $S(f,0)$.

Note that $S(f,\varepsilon)$ is a set. This formulation allows the exact solution $S(f,0)$ to be a set, i.e., a problem may have multiple solutions. In addition, $S(f,\varepsilon)$ a set means that we are willing to accept any element of $S(f,\varepsilon)$ as an ε-approximation. These axioms are very natural: the first says that every problem has a solution, while the second says that increasing the uncertainty cannot decrease the family of ε-approximations.

In order to clarify these notions we give three examples.

<u>Example 2.1.</u> Let F be a set and let G be a normed linear space. Let $\bar{S} : F \to G$ be an operator. Define $S : F \times \mathbb{R}^+ \to 2^G$ by

$$S(f,\varepsilon) = \{g \in G : \|\bar{S}f - g\| \leq \varepsilon\}.$$

Then S is a solution operator, and $g \in G$ is an ε-approximation to $\bar{S}f$ precisely when $\|g - \bar{S}f\| \leq \varepsilon$. (This is the setting extensively studied in [4].)

Example 2.2. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, let

$$Z(f) := \{x \in \mathbb{R} : f(x) = 0\}$$

denote the zeroset of f. Now let

$$F = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } Z(f) \neq \emptyset\},$$

choose $G = \mathbb{R}$, and define $S : F \times \mathbb{R}^+ \to 2^G$ by

$$S(f,\varepsilon) := \{x \in \mathbb{R} : |f(x)| \leq \varepsilon\}. \tag{2.3}$$

Then S is a solution operator, and $x \in S(f,\varepsilon)$ precisely when $|f(x)| \leq \varepsilon$, i.e., the residual of f at x is at most ε.

Before introducing the last example, we recall that a pseudometric d on G is a map $d : G \times G \to \mathbb{R}^+$ satisfying

$$\begin{aligned} &d(g,g) = 0 && \forall\, g \in G, \\ &d(g_1,g_2) = d(g_2,g_1) && \forall\, g_1,g_2 \in G, \\ &d(g_1,g_3) \leq d(g_1,g_2) + d(g_2,g_3) && \forall\, g_1,g_2,g_3 \in G. \end{aligned} \tag{2.4}$$

(This terminology is standard in topology, see [2, pg. 198]. However, Collatz [1, pg. 21] refers to such a map as a "quasimetric," letting "pseudometric" refer to another concept entirely [1, pg. 51].) If d is a pseudometric on G, the set

$$B(g,d,\varepsilon) := \{x \in G : d(x,g) \leq \varepsilon\}$$

is called the d-ball of radius ε centered at g.

<u>Example 2.3.</u> Let F and G be sets. Let $\mathcal{D}$ be an F-indexed family of pseudometrics on G. Let $\bar{S} : F \to G$ be an operator. For $f \in F$, choose $d_f \in \mathcal{D}$, and define

$$S(f,\varepsilon) := B(\bar{S}f, d_f, \varepsilon) \qquad \forall\ \varepsilon \geq 0.$$

Then it is easy to see that $S : F \times \mathbb{R}^+ \to 2^G$ is a solution operator.

We now show that, roughly speaking, Example 3 is the most general example of a solution operator.

<u>Theorem 2.1.</u> Let F, G be sets, and let $S : F \times \mathbb{R}^+ \to 2^G$ be a solution operator. Then for any $\varepsilon_0 > 0$, there is an operator $\bar{S} : F \to G$ and a family $\mathcal{D} = \{d_f : f \in F\}$ of pseudometrics on G such that

$$S(f,\varepsilon) \subseteq B(\bar{S}f, d_f, \varepsilon) \subseteq S(f,\varepsilon') \tag{2.5}$$

for any $f \in F$, any $\varepsilon \in [0,\varepsilon_0)$, and any $\varepsilon' \in (\varepsilon,\varepsilon_0]$.

<u>Proof</u>: Let $f \in F$. For $g \in G$, let

$$D_f(g) := \{\varepsilon \geq 0 : g \in S(f,\varepsilon)\},$$

and now define $d_f : G \times G \to \mathbb{R}^+$ by

$$d_f(g_1,g_2) := \min\{\varepsilon_0, |\inf D_f(g_2) - \inf D_f(g_1)|\}, \tag{2.6}$$

where the inf of an empty set is defined to be ∞, and $\infty - \infty = 0$.

We first show that d_f is a pseudometric on G. Clearly, the first two properties in (2.4) hold for d_f; we need only check the third (the triangle inequality). Let $g_1, g_2, g_3 \in G$, and let

$\delta_i = \inf D_f(g_i)$. Then

$$|\delta_3 - \delta_1| \leq |\delta_2 - \delta_1| + |\delta_3 - \delta_2|. \tag{2.17}$$

Arguing by cases if necessary, it is easy to see that (2.6) and (2.7) yield the triangle inequality.

We now define $\bar{S} : F \to G$ to be any map such that

$$\bar{S}f \in S(f,0) \qquad \forall\ f \in F.$$

This is possible because $S(f,0) \neq \emptyset$.

We now must prove (2.5). Let $f \in F$ and $\varepsilon \in [0,\varepsilon_0)$. We first claim that

$$d_f(g,\bar{S}f) = \min\{\varepsilon_0, \inf D_f(g)\}. \tag{2.8}$$

Indeed, since $\bar{S}f \in S(f,0)$, we have

$$\inf D_f(\bar{S}f) = 0,$$

so that (2.8) follows from (2.6).

To see that $S(f,\varepsilon) \subseteq B(\bar{S}f,d_f,\varepsilon)$, let $g \in S(f,\varepsilon)$. We then have

$$\inf D_f(g) \leq \varepsilon;$$

since $\varepsilon < \varepsilon_0$, we use (2.8) to find

$$d_f(g,\bar{S}f) \leq \varepsilon,$$

and so $g \in B(\bar{S}f,d_f,\varepsilon)$.

Now let $\varepsilon' \in (\varepsilon,\varepsilon_0]$. We wish to show that $B(\bar{S}f,d_f,\varepsilon) \subseteq S(f,\varepsilon')$. Let $g \in B(\bar{S}f,d_f,\varepsilon)$. Since $\varepsilon < \varepsilon_0$, we use (2.8) to find

$$\text{(2.9)} \qquad \inf D_f(g) = d_f(g,\bar{S}f) \leq \varepsilon.$$

The first part of (2.9) and the definition of infimum yield

$$g \in S(f, d_f(g,\bar{S}f) + \delta)$$

for all $\delta > 0$, no matter how small. Setting $\delta = \varepsilon' - \varepsilon > 0$, we then have

$$\begin{aligned} g &\in S(f, d_f(g,\bar{S}f) + \varepsilon' - \varepsilon) \\ &\subseteq S(f, \varepsilon + \varepsilon' - \varepsilon) \\ &= S(f,\varepsilon'), \end{aligned}$$

where the inclusion follows from the second part of (2.9) and the monotonicity condition (2.2).

<u>Remark 2.1.</u> We comment on the role played by ε_0. It is possible to describe problems with $D_f(g)$ empty for some $f \in F$ and $g \in G$. (For example, take $\bar{S} : F \to G$ to be any operator where G is not a singleton, and define $S(f,\varepsilon) := \{\bar{S}f\}$ $\forall$ $f \in F$, $\varepsilon \geq 0$; then for any $f \in F$, $g \in G$ with $g \neq \bar{S}f$, and $\varepsilon \geq 0$, $g \notin S(f,\varepsilon)$, so that $D_f(g) = \emptyset$.) If we were to define

$$d_f^*(g_1,g_2) := |\inf D_f(g_2) - \inf D_f(g_1)|,$$

we would then find $d_f^*(g,\bar{S}f) = \infty$ for such f and g. Hence, d_f^* is not a pseudometric (since the value of a pseudometric must be finite).

Hence, ε_0 is used to force d_f to take finite values. It may be thought of as the maximal uncertainty to be tolerated, the motivation being that we want "good" approximations, i.e.,

ε-approximations for small values of ε.

On the other hand, if for any $f \in F$ and $g \in G$, there is an $\varepsilon \geq 0$ such that $g \in S(f,\varepsilon)$ (i.e., the "distance" between any solution and any point in G is finite), then d_f^* is always finite. Hence d_f^* is a pseudometric, and (2.5) holds for all $\varepsilon \geq 0$, with d_f^* replacing d_f.

Remark 2.2. It would be more satisfying to be able to say that

$$S(f,\varepsilon) = B(\bar{S}f,d_f,\varepsilon) \tag{2.10}$$

in the conclusion of Theorem 2.1. However, we cannot do this in general. To see this, let d be a pseudometric on G, where G is not a singleton, and let $\hat{S} : F \to G$ be any operator. Define a solution operator $S : F \times \mathbb{R}^+ \to 2^G$ by

$$S(f,\varepsilon) := \begin{cases} \{g \in G : d(g,\hat{S}f) < \varepsilon\} & \text{if } \varepsilon > 0 \\ \{\hat{S}f\} & \text{if } \varepsilon = 0 \end{cases} .$$

Suppose there exists $\bar{S} : F \to G$ and a family $\{d_f : f \in F\}$ of pseudometrics such that (2.10) holds.

We first note that $\bar{S} = \hat{S}$. Indeed, since $d_f(\bar{S}f,\bar{S}f) = 0$, we have

$$\bar{S}f \in B(\bar{S}f,d_f,0) = S(f,0) = \{\hat{S}f\},$$

i.e., $\bar{S}f = \hat{S}f$.

Next, we show that for any $f \in F$ and $g \in G$,

$$d_f(g,\bar{S}f) \leq d(g,\bar{S}f). \tag{2.11}$$

Indeed, let $\delta > 0$. Since $d(g,\overline{S}f) < d(g,\overline{S}f) + \delta$ and $\hat{S}f = \overline{S}f$, we have

$$g \in S(f,d(g,\overline{S}f) + \delta) = B(\overline{S}f,d_f,d(g,\overline{S}f) + \delta),$$

so that

$$d_f(g,\overline{S}f) \leq d(g,\overline{S}f) + \delta.$$

Since $\delta > 0$ is arbitrary, (2.11) follows.

We claim that there exist $f \in F$ and $g \in G$ such that

$$d_f(g,\overline{S}f) > 0. \tag{2.12}$$

Indeed, if $d_f(g,\overline{S}f) = 0$ for all $f \in F$ and $g \in G$, we would have

$$g \in B(\overline{S}f,d_f,0) = S(f,0) = \{\overline{S}f\},$$

which would mean that

$$g = \overline{S}f \quad \forall\ f \in F,\ g \in G.$$

Fixing f and letting g vary, this would imply that G is a singleton, a contradiction.

Finally, we choose $f \in F$ and $g \in G$ such that (2.12) holds. Then $d_f(g,\overline{S}f) \leq d_f(g,\overline{S}f)$ yields

$$g \in B(\overline{S}f,d_f,d_f(g,\overline{S}f)) = S(f,d_f(g,\overline{S}f)).$$

Since (2.12) holds, we have

$$d(g,\overline{S}f) < d_f(g,\overline{S}f),$$

contradicting (2.11).

3. A Family of Pseudometrics is Necessary

In this section, we reconsider Example 2.2, the solution of nonlinear equations. We show explicitly how to construct a family of pseudometrics such that (2.10) holds for all $\varepsilon \geq 0$. Moreover, we show that a family of pseudometrics is necessary.

We first define $\bar{S} : F \to \mathbb{R}$ by letting $\bar{S}f$ be the zero of f that is smallest in magnitude; if there are two such zeros, choose the positive one. (Such a zero exists because f is continuous.)

Next, define $d_f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ (for $f \in F$) by

$$d_f(x,y) := |f(x) - f(y)|.$$

Then d_f is a pseudometric.

We then have

Theorem 3.1. $S(f,\varepsilon) = B(\bar{S}f, d_f, \varepsilon) \quad \forall\ f \in F,\ \varepsilon \geq 0.$

Proof: Let $f \in F$, $\varepsilon \geq 0$. Since $f(\bar{S}f) = 0$, we have

$$\begin{aligned} x \in S(f,\varepsilon) &\Leftrightarrow |f(x)| \leq \varepsilon \\ &\Leftrightarrow |f(x) - f(\bar{S}f)| \leq \varepsilon \\ &\Leftrightarrow d_f(x,\bar{S}f) \leq \varepsilon \\ &\Leftrightarrow x \in B(\bar{S}f, d_f, \varepsilon). \end{aligned}$$

Hence, Example 2.2 generates a family $\{d_f : f \in F\}$ of pseudometrics, and the d_f-ball of radius ε about $\bar{S}f$ is the set of ε-approximations to the zeroset of f, for any $f \in F$ and $\varepsilon \geq 0$.

We now show that Example 2.2 cannot be generated by a single pseudometric.

<u>Theorem 3.2.</u> There does not exist an operator $\bar{S} : F \to \mathbb{R}$ and a single pseudometric d on $\mathbb{R}$ such that

$$(3.1) \qquad S(f,\varepsilon) = B(\bar{S}f,d,\varepsilon) \qquad \forall\ f \in F, \quad \varepsilon \geq 0.$$

<u>Proof</u>: Suppose there exists $\bar{S}$ and d such that (3.1) holds. We first note that $\bar{S}f \in Z(f)$ for all $f \in F$. To see this, let $f \in F$. Since $d(\bar{S}f,\bar{S}f) = 0$, we have $\bar{S}f \in B(\bar{S}f,d,0) = S(f,0)$. Hence $|f(\bar{S}f)| \leq 0$ by (2.3), so that $f(\bar{S}f) = 0$ and $\bar{S}f \in Z(f)$, as claimed.

We next claim that

$$(3.2) \qquad d(x,\bar{S}f) = |f(x)| \qquad \forall\ x \in \mathbb{R}, \quad f \in F.$$

Indeed, let $f \in F$, and $x \in \mathbb{R}$. Using (2.3) and (3.1), we have

$$\begin{aligned} |f(x)| \leq |f(x)| &\Rightarrow x \in S(f,|f(x)|) = B(\bar{S}f,d,|f(x)|) \\ &\Rightarrow d(x,\bar{S}f) \leq |f(x)|, \end{aligned}$$

and

$$\begin{aligned} d(x,\bar{S}f) \leq d(x,\bar{S}f) &\Rightarrow x \in B(\bar{S}f,d,d(x,\bar{S}f)) = S(f,d(x,\bar{S}f)) \\ &\Rightarrow |f(x)| \leq d(x,\bar{S}f), \end{aligned}$$

yielding (3.2).

We now let $x,y \in \mathbb{R}$ with $x \neq y$. Define $f_\alpha \in F$ by

$$(3.3) \qquad f_\alpha(t) := \alpha(t - y) \qquad \forall\ \alpha \in \mathbb{R}.$$

Then the first paragraph of this proof yields

$$\bar{S}f_\alpha \in Z(f_\alpha) = \{y\} \qquad \forall\ \alpha \in \mathbb{R},$$

i.e.,

$$\bar{S}f_\alpha = y \qquad \forall\, \alpha \in \mathbb{R}. \tag{3.4}$$

Hence (3.4), (3.2), and (3.3) yield

$$\left.\begin{aligned} d(x,y) &= d(x,\bar{S}f_\alpha) \\ &= |f_\alpha(x)| \\ &= |\alpha(x - y)| \\ &= |\alpha||x - y| \end{aligned}\right\} \quad \forall\, \alpha \in \mathbb{R}.$$

Since $x \neq y$, this means that $d(x,y)$ must be multiple-valued, a contradiction.

Note that the proof of Theorem 3.2 did not require the use of functions with multiple zeros. Hence, even if we consider Example 2.2 with F replaced by

$$F' := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and has exactly one zero}\}$$

we still cannot use a single pseudometric to generate this example.

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References

[1] L. Collatz, Functional Analysis and Numerical Mathematics, Academic Press, New York, 1966.

[2] J. Dugundji, Topology, Allyn and Bacon, Boston, 1970.

[3] J. F. Traub, G. Wasilkowski, and H. Woźniakowski, Information, Uncertainty, Complexity, Addison-Wesley, Reading, MA, 1982.

[4] J. F. Traub and H. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.