A Note on Identification Test Procedures

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Abstract

This paper discusses the meaning of "identification" tests in the context of the General Linear Structural Econometric Model (GLSEM). Its major contribution is to show that, contrary to some implications of the literature, it is not possible to devise **independent tests for all prior restrictions**, and to clarify the meaning of results from such tests.

Key words: **Specification Test, Identification Test, Lagrange Multiplier Test, Conformity Test, Hausman Test, Simultaneous Equations, Two Stage Least Squares, Three Stage Least Squares.**

1 Introduction and Summary

Ever since the classic papers of Anderson and Rubin (1949), (1950) an important problem has been the test (and verification) of prior restrictions, and the proper interpretation of the results one obtains. This problem is intimately related to the interpretation of test results involving singular covariance matrices as is evdent, for example, in the tests introduced by Hausman (1978), (1982), and Dhrymes (1989), (1991) and, indirectly, in the tests proposed in Byron (1974), and recently extended by Anderson and Kunitomo (1992). We shall give an example of the resolution proposed by this paper for the tests

* I wish to thank T.W. Anderson for interesting conversations on this subject.
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introduced in Dhrymes (1989). Although such tests can be carried out in a multi-equation, or in a systemwide context, as in the three state least squares (3SLS) case, we confine our discussion to the context of single equation two stage least squares (2SLS).

Suppose

$$Y_i^{\circ}\beta_{\cdot i}^{\circ} + Y_i^{*}\beta_{\cdot i}^{*} = X_i\gamma_{\cdot i} + X_i^{*}\gamma_{\cdot i}^{*} + u_{\cdot i}, \quad i = 1, 2, \ldots, m, \tag{1}$$

is the i^{th} equation of a complete system containing $(Y_i^{\circ},\ Y_i^{*})$ and (X_i, X_i^{*}), which are the matrices, respectively, of the T observations on the m jointly dependent and G predetermined variables. The "prior information" is summarized in the statement

$$\beta_{\cdot i}^{*} = 0, \quad \gamma_{\cdot i}^{*} = 0. \tag{2}$$

Thus, the unknown (structural) parameters in the i^{th} equation above are the $m_i + 1$ elements[1] in $\beta_{\cdot i}^{\circ}$, and the G_i elements in $\gamma_{\cdot i}$. Writing the (relevant) reduced form as

$$Y_i^{\circ} \ = \ X_i\Pi_{1i}^{\circ} + X_i^{*}\Pi_{2i}^{\circ} + V_i^{\circ}, \tag{3}$$

we note that, by substitution in Eq. (1), the prior restrictions imply

$$\Pi_{1i}^{\circ}\beta_{\cdot i}^{\circ} = \gamma_{\cdot i}, \quad \Pi_{2i}^{\circ}\beta_{\cdot i}^{\circ} = 0. \tag{4}$$

These relationships have formed the basis of several papers dealing with issues of identifiability and prior restrictions. Thus, in the classic Anderson Rubin (AR) test, the identification of the i^{th} equation is carried out in the framework

$H_0:\ \text{rank}(\Pi_{2i}^{\circ}) \leq m_i$, as against the alternative, $H_1:\ \text{rank}(\Pi_{2i}^{\circ}) = m_i + 1$.

This test is implemented in terms of the smallest characteristic root of a certain matrix in the metric of another matrix. Subsequently, Koopmans and Hood (1954), (KH), introduced another test based on the **two smallest roots**; Kadane (1974) purports to show that, in the context of small-σ asymptotics, the AR test is equivalent to the test of the conditions in Eq.

[1] Actuallly, one of these elements is to be normalized to unity, so that operationally there are only m_i parameters to be estimated therein.

(2); similarly, Kadane and Anderson (1977) show that the existence of a non-singular matrix of transformation such that the conditions in Eq. (2) hold, is equivalent to the rank condition above. The question posed by these papers is what alternative is to be accepted, should the null be rejcted. For example, in Byron (1974), a test of prior restrictions is carried out, in the context of Eq. (4), by **discarding** some of the prior restrictions and expressing the remainder in the form $f(\Pi) = 0$, where f is an appropriate nonlinear function of reduced form parameters. Byron notes that the result is invariant, within certain limits, with respect to the discarded prior restrictions. The same is noted in various contexts by Hwang (1980), Wegge (1978), and more recently by Anderson and Kunitomo (1992). In the discussion below we shall provide an explanation for this phenomenon and illustrate it in the context of single equation 2SLS estimation.

2 Reformulation of the Problem

In a previous paper, Dhrymes (1989), (1991), the author has restated the 2SLS problem as

$$\min_{a_{\cdot i}} \frac{1}{T}(w_{\cdot i} - Qa_{\cdot i})' w_{\cdot i} - Qa_{\cdot i}), \quad \text{subject to} \quad L_i^{*\prime} a_{\cdot i} = 0,$$

where

$$\begin{aligned} Q &= R^{-1}X'Z, \quad Z = (Y, X), \quad X'X = RR' \\ w_{\cdot i} &= R^{-1}X'y_{\cdot i}, \quad L_i^* = \begin{bmatrix} L_{1i}^* & 0 \\ 0 & L_{2i}^* \end{bmatrix}, \quad L_{i\cdot} = \begin{bmatrix} L_{1i} & 0 \\ 0 & L_{2i} \end{bmatrix}, \\ a_{\cdot i} &= \begin{pmatrix} b_{\cdot i} \\ c_{\cdot i} \end{pmatrix}, \quad L_{1i}^{*\prime} b_{\cdot i} = \beta_{\cdot i}^*, \quad L_{2i}^{*\prime} c_{\cdot i} = \gamma_{\cdot i}^*, \end{aligned} \tag{5}$$

where L_i is the **complement of** L_i^*. The first order conditions are given by

$$\begin{bmatrix} \frac{Q'Q}{T} & L_i^* \\ L_i^{*\prime} & 0 \end{bmatrix} \begin{bmatrix} a_{\cdot i} \\ \lambda_{\cdot i} \end{bmatrix} = \begin{bmatrix} \frac{Q'w_{\cdot i}}{T} \\ 0 \end{bmatrix}, \tag{6}$$

where $\lambda_{\cdot i}$ is the ($m_i^* + G_i^*$-element) vector of Lagrange multipliers, and $G_i^* = G - G_i$, $m_i^* = m - m_i - 1$. It may be shown, moreover, that the system above has a unique solution that may be expressed as

$$\begin{aligned} \tilde{a}_{\cdot i} &= \frac{1}{T}\tilde{E}Q'w_{\cdot i} \\ \tilde{\lambda}_{\cdot i} &= \frac{1}{T}\tilde{J}_{22}^{-1}L_i^{*\prime}\tilde{V}_{11}Q'w_{\cdot i}, \end{aligned} \tag{7}$$

where

$$\tilde{E} = \tilde{V}_{11} - \tilde{V}_{11}L_i^*\tilde{J}_{22}^{-1}L_i^{*\prime}\tilde{V}_{11}, \quad \tilde{J}_{22}^{-1} = (L_i^{*\prime}\tilde{V}_{11}L_i^*)^{-1},$$

and

$$\tilde{V}_{11} = [\frac{Q'Q}{T} + L_i^*L_i^{*\prime}]^{-1}.$$

The reader may verify that this is indeed the single equation 2SLS by verifying that $L_i'\hat{a}_{\cdot i(2SLS)} = \hat{\delta}_{\cdot i(2SLS)}$, as defined, for example, in Dhrymes (1978).

Substituting, in the equations above, we find

$$\begin{aligned} \sqrt{T}\,(\tilde{a}_{\cdot i} - a_{\cdot i})_{2SLS} &= \frac{1}{\sqrt{T}}\tilde{E}Q'r_{\cdot i} \\ \tilde{\lambda}_{\cdot i} &= \left(\tilde{J}_{22}^{-1} - I\right)L_i^{*\prime}a_{\cdot i} + \frac{1}{T}\tilde{J}_{22}^{-1}L_i^{*\prime}\tilde{V}_{11}Q'r_{\cdot i}, \end{aligned} \tag{8}$$

from which it follows that, under the null $L_i^{*\prime}a_{\cdot i} = 0$,

$$\sqrt{T}\begin{pmatrix} \tilde{a}_{\cdot i} - a_{\cdot i} \\ \tilde{\lambda}_{\cdot i} \end{pmatrix}_{2SLS} \xrightarrow{\text{d}} N(0, C_{2SLS}), \tag{9}$$

where

$$C_{2SLS} = \text{diag}(C_{1i}, J_{22}^{-1} - I). \tag{10}$$

It is fairly simple to show that $J_{22}^{-1} - I$ is singular. Specifically, we have

Proposition 1. The covariance matrix of the limiting distribution of the Lagrange multiplier estimators is singular and has m zero roots.

Proof: Define $J = H_i'V_{11}H_i$, where $H_i = (L_i, L_i^*)$, and note that H_i is orthogonal and of dimension $m + G$. It follows, therefore, that

$$J^{-1} = H_i'V_{11}^{-1}H_i = \begin{bmatrix} L_i'KL_i & L_i'KL_i^* \\ L_i^{*\prime}KL_i & L_i^{*\prime}KL_i^* + I \end{bmatrix} = \begin{bmatrix} J^{11} & J^{12} \\ J^{21} & J^{22} \end{bmatrix}. \tag{11}$$

By the partitioned inverse formula, we obtain

$$J_{22}^{-1} = I + L_i^{*\prime}KL_i^* - L_i^{*\prime}KL_i(L_i'KL_i)^{-1}L_i'KL_i^*, \tag{12}$$

and we observe that the components of $J_{22}^{-1} - I$ consist of the blocks of

$$H_i' K H_i = \begin{bmatrix} L_i' K L_i & L_i' K L_i^* \\ L_i^{*\prime} K L_i & L_i^{*\prime} K L_i^* \end{bmatrix}, \tag{13}$$

which is a matrix of rank G. Define, further, the (nonsingular) matrix

$$D = \begin{bmatrix} (L_i' K L_i)^{-1} & -(L_i' K L_i)^{-1} L_i' K L_i^* \\ 0 & I \end{bmatrix}, \tag{14}$$

and note that

$$V^* = L_i^{*\prime} K L_i^* - L_i^{*\prime} K L_i (L_i' K L_i)^{-1} L_i' K L_i^*$$

$$H_i' K H_i D = \begin{bmatrix} I & 0 \\ & \\ L_i^{*\prime} K L_i (L_i' K L_i)^{-1} & V^* \end{bmatrix}. \tag{15}$$

Since the rank of the left matrix is G and the rank of the identity matrix is $G_i + m_i + 1$, it follows that $J_{22}^{-1} - I$, is a matrix of **dimension** $G + m - G_i - m_i - 1$ and rank $G - G_i - m_i - 1$, i.e., it has m zero roots.

q.e.d.

The fact that the covariance matrix of the limiting distribution of the Lagrange multipliers is **singular** has certain implications, which throw considerable light on the results of Byron, Hwang, Anderson and Kunitomo etc. These implications are best outlined in terms of a purely mathematical result, which will provide a context-neutral explanation.

Lemma 1. Let $x \sim N(\mu, \Sigma)$ be an m-element column vector of random variables and suppose

$$\text{rank}(\Sigma) = r \leq m. \tag{16}$$

Without loss of generality, partition (conformably)

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

such that Σ_{11} is a **nonsingular matrix of rank** r. Let Σ_g be the generalized inverse of Σ. Then,

$$(x - \mu)' \Sigma_g (x - \mu) = (x_1 - \mu_1)' \Sigma_{11}^{-1} (x_1 - \mu_1). \tag{18}$$

Proof: Since Σ is positive semidefinite, its characteristic vectors form an orthogonal matrix, say P; partition $P = (P_1, P_2)$ such that P_1 corresponds to the **positive** roots and P_2 to the zero roots; further, partition the (diagonal) matrix of characteristic roots, $\Lambda = \text{diag}(\Lambda_1, 0)$ so that we have the representation

$$\Sigma = P_1 \Lambda_1 P_1', \quad \Sigma_g = P_1 \Lambda_1^{-1} P_1'. \tag{19}$$

Further partition

$$P_1 = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix}, \quad P_2 = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}, \tag{20}$$

and note that P_{11}, P_{22} are nonsingular matrices of rank r, $m - r$ respectively and, moreover, that

$$\Sigma_{11} = P_{11} \Lambda_1 P_{11}', \quad \Sigma_{11}^{-1} = P_{11}'^{-1} \Lambda_1^{-1} P_{11}^{-1}. \tag{21}$$

Since

$$0 = \Sigma P_2 = \begin{bmatrix} \Sigma_{11} P_{12} + \Sigma_{12} P_{22} \\ \Sigma_{21} P_{12} + \Sigma_{22} P_{22} \end{bmatrix}, \tag{22}$$

we conclude that

$$(\Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}) P_{22} = 0. \tag{23}$$

But this implies

$$\Sigma_{22} = \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}. \tag{24}$$

Since $x - \mu$ has mean zero the equation above implies

$$x_2 - \mu_2 = \Sigma_{21} \Sigma_{11}^{-1} (x_1 - \mu_1), \tag{25}$$

and an easy substitution from Eq. (21) yields,

$$x_2 - \mu_2 = P_{21} P_{11}^{-1} (x_1 - \mu_1). \tag{26}$$

Consequently,

$$\begin{aligned} (x - \mu)' \Sigma_g x &= (x_1 - \mu_1)' (I, P_{11}'^{-1} P_{21}') \Sigma_g \begin{pmatrix} I \\ P_{21} P_{11}^{-1} \end{pmatrix} (x_1 - \mu_1) \qquad (27) \\ &= (x_1 - \mu_1)' P_{11}'^{-1} \Lambda_1^{-1} P_{11}^{-1} (x_1 - \mu_1) = (x_1 - \mu_1)' \Sigma_{11}^{-1} (x_1 - m_1). \end{aligned}$$

q.e.d.

Corollary 1. Under the null hypothesis $\mu = 0$,

$$x^{'}\Sigma_g x \;=\; x_1^{'}\Sigma_{11}^{-1}x_1 \sim \chi_r(\theta), \tag{28}$$

where θ is the noncentrality parameter (NCP), which is **zero** under the null, and $x_2 = P_{21}P_{11}^{-1}x_1 = \Sigma_{21}\Sigma_{11}^{-1}x_1$.

Proof: The result is obvious from the conclusion of the Lemma.

The next obvious question concerns the alternative hypothesis. Is the appropriate alternative $\mu \neq 0$, or $\mu_1 \neq 0$. One may interpret the preponderance of the literature cited above as indicating that the proper alternative is the former, with μ **an arbitrary vector in the admissible parameter space.** The following, however, produces quite a different insight.

Corollary 2. Under the alternative hypothesis, there is a difference between the NCP produced by the left and right members of the test statistic of the Lemma, Eq. (28), if $\mu \neq 0$ is **an arbitrary vector**. Specifically, the NCP q and q_1 are given,[2] respectively, by

$$\begin{aligned} q_1 &= \mu_1^{'}\Sigma_{11}^{-1}\mu_1, \quad \hat{\mu}_2 = P_{21}P_{11}^{-1}\mu_1, \\ q &= \mu^{'}\Sigma_g\mu \;=\; q_1 + (\hat{\mu}_2 - \mu_2)^{'}(P_{21}\Lambda_1^{-1}P_{21}^{'}(\hat{\mu}_2 - \mu_2) \\ &\quad -2(\hat{\mu}_2 - \mu_2)^{'}(P_{21}\Lambda_1^{-1}P_{11}^{-1})\mu_1 + 2q_1 \end{aligned} \tag{29}$$

and the two coincide if and only if $\mu_2 = \hat{\mu}_2$.

Proof: Evidently, $q = q_1$ if and only if the two remaining terms in the right member of the second equation above (excluding q_1) sum to zero, for arbitrary vectors $\mu \neq 0$. The first term is invariably nonnegative and is null if and only if $(\hat{\mu}_2 - \mu_2)$ is in the null space of $P_{21}\Lambda_1^{-1}P_{21}^{'}$; this is generally ruled out because the mean and covariance parameters of the multivariate normal are independent. That the two terms do not cancel each other for arbitrary parameter μ is clear, since whenever the second term is negative it can be made positive through the replacement of μ_1 by $-\mu_1$. Thus, the two terms must be separately null, which implies $q = q_1$ if and only if $\mu_2 = \hat{\mu}_2$.

[2] Note that for notational convenience we define the q 's to be **twice the corresponding NCP**.

q.e.d.

Remark 1. Since the test statistic **under the null** is the same **on both sides** of Eq. (28) it is certainly odd, to say the least, that under the general alternative $\mu \neq 0$ we get two different distributions depending on which side we contemplate! Note further that, under the null, we have the **homogeneous** linear dependency, $x_2 = \Sigma_{21}\Sigma_{11}^{-1}x_1$. If, under the alternative, we allow **an independent specfication of** μ_1 **and** μ_2 the relationship **should become inhomogeneous**, i.e., we should have $x_2 - \mu_2 = \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$. Thus, what is suggested by Corollary 2 is that under the null we think solely of the statistic $x_1'\Sigma_{11}^{-1}x_1$ and we consider that the alternative is simply $\mu_1 \neq 0$. This further suggests that the only **subset** on which an independent test of parametric structrure can take place is given by μ_1, i.e., that the null is $\mu = 0$, and the alternative is $\mu_1 \neq 0$, it being understood that $\mu_2 = 0$ is a **maintained hypothesis.** Finally, the Lemma and the Corollaries indicate that the choice of the maintained hypothesis **may not be unique**, depending on the number of **nonsingular principal submatrices of dimension** r contained in the covariance matrix Σ.

3 The LMT for Prior Restrictions

It is shown in Dhrymes (1989), (1991) that the LMT (Lagrange Multiplier Test) test statistic for all prior restrictions is given by

$$\phi = T\tilde{\lambda}_{\cdot i}'(\tilde{J}_{22}^{-1} - I)_g\tilde{\lambda}_{\cdot i}, \tag{30}$$

which under the null hypothesis, $H_0 : L_i^{*\prime}a_{\cdot i} = 0$, is chi-squared with $G - G_i - m_i$ degrees of freedom.

In view of the previous discussion, it would not be appropriate to interpret rejection as **encompassing the alternative of lack of identification.** Rather it is to be interpreted that the investigator, not wishing to specify **which** (m) **of the prior restrictions** are to be taken as a maintained hypothesis, subjects **all** prior restrictions with the understanding that rejection **only means** that **at least** m of the restrictions hold and others may well not hold, i.e., it is a test for all the imposed restrictions **as against the alternative of just identification.**

An alternative procedure is that the investigator having determined that m of the restrictions are not to be subject to test, (presumbalby because they are manifestly true?) **selects** F_{s_i} to be an appropriate permutation of $s_i = G - G_i - m_i$ of the columns of $I_{G+m-G_i-m_i}$; the set, $F_{s_i}' L^{*\prime} a_{\cdot i}$ represents the **particular subset** of restrictions subject to test. It is also shown in Dhrymes (1989), (1991) that

$$\phi_1 = T(F_{s_i}' L^{*\prime} a_{\cdot i})' \left[F_{s_i}' (\tilde{J}_{22}^{-1} - I) F_{s_i} \right]^{-1} (F_{s_i}' L_i^{*\prime} a_{\cdot i}) \quad (31)$$

is the test statistic for testing the null hypothesis that **all overidentifying restrictions hold**. The preceding discussion shows that $\phi_1 = \phi$, so that in Eq. (31) we have a test on the overidentifying restrictions **alone**, only because we are asserting that (certain) m restrictions hold as **a maintained hypothesis**. The difference between the two tests is only hypothetical. In the first test statistic (ϕ) **any** of the $G - G_i - m_i$ principal submatrices of $\tilde{J}_{22}^{-1} - I$ that are **nonsingular** can serve as the Σ_{11} matrix of Lemma 1; in the second test statistic (ϕ_1) the investigator chooses **a specific submatrix.** Since the generalized inverse Σ_g is **unique** both statistics have the same properties and numerical values!

The import of this discussion is that **only the overidentifying restrictions can be tested unambiguously**; all other results beyond this are simply obtained by convention!

4 Conclusion

In this paper we have attempted to elucidate a number of results in the literature regarding tests of the validity of prior restrictions. In several previous studies it was found by many authors that certain test results were invariant to the choice of restrictions subjected to formal test. This issue has been clarified by Lemma 1 of this paper, which shows that this is a consequence of the singularity imposed on the distribution of test statistics designed to test the validity of a priori restrictions. This aspect has also been illustrated in the LMT framework for testing directly the prior restrictions of the GLSEM. We have shown that unambiguous tests are available for overidentifying restrictions alone, owing to the **singularity** of the covariance matrix of the limiting distribution of the relevant Lagrange multipliers. In the absence of prior information regarding the (hierarchical) truth value of the various restrictions, rejection of the null questions the validity of a subset of **prior restrictions**, but **does not raise the question of identifiability**, i.e., the alternative can be interpreted at its worst as **just identification.**

If a hierarchical structure can be imposed, so that m restrictions are held to be true as **a maintained hypothesis, owing presumably to their higher "truth value"**, rejection of the null on the overidentifying restrictions does not cast doubt on the identification of the equation in question and, moreover, multiple comparison tests may aid us in determining which of the restrictions may be responsible for rejection.

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